

Existence of Large-Data Global Weak Solutions to Kinetic Models of Nonhomogeneous Dilute Polymeric Fluids

Chuhui He and Endre Süli

Mathematical Institute, University of Oxford,
Woodstock Road, Oxford OX2 6GG, UK

Abstract

We prove the existence of large-data global-in-time weak solutions to a general class of coupled bead-spring chain models with finitely extensible nonlinear elastic (FENE) type spring potentials for nonhomogeneous incompressible dilute polymeric fluids in a bounded domain in \mathbb{R}^d , $d = 2$ or 3 . The class of models under consideration involves the Navier–Stokes system with variable density, where the viscosity coefficient depends on both the density and the polymer number density, coupled to a Fokker–Planck equation with a density-dependent drag coefficient. The proof is based on combining a truncation of the probability density function with a two-stage Galerkin approximation and weak compactness and compensated compactness techniques to pass to the limits in the sequence of Galerkin approximations and in the truncation level.

1 Introduction

The aim of this paper is to prove the existence of global-in-time large-data weak solutions to a Navier–Stokes–Fokker–Planck system that arises in models of nonhomogeneous dilute polymeric fluids. The paper extends the results presented in [4] to a more general class of models by using a different proof, which is also considerably simpler than the original proof presented in [4]. We assume that the solvent, occupying a bounded open Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3 , with boundary $\partial\Omega$, is an incompressible, viscous, isothermal Newtonian fluid with variable density ρ and dynamic viscosity $\mu = \mu(\rho, \varrho)$, where ϱ is the variable polymer number density. Let $Q := \Omega \times (0, T)$ denote the space-time domain under consideration, where $T \in \mathbb{R}_{>0}$ is fixed. We consider the following system of equations:

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\mathbf{v}\rho) = 0 \quad \text{in } Q, \quad (1.1)$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla_x \cdot (\rho\mathbf{v} \otimes \mathbf{v}) - \nabla_x \cdot (\mu(\rho, \varrho)D(\mathbf{v})) + \nabla_x p = \rho\mathbf{f} + \nabla_x \cdot \boldsymbol{\tau} \quad \text{in } Q, \quad (1.2)$$

$$\nabla_x \cdot \mathbf{v} = 0 \quad \text{in } Q, \quad (1.3)$$

subject to the initial conditions

$$\begin{aligned} \rho(\cdot, 0) &= \rho_0(\cdot) & \text{in } \Omega, \\ (\rho\mathbf{v})(\cdot, 0) &= (\rho_0\mathbf{v}_0)(\cdot) & \text{in } \Omega, \end{aligned} \quad (1.4)$$

and the no-slip boundary condition

$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, T). \quad (1.5)$$

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It is assumed that each of the equations above has been written in a nondimensional form; $\rho : Q \rightarrow \mathbb{R}$ denotes the solvent density, $\mathbf{v} : Q \rightarrow \mathbb{R}^d$ denotes the solvent velocity, $\tau : Q \rightarrow \mathbb{R}^{d \times d}$ denotes the elastic extra-stress tensor (i.e., the polymeric part of the Cauchy stress tensor), $\mathbf{f} : Q \rightarrow \mathbb{R}^d$ represents the density of the external body forces, and $p : Q \rightarrow \mathbb{R}$ denotes the pressure. Here, $D(\mathbf{v}) := \frac{1}{2}(\nabla_x \mathbf{v} + (\nabla_x \mathbf{v})^T)$ is the symmetric part of the velocity gradient.

In a bead-spring chain model for dilute polymers, consisting of $K + 1$ beads coupled with K elastic springs to represent a polymer chain, the elastic extra-stress tensor τ is defined by a version of the Kramers expression, depending on the probability density function ψ of the (random) conformation $\mathbf{q} := ((\mathbf{q}^1)^T, \dots, (\mathbf{q}^K)^T)^T \in \mathbb{R}^{d \times K}$ of the chain, where the column vector $\mathbf{q}^j := (q_1^j, \dots, q_d^j)^T$ denotes the conformation vector of the j -th spring in the bead-spring chain. Let $D := D^1 \times \dots \times D^K \subset \mathbb{R}^{d \times K}$ be the domain of admissible conformation vectors. Typically D^j is the whole space \mathbb{R}^d or a bounded open ball centred at the origin $\mathbf{0}$ in \mathbb{R}^d , for each $j = 1, \dots, K$. When $K = 1$, the model is referred to as the dumbbell model. We focus on the finitely extensible nonlinear elastic (FENE) type models where $D^j = B(0, b_j^{\frac{1}{2}})$, a ball centred at the origin $\mathbf{0}$ in \mathbb{R}^d and of radius $b_j^{\frac{1}{2}}$, with $b_j > 0$ for each $j \in \{1, \dots, K\}$. The j -th spring in the bead-spring chain is assigned a (sufficiently smooth) spring potential function $U^j : [0, \frac{b_j}{2}) \rightarrow [0, \infty)$, such that $U^j(0) = 0$, $\lim_{s \rightarrow (b_j/2)^-} U^j(s) = +\infty$, $j = 1, \dots, K$.

Example 1.1. A typical example is the classical FENE potential introduced by Warner [19], defined by

$$U_j(s) = -\frac{1}{2}Hb_j \log \left(1 - \frac{2s}{b_j} \right), \quad s \in [0, \frac{b_j}{2}), \quad j = 1, \dots, K,$$

where $H > 0$ is the spring constant and $b_j^{\frac{1}{2}}$ is a strict upper bound on the length of the j -th spring in the bead-spring chain.

The polymeric extra-stress tensor τ is defined by the formula:

$$\tau(x, t) := k \left(\sum_{j=1}^K \int_D \psi(x, \mathbf{q}, t) \mathbf{q}^j \mathbf{q}^{jT} (U^j)' \left(\frac{1}{2} |\mathbf{q}^j|^2 \right) d\mathbf{q} - K \varrho(x, t) I \right), \quad (1.6)$$

with I denoting the $d \times d$ identity matrix, $d\mathbf{q} := d\mathbf{q}^1 \dots d\mathbf{q}^K$, and the density of polymer chains (referred to as *polymer number density*) located at x at time t given by

$$\varrho(x, t) := \int_D \psi(x, \mathbf{q}, t) d\mathbf{q}.$$

The (normalized) partial Maxwellian $M^j : D^j \rightarrow [0, \infty)$ is given by

$$M^j(\mathbf{q}^j) = \frac{1}{Z^j} e^{-U^j(\frac{1}{2}|\mathbf{q}^j|^2)}, \quad \text{where } Z^j := \int_{D^j} e^{-U^j(\frac{1}{2}|\mathbf{q}^j|^2)} d\mathbf{q}^j,$$

where $d\mathbf{q}^j := dq_1^j \dots dq_d^j$, $j = 1, \dots, K$. Then we define the full Maxwellian $M : D \rightarrow [0, \infty)$ in the model by

$$M(\mathbf{q}) := \prod_{j=1}^K M^j(\mathbf{q}^j) \quad \forall \mathbf{q} := (\mathbf{q}^1, \dots, \mathbf{q}^K) \in D := D^1 \times \dots \times D^K.$$

Observe that, for $\mathbf{q} \in D$ and $j = 1, \dots, K$,

$$M(\mathbf{q}) \nabla_{\mathbf{q}^j} (M(\mathbf{q}))^{-1} = -(M(\mathbf{q}))^{-1} \nabla_{\mathbf{q}^j} M(\mathbf{q}) = \nabla_{\mathbf{q}^j} U^j \left(\frac{1}{2} |\mathbf{q}^j|^2 \right) = (U^j)' \left(\frac{1}{2} |\mathbf{q}^j|^2 \right) \mathbf{q}^j,$$

and, by definition,

$$\int_D M(\mathbf{q}) d\mathbf{q} = 1.$$

The probability density function ψ satisfies the following Fokker–Planck equation:

$$\begin{aligned} \frac{\partial \psi}{\partial t} + \nabla_x \cdot (\mathbf{v}\psi) + \sum_{j=1}^K \nabla_{\mathbf{q}^j} \cdot ((\nabla_x \mathbf{v}) \mathbf{q}^j \psi) \\ = \varepsilon \Delta_x \left(\frac{\psi}{\zeta(\rho)} \right) + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{\mathbf{q}^j} \cdot \left(M \nabla_{\mathbf{q}^i} \left(\frac{\psi}{\zeta(\rho)M} \right) \right), \end{aligned} \quad (1.7)$$

in $\mathcal{O} \times (0, T)$, with $\mathcal{O} := \Omega \times D$. In the above equation $\zeta(\cdot) \in \mathbb{R}_{>0}$ is a density-dependent scaled drag coefficient. Let $\partial \bar{D}^j := D^1 \times \dots \times D^{j-1} \times \partial D^j \times D^{j+1} \times \dots \times D^K$. We impose the following boundary and initial conditions, for all $j = 1, \dots, K$:

$$\left[\frac{1}{4\lambda} \sum_{i=1}^K A_{ij} M \nabla_{\mathbf{q}^i} \left(\frac{\psi}{\zeta(\rho)M} \right) - (\nabla_x \mathbf{v}) \mathbf{q}^j \psi \right] \cdot \frac{\mathbf{q}^j}{|\mathbf{q}^j|} = 0 \quad \text{on } \Omega \times \partial \bar{D}^j \times (0, T), \quad (1.8)$$

$$\varepsilon \nabla_x \left(\frac{\psi}{\zeta(\rho)} \right) \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega \times D \times (0, T), \quad (1.9)$$

$$\psi(x, \mathbf{q}, 0) = \psi_0(x, \mathbf{q}) \quad \text{in } \Omega \times D, \quad (1.10)$$

where \mathbf{n} is the unit outward normal to $\partial \Omega$. In (1.6), the dimensionless constant $k > 0$ is a constant multiple of the product of the Boltzmann constant k_B and the absolute temperature κ . In (1.7), the centre-of-mass diffusion coefficient $\varepsilon > 0$ is defined as $\varepsilon := (l_0/L_0)^2/(4(K+1)\lambda)$ with $l_0 := \sqrt{k_B \kappa / H}$ signifying the characteristic microscopic length-scale and $\lambda := (\zeta_0/4H)(U_0/L_0)$, where $\zeta_0 > 0$ is a characteristic drag coefficient and $H > 0$ is a spring-constant. The dimensionless positive parameter λ is called the Deborah number, which characterizes the elastic relaxation property of the fluid. The constant matrix $A = (A_{ij})_{i,j=1}^K$, called the Rouse matrix, is symmetric and positive definite. Furthermore, we associate with A the linear mapping $\mathbb{A} : \mathbb{R}^{d \times K} \rightarrow \mathbb{R}^{d \times K}$ defined, for any $B = (B_i^j)_{i=1, \dots, d}^{j=1, \dots, K} \in \mathbb{R}^{d \times K}$, by $(\mathbb{A}(B))_i^j := \sum_{k=1}^K B_i^k A_{kj}$, and let $\mathbb{A}^j : \mathbb{R}^{d \times K} \rightarrow \mathbb{R}^d$ be the linear mapping defined by $(\mathbb{A}^j(B))_i := (\mathbb{A}(B))_i^j$, for $i = 1, \dots, d$ and $j = 1, \dots, K$. By the positive definiteness of the Rouse matrix $A \in \mathbb{R}_{sym}^{K \times K}$, there exist positive constants C_1 and C_2 such that

$$C_1 |B|^2 \leq \mathbb{A}(B) : B \leq C_2 |B|^2 \quad \forall B \in \mathbb{R}^{d \times K}. \quad (1.11)$$

For simplicity, we set $\varepsilon = 1$ and $\lambda = 1/4$ in (1.7) since the particular choices of constants will not affect the results.

Before embarking on the proof of our main result we provide a brief literature survey. Unless otherwise stated, there is no centre-of-mass diffusion term in the model in the cited reference, i.e., $\varepsilon = 0$; also, unless otherwise stated, the density ρ is assumed to be constant; and all cited references only consider a simple dumbbell model ($K = 1$) with a pair of beads connected by one single spring, rather than a bead-spring chain model.

In [17], Renardy proved a local existence and uniqueness result for a family of Hookean-type dumbbell models. Subsequently, E, Li & Zhang [9] and Li, Zhang & Zhang [13] revisited the question of local existence of solutions for dumbbell models, while Zhang and Zhang [20] showed a local existence and uniqueness result for regular solutions to FENE-type dumbbell models. All of these papers required high regularity of the initial data. In [7], Constantin considered the Navier–Stokes equations coupled to nonlinear Fokker–Planck equations modelling the probability distribution of particles interacting with the fluid, and in [8] Constantin and Seregin proved the global regularity of solutions of the incompressible Navier–Stokes–Fokker–Planck system in \mathbb{R}^2 , in the absence of boundaries.

In [14], Lions and Masmoudi proved the global existence of weak solutions for the corotational FENE dumbbell model and the Doi model (also called the rod model) using a propagation-of-compactness argument, i.e., the property that if one takes a sequence of weak solutions, which converges weakly and

such that the initial data converge strongly, then the weak limit is also a solution. In [15], Masmoudi explored the FENE dumbbell model for a general class of potentials; he proved local existence in Sobolev spaces, global existence if the initial data are close to equilibrium, and global existence in two dimensions for the corotational FENE model.

In [1], Barrett, Schwab & Süli showed the existence of global-in-time weak solutions to the coupled microscopic-macroscopic bead-spring chain model. The paper admitted a large class of potentials U , including the Hookean dumbbell model and general FENE-type dumbbell models in the general non-corotational case, however the velocity field \mathbf{v} in the drift-term of the Fokker–Planck equation and the extra stress tensor had to be mollified.

Subsequently, in [2] and [3] Barrett and Süli managed to prove the existence of global-in-time weak solutions to general noncorotational Hookean-type bead-spring chain models and FENE-type bead-spring chain models respectively, with centre-of-mass diffusion $\varepsilon > 0$, but without mollification and in the general case of $K \geq 1$ coupled beads in the bead-spring chain. This was achieved by introducing a cut-off parameter L , discretizing the resulting model in time, and then passing to the limit as $L \rightarrow \infty$ by requiring that the time step $\Delta t = o(L^{-1})$. The papers also provided rigorous proofs, for both FENE-type and Hookean-type models, of the convergence of weak solutions to their respective equilibria: $\mathbf{v}_\infty = \mathbf{0}$ and $\psi_\infty = M$, as $t \rightarrow \infty$. A key contribution to the field has been Masmoudi’s paper [16], which proved global existence of weak solutions to the FENE dumbbell model, in the absence of a centre-of-mass diffusion term.

All of the papers cited above were concerned with homogeneous fluids (i.e., fluids with constant density). In this paper, we study a model similar to the one in [4]; there, the existence of global-in-time weak solutions to FENE-type bead-spring chain models with variable density ρ , density-dependent dynamic viscosity $\mu(\rho)$ and density-dependent drag coefficient $\zeta(\rho)$ was shown in a bounded domain in \mathbb{R}^d , $d = 2, 3$. Here, in contrast with [4], we permit dependence of the dynamic viscosity on both the density and the polymer number density, i.e., $\mu = \mu(\rho, \varrho)$; this simple extension of the model considered in [4] introduces nontrivial technical difficulties. Thus, instead of using a sequence of approximating problems based on time-discretization as in [4], here, motivated by the approach in [6], we perform a Galerkin semi-discretization on the spatial domains. This approach shortens and simplifies the proof of existence of weak solutions compared to the proof in [4]; and, by admitting the dependence of the dynamic viscosity on both the density and the polymer number density, it also generalises the results of [4] to a wider and physically more realistic class of models.

The paper is structured as follows. In the next section, we shall derive a formal energy identity, which is at the heart of our proof. We shall introduce the necessary notation and the relevant function spaces, and we state the assumptions on the data. In Section 3, we formulate the main result; the rest of the paper is devoted to the proof of our main result, which guarantees the existence of large-data global weak solutions to the model under consideration.

In Section 4, we begin our proof by first introducing a truncation of the extra-stress tensor with truncation parameter ℓ in Subsection 4.1. We also truncate the probability density function correspondingly in the Fokker–Planck equation to maintain the energy identity. The initial probability density is also truncated for boundedness. In Subsection 4.2, we perform a spatial Galerkin semi-discretization of the velocity field and the probability density function with parameters n and m . Given a sufficiently smooth velocity field, i.e., $\mathbf{v} \in L^1(0, T; W^{1,1}(\Omega; \mathbb{R}^d))$, the proofs of existence and uniqueness of the weak solution to the transport equation (1.1) can be found in [5], for example. By rewriting the truncated Navier–Stokes equation and the truncated Fokker–Planck equation in non-conservative form, we then use Schauder’s fixed point theorem to prove the existence of solutions to our partially Galerkin discretized system in Subsection 4.3. In Subsection 4.4, we study the limit $n \rightarrow \infty$ by deriving uniform a priori estimates independent of the parameter n . In Subsection 4.5, we first prove the boundedness of the sequence of approximate densities ρ^m and the nonnegativity of the Galerkin approximations $\hat{\psi}^m$. Then we derive an m -independent a priori estimate. We use a Nikolskiĭ norm estimate and the Aubin–Lions Lemma to deduce strong convergence of $(\mathbf{v}^m)_{m=1}^\infty$. To deduce the strong convergence of $(\hat{\psi}^m)_{m=1}^\infty$, we first apply the Dunford–Pettis theorem to deduce its weak convergence in $L_{loc}^1(\mathcal{O} \times (0, T))$. Then we use the

Div-Curl lemma to show that $(\hat{\psi}^m)_{m=1}^\infty$ converges almost everywhere in $\mathcal{O} \times (0, T)$, followed by Vitali's convergence theorem, which gives the strong convergence of the sequence $(\hat{\psi}^m)_{m=1}^\infty$ in $L^1(0, T; L^1(\mathcal{O}))$ as $m \rightarrow \infty$. Finally, in Subsection 4.6, we derive ℓ -independent estimates and apply similar techniques as in Subsection 4.5 to pass to the limit as $\ell \rightarrow \infty$. Finally, we show the weak attainment of the initial conditions.

2 Notational conventions, and assumptions on the data

In this section, we shall first introduce the necessary notational conventions used throughout the paper, and then introduce our assumptions on the data under which the existence result is subsequently proved.

First we shall summarise the definitions of Lebesgue spaces, Sobolev spaces and Bochner spaces. Let O be a measurable set in \mathbb{R}^d and $p \in [1, \infty)$. The standard Lebesgue space of p -integrable functions is denoted by $(L^p(O), \|\cdot\|_{L^p(O)})$. When $p = \infty$, $(L^\infty(O), \|\cdot\|_{L^\infty(O)})$ denotes the space of essentially bounded functions. For $s \in \mathbb{N}$, let $(W^{s,p}(O), \|\cdot\|_{W^{s,p}(O)})$ be the standard Sobolev spaces and denote by $|\cdot|_{W^{s,p}(O)}$ the corresponding semi-norm. We define the weighted Lebesgue and Sobolev spaces by

$$L_N^p(O) := \{f \in L_{loc}^p(O) : \int_O N(z)|f(z)|^p dz < \infty\},$$

$$W_N^{1,p}(O) := \{f \in W_{loc}^{1,p}(O) : \int_O N(z)(|f(z)|^p + |\nabla_z f(z)|^p) dz < \infty\},$$

for a nonnegative weight-function $N \in L_{loc}^\infty(O)$. For any pair of functions f, g , with $f \in L^p(O)$ and $g \in L^{p'}(O)$, where $1/p + 1/p' = 1$ and $p, p' \in [1, \infty]$, we set

$$(f, g)_O := \int_O f(z)g(z) dz.$$

Note that we set $1' := \infty$ and $\infty' := 1$. If $O = \Omega$, we omit the subscript Ω from the inner product $(f, g)_\Omega$ for simplicity. The notation for vector-valued and tensor-valued functions is analogous. For a general Banach space $(X, \|\cdot\|_X)$, the dual space consisting of all continuous linear functionals on X is denoted by X' and the dual pairing is denoted by $\langle f, g \rangle_X$ if $f \in X'$ and $g \in X$. Similarly, we shall write $\langle \cdot, \cdot \rangle_X$ to denote the dual pairing between a locally convex topological vector space X and its dual space X' . The subscript $_X$ will be omitted when the choice of X and its dual X' is clear from the context. For the Sobolev space $W^{1,p}(O)$ where $1 < p < \infty$, we denote its dual space by $(W^{1,p}(O))'$. We define the negative order Sobolev space $W^{-1,1}(O)$ by $W^{-1,1}(O) := \{\operatorname{div} \mathbf{f} : \mathbf{f} \in L^1(O; \mathbb{R}^d)\}$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain, with $d \in \{2, 3\}$. $C(\bar{\Omega})$ denotes the set of all continuous real-valued functions on $\bar{\Omega}$. Let $C^\infty(\Omega)$ be the set of all smooth functions on Ω and let $C_0^\infty(\Omega)$ be the set of all functions in $C^\infty(\Omega)$ that are compactly supported in Ω . Then we define for $p \in (1, \infty)$ the following function spaces:

$$W_0^{1,p}(\Omega; \mathbb{R}^d) := \overline{C_0^\infty(\Omega; \mathbb{R}^d)}^{\|\cdot\|_{W^{1,p}(\Omega; \mathbb{R}^d)}},$$

$$W_{0,\operatorname{div}}^{1,p}(\Omega; \mathbb{R}^d) := \overline{\{\mathbf{v} \in C_0^\infty(\Omega; \mathbb{R}^d) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\}}^{\|\cdot\|_{W^{1,p}(\Omega; \mathbb{R}^d)}},$$

$$L_{0,\operatorname{div}}^2(\Omega; \mathbb{R}^d) := \overline{W_{0,\operatorname{div}}^{1,2}(\Omega; \mathbb{R}^d)}^{\|\cdot\|_{L^2(\Omega; \mathbb{R}^d)}},$$

$$W^{-1,p'}(\Omega; \mathbb{R}^d) := (W_0^{1,p}(\Omega; \mathbb{R}^d))',$$

$$W_{\operatorname{div}}^{-1,p'}(\Omega; \mathbb{R}^d) := (W_{0,\operatorname{div}}^{1,p}(\Omega; \mathbb{R}^d))'.$$

We for $p \in [1, \infty]$ denote by $L^p(0, T; X)$ the standard Bochner space of p -integrable for $p \in [1, \infty)$, and essentially bounded, when $p = \infty$, X -valued functions defined on the interval $(0, T)$. If X is separable and reflexive and $p \in (1, \infty)$, then $L^p(0, T; X)$ is separable and reflexive and $(L^p(0, T; X))' = L^{p'}(0, T; X')$.

Next, we recall some inequalities which shall be frequently used in the proof. Let $r \in [2, \infty)$ if $d = 2$, and $r \in [2, 6]$ if $d = 3$ and $\theta = d(\frac{1}{2} - \frac{1}{r})$. Then, we have the following Gagliardo–Nirenberg inequality:

$$\|f\|_{L^r(\Omega)} \leq C \|f\|_{L^2(\Omega)}^{1-\theta} \|f\|_{W^{1,2}(\Omega)}^\theta \quad \forall f \in W^{1,2}(\Omega), \quad (2.1)$$

where $C > 0$ is a constant that may depend on Ω, r, d . We also recall the following version of Korn's inequality: for all $\mathbf{w} \in W_0^{1,2}(\Omega; \mathbb{R}^d)$, we have

$$\int_{\Omega} |D(\mathbf{w})|^2 dx \geq c_0 \|\mathbf{w}\|_{W^{1,2}(\Omega; \mathbb{R}^d)}^2, \quad (2.2)$$

where $c_0 > 0$ is a constant.

Next, we need to make a few assumptions. First we introduce the notation:

$$\hat{\psi} := \frac{\psi}{\zeta(\rho)M}, \quad \hat{\psi}_0 := \frac{\psi_0}{\zeta(\rho_0)M}.$$

We assume that $\partial\Omega \in C^{0,1}$. For the Maxwellian M we assume that

$$M \in C_0(\overline{D}) \cap C_{loc}^{0,1}(D) \cap W_0^{1,1}(D). \quad (2.3)$$

For the initial density ρ_0 we assume that

$$\rho_0 \in [\rho_{\min}, \rho_{\max}], \quad \text{with } \rho_{\min} > 0. \quad (2.4)$$

For the initial velocity \mathbf{v}_0 , we assume that

$$\mathbf{v}_0 \in L_{0,\text{div}}^2(\Omega; \mathbb{R}^d). \quad (2.5)$$

For the initial probability density ψ_0 , we assume that

$$\begin{aligned} \psi_0 &\geq 0 \quad \text{a.e. on } \Omega \times D, \quad \hat{\psi}_0 \log \hat{\psi}_0 \in L_M^1(\Omega \times D), \\ 0 \leq \varrho_0(x) &:= \int_D \psi_0(\cdot, \mathbf{q}) d\mathbf{q} \leq \varrho_{\max} \quad \text{a.e. on } \Omega, \quad \int_{\Omega \times D} \psi_0(x, \mathbf{q}) d\mathbf{q} dx = 1. \end{aligned} \quad (2.6)$$

We shall further assume that

$$\mu \in C^1([\rho_{\min}, \rho_{\max}] \times [0, \infty), [\mu_{\min}, \mu_{\max}]), \quad \zeta \in C^1([\rho_{\min}, \rho_{\max}], [\zeta_{\min}, \zeta_{\max}]), \quad \text{with } \mu_{\min}, \zeta_{\min} > 0. \quad (2.7)$$

Finally, we assume that

$$\mathbf{f} \in L^2(0, T; L^2(\Omega; \mathbb{R}^d)). \quad (2.8)$$

3 The main result

In this section, we first formulate the notion of weak solutions to the coupled Navier–Stokes–Fokker–Planck system introduced in Section 1.

Definition 3.1. *We say that the triple (ρ, \mathbf{v}, ψ) , with $\psi(x, \mathbf{q}, t) = \zeta(\rho(x, t))M(\mathbf{q})\hat{\psi}(x, \mathbf{q}, t)$, is a weak solution to the system of nonlinear partial differential equations (1.1)–(1.5), (1.7)–(1.10), if the following conditions hold:*

(i) *The functions $(\rho, \mathbf{v}, \hat{\psi})$ belong to the following function spaces:*

$$\begin{aligned} \rho &\in L^\infty(\Omega \times (0, T)) \cap C([0, T]; L^p(\Omega)), \quad \text{where } p \in [1, \infty), \\ \mathbf{v} &\in L^\infty(0, T; L_{0,\text{div}}^2(\Omega; \mathbb{R}^d)) \cap L^2(0, T; W_{0,\text{div}}^{1,2}(\Omega; \mathbb{R}^d)), \\ \hat{\psi} &\in L^\infty(\Omega \times (0, T); L_M^1(D)) \cap L^2(0, T; W_M^{1,1}(\mathcal{O})), \quad \hat{\psi} \geq 0 \text{ a.e. in } \mathcal{O} \times (0, T), \end{aligned}$$

where $\mathcal{O} := \Omega \times D$.

(ii) The system (1.1)–(1.5), (1.7)–(1.10) is satisfied in the following sense:

$$\int_0^T [\langle \partial_t \rho, \eta \rangle - (\mathbf{v} \rho, \nabla_x \eta)] dt = 0, \quad \text{for all } \eta \in L^1(0, T; W^{1, \frac{q}{q-1}}(\Omega)), \quad (3.1)$$

where $q \in (2, \infty)$ when $d = 2$ and $q \in [3, 6]$ when $d = 3$,

$$\begin{aligned} & \int_0^T \langle \partial_t(\rho \mathbf{v}), \mathbf{w} \rangle dt + \int_0^T [-(\rho \mathbf{v} \otimes \mathbf{v}, \nabla_x \mathbf{w}) + (\mu(\rho, \varrho) D(\mathbf{v}), \nabla_x \mathbf{w})] dt \\ &= \int_0^T [-(\tau, \nabla_x \mathbf{w}) + (\rho \mathbf{f}, \mathbf{w})] dt, \quad \text{for all } \mathbf{w} \in L^s(0, T; W_{0, \text{div}}^{1, s}(\Omega; \mathbb{R}^d)) \text{ with } s > 2, \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} & \int_0^T \left\langle \partial_t(M\zeta(\rho)\hat{\psi}), \varphi \right\rangle_{\mathcal{O}} - \left(M\zeta(\rho)\mathbf{v}\hat{\psi}, \nabla_x \varphi \right)_{\mathcal{O}} - \left(M\zeta(\rho)\hat{\psi}(\nabla_x \mathbf{v})\mathbf{q}, \nabla_{\mathbf{q}} \varphi \right)_{\mathcal{O}} dt \\ &+ \int_0^T (M\nabla_x \hat{\psi}, \nabla_x \varphi)_{\mathcal{O}} + \left(M\mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}), \nabla_{\mathbf{q}} \varphi \right)_{\mathcal{O}} dt = 0, \quad \text{for all } \varphi \in L^\infty(0, T; W^{1, \infty}(\mathcal{O})), \end{aligned} \quad (3.3)$$

where the polymer number density ϱ is given by

$$\varrho(x, t) := \zeta(\rho) \int_D M(\mathbf{q}) \hat{\psi}(x, \mathbf{q}, t) d\mathbf{q} \quad \text{for a.e. } (x, t) \in \Omega \times (0, T) \quad (3.4)$$

and the extra-stress tensor τ is given by

$$\tau(x, t) := k \sum_{j=1}^K \int_D M\zeta(\rho) \nabla_{\mathbf{q}^j} \hat{\psi}(x, \mathbf{q}, t) \otimes \mathbf{q}^j d\mathbf{q} \quad \text{for a.e. } (x, t) \in \Omega \times (0, T). \quad (3.5)$$

(iii) The following weak continuity properties hold:

$$\begin{aligned} & t \mapsto \int_\Omega \rho(x, t) \mathbf{v}(x, t) \cdot \mathbf{u} dx \in C([0, T]) \quad \text{for any } \mathbf{u} \in W_{0, \text{div}}^{1, s}(\Omega; \mathbb{R}^d), \\ & t \mapsto \int_{\mathcal{O}} M(\mathbf{q}) \zeta(\rho(x, t)) (t) \hat{\psi}(x, \mathbf{q}, t) \phi(x, \mathbf{q}) dx d\mathbf{q} \in C([0, T]) \quad \text{for any } \phi \in W^{1, \infty}(\mathcal{O}), \end{aligned} \quad (3.6)$$

and the initial data are attained in the following sense:

$$\begin{aligned} & \lim_{t \rightarrow 0_+} ((\rho \mathbf{v})(t), \mathbf{u}) = (\rho_0 \mathbf{v}_0, \mathbf{u}) \quad \text{for all } \mathbf{u} \in W_{0, \text{div}}^{1, s}(\Omega; \mathbb{R}^d) \text{ where } s > 2, \\ & \lim_{t \rightarrow 0_+} (M(\zeta(\rho)\hat{\psi})(t), \phi)_{\mathcal{O}} = (M\zeta(\rho_0)\hat{\psi}_0, \phi)_{\mathcal{O}} \quad \text{for all } \phi \in W^{1, \infty}(\mathcal{O}). \end{aligned} \quad (3.7)$$

Next we state our main result, which we shall prove in the following section.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded open Lipschitz domain. Let $K \in \mathbb{N}$ be arbitrary and let $D^i \subset \mathbb{R}^d$, $i = 1, \dots, K$, be bounded open balls centred at the origin. Suppose that $\mathbf{f} \in L^2(0, T; L^2(\Omega; \mathbb{R}^d))$. Assume that the map $\mathbb{A} : B \in \mathbb{R}^{d \times K} \mapsto \mathbb{A}(B) \in \mathbb{R}^{d \times K}$ is linear and satisfies (1.11), the Maxwellian $M : D \rightarrow \mathbb{R}$ satisfies (2.3), $\mu(\cdot, \cdot)$ and $\zeta(\cdot)$ satisfy (2.7), and the initial data $(\rho_0, \mathbf{v}_0, \psi_0)$ satisfy (2.4)–(2.6). Then, there exists a triple (ρ, \mathbf{v}, ψ) which is a weak solution to the system*

(1.1)–(1.5), (1.7)–(1.10) in the sense of Definition 3.1. Moreover, for a.e. $t \in (0, T)$, the following energy inequality holds:

$$\begin{aligned} & k \int_{\mathcal{O}} M\zeta(\rho(\cdot, t))\mathcal{F}(\hat{\psi}(\cdot, t)) \, dx \, d\mathbf{q} + \frac{1}{2} \int_{\Omega} \rho(\cdot, t) |\mathbf{v}(\cdot, t)|^2 \, dx \\ & + \int_0^t \int_{\Omega} \mu(\rho, \varrho) |D(\mathbf{v})|^2 \, dx \, ds + 4kC_1 \int_0^t \int_{\mathcal{O}} M \left| \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}} \right|^2 \, dx \, d\mathbf{q} \, ds \\ & \leq k \int_{\mathcal{O}} M\zeta(\rho_0)\mathcal{F}(\hat{\psi}_0) \, dx \, d\mathbf{q} + \frac{1}{2} \int_{\Omega} \rho_0 |\mathbf{v}_0|^2 \, dx + \int_0^t (\rho \mathbf{f}, \mathbf{v}) \, ds, \end{aligned} \quad (3.8)$$

where C_1 is the same constant as in (1.11), $\mathcal{F}(s) := s \log s + 1$ for $s > 0$ and $\mathcal{F}(0) := \lim_{s \rightarrow 0^+} \mathcal{F}(s) = 1$.

