

An Optimal Auction with Correlated Values and Risk Aversion *

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Abstract:

We consider an auction setting where the buyers are risk averse with correlated private valuations (CARA preferences, binary types), and characterize the optimal mechanism for a risk neutral seller. We show that the optimal auction extracts all buyer surplus whenever the correlation is sufficiently strong (greater than $1/3$ in absolute value), no matter how risk averse the buyers are. In contrast, we note that a sufficiently risk-averse seller would not use a full rent extracting mechanism for any positive correlation of the valuations even if the buyers were risk neutral.

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1 Introduction

The classic single-object auction model (Vickrey [9]) has three important restrictive assumptions: the buyers are risk neutral, and their valuations are purely private and statistically independent. In this note, we simultaneously relax the risk neutrality and independence assumptions and characterize the seller’s expected revenue maximizing mechanism in a simple environment.

The two modifications that we consider have already been studied separately. Crémer and McLean [1], [2] found revenue maximizing auctions for correlated values with risk neutral buyers. Maskin and Riley [4] characterized the optimal auction with risk-averse buyers under the assumption that the valuations are statistically independent. By simultaneously relaxing both assumptions we expect to find interesting tradeoffs.

If the buyers’ valuations are correlated and they are risk neutral then it is possible to construct an efficient mechanism in which the seller extracts all the information rents from the buyers (see Crémer and McLean [2]). However, this auction form imposes a certain kind of risk on the buyers: their payments are sensitive to the announcements of the others. If the buyers are risk averse, they need to be compensated. This is a previously not analyzed tradeoff between certain screening and insurance incentives.

It is also an interesting theoretical question whether the Crémer–McLean result is an “isolated example.” That is, whether full rent extraction is peculiar to the optimal mechanism with risk neutral bidders, or full rent extraction is optimal with any (potentially strong) risk aversion if the signals are sufficiently correlated.

In the next section we introduce a simple auction model exhibiting risk averse buyers with correlated valuations (CARA preferences, two, symmetrically distributed types), and derive the following results. First, we show that the seller’s auction design problem is a “regular” screening problem: In the optimal mechanism the incentive compatibility

constraints bind downward, there is no distortion for the high type (he may only lose to another high type and is fully insured against not winning), and the low type may be rationed and earns no surplus. Second, we show that no matter how risk averse the buyers are, if their types are sufficiently strongly correlated (specifically, if the correlation coefficient is larger than $1/3$ in absolute value), then *neither type enjoys any surplus*. We also show that when valuations are correlated, the seller’s revenue is generally non-monotonic in the degree of the buyers’ risk aversion. At the end of our discussion we briefly consider the case of a risk-averse seller.¹ We note that even if the buyers are risk-neutral and hence Crémer–McLean mechanisms could be used, a sufficiently risk-averse seller would prefer not to use such (full-rent-extracting) mechanisms. This is so because such mechanisms provide the seller with high but also *highly risky* revenue.

Throughout the analysis we make two modelling assumptions. First, we only consider binary types (high or low) and symmetric type-distributions. This assumption makes it possible to parametrize the correlation between the buyers’ valuations with just a single number (the correlation coefficient). Second, we model risk aversion by assuming that the bidders’ preferences exhibit constant absolute risk aversion.² This assumption rules out random transfers in the optimal mechanism, and makes certain derivations simpler and more intuitive.

2 The Model

There are two buyers with private valuations (also called types). A buyer’s valuation can take one of two values, either $v^n = H$ (high) or $v^n = L$ (low), for $n = 1, 2$. We

¹In the classical (independent private values, risk neutral buyers) setup the case of a risk averse seller is considered by Waehrer et al. [10] and Eső and Futó [3].

²In the independent values case, Matthews [5] makes the same assumption.

normalize the seller's commonly known valuation to zero, and assume $H > L > 0$.

The joint distribution of the buyers' valuations is symmetric described by the following matrix with $\alpha \in (0, 1)$.

	$v^2 = L$	$v^2 = H$
$v^1 = L$	$\alpha/2$	$(1 - \alpha)/2$
$v^1 = H$	$(1 - \alpha)/2$	$\alpha/2$

That is, $\Pr(v^n = L) = 1/2$, and $\Pr(v^n = L | v^{-n} = L) = \Pr(v^n = H | v^{-n} = H) = \alpha$. Note that the coefficient of correlation between the two buyers' valuations is exactly $2\alpha - 1$. We will refer to the entries of this matrix as $\Pi_{ij} = \Pr(v^n = i, v^{-n} = j)$.

