Aspects of the Class $\mathcal{S}$ Superconformal Index, and Gauge/Gravity Duality in five/six dimensions

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A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2015

Acknowledgements

First and foremost I am immensely grateful to my supervisor, Fernando Alday. Without his constant support and guidance throughout the years, this thesis would have not been possible in the present form. I am also deeply indebted to James Sparks, who has been like a second supervisor to me. It has been a privilege and a great pleasure to work with Fernando and James and I would like to thank them for illuminating discussions, and for sharing their deep understanding of physics with me.

I would also especially like to express my most sincere gratitude and appreciation to Mathew Bullimore, Lotte Hollands and Paul Richmond for countless enlightening discussions, for interesting collaborations and for sharing their vast knowledge with me. Furthermore I would like to thank Carolina Matte Gregory and Pietro Benetti Genolini for interesting collaborations as well as the Oxford Mathematics Department and the Oxford String Theory Group for providing an excellent research environment.

Finally I would like to thank all my friends and especially my parents and my sister for their constant support throughout my studies. None of this would have been possible without them.

Abstract

In the first part of this thesis, we discuss some aspects of the four-dimensional $\mathcal{N} = 2$ superconformal index of theories of class $\mathcal{S}$. We first consider a generalized index on $S^1 \times S^3/\mathbb{Z}_r$, and prove S-duality in a particular fugacity slice. We then go on to study the (round) superconformal index in the presence of surface defects. We develop a systematic prescription to compute surface defects labeled by arbitrary irreducible representations of the gauge group and subject those defects to various tests in several different limits. Each of these limits is interesting in its own right, and we go on to explore them in some depth.

In the second part of this thesis, we construct the gravity duals of large N supersymmetric gauge theories defined on squashed five-spheres with $SU(3) \times U(1)$ symmetry. The gravity duals are constructed in Euclidean Romans $F(4)$ gauged supergravity in six-dimensions, and uplift to massive type IIA supergravity. We compute the partition function and Wilson loop in the large N limit of the gauge theory and compare them to their corresponding supergravity dual quantities. As expected from AdS/CFT, both sides agree perfectly. Based on these results, we conjecture a general formula for the partition function and Wilson loop on any five-sphere background, which for fixed gauge theory depends only on a certain supersymmetric Killing vector. We then go on to construct rigid supersymmetric gauge theories on more general Riemannian five-manifolds. We follow a holographic approach, realizing the manifold as the conformal boundary of the six-dimensional bulk supergravity solution. This leads to a systematic classification of five-dimensional supersymmetric backgrounds with gravity duals.

The first part of this thesis is based on original material in [1, 2] as well as [3]. The second part is based on [4, 5] and [6]. The author's graduate work also included [7], which is only partially related to the present topics, as well as [8], which would go beyond the scope of this thesis.

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Chapter 1

Introduction and overview

Supersymmetric gauge theories were originally intended to describe the nature of elementary particle physics. However with their development it became apparent that they serve as much more than that. In particular, dualities of supersymmetric gauge theories have been a cornerstone of progress in theoretical physics, and have motivated some of the most interesting and nontrivial new relations between physics and mathematics. Nowadays their study is a fully established branch of research in theoretical high energy physics. The reason for this is severalfold. On the one hand it became apparent that supersymmetric gauge theories describe the low energy worldvolume dynamics of certain branes in string and M-theory. This realisation was a crucial ingredient in the development of the famous AdS/CFT correspondence [9]. On the other hand supersymmetric gauge theories have a rich mathematical structure which goes hand in hand with their physical interpretation. One famous example is the seminal discovery by N. Seiberg and E. Witten [10, 11], which revolutionised the study of four-manifolds in a purely mathematical setting.

More recently other prominent examples of the synergy between physics and mathematics have been uncovered. An important example is the AGT correspondence [12], which stems from the discovery of a duality between supersymmetric gauge theory in four dimensions and conformal field theory in two dimensions. The aim of the present thesis is to further study two particular cases of such dualities.

In part I of this thesis, we will focus on the study of a particular $4d/2d$ duality similar to the AGT correspondence. More precisely we will study a BPS index, called the superconformal index. This object, defined for a four-dimensional $\mathcal{N} = 2$ superconformal field theory of class $\mathcal{S}$ is conjectured to be dual to a topological quantum field theory. In analogy to the AGT correspondence, this duality is motivated by the compactification of an elusive six-dimensional $\mathcal{N} = (2, 0)$ superconformal field theory.

In part II, we will concentrate on the two sides of a particular example of the AdS/CFT correspondence. We will be working on five-dimensional gauge theories and their six-dimensional holographic duals. The idea is to compute exact quantities via localization for gauge theories with rigid supersymmetry on the five-dimensional background and compare them to the corresponding quantities in supergravity. Furthermore we shall use this correspondence as a tool to extract important results for the dual five-dimensional supersymmetric gauge theory.

1.1 Dualities of four-dimensional supersymmetric gauge theories

Four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories have proved a fertile arena from both the mathematical and physical perspectives and have been a constant focus of research for the past few decades. Several years ago, it was observed [13, 14] that a large subset of them, called of *class* $\mathcal{S}$, can be viewed in a more unified manner. The authors argue that based on the Seiberg-Witten construction there is a way to describe and classify theories of class $\mathcal{S}$ in terms of the data on Riemann surfaces.

The idea is to describe four-dimensional supersymmetric gauge theories in the setting of M-theory. M-theory is an 11-dimensional theory with membranes as fundamental objects. It was introduced as an extension of string theory, such that known string theories can be obtained as a limit of it. It contains three- and six-dimensional objects called M2- and M5-branes respectively. On the worldvolume of a stack of parallel M5-branes lives a six-dimensional $\mathcal{N} = (2,0)$ superconformal field theory (SCFT) [15–18].[1] This is an always strongly coupled theory, which at present does not have a Lagrangian description, which makes it extremely difficult to work with directly.

To gain control over this theory, it is convenient to study its dimensional reductions. By doing so, we can essentially work with the resulting lower dimensional effective theory of which we usually have a more thorough grasp. For example we can consider the dimensional reduction on $\mathcal{C}_g$, a Riemann surface of genus g, to obtain a four-dimensional supersymmetric gauge theory.

Let us for simplicity focus on a six-dimensional theory of type $\mathfrak{g} = A_1$. The spinor bundle over $\mathcal{C}_g$ is nontrivial, *i.e.* the supercharges are charged under the $SO(2)$ metric curvature of $\mathcal{C}_g$. In order to preserve a subset of supersymmetries in four dimensions, we have to perform a partial twist in the dimensional reduction. The idea is to pick an $SO(2)_R$ subgroup of the $SO(5)_R$ flavor symmetry of the six-dimensional parent theory to compensate for the metric curvature of $\mathcal{C}_g$. An obvious choice is $SO(2)_R \times SO(3)_R \subset SO(5)_R$, where one uses the $SO(2)_R$ for the partial twisting. Then the resulting four-dimensional theory preserves $\mathcal{N} = 2$ supersymmetry. An important consequence from the partial twist is that most of the four-dimensional physics does not depend on the particular choice of metric, but only on the total area and the complex structure of $\mathcal{C}_g$. For example the Riemann surface associated to maximally ($\mathcal{N} = 4$) supersymmetric Yang-Mills (sYM) theory in four dimensions is a torus and the complexified coupling constant of the gauge theory corresponds to the complex structure modulus of the associated torus. Furthermore the well-known Montonen-Olive duality [19] of $\mathcal{N} = 4$ sYM is geometrically realized as the modular transformation of the torus.

A Riemann surface can also be decorated with punctures.[2] Let us denote a general Riemann surface of genus g and with n punctures by $\mathcal{C}_{g,n}$. A puncture locally looks like a cylinder $S^1 \times I$, where I is a finite interval. Hence we can study a puncture by performing

[1] This is strictly only true for $\mathcal{N} = (2,0)$ theories of type $\mathfrak{g} = A_N$.

[2] In general there is a label associated to each puncture – see Chapter 2. For now we will only focus on type A_1 theories with full punctures.

a Kaluza-Klein reduction of the six-dimensional theory on S^1. The $\mathcal{N} = (2,0)$ SCFT compactified on a circle becomes maximally sYM in five dimensions [20, 21].[3] We thus get five-dimensional $\mathcal{N} = 2$ sYM on an interval, which corresponds to an $\mathcal{N} = 2$, $SU(2)$ vectormultiplet in four dimensions with coupling constant related to the length of the cylinder.

Since a Riemann surface $\mathcal{C}_{g,n}$ is categorised by its genus g and number of punctures n, the above observations naturally imply a classification of four-dimensional $\mathcal{N} = 2$ SCFTs of class $\mathcal{S}$ by the set (g, n). We shall denote such theories by $\mathcal{T}_{g,n}$. A Riemann surface can be decomposed into different pairs-of-pants, *i.e.* three-punctured spheres $\mathcal{C}_{0,3}$, connected by thin tubes. Similarly the field theory $\mathcal{T}_{g,n}$ can be decomposed into different building blocks $\mathcal{T}_{0,3}$, where one has to appropriately define "sewing pairs-of-pants" in the gauge theory language [13]. However, following the Riemann surface logic, various pairs-of-pants decompositions of $\mathcal{T}_{g,n}$ give generically different descriptions of the same theory. More precisely, a choice of decomposition corresponds to a particular degeneration limit in the parameter space of the theory. All such descriptions have to be related to one another by intricate dualities, such as generalized Seiberg- and Argyres-Seiberg dualitites [22, 23] – see Figure 2.2 for example.

Now consider a physical observable in the six-dimensional, $\mathcal{N} = (2,0)$ SCFT on $S^4 \times \mathcal{C}_{g,n}$, and assume that this observable is independent of the ratio of the "sizes" of S^4 and $\mathcal{C}_{g,n}$. Intuitively it makes sense that a (twisted) Kaluza-Klein compactification of the theory on S^4 should give a physical quantity on a field theory living on $\mathcal{C}_{g,n}$. Similarly a reduction on $\mathcal{C}_{g,n}$, should result in a physical observable in the theory $\mathcal{T}_{g,n}$ on S^4. This is exactly the idea behind the *AGT correspondence* [12], which states that the partition function of $\mathcal{N} = 2$, $SU(2)$ gauge theory on S^4 corresponds to the n-point correlation function of Liouville conformal field theory (CFT) on $\mathcal{C}_{g,n}$. It was subsequently extended to $SU(N)$ gauge theories and A_{N-1} Toda CFT [12, 24].

[3] The precise reduction is not clear as of this point. For example the Kaluza-Klein modes seem to be in correspondence with the $5d$ instanton numbers, which would naturally lead to double-counting in $5d$ computations.

Following this discovery a variety of new dualities and relations among field theories in various dimensions was found. Here are some examples:

- ⋄ The superconformal index (SCI) [25, 26], which is a twisted partition function of theories on $S^1 \times S^3$, was shown to have an interpretation in terms of a two-dimensional topological quantum field theory (TQFT) [27]. The TQFT was shown to be a (p, q, t)-deformed version of two-dimensional Yang-Mills in the zero-area limit [28–30].

- ⋄ A class $\mathcal{T}[\mathcal{M}_3]$ of three-dimensional $\mathcal{N} = 2$ theories labeled by three-manifolds $\mathcal{M}_3$ was discovered in [31–33]; such theories are said to be of *class* $\mathcal{R}$. The $3d/3d$ correspondence states that the partition function of the $\mathcal{T}[\mathcal{M}_3]$ theory on S^3 with gauge group $SU(N)$ is equivalent to the partition function of the $SL(N, \mathbb{C})$ Chern-Simons theory on $\mathcal{M}_3$.

- ⋄ Similarly a class of two-dimensional theories $\mathcal{T}[\mathcal{M}_4]$ was found to be labeled by four-manifolds $\mathcal{M}_4$ and are said to be of *class* $\mathcal{H}$ [34–36].

In the present thesis we shall mainly focus on the first example in this list, namely the superconformal index and its relation to two-dimensional TQFT. The superconformal index computes a trace over BPS states of a superconformal field theory in radial quantization.[4] It is a comparatively simple quantity, because it does not depend on exactly marginal deformations of the four-dimensional SCFT. From a six-dimensional perspective this means that the SCI does not depend on the complex structure of $\mathcal{C}_{g,n}$, and so one expects that the dual two-dimensional theory on $\mathcal{C}_{g,n}$ is topological [27]. The authors of [28] showed that in a particular limit, the normalized superconformal index of the $SU(2)$, $\mathcal{T}_{0,3}$ SCFT (*i.e.* the theory corresponding to a three-punctured sphere) can be written as

$$\mathrm{SCI}[\mathcal{T}_{0,3}](a, b, c) \quad \propto \quad \sum_R \frac{\chi_R(a)\chi_R(b)\chi_R(c)}{\dim_q R}\,, \tag{1.1.1}$$

where a, b, c are the fugacities at a puncture, the sum is over irreducible representations R of $SU(2)$, χ_R is a Schur polynomial and $\dim_q$ is the quantum dimension. This has precisely

[4] See Chapter 2 for a definition and a detailed discussion of its properties.

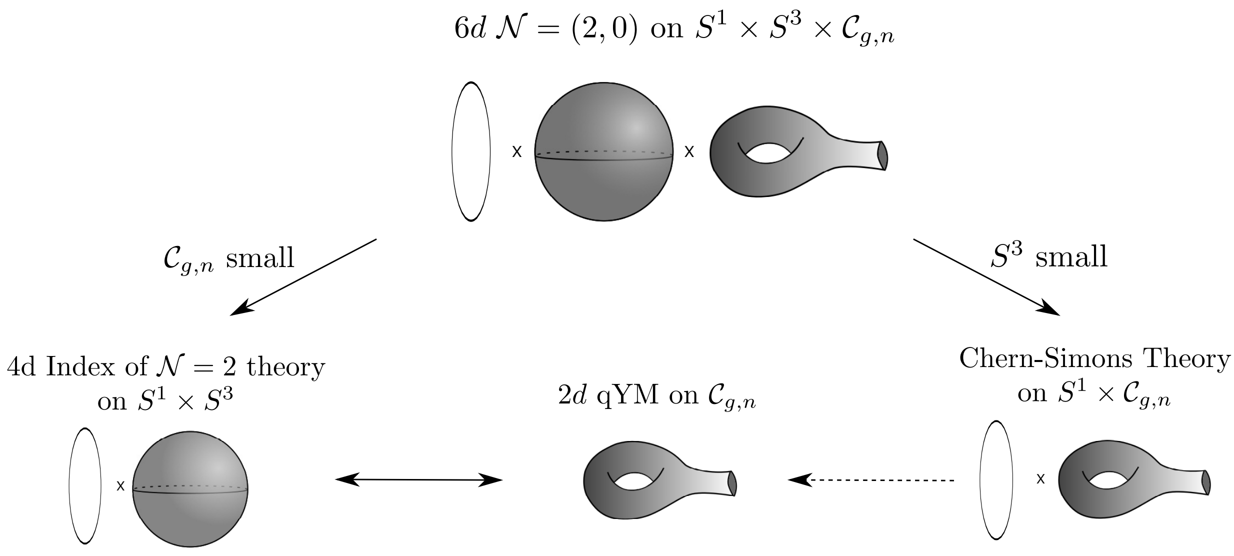


Figure 1.1: A $4d/2d$ correspondence relating the four-dimensional superconformal index to some two-dimensional TQFT, which can be described as an analytic continuation of refined Chern-Simons theory on $S^1 \times \mathcal{C}_{g,n}$.

the form of a three-point correlator of two-dimensional q-deformed Yang-Mills theory in the zero-area limit [37–40].[5] An alternative way of thinking of this two-dimenisonal TQFT is as an analytic continuation of refined Chern-Simons theory on $S^1 \times \mathcal{C}_{g,n}$, where the Chern-Simons coupling is related to q – see Figure 1.1.

Alternatively in [42], the authors explicitly reduce the six-dimensional $\mathcal{N} = (2,0)$ parent theory of type A_1 to two-dimensional q-deformed YM. They do so by first reducing to five-dimensional $\mathcal{N} = 2$ sYM on $S^3 \times \mathcal{C}_{g,n}$. This is a well-known theory with a Lagrangian description and one can localize its partition function. The path integral decomposes into constant modes along S^3, which give rise to two-dimensional YM on the remaining two directions, and fluctuations about the constant modes, which induce the q-deformation. In Chapter 2 of this thesis we review the most important aspects of this duality relevant for the remainder of part I, Chapters 3 – 5.

An interesting way to extend this particular $4d/2d$ correspondence is by studying the *lens index*. This is a twisted partition function on $S^1 \times S^3/\mathbb{Z}_r$. It is a particularly interesting object, since it is sensitive to the global properties of the gauge group [43]. The localized

[5]For an extensive review of two-dimensional Yang-Mills, see [41].

partition function contains a sum over flat connections [44]. Consequently every puncture is labeled by additional discrete data corresponding to a choice of holonomy around the Hopf fibre of $S^3/\mathbb{Z}_r$. However, the TQFT structure should still be inherent. In Chapter 3, we focus on a particular limit of the lens index and prove this, by explicitly showing that S-duality holds. Our strategy is to find a similar factorization as in the round case – see equation (1.1.1). We find that the index corresponding to a sphere with three punctures labeled by (a_i, m_i) can schematically be written as

$$\sum_R \left[\sum_{i,j,k=1}^{2} f_R^{(ijk)}(q) U_{R,i}^{(m_1)}(a_1) U_{R,j}^{(m_2)}(a_2) U_{R,k}^{(m_3)}(a_3) \right] , \tag{1.1.2}$$

where $f_R^{(ijk)}(q)$ and $U_{R,i}^{(m)}(a)$ are some functions, we shall specify in Chapter 3. From this factorization, S-duality follows from some orthogonality relations of $U_{R,i}^{(m)}(a)$ under the integration measure. Subsequent to our work, it was found that in a more general limit the two-dimensional correlation functions are described by a combination of so-called non-symmetric Macdonald polynomials [45].

Gauge theories can be decorated with general global defects, such as Wilson and 't Hooft loops as well as higher dimensional analogues. For example 1/2 BPS surface defects in $\mathcal{N} = 4$ sYM were defined and studied in [46, 47], by specifying a codimension-two singularity in the field configuration, such that half of the supersymmetries are preserved. An alternative way to define surface defects is by coupling the four-dimensional theory to two-dimensional degrees of freedom on the supported surface and subsequently integrating them out in the path integral. It has become apparent that global defect operators are crucial in the study and classification of gauge theories. For example, there are known cases of disparate field theories that can only be distinguished by the insertion of defects [48]. Therefore, in order to properly define a quantum field theory it is crucial to carefully analyse the spectrum of extended objects in the theory.

From the six-dimensional M-theory perspective there are two distinct constructions of surface defects in four dimensions. On the one hand they can descend from codimension-four defects in the parent theory arising from M2-branes ending on the stack of M5-branes. On the other hand, they can also descend from codimension-two defects in six dimensions, which come from M5-branes intersecting the stack of coincident M5-branes. This is summarized in Table 1.1.

	$\mathcal{M}_4$	$\mathcal{C}_{g,n}$	Name	Type
(i)	4	0	flavor puncture	M5-brane
(ii)	2	2	surface defect	M5-brane
(iii)	2	0	surface defect	M2-brane

Table 1.1: Summary of the defects in the six-dimensional $\mathcal{N} = (2,0)$ theory on $\mathcal{M}_4 \times \mathcal{C}_{g,n}$. $\mathcal{M}_4$ is the four dimensional space-time and $\mathcal{C}_{g,n}$ is a decorated Riemann surface. (i) and (ii) show configurations of codimension-two defects while (iii) shows the configuration of a codimension-four defect.

Let us first discuss codimension-two defects of the $\mathcal{N} = (2,0)$, type $\mathfrak{g}$ theory in six-dimensions, which are labeled by embeddings $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. These defects play an important role in the construction of theories of class $\mathcal{S}$: a codimension-two defect that is inserted at a point on the Riemann surface $\mathcal{C}_{g,n}$ and spans all four space-time dimensions corresponds to a flavor puncture in the construction of [13, 14] - see (i) of Table 1.1. Alternatively, wrapping the same codimension-two defect on the whole Riemann surface $\mathcal{C}_{g,n}$ leads to a surface defect in the four-dimensional theory - see (ii) of Table 1.1. This class of surface defects has been studied, for example, in [49, 50].

On the other hand, there are codimension-four defects in the $\mathcal{N} = (2,0)$ theory in six-dimensions, which are expected to be labeled by an irreducible representation of $\mathfrak{g}$, see for example [51] and references therein. Inserting a codimension-four defect at a point on the Riemann surface $\mathcal{C}_{g,n}$ engineers another class of surface defects in the four-dimensional theory - see (iii) of Table 1.1. In Chapters 4 and 5, we focus on the study of this second class of surface defects in four-dimensional $\mathcal{N} = 2$ theories of class $\mathcal{S}$.

A central ingredient in our analysis is the assumption that such codimension-four defects

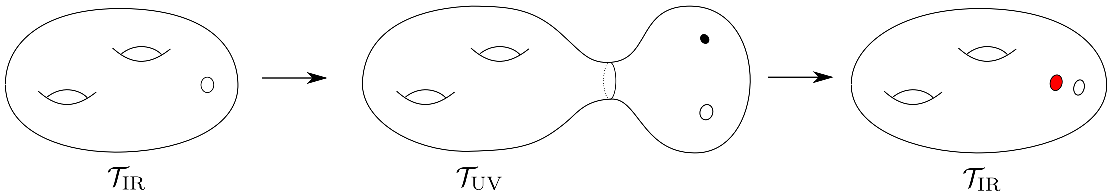


Figure 1.2: Schematic illustration of the renormalization group flow $\mathcal{T}_{UV} \to \mathcal{T}_{IR}$ that can be used to introduce surface defects. The white dots represent full punctures with $SU(N)$ symmetry while the black dot is a simple puncture with $U(1)$ symmetry. The red dot represents a codimension-four defect engineering a surface defect in four dimensions.

are labeled by irreducible representations of $\mathfrak{g}$. Important evidence for that comes from the AGT correspondence. In this correspondence, flavor punctures are represented by vertex operators labeled by non-degenerate and semi-degenerate representations of the Virasoro or W_N-algebra. There are also completely degenerate representations labeled by two dominant integral weights of $\mathfrak{g}$, or equivalently, by two irreducible representations R_1 and R_2 of $\mathfrak{g}$. Correlation functions with additional insertions of completely degenerate vertex operators compute the four-sphere partition function in the presence of surface defects [52]. In particular, the labels R_1 and R_2 characterize the surface defects supported on orthogonal two-spheres.[6]

Inspired by the connection to degenerate vertex operators and the analytic structure of Virasoro/W_N-algebra conformal blocks, the authors of [30] introduced a renormalization group flow that can be used to construct the surface defects from vortex configurations in a larger theory. Let us consider the simplest example of this procedure illustrated in Figure 1.2.

The starting point is a theory $\mathcal{T}_{IR}$ with a full flavor puncture encoding an $SU(N)$ flavor symmetry. We then form the larger theory $\mathcal{T}_{UV}$ by adding a simple puncture nearby with $U(1)$ flavor symmetry. This corresponds to adding an additional hypermultiplet in the bifundamental of $SU(N) \times SU(N)$ by gauging the diagonal $SU(N)$. The extra $U(1)$ symmetry corresponds to the baryonic symmetry of the bifundamental hypermultiplet and the position of the simple puncture controls the gauge coupling of the gauged $SU(N)$.

[6]Recently this was made precise by explicitly computing the coupled $4d-2d$ partition function [53].

The theories $\mathcal{T}_{IR}$ and $\mathcal{T}_{UV}$ are connected by a renormalization group flow that is initiated by turning on a constant vacuum expectation value for the hypermultiplet scalar. By turning on a position-dependent vacuum expectation value corresponding to a 1/2-BPS vortex configuration in $\mathcal{T}_{UV}$, the endpoint of the renormalization group flow is a surface defect in the original theory $\mathcal{T}_{IR}$. These surface defects are labeled by a pair of positive integers (r_1, r_2) corresponding to the vortex numbers in orthogonal two-planes. This construction is analogous to the Toda construction of codimension-four surface operators [52]. Hence our working conjecture is that they give a representation of codimension-four surface defects labeled by a pair of symmetric tensor representations of $\mathfrak{g}$.

A concrete prescription was given in [30] to implement this renormalization group flow at the level of the superconformal index. In full generality, the $\mathcal{N}=2$ superconformal index depends on three parameters denoted by $\{p, q, t\}$ that are associated to combinations of bosonic conserved charges commuting with a chosen supercharge. It also depends on flavor parameters $\{a_1, \ldots, a_N\}$, such that $\prod_j a_j = 1$, for each global $SU(N)$ symmetry and an additional parameter b for each $U(1)$ flavor symmetry. The superconformal index is thus denoted by

$$\mathcal{I}(p, q, t, a_j, b, \ldots)\,. \tag{1.1.3}$$

The superconformal index of the theory $\mathcal{T}_{IR}$ with surface defects is obtained by computing a residue of the superconformal index of the theory $\mathcal{T}_{UV}$ in the additional fugacity b associated to the additional $U(1)$ symmetry. The result is a difference operator G_{r_1,r_2} that acts on the superconformal index of the original theory $\mathcal{T}_{IR}$ by shifting the fugacities of the $SU(N)$ flavor symmetry. Schematically, the difference operator is defined by

$$G_{r_1,r_2} \cdot \mathcal{I}_{IR}(a_j, \ldots) \;\sim \underset{b=t^{\frac{1}{2}} p^{r_1/N} q^{r_2/N}}{\mathrm{Res}} \left[\frac{1}{b}\, \mathcal{I}_{UV}(a_j, b, \ldots) \right], \tag{1.1.4}$$

where the proportionality constant is discussed in section 2.4. The difference operator G_{r_1,r_2} corresponds to inserting a surface defect in the original theory $\mathcal{T}_{IR}$ that is labeled by the pair (r_1, r_2).

In what follows we concentrate on the case $r_1 = 0$ and simply label the difference operators by G_r, where $r \in \mathbb{Z}_{\geq 0}$. The label r can be thought of as denoting a symmetric tensor representation of rank r. The resulting expression for G_r is

$$G_r \cdot \mathcal{I}(a_j) \;=\; \sum_{\sum_{k=1}^N m_k = r} \; \prod_{j,k=1}^{N} \left[\prod_{m=0}^{m_k-1} \frac{\theta\left(q^{m+m_k-m_j} t a_j/a_k;p\right)}{\theta\left(q^{m-m_k} a_k/a_j;p\right)} \right] \mathcal{I}\left(a_j \mapsto q^{\frac{r}{N}-m_j}\, a_j\right) \,, \quad (1.1.5)$$

where the theta-function $\theta(z,p)$ is defined in section 2.3.1.

Following our arguments above, we expect that there exist difference operators G_R corresponding to surface defects labeled by all irreducible representations R of $\mathfrak{g}$. In principle, they could be constructed by starting from a theory $\mathcal{T}_{UV}$ with an additional puncture with a larger flavor symmetry.[7] However, in Chapter 4, we follow a different approach. Namely we complete the algebra of difference operators. For the difference operator associated to the representation R we make an ansatz

$$G_R \cdot \mathcal{I}(a_j) \;=\; \sum_{\lambda} C_{R,\lambda}(p,q,t,a_j)\, \mathcal{I}(q^{-(\lambda,h_j)} a_j)\,, \quad (1.1.6)$$

where the sum is over the weights λ of the representation R, $(\,,\,)$ is the standard inner product on the Cartan subalgebra of $\mathfrak{g}$, and h_j are the weights of the fundamental representation. This ansatz is compatible with what we already know about difference operators G_r for symmetric tensor representations $R = (r)$.

The coefficients $C_{R,\lambda}(p,q,t,a_j)$ are then determined by imposing that the full set of difference operators G_R is closed under composition

$$G_{R_1} \circ G_{R_2} \;=\; \sum_{R_3} \mathcal{N}_{R_1,R_2}{}^{R_3}(p,q,t)\, G_{R_3}\,, \quad (1.1.7)$$

and forms a commutative algebra. Since the symmetric tensor representations form an over-complete basis, there are many compatibility conditions for the system (1.1.7) to be solved consistently. It is thus non-trivial that a solution exists. Nevertheless, we can find

[7]However that would include non-Lagrangian ingredients that are hard to deal with.

a solution using the following method.

First, we notice that all irreducible representations in the case $\mathfrak{g} = \mathfrak{su}(2)$ are symmetric tensor representations, so that there are no additional difference operators. Even though it is not obvious and requires numerous functional identities for theta-functions, the system (1.1.7) can be solved uniquely in this case. The structure coefficients ${\mathcal{N}_{R_1,R_2}}^{R_3}(p,q,t)$ turn out to be an elliptic generalization of the (q,t)-deformed Littlewood-Richardson coefficients.

If we then assume that for any rank of the gauge group the structure coefficients ${\mathcal{N}_{R_1,R_2}}^{R_3}(p,q,t)$ are given by this elliptic generalization of the Littlewood-Richardson coefficients, the system (1.1.7) can be solved consistently and uniquely for all of the difference operators G_R. In particular, we find that the difference operators $G_{(1^r)}$ labeled by the rank r antisymmetric tensor representations, can be conjugated to the Hamiltonians of the N-body elliptic Ruijsenaars-Schneider integrable system. We shall discuss this in more details in section 4.1.

The superconformal index of $\mathcal{N} = 2$ theories of class $\mathcal{S}$ has a dual description in terms of a two-dimensional topological quantum field theory on the surface $\mathcal{C}_{g,n}$. We show that the difference operators G_R are natural objects in this two-dimensional TQFT. In a truncation – called the Macdonald slice (see section 2.3.3) – where we take $p = 0$, the operators $G_{(1^r)}$, labeled by antisymmetric tensor representations, can be conjugated to the so-called Macdonald operators, whose eigenfunctions are the Macdonald polynomials $P_S(a,q,t)$ labeled by an irreducible representation S. We find that the eigenvalue of a general, conjugated, difference operator $\bar{G}_R$ in the Macdonald limit is given by

$$\bar{G}_R \cdot P_S(a_j,q,t) \;=\; \frac{S_{R,S}}{S_{0,S}}\, P_S(a_j,q,t)\,, \tag{1.1.8}$$

where $S_{R,S}$ is an analytic continuation of the modular S-matrix of refined Chern-Simons theory, which depends on q and t. A consequence is that the surface defect introduced by the operator $\bar{G}_R$ is equivalent to a Wilson loop wrapping around the S^1 of the three-manifold $S^1 \times \mathcal{C}_{g,n}$. In the Macdonald limit, the structure constants ${\mathcal{N}_{R_1,R_2}}^{R_3}(q,t)$ become

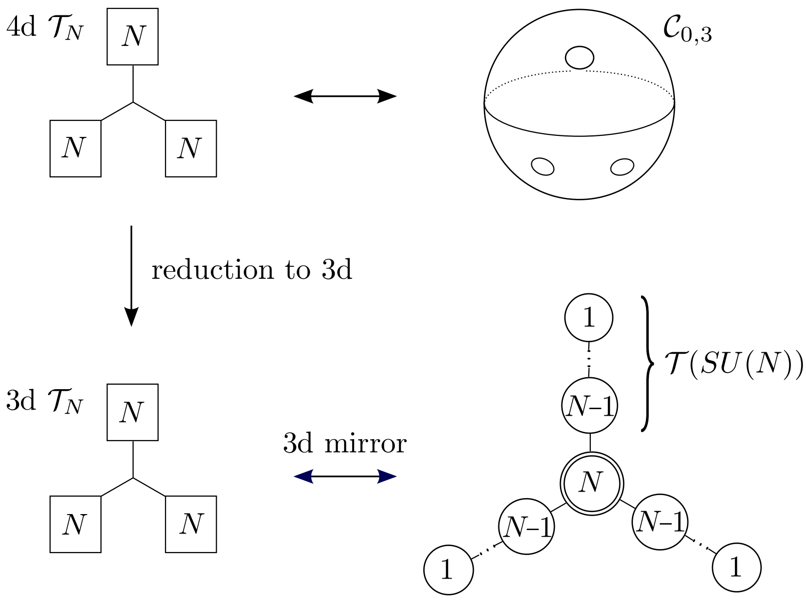


Figure 1.3: Sequence of dualities that maps the four-dimensional $\mathcal{T}_N$ theory (upper-left) to the three-dimensional star-shaped quiver theory (lower-right).

the (q,t)-deformed Littlewood-Richardson coefficients and the algebra of difference operators G_R is identified with the Verlinde algebra. We expect that this Verlinde algebra has a natural interpretation in the (analytically continued) chiral boundary theory on the two-torus boundary near a puncture of $\mathcal{C}_{g,n}$. We discuss these aspects in section 4.2.

In Chapter 5, we find further confirmation of the physical relevance of the difference operators G_R by reducing the superconformal index to the squashed three-sphere partition function, following [54–56]. In particular, we consider the dimensional reduction of the four-dimensional $\mathcal{T}_N$ theory, which is obtained by compactifying the six-dimensional $\mathcal{N}=(2,0)$ theory on a three-punctured sphere with three full punctures. The dimensionally reduced $\mathcal{T}_N$ theory has a Lagrangian mirror description as a star-shaped quiver theory [57]. This is illustrated in Figure 1.3. In particular, each full puncture of the three-punctured sphere is represented by a three-dimensional linear quiver theory called $\mathcal{T}(SU(N))$.

It is expected that the surface defects G_R introduced above dimensionally reduce to supersymmetric Wilson loops in the representation R of $\mathfrak{g}$ for the central node of the star-shaped quiver. This is in fact equivalent to the statement that the partition function of the $\mathcal{T}(SU(N))$ theory is an eigenfunction of the dimensionally reduced operators $G_R^{(3\mathrm{d})}$. The three-dimensional partition function $\mathcal{Z}(x,y)$ of the $\mathcal{T}(SU(N))$ theory depends on two mass parameters x and y associated to the Higgs branch and the Coulomb branch respectively, and is symmetric under $x \leftrightarrow y$. For the case of a round four-sphere, we show that indeed

$$G^{(3\mathrm{d})}_{(1^r)}(y) \cdot \mathcal{Z}(x,y) \;=\; W_{(1^r)}(x)\, \mathcal{Z}(x,y), \tag{1.1.9}$$

where $W_{(1^r)}(x)$ is a supersymmetric Wilson loop in the r-th antisymmetric tensor representation.

For other (non-minuscule) representations we find that this is not quite correct. In particular, the Wilson loops obey the algebra

$$W_{R_1} \cdot W_{R_2} \;=\; \sum_{R_3} N_{R_1,R_2}{}^{R_3}\, W_{R_3}\,, \tag{1.1.10}$$

where $N_{R_1,R_2}{}^{R_3}$ are the ordinary Littlewood-Richardson coefficients, whereas the algebra of the three-dimensional operators $G^{(3\mathrm{d})}_R$ is not of this form. Instead, we find that when the representation R is non-minuscule, the dimensionally reduced operators $G^{(3\mathrm{d})}_R$ are linear combinations of operators $\tilde{G}^{(3\mathrm{d})}_S$, with $|S| \le |R|$, that *are* dual to Wilson loop operators[8]. This gives a simple invertible linear transformation on the algebra of difference operators.

Finally, in section 5.2, we embed the three-dimensional $\mathcal{T}(SU(N))$ theory as an S-duality domain wall in the four-dimensional $\mathcal{N}=2^*$ theory, we interpret the dimensionally reduced difference operators $G^{(3\mathrm{d})}_R$ as operators that introduce 't Hooft defects, labeled by irreducible representations R, into the four-sphere partition function of the $\mathcal{N}=2^*$ theory. Again, when the representation R is an antisymmetric tensor representation, we find perfect agreement with both localization [58] and (in the case of the fundamental representation) computations of Verlinde operators in Liouville/Toda conformal field theory [52, 59–61], while for other representations we once more find an invertible linear transformation on the algebra of operators.

[8]We define the partial ordering of representation by $|R_1| < |R_2| \iff \dim R_1 < \dim R_2$.

1.2 The gauge/gravity correspondence in five/six dimensions

In the second part of this thesis we will focus on a particular case of the AdS/CFT correspondence. The AdS/CFT correspondence is an instance of the holographic principle [62, 63]. This is the idea that quantum gravity in a volume is naturally formulated in terms of degrees of freedom on its surface. The Bekenstein-Hawking formula is one of the main examples, relating the entropy of a black hole to the area of its surface [64]. The AdS/CFT correspondence [9] predicts an equivalence between string theory living on particular backgrounds of the form $\mathrm{AdS}_d \times X$ and quantum field theory on the $(d-1)$-dimensional boundary of AdS_d, where AdS_d is the d-dimensional anti-de Sitter space.[9] The original example considered by J. Maldacena consists of a duality between superstring theory on $\mathrm{AdS}_5 \times S^5$ and four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

The AdS/CFT duality was introduced in the setting of superstring theory.[10] Let us consider N coincident D3-branes[11] in type IIB string theory with string coupling g. At $gN \ll 1$, string perturbation theory is a good approximation and by taking the low energy limit, we arrive at a theory of massless open and closed strings. The dynamics of the D3-branes is then described by a four-dimensional $\mathcal{N} = 4$, $SU(N)$ gauge theory on the brane worldvolume [77] and the closed strings decouple.

At $gN \gg 1$, we can describe the same setup as a black brane [78] system in supergravity [79]. Taking the low energy limit again leads to closed strings in the bulk as well as states near the horizon. This is because of the redshift; an object being brought closer and closer to the horizon of the black brane would appear to have lower and lower energy for the observer at infinity. In the near horizon limit, the metric can be shown to be given by

[9] We refer to [65–68] for excellent reviews of this vast subject.

[10] It can also be understood without the explicit use of string theory, by applying the holographic principle to a quantum field theory. The idea is to write the five-dimensional graviton as a two boson state in four-dimensions [68, 69].

[11] D-Branes were originally introduced in [70, 71]. For general reviews about branes in string theory, see [72–76].

$\mathrm{AdS}_5 \times S^5$.

Together – if we decouple the massless closed strings in the bulk for both limits – we arrive at an astounding result: we obtain a gauge theory at weak coupling and at strong coupling we have the string states in the $\mathrm{AdS}_5 \times S^5$ region. Assuming that the low energy limit commutes with the limits $gN \ll 1$ and $gN \gg 1$, it follows that the strongly coupled gauge theory is identical to the full string theory in the $\mathrm{AdS}_5 \times S^5$ region.

Ever since its discovery, the AdS/CFT correspondence has been subjected to innumerable tests. For example a simple consistency check is to match the global symmetries on both sides of the duality. More involved tests include the matching of states or supersymmetric amplitudes on both sides.

The correspondence has also been generalized to different dimensions and brane systems. An important extension consists of looking at M-theory brane configurations. For example we can consider the low energy limit of a stack of M2-branes. The near horizon geometry is then $\mathrm{AdS}_4 \times X^7$, where X^7 is a seven-dimensional compactification manifold in 11-dimensional M-theory. On the M2-brane worldvolume lives the ABJM-theory, a three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons matter theory [80]. A consequence of the AdS/CFT is that the partition function of the ABJM theory on S^3 is supposed to match the partition function of M-theory on $\mathrm{AdS}_4 \times X^7$ [81, 82]

$$Z_{\mathrm{ABJM}}\left[S^3\right] \;=\; Z\left[\mathrm{AdS}_4 \times X^7\right] \,. \tag{1.2.1}$$

When N is large, the M-theory partition function can be computed in the genus-zero limit, and at strong coupling, the supergravity approximation is sufficient. Therefore by taking the large N and strong coupling limits on both sides, we end up with

$$Z_{\mathrm{ABJM}}\left[S^3\right] \;\sim\; \exp\big(-I\left[\mathrm{AdS}_4\right]\big)\,, \tag{1.2.2}$$

where $I\left[\mathrm{AdS}_4\right]$ is the classical gravity action evaluated on the AdS_4 metric. The left hand side of this equation can be computed explicitly at large N using localization [83, 84] and matrix model techniques [85–87].[12]

Localization as a tool to compute exact quantities in gauge theories on compact manifolds with rigid supersymmetry was introduced rather recently by V. Pestun in [83]. Following his seminal work a plethora of new exact observables were computed in supersymmetric gauge theories in various dimensions. As a consequence this led to a variety of new checks of the AdS/CFT correspondence by comparing the large N limit of exact results on the gauge theory side to the corresponding quantities in the holographic dual supergravity. A lot of effort was put into understanding and extending this three-dimensional case culminating in the computation of the partition function on a large class of three-manifolds [89, 90] and comparing it to an equally large class of gravity duals [91].