We will prove this result by constructing a sequence of approximations to the problem under consideration. We shall then pass to the limits in the various approximation parameters — first in the dimensions n and m of the Galerkin subspaces for the velocity field and the probability density function, respectively, and then in the parameter ℓ in the truncation process that we shall next introduce, to deduce the convergence of the sequence of approximations to a global-in-time weak solution of the problem.

4 Proof of existence

4.1 The first level of approximation: truncation

To approximate our original Navier–Stokes–Fokker–Planck system, we begin by introducing a smooth nonnegative function $\Gamma \in C_0^\infty((-2, 2))$, such that $\Gamma(s) = 1$ for all $s \in [-1, 1]$. We define $\Gamma_\ell(s) := \Gamma(\frac{s}{\ell})$ for an arbitrary $\ell \in \mathbb{N}$. As in [6], the primitive function to Γ_ℓ is the truncation function defined by

$$T_\ell(s) := \int_0^s \Gamma_\ell(r) \, dr.$$

The ℓ -th approximation of (1.1) is given by

$$\frac{\partial \rho^\ell}{\partial t} + \nabla_x \cdot (\mathbf{v}^\ell \rho^\ell) = 0 \quad \text{in } Q, \quad (4.1)$$

subject to the following initial condition:

$$\rho^\ell(\cdot, 0) = \rho_0(\cdot) \quad \text{in } \Omega. \quad (4.2)$$

We define the ℓ -th approximation of (1.2) and (1.3) as follows:

$$\begin{aligned} \frac{\partial(\rho^\ell \mathbf{v}^\ell)}{\partial t} + \nabla_x \cdot (\rho^\ell \mathbf{v}^\ell \otimes \mathbf{v}^\ell) - \nabla_x \cdot (\mu(\rho^\ell, \varrho^\ell) D(\mathbf{v}^\ell)) + \nabla_x p^\ell &= \rho^\ell \mathbf{f} + \nabla_x \cdot \tau^\ell & \text{in } Q, \\ \nabla_x \cdot \mathbf{v}^\ell &= 0 & \text{in } Q, \end{aligned} \quad (4.3)$$

with initial and boundary conditions given by

$$\begin{aligned} \mathbf{v}^\ell(\cdot, 0) &= \mathbf{v}_0(\cdot) & \text{in } \Omega, \\ \mathbf{v}^\ell &= \mathbf{0} & \text{on } \partial\Omega \times (0, T). \end{aligned} \quad (4.4)$$

The ℓ -th approximation ϱ^ℓ of the polymer number density ϱ is defined by

$$\varrho^\ell(x, t) := \zeta(\rho^\ell(x, t)) \int_D M(\mathbf{q}) \hat{\psi}^\ell(x, \mathbf{q}, t) \, d\mathbf{q}, \quad (4.5)$$

and the ℓ -th approximation τ^ℓ of the polymeric extra stress tensor τ is given by

$$\tau^\ell(x, t) := k \sum_{j=1}^K \int_D M(\mathbf{q}) \zeta(\rho^\ell(x, t)) \nabla_{\mathbf{q}^j} T_\ell(\hat{\psi}^\ell(x, \mathbf{q}, t)) \otimes \mathbf{q}^j d\mathbf{q}, \quad (4.6)$$

where $\hat{\psi}^\ell$ is the solution of the initial-boundary-value problem (4.8)–(4.11) stated below.

By partial integration we have that

$$\tau^\ell(x, t) = -k \int_D \left[KM(\mathbf{q}) \zeta(\rho^\ell) T_\ell(\hat{\psi}^\ell(x, \mathbf{q}, t)) I + \sum_{j=1}^K \zeta(\rho^\ell) T_\ell(\hat{\psi}^\ell(x, \mathbf{q}, t)) \nabla_{\mathbf{q}^j} M(\mathbf{q}) \otimes \mathbf{q}^j \right] d\mathbf{q}, \quad (4.7)$$

where the boundary term vanishes since $M = 0$ on ∂D . We shall also modify the Fokker–Planck equation (1.7) in order to ensure that the truncated system satisfies a formal energy equality. We first set

$$\Lambda_\ell(s) := s \Gamma_\ell(s).$$

We define the ℓ -th approximation of (1.7) as follows:

$$\begin{aligned} \frac{\partial(M\zeta(\rho^\ell)\hat{\psi}^\ell)}{\partial t} + \nabla_x \cdot (M\zeta(\rho^\ell)\hat{\psi}^\ell \mathbf{v}^\ell) + \operatorname{div}_{\mathbf{q}}(M\zeta(\rho^\ell)\Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \mathbf{v}^\ell) \mathbf{q}) \\ - \Delta_x(M\hat{\psi}^\ell) - \operatorname{div}_{\mathbf{q}} \mathbb{A}(M\nabla_{\mathbf{q}} \hat{\psi}^\ell) = 0 \end{aligned} \quad (4.8)$$

on $\mathcal{O} \times (0, T)$, where $\mathcal{O} := \Omega \times D$, subject to the following boundary conditions:

$$\left[\mathbb{A}^j(M\nabla_{\mathbf{q}} \hat{\psi}^\ell) - M\zeta(\rho^\ell)\Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \mathbf{v}^\ell) \mathbf{q}^j \right] \cdot \mathbf{n}^j = 0 \quad \text{on } \Omega \times \partial \bar{D}^j \times (0, T), \quad (4.9)$$

$$M\nabla_x \hat{\psi}^\ell \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega \times D \times (0, T), \quad (4.10)$$

for all $j = 1, \dots, K$. The initial condition for $\hat{\psi}^\ell$ is truncated in the following way:

$$\hat{\psi}^\ell(x, \mathbf{q}, 0) = T_\ell(\hat{\psi}_0(x, \mathbf{q})) \quad \text{for } (x, \mathbf{q}) \in \Omega \times D. \quad (4.11)$$

For simplicity, we shall omit the superscript ℓ temporarily in the following discussions; we shall reinstate it later and will then pass to the limit $\ell \rightarrow \infty$. We need to show first, however, that this approximating problem has a solution $(\rho^\ell, \mathbf{v}^\ell, \hat{\psi}^\ell, \varrho^\ell)$ for each $\ell \geq 1$; we shall do so by constructing a two-stage Galerkin approximation to it and passing to the limits in the sequences of Galerkin approximations.

4.2 A two-stage Galerkin approximation

First, as in [6], we approximate the Maxwellian by fixing a sequence of positive functions $(\bar{M}^m)_{m \in \mathbb{N}} \subset C_0^{0,1}(\bar{D})$ such that for each compact set $\varkappa \subset D$, we have

$$\lim_{m \rightarrow \infty} \|\bar{M}^m - M\|_{C(\bar{D}) \cap W_0^{1,1}(D)} + \|(\bar{M}^m)^{-1} - M^{-1}\|_{C(\varkappa)} = 0. \quad (4.12)$$

Then, the approximate Maxwellian M^m is defined by

$$M^m := \bar{M}^m + \frac{1}{m}, \quad \text{for } m = 1, 2, \dots$$

Let $(\mathbf{f}^m)_{m=1}^\infty$ be a sequence of functions in $C([0, T]; L^2(\Omega; \mathbb{R}^d))$ converging to \mathbf{f} in $L^2(0, T; L^2(\Omega; \mathbb{R}^d))$. Let further $(\rho_0^m)_{m=1}^\infty$ be a sequence of functions in $C^1(\bar{\Omega})$ such that $\rho_0^m \in [\rho_{\min}, \rho_{\max}]$, which converges to ρ_0 in $L^1(\Omega)$; such a sequence can be constructed by extending ρ_0 from Ω to \mathbb{R}^d by 0 and convolving

the resulting function, still denoted by ρ_0 , with θ^m , where $\theta^m(x) := m^d \theta(mx)$, $\theta \in C_0^\infty(\mathbb{R}^d)$, $\theta \geq 0$, and $\int_{\mathbb{R}^d} \theta(x) dx = 1$, and observing that

$$\rho_0^m(x) - \rho_{\min} = \int_{\mathbb{R}^d} (\rho_0(x-y) - \rho_{\min}) \theta^m(y) dy \geq 0,$$

and

$$\rho_0^m(x) - \rho_{\max} = \int_{\mathbb{R}^d} (\rho_0(x-y) - \rho_{\max}) \theta^m(y) dy \leq 0.$$

Next, we shall introduce the Galerkin basis functions for the velocity field and the probability density function respectively. By the Hilbert–Schmidt theorem (see, for example, Lemma 5.1 and Lemma 5.2 in [11]), there exists a sequence $(\mathbf{w}_i)_{i=1}^\infty$ of eigenfunctions in $W_{0,\text{div}}^{1,2} \cap W^{d+1,2}(\Omega; \mathbb{R}^d)$ whose linear span is dense in $L_{0,\text{div}}^2(\Omega; \mathbb{R}^d)$. Moreover, we can choose \mathbf{w}_i , $i = 1, 2, \dots$, in a way that they are orthogonal in $W^{d+1,2}(\Omega; \mathbb{R}^d)$ and orthonormal in $L^2(\Omega; \mathbb{R}^d)$. By Sobolev embedding, it follows that $\mathbf{w}_i \in C^1(\bar{\Omega}; \mathbb{R}^d)$ for all $i = 1, 2, \dots$. Similarly, for each $m \in \mathbb{N}$, there exists a sequence $(\varphi_i^m)_{i=1}^\infty$ of eigenfunctions in $W^{(K+1)d+1,2}(\mathcal{O})$ that are orthogonal in $W_{M^m}^{1,2}(\mathcal{O})$ and orthonormal in $L_{M^m}^2(\mathcal{O})$. As $\mathcal{O} = \Omega \times D \subset \mathbb{R}^{(K+1)d}$, by Sobolev embedding we deduce that $\varphi_i^m \in C^1(\mathcal{O})$.

For fixed $m, n \in \mathbb{N}$, we seek $(\rho^{m,n}, \mathbf{v}^{m,n}, \hat{\psi}^{m,n})$, where $\mathbf{v}^{m,n}$ and $\hat{\psi}^{m,n}$ are of the form

$$\mathbf{v}^{m,n}(x, t) := \sum_{i=1}^m c_i^{m,n}(t) \mathbf{w}_i(x), \quad (4.13)$$

$$\hat{\psi}^{m,n}(x, \mathbf{q}, t) := \sum_{i=1}^n d_i^{m,n}(t) \varphi_i^m(x, \mathbf{q}), \quad (4.14)$$

that solve

$$(\partial_t \rho^{m,n}, \eta) - (\mathbf{v}^{m,n} \rho^{m,n}, \nabla_x \eta) = 0 \quad \text{for all } \eta \in C^{0,1}(\bar{\Omega}) \text{ and a.e. } t \in (0, T), \quad (4.15)$$

$$\begin{aligned} (\partial_t (\rho^{m,n} \mathbf{v}^{m,n}), \mathbf{w}_i) - (\rho^{m,n} \mathbf{v}^{m,n} \otimes \mathbf{v}^{m,n}, \nabla_x \mathbf{w}_i) + (\mu(\rho^{m,n}, \varrho^{m,n}) D(\mathbf{v}^{m,n}), \nabla_x \mathbf{w}_i) \\ = -(\tau^{m,n}, \nabla_x \mathbf{w}_i) + (\rho^{m,n} \mathbf{f}^m, \mathbf{w}_i) \quad \text{for all } i = 1, \dots, m \text{ and a.e. } t \in (0, T), \end{aligned} \quad (4.16)$$

$$\begin{aligned} \left(\partial_t (M^m \zeta(\rho^{m,n}) \hat{\psi}^{m,n}), \varphi_i^m \right)_{\mathcal{O}} - \left(M^m \zeta(\rho^{m,n}) \mathbf{v}^{m,n} \hat{\psi}^{m,n}, \nabla_x \varphi_i^m \right)_{\mathcal{O}} \\ - \left(M \zeta(\rho^{m,n}) \Lambda_\ell(\hat{\psi}^{m,n})(\nabla_x \mathbf{v}^{m,n}) \mathbf{q}, \nabla_{\mathbf{q}} \varphi_i^m \right)_{\mathcal{O}} + (M^m \nabla_x \hat{\psi}^{m,n}, \nabla_x \varphi_i^m)_{\mathcal{O}} \\ + \left(M^m \mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^{m,n}), \nabla_{\mathbf{q}} \varphi_i^m \right)_{\mathcal{O}} = 0 \quad \text{for all } i = 1, \dots, n \text{ and a.e. } t \in (0, T). \end{aligned} \quad (4.17)$$

Here $\varrho^{m,n}$ is defined by

$$\varrho^{m,n}(x, t) := \zeta(\rho^{m,n}(x, t)) \int_D M^m(\mathbf{q}) [\hat{\psi}^{m,n}(x, \mathbf{q}, t)]_+ d\mathbf{q} \quad \text{for a.e. } (x, t) \in Q, \quad (4.18)$$

where, for a real number s , $[s]_+ := \max(0, s)$, and the expression $\tau^{m,n}$ is defined as follows:

$$\tau^{m,n} := -k \int_D \left[KM \zeta(\rho^{m,n}) T_\ell(\hat{\psi}^{m,n}) I + \sum_{j=1}^K \zeta(\rho^{m,n}) T_\ell(\hat{\psi}^{m,n}) \nabla_{\mathbf{q}^j} M \otimes \mathbf{q}^j \right] d\mathbf{q} \quad \text{a.e. in } Q. \quad (4.19)$$

The initial data are given by

$$\begin{aligned} \rho^{m,n}(x, 0) &= \rho_0^m(x) && \text{on } \Omega, \\ \mathbf{v}^{m,n}(x, 0) &= \mathbf{v}_0^m(x) := \sum_{i=1}^m (\mathbf{v}_0, \mathbf{w}_i) \mathbf{w}_i(x) && \text{on } \Omega, \\ \hat{\psi}^{m,n}(x, \mathbf{q}, 0) &= \hat{\psi}_0^{m,n}(x, \mathbf{q}) := \sum_{i=1}^n (T_\ell(\hat{\psi}_0^m), \varphi_i^m)_{\mathcal{O}} \varphi_i^m(x, \mathbf{q}) && \text{on } \Omega \times D, \end{aligned} \quad (4.20)$$

where

$$\hat{\psi}_0^m := \hat{\psi}_0 \frac{M}{M^m}. \quad (4.21)$$

Note that in the third term on the left-hand side of (4.17) and in (4.19) the Maxwellian M has, intentionally, not been replaced by the approximate Maxwellian M^m . Note also that we do not perform Galerkin discretizations of the density ρ and of the polymer number density ϱ in the above system (4.15)–(4.21), so at this point we have no guarantee that solutions to this system exist. Our aim in the next section is therefore to show that solutions to this partially Galerkin-discretized system do in fact exist. Having done so, we shall pass to the limit $n \rightarrow \infty$, then to the limit $m \rightarrow \infty$, and finally we shall let $\ell \rightarrow \infty$.

4.3 Existence of solutions to the partially Galerkin-discretized system

In this subsection, we will first show that solutions $(\rho^{m,n}, \mathbf{v}^{m,n}, \hat{\psi}^{m,n})$ exist for the system (4.15)–(4.21) using Schauder's fixed-point theorem.

For any integers $m, n \geq 1$, let $V^m = \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ and $X^n = \text{span}\{\varphi_1^m, \dots, \varphi_n^m\}$ be the finite-dimensional Galerkin approximation spaces under consideration. Let $\mathbf{u}^{m,n} \in C([0, T]; V^m)$ and $\xi^{m,n} \in C([0, T]; X^n)$ be given. First let us consider the following transport problem:

$$\frac{\partial \rho^{m,n}}{\partial t} + \text{div}_x(\rho^{m,n} \mathbf{u}^{m,n}) = 0, \quad (4.22)$$

subject to the initial condition

$$\rho^{m,n}(0) = \rho_0^m. \quad (4.23)$$

Since $\mathbf{u}^{m,n} \in L^1(0, T; W_{0,\text{div}}^{1,1}(\Omega; \mathbb{R}^d))$ and $\text{div}_x \mathbf{u}^{m,n} = 0$, one can show, by following the arguments in Chapter VI in [5], that there exists a unique solution $\rho^{m,n}$ to (4.22), (4.23) which satisfies

$$0 < \rho_{\min} \leq \rho^{m,n} \leq \rho_{\max} \quad (4.24)$$

and, in addition, $\rho^{m,n} \in C([0, T]; L^p(\Omega))$ for $1 \leq p < \infty$. We also define

$$\lambda^{m,n}(x, t) := \zeta(\rho^{m,n}(x, t)) \int_D M^m(\mathbf{q}) [\xi^{m,n}(x, \mathbf{q}, t)]_+ d\mathbf{q}.$$

Now that we have built $\rho^{m,n}$ and $\lambda^{m,n}$, we seek $\mathbf{v}^{m,n} \in C^1([0, T]; V^m)$ and $\hat{\psi}^{m,n} \in C^1([0, T]; X^n)$ satisfying

$$\begin{aligned} & \int_{\Omega} \rho^{m,n} \left(\frac{\partial \mathbf{v}^{m,n}}{\partial t} + (\mathbf{u}^{m,n} \cdot \nabla_x) \mathbf{v}^{m,n} \right) \cdot \mathbf{w} dx + \int_{\Omega} \mu(\rho^{m,n}, \lambda^{m,n}) D(\mathbf{v}^{m,n}) : D(\mathbf{w}) dx \\ &= \int_{\Omega} \rho^{m,n} \mathbf{f}^m \cdot \mathbf{w} dx - \int_{\Omega} \tau^{m,n}(\xi^{m,n}) : \nabla_x \mathbf{w} dx, \end{aligned} \quad (4.25)$$

$$\begin{aligned} & \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) \frac{\partial \hat{\psi}^{m,n}}{\partial t} \varphi dx d\mathbf{q} + \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) (\nabla_x \hat{\psi}^{m,n} \cdot \mathbf{u}^{m,n}) \varphi dx d\mathbf{q} + \int_{\mathcal{O}} M^m \nabla_x \hat{\psi}^{m,n} \cdot \nabla_x \varphi dx d\mathbf{q} \\ &+ \int_{\mathcal{O}} M^m \mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^{m,n}) : \nabla_{\mathbf{q}} \varphi dx d\mathbf{q} = \int_{\mathcal{O}} M \zeta(\rho^{m,n}) \Lambda_{\ell}(\xi^{m,n}) (\nabla_x \mathbf{u}^{m,n}) \mathbf{q} : \nabla_{\mathbf{q}} \varphi dx d\mathbf{q} \end{aligned} \quad (4.26)$$

for all $\mathbf{w} \in V^m$ and $\varphi \in X^n$ and a.e. $t \in (0, T)$, where $\tau^{m,n}(\xi^{m,n})$ is defined by

$$\tau^{m,n}(\xi^{m,n}) := -k \int_D \left[KM \zeta(\rho^{m,n}) T_{\ell}(\xi^{m,n}) I + \sum_{j=1}^K \zeta(\rho^{m,n}) T_{\ell}(\xi^{m,n}) \nabla_{\mathbf{q}^j} M \otimes \mathbf{q}^j \right] d\mathbf{q}. \quad (4.27)$$

First we note that since by hypothesis ζ is a continuous function of its argument and $\rho^{m,n}$ is bounded, $\zeta(\rho^{m,n})$ is bounded above. Thanks to the presence of the truncation function T_{ℓ} we deduce that

$$|\tau^{m,n}(\xi^{m,n})| \leq C(\ell, M, \zeta_{\max}). \quad (4.28)$$

Since we have replaced the unknown advection vector field $\mathbf{v}^{m,n}$ in the convective term by the known vector field $\mathbf{u}^{m,n}$, (4.25) is now a linear ordinary differential equation. Also, since we have replaced $\hat{\psi}^{m,n}$ in the drag term by $\xi^{m,n}$, (4.26) is a linear ordinary differential equation.

Claim 4.1. *Problem (4.25) has a unique solution $\mathbf{v}^{m,n} \in C^1([0, T]; V^m)$ subject to the initial condition*

$$\mathbf{v}^{m,n}(\cdot, 0) = \mathbf{v}_{0,m} := \sum_{i=1}^m (\mathbf{v}_0, \mathbf{w}_i) \mathbf{w}_i.$$

Proof. We shall seek the solution in the form $\mathbf{v}^{m,n}(x, t) = \sum_{i=1}^m \alpha_i^{m,n}(t) \mathbf{w}_i(x)$. Thereby the equation (4.25) can be rewritten as

$$M(t) \frac{d\boldsymbol{\alpha}}{dt}(t) = A(t) \boldsymbol{\alpha}(t) + B(t), \quad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0, \quad (4.29)$$

where $\boldsymbol{\alpha}(t) = (\alpha_1^{m,n}(t), \dots, \alpha_m^{m,n}(t))^T$ is the unknown m -component coefficient vector of $\mathbf{v}^{m,n}$ at time t , with associated initial datum $\boldsymbol{\alpha}_0 = ((\mathbf{v}_0, \mathbf{w}_0), \dots, (\mathbf{v}_0, \mathbf{w}_m))^T \in \mathbb{R}^m$. In the above equation

$$\begin{aligned} (M(t))_{ij} &:= \int_{\Omega} \rho^{m,n} \mathbf{w}_i \cdot \mathbf{w}_j \, dx, \\ (A(t))_{ij} &:= - \left(\int_{\Omega} \rho^{m,n} (\mathbf{u}^{m,n} \cdot \nabla_x) \mathbf{w}_i \cdot \mathbf{w}_j \, dx + \int_{\Omega} \mu(\rho^{m,n}, \lambda^{m,n}) D(\mathbf{w}_i) : D(\mathbf{w}_j) \, dx \right), \\ (B(t))_j &:= \int_{\Omega} \rho^{m,n} \mathbf{f}^m \cdot \mathbf{w}_j \, dx - \int_{\Omega} \tau^{m,n} : \nabla_x \mathbf{w}_j \, dx, \end{aligned}$$

where $i, j \in \{1, \dots, m\}$ and $\tau^{m,n}$ is a function of $\xi^{m,n}$, as given in (4.27). Note that $M(t)$, $A(t)$ and $B(t)$ are continuous with respect to t . Since $M(t) \in \mathbb{R}^{m \times m}$ is the Gram matrix associated with the basis $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ with respect to the inner product (note the two-sided bound (4.24) on $\rho^{m,n}$) defined by

$$[\mathbf{v}, \mathbf{w}]_{\rho^{m,n}(t)} := \int_{\Omega} \rho^{m,n}(t, x) \mathbf{v}(x) \cdot \mathbf{w}(x) \, dx,$$

it follows that $M(t)$ is invertible for all $t \in [0, T]$. By the Cauchy–Lipschitz theorem the initial-value problem for the system of linear differential equations (4.29) has a unique global solution. Hence, (4.25) has a unique solution $\mathbf{v}^{m,n} \in C^1([0, T]; V^m)$ subject to the initial condition $\mathbf{v}^{m,n}(0) = \mathbf{v}_{0,m}$. \square

Claim 4.2. *Problem (4.26) has a unique solution $\hat{\psi}^{m,n} \in C^1([0, T]; X^n)$ subject to the initial condition*

$$\hat{\psi}^{m,n}(\cdot, \cdot, 0) = \hat{\psi}_{0,n} := \sum_{i=1}^n (T_\ell(\hat{\psi}_0^m), \varphi_i^m) \varphi_i^m.$$

Proof. Note that (4.26) is a system of linear ordinary differential equations. By writing $\hat{\psi}^{m,n}(x, \mathbf{q}, t) = \sum_{i=1}^n \beta_i^{m,n}(t) \varphi_i^m(x, \mathbf{q})$, and proceeding analogously as in the proof of Claim 4.1, we deduce that (4.26) has a unique solution $\hat{\psi}^{m,n} \in C^1([0, T]; X^n)$ subject to the initial condition $\hat{\psi}^{m,n}(0) = \hat{\psi}_{0,n}$. \square

Let $\|\cdot\|_{V^m}$ and $\|\cdot\|_{X^n}$ be norms on V^m and X^n , respectively. Since V^m and X^n are finite-dimensional linear spaces, and all norms on finite-dimensional linear spaces are equivalent, the precise choice of these norms is of no relevance in the discussion that follows.

Claim 4.3. *Let*

$$\mathcal{K} := \left\{ \mathbf{v} \in C^1([0, T]; V^m); \sup_{t \in [0, T]} \|\mathbf{v}(t)\|_{L^2(\Omega; \mathbb{R}^d)} \leq C, \quad \sup_{t \in [0, T]} \left\| \frac{\partial \mathbf{v}}{\partial t} \right\|_{V^m} \leq C(m, n) \right\} \subset C([0, T]; V^m)$$

and

$$\mathcal{S} := \left\{ \hat{\psi} \in C^1([0, T]; X^n); \sup_{t \in [0, T]} \|\hat{\psi}(t)\|_{L^2_{M^m}(\mathcal{O})} \leq C, \quad \sup_{t \in [0, T]} \left\| \frac{\partial \hat{\psi}}{\partial t} \right\|_{X^n} \leq C(m, n) \right\} \subset C([0, T]; X^n).$$

Let $\Theta : \overline{\mathcal{K} \times \mathcal{S}} \rightarrow \overline{\mathcal{K} \times \mathcal{S}}$ denote the map that takes the pair $(\mathbf{u}^{m,n}, \xi^{m,n})$ to $(\mathbf{v}^{m,n}, \hat{\psi}^{m,n}) =: \Theta(\mathbf{u}^{m,n}, \xi^{m,n})$ via the procedure (4.25) and (4.26); then, for C and $C(m, n)$ sufficiently large, the mapping Θ has a fixed point in $\overline{\mathcal{K} \times \mathcal{S}}$.

Proof. To show that Θ has a fixed point we apply Schauder's fixed-point theorem. First we note that $\overline{\mathcal{K} \times \mathcal{S}}$ is obviously non-empty and it is easy to show that $\overline{\mathcal{K} \times \mathcal{S}}$ is convex. Then, it remains to show that: (i) Θ maps $\mathcal{K} \times \mathcal{S}$ into itself; (ii) $\mathcal{K} \times \mathcal{S}$ is relatively compact in $C([0, T]; V^m) \times C([0, T]; X^n)$; then $\overline{\mathcal{K} \times \mathcal{S}}$ is compact in $C([0, T]; V^m) \times C([0, T]; X^n)$; (iii) $\Theta : \overline{\mathcal{K} \times \mathcal{S}} \rightarrow \overline{\mathcal{K} \times \mathcal{S}}$ is continuous.