The buyers are risk averse, and their von Neumann–Morgenstern utility functions exhibit constant absolute risk aversion (CARA), i.e., $u(w) = -e^{-rw}/r$, $r > 0$. Denote $c_H = e^{-rH}$ and $c_L = e^{-rL}$, and observe that $0 < c_H < c_L < 1$.

The Revelation Principle applies, therefore we can confine our attention to truth-telling Nash equilibria of direct mechanisms. A direct mechanism is given by, for each buyer, his probability of winning conditional on the type-announcements, and his transfers conditional on the type-announcements and a binary variable showing whether he has won the object. Formally, for all $i, j \in \{L, H\}$, $n \in \{1, 2\}$, let p_{ij}^n denote the probability of buyer n winning when he announces type i and the other buyer announces type j . Also, for all $i, j \in \{L, H\}$ and $n \in \{1, 2\}$, let t_{ij}^n denote the transfer from buyer n to the seller when the type-announcements are i (from buyer n) and j (from buyer not n) and buyer n gets the good; let \hat{t}_{ij}^n denote the transfer from buyer n to the seller under the same circumstances but when buyer n does not win. In order to simplify notation we define y_{ij}^n and \hat{y}_{ij}^n , the “disutility factor” corresponding to the transfers, as $y_{ij}^n = e^{rt_{ij}^n}$ and $\hat{y}_{ij}^n = e^{r\hat{t}_{ij}^n}$.

3 The Optimal Auction

3.1 Participation and Incentive Compatibility Constraints

If the other buyer plays according to the truth-telling equilibrium in an incentive compatible direct mechanism, then buyer n 's expected utility when he has type $i \in \{L, H\}$ and announces $k \in \{L, H\}$ is,

$$\sum_{j=L,H} 2\Pi_{ij} [p_{kj}^n u(w + i - t_{kj}^n) + (1 - p_{kj}^n)u(w - \hat{t}_{kj}^n)]. \quad (1)$$

Participation requires that this expression meet or exceed $u(w)$ for $k = i$. Using $\Pi_{ij} = \alpha/2$ for $i = j$ and $\Pi_{ij} = (1 - \alpha)/2$ for $i \neq j$, the specification of utilities and the notation introduced above, the participation constraints become

$$\begin{aligned} \alpha (p_{LL}^n c_L y_{LL}^n + (1 - p_{LL}^n) \hat{y}_{LL}^n) + (1 - \alpha) (p_{LH}^n c_L y_{LH}^n + (1 - p_{LH}^n) \hat{y}_{LH}^n) &\leq 1, \\ (1 - \alpha) (p_{HL}^n c_H y_{HL}^n + (1 - p_{HL}^n) \hat{y}_{HL}^n) + \alpha (p_{HH}^n c_H y_{HH}^n + (1 - p_{HH}^n) \hat{y}_{HH}^n) &\leq 1, \end{aligned}$$

for types L and H respectively. We will refer to these inequalities (individual rationality for the low and high types) by IR_L^n and IR_H^n , and denote their left-hand sides by D_L^n and D_H^n , respectively.

Incentive compatibility requires that (1) be weakly greater for $k = i$ than it is for $k \neq i$. That is,

$$\begin{aligned} D_L^n &\leq \alpha (p_{HL}^n c_L y_{HL}^n + (1 - p_{HL}^n) \hat{y}_{HL}^n) + (1 - \alpha) (p_{HH}^n c_L y_{HH}^n + (1 - p_{HH}^n) \hat{y}_{HH}^n), \\ D_H^n &\leq (1 - \alpha) (p_{LL}^n c_H y_{LL}^n + (1 - p_{LL}^n) \hat{y}_{LL}^n) + \alpha (p_{LH}^n c_H y_{LH}^n + (1 - p_{LH}^n) \hat{y}_{LH}^n). \end{aligned}$$

We will refer to these inequalities (incentive compatibility for the low and high types)

by IC_L^n and IC_H^n , respectively.

3.2 Expected Revenue Maximization

The seller's ex ante expected revenue from buyer n can be written as

$$R^n = \frac{\alpha}{2r} [p_{LL}^n \ln y_{LL}^n + (1 - p_{LL}^n) \ln \hat{y}_{LL}^n + p_{HH}^n \ln y_{HH}^n + (1 - p_{HH}^n) \ln \hat{y}_{HH}^n] \\ + \frac{1 - \alpha}{2r} [p_{LH}^n \ln y_{LH}^n + (1 - p_{LH}^n) \ln \hat{y}_{LH}^n + p_{HL}^n \ln y_{HL}^n + (1 - p_{HL}^n) \ln \hat{y}_{HL}^n]. \quad (2)$$

For a risk neutral seller, the auction design problem is to maximize $R^1 + R^2$ subject to IR_L^n , IR_H^n , IC_L^n , IC_H^n for $n = 1, 2$, and the feasibility constraints, $p_{ij}^1 + p_{ji}^2 \leq 1$ for $i, j \in \{L, H\}$. The choice variables are $p_{ij}^n \in [0, 1]$ and $y_{ij}^n, \hat{y}_{ij}^n \in (0, \infty)$ for $i, j \in \{L, H\}$ and $n = 1, 2$.