In part II of this thesis we will discuss the gauge/gravity correspondence in five/six dimensions. The status in five/six dimensions is in a much more elementary phase than its lower dimensional cousins. It is a well-known fact that five dimensional gauge theories are generically non-renormalizable, and therefore they generally do not exist as a microscopic theory. In some cases theories with specific gauge groups and matter contents can be shown to have a strongly coupled five-dimensional superconformal fixed point in the UV [92–94]. Such theories usually stem from brane constructions in string theory. There is a particular class of five-dimensional superconformal gauge theories, with gauge group $USp(2N)$, which arises from a D4-D8-brane system. In the context of AdS/CFT, this is expected to have a large N description in terms of massive type IIA supergravity [95–97].

Let us consider type I' string theory with N D4-branes, two orientifold planes O8 and 16 D8-branes.

	0	1	2	3	4	5	6	7	8	9
O8	×	×	×	×	×	×	×	×	×	
D8	×	×	×	×	×	×	×	×	×	
D4	×	×	×	×	×					

Let one of two O8 branes be located at $x^9 = 0$ and the other far away at some $x^9 = b \gg 0$.

[12]For a nice review, see also [88].

We are only interested in the dynamics near one of the O8-branes and so we take N_f D8-branes at $x^9 = 0$ and the remaining $16 - N_f$ at $x^9 = b$. The low energy dynamics of the theory on the D4-branes located at $x^9 = 0$ is given by taking the field theory limit. It was shown in [92] that this theory at the origin of the Coulomb branch with $N_f < 8$ massless hypermultiplets sits a non-trivial fixed point of the renormalization group of a five-dimensional supersymmetric field theory. More precisely, the worldvolume theory is a five-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $USp(2N)$, a single hypermultiplet in the antisymmetric representation and N_f hypermultiplets in the fundamental representation arising from D4-D8 strings.

On the other hand, we can view this system from a black brane perspective. One considers N_f D8-branes on top of the O8-plane being probed by N D4-branes. It was shown in [96] that the near horizon geometry of this system is described by a fibration of AdS_6 over S^4. Taking N large, one enters a region where the near horizon theory can be characterized by classical massive type IIA supergravity.[13] The warped $\mathrm{AdS}_6 \times S^4$ that arises in the near horizon treatment is precisely the 10-dimensional background considered in [99]. In this reference, the authors showed that compactifying type IIA supergravity on the warped S^4 gives rise to Romans $F(4)$ gauged supergravity [100].

Hence analogous to the prototypical AdS/CFT example, this leads to a duality between massive type IIA supergravity on a warped product of $\mathrm{AdS}_6 \times S^4$ and the superconformal fixed point of a five-dimensional gauge theory with gauge group $USp(2N)$, N_f hypermultiplets in the fundamental representation, and a single hypermultiplet in the antisymmetric representation of the gauge group. Ultimately one wants to test this duality for example by comparing the free energies on both sides, analogous to (1.2.2).

In [101] the large N limit of the partition function of these $USp(2N)$ theories on the round sphere was computed and successfully compared to the entanglement entropy of the dual warped $\mathrm{AdS}_6 \times S^4$ supergravity solution. In Chapter 6, we shall present the first construction of gravity duals to gauge theories on non-conformally flat backgrounds (specif-

[13] Note that – as discussed in [79] – the theory between two D8-branes in type I' is given by IIA supergravity [98].

ically, certain families of squashed five-spheres). We effectively work in six-dimensional Romans $F(4)$ supergravity [100], being a consistent truncation of massive IIA supergravity on S^4 [99] – see also section 6.2.1. Having constructed supergravity solutions that have squashed five-sphere conformal boundaries, we compute the holographic free energy

$$\mathcal{F} \;=\; -\log Z$$

by holographically renormalizing the on-shell Euclidean action. More specifically, we construct families of solutions with different numbers of preserved supercharges. Two of these families are shown to be dual to the 1/4 BPS and 3/4 BPS gauge theories defined in [102]. The perturbative partition function for these theories has been computed in [103] and we explicitly show that the large N limit of these partition functions is in precise agreement with the holographic free energies of our supergravity solutions. We also present more general solutions (and in particular a 1/2 BPS solution) which have not previously been considered from the gauge theory side.

From the Killing spinors of a supersymmetric supergravity solution one can always construct a certain Killing vector K. For all solutions found in Chapter 6 the free energy is only sensitive to this Killing vector

$$\mathcal{F} \;=\; \mathcal{F}(K)\,,$$

and not to other parameters of the solution. It is natural to conjecture that this is also the case for more general solutions, extending what happens in four dimensions [91]. In addition in section 6.5 we compute the expectation values of BPS Wilson loops in these backgrounds, both in supergravity and in the large N matrix model, finding precise agreement. Again the expectation value depends only on the Killing vector K.

An interesting consequence of these results is that from the general local supersymmetry constraints in the six-dimensional supergravity we obtain constraints for rigid supersymmetry on the conformal boundary M_5 – see section 7.1. Similar to [104] in lower dimensions, in Chapter 7 we shall realize M_5 as the conformal boundary of the six-dimensional bulk solution of Romans $F(4)$ gauged supergravity. We start with a general supersymmetric asymptotically locally AdS solution to the Romans theory, and extract the conditions this imposes on the five-dimensional conformal boundary. Although the resulting spinor equations are quite complicated, we will show they are completely equivalent to a very simple geometric structure.[14] We shall call this the *holographic* approach to rigid supersymmetry.

Five-dimensional supersymmetric gauge theories are currently not as well-developed as their lower-dimensional cousins. They were constructed and studied on the round S^5 in [105–108]. The product background $S^1 \times S^4$ studied in [109, 110] leads to the superconformal index. As in lower dimensions, the first constructions of non-conformally flat backgrounds were produced via various *ad hoc* methods. These include the squashed S^5 geometries of [102,103], and the product backgrounds $S^3 \times \Sigma_2$ [42,111] and $S^2 \times M_3$ [112–114]. In the latter two cases the spheres are round, while supersymmetry on the Riemann surface Σ_2 or three-manifold M_3 is achieved via a topological twist utilizing the $SU(2)_R$ symmetry of the theory.

A systematic method for constructing rigid supersymmetric field theories on curved backgrounds, in any dimension d, was initiated in [115]. Here one first couples the field theory to off-shell supergravity, and then takes a decoupling limit in which the gravity multiplet becomes a non-dynamical background field. This approach was applied to five-dimensional Poincaré supergravity [116–118] in the series of papers [119–121]. Supersymmetry of the background requires a certain generalized Killing spinor equation to hold, whose related geometry was investigated in [119], together with an algebraic "dilatino" equation which was studied in [120]. The latter reference recasts these conditions into local geometric constraints on the five-manifold M_5. As in lower dimensions, one finds that the background is parametrized by various arbitrary functions/tensors. In particular (M_5, g) is equipped with a Killing vector field $\xi = \partial_\psi$, with dual one-form $S^2(\mathrm{d}\psi + \rho)$ and transverse four-dimensional metric $g^{(4)}$, where locally the function $S = \|\xi\|$ and tensors ρ

[14] We note that in [6], we also analyse the supersymmetry transformations and Lagrangians of theories on such general backgrounds.

and $g^{(4)}$ are ξ-invariant but otherwise freely specifiable.[15] The authors of [120] furthermore show that *locally* all deformations of the background fields lead to Q-exact deformations of the action, where Q is the supercharge. Despite this generality, these backgrounds do not include the conformally flat geometry $S^1 \times S^4$ [120].

In Chapter 7 we take the holographic approach instead. We find that the following conditions for M_5 are *necessary* and *sufficient* for rigid supersymmetry coming from holography. Namely M_5 is equipped with a conformal Killing vector $\xi = \partial_\psi$ which generates a *transversely holomorphic foliation*. This is compatible with an almost contact form $\eta = \mathrm{d}\psi + \rho$, where – up to global constraints that we describe – the norm $S = \|\xi\|$ and ρ are arbitrary, and the transverse metric $g^{(4)}$ is *Hermitian*. The only other remaining freedom is an arbitrary function α (such that $S\alpha$ is ξ-invariant), which together with the metric determines all the remaining background data. This structure is similar to the rigid limit of Poincaré supergravity described above, but with the addition of an integrable transverse complex structure and Hermitian metric. In fact it is a natural hybrid of the "real" three-dimensional rigid supersymmetric geometry studied in [89, 122] and the four-dimensional supersymmetric geometry of [123, 124] (where the four-manifold is complex with a compatible Hermitian metric).

[15] There are also additional freely specifiable fields, which determine the rest of the background.

Part I

The superconformal index and surface defects

Chapter 2

The superconformal index and theories of class $\mathcal{S}$

In this chapter we introduce the most important notions and concepts relevant for the remainder of part I of this thesis. This is a very large topic and so we cannot deliver a complete and self-contained treatment. However we do attempt to introduce most relevant ideas in a concise manner. For more details we refer to the excellent reviews [125–136].

2.1 Definition of the superconformal index

The superconformal index is a trace over states of a superconformal field theory in radial quantization. We will focus here on $\mathcal{N} = 2$ theories in four dimensions. The bosonic part of the four-dimensional $\mathcal{N} = 2$ superconformal algebra consists of the conformal algebra, $\mathfrak{so}(2, 4)$, times the $SU(2)_R \times U(1)_r$ algebra. We denote by E the dilatation generator, by $j_z = j_2 - j_1$ and $j_w = j_2 + j_1$ the generators of the $SO(4) \sim SU(2)_1 \times SU(2)_2$ isometry group of the sphere[1] and by R and r the Cartan generators of the $SU(2)_R \times U(1)_r$ symmetry. The fermionic part of the algebra is generated by eight supersymmetry generators $\mathcal{Q}^{\mathcal{I}}_{\alpha}$, $\tilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}$, and their superconformal counterparts $\mathcal{S}^{\alpha}_{\mathcal{I}}$ and $\tilde{\mathcal{S}}^{\mathcal{I}\dot{\alpha}}$, with $\mathcal{S} = \mathcal{Q}^{\dagger}$ and $\tilde{\mathcal{S}} = \tilde{\mathcal{Q}}^{\dagger}$. Here $\alpha = \pm$,

[1] We are parametrizing S^3 by two complex coordinates (z, w) obeying $|z|^2 + |w|^2 = 1$, and the generators j_z and j_w are rotations in the orthogonal z- and w-planes respectively.

$\dot{\alpha} = \dot{\pm}$ is an $SU(2)_1$, $SU(2)_2$ index respectively and $\mathcal{I} = 1, 2$ is an $SU(2)_R$ index [25,137].

We define the superconformal index as

$$\mathcal{I} \;=\; \mathrm{Tr}(-1)^F p^{j_z - r} q^{j_w - r} t^{r+R} \prod_j a_j^{f_j} e^{-\beta \delta_{2,\dot{-}}} \,, \tag{2.1.1}$$

where

$$2\delta_{2,\dot{-}} \;=\; \{\tilde{\mathcal{Q}}_{2,\dot{-}}, \tilde{\mathcal{Q}}^{\dagger}_{2,\dot{-}}\} \;=\; E - j_z - j_w - 2R + r \,. \tag{2.1.2}$$

The f_j are generators of the Cartan subalgebra of the flavor symmetry group. The combinations of generators appearing in the powers of (p, q, t, a_j) in equation (2.1.1) are those combinations that commute with the supercharge $\tilde{\mathcal{Q}}_{2,\dot{-}}$. The letters p, q, t and a_i are fugacities for these symmetries and obey

$$|p|, |q|, |t|, |pq/t| \;<\; 1, \qquad |a_j| \;=\; 1 \,, \tag{2.1.3}$$

ensuring that the index is well-defined.

In the definition (2.1.1), the trace is taken over states of the theory in radial quantization satisfying $\delta_{2,\dot{-}} = 0$. The superconformal index is a Witten index [138]. Hence, by the usual arguments, the index counts the short multiplets of the superconformal algebra with energies at the unitarity bound given by $E = j_z + j_w + 2R - r$, up to those states which are recombined into long multiplets once the energy is perturbed away from the unitarity bound [25]. As a consequence the index is invariant under the continuous variation of exactly marginal couplings of the theory, and in particular it is independent of β.

If there exists a weakly coupled Lagrangian description of the theory, the superconformal index can be computed in the free limit. In practice this is done by summing up all the gauge invariant states of the theory with $\delta_{2,\dot{-}} = 0$ or by computing the path integral in the free limit. The basic ingredients are the single letter or single particle indices of a half-hypermultiplet and vectormultiplet, which are given by

$$\begin{aligned} i_H(p,q,t) &= \frac{\sqrt{t}-\frac{pq}{\sqrt{t}}}{(1-p)(1-q)}\,, \\ i_V(p,q,t) &= -\frac{p}{1-p}-\frac{q}{1-q}+\frac{\frac{pq}{t}-t}{(1-p)(1-q)}\,. \end{aligned} \tag{2.1.4}$$

The full index is then the plethystic exponential of those single-particle contributions, which accounts for higher order oscillations [139, 140].

We shall focus on $G = SU(N)$ gauge theories. Then the superconformal index of a free hypermultiplet in the bifundamental representation of $SU(N) \times SU(N)$ is

$$\mathcal{I}(a_j, b_j, c) = \mathrm{PE}\left[i_H(p,q,t) \sum_{i,j=1}^{N} \left(a_i b_j c + \frac{1}{a_i b_j c} \right) \right] = \prod_{i,j=1}^{N} \Gamma\left(\sqrt{t}(a_i b_j c)^{\pm}; p, q \right)\,, \tag{2.1.5}$$

where PE stands for the plethystic exponential, defined as

$$\mathrm{PE}\left[f(x_1, x_2, \ldots)\right] \equiv \exp\left(\sum_{i=1}^{\infty} \frac{1}{n} f(x_1^n, x_2^n, \ldots) \right)\,. \tag{2.1.6}$$

The parameters $\{a_i\}$ and $\{b_j\}$ are fugacities for the $SU(N) \times SU(N)$ symmetry and c is the fugacity for the overall $U(1)$ symmetry under which the two half-hypers have opposite charges. Furthermore, the elliptic gamma function $\Gamma(z; p, q)$ is defined as

$$\Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{(1 - z^{-1} p^{i+1} q^{j+1})}{(1 - z p^i q^j)}\,. \tag{2.1.7}$$

Similarly, the superconformal index of a vectormultiplet is given by

$$\mathcal{I}_V(a) = \mathrm{PE}\left[i_V(p,q,t) \left(\sum_{i,j=1}^{N} \frac{a_i}{a_j} - 1 \right) \right]\,. \tag{2.1.8}$$

An important operation on the superconformal index is that of *gauging* a global symmetry. If the theory has a Lagrangian description, a global symmetry can be gauged in the limit of vanishing coupling constant. Then, the only effect of the gauging action is Gauss' law requiring the projection onto gauge-invariant states. Given the superconformal index $\mathcal{I}(a)$ of a theory with $SU(N)$ flavor symmetry and corresponding fugacity a, the superconformal index of the theory where the flavor symmetry has been gauged is given by

$$\oint \Delta(a)\,\mathcal{I}_V(a)\,\mathcal{I}(a)\,, \tag{2.1.9}$$

where $\mathcal{I}_V$ is the superconformal index of a vectormultiplet in the adjoint representation of $SU(N)$ and Δ is the invariant Haar measure. This formula precisely enforces the projection over gauge-singlets. For $SU(N)$ gauge theories the Haar measure is

$$\Delta(a) \;=\; \left[\prod_{j=1}^{N-1} \frac{\mathrm{d}a_j}{2\pi\mathrm{i}a_j}\right] \frac{1}{N!} \prod_{i\neq j}^{N} \left(1 - \frac{a_i}{a_j}\right). \tag{2.1.10}$$

2.2 Class $\mathcal{S}$ theories

Let us review theories of class $\mathcal{S}$ [13, 14]. These four-dimensional SCFTs are defined by a twisted compactification of a six-dimensional $\mathcal{N} = (2, 0)$ SCFT on a Riemann surface. They are constructed by the following procedure:

(i) Pick a simply-laced Lie Algebra $\mathfrak{g} = \{A_N, D_N, E_6, E_7, E_8\}$ of the six-dimensional parent theory.

(ii) Choose an *ultraviolet* curve (also known as *Gaiotto curve*) $\mathcal{C}_{g,n}$ of genus g and with n punctures of various types, on which to put this theory. A puncture can be interpreted as a codimension-two defect of the six-dimensional $\mathcal{N} = (2, 0)$ theory.

(iii) Go to the low energy limit and decouple the higher dimensional degrees of freedom.

The resulting four-dimensional theories are called class $\mathcal{S}$ theories. The data defining them is encoded in the corresponding curve $\mathcal{C}_{g,n}$. In particular the complex structure moduli of $\mathcal{C}_{g,n}$ correspond to the exactly marginal gauge couplings of the four-dimensional superconformal field theory.

We will solely focus on $\mathfrak{g} = A_N$ type theories. Such theories are defined by a Seiberg-Witten curve Σ encoding the low energy effective action of the theory [10, 11]. It can be presented as an N-fold covering of the ultraviolet curve $\mathcal{C}_{g,n}$ given by the equation

$$\lambda^N = \sum_{k=2}^{N} \phi_k(z)\lambda^{N-k}, \tag{2.2.1}$$

where λ is the Seiberg-Witten differential and $\phi_k(z)$ are meromorphic differentials of order k with appropriate poles at the punctures of $\mathcal{C}_{g,n}$.

If the order of a pole of $\phi_k(z)$ satisfies $\deg \phi_k \leq k$, the puncture is called *regular* or *tame*, otherwise it is called *irregular* or *wild*. We shall mainly restrict to regular punctures, which are labeled by a choice of embedding $\Lambda : \mathfrak{su}(2) \to \mathfrak{g}$. The centralizer of the image $\Lambda(\mathfrak{su}(2)) \subset \mathfrak{g}$ then corresponds to the flavor symmetry associated to the puncture. In the case of $\mathfrak{g} = A_{N-1}$, such embeddings can be described by partitions of N, $(\ell_1, \ldots, \ell_N)$ such that $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_N = 0$, and $\sum_{i=1}^{N} \ell_i = N$. We call a puncture *full* or *maximal* if the defining embedding is trivial $\Lambda = 0$. Consequently the corresponding flavor symmetry is the full $SU(N)$. In terms of Young diagrams, this case corresponds to the partition $(N, 0, \ldots, 0)$. Similarly a puncture is essentially removed (*i.e.* the associated flavor symmetry is $\{0\}$) by choosing the principal embedding, or equivalently a partition of the form $(1, \ldots, 1)$. Lastly a puncture is called *simple* or *minimal* if the defining embedding is associated to the partition $(2, 1^{N-2}, 0)$. This leads to an associated $U(1)$ flavor symmetry in the field theory – see Figure 2.1.

In general, a singularity at a point on the UV curve associated to a Young diagram labeled by $(\ell_1, \ldots, \ell_N)$ gives rise to the flavor symmetry

$$S[U(n_1) \times U(n_2) \times \cdots \times U(n_k)],$$

where n_j is the number of columns of the Young diagram of height j. In the Seiberg-Witten description $p_k = k - j_k$ is the order of the pole of $\phi_k(z)$ at the corresponding singularity.

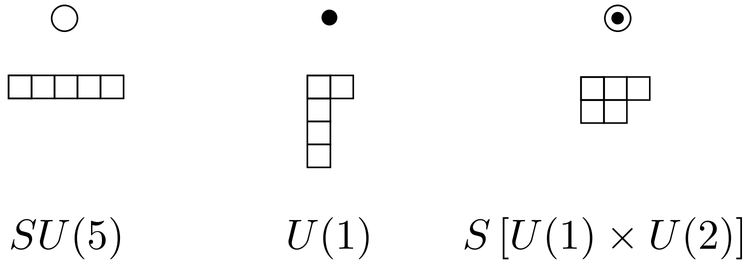


Figure 2.1: Three different types of punctures of an $SU(5)$ theory. We denote by a white dot a full puncture with associated $SU(5)$ flavor symmetry, by a black dot a simple puncture with $U(1)$ flavor symmetry and by a circle with a dot a generic puncture, in this case with $S[U(1) \times U(2)]$ flavor symmetry.

Here we have defined

$$(j_1, \dots j_N) \;=\; (\underbrace{1, \dots, 1}_{\ell_1}, \underbrace{2, \dots, 2}_{\ell_2}, \dots)\,.$$

A surface $\mathcal{C}_{g,n}$ can be constructed by gluing together elementary building blocks, so called pairs-of-pants, *i.e.* spheres with three punctures. More precisely, we pick local coordinate patches G_1 and G_2 with coordinates z_1 and z_2 around two full punctures and glue these two patches together via the prescription $z_1 z_2 = q$. In the gauge theory language this operation corresponds to coupling an $SU(N)$ vectormultiplet to the diagonal $SU(N) \subset SU(N) \times SU(N)$ flavor symmetry. The corresponding $SU(N)$ gauge theory has complexified UV coupling constant

$$\tau \;=\; \frac{4\pi \mathrm{i}}{g^2} + \frac{\theta}{2\pi}$$

related to the sewing parameter by $q = \exp(2\pi \mathrm{i}\tau)$. This is the process of *gauging* described in equation (2.1.9) for the superconformal index. Taking the limit in which the cylinder becomes very long corresponds to the weak coupling (or rather decoupling) limit. In the following we shall illustrate this for several interesting examples.

2.2.1 $SU(2)$ theories

$SU(2)$ theory with $N_f = 4$

We start by considering an $\mathcal{N} = 2$, $SU(2)$ theory with $N_f = 4$ hypermultiplets in the fundamental representation of $SU(2)$. This theory is superconformal and the Seiberg-Witten curve Σ has been computed a long time ago [10,11]. In the spirit of (2.2.1), it can be rewritten as a double-cover of $\mathcal{C}_{0,4}$

$$\lambda^2 - \phi_2(z) \;=\; 0\,, \qquad \phi_2(z) \;=\; \frac{Q(z)}{(z-1)^2(z-q)^2}\frac{\mathrm{d}z^2}{z^2}\,. \tag{2.2.2}$$

Here z is a coordinate on a sphere, $Q(z)$ is some quartic polynomial which explicitly depends on the Coulomb branch parameter as well as the hypermultiplet masses, and the quadratic form $\phi_2(z)$ has poles of order two at $z = 0, 1, q, \infty$. From the Seiberg-Witten differential we can compute the coupling constant in the weak coupling limit. This explicitly shows that $q = \exp(2\pi\mathrm{i}\tau)$, where $\tau \equiv \tau_{UV}$, the coupling constant in the ultraviolet.

The masses of the hypermulitplets are encoded in $Q(z)$. In particular from the treatment of [10, 11], it is known that the Seiberg-Witten differential has simple poles with residues proportional to the mass parameters. It is thus clear that the double poles of (2.2.2) – or equivalently four coefficients of the polynomial $Q(z)$ – encode the same masses.

Since the doublet and anti-doublet representation of $SU(2)$ are identical, we can rearrange the four hypermultiplet scalars $(Q_i^a, \tilde{Q}_a^i)$ with $i = 1, \ldots, 4$, into q_I^a with $I = 1, \ldots, 8$. The mass term in the superpotential $\mu_i Q_i \tilde{Q}^i$ is then written as $M^{IJ} q_I^a q_J^b \epsilon_{ab}$, which is manifestly gauge invariant, and where

$$M^{IJ} \;=\; \mathrm{diag}(-\mathrm{i}\mu_1\sigma_2, \ldots, -\mathrm{i}\mu_4\sigma_2)\,,$$

with second Pauli matrix σ_2. This is a constant matrix with anti-symmetric $SO(8)$ index.

It is convenient to decompose the flavor symmetry

$$SO(8) \supset SO(4) \times SO(4) \sim SU(2)_A \times SU(2)_B \times SU(2)_C \times SU(2)_D\,.$$

In this decomposition M^{IJ} reduces to diagonal matrices for each of the $SU(2)$ factors with eigenvalues

$$\pm\tfrac{\mu_1+\mu_2}{2} \text{ for } SU(2)_A\,,\ \pm\tfrac{\mu_1-\mu_2}{2} \text{ for } SU(2)_B\,,\ \pm\tfrac{\mu_3+\mu_4}{2} \text{ for } SU(2)_C\,,\ \pm\tfrac{\mu_3-\mu_4}{2} \text{ for } SU(2)_D\,.$$

These are precisely the residues one finds for λ in (2.2.2), where $SU(2)_A$ is associated to $z = 0$, $SU(2)_B$ to $z = q$, $SU(2)_C$ to $z = 1$, and $SU(2)_D$ to $z = \infty$. The masses μ_i are precisely the ones in the BPS mass formula. In summary, the behaviour of λ near a singularity is governed by an $SU(2)$ matrix M. This matrix has eigenvalues related to the BPS masses μ in the superpotential.

We can take the weak coupling limit $q \to 0$. Graphically that means that the two punctures A and B are very close together and decouple from the other two punctures. The gauge group becomes non-dynamical and we are left with two three-punctured spheres. They correspond to two free hypermultiplets $(Q_i^a, \tilde{Q}_a^i)$, for $a, i = 1, 2$. Once again one can recombine them into q_I^a with $I = 1, \ldots, 4$ an $SO(4)$ index and decompose $SO(4) \sim SU(2) \times SU(2)$ to get eight half-hypermultiplets $q_{a\alpha u}$ in the trifundamental of $SU(2)^3$. Note that this decomposition will not work in general; the $SU(2)$ case is special because the fundamental representation is pseudoreal.

We can also take the strong coupling limit $q \to \infty$. This can equivalently be described as the weak coupling limit of the theory with $\tilde{q} = 1/q \to 0$. With regards to the weak coupling limit above, this exchanges the singularities B and C. There is a third limit we can take, namely $q \to 1$. Again this can be recast into the weak coupling limit of $\hat{q} = 1-q \to 0$. This has the effect of interchanging A and C. These different duality frames are precisely the famous $SL(2, \mathbb{Z})$ strong-weak duality of the $SU(2)$ theory with $N_f = 4$ accompanied by triality of $SO(8)$, which exchanges the three fundamental representations of $SO(8)$,

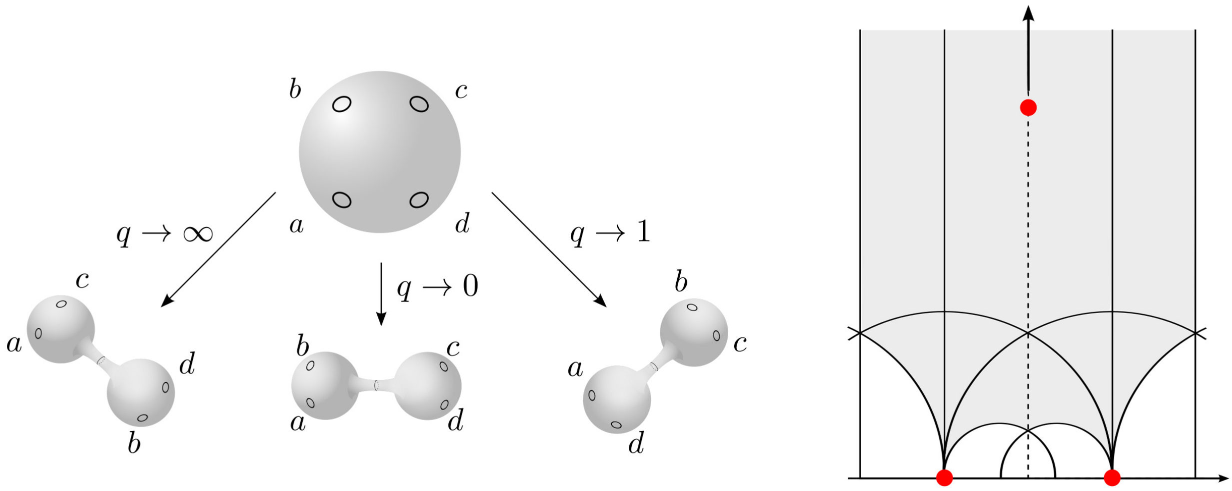


Figure 2.2: *Left*: Three different degeneration limits of an $SU(2)$, $N_f = 4$ theory, corresponding to a sphere with four punctures. They are related to weak coupling limits of the theory via S-duality and triality. *Right*: The space of coupling constant τ modulo S-duality. The different cusps correspond to the three degeneration limits of the picture on the left. It also coincides with the complex structure moduli space of a sphere with four labeled (inequivalent) punctures.

$\mathbf{8}_v \leftrightarrow \mathbf{8}_s \leftrightarrow \mathbf{8}_c$ – see Figure 2.2. This duality action was first discovered in [11] and it is an instance of the more general principle of S-duality.

The fundamental domain of the complex structure parameter of a sphere with four inequivalent punctures is given in Figure 2.2. An important observation is that the fundamental domain of the marginal UV coupling of the $N_f = 4$ theory modulo S-duality is precisely the same, and we can identify the former with the latter. This is an important observation, which also more generally true.

$\mathcal{N} = 2^*$ theory

In the above example we have seen that the trifundamental hypermultiplet – usually denoted by $\mathcal{T}_2$ (more generally $\mathcal{T}_N$ for the generalization to $SU(N)$) – can be viewed as the main building block in an $SU(2)$ generalized quiver gauge theory. Its UV curve corresponds to a sphere with three punctures and the Seiberg-Witten curve is given by

$$\lambda^2 = \phi_2(z) , \qquad (2.2.3)$$

where the precise form of $\phi_2(z)$ follows from the requirement that the three double poles have coefficients related to the hypermultiplet masses.

Let us apply this to another important example, the $\mathcal{N} = 2^*$ theory. This is the mass deformation of the $\mathcal{N} = 4$ gauge theory, *i.e.* an $SU(2)$ gauge theory with a single hypermultiplet in the adjoint representation with nonzero mass. We shall construct it by coupling two $SU(2)$ flavor symmetries of a single trifundamental hypermultiplet to a vectormultiplet. More precisely we start with the trifundamental $q_{a\alpha u}$ and couple a vectormultiplet to the indices a and α. This leads to the tensor product of the two fundamental representations which decomposes into a triplet plus a singlet $q_{a\alpha u} \to q^{(1)}_{k,u} \oplus q^{(2)}_{u}$, with $k = 1, 2, 3$. The singlet hypermultiplet is completely decoupled and we (essentially) end up with a hypermultiplet $q^{(1)}_{k,u}$ in the adjoint representation with remaining $SU(2)$ flavor symmetry, to which we can associate a mass deformation.

It follows from this construction that the UV curve is a single-punctured torus. The Seiberg-Witten curve is as in equation (2.2.3), however z is now a coordinate on the torus. Putting the puncture at $z = 0$, $\phi_2(z)$ is given by the requirement that the double pole is given by the squared hypermultiplet mass. Again it turns out that the space of the coupling constant can be identified with the moduli space of the torus.

Relation to theories with $N_f < 4$

So far we have only discussed theories with exactly marginal gauge couplings. It is natural to ask whether we can use this formalism for more general $\mathcal{N} = 2$ theories. Let us for example consider the pure $N_f = 0$ theory. The Seiberg-Witten curve is given by

$$\frac{\Lambda}{z} + \Lambda^2 z \; = \; x^2 - u \, , \qquad \lambda \; = \; x \frac{\mathrm{d}z}{z} \, . \tag{2.2.4}$$

Here Λ is the dynamical scale and u is the Coulomb branch parameter $u \equiv \langle \mathrm{Tr}\Phi^2 \rangle / 2$. We can rewrite this in a form as above

$$\lambda^2 \; = \; \phi_2(z) \, , \qquad \phi_2(z) \; = \; \frac{\Lambda^2 \, (1 + z^2) + uz}{z} \frac{\mathrm{d}z^2}{z^2} \, . \tag{2.2.5}$$

However now $\phi_2(z)$ has poles of order three at $z = 0$ and $z = \infty$. This is an example of a theory with irregular or wild punctures.

Similarly the $N_f = 1$ theory leads to a pole of order four at $z = 0$, indicating that there is a hypermultiplet in the fundamental representation at this puncture, and a second pole of order three at $z = \infty$, which means there is no matter at this puncture. Of course an order four pole will have its own $SU(2)$ flavor symmetry, whereas an order three pole will have none. By carefully keeping track of the order of the irregular punctures and the dynamical scales of each gauge group, one can extend the above formalism to these cases.

2.2.2 $SU(N)$ theories

$SU(N)$ **theories with** $N_f = 2N$

Let us consider an $SU(N)$ gauge theory with $N_f = 2N$ flavors in the fundamental representation of $SU(N)$. This theory is superconformal and the Seiberg-Witten curve can be viewed as an N-fold covering of a UV curve with four punctures

$$\lambda^N + \phi_2(z)\lambda^{N-2} + \cdots + \phi_N(z) \ = \ 0 \,. \tag{2.2.6}$$

Then $\phi_j(z)$ has poles of order less than or equal to k at $z = 0, q, 1, \infty$. Generically the residues of λ at these singularities are given by

$$\begin{array}{lll} (\mu_1, \ldots, \mu_N) \,, & \text{at} & z \ = \ 0 \,, \\ (\mu, \ldots, \mu, -\mu(N-1)) \,, & \text{at} & z \ = \ q \,, \\ (\nu, \ldots, \nu, -\nu(N-1)) \,, & \text{at} & z \ = \ 1 \,, \\ (\nu_1, \ldots, \nu_N) \,, & \text{at} & z \ = \ \infty \,, \end{array} \tag{2.2.7}$$

where μ_i, μ, ν and ν_j are the mass parameters appearing in the BPS mass formula, which are related to the bare mass parameters via finite renormalization, and $\sum_{i=1}^{N} \mu_i = \sum_{i=1}^{N} \nu_i = 0$.

Similar to the $N = 2$ case, it is convenient to decompose the flavor symmetry

$$U(2N) \supset U(N) \times U(N) \sim SU(N)_1 \times U(1)_2 \times U(1)_3 \times SU(N)_4 .$$

The singularities at $z = q$ and $z = 1$ encode the $U(1)$ mass parameters associated to $U(1)_2$ and $U(1)_3$. Similarly the singularities at $z = 0, \infty$ encode the $SU(N)_1$, $SU(N)_4$ mass parameters respectively. In the nomenclature introduced above, the punctures of the UV curve associated to an $SU(N)$ flavor symmetry are full punctures and the ones associated to the $U(1)$ flavor symmetry are simple punctures.

Again in analogy to the $SU(2)$ discussion, one can take several limits in q. In the decoupling limit, we find that cutting the tube leaves a full puncture and we end up with two spheres with each two full and one simple puncture. Each of those corresponds to N flavors $(Q^a_i, \tilde{Q}^i_a)$ with $a, i = 1, \ldots, N$, in the bifundamental of $SU(N) \times SU(N)$. The remaining $U(1)$ flavor symmetry is the one acting with charge -1 on Q^a_i and $+1$ on $\tilde{Q}^i_a$. We obtain the original $N_f = 2N$ gauge theory by coupling an $SU(N)$ vectormultiplet the hypermultiplets.

We can also take the $q \to \infty$ strong coupling or equivalently the $\tilde{q} = 1/q \to 0$ weak coupling limit. The resulting theory is again an $SU(N)$ theory with $N_f = 2N$ flavors, where now the two $U(1)$ punctures are exchanged. Things get more complicated when we take $q \to 1$. The decoupling limit leads to a conformal field theory of which we do not have a Lagrangian description in general [141]. In the case of $SU(3)$ this is part of the famous Argyres-Seiberg duality [23].

Higher rank $\mathcal{N} = 2^*$ theory

Let us also extend our description of the $\mathcal{N} = 2^*$ theory to higher rank gauge groups. The construction is the same as in $N = 2$; we start with a free theory of N flavors $(Q^a_i, \tilde{Q}^i_a)$ and couple them to an $SU(N)$ gauge group. Correspondingly we obtain a singlet, which completely decouples plus a hypermultiplet in the adjoint representation. Hence pictorially we are left with a torus with one single simple puncture encoding the $U(1)$ flavor symmetry of the adjoint hypermultiplet (notice that this was enhanced to $SU(2)$ in the rank-one case). We can add a mass, deforming the $\mathcal{N}=4$ to the $\mathcal{N}=2^*$ theory.

2.2.3 Higgsing

Let us briefly introduce the notion of *Higgsing.* This gives a way of obtaining theories with general reduced (regular) punctures from theories with full punctures. The point is that a puncture can be (partially) closed by (partially) Higgsing the full $SU(N)$ flavor symmetry and flowing to the infrared.

Let us start by considering a bifundamental hypermultiplet $(Q^i_a, \tilde{Q}^a_i)$, with $SU(N)_a \times SU(N)_i$ flavor symmetry. The moment maps for the $SU(N)$ action on the hyperkähler Higgs branch are given by

$$\mu^+ \; = \; Q^i_a \tilde{Q}^b_i \,, \qquad \mu^0 \; = \; (Q^\dagger)^a_i Q^i_b - \tilde{Q}^a_i (\tilde{Q}^\dagger)^i_b \,, \qquad \mu^- \; = \; (Q^\dagger)^a_i (\tilde{Q}^\dagger)^i_b \,. \tag{2.2.8}$$

The existence of these operators is guaranteed by the $\mathcal{N}=2$ superconformal algebra; in particular they sit in a supermultiplet with the flavor symmetry currents. We shall now assign a vacuum expectation value to μ^+ given by

$$\langle \mu^+ \rangle \; = \; J_Y \; = \; J_{\ell_1} \oplus \cdots \oplus J_{\ell_n} \,, \qquad \text{with} \qquad \sum_{i=1}^{n} \ell_i \; = \; N \,, \tag{2.2.9}$$

where J_ℓ is the $\ell \times \ell$ Jordan matrix. This breaks the original flavor symmetry as

$$SU(N) \; \longrightarrow \; G_Y \; \equiv \; S\Big[\prod_j U(n_j) \Big] \,. \tag{2.2.10}$$

We can think of $(\ell_1, \ldots, \ell_n)$ as defining a Young diagram. The number n_i are then the number of columns of length i. By introducing this VEV, the original theory flows to a theory with reduced flavor symmetry plus some Nambu-Goldstone bosons

$$\mathcal{T}\,[\text{flavors} \times U(1) \times SU(N)] \; \xrightarrow{\langle \mu^+ \rangle \, = \, J_Y} \; \mathcal{T}\,[\text{flavors} \times G_Y] + \text{NG modes} \,. \tag{2.2.11}$$

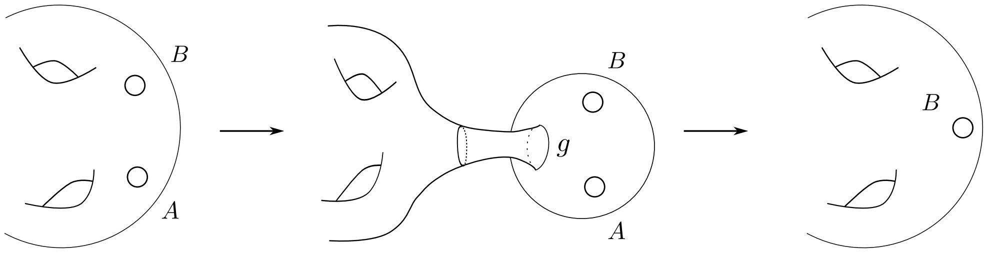


Figure 2.3: Closing a puncture of an $SU(2)$ gauge theory. We start with a theory with two $SU(2)$ flavor punctures A and B. By taking an appropriate weakly coupled frame, we can essentially treat the system as gauging a trifundamental hypermultiplet. By giving a non-zero VEV to the hypermultiplet, and flowing to the IR, we essentially close the A puncture.

We now have a theory with a mass scale set by the VEV, and so take the IR limit, as we are interested in conformal field theories. We decouple the Nambu-Goldstone modes from the IR limit and define the remanent as the partially closed theory. Notice that in order to access the Higgs branch in the first place, we may have to tune some of the Coulomb branch parameters. Furthermore it is important to realize that via the F-term constraints, the VEV of μ^+ may cause some of the other hypermultiplets in the theory to obtain non-zero VEVs. This may break the original gauge group to some subgroup.