We start by showing suitable energy estimates. Taking the test function $\mathbf{w} = \mathbf{v}^{m,n}$ in (4.25) and integrating with respect to time over $(0, t)$, where $t \in (0, T]$, we deduce that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \rho^{m,n}(t) |\mathbf{v}^{m,n}(t)|^2 dx - \frac{1}{2} \int_0^t \int_{\Omega} \frac{\partial \rho^{m,n}}{\partial t} |\mathbf{v}^{m,n}|^2 dx ds + \frac{1}{2} \int_0^t \int_{\Omega} \rho^{m,n} \mathbf{u}^{m,n} \cdot \nabla_x (|\mathbf{v}^{m,n}|^2) dx ds \\ & + \int_0^t \int_{\Omega} \mu(\rho^{m,n}, \lambda^{m,n}) |D(\mathbf{v}^{m,n})|^2 dx ds \\ & = \frac{1}{2} \int_{\Omega} \rho_0^m |\mathbf{v}_{0,m}|^2 dx + \int_0^t \int_{\Omega} \rho^{m,n} \mathbf{f}^m \cdot \mathbf{v}^{m,n} dx ds - \int_0^t \int_{\Omega} \tau^{m,n}(\xi^{m,n}) : \nabla_x \mathbf{v}^{m,n} dx ds. \end{aligned} \quad (4.30)$$

Since $\mathbf{v}^{m,n} \in C^1([0, T]; V^m)$, we can test the transport equation (4.22) with $|\mathbf{v}^{m,n}|^2$ and we see that the second and third term in the above identity add up to 0. From the bounds (4.24) and (4.28), the assumption (2.7), Young's inequality and Korn's inequality (2.2), we deduce from (4.30) that

$$\int_{\Omega} \rho^{m,n}(t) |\mathbf{v}^{m,n}(t)|^2 dx + \mu_{\min} \int_0^t \int_{\Omega} |D(\mathbf{v}^{m,n})|^2 dx ds \leq C + C \int_0^t \int_{\Omega} |\mathbf{v}^{m,n}|^2 dx ds, \quad (4.31)$$

where C is a constant depending on the data $\ell, \mathbf{f}, \mathbf{v}_0, \mu_{\min}, M, \zeta_{\max}$. On noting that $\rho^{m,n} \geq \rho_{\min} > 0$, we have in particular that

$$\int_{\Omega} |\mathbf{v}^{m,n}(t)|^2 dx \leq C + C \int_0^t \int_{\Omega} |\mathbf{v}^{m,n}|^2 dx ds.$$

By Gronwall's inequality we then have that

$$\sup_{t \in [0, T]} \|\mathbf{v}^{m,n}(t)\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \mu_{\min} \int_0^T \int_{\Omega} |D(\mathbf{v}^{m,n})|^2 dx dt \leq C. \quad (4.32)$$

Similarly, by taking the test function $\varphi = \hat{\psi}^{m,n}$ in (4.26) and integrating with respect to time over $(0, t)$, where $t \in (0, T]$, we get

$$\begin{aligned} & \int_0^t \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) \left(\frac{\partial (\hat{\psi}^{m,n})^2}{\partial t} + \mathbf{u}^{m,n} \cdot \nabla_x (\hat{\psi}^{m,n})^2 \right) dx d\mathbf{q} ds + 2 \int_0^t \int_{\mathcal{O}} M^m |\nabla_x \hat{\psi}^{m,n}|^2 dx d\mathbf{q} ds \\ & + 2 \int_0^t \int_{\mathcal{O}} M^m \mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^{m,n}) : \nabla_{\mathbf{q}} \hat{\psi}^{m,n} dx d\mathbf{q} ds \\ & = 2 \int_0^t \int_{\mathcal{O}} M \zeta(\rho^{m,n}) \Lambda_{\ell}(\xi^{m,n}) (\nabla_x \mathbf{u}^{m,n}) \mathbf{q} : \nabla_{\mathbf{q}} \hat{\psi}^{m,n} dx d\mathbf{q} ds. \end{aligned} \quad (4.33)$$

Since ζ is a C^1 function of the density, by the renormalization property we have that

$$\int_0^t \int_{\mathcal{O}} \zeta(\rho^{m,n}) \left(\frac{\partial \phi}{\partial t} + \mathbf{u}^{m,n} \cdot \nabla_x \phi \right) dx d\mathbf{q} ds - \int_{\mathcal{O}} \zeta(\rho^{m,n}(t)) \phi(t) dx d\mathbf{q} + \int_{\mathcal{O}} \zeta(\rho_0^m) \phi(0) dx d\mathbf{q} = 0 \quad (4.34)$$

for any $\phi \in C^{0,1}([0, T] \times \overline{\mathcal{O}})$. As $\hat{\psi}^{m,n} \in C^1([0, T]; W^{(K+1)d+1,2}(\mathcal{O})) \hookrightarrow C^1([0, T]; C^1(\overline{\mathcal{O}}))$, it follows that $|\hat{\psi}^{m,n}|^2 \in C^1([0, T]; C^1(\overline{\mathcal{O}}))$. Thanks to the assumed smoothness of M (and thereby also of M^m), we take the test function $\phi = M^m |\hat{\psi}^{m,n}|^2$ in (4.34) and subtract the resulting equation from (4.33) to obtain

$$\begin{aligned} & \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}(t)) |\hat{\psi}^{m,n}(t)|^2 dx d\mathbf{q} + 2 \int_0^t \int_{\mathcal{O}} M^m |\nabla_x \hat{\psi}^{m,n}|^2 dx d\mathbf{q} ds \\ & + 2 \int_0^t \int_{\mathcal{O}} M^m \mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^{m,n}) : \nabla_{\mathbf{q}} \hat{\psi}^{m,n} dx d\mathbf{q} ds \\ & = \int_{\mathcal{O}} M^m \zeta(\rho_0^m) |\hat{\psi}_0^{m,n}|^2 dx d\mathbf{q} + 2 \int_0^t \int_{\mathcal{O}} M \zeta(\rho^{m,n}) \Lambda_\ell(\xi^{m,n})(\nabla_x \mathbf{u}^{m,n}) \mathbf{q} : \nabla_{\mathbf{q}} \hat{\psi}^{m,n} dx d\mathbf{q} ds. \end{aligned}$$

On noting (1.11), (2.7) and the presence of the truncation function $\Lambda_\ell(\cdot)$ we can apply Hölder's inequality and Young's inequality to get that

$$\zeta_{\min} \int_{\mathcal{O}} M^m |\hat{\psi}^{m,n}(t)|^2 dx d\mathbf{q} + \int_0^t \int_{\mathcal{O}} M^m |\nabla_{x,\mathbf{q}} \hat{\psi}^{m,n}|^2 dx d\mathbf{q} ds \leq C(\ell, M, \zeta_{\max}, \hat{\psi}_0). \quad (4.35)$$

From the estimates (4.32) and (4.35) we deduce that

$$\sup_{t \in [0, T]} \|\mathbf{v}^{m,n}(t)\|_{L^2(\Omega; \mathbb{R}^d)} \leq C, \quad (4.36)$$

$$\sup_{t \in [0, T]} \|\hat{\psi}^{m,n}(t)\|_{L^2_{M^m}(\mathcal{O})} \leq C. \quad (4.37)$$

Next, we shall derive estimates for the norms of $\partial_t \mathbf{v}^{m,n}$ and $\partial_t \hat{\psi}^{m,n}$. We take the test function $\mathbf{w} = \partial_t \mathbf{v}^{m,n}$ in (4.25). Since V^m is finite-dimensional, all norms on V^m are equivalent (with constants depending on m). We obtain, using Hölder's inequality and the bound (4.28), that

$$\begin{aligned} \int_{\Omega} \rho^{m,n} \left| \frac{\partial \mathbf{v}^{m,n}}{\partial t} \right|^2 dx & \leq C \|\mathbf{f}^m\|_{L^2(\Omega; \mathbb{R}^d)} \left\| \frac{\partial \mathbf{v}^{m,n}}{\partial t} \right\|_{L^2(\Omega; \mathbb{R}^d)} \\ & + C \|\mathbf{u}^{m,n}\|_{L^3(\Omega; \mathbb{R}^d)} \|\nabla_x \mathbf{v}^{m,n}\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \left\| \frac{\partial \mathbf{v}^{m,n}}{\partial t} \right\|_{L^6(\Omega; \mathbb{R}^d)} \\ & + C \left\| \frac{\partial \mathbf{v}^{m,n}}{\partial t} \right\|_{W^{1,2}(\Omega; \mathbb{R}^d)} + C \|\nabla_x \mathbf{v}^{m,n}\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \left\| \frac{\partial \mathbf{v}^{m,n}}{\partial t} \right\|_{W^{1,2}(\Omega; \mathbb{R}^d)} \\ & \leq C(m, n) \left\| \frac{\partial \mathbf{v}^{m,n}}{\partial t} \right\|_{L^2(\Omega; \mathbb{R}^d)}. \end{aligned} \quad (4.38)$$

On noting that $\rho^{m,n} \geq \rho_{\min}$ it follows that

$$\left\| \frac{\partial \mathbf{v}^{m,n}}{\partial t} \right\|_{L^2(\Omega; \mathbb{R}^d)} \leq \frac{C(m, n)}{\rho_{\min}}.$$

Since all norms on V^m are equivalent, we have that

$$\sup_{t \in [0, T]} \left\| \frac{\partial \mathbf{v}^{m,n}}{\partial t} \right\|_{V^m} \leq C(m, n). \quad (4.39)$$

Now let us take the test function $\varphi = \partial_t \hat{\psi}^{m,n}$ in (4.26). Since X^n is finite-dimensional, all norms on X^n

are equivalent (with constants depending on n). We obtain, using Hölder's inequality, that

$$\begin{aligned}
\int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) \left| \frac{\partial \hat{\psi}^{m,n}}{\partial t} \right|^2 dx d\mathbf{q} &\leq C(M, \zeta_{\max}) \|\nabla_x \hat{\psi}^{m,n}\|_{L^2(\mathcal{O}; \mathbb{R}^d)} \|\mathbf{v}^{m,n}\|_{L^3(\Omega; \mathbb{R}^d)} \left\| \frac{\partial \hat{\psi}^{m,n}}{\partial t} \right\|_{L^6(\mathcal{O})} \\
&\quad + C(M) \|\nabla_{x,\mathbf{q}} \hat{\psi}^{m,n}\|_{L^2(\mathcal{O}; \mathbb{R}^{d(K+1)})} \left\| \frac{\partial \hat{\psi}^{m,n}}{\partial t} \right\|_{W^{1,2}(\mathcal{O})} \\
&\quad + C(\ell, M, \zeta_{\max}) \|\nabla_x \mathbf{u}^{m,n}\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \left\| \frac{\partial \hat{\psi}^{m,n}}{\partial t} \right\|_{W^{1,2}(\mathcal{O})} \\
&\leq C(n) \left\| \frac{\partial \hat{\psi}^{m,n}}{\partial t} \right\|_{L^2(\mathcal{O})}.
\end{aligned} \tag{4.40}$$

Since $\zeta(\rho^{m,n}) \geq \zeta_{\min}$ and $M^m \geq 1/m$, we have

$$\left\| \frac{\partial \hat{\psi}^{m,n}}{\partial t} \right\|_{L^2(\mathcal{O})} \leq \frac{mC(n)}{\zeta_{\min}}.$$

Since all norms on X^n are equivalent, we have that

$$\sup_{t \in [0, T]} \left\| \frac{\partial \hat{\psi}^{m,n}}{\partial t} \right\|_{X^n} \leq C(m, n). \tag{4.41}$$

From the estimates (4.39) and (4.41) we obtain that

$$\sup_{t \in [0, T]} \left\| \frac{\partial \mathbf{v}^{m,n}}{\partial t} \right\|_{V^m} \leq C(m, n), \tag{4.42}$$

$$\sup_{t \in [0, T]} \left\| \frac{\partial \hat{\psi}^{m,n}}{\partial t} \right\|_{X^n} \leq C(m, n). \tag{4.43}$$

From (4.36), (4.37), (4.42) and (4.43) we see that if we take $(\mathbf{u}^{m,n}, \xi^{m,n}) \in \mathcal{K} \times \mathcal{S}$, then $\Theta(\mathbf{u}^{m,n}, \xi^{m,n}) = (\mathbf{v}^{m,n}, \hat{\psi}^{m,n})$ still belongs to $\mathcal{K} \times \mathcal{S}$. Therefore, we have shown that (i) Θ maps $\mathcal{K} \times \mathcal{S}$ into itself.

(ii) From (4.36) we obtain that, for any $t \in [0, T]$, $\|\mathbf{v}^{m,n}(t)\|_{V^m} \leq C(m)$, thanks to the fact that all norms are equivalent in finite-dimensional spaces. Then, the subset $\mathcal{K}(t) := \{\mathbf{v}(t); \mathbf{v} \in \mathcal{K}\}$ is relatively compact in V^m . Also, since $\mathbf{v} \in C^1([0, T]; V^m)$ and \mathbf{v} satisfies (4.42), we deduce that

$$\|\mathbf{v}(t_1) - \mathbf{v}(t_2)\|_{V^m} \leq C(m, n)|t_1 - t_2|.$$

Then, for all $t_1 \in [0, T]$ and for all $\varepsilon > 0$, there exists a $\delta = \varepsilon/2C(m, n) > 0$ such that

$$\|\mathbf{v}(t_1) - \mathbf{v}(t_2)\|_{V^m} < \varepsilon$$

for all $t_2 \in [0, T]$ with $|t_1 - t_2| < \delta$ and for all $\mathbf{v} \in \mathcal{K}$. Therefore, by the Arzelà–Ascoli theorem, we deduce that \mathcal{K} is relatively compact in $C([0, T]; V^m)$.

Since $M^m \geq 1/m$, we have from (4.37) that, for any $t \in [0, T]$, $\|\hat{\psi}^{m,n}(t)\|_{L^2(\mathcal{O})} \leq C(m)$. This then gives $\|\hat{\psi}^{m,n}(t)\|_{X^n} \leq C(m, n)$, since all norms are equivalent in finite-dimensional spaces. Thus the subset $\mathcal{S}(t) := \{\hat{\psi}(t); \hat{\psi} \in \mathcal{S}\}$ is relatively compact in X^n . Also, since $\hat{\psi} \in C^1([0, T]; X^n)$ and $\hat{\psi}$ satisfies (4.43), we deduce that

$$\|\hat{\psi}(t_3) - \hat{\psi}(t_4)\|_{X^n} \leq C(m, n)|t_3 - t_4|.$$

Then, for all $t_3 \in [0, T]$ and for all $\varepsilon' > 0$, there exists a $\delta' = \varepsilon'/2C(m, n) > 0$ such that

$$\|\hat{\psi}(t_3) - \hat{\psi}(t_4)\|_{X^n} < \varepsilon'$$

for all $t_4 \in [0, T]$ with $|t_3 - t_4| < \delta'$ and for all $\hat{\psi} \in \mathcal{S}$. Therefore, by the Arzelà–Ascoli theorem, \mathcal{S} is relatively compact in $C([0, T]; X^n)$. Hence, we have shown that $\mathcal{K} \times \mathcal{S}$ is relatively compact in $C([0, T]; V^m) \times C([0, T]; X^n)$.

(iii) Next we will show that Θ is continuous; to this end it suffices to show that Θ is sequentially continuous. Let $(\mathbf{u}_{(r)}^{m,n})_{r=1}^\infty$ be a sequence in $C([0, T]; V^m)$ which converges to some $\mathbf{u}^{m,n}$ in $C([0, T]; V^m)$ as $r \rightarrow \infty$ and let $(\xi_{(r)}^{m,n})_{r=1}^\infty$ be a sequence in $C([0, T]; X^n)$ which converges to some $\xi^{m,n}$ in $C([0, T]; X^n)$ as $r \rightarrow \infty$. To show that Θ is sequentially continuous we shall show that $(\mathbf{v}_{(r)}^{m,n}, \hat{\psi}_{(r)}^{m,n}) = \Theta(\mathbf{u}_{(r)}^{m,n}, \xi_{(r)}^{m,n})$ converges to $\Theta(\mathbf{u}^{m,n}, \xi^{m,n})$ as $r \rightarrow \infty$ and $\Theta(\mathbf{u}^{m,n}, \xi^{m,n}) = (\mathbf{v}^{m,n}, \hat{\psi}^{m,n})$.

For any r , let $\rho_{(r)}^{m,n}$ be the unique solution to the transport equation (4.22) corresponding to the velocity field $\mathbf{u}_{(r)}^{m,n}$. Let $\rho^{m,n}$ be the unique solution to (4.22) corresponding to the velocity field $\mathbf{u}^{m,n}$. All of these transport problems are associated with the same initial datum $\rho_0^m \in W^{d+1,2}(\Omega)$ satisfying (2.4) (and converging to ρ_0 in $L^1(\Omega)$ as $m \rightarrow \infty$; here though, $m \geq 1$ and $n \geq 1$ are fixed, and we are interested, instead, in the limit $r \rightarrow \infty$).

We further define $\lambda_{(r)}^{m,n}$ and $\lambda^{m,n}$ by

$$\begin{aligned}\lambda_{(r)}^{m,n}(x, t) &:= \zeta(\rho_{(r)}^{m,n}(x, t)) \int_D M^m(\mathbf{q}) [\xi_{(r)}^{m,n}(x, \mathbf{q}, t)]_+ d\mathbf{q}, \\ \lambda^{m,n}(x, t) &:= \zeta(\rho^{m,n}(x, t)) \int_D M^m(\mathbf{q}) [\xi^{m,n}(x, \mathbf{q}, t)]_+ d\mathbf{q}.\end{aligned}$$

By Theorem VI.1.9 in [5] we deduce that, as $r \rightarrow \infty$,

$$\rho_{(r)}^{m,n} \longrightarrow \rho^{m,n} \quad \text{strongly in } C([0, T]; L^p(\Omega)) \text{ for any } p \in [1, \infty). \quad (4.44)$$

From the assumptions (2.7) on ζ we then deduce that, as $r \rightarrow \infty$,

$$\zeta(\rho_{(r)}^{m,n}) \longrightarrow \zeta(\rho^{m,n}) \quad \text{strongly in } C([0, T]; L^p(\Omega)) \text{ for any } p \in [1, \infty). \quad (4.45)$$

Hence, and thanks to the global Lipschitz continuity of the mapping $s \in \mathbb{R} \mapsto [s]_+ \in \mathbb{R}_{\geq 0}$, we have that

$$\lambda_{(r)}^{m,n} \longrightarrow \lambda^{m,n} \quad \text{strongly in } C([0, T]; L^p(\Omega)) \text{ for any } p \in [1, \infty). \quad (4.46)$$

From the assumptions (2.7) on μ we then deduce that, as $r \rightarrow \infty$,

$$\mu(\rho_{(r)}^{m,n}, \lambda_{(r)}^{m,n}) \longrightarrow \mu(\rho^{m,n}, \lambda^{m,n}) \quad \text{strongly in } L^\infty(0, T; L^p(\Omega)) \text{ for any } p \in [1, \infty). \quad (4.47)$$

For any r , we take the test function in (4.25) and (4.26) to be $\mathbf{v}_{(r)}^{m,n}$ and $\hat{\psi}_{(r)}^{m,n}$ respectively and perform a similar procedure as in (4.30)–(4.35). Then, we deduce that

$$\sup_r \|\mathbf{v}_{(r)}^{m,n}\|_{C([0, T]; L^2(\Omega; \mathbb{R}^d))} \leq C, \quad (4.48)$$

$$\sup_r \|\nabla_x \mathbf{v}_{(r)}^{m,n}\|_{L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d}))} \leq C, \quad (4.49)$$

$$\sup_r \|\hat{\psi}_{(r)}^{m,n}\|_{C([0, T]; L^2_{M^m}(\mathcal{O}))} \leq C, \quad (4.50)$$

$$\sup_r \|\nabla_{x, \mathbf{q}} \hat{\psi}_{(r)}^{m,n}\|_{L^2(0, T; L^2_{M^m}(\mathcal{O}; \mathbb{R}^{d(K+1)}))} \leq C. \quad (4.51)$$

Taking the test function in (4.25) to be $\partial_t \mathbf{v}_{(r)}^{m,n}$ we perform a similar argument as in (4.38), (4.39) to deduce that

$$\sup_r \left\| \frac{\partial \mathbf{v}_{(r)}^{m,n}}{\partial t} \right\|_{C([0, T]; V^m)} \leq C(m, n).$$

Similarly, taking the test function in (4.26) to be $\partial_t \hat{\psi}_{(r)}^{m,n}$, as in (4.40), (4.41), we get

$$\sup_r \left\| \frac{\partial \hat{\psi}_{(r)}^{m,n}}{\partial t} \right\|_{C([0,T];X^n)} < C(m,n).$$

Now since $\mathbf{v}_{(r)}^{m,n} \in \mathcal{K}$ for all r and \mathcal{K} is relatively compact in $C([0,T];V^m)$, there exists a subsequence (not relabelled) such that, as $r \rightarrow \infty$,

$$\mathbf{v}_{(r)}^{m,n} \longrightarrow \mathbf{v}' \quad \text{strongly in } C([0,T];V^m). \quad (4.52)$$

From the bounds (4.48) and (4.49) we deduce the following weak convergence:

$$\mathbf{v}_{(r)}^{m,n} \rightharpoonup \mathbf{v}' \quad \text{weakly in } L^2(0,T;W_{0,\text{div}}^{1,2}(\Omega;\mathbb{R}^d)). \quad (4.53)$$

With the convergence results (4.44), (4.46), (4.47), (4.45), (4.52) and (4.53) we can pass to the limit as $r \rightarrow \infty$ in the equation satisfied by $\mathbf{v}_{(r)}^{m,n}$ and the limit \mathbf{v}' satisfies (4.25). However, with $\rho^{m,n}$, $\mathbf{u}^{m,n}$ and $\xi^{m,n}$ given, (4.25) can be solved with a unique solution $\mathbf{v}^{m,n} \in C^1([0,T];V^m)$, which then implies that $\mathbf{v}' \equiv \mathbf{v}^{m,n}$.

Meanwhile, since $\hat{\psi}_{(r)}^{m,n} \in \mathcal{S}$ for all r and \mathcal{S} is relatively compact in $C([0,T];X^n)$, there exists a subsequence (not relabelled) such that, as $r \rightarrow \infty$,

$$\hat{\psi}_{(r)}^{m,n} \longrightarrow \hat{\psi}' \quad \text{strongly in } C([0,T];X^n). \quad (4.54)$$

From the bounds (4.50) and (4.51) we deduce the following weak convergence:

$$\sqrt{M^m} \nabla_{x,\mathbf{q}} \hat{\psi}_{(r)}^{m,n} \rightharpoonup \sqrt{M^m} \nabla_{x,\mathbf{q}} \hat{\psi}' \quad \text{weakly in } L^2(0,T;L^2(\mathcal{O};\mathbb{R}^{d(K+1)})), \quad (4.55)$$

With the convergence results (4.45), (4.54) and (4.55) we can pass to the limit as $r \rightarrow \infty$ in the equation satisfied by $\hat{\psi}_{(r)}^{m,n}$ and the limit $\hat{\psi}'$ satisfies (4.26). However, with $\rho^{m,n}$, $\mathbf{u}^{m,n}$ and $\xi^{m,n}$ given, (4.26) can be solved uniquely with $\hat{\psi}^{m,n} \in C^1([0,T];X^n)$, which implies that $\hat{\psi}' \equiv \hat{\psi}^{m,n}$. Hence, we have shown that the mapping Θ is continuous.

Finally, by Schauder's fixed-point theorem, we deduce that $\Theta : \overline{\mathcal{K}} \times \overline{\mathcal{S}} \rightarrow \overline{\mathcal{K}} \times \overline{\mathcal{S}}$ has a fixed point $(\mathbf{v}^{m,n}, \hat{\psi}^{m,n})$ in $\overline{\mathcal{K}} \times \overline{\mathcal{S}}$. Thus, also,

$$\varrho^{m,n}(x,t) = \zeta(\rho^{m,n}(x,t)) \int_D M^m(\mathbf{q}) [\hat{\psi}^{m,n}(x,\mathbf{q},t)]_+ d\mathbf{q},$$

and

$$\begin{aligned} \tau^{m,n}(x,t) = & -k \int_D \left[KM(\mathbf{q}) \zeta(\rho^{m,n}(x,t)) T_\ell(\hat{\psi}^{m,n}(x,\mathbf{q},t)) I \right. \\ & \left. + \sum_{j=1}^K \zeta(\rho^{m,n}(x,t)) T_\ell(\hat{\psi}^{m,n}(x,\mathbf{q},t)) \nabla_{\mathbf{q}^j} M(\mathbf{q}) \otimes \mathbf{q}^j \right] d\mathbf{q}. \end{aligned}$$

That completes the proof of the existence of a solution $(\rho^{m,n}, \mathbf{v}^{m,n}, \hat{\psi}^{m,n})$ to the partially Galerkin discretized system (4.15)–(4.21). \square

In the following sections we shall derive uniform bounds independent of the parameters n , m and ℓ , and then use those to successively pass to the limits with $n, m, \ell \rightarrow \infty$.

4.4 Passage to the limit with n

The goal of this section is to pass to the limit as $n \rightarrow \infty$. To achieve this we shall first derive uniform bounds independent of n . We note that from the definition (4.19) of $\tau^{m,n}$ we deduce that

$$|\tau^{m,n}| \leq C(\ell, \zeta_{\max}, M), \quad (4.56)$$

which implies that there exists a subsequence (not relabelled) such that

$$\tau^{m,n} \rightharpoonup \tau^m \quad \text{weak}^* \text{ in } L^\infty(\Omega \times (0, T); \mathbb{R}^{d \times d}). \quad (4.57)$$

By Theorem VI.1.6 in [5] we find that

$$\sup_{t \in (0, T)} \|\rho^{m,n}(t)\|_{L^\infty(\Omega)} \leq \|\rho_0^m\|_{L^\infty(\Omega)} \leq \rho_{\max}.$$

Moreover, Proposition VI.1.8 in [5] gives

$$\rho^{m,n} \geq \rho_{\min}. \quad (4.58)$$

We deduce the existence of subsequences (not relabelled) such that, as $n \rightarrow \infty$,

$$\rho^{m,n} \rightharpoonup \rho^m \quad \text{weak}^* \text{ in } L^\infty(\Omega \times (0, T)), \quad (4.59)$$

$$(\rho^{m,n})^2 \rightharpoonup \vartheta^m \quad \text{weak}^* \text{ in } L^\infty(\Omega \times (0, T)). \quad (4.60)$$

By the renormalization property, for any $\beta \in C^1(\mathbb{R})$, $\beta(\rho^{m,n})$ satisfies

$$\frac{\partial \beta(\rho^{m,n})}{\partial t} + \operatorname{div}_x(\beta(\rho^{m,n}) \mathbf{v}^{m,n}) = 0 \quad (4.61)$$

in the distributional sense. We recall from the previous section that $\mathbf{v}^{m,n} \in C^1([0, T]; V^m)$; it will be shown below that the sequence $(\mathbf{v}^{m,n})_{m,n=1}^\infty$ is uniformly bounded in $L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^d))$, independent of m and n .

Multiplying the i -th equation in (4.16) by $c_i^{m,n}(t)$ and summing with respect to $i = 1, \dots, m$ give the following identity

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho^{m,n} |\mathbf{v}^{m,n}|^2 dx &+ \frac{1}{2} \int_{\Omega} \frac{\partial \rho^{m,n}}{\partial t} |\mathbf{v}^{m,n}|^2 dx - \frac{1}{2} \int_{\Omega} \rho^{m,n} \mathbf{v}^{m,n} \cdot \nabla_x (|\mathbf{v}^{m,n}|^2) dx \\ &+ \int_{\Omega} \mu(\rho^{m,n}, \varrho^{m,n}) |D(\mathbf{v}^{m,n})|^2 dx = - \int_{\Omega} \tau^{m,n} : D(\mathbf{v}^{m,n}) dx + (\rho^{m,n} \mathbf{f}^m, \mathbf{v}^{m,n}). \end{aligned}$$

The second term and the third term in the above equality add up to 0, since we can take the test function $\eta = |\mathbf{v}^{m,n}|^2$ in (4.15). Using (2.7), (4.56), (4.58), Young's inequality, Korn's inequality (2.2) and Gronwall's inequality, we find that

$$\sup_{t \in (0, T)} \|\mathbf{v}^{m,n}(t)\|_{L^2(\Omega; \mathbb{R}^d)}^2 + c_0 \mu_{\min} \int_0^T \|\mathbf{v}^{m,n}\|_{W^{1,2}(\Omega; \mathbb{R}^d)}^2 dt \leq C(\ell, M, \mathbf{v}_0, \mathbf{f}, \mu_{\min}, \rho_{\max}).$$

The orthogonality of the basis and the embedding $W^{d+1,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ imply that

$$\sup_{t \in (0, T); i=1, \dots, m} |c_i^{m,n}(t)| + \left| \frac{dc_i^{m,n}(t)}{dt} \right| \leq C(m, \ell). \quad (4.62)$$

By the definition of $\mathbf{v}^{m,n}$ it is easy to see that

$$\sup_{t \in (0, T)} \|\mathbf{v}^{m,n}(t)\|_{W^{1,\infty}(\Omega; \mathbb{R}^d)} \leq C(m, \ell). \quad (4.63)$$

By using (4.63) and the Sobolev embedding theorem it then follows from (4.61) that

$$\left\| \frac{\partial \beta(\rho^{m,n})}{\partial t} \right\|_{L^2(0,T;(W^{1,p'}(\Omega))')} \leq \|\beta(\rho^{m,n})\mathbf{v}^{m,n}\|_{L^2(0,T;L^p(\Omega;\mathbb{R}^d))} \leq C\|\mathbf{v}^{m,n}\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d))} \leq C \quad (4.64)$$

for any $p \in (1, \infty)$ when $d = 2$ and $p \in (1, 6]$ when $d = 3$; the constant C is independent of m and n .