Since the buyers' preferences are CARA, transfers in the optimal mechanism (conditional on the buyers' announcements and the allocation) are deterministic. This is so because if some y_{ij} 's were random then the seller could replace them by their expected values; this would not alter incentives (as utilities are linear in y_{ij}), but would lead to a higher expected revenue (as revenue is a concave function of y_{ij}), which is a contradiction.

The optimal mechanism is symmetric in n , that is, $p_{ij}^1 = p_{ij}^2$, $y_{ij}^1 = y_{ij}^2$, and $\hat{y}_{ij}^1 = \hat{y}_{ij}^2$, for all $i, j \in \{L, H\}$. If it was not then we could define a symmetric ("average") mechanism by

$$\bar{p}_{ij}^n = \frac{p_{ij}^1 + p_{ij}^2}{2}, \bar{y}_{ij}^n = \frac{p_{ij}^1 y_{ij}^1 + p_{ij}^2 y_{ij}^2}{2\bar{p}_{ij}^n}, \bar{\hat{y}}_{ij}^n = \frac{(1 - p_{ij}^1) \hat{y}_{ij}^1 + (1 - p_{ij}^2) \hat{y}_{ij}^2}{2(1 - \bar{p}_{ij}^n)}, \quad n = 1, 2.$$

Since the original mechanism satisfies the IR and IC constraints, this new one satisfies them too (this can be seen by adding the corresponding constraints for $n = 1, 2$), and

it is clearly also feasible. However, because the seller's objective function is strictly concave in the y 's the new mechanism is strictly preferred by the seller. Therefore, in the remainder of the analysis we will drop the reference to the buyers' identities (the superscript n).

A first inspection of the incentive and participation constraints yields the following result (the proof is relegated to the Appendix).

Lemma 1 *In the mechanism that maximizes the seller's expected revenue, from a pair of constraints (IR_i, IC_i) at least one is binding. Moreover, either IR_L or IR_H or both are binding.*

In the rest of the subsection we will establish that the low type's incentive constraint, IC_L , is not binding in the seller's expected revenue maximizing mechanism. In order to do this we will first analyze the seller's problem when IC_L is relaxed.

We can write the Lagrangian to the relaxed problem (when IC_L is ignored) as

$$\begin{aligned} \mathcal{L} = & 2R - \lambda_L(D_L - 1) - \lambda_H(D_H - 1) \\ & - \mu_H[D_H - (1 - \alpha)(p_{LL}c_{HYLL} + (1 - p_{LL})\hat{y}_{LL}) + \alpha(p_{LH}c_{HYLH} + (1 - p_{LH})\hat{y}_{LH})] \\ & - \phi_{LL}(p_{LL} - 1/2) - \phi_{HL}(p_{LH} + p_{HL} - 1) - \phi_{HH}(p_{HH} - 1/2), \end{aligned}$$

where R is the seller's revenue from (2), D_L and D_H are the left-hand sides of IR_L and IR_H , respectively, λ_L , λ_H , and μ_H are the multipliers on IR_L , IR_H and IC_H , respectively, and ϕ_{LL} , ϕ_{HL} , and ϕ_{HH} are the multipliers on the feasibility constraints.

Lemma 2 *In the optimal solution when IC_L is relaxed we have $p_{HH} = 1/2$, $p_{HL} = 1$, and $p_{LH} = 0$.*

The proof is based on perturbations of the Lagrangian and can be found in the Appendix. This result implies that in the optimal auction, only the low type may be

rationed—if either of the two buyers reports a high type then the good is sold efficiently. The low type is rationed if $p_{LL} < 1/2$.

With this much preparation we are ready to prove our first main result.

Proposition 1 *In the mechanism that maximizes the seller’s revenue, IR_L and IC_H are binding and IC_L is slack.*

Proof. We will argue that in the solution where IC_L is neglected the incentive compatibility constraint for the low type is satisfied (IC_L holds strictly), therefore IC_L must be slack in the solution. This implies that IR_L is binding (by Lemma 1) and that IC_H is also binding (otherwise the first-best would be feasible).