Let us look at the example of closing a puncture of an $SU(2)$ gauge theory. Consider a theory with two punctures A and B with flavor symmetry $SU(2)_A \times SU(2)_B$. We can take a limit in which we have a hypermultiplet in the trifundamental q_{abg}, with flavor symmetry $SU(2)_A \times SU(2)_B \times SU(2)_g$, where $SU(2)_g$ is gauged – see Figure 2.3. The moment map can only take one possible non-trivial VEV,

$$\langle q_{abg} \rangle \;=\; \delta_{a1}\epsilon_{bg}\,. \tag{2.2.12}$$

This is equivalent to the above description with $\langle \mu^+ \rangle \equiv J_2$. We see that only a diagonal subgroup $SU(2)_B \subset SU(2)_B \times SU(2)_g$ survives. Furthermore by acting with $SU(2)_A$ on the above VEV, $\delta_{a1} \to \tilde{q}_a$, which is precisely the expected Nambu-Goldstone hypermultiplet. Together we end up with a theory with the A-puncture removed – see Figure 2.3.

2.3 Limits and TQFT interpretation of the superconformal index

As we remarked above, the complex structure moduli of the surface $\mathcal{C}_{g,n}$ are related to the exactly marginal deformations of SCFTs of class $\mathcal{S}$. Since the index is independent under S-dualities and exactly marginal deformations, it is independent of the complex structure moduli of the associated UV curve. Hence it may be regarded as a correlation function of a topological quantum field theory defined on the UV curve [27]. A TQFT is a field theory whose observables do not depend on the metric (g_{ij}), but only on the topological data of the underlying space, *i.e.* in two dimensions on the genus and punctures of the Riemann surface. To make this more precise, let us look at two particular limits of the index. The general case can be treated in a similar way.

2.3.1 The Schur index

First we notice that using the superconformal algebra, the index in (2.1.1) can be rewritten in a more suggestive manner

$$\mathcal{I} \;=\; \mathrm{Tr}(-1)^F p^{\frac{1}{2}\delta^1_-} q^{\frac{1}{2}\delta^1_+} t^{r+R} \prod_j a_j^{f_j} e^{-\beta\delta_{2,\dot{-}}} \,, \tag{2.3.1}$$

where

$$\begin{aligned} 2\delta^1_+ &= \{\mathcal{Q}^1_+, (\mathcal{Q}^1_+)^\dagger\} &&= E + j_w - j_z - 2R - r \,, \\ 2\delta^1_- &= \{\mathcal{Q}^1_-, (\mathcal{Q}^1_-)^\dagger\} &&= E - j_w + j_z - 2R - r \,, \\ 2\delta_{2,\dot{-}} &= \{\tilde{\mathcal{Q}}_{2,\dot{-}}, \tilde{\mathcal{Q}}^\dagger_{2,\dot{-}}\} &&= E - j_z - j_w - 2R + r \,. \end{aligned} \tag{2.3.2}$$

Let us take a particularly nice truncation. In these variables this is given by $q = t$ and p arbitrary. Then the definition in equation (2.3.1) reduces to

$$\mathcal{I} \;=\; \mathrm{Tr}(-1)^F q^{j_w - j_z + R + r} \prod_j a_j^{f_j} \,, \tag{2.3.3}$$

where now the trace is over states in theory that are annihilated by two supercharges $\mathcal{Q}^1_-$ and $\tilde{\mathcal{Q}}_{2,\dot{-}}$, and their Hermitian conjugates. The p-dependence is $\mathcal{Q}_1^-$-exact and therefore drops out.

In the Schur limit, the index of an SU(N) vectormultiplet is

$$\mathcal{I}_V(a_i) \;=\; (q\,;q)^{2N-2}\prod_{i\neq j}^{N}(qa_i/a_j\,;q)^2\,, \tag{2.3.4}$$

where $(a;\,q)=\prod_{i=0}^{\infty}(1-q^ia)$ is the q-Pochhammer symbol. A free hypermultiplet in the bifundamental representation of $SU(N)\times SU(N)$ is given by

$$\mathcal{I}_H(a_i,b_j,c) \;=\; \prod_{i,j=1}^{N}\frac{1}{\theta(q^{\frac{1}{2}}\,a_ib_jc\,;q)}\,, \tag{2.3.5}$$

where the theta-function is defined as $\theta(a;q)=(a;q)(q/a;q)$. Gauging is done using the prescription in equation (2.1.9).

The relation to two-dimensional TQFT becomes apparent if we rewrite the index of an $SU(N)$ SCFT corresponding to a sphere with three full punctures (*i.e.* the $\mathcal{T}_N$ theory) in a more suggesting form [28, 29]

$$\mathcal{I}[\mathcal{T}_N](a,b,c) \;=\; \frac{(q;q)^{N-1}}{\prod_{j=1}^{N-1}(1-q^j)^{N-j}}\,[\mathcal{I}_V(a)\mathcal{I}_V(b)\mathcal{I}_V(c)]^{-\frac{1}{2}}\sum_R\frac{\chi_R(a)\chi_R(b)\chi_R(c)}{\dim_q R}\,, \tag{2.3.6}$$

where the sum is over irreducible representations of $SU(N)$, χ_R are the Schur polynomials as defined in Appendix A. Furthermore

$$\dim_q R \;=\; \chi_R\left(q^{(N-1)/2},q^{(N-3)/2},\ldots,q^{-(N-1)/2}\right)$$

is the quantum dimension of the representation R and by $a=(a_1,\ldots,a_N)$ we label the $SU(N)$ flavor fugacity with $\prod_{i=1}^{N}a_i=1$. The important point is that by an appropriate normalization of the index, the gauging measure $\Delta(a)\mathcal{I}_V(a)$ simply reduces to the Haar measure. By construction the Schur polynomials are orthonormal under the Haar measure.

The relation to TQFT is now apparent: by picking an appropriate normalization, such that gauging is performed by the Haar measure, we can precisely match (2.3.6) to a three-point function of q-deformed Yang-Mills in the zero-area limit. By gluing together different pairs-of-pants and by Higgising punctures, one can generate the SCFT corresponding to any Riemann surface with arbitrary punctures. For example the free bifundamental $SU(N)$ hypermultiplet corresponding to a sphere with two full and a simple puncture is given by

$$\begin{aligned}\mathcal{I}_H(a,b,c) &= \frac{(q;q)^{N-1}}{\prod_{j=1}^{N-1}(1-q^j)}\left[\mathcal{I}_V(a)\mathcal{I}_V(b)\right]^{-\frac{1}{2}}\exp\left(\sum_{n=1}^{\infty}\frac{q^{Nn/2}}{1-q^n}\frac{c^{Nn}+c^{-Nn}}{n}\right)\\ &\quad\times\sum_R\frac{\chi_R(a)\chi_R(b)\chi_R(cq^{(N-2)/2},\ldots,cq^{-(N-2)/2},c^{1-N})}{\dim_q R}\,.\end{aligned} \tag{2.3.7}$$

We see that the full puncture is reduced to a simple puncture by Higgsing and the TQFT structure is still evident.

Let us elaborate on the Higgsing procedure for the $SU(2)$-case. As we have seen in the previous section, by assigning a VEV to μ^+, we can (partially) close a puncture. Picking the VEV in (2.2.12) for an $SU(2)_C$ flavor symmetry preserves a diagonal subgroup $SU(2)_{R'}\subset SU(2)_R\times SU(2)_C$. In the IR $SU(2)_{R'}$ is the diagonal combination of the original fugacities related to $SU(2)_R$ and $SU(2)_C$. In the superconformal index this corresponds to the replacement $c\to q^{1/2}$. However, this leads to a singularity in the prefactor $\mathcal{I}_V(c)$, coming from the Nambu-Goldstone contributions. Removing these contributions, we precisely go from a three-punctured to a two-punctured sphere.

2.3.2 Relation to q-deformed Yang-Mills

The q-deformed version of Yang-Mills can be understood as an analytic continuation of Chern-Simons theory on $\mathcal{C}_{g,n}\times S^1$ [40, 142]. The fundamental variables are the connection A on $\mathcal{C}_{g,n}$ and a periodic adjoint valued scalar ϕ given by the holonomy of the Chern-Simons connection around the S^1,

$$e^{\mathrm{i}\phi} = \mathrm{P}\exp\left(\mathrm{i}\oint_{S^1}A\right)\,. \tag{2.3.8}$$

The gauge fixed path integral is

$$Z \sim \int \prod \mathrm{d}\phi_i \, (\Delta(\phi))^{\chi(\mathcal{C}_{g,n})} \exp\left(-\frac{1}{g_s} \int_{\mathcal{C}_{g,n}} \sum_i \phi_i \, F_i \right) , \tag{2.3.9}$$

where the path integral measure $\Delta(\phi) = \prod_{1 \leq i < j \leq N} 2 \sin\left(\frac{\phi_i - \phi_j}{2}\right)$ takes into account the periodicity of ϕ and leads to the deformation with parameter $q = e^{-g_s}$. This provides an analytic continuation of Chern-Simons theory away from integer level k by moving q away from rational points $e^{2\pi \mathrm{i}/(k+N)}$ on the unit circle.

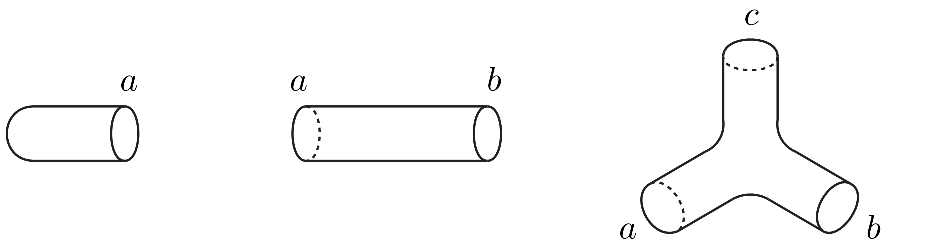
The partition function on a Riemann surface $\mathcal{C}_{g,n}$ with boundaries can be evaluated by surgery. The starting point for this construction is the Hilbert space obtained by Hamiltonian quantization on $\mathbb{R} \times S^1$. This is given by gauge invariant functions of the connection A, which are symmetric polynomials in the holonomy eigenvalues $a = (a_1, \ldots, a_{N-1})$ of the connection around the S^1. The path integral on a Riemann surface with a boundary where the holonomy eigenvalues are held fixed at a defines a wavefunction $\psi(a)$ in the Hilbert space associated to that boundary.

A convenient basis is given by the Schur polynomials $\chi_S(a)$ labeled by irreducible representations S. The Schur polynomials are orthonormal in the Haar measure and any wavefunction can be expanded in terms of those

$$\psi(a) = \sum_S \psi_S \, \chi_S(a) \, , \qquad \psi_S = \int [\mathrm{d}a] \, \chi_S(a) \, \psi(a^{-1}) \, . \tag{2.3.10}$$

The amplitudes for Riemann surfaces with boundaries can be glued by identifying the holonomy eigenvalues and integrating with respect to the Haar measure.

The partition function on any Riemann surface with boundaries can be computed by gluing together the basic amplitudes with one, two and three boundaries, shown in Figure 2.4. The general result for a Riemann surface of genus g with n punctures with



$$\sum_S S_{S,0}\chi_S(a) \qquad \sum_S \chi_S(a)\chi_S(b) \qquad \sum_S S_{S,0}^{-1}\chi_S(a)\chi_S(b)\chi_S(c)$$

Figure 2.4: Partition functions for the sphere with one, two and three punctures respectively, for q-deformed YM in the zero area limit. Here $S_{S,S'}$ is the modular S-matrix as defined in Appendix A.

boundary conditions fixed to $a_1, \ldots, a_n$ is given by

$$\sum_S S_{S,0}^{2-2g-n}\,\chi_S(a_1)\ldots\chi_S(a_n)\,. \tag{2.3.11}$$

Clearly by an appropriate normalization one can match this to the index in (2.3.6).

2.3.3 The Macdonald index

Let us look at another particular limit, which will be important further below. We set $p \to 0$ in the index (2.3.1) to obtain

$$\mathcal{I} \;=\; \mathrm{Tr}(-1)^F q^{2j_1} t^{R+r} \prod_j a_j^{f_j}\,, \tag{2.3.12}$$

where we are restricting the trace to states with $2\delta^1_- = \{\mathcal{Q}^1_-, (\mathcal{Q}^1_-)^\dagger\} = 0$ and so we are summing over states annihilated by $\mathcal{Q}^1_-$ as well as $\tilde{\mathcal{Q}}_{2,\dot{-}}$ and their Hermitian conjugates. Hence we are adding up the same states as in the Schur index but in addition we are keeping track of the quantum numbers of $R+r$ now. Accordingly the Schur limit can be reproduced by taking $q = t$.

The discussion is essentially the same as above. The important difference is however that by an appropriate normalization, the vectormultiplet measure is given by the Macdonald measure

$$\Delta_{q,t}^{(n)}(a) \;=\; \frac{1}{N!}\prod_{i=1}^{N-1}\frac{\mathrm{d}a_i}{2\pi\mathrm{i}a_i}\prod_{i\neq j}\frac{(a_i/a_j;\,q)}{(ta_i/a_j;\,q)}\,. \tag{2.3.13}$$

The discussion is now analogous to the previous sections if we replace the Schur polynomials with the Macdonald polynomials $P_R(x, q, t)$, which are orthogonal under the Macdonald measure. We refer to Appendix A.3 for more details on these special functions. One now obtains a factorisation as in (2.3.6) and (2.3.7) by replacing the Schur polynomials with the Macdonald ones. Of course the Macdonald polynomials reduce to the Schur ones in the limit $q = t$, which makes the story consistent.

The TQFT associated to the superconformal index in the Macdonald limit is a certain deformation of q-deformed Yang-Mills, which is closely related to an analytic continuation of refined Chern-Simons theory on $\mathcal{C}_{g,n}\times S^1$ [142]. The refinement of the theory is associated to the change of measure in the path integral.

2.3.4 General case

The general index has been shown to have a similar factorization. However the functions diagonalizing it are not known and have so far not been found explicitly. In [30] the authors argue that a factorization such as (2.3.6) and (2.3.7) has to exist in the general case as well by looking at surface operators acting on the superconformal index. It can be shown that Schur and Macdonald polynomials are eigenfunctions of these operators in their respective limits and the same is expected to be true in the general case. A further argument then shows that the index has to be factorized by such unknown eigenfunctions. In order to understand this argument, let us first introduce a way to obtain the index in the presence of surface defects.

2.4 The superconformal index in the presence of surface defects

In this section, we review the construction of the superconformal index in the presence of a certain class of surface defects, which arise as the infinite tension limit of background vortex configurations [30]. They are labeled by a nonnegative integer r, the vortex number, which may be interpreted as the magnetic flux through the vortex core.

The starting point is any superconformal field theory $\mathcal{T}_{IR}$ with a global $SU(N)$ flavor symmetry. By gauging this flavor symmetry, the theory may be coupled to a hypermultiplet in the bifundamental representation of $SU(N) \times SU(N)$. The resulting superconformal field theory $\mathcal{T}_{UV}$ has an additional baryonic $U(1)$ symmetry acting on the bifundamental hypermultiplet Q and $\tilde{Q}$ with charges -1 and $+1$ respectively.

The two theories $\mathcal{T}_{IR}$ and $\mathcal{T}_{UV}$ are related by a renormalization flow initiated by turning on a Higgs branch vacuum expectation value for the bifundamental scalar field q. When this expectation value is a constant, the RG flow brings us back to the theory $\mathcal{T}_{IR}$. When the expectation value is taken to be coordinate-dependent, the theory $\mathcal{T}_{IR}$ is modified along a surface and in the low energy limit we recover the theory $\mathcal{T}_{IR}$ in the presence of a surface defect.

More precisely, we can introduce a vacuum expectation value for the baryon operator $B = \det Q$ of the form

$$B(z) \;=\; \prod_{i=1}^{r}(z - z_i)\,, \tag{2.4.1}$$

where z is a complex coordinate in a two-plane, the degree r corresponds to the vortex number, and the parameters z_i are the positions of the vortex strings. Taking the $z_i = 0$ leads to r coincident vortices. This construction then leads to surface defects labeled by $r \in \mathbb{Z}_{\geq 0}$. For $\mathcal{N} = 2$ superconformal field theories of class $\mathcal{S}$, this construction has an elegant interpretation in terms of the curve $\mathcal{C}_{g,n}$ – see Figure 2.5.

This field theoretic construction of surface defects can be implemented concretely in the

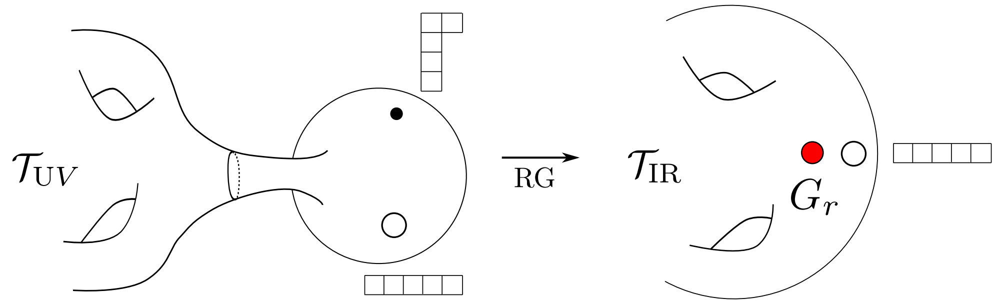


Figure 2.5: The left picture illustrates the Riemann surface $\mathcal{C}_{g,n}$ corresponding to a theory $\mathcal{T}_{UV}$, which is obtained by coupling the theory $\mathcal{T}_{IR}$ to a bifundamental field. An RG flow, that is initiated by turning on a Higgs vev for the bifundamental scalar, relates the theory $\mathcal{T}_{UV}$ to the original theory $\mathcal{T}_{IR}$ with a surface defect G_r. This is illustrated on the right.

superconformal index for surface defects supported on the $S^1 \times S^1$ defined by the locus $\{z = 0\}$. Denoting the superconformal index of $\mathcal{T}_{IR}$ by $\mathcal{I}_{IR}(a_j, \ldots)$, then the superconformal index of $\mathcal{T}_{UV}$ is

$$\mathcal{I}_{UV}(b_j, c, \ldots) \;=\; \oint \Delta(a_i)\, \mathcal{I}_V(a_i)\, \mathcal{I}_H(a_i, b_j, c)\, \mathcal{I}_{\text{IR}}(a_i^{-1}, \ldots)\,. \tag{2.4.2}$$

This has simple poles that originate from simple poles in the integrand pinching the contour. We consider the simple poles of the integrand coming from the bifundamental hypermultiplet index at

$$a_i \;=\; t^{\frac{1}{2}} q^{m_i} \frac{1}{b_{\sigma(i)} c}\,, \tag{2.4.3}$$

where σ is a permutation of $\{1, \ldots, N\}$ and $\sum_i m_i = r$ where $r \in \mathbb{Z}_{\geq 0}$. They correspond to the chiral ring generated by derivatives of components of the bifundamental scalar field, $(\partial_w)^{m_i} Q_i^{\sigma(i)}$. For each permutation σ, these poles pinch the contour when

$$c \;=\; t^{\frac{1}{2}} q^{\frac{r}{N}}\,, \tag{2.4.4}$$

leading to a simple pole in the integral at this point. This pole then corresponds to the chiral ring generated by derivatives of the baryon operator $(\partial_w)^r B$ where $B = \det Q$, which is charged only under the $U(1)$. The residue at this pole corresponds to the index of $\mathcal{T}_{IR}$ in the presence of a surface defect obtained by giving an expectation value $B = z^r$ to the baryon operator of $\mathcal{T}_{UV}$ and flowing to the IR.

As demonstrated in [30], the residue takes the form of a difference operator G_r acting on the superconformal index of $\mathcal{T}_{IR}$. There is one term in the operator for each distinct set of integers $\{m_1, \ldots, m_N\}$ such that $\sum_i m_i = r$. The precise prescription defining the difference operator is

$$G_r \cdot \mathcal{I}_{IR}(b_i, \ldots) = N \, \mathcal{I}_V(b_i) \operatorname*{Res}_{c = t^{\frac{1}{2}} q^{\frac{r}{N}}} \left[\frac{1}{c} \mathcal{I}_{\mathrm{UV}}(c, b_i, \ldots) \right] . \tag{2.4.5}$$

The result of the computation is

$$G_r \cdot \mathcal{I}(b_i) = \sum_{\sum_{j=1}^N m_j = r} \prod_{i,j=1}^{N} \left[\prod_{m=0}^{m_j - 1} \frac{\theta\left(q^{m + m_j - m_i} t b_i / b_j ; p\right)}{\theta\left(q^{m - m_j} b_j / b_i ; p\right)} \right] \mathcal{I}\left(b_i \mapsto q^{\frac{r}{N} - m_i} \, b_i\right) , \tag{2.4.6}$$

where the theta-function is defined as above.

The difference operators G_r constructed by this method are formally self-adjoint with respect to the measure $\Delta(a) \, \mathcal{I}_V(a)$ used for gauging. They are labeled by a nonnegative integer $r \in \mathbb{Z}_{\geq 0}$. Furthermore each term in the operator can be identified with a weight of the r-th symmetric tensor representation of $\mathfrak{su}(N)$. In particular, the numbers $\{m_1, m_2, \ldots, m_N\}$ denote the number of times the integers $\{1, \ldots, N\}$ appear in the corresponding Young tableau. Based on this observation, we associate these operators to surface defects labeled by the symmetric tensor representations of $\mathfrak{su}(N)$.

2.5 Bootstrapping the superconformal index

Recall that for any superconformal field theory of class $\mathcal{S}$ the superconformal index is related to some two-dimensional TQFT on the associated Riemann surface $\mathcal{C}_{g,n}$. In the Schur and Macdonald limit, this was observed by rewriting the index in terms of correlation functions of the TQFT in equations (2.3.6) and (2.3.7). We present here the argument that this can be done in general.

In order verify this, it is necessary to extract a certain function $K(a)$ from the superconformal index for each full $SU(N)$ flavor puncture. In what follows, we define the normalized index $\mathcal{I}^{(n)}$ through the equation

$$\mathcal{I}(a,b,\ldots) \;=\; \left(K(a)K(b)\cdots\right)\mathcal{I}^{(n)}(a,b,\ldots)\,, \tag{2.5.1}$$

where

$$K(a) \;=\; \prod_{i\neq j}^{N}\Gamma(ta_i/a_j,p,q)\,. \tag{2.5.2}$$

The normalized index $\mathcal{I}^{(n)}$ is now gauged using the measure[2]

$$\Delta^{(n)}_{p,q,t}(a) \;=\; K(a)^2\Delta_{p,q,t}(a) \;=\; \frac{1}{N!}\left(\frac{(p,p)(q,q)}{\Gamma(t,p,q)}\right)^{N-1}\prod_{i\neq j}\frac{\Gamma(ta_i/a_j,p,q)}{\Gamma(a_i/a_j,p,q)}\,. \tag{2.5.3}$$

The difference operators $\bar{G}_r$ acting on the normalized index are thus obtained by conjugation with $K(a)$ and are given by

$$\bar{G}_r\cdot\mathcal{I}^{(n)}(a_i) \;=\; \mathcal{N}_r\sum_{\sum_{j=1}^{N}m_j=r}\;\prod_{i,j=1}^{N}\prod_{m=0}^{m_j-1}\frac{\theta\left(tq^{m}a_i/a_j,p\right)}{\theta\left(q^{m-m_i}a_i/a_j,p\right)}\,\mathcal{I}^{(n)}\left(q^{\frac{r}{N}-m_i}a_i\right)\,. \tag{2.5.4}$$

Comparing with (2.4.6) and noting the reflection property $\theta(z,p)=\theta(p/z,p)$ we see that the effect of the conjugation is simply to interchange $t\leftrightarrow pq/t$.

We will assume that the difference operators $\bar{G}_r$ admit a complete set of eigenfunctions $\{\psi_S(a_i)\}$, indexed by irreducible representations S of $\mathfrak{su}(N)$. Suppose the eigenvalues are non-degenerate (this can explicitly be checked), the eigenfunctions are then automatically orthogonal under the measure $\Delta^{(n)}_{p,q,t}(a)$. Now the TQFT structure of the superconformal index can be made very explicit [30].

Consider for instance the sphere with three maximal punctures. The corresponding four-dimensional superconformal field theory is known as $\mathcal{T}_N$. It has at least $SU(N)^3$ flavor symmetry. The superconformal index of this theory can be expanded in terms of the

[2]Notice that in the Schur, Macdonald limit the normalized measure precisely reduces to the Haar, Macdonald measure respectively.

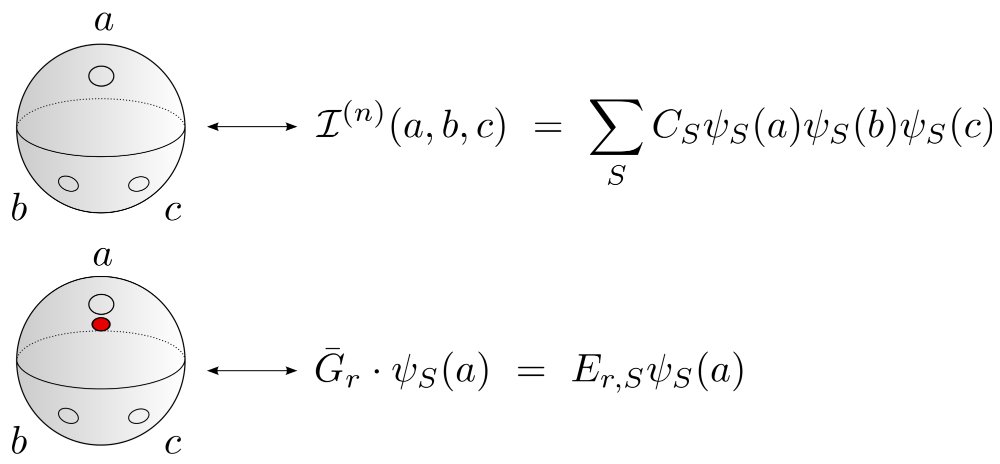


Figure 2.6: The superconformal index can be written as a TQFT correlator. This correlator is diagonal in the eigenfunctions $\psi_S(a_i)$ of the difference operators G_R.

set of eigenfunctions $\{\psi_S(a_i)\}$ as

$$\mathcal{I}^{(n)}(a,b,c) \;=\; \sum_{S_1,S_2,S_3} C_{S_1,S_2,S_3}\, \psi_{S_1}(a)\psi_{S_2}(b)\psi_{S_3}(c)\,, \tag{2.5.5}$$

where C_{S_1,S_2,S_3} are the structure constants of the two-dimensional TQFT. By generalised S-duality, acting on different punctures must give the same answer. Since the eigenvalues are non-degenerate, the superconformal index is in fact diagonal in this basis, *i.e.* the structure constants are nonzero only for $S_1 = S_2 = S_3$, and we end up with the general result

$$\mathcal{I}^{(n)}(a,b,c) \;=\; \sum_S C_S\, \psi_S(a)\psi_S(b)\psi_S(c)\,. \tag{2.5.6}$$

Chapter 3

The lens index and its TQFT structure

In this chapter we study the supersymmetric partition function on $S^1 \times L(r,1) = S^1 \times S^3/\mathbb{Z}_r$. This is an interesting extension of the superconformal index, which we introduced in the previous chapter. The lens index is a particularly interesting object, since it is sensitive to global properties of the theory [43]. It was originally computed in [44], where S-duality was checked up to a few orders in a fugacities expansion. The aim of the present chapter is to prove S-duality in a subspace of the space of fugacities resembling the Schur limit. We shall follow a strategy similar to [28], *i.e.* we uncover the structure behind S-duality by rewriting the index in a way that resembles the correlation function of a two-dimensional topological quantum field theory, as in equation (2.5.6). The structure of the index on lens spaces however, is much more complicated and we are not able to identify the relevant two-dimensional TQFT.

In interesting follow-up work to our paper [1], it was found that the two-dimensional TQFT dual to the lens index in a limit reminiscent of the Macdonald limit, has correlation functions described by a combination of non-symmetric Macdonald polynomials [45]. The authors follow the bootstrapping approach which was introduced in [30] and outlined at the end of the previous chapter for the round index.

3.1 The superconformal index on $S^1 \times L(r,1)$

The lens space $L(r,1) = S^3/\mathbb{Z}_r$ is defined as the orbifold of $S^3 \subset \mathbb{C}^2$ under the identification

$$(z_1, z_2) \sim (e^{2\pi \mathrm{i}/r} z_1, e^{-2\pi \mathrm{i}/r} z_2)\,, \quad \text{where} \quad |z_1|^2 + |z_2|^2 \;=\; 1\,.$$

The isometry $SU(2)_1$ of S^3 acts on (z_1, z_2) as a doublet and $\mathbb{Z}_r$ acts on the Hopf fibre S^1_H of S^3. The lens index is then defined as the round index in equation (2.1.1), where the trace is now taken over the Hilbert space of the theory on $S^3/\mathbb{Z}_r$ in radial quantisation. In [44], it was computed as a path integral of the theory on $S^1 \times L(r,1)$ in the free field limit.

The important new feature is that the orbifold theory has a set of vacua labeled by a non-trivial holonomy V along S^1_H with $V^r = 1$. In the remainder of this chapter, we will only treat $\mathfrak{g} = A_1$ type theories, in which case the holonomy can be taken of the form

$$V \;=\; \mathrm{diag}\left(e^{2\pi \mathrm{i} m/r}, e^{2\pi \mathrm{i}(r-m)/r}\right)\,, \qquad m \;=\; 0, 1, ..., \lfloor r/2 \rfloor\,, \tag{3.1.1}$$

and the flavor fugacities $a = (a_1, a_2)$ can be expressed in terms of a single parameter $a = a_1 = a_2^{-1}$. Different sectors of the theory are then labeled by the integer m. Furthermore punctures are now labeled by the data (a, m) consisting of a continuous fugacity a and a discrete choice of $m \in \{0, \ldots, \lfloor r/2 \rfloor\}$. As a consequence, the gauging action involves a finite sum over all possible choices of m.

As discussed in the previous chapter, the total index of an $SU(2)$ theory of class $\mathcal{S}$ is constructed from the following building blocks: a trifundamental hypermultiplet and an $SU(2)$ vectormultiplet, which can be interpreted as a three-point function and a propagator of a two-dimensional TQFT. For the lens index they are given by

$$\mathcal{I}_{trif}^{(m_1,m_2,m_3)}(a_1, a_2, a_3) = \mathcal{I}_0^{(m_1,m_2,m_3)} \exp\left[\sum_{n=1}^{\infty} \sum_{s_i=\pm} \frac{1}{n} i_H \left(t^n, p^n, q^n; \left[\textstyle\sum_{i=1}^3 m_i s_i\right]_r\right) a_1^{s_1 n} a_2^{s_2 n} a_3^{s_3 n}\right],$$

$$\eta^{(m)}(a) = \eta_0^{(m)} \exp\left[\sum_{n=1}^{\infty} \sum_{s=\pm 2,0} \frac{1}{n} i_V \left(t^n, p^n, q^n; [ms]_r\right) a^{sn}\right],$$

where $[x]_r = x \bmod r$, the zero-point contributions are

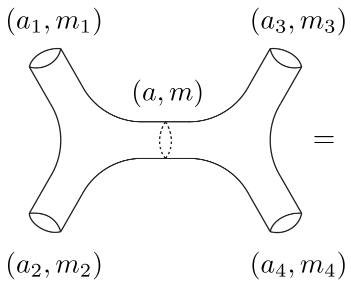
$$\begin{aligned} \mathcal{I}_0^{(m_1,m_2,m_3)} &= \left(\frac{t}{pq}\right)^{-\frac{1}{4}\sum_{s_i=\pm}\left([m_1s_1+m_2s_2+m_3s_3]_r - [m_1s_1+m_2s_2+m_3s_3]_r^2/r\right)}, \\ \eta_0^{(m)} &= \left(\frac{t}{pq}\right)^{[2m]_r - [2m]_r^2/r}, \end{aligned} \tag{3.1.2}$$

and the single letter contributions are given by

$$i_H(t,p,q;m) = \frac{\frac{p^m}{1-p^r} + \frac{q^{r-m}}{1-q^r}}{1-pq}\left(pq\left(1+\frac{1}{t}\right) - 1 - t\right) + \delta_{m,0}\,, \tag{3.1.3}$$

$$i_V(t,p,q;m) = \frac{\frac{p^m}{1-p^r} + \frac{q^{r-m}}{1-q^r}}{1-pq}\left(\sqrt{t} - \frac{pq}{\sqrt{t}}\right). \tag{3.1.4}$$

The total index is then obtained by gluing such basic building blocks. For instance, the four-point function is obtained by joining two three-point functions with the corresponding propagators[1]



$$= \sum_{m=0}^{\lfloor r/2 \rfloor} \int [\mathrm{d}a]^m \mathcal{I}_{trif}^{(m_1,m_2,m)}(a_1,a_2,a)\eta^{(m)}(a)\mathcal{I}_{trif}^{(-m,m_3,m_4)}(a^{-1},a_3,a_4)\,,$$

where the integration measure is given by

$$[\mathrm{d}a]^m = \begin{cases} \frac{2-a^2-a^{-2}}{4\pi}\frac{\mathrm{d}a}{a}\,, & [2m]_r = 0\,, \\ \frac{1}{4\pi}\frac{\mathrm{d}a}{a}\,, & [2m]_r \neq 0\,. \end{cases} \tag{3.1.5}$$

S-duality implies that the four-point function is symmetric under the interchange of any two pairs $(a_i, m_i) \leftrightarrow (a_j, m_j)$.

[1]Note that $\mathcal{I}_{trif}$ is invariant under exchange $m \leftrightarrow -m$ and $a \leftrightarrow a^{-1}$.

3.2 A particular slice

In the Macdonald limit (*i.e.* $p \to 0$) we find the following expressions

$$\mathcal{I}_{trif}^{(m_1,m_2,m_3)}(a_1,a_2,a_3) = \mathcal{I}_0^{(m_1,m_2,m_3)} \prod_{s_i=\pm} \frac{1}{(\sqrt{t}\, a_1^{s_1} a_2^{s_2} a_3^{s_3} q^{[-m_1 s_1 - m_2 s_2 - m_3 s_3]_r}; q^r)}\,, \quad (3.2.1)$$

$$\eta^{(m)}(a) = \eta_0^{(m)} \left(\frac{-a^2}{(1-a^2)^2}\right)^{\delta_{m,0}} (q^r;q^r)(t;q^r) \prod_{s=\pm} (a^{2s} q^{[-2ms]_r}; q^r)(a^{2s} t q^{[-2ms]_r}; q^r)\,, \quad (3.2.2)$$

where $(a;\, q) = \prod_{i=0}^{\infty}(1-q^i a)$ is the q-Pochhammer symbol. The zero-point contributions will be discussed momentarily. Mimicking the discussion of the round index in [27] and reviewed in section 2.3, we define the rescaled structure constants

$$\hat{C}^{(m_1,m_2,m_3)}(a_1,a_2,a_3) = \sqrt{\eta^{(m_1)}(a_1)\eta^{(m_2)}(a_2)\eta^{(m_3)}(a_3)}\; \mathcal{I}_{trif}^{(m_1,m_2,m_3)}(a_1,a_2,a_3)\,. \quad (3.2.3)$$

Note that the zero-point contributions are subtle in this limit, since they are proportional to p to some power. It can be checked that this power is always bigger than or equal to zero. Furthermore, quite remarkably, this power is exactly zero (and so the contribution does not vanish in the $p \to 0$ limit), provided m_1, m_2 and m_3 satisfy a selection rule. Namely, for fixed m_1 and m_2, m_3 should run between $|m_1 - m_2|$ and $min(|r - m_1 - m_2|, m_1 + m_2)$. Note that this agrees with the selection rules for $\hat{\mathfrak{su}}(2)_r$, the affine algebra at level r.

In what follows, we will make a further simplification in the space of fugacities, and we will consider the limit $t = q^r$. Note that this reduces to the Schur limit for $r = 1$ and so the two-dimensional TQFT we are after should reduce to q-deformed YM in the zero area limit [28]. It is the natural extension of the Schur limit to the lens case, since it has the same enhanced supersymmetry – see section 2.3.1.

3.3 Two-dimensional TQFT interpretation

We would like to interpret the rescaled structure constants as the three-point correlation functions of some two-dimensional TQFT. Let us start by defining

$$C^{(m_1,m_2,m_3)}(a_1,a_2,a_3) \equiv \frac{1-q^r}{(q^r;q^r)}\hat{C}^{(m_1,m_2,m_3)}(a_1,a_2,a_3)\,. \tag{3.3.1}$$

Using the results of [28] we can immediately write

$$C^{(0,0,0)}(a_1,a_2,a_3) = \sum_{\ell=1}^{\infty}\frac{\chi_\ell(a_1)\chi_\ell(a_2)\chi_\ell(a_3)}{\dim_{q^r}\ell}\,, \tag{3.3.2}$$

where we introduced the Schur polynomials and the q^r-deformed dimension as in the previous chapter and Appendix A.

In [1], we find that for general $m_i \neq 0$, the structure constants can be expanded as

$$C^{(m_1,m_2,m_3)}(a_1,a_2,a_3) = \sum_{\ell=1}^{\infty}C_\ell^{(m_1,m_2,m_3)}(a_1,a_2,a_3)\,, \tag{3.3.3}$$

and we define the three-point correlation functions of our putative TQFT as

$$\langle\mathcal{O}^{(m_1)}(a_1)\mathcal{O}^{(m_2)}(a_2)\mathcal{O}^{(m_3)}(a_3)\rangle_{\mathcal{C}_{0,3}} = \sum_{\ell=1}^{\infty}\langle\mathcal{O}^{(m_1)}(a_1)\mathcal{O}^{(m_2)}(a_2)\mathcal{O}^{(m_3)}(a_3)\rangle_\ell\,, \tag{3.3.4}$$

$$\langle\mathcal{O}^{(m_1)}(a_1)\mathcal{O}^{(m_2)}(a_2)\mathcal{O}^{(m_3)}(a_3)\rangle_\ell = \frac{C_\ell^{(m_1,m_2,m_3)}(a_1,a_2,a_3)}{\sqrt{\mathcal{N}^{(m_1)}(a_1)\mathcal{N}^{(m_2)}(a_2)\mathcal{N}^{(m_3)}(a_3)}}\,, \tag{3.3.5}$$

where the normalization factor is such that in the two-dimensional TQFT the gluing is done with the measure factor

$$[\mathrm{d}a] = \frac{\mathrm{d}a}{2\pi\mathrm{i}a}\mathcal{N}^{(m)}(a)\,, \quad \text{with} \quad \mathcal{N}^{(m)}(a) = 1-\frac{a^2q^{r-2m}+a^{-2}q^{2m}}{1+q^r}\,. \tag{3.3.6}$$

Note that the m-dependence of the measure factor can be removed by taking $a \to aq^m$.