The uniform bound (4.62) implies the following convergence results:

$$c_i^{m,n} \rightharpoonup c_i^m \quad \text{weak* in } W^{1,\infty}((0,T)), \quad (4.65)$$

$$c_i^{m,n} \rightarrow c_i^m \quad \text{strongly in } C([0,T]), \quad (4.66)$$

which then imply that

$$\mathbf{v}^{m,n} \rightarrow \mathbf{v}^m \quad \text{strongly in } C([0,T]; W_{0,\text{div}}^{1,2}(\Omega; \mathbb{R}^d)). \quad (4.67)$$

Noting (4.59) and (4.60) we deduce that

$$\rho^{m,n}\mathbf{v}^{m,n} \rightharpoonup \rho^m\mathbf{v}^m \quad \text{weakly in } L^p(Q; \mathbb{R}^d), \quad (4.68)$$

$$(\rho^{m,n})^2\mathbf{v}^{m,n} \rightharpoonup \vartheta^m\mathbf{v}^m \quad \text{weakly in } L^p(Q; \mathbb{R}^d), \quad (4.69)$$

where $p \in [1, \infty)$ when $d = 2$ and $p \in [1, 6]$ when $d = 3$. From the estimate (4.64) we deduce that

$$\frac{\partial \rho^{m,n}}{\partial t} \rightharpoonup \frac{\partial \rho^m}{\partial t} \quad \text{weakly in } L^2(0,T; (W^{1,p'}(\Omega))'), \quad (4.70)$$

$$\frac{\partial (\rho^{m,n})^2}{\partial t} \rightharpoonup \frac{\partial \vartheta^m}{\partial t} \quad \text{weakly in } L^2(0,T; (W^{1,p'}(\Omega))'), \quad (4.71)$$

where $p \in (1, \infty)$ when $d = 2$ and $p \in (1, 6]$ when $d = 3$. With the convergence results (4.70) and (4.68) we pass to the limit as $n \rightarrow \infty$ and deduce that ρ^m is the (unique) weak solution of

$$\frac{\partial \rho^m}{\partial t} + \text{div}_x(\rho^m\mathbf{v}^m) = 0 \quad (4.72)$$

with the initial condition

$$\rho^m(0) = \rho_0^m. \quad (4.73)$$

By the renormalization property, if we define $P^{m,n} = (\rho^{m,n})^2$, then $P^{m,n}$ solves the following initial-value problem:

$$\begin{aligned} \frac{\partial P^{m,n}}{\partial t} + \text{div}_x(P^{m,n}\mathbf{v}^{m,n}) &= 0, \\ P^{m,n}(0) &= (\rho_0^m)^2. \end{aligned}$$

Similarly, $P^m = (\rho^m)^2$ solves the following problem:

$$\begin{aligned} \frac{\partial P^m}{\partial t} + \text{div}_x(P^m\mathbf{v}^m) &= 0, \\ P^m(0) &= (\rho_0^m)^2. \end{aligned}$$

With the convergence results (4.71) and (4.69) we deduce that ϑ^m also solves the following initial-value problem

$$\begin{aligned} \frac{\partial \vartheta^m}{\partial t} + \text{div}_x(\vartheta^m\mathbf{v}^m) &= 0, \\ \vartheta^m(0) &= (\rho_0^m)^2. \end{aligned} \quad (4.74)$$

However, since (4.74) is of the same form as (4.72), the solution to (4.74) is unique. Hence, $\vartheta^m = (\rho^m)^2$. Then, the convergence result (4.60) gives

$$\int_0^T \int_{\Omega} |\rho^{m,n}|^2 dx dt \rightarrow \int_0^T \int_{\Omega} |\rho^m|^2 dx dt,$$

which then implies that

$$\rho^{m,n} \rightarrow \rho^m \quad \text{strongly in } L^2(\Omega \times (0, T)).$$

It then follows from (4.59) that

$$\rho^{m,n} \rightarrow \rho^m \quad \text{strongly in } L^p(\Omega \times (0, T)), \quad (4.75)$$

for any $p \in [1, \infty)$. With the convergence result (4.67) for $\mathbf{v}^{m,n}$ we can perform a similar argument as in Theorem VI.1.9 in [5] and strengthen the above convergence to get

$$\rho^{m,n} \rightarrow \rho^m \quad \text{strongly in } C([0, T]; L^p(\Omega)), \quad (4.76)$$

for any $p \in [1, \infty)$. Thanks to the assumption (2.7) on ζ we then have that

$$\zeta(\rho^{m,n}) \rightarrow \zeta(\rho^m) \quad \text{strongly in } C([0, T]; L^p(\Omega)). \quad (4.77)$$

Multiplying the i -th equation in (4.17) by $d_i^{m,n}$ and summing with respect to $i = 1, \dots, n$, we obtain the following identity

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) (\hat{\psi}^{m,n})^2 dx d\mathbf{q} + \frac{1}{2} \int_{\mathcal{O}} M^m \frac{\partial \zeta(\rho^{m,n})}{\partial t} (\hat{\psi}^{m,n})^2 dx d\mathbf{q} \\ & - \frac{1}{2} \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) \mathbf{v}^{m,n} \cdot \nabla_x (\hat{\psi}^{m,n})^2 dx d\mathbf{q} + \int_{\mathcal{O}} M^m |\nabla_x \hat{\psi}^{m,n}|^2 dx d\mathbf{q} \\ & + \int_{\mathcal{O}} M^m \mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^{m,n}) : \nabla_{\mathbf{q}} \hat{\psi}^{m,n} dx d\mathbf{q} = \left(M \zeta(\rho^{m,n}) \Lambda_{\ell}(\hat{\psi}^{m,n}) (\nabla_x \mathbf{v}^{m,n}) \mathbf{q}, \nabla_{\mathbf{q}} \hat{\psi}^{m,n} \right)_{\mathcal{O}}. \end{aligned} \quad (4.78)$$

Note that $\zeta(\rho^{m,n})$ is a renormalized solution that satisfies

$$\langle \partial_t \zeta(\rho^{m,n}), \eta \rangle - (\mathbf{v}^{m,n} \zeta(\rho^{m,n}), \nabla_x \eta) = 0 \quad \forall \eta \in C^{0,1}([0, T]; C^{0,1}(\overline{\Omega})).$$

Taking the test function $\eta = \int_D M^m (\hat{\psi}^{m,n})^2 d\mathbf{q}$ we see that the second term and the third term in (4.78) add up to 0. By Young's inequality, the definition of Λ_{ℓ} , (2.7) and (4.63), we deduce that

$$\begin{aligned} \left(M \zeta(\rho^{m,n}) \Lambda_{\ell}(\hat{\psi}^{m,n}) (\nabla_x \mathbf{v}^{m,n}) \mathbf{q}, \nabla_{\mathbf{q}} \hat{\psi}^{m,n} \right)_{\mathcal{O}} & \leq \frac{C_1}{2} \int_{\mathcal{O}} M^m |\nabla_{\mathbf{q}} \hat{\psi}^{m,n}|^2 dx d\mathbf{q} \\ & + C(m, \ell) \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) (\hat{\psi}^{m,n})^2 dx d\mathbf{q}. \end{aligned} \quad (4.79)$$

Inserting (4.79) into (4.78) and using (1.11) we deduce that

$$\frac{d}{dt} \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) (\hat{\psi}^{m,n})^2 dx d\mathbf{q} + \int_{\mathcal{O}} M^m |\nabla_{x, \mathbf{q}} \hat{\psi}^{m,n}|^2 dx d\mathbf{q} \leq C(m, \ell) \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) (\hat{\psi}^{m,n})^2 dx d\mathbf{q}. \quad (4.80)$$

Gronwall's inequality gives that

$$\sup_{t \in (0, T)} \|\sqrt{\zeta(\rho^{m,n}(t))} \hat{\psi}^{m,n}(t)\|_{L^2_{M^m}(\mathcal{O})}^2 + \int_0^T \int_{\mathcal{O}} M^m |\nabla_{x, \mathbf{q}} \hat{\psi}^{m,n}|^2 dx d\mathbf{q} dt \leq C(m, \ell).$$

By noting that $M^m \geq 1/m$ and (2.7) we deduce that

$$\sup_{t \in (0, T)} \|\hat{\psi}^{m,n}(t)\|_{L^2(\mathcal{O})}^2 + \int_0^T \int_{\mathcal{O}} |\nabla_{x, \mathbf{q}} \hat{\psi}^{m,n}|^2 dx d\mathbf{q} dt \leq C(m, \ell). \quad (4.81)$$

Since M^m is Lipschitz continuous, we can further deduce that

$$\int_0^T \|M^m \hat{\psi}^{m,n}\|_{W^{1,2}(\mathcal{O})}^2 dt \leq C(m, \ell).$$

Then, using (4.17) and a standard calculation, we obtain that

$$\int_0^T \|\partial_t(M^m \zeta(\rho^{m,n}) \hat{\psi}^{m,n})\|_{(W^{1,2}(\mathcal{O}))'}^2 dt \leq C(m, \ell). \quad (4.82)$$

Next, we shall focus on the fractional time derivative of $\hat{\psi}^{m,n}$. Integrating (4.17) with respect to time over $(s, s+h)$, with $s < T-h$, we have

$$\begin{aligned} & \int_{\mathcal{O}} M^m [(\zeta(\rho^{m,n}) \hat{\psi}^{m,n})(s+h) - (\zeta(\rho^{m,n}) \hat{\psi}^{m,n})(s)] \varphi_i^m dx d\mathbf{q} \\ &= \int_s^{s+h} \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) \hat{\psi}^{m,n} \mathbf{v}^{m,n} \cdot \nabla_x \varphi_i^m dx d\mathbf{q} dt \\ & \quad - \int_s^{s+h} \int_{\mathcal{O}} M^m \nabla_x \hat{\psi}^{m,n} : \nabla_x \varphi_i^m dx d\mathbf{q} dt \\ & \quad - \int_s^{s+h} \int_{\mathcal{O}} M^m \mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^{m,n}) : \nabla_{\mathbf{q}} \varphi_i^m dx d\mathbf{q} dt \\ & \quad + \int_s^{s+h} \int_{\mathcal{O}} M \zeta(\rho^{m,n}) \Lambda_{\ell}(\hat{\psi}^{m,n}) (\nabla_x \mathbf{v}^{m,n}) \mathbf{q} : \nabla_{\mathbf{q}} \varphi_i^m dx d\mathbf{q} dt \\ &=: U_1 + U_2 + U_3 + U_4. \end{aligned}$$

For U_1 , we have by Hölder's inequality and the Gagliardo–Nirenberg inequality (2.1) that

$$\begin{aligned} |U_1| &\leq C(M, \zeta_{\max}) \|\nabla_x \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^d)} \int_s^{s+h} \|\hat{\psi}^{m,n}\|_{L^q(\Omega; L^2(D))} \|\mathbf{v}^{m,n}\|_{L^{\frac{2q}{q-2}}(\Omega; \mathbb{R}^d)} dt \\ &\leq C(M, \zeta_{\max}) \|\nabla_x \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^d)} \int_s^{s+h} \|\hat{\psi}^{m,n}\|_{L^q(\Omega; L^2(D))} \|\mathbf{v}^{m,n}\|_{L^2(\Omega; \mathbb{R}^d)}^{1-\frac{d}{q}} \|\mathbf{v}^{m,n}\|_{W^{1,2}(\Omega; \mathbb{R}^d)}^{\frac{d}{q}} dt \\ &\leq C(M, \zeta_{\max}) h^{\frac{q-d}{2q}} \|\nabla_x \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^d)} \|\mathbf{v}^{m,n}\|_{L^\infty(0,T; L^2(\Omega; \mathbb{R}^d))}^{1-\frac{d}{q}} \|\mathbf{v}^{m,n}\|_{L^2(s, s+h; W^{1,2}(\Omega; \mathbb{R}^d))}^{\frac{d}{q}} \\ & \quad \times \left(\int_s^{s+h} \|\hat{\psi}^{m,n}\|_{L^q(\Omega; L^2(D))}^2 dt \right)^{\frac{1}{2}} \\ &\leq Ch^{\frac{q-d}{2q}} \|\nabla_{x, \mathbf{q}} \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^{d(K+1)})}, \end{aligned}$$

where $q \in (2, \infty)$ when $d = 2$ and $q \in (3, 6]$ when $d = 3$. For U_2 , we have that

$$|U_2| \leq C(M) h^{\frac{1}{2}} \|\nabla_x \hat{\psi}^{m,n}\|_{L^2(s, s+h; L^2(\mathcal{O}; \mathbb{R}^d))} \|\nabla_x \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^d)} \leq Ch^{\frac{1}{2}} \|\nabla_{x, \mathbf{q}} \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^{d(K+1)})}.$$

Similarly as above, for U_3 we have that

$$|U_3| \leq Ch^{\frac{1}{2}} \|\nabla_{x, \mathbf{q}} \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^{d(K+1)})}.$$

For U_4 , thanks to the presence of the truncation function Λ_{ℓ} , we obtain that

$$|U_4| \leq Ch^{\frac{1}{2}} \|\nabla_{\mathbf{q}} \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^{K+1})} \|\nabla_x \mathbf{v}^{m,n}\|_{L^2(s, s+h; L^2(\Omega; \mathbb{R}^{d \times d}))} \leq Ch^{\frac{1}{2}} \|\nabla_{x, \mathbf{q}} \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^{d(K+1)})}.$$

Hence,

$$\left| \int_{\mathcal{O}} M^m [(\zeta(\rho^{m,n}) \hat{\psi}^{m,n})(s+h) - (\zeta(\rho^{m,n}) \hat{\psi}^{m,n})(s)] \varphi_i^m dx d\mathbf{q} \right| \leq C(h^{\frac{1}{2}} + h^{\frac{q-d}{2q}}) \|\nabla_{x, \mathbf{q}} \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^{d(K+1)})}, \quad (4.83)$$

where $q \in (2, \infty)$ when $d = 2$ and $q \in (3, 6]$ when $d = 3$. By the renormalization property $\zeta(\rho^{m,n})$ satisfies

$$\int_0^T \int_{\Omega} \left(\frac{\partial \zeta(\rho^{m,n})}{\partial t} \eta - \zeta(\rho^{m,n}) \mathbf{v}^{m,n} \cdot \nabla_x \eta \right) dx dt = 0.$$

Taking $\eta = \chi_{[s, s+h]} \int_D M^m \hat{\psi}^{m,n}(s) \varphi_i^m d\mathbf{q}$ in the above equation we obtain that

$$\int_{\mathcal{O}} M^m [\zeta(\rho^{m,n})(s+h) - \zeta(\rho^{m,n})(s)] \hat{\psi}^{m,n} \varphi_i^m dx d\mathbf{q} = \int_s^{s+h} \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) \mathbf{v}^{m,n} \cdot \nabla_x (\hat{\psi}^{m,n}(s) \varphi_i^m) dx d\mathbf{q} dt.$$

By Hölder's inequality we obtain that

$$\begin{aligned} & \left| \int_{\mathcal{O}} M^m [\zeta(\rho^{m,n})(s+h) - \zeta(\rho^{m,n})(s)] \hat{\psi}^{m,n} \varphi_i^m dx d\mathbf{q} \right| \\ & \leq C(M, \zeta_{\max}) h^{\frac{1}{2}} \|\mathbf{v}^{m,n}\|_{L^2(s, s+h; L^q(\Omega; \mathbb{R}^d))} \|\nabla_x (\hat{\psi}^{m,n}(s) \varphi_i^m)\|_{L^{\frac{q}{q-1}}(\mathcal{O}; \mathbb{R}^d)} \\ & \leq C(M, \zeta_{\max}) h^{\frac{1}{2}} \|\mathbf{v}^{m,n}\|_{L^2(s, s+h; L^q(\Omega; \mathbb{R}^d))} \left(\|\nabla_x \hat{\psi}^{m,n}(s)\|_{L^2(\mathcal{O}; \mathbb{R}^d)} \|\varphi_i^m\|_{L^{\frac{2q}{q-2}}(\mathcal{O})} \right. \\ & \quad \left. + \|\nabla_x \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^d)} \|\hat{\psi}^{m,n}(s)\|_{L^{\frac{2q}{q-2}}(\mathcal{O})} \right), \end{aligned} \quad (4.84)$$

where $q \in (2, 2(K+1)]$ when $d = 2$ and $q \in (3, 6]$ when $d = 3$. Since

$$\begin{aligned} & (\zeta(\rho^{m,n}) \hat{\psi}^{m,n})(s+h) - (\zeta(\rho^{m,n}) \hat{\psi}^{m,n})(s) \\ & = \zeta(\rho^{m,n})(s+h) [\hat{\psi}^{m,n}(s+h) - \hat{\psi}^{m,n}(s)] + [\zeta(\rho^{m,n})(s+h) - \zeta(\rho^{m,n})(s)] \hat{\psi}^{m,n}(s), \end{aligned}$$

it follows from (4.83) and (4.84) that

$$\begin{aligned} & \left| \int_{\mathcal{O}} M^m \zeta(\rho^{m,n})(s+h) [\hat{\psi}^{m,n}(s+h) - \hat{\psi}^{m,n}(s)] \varphi_i^m dx d\mathbf{q} \right| \leq C(h^{\frac{1}{2}} + h^{\frac{q-d}{2q}}) \|\nabla_{x, \mathbf{q}} \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^{d(K+1)})} \\ & \quad + Ch^{\frac{1}{2}} \left(\|\nabla_x \hat{\psi}^{m,n}(s)\|_{L^2(\mathcal{O}; \mathbb{R}^d)} \|\varphi_i^m\|_{L^{\frac{2q}{q-2}}(\mathcal{O})} + \|\nabla_x \varphi_i^m\|_{L^2(\mathcal{O}; \mathbb{R}^d)} \|\hat{\psi}^{m,n}(s)\|_{L^{\frac{2q}{q-2}}(\mathcal{O})} \right), \end{aligned}$$

where $q \in (2, 2(K+1)]$ when $d = 2$ and $q \in (3, 6]$ when $d = 3$. Taking $\varphi_i^m = \hat{\psi}^{m,n}(s+h) - \hat{\psi}^{m,n}(s)$ in the above inequality and integrating with respect to s we obtain that

$$\begin{aligned} & \int_0^{T-h} \int_{\mathcal{O}} M^m \zeta(\rho^{m,n})(s+h) [\hat{\psi}^{m,n}(s+h) - \hat{\psi}^{m,n}(s)]^2 dx d\mathbf{q} ds \\ & \leq 2C(h^{\frac{1}{2}} + h^{\frac{q-d}{2q}}) \left(\int_0^T \|\nabla_{x, \mathbf{q}} \hat{\psi}^{m,n}(s)\|_{L^2(\mathcal{O}; \mathbb{R}^{d(K+1)})}^2 ds \right)^{\frac{1}{2}} \\ & \quad + 4Ch^{\frac{1}{2}} \left(\int_0^T \|\nabla_x \hat{\psi}^{m,n}(s)\|_{L^2(\mathcal{O}; \mathbb{R}^d)}^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \|\hat{\psi}^{m,n}(s)\|_{L^{\frac{2q}{q-2}}(\mathcal{O})}^2 ds \right)^{\frac{1}{2}} \\ & \leq Ch^{\frac{q-d}{2q}}, \end{aligned}$$

where $q \in (2, 2(K+1)]$ when $d = 2$ and $q \in (3, 6]$ when $d = 3$. The above inequality follows from Hölder's inequality, the uniform estimate (4.81) and the Sobolev embedding $W^{1,2}(\mathcal{O}) \hookrightarrow L^p(\mathcal{O})$, $p \in \left[1, \frac{2(K+1)d}{(K+1)d-2}\right]$. Since $\zeta(\rho^{m,n}) \geq \zeta_{\min}$ and $M^m \geq 1/m$, we have that

$$\|\hat{\psi}^{m,n}(\cdot + h) - \hat{\psi}^{m,n}(\cdot)\|_{L^2(0, T-h; L^2(\mathcal{O}))} \leq Ch^\gamma,$$

where $0 < \gamma \leq \frac{K}{4(K+1)}$ when $d = 2$ and $0 < \gamma \leq 1/8$ when $d = 3$. We have therefore shown the following Nikolskiĭ norm estimate:

$$\|\hat{\psi}^{m,n}\|_{N_2^\gamma(0, T; L^2(\mathcal{O}))} := \sup_{0 < h < T} h^{-\gamma} \|\hat{\psi}^{m,n}(\cdot + h) - \hat{\psi}^{m,n}(\cdot)\|_{L^2(0, T-h; L^2(\mathcal{O}))} \leq C, \quad (4.85)$$

where $0 < \gamma \leq \frac{K}{4(K+1)}$ when $d = 2$ and $0 < \gamma \leq 1/8$ when $d = 3$.

From (4.81), (4.82) and (4.85) and by an application of the Aubin–Lions Lemma we deduce that there exists a subsequence (not relabelled) such that

$$\hat{\psi}^{m,n} \rightharpoonup \hat{\psi}^m \quad \text{weakly in } L^2(0, T; W^{1,2}(\mathcal{O})), \quad (4.86)$$

$$\partial_t(M^m \zeta(\rho^{m,n}) \hat{\psi}^{m,n}) \rightharpoonup \partial_t(M^m \zeta(\rho^m) \hat{\psi}^m) \quad \text{weakly in } L^2(0, T; (W^{1,2}(\mathcal{O}))'), \quad (4.87)$$

$$\hat{\psi}^{m,n} \rightarrow \hat{\psi}^m \quad \text{strongly in } L^2(0, T; L^2(\mathcal{O})). \quad (4.88)$$

It then follows from (4.77), (4.88) and the global Lipschitz continuity of the mapping $s \in \mathbb{R} \mapsto [s]_+ \in \mathbb{R}_{\geq 0}$ that

$$\varrho^{m,n} \rightarrow \varrho^m \quad \text{strongly in } L^2(0, T; L^2(\Omega)),$$

where

$$\varrho^m = \zeta(\rho^m) \int_D M^m [\hat{\psi}^m]_+ d\mathbf{q},$$

and therefore, thanks to (4.76) and the assumption (2.7) on μ , the dominated convergence theorem implies that

$$\mu(\rho^{m,n}, \varrho^{m,n}) \rightarrow \mu(\rho^m, \varrho^m) \quad \text{strongly in } L^1(Q).$$

Hence, because $0 \leq \mu_{\min} \leq \mu(\cdot, \cdot) \leq \mu_{\max} < \infty$, it follows that

$$\mu(\rho^{m,n}, \varrho^{m,n}) \rightarrow \mu(\rho^m, \varrho^m) \quad \text{strongly in } L^p(Q), \quad (4.89)$$

for any $p \in [1, \infty)$.

Now the above convergence results (4.68), (4.70) and (4.75) for $\rho^{m,n}$, (4.89) for $\mu(\rho^{m,n}, \varrho^{m,n})$, (4.77) for $\zeta(\rho^{m,n})$, (4.86), (4.87) and (4.88) for $\hat{\psi}^{m,n}$, (4.65) and (4.66) for $c_i^{m,n}$, (4.67) for $\mathbf{v}^{m,n}$ and (4.57) for $\tau^{m,n}$ enable us to pass to the limit $n \rightarrow \infty$ in (4.15)–(4.17) to obtain the following:

$$\langle \partial_t \rho^m, \eta \rangle - (\mathbf{v}^m \rho^m, \nabla_x \eta) = 0, \quad \text{for all } \eta \in C^{0,1}(\bar{\Omega}) \text{ and a.e. } t \in (0, T), \quad (4.90)$$

$$\begin{aligned} \langle \partial_t(\rho^m \mathbf{v}^m), \mathbf{w}_i \rangle - (\rho^m \mathbf{v}^m \otimes \mathbf{v}^m, \nabla_x \mathbf{w}_i) + (\mu(\rho^m, \varrho^m) D(\mathbf{v}^m), \nabla_x \mathbf{w}_i) \\ = -(\tau^m, \nabla_x \mathbf{w}_i) + (\rho^m \mathbf{f}^m, \mathbf{w}_i) \quad \text{for all } i = 1, \dots, m \text{ and a.e. } t \in (0, T), \end{aligned} \quad (4.91)$$

and

$$\begin{aligned} \left\langle \partial_t(M^m \zeta(\rho^m) \hat{\psi}^m), \varphi \right\rangle_{\mathcal{O}} - \left(M^m \zeta(\rho^m) \mathbf{v}^m \hat{\psi}^m, \nabla_x \varphi \right)_{\mathcal{O}} - \left(M \zeta(\rho^m) \Lambda_\ell(\hat{\psi}^m) (\nabla_x \mathbf{v}^m) \mathbf{q}, \nabla_{\mathbf{q}} \varphi \right)_{\mathcal{O}} \\ + (M^m \nabla_x \hat{\psi}^m, \nabla_x \varphi)_{\mathcal{O}} + \left(M^m \mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^m), \nabla_{\mathbf{q}} \varphi \right)_{\mathcal{O}} = 0 \quad \text{for all } \varphi \in W^{1,2}(\mathcal{O}) \text{ and a.e. } t \in (0, T). \end{aligned} \quad (4.92)$$

It is obvious that $\mathbf{v}^m(x, 0) = \mathbf{v}_0^m(x)$. We can deduce using standard calculations that

$$\lim_{t \rightarrow 0_+} \|\hat{\psi}^m(\cdot, t) - T_\ell(\hat{\psi}_0^m(\cdot))\|_{L^2(\mathcal{O})} = 0.$$

Also, from (4.77) we deduce that

$$\zeta(\rho^{m,n}) \rightarrow \zeta(\rho^m) \quad \text{a.e. in } Q. \quad (4.93)$$

Similarly, from (4.88) we deduce that

$$\hat{\psi}^{m,n} \rightarrow \hat{\psi}^m \quad \text{a.e. in } \mathcal{O} \times (0, T).$$

Therefore,

$$\zeta(\rho^{m,n}) \hat{\psi}^{m,n} \rightarrow \zeta(\rho^m) \hat{\psi}^m \quad \text{a.e. in } \mathcal{O} \times (0, T).$$

Finally, the dominated convergence theorem gives that

$$\tau^m = -k \int_D \left[K M \zeta(\rho^m) T_\ell(\hat{\psi}^m) I + \sum_{j=1}^K \zeta(\rho^m) T_\ell(\hat{\psi}^m) \nabla_{\mathbf{q}^j} M \otimes \mathbf{q}^j \right] d\mathbf{q} \quad \text{a.e. in } Q. \quad (4.94)$$

4.5 Passage to the limit with m

Since $\mathbf{v}^m \in L^2(0, T; W_{0,\text{div}}^{1,2}(\Omega; \mathbb{R}^d)) \hookrightarrow L^1(0, T; W_{0,\text{div}}^{1,1}(\Omega; \mathbb{R}^d))$ and the initial data $\rho^m(0) = \rho_0^m \in C^1(\overline{\Omega}) \subset L^\infty(\Omega)$, we deduce from Theorem VI.1.6 in [5] that there exists a unique weak solution $\rho^m \in L^\infty(\Omega \times (0, T))$ that solves (4.90). Moreover, by Theorem VI.1.3 in [5] we have that $\rho^m \in C([0, T]; L^p(\Omega))$, where $1 \leq p < \infty$, and

$$\begin{aligned} \sup_{t \in (0, T)} \|\rho^m(t)\|_{L^\infty(\Omega)} &\leq \|\rho_0^m\|_{L^\infty(\Omega)} \leq \rho_{\max}, \\ \rho^m &\geq \rho_{\min} \quad \text{a.e. in } Q. \end{aligned} \quad (4.95)$$

By the renormalization property, for any $\beta \in C^1(\mathbb{R})$, $\beta(\rho^m)$ satisfies

$$\frac{\partial \beta(\rho^m)}{\partial t} + \text{div}_x(\beta(\rho^m) \mathbf{v}^m) = 0 \quad (4.96)$$

in the distributional sense. We note that $\zeta(\rho^m)$ is a renormalized solution in the sense that (4.96) is satisfied with $\beta = \zeta$.

We note further that thanks to (4.63) it follows by weak lower semicontinuity and passage to the limit as $n \rightarrow \infty$ that

$$\|\mathbf{v}^m\|_{L^\infty(0, T; W^{1,\infty}(\Omega; \mathbb{R}^d))} \leq C(m, \ell). \quad (4.97)$$

We would now like to show that $\hat{\psi}^m \geq 0$ a.e. in $\mathcal{O} \times (0, T)$. Before doing so, we shall revisit the question of regularity of ρ^m and will show that, thanks to the regularity of \mathbf{v}^m and ρ_0^m , in fact $\partial_t \rho^m \in L^\infty(Q)$ and $\nabla_x \rho^m \in L^\infty(Q)$, and therefore also $\partial_t \zeta(\rho^m) \in L^\infty(Q)$ and $\nabla_x \zeta(\rho^m) \in L^\infty(Q)$, whereby the renormalized equation (4.96) does in fact hold in the sense that

$$\partial_t \zeta(\rho^m) + \nabla_x \cdot (\mathbf{v}^m \zeta(\rho^m)) = 0 \quad \text{a.e. on } Q. \quad (4.98)$$

To this end we note that after passing to the limit $n \rightarrow \infty$ in (4.13),

$$\mathbf{v}^m(x, t) := \sum_{i=1}^m c_i^m(t) \mathbf{w}_i(x), \quad (4.99)$$

where $c_i^{m,n} \rightharpoonup c_i^m \in W^{1,\infty}((0, T))$ for each $m \geq 1$ as $n \rightarrow \infty$ by (4.65) and $\mathbf{w}_i \in C^1(\overline{\Omega}; \mathbb{R}^d)$ for each $i = 1, \dots, m$. Thanks to the orthonormality of $(\mathbf{w}_i)_{i=1}^m$ in $L^2(\Omega; \mathbb{R}^d)$ it therefore follows that

$$\|\mathbf{v}^m(\cdot)\|_{L^2(\Omega; \mathbb{R}^d)} = \left(\sum_{i=1}^m |c_i^m(\cdot)|^2 \right)^{\frac{1}{2}} \in W^{1,\infty}((0, T)).$$

For $x \in \overline{\Omega}$ and $t \in [0, T]$, consider the following initial-value problem for a system of ordinary differential equations:

$$\begin{aligned} \frac{dX^m}{ds}(x, t; s) &= \mathbf{v}^m(X^m(x, t; s), s), \quad s \in [0, T], \\ X^m(x, t; t) &= x. \end{aligned} \quad (4.100)$$

By the Cauchy–Lipschitz theorem, whose assumptions are satisfied thanks to the regularity properties of c_i^m and \mathbf{w}_i , $i = 1, \dots, m$, we deduce that, for each $(x, t) \in \overline{\Omega} \times [0, T]$, the system (4.100) has a unique solution $s \in [0, T] \mapsto X^m(x, t; s) \in C^1([0, T]; \mathbb{R}^d)$. Furthermore, thanks to the specific form of (4.99) and the fact that $\mathbf{w}_i \in C^1(\overline{\Omega}; \mathbb{R}^n)$ for $i = 1, \dots, m$, it follows that X^m is a differentiable function of x , and, for every $x \in \overline{\Omega}$ and $t \in [0, T]$,

$$\begin{aligned}\frac{d}{ds}(\nabla_x X^m(x, t; s)) &= \nabla_x \mathbf{v}^m(X^m(x, t; s), s) \nabla_x X^m(x, t; s), \quad s \in [0, T], \\ \nabla_x X^m(x, t; t) &= I,\end{aligned}$$

where I is the $d \times d$ identity matrix. Thus, upon integration and recalling the form of \mathbf{v}^m from (4.99), we deduce that

$$\begin{aligned}\nabla_x X^m(x, t; s) &= \nabla_x X^m(x, t; t) \\ &\quad + \int_t^s \sum_{i=1}^m c_i^m(\sigma) (\nabla_x \mathbf{w}_i)(X^m(x, t; \sigma)) \nabla_x X^m(x, t; \sigma) d\sigma.\end{aligned}$$

By Liouville's formula for matrix-differential equations, and recalling that the functions \mathbf{w}_i , $i = 1, 2, \dots$ are divergence-free, whereby the matrix $\nabla_x \mathbf{v}^m$ is trace-free, we have that

$$\det(\nabla_x X^m(x, t; s)) = \det(\nabla_x X^m(x, t; t)) \exp\left(\int_t^s \operatorname{tr}[(\nabla_x \mathbf{v}^m)(X^m(x, t; \sigma), \sigma)] d\sigma\right) = \det(I) \exp(0) = 1$$

for all $x \in \overline{\Omega}$ and all $t, s \in [0, T]$. Thus, the mapping $x \in \overline{\Omega} \mapsto X^m(x, t; s) \in \overline{\Omega}$ has a nonvanishing Jacobian for all $x \in \overline{\Omega}$ and all $t, s \in [0, T]$ and is therefore an invertible C^1 bijection from $\overline{\Omega}$ onto $\overline{\Omega}$, with C^1 inverse $y \in \overline{\Omega} \mapsto X^m(y, s; t)$ for all $t, s \in [0, T]$. In what follows we require a bound on $\nabla_x X^m$.