In the solution where IC_L is neglected, we know that $p_{HH} = 1/2$, $p_{HL} = 1$, and $p_{LH} = 0$. Moreover, in the proof of Lemma 2 we found that the high type is insured against the loss of the good, $c_H y_{HL} = c_H y_{HH} = \hat{y}_{HH} \equiv K$, and by IR_H , $K \leq 1$.

Consider as a candidate mechanism a contract where the low type is offered complete insurance such that IR_L is binding, $c_L y_{LL} = \hat{y}_{LH} = \hat{y}_{LL} = 1$, and K is set so that the high type’s incentive constraint binds,

$$K = (1 - \alpha)(p_{LL}c + 1 - p_{LL}) + \alpha, \tag{3}$$

where $c = c_H/c_L$. Since $c < 1$, we have $K < 1$, therefore IR_H is slack. Since $p_{LL} \leq 1/2$, we also have $K \geq (c + 1)/2$.

We will now show that the candidate mechanism is feasible, that is, IC_L holds. The left-hand side of IC_L equals 1, while the right-hand side can be written as

$$rhs(IC_L) = \alpha \frac{K}{c} + \frac{(1 - \alpha)}{2} \left(\frac{K}{c} + K \right).$$

At $\alpha = 0$, this expression becomes $(K/c + K)/2$, which, by $K \geq (c + 1)/2$, is weakly

greater than

$$\frac{1}{2} \left(\frac{c+1}{2} \frac{1}{c} + \frac{c+1}{2} \right) = \frac{1}{2} \left(1 + \frac{1/c + c}{2} \right) > 1.$$

That is, IC_L holds as a strict inequality at $\alpha = 0$. However,

$$\frac{\partial rhs(IC_L)}{\partial \alpha} = \frac{1}{2} \left(\frac{K}{c} - K \right) > 0.$$

Therefore IC_L holds strictly for all $\alpha \in [0, 1]$. The candidate mechanism indeed satisfied all (incentive, participation, and feasibility) constraints.

Starting from the candidate mechanism (both types fully insured, IR_L and IC_H binding, $K < 1$), the optimal mechanism is obtained by distorting the low type's full insurance. In order to ensure his participation, the low type's expected payment must be reduced. Then, the only way to increase the seller's revenue is by requiring the high type to pay more in the optimal mechanism than in the candidate one. This means that a low type imitating the high type would pay uniformly more in the optimal mechanism than he would have in the candidate one, therefore IC_L must hold as a strict inequality in the optimal mechanism as well. ■

Remark 1 *It is clear from the proof of Proposition 1 that the same result holds for the expected revenue maximizing efficient auction as well (i.e., when $p_{LL} = 1/2$ instead of being set optimally).*

There are three types of risk that the buyers may face in the optimal auction. First, their payments conditional on the type-announcements and the allocation could be random—this is excluded by assuming constant absolute risk aversion. Second, for a given allocation, a buyer may face risk arising from the other buyer's type announcement. Third, for a given type-profile, a buyer's marginal utility of income may be

different when he wins and when he loses.³ In the optimal auction, as we have shown, the high type faces no risk of the second kind because $y_{HH} = y_{HL}$, and he is insured against risk of the third kind because $c_H y_{HH} = \hat{y}_{HH}$. On the other hand, the low type almost always incurs both types of risks. Only if the buyers types are independent and he does not win will the low type's marginal utility of income be independent of the other buyer's valuation.

The significance of Proposition 1 is that it shows that our optimal auction design problem is “regular” in the sense that in the optimum the low type's incentive compatibility constraint is not binding (while the high type's is), and the low type enjoys no rents. (The proof is different from that in the risk neutral case because of interesting insurance considerations.) In the next subsection we will show that when the buyers' types are sufficiently strongly correlated, neither type earns any information rents in the optimal auction.

3.3 Full Rent Extraction in the Optimal Auction

We have established that IR_L and IC_H are binding, and IC_L is slack in the optimal auction. Whether IR_H is binding (whether the high type has no surplus either), it turns out, depends on how strong the correlation between the types is. Recall that in our model, the correlation coefficient between high and low types is $2\alpha - 1$.

Proposition 2 *If the correlation between high and low valuations is greater than 1/3 in absolute value (i.e., $\alpha \notin [1/3, 2/3]$), then no matter how risk-averse the buyers are (i.e., for all $r > 0$), in the optimal auction neither type of the buyers earns any surplus, that is, both IR_L and IR_H are binding.*

³The latter possibility arises because ties happen with positive probability in a discrete-type model.