Furthermore, note that for $q \to 1$, the measure factor reduces to the usual $SU(2)$ Haar measure, as given in (2.1.10). Even though the correlation functions of the two-dimensional TQFT do not factorize into functions of the a_i, we find the following general form

$$\langle \mathcal{O}^{(m_1)}(a_1)\mathcal{O}^{(m_2)}(a_2)\mathcal{O}^{(m_3)}(a_3)\rangle_\ell \;=\; \sum_{i,j,k=1}^{2} c^{(\mathcal{R})}(q) f^{(\mathcal{R})}_{\ell,ijk}(q) U^{(m_1)}_{\ell,i}(a_1) U^{(m_2)}_{\ell,j}(a_2) U^{(m_3)}_{\ell,k}(a_3)\,, \tag{3.3.7}$$

where $U^{(m)}_{\ell,i}(a)$ and $U^{(m)}_{\ell,j}(a)$ are some functions, which satisfy orthonormality properties with respect to the above measure (3.3.6)

$$\frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}a}{a} \mathcal{N}^{(m)}(a)\, U^{(m)}_{\ell,i}(a) U^{(m)}_{\ell',j}(a) \;=\; \delta_{\ell\ell'}\delta_{ij}\,. \tag{3.3.8}$$

Furthermore in (3.3.7) we have stressed the fact that the functions $c^{(\mathcal{R})}(q)$ and $f^{(\mathcal{R})}_{\ell,ijk}(q)$ will depend on q. Besides the explicit dependence on m_i in the functions $U^{(m_i)}_{\ell,i}(a_i)$, the value of $f^{(\mathcal{R})}_{\ell,ijk}(q)$ can change as we jump from one "region" to another. Here $\mathcal{R}$ runs over the possible regions and the values of the m_i fix the region we are in. Let's for simplicity consider the case in which r is odd and assume $0 < m_1 \leq m_2 \leq m_3 < r/2$, then

$$\mathcal{R} \;=\; \begin{cases} I\,, & m_1 + m_2 + m_2 = r\,, \\ II\,, & m_1 + m_2 = m_3\,, \\ III\,, & \text{other cases}\,. \end{cases} \tag{3.3.9}$$

This division into regions is of course expected, due to the presence of $\left[\sum_i m_i s_i\right]_r$ in (3.2.1). Within a region we have a "continuous dependence" on the m_i, but the expression "jumps" when we cross to a different region. The normalization factor $c^{(\mathcal{R})}$ has been pulled out in order to have $f^{(\mathcal{R})}_{\ell=1,111} = 1$, and is given by

$$c^{(I)}(q) \;=\; c^{(II)}(q) \;=\; \frac{1}{\sqrt{1+q^r}}\,, \qquad c^{(III)}(q) \;=\; \frac{1-q^r}{\sqrt{1+q^r}}\,. \tag{3.3.10}$$

The structure constants have certain symmetry properties under permutation of the holonomies m_i. This implies certain symmetries among the functions $f^{(\mathcal{R})}_{\ell,ijk}$, namely $f^{(I)}_{\ell,ijk}$ and $f^{(III)}_{\ell,ijk}$ are invariant under permutation of i, j, k and $f^{(II)}_{\ell,ijk}$ is invariant under the interchange $i \leftrightarrow j$. Furthermore, up to a very high order in the q-expansion, we have checked the additional symmetries

$$
\begin{aligned}
&f^{(II)}_{\ell,111} = f^{(II)}_{\ell,122} = f^{(II)}_{\ell,212} = f^{(III)}_{\ell,111} = f^{(III)}_{\ell,122} = f^{(III)}_{\ell,212} = f^{(III)}_{\ell,221}\,,\\
&f^{(II)}_{\ell,112} = f^{(II)}_{\ell,121} = f^{(II)}_{\ell,211} = f^{(III)}_{\ell,112} = f^{(III)}_{\ell,121} = f^{(III)}_{\ell,211} = 0\,.
\end{aligned}
$$

All in all, at each order ℓ and for generic r, the structure constants depend on eight functions of the fugacity q (for small values of r not all the regions are present). Let us introduce the following notation

$$
\begin{aligned}
h^{(1)}_\ell(q) &= f^{(I)}_{\ell,111}\,,\\
h^{(2)}_\ell(q) &= f^{(I)}_{\ell,112} \quad \text{(plus permutations)}\,,\\
h^{(3)}_\ell(q) &= f^{(I)}_{\ell,122} \quad \text{(plus permutations)}\,,\\
h^{(4)}_\ell(q) &= f^{(I)}_{\ell,222}\,,\\
h^{(5)}_\ell(q) &= f^{(II)}_{\ell,111} = f^{(II)}_{\ell,122} = f^{(II)}_{\ell,212} = f^{(III)}_{\ell,111} = f^{(III)}_{\ell,122} \quad \text{(plus permutations)}\,,\\
h^{(6)}_\ell(q) &= f^{(II)}_{\ell,221}\,,\\
h^{(7)}_\ell(q) &= f^{(II)}_{\ell,222}\,,\\
h^{(8)}_\ell(q) &= f^{(III)}_{\ell,222}\,.
\end{aligned}
$$

Up to very high powers in the fugacity, and for several values of ℓ and r, we have found the following expressions for these functions

$$
\begin{aligned}
h^{(1)}_\ell(q) &= q^{-1/2(\ell-1)r}\frac{(1-q^r)(1+q^{\ell r}+q^{2\ell r}-q^{(1+\ell)r})}{(1+q^r)(1-q^{\ell r})}\,,\\
h^{(2)}_\ell(q) &= \mathrm{i}q^{-1/2(\ell-1)r}\frac{(1-q^r)(1+q^{\ell r})}{(1+q^r)(1-q^{\ell r})}\sqrt{1-q^{(\ell-1)r}+q^{2\ell r}-q^{(\ell+1)r}}\,,\\
h^{(3)}_\ell(q) &= -q^{-1/2(\ell-1)r}\frac{(1-q^r)(1-q^{(\ell-1)r}+q^{\ell r}+q^{2\ell r})}{(1+q^r)(1-q^{\ell r})}\,,
\end{aligned}
$$

$$
\begin{aligned}
h_\ell^{(4)}(q) &= -\mathrm{i}q^{-1/2(\ell-1)r}\frac{(1-q^r)(1+q^{\ell r})}{(1+q^r)(1-q^{\ell r})}\frac{1-2q^{(\ell-1)r}+q^{2\ell r}}{\sqrt{1-q^{(\ell-1)r}+q^{2\ell r}-q^{(\ell+1)r}}}\,,\\
h_\ell^{(5)}(q) &= q^{1/2(\ell-1)r}\frac{1-q^r}{1-q^{\ell r}}\,,\\
h_\ell^{(6)}(q) &= -q^{1/2(\ell+1)r}\frac{1-q^r}{1-q^{\ell r}}\,,\\
h_\ell^{(7)}(q) &= \mathrm{i}q^{1/2(\ell-1)r}\frac{(1-q^r)^2(1+q^{\ell r})}{(1-q^{\ell r})}\frac{1}{\sqrt{1-q^{(\ell-1)r}+q^{2\ell r}-q^{(\ell+1)r}}}\,,\\
h_\ell^{(8)}(q) &= 2\mathrm{i}q^{1/2(\ell-1)r}\frac{(1-q^r)(1+q^{\ell r})}{(1-q^{\ell r})\sqrt{(1-q^{(\ell-1)r})(1-q^{(\ell+1)r})}}\,.
\end{aligned}
$$

These expressions are also valid for r even, except the case where $m_1+m_2=m_3=r/2$ (which can only exist for r even). If we call that region $\mathcal{R}=IV$, we find

$$
\begin{aligned}
f_{\ell,111}^{(IV)} &= -f_{\ell,221}^{(IV)} = \frac{(1-q^r)(1+q^{\ell r})}{(1+q^r)(1-q^{\ell r})}\,,\\
f_{\ell,121}^{(IV)} &= f_{\ell,211}^{(IV)} = \mathrm{i}\frac{(1-q^r)\sqrt{1-q^{(\ell-1)r}-q^{(\ell+1)r}+q^{2\ell r}}}{(1+q^r)(1-q^{\ell r})}\,,
\end{aligned}
\tag{3.3.11}
$$

as well as $c^{(IV)}(q)=1$. Remember that if $m_3=r/2$, the third index of $f_{\ell,ijk}$ is forced to be one. Hence, we have fully defined the structure constants of the putative two-dimensional TQFT. Equivalently, we can say that these are the partition functions of the two-dimensional TQFT on a sphere with three punctures, where the punctures have labels (a_i,m_i). It turns out to be impossible to diagonalise the structure constants. This is because in region $\mathcal{R}=II$ they are not invariant under permutations, *i.e.* $f_{\ell,221}^{(II)}\neq f_{\ell,212}^{(II)}=f_{\ell,122}^{(II)}$, whereas an ansatz $\tilde{U}_{\ell,i}^{(m)}=\alpha_i U_{\ell,1}^{(m)}+\beta_i U_{\ell,2}^{(m)}$ leading to a purely diagonal form, would require them to be.

3.4 Crossing symmetry

As mentioned above, the structure constants correspond to the partition function of a two-dimensional TQFT on a sphere with three punctures. The partition function on a generic Riemann surface with punctures can be computed by the gluing procedure, decomposing the Riemann surface into pairs of pants joined by tubes. A novel feature, not present in a usual two-dimensional TQFT, is the sum over the holonomy m of the intermediate state. The factorization (3.3.7) implies a result of the form

$$\langle \mathcal{O}^{(m_1)}(a_1)\cdots\mathcal{O}^{m_n}(a_n)\rangle_{\mathcal{C}_{g,n}} = \sum_{\ell}\sum_{i_1,\dots,i_n} f^{m_1,\dots,m_n}_{\ell,i_1,\dots,i_n}(q)U^{(m_1)}_{\ell,i_1}(a_1)...U^{(m_n)}_{\ell,i_n}(a_n)\,, \qquad (3.4.1)$$

where the q-dependent factors $f^{m_1,\dots,m_n}_{\ell,i_1,\dots,i_n}(q)$ can be easily computed from the formulae given in this chapter. For the gluing procedure to be consistent, it should not matter which particular pants decomposition we choose in order to do the computation. From the point of view of the four-dimensional index, this is true once we assume S-duality. From the two-dimensional TQFT perspective, this happens provided the four-point correlation function satisfies crossing symmetry, see Figure 3.1.

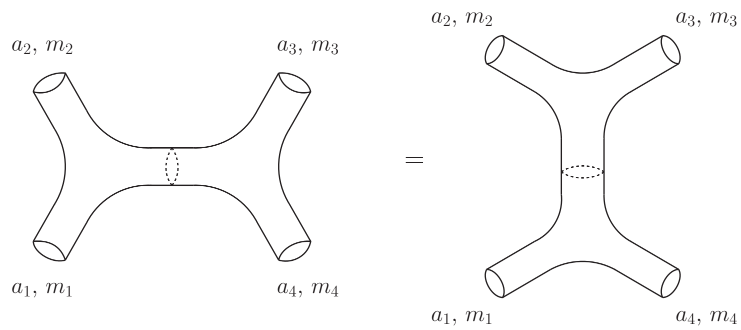


Figure 3.1: The four point correlation function should be crossing symmetric. On both sides one should sum over the holonomy m of the internal state.

More precisely, we should have

$$\sum_m \oint [\mathrm{d}a]^m \langle \mathcal{O}^{(m_1)}(a_1)\mathcal{O}^{(m_2)}(a_2)\mathcal{O}^{(m)}(a)\rangle \, \langle \mathcal{O}^{(m)}(a)\mathcal{O}^{(m_3)}(a_3)\mathcal{O}^{(m_4)}(a_4)\rangle =$$
$$= \sum_m \oint [\mathrm{d}a]^m \langle \mathcal{O}^{(m_1)}(a_1)\mathcal{O}^{(m_3)}(a_3)\mathcal{O}^{(m)}(a)\rangle \, \langle \mathcal{O}^{(m)}(a)\mathcal{O}^{(m_2)}(a_2)\mathcal{O}^{(m_4)}(a_4)\rangle\,, \qquad (3.4.2)$$

where the sums over m run over the holonomies compatible with the selection rules of the putative two-dimensional TQFT, discussed above, and the integration over a is done with the measure defined in (3.3.6). The factorization (3.3.7) implies that crossing symmetry has to hold at each level ℓ independently. For a fixed choice of the external holonomies $m_{1,2,3,4}$, crossing symmetry holds provided the functions $h_\ell^{(i)}(q)$ satisfy specific quadratic constraints. When r is large enough, the number of constraints is much larger than the number of functions. It turns out that one can isolate seven "basic" constraints:

$$\begin{aligned}
h_\ell^{(1)}(q)+h_\ell^{(3)}(q) &= \frac{1-q^r}{|\ell|_{q^r}}\,, & h_\ell^{(2)}(q)^2 &= \frac{(1-|\ell|_{q^r}h_\ell^{(3)}(q))(q^r+|\ell|_{q^r}h_\ell^{(3)}(q))}{|\ell|_{q^r}^2} \\
h_\ell^{(3)}(q)^2-h_\ell^{(4)}(q)h_\ell^{(2)}(q) &= \frac{1}{|\ell|_{q^r}^2}\,, & h_\ell^{(5)}(q) &= \frac{1}{|\ell|_{q^r}}\,, \\
h_\ell^{(6)}(q) &= -\frac{q^r}{|\ell|_{q^r}}\,, & h_\ell^{(7)}(q) &= \frac{1}{2}(1-q^r)h_\ell^{(8)}(q)\,, \\
\frac{1-h_\ell^{(3)}(q)|\ell|_{q^r}}{q^r+h_\ell^{(3)}(q)|\ell|_{q^r}} &= \frac{h_\ell^{(7)}(q)^2|\ell|_{q^r}^2}{(1-q^r)^2}\,,
\end{aligned}$$

which imply all the constraints for any values of $0 \le m_{1,2,3,4} < r/2$. Here we have used the shorthand notation $|\ell|_q$ for the quantum dimension

$$|\ell|_q \;\equiv\; \mathrm{dim}_q\ell \;=\; \frac{q^{-\ell/2}-q^{\ell/2}}{q^{-1/2}-q^{1/2}}\,.$$

One can explicitly check that the functions $h_\ell^{(i)}(q)$ found in this chapter indeed satisfy these constraints. As a result, correlation functions of this two-dimensional TQFT possess crossing symmetry and the superconformal lens index possesses S-duality in the particular fugacity slice of $p=0$ and $t=q^r$.

Chapter 4

An elliptic algebra of surface defects and their TQFT interpretation

In this chapter we continue the study of the (round) superconformal index of four-dimensional $\mathcal{N}=2$ theories of class $\mathcal{S}$ in the presence of surface defects. Our starting point is the prescription of the index in the presence of surface operators coming from the infinite-tension limit of vortex loops as first obtained in [30], and reviewed in section 2.4. Let us for convenience state the main result again: we consider any theory of class $\mathcal{S}$ in the presence of a surface defect labeled by an integer r. This defect acts as a difference operator on a full $SU(N)$ puncture of the index given by the formula

$$G_r \cdot \mathcal{I}(b_i, \ldots) \; = \sum_{\sum_{j=1}^{N} m_j = r} \; \prod_{i,j=1}^{N} \left[\prod_{m=0}^{m_j-1} \frac{\theta\left(q^{m+m_j-m_i} t b_i/b_j; p\right)}{\theta\left(q^{m-m_j} b_j/b_i; p\right)} \right] \mathcal{I}\left(b_i \mapsto q^{\frac{r}{N}-m_i}\, b_i, \ldots\right) .$$

In the present chapter, we introduce surface operators for any irreducible representations of $\mathfrak{su}(N)$. We do so by constructing an algebra of difference operators by composition. For the fully antisymmetric tensor representations these difference operators are the Hamiltonians of the elliptic Ruijsenaars-Schneider system. Furthermore, the structure constants of the algebra are elliptic generalizations of the Littlewood-Richardson coefficients. In the Schur and Macdonald limit, we identify the difference operators with local operators in the two-dimensional TQFT interpretation of the superconformal index.

4.1 Elliptic algebra of four-dimensional surface defects

4.1.1 Composition of difference operators

We start by considering and analysing the composition of two difference operators, $G_{r_1} \circ G_{r_2}$. This can be given a physical interpretation by coupling the theory $\mathcal{T}_{IR}$ to a single hypermultiplet Q_1 in the bifundamental representation of $SU(N) \times SU(N)$ and then to another bifundamental hypermultiplet Q_2. The resulting theory $\mathcal{T}'_{UV}$ is illustrated in Figure 4.1. It has two additional flavor symmetries $U(1)_{f,1}$ and $U(1)_{f,2}$ that act on the two bifundamental hypermultiplets Q_1 and Q_2 respectively. The original theory $\mathcal{T}_{IR}$ is reached by turning on constant vacuum expectation values for both baryon operators $B_1 = \det Q_1$ and $B_2 = \det Q_2$ charged under the additional flavor symmetries $U(1)_{f,1}$ and $U(1)_{f,2}$. In the superconformal index, this corresponds to the residues of $\mathcal{I}_{UV}$ at the simple poles $c_1 = t^{1/2}$ and $c_2 = t^{1/2}$ in the fugacities associated to $U(1)_{f,1}$ and $U(1)_{f,2}$ respectively. Turning on position dependent vacuum expectation values $B_1 = z^{r_1}$ and $B_2 = z^{r_2}$ corresponds to computing the residues at simple poles $c_1 = t^{1/2}q^{r_1}$ and $c_2 = t^{1/2}q^{r_2}$. The order in which the residues are computed is irrelevant and the result

$$G_{r_1} \cdot (G_{r_2} \cdot \mathcal{I}_{IR}) \; = \; G_{r_2} \cdot (G_{r_1} \cdot \mathcal{I}_{IR}) \; , \tag{4.1.1}$$

defines the (commutative) composition $G_{r_1} \circ G_{r_2}$. This construction again has an interpretation in terms of the curve $\mathcal{C}_{g,n}$ for theories of class $\mathcal{S}$, shown in Figure 4.1.

4.1.2 The algebra of surface defects

The operators G_r constructed above do not form a closed algebra under composition and addition. More precisely: except for $\mathfrak{su}(2)$ the composition $G_{r_1} \circ G_{r_2}$ cannot be written as a sum of other operators G_{r_3} with coefficients that are independent of the flavor fugacities

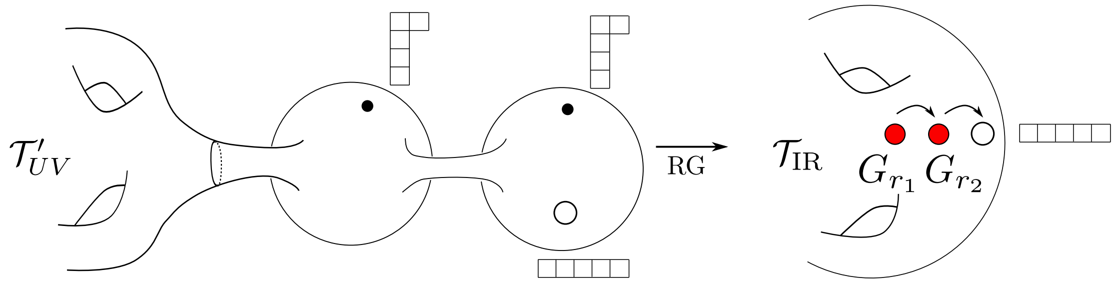


Figure 4.1: The left picture illustrates the Riemann surface $\mathcal{C}_{g,n}$ corresponding to the theory $\mathcal{T}'_{UV}$, which is obtained by coupling the theory $\mathcal{T}_{IR}$ to two bifundamental fields. An RG flow, that is initiated by turning on Higgs vevs for both bifundamental scalars, relates the theory $\mathcal{T}'_{\mathrm{UV}}$ to the original theory $\mathcal{T}_{\mathrm{IR}}$ with two surface defects G_{r_1} and G_{r_2}. This is illustrated on the right.

$\{a_j\}$ acted on by the operators.

In order to close the algebra, we need to enlarge the set of difference operators G_r. Having identified the label r with the r-th symmetric tensor representation of $\mathfrak{su}(N)$, it is natural to introduce operators G_R for any irreducible representation R of $\mathfrak{su}(N)$ and to force them to obey the algebra

$$G_{R_1} \circ G_{R_2} \;=\; \sum_{R_3} \mathcal{N}_{R_1,R_2}{}^{R_3}\; G_{R_3}\,, \tag{4.1.2}$$

where the coefficients $\mathcal{N}_{R_1,R_2}{}^{R_3}$ are non-zero only when the representation R_3 appears in the direct sum decomposition of the tensor product $R_1 \otimes R_2$. Indeed, it turns out that this determines the operators G_R and the algebra coefficients $\mathcal{N}_{R_1,R_2}{}^{R_3}$ (essentially) uniquely, in a sense we explain in detail below. The closure of the algebra is a highly non-trivial statement, since it is depending on intricate theta-function identities.

Let us explain the procedure is some more detail. For each irreducible representation R of $\mathfrak{su}(N)$, we make an ansatz for the operator G_R. The ansatz is a sum over the weights λ of the representation R,

$$G_R \cdot \mathcal{I}(a_i) \;=\; \sum_{\lambda} C_{R,\lambda}(p,q,t,a_j)\, \mathcal{I}(q^{-(\lambda,h_i)} a_i)\,, \tag{4.1.3}$$

for some unknown functions $C_{R,\lambda}(p,q,t,a_j)$. Here, the bracket $(\,,\,)$ denotes the standard inner product on the Cartan subalgebra normalized such that $(e_i, e_i) = 2$ for all simple roots. Furthermore, h_i are the weights of the fundamental representation, which obey $(h_i, h_j) = \delta_{i,j} - 1/N$.

The weights of an irreducible representation R of $\mathfrak{su}(N)$ can be represented by semi-standard Young tableaux that are obtained by placing a number $1, \ldots, N$ in each box of the Young diagram (as we review in Appendix A). Each weight can be written as a sum

$$\lambda \;=\; \sum_{j=1}^{N} m_j\, h_j\,, \tag{4.1.4}$$

where m_j are the filling numbers of the corresponding semi-standard Young tableau. In particular, the weights of the r-th symmetric tensor representation are given by

$$\lambda \;=\; \sum_{j=1}^{N} m_j\, h_j\,, \tag{4.1.5}$$

where the numbers m_i are such that $\sum_j m_j = r$. Since $(\lambda, h_i) = m_i - \frac{r}{N}$, the chosen ansatz is compatible with the symmetric tensor operators G_r.

Now we substitute the coefficients $C_{R,\lambda}(p,q,t,a_j)$ for the symmetric tensor operators, as well as our ansatz for the remaining representations, into the algebra relations

$$G_{R_1} \circ G_{R_2} \;=\; \sum_{R_3} \mathcal{N}_{R_1,R_2}{}^{R_3}\; G_{R_3}\,. \tag{4.1.6}$$

We first solve these relations for the $\mathfrak{su}(2)$ coefficients $\mathcal{N}_{r_1,r_2}{}^{r_3}(p,q,t)$, and propose a generalization for the $\mathfrak{su}(N)$ coefficients $\mathcal{N}_{R_1,R_2}{}^{R_3}(p,q,t)$. We then find that the remaining coefficients $C_{R,\lambda}(p,q,t,a_j)$ are determined uniquely. The fact that this procedure works requires intricate theta-function identities, providing a strong self-consistency check of our ansatz.

As a preliminary step, we introduce a normalization of the operators G_r labeled by r-th symmetric tensor representations. We redefine the operators by multiplying them by the factor

$$\mathcal{N}_r \;=\; t^{-r(N-1)/2} \prod_{i=0}^{r-1} \frac{\theta(q^{-1-i},p)}{\theta(tq^i,p)}\,. \tag{4.1.7}$$

The purpose of the normalization is to render the leading algebra coefficient[1] equal to one.

Rank 1

A good starting point is $\mathfrak{su}(2)$, since its irreducible representations are exhausted by r-fold symmetric products of the fundamental representation. Thus, the algebra of difference operators should close without introducing any new operators. In particular, we expect that the product $G_{r_1} \circ G_{r_2}$ can be decomposed according to the tensor product of the corresponding irreducible representations

$$G_{r_1} \circ G_{r_2} \;=\; \sum_{r=|r_1-r_2|}^{r_1+r_2} \mathcal{N}_{r_1,r_2}{}^{r_3}\, G_{r_3}\,, \tag{4.1.8}$$

where we can compute the OPE coefficients $\mathcal{N}_{r_1,r_2}{}^{r_3}(p,q,t)$. Consistency of this structure demands that the coefficients $\mathcal{N}_{r_1,r_2}{}^{r_3}$ constructed in this way are independent of the fugacity parameter a_i.

For simplicity, let us first consider the Macdonald limit $p \to 0$. In this limit, the ratios of theta-functions in the operators are replaced by rational functions of the remaining variables q and t. The operators G_r become

$$G_r \cdot \mathcal{I}(a_i) \;=\; \mathcal{N}_r \sum_{m_1+m_2=r} \prod_{i,j=1}^{2} \left[\prod_{m=0}^{m_j-1} \frac{\left(1-q^{m+m_j-m_i}\frac{ta_i}{a_j}\right)}{\left(1-q^{m-m_j}\frac{a_j}{a_i}\right)} \right] \mathcal{I}\left(a_i \mapsto q^{\frac{r}{N}-m_i}\, a_i\right), \tag{4.1.9}$$

where $a_1 = a$ and $a_2 = a^{-1}$.

When composing any two such rational operators G_{r_1} and G_{r_2}, we indeed find that the product $G_{r_1} \circ G_{r_2}$ decomposes according to the tensor product of the corresponding irreducible representations, in such a way that the structure constants $\mathcal{N}_{r_1,r_2}{}^{r_3}(q,t)$ are

[1] Here we mean leading in the sense of the partial ordering $|R_1| < |R_2| \iff \mathrm{dim}R_1 < \mathrm{dim}R_2$.

rational functions of q and t.

As mentioned above, we have normalized the difference operators such that the structure constant for the leading OPE coefficient ${\mathcal{N}_{r_1,r_2}}^{r_1+r_2} = 1$. The remaining structure constants can be computed straightforwardly in each case. For example, $G_1 \circ G_1 = G_2 + {\mathcal{N}_{1,1}}^0\, G_0$, where

$$ {\mathcal{N}_{1,1}}^0(q,t) \;=\; \frac{(1+t)(1-q)}{(1-qt)}\,. \tag{4.1.10} $$

This is a particular case of the more general decomposition

$$ G_1 \circ G_r \;=\; G_{r+1} + {\mathcal{N}_{1,r}}^{r-1}\, G_{r-1}, \tag{4.1.11} $$

where

$$ {\mathcal{N}_{1,r}}^{r-1}(q,t) \;=\; \frac{(1-t^2q^{r-1})(1-q^r)}{(1-tq^{r-1})(1-tq^r)}\,. \tag{4.1.12} $$

Similar formulae can be derived for any other example.

Remarkably, we observe that the structure constants ${\mathcal{N}_{r_1,r_2}}^r(q,t)$ are equal to the (q,t)-deformed Littlewood-Richardson coefficients. In other words, the operators G_r in the limit $p \to 0$ obey the same algebra as the Macdonald polynomials $P_r(a,q,t)$ for $\mathfrak{su}(2)$.[2]

It turns out that the structure constants of the general elliptic operator algebra can be obtained in a canonical way by *lifting* the structure constants ${\mathcal{N}_{r_1,r_2}}^{r_3}(q,t)$ of the Macdonald algebra. This works as follows.

(i) First we express the (q,t)-deformed Littlewood-Richardson coefficients as rational functions consisting of factors of the form $(1-x)$, where x is a monomial of the form $q^\alpha t^\beta$.

(ii) We then *lift* each factor to an elliptic function $\theta(x,p)$ whose second argument is the additional parameter p. The original coefficients are obtained in the limit $p \to 0$.

Note that apparent ambiguities in writing the (q,t)-deformed Littlewood-Richardson coefficients as rational functions of the form $(1-x)$ lead to a well-defined elliptic lift, because

[2]We refer to Appendix A for more details regarding Macdonald polynomials and (q,t)-deformed Littlewood-Richardson coefficients.

of the theta-function identity

$$\theta(z^{-1};p) \;=\; -\frac{1}{z}\,\theta(z;p)\,. \tag{4.1.13}$$

For example in

$${\mathcal{N}_{1,1}}^{0}(q,t) \;=\; \frac{(1-t^2)(1-q)}{(1-t)(1-qt)} \;=\; \frac{(1-\frac{1}{t^2})(1-\frac{1}{q})}{(1-\frac{1}{t})(1-\frac{1}{qt})}\,, \tag{4.1.14}$$

the elliptic lift

$${\mathcal{N}_{1,1}}^{0}(p,q,t) \;=\; \frac{\theta(t^2,p)\,\theta(q,p)}{\theta(t,p)\,\theta(qt,p)}\,, \tag{4.1.15}$$

is uniquely defined by the above procedure due to the relation (4.1.13).

Verifying the composition rules for the elliptic difference operators G_r now requires numerous theta-function identities. For instance, checking that $G_1 \circ G_1 = G_2 + {\mathcal{N}_{1,1}}^{0}(p,q,t)\,G_0$ requires

$$\begin{aligned}\frac{\theta(t^2,p)\theta(q,p)}{\theta(t,p)\theta(qt,p)} \;&=\; +\frac{\theta\left(q^{-2},p\right)\theta\left(t^{-1},p\right)\theta\left(ta^{-2},p\right)\theta\left(ta^{2},p\right)}{\theta\left(q^{-1},p\right)\theta\left(q^{-1}a^{-2},p\right)\theta\left(q^{-1}a^{2},p\right)\theta(qt,p)} \\ &\quad -\frac{\theta\left(t^{-1},p\right)\theta\left(ta^{-2},p\right)\theta\left(tq^{-1}a^{2},p\right)}{\theta\left(a^{-2},p\right)\theta(t,p)\theta\left(q^{-1}a^{2},p\right)} - \frac{\theta\left(t^{-1},p\right)\theta\left(ta^{2},p\right)\theta\left(tq^{-1}a^{-2},p\right)}{\theta\left(a^{2},p\right)\theta(t,p)\theta\left(q^{-1}a^{-2},p\right)}\,,\end{aligned} \tag{4.1.16}$$

which can be checked for instance by expanding around $p=0$.

Similarly, when composing the fundamental operator G_1 with the operator G_r for any other irreducible representation of $\mathfrak{su}(2)$, we find that another elliptic theta-function identity brings the non-trivial structure constant into the form

$${\mathcal{N}_{1,r}}^{r-1}(p,q,t) \;=\; \frac{\theta(t^2q^{r-1},p)\,\theta(q^r,p)}{\theta(tq^{r-1},p)\,\theta(tq^r,p)}\,. \tag{4.1.17}$$

In fact, for any other check we did, we find that the structure constants ${\mathcal{N}_{r_1,r_2}}^{r_3}$ are independent of the fugacity parameter a and can be expressed as ratios of theta-functions. Even better, we find that they are elliptic (lifts of (q,t)-deformed) Littlewood-Richardson coefficients, in the sense explained above.

The elliptic operators G_r thus obey an elliptic version of the Macdonald polynomial algebra. In particular, this provides evidence for the conjecture that the surface defects labeled by $r \in \mathbb{Z}_{\geq 0}$ are to be identified with irreducible representations of $\mathfrak{su}(2)$.

Higher rank

For $\mathfrak{su}(N)$, with $N > 2$, the algebra of the difference operators G_r is not closed. We introduce a new set of operators G_R labeled by irreducible representations of $\mathfrak{su}(N)$. By doing so we identify the difference operators G_r with the operators $G_{(r)}$ labeled by the rank r symmetric tensor representation.[3] We systematically find expressions for the novel operators by imposing the algebra relation

$$G_{R_1} \circ G_{R_2} \;=\; \sum_{R_3} \mathcal{N}_{R_1,R_2}{}^{R_3}\, G_{R_3}\,, \tag{4.1.18}$$

where we assume that the coefficients $\mathcal{N}_{R_1,R_2}{}^{R_3}(p,q,t)$ are given by the elliptic (lifts of (q,t)-refined) Littlewood-Richardson coefficients, which can be found uniquely for any triple of representations R_1, R_2 and R_3.

In the rank 2 and 3 cases, we have explicitly computed a large set of elliptic difference operators G_R, and performed ample consistency checks amongst them. These computations reveal several structures amongst the difference operators, and we are to make some proposals for general N. Let us give a few examples here.

First, consider the composition of two operators each labeled by the fundamental representation, $G_{(1)} \circ G_{(1)}$. This representation $(1) \otimes (1)$ decomposes into the symmetric tensor (2) and the antisymmetric tensor $(1,1)$ representations. The coefficient of the operator $G_{(2)}$ labeled by the symmetric tensor representation is one, following from our choice of normalization. Choose the coefficient

$$\mathcal{N}_{(1),(1)}{}^{(1,1)}(p,q,t) \;=\; \frac{\theta(q,p)\,\theta(t^2,p)}{\theta(t,p)\,\theta(qt,p)} \tag{4.1.19}$$

[3]We label by $(\ell_1,\ldots,\ell_{N-1})$, with $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_{N-1} \geq 0$ the representation associated to the Young diagram whose j-th row has length ℓ_j.

to be the uplift of the corresponding (q,t)-deformed Littlewood-Richardson coefficient. The difference operator $G_{(1,1)}$ labeled by the rank-two antisymmetric tensor representation of $\mathfrak{su}(N)$ can then be determined from the equation

$$G_{(1)} \circ G_{(1)} \;=\; G_{(2)} + \frac{\theta(q,p)\,\theta(t^2,p)}{\theta(t,p)\,\theta(qt,p)}\, G_{(1,1)}\,. \tag{4.1.20}$$

By this method, we find that the elliptic difference operator $G_{(1,1)}$ for the antisymmetric tensor representation is given by

$$G_{(1,1)} \cdot \mathcal{I}(a_i) \;=\; t^{-1} \sum_{j_1<j_2} \prod_{k\neq\{j_1,j_2\}} \frac{\theta\left(\frac{t}{q}a_{j_1}/a_k,p\right)\theta\left(\frac{t}{q}a_{j_2}/a_k,p\right)}{\theta\left(a_k/a_{j_1},p\right)\theta\left(a_k/a_{j_2},p\right)}\, \mathcal{I}\left(q^{\frac{2}{N}-\delta_{i,\{j_1,j_2\}}}a_i\right). \tag{4.1.21}$$

The term in the sum labeled by $j_1 < j_2$ corresponds to the weight $\lambda = h_{j_1} + h_{j_2}$ in the antisymmetric tensor representation $(1,1)$.

Next, we determine the difference operator $G_{(2,1)}$ from the equation

$$\left(G_{(2)} \circ G_{(1)}\right) \cdot \mathcal{I} \;=\; G_{(3)} \cdot \mathcal{I} + \frac{\theta(q^2,p)\,\theta(qt^2,p)}{\theta(qt,p)\,\theta(q^2t,p)}\, G_{(2,1)} \cdot \mathcal{I}\,, \tag{4.1.22}$$

where

$$\mathcal{N}_{(2),(1)}{}^{(2,1)}(p,q,t) \;=\; \frac{\theta(q^2,p)\,\theta(qt^2,p)}{\theta(qt,p)\,\theta(q^2t,p)} \tag{4.1.23}$$

is the elliptic lift of the (q,t)-deformed Littlewood-Richardson coefficient $\mathcal{N}_{(2),(1)}{}^{(2,1)}(q,t)$.

We verify that the difference operator $G_{(2,1)}$ can indeed be written as a sum over the weights $\lambda = \sum_i m_i h_i$ with $\sum_i m_i = 3$, *i.e.* as a sum over the weights in the representation labeled by the Young diagram $(2,1)$. These weights can be divided into two groups. The weights $\{m_{i_1} = m_{i_2} = m_{i_3} = 1\}$ occur with multiplicity two, whereas the weights $\{m_{j_1} = 2,\, m_{j_2} = 1\}$ occur with multiplicity one.

We then expand the resulting operator to lowest order in p, read off its elliptic lift and check this in an expansion in p. For instance, for $\mathfrak{su}(3)$ we find that

$$\begin{aligned} G_{(2,1)} \cdot \mathcal{I}(a_1,a_2,a_3) &= \sum_{\sigma\in S_3} C_{210}(a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)})\,\mathcal{I}\left(\frac{a_{\sigma(1)}}{q},a_{\sigma(2)},qa_{\sigma(3)}\right) \\ &\quad + C_{111}(a_1,a_2,a_3)\,\mathcal{I}\big(a_1,a_2,a_3\big)\,. \end{aligned} \tag{4.1.24}$$

The first group of terms in this sum correspond to weights $\lambda = 2h_{\sigma(1)}+h_{\sigma(2)}$ that occur with multiplicity one. These terms are given by a single product over ratios of theta-functions:

$$C_{210}(a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}) = t^{-2}\,\frac{\theta\left(\frac{ta_{\sigma(1)}}{qa_{\sigma(2)}},p\right)\theta\left(\frac{ta_{\sigma(1)}}{qa_{\sigma(3)}},p\right)\theta\left(\frac{ta_{\sigma(2)}}{qa_{\sigma(3)}},p\right)\theta\left(\frac{ta_{\sigma(1)}}{q^2a_{\sigma(3)}},p\right)}{\theta\left(\frac{a_{\sigma(2)}}{a_{\sigma(1)}},p\right)\theta\left(\frac{a_{\sigma(3)}}{a_{\sigma(1)}},p\right)\theta\left(\frac{a_{\sigma(3)}}{a_{\sigma(2)}},p\right)\theta\left(\frac{qa_{\sigma(3)}}{a_{\sigma(1)}},p\right)}\,. \tag{4.1.25}$$

The last term in (4.1.24) corresponds to the weight $\lambda = h_1+h_2+h_3$, which occurs with multiplicity two. Its contribution is given by

$$\begin{aligned} C_{111}(a_1,a_2,a_3) &= -t^{-3}\,\frac{\theta\left(t,p\right)\theta\left(q^2t,p\right)}{\theta\left(q^{-1},p\right)\theta\left(qt^2,p\right)}\times \\ \times\Bigg(&\sum_{\sigma\in S_3}\frac{\theta\left(\frac{ta_{\sigma(1)}}{a_{\sigma(2)}},p\right)\theta\left(\frac{ta_{\sigma(2)}}{a_{\sigma(1)}},p\right)\theta\left(\frac{ta_{\sigma(1)}}{a_{\sigma(3)}},p\right)\theta\left(\frac{ta_{\sigma(2)}}{a_{\sigma(3)}},p\right)\theta\left(\frac{ta_{\sigma(3)}}{qa_{\sigma(1)}},p\right)\theta\left(\frac{ta_{\sigma(3)}}{qa_{\sigma(2)}},p\right)}{\theta\left(\frac{a_{\sigma(1)}}{qa_{\sigma(2)}},p\right)\theta\left(\frac{a_{\sigma(2)}}{qa_{\sigma(1)}},p\right)\theta\left(\frac{a_{\sigma(3)}}{a_{\sigma(1)}},p\right)\theta\left(\frac{qa_{\sigma(1)}}{a_{\sigma(3)}},p\right)\theta\left(\frac{a_{\sigma(3)}}{a_{\sigma(2)}},p\right)\theta\left(\frac{qa_{\sigma(2)}}{a_{\sigma(3)}},p\right)} \\ &+\frac{\theta(\frac{1}{t},p)^2\,\theta(q^3,p)}{\theta(q,p)^2\,\theta(\frac{1}{qt^2},p)}\,\frac{\theta(\frac{ta_1}{a_2},p)\,\theta(\frac{ta_2}{a_1},p)\,\theta(\frac{ta_1}{a_3},p)\,\theta(\frac{ta_2}{a_3},p)\,\theta(\frac{ta_3}{a_1},p)\,\theta(\frac{ta_3}{a_2},p)}{\theta(\frac{a_1}{qa_2},p)\,\theta(\frac{a_2}{qa_1},p)\,\theta(\frac{qa_3}{a_1},p)\,\theta(\frac{qa_1}{a_3},p)\,\theta(\frac{qa_3}{a_2},p)\,\theta(\frac{qa_2}{a_3},p)}\Bigg)\,. \end{aligned}$$

Continuing this strategy, one can systematically find the elliptic difference operators for any given representation R of $\mathfrak{su}(N)$ and perform consistency checks on it. We have explicitly computed all $\mathfrak{su}(3)$ and $\mathfrak{su}(4)$ difference operators labeled by Young diagrams with up to four boxes. From these results we infer that the difference operator $G_{(1^r)}$, corresponding to the rank r antisymmetric representation of $\mathfrak{su}(N)$, is given by

$$G_{(1^r)}\cdot\mathcal{I}(a_i) = t^{r(r-N)/2}\sum_{|I|=r}\prod_{\substack{j\in I\\ k\notin I}}\frac{\theta(\frac{t}{q}a_j/a_k,p)}{\theta(a_k/a_j,p)}\,\mathcal{I}\left(q^{\frac{r}{N}-\delta_{i,I}}a_i\right)\,, \tag{4.1.26}$$

where the summation is over subsets $I\subset\{1,\ldots,N\}$ of length $|I|=r$ and where the symbol $\delta_{i,I}$ is one if $i \in I$ and zero if $i \notin I$. As we will show in more detail in the next section, these operators are related by conjugation to the Hamiltonians of the elliptic Ruijsenaars-Schneider model.

4.1.3 Properties of difference operators

Let us summarize a few properties of the resulting difference operators G_R:

(i) They are formally self-adjoint with respect to the vectormultiplet measure $\Delta_{p,q,t}(a)\,\mathcal{I}_V(a)$ on the unit circle $|a| = 1$.

(ii) The composition $G_{R_1} \circ G_{R_2}$ is commutative.

(iii) The difference operators G_R obey the algebra

$$G_{R_1} \circ G_{R_2} \;=\; \sum_{R_3} \mathcal{N}_{R_1,R_2}{}^{R_3}\, G_{R_3}\,, \tag{4.1.27}$$

where the coefficients $\mathcal{N}_{R_1,R_2}{}^{R_3}$ are elliptic lifts of the (q,t)-deformed Littlewood-Richardson coefficients.