To this end, let $\|\cdot\|$ be a matrix norm on $\mathbb{R}^{d \times d}$ induced by a vector norm on \mathbb{R}^d . Then, $\|I\| = 1$ and therefore, because $X^m(x, t; t) = I$, also

$$\begin{aligned}\|\nabla_x X^m(x, t; s)\| &\leq 1 + \left| \int_t^s \sum_{i=1}^m |c_i^m(\sigma)| \|(\nabla_x \mathbf{w}_i)(X^m(x, t; \sigma))\| \|\nabla_x X^m(x, t; \sigma)\| d\sigma \right| \\ &\leq 1 + \left| \int_t^s \left(\sum_{i=1}^m |c_i^m(\sigma)|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^m \|\nabla_x \mathbf{w}_i\|_{C(\overline{\Omega}; \mathbb{R}^{d \times d})}^2 \right)^{\frac{1}{2}} \|\nabla_x X^m(x, t; \sigma)\| d\sigma \right| \\ &= 1 + \left(\sum_{i=1}^m \|\nabla_x \mathbf{w}_i\|_{C(\overline{\Omega}; \mathbb{R}^{d \times d})}^2 \right)^{\frac{1}{2}} \left| \int_t^s \|\mathbf{v}^m(\sigma)\|_{L^2(\Omega; \mathbb{R}^d)} \|\nabla_x X^m(x, t; \sigma)\| d\sigma \right| \\ &\leq 1 + \left(\sum_{i=1}^m \|\nabla_x \mathbf{w}_i\|_{C(\overline{\Omega}; \mathbb{R}^{d \times d})}^2 \right)^{\frac{1}{2}} \|\mathbf{v}^m\|_{L^\infty(0, T; L^2(\Omega; \mathbb{R}^d))} \left| \int_t^s \|\nabla_x X^m(x, t; \sigma)\| d\sigma \right|.\end{aligned}$$

Thus, by Gronwall's lemma, we have that

$$\|\nabla_x X^m(x, t; s)\| \leq \exp\left(|s - t| \left(\sum_{i=1}^m \|\nabla_x \mathbf{w}_i\|_{C(\overline{\Omega}; \mathbb{R}^{d \times d})}^2 \right)^{\frac{1}{2}} \|\mathbf{v}^m\|_{L^\infty(0, T; L^2(\Omega; \mathbb{R}^d))}\right).$$

Consequently,

$$\max_{x \in \overline{\Omega}, t, s \in [0, T]} \|\nabla_x X^m(x, t; s)\| \leq C_m := \exp\left(T \left(\sum_{i=1}^m \|\nabla_x \mathbf{w}_i\|_{C(\overline{\Omega}; \mathbb{R}^{d \times d})}^2 \right)^{\frac{1}{2}} \|\mathbf{v}^m\|_{L^\infty(0, T; L^2(\Omega; \mathbb{R}^d))}\right).$$

We shall show that the unique weak solution $\rho^m \in L^\infty(\Omega \times (0, T))$ of (4.90) is given by the expression

$$\rho^m(x, t) = \rho_0^m(X^m(x, t; 0)), \quad x \in \overline{\Omega}, t \in [0, T].$$

Indeed, for any $\eta \in \mathcal{D}(\Omega) := C_0^\infty(\Omega)$, we have as equalities in $\mathcal{D}'(0, T)$, that

$$\begin{aligned}
\langle \partial_t \rho^m(\cdot, t), \eta \rangle_{\mathcal{D}(\Omega)} &= \frac{d}{dt} \langle \rho^m(\cdot, t), \eta \rangle_{\mathcal{D}(\Omega)} \\
&= \frac{d}{dt} \int_{\Omega} \rho^m(x, t) \eta(x) dx = \frac{d}{dt} \int_{\Omega} \rho_0^m(X^m(x, t; 0)) \eta(x) dx \\
&= \frac{d}{dt} \int_{\Omega} \rho_0^m(y) \eta(X^m(y, 0; t)) \det[(\nabla_y X^m)(y, 0; t)] dy \\
&= \frac{d}{dt} \int_{\Omega} \rho_0^m(y) \eta(X^m(y, 0; t)) dy = \int_{\Omega} \rho_0^m(y) (\nabla_x \eta)(X^m(y, 0; t)) \cdot \frac{dX^m}{dt}(y, 0; t) dy.
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle \partial_t \rho^m, \eta \rangle_{\mathcal{D}(\Omega)} &= \int_{\Omega} \rho_0^m(y) (\nabla_x \eta)(X^m(y, 0; t)) \cdot \mathbf{v}^m(X^m(y, 0; t)) dy \\
&= \int_{\Omega} \rho_0^m(X^m(x, t; 0)) \nabla_x \eta(x) \cdot \mathbf{v}^m(x) \det[(\nabla_x X^m)(x, t; 0)] dx \\
&= \int_{\Omega} \rho_0^m(X^m(x, t; 0)) \nabla_x \eta(x) \cdot \mathbf{v}^m(x) dx \\
&= \int_{\Omega} \rho^m(x, t) \nabla_x \eta(x) \cdot \mathbf{v}^m(x) dx \\
&= \int_{\Omega} \rho^m(x, t) \mathbf{v}^m(x) \cdot \nabla_x \eta(x) dx \\
&= \langle \rho^m(\cdot, t) \mathbf{v}^m, \nabla_x \eta \rangle_{\mathcal{D}(\Omega)}.
\end{aligned}$$

Thus,

$$\langle \partial_t \rho^m, \eta \rangle_{\mathcal{D}(\Omega)} = \langle \rho^m(\cdot, t) \mathbf{v}^m, \nabla_x \eta \rangle_{\mathcal{D}(\Omega)} = \langle -\nabla_x \cdot (\rho^m(\cdot, t) \mathbf{v}^m), \eta \rangle_{\mathcal{D}(\Omega)},$$

as equalities in $\mathcal{D}'(0, T)$. Thus we have shown that ρ^m defined by $\rho^m(x, t) = \rho_0^m(X^m(x, t; 0))$ for $x \in \overline{\Omega}$ and $t \in [0, T]$ satisfies the equation

$$\frac{\partial \rho^m}{\partial t} + \nabla_x \cdot (\mathbf{v}^m \rho^m) = 0$$

in $\mathcal{D}'(\Omega \times (0, T))$; in addition $\rho^m(x, 0) = \rho_0^m(x)$ for all $x \in \Omega$. By the uniqueness of the weak solution to the problem (4.90) asserted by Theorem VI.1.3 in [5], it follows that the weak solution in question is given by $\rho^m(x, t) = \rho_0^m(X^m(x, t; 0))$ for $x \in \overline{\Omega}$ and $t \in [0, T]$.

Using this representation of ρ^m we are now in a position to show that, in addition to the regularity $\rho^m \in C([0, T]; L^p(\Omega))$, where $1 \leq p < \infty$, guaranteed by Theorem VI.1.3 in [5], in our case, thanks to the regularity of \mathbf{v}^m and ρ_0^m , ρ^m has additional regularity.

Indeed, because $\rho_0^m \in C^1(\overline{\Omega})$ and the mapping $x \in \overline{\Omega} \mapsto X^m(x, t; 0) \in \overline{\Omega}$ is a C^1 mapping, it follows by the chain rule that

$$\nabla_x \rho^m(x, t) = \nabla_x [\rho_0^m(X^m(x, t; 0))] = [(\nabla_x X^m)(x, t; 0)]^T (\nabla \rho_0^m)(X^m(x, t; 0))$$

for all $x \in \overline{\Omega}$ and all $t \in [0, T]$, whereby

$$\|\nabla_x \rho^m\|_{L^\infty(Q; \mathbb{R}^d)} \leq C_m \|\nabla_x \rho_0^m\|_{L^\infty(\Omega; \mathbb{R}^d)} \leq C(m).$$

As $\mathbf{v}^m \in W^{1, \infty}(Q; \mathbb{R}^d)$, it then also follows that

$$\left\| \frac{\partial \rho^m}{\partial t} \right\|_{L^\infty(Q)} \leq \|\mathbf{v}^m\|_{L^\infty(Q; \mathbb{R}^d)} \|\nabla_x \rho^m\|_{L^\infty(Q; \mathbb{R}^d)} \leq C_m \|\nabla_x \rho_0^m\|_{L^\infty(\Omega; \mathbb{R}^d)} \|\mathbf{v}^m\|_{L^\infty(Q; \mathbb{R}^d)} \leq C(m).$$

With these bounds it follows that

$$\frac{\partial \rho^m}{\partial t} + \nabla_x \cdot (\mathbf{v}^m \rho^m) = \frac{\partial \rho^m}{\partial t} + \mathbf{v}^m \cdot \nabla_x \rho^m = 0 \quad \text{a.e. in } Q,$$

and $\rho^m(x, 0) = \rho_0^m(x)$ for all $x \in \Omega$. Hence, the renormalized equation also holds in the sense that

$$\frac{\partial \beta(\rho^m)}{\partial t} + \nabla_x \cdot (\mathbf{v}^m \beta(\rho^m)) = \frac{\partial \beta(\rho^m)}{\partial t} + \mathbf{v}^m \cdot \nabla_x \beta(\rho^m) = 0 \quad \text{a.e. in } Q, \quad (4.101)$$

for any $\beta \in C^1(\mathbb{R})$, and $\beta(\rho^m(x, 0)) = \beta(\rho_0^m(x))$ for all $x \in \Omega$.

We are now ready to return to the proof of the nonnegativity of $\hat{\psi}^m$. By testing (4.92) with $(\hat{\psi}^m)_-$, using the renormalization property in the form (4.101), and bearing in mind that

$$\nabla_x \mathbf{v}^m \in L^\infty(Q; \mathbb{R}^{d \times d}), \quad \nabla_x \rho^m \in L^\infty(Q; \mathbb{R}^d) \quad \text{and} \quad \frac{\partial \rho^m}{\partial t} \in L^\infty(Q), \quad (4.102)$$

we obtain from (4.92) that

$$\frac{d}{dt} \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) ((\hat{\psi}^m)_-)^2 dx d\mathbf{q} + \int_{\mathcal{O}} M^m |\nabla_{x,\mathbf{q}} (\hat{\psi}^m)_-|^2 dx d\mathbf{q} \leq C(m, \ell) \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) ((\hat{\psi}^m)_-)^2 dx d\mathbf{q}.$$

Since $\hat{\psi}^m(0) = T_\ell(\hat{\psi}_0^m) \geq 0$ and $\zeta(\cdot) \geq \zeta_{\min} > 0$, we apply Gronwall's inequality to deduce that $(\hat{\psi}^m)_- \equiv 0$ in $\mathcal{O} \times (0, T)$. Therefore, we have

$$\hat{\psi}^m \geq 0 \quad \text{a.e. in } \mathcal{O} \times (0, T).$$

Next, we will derive m -independent estimates for \mathbf{v}^m and $\hat{\psi}^m$. We define

$$\begin{aligned} \mathcal{F}_\delta(s) &:= (s + \delta) \log(s + \delta) + 1, \quad \mathcal{F}(s) := s \log s + 1, \\ T_{\delta, \ell}(s) &:= \int_0^s \frac{\Lambda_\ell(t)}{t + \delta} dt = \int_0^s \frac{t \Gamma_\ell(t)}{t + \delta} dt. \end{aligned}$$

We set $\varphi := \log(\hat{\psi}^m + \delta) + 1$ in (4.92), where $\delta > 0$ is arbitrary, to obtain the following identity:

$$\begin{aligned} & \frac{d}{dt} \int_{\mathcal{O}} M^m \zeta(\rho^m) \mathcal{F}_\delta(\hat{\psi}^m) dx d\mathbf{q} - \int_{\mathcal{O}} M^m \partial_t \zeta(\rho^m) (\delta \log(\hat{\psi}^m + \delta) + 1 - \hat{\psi}^m) dx d\mathbf{q} \\ & - \int_{\mathcal{O}} M^m \zeta(\rho^m) \frac{\hat{\psi}^m}{\hat{\psi}^m + \delta} (\mathbf{v}^m \cdot \nabla_x \hat{\psi}^m) dx d\mathbf{q} + \int_{\mathcal{O}} \frac{M^m}{\hat{\psi}^m + \delta} \nabla_x \hat{\psi}^m : \nabla_x \hat{\psi}^m dx d\mathbf{q} \\ & + \int_{\mathcal{O}} \frac{M^m}{\hat{\psi}^m + \delta} \mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^m) : \nabla_{\mathbf{q}} \hat{\psi}^m dx d\mathbf{q} = \left(M \zeta(\rho^m) (\nabla_x \mathbf{v}^m) \mathbf{q}, \nabla_{\mathbf{q}} T_{\delta, \ell}(\hat{\psi}^m) \right)_{\mathcal{O}}. \end{aligned} \quad (4.103)$$

Since $\hat{\psi}^m \in W^{1,2}(\mathcal{O})$, we can test the equation (4.98) with the function

$$\eta = \int_D M^m (\delta \log(\hat{\psi}^m + \delta) + 1 - \hat{\psi}^m) d\mathbf{q}, \quad \nabla_x \eta = - \int_D \frac{M^m \hat{\psi}^m \nabla_x \hat{\psi}^m}{\hat{\psi}^m + \delta} d\mathbf{q}$$

to deduce that the second term and the third term in (4.103) add up to 0. Next, we integrate the resulting equation with respect to time over $(0, t)$ and use the assumption (1.11), to obtain that

$$\begin{aligned} & \int_{\mathcal{O}} M^m \zeta(\rho^m(\cdot, t)) \mathcal{F}_\delta(\hat{\psi}^m(\cdot, t)) dx d\mathbf{q} + C_1 \int_0^t \int_{\mathcal{O}} \frac{M^m}{\hat{\psi}^m + \delta} |\nabla_{x,\mathbf{q}} \hat{\psi}^m|^2 dx d\mathbf{q} ds \\ & \leq \int_{\mathcal{O}} M^m \zeta(\rho_0^m) \mathcal{F}_\delta(T_\ell(\hat{\psi}_0^m)) dx d\mathbf{q} + \int_0^t \left(M \zeta(\rho^m) (\nabla_x \mathbf{v}^m) \mathbf{q}, \nabla_{\mathbf{q}} T_{\delta, \ell}(\hat{\psi}^m) \right)_{\mathcal{O}} ds. \end{aligned}$$

Taking the limit $\delta \rightarrow 0_+$ in the above inequality, the first term on the left-hand side and the first-term on the right-hand side can be easily dealt with. For the second term on the left-hand side we apply the monotone convergence theorem. Therefore we get

$$\begin{aligned} & \int_{\mathcal{O}} M^m \zeta(\rho^m(\cdot, t)) \mathcal{F}(\hat{\psi}^m(\cdot, t)) dx d\mathbf{q} + 4C_1 \int_0^t \int_{\mathcal{O}} M^m \left| \nabla_{x,\mathbf{q}} \sqrt{\hat{\psi}^m} \right|^2 dx d\mathbf{q} ds \\ & \leq \int_{\mathcal{O}} M^m \zeta(\rho_0^m) \mathcal{F}(T_\ell(\hat{\psi}_0^m)) dx d\mathbf{q} + \limsup_{\delta \rightarrow 0_+} \int_0^t \left(M \zeta(\rho^m) (\nabla_x \mathbf{v}^m) \mathbf{q}, \nabla_{\mathbf{q}} T_{\delta, \ell}(\hat{\psi}^m) \right)_{\mathcal{O}} ds. \end{aligned} \quad (4.104)$$

For the last term on the right-hand side we use integration by parts (the boundary term vanishes since $M = 0$ on ∂D) and note that $\operatorname{div}_x \mathbf{v}^m = 0$. We obtain

$$\begin{aligned} \int_0^t \left(M \zeta(\rho^m)(\nabla_x \mathbf{v}^m) \mathbf{q}, \nabla_{\mathbf{q}} T_{\delta, \ell}(\hat{\psi}^m) \right)_{\mathcal{O}} ds &= - \int_0^t \left(\operatorname{div}_{\mathbf{q}} (M \zeta(\rho^m)(\nabla_x \mathbf{v}^m) \mathbf{q}), T_{\delta, \ell}(\hat{\psi}^m) \right)_{\mathcal{O}} ds \\ &= - \sum_{j=1}^K \int_0^t \left(\zeta(\rho^m)(\nabla_x \mathbf{v}^m), T_{\delta, \ell}(\hat{\psi}^m) \nabla_{\mathbf{q}^j} M \otimes \mathbf{q}^j \right)_{\mathcal{O}} ds. \end{aligned} \quad (4.105)$$

Since $|T_{\delta, \ell}(s) - T_{\ell}(s)| \leq \delta \log(1 + \frac{\ell}{\delta})$ for all $s \in [0, \infty)$, using (4.105) in (4.104) we can pass to the limit as $\delta \rightarrow 0_+$ in (4.104) to obtain

$$\begin{aligned} &\int_{\mathcal{O}} M^m \zeta(\rho^m(\cdot, t)) \mathcal{F}(\hat{\psi}^m(\cdot, t)) dx d\mathbf{q} + 4C_1 \int_0^t \int_{\mathcal{O}} M^m \left| \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m} \right|^2 dx d\mathbf{q} ds \\ &\leq \int_{\mathcal{O}} M^m \zeta(\rho_0^m) \mathcal{F}(T_{\ell}(\hat{\psi}_0^m)) dx d\mathbf{q} - \sum_{j=1}^K \int_0^t \left(\zeta(\rho^m)(\nabla_x \mathbf{v}^m), T_{\ell}(\hat{\psi}^m) \nabla_{\mathbf{q}^j} M \otimes \mathbf{q}^j \right)_{\mathcal{O}} ds. \end{aligned} \quad (4.106)$$

Multiplying the i -th equation in (4.91) by $c_i^m(t)$ and summing with respect to $i = 1, \dots, m$, we deduce the following energy identity:

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho^m |\mathbf{v}^m|^2 dx + \int_{\Omega} \mu(\rho^m, \varrho^m) |D(\mathbf{v}^m)|^2 dx = - \int_{\Omega} \tau^m : \nabla_x \mathbf{v}^m dx + \int_{\Omega} \rho^m \mathbf{f}^m \cdot \mathbf{v}^m dx. \quad (4.107)$$

Using $\operatorname{div}_x \mathbf{v}^m = 0$ and (4.94) we have that

$$\int_{\Omega} \tau^m : \nabla_x \mathbf{v}^m dx = -k \sum_{j=1}^K \left(\zeta(\rho^m)(\nabla_x \mathbf{v}^m), T_{\ell}(\hat{\psi}^m) \nabla_{\mathbf{q}^j} M \otimes \mathbf{q}^j \right)_{\mathcal{O}}.$$

Integrating (4.107) with respect to time over $(0, t)$ and multiplying (4.106) by k and adding the results we get

$$\begin{aligned} &k \int_{\mathcal{O}} M^m \zeta(\rho^m(\cdot, t)) \mathcal{F}(\hat{\psi}^m(\cdot, t)) dx d\mathbf{q} + \frac{1}{2} \int_{\Omega} \rho^m(\cdot, t) |\mathbf{v}^m(\cdot, t)|^2 dx \\ &\quad + \int_0^t \int_{\Omega} \mu(\rho^m, \varrho^m) |D(\mathbf{v}^m)|^2 dx ds + 4kC_1 \int_0^t \int_{\mathcal{O}} M^m \left| \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m} \right|^2 dx d\mathbf{q} ds \\ &\leq k \int_{\mathcal{O}} M^m \zeta(\rho_0^m) \mathcal{F}(T_{\ell}(\hat{\psi}_0^m)) dx d\mathbf{q} + \frac{1}{2} \int_{\Omega} \rho_0^m |\mathbf{v}_0|^2 dx + \int_0^t \int_{\Omega} \rho^m \mathbf{f}^m \cdot \mathbf{v}^m dx ds. \end{aligned} \quad (4.108)$$

Using the assumption (2.7), Korn's inequality (2.2) and Gronwall's inequality we arrive at the following uniform estimate:

$$\begin{aligned} &k \zeta_{\min} \sup_{t \in (0, T)} \|M^m \mathcal{F}(\hat{\psi}^m(\cdot, t))\|_{L^1(\mathcal{O})} + \frac{\rho_{\min}}{2} \sup_{t \in (0, T)} \|\mathbf{v}^m(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^d)}^2 \\ &\quad + \frac{\mu_{\min} c_0}{2} \int_0^T \|\mathbf{v}^m\|_{W^{1,2}(\Omega; \mathbb{R}^d)}^2 dt + 4kC_1 \int_0^T \int_{\mathcal{O}} M^m \left| \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m} \right|^2 dx d\mathbf{q} dt \\ &\leq k \zeta_{\max} \int_{\mathcal{O}} M^m \mathcal{F}(T_{\ell}(\hat{\psi}_0^m)) dx d\mathbf{q} + \frac{\rho_{\max}}{2} \|\mathbf{v}_0\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \frac{\rho_{\max}^2}{2\mu_{\min} c_0} \int_0^T \|\mathbf{f}^m\|_{L^2(\Omega; \mathbb{R}^d)}^2 dt \\ &\leq C(\ell, \zeta_{\max}, \rho_{\max}, \hat{\psi}_0, \mathbf{v}_0, \mathbf{f}). \end{aligned} \quad (4.109)$$

By interpolation we have the following estimate for \mathbf{v}^m :

$$\|\mathbf{v}^m\|_{L^{\frac{2(d+2)}{d}}(Q; \mathbb{R}^d)} \leq C(\ell). \quad (4.110)$$

Thanks to the presence of the truncation function $T_\ell(\cdot)$ we have that

$$|\tau^m| \leq C(\ell). \quad (4.111)$$

Using (4.91), the uniform bounds (2.7), (4.109), (4.110) and (4.111), we obtain by an application of Hölder's inequality that

$$\begin{aligned} \int_0^T \|\partial_t(\rho^m \mathbf{v}^m)\|_{W_{\text{div}}^{-1,p}(\Omega; \mathbb{R}^d)}^p dt &= \int_0^T \left(\sup_{\mathbf{w} \in W_{0,\text{div}}^{1,p'}(\Omega; \mathbb{R}^d)} \frac{|\langle \partial_t(\rho^m \mathbf{v}^m), \mathbf{w} \rangle|}{\|\mathbf{w}\|_{W_{0,\text{div}}^{1,p'}(\Omega; \mathbb{R}^d)}} \right)^p dt \\ &\leq C \int_0^T \left(\sup_{\mathbf{w} \in W_{0,\text{div}}^{1,p'}(\Omega; \mathbb{R}^d)} \frac{|(\rho^m \mathbf{v}^m \otimes \mathbf{v}^m, \nabla_x \mathbf{w})|}{\|\nabla_x \mathbf{w}\|_{L^{p'}(\Omega; \mathbb{R}^{d \times d})}} \right)^p dt \\ &\quad + C \int_0^T \left(\sup_{\mathbf{w} \in W_{0,\text{div}}^{1,p'}(\Omega; \mathbb{R}^d)} \frac{(\mu(\rho^m, \varrho^m) D(\mathbf{v}^m), \nabla_x \mathbf{w})}{\|\nabla_x \mathbf{w}\|_{L^{p'}(\Omega; \mathbb{R}^{d \times d})}} \right)^p dt \\ &\quad + C \int_0^T \left(\sup_{\mathbf{w} \in W_{0,\text{div}}^{1,p'}(\Omega; \mathbb{R}^d)} \frac{|(\tau^m, \nabla_x \mathbf{w})|}{\|\nabla_x \mathbf{w}\|_{L^{p'}(\Omega; \mathbb{R}^{d \times d})}} \right)^p dt + C \int_0^T \left(\sup_{\mathbf{w} \in W_{0,\text{div}}^{1,p'}(\Omega; \mathbb{R}^d)} \frac{|(\rho^m \mathbf{f}^m, \mathbf{w})|}{\|\mathbf{w}\|_{L^{p'}(\Omega; \mathbb{R}^{d \times d})}} \right)^p dt \\ &\leq C(\rho_{\max}) \|\mathbf{v}^m\|_{L^{2p}(Q; \mathbb{R}^d)}^{2p} + C(\mu_{\max}) \|\nabla_x \mathbf{v}^m\|_{L^2(Q; \mathbb{R}^{d \times d})}^p + C\|\tau^m\|_{L^p(Q; \mathbb{R}^{d \times d})}^p + C(\rho_{\max}, \mathbf{f}) \\ &\leq C(\ell), \end{aligned} \quad (4.112)$$

where $p \in (1, \frac{d+2}{d}]$. By a similar calculation as in (4.64) we have that

$$\|\partial_t \rho^m\|_{L^2(0,T;(W^{1,p'}(\Omega))')} \leq C(\ell), \quad (4.113)$$

where $p \in (1, \infty)$ when $d = 2$ and $p \in (1, 6]$ when $d = 3$.

Next, we shall focus on the fractional time derivative of \mathbf{v}^m . Integrating (4.91) with respect to time over $(s, s+h)$, with $s < T-h$, we have that

$$\begin{aligned} \int_\Omega [(\rho^m \mathbf{v}^m)(s+h) - (\rho^m \mathbf{v}^m)(s)] \cdot \mathbf{w}_i dx &= \int_s^{s+h} \int_\Omega \rho^m \mathbf{v}^m \otimes \mathbf{v}^m : \nabla_x \mathbf{w}_i dx dt \\ &\quad - \int_s^{s+h} \int_\Omega \mu(\rho^m, \varrho^m) D(\mathbf{v}^m) : D(\mathbf{w}_i) dx dt \\ &\quad - \int_s^{s+h} \int_\Omega \tau^m : \nabla_x \mathbf{w}_i dx dt + \int_s^{s+h} \rho^m \mathbf{f}^m \cdot \mathbf{w}_i dx dt \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (4.114)$$

For I_1 , we have by Hölder's inequality, the Gagliardo–Nirenberg inequality (2.1) and Korn's inequality (2.2) that

$$\begin{aligned} |I_1| &\leq \rho_{\max} \left(\int_s^{s+h} \|\mathbf{v}^m\|_{L^q(\Omega; \mathbb{R}^d)} \|\mathbf{v}^m\|_{L^{\frac{2q}{q-2}}(\Omega; \mathbb{R}^d)} dt \right) \|\nabla_x \mathbf{w}_i\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \\ &\leq C \rho_{\max} \left(\int_s^{s+h} \|\mathbf{v}^m\|_{L^q(\Omega; \mathbb{R}^d)} \|\mathbf{v}^m\|_{L^2(\Omega; \mathbb{R}^d)}^{1-\frac{d}{q}} \|\mathbf{v}^m\|_{W^{1,2}(\Omega; \mathbb{R}^d)}^{\frac{d}{q}} dt \right) \|\nabla_x \mathbf{w}_i\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \\ &\leq C \rho_{\max} \|\nabla_x \mathbf{w}_i\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \|\mathbf{v}^m\|_{L^\infty(0,T;L^2(\Omega; \mathbb{R}^d))}^{1-\frac{d}{q}} \left(\int_s^{s+h} \|\mathbf{v}^m\|_{W^{1,2}(\Omega; \mathbb{R}^d)}^2 dt \right)^{\frac{d}{2q}} \\ &\quad \times \left(\int_s^{s+h} \|\mathbf{v}^m\|_{L^q(\Omega; \mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \left(\int_s^{s+h} 1^{\frac{2q}{q-d}} dt \right)^{\frac{q-d}{2q}} \\ &\leq C h^{\frac{q-d}{2q}} \|D(\mathbf{w}_i)\|_{L^2(\Omega; \mathbb{R}^{d \times d})}, \end{aligned}$$

where $q \in (2, \infty)$ when $d = 2$ and $q \in (3, 6]$ when $d = 3$. For I_2 , we have by (4.109) that

$$|I_2| \leq \mu_{\max} h^{\frac{1}{2}} \|D(\mathbf{v}^m)\|_{L^2(s, s+h; L^2(\Omega; \mathbb{R}^{d \times d}))} \|D(\mathbf{w}_i)\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \leq Ch^{\frac{1}{2}} \|D(\mathbf{w}_i)\|_{L^2(\Omega; \mathbb{R}^{d \times d})}.$$

For I_3 , we have by (4.111) that

$$|I_3| \leq Ch^{\frac{1}{2}} \|D(\mathbf{w}_i)\|_{L^2(\Omega; \mathbb{R}^{d \times d})}.$$

For I_4 , we have by Hölder's inequality and Korn's inequality (2.2) that

$$|I_4| \leq \rho_{\max} h^{\frac{1}{2}} \|\mathbf{f}^m\|_{L^2(s, s+h; L^2(\Omega; \mathbb{R}^d))} \|\mathbf{w}_i\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \leq Ch^{\frac{1}{2}} \|D(\mathbf{w}_i)\|_{L^2(\Omega; \mathbb{R}^{d \times d})}.$$

Hence,

$$\left| \int_{\Omega} [(\rho^m \mathbf{v}^m)(s+h) - (\rho^m \mathbf{v}^m)(s)] \cdot \mathbf{w}_i \, dx \right| \leq C(h^{\frac{1}{2}} + h^{\frac{q-d}{2q}}) \|D(\mathbf{w}_i)\|_{L^2(\Omega; \mathbb{R}^{d \times d})}, \quad (4.115)$$

where $q \in (2, \infty)$ when $d = 2$ and $q \in (3, 6]$ when $d = 3$. Next, we take the test function $\eta = \chi_{[s, s+h]}(\mathbf{v}^m(s) \cdot \mathbf{w}_i)$ in

$$\int_0^T \int_{\Omega} \left(\frac{\partial \rho^m}{\partial t} \eta - \rho^m \mathbf{v}^m \cdot \nabla_x \eta \right) \, dx \, dt = 0,$$

to deduce that

$$\int_{\Omega} [\rho^m(s+h) - \rho^m(s)] (\mathbf{v}^m(s) \cdot \mathbf{w}_i) \, dx = \int_s^{s+h} \int_{\Omega} \rho^m \mathbf{v}^m \cdot \nabla_x (\mathbf{v}^m(s) \cdot \mathbf{w}_i) \, dx \, dt.$$

By Hölder's inequality we have that

$$\begin{aligned} & \left| \int_{\Omega} [\rho^m(s+h) - \rho^m(s)] (\mathbf{v}^m(s) \cdot \mathbf{w}_i) \, dx \right| \\ & \leq \rho_{\max} h^{\frac{1}{2}} \|\mathbf{v}^m\|_{L^2(s, s+h; L^q(\Omega; \mathbb{R}^d))} \|\nabla_x (\mathbf{v}^m(s) \cdot \mathbf{w}_i)\|_{L^{\frac{q}{q-1}}(\Omega; \mathbb{R}^d)} \\ & \leq \rho_{\max} h^{\frac{1}{2}} \|\mathbf{v}^m\|_{L^2(s, s+h; L^q(\Omega; \mathbb{R}^d))} \\ & \quad \times \left(\|\nabla_x \mathbf{v}^m(s)\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \|\mathbf{w}_i\|_{L^{\frac{2q}{q-2}}(\Omega; \mathbb{R}^d)} + \|\nabla_x \mathbf{w}_i\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \|\mathbf{v}^m(s)\|_{L^{\frac{2q}{q-2}}(\Omega; \mathbb{R}^d)} \right). \end{aligned} \quad (4.116)$$

Since

$$(\rho^m \mathbf{v}^m)(s+h) - (\rho^m \mathbf{v}^m)(s) = \rho^m(s+h) [\mathbf{v}^m(s+h) - \mathbf{v}^m(s)] + [\rho^m(s+h) - \rho^m(s)] \mathbf{v}^m(s),$$

it follows from (4.115) and (4.116) that

$$\begin{aligned} & \left| \int_{\Omega} \rho^m(s+h) [\mathbf{v}^m(s+h) - \mathbf{v}^m(s)] \cdot \mathbf{w}_i \, dx \right| \leq C(h^{\frac{1}{2}} + h^{\frac{q-d}{2q}}) \|D(\mathbf{w}_i)\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \\ & \quad + Ch^{\frac{1}{2}} \left(\|\nabla_x \mathbf{v}^m(s)\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \|\mathbf{w}_i\|_{L^{\frac{2q}{q-2}}(\Omega; \mathbb{R}^d)} + \|\nabla_x \mathbf{w}_i\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \|\mathbf{v}^m(s)\|_{L^{\frac{2q}{q-2}}(\Omega; \mathbb{R}^d)} \right), \end{aligned}$$

where $q \in (2, \infty)$ when $d = 2$ and $q \in (3, 6]$ when $d = 3$. Taking $\mathbf{w}_i = \mathbf{v}^m(s+h) - \mathbf{v}^m(s)$ in the above inequality and integrating with respect to s we obtain that

$$\begin{aligned} & \left| \int_0^{T-h} \int_{\Omega} \rho^m(s+h) [\mathbf{v}^m(s+h) - \mathbf{v}^m(s)]^2 \, dx \, ds \right| \\ & \leq 2C(h^{\frac{1}{2}} + h^{\frac{q-d}{2q}}) \left(\int_0^T \|D(\mathbf{v}^m(s))\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2 \, ds \right)^{\frac{1}{2}} \\ & \quad + 4Ch^{\frac{1}{2}} \left(\int_0^T \|\nabla_x \mathbf{v}^m(s)\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^T \|\mathbf{v}^m(s)\|_{L^{\frac{2q}{q-2}}(\Omega; \mathbb{R}^d)}^2 \, ds \right)^{\frac{1}{2}} \\ & \leq Ch^{\frac{q-d}{2q}}, \end{aligned}$$

where $q \in (2, \infty)$ when $d = 2$ and $q \in (3, 6]$ when $d = 3$. The above inequality follows from Hölder's inequality, the uniform estimate (4.109) and the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, $p \in [1, \infty)$ when $d = 2$ and $p \in [1, 6]$ when $d = 3$. Since $\rho^m \geq \rho_{\min}$, we have

$$\|\mathbf{v}^m(\cdot + h) - \mathbf{v}^m(\cdot)\|_{L^2(0, T-h; L^2(\Omega; \mathbb{R}^d))} \leq Ch^\gamma,$$

where $0 < \gamma < 1/4$ when $d = 2$ and $0 < \gamma \leq 1/8$ when $d = 3$. Therefore, we obtain the following Nikolskiĭ norm estimate:

$$\|\mathbf{v}^m\|_{N_2^\gamma(0, T; L^2(\Omega; \mathbb{R}^d))} := \sup_{0 < h < T} h^{-\gamma} \|\mathbf{v}^m(\cdot + h) - \mathbf{v}^m(\cdot)\|_{L^2(0, T-h; L^2(\Omega; \mathbb{R}^d))} \leq C, \quad (4.117)$$

where $0 < \gamma < 1/4$ when $d = 2$ and $0 < \gamma \leq 1/8$ when $d = 3$. It follows from the m -independent estimates (4.95), (4.109), (4.117), (4.110), (4.112), (4.113) and (4.111) and an application of the Aubin–Lions Lemma that there exist subsequences (not relabelled) such that, as $m \rightarrow \infty$,

$$\rho^m \rightharpoonup \rho \quad \text{weak* in } L^\infty(\Omega \times (0, T)), \quad (4.118)$$

$$\partial_t \rho^m \rightharpoonup \partial_t \rho \quad \text{weakly in } L^2(0, T; (W^{1,p'}(\Omega))'), \quad (4.119)$$

$$\tau^m \rightharpoonup \tau \quad \text{weak* in } L^\infty(\Omega \times (0, T); \mathbb{R}^{d \times d}), \quad (4.120)$$

$$\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weak* in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^d)), \quad (4.121)$$

$$\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^d)), \quad (4.122)$$

$$\partial_t(\rho^m \mathbf{v}^m) \rightharpoonup \partial_t(\rho \mathbf{v}) \quad \text{weakly in } L^q(0, T; W_{\text{div}}^{-1,q}(\Omega; \mathbb{R}^d)), \quad (4.123)$$

$$\mathbf{v}^m \rightarrow \mathbf{v} \quad \text{strongly in } L^2(0, T; L^p(\Omega; \mathbb{R}^d)), \quad (4.124)$$

where $p \in [1, \infty)$ when $d = 2$ and $p \in [1, 6]$ when $d = 3$, and $q \in (1, \frac{d+2}{d})$. By the strong convergence of \mathbf{v}^m , we can perform a similar argument as in (4.72)–(4.75) to deduce that,

$$\rho^m \rightarrow \rho \quad \text{strongly in } L^p(\Omega \times (0, T)), \quad (4.125)$$

for any $p \in [1, \infty)$. With the convergence result (4.124) for \mathbf{v}^m we can perform a similar argument as in Theorem VI.1.9 in [5] and strengthen the convergence above to get

$$\rho^m \rightarrow \rho \quad \text{strongly in } C([0, T]; L^p(\Omega)), \quad (4.126)$$

for any $p \in [1, \infty)$. Therefore,

$$\zeta(\rho^m) \rightarrow \zeta(\rho) \quad \text{strongly in } C([0, T]; L^p(\Omega)), \quad (4.127)$$

for any $p \in [1, \infty)$.