Proof. Suppose that IR_H is slack. Then $\lambda_H = 0$ in the Lagrangian, and the Kuhn-Tucker conditions of maximization with respect to the y -variables become

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y_{LL}} &= \frac{\alpha p_{LL}}{r y_{LL}} - \lambda_L \alpha p_{LL} c_L + \mu_H (1 - \alpha) p_{LL} c_H = 0, \\ \frac{\partial \mathcal{L}}{\partial \hat{y}_{LL}} &= \frac{\alpha (1 - p_{LL})}{r \hat{y}_{LL}} - \lambda_L \alpha (1 - p_{LL}) + \mu_H (1 - \alpha) (1 - p_{LL}) = 0, \\ \frac{\partial \mathcal{L}}{\partial \hat{y}_{LH}} &= \frac{1 - \alpha}{r} \frac{1}{\hat{y}_{LH}} - \lambda_L (1 - \alpha) + \mu_H \alpha = 0, \\ \frac{\partial \mathcal{L}}{\partial y_{HH}} &= \frac{\alpha}{r} \frac{1/2}{y_{HH}} - \mu_H \alpha \frac{1}{2} c_H = 0, \\ \frac{\partial \mathcal{L}}{\partial \hat{y}_{HH}} &= \frac{\alpha}{r} \frac{1/2}{\hat{y}_{HH}} - \mu_H \frac{1}{2} \alpha = 0, \\ \frac{\partial \mathcal{L}}{\partial y_{HL}} &= \frac{1 - \alpha}{r} \frac{1}{y_{HL}} - \mu_H (1 - \alpha) c_H = 0.\end{aligned}$$

Cross-multiplying the first three equations by y_{LL} , \hat{y}_{LL} , and \hat{y}_{LH} , respectively, and adding them up yields

$$\begin{aligned}\frac{1}{r} &= [\alpha p_{LL} c_L y_{LL} + \alpha (1 - p_{LL}) \hat{y}_{LL} + (1 - \alpha) \hat{y}_{LH}] \lambda_L \\ &\quad - [(1 - \alpha) p_{LL} c_H y_{LL} + (1 - \alpha) (1 - p_{LL}) \hat{y}_{LL} + \alpha \hat{y}_{LH}] \mu_H. \quad (4)\end{aligned}$$

From the second three equations,

$$\frac{1}{r \mu_H} = c_H y_{HH} = c_H y_{HL} = \hat{y}_{HH}.$$

Note that $D_H \equiv 1/r \mu_H$, hence $1/r \mu_H \leq 1$. The high type's incentive constraint is binding because IR_H is slack. Hence IC_H can be written as

$$\frac{1}{r \mu_H} = (1 - \alpha) (p_{LL} c_H y_{LL} + (1 - p_{LL}) \hat{y}_{LL}) + \alpha \hat{y}_{LH}.$$

In equation (4), the coefficient on λ_L equals 1 by IR_L , and the coefficient on μ_H equals $-1/r\mu_H$ by IC_H . Therefore $\lambda_L = 2/r$.

From the second and third Kuhn-Tucker conditions used above,

$$\hat{y}_{LL} = \frac{\alpha}{r\lambda_L\alpha - r\mu_H(1-\alpha)},$$

$$\hat{y}_{LH} = \frac{1-\alpha}{r\lambda_L(1-\alpha) - r\mu_H\alpha}.$$

Recall $\lambda_L = 2/r$ and $r\mu_H \geq 1$. If $\alpha < 1/3$ then the denominator in \hat{y}_{LL} , $r\lambda_L\alpha - r\mu_H(1-\alpha) \leq 2\alpha - (1-\alpha) = 3\alpha - 1$, is negative, so $\hat{y}_{LL} < 0$, a contradiction. If $\alpha > 2/3$ then the denominator in \hat{y}_{LH} , $r\lambda_L(1-\alpha) - r\mu_H\alpha \leq 2(1-\alpha) - \alpha = 2 - 3\alpha$, is negative, so $\hat{y}_{LH} < 0$, a contradiction. Therefore, if IR_H is slack then α cannot lie outside the interval $[1/3, 2/3]$. ■

Proposition 2 shows that the well-known property of optimal auctions under correlation and risk neutrality (Cr mer–McLean mechanisms) that they leave no surplus with the buyers is not so peculiar: even if the buyers are risk-averse (in fact, for *any* level of constant absolute risk aversion), the seller’s optimal mechanism exhibits the same feature provided the correlation of valuations is sufficiently strong.⁴

An interesting comparative statics result is that for $\alpha \neq 1/2$ (with valuations that are not independent), the seller’s expected revenue is not monotonic in r (the buyers’ risk-aversion parameter).⁵ This is so because in both limits, as $r \rightarrow 0$ and as $r \rightarrow \infty$, the seller’s revenue converges to the first-best. Intuitively, the non-monotonicity is the result of two effects. Start with risk-neutral buyers and correlated values: there exists a

⁴Note, however, that the seller’s expected revenue is not equal to the first-best when the buyers are risk-averse.