(iv) They can be expanded as

$$G_R \cdot \mathcal{I}(a_i) \;=\; \sum_{\lambda} C_{R,\lambda}(p,q,t,a_j)\,\mathcal{I}(q^{-(\lambda,h_i)}a_i)\,,$$

where the summation is over weights λ in the representation R.

4.1.4 The Schur limit

We have not found a closed expression for the coefficients $C_{R,\lambda}(p,q,t,a_j)$ in general. However, we note that the elliptic lift of the (q,t)-deformed Littlewood-Richardson coefficients $\mathcal{N}_{R_1,R_2}{}^{R_3}(p,q,t)$ have the same number of terms in the numerator and denominator. In the Schur limit (*i.e.* $q = t$) these terms all cancel each other and the elliptic algebra reduces to the Schur algebra. By inspecting many examples we can conjecture a closed formula for the surface operators for any irreducible representation of $\mathfrak{su}(N)$ in this limit

$$G_R \cdot \mathcal{I}(a_i) \; = \; \sum_{\lambda} \left[\prod_{i=1}^{N} a_i^{N n_i} \right] q^{-\frac{N-1}{2}\sum_i n_i + \sum_{i<j} n_i n_j} \, \mathcal{I}\left(q^{|R|/N - n_i}\, a_i\right) . \qquad (4.1.28)$$

Here the $n_i = (h_i, \lambda) + \frac{|R|}{N}$ denote the filling numbers of the Young tableau of the state λ and $|R| = \sum_i \ell_i$ is the number of boxes in the Young diagram.

4.2 Two-dimensional TQFT and Verlinde algebra

In this section we identify the difference operators G_R with local operators in a topological quantum field theory on the Riemann surface $\mathcal{C}_{g,n}$. In the case $p = 0$, this can be identified with an analytic continuation of refined Chern-Simons theory on $S^1 \times \mathcal{C}_{g,n}$ and the relevant local operators arise from Wilson loops in the representation R wrapping the S^1.

4.2.1 TQFT structure

Let us start by defining conjugated difference operators by $\bar{G}_R = \frac{1}{K(a)}(G_R \cdot K(a))$, where $K(a) = \prod_{i \neq j}^{N} \Gamma(t a_i/a_j, p, q)$ as given in equation (2.5.2) – see section 2.5. We have seen there that this operation simply reduces to replacing $t \to pq/t$. The conjugated surface operators for fully antisymmetric representations $R = (1^r)$ acting on the normalized index $\mathcal{I}^{(n)}$ are then

$$\bar{G}_{(1^r)} \cdot \mathcal{I}^{(n)}(a_i) \; = \; t^{r(r-N)/2} \sum_{|I|=r} \prod_{\substack{k \in I \\ j \notin I}} \frac{\theta\left(t a_j/a_k, p\right)}{\theta\left(a_j/a_k, p\right)} \mathcal{I}^{(n)}(q^{\frac{r}{N} - \delta_{i,I}} a_i) \, , \qquad (4.2.1)$$

where the summation is over all subsets $I \subset \{1, 2, \dots, N\}$ of length r. Remarkably, the conjugated antisymmetric tensor operators $\bar{G}_{(1^r)}$ are precisely the Hamiltonians of the elliptic Ruijsenaars-Schneider model, extending the observation made in [30].

In the remainder of this section we will restrict ourselves to the Macdonald slice

$(p,q,t) = (0,q,t)$. In this limit, the antisymmetric difference operators $\bar{G}_{(1^r)}$ turn into the Macdonald operators

$$\bar{G}_{(1^r)} \cdot \mathcal{I}^{(n)}(a_i) \;=\; t^{r(r-N)/2} \sum_{|I|=r} \prod_{\substack{k\in I\\ j\notin I}} \frac{(qa_j/a_k;p)}{(a_j/a_k;p)} \mathcal{I}^{(n)}(q^{\frac{r}{N}-\delta_{i,I}}a_i)\,, \tag{4.2.2}$$

and the normalized vectormultiplet measure becomes the Macdonald measure $\Delta^{(n)}_{p,q}(a)$. The operators $\bar{G}_{(1^r)}$ are self-adjoint with respect to this measure on the unit circle $|a|=1$ and their common eigenfunctions are the Macdonald polynomials $P_S(a_i;q,t)$, which are labeled by irreducible representations S of $\mathfrak{su}(N)$. They are by construction orthogonal with respect to the measure $\Delta^{(n)}_{p,q}(a)$ and are normalized such that

$$P_S(a_i;q,t) \;=\; \chi_S(a_i) + \sum_{T<S} c_{S,T}(q,t)\, \chi_T(a_i)\,. \tag{4.2.3}$$

In this equation, the $c_{S,T}$ are rational functions of q and t that are uniquely fixed by ensuring that $P_S(a_i;q,t)$ is an eigenfunction of the Macdonald operators $\bar{G}_{(1^r)}$ for $r=1,\ldots,N-1$. In this limit the structure constants C_S in the factorization (2.5.6) are simply given by

$$C_S \;=\; \frac{1}{S_{0,S}}\,, \tag{4.2.4}$$

where $S_{R,S}$ is an analytic continuation of the modular S-matrix of refined Chern-Simons theory.

4.2.2 q-Deformed Yang-Mills: Defect punctures

Let us first discuss the case of the Schur limit. The corresponding TQFT is then q-deformed Yang-Mills in the zero-area limit. The partition function can be enriched by the insertion of gauge invariant operators constructed from the scalar

$$\Omega_R \;\equiv\; \chi_R(e^{\mathrm{i}\phi})\,, \tag{4.2.5}$$

whose correlators are independent of their position on $\mathcal{C}_{g,n}$ [40]. In Chern-Simons theory on $S^1 \times \mathcal{C}_{g,n}$, an insertion of the operator Ω_R on $\mathcal{C}_{g,n}$ corresponds to inserting a Wilson loop around the S^1 in the representation R. In the following, we will refer to insertions of such operators as *defect punctures* to distinguish them from normal flavor punctures.

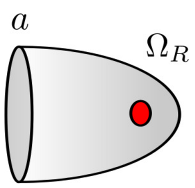
To compute correlation functions with defect punctures, we return to the cylinder amplitude with holonomy eigenvalues a and b respectively – see Figure 2.4. In the Chern-Simons theory on $S^1 \times \mathcal{C}_{g,n}$, the boundary with holonomy eigenvalues b becomes a boundary torus $S^1 \times \tilde{S}^1$. The first step is to interchange those circles using the modular S matrix

$$\sum_S \chi_S(a)\,\chi_S(b) \quad \xrightarrow{\tilde{S}^1 \leftrightarrow S^1} \quad \sum_{S,S'} \chi_S(a)\,S_{S,S'}\,\chi_{S'}(b)\,, \tag{4.2.6}$$

so that we are now fixing the holonomy eigenvalues b on S^1. To insert a Wilson line in the representation R, we multiply by $\chi_R(b)$ and integrate over the holonomy b. From the orthonormality of Schur polynomials with respect to the Haar measure we find

$$\langle \Omega_R \rangle_{0,1} \;=\; \sum_S \chi_S(a)\,S_{S,R}\,. \tag{4.2.7}$$

This is the amplitude for a disk with holonomy eigenvalues a and a defect puncture labeled by the irreducible representation R – see Figure 4.2.



$$= \sum_S \chi_S(a) S_{S,R}$$

Figure 4.2: Disk amplitude in the presence of a defect puncture.

Any amplitude with defect punctures can now be calculated by gluing the above amplitude to fixed holonomy boundaries. For example, starting from the sphere with three boundaries we can obtain the amplitude for three defect punctures in representations R_1, R_2

and R_3

$$\langle \Omega_{R_1}\Omega_{R_2}\Omega_{R_3}\rangle_{\mathcal{C}_{0,0}} \;=\; \sum_S \frac{S_{S,R_1}\,S_{S,R_2}\,S_{S,R_3}}{S_{S,0}} \;=\; N_{R_1,R_2}{}^{\bar{R}_3}\,, \tag{4.2.8}$$

where the numbers $N_{R_1,R_2}{}^{R_3}$ are simply the Littlewood-Richardson coefficients for $SU(N)$. This is an analytic continuation of the Verlinde formula.

In full generality, for the amplitude of a Riemann surface of genus g, with n boundaries with fixed holonomies $(a_1,\ldots,a_n)$ and defect punctures in representations $(R_1,\ldots,R_l)$ we have

$$\langle \Omega_{R_1}\ldots\Omega_{R_l}\rangle_{\mathcal{C}_{g,n}} \;=\; \sum_S S_{S,0}^{2-2g-n}\prod_{i=1}^{n}\chi_S(a_i)\prod_{j=1}^{l}\frac{S_{S,R_j}}{S_{S,0}}\,. \tag{4.2.9}$$

Therefore, adding a defect puncture labeled by the irreducible representation R inserts a factor $S_{S,R}/S_{S,0}$ into the sum over representations. From the formula

$$S_{S,R_1}\,S_{S,R_2} \;=\; S_{S,0}\sum_R \mathcal{N}_{R_1,R_2}{}^{R}\,S_{S,R} \tag{4.2.10}$$

we derive the operator product expansion

$$\Omega_{R_1}\cdot\Omega_{R_2} \;=\; \sum_R \mathcal{N}_{R_1R_2}{}^{R}\,\Omega_R \tag{4.2.11}$$

inside the correlation function. This is (an analytic continuation of) the representation of the Verlinde algebra for Chern-Simons theory on $S^1\times\mathcal{C}_{g,n}$.

Now we are ready to spell out the precise dictionary in the Schur. First recall that the rescaled index without surface defects is given by the expression (2.3.11) involving a sum over irreducible representations S. When acting with the transformed difference operator $\bar{G}_R$ on a flavor puncture, each term in the sum picks up a factor of $S_{S,R}/S_{S,0}$. This is equivalent to the insertion of a defect puncture Ω_R in the q-deformed YM correlator, see (4.2.9). Hence, surface defects in the four-dimensional theory correspond to defect punctures in q-deformed YM, see Figure 4.3. Finally, let us mention that the OPE expansion (4.2.11) guarantees consistency if we add several defect punctures.

$$\mathcal{I}^{(n)}(a,b,c,\ldots) \;=$$

$$\bar{G}_R \cdot \mathcal{I}^{(n)}(a,b,c,\ldots) \;=$$

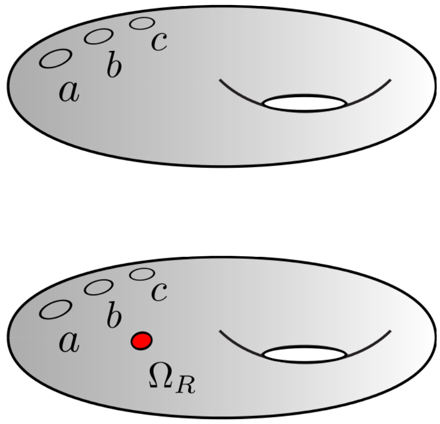


Figure 4.3: The index without surface defects corresponds to a correlator of q-YM. Inserting a surface defect/acting with a difference operator in the $4d$ theory corresponds to inserting a defect puncture on the $2d$ side.

4.2.3 (q,t)-Deformed Yang-Mills

Let us now have a brief look at the more general Macdonald limit. The discussion goes exactly through by replacing the Schur with the more general Macdonald polynomials. In this limit the dual two-dimensional TQFT is a (q,t)-deformation of Yang-Mills theory in the zero-area limit. Equivalently this can be interpreted as the analytic continuation of refined Chern-Simons theory on $S^1 \times \mathcal{C}_{g,n}$. Hence a surface operator insertion into the superconformal index corresponds to inserting a defect puncture into the $2d$ theory. This is given by the insertion the analytically continued modular S-matrix.

We now briefly outline the reasoning of why we obtain (q,t)-deformed Littlewood Richardson coefficients in the operator algebra,

$$\bar{G}_{R_1} \circ \bar{G}_{R_2} \;=\; \sum_R \mathcal{N}_{R_1,R_2}{}^R \, \bar{G}_R \,. \qquad (4.2.12)$$

Operator algebra from Macdonald polynomials

The Macdonald polynomials obey

$$P_{S_1} \cdot P_{S_2} \;=\; \sum_{S} \mathcal{N}_{S_1,S_2}{}^{S_3} P_{S_3}\,, \tag{4.2.13}$$

where $\mathcal{N}_{S_1,S_2}{}^{S_3}$ are the (q,t)-deformed Littlewood-Richardson coefficients. Remarkably, we have found that the conjugated difference operators $\bar{G}_R$ obey the same algebra. Let us try to understand this fact.

Consider for instance the case of $SU(2)$. The eigenvalues of the difference operators $\bar{G}_r$ can be computed from explicit formulae that we have found. They are given by

$$\bar{G}_{r_1} \cdot P_{r_2}(a) \;=\; \frac{S_{r_1,r_2}}{S_{0,r_2}}\, P_{r_2}\,, \tag{4.2.14}$$

where S_{r_1,r_2} is an analytic continuation of the modular S-matrix of refined Chern-Simons theory (see Appendix A for the construction of this quantity). This S-matrix is known to obey the (q,t)-deformed Verlinde formula

$$S_{r_1,s} \cdot S_{r_2,s} \;=\; S_{0,s} \sum_{r} \mathcal{N}_{r_1,r_2}{}^{r} S_{r,s}\,. \tag{4.2.15}$$

Let us now act with the composition of the operators $\bar{G}_{r_1}$ and $\bar{G}_{r_2}$ on the Macdonald polynomial P_s and apply the refined Verlinde formula

$$(\bar{G}_{r_1} \circ \bar{G}_{r_2}) \cdot P_s \;=\; \frac{S_{r_1,s}}{S_{0,s}} \frac{S_{r_2,s}}{S_{0,s}}\, P_s \;=\; \sum_{r} \mathcal{N}_{r_1,r_2}{}^{r} \frac{S_{r,s}}{S_{0,s}}\, P_s \;=\; \sum_{r} \mathcal{N}_{r_1,r_2}{}^{r} \bar{G}_r \cdot P_s\,. \tag{4.2.16}$$

Since the Macdonald polynomials form a complete basis of symmetric functions, we find that the structure constants of the difference operator algebra are indeed the (q,t)-deformed Littlewood-Richardson coefficients.

Similarly, we have verified that the generalized difference operators $\bar{G}_R$, labeled by irreducible representations R of $\mathfrak{su}(N)$, satisfy the eigenvalue equation

$$\bar{G}_{R_1} \cdot P_{R_2} \;=\; \frac{S_{R_1,R_2}}{S_{0,R_2}}\, P_{R_2} \tag{4.2.17}$$

in the Macdonald slice.

Chapter 5

Relation to three- and four-dimensional line defects

5.1 Algebra of three-dimensional line defects

In Chapter 4, we constructed the superconformal index of $\mathcal{N} = 2$ theories on $S^1 \times S^3$ in the presence of certain surface defects supported on $S^1 \times S^1$. These surface defects were labeled by an irreducible representation R of $\mathfrak{su}(N)$ and could be added to any superconformal theory with an $SU(N)$ flavor symmetry. In this section, we consider the reduction of the four-dimensional superconformal index to a partition function on a squashed three-sphere S^3, following [54–56]. In this limit, the surface defects become codimension-two defects in the three-dimensional theory wrapping an $S^1 \subset S^3$.

For four-dimensional theories of class $\mathcal{S}$, upon dimensionally reducing on S^1 the theory flows to an $\mathcal{N} = 4$ superconformal field theory in three-dimensions. Moreover, this has a mirror description in terms of a star-shaped quiver theory [57]. It is expected that the surface defects introduced by the difference operators G_R become supersymmetric Wilson loops in the representation R for the central node of this star-shaped quiver upon dimensional reduction. We demonstrate this explicitly for antisymmetric tensor representations $R = (1^r)$ and the case of a round three-sphere. For non-minuscule representations R, however, we find that the difference operators G_R introduce a linear combination of Wilson loops in irreducible representations S with $|S| \leq |R|$.

5.1.1 From the superconformal index to a $3d$ partition function

The four-dimensional superconformal index on $S^1 \times S^3$ can be reduced to a partition function on the squashed three-sphere S^3, as demonstrated in [54–56]. This limit is taken by parametrizing the fugacities by

$$p = e^{-\beta b^{-1}}, \qquad q = e^{-\beta b}, \qquad t = e^{-\beta \epsilon}, \qquad a_j = e^{-\mathrm{i}\beta x_j}, \tag{5.1.1}$$

with $\beta > 0$ and then taking the limit $\beta \to 0^+$. Here we have introduced the convenient notation $\epsilon = \frac{q}{2} + \mathrm{i}m$ where $q = b + b^{-1}$.

The real parameter $b > 0$ encodes the geometry of the three-sphere, defined by the embedding

$$b^{-2}|z|^2 + b^2|w|^2 = 1, \qquad (z, w) \in \mathbb{C}^2. \tag{5.1.2}$$

The parameters x_i with $\sum_{i=1}^N x_i = 0$ are real mass parameters for the global $SU(N)$ symmetry that is inherited by the three-dimensional theory. It is convenient to repackage the components x_j into a vector x such that $x_j = (x, h_j)$. In addition, the real parameter m gives a mass to the adjoint chiral multiplet inside the background $\mathcal{N} = 4$ vectormultiplet, breaking the supersymmetry to $\mathcal{N} = 2$ in three dimensions.

Let us consider two important examples. Firstly, the three-dimensional limit (5.1.1) of the superconformal index of a free hypermultiplet is given by

$$\mathcal{Z}_H(x) = S_b\left(\frac{\epsilon}{2} \pm \mathrm{i}x\right). \tag{5.1.3}$$

Secondly, the superconformal index of an $SU(N)$ vectormultiplet combined with the Haar measure becomes the partition function of a three-dimensional $\mathcal{N}=4$ vectormultiplet

$$\mathcal{Z}_V(x) \;=\; \prod_{i<j}^{N} 2\sin\left(\mathrm{i}\pi b^{\pm} x_{ij}\right)\,\mathcal{K}(x)\,, \tag{5.1.4}$$

where

$$\mathcal{K}(x) \;=\; \frac{1}{S_b(\epsilon^*)}\prod_{i,j=1}^{N} S_b(\epsilon^* + \mathrm{i}x_{ij})\,, \tag{5.1.5}$$

with $\epsilon^* = \frac{q}{2} - \mathrm{i}m$. Note that $\mathcal{K}(x)$ is the contribution from the $\mathcal{N}=2$ adjoint chiral multiplet inside the three-dimensional $\mathcal{N}=4$ vectormultiplet. We use the double sine function that obeys the difference equation $S_b(x+b^{\pm}) = 2\sin(\pi b^{\pm} x) S_b(x)$ and the reflection property $S_b(x)S_b(q-x) = 1$. Further properties of this special function can be found in Appendix C.

Let us now consider the three-dimensional limit of the difference operators G_R that introduce surface defects into the four-dimensional $\mathcal{N}=2$ theory. The three-dimensional limit can be evaluated using the fact that the ratio of theta-functions with a common second argument reduces to a ratio of sine-functions,

$$\frac{\theta\left(e^{\alpha\rho}, e^{\beta\rho}\right)}{\theta\left(e^{\gamma\rho}, e^{\beta\rho}\right)} \quad \xrightarrow{\rho\to 0} \quad \frac{\sin\left(\pi\alpha/\beta\right)}{\sin\left(\pi\gamma/\beta\right)}\,. \tag{5.1.6}$$

Let us from now on focus on the rank r antisymmetric tensor representation (1^r) of $\mathfrak{su}(N)$. In section 4.1 we found that up to a power of t the corresponding difference operator is

$$G_{(1^r)}\cdot\mathcal{I}(a_i) \;=\; \sum_{|I|=r}\prod_{\substack{j\in I\\ k\notin I}} \frac{\theta(\frac{t}{q}a_j/a_k, p)}{\theta(a_k/a_j, p)}\,\mathcal{I}\left(q^{\frac{r}{N}-\delta_{i,I}} a_i\right)\,, \tag{5.1.7}$$

where the summation is over subsets $I\subset\{1,\ldots,N\}$ of length $|I|=r$ and where the symbol $\delta_{i,I}$ is one if $i\in I$ and zero if $i\notin I$. In the three-dimensional limit, we obtain the operator

$$G^{(3\mathrm{d})}_{(1^r)}\cdot\mathcal{Z}(x) \;=\; \sum_{|I|=r}\prod_{\substack{j\in I\\ k\notin I}} \frac{\sin\pi b\left(\epsilon^* - \mathrm{i}x_{jk}\right)}{\sin\pi b\left(-\mathrm{i}x_{jk}\right)}\,\mathcal{Z}\left(x + \mathrm{i}b\sum_{j\in I} h_j\right)\,, \tag{5.1.8}$$

acting on the squashed three-sphere partition function $\mathcal{Z}(x)$. We use the shorthand $x_{jk} = x_j - x_k$ and remember that the weights obey $(h_i, h_j) = \delta_{ij} - 1/N$. Similar computations can be performed for the four-dimensional difference operators G_R corresponding to any irreducible representation R of $\mathfrak{su}(N)$.

The dimensionally reduced operators $G_R^{(3\mathrm{d})}$ have similar properties to their four-dimensional ancestors. We define an inner product in three dimension with respect to the three-dimensional $\mathcal{N} = 4$ vectormultiplet measure (5.1.4) on $\mathbb{R}^{N-1}$,

$$\langle f_1, f_2 \rangle \;=\; \int_{\mathbb{R}^{N-1}} \left[\frac{\mathrm{d}^{N-1}x}{N!} \prod_{i<j}^{N} 2\sin\left(\mathrm{i}\pi b^{\pm} x_{ij}\right) \mathcal{K}(x) \right] \cdot f_1(x) f_2(x)\,. \tag{5.1.9}$$

The adjoint operator $\left(G_R^{(3\mathrm{d})}\right)^\dagger$ of $G_R^{(3\mathrm{d})}$ is then given by $G_{R^*}^{(3\mathrm{d})}$, where R^* is the conjugate representation of R. Furthermore, the operators $G_R^{(3\mathrm{d})}$ generate a commutative algebra that can be derived by applying the limit (5.1.1) to the general structure constants $\mathcal{N}_{R_1,R_2}{}^{R_3}(p,q,t)$ that we found in the four-dimensional case.

Finally, in section 4.2.1 we pointed out that the four-dimensional difference operators with the replacement $t \to pq/t$ were related by a similarity transformation. In the three-dimensional limit, this corresponds to $m \to -m$ or quivalently $\epsilon \to \epsilon^*$. Thus, either by direct computation or by taking the three-dimensional limit of the the result in section 4.2.1 we find that

$$G_R^{(3d)} \;=\; \frac{1}{\mathcal{K}(x)} \cdot \bar{G}_R^{(3d)} \cdot \mathcal{K}(x)\,, \tag{5.1.10}$$

where the operator $\bar{G}_R^{(3d)}$ is related to $G_R^{(3d)}$ by the replacement $\epsilon \to \epsilon^*$ and $\mathcal{K}(x)$ is the partition function of an adjoint $\mathcal{N} = 2$ chiral multiplet inside an $\mathcal{N} = 4$ vectormultiplet. A consequence is that the eigenfunctions of the two sets of operators are proportional.

5.1.2 Wilson loops in three-dimensional star-shaped quivers

Since the dimensional reduction is performed along a circle on which the surface defect is supported, we expect that the difference operators (5.1.8) introduce defects in the three-dimensional theory supported on the circle $|z|^2 = b^2$. In the following we will perform indirect checks of this prediction by exploiting three-dimensional mirror symmetry to relate these defects to supersymmetric Wilson loops.

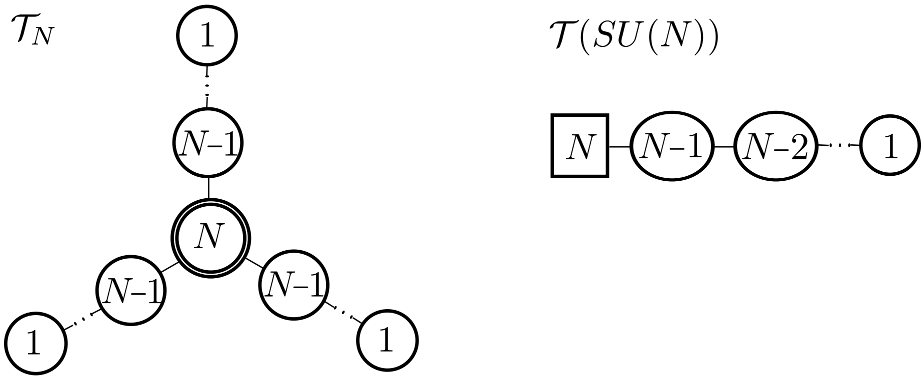


Figure 5.1: *Left*: Star-shaped quiver corresponding to the mirror of the three-dimensional $\mathcal{T}_N$ theory. *Right*: Linear quiver description of the three-dimensional $\mathcal{T}(SU(N))$ theory.

Upon dimensional reduction, a four-dimensional $\mathcal{N} = 2$ theory of class $\mathcal{S}$ flows to an $\mathcal{N} = 4$ superconformal field theory in three dimensions, which is related by mirror symmetry to a star-shaped quiver gauge theory [57]. For example, the mirror of the three-dimensional reduction of the $\mathcal{T}_N$ theory is given by a star-shaped quiver with three legs, shown in Figure 5.1. More generally, each full puncture in four dimensions gives rise to one copy of the linear quiver illustrated on the right in Figure 5.1. The corresponding field theory is known as the $\mathcal{T}(SU(N))$ theory [143, 144]. This theory contains $\mathcal{N} = 4$ vectormultiplets for the gauge groups $U(1), \ldots, U(N-1)$. These gauge groups are coupled linearly through $U(k) - U(k+1)$ bifundamental hypermultiplets. Lastly, there are N hypermultiplets in the fundamental representation of the largest gauge group $U(N-1)$.

The $\mathcal{T}(SU(N))$ theory has a manifest $SU(N)$ Higgs branch symmetry acting on the N hypermultiplets whose mass parameters are denoted by the vector x. In addition, we can introduce $N-1$ Fayet-Illiopoulos parameters $t_1, \ldots, t_{N-1}$, which are mass parameters for the topological $U(1)$ symmetries associated to each node of the quiver. Let us express these parameters in terms of a new vector y such that $t_k = y_k - y_{k+1}$ and let us denote the partition function of $\mathcal{T}(SU(N))$ by $\mathcal{Z}_\epsilon(x, y)$. This partition function is expected to be invariant under exchanging $x \leftrightarrow y$ and $\epsilon \leftrightarrow \epsilon^*$. This symmetry reflects an enhancement of the Coulomb branch symmetry to ${}^L SU(N)$ in the infrared, as well as the presence of mirror symmetry exchanging $SU(N) \leftrightarrow {}^L SU(N)$.

The partition function of the three-dimensional mass-deformed $\mathcal{T}_N$ theory is given by

$$\mathcal{Z}_{\mathcal{T}_N}(x, y, z) \; = \; \int \frac{\mathrm{d}^{N-1} w}{N!} \mathcal{Z}_V(w) \, \mathcal{Z}_\epsilon(w, x) \, \mathcal{Z}_\epsilon(w, y) \, \mathcal{Z}_\epsilon(w, z) \, , \tag{5.1.11}$$

where $\mathcal{Z}_V(w)$ is the partition function of an $\mathcal{N} = 4$ vectormultiplet. This multiplet is used to gauge the diagonal combination of $SU(N)$ Higgs branch symmetries.

Let us now consider the action of the three-dimensional operators $G_R^{(3\mathrm{d})}$ on the partition function $\mathcal{Z}(x, y, z)$. Similar as in four dimensions, the result should be independent of which puncture we act on. This condition is guaranteed if the partition function of each quiver tail $\mathcal{T}(SU(N))$ is an eigenfunction of the operator $G_{(1^r)}^{(3\mathrm{d})}$. We will now show that in fact we have

$$G_{(1^r)}^{(3\mathrm{d})}(y) \cdot \mathcal{Z}_\epsilon(x, y) \; = \; W_{(1^r)}(x) \, \mathcal{Z}_\epsilon(x, y) \, , \tag{5.1.12}$$

where

$$W_{(1^r)}(x) \; = \sum_{j_1 < \ldots < j_r} e^{-2\pi b (x_{j_1} + \cdots + x_{j_r})} \tag{5.1.13}$$

is the expectation value of a supersymmetric Wilson loop in the rank r antisymmetric representation of the $SU(N)$ flavor symmetry[1] with mass parameter x.

However let us first remark that equation (5.1.12) tells us that introducing a background defect in the $\mathcal{T}(SU(N))$ theory for the Coulomb branch symmetry, through the operator $G_{(1^r)}^{(3\mathrm{d})}(y)$, is equivalent to introducing a background Wilson loop $W_{(1^r)}(x)$ for the Higgs branch symmetry. This means that mirror symmetry interchanges the defects introduced by the operators $G_{(1^r)}^{(3\mathrm{d})}$ and supersymmetric Wilson loops in the r-the antisymmetric representation of $SU(N)$. In the context of the mirror description of the three-dimensional $\mathcal{T}_N$ theory, the operators $G_{(1^r)}^{(3\mathrm{d})}$ therefore introduce a dynamical Wilson loop for the central node of the star-shaped quiver theory.

[1] Recall that Wilson loops are labeled by irreducible representations of the gauge group.

$\mathcal{T}(SU(2))$

We first show equation (5.1.12) in complete generality for the partition function of the mass deformed $\mathcal{T}(SU(2))$ theory on a squashed three-sphere. This partition function is given by

$$\mathcal{Z}_\epsilon(x,y) \;=\; \frac{1}{S_b(\epsilon^*)}\int \mathrm{d}z\, \mathcal{Q}(x,z)\, e^{4\pi \mathrm{i} yz}\,, \tag{5.1.14}$$

where

$$\mathcal{Q}(x,z) \;=\; \frac{S_b\left(\frac{\epsilon^*}{2} \pm \mathrm{i}x - \mathrm{i}z\right)}{S_b\left(q - \frac{\epsilon^*}{2} \pm \mathrm{i}x - \mathrm{i}z\right)}\,, \tag{5.1.15}$$

x is the $SU(2)$ mass parameter and y the FI parameter and the contour is given by $z \in \mathbb{R} + \mathrm{i}\delta$ with $\delta > 0$. Note that the $\mathcal{N} = 2$ mass deformation m in the hypermultiplet contribution appears with the opposite sign compared to equation (5.1.3). The reason is that after dimensional reduction, there is a mirror symmetry required to reach the star-shaped quiver description.

It is expected that the partition function has the following properties

$$\begin{aligned} \mathcal{Z}_\epsilon(x,y) &= \mathcal{Z}_\epsilon(-x,y) = \mathcal{Z}_\epsilon(x,-y)\,, \\ \mathcal{Z}_\epsilon(x,y) &= \mathcal{Z}_{\epsilon^*}(y,x)\,, \\ G^{(3d)}_{(1)}(y)\, \mathcal{Z}_\epsilon(x,y) &= W_{(1)}(x)\, \mathcal{Z}_\epsilon(x,y)\,, \end{aligned} \tag{5.1.16}$$

where

$$G^{(3d)}_{(1)}(x) \;=\; \frac{\sin \pi b\,(\epsilon^* - 2\mathrm{i}x)}{\sin \pi b(-2\mathrm{i}x)} e^{\frac{\mathrm{i}b}{2}\partial_x} + \frac{\sin \pi b\,(\epsilon^* + 2\mathrm{i}x)}{\sin \pi b(2\mathrm{i}x)} e^{-\frac{\mathrm{i}b}{2}\partial_x} \tag{5.1.17}$$

is the fundamental difference operator for $SU(2)$, and $W_{(1)}(x) = e^{2\pi bx} + e^{-2\pi bx}$ is the fundamental Wilson loop expectation value.

The first line of equation (5.1.16) represents the enhancement of the Higgs and Coulomb branch symmetry to $SU(2) \times {}^{L}SU(2)$ in the infrared. The second line illustrates the mirror symmetry of the mass-deformed $\mathcal{T}(SU(2))$ theory. These properties were demonstrated in [145]. Here we would like to prove the final line of equation (5.1.16). Using mirror symmetry this line is equivalent to

$$\bar{G}^{(3d)}_{(1)}(x)\,\mathcal{Z}_\epsilon(x,y) \;=\; W_{(1)}(y)\,\mathcal{Z}_\epsilon(x,y)\,, \tag{5.1.18}$$

where $\bar{G}^{(3d)}_{(1)}(x)$ is obtained from $G^{(3d)}_{(1)}(x)$ by the replacement $m \to -m$. Let us prove the intertwining property in this equivalent form.

As a preliminary step, we derive a few properties of the function $\mathcal{Q}(x,z)$ defined in equation (5.1.15). From the difference equation and the reflection property obeyed by the double sine function $S_b(x)$, it is straightforward to show that

$$e^{\mathrm{i}b\partial_z}\mathcal{Q}(x,z) \;=\; \frac{\sin\pi b(\frac{\epsilon^*}{2}\pm\mathrm{i}x-\mathrm{i}z)}{\sin\pi b(q-\frac{\epsilon^*}{2}\pm\mathrm{i}x-\mathrm{i}z)}\mathcal{Q}(x,z) \tag{5.1.19}$$

and

$$e^{\frac{\mathrm{i}b}{2}\partial_x}\mathcal{Q}(x,z) \;=\; e^{-\frac{\mathrm{i}b}{2}\partial_z}\left[\frac{\sin\pi b(\frac{\epsilon^*}{2}-\mathrm{i}x-\mathrm{i}z)}{\sin\pi b(q-\frac{\epsilon^*}{2}-\mathrm{i}x-\mathrm{i}z)}\,\mathcal{Q}(x,z)\right], \tag{5.1.20}$$

$$e^{-\frac{\mathrm{i}b}{2}\partial_x}\mathcal{Q}(x,z) \;=\; e^{-\frac{\mathrm{i}b}{2}\partial_z}\left[\frac{\sin\pi b(\frac{\epsilon^*}{2}+\mathrm{i}x-\mathrm{i}z)}{\sin\pi b(q-\frac{\epsilon^*}{2}+\mathrm{i}x-\mathrm{i}z)}\,\mathcal{Q}(x,z)\right]. \tag{5.1.21}$$

Using these results we can now compute the action of the difference operator $\bar{G}^{(3d)}_{(1)}(x)$ on this function,

$$\begin{aligned}\bar{G}^{(3d)}_{(1)}(x)\cdot\mathcal{Q}(x,z) \;&=\; e^{-\frac{\mathrm{i}b}{2}\partial_z}\Bigg[\frac{\sin\pi b\,(\epsilon-2\mathrm{i}x)}{\sin\pi b(-2\mathrm{i}x)}\frac{\sin\pi b(\frac{\epsilon^*}{2}-\mathrm{i}x-\mathrm{i}z)}{\sin\pi b(q-\frac{\epsilon^*}{2}-\mathrm{i}x-\mathrm{i}z)}\\ &\qquad+\frac{\sin\pi b\,(\epsilon+2\mathrm{i}x)}{\sin\pi b(2\mathrm{i}x)}\frac{\sin\pi b(\frac{\epsilon^*}{2}+\mathrm{i}x-\mathrm{i}z)}{\sin\pi b(q-\frac{\epsilon^*}{2}+\mathrm{i}x-\mathrm{i}z)}\Bigg]\mathcal{Q}(x,z)\\ &=\; e^{-\frac{\mathrm{i}b}{2}\partial_z}\left[1+\frac{\sin\pi b(\frac{\epsilon^*}{2}\pm\mathrm{i}x-\mathrm{i}z)}{\sin\pi b(q-\frac{\epsilon^*}{2}\pm\mathrm{i}x-\mathrm{i}z)}\right]\mathcal{Q}(x,z)\\ &=\; (e^{\frac{\mathrm{i}b}{2}\partial_z}+e^{-\frac{\mathrm{i}b}{2}\partial_z})\mathcal{Q}(z,x)\,.\end{aligned} \tag{5.1.22}$$

In going from the first to the second line we have applied a simple trigonometric identity.

Armed with this result, we now consider the action of the difference operator $\bar{G}^{(3d)}_{(1)}(x)$

on the full partition function (5.1.14) of the $\mathcal{T}(SU(2))$ theory. The difference operator can be brought inside the integral to act on $\mathcal{Q}(x,z)$ as in equation (5.1.22). By shifting the contour of integration by $z \to z \pm \frac{ib}{2}$, we find

$$\begin{aligned} \bar{G}^{(3\mathrm{d})}_{(1)}(x) \cdot \mathcal{Z}_\epsilon(x,y) &= \frac{1}{S_b(\frac{q}{2}-\mathrm{i}m)} \int \mathrm{d}z \left[(e^{\frac{\mathrm{i}b}{2}\partial_z} + e^{-\frac{\mathrm{i}b}{2}\partial_z})\, \mathcal{Q}(x,z) \right] e^{4\pi \mathrm{i} yz} \\ &= \frac{1}{S_b(\frac{q}{2}-\mathrm{i}m)} \int \mathcal{Q}(x,z) \, \left[(e^{\frac{\mathrm{i}b}{2}\partial_z} + e^{-\frac{\mathrm{i}b}{2}\partial_z})\, e^{4\pi \mathrm{i} yz} \right] \\ &= W_{(1)}(y)\, \mathcal{Z}_\epsilon(x,y)\,. \end{aligned} \tag{5.1.23}$$

Using the analytic structure of the double sine function $S_b(x)$, it is straightforward to check that no poles are crossed in shifting the contours provided the mass is real. Now, applying mirror symmetry we have

$$G^{(3\mathrm{d})}_{(1)}(y)\, \mathcal{Z}_\epsilon(x,y) \;=\; W_{(1)}(x) \mathcal{Z}_\epsilon(x,y)\,, \tag{5.1.24}$$

which is the required result.

An important consequence of this result, together with the similarity transformation (5.1.10) relating the operators $G^{(3d)}_{(1)}$ and $\bar{G}^{(3d)}_{(1)}$, is that the partition function of mass deformed $\mathcal{T}(SU(2))$ theory obeys

$$\mathcal{Z}_{\epsilon^*}(x,y) \;=\; \mathcal{K}(y)\, \mathcal{Z}_\epsilon(x,y)\,, \tag{5.1.25}$$

which is rather non-obvious from the integral representation.

$\mathcal{T}(SU(N))$

Let us now consider equation (5.1.12) for the general $\mathcal{T}(SU(N))$ theory. In this case, we simplify the problem and prove a weaker result by taking the limit of $\mathcal{N}=4$ supersymmetry ($m=0$) and a round three-sphere ($b=0$).

In this limit, the operators for the fully antisymmetric representations are given by

$$G^{(3\mathrm{d})}_{(1^r)} \cdot \mathcal{Z}(x) \;=\; (-1)^{r(N-r)} \sum_{j_1<\cdots<j_r} \mathcal{Z}(x + \mathrm{i}(h_{j_1} + \cdots + h_{j_r}))\,. \tag{5.1.26}$$

Hence, up to a sign, the operators are simply a sum of shift operators with coefficients one and shifts determined by the weights of the representation $R = (1^r)$.