Next, we shall prove the strong convergence of $\hat{\psi}^m$. First, by setting $\varphi \equiv 1$ in (4.92) we deduce that

$$\begin{aligned} 0 &\leq \int_{\mathcal{O}} M^m(\mathbf{q}) \zeta(\rho^m(x, t)) \hat{\psi}^m(x, \mathbf{q}, t) \, dx \, d\mathbf{q} = \int_{\mathcal{O}} M^m \zeta(\rho_0^m) T_\ell(\hat{\psi}_0^m) \, dx \, d\mathbf{q} \\ &\leq \zeta_{\max} \int_{\mathcal{O}} M^m \hat{\psi}_0^m \, dx \, d\mathbf{q} = \zeta_{\max} \int_{\mathcal{O}} M \hat{\psi}_0 \, dx \, d\mathbf{q} \leq C. \end{aligned}$$

As $\hat{\psi}^m \geq 0$ a.e. on $\mathcal{O} \times (0, T)$, it follows that

$$\varrho^m(x, t) = \zeta(\rho^m(x, t)) \int_D M^m(\mathbf{q}) [\hat{\psi}^m(x, \mathbf{q}, t)]_+ \, d\mathbf{q} = \zeta(\rho^m(x, t)) \int_D M^m(\mathbf{q}) \hat{\psi}^m(x, \mathbf{q}, t) \, d\mathbf{q} \geq 0. \quad (4.128)$$

By defining

$$\lambda^m(x, t) := \int_D M^m(\mathbf{q}) \hat{\psi}^m(x, \mathbf{q}, t) \, d\mathbf{q} \quad (4.129)$$

and setting $\varphi(x, \mathbf{q}, t) := \bar{\varphi}(x, t)$ in (4.92), we have the following equation satisfied by λ^m :

$$\langle \partial_t(\zeta(\rho^m)\lambda^m), \bar{\varphi} \rangle - (\zeta(\rho^m)\lambda^m \mathbf{v}^m, \nabla_x \bar{\varphi}) + (\nabla_x \lambda^m, \nabla_x \bar{\varphi}) = 0 \quad \text{for all } \bar{\varphi} \in W^{1,2}(\Omega) \text{ and a.e. } t \in (0, T), \quad (4.130)$$

supplemented by the initial condition $\lambda^m(0) := \lambda_0^m$, where, thanks to (2.6),

$$0 \leq \lambda_0^m(x) := \int_D M^m T_\ell(\hat{\psi}_0^m) d\mathbf{q} \leq \int_D M \hat{\psi}_0 d\mathbf{q} = \frac{1}{\zeta(\rho_0)} \int_D \psi_0 d\mathbf{q} \leq \frac{\varrho_{\max}}{\zeta_{\min}}.$$

Let $\omega := \sup_{x \in \Omega} \lambda_0^m(x)$, test (4.98) with the function $\eta = \omega \bar{\varphi}$, and subtract (4.130) from the resulting equation; this gives that

$$\langle \partial_t(\zeta(\rho^m)(\omega - \lambda^m)), \bar{\varphi} \rangle - (\zeta(\rho^m)(\omega - \lambda^m) \mathbf{v}^m, \nabla_x \bar{\varphi}) + (\nabla_x(\omega - \lambda^m), \nabla_x \bar{\varphi}) = 0 \quad \forall \bar{\varphi} \in W^{1,2}(\Omega) \quad (4.131)$$

for a.e. $t \in (0, T)$. Note that $\hat{\psi}^m \in L^2(0, T; W^{1,2}(\mathcal{O}))$ by (4.86). It therefore follows from (4.129) that $\lambda^m \in L^2(0, T; W^{1,2}(\Omega))$. Hence, because $\zeta(\rho^m) \in W^{1,\infty}(Q)$, which follows from (4.102), we have that

$$\zeta(\rho^m)(\omega - \lambda^m) \in L^2(0, T; W^{1,2}(\Omega)).$$

Furthermore, because $\zeta(\rho^m)(\omega - \lambda^m) \mathbf{v}^m \in L^2(0, T; L^2(\Omega))$ and $\nabla_x(\omega - \lambda^m) \in L^2(0, T; L^2(\Omega))$, it follows from (4.130) that

$$\partial_t(\zeta(\rho^m)(\omega - \lambda^m)) \in L^2(0, T; (W^{1,2}(\Omega))'),$$

where $(W^{1,2}(\Omega))'$ is the dual space of $W^{1,2}(\Omega)$. In addition, since $\zeta(\rho^m) \in W^{1,\infty}(Q)$ and $\lambda^m \in L^2(0, T; W^{1,2}(\Omega))$, we have that $\varrho^m \in L^2(0, T; W^{1,2}(\Omega))$. By (4.87) and (4.102) we also have that $\partial_t(M^m \zeta(\rho^m) \hat{\psi}^m) \in L^2(0, T; (W^{1,2}(\mathcal{O}))')$, and therefore $\partial_t \varrho^m \in L^2(0, T; (W^{1,2}(\Omega))')$. Thus, in summary,

$$\varrho^m \in L^2(0, T; W^{1,2}(\Omega)) \quad \text{and} \quad \partial_t \varrho^m \in L^2(0, T; (W^{1,2}(\Omega))'). \quad (4.132)$$

We shall now show that $\partial_t \lambda^m \in L^2(0, T; (W^{1,2}(\Omega))')$. For any $\phi \in W^{1,2}(\Omega)$ and $\theta \in C_0^\infty((0, T))$, we have

$$\begin{aligned} \int_0^T \langle \partial_t \lambda^m, \phi \rangle \theta dt &= - \int_0^T \int_\Omega \lambda^m \phi \partial_t \theta dx dt = - \int_0^T \int_\Omega \varrho^m \frac{\phi}{\zeta(\rho^m)} \partial_t \theta dx dt \\ &= - \int_0^T \int_\Omega \varrho^m \phi \left[\partial_t \left(\frac{\theta}{\zeta(\rho^m)} \right) - \theta \partial_t \left(\frac{1}{\zeta(\rho^m)} \right) \right] dx dt \\ &= \int_0^T \left\langle \partial_t \varrho^m, \frac{\phi}{\zeta(\rho^m)} \right\rangle \theta dt - \int_0^T \left(\int_\Omega \varrho^m \phi \frac{\zeta'(\rho^m) \partial_t \rho^m}{\zeta(\rho^m)^2} dx \right) \theta dt. \end{aligned}$$

Note that $\partial_t \rho^m$ is integrable since $\rho^m \in W^{1,\infty}(Q)$ by (4.102). From the last equality, we deduce that

$$\langle \partial_t \lambda^m, \phi \rangle = \left\langle \partial_t \varrho^m, \frac{\phi}{\zeta(\rho^m)} \right\rangle - \int_\Omega \varrho^m \phi \frac{\zeta'(\rho^m) \partial_t \rho^m}{\zeta(\rho^m)^2} dx \quad \text{a.e. on } \Omega$$

for all $\phi \in W^{1,2}(\Omega)$. It follows that

$$|\langle \partial_t \lambda^m, \phi \rangle| \leq \|\partial_t \varrho^m\|_{(W^{1,2}(\Omega))'} \left\| \frac{\phi}{\zeta(\rho^m)} \right\|_{W^{1,2}(\Omega)} + \|\varrho^m\|_{L^2(\Omega)} \left\| \phi \frac{\zeta'(\rho^m) \partial_t \rho^m}{\zeta(\rho^m)^2} \right\|_{L^2(\Omega)}.$$

Note that by (4.132) we have that $\|\partial_t \varrho^m\|_{(W^{1,2}(\Omega))'} \leq C(m)$ and $\|\varrho^m\|_{L^2(\Omega)} \leq C(m)$. Also, by using $\|\nabla_x \rho^m\|_{L^\infty(Q)} \leq C(m)$ and $\|\partial_t \rho^m\|_{L^\infty(Q)} \leq C(m)$, we obtain the following bounds

$$\begin{aligned} \left\| \frac{\phi}{\zeta(\rho^m)} \right\|_{W^{1,2}(\Omega)} &\leq C(m) \|\phi\|_{W^{1,2}(\Omega)}, \\ \left\| \phi \frac{\zeta'(\rho^m) \partial_t \rho^m}{\zeta(\rho^m)^2} \right\|_{L^2(\Omega)} &\leq C(m) \|\phi\|_{W^{1,2}(\Omega)}. \end{aligned}$$

Hence,

$$|\langle \partial_t \lambda^m, \phi \rangle| \leq C(m) \|\phi\|_{W^{1,2}(\Omega)},$$

which then implies that

$$\|\partial_t \lambda^m\|_{L^2(0,T;(W^{1,2}(\Omega))')}^2 = \int_0^T \|\partial_t \lambda^m\|_{(W^{1,2}(\Omega))'}^2 dt \leq C(m).$$

We have shown that

$$\lambda^m \in L^2(0,T;W^{1,2}(\Omega)) \quad \text{and} \quad \partial_t \lambda^m \in L^2(0,T;(W^{1,2}(\Omega))'),$$

whereby, upon defining $\alpha^m := \omega - \lambda^m$, also

$$\alpha^m \in L^2(0,T;W^{1,2}(\Omega)) \quad \text{and} \quad \partial_t \alpha^m \in L^2(0,T;(W^{1,2}(\Omega))'). \quad (4.133)$$

It then follows from (4.131) that, for any $\bar{\varphi} \in W^{1,2}(\Omega)$ and any $\theta \in C_0^\infty((0,T))$, we have

$$\begin{aligned} 0 &= - \int_0^T \int_\Omega \zeta(\rho^m) \alpha^m \bar{\varphi} \partial_t \theta \, dx \, dt - \int_0^T \int_\Omega \zeta(\rho^m) \alpha^m \mathbf{v}^m \cdot (\nabla_x \bar{\varphi}) \theta \, dx \, dt + \int_0^T \int_\Omega \nabla_x \alpha^m \cdot (\nabla_x \bar{\varphi}) \theta \, dx \, dt \\ &= - \int_0^T \int_\Omega \alpha^m [\partial_t(\zeta(\rho^m) \theta) - (\partial_t \zeta(\rho^m)) \theta] \bar{\varphi} \, dx \, dt \\ &\quad - \int_0^T \int_\Omega \alpha^m [\nabla_x \cdot (\zeta(\rho^m) \mathbf{v}^m \bar{\varphi}) - (\nabla_x \cdot (\zeta(\rho^m) \mathbf{v}^m)) \bar{\varphi}] \theta \, dx \, dt + \int_0^T \int_\Omega \nabla_x \alpha^m \cdot (\nabla_x \bar{\varphi}) \theta \, dx \, dt \\ &= - \int_0^T \int_\Omega \alpha^m [\partial_t(\zeta(\rho^m) \theta) \bar{\varphi} + \nabla_x \cdot (\zeta(\rho^m) \mathbf{v}^m \bar{\varphi}) \theta] \, dx \, dt \\ &\quad + \int_0^T \int_\Omega \alpha^m [\partial_t \zeta(\rho^m) + \nabla_x \cdot (\zeta(\rho^m) \mathbf{v}^m)] \bar{\varphi} \theta \, dx \, dt + \int_0^T \int_\Omega \nabla_x \alpha^m \cdot (\nabla_x \bar{\varphi}) \theta \, dx \, dt \\ &= - \int_0^T \int_\Omega \alpha^m [\partial_t(\zeta(\rho^m) \theta) \bar{\varphi} + \nabla_x \cdot (\zeta(\rho^m) \mathbf{v}^m \bar{\varphi}) \theta] \, dx \, dt + \int_0^T \int_\Omega \nabla_x \alpha^m \cdot (\nabla_x \bar{\varphi}) \theta \, dx \, dt, \end{aligned}$$

where in the transition to the last line we made use of the fact that the renormalized equation (4.101) satisfied by ρ^m , with $\beta = \zeta$, holds almost everywhere on $Q = \Omega \times (0,T)$. We then deduce from the last equality that

$$\int_0^T \langle \partial_t \alpha^m, \zeta(\rho^m) \bar{\varphi} \rangle \theta \, dt + \int_0^T \left(\int_\Omega \nabla_x \alpha^m \cdot \zeta(\rho^m) \mathbf{v}^m \bar{\varphi} \, dx \right) \theta \, dt + \int_0^T \left(\int_\Omega \nabla_x \alpha^m \cdot (\nabla_x \bar{\varphi}) \, dx \right) \theta \, dt = 0,$$

for all $\bar{\varphi} \in W^{1,2}(\Omega)$ and all $\theta \in C_0^\infty((0,T))$. Hence, also

$$\langle \partial_t \alpha^m, \zeta(\rho^m) \bar{\varphi} \rangle + \int_\Omega \nabla_x \alpha^m \cdot \zeta(\rho^m) \mathbf{v}^m \bar{\varphi} \, dx + \int_\Omega \nabla_x \alpha^m \cdot \nabla_x \bar{\varphi} \, dx = 0 \quad (4.134)$$

for all $\bar{\varphi} \in W^{1,2}(\Omega)$ and for a.e. $t \in (0,T)$. Our objective is to show that $\alpha^m = \omega - \varrho^m \geq 0$ a.e. on Q . To this end, it seems tempting to take $\bar{\varphi} = [\alpha^m]_-$ in (4.134); however, the calculation that would by use of the renormalized equation (4.101) satisfied by ρ^m with $\beta = \zeta$ result in the desired inequality

$$\int_\Omega \zeta(\rho) ([\alpha^m(x,t)]_-)^2 \, dx \leq \int_\Omega \zeta(\rho) ([\alpha^m(x,0)]_-)^2 \, dx = \int_\Omega \zeta(\rho) ([\omega - \varrho_0^m(x)]_-)^2 \, dx = 0,$$

which would then ultimately imply that $[\omega - \varrho^m(x,t)]_- = 0$, a.e. on Q , is difficult to justify rigorously. The main obstacle in the approach is the limited regularity of α^m in conjunction with the fact that the function $s \in \mathbb{R} \mapsto [s]_+ \in \mathbb{R}_+$ is only Lipschitz continuous. An equivalent way of phrasing this would be

to say that we define $G(s) := \frac{1}{2}([s]_+)^2$ for $s \in \mathbb{R}$, and we take as our test function in (4.134) the function $\bar{\varphi} = G'(\alpha^m)$. We shall overcome this difficulty by making a different choice of the function G .

For $\delta \in (0, 1)$, let

$$G_\delta(s) := \begin{cases} \frac{s^2 - \delta^2}{2\delta} + s(\log \delta - 1) + 1 & \text{for } s \leq \delta, \\ s(\log s - 1) + 1 & \text{for } s \geq \delta. \end{cases}$$

It then follows that

$$G'_\delta(s) = \begin{cases} \frac{s}{\delta} + \log \delta - 1 & \text{for } s \leq \delta, \\ \log s & \text{for } s \geq \delta, \end{cases}$$

and

$$G''_\delta(s) = \begin{cases} 1/\delta & \text{for } s \leq \delta, \\ 1/s & \text{for } s \geq \delta. \end{cases}$$

Clearly $G_\delta \in C^{2,1}(\mathbb{R})$, $G_\delta(s) \geq 0$ for all $s \in \mathbb{R}$, $G_\delta(1) = 0$, G_δ is strictly convex, and in addition

$$G_\delta(s) \geq \begin{cases} \frac{s^2}{2\delta} & \text{for } s \leq 0, \\ 0 & \text{for } s \geq 0. \end{cases}$$

We shall choose $\bar{\varphi} = G'_\delta(\alpha^m)$ in (4.134), and will at the end of the calculation pass to the limit $\delta \rightarrow 0_+$. Hence, a.e. on $(0, T)$,

$$\langle \partial_t \alpha^m, \zeta(\rho^m) G'_\delta(\alpha^m) \rangle + \int_\Omega \nabla_x \alpha^m \cdot \zeta(\rho^m) \mathbf{v}^m G'_\delta(\alpha^m) dx + \int_\Omega \nabla_x \alpha^m \cdot \nabla_x G'_\delta(\alpha^m) dx = 0.$$

Equivalently,

$$\langle \partial_t \alpha^m, G'_\delta(\alpha^m) \zeta(\rho^m) \rangle + \int_\Omega \zeta(\rho^m) \mathbf{v}^m \cdot \nabla_x G_\delta(\alpha^m) dx + \int_\Omega G''_\delta(\alpha^m) |\nabla_x \alpha^m|^2 dx = 0 \quad \text{a.e. on } (0, T).$$

Thus, after partial integration in the second term on the left-hand side noting that $\mathbf{v}^m|_{\partial\Omega \times (0, T)} = 0$ and $\operatorname{div} \mathbf{v}^m = 0$ on Q we have that

$$\langle \partial_t \alpha^m, G'_\delta(\alpha^m) \zeta(\rho^m) \rangle - \int_\Omega (\mathbf{v}^m \cdot \nabla_x \zeta(\rho^m)) G_\delta(\alpha^m) dx + \int_\Omega G''_\delta(\alpha^m) |\nabla_x \alpha^m|^2 dx = 0 \quad \text{a.e. on } (0, T).$$

Next, we invoke the renormalized equation (4.101) to transform the second integral on the left-hand side further, resulting in

$$\langle \partial_t \alpha^m, G'_\delta(\alpha^m) \zeta(\rho^m) \rangle + \int_\Omega (\partial_t \zeta(\rho^m)) G_\delta(\alpha^m) dx + \int_\Omega G''_\delta(\alpha^m) |\nabla_x \alpha^m|^2 dx = 0 \quad \text{a.e. on } (0, T). \quad (4.135)$$

Our objective is now to show that the first term of the left-hand side can be rewritten as follows:

$$\langle \partial_t \alpha^m, G'_\delta(\alpha^m) \zeta(\rho^m) \rangle = \langle \partial_t G_\delta(\alpha^m), \zeta(\rho^m) \rangle \quad \text{a.e. on } (0, T).$$

To this end, we extend α^m by 0 from $\Omega \times [0, T]$ to $\Omega \times \mathbb{R}$, and we mollify the resulting function, still denoted by α^m , with respect to t ; to be more specific, for a nonnegative function $\chi \in C_0^\infty((0, T))$ such that $\int_{\mathbb{R}} \chi(t) dt = 1$ and $\varepsilon \in (0, 1)$, we let $\chi_\varepsilon(t) = \frac{1}{\varepsilon} \chi(\frac{t}{\varepsilon})$, and define

$$\alpha_\varepsilon^m := \alpha^m *_t \chi_\varepsilon \in C_0^\infty(\mathbb{R}; W^{1,2}(\Omega)),$$

where $*_t$ denotes convolution with respect to t . It then follows from (4.133) that

$$\|\alpha_\varepsilon^m - \alpha^m\|_{L^2(0, T; W^{1,2}(\Omega))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0_+, \quad (4.136)$$

and, by Young's inequality for convolutions,

$$\|\partial_t \alpha_\varepsilon^m\|_{L^2(0,T;(W^{1,2}(\Omega))')} \leq \|\partial_t \alpha^m\|_{L^2(0,T;(W^{1,2}(\Omega))')}. \quad (4.137)$$

Now, because $\alpha_\varepsilon^m \in C_0^\infty(\mathbb{R}; W^{1,2}(\Omega))$, it follows by the chain rule and an application of Hölder's inequality that $\partial_t G_\delta(\alpha_\varepsilon^m) = G'_\delta(\alpha_\varepsilon^m) \partial_t \alpha_\varepsilon^m \in C([0, T]; L^2(\Omega))$, and therefore, for any $\theta \in C_0^\infty((0, T))$,

$$\begin{aligned} \int_0^T \langle \partial_t G_\delta(\alpha_\varepsilon^m), \zeta(\rho^m) \rangle \theta(t) dt &= \int_0^T \int_\Omega \partial_t G_\delta(\alpha_\varepsilon^m) \zeta(\rho^m) \theta(t) dx dt \\ &= \int_0^T \int_\Omega G'_\delta(\alpha_\varepsilon^m) \partial_t \alpha_\varepsilon^m \zeta(\rho^m) \theta(t) dx dt \\ &= \int_0^T \int_\Omega G'_\delta(\alpha^m) \partial_t \alpha_\varepsilon^m \zeta(\rho^m) \theta(t) dx dt \\ &\quad + \int_0^T \int_\Omega (G'_\delta(\alpha_\varepsilon^m) - G'_\delta(\alpha^m)) \partial_t \alpha_\varepsilon^m \zeta(\rho^m) \theta(t) dx dt \\ &=: T_{1,\varepsilon} + T_{2,\varepsilon}. \end{aligned} \quad (4.138)$$

We shall now pass to the limit $\varepsilon \rightarrow 0_+$ in this equality. First, because, by (4.137), $\partial_t \alpha_\varepsilon^m$ is, for each $m \geq 1$, uniformly bounded in $L^2(0, T; (W^{1,2}(\Omega))')$ and because $L^2(0, T; (W^{1,2}(\Omega))')$ is, by definition, the dual space of the normed linear space $L^2(0, T; W^{1,2}(\Omega))$, thanks to the Banach–Alaoglu theorem we can extract a weak* convergent subsequence, for which (without indicating the subsequence in our notation) we have by the uniqueness of the weak* limit that

$$\partial_t \alpha_\varepsilon^m \rightharpoonup^* \partial_t \alpha^m \quad \text{in } L^2(0, T; (W^{1,2}(\Omega))').$$

Because $G'_\delta(\alpha^m) \zeta(\rho^m) \theta \in L^2(0, T; W^{1,2}(\Omega))$, the predual of $L^2(0, T; (W^{1,2}(\Omega))')$, it therefore follows that

$$\lim_{\varepsilon \rightarrow 0_+} T_{1,\varepsilon} = \int_0^T \langle \partial_t \alpha^m, G'_\delta(\alpha^m) \zeta(\rho^m) \rangle \theta(t) dt.$$

Next, we will show that $T_{2,\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0_+$. We begin by noting that

$$\begin{aligned} |T_{2,\varepsilon}| &\leq \|\partial_t \alpha_\varepsilon^m\|_{L^2(0,T;(W^{1,2}(\Omega))')} \| \zeta(\rho^m) \theta (G'_\delta(\alpha_\varepsilon^m) - G'_\delta(\alpha^m)) \|_{L^2(0,T;W^{1,2}(\Omega))} \\ &\leq \sqrt{2} \left(\|\zeta(\rho^m) \theta\|_{L^\infty(Q)}^2 + \|\nabla_x(\zeta(\rho^m)) \theta\|_{L^\infty(Q;\mathbb{R}^d)}^2 \right)^{\frac{1}{2}} \|G'_\delta(\alpha_\varepsilon^m) - G'_\delta(\alpha^m)\|_{L^2(0,T;W^{1,2}(\Omega))}. \end{aligned} \quad (4.139)$$

It therefore remains to show that

$$\|G'_\delta(\alpha_\varepsilon^m) - G'_\delta(\alpha^m)\|_{L^2(0,T;W^{1,2}(\Omega))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0_+. \quad (4.140)$$

First, observe that

$$\begin{aligned} \|G'_\delta(\alpha_\varepsilon^m) - G'_\delta(\alpha^m)\|_{L^2(0,T;L^2(\Omega))} &\leq \|G''_\delta\|_{L^\infty(\mathbb{R})} \|\alpha_\varepsilon^m - \alpha^m\|_{L^2(0,T;L^2(\Omega))} \\ &\leq \frac{1}{\delta} \|\alpha_\varepsilon^m - \alpha^m\|_{L^2(0,T;L^2(\Omega))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0_+. \end{aligned} \quad (4.141)$$

Next we shall show that $\|\nabla_x(G'_\delta(\alpha_\varepsilon^m)) - \nabla_x(G'_\delta(\alpha^m))\|_{L^2(0,T;L^2(\Omega))} \rightarrow 0$. We begin by observing that

$$\nabla_x(G'_\delta(\alpha_\varepsilon^m)) = G''_\delta(\alpha_\varepsilon^m) \nabla_x \alpha_\varepsilon^m.$$

Thanks to the strong convergence (4.136), for m fixed, there exists a subsequence (with respect to ε , not indicated) such that $\alpha_\varepsilon^m \rightarrow \alpha^m$ a.e. on Q . Because $G''_\delta \in C^{0,1}(\mathbb{R})$, it then follows that $G''_\delta(\alpha_\varepsilon^m) \rightarrow G''_\delta(\alpha^m)$ a.e. on Q . Furthermore, $0 \leq G''_\delta(\alpha_\varepsilon^m) \leq 1/\delta$ a.e. on Q . In addition, thanks to a ‘converse’ of the dominated convergence theorem, which asserts that each subsequence of a strongly convergent sequence in $L^1(Q)$ contains a dominated sub-subsequence (c.f., for example, Theorem 1 in [18]), because by (4.136)

$|\alpha_\varepsilon^m - \alpha^m|^2 + |\nabla_x \alpha_\varepsilon^m - \nabla_x \alpha^m|^2 \rightarrow 0$ in $L^1(Q)$ as $\varepsilon \rightarrow 0_+$, there exists a nonnegative function $g \in L^1(Q)$ such that for a particular sub-subsequence (not indicated)

$$|\alpha_\varepsilon^m(x, t) - \alpha^m(x, t)|^2 + |\nabla_x \alpha_\varepsilon^m(x, t) - \nabla_x \alpha^m(x, t)|^2 \leq g(x, t) \quad \text{for a.e. } (x, t) \in Q. \quad (4.142)$$

For this same sub-subsequence,

$$G_\delta''(\alpha_\varepsilon^m) \nabla_x \alpha_\varepsilon^m \rightarrow G_\delta''(\alpha^m) \nabla_x \alpha^m \quad \text{a.e. on } Q$$

and

$$|G_\delta''(\alpha_\varepsilon^m(x, t)) \nabla_x \alpha_\varepsilon^m(x, t) - G_\delta''(\alpha^m(x, t)) \nabla_x \alpha^m(x, t)|^2 \leq \frac{2}{\delta^2} g(x, t) + \frac{4}{\delta^2} |\nabla_x \alpha^m(x, t)|^2 \quad \text{a.e. on } Q.$$

Thus, we can pass to the limit over this sub-subsequence to deduce by the dominated convergence theorem that

$$\begin{aligned} & \|\nabla_x(G_\delta'(\alpha_\varepsilon^m)) - \nabla_x(G_\delta'(\alpha^m))\|_{L^2(0, T; L^2(\Omega; \mathbb{R}^d))}^2 \\ &= \int_Q |G_\delta''(\alpha_\varepsilon^m(x, t)) \nabla_x \alpha_\varepsilon^m(x, t) - G_\delta''(\alpha^m(x, t)) \nabla_x \alpha^m(x, t)|^2 dx dt \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0_+. \end{aligned} \quad (4.143)$$

By passing to the limit over this sub-subsequence, (4.141) and (4.143) imply (4.140), and then using (4.140) in (4.139), again by passage to the limit with $\varepsilon \rightarrow 0_+$ over this sub-subsequence implies that $\lim_{\varepsilon \rightarrow 0_+} |\mathbf{T}_{2, \varepsilon}| = 0$. Hence, from (4.138) we have, by passage to the limit over this sub-subsequence,

$$\lim_{\varepsilon \rightarrow 0_+} \int_0^T \langle \partial_t G_\delta(\alpha_\varepsilon^m), \zeta(\rho^m) \rangle \theta(t) dt = \int_0^T \langle \partial_t \alpha^m, G_\delta'(\alpha^m) \zeta(\rho^m) \rangle \theta(t) dt. \quad (4.144)$$

On the other hand, thanks to the smoothness of α_ε^m with respect to t , use of the chain rule and since $\partial_t G_\delta(\alpha_\varepsilon^m) \in C([0, T]; L^2(\Omega))$, the expression on the left-hand side of (4.144) can be rewritten as follows:

$$\begin{aligned} \int_0^T \langle \partial_t G_\delta(\alpha_\varepsilon^m), \zeta(\rho^m) \rangle \theta(t) dt &= \int_0^T \int_\Omega \partial_t G_\delta(\alpha_\varepsilon^m) \zeta(\rho^m) \theta(t) dx dt \\ &= - \int_0^T \int_\Omega G_\delta(\alpha_\varepsilon^m) \partial_t (\zeta(\rho^m) \theta(t)) dx dt. \end{aligned} \quad (4.145)$$

Now, with the sub-subsequence under consideration $G_\delta(\alpha_\varepsilon^m) \rightarrow G_\delta(\alpha^m)$ as $\varepsilon \rightarrow 0_+$ a.e. on Q . Also, because $0 \leq G_\delta(s) \leq C_\delta(s^2 + 1)$, it follows from (4.142) that

$$0 \leq G_\delta(\alpha_\varepsilon^m(x, t)) \leq C_\delta(|\alpha_\varepsilon^m(x, t)|^2 + 1) \leq C_\delta + 2C_\delta g(x, t) + 2C_\delta |\alpha^m(x, t)|^2.$$