⁵In contrast, in the independent-values case Matthews [5] shows that in the optimal auction the seller’s revenue is increasing with r .

family of full-rent extracting mechanisms. As r increases, it becomes more costly for the seller to compensate the buyers for the risks of any of these mechanisms. However, other auction forms, such as the first-price auction, which exploit the buyers' risk aversion, become more profitable. As r becomes sufficiently high, these auctions become optimal.

Another observation is that the outcome of the optimal auction is risky not only for the buyers but the seller as well. Consequently, one may ask how to design an optimal auction when the seller is risk-averse. In the classical setup (independent private values, risk neutral buyers) Esó and Futó [3] show that the seller can get full insurance against the revenue-risk of any auction at no cost. Therefore, in each expected-revenue equivalence class there is a mechanism with a deterministic revenue, which is optimal for the risk-averse seller. However, when the buyers' valuations are correlated, there may be a trade-off between the expected value and the riskiness of the seller's revenue.

To see this point, consider our model with binary types, positive correlation ($\alpha > 1/2$), and risk-neutral buyers ($r \rightarrow 0$). Suppose that the allocation rule is such that the low type won't win against the high type but may be rationed ($p_{LL} \leq 1/2$, $p_{LH} = 0$), just like in the optimal auction of Propositions 1 and 2. Denote the expected transfers from a buyer to the seller when he announces low and the other buyer announces low and high by t_{LL} and t_{LH} , respectively. In any Crémer–McLean mechanism neither type enjoys any surplus. The participation constraint of the low type is

$$\alpha(p_{LL}L - t_{LL}) - (1 - \alpha)t_{LH} = 0,$$

while the high type's incentive compatibility constraint is

$$0 \geq (1 - \alpha)(p_{LL}H - t_{LL}) - \alpha t_{LH}.$$

(The left-hand side of the latter constraint is zero because the high type makes no profit in a Crémer–McLean auction.) From the former equation, $t_{LH} = \alpha/(1 - \alpha)(p_{LL}L - t_{LL})$, which substituted into the latter inequality yields

$$\frac{\alpha}{1 - \alpha}(p_{LL}L - t_{LL}) \geq \frac{1 - \alpha}{\alpha}(p_{LL}H - t_{LL}).$$

If $p_{LL}L < t_{LL}$ then both sides of the above inequality would be negative; but then since $L < H$ and $\alpha/(1 - \alpha) > 1 > (1 - \alpha)/\alpha$ (because of $\alpha > 1/2$), the inequality would be violated. Therefore $t_{LL} \leq p_{LL}L$, so in the state of nature when both buyers report L the seller’s revenue does not exceed $2p_{LL}L \leq L$. However, an ordinary English auction would yield a revenue of at least L in every state of nature, which is preferred by a sufficiently risk averse seller. This shows that a sufficiently risk averse seller would not choose a full-rent-extracting mechanisms when the correlation of valuations is positive even if the buyers were risk neutral.

4 Conclusions

We studied the expected revenue maximizing mechanism in a simple auction environment with risk-averse buyers (CARA preferences) and correlated private values (binary types, symmetric distribution). Correlation of the valuations makes it possible for the seller to extract all rents from risk-neutral buyers (Crémer and McLean [1], [2]); however, risk-averse buyers require compensation for the risks involved in such mechanisms. On the other hand, we know from Matthews [5] and Maskin and Riley [4] that risk aversion of the buyers may also help the seller to screen them and hence it may increase the seller’s expected revenue.

We found that the optimal auction with risk-averse bidders retains the full-rent-

extracting property of Crémer–McLean mechanisms, provided that the correlation of the buyers’ valuations is sufficiently strong. (In our model with CARA preferences, if the correlation is sufficiently strong then both types have zero surplus, no matter how risk-averse the buyers are.) We also showed that the same result does not hold if the seller is sufficiently risk-averse because full-rent-extracting mechanisms generate highly risky revenue for the seller even if the buyers are risk neutral.

Among other questions, the generalization of the model to more than two types and a fuller analysis of the case of a risk-averse seller may be interesting for future research.