Furthermore, the zero mode integrals in the partition function of the $\mathcal{T}(SU(N))$ theory can be performed explicitly [146, 147]. The result is[2]

$$\mathcal{Z}(x,y) \;=\; \frac{\sum_{\rho\in S_N} (-1)^\rho e^{2\pi\mathrm{i}\sum_{j=1}^N x_{\rho(j)} y_j}}{\prod_{i<j} 2\sinh\pi(x_i - x_j)\, 2\sinh\pi(y_i - y_j)}\,, \tag{5.1.27}$$

where the summation in the numerator is over the group S_N of permutations of $\{1,\ldots,N\}$. In this case, mirror symmetry acting as $x \leftrightarrow y$ follows from the identity

$$\sum_{\rho\in S^N} (-1)^\rho e^{2\pi\mathrm{i}\sum_{j=1}^N x_{\rho(j)} y_j} \;=\; \sum_{\rho\in S^N} (-1)^\rho e^{2\pi\mathrm{i}\sum_{j=1}^N x_j y_{\rho(j)}}\,. \tag{5.1.28}$$

Let us now act with the operator $G^{(3\mathrm{d})}_{(1^r)}$ on the partition function $\mathcal{Z}(x,y)$. First, note that up to a factor of $(-1)^{r(N-r)}$, each term in the operator $G^{(3\mathrm{d})}_{(1^r)}$ leaves the denominator invariant. This factor cancels the overall sign of the operator. Thus we can concentrate on the numerator of $\mathcal{Z}(x,y)$ and find

$$G^{(3\mathrm{d})}_{(1^r)}(y) \cdot \sum_{\rho\in S^N} (-1)^\rho e^{2\pi\mathrm{i}\sum_{j=1}^N y_{\rho(j)} x_j} = \sum_{|I|=r} \left[\sum_{\rho\in S^N} (-1)^\rho e^{2\pi\mathrm{i}\sum_{k=1}^N y_{\rho(k)} x_k} e^{-2\pi\sum_{\rho(i)\in I} x_i} \right]\,,$$

whereby the overall background shifts vanish due to $\sum_{i=1}^N x_i = 0$. Now label each subset $I \subset \{1,\ldots,N\}$ with $|I| = r$ by I_ℓ for $1 \leq \ell \leq N_r = \binom{N}{r}$. Furthermore split S^N into $\mathbb{Z}_{N_r} \otimes (S^{N-r} \otimes S^r)$, where $S^{(N-r)} \otimes S^r$ gives all the permutations satisfying $\rho(I_\ell) = I$ for a fixed choice of I and ℓ, and $\mathbb{Z}_{N_r}$ gives the different choices of ℓ (or I). Since this splitting

[2]Here we drop the subscript m on the partition function of $\mathcal{T}(SU(N))$ because we have set $m = 0$.

is an isomorphism,[3] we can write the sum over all $\rho \in S^N$ as a double sum over sets I with $|I| = r$ and permutations in $S^{N-r} \otimes S^r$ preserving $\rho(I_\ell) = I$. With that in mind, we can rewrite the above as

$$G^{(3d)}_{(1^r)}(y) \cdot \sum_{\rho \in S^N} (-1)^\rho e^{2\pi \mathrm{i} \sum_{j=1}^N y_{\rho(j)} x_j} = \sum_{\ell=1}^{N_r} e^{-2\pi \sum_{i \in I_\ell} x_i} \sum_{|I|=r} \sum_{\substack{\rho \in S^N \\ \rho(I_\ell)=I}} (-1)^\rho e^{2\pi \mathrm{i} \sum_{k=1}^N y_{\rho(k)} x_k} ,$$

which is equal to the Wilson loop vacuum expectation value $W_{(1^r)}(x)$ times the numerator of $\mathcal{Z}(x,y)$. The eigenvalue of the operator $G^{(3d)}_{(1^r)}$ acting on the full partition function $\mathcal{Z}(x,y)$ is thus precisely the localization expression for a supersymmetric Wilson loop in the antisymmetric representation $R = (1^r)$ of $SU(N)$.

5.1.3 Three-dimensional algebra

In the above, we have shown that the defect operator $G^{(3d)}_{(1^r)}$ is dual to a Wilson loop in the rank r antisymmetric representation of $SU(N)$ under mirror symmetry. This turns out *not* to be the case for non-minuscule representations.

One immediate way to see this is the following. Denote the operator that *is* exactly dual to a Wilson loop in the representation R by $\tilde{G}^{(3d)}_R$. The operators $\tilde{G}^{(3d)}_R$ must obey the algebra

$$\tilde{G}^{(3d)}_{R_1} \circ \tilde{G}^{(3d)}_{R_2} = \sum_{R_1} N_{R_1,R_2}{}^{R_3} \, \tilde{G}^{(3d)}_{R_3} , \qquad (5.1.29)$$

where $N_{R_1,R_2}{}^{R_3}$ are the (plain) Littlewood-Richardson coefficients. Indeed, the supersymmetric Wilson loops are characters and hence obey this algebra. Instead, the elliptic Littlewood-Richardson coefficients $\mathcal{N}_{R_1,R_2}{}^{R_3}(p,q,t)$ reduce in general to non-integer coefficients in three dimensions.

For example, let us consider $SU(2)$ and the composition of two operators in the funda-

[3] Any $\rho \in S^N$ can be uniquely characterized by $\rho(\sigma(I_\ell)) = \pi(I)$ for a unique I_ℓ or I and $\sigma \in S^r$, $\pi \in S^{N-r}$ permutations of I, $\mathbb{Z}_N \setminus I$ respectively.

mental representation. In the three-dimensional limit we find that

$$G^{(3d)}_{(1)} \circ G^{(3d)}_{(1)} = G^{(3d)}_{(2)} + \left(\frac{1}{2\cos(\pi b^2) - 1} - 1 \right) G^{(3d)}_{(0)} . \tag{5.1.30}$$

Since $\tilde{G}^{(3d)}_{(1)} = G^{(3d)}_{(1)}$ and $\tilde{G}^{(3d)}_{(0)} = G^{(3d)}_{(0)} = 1$, we read off from equation (5.1.29) that

$$\tilde{G}^{(3d)}_{(2)} = G^{(3d)}_{(2)} + \left(\frac{1}{2\cos(\pi b^2) - 1} - 2 \right) G^{(3d)}_{(0)} . \tag{5.1.31}$$

The operator $\tilde{G}^{(3d)}_{(2)}$ that *is* dual to a Wilson loop thus differs from the difference operator $G^{(3d)}_{(2)}$ by lower order contributions.

In general, the relation between the operators $G^{(3d)}_{R}$ appearing in the vortex construction and the operators $\tilde{G}^{(3d)}_{R}$ that are exactly dual to Wilson loops in the three-dimensional limit is given by

$$\tilde{G}^{(3d)}_{R} = G^{(3d)}_{R} + \sum_{|S|<|R|} c_S \, G^{(3d)}_{S} . \tag{5.1.32}$$

Even though the difference operators $G^{(3d)}_{R}$ are thus not exactly dual to Wilson loops, this is merely an invertible linear transformation on the algebra that these operators obey.

The original basis of operators G_R appears to be more fundamental from a four-dimensional perspective, since in the limit $p \to 0$ they are precisely dual to Wilson loop operators in refined Chern-Simons theory on $S^1 \times \mathcal{C}_{g,n}$. On the other hand, in the three-dimensional limit, the basis $\tilde{G}^{(3d)}_{R}$ seems more fundamental since it is dual to a basis of Wilson loop operators in the star-shaped quiver theories.

5.2 't Hooft loops in the four-dimensional $\mathcal{N} = 2^*$ theory

In this section, we realize the mass-deformed theory $\mathcal{T}(SU(N))$ on a squashed three-sphere as an S-duality domain wall in four-dimensional $\mathcal{N} = 2^*$ theory on an ellipsoid, as described in [145, 148]. We then use this observation to interpret the three-dimensional difference operators $G^{(3d)}_{R}$ as operators that introduce supersymmetric 't Hooft loops in the $\mathcal{N} = 2^*$

theory partition function on a four-sphere.

The four-dimensional $\mathcal{N} = 2^*$ theory can also be obtained by compactifying the six-dimensional $\mathcal{N} = (2,0)$ theory of type A_{N-1} on a torus with a simple puncture. A consequence of this construction is that via the AGT correspondence [12, 24], the four-sphere partition function of the $\mathcal{N} = 2^*$ theory can also be computed as a correlation function in Liouville or Toda CFT on the punctured torus. The difference operators $G_R^{(3\mathrm{d})}$ can then be interpreted as Verlinde loop operators that act on suitably normalized Virasoro or W_N-algebra conformal blocks on a punctured torus.

5.2.1 Four-sphere partition function

The exact partition function of $\mathcal{N} = 2$ supersymmetric gauge theories on an ellipsoid has been computed by supersymmetric localization in [149], extending the computation of Pestun for the round four-sphere S^4 [83]. The ellipsoid geometry can be embedded into five-dimensional Euclidean space as

$$x_0^2 + \frac{1}{b^2}(x_1^2 + x_2^2) + b^2(x_3^2 + x_4^2) \ = \ 1 \,, \qquad b \in \mathbb{R}_{\geq 0} \,. \tag{5.2.1}$$

The equator $\{x_0 = 0\}$ is identified with the squashed three-sphere geometry considered in the previous section by setting $z = x_1 + \mathrm{i}x_2$ and $w = x_3 + \mathrm{i}x_4$.

Let us concentrate on the $\mathcal{N} = 2^*$ theory and denote the real hypermultiplet mass parameter by m and the complexified gauge coupling by τ. The result of the localization computation can be written as a matrix integral

$$\mathcal{Z}_{S_b^4}(m,\tau) \ = \ \int \mathrm{d}a \, \big| \, \mathcal{Z}(a,m;\tau) \, \big|^2 \tag{5.2.2}$$

over a real slice of the Coulomb branch. In this integral $\mathcal{Z}(a,m;\tau)$ is the Nekrasov partition function for the four-dimensional $\mathcal{N} = 2^*$ theory in the Omega-background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$, with equivariant parameters $\epsilon_1 = b$ and $\epsilon_2 = b^{-1}$ [150, 151]. It can be split into a classical, 1-loop and instanton piece as

$$\mathcal{Z}(a, m; \tau) \;\equiv\; \mathcal{Z}_{\rm cl}(a;\tau)\, \mathcal{Z}_{1-\rm loop}(a, m)\, \mathcal{Z}_{\rm inst}(a, m; \tau)\,. \tag{5.2.3}$$

In this chapter we advertise an alternative factorization of the ellipsoid partition function $\mathcal{Z}_{S^4_b}$. We find it insightful to rewrite the matrix integral (5.2.2) in the form (for a derivation of this representation see Appendix C)

$$\mathcal{Z}_{S^4_b}(m;\tau) \;=\; \int {\rm d}a\, \mu(a)\, \big|\, \mathcal{G}(a, m; \tau)\, \big|^2\,, \tag{5.2.4}$$

where

$$\mu(a) \;=\; \prod_{e>0} 2\sinh\left(\pi b (e, a)\right)\, 2\sinh\left(\pi b^{-1}(e, a)\right) \tag{5.2.5}$$

is the Haar measure times the partition function of a three-dimensional $\mathcal{N} = 2$ vectormultiplet on the squashed three-sphere at the equator $\{x_0 = 0\}$. Here we take the product over the positive roots $e > 0$ of the gauge group.

We expect that the factorization (5.2.4) has the following interpretation [60]. We can cut the ellipsoid into two half-spheres $\{x_0 > 0\}$ and $\{x_0 < 0\}$ and impose Dirichlet boundary conditions on the fields in the $\mathcal{N} = 2^*$ theory at the boundary $\{x_0 = 0\}$. This decouples the dynamics on both half-spheres. Restricting the gauge transformations to the identity on the boundary, leaves a flavor symmetry group $SU(N)$ acting on the values of the fields at $x_0 = 0$. We can reconstruct the partition function of an ellipsoid by inserting a three-dimensional $\mathcal{N} = 2$ $SU(N)$ vectormultiplet on the boundary $\{x_0 = 0\}$ and gauging the diagonal $SU(N)$ symmetry. We thus claim that $\mathcal{G}(a, m; \tau)$ in the matrix integral (5.2.4) is the partition function of $\mathcal{N} = 2^*$ theory on the upper half of the ellipsoid $\{x_0 > 0\}$ with Dirichlet boundary conditions, and similarly for $\overline{\mathcal{G}(a, m; \tau)}$ on the lower half $\{x_0 < 0\}$.

Note that $\mathcal{G}(a, m; \tau)$ can be split into classical, one-loop and instanton contributions just like the Nekrasov partition function in (5.2.3). Whereas we take its classical and instanton contributions to be the same as those of $\mathcal{Z}(a, m; \tau)$, *i.e.* $\mathcal{G}_{\rm cl}(a;\tau) \equiv \mathcal{Z}_{\rm cl}(a;\tau)$ and $\mathcal{G}_{\rm inst}(a, m; \tau) \equiv \mathcal{Z}_{\rm inst}(a, m; \tau)$, the one-loop factor $\mathcal{G}_{1-\rm loop}$ is not canonically determined.

We claim that it is fixed by imposing Dirichlet boundary conditions on the half-sphere, in such a way that

$$\mathcal{G}_{1-\text{loop}}(a,m) = \frac{\prod_{w\in\text{adj}} \Gamma_b\left(\frac{q}{2} + \mathrm{i}(a,w) + \mathrm{i}m\right)}{\prod_{e>0} \Gamma_b(q+\mathrm{i}(a,e))\Gamma_b(q-\mathrm{i}(a,e))}, \tag{5.2.6}$$

where $q = b + b^{-1}$ and $\Gamma_b(x)$ is the Barnes' double gamma function. The numerator contains the contribution from the vectormultiplet and the denominator that from the adjoint hypermultiplet with mass m in the $\mathcal{N} = 2^*$ theory.

Let us mention that for $SU(2)$, via the AGT correspondence this choice is equivalent to a commonly used normalization of Virasoro conformal blocks in Liouville theory [59]. For this choice of normalization, we will show that the expectation values of 't Hooft loop operators in the $\mathcal{N} = 2^*$ theory are given by acting on $\mathcal{G}(a,m;\tau)$ with the three-dimensional difference operators $G_R^{(3d)}$, constructed in section 5.1.

5.2.2 S-duality domain wall

The three-dimensional theory $\mathcal{T}(SU(N))$ appears as an S-duality domain wall between two four-dimensional $\mathcal{N} = 4$ SYM theories with gauge groups $SU(N)$ and ${}^LSU(N)$ respectively and equal holomorphic gauge coupling τ [143,144]. Furthermore, the mass deformation m of the domain wall theory can be identified with the canonical mass deformation of the bulk theory to $\mathcal{N} = 2^*$ by giving a mass to the adjoint $\mathcal{N} = 2$ hypermultiplet.

On the ellipsoid S_b^4, one can introduce the S-duality domain wall at the equator $\{x_0 = 0\}$ in a way that preserves half of the supersymmetries of the bulk [60]. As above, let us assume that the normalized function $\overline{\mathcal{G}(a,m;\tau)}$ corresponds to the partition function of the $\mathcal{N} = 2^*$ theory with gauge group $SU(N)$ on $\{x_0 < 0\}$ with Dirichlet boundary conditions for the vectormultiplet, and similarly that $\mathcal{G}(a',m;\tau)$ corresponds to the partition function of the $\mathcal{N} = 2^*$ theory with gauge group ${}^LSU(N)$ on $\{x_0 > 0\}$. Let us also denote the partition function of the $\mathcal{T}(SU(N))$ theory on the squashed three-sphere at the equator $\{x_0 = 0\}$ by $\mathcal{Z}(a,a',m)$, where a and a' are mass parameters for the $SU(N)\times{}^LSU(N)$ global symmetry

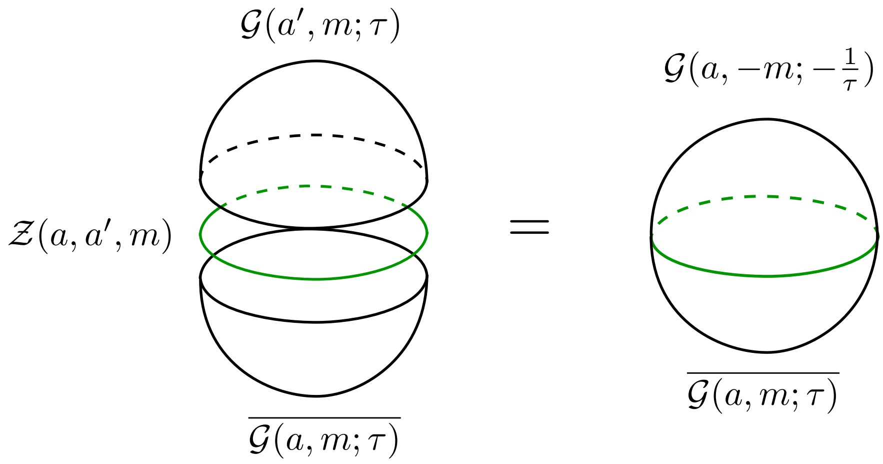


Figure 5.2: *Left*: The ellipsoid partition function in the presence of an S-duality domain wall can be constructed by gluing the domain wall partition function $\mathcal{Z}(a, a', m)$ in between the half-sphere partition functions $\overline{\mathcal{G}(a, m; \tau)}$ and $\mathcal{G}(a', m; \tau)$, while gauging their flavor symmetries. *Right*: The same ellipsoid partition function can be constructed by gluing the half-sphere partition functions $\overline{\mathcal{G}(a, m; \tau)}$ and $\mathcal{G}(a, -m; -\frac{1}{\tau})$.

as in section 5.1. Then the combined partition function in the presence of the S-duality domain wall is

$$\int \mathrm{d}a\, \mathrm{d}a'\, \mu(a)\, \mu(a')\, \overline{\mathcal{G}(a, m; \tau)}\, \mathcal{Z}(a, a', m)\, \mathcal{G}(a', m; \tau)\,, \tag{5.2.7}$$

where $\mu(a)\, \mu(a')$ is the partition function of three-dimensional $\mathcal{N} = 2$ vectormultiplets on the equator $\{x_0 = 0\}$ gauging the symmetry $SU(N) \times {}^L SU(N)$ – see Figure 5.2.

Another interpretation of the same domain wall is as a Janus domain wall interpolating between holomorphic gauge coupling τ for $\{x_0 < 0\}$ and $-1/\tau$ for $\{x_0 > 0\}$. The two pictures are related by an S-duality transformation of the theory on $\{x_0 > 0\}$. Another way of saying this is that the partition function $\mathcal{Z}(a, a', m)$ should be an S-duality kernel relating the functions $\mathcal{G}(a, m; \tau)$ and $\mathcal{G}(a, -m; -1/\tau)$ through the measure $\mathrm{d}\mu(a)$. This statement is rather hard to check in field theory because there is in general no closed expression for the $\mathcal{G}(a, m; \tau)$.[4]

On the other hand, in the context of the AGT correspondence, it has been checked

[4]We demonstrate this explicitly in the limit $m = 0$ and $b = 1$ in Appendix B.

in [145, 148] that the partition function $\mathcal{Z}(a, a', m)$ on a squashed three-sphere of the $\mathcal{T}(SU(2))$ theory is precisely equal to the S-duality kernel of a normalized conformal block in Liouville theory [152] under the relevant identification of parameters.

5.2.3 Supersymmetric loop operators

Since we can embed the mass-deformed $\mathcal{T}(SU(N))$ theory as a domain wall in the four-dimensional $\mathcal{N} = 2^*$ theory on an ellipsoid, it is natural to think that supersymmetric loop operators in the two theories on $\{x < 0\}$ and $\{x > 0\}$ are related. In particular, one can introduce a loop operator on one hemisphere and push it through the domain wall to find another loop operator on the other hemisphere. For an S-duality wall one expects that this process turns a Wilson loop operator in the four-dimensional $\mathcal{N} = 2^*$ theory into an 't Hooft loop operator.

Let us briefly summarize a few facts that are known about supersymmetric loop operators on the four-sphere. The four-sphere partition function can for instance be enriched with Wilson and 't Hooft loop operators. To preserve half of the supersymmetries such loop operators should be supported on the circle

$$x_0 = \cos\rho\,, \quad x_1 = b\sin\rho\cos\varphi\,, \quad x_2 = b\sin\rho\sin\varphi\,, \quad x_3 = x_4 = 0\,, \tag{5.2.8}$$

where $0 < \varphi < 2\pi$ and $0 < \rho < \pi$, or alternatively supported on the circle obtained by interchanging $b \leftrightarrow b^{-1}$ and $\{x_1, x_2\} \leftrightarrow \{x_3, x_4\}$. The support of the loop operator lies in the squashed three-sphere at the equator $\{x_0 = 0\}$ when $\rho = \pi/2$. However, the expectation value of the loop operator is independent of ρ.

Wilson loops

Supersymmetric Wilson loops in the four-dimensional $\mathcal{N} = 2^*$ theory are labeled by irreducible representations of the gauge group G. The expectation values of supersymmetric Wilson loops on the ellipsoid have been computed in [149].

The expectation value for a supersymmetric Wilson loop in the irreducible representation R around a circle in the (x_1, x_2)-plane is obtained by inserting the factor

$$W_R(a) \;=\; \sum_{w \in R} e^{-2\pi b(w,a)} \tag{5.2.9}$$

into the matrix integral. For example, for a rank r antisymmetric tensor representation of $SU(N)$ we insert the factor

$$W_{(1^r)}(a) \;=\; \sum_{\{j_1 < \ldots < j_r\}} e^{-2\pi b\,(a_{j_1} + \cdots + a_{j_r})} \,. \tag{5.2.10}$$

The expectation value for supersymmetric Wilson loops in the (x_3, x_4)-plane is obtained by replacing $b \to b^{-1}$.

't Hooft loops

A supersymmetric 't Hooft loop is defined by computing the path integral in the presence of a singular boundary condition along a circle that preserves half of the supersymmetries. The boundary condition is specified by the image of an abelian 't Hooft monopole under a homomorphism $\rho : U(1) \to G$, with gauge transformations acting by conjugation on ρ. These configurations are classified by irreducible representations R of the Langlands dual ${}^L G$ [153].

The expectation values of supersymmetric 't Hooft loop operators in the $\mathcal{N} = 2^*$ theory on the round four-sphere have been computed in [58]. It was found that it can be expressed as (where now $b = 1$)

$$\int \mathrm{d}a\, \overline{\mathcal{Z}(a, m; \tau)} \left[T_R \cdot \mathcal{Z}(a, m; \tau) \right] , \tag{5.2.11}$$

where T_R is a difference operator that acts on the Coulomb branch parameters a. The difference operator takes the general form

$$T_R \cdot \mathcal{Z}(a) \;=\; \sum_{\nu} C_\nu(a, m)\, \mathcal{Z}(a + \mathrm{i}\nu) \,, \tag{5.2.12}$$

where the sum is taken over the weights ν of the representation R.

For the antisymmetric tensor representations $R = (1^r)$ the coefficients $C_\nu(a,m)$ only receive one-loop contributions. In this case

$$C_\nu(a,m) \;=\; \prod_{\substack{j\in I\\ k\notin I}} \left[\frac{\sinh\pi(a_{jk}-m)\sinh\pi(-a_{jk}-m)}{\sinh\pi(a_{jk})\sinh\pi(-a_{jk})}\right]^{1/2}, \tag{5.2.13}$$

where we have denoted the weights of the r-th antisymmetric tensor representation by $\nu = \sum_{j\in I} h_j$ for $I = \{j_1 < \ldots < j_r\}$. For general representations R there are additional non-perturbative monopole bubbling contributions to the coefficients $C_\nu(a,m)$.

Here, we want to re-express the expectation value of the 't Hooft operator in terms of a difference operator $\widetilde{T}_R$ acting on the half-sphere partition function $\mathcal{G}(a,m;\tau)$ in the case $b = 1$, *i.e.*

$$\int \mathrm{d}a\, \mu(a)\, \overline{\mathcal{G}(a,m;\tau)}\, \left[\widetilde{T}_R \cdot \mathcal{G}(a,m;\tau)\right]. \tag{5.2.14}$$

The difference operator $\widetilde{T}_R$ is related to T_R by conjugating with the one-loop factor that relates the Nekrasov partition function $\mathcal{Z}(a,m;\tau)$ to the half-sphere partition function $\mathcal{G}(a,m;\tau)$. Later it will be important that $\widetilde{T}_R$ is self-adjoint with respect to the measure $\mu(a)\mathrm{d}a$.

In Appendix C, we perform this conjugation explicitly for the antisymmetric tensor representations to find

$$\widetilde{T}_{(1^r)} \cdot \mathcal{G}(a) \;=\; \sum_{|I|=r} \prod_{\substack{j\in I\\ k\notin I}} \frac{\sin\pi(-\mathrm{i}a_{jk}-\mathrm{i}m)}{\sin\pi(\mathrm{i}a_{jk})}\, \mathcal{G}\left(a + \mathrm{i}\sum_{j\in I} h_j\right). \tag{5.2.15}$$

Remarkably, this difference operator is in agreement with the difference operators $G^{(3\mathrm{d})}_{(1^r)}$ that introduce codimension-two defects in the $\mathcal{T}(SU(N))$ theory by acting on the three-dimensional partition function $\mathcal{Z}(a,a',m)$.

5.2.4 Intertwining Wilson and 't Hooft loops

Let us now explain why the difference operators $G_R^{(3d)}$ are related to 't Hooft operators $\widetilde{T}_R$. We consider the four-sphere partition function in the presence of both an S-duality wall and a supersymmetric loop operator. Recall that on the lower half-sphere $\{x_0 < 0\}$ we have the gauge group $SU(N)$ for which the Wilson loops are labeled by irreducible representations of $SU(N)$. On the upper half-sphere $\{x_0 > 0\}$, we have the gauge group ${}^L SU(N)$ for which the 't Hooft operators are labeled by irreducible representations of $SU(N)$.

Thus, let us now consider an 't Hooft loop labeled by an irreducible representation of $SU(N)$ inserted at some point $\rho > \pi/2$ in the upper half-sphere $\{x_0 > 0\}$. The expectation value of this system takes the form

$$\int \mathrm{d}a\, \mathrm{d}a'\, \mu(a)\, \mu(a')\, \overline{\mathcal{G}(a,m;\tau)}\, \mathcal{Z}(a,a',m)\, \left[\widetilde{T}_{(1^r)}(a') \cdot \mathcal{G}(a',m;\tau)\right] , \tag{5.2.16}$$

where for simplicity we focus on antisymmetric tensor representations.

The expectation value is independent of the position ρ. Thus we can imagine moving the 't Hooft loop through the S-duality domain wall to some point $\rho < \pi/2$ in the region $\{x_0 < 0\}$. According to the transformation of loop operators under S-duality, it should become a Wilson loop in the antisymmetric tensor representation $R = (1^r)$. At the level of the partition function, since the operator $\widetilde{T}_{(1^r)}(a')$ is self-adjoint with respect to the measure $\mu(a')$, we can bring it to act on $\mathcal{Z}(a,a',m)$. Provided that $\mathcal{Z}(a,a',m)$ is an eigenfunction such that

$$\widetilde{T}_{(1^r)}(a') \cdot \mathcal{Z}(a,a',m) \;=\; W_{(1^r)}(a)\, \mathcal{Z}(a,a',m)\,, \tag{5.2.17}$$

we find

$$\begin{aligned}
&\int \mathrm{d}a\, \mathrm{d}a'\, \mu(a)\, \mu(a')\, \overline{\mathcal{G}(a,m;\tau)}\, \mathcal{Z}(a,a',m)\, \left[\widetilde{T}_{(1^r)}(a') \cdot \mathcal{G}(a',m;\tau)\right] = \\
&\quad = \int \mathrm{d}a\, \mathrm{d}a'\, \mu(a)\, \mu(a')\, \overline{\mathcal{G}(a,m;\tau)}\, \left[\widetilde{T}_{(1^r)}(a') \cdot \mathcal{Z}(a,a',m)\right]\, \mathcal{G}(a',m;\tau) \\
&\quad = \int \mathrm{d}a\, \mathrm{d}a'\, \mu(a)\, \mu(a')\, \left[\overline{W_{(1^r)}(a)\, \mathcal{G}(a,m;\tau)}\right]\, \mathcal{Z}(a,a',m)\, \mathcal{G}(a',m;\tau)\,,
\end{aligned}$$

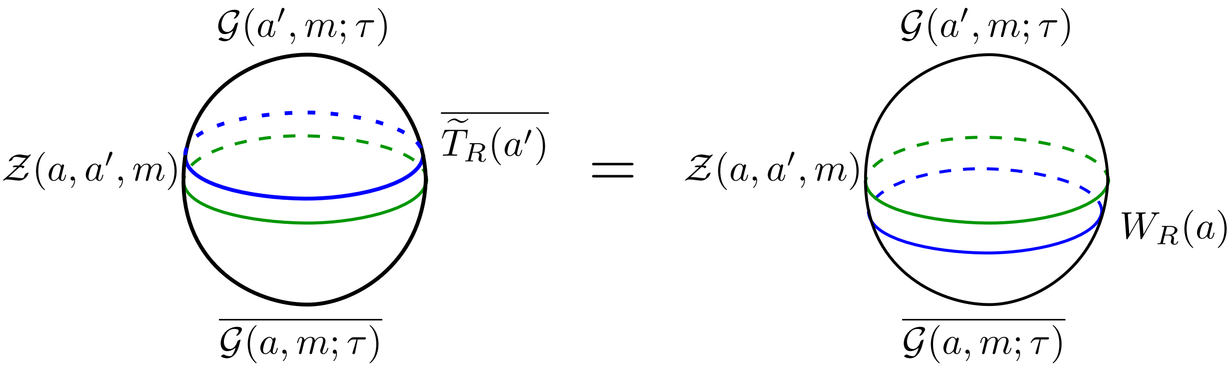


Figure 5.3: An 't Hooft loop operator T_R can be moved through the S-duality domain wall to obtain a Wilson loop operator W_R.

which is the expectation value of an S-duality domain wall together with a Wilson loop in the representation (1^r) at some point $\rho < \pi/2$. Thus compatibility with S-duality demands that $\mathcal{Z}(a, a', m)$ intertwines 't Hooft loops and Wilson loops according to equation (5.2.17) – see Figure 5.3. In section 5.1 we have argued that three-dimensional mirror symmetry requires $\mathcal{Z}(a, a', m)$ to obey the same intertwining property with respect to the three-dimensional limit of the surface defect operators $G^{(3d)}_{(1^r)}$. Thus the corresponding operators should agree. Above we checked that this is indeed the case for a round four-sphere.

Let us now make some comments on non-minuscule representations R. Since Wilson loops are defined by a trace over the representation R, they obey the character algebra

$$W_{R_1} \circ W_{R_2} \;=\; \sum_{R_3} N_{R_1,R_2}{}^{R_3}\, W_{R_3}\,, \tag{5.2.18}$$

where $N_{R_1,R_2}{}^{R_3}$ are the standard Littlewood-Richardson coefficients.

Therefore, we can define a new set of operators $\hat{T}_R$ by taking $\hat{T}_{(1^r)} \equiv \widetilde{T}_{(1^r)}$ – or equivalently $\hat{T}_{(1^r)} \equiv G^{(3d)}_{(1^r)}$ – for antisymmetric representations and imposing the character algebra

$$\hat{T}_{R_1} \circ \hat{T}_{R_2} \;=\; \sum_{R_3} N_{R_1,R_2}{}^{R_3}\, \hat{T}_{R_3}\,. \tag{5.2.19}$$

The resulting operators $\hat{T}_R$ automatically transform in the expected way under S-duality, and it is natural to expect that these operators encode the expectation value of 't Hooft

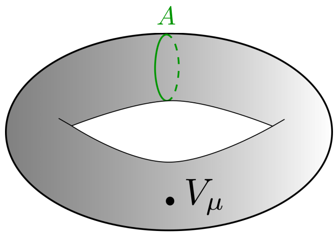


Figure 5.4: The four-sphere partition function of the $\mathcal{N} = 2^*$ theory is equal to a Toda correlation function on the punctured torus with a semi-degenerate vertex operator V_μ, with momentum μ, inserted at the puncture.

loops for general representations.

However, we emphasize that the $\hat{T}_R$ do not seem to correspond to the expectation value of an 't Hooft loop with magnetic weight given by the highest weight of the representation R, when the representation is non-minuscule. For example, for $SU(2)$ the 't Hooft loop whose magnetic weight is double that of the 't Hooft loop of minimal charge is given by $T_1 \circ T_1$ rather than $T_1 \circ T_1 - T_0$. This is again an invertible linear transformation on the algebra of the operators. In this case, the origin of the basis transformation is a natural resolution of the Bogomolnyi moduli space that arises for representations with non-perturbative monopole bubbling effects [58]. Once again, we emphasize that the simplest and unambiguous operators are those in antisymmetric tensor representations.

5.2.5 Verlinde operators in Toda CFT

Everything we have discussed so far in this section can also be framed in the language of Liouville or Toda conformal field theory. This approach has the benefit that, at least for the 't Hooft loop in the fundamental representation, we can compute the required operator for general squashing parameter b.

Let us briefly review aspects of this correspondence. For the $\mathcal{N} = 2^*$ theory with gauge group $SU(N)$, the ellipsoid partition function is related to a Liouville or type A_{N-1} Toda correlator on the punctured torus with an insertion of a semi-degenerate primary field. The parameters on both sides of the correspondence are related as follows:

(i) The geometric parameter b is a dimensionless coupling in the conformal field theory and gives the central charge $c = (N-1)(1 + N(N+1)q^2)$, where $q = b + b^{-1}$.

(ii) The holomorphic gauge coupling τ is the complex structure parameter of the punctured torus.

(iii) The mass m of the adjoint hypermultiplet is encoded in the momentum of the semi-degenerate primary field,

$$\mu \;=\; N\left(\frac{q}{2} + \mathrm{i}m\right)\omega_{N-1}\,. \tag{5.2.20}$$

Choosing a pants decomposition, the correlation function of the primary field on the punctured torus can be written as an expansion in Liouville or W_N-algebra conformal blocks

$$\int \mathrm{d}z\, C(\mu, z, 2Q - z)\,\overline{\mathcal{F}(z,\mu;\tau)}\mathcal{F}(z,\mu;\tau)\,, \tag{5.2.21}$$

where the integral is over non-degenerate momenta $z = Q + \mathrm{i}a$, with $a \in \mathbb{R}^{N-1}$ and $Q = q\rho$, where ρ is the Weyl vector of A_{N-1}.

The conformal blocks $\mathcal{F}(a, \mu; \tau)$ are normalized to contain the classical and instanton contributions to the Nekrasov partition function. The three-point function $C(\mu, z, 2Q - z)$ is proportional (up to an m-dependent piece that can be absorbed in the normalization of the primary field) to the modulus squared of the 1-loop contribution $|\,\mathcal{G}_{1-\mathrm{loop}}\,|^2$ times the measure $\mu(a)$.

Loop operators in the four-dimensional gauge theory are realized as Verlinde operators in the dual conformal field theory [52, 59]. The Verlinde operators act on the space of Virasoro or W_N-algebra conformal blocks by transporting a chiral primary field around a simple closed curve C on the Riemann surface. The operators constructed in this way depend only on the homotopy class of the curve C up to a choice of 'framing' that will not be important here.

Let us choose the pants decomposition of the punctured torus determined by the A-cycle in Figure 5.4. Then a supersymmetric Wilson loop in the $\mathcal{N} = 2^*$ theory in the rank r antisymmetric tensor representation corresponds to transporting a degenerate chiral primary with momentum $\eta = -b\omega_j$ around that A-cycle. The resulting expression changes from the original conformal block by the factor

$$W^{\mathrm{CFT}}_{(1^r)} = \sum_{\{j_1<\ldots<j_r\}} e^{-2\pi b\,(a_{j_1}+\cdots+j_r)}\,, \tag{5.2.22}$$

which is in agreement with the localization computation.

An 't Hooft loop in the r-th fundamental representation corresponds to transporting the same chiral primary around the B-cycle of the punctured torus. This Verlinde operator has been computed directly in Toda theory for the fundamental representation in [61]. Acting on the conformal blocks $\mathcal{F}(z,\mu;\tau)$, the operator is given by

$$T^{\mathrm{CFT}}_{(1)}\cdot\mathcal{F}(z) = \sum_{j=1}^{N}\prod_{k\neq j}^{N}\frac{\Gamma(\mathrm{i}ba_{\mathrm{i}k})\Gamma\left(bq+\mathrm{i}ba_{jk}\right)}{\Gamma\left(\frac{bq}{2}+\mathrm{i}ba_{jk}-\mathrm{i}bm\right)\Gamma\left(\frac{bq}{2}+\mathrm{i}ba_{jk}+\mathrm{i}bm\right)}\mathcal{F}(z-bh_j)\,, \tag{5.2.23}$$

where $z=Q+\mathrm{i}a$ is the momentum around the loop that defines the pants decomposition.

To construct an operator that acts on the normalized conformal blocks $\mathcal{G}(a,m;\tau)$, we have to conjugate by the one-loop contribution (5.2.6). In Appendix C we perform this conjugation to find

$$\widetilde{T}^{\mathrm{CFT}}_{(1)}\cdot\mathcal{G}(a) = \sum_{j=1}^{N}\prod_{k\neq j}^{N}\frac{\sin\pi b\left(\frac{q}{2}-\mathrm{i}a_{jk}-\mathrm{i}m\right)}{\sin\pi b\left(-\mathrm{i}a_{jk}\right)}\,\mathcal{G}\left(a+\mathrm{i}bh_j\right), \tag{5.2.24}$$

which is precisely equal to the three-dimensional operator $G^{(3\mathrm{d})}_{(1)}$ for any real b (see equation (5.1.8)). This provides another check on the relation of the difference operators $G^{(3\mathrm{d})}_{R}$ to the 't Hooft loop operators for the fundamental representation.

Part II

Supersymmetric gauge theories in five dimensions and their gravity duals

Chapter 6

Supersymmetric gauge theories on squashed five-spheres and their gravity duals

In this chapter we construct the gravity duals of large N supersymmetric gauge theories defined on squashed five-spheres with $SU(3) \times U(1)$ symmetry. These five-sphere backgrounds are continuously connected to the round sphere, and we find a one-parameter family of 3/4 BPS deformations and a two-parameter family of (generically) 1/4 BPS deformations. The gravity duals are constructed in Euclidean Romans $F(4)$ gauged supergravity in six dimensions, and uplift to massive type IIA supergravity. We holographically renormalize the Romans theory, and use our general result to compute the renormalized on-shell actions for the solutions. The results agree perfectly with the large N limit of the dual gauge theory partition function, which we compute using large N matrix model techniques. In addition we compute BPS Wilson loops in these backgrounds, both in supergravity and in the large N matrix model, again finding precise agreement. Finally, we conjecture a general formula for the partition function on any five-sphere background, which for fixed gauge theory depends only on a certain supersymmetric Killing vector.

6.1 Supersymmetric gauge theories on squashed five-spheres

We begin first by describing the squashed five-sphere backgrounds of interest [103]. One can define a supersymmetric gauge theory with general matter content on such a background, and in [102] the perturbative partition function was computed via a twisted reduction of the supersymmetric index in six dimensions,[1] that we summarize in section 6.1.2. As discussed in the introduction to this thesis, a particular class of five-dimensional gauge theories, with gauge group $USp(2N)$ arising from a D4-D8-brane system in massive type IIA string theory, is expected to have a large N limit with a gravity dual. In section 6.1.3 we compute the large N limit of the partition function for these theories using matrix model techniques.

6.1.1 $SU(3) \times U(1)$ squashed five-sphere

The squashed S^5 backgrounds of interest are homogeneous spaces with symmetry $SU(3) \times U(1)$. In particular this is the isometry group of the metric

$$\begin{aligned} \mathrm{d}s_5^2 &= \frac{1}{s^2}(\mathrm{d}\tau + C)^2 + \mathrm{d}\sigma^2 + \frac{1}{4}\sin^2\sigma(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \\ &\quad + \frac{1}{4}\cos^2\sigma\sin^2\sigma(\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi)^2 \,, \end{aligned} \tag{6.1.1}$$

where we have defined the (local) one-form

$$C = -\frac{1}{2}\sin^2\sigma(\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi)\,. \tag{6.1.2}$$

We refer to the parameter s as a *squashing parameter*, and note that $s = 1$ is the round sphere. The coordinates in (6.1.1) realize the five-sphere as the total space of the Hopf circle bundle over $\mathbb{CP}^2$, where τ is a 2π-period coordinate along the circle fibre. The coordinates

[1] See also [154].

$\sigma, \psi, \theta, \varphi$ are then coordinates on the base $\mathbb{CP}^2$, with ψ having period 4π, φ having period 2π, while $\sigma \in [0, \frac{\pi}{2}]$, $\theta \in [0, \pi]$. The local one-form C in (6.1.2) satisfies

$$\mathrm{d}C \;\equiv\; 2\omega \;=\; -\sin\sigma\cos\sigma\mathrm{d}\sigma\wedge(\mathrm{d}\psi+\cos\theta\mathrm{d}\varphi)+\frac{1}{2}\sin^2\sigma\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi\,, \qquad (6.1.3)$$

where ω is the Kähler two-form on $\mathbb{CP}^2$.

In order to preserve supersymmetry one must also turn on other backgrounds fields. In particular in [103] it was shown that one can define general supersymmetric gauge theories on the above squashed five-sphere, provided one turns on a background $SU(2)_R$ gauge field

$$\mathcal{A} \;=\; \frac{(1+Q\sqrt{1-s^2})\sqrt{1-s^2}}{s^2}(\mathrm{d}\tau + C)\,, \qquad (6.1.4)$$

where we have embedded $U(1)_R \subset SU(2)_R$. More precisely, writing the $SU(2)_R \sim SO(3)_R$ gauge field as a triplet of one-forms $\mathcal{A}^i$, $i = 1, 2, 3$, we have $\mathcal{A}^1 = \mathcal{A}^2 = 0$, while $\mathcal{A}^3 = \mathcal{A}$ is given by (6.1.4). For supersymmetric backgrounds the parameter Q takes the values $Q = 1$ and $Q = -3$, which lead to 3/4 BPS and 1/4 BPS solutions, respectively. Notice that the gauge field (6.1.4) is also invariant under $SU(3) \times U(1)$, and is real when $|s| < 1$ but complex for $|s| > 1$.