As the right-hand side of this is a function in $L^1(Q)$, by the dominated convergence theorem we can pass to the limit over the sub-subsequence under consideration to deduce strong convergence of $G_\delta(\alpha_\varepsilon^m)$ to $G_\delta(\alpha^m)$ in $L^1(Q)$ as $\varepsilon \rightarrow 0_+$. It then follows from (4.144) and (4.145) that

$$- \int_0^T \int_\Omega G_\delta(\alpha^m) \partial_t (\zeta(\rho^m) \theta(t)) dx dt = \int_0^T \langle \partial_t \alpha^m, G_\delta'(\alpha^m) \zeta(\rho^m) \rangle \theta(t) dt. \quad (4.146)$$

Next, we multiply (4.135) by θ , integrate this over $(0, T)$, and use (4.146) to rewrite the resulting first term on the left-hand side; hence

$$\begin{aligned} & - \int_0^T \int_\Omega G_\delta(\alpha^m) \partial_t (\zeta(\rho^m) \theta(t)) dx dt + \int_0^T \int_\Omega (\partial_t \zeta(\rho^m)) G_\delta(\alpha^m) \theta(t) dx dt \\ & + \int_0^T \int_\Omega G_\delta''(\alpha^m) |\nabla_x \alpha^m|^2 \theta(t) dx dt = 0. \end{aligned} \quad (4.147)$$

As $\partial_t(\zeta(\rho^m)\theta) = \partial_t(\zeta(\rho^m))\theta + \zeta(\rho^m)\partial_t\theta$, the first integral on the left-hand side of (4.147) can be written as a sum of two integrals, the first of which cancels with the penultimate integral on the left-hand side of (4.147); hence,

$$-\int_0^T \int_{\Omega} G_{\delta}(\alpha^m) \zeta(\rho^m) \partial_t \theta \, dx \, dt + \int_0^T \int_{\Omega} G_{\delta}''(\alpha^m) |\nabla_x \alpha^m|^2 \theta \, dx \, dt = 0.$$

Equivalently, since θ is independent of x ,

$$-\int_0^T \left(\int_{\Omega} G_{\delta}(\alpha^m) \zeta(\rho^m) \, dx \right) \partial_t \theta \, dt + \int_0^T \left(\int_{\Omega} G_{\delta}''(\alpha^m) |\nabla_x \alpha^m|^2 \, dx \right) \theta \, dt = 0 \quad (4.148)$$

for all $\theta \in C_0^{\infty}((0, T))$. Consequently,

$$\frac{d}{dt} \int_{\Omega} G_{\delta}(\alpha^m) \zeta(\rho^m) \, dx + \int_{\Omega} G_{\delta}''(\alpha^m) |\nabla_x \alpha^m|^2 \, dx = 0 \quad \text{a.e. on } (0, T).$$

This implies, upon integration with respect to t and thanks to the properties of G_{δ} that

$$\begin{aligned} & \int_{\Omega} G_{\delta}(\alpha^m(x, t)) \zeta(\rho^m(x, t)) \, dx + \frac{1}{\delta} \int_0^t \int_{\Omega} |\nabla_x \alpha^m(x, s)|^2 \, dx \, ds \\ & \leq \int_{\Omega} G_{\delta}(\alpha^m(x, 0)) \zeta(\rho^m(x, 0)) \, dx = \int_{\Omega} G_{\delta}(\omega - \lambda_0^m(x)) \zeta(\rho_0^m(x)) \, dx \quad \text{for all } t \in (0, T). \end{aligned}$$

Thus,

$$\int_{\Omega} G_{\delta}(\alpha^m(x, t)) \zeta(\rho^m(x, t)) \, dx \leq \int_{\Omega} G_{\delta}(\omega - \lambda_0^m(x)) \zeta(\rho_0^m(x)) \, dx \quad \text{for all } t \in (0, T).$$

Let us now denote by $\Omega_-(t)$, for $t \in (0, T)$, the set of all $x \in \Omega$ such that $\alpha^m(x, t) < 0$. Once again, appealing to the properties of G_{δ} , we have that $G_{\delta}(\alpha^m(x, t)) \geq |\alpha^m(x, t)|^2/(2\delta)$ for all $x \in \Omega_-(t)$. Therefore,

$$\zeta_{\min} \int_{\Omega_-(t)} |\alpha^m(x, t)|^2 \, dx \leq 2\delta \int_{\Omega} G_{\delta}(\omega - \lambda_0^m(x)) \zeta(\rho_0^m(x)) \, dx \quad \text{for all } t \in (0, T). \quad (4.149)$$

On the other hand, because $\omega - \lambda_0^m(x) \geq 0$ on Ω , by passing to the limit $\delta \rightarrow 0_+$ it follows that

$$\lim_{\delta \rightarrow 0_+} \int_{\Omega} G_{\delta}(\omega - \lambda_0^m(x)) \zeta(\rho_0^m(x)) \, dx = \int_{\Omega} [(\omega - \lambda_0^m(x)) (\log(\omega - \lambda_0^m(x)) - 1) + 1] \zeta(\rho_0^m(x)) \, dx.$$

Hence, by passing to the limit $\delta \rightarrow 0_+$ in (4.149) it follows that

$$\int_{\Omega_-(t)} |\alpha^m(x, t)|^2 \, dx \leq 0 \quad \text{for all } t \in (0, T).$$

Therefore $\text{meas}(\Omega_-(t)) = 0$ for all $t \in (0, T)$. In other words, $0 \leq \lambda^m(x, t) \leq \omega$ for a.e. $(x, t) \in Q$. Consequently,

$$\|\lambda^m\|_{L^{\infty}(\Omega \times (0, T))} \leq \|\lambda_0^m\|_{L^{\infty}(\Omega \times (0, T))} \leq \frac{\varrho_{\max}}{\zeta_{\min}} \leq C. \quad (4.150)$$

Noting that $\zeta(\cdot) \leq \zeta_{\max}$, we obtain from (4.128) and (4.150) that

$$\|\varrho^m\|_{L^{\infty}(\Omega \times (0, T))} \leq \zeta_{\max} \|\lambda^m\|_{L^{\infty}(\Omega \times (0, T))} \leq C. \quad (4.151)$$

By setting $\bar{\varphi} = \lambda^m$ in (4.130) and using that $\zeta(\rho^m)$ satisfies (4.98) we further deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \zeta(\rho^m(x, t)) [\lambda^m(x, t)]^2 \, dx + \int_{\Omega} |\nabla_x \lambda^m(x, t)|^2 \, dx = 0,$$

and therefore, upon integration with respect to t , also

$$\int_0^T \int_{\Omega} |\nabla_x \lambda^m|^2 dx dt \leq C.$$

From the uniform estimate (4.109) we deduce that

$$\sup_{t \in (0, T)} \int_{\mathcal{O}} \tilde{\psi}^m(x, \mathbf{q}, t) \log(1 + \tilde{\psi}^m(x, \mathbf{q}, t)) dx d\mathbf{q} \leq C(\ell), \quad (4.152)$$

where $\tilde{\psi}^m := M^m \hat{\psi}^m$. By de la Vallée Poussin's theorem (Theorem 2.29 in [12]) the sequence $\tilde{\psi}^m$ is uniformly equi-integrable. Hence, by the Dunford–Pettis theorem (Theorem 2.54 in [12]), the sequence $(\tilde{\psi}^m)_{m \geq 1}$ is weakly relatively compact in $L^1(\mathcal{O} \times (0, T))$, which implies the existence of a subsequence (not relabelled) such that

$$\tilde{\psi}^m \rightharpoonup \tilde{\psi} \quad \text{weakly in } L^1(\mathcal{O} \times (0, T)).$$

Since M^m converges to M uniformly in $C(\overline{D})$, we deduce that

$$\hat{\psi}^m \rightharpoonup \frac{\tilde{\psi}}{M} =: \hat{\psi} \quad \text{weakly in } L^1_{loc}(\mathcal{O} \times (0, T)).$$

Next, we shall show that

$$\hat{\psi}^m \rightarrow \hat{\psi} \quad \text{a.e. in } \mathcal{O} \times (0, T).$$

Let \mathcal{O}_0 be a Lipschitz subdomain of \mathcal{O} such that $\mathcal{O}_0 \subset \overline{\mathcal{O}_0} \subset \mathcal{O}$. Since $\mathcal{F}(s) = s \log s + 1 \geq s$ for all $s \in \mathbb{R}_{\geq 0}$ and M^m is bounded below on \mathcal{O}_0 by a positive constant (which may depend on \mathcal{O}_0), we have from (4.109) that

$$\sup_{t \in (0, T)} \|\sqrt{\hat{\psi}^m(\cdot, t)}\|_{L^2(\mathcal{O}_0)}^2 + \int_0^T \|\sqrt{\hat{\psi}^m}\|_{W^{1,2}(\mathcal{O}_0)}^2 dt \leq C(\mathcal{O}_0). \quad (4.153)$$

Since $\mathcal{O}_0 \subset \mathcal{O} \subset \mathbb{R}^{(K+1)d}$, standard function-space interpolation gives that

$$\int_0^T \int_{\mathcal{O}_0} |\hat{\psi}^m|^{\frac{(K+1)d+2}{d(K+1)}} dx d\mathbf{q} dt = \int_0^T \int_{\mathcal{O}_0} \left| \sqrt{\hat{\psi}^m} \right|^{\frac{2((K+1)d+2)}{d(K+1)}} dx d\mathbf{q} dt \leq C(\mathcal{O}_0). \quad (4.154)$$

The application of Hölder's inequality then implies that

$$\begin{aligned} \int_0^T \int_{\mathcal{O}_0} |\nabla_{x, \mathbf{q}} \hat{\psi}^m|^p dx d\mathbf{q} dt &= 2^p \int_0^T \int_{\mathcal{O}_0} \left| \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m} \right|^p \left| \sqrt{\hat{\psi}^m} \right|^p dx d\mathbf{q} dt \\ &\leq 2^p \left(\int_0^T \int_{\mathcal{O}_0} \left| \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m} \right|^2 dx d\mathbf{q} dt \right)^{\frac{p}{2}} \left(\int_0^T \int_{\mathcal{O}_0} \left| \sqrt{\hat{\psi}^m} \right|^{\frac{2p}{2-p}} dx d\mathbf{q} dt \right)^{\frac{2-p}{2}} \\ &\leq C(\mathcal{O}_0), \end{aligned}$$

provided that

$$\frac{2p}{2-p} \leq \frac{2((K+1)d+2)}{d(K+1)}. \quad (4.155)$$

By selecting $p = \frac{d(K+1)+2}{d(K+1)+1}$, which is the largest value satisfying (4.155), we obtain

$$\int_0^T \int_{\mathcal{O}_0} |\nabla_{x, \mathbf{q}} \hat{\psi}^m|^{\frac{d(K+1)+2}{d(K+1)+1}} dx d\mathbf{q} dt \leq C(\mathcal{O}_0). \quad (4.156)$$

It directly follows from (4.150) and the definition of λ^m stated in (4.129) that

$$\|\tilde{\psi}^m\|_{L^\infty(Q; L^1(D))} \leq C,$$

and therefore,

$$\|\hat{\psi}^m\|_{L^\infty(\Omega_0 \times (0,T); L^1(D_0))} \leq C, \quad (4.157)$$

where $\mathcal{O}_0 = \Omega_0 \times D_0$. Interpolating between (4.154), (4.156) and (4.157) we see that for any two real numbers q_1 and q_2 , with

$$\frac{(K+1)d+2}{(K+1)d} \leq q_1 < \infty, \quad 1 < q_2 \leq \frac{(K+1)d+2}{(K+1)d},$$

and satisfying the relation

$$q_1 \left(1 - \frac{1}{q_2}\right) = \frac{2}{(K+1)d},$$

we have that

$$\|\hat{\psi}^m\|_{L^{q_1}(\Omega_0 \times (0,T); L^{q_2}(D_0))} \leq C(\mathcal{O}_0).$$

Since $\zeta(\cdot) \leq \zeta_{\max}$, using (4.110) and Hölder's inequality we deduce that

$$\|\zeta(\rho^m) \mathbf{v}^m \hat{\psi}^m\|_{L^{1+\delta}(\mathcal{O}_0 \times (0,T))} \leq C(\mathcal{O}_0), \quad (4.158)$$

for some $\delta \in (0, 1)$. Similarly, from (4.109), we deduce that

$$\|\zeta(\rho^m) \Lambda_\ell(\hat{\psi}^m)(\nabla_x \mathbf{v}^m) \mathbf{q}\|_{L^{1+\delta}(\mathcal{O}_0 \times (0,T))} \leq C(\mathcal{O}_0). \quad (4.159)$$

We proceed by applying the Div-Curl lemma (Theorem 10.21 in [10]); to this end, we define the following sequences of $(1 + d + Kd)$ -component vector fields:

$$\begin{aligned} H^m &:= (M^m \zeta(\rho^m) \hat{\psi}^m, M^m \zeta(\rho^m) \hat{\psi}^m \mathbf{v}^m - M^m \nabla_x \hat{\psi}^m, M \zeta(\rho^m) \Lambda_\ell(\hat{\psi}^m)(\nabla_x \mathbf{v}^m) \mathbf{q} - M^m \mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^m)), \\ Q^m &:= ((1 + \hat{\psi}^m)^\alpha, \underbrace{0, \dots, 0}_{(d + Kd)\text{-times}}), \end{aligned}$$

for some $\alpha \in (0, 1/2)$. Consequently, using (4.109), (4.158) and (4.159), we deduce that there exist subsequences (not relabelled) such that, as $m \rightarrow \infty$,

$$\begin{aligned} H^m &\rightharpoonup H \quad \text{weakly in } L^{1+\delta}(\mathcal{O}_0 \times (0, T); \mathbb{R}^{1+d+Kd}), \\ Q^m &\rightharpoonup Q \quad \text{weakly in } L^{\frac{1}{\alpha}}(\mathcal{O}_0 \times (0, T); \mathbb{R}^{1+d+Kd}), \end{aligned}$$

where, noting the uniform convergence of M^m and the strong convergences of $\zeta(\rho^m)$ and \mathbf{v}^m ,

$$\begin{aligned} H &:= (M \zeta(\rho) \hat{\psi}, M \zeta(\rho) \hat{\psi} \mathbf{v} - M \nabla_x \hat{\psi}, M \zeta(\rho) \overline{\Lambda_\ell(\hat{\psi})}(\nabla_x \mathbf{v}) \mathbf{q} - M \mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi})), \\ Q &:= ((1 + \hat{\psi})^\alpha, 0, \dots, 0). \end{aligned}$$

It follows from (4.92) that

$$\operatorname{div}_{t,x,\mathbf{q}} H^m = 0 \quad \text{in } \mathcal{O}_0 \times (0, T).$$

Since $\alpha \in (0, 1/2)$, we obtain by using (4.153) that

$$\begin{aligned} \int_0^T \int_{\mathcal{O}_0} |\operatorname{curl}_{t,x,\mathbf{q}} Q^m|^2 dx d\mathbf{q} dt &= \int_0^T \int_{\mathcal{O}_0} |\nabla_{t,x,\mathbf{q}} Q^m - (\nabla_{t,x,\mathbf{q}} Q^m)^T|^2 dx d\mathbf{q} dt \\ &\leq C \int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\mathbf{q}} (1 + \hat{\psi}^m)^\alpha|^2 dx d\mathbf{q} dt \\ &\leq C \int_0^T \int_{\mathcal{O}_0} \left| \nabla_{x,\mathbf{q}} \sqrt{\hat{\psi}^m} \right|^2 dx d\mathbf{q} dt \\ &\leq C(\mathcal{O}_0). \end{aligned}$$

Hence, $(\operatorname{div}_{t,x,\mathbf{q}} H^m)_{m=1}^\infty$ is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0, T))$ and $(\operatorname{curl}_{t,x,\mathbf{q}} Q^m)_{m=1}^\infty$ is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0, T))$. By choosing a positive number $\alpha < \frac{\delta}{1+\delta}$, the Div-Curl lemma gives that

$$H^m \cdot Q^m \rightharpoonup H \cdot Q \quad \text{weakly in } L^1(\mathcal{O}_0 \times (0, T)).$$

In particular, we have that

$$M^m \zeta(\rho^m) \hat{\psi}^m (1 + \hat{\psi}^m)^\alpha \rightharpoonup M \zeta(\rho) \overline{\hat{\psi} (1 + \hat{\psi})^\alpha}.$$

Since M^m converges to M uniformly and $\zeta(\rho^m)$ converges to $\zeta(\rho)$ strongly in $L^\infty(0, T; L^p(\Omega))$, $1 < p < \infty$, the above implies that

$$\hat{\psi}^m (1 + \hat{\psi}^m)^\alpha \rightharpoonup \overline{\hat{\psi} (1 + \hat{\psi})^\alpha}. \quad (4.160)$$

Since $(1 + \hat{\psi}^m)^\alpha \rightharpoonup \overline{(1 + \hat{\psi})^\alpha}$ in $L^1(\mathcal{O}_0 \times (0, T))$, we can add this to (4.160), which gives

$$(1 + \hat{\psi}^m)^{\alpha+1} = (1 + \hat{\psi}^m)(1 + \hat{\psi}^m)^\alpha \rightharpoonup (1 + \hat{\psi}) \overline{(1 + \hat{\psi})^\alpha}.$$

Since the function $s \in [0, \infty) \mapsto s^{\alpha+1} \in [0, \infty)$ is continuous and convex, by weak lower-semicontinuity we have that

$$(1 + \hat{\psi})^{\alpha+1} \leq (1 + \hat{\psi}) \overline{(1 + \hat{\psi})^\alpha},$$

which, thanks to the nonnegativity of $\hat{\psi}$, is equivalent to

$$(1 + \hat{\psi})^\alpha \leq \overline{(1 + \hat{\psi})^\alpha}.$$

On the other hand, the function $s \in [0, \infty) \mapsto s^\alpha \in [0, \infty)$ is concave. Again, by the weak lower-semicontinuity of the continuous convex function $s \in [0, \infty) \mapsto -s^\alpha \in [0, \infty)$, we deduce that

$$-(1 + \hat{\psi})^\alpha \leq -\overline{(1 + \hat{\psi})^\alpha}.$$

Therefore,

$$(1 + \hat{\psi})^\alpha = \overline{(1 + \hat{\psi})^\alpha}.$$

Since $s \in [0, \infty) \mapsto s^\alpha \in [0, \infty)$, for $\alpha \in (0, \frac{\delta}{\delta+1})$, is strictly concave, by using Theorem 10.20 in [10], we deduce the existence of a subsequence (not relabelled), such that

$$\hat{\psi}^m \rightarrow \hat{\psi} \quad \text{a.e. in } \mathcal{O}_0 \times (0, T).$$

Now we want to extend the pointwise convergence result of $\hat{\psi}^m$ to the whole of our domain $\mathcal{O} \times (0, T)$. For this purpose we choose a nondecreasing sequence of nested sets $(\mathcal{O}_0^k)_{k=1}^\infty$, i.e., $\mathcal{O}_0^1 \subset \mathcal{O}_0^2 \subset \dots \subset \mathcal{O}_0^k \subset \dots$, satisfying $\cup_{k=1}^\infty \mathcal{O}_0^k = \mathcal{O}$. For each $k \in \mathbb{N}$, we deduce the existence of a subsequence of $(\hat{\psi}^m)_{m=1}^\infty$ that is pointwise convergent to $\hat{\psi}$ a.e. in $\mathcal{O}_0^k \times (0, T)$. Arguing by a diagonal procedure we deduce that there exists a subsequence such that

$$\hat{\psi}^m \rightarrow \hat{\psi} \quad \text{a.e. in } \mathcal{O} \times (0, T). \quad (4.161)$$

Since M^m converges uniformly to M , we have that

$$\tilde{\psi}^m \rightarrow \tilde{\psi} \quad \text{a.e. in } \mathcal{O} \times (0, T).$$

Since $\tilde{\psi}^m$ is uniformly equi-integrable (cf. the sentence following the bound (4.152)), using Vitali's convergence theorem (Theorem 2.24 in [12]), we obtain that

$$\tilde{\psi}^m \rightarrow \tilde{\psi} \quad \text{strongly in } L^1(0, T; L^1(\mathcal{O})). \quad (4.162)$$

It directly follows from (4.150) that

$$\|\tilde{\psi}^m\|_{L^\infty(Q; L^1(D))} \leq C. \quad (4.163)$$

Then, interpolating between (4.163) and (4.162) gives that

$$M^m \hat{\psi}^m = \tilde{\psi}^m \rightarrow \tilde{\psi} = M \hat{\psi} \quad \text{strongly in } L^p(Q; L^1(D)), \text{ for all } p \in [1, \infty). \quad (4.164)$$

Thus, by recalling (4.127) and (4.129) it follows that

$$\varrho^m \rightarrow \varrho \quad \text{strongly in } L^p(Q) \text{ for all } p \in [1, \infty), \quad \text{where} \quad \varrho(x, t) = \zeta(\rho) \int_D M(\mathbf{q}) \hat{\psi}(x, \mathbf{q}, t) d\mathbf{q}.$$

Hence, and by recalling (4.126), we have that

$$\mu(\rho^m, \varrho^m) \rightarrow \mu(\rho, \varrho) \quad \text{strongly in } L^p(Q) \text{ for all } p \in [1, \infty). \quad (4.165)$$

For any measurable set $U \subset \mathcal{O} \times (0, T)$, with $|U| \leq \delta$, Hölder's inequality implies that

$$\begin{aligned} \int_U M^m |\nabla_{x, \mathbf{q}} \hat{\psi}^m| dx d\mathbf{q} dt &= 2 \int_U M^m \left| \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m} \right| \left| \sqrt{\hat{\psi}^m} \right| dx d\mathbf{q} dt \\ &\leq 2 \left(\int_U M^m \left| \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m} \right|^2 dx d\mathbf{q} dt \right)^{\frac{1}{2}} \left(\int_U \tilde{\psi}^m dx d\mathbf{q} dt \right)^{\frac{1}{2}} \\ &\leq C \varepsilon^{\frac{1}{2}}, \end{aligned} \quad (4.166)$$

which follows from the uniform estimate (4.109) and the uniform equi-integrability of $\tilde{\psi}^m$. By the Dunford–Pettis theorem we can extract a subsequence such that

$$M^m \nabla_{x, \mathbf{q}} \hat{\psi}^m \rightharpoonup \overline{M^m \nabla_{x, \mathbf{q}} \hat{\psi}^m} \quad \text{weakly in } L^1(\mathcal{O} \times (0, T); \mathbb{R}^{d(K+1)}). \quad (4.167)$$

From (4.109), we deduce using Hölder's inequality and standard interpolation that $\nabla_{x, \mathbf{q}} \hat{\psi}^m$ weakly converges to $\nabla_{x, \mathbf{q}} \hat{\psi}$ in $L^1_{loc}(\mathcal{O} \times (0, T); \mathbb{R}^{d(K+1)})$. Since M^m converges uniformly to M , we can identify the weak limit $\overline{M^m \nabla_{x, \mathbf{q}} \hat{\psi}^m}$ as $M \nabla_{x, \mathbf{q}} \hat{\psi}$. Analogously as in (4.167) it follows from (4.109) that

$$\sqrt{M^m} \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m} \rightharpoonup \sqrt{M} \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}} \quad \text{weakly in } L^2(\mathcal{O} \times (0, T); \mathbb{R}^{d(K+1)}). \quad (4.168)$$

Since $M^m \nabla_{x, \mathbf{q}} \hat{\psi}^m = 2 \sqrt{M^m} \sqrt{\hat{\psi}^m} \sqrt{M^m} \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m} = 2 \sqrt{\tilde{\psi}^m} \sqrt{M^m} \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m}$, by using a similar calculation as in (4.166) we also see that

$$\begin{aligned} \int_Q \left(\int_D M^m |\nabla_{x, \mathbf{q}} \hat{\psi}^m| d\mathbf{q} \right)^2 dx dt &= 4 \int_Q \left(\int_D \sqrt{\tilde{\psi}^m} \left| \sqrt{M^m} \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m} \right| d\mathbf{q} \right)^2 dx dt \\ &\leq 4 \int_Q \|\tilde{\psi}^m\|_{L^1(D)}^2 \|\sqrt{M^m} \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m}\|_{L^2(D)}^2 dx dt \\ &\leq 4 \|\lambda^m\|_{L^\infty(\Omega \times (0, T))}^2 \int_Q \|\sqrt{M^m} \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^m}\|_{L^2(D)}^2 dx dt \\ &\leq C, \end{aligned}$$

where we have used the estimates (4.109) and (4.150). Therefore, we can strengthen (4.167) as follows:

$$M^m \nabla_{x, \mathbf{q}} \hat{\psi}^m \rightharpoonup M \nabla_{x, \mathbf{q}} \hat{\psi} \quad \text{weakly in } L^2(Q; L^1(D; \mathbb{R}^{d(K+1)})). \quad (4.169)$$

With the strong convergence results (4.124), (4.127) and (4.164) for \mathbf{v}^m , $\zeta(\rho^m)$ and $\tilde{\psi}^m$ we deduce that

$$M^m \zeta(\rho^m) \mathbf{v}^m \hat{\psi}^m \rightarrow M \zeta(\rho) \mathbf{v} \hat{\psi} \quad \text{strongly in } L^p(Q; L^1(D; \mathbb{R}^d)), \quad (4.170)$$

where $p \in [1, \frac{2(d+2)}{d})$. It follows from (4.161) and the fact that $\Gamma_\ell \in C_0^\infty((-2\ell, 2\ell))$ that

$$\hat{\psi}^m \Gamma_\ell(\hat{\psi}^m) = \Lambda_\ell(\hat{\psi}^m) \rightarrow \Lambda_\ell(\hat{\psi}) = \hat{\psi} \Gamma_\ell(\hat{\psi}) \quad \text{a.e. in } \mathcal{O} \times (0, T).$$

Hence, using (4.127) and the boundedness of the function $\Lambda_\ell(\cdot)$,

$$\zeta(\rho^m) \Lambda_\ell(\hat{\psi}^m) \rightarrow \zeta(\rho) \Lambda_\ell(\hat{\psi}) \quad \text{strongly in } L^p(\mathcal{O} \times (0, T)), \text{ for all } p \in [1, \infty).$$

It then follows from (4.122) that

$$M\zeta(\rho^m) \Lambda_\ell(\hat{\psi}^m) (\nabla_x \mathbf{v}^m) \rightharpoonup M\zeta(\rho) \Lambda_\ell(\hat{\psi}) (\nabla_x \mathbf{v}) \quad \text{weakly in } L^p(Q \times D; \mathbb{R}^{d \times d}), \text{ for all } p \in [1, 2). \quad (4.171)$$

Using (4.92) and the convergence results (4.169), (4.170) and (4.171) we obtain that

$$\partial_t(M^m \zeta(\rho^m) \hat{\psi}^m) \rightharpoonup \partial_t(M\zeta(\rho) \hat{\psi}) \quad \text{weakly in } L^p(0, T; W^{-1,1}(\mathcal{O})), \text{ for all } p \in [1, 2). \quad (4.172)$$

From (4.127) we deduce that

$$\zeta(\rho^m) \rightarrow \zeta(\rho) \quad \text{a.e. in } Q.$$

Therefore, using (4.161) and the dominated convergence theorem (thanks to the presence of the truncation T_ℓ), we can let $m \rightarrow \infty$ in (4.94) to deduce that

$$\tau^m \rightarrow \tau \quad \text{strongly in } L^1(Q; \mathbb{R}^{d \times d}), \quad (4.173)$$

where

$$\tau = -k \int_D \left[KM\zeta(\rho) T_\ell(\hat{\psi}) I + \sum_{j=1}^K \zeta(\rho) T_\ell(\hat{\psi}) \nabla_{\mathbf{q}^j} M \otimes \mathbf{q}^j \right] d\mathbf{q} \quad \text{a.e. in } Q. \quad (4.174)$$

Now let us reinstate the superscript ℓ . Collecting the convergence results (4.119), (4.122), (4.124), (4.125), (4.165), (4.127), (4.162), (4.169), (4.170), (4.171), (4.172) and (4.173), and using the fact that \mathbf{f}^m converges to \mathbf{f} in $L^2(0, T; L^2(\Omega; \mathbb{R}^d))$, we can pass to the limit as $m \rightarrow \infty$ in (4.90)–(4.92) to obtain the following:

$$\int_0^T [\langle \partial_t \rho^\ell, \eta \rangle - (\mathbf{v}^\ell \rho^\ell, \nabla_x \eta)] dt = 0, \quad \text{for all } \eta \in L^1(0, T; W^{1, \frac{q}{q-1}}(\Omega)), \quad (4.175)$$

where $q \in (2, \infty)$ when $d = 2$ and $q \in [3, 6]$ when $d = 3$,

$$\begin{aligned} & \int_0^T \langle \partial_t(\rho^\ell \mathbf{v}^\ell), \mathbf{w} \rangle dt + \int_0^T [-(\rho^\ell \mathbf{v}^\ell \otimes \mathbf{v}^\ell, \nabla_x \mathbf{w}) + (\mu(\rho^\ell, \varrho^\ell) D(\mathbf{v}^\ell), \nabla_x \mathbf{w})] dt \\ &= \int_0^T [-(\tau^\ell, \nabla_x \mathbf{w}) + (\rho^\ell \mathbf{f}, \mathbf{w})] dt \quad \text{for all } \mathbf{w} \in L^s(0, T; W_{0, \text{div}}^{1, s}(\Omega; \mathbb{R}^d)) \quad \text{with } s > 2, \end{aligned} \quad (4.176)$$

and

$$\begin{aligned} & \int_0^T \left\langle \partial_t(M\zeta(\rho^\ell) \hat{\psi}^\ell), \varphi \right\rangle_{\mathcal{O}} - \left(M\zeta(\rho^\ell) \mathbf{v}^\ell \hat{\psi}^\ell, \nabla_x \varphi \right)_{\mathcal{O}} - \left(M\zeta(\rho^\ell) \Lambda_\ell(\hat{\psi}^\ell) (\nabla_x \mathbf{v}^\ell) \mathbf{q}, \nabla_{\mathbf{q}} \varphi \right)_{\mathcal{O}} dt \\ &+ \int_0^T (M \nabla_x \hat{\psi}^\ell, \nabla_x \varphi)_{\mathcal{O}} + \left(M \mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^\ell), \nabla_{\mathbf{q}} \varphi \right)_{\mathcal{O}} dt = 0 \quad \text{for all } \varphi \in L^\infty(0, T; W^{1, \infty}(\mathcal{O})). \end{aligned} \quad (4.177)$$

Letting $m \rightarrow \infty$ in (4.108), we deduce the following energy inequality:

$$\begin{aligned} & k \int_{\mathcal{O}} M\zeta(\rho^\ell(\cdot, t)) \mathcal{F}(\hat{\psi}^\ell(\cdot, t)) dx d\mathbf{q} + \frac{1}{2} \int_{\Omega} \rho^\ell(\cdot, t) |\mathbf{v}^\ell(\cdot, t)|^2 dx \\ &+ \int_0^t \int_{\Omega} \mu(\rho^\ell) |D(\mathbf{v}^\ell)|^2 dx ds + 4kC_1 \int_0^t \int_{\mathcal{O}} M \left| \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}^\ell} \right|^2 dx d\mathbf{q} ds \\ &\leq k \int_{\mathcal{O}} M\zeta(\rho_0) \mathcal{F}(T_\ell(\hat{\psi}_0)) dx d\mathbf{q} + \frac{1}{2} \int_{\Omega} \rho_0 |\mathbf{v}_0|^2 dx + \int_0^t (\rho^\ell \mathbf{f}, \mathbf{v}^\ell) ds. \end{aligned} \quad (4.178)$$

Letting $m \rightarrow \infty$ in (4.95), (4.151) and (4.109), by the weak lower semicontinuity of norms, we deduce the following a priori estimate:

$$\begin{aligned}
& \sup_{t \in (0, T)} \left(\|\rho^\ell(\cdot, t)\|_{L^\infty(\Omega)} + \|\varrho^\ell(\cdot, t)\|_{L^\infty(\Omega)} \right) + 2k\zeta_{\min} \sup_{t \in (0, T)} \|\mathcal{F}(\hat{\psi}^\ell(\cdot, t))\|_{L_M^1(\mathcal{O})} \\
& + \rho_{\min} \sup_{t \in (0, T)} \|\mathbf{v}^\ell(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \mu_{\min} c_0 \int_0^T \|\mathbf{v}^\ell\|_{W^{1,2}(\Omega; \mathbb{R}^d)}^2 dt + 8kC_1 \int_0^T \left\| \sqrt{\hat{\psi}^\ell} \right\|_{W_M^{1,2}(\mathcal{O})}^2 dt \\
& \leq 2k\zeta_{\max} \int_{\mathcal{O}} M\mathcal{F}(T_\ell(\hat{\psi}_0)) dx d\mathbf{q} + \rho_{\max} \|\mathbf{v}_0\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \frac{\rho_{\max}^2}{\mu_{\min} c_0} \int_0^T \|\mathbf{f}\|_{L^2(\Omega; \mathbb{R}^d)}^2 dt \\
& \leq C,
\end{aligned} \tag{4.179}$$

where C is a positive constant independent of ℓ and

$$\varrho^\ell(x, t) = \zeta(\rho^\ell(x, t)) \int_D M(\mathbf{q}) \hat{\psi}^\ell(x, \mathbf{q}, t) d\mathbf{q}. \tag{4.180}$$

The next section is devoted to passing to the final limit, $\ell \rightarrow \infty$.