5 Appendix: Omitted Proofs

Proof of Lemma 1. In order to prove the first claim, suppose towards contradiction that both IR_i and IC_i are slack. Then, the seller can increase one of the variables y_{ij} or \hat{y}_{ij} that appears with a positive coefficient in D_i by $\varepsilon > 0$. For a sufficiently small ε both IR_i and IC_i continue to hold while the other IR and IC constraints are relaxed and the seller’s revenue increases, which is a contradiction.

Second, suppose towards contradiction that both IR_L or IR_H are slack. Increase \hat{y}_{LL} by $\varepsilon/(1 - p_{LL}) > 0$ (note $p_{LL} \leq 1/2$ by symmetry and feasibility). This change increases D_L (the left-hand sides of both IR_L and IC_L) by $\alpha\varepsilon$ and the right-hand side of IC_H by $(1 - \alpha)\varepsilon$. Also, increase either y_{HL} by $\varepsilon/(c_H p_{HL})$ (if $p_{HL} > 0$), or \hat{y}_{HL} by ε (if $p_{HL} = 0$). This increases D_H (the left-hand sides of IR_H and IC_H) by $(1 - \alpha)\varepsilon$ and the right-hand side of IC_L by at least $\alpha\varepsilon$ (since $c_H < c_L$). For $\varepsilon > 0$ sufficiently small this perturbation is feasible, and the seller’s revenue increases, which is a contradiction. ■

Proof of Lemma 2. First, we claim that in the solution $p_{HH} > 0$. Suppose towards contradiction that $p_{HH} = 0$. Increase p_{HH} to $\varepsilon > 0$ and set $y_{HH} = \hat{y}_{HH}/c_H$. No

constraint is affected as D_H , the left-hand sides of IR_H and IC_H , is unchanged. The seller's revenue, $2R$, changes by the amount $\alpha\varepsilon(\ln y_{HH} - \ln \hat{y}_{HH})/r$, which is positive by $c_H < 1$, a contradiction.

Now we will show $p_{HH} = 1/2$. The necessary Kuhn-Tucker conditions of maximization with respect to y_{HH} and \hat{y}_{HH} are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y_{HH}} &= \frac{\alpha p_{HH}}{r y_{HH}} - (\lambda_H + \mu_H)\alpha p_{HH} c_H = 0, \\ \frac{\partial \mathcal{L}}{\partial \hat{y}_{HH}} &= \frac{\alpha(1 - p_{HH})}{r \hat{y}_{HH}} - (\lambda_H + \mu_H)\alpha(1 - p_{HH}) = 0.\end{aligned}$$

(Note that the first-order conditions with respect to y_{ij} hold as equalities because $y_{ij} > 0$.) From $p_{HH} \in (0, 1)$ we get $c_H y_{HH} = \hat{y}_{HH} = 1/(r\lambda_H + r\mu_H)$. Moreover, by the Kuhn-Tucker conditions with respect to p_{HH} and ϕ_{HH} ,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial p_{HH}} &= \frac{\alpha}{r}(\ln y_{HH} - \ln \hat{y}_{HH}) - (\lambda_H + \mu_H)\alpha(c_H y_{HH} - \hat{y}_{HH}) - \phi_{HH} = 0, \\ -\frac{\partial \mathcal{L}}{\partial \phi_{HH}} &= p_{HH} - \frac{1}{2} \leq 0, \quad \phi_{HH} \geq 0, \quad \text{and} \quad \left(p_{HH} - \frac{1}{2}\right)\phi_{HH} = 0.\end{aligned}$$

($\partial \mathcal{L}/\partial p_{HH} = 0$ because $p_{HH} > 0$). Using $c_H y_{HH} = \hat{y}_{HH}$ the former condition becomes $\alpha \ln(1/c_H)/r = \phi_{HH}$, hence $\phi_{HH} > 0$. But then by the latter condition $p_{HH} = 1/2$ as claimed.

In order to show that in the optimum $p_{HL} = 1$ and $p_{LH} = 0$, first fix any $p_{LH}, p_{HL} \in (0, 1)$ with $p_{LH} + p_{HL} \leq 1$, and choose $y_{HL}, \hat{y}_{HL}, y_{LH}$, and \hat{y}_{LH} optimally. The Kuhn-Tucker conditions with respect to y_{HL} and \hat{y}_{HL} are,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y_{HL}} &= \frac{1 - \alpha p_{HL}}{r y_{HL}} - (\lambda_H + \mu_H)(1 - \alpha)p_{HL} c_H = 0, \\ \frac{\partial \mathcal{L}}{\partial \hat{y}_{HL}} &= \frac{1 - \alpha(1 - p_{HL})}{r \hat{y}_{HL}} - (\lambda_H + \mu_H)(1 - \alpha)(1 - p_{LH}) = 0.\end{aligned}$$