A supersymmetric background of course admits an appropriate Killing spinor, which then enters the supersymmetry transformations of a supersymmetric gauge theory defined on the background. Recall that a Killing spinor χ on the round S^5 with $s = 1$, solving $\nabla_m\chi = -\frac{\mathrm{i}}{2}\gamma_m\chi$ where γ_m generate the Clifford algebra Cliff(5, 0) in an orthonormal frame, transforms in the $\mathbf{4}$ of the $SU(4) \sim SO(6)$ isometry. The squashing breaks this symmetry to $SU(3) \times U(1)$, and for $Q = 1$ the resulting Killing spinor transforms as $\mathbf{3}_{+1}$, while for $Q = -3$ the resulting Killing spinor instead transforms as $\mathbf{1}_{-3}$. Similarly, solutions to $\nabla_m\chi = \frac{\mathrm{i}}{2}\gamma_m\chi$ transform in the $\overline{\mathbf{4}}$ of $SU(4)$, which is broken to $\overline{\mathbf{3}}_{-1}$ and $\mathbf{1}_{+3}$ in the two cases, respectively.

The corresponding Killing spinor equation for the squashed S^5 was obtained in [103] via a twisted reduction (described in the next subsection) of a standard Killing spinor equation in six dimensions. In order to write this down, we first introduce an orthonormal frame for the metric (6.1.1)

$$\begin{aligned} e^1_{(5)} &= \frac{1}{s}(\mathrm{d}\tau + C)\,, \qquad & e^2_{(5)} &= \mathrm{d}\sigma\,, \qquad & e^3_{(5)} &= \frac{1}{2}\sin\sigma\cos\sigma\,\tau_3\,, \\ e^4_{(5)} &= \frac{1}{2}\sin\sigma\,\tau_2\,, & e^5_{(5)} &= \frac{1}{2}\sin\sigma\,\tau_1\,, \end{aligned} \tag{6.1.5}$$

where τ_i, $i = 1, 2, 3$, are left-invariant one-forms on $SU(2)$. These are parametrized in terms of the Euler angles as

$$\tau_1 + \mathrm{i}\tau_2 \;=\; \mathrm{e}^{-\mathrm{i}\psi}(\mathrm{d}\theta + \mathrm{i}\sin\theta\mathrm{d}\varphi)\,, \qquad \tau_3 \;=\; \mathrm{d}\psi + \cos\theta\,\mathrm{d}\varphi\,. \tag{6.1.6}$$

The Killing spinor equation then reads

$$\begin{aligned} \nabla_m\chi_I + \tfrac{\mathrm{i}}{2}\mathcal{A}^i_m(\sigma^i)_I{}^J\chi_J \;&=\; -\frac{\mathrm{i}\left(1 + Q\sqrt{1-s^2}\right)}{2s}(\sigma^3)_I{}^J\gamma_m\chi_J \\ &\quad + \frac{\sqrt{1-s^2}}{4s}\left(3\gamma_m\not{\omega} - \not{\omega}\gamma_m\right)\chi_I\,, \end{aligned} \tag{6.1.7}$$

which is supplemented by the following algebraic equation

$$Q\sqrt{1-s^2}\chi_I \;=\; -\sqrt{1-s^2}\gamma_1\chi_I - \mathrm{i}\sqrt{1-s^2}(\sigma^3)_I{}^J\not{\omega}\chi_J\,. \tag{6.1.8}$$

Here χ_I, $I = 1, 2$, form a doublet under the $SU(2)_R$ symmetry, γ_m generate the Clifford algebra Cliff$(5, 0)$ in the orthonormal frame (6.1.5), and $(\sigma^i)_I{}^J$ denote the Pauli matrices. Recall also that ω denotes the Kähler form on $\mathbb{CP}^2$, given by (6.1.3), and if α is a p-form we denote $\not{\alpha} \equiv \frac{1}{p!}\alpha_{m_1\cdots m_p}\gamma^{m_1\cdots m_p}$.

Of course in the case at hand we have that the $SU(2)_R$ gauge field $\mathcal{A}^i$ is only turned on in the $i = 3$ direction, with $\mathcal{A}^3 = \mathcal{A}$ given by (6.1.4), and we may also write (6.1.7) and

(6.1.8) as

$$\nabla_m \chi_\pm \pm \tfrac{\mathrm{i}}{2}\mathcal{A}_m \chi_\pm \;=\; \mp \frac{\mathrm{i}\left(1+Q\sqrt{1-s^2}\right)}{2s}\gamma_m \chi_\pm + \frac{\sqrt{1-s^2}}{4s}\left(3\gamma_m \not{b} - \not{b}\gamma_m\right)\chi_\pm\,, \tag{6.1.9}$$

$$Q\sqrt{1-s^2}\chi_\pm \;=\; -\sqrt{1-s^2}\,\gamma_1 \chi_\pm \mp \mathrm{i}\sqrt{1-s^2}\,\not{b}\chi_\pm\,, \tag{6.1.10}$$

where $\chi_+ = \chi_1$, $\chi_- = \chi_2$. Provided the background fields are real, meaning in particular that the metric and $\mathcal{A}$ are real and $|s| < 1$, then notice that the equations for χ_- are simply the charge conjugates of the χ_+ equations, where we define the charge conjugate as

$$\chi^c \;\equiv\; \mathcal{C}_5 \chi^*\,, \tag{6.1.11}$$

and the charge conjugation matrix $\mathcal{C}_5$ satisfies $\mathcal{C}_5^{-1}\gamma_m\mathcal{C}_5 = \gamma_m^*$. In particular it is then consistent to impose the symplectic Majorana condition $\chi_- = \chi_+^c$, or equivalently $\varepsilon_I{}^J\chi_J = \mathcal{C}_5\chi_I^*$, as we shall see below.

Notice that in setting $s = 1$ to obtain the round sphere one has that (6.1.8) is trivially satisfied, while the Killing spinor equation (6.1.7) implies that χ_1 and χ_2 transform in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of the enhanced $SU(4) \sim SO(6)$ symmetry, respectively. In order to present the general solution to (6.1.7), (6.1.8), we first introduce the following basis of Cliff(5, 0)

$$\gamma_1 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & \mathrm{i}\sigma^3 \\ -\mathrm{i}\sigma^3 & 0 \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} 0 & \mathrm{i}\sigma^2 \\ -\mathrm{i}\sigma^2 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & \mathrm{i}\sigma^1 \\ -\mathrm{i}\sigma^1 & 0 \end{pmatrix},$$

where as above σ^i, $i = 1, 2, 3$ denote the Pauli matrices, and 1_2 is the 2×2 identity matrix. A choice of the charge conjugation matrix in this basis is

$$\mathcal{C}_5 \;=\; \begin{pmatrix} -\mathrm{i}\sigma_2 & 0 \\ 0 & -\mathrm{i}\sigma_2 \end{pmatrix}. \tag{6.1.12}$$

Then for the 1/4 BPS background we find the general solution to (6.1.7), (6.1.8) (or equivalently (6.1.9), (6.1.10)) is given by

$$\chi_+ \;=\; c_+ \mathrm{e}^{-\frac{3\mathrm{i}\tau}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \chi_- \;=\; c_- \mathrm{e}^{\frac{3\mathrm{i}\tau}{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{6.1.13}$$

where $c_\pm$ are constants of integration. In particular then notice that the symplectic Majorana condition $\chi_- = \chi_+^c$ simply imposes $c_- = c_+^*$.

For the 3/4 BPS background the solution is a little more complicated. One finds

$$\chi_+ \;=\; a_+^{(1)}\,\mathrm{e}^{\mathrm{i}\frac{\tau}{2}} \begin{pmatrix} \cos\sigma + \mathrm{i}\lambda_+(s)\mathrm{e}^{\mathrm{i}\frac{\psi}{2}} S_+^{(1)} \sin\sigma \\ 0 \\ \mathrm{i}\lambda_-(s)\sin\sigma - \mathrm{e}^{\mathrm{i}\frac{\psi}{2}} S_+^{(1)}\cos\sigma \\ -\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}} S_+^{(2)} \end{pmatrix}, \tag{6.1.14}$$

where

$$\begin{aligned} S_\pm^{(1)} \;&=\; S_\pm^{(1)}(\theta,\varphi) \;=\; a_\pm^{(3)}\mathrm{e}^{\pm\mathrm{i}\frac{\varphi}{2}}\cos\frac{\theta}{2} - a_\pm^{(2)}\mathrm{e}^{\mp\mathrm{i}\frac{\varphi}{2}}\sin\frac{\theta}{2}, \\ S_\pm^{(2)} \;&=\; S_\pm^{(2)}(\theta,\varphi) \;=\; a_\pm^{(2)}\mathrm{e}^{\mp\mathrm{i}\frac{\varphi}{2}}\cos\frac{\theta}{2} + a_\pm^{(3)}\mathrm{e}^{\pm\mathrm{i}\frac{\varphi}{2}}\sin\frac{\theta}{2}, \end{aligned} \tag{6.1.15}$$

and where we have introduced $\lambda_\pm(s) \equiv (\pm 1 + \sqrt{1-s^2})/s$. As expected, the solution depends on three integration constants $a_+^{(1)}, a_+^{(2)}, a_+^{(3)}$. Similarly, one finds

$$\chi_- \;=\; a_-^{(1)}\,\mathrm{e}^{-\mathrm{i}\frac{\tau}{2}} \begin{pmatrix} 0 \\ \cos\sigma - \mathrm{i}\lambda_+(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}} S_-^{(1)}\sin\sigma \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\psi}{2}} S_-^{(2)} \\ -\mathrm{i}\lambda_-(s)\sin\sigma - \mathrm{e}^{-\mathrm{i}\frac{\psi}{2}} S_-^{(1)}\cos\sigma \end{pmatrix}, \tag{6.1.16}$$

where $a_-^{(i)}$ are integration constants. One can once again impose the symplectic Majorana condition, which leads to the relation $(a_-^{(i)})^* = a_+^{(i)}$ for $i = 1, 2, 3$.

6.1.2 Twisted reduction and the partition function

The backgrounds above may be obtained via a twisted reduction of $\mathbb{R}\times S^5$, starting from the *round* metric on S^5. This is important, as the perturbative partition function on the squashed five-spheres was computed in [102] indirectly, by taking a limit of the supersymmetric index of a corresponding six-dimensional theory on $\mathbb{R}\times S^5$.

We thus begin with the product metric on $\mathbb{R}$ times the round S^5

$$\mathrm{d}s^2_{\mathbb{R}\times S^5} \;=\; \mathrm{d}t^2+\sum_{i=1}^{3}|\mathrm{d}w_i|^2\,, \tag{6.1.17}$$

where the complex coordinates w_i on $\mathbb{C}^3\cong\mathbb{R}^6$, $i=1,2,3$, satisfy the constraint $\sum_{i=1}^3|w_i|^2=1$. We then compactify this space by identifying

$$(t,w_i)\sim(t+\beta,\mathrm{e}^{\mathrm{i}\mu_i\beta}w_i)\,, \tag{6.1.18}$$

where $\beta>0$ and the μ_i are also sometimes referred to as squashing parameters. Notice that (6.1.18) is an isometry for $\mu_i\in\mathbb{R}$. We may then change coordinates

$$\rho_i\,\mathrm{e}^{\mathrm{i}\varphi_i} \;\equiv\; \mathrm{e}^{-\mathrm{i}\mu_i t}w_i\,, \tag{6.1.19}$$

where $\rho_i\geq 0$ and the φ_i have period 2π. In terms of these new coordinates the identification (6.1.18) reads $(t,\rho_i,\varphi_i)\sim(t+\beta,\rho_i,\varphi_i)$. We then dimensionally reduce along the t-direction to obtain the five-dimensional metric

$$\mathrm{d}s^2_5 \;=\; \sum_{i=1}^{3}(\mathrm{d}\rho_i^2+\rho_i^2\mathrm{d}\varphi_i^2)-\frac{1}{1+\sum_{i=1}^3\mu_i^2\rho_i^2}\left(\sum_{i=1}^{3}\mu_i\rho_i^2\mathrm{d}\varphi_i\right)^2\,. \tag{6.1.20}$$

One then makes contact with the previous section by choosing

$$\begin{aligned} -\mu_1=\;\mu_2\;=\;\mu_3\;&=\;\mathrm{i}\sqrt{1-s^2}\,, && 3/4\text{ BPS}\,,\\ \mu_1=\;\mu_2\;=\;\mu_3\;&=\;-\mathrm{i}\sqrt{1-s^2}\,, && 1/4\text{ BPS}\,. \end{aligned} \tag{6.1.21}$$

Notice these are real only if $|s| \geq 1$. The metric (6.1.20) then agrees with the metric (6.1.1) on making the standard polar coordinate identifications

$$\rho_1 \;=\; \cos\sigma\,, \qquad \rho_2 \;=\; \sin\sigma\cos\frac{\theta}{2}\,, \qquad \rho_3 \;=\; \sin\sigma\sin\frac{\theta}{2}\,, \tag{6.1.22}$$

together with

$$\begin{aligned} \varphi_1 = \; -\tau\,, \quad \varphi_2 \;&=\; \tau - \frac{1}{2}(\psi+\varphi)\,, \quad \varphi_3 \;=\; \tau - \frac{1}{2}(\psi-\varphi)\,, \quad 3/4 \text{ BPS}\,, \\ \varphi_1 = \; \tau\,, \quad \varphi_2 \;&=\; \tau - \frac{1}{2}(\psi+\varphi)\,, \quad \varphi_3 \;=\; \tau - \frac{1}{2}(\psi-\varphi)\,, \quad 1/4 \text{ BPS}\,. \end{aligned} \tag{6.1.23}$$

The Killing spinor equation (6.1.7) and algebraic equation (6.1.8) were then obtained in [103] by dimensionally reducing a standard Killing spinor equation on the $\mathbb{R}\times S^5$ background (6.1.17).

In practice the perturbative contribution to the squashed S^5 partition function, with more general squashed metric (6.1.20), was computed in [102] by dimensionally reducing the superconformal index of a corresponding six-dimensional theory on the $\mathbb{R}\times S^5$ background (6.1.17) with twisted identification (6.1.18), and then taking the limit $\beta \to 0$, so that the radius of the circle we reduced on to obtain (6.1.20) is sent to zero. For a gauge theory with gauge group G, prepotential $\mathscr{F}$ – a cubic polynomial in the vectormultiplet scalar σ – and matter in the real representation $R \oplus \bar{R}$ of G, the result is

$$Z_{\text{pert}} \;=\; C(\mathbf{b}) \prod_{a=1}^{\text{rank } G} \int_{-\infty}^{\infty} \mathrm{d}\sigma_a \, \mathrm{e}^{-\frac{(2\pi)^3}{b_1 b_2 b_3}\mathscr{F}(\sigma)} \frac{\prod_\alpha S_3\left(-\mathrm{i}\alpha(\sigma) \mid \mathbf{b}\right)}{\prod_\rho S_3\left(-\mathrm{i}\rho(\sigma) + \frac{1}{2}(b_1+b_2+b_3) \mid \mathbf{b}\right)}\,. \tag{6.1.24}$$

Here we have introduced

$$\mathbf{b} \;=\; (b_1, b_2, b_3)\,, \qquad \text{with} \qquad b_i \;=\; 1 + \mathrm{i}\mu_i\,, \tag{6.1.25}$$

and the prefactor $C(\mathbf{b})$ in (6.1.24) depends only on (b_1, b_2, b_3), and in particular will not contribute to the large N limit of interest in the next section.[2] The perturbative partition

[2]The precise formula for $C(\mathbf{b})$ may be found in [102].

function thus localizes onto field configurations in which the only non-zero field is a constant mode for the scalar σ in the vector multiplet, and this is then integrated over in (6.1.24). As usual in such expressions the product over α in the numerator is over roots of G, while the product over ρ in the denominator is over the weights of R. Finally, $S_3\,(z \mid \mathbf{b})$ is the triple sine function, which is a special case of the multiple sine functions defined by

$$S_{\mathcal{N}}\,(z \mid \mathbf{b}) \;\equiv\; \Gamma_{\mathcal{N}}(z \mid \mathbf{b})^{-1}\;\Gamma_{\mathcal{N}}(b_{\rm tot} - z \mid \mathbf{b})^{(-1)^{\mathcal{N}}}\,, \qquad (6.1.26)$$

where we have written $\mathbf{b} = (b_1, \ldots, b_{\mathcal{N}})$ and defined $b_{\rm tot} = \sum_{i=1}^{\mathcal{N}} b_i$. The function $\Gamma_{\mathcal{N}}(z \mid \mathbf{b})$ is the so-called Barnes' multiple gamma function

$$\Gamma_{\mathcal{N}}(z \mid \mathbf{b}) \;\;\equiv\;\; \prod_{n_1,\ldots,n_{\mathcal{N}}=0}^{\infty} \left[\sum_{i=1}^{\mathcal{N}} n_i b_i + z\right]^{-1} . \qquad (6.1.27)$$

We conclude this section by noting from (6.1.21) and (6.1.25) that for the $SU(3)\times U(1)$ squashed five-spheres in section 6.1.1

$$\begin{aligned} b_1 &= \;\; 1+\sqrt{1-s^2}\,, \qquad b_2 \;=\; b_3 \;=\; 1-\sqrt{1-s^2}\,, \qquad && 3/4 \text{ BPS}\,, \\ b_1 &= \;\; b_2 \;=\; b_3 \;=\; 1+\sqrt{1-s^2}\,, && 1/4 \text{ BPS}\,. \end{aligned} \qquad (6.1.28)$$

In particular it is straightforward to see [102] that in the 1/4 BPS case the perturbative partition function (6.1.24) is independent of the squashing parameter s.

6.1.3 The large N limit

The result for the perturbative partition function (6.1.24) in the previous section is valid for a general supersymmetric gauge theory in five dimensions. We now focus on a particular class of theories with gauge group $G = USp(2N)$, that arises from a system of N D4-branes and some number of D8-branes and orientifold planes in massive type IIA string theory. As briefly reviewed in the introduction, these theories are expected to have a large N limit that has a dual description in massive type IIA supergravity [95–97]. Indeed, in [101] the large

N limit of the partition function of these theories on the *round* five-sphere was computed and successfully compared to the entanglement entropy of the dual warped $\mathrm{AdS}_6 \times S^4$ supergravity solution. Here the gauge theories flow to a UV superconformal fixed point, and in particular the localization computation in the IR supersymmetric Yang-Mills theory coupled to matter theory successfully reproduces the expected $N^{5/2}$ scaling of the number of degrees of freedom.

In general one certainly expects non-perturbative contributions to the full partition function Z, in addition to the perturbative result (6.1.24). In particular in the localization computation of [107] on the round five-sphere one finds that the gauge multiplet localizes onto instanton configurations on $\mathbb{CP}^2$. There is thus a non-perturbative contribution to Z involving a sum over the instanton number. For fixed instanton number $n \neq 0$ and fixed choice of instanton, in addition to the classical instanton action there will also be one-loop determinant contributions around that instanton, plus an integral over the instanton moduli space with fixed n. In general this expression will be very difficult to evaluate. However, in [101] it was argued that in the large N limit these instanton contributions should be suppressed. We shall also assume this to be the case on the squashed five-sphere, although clearly this issue deserves further study. In particular, for general choice of the vector $\mathbf{b} = (b_1, b_2, b_3)$ we expect to find instantons not on $\mathbb{CP}^2$, but rather instantons transverse to the Killing vector $K = \sum_{i=1}^{3} b_i \partial_{\varphi_i}$, as in [155]. These *contact instantons* were discussed in the latter reference in the context of the partition function on Sasaki-Einstein manifolds.

Our task thus reduces to computing the large N limit of the perturbative result (6.1.24), for the $USp(2N)$ gauge theories of interest. This may be carried out using the matrix model saddle point method originally introduced in [86], and subsequently applied to the round S^5 partition function in [101]. As in the latter reference, we also set the Chern-Simons level for the theory $k = 0$ (thus setting the cubic terms in the prepotential $\mathscr{F}(\sigma)$ to zero). The quadratic and linear terms of $\mathscr{F}(\sigma)$ will only contribute to subleading order in the large N limit. This is because the leading contribution to the free energy arises from the scaling $\sigma \sim \mathcal{O}(N^{1/2})$. Such a behaviour for σ leads to an $\mathcal{O}(N^2)$ contribution for the classical parts in the perturbative partition function (6.1.24). Thus in the limit of large N we only have to analyse the behaviour of the two one-loop determinants from the vector and matter multiplets. In particular, for a given theory we will have to find the expansion of the logarithm of the triple sine function entering (6.1.24).

The $USp(2N)$ gauge theories have N_f matter fields in the fundamental and a single hypermultiplet in the antisymmetric representation of the gauge group. Let us denote an element in the Cartan subalgebra for $USp(2N)$ as $\{\lambda_1, \ldots, \lambda_N\}$, so that $\sigma = \mathrm{diag}(\lambda_1, \ldots, \lambda_N, -\lambda_1, \ldots, -\lambda_N)$. The Weyl group acts as $\lambda_i \to -\lambda_i$ for each i, and also permutes the λ_i. If the normalized weights of the fundamental representation are given by $\pm e_i$, where $\{e_1, \ldots, e_N\}$ is a basis of $\mathbb{R}^N$, then the antisymmetric representation has weights $\{e_i \pm e_j\}_{i\neq j}$ and the adjoint representation has weights $\{e_i \pm e_j\}_{i\neq j} \cup \{\pm 2e_i\}_{i=1}^N$. Therefore we can write the free energy for this theory as

$$\begin{aligned}\mathcal{F}(\lambda_i) \;=\; & \sum_{\substack{i,j=1\\ i\neq j}}^{N} G_V(\lambda_i+\lambda_j \mid \mathbf{b}) + G_V(\lambda_i-\lambda_j \mid \mathbf{b}) + G_H(\lambda_i+\lambda_j \mid \mathbf{b}) + G_H(\lambda_i-\lambda_j \mid \mathbf{b}) \\ & + \sum_{i=1}^{N} G_V\left(2\lambda_i \mid \mathbf{b}\right) + G_V\left(-2\lambda_i \mid \mathbf{b}\right) + N_f\big[G_H\left(\lambda_i \mid \mathbf{b}\right) + G_H\left(-\lambda_i \mid \mathbf{b}\right)\big]\,, \qquad (6.1.29)\end{aligned}$$

where G_V and G_H are the logarithms of the triple sine functions in the numerator and denominator of (6.1.24) for the vector and the hypermultiplets, respectively. We are interested in their asymptotics for large λ_i only, because we assume that the eigenvalues scale with N^α for some $\alpha > 0$. These asymptotics are explicitly computed in [5], and here we simply quote the results:

$$\begin{aligned}G_V(x \mid \mathbf{b}) + G_V(-x \mid \mathbf{b}) \;=\; & -\log S_3\left(-\mathrm{i}x \mid \mathbf{b}\right) - \log S_3\left(\mathrm{i}x \mid \mathbf{b}\right) \\ \sim\; & \frac{\pi}{3\,b_1b_2b_3}|x|^3 - \frac{\pi\left(b_{\mathrm{tot}}^2 + b_1b_2 + b_1b_3 + b_2b_3\right)}{6\,b_1b_2b_3}|x|\,, \qquad (6.1.30)\end{aligned}$$

where we have expanded in the limit $|x| \to \infty$. Here we have used the fact that $b_i > 0$ for each $i = 1, 2, 3$, as this is the case of interest. Similarly, for the free energy contribution of the hypermultiplet we obtain

$$G_H(x \mid \mathbf{b}) \;=\; \log S_3\left(\tfrac{1}{2}b_{\rm tot} - {\rm i}x \mid \mathbf{b}\right) \;\sim\; -\frac{\pi}{6\, b_1 b_2 b_3}|x|^3 - \frac{\pi\,(b_1^2+b_2^2+b_3^2)}{24\, b_1 b_2 b_3}|x| \tag{6.1.31}$$

in the asymptotic limit $|x| \to \infty$.

Using the Weyl symmetry of $USp(2N)$ we may take $\lambda_i \geq 0$, and we shall furthermore assume that these eigenvalues scale as $\lambda_i = N^\alpha x_i$ to leading order in the large N limit, with $\alpha > 0$. We next introduce the density $\rho(x) = \frac{1}{N}\sum_{i=1}^N \delta\left(x - x_i\right)$, which becomes an integrable function with

$$\int \rho(x){\rm d}x \;=\; 1\,, \tag{6.1.32}$$

upon taking $N \to \infty$. In that limit, the discrete sums in (6.1.29) become Riemann integrals

$$\frac{1}{N}\sum_{i=1}^{N} \;\longrightarrow\; \int_0^{x_\star} \rho(x){\rm d}x\,. \tag{6.1.33}$$

Hence taking the large N limit of (6.1.29), we obtain to leading order

$$\begin{aligned}\mathcal{F} \;\approx\;& N^2\int_0^{x_\star}\rho(x)\int_0^{x_\star}\rho(y)\Big[G_V(\lambda(x)\pm\lambda(y)\mid\mathbf{b}) + G_H(\lambda(x)\pm\lambda(y)\mid\mathbf{b})\Big]{\rm d}y\,{\rm d}x\\ &+N\int_0^{x_\star}\rho(x)\Big[G_V(\pm 2\lambda(x)\mid\mathbf{b}) + N_f\, G_H(\pm\lambda(x)\mid\mathbf{b})\Big]{\rm d}x\,.\end{aligned} \tag{6.1.34}$$

By assumption we have $\lambda(x) = N^\alpha x$ to leading order in the continuum limit, and hence we may use the above expansions for the vector and hypermultiplet contributions (6.1.30), (6.1.31) respectively. Then the leading order term in the first line of (6.1.34) scales as $N^{2+\alpha}$, because the cubic terms in the asymptotic expansion of G_H and G_V cancel. The leading order term of the second line in (6.1.34) however does not cancel, and is given by $N^{1+3\alpha}$. In order to obtain a non-trivial saddle point, both terms must contribute and we deduce that $\alpha = 1/2$.[3] Putting everything together we obtain

$$\mathcal{F} = -N^{5/2}\int_0^{x_\star}\int_0^{x_\star}\rho(x)\rho(y)\left[\frac{\pi b_{\rm tot}^2}{8b_1b_2b_3}\left(|x+y|+|x-y|\right)-\frac{(8-N_f)\pi}{3b_1b_2b_3}|x|^3\right]\mathrm{d}y\mathrm{d}x+\mathcal{O}(N^{3/2})\,.$$

It thus remains to solve a simple variational problem for $\rho(x)$ extremizing the free energy. We add a Lagrange multiplier term to impose the constraint (6.1.32), namely $\mu\left(\int_0^{x_\star}\rho(x)\mathrm{d}x-1\right)$, and then solve $\frac{\partial F}{\partial\rho}=0$ for $\rho(x)$. Doing so we find (with $N_f<8$)

$$\rho(x) \;=\; \frac{4(8-N_f)}{b_{\rm tot}^2}\,|x|\,, \qquad (6.1.35)$$

inside the interval $[0,x_\star]$, with ρ identically zero outside this interval, and where extremizing $\mathcal{F}$ over the end-point $x_\star$ gives

$$x_\star^2 \;=\; \frac{b_{\rm tot}^2}{2(8-N_f)}\,. \qquad (6.1.36)$$

We may then evaluate the free energy by substituting these saddle point configurations back into (6.1.34) to obtain

$$\mathcal{F} \;=\; -\frac{\sqrt{2}\pi b_{\rm tot}^3}{15\sqrt{8-N_f}\;b_1b_2b_3}N^{5/2}+\mathcal{O}\left(N^{3/2}\right)\,, \qquad (6.1.37)$$

which may be rewritten as (where recall we have assumed that $b_i>0$ for each $i=1,2,3$)

$$\mathcal{F} \;=\; \frac{(b_1+b_2+b_3)^3}{27b_1b_2b_3}\mathcal{F}_{S^5_{\rm round}}\,, \qquad (6.1.38)$$

where $\mathcal{F}_{S^5_{\rm round}}$ is the large N limit of the free energy on the round five-sphere computed in reference [101]

$$\mathcal{F}_{S^5_{\rm round}} \;=\; -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}}+\mathcal{O}\left(N^{3/2}\right)\,. \qquad (6.1.39)$$

[3]This can be confirmed by a numerical analysis.

We note that the above result has a very similar structure to that obtained in three dimensions [156]. Also notice that we get the same result, (6.1.38), for the orbifold theories discussed in [97, 101].

We conclude this section by noting that for the $SU(3) \times U(1)$ squashed five-spheres, with the vector $\mathbf{b} = (b_1, b_2, b_3)$ given by (6.1.28), we obtain the large N free energies

$$\mathcal{F} = \begin{cases} \dfrac{1}{27s^2}\dfrac{(3-\sqrt{1-s^2})^3}{1-\sqrt{1-s^2}}\mathcal{F}_{S^5_{\text{round}}}, & 3/4 \text{ BPS} \quad , \\ \mathcal{F}_{S^5_{\text{round}}}, & 1/4 \text{ BPS} \quad . \end{cases} \tag{6.1.40}$$

6.2 Romans $F(4)$ supergravity

When the $USp(2N)$ superconformal theories discussed in section 6.1 are put on the round S^5, they are conjectured to be dual in the large N limit to the $\text{AdS}_6 \times S^4$ solution of massive type IIA supergravity [95–97]. In order to find gravity duals to the same superconformal theories put on different background five-manifolds, it is then natural to work in the six-dimenional Romans $F(4)$ supergravity theory [100]. The key here is that, as shown in [99], the Romans theory is a consistent truncation of massive type IIA supergravity on S^4. In the next subsection we shall review this uplift to ten dimensions, and then present the Romans theory in Euclidean signature in section 6.2.2.

6.2.1 Uplift to massive type IIA

The Romans theory [100] is a six-dimensional gauged supergravity that admits an AdS_6 vacuum. The bosonic fields consist of the metric, a dilaton ϕ, a two-form potential B, a one-form potential A, together with an $SU(2) \sim SO(3)$ gauge field A^i, $i = 1, 2, 3$. It is convenient to introduce the scalar field $X \equiv \exp(-\phi/2\sqrt{2})$, and we define the field strengths as $H = \mathrm{d}B$, $F = \mathrm{d}A + \frac{2}{3}gB$, $F^i = \mathrm{d}A^i - \frac{1}{2}g\varepsilon_{ijk}A^j \wedge A^k$. Here g denotes the gauge coupling constant. Notice that B appears in the field strength for A.

As shown in [99], this Romans theory is a consistent truncation of massive type IIA supergravity on S^4. This means that any solution to the Romans theory automatically uplifts, via the non-linear Kaluza-Klein ansatz of [99] presented in (6.2.1) below, to a solution of massive type IIA. Moreover, the $\mathrm{AdS}_6 \times S^4$ solution of the latter is the uplift of the AdS_6 vacuum of the Romans theory.

In section 6.5, we shall need some details of how the six-dimensional solutions uplift to ten dimensions. The gauge coupling constant g is related to the ten-dimensional mass parameter by $m_{\mathrm{IIA}} = \frac{\sqrt{2}}{3} g$, while the remaining fields uplift via

$$\begin{aligned}
\mathrm{d}s^2_{10} &= (\sin\xi)^{\frac{1}{12}} X^{\frac{1}{8}} \left[\Delta^{\frac{3}{8}} \mathrm{d}s^2_6 + 2g^{-2}\Delta^{\frac{3}{8}} X^2 \mathrm{d}\xi^2 + \tfrac{1}{2} g^{-2}\Delta^{-\frac{5}{8}} X^{-1}\cos^2\xi \sum_{i=1}^{3} (\hat{\tau}^i - gA^i)^2 \right] , \\
F_{(4)} &= -\tfrac{\sqrt{2}}{6} g^{-3} s^{1/3} c^3 \Delta^{-2} U \,\mathrm{d}\xi \wedge \mathrm{vol}_3 - \sqrt{2} g^{-3} s^{4/3} c^4 \Delta^{-2} X^{-3}\,\mathrm{d}X \wedge \mathrm{vol}_3 \\
&\quad + \sqrt{2} g^{-1} s^{1/3} c X^4 *H \wedge \mathrm{d}\xi - \tfrac{1}{\sqrt{2}} s^{4/3} X^{-2} *F + \tfrac{1}{\sqrt{2}} g^{-2} s^{1/3} c\, F^i h^i \wedge \mathrm{d}\xi \\
&\quad - \tfrac{1}{4\sqrt{2}} g^{-2} s^{4/3} c^2 \Delta^{-1} X^{-3} F^i \wedge h^j \wedge h^k\, \varepsilon_{ijk} , \\
F_{(3)} &= s^{2/3} H + g^{-1} s^{-1/3} c\, F \wedge \mathrm{d}\xi , \\
F_{(2)} &= \tfrac{1}{\sqrt{2}} s^{2/3} F , \\
\mathrm{e}^{\Phi} &= s^{-5/6} \Delta^{1/4} X^{-5/4} ,
\end{aligned} \qquad (6.2.1)$$

where

$$\begin{aligned}
\Delta &\equiv X\cos^2\xi + X^{-3}\sin^2\xi , \\
U &\equiv X^{-6} s^2 - 3X^2 c^2 + 4X^{-2} c^2 - 6X^{-2} .
\end{aligned} \qquad (6.2.2)$$

Here $\mathrm{d}s^2_{10}$ is the ten-dimensional metric in Einstein frame, Φ is the ten-dimensional dilaton, $F_{(3)}$ is the NS-NS three-form field strength, while $F_{(4)}$ and $F_{(2)}$ are the RR four-form and two-form field strengths, respectively. The $\hat{\tau}^i$, $i = 1, 2, 3$, are left-invariant one-forms on a copy of $SU(2) \cong S^3$. These are defined precisely as in (6.1.6), except here this S^3 is in the internal space (hence the hats). We have also defined $h^i \equiv \hat{\tau}^i - gA^i$, $\mathrm{vol}_3 \equiv h^1 \wedge h^2 \wedge h^3$, and $s = \sin\xi$ and $c = \cos\xi$. The Hodge duals in (6.2.1) are computed with respect to the six-dimensional metric $\mathrm{d}s^2_6$. This is defined on some six-manifold M_6, and the ten-dimensional metric in (6.2.1) then describes a warped product $M_6 \times S^4$. More precisely, the solution only describes "half" of a four-sphere, where the coordinate $\xi \in (0, \frac{\pi}{2}]$ is a polar coordinate for which constant $\xi \in (0, \frac{\pi}{2})$ slices are three-spheres, parametrized by Euler angles on S^3 as in (6.1.6). The solution is smooth at the north pole $\xi = \frac{\pi}{2}$, where the S^3 slices of S^4 collapse to zero size, but singular on the equator $\xi = 0$. Nevertheless, it is argued in [96,97] that the supergravity solution (6.2.1) can be trusted away from this singularity.

6.2.2 Euclidean theory

The equations of motion and action for the Romans theory in Lorentz signature appear in [99, 100]. However, the gravity duals to the large N field theories on the squashed five-sphere of section 6.1 will be constructed in Euclidean signature. The corresponding Wick rotation is not entirely straightforward because the Romans theory contains Chern-Simons-type couplings, that become purely imaginary in Euclidean signature in order that the theory is gauge invariant. The associated factors of i are also crucial for supersymmetry in Euclidean signature. The Euclidean equations of motion for the Romans supergravity fields are

$$\begin{aligned} \mathrm{d}\left(X^4 * H\right) &= \tfrac{\mathrm{i}}{2} F \wedge F + \tfrac{\mathrm{i}}{2} F^i \wedge F^i + \tfrac{2}{3} g X^{-2} * F\,, \\ \mathrm{d}\left(X^{-2} * F\right) &= -\,\mathrm{i} F \wedge H\,, \\ D\left(X^{-2} * F^i\right) &= -\,\mathrm{i} F^i \wedge H\,, \\ \mathrm{d}\left(X^{-1} * \mathrm{d}X\right) &= -\,g^2\left(\tfrac{1}{6}X^{-6} - \tfrac{2}{3}X^{-2} + \tfrac{1}{2}X^2\right) * 1 \\ &\quad -\tfrac{1}{8}X^{-2}\left(F \wedge *F + F^i \wedge *F^i\right) + \tfrac{1}{4}X^4 H \wedge *H\,. \end{aligned} \tag{6.2.3}$$

Here $D\omega^i = \mathrm{d}\omega^i - g\varepsilon_{ijk}A^j \wedge \omega^k$ is the $SO(3)$ covariant derivative, and our convention for the Hodge duality operator is fixed via $\alpha \wedge *\beta = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}\beta^{\mu_1\cdots\mu_p} * 1$,where α and β are p-forms.[4]

[4] In particular this convention differs from that in [99].

The Einstein equation is

$$\begin{aligned} R_{\mu\nu} &= 4X^{-2}\partial_\mu X\partial_\nu X + g^2\left(\tfrac{1}{18}X^{-6} - \tfrac{2}{3}X^{-2} - \tfrac{1}{2}X^2\right)g_{\mu\nu} + \tfrac{1}{4}X^4\left(H^2_{\mu\nu} - \tfrac{1}{6}H^2 g_{\mu\nu}\right) \\ &\quad + \tfrac{1}{2}X^{-2}\left(F^2_{\mu\nu} - \tfrac{1}{8}F^2 g_{\mu\nu}\right) + \tfrac{1}{2}X^{-2}\left((F^i)^2_{\mu\nu} - \tfrac{1}{8}(F^i)^2 g_{\mu\nu}\right), \end{aligned} \tag{6.2.4}$$

where $F^2_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho$ and $H^2_{\mu\nu} = H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}$.

The Euclidean action which gives rise to these field equations is

$$\begin{aligned} I_E &= -\frac{1}{16\pi G_N}\int\Big[R * 1 - 4X^{-2}\mathrm{d}X\wedge *\mathrm{d}X - g^2\left(\tfrac{2}{9}X^{-6} - \tfrac{8}{3}X^{-2} - 2X^2\right) * 1 \\ &\qquad - \tfrac{1}{2}X^{-2}\left(F\wedge *F + F^i\wedge *F^i\right) - \tfrac{1}{2}X^4 H\wedge *H \\ &\qquad - \mathrm{i}B\wedge\left(\tfrac{1}{2}\mathrm{d}A\wedge\mathrm{d}A + \tfrac{1}{3}B\wedge\mathrm{d}A + \tfrac{2}{27}g^2 B\wedge B + \tfrac{1}{2}F^i\wedge F^i\right)\Big]. \end{aligned} \tag{6.2.5}$$

In particular notice that the final term is a Chern-Simons-type coupling, and is accompanied by a factor of i. This is required for gauge-invariance in the path integral with Euclidean measure $\exp(-I_E)$. It is also implied by supersymmetry. Indeed, a solution to the above equations of motion is supersymmetric provided the following Killing spinor equation and dilatino equation hold:

$$\begin{aligned} D_\mu\epsilon_I &= \frac{\mathrm{i}}{4\sqrt{2}}g(X + \tfrac{1}{3}X^{-3})\Gamma_\mu\Gamma_7\epsilon_I - \frac{\mathrm{i}}{16\sqrt{2}}X^{-1}F_{\nu\rho}(\Gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^\nu\Gamma^\rho)\epsilon_I \qquad (6.2.6) \\ &\quad - \frac{1}{48}X^2 H_{\nu\rho\sigma}\Gamma^{\nu\rho\sigma}\Gamma_\mu\Gamma_7\epsilon_I + \frac{1}{16\sqrt{2}}X^{-1}F^i_{\nu\rho}(\Gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^\nu\Gamma^\rho)\Gamma_7(\sigma^i)_I{}^J\epsilon_J, \\ 0 &= -\mathrm{i}X^{-1}\partial_\mu X\Gamma^\mu\epsilon_I + \frac{1}{2\sqrt{2}}g\left(X - X^{-3}\right)\Gamma_7\epsilon_I + \frac{\mathrm{i}}{24}X^2 H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_7\epsilon_I \\ &\quad - \frac{1}{8\sqrt{2}}X^{-1}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon_I - \frac{\mathrm{i}}{8\sqrt{2}}X^{-1}F^i_{\mu\nu}\Gamma^{\mu\nu}\Gamma_7(\sigma^i)_I{}^J\epsilon_J. \qquad (6.2.7) \end{aligned}$$

Here ϵ_I, $I = 1, 2$, are two Dirac spinors, Γ_μ generate the Clifford algebra Cliff$(6, 0)$ in an orthonormal frame, and we have defined the chirality operator $\Gamma_7 = \mathrm{i}\Gamma_{012345}$, which satisfies $\Gamma_7^2 = 1$. The $SO(3) \sim SU(2)$ gauge field A^i is an R-symmetry gauge field, with the spinor ϵ_I transforming in the two-dimensional representation via the Pauli matrices $(\sigma^i)_I{}^J$. Thus the covariant derivative acting on the spinor is $D_\mu\epsilon_I = \nabla_\mu\epsilon_I + \frac{\mathrm{i}}{2}gA^i_\mu(\sigma^i)_I{}^J\epsilon_J$.