4.6 Passage to the limit with ℓ

From the uniform estimate (4.179) (noting that the constant C does not depend on ℓ), we deduce that there exists a subsequence (not relabelled) such that, as $\ell \rightarrow \infty$,

$$\begin{aligned}
\mathbf{v}^\ell &\rightharpoonup \mathbf{v} && \text{weak* in } L^\infty(0, T; L_{0, \text{div}}^2(\Omega; \mathbb{R}^d)), \\
\mathbf{v}^\ell &\rightharpoonup \mathbf{v} && \text{weakly in } L^2(0, T; W_{0, \text{div}}^{1,2}(\Omega; \mathbb{R}^d)).
\end{aligned} \tag{4.181}$$

By standard interpolation

$$\mathbf{v}^\ell \rightharpoonup \mathbf{v} \quad \text{weakly in } L^{\frac{2(d+2)}{d}}(Q; \mathbb{R}^d). \tag{4.182}$$

From the definition (4.174) of τ^ℓ and the definition of the truncation T_ℓ we deduce that

$$\|\tau^\ell\|_{L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d}))}^2 \leq C, \tag{4.183}$$

where C is a positive constant independent of ℓ , which then implies the existence of a subsequence (not relabelled) such that

$$\tau^\ell \rightharpoonup \tau \quad \text{weakly in } L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d})). \tag{4.184}$$

We can perform a similar argument as in (4.114)–(4.117) to deduce that

$$\|\mathbf{v}^\ell\|_{N_2^\gamma(0, T; L^2(\Omega; \mathbb{R}^d))} \leq C,$$

where $0 < \gamma < 1/4$ when $d = 2$ and $0 < \gamma \leq 1/8$ when $d = 3$. By the Aubin–Lions Lemma it follows that

$$\mathbf{v}^\ell \rightarrow \mathbf{v} \quad \text{strongly in } L^2(0, T; L^p(\Omega; \mathbb{R}^d)), \tag{4.185}$$

where $p \in [1, \infty)$ when $d = 2$ and $p \in [1, 6)$ when $d = 3$.

From the uniform estimate (4.179) we deduce that there exists a subsequence (not relabelled) such that, as $\ell \rightarrow \infty$,

$$\rho^\ell \rightharpoonup \rho \quad \text{weak* in } L^\infty(Q).$$

Using (4.175), (4.179) and Sobolev embedding we have that

$$\|\partial_t \rho^\ell\|_{L^2(0, T; (W^{1, p'}(\Omega))')} \leq C,$$

where $p \in (1, \infty)$ when $d = 2$ and $p \in (1, 6]$ when $d = 3$. Therefore, we have that

$$\partial_t \rho^\ell \rightharpoonup \partial_t \rho \quad \text{weakly in } L^2(0, T; (W^{1,p'}(\Omega))'), \quad (4.186)$$

where $p \in (1, \infty)$ when $d = 2$ and $p \in (1, 6]$ when $d = 3$. We can also show that

$$\rho^\ell \rightarrow \rho \quad \text{strongly in } L^p(Q), \text{ for any } p \in [1, \infty); \quad (4.187)$$

the proof proceeds similarly as in (4.72)–(4.75). With the convergence result (4.185) for \mathbf{v}^ℓ we can perform a similar argument as in Theorem VI.1.9 in [5] and strengthen the above convergence to get

$$\rho^\ell \rightarrow \rho \quad \text{strongly in } C([0, T]; L^p(\Omega)), \text{ for any } p \in [1, \infty). \quad (4.188)$$

From the assumption (2.7) we further deduce that

$$\zeta(\rho^\ell) \rightarrow \zeta(\rho) \quad \text{strongly in } C([0, T]; L^p(\Omega)), \text{ for any } p \in [1, \infty). \quad (4.189)$$

Using (4.176), (4.179), (4.182) and (4.183) we have that

$$\int_0^T \|\partial_t(\rho^\ell \mathbf{v}^\ell)\|_{W_{\text{div}}^{-1,p}(\Omega; \mathbb{R}^d)}^p dt \leq C,$$

where C is a positive constant independent of ℓ and $p \in (1, \frac{d+2}{d})$, which then implies that

$$\partial_t(\rho^\ell \mathbf{v}^\ell) \rightharpoonup \partial_t(\rho \mathbf{v}) \quad \text{weakly in } L^p(0, T; W_{\text{div}}^{-1,p}(\Omega; \mathbb{R}^d)), \text{ for all } p \in \left(1, \frac{d+2}{d}\right). \quad (4.190)$$

Next, we will show the strong convergence of $\hat{\psi}^\ell$. We shall, again, use the Div-Curl lemma. Since the argument is similar to the one presented in the previous section, we shall confine ourselves to indicating the key differences. From (4.179) we have that

$$\sup_{t \in (0, T)} \int_{\mathcal{O}} M(\mathbf{q}) \hat{\psi}^\ell(x, \mathbf{q}, t) \log(1 + \hat{\psi}^\ell(x, \mathbf{q}, t)) dx d\mathbf{q} \leq C,$$

where C is a positive constant independent of ℓ , which implies the uniform equi-integrability of the sequence $(\hat{\psi}^\ell)_{\ell \geq 0}$. By the Dunford–Pettis theorem the sequence $(\hat{\psi}^\ell)_{\ell \geq 0}$ is weakly relatively compact in $L_M^1(\mathcal{O} \times (0, T))$. Hence, there exists a $\hat{\psi} \in L_M^1(\mathcal{O} \times (0, T))$ and a subsequence (not relabelled) such that

$$\hat{\psi}^\ell \rightharpoonup \hat{\psi} \quad \text{weakly in } L_M^1(\mathcal{O} \times (0, T)).$$

The next step is to show that

$$\hat{\psi}^\ell \rightarrow \hat{\psi} \quad \text{a.e. in } \mathcal{O} \times (0, T).$$

The proof now proceeds similarly to the one in the previous section. Let \mathcal{O}_0 be a Lipschitz subdomain of \mathcal{O} such that $\mathcal{O}_0 \subset \overline{\mathcal{O}_0} \subset \mathcal{O}$. Since M is bounded below in \mathcal{O}_0 , we have that

$$\sup_{t \in (0, T)} \|\sqrt{\hat{\psi}^\ell(\cdot, t)}\|_{L^2(\mathcal{O}_0)}^2 + \int_0^T \|\sqrt{\hat{\psi}^\ell}\|_{W^{1,2}(\mathcal{O}_0)}^2 dt \leq C(\mathcal{O}_0).$$

Since $\mathcal{O}_0 \subset \mathcal{O} \subset \mathbb{R}^{(K+1)d}$, standard function-space interpolation gives that

$$\int_0^T \int_{\mathcal{O}_0} |\hat{\psi}^\ell|^{\frac{(K+1)d+2}{d(K+1)}} dx d\mathbf{q} dt = \int_0^T \int_{\mathcal{O}_0} \left| \sqrt{\hat{\psi}^\ell} \right|^{\frac{2((K+1)d+2)}{d(K+1)}} dx d\mathbf{q} dt \leq C(\mathcal{O}_0).$$

We apply Hölder's inequality to deduce that

$$\int_0^T \int_{\mathcal{O}_0} |\nabla_{x, \mathbf{q}} \hat{\psi}^\ell|^{\frac{d(K+1)+2}{d(K+1)+1}} dx d\mathbf{q} dt \leq C(\mathcal{O}_0). \quad (4.191)$$

Thanks to (4.179) we have that

$$\|\hat{\psi}^\ell\|_{L^\infty(Q; L_M^1(D))} \leq C,$$

where C is a positive constant independent of ℓ . It then follows that for any $q_1 \in (1, \infty)$ there exists a $q_2 > 1$ such that

$$\|\hat{\psi}^\ell\|_{L^{q_1}(\Omega_0 \times (0, T); L^{q_2}(D_0))} \leq C(\mathcal{O}_0). \quad (4.192)$$

Hence, using (4.179), (4.192), the fact that $\zeta(\cdot) \leq \zeta_{\max}$ and Hölder's inequality, we deduce that

$$\|\zeta(\rho^\ell) \mathbf{v}^\ell \hat{\psi}^\ell\|_{L^{1+\delta}(\mathcal{O}_0 \times (0, T))} + \|\zeta(\rho^\ell) \Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \mathbf{v}^\ell) \mathbf{q}\|_{L^{1+\delta}(\mathcal{O}_0 \times (0, T))} \leq C(\mathcal{O}_0), \quad (4.193)$$

for some $\delta \in (0, 1)$. To apply the Div-Curl lemma we define

$$\begin{aligned} H^\ell &:= (M\zeta(\rho^\ell) \hat{\psi}^\ell, M\zeta(\rho^\ell) \hat{\psi}^\ell \mathbf{v}^\ell - M\nabla_x \hat{\psi}^\ell, M\zeta(\rho^\ell) \Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \mathbf{v}^\ell) \mathbf{q} - M\mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^\ell)), \\ Q^\ell &:= ((1 + \hat{\psi}^\ell)^\alpha, \underbrace{0, \dots, 0}_{(d+Kd)\text{-times}}), \end{aligned}$$

for some $\alpha \in (0, \frac{\delta}{1+\delta})$. Consequently, using (4.179), (4.191) and (4.193), we deduce that there exist subsequences (not relabelled) such that

$$\begin{aligned} H^\ell &\rightharpoonup H \quad \text{weakly in } L^{1+\delta}(\mathcal{O}_0 \times (0, T); \mathbb{R}^{1+d+Kd}), \\ Q^\ell &\rightharpoonup Q \quad \text{weakly in } L^{\frac{1}{\alpha}}(\mathcal{O}_0 \times (0, T); \mathbb{R}^{1+d+Kd}), \end{aligned}$$

where, noting the strong convergences of $\zeta(\rho^\ell)$ and \mathbf{v}^ℓ ,

$$\begin{aligned} H &:= (M\zeta(\rho) \hat{\psi}, M\zeta(\rho) \hat{\psi} \mathbf{v} - M\nabla_x \hat{\psi}, M\zeta(\rho) \Lambda_\ell(\hat{\psi})(\nabla_x \mathbf{v}) \mathbf{q} - M\mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi})), \\ Q &:= ((1 + \hat{\psi})^\alpha, 0, \dots, 0). \end{aligned}$$

Similarly as in the previous section, we can show that $\text{div}_{t,x,\mathbf{q}} H^\ell$ and $\text{curl}_{t,x,\mathbf{q}} Q^\ell$ are precompact in $W^{-1,2}(\mathcal{O}_0 \times (0, T))$. Thanks to our choice of α , the Div-Curl lemma then implies that

$$H^\ell \cdot Q^\ell \rightharpoonup H \cdot Q \quad \text{weakly in } L^1(\mathcal{O}_0 \times (0, T)).$$

In particular, we have that

$$M\zeta(\rho^\ell) \hat{\psi}^\ell (1 + \hat{\psi}^\ell)^\alpha \rightharpoonup M\zeta(\rho) \hat{\psi} \overline{(1 + \hat{\psi})^\alpha}.$$

Because of the strong convergence of $\zeta(\rho^\ell)$ to $\zeta(\rho)$ and the assumed positivity of ζ (cf. (2.7)), it then follows that

$$\hat{\psi}^\ell (1 + \hat{\psi}^\ell)^\alpha \rightharpoonup \hat{\psi} \overline{(1 + \hat{\psi})^\alpha},$$

which then implies that

$$\hat{\psi}^\ell \rightarrow \hat{\psi} \quad \text{a.e. in } \mathcal{O}_0 \times (0, T).$$

By selecting a nondecreasing sequence of nested set $(\mathcal{O}_0^k)_{k \geq 1}$ such that $\cup_{k=1}^\infty \mathcal{O}_0^k = \mathcal{O}$ and deducing pointwise convergence on each \mathcal{O}_0^k , we can, again, use a diagonal procedure to deduce that there exists a subsequence such that

$$\hat{\psi}^\ell \rightarrow \hat{\psi} \quad \text{a.e. in } \mathcal{O} \times (0, T).$$

Then, by Vitali's convergence theorem we obtain that

$$\hat{\psi}^\ell \rightarrow \hat{\psi} \quad \text{strongly in } L^1(0, T; L_M^1(\mathcal{O})).$$

By standard interpolation with (4.179) we deduce that

$$\hat{\psi}^\ell \rightarrow \hat{\psi} \quad \text{strongly in } L^p(Q; L_M^1(D)) \text{ for all } p \in [1, \infty). \quad (4.194)$$

We can now use (4.189) and (4.194) to pass to the limit $\ell \rightarrow \infty$ in (4.180) to deduce that

$$\varrho^\ell \rightarrow \varrho \quad \text{strongly in } L^p(Q) \text{ for all } p \in [1, \infty), \quad \text{where } \varrho(x, t) = \zeta(\rho(x, t)) \int_D M(\mathbf{q}) \hat{\psi}(x, \mathbf{q}, t) d\mathbf{q}.$$

Thus, by noting (2.7) and (4.188), we have that

$$\mu(\rho^\ell, \varrho^\ell) \rightarrow \mu(\rho, \varrho) \quad \text{strongly in } L^p(Q) \text{ for all } p \in [1, \infty). \quad (4.195)$$

Following a similar argument as in (4.166)–(4.169) we deduce the following convergence results:

$$\begin{aligned} \nabla_{x,\mathbf{q}} \hat{\psi}^\ell &\rightharpoonup \nabla_{x,\mathbf{q}} \hat{\psi} && \text{weakly in } L^1(0, T; L_M^1(\mathcal{O}; \mathbb{R}^{d(K+1)})), \\ \nabla_{x,\mathbf{q}} \sqrt{\hat{\psi}^\ell} &\rightharpoonup \nabla_{x,\mathbf{q}} \sqrt{\hat{\psi}} && \text{weakly in } L^2(0, T; L^2(\mathcal{O}; \mathbb{R}^{d(K+1)})), \\ \nabla_{x,\mathbf{q}} \hat{\psi}^\ell &\rightharpoonup \nabla_{x,\mathbf{q}} \hat{\psi} && \text{weakly in } L^2(Q; L_M^1(D; \mathbb{R}^{d(K+1)})). \end{aligned} \quad (4.196)$$

From the Lipschitz continuity of Γ , and therefore the Lipschitz continuity of Λ_ℓ , we obtain for any $p \in [1, \infty)$ that

$$\begin{aligned} \|\Lambda_\ell(\hat{\psi}^\ell) - \hat{\psi}\|_{L^p(Q; L_M^1(D))} &\leq \|\Lambda_\ell(\hat{\psi}^\ell) - \Lambda_\ell(\hat{\psi})\|_{L^p(Q; L_M^1(D))} + \|\Lambda_\ell(\hat{\psi}) - \hat{\psi}\|_{L^p(Q; L_M^1(D))} \\ &\leq C\|\hat{\psi}^\ell - \hat{\psi}\|_{L^p(Q; L_M^1(D))} + \|\Lambda_\ell(\hat{\psi}) - \hat{\psi}\|_{L^p(Q; L_M^1(D))}. \end{aligned}$$

The first term in the above inequality converges to 0 as $\ell \rightarrow \infty$ on noting (4.194). Since $\Lambda_\ell(\hat{\psi})$ converges to $\hat{\psi}$ almost everywhere in $\mathcal{O} \times (0, T)$, we apply the dominated convergence theorem to deduce that the second term in the above inequality also converges to 0 as $\ell \rightarrow \infty$. Therefore,

$$\Lambda_\ell(\hat{\psi}^\ell) \rightarrow \hat{\psi} \quad \text{strongly in } L^p(Q; L_M^1(D)) \text{ for all } p \in [1, \infty).$$

Moreover, using (4.181) and (4.189) we deduce that

$$\zeta(\rho^\ell) \Lambda_\ell(\hat{\psi}^\ell) (\nabla_x \mathbf{v}^\ell) \rightharpoonup \zeta(\rho) \hat{\psi} (\nabla_x \mathbf{v}) \quad \text{weakly in } L^p(Q; L_M^1(D; \mathbb{R}^{d \times d})) \text{ for any } p \in [1, 2). \quad (4.197)$$

With the convergence results (4.181), (4.189) and (4.194) we deduce that

$$\zeta(\rho^\ell) \mathbf{v}^\ell \hat{\psi}^\ell \rightarrow \zeta(\rho) \mathbf{v} \hat{\psi} \quad \text{strongly in } L^p(Q; L_M^1(D; \mathbb{R}^d)) \text{ for any } p \in \left[1, \frac{2(d+2)}{d}\right). \quad (4.198)$$

Consequently, using the identity (4.177) and the convergence results (4.196), (4.197) and (4.198), it follows that

$$\partial_t(M\zeta(\rho^\ell) \hat{\psi}^\ell) \rightharpoonup \partial_t(M\zeta(\rho) \hat{\psi}) \quad \text{weakly in } L^p(0, T; W^{-1,1}(\mathcal{O})) \text{ for any } p \in [1, 2). \quad (4.199)$$

Next, we shall deduce the expression for the extra-stress tensor τ . Performing partial integration on (4.174) and noting the fact that M has zero trace, τ^ℓ can be rewritten as follows:

$$\begin{aligned} \tau^\ell &= -k \int_D \left[KM\zeta(\rho^\ell) T_\ell(\hat{\psi}^\ell) I + \sum_{j=1}^K \zeta(\rho^\ell) T_\ell(\hat{\psi}^\ell) \nabla_{\mathbf{q}^j} M \otimes \mathbf{q}^j \right] d\mathbf{q} \\ &= k \sum_{j=1}^K \int_D M\zeta(\rho^\ell) \nabla_{\mathbf{q}^j} T_\ell(\hat{\psi}^\ell) \otimes \mathbf{q}^j d\mathbf{q}. \end{aligned}$$

Using (4.189), (4.194) and (4.196) we pass to the limit as $\ell \rightarrow \infty$ in this expression to identify the weak limit τ in (4.184) as

$$\tau = k \sum_{j=1}^K \int_D M\zeta(\rho) \nabla_{\mathbf{q}^j} \hat{\psi} \otimes \mathbf{q}^j d\mathbf{q}.$$

With the convergence results (4.181), (4.184), (4.185), (4.186), (4.187), (4.190), (4.194), (4.195), (4.196), (4.197), (4.198) and (4.199), we pass to the limit as $\ell \rightarrow \infty$ in (4.175)–(4.177) to deduce that

$$\int_0^T [\langle \partial_t \rho, \eta \rangle - (\mathbf{v} \rho, \nabla_x \eta)] dt = 0, \quad \text{for all } \eta \in L^1(0, T; W^{1, \frac{q}{q-1}}(\Omega)),$$

where $q \in (2, \infty)$ when $d = 2$ and $q \in [3, 6]$ when $d = 3$,

$$\begin{aligned} & \int_0^T \langle \partial_t(\rho \mathbf{v}), \mathbf{w} \rangle dt + \int_0^T [-(\rho \mathbf{v} \otimes \mathbf{v}, \nabla_x \mathbf{w}) + (\mu(\rho, \varrho) D(\mathbf{v}), \nabla_x \mathbf{w})] dt \\ &= \int_0^T [-(\tau, \nabla_x \mathbf{w}) + (\rho \mathbf{f}, \mathbf{w})] dt \quad \text{for all } \mathbf{w} \in L^s(0, T; W_{0, \text{div}}^{1, s}(\Omega; \mathbb{R}^d)) \quad \text{where } s > 2, \end{aligned}$$

and

$$\begin{aligned} & \int_0^T \left\langle \partial_t(M\zeta(\rho)\hat{\psi}), \varphi \right\rangle_{\mathcal{O}} - \left(M\zeta(\rho)\mathbf{v}\hat{\psi}, \nabla_x \varphi \right)_{\mathcal{O}} - \left(M\zeta(\rho)\hat{\psi}(\nabla_x \mathbf{v})\mathbf{q}, \nabla_{\mathbf{q}} \varphi \right)_{\mathcal{O}} dt \\ &+ \int_0^T (M\nabla_x \hat{\psi}, \nabla_x \varphi)_{\mathcal{O}} + \left(M\mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}), \nabla_{\mathbf{q}} \varphi \right)_{\mathcal{O}} dt = 0 \quad \text{for all } \varphi \in L^\infty(0, T; W^{1, \infty}(\mathcal{O})). \end{aligned}$$

Letting $\ell \rightarrow \infty$ in (4.178) we deduce the following energy inequality, stated in (3.8):

$$\begin{aligned} & k \int_{\mathcal{O}} M\zeta(\rho(\cdot, t)) \mathcal{F}(\hat{\psi}(\cdot, t)) dx d\mathbf{q} + \frac{1}{2} \int_{\Omega} \rho(\cdot, t) |\mathbf{v}(\cdot, t)|^2 dx \\ &+ \int_0^t \int_{\Omega} \mu(\rho, \varrho) |D(\mathbf{v})|^2 dx ds + 4kC_1 \int_0^t \int_{\mathcal{O}} M \left| \nabla_{x, \mathbf{q}} \sqrt{\hat{\psi}} \right|^2 dx d\mathbf{q} ds \\ &\leq k \int_{\mathcal{O}} M\zeta(\rho_0) \mathcal{F}(\hat{\psi}_0) dx d\mathbf{q} + \frac{1}{2} \int_{\Omega} \rho_0 |\mathbf{v}_0|^2 dx + \int_0^t (\rho \mathbf{f}, \mathbf{v}) ds, \end{aligned}$$

where, as before, $\mathcal{F}(s) := s \log s + 1$ for $s > 0$ and $\mathcal{F}(0) := \lim_{s \rightarrow 0+} \mathcal{F}(s) = 1$.

It remains to prove the weak continuity properties stated in (3.6) and the attainment of the initial data asserted in (3.7). First, we set $\mathbf{w}(x, t) := \chi_{[0, t]} \mathbf{u}(x)$ in (4.176), where $\mathbf{u} \in W_{0, \text{div}}^{1, s}(\Omega; \mathbb{R}^d)$ is arbitrary, with $s > 2$, to deduce that

$$\begin{aligned} & ((\rho^\ell \mathbf{v}^\ell)(t), \mathbf{u}) + \int_0^t [-(\rho^\ell \mathbf{v}^\ell \otimes \mathbf{v}^\ell, \nabla_x \mathbf{u}) + (\mu(\rho^\ell, \varrho^\ell) D(\mathbf{v}^\ell), \nabla_x \mathbf{u})] d\tau \\ &= \int_0^t [-(\tau^\ell, \nabla_x \mathbf{u}) + (\rho^\ell \mathbf{f}, \mathbf{w})] d\tau + (\rho_0 \mathbf{v}_0, \mathbf{u}). \end{aligned} \tag{4.200}$$

Letting $\ell \rightarrow \infty$ in (4.200) and using the convergence results above we obtain, for almost all $t \in (0, T)$, that, for each $\mathbf{u} \in W_{0, \text{div}}^{1, s}(\Omega; \mathbb{R}^d)$ with $s > 2$,

$$\begin{aligned} & ((\rho \mathbf{v})(t), \mathbf{u}) + \int_0^t [-(\rho \mathbf{v} \otimes \mathbf{v}, \nabla_x \mathbf{u}) + (\mu(\rho, \varrho) D(\mathbf{v}), \nabla_x \mathbf{u})] d\tau \\ &= \int_0^t [-(\tau, \nabla_x \mathbf{u}) + \langle \rho \mathbf{f}, \mathbf{w} \rangle] d\tau + (\rho_0 \mathbf{v}_0, \mathbf{u}). \end{aligned} \tag{4.201}$$

After redefining $\rho \mathbf{v}$ on a set of measure zero, the above equation holds for all $t \in (0, T)$. Thus, by letting $t \rightarrow 0_+$, we deduce that

$$\lim_{t \rightarrow 0_+} ((\rho \mathbf{v})(t), \mathbf{u}) = (\rho_0 \mathbf{v}_0, \mathbf{u}) \quad \text{for all } \mathbf{u} \in W_{0, \text{div}}^{1, s}(\Omega; \mathbb{R}^d) \text{ with } s > 2. \tag{4.202}$$

Replacing t with t' in (4.201) and subtracting the resulting equality from (4.201) we deduce, for almost every $t, t' \in (0, T)$ that, for each $\mathbf{u} \in W_{0, \text{div}}^{1, s}(\Omega; \mathbb{R}^d)$ with $s > 2$,

$$\begin{aligned} & ((\rho \mathbf{v})(t), \mathbf{u}) + \int_{t'}^t [-(\rho \mathbf{v} \otimes \mathbf{v}, \nabla_x \mathbf{u}) + (\mu(\rho, \varrho) D(\mathbf{v}), \nabla_x \mathbf{u})] d\tau \\ &= \int_{t'}^t [-(\tau, \nabla_x \mathbf{u}) + \langle \rho \mathbf{f}, \mathbf{w} \rangle] d\tau + ((\rho \mathbf{v})(t'), \mathbf{u}). \end{aligned} \tag{4.203}$$

As the integrands in the integrals appearing on the left-hand side and right-hand side of (4.203) belong to $L^1(0, T)$, it follows by the fundamental theorem of Calculus for Lebesgue integral, again, after redefining $\rho \mathbf{v}$ on a set of measure zero, that $t \mapsto ((\rho \mathbf{v})(t), \mathbf{u})$ is, for each $\mathbf{u} \in W_{0,\text{div}}^{1,s}(\Omega; \mathbb{R}^d)$ with $s > 2$, absolutely continuous on $[0, T]$.

Similarly, setting $\varphi := \chi_{[0,t]} \phi(x, \mathbf{q})$ in (4.177), where $\phi \in W^{1,\infty}(\mathcal{O})$ is arbitrary, we obtain that

$$\begin{aligned} & (M(\zeta(\rho^\ell) \hat{\psi}^\ell)(t), \phi)_{\mathcal{O}} - \int_0^t \left[\left(M\zeta(\rho^\ell) \mathbf{v}^\ell \hat{\psi}^\ell, \nabla_x \phi \right)_{\mathcal{O}} + \left(M\zeta(\rho^\ell) \Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \mathbf{v}^\ell) \mathbf{q}, \nabla_{\mathbf{q}} \phi \right)_{\mathcal{O}} \right] d\tau \\ & + \int_0^t \left[\left(M\nabla_x \hat{\psi}^\ell, \nabla_x \phi \right)_{\mathcal{O}} + \left(M\mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}^\ell), \nabla_{\mathbf{q}} \phi \right)_{\mathcal{O}} \right] d\tau = (M\zeta(\rho_0) \hat{\psi}_0, \phi)_{\mathcal{O}}. \end{aligned} \quad (4.204)$$

Letting $\ell \rightarrow \infty$ in (4.204) and using the convergence results above we obtain, for almost all $t \in (0, T)$, that, for each $\phi \in W^{1,\infty}(\mathcal{O})$,

$$\begin{aligned} & (M(\zeta(\rho) \hat{\psi})(t), \phi)_{\mathcal{O}} - \int_0^t \left[\left(M\zeta(\rho) \mathbf{v} \hat{\psi}, \nabla_x \phi \right)_{\mathcal{O}} + \left(M\zeta(\rho) \hat{\psi}(\nabla_x \mathbf{v}) \mathbf{q}, \nabla_{\mathbf{q}} \phi \right)_{\mathcal{O}} \right] d\tau \\ & + \int_0^t \left[\left(M\nabla_x \hat{\psi}, \nabla_x \phi \right)_{\mathcal{O}} + \left(M\mathbb{A}(\nabla_{\mathbf{q}} \hat{\psi}), \nabla_{\mathbf{q}} \phi \right)_{\mathcal{O}} \right] d\tau = (M\zeta(\rho_0) \hat{\psi}_0, \phi)_{\mathcal{O}}. \end{aligned}$$

After redefining $\zeta(\rho) \hat{\psi}$ on a set of measure zero, the above equation holds for all $t \in (0, T)$. Letting $t \rightarrow 0_+$, we deduce that

$$\lim_{t \rightarrow 0_+} (M(\zeta(\rho) \hat{\psi})(t), \phi)_{\mathcal{O}} = (M\zeta(\rho_0) \hat{\psi}_0, \phi)_{\mathcal{O}} \quad \text{for all } \phi \in W^{1,\infty}(\mathcal{O}). \quad (4.205)$$

Similarly as in the case of $\rho \mathbf{v}$ above, after a possible redefinition on a set of measure zero, the function $t \mapsto (M(\zeta(\rho) \hat{\psi})(t), \phi)_{\mathcal{O}}$ is absolutely continuous on $[0, T]$ for each $\phi \in W^{1,\infty}(\mathcal{O})$. This completes the proofs of the assertions (3.6) and (3.7), and thereby the proof of Theorem 3.2 is also complete.

5 Conclusion

We have proved the existence of large-data global-in-time weak solutions to a Navier–Stokes–Fokker–Planck system, which arises in models of incompressible dilute polymeric fluids with variable density. A key feature of the model is the presence of a density-dependent drag coefficient in the Fokker–Planck equation. We have extended the model considered in [4] to a more general class by allowing dependence of the viscosity coefficient on both the density and the polymer number density. This is a nontrivial extension from the point of view of the analysis of the resulting model, as the additional dependence of the viscosity coefficient on the polymer number density complicates the proof of existence of solutions in an essential manner. Nevertheless, our proof of large-data global-in-time weak solutions, based on a two-stage Galerkin approximation on the spatial domain and on the configuration space domain, is shorter and more direct than the one presented in [4], based on time-discretization, in the simpler setting where the viscosity coefficient was assumed to be independent of the polymer number density.

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E-mail address: chuhui.he@maths.ox.ac.uk

E-mail address: endre.suli@maths.ox.ac.uk

Proposed running head: Existence of Weak Solutions to Nonhomogeneous Dilute Polymeric Fluids

Corresponding author: Chuhui He (chuhui.he@maths.ox.ac.uk)