The two conditions imply $c_H y_{HL} = \hat{y}_{HL}$. Similarly, the Kuhn-Tucker conditions with respect to y_{LH} and \hat{y}_{LH} are,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y_{LH}} &= \frac{1-\alpha}{r} \frac{p_{LH}}{y_{LH}} - \lambda_L(1-\alpha)p_{LH}c_L + \mu_H\alpha p_{LH}c_H = 0, \\ \frac{\partial \mathcal{L}}{\partial \hat{y}_{LH}} &= \frac{1-\alpha}{r} \frac{1-p_{LH}}{\hat{y}_{LH}} - \lambda_L(1-\alpha)(1-p_{LH}) + \mu_H\alpha(1-p_{LH}) = 0.\end{aligned}$$

If we replace c_L by $c_H < c_L$ in the first condition then the resulting inequality and the second condition together imply $c_H y_{LH} < \hat{y}_{LH}$. Multiplying the original conditions by y_{LH}/p_{LH} and $-\hat{y}_{LH}/(1-p_{LH})$, respectively, and adding them up yields,

$$-\lambda_L(1-\alpha)(c_L y_{LH} - \hat{y}_{LH}) + \mu_H\alpha(c_H y_{LH} - \hat{y}_{LH}) = 0. \quad (5)$$

Now consider perturbing p_{HL} and p_{LH} by dp and $-dp$, respectively. The corresponding change in the Lagrangian is,

$$\begin{aligned}d\mathcal{L} &= \left[\frac{1-\alpha}{r} (\ln y_{HL} - \ln \hat{y}_{HL}) - (\lambda_H + \mu_H)(1-\alpha)(c_H y_{HL} - \hat{y}_{HL}) - \phi_{HL} \right] dp \\ &\quad - \left[\frac{1-\alpha}{r} (\ln y_{LH} - \ln \hat{y}_{LH}) - \lambda_L(1-\alpha)(c_L y_{LH} - \hat{y}_{LH}) + \mu_H\alpha(c_H y_{LH} - \hat{y}_{LH}) - \phi_{HL} \right] dp.\end{aligned}$$

Using $\hat{y}_{HL} = c_H y_{HL}$ and equation (5), $d\mathcal{L}$ can be rewritten as

$$d\mathcal{L} = \left[\frac{1-\alpha}{r} \ln(1/c_H) - \frac{1-\alpha}{r} (\ln y_{LH} - \ln \hat{y}_{LH}) \right] dp.$$

By $c_H y_{LH} < \hat{y}_{LH}$, the perturbation increases the Lagrangian, therefore $p_{LH} = 0$. Moreover, if p_{HL} is less than 1 in the optimum then ϕ_{HL} must be zero (by complementary slackness), and hence $\partial \mathcal{L} / \partial p_{HL} = (1-\alpha) \ln(1/c_H) / r > 0$. Therefore $p_{HL} = 1$. ■

References

- [1] J. Crémer, R. P. McLean, Optimal Selling Strategies under Uncertainty for a Discriminating Monopolist, *Econometrica* 53 (1985) 345-361.
- [2] J. Crémer, R. P. McLean, Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions, *Econometrica* 56 (1988) 345-361.
- [3] P. Esó, G. Futó, Auction Design with a Risk Averse Seller, *Econ. Letters* 65 (1999) 71-74.
- [4] E. Maskin, J. Riley, Optimal Auctions with Risk Averse Buyers, *Econometrica* 52 (1984) 1473-1518.
- [5] S. Matthews, Selling to Risk-Averse Buyers with Unobservable Tastes, *J. Econ. Theory* 30 (1983) 370-400.
- [6] P. Milgrom, R. Weber, A Theory of Auctions and Competitive Bidding, *Econometrica* 50 (1982) 1089-1122.
- [7] R. Myerson, Optimal Auction Design, *Mathematics of Operations Research* 6 (1981) 58-73.
- [8] J. Riley, W. Samuelson, Optimal Auctions, *Amer. Econ. Rev.* 71 (1981) 381-92.
- [9] W. Vickrey, Counterspeculation, Auctions, and Competitive Sealed Tenders, *J. Finance* 16 (1961) 8-37.
- [10] K. Waehrer, R. M. Harstad, M. H. Rothkopf, Auction Form Preferences of Risk-Averse Bid Takers, *RAND J. Econ.* 29 (1998) 179-192.