Returning to the equations of motion (6.2.3), notice that the exterior derivative of the first equation (the equation of motion for B) implies the second equation on using the Bianchi identities for F and F^i, where note that $\mathrm{d}F = \frac{2}{3}gH$. This is related to the fact that the theory possesses a gauge invariance $A \to A + \frac{2}{3}g\lambda$, $B \to B - \mathrm{d}\lambda$, where λ is an arbitrary one-form. Using this freedom one can then gauge away $A = 0$, leaving $F = \frac{2}{3}gB$. The kinetic term for F in the action (6.2.5) then becomes a mass term for the B-field; that is, the B-field "eats" the $U(1)$ gauge field A in a Higgs-like mechanism. Notice that there is also a cubic Chern-Simons coupling for B in (6.2.5), making it a somewhat exotic field. We may also make a simple rescaling of the fields via $g_{\mu\nu} \to \frac{1}{g^2}g_{\mu\nu}$, $B \to \frac{1}{g^2}B$, $A \to \frac{1}{g}A$, $A^i \to \frac{1}{g}A^i$, after which one sees that the coupling constant g only appears in the action as an overall constant $1/g^4$ factor. Thus we may without loss of generality set $g = 1$, which we henceforth will do.

6.2.3 Killing vector bilinear

Given a supersymmetric solution to the Euclidean Romans theory, one can verify that the bilinear

$$K_\mu \;\equiv\; \varepsilon^{IJ}\epsilon_I^T\mathcal{C}\Gamma_\mu\epsilon_J \tag{6.2.8}$$

is a Killing one-form. Here $\mathcal{C}$ is the charge conjugation matrix, satisfying $\Gamma_\mu^T = \mathcal{C}^{-1}\Gamma_\mu\mathcal{C}$. In our conventions it is antisymmetric satisfying $\mathcal{C}^2 = -1$. If we also impose a symplectic Majorana condition

$$\mathcal{C}\epsilon_I^* \;=\; \varepsilon_I{}^J\epsilon_J\,, \tag{6.2.9}$$

then this Killing one-form may be rewritten as

$$K_\mu \;=\; \epsilon_I^\dagger\Gamma_\mu\epsilon_I\,, \tag{6.2.10}$$

which is then manifestly real. In particular we will be able to impose this symplectic Majorana condition for the solutions we construct in section 6.3.

6.3 Supergravity solutions

In this section we present supergravity duals to the $SU(3) \times U(1)$ squashed five-sphere backgrounds of section 6.1. Via the consistent truncation to the Romans theory in the previous section, this effectively becomes a filling problem in six-dimensional gauged supergravity: one seeks a smooth, asymptotically locally Euclidean AdS_6 supergravity solution, with conformal boundary data given by the squashed five-sphere background in section 6.1. In particular this means that the bulk supergravity solution is equipped with an $SU(2)_R$ doublet of Killing spinors ϵ_I, $I = 1, 2$, solving (6.2.6) and (6.2.7), which should suitably approach the boundary Killing spinors in section 6.1.1. We shall indeed find such fillings for both the 3/4 BPS and 1/4 BPS solutions.The 1/4 BPS solution corresponds to a two-parameter family, containing a one-parameter 1/2 BPS subfamily of new solutions.

6.3.1 $SU(3) \times U(1)$ invariant ansatz

The squashed five-sphere backgrounds of section 6.1.1 have $SU(3) \times U(1)$ symmetry, and one expects this symmetry to be preserved by the bulk supergravity filling. Indeed, for asymptotically locally Euclidean AdS solutions of the *vacuum* Einstein equations this is a theorem [157]. This leads to the following ansatz for the Romans supergravity fields

$$
\begin{aligned}
\mathrm{d}s_6^2 &= \alpha^2(r)\mathrm{d}r^2 + \gamma^2(r)(\mathrm{d}\tau + C)^2 + \beta^2(r)\Big[\mathrm{d}\sigma^2 + \frac{1}{4}\sin^2\sigma(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \\
&\quad + \frac{1}{4}\cos^2\sigma\sin^2\sigma(\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi)^2\Big]\,, \\
B &= p(r)\mathrm{d}r \wedge (\mathrm{d}\tau + C) + \frac{1}{2}q(r)\mathrm{d}C\,, \\
A^i &= f^i(r)(\mathrm{d}\tau + C)\,, \quad i \;=\; 1, 2, 3\,,
\end{aligned}
\tag{6.3.1}
$$

together with $X = X(r)$. Recall here that we have used the gauge freedom to set the $U(1)$ gauge field (which is really a *Stueckelberg* field) to $A = 0$. The additional coordinate r is a radial coordinate, and we shall choose a parametrization in which the conformal boundary is at $r = \infty$. For fixed r, provided $\gamma(r)$ and $\beta(r)$ are non-zero the constant r surfaces in (6.3.1) are squashed five-spheres. We shall seek solutions with the topology of a ball, so that $r \in [r_0, \infty)$ with $r = r_0$ being the origin. At this point the squashed five-spheres must become *round* in order that the metric extends smoothly to the origin of the ball. Similarly, in order for the gauge fields B, A^i in (6.3.1) to be non-singular at the origin they must tend to zero sufficiently quickly at $r = r_0$. In writing the ansatz (6.3.1) we have used the fact that the only $SU(3) \times U(1)$ invariant one-form on the squashed five-sphere is the global angular form $\mathrm{d}\tau + C$ for the Hopf fibration $S^1 \hookrightarrow S^5 \to \mathbb{CP}^2$, while the only invariant two-form is the pull-back $\frac{1}{2}\mathrm{d}C = \omega$ of the Kähler form on $\mathbb{CP}^2$.

Substituting the ansatz (6.3.1) into the equations of motion (6.2.3) and Einstein equation (6.2.4) leads to a rather complicated coupled system of ODEs. The equations of motion for the background $SU(2)_R$ gauge field imply $f^i(r) = \kappa_i f(r)$, $i = 1, 2, 3$. The equations for the other fields then depend only on the $SU(2) \sim SO(3)$ invariant combination $\kappa_1^2 + \kappa_2^2 + \kappa_3^2$, which we can set to one by rescaling $f(r)$. The equations of motion then result in the coupled ODEs for the functions $\alpha(r)$, $\beta(r)$, $\gamma(r)$, $p(r)$, $q(r)$, $f(r)$, $X(r)$.

Since the solutions we find are continuously connected to Euclidean AdS_6, we first present the latter in these coordinates:

$$\begin{aligned} \alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}}\,, & \beta(r) &= \gamma(r) = \frac{3\sqrt{6r^2-1}}{\sqrt{2}}\,, \\ X(r) &= 1\,, & p(r) &= q(r) = f(r) = 0\,. \end{aligned} \tag{6.3.2}$$

Here only the metric is non-trivial, and (6.3.2) realizes Euclidean AdS_6 as a hyperbolic ball with radial coordinate $r \in [\frac{1}{\sqrt{6}}, \infty)$, and the conformal boundary at infinity $r = \infty$. Thus the origin is at $r_0 = \frac{1}{\sqrt{6}}$. Notice in particular that the conformal boundary at $r = \infty$ is equipped with a *round* metric on S^5, which is conformally flat. We would like to find families of solutions that generalize (6.3.2) by allowing for a squashed five-sphere boundary, keeping the metric asymptotically locally Euclidean AdS near $r = \infty$. That is, near $r = \infty$ the metric should approach

$$\mathrm{d}s_6^2 \simeq \frac{9\mathrm{d}r^2}{2r^2} + 27r^2\mathrm{d}s_5^2\,, \tag{6.3.3}$$

where $\mathrm{d}s_5^2$ is the squashed five-sphere (6.1.1). For such solutions we may thus define the squashing parameter by

$$\lim_{r\to\infty} \frac{\gamma(r)}{r} \;=\; 3\sqrt{3}\,\frac{1}{s}\,, \tag{6.3.4}$$

so that $s = 1$ for the round sphere. Even though we did not manage to find supersymmetric solutions in closed form, the solutions can nevertheless be given as expansions around different limits. In general notice that we can use reparametrization invariance to set

$$\beta(r) \;=\; \frac{3\sqrt{6r^2-1}}{\sqrt{2}}\,, \tag{6.3.5}$$

which we assume henceforth. In particular this fixes the origin of the ball to be at $r_0 = \frac{1}{\sqrt{6}}$.

In the following we summarize the various families of supersymmetric solutions we have constructed with the ansatz (6.3.1). Some more details of the computations may be found in Appendix B of [5].

6.3.2 3/4 BPS solutions

There is a one-parameter family of 3/4 BPS solutions parametrized by the squashing parameter s. The solution expanded around the conformal boundary is given by

$$\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r} + \frac{8+s^2}{36\sqrt{2}s^2}\frac{1}{r^3} + \dots\,, \qquad (6.3.6)\\
\gamma(r) &= \frac{3\sqrt{3}}{s}r + \frac{-16+7s^2}{12\sqrt{3}s^3}\frac{1}{r} - \frac{-1280+1120s^2+241s^4}{2592\sqrt{3}s^5}\frac{1}{r^3} + \dots\,,\\
X(r) &= 1 + \frac{1-s^2-3\sqrt{1-s^2}}{54s^2}\frac{1}{r^2} + \frac{s^2\sqrt{1-s^2}\kappa}{12\left(1-s^2+\sqrt{1-s^2}\right)}\frac{1}{r^3} + \dots\,,\\
p(r) &= -\frac{\mathrm{i}\sqrt{\frac{2}{3}}\left(s^2+3\sqrt{1-s^2}-1\right)}{s^3}\frac{1}{r^2} + \dots\,,\\
q(r) &= -\frac{3\mathrm{i}\left(\sqrt{6}\sqrt{1-s^2}\right)}{s}r + \frac{\sqrt{\frac{2}{3}}\mathrm{i}\sqrt{1-s^2}\left(5s^2+9\sqrt{1-s^2}-5\right)}{3s^3}\frac{1}{r} + \dots\,,\\
f(r) &= \frac{1-s^2+\sqrt{1-s^2}}{s^2} + \frac{2\left(-2+2s^2-(2+s^2)\sqrt{1-s^2}\right)}{9s^4}\frac{1}{r^2} + \frac{\kappa}{r^3} + \dots\,,
\end{aligned}$$

where we have computed this expansion up to $\mathcal{O}(1/r^9)$. The extra parameter κ is fixed by requiring regularity at the origin $r=\frac{1}{\sqrt{6}}$ (see (6.3.8) below). Notice that the $SU(2)_R$ gauge field at the conformal boundary agrees with the gauge field (6.1.4) with $Q=1$. We may also expand the solution around Euclidean AdS_6, which has $s=1$:

$$\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} + \frac{\left(-5\sqrt{6}+330\sqrt{6}r^2-3744r^3+1620\sqrt{6}r^4+8640r^5-7560\sqrt{6}r^6+5184\sqrt{6}r^8\right)(1-s)}{9\sqrt{2}r^2(6r^2-1)^{9/2}} + \dots\,,\\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} - \frac{\left(55\sqrt{2}-384\sqrt{3}r+1080\sqrt{2}r^2+768\sqrt{3}r^3-5400\sqrt{2}r^4+11232\sqrt{2}r^6-11664\sqrt{2}r^8\right)(1-s)}{6(6r^2-1)^{7/2}} + \dots\,,\\
X(r) &= 1-\frac{\sqrt{2}\left(1-2\sqrt{6}r+6r^2\right)}{3\left(6r^2-1\right)^2}\sqrt{1-s}+\dots\,,\\
p(r) &= \frac{18\mathrm{i}\sqrt{2}\left(\sqrt{6}-16r+12\sqrt{6}r^2-12\sqrt{6}r^4\right)}{\left(6r^2-1\right)^3}\sqrt{1-s}+\dots\,,\\
q(r) &= -\frac{3\mathrm{i}\sqrt{2}\left(-4+9\sqrt{6}r-24r^2-12\sqrt{6}r^3+36\sqrt{6}r^5\right)}{\left(6r^2-1\right)^2}\sqrt{1-s}+\dots\,,\\
f(r) &= \frac{\sqrt{2}\left(-3+8\sqrt{6}r-36r^2+36r^4\right)}{\left(6r^2-1\right)^2}\sqrt{1-s}+\dots\,.
\end{aligned} \tag{6.3.7}$$

In particular one can check that these functions lead to a regular solution at the origin $r=\frac{1}{\sqrt{6}}$, although this is not manifest in the formulas presented above. Indeed, we have computed this expansion up to sixth order, and by comparing the two expansions we find that regularity at the origin fixes the parameter κ in (6.3.6) via

$$\frac{3\sqrt{3}}{4}\kappa \;=\; \delta+\frac{\sqrt{2}}{3}\delta^2+\frac{113}{36}\delta^3+\frac{25}{9\sqrt{2}}\delta^4+\frac{1127}{288}\delta^5+\frac{35}{9\sqrt{2}}\delta^6+\dots\,, \tag{6.3.8}$$

where we have introduced

$$\delta^2 \;\equiv\; \frac{1}{s}-1\,. \tag{6.3.9}$$

The explicit solution ϵ_I to the Killing spinor (6.2.6) and dilatino equation (6.2.7) for this solution may be found in Appendix D. In particular there are three independent constants of integration after imposing the symplectic Majorana condition (6.2.9). Using this solution one can compute the Killing vector bilinear (6.2.8). Requiring that this Killing vector lies in the Lie algebra of the maximal torus $U(1)^3 \subset SU(3) \times U(1)$ fixes the constants of integration, up to an overall irrelevant scaling. In this case we obtain

$$K \;=\; b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} + b_3\partial_{\varphi_3}\,, \tag{6.3.10}$$

where $b_1 = 1 + \sqrt{1-s^2}$, $b_2 = b_3 = 1 - \sqrt{1-s^2}$ and the coordinates φ_i are related to τ, ψ and φ via (6.1.23).

6.3.3 1/4 BPS solutions

We also find a two-parameter family of 1/4 BPS solutions, parametrized by the squashing parameter s and the background $SU(2)_R$ field at the conformal boundary, which is parametrized by f_0. The solution expanded around the conformal boundary is given by

$$\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r} - \frac{f_0^2 s^2 + 9\left(-2+s^2\right) - 6f_0\left(-1+s^2\right)}{36\sqrt{2}}\frac{1}{r^3} + \dots\,, \\
\gamma(r) &= \frac{3\sqrt{3}}{s}r + \frac{2f_0^2 s^2 - 12f_0\left(-1+s^2\right) + 9\left(-3+2s^2\right)}{12\sqrt{3}s}\frac{1}{r} + \dots\,, \\
X(r) &= 1 + \frac{18 - 3f_0 - 18s^2 + 12f_0 s^2 - 2f_0^2 s^2}{54}\frac{1}{r^2} + \dots\,, \\
p(r) &= \frac{\mathrm{i}\sqrt{\frac{2}{3}}(-3+f_0)\left(3+(-3+f_0)s^2\right)}{s}\frac{1}{r^2} + \dots\,, \\
q(r) &= -\frac{3\mathrm{i}\sqrt{6}\left(3+(-3+f_0)s^2\right)}{s}r + \frac{\mathrm{i}\left(3+(-3+f_0)s^2\right)\left(f_0^2 s^2+9\left(-1+s^2\right)-6f_0\left(1+s^2\right)\right)}{6\sqrt{6}s}\frac{1}{r} + \frac{\xi_1}{r^2} + \dots\,, \\
f(r) &= f_0 + \frac{2(-3+f_0)f_0}{9}\frac{1}{r^2} + \frac{\xi_2}{r^3} + \dots\,.
\end{aligned} \tag{6.3.11}$$

Again, we have found this solution up to $\mathcal{O}(1/r^9)$. The constants ξ_1 and ξ_2 are again fixed by requiring regularity at the origin.

There are a number of interesting special cases. First, we obtain the one-parameter family of 1/4 BPS squashed five-spheres of section 6.1.1 by choosing the constant f_0 so as to reproduce (6.1.4) with $Q = -3$. That is, $f_0 = (1 - 3\sqrt{1-s^2})\sqrt{1-s^2}/s^2$. Another interesting case is $f_0 = 0$. In this case the $SU(2)_R$ background gauge field is completely switched off, but the solution is still supersymmetric with a squashed five-sphere at the conformal boundary. This solution has enhanced supersymmetry – as we show in Appendix D, it is 1/2 BPS. On the other hand we may also set $s = 1$, so that the conformal boundary is the *round* five-sphere, but keep the parameter f_0. This shows that one can define non-trivial Killing spinors on the *round* S^5 by turning on other fields.

We may also expand the solution around Euclidean AdS$_6$ with $s = 1$

$$\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} + \frac{\sqrt{3}\left(1-54r^2+96\sqrt{6}r^3-324r^4+216r^6\right)}{2r^2\left(6r^2-1\right)^{7/2}}(1-s)+\ldots\,, \\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} + \frac{\left(15-48\sqrt{6}r+270r^2-540r^4+648r^6\right)}{\sqrt{2}\left(6r^2-1\right)^{5/2}}(1-s)+\ldots\,, \\
X(r) &= 1+\frac{\left(1-2\sqrt{6}r+6r^2\right)(4+\omega)}{\left(6r^2-1\right)^2}(1-s)+\ldots\,, \\
p(r) &= -\frac{18\mathrm{i}\sqrt{2}\left(-\sqrt{3}+8\sqrt{2}r-12\sqrt{3}r^2+12\sqrt{3}r^4\right)(6+\omega)}{\left(6r^2-1\right)^3}(1-s)+\ldots\,, \\
q(r) &= -\frac{3\mathrm{i}\left(-4+9\sqrt{6}r-24r^2-12\sqrt{6}r^3+36\sqrt{6}r^5\right)(6+\omega)}{\left(6r^2-1\right)^2}(1-s)+\ldots\,, \\
f(r) &= \frac{\left(-3+8\sqrt{6}r-36r^2+36r^4\right)\omega}{\left(6r^2-1\right)^2}(1-s)+\ldots\,,
\end{aligned} \tag{6.3.12}$$

where we have introduced the parameter ω via $(1-s)\omega = f_0$. As before it can be checked explicitly that the solution is regular at $r = \frac{1}{\sqrt{6}}$. We have checked this up to fourth order in the expansion variable $\delta \equiv \frac{1}{s} - 1$, and we deduce

$$\begin{aligned}
\xi_1 &= 2\mathrm{i}(6+\omega)\delta - \frac{\mathrm{i}\left(144+98\omega+13\omega^2\right)}{5}\delta^2+\ldots\,, \\
\xi_2 &= \frac{2}{3}\sqrt{\frac{2}{3}}\omega\delta - \frac{2\left(-\sqrt{6}\omega+2\sqrt{6}\omega^2\right)}{45}\delta^2+\ldots\,.
\end{aligned} \tag{6.3.13}$$

The explicit solution ϵ_I to the dilatino and Killing spinor equation (6.2.7), (6.2.6) for this solution may also be found in Appendix D. In this case there is a single integration constant. The Killing vector automatically lies in the Lie algebra of the torus $U(1)^3 \subset SU(3)\times U(1)$, and with an appropriate scaling we obtain

$$K \;=\; \partial_\tau \;=\; b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} + b_3\partial_{\varphi_3}\,, \tag{6.3.14}$$

where $b_1 = b_2 = b_3 = 1$ and the coordinates φ_i are related to τ, ψ and φ via (6.1.23).

6.4 Holographic free energy

In this section we describe how the on-shell action for the Euclidean Romans theory given in section 6.2 can be computed. For asymptotically locally Euclidean AdS solutions one has to holographically renormalize the action by adding boundary counterterms [158–160]. We evaluate the renormalized on-shell action for the supersymmetric solutions presented in section 6.3, and determine their holographic free energies.

6.4.1 On-shell action

We will work in the gauge $A = 0$. Starting from the Euclidean action (6.2.5) and using the equations of motion (6.2.3) together with the Einstein equation (6.2.4) and its trace, we find the following for the on-shell action defined on a manifold M_6 with boundary ∂M_6

$$I_{\text{on-shell}} \;=\; I_{\text{bulk}} + I_{\text{boundary}}\,, \tag{6.4.1}$$

where

$$I_{\text{bulk}} \;=\; \frac{1}{16\pi G_N}\int_{M_6} \tfrac{4}{9}X^{-2}\left(2+3X^4\right) * 1 + \tfrac{1}{3}X^{-2}F^i\wedge *F^i + \tfrac{\mathrm{i}}{3}B\wedge F^i\wedge F^i\,, \tag{6.4.2}$$

$$I_{\text{boundary}} \;=\; \frac{1}{16\pi G_N}\int_{\partial M_6} \tfrac{2}{3}\left(X^{-1} * \mathrm{d}X\right) + \tfrac{1}{3}(B\wedge X^4 * H)\,. \tag{6.4.3}$$

Here we have used Stokes' theorem to write a total derivative as a boundary integral. In particular this assumes that the potentials B and A^i are globally defined, which is the case for our supergravity solutions. The Hodge duals in (6.4.3) are defined on M_6, and then restricted to the boundary. The on-shell action is divergent due to the infinite volume of M_6 and ∂M_6, and from divergences in the supergravity fields as the conformal boundary $r \to \infty$ is approached. Consequently, $I_{\rm bulk}$ should be understood as integrated up to a finite cut-off which is then sent to infinity only after adding counterterms which regularize the divergences. In addition, because of the presence of boundary terms in the on-shell action, one should add a Gibbons-Hawking term [161]

$$I_{\rm GH} \;=\; -\frac{1}{8\pi G_N}\int_{\partial M_6} \mathcal{K}\sqrt{\det h}\,{\rm d}^5x\,. \tag{6.4.4}$$

This involves the trace $\mathcal{K}$ of the extrinsic curvature of the boundary, and where h_{mn} is the induced boundary metric, and also leads to divergences. Hence the finite on-shell action is

$$I_{\rm renormalized} \;=\; I_{\rm on-shell} + I_{\rm GH} + I_{\rm counterterms}\,. \tag{6.4.5}$$

In the next subsection we discuss how to determine the precise form of the counterterms.

6.4.2 Boundary counterterms

We now very briefly outline the construction of the counterterms needed to regularize the action of the Euclidean Romans $F(4)$ theory (for more details we refer to [5]). We assume a general expansion of the fields for an asymptotically locally Euclidean AdS$_6$ solution. In particular, we take the metric to be given in Fefferman-Graham form [162, 163]

$${\rm d}s^2_6 \;=\; \frac{\ell^2}{z^2}{\rm d}z^2 + \frac{1}{z^2}\gamma_{mn}(z,x){\rm d}x^m{\rm d}x^n\,, \tag{6.4.6}$$

where $\ell = 3/\sqrt{2}$ is the AdS$_6$ radius, and

$$\gamma_{mn}(z,x) \;=\; \gamma^0_{mn} + z^2\gamma^2_{mn} + z^4\gamma^4_{mn} + \mathcal{O}(z^5)\,. \tag{6.4.7}$$

Here $\gamma^0_{mn}(x)$ is the metric induced on the conformal boundary which, due to the radial coordinate transformation $r \to \frac{1}{z}$, is now at $z = 0$. The Gibbons-Hawking term is then

$$I_{\text{GH}} = \frac{1}{8\pi G_N} \int_{\partial M_6} \frac{z}{\ell} \partial_z \sqrt{\det h}\ \mathrm{d}^5 x\,, \tag{6.4.8}$$

and $h_{mn} = \frac{1}{z^2}\gamma_{mn}$ is the induced metric on the boundary. We also assume an asymptotic expansion for bulk scalar and gauge fields, namely

$$\begin{aligned} X &= 1 + zX_1 + z^2 X_2 + \cdots\,, \\ B &= \frac{1}{z} b + \mathrm{d}z \wedge A_0 + B_0 + z\mathrm{d}z \wedge A_1 + zB_1 + \cdots\,, \\ H = \mathrm{d}B &= -\frac{1}{z^2}\mathrm{d}z \wedge b + \frac{1}{z}\mathrm{d}b - \mathrm{d}z \wedge \mathrm{d}A_0 + \mathrm{d}B_0 + \mathrm{d}z \wedge B_1 - z\mathrm{d}z \wedge \mathrm{d}A_1 \cdots\,, \\ F^i &= f^i + \mathrm{d}z \wedge A^i_0 + z\mathrm{d}z \wedge A^i_1 + zF^i_1 + \cdots\,. \end{aligned} \tag{6.4.9}$$

The basic principle is to plug this expansion into the Romans field equations, leading to relations among the fields, and then plug them into the on-shell action. For example at the lowest order term in z in each of the equations of motion and and Einstein equations, the leading order term in the X and B equation of motion dictates $X_1 = 0$ as well as $\ell = 3/\sqrt{2}$.

Hence, setting $X_1 = 0$ and $\ell = 3/\sqrt{2}$, the first divergence we encounter is at order $\mathcal{O}(1/\epsilon^5)$, where $z = \epsilon$ is the finite cut-off. The divergence comes from expanding the $\frac{4}{9}X^{-2}\left(2 + 3X^4\right) * 1$ integrand in I_{bulk} and the Gibbons-Hawking term. It is given by

$$I^{\text{div}}_{\mathcal{O}(1/\epsilon^5)} = \frac{1}{8\pi G_N}\frac{1}{\epsilon^5} \int_{\partial M_6} -\frac{4\sqrt{2}}{3}\sqrt{\det \gamma^0}\ \mathrm{d}^5 x\,, \tag{6.4.10}$$

and hence it is simply cancelled by adding the counterterm

$$I^{\text{counterterm}}_5 = \frac{1}{8\pi G_N} \cdot \frac{4\sqrt{2}}{3} \int_{\partial M_6} \sqrt{\det h}\ \mathrm{d}^5 x\,. \tag{6.4.11}$$

Continuing in this way, a long and strenuous but straightforward calculation for the divergences of order $\mathcal{O}(1/\epsilon^4)$ up to order $\mathcal{O}(1/\epsilon)$ leads to explicit expressions for the counterterms $I_4^{\text{counterterm}}$ up to $I_1^{\text{counterterm}}$. Collating all these expressions for the counterterms we finally arrive at [4, 5]

$$\begin{aligned} I_{\text{counterterms}} &= \frac{1}{8\pi G_N}\int_{\partial M_6}\Bigg\{\Bigg[\frac{4\sqrt{2}}{3}+\frac{1}{2\sqrt{2}}R(h)-\frac{1}{6\sqrt{2}}\|B\|_h^2+\frac{3}{4\sqrt{2}}R(h)_{mn}R(h)^{mn} \\ &\quad -\frac{15}{64\sqrt{2}}R(h)^2-\frac{3}{4\sqrt{2}}\|F^i\|_h^2+\frac{1}{12\sqrt{2}}\mathrm{Tr}_hB^4-\frac{13}{192\sqrt{2}}\|B\|_h^4-\frac{1}{\sqrt{2}}\|\mathrm{d}B\|_h^2 \\ &\quad +\frac{5}{8\sqrt{2}}\|\mathrm{d}*_h B+\tfrac{\mathrm{i}\sqrt{2}}{3}B\wedge B\|_h^2-\frac{1}{4\sqrt{2}}\langle B,\mathrm{d}\delta_hB+\tfrac{\mathrm{i}\sqrt{2}}{3}\mathrm{d}*_h B\wedge B\rangle_h \\ &\quad +\frac{4\sqrt{2}}{3}(1-X)^2-\frac{1}{\sqrt{2}}\langle \mathrm{Ric}(h)\circ B,B\rangle_h+\frac{9}{32\sqrt{2}}R(h)\|B\|_h^2\Bigg]\sqrt{\det h}\,\mathrm{d}^5x \\ &\quad -\frac{1}{4\sqrt{2}}B\wedge\left[\mathrm{d}*_h\mathrm{d}B+\frac{\sqrt{2}\mathrm{i}}{3}B\wedge\delta_hB-\frac{2}{9}B\wedge *_h(B\wedge B)\right]\Bigg\}. \end{aligned} \tag{6.4.12}$$

Here $\mathrm{Ric}(h)_{ij} = R(h)_{ij}$ denotes the Ricci tensor of the metric h_{ij}, with $R(h)$ the Ricci scalar. The inner product of two p-forms ν_1, ν_2 is defined by $\langle\nu_1,\nu_2\rangle_h\sqrt{\det h}\,\mathrm{d}^5x = \nu_1\wedge *_h\nu_2$, which then also defines the square norm via $\|\nu\|_h^2 = \langle\nu,\nu\rangle_h$. The adjoint δ_h of d with respect to h_{ij} acting on the two-form B is $\delta_hB = *_h\mathrm{d}*_h B$, and we have also defined $\mathrm{Tr}_hB^4 \equiv B_i{}^jB_j{}^kB_k{}^lB_l{}^i$. Finally, we have defined the p-form $(S\circ\nu)_{i_1\cdots i_p} \equiv S_{[i_1}{}^j\nu_{|j|i_2\cdots i_p]}$, where S_{ij} is any symmetric 2-tensor, and ν is any p-form.

6.4.3 Free energy of the solutions

The renormalized on-shell action determined in the previous subsection holds for all Romans supergravity solutions which are asymptotically locally AdS. In particular we may use these results to compute the holographic free energy for the supersymmetric solutions of section 6.3. In order to present the results, we first split the renormalized action as

$$I_{\text{renormalized}} = I_{\text{bulk}} + I_{\text{non-bulk}}\,, \tag{6.4.13}$$

where $I_{\rm bulk}$ is the bulk integral given by (6.4.2), while

$$I_{\rm non-bulk} = I_{\rm boundary} + I_{\rm GH} + I_{\rm counterterms}\,, \tag{6.4.14}$$

where $I_{\rm boundary}$ is the boundary contribution to the on-shell action (6.4.3), $I_{\rm GH}$ is the Gibbons-Hawking term, while $I_{\rm counterterms}$ is the full counterterm (6.4.12). For our $SU(3)\times U(1)$ ansatz (6.3.1), with $f^1(r)\equiv f^2(r)\equiv 0$ and $f^3(r)=f(r)$, we have in particular

$$\begin{aligned} I_{\rm bulk} &= \frac{\pi^2}{36G_N}\int_{r=\frac{1}{\sqrt{6}}}^{\Lambda}\Bigg[3X^2(r)\alpha(r)\beta^4(r)\gamma(r)+6{\rm i}f(r)\left[f(r)p(r)+q(r)f'(r)\right] \\ &+\frac{24f^2(r)\alpha^2(r)\gamma^2(r)+8\alpha^2(r)\beta^4(r)\gamma^2(r)+3\beta^4(r)(f'(r))^2}{4X^2(r)\alpha(r)\gamma(r)}\Bigg]{\rm d}r\,, \end{aligned} \tag{6.4.15}$$

where Λ is the cut-off for the r coordinate.

3/4 BPS solution

For the one-parameter family of 3/4 BPS solutions in section 6.3.2, by taking $\Lambda\to\infty$ we obtain the following finite result

$$I_{\rm renormalized} = -\frac{27\pi^2}{4G_N}\left(1+\frac{8}{3}\delta^2+\frac{16\sqrt{2}}{27}\delta^3+\frac{68}{27}\delta^4+\frac{28\sqrt{2}}{27}\delta^5+\frac{32}{27}\delta^6+\dots\right)\,, \tag{6.4.16}$$

where the six-dimensional Newton constant is given by[5]

$$G_N = \frac{15\pi\sqrt{8-N_f}}{4\sqrt{2}N^{5/2}}. \tag{6.4.17}$$

The holographic free energy is identified with $I_{\rm renormalized}$ and agrees precisely with the series expansion of the large N field theory result (6.1.40)

$$\mathcal{F} = \frac{1}{27s^2}\frac{(3-\sqrt{1-s^2})^3}{1-\sqrt{1-s^2}}\mathcal{F}_{{\rm round}\,S^5}\,, \tag{6.4.18}$$

[5]This was effectively calculated in [101] by identifying the holographic free energy of Euclidean AdS_6 with an entanglement entropy. The $N^{5/2}$ scaling of the free energy had previously been predicted in [96].

where recall that $s = 1/(1+\delta^2)$.

1/4 **BPS Solution**

We may similarly compute the holographic free energy of the two-parameter family of 1/4 BPS solutions in section 6.3.3. Again we obtain two divergent contributions whose divergences cancel. The finite piece may be computed as an expansion in $\delta = \frac{1}{s} - 1$ using the series expansions of the parameters ξ_1, ξ_2 in (6.3.13). Putting everything together we obtain

$$I_{\text{renormalized}} = -\frac{27\pi^2}{4G_N}\left(1 + \mathcal{O}(\delta^5)\right) . \qquad (6.4.19)$$

This again agrees with large N field theory result (6.1.40). Notice that this result holds for any values of f_0.

6.5 Wilson loops

In this section we compute the expectation values of certain BPS Wilson loops, both in the large N matrix model of section 6.1.3 and also in the supergravity dual solutions of section 6.3. More precisely it will be important to uplift these solutions to massive type IIA supergravity, where the Wilson loop in the fundamental representation is dual to a fundamental string. Minus the action of this string precisely matches the logarithm of the Wilson loop VEV in the large N limit, as a function of the parameters of the solutions.

6.5.1 Large N field theory

An interesting observable to consider is the VEV of the Wilson loop in a representation R of the gauge group G:

$$\langle W_R \rangle = \frac{1}{\dim R}\left\langle \mathrm{Tr}_R \mathcal{P} \exp \int (\mathscr{A}_m \dot{x}^m + \sigma|\dot{x}|)\, \mathrm{d}t \right\rangle . \qquad (6.5.1)$$

Here $\mathscr{A}$ denotes the dynamical gauge field for the gauge group G, σ is the scalar in the corresponding vector multipet, and the worldline is parametrized by $x^m(t)$. It is straightforward to see that (6.5.1) is invariant under the supersymmetry transformations for the squashed five-sphere provided the Wilson loop wraps an orbit of the five-dimensional Killing vector bilinear

$$K_m \;=\; \varepsilon^{IJ}\chi_I{}^T\mathcal{C}_{(5)}\gamma_m\chi_J\,. \tag{6.5.2}$$

That is, we take $x^m(t)$ to be an integral curve of K. The supersymmetry variations of the two terms in (6.5.1) then cancel each other.

The large N limit of (6.5.1) for the $USp(2N)$ gauge theories described in section 6.1.3 was computed for the round five-sphere in [164]. It is straightforward to extend this to the more general squashed sphere matrix model in section 6.1.3. The key point is that the insertion of the Wilson loop into the path integral does not affect the leading order saddle point configuration because its logarithm scales as $N^{1/2}$, while the free energy instead scales as $N^{5/2}$. The dynamical gauge field $\mathscr{A}$ localizes to zero, so only the constant scalar σ contributes to the Wilson loop (6.5.1) in the localization computation. Thus the VEV (6.5.1), for the fundamental representation of $USp(2N)$, is effectively computed in the large N matrix model as

$$\langle\, W_{\text{fund}}\,\rangle \;=\; \int_0^{x_\star} \mathrm{e}^{2\pi\mathscr{L}\lambda(x)}\,\rho(x)\mathrm{d}x\,, \tag{6.5.3}$$

where $\rho(x)$ is the saddle point eigenvalue density (6.1.35), with the eigenvalues supported on $[0, x_\star]$ with $x_\star$ given by (6.1.36). We have also denoted by $2\pi\mathscr{L} = \int |\dot{x}|\,\mathrm{d}t$ the length of the integral curve of K which is wrapped by the Wilson loop, and recall that $\lambda(x) = N^{1/2}x$ to leading order. Thus we find the large N result

$$\begin{aligned}\log\langle\, W_{\text{fund}}\,\rangle \;&=\; \frac{(b_1+b_2+b_3)\sqrt{2}\pi\mathscr{L}}{\sqrt{8-N_f}}N^{1/2} + o(N^{1/2})\\ &=\; \frac{(b_1+b_2+b_3)\mathscr{L}}{3}\log\langle\, W_{\text{fund}}\,\rangle_{\text{round}}\,.\end{aligned} \tag{6.5.4}$$

Indeed, recalling that

$$K \;=\; b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} + b_3\partial_{\varphi_3}\,, \tag{6.5.5}$$

in terms of the standard $U(1)^3$ action on $S^5 \subset \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$, then the orbits of K are always closed circles at the origins of any two copies of $\mathbb{R}^2$. If we call these $U(1)^3$ invariant circles S^1_i, $i = 1,2,3$, then $\mathscr{L} = 1/b_i$ and we may write

$$\log\langle\, W_{\mathrm{fund},\,S^1_i}\,\rangle \;=\; \frac{(b_1+b_2+b_3)}{3b_i}\log\langle\, W_{\mathrm{fund}}\,\rangle_{\mathrm{round}}\,. \tag{6.5.6}$$

We now explain how to reproduce this large N result from the dual supergravity solutions.

6.5.2 Dual fundamental strings

The supergravity dual of the Wilson loop W_{fund} was studied in [164] for the round five-sphere. The supergravity background is in this case the massive type IIA uplift $\mathrm{AdS}_6 \times S^4$ of the AdS_6 vacuum of the Romans theory of section 6.2. The Wilson loop maps to a fundamental string sitting at the north pole $\xi = \frac{\pi}{2}$ of the internal S^4–hemisphere, in the notation of section 6.2.1. The string then wraps a copy of $\mathbb{R}^2 \subset \mathrm{AdS}_6$ parametrized by the radial direction r in AdS together with the Wilson loop curve $S^1 \subset S^5$.

We now generalize this to our supergravity backgrounds in section 6.3. Here the type IIA background is a warped and fibred product $M_6 \times S^4$, together with various non-trivial background fluxes. However, M_6 still has the topology of a ball, with a natural radial direction r. Thus the candidate dual of the Wilson loops computed in the previous section is a fundamental string sitting at $\xi = \frac{\pi}{2}$ in the internal S^4 of (6.2.1), together with the Wilson loop curve $S^1 \subset S^5_{\mathrm{squashed}}$ and the radial direction r – see Figure 6.1. This is then a copy of $\Sigma_2 \cong \mathbb{R}^2 \subset M_6$, and we would like to compute the regularized action of a fundamental string wrapping this submanifold.

In order to compute the string action we must first convert to the string frame metric in (6.2.1), which introduces a factor of $\mathrm{e}^{\Phi/2}$, where Φ is the ten-dimensional dilaton. The

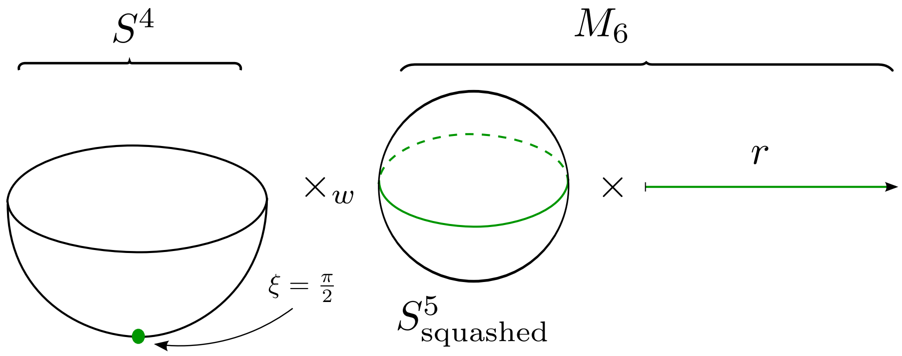


Figure 6.1: The fundamental string is located at a point on the internal S^4 hemisphere and wraps the radial direction of M_6 together with the Wilson line in the ten dimensional supergravity background given by a warped product of $M_6 \times S^4$.

induced string frame metric on M_6 at the north pole $\xi = \frac{\pi}{2}$ of S^4 is then

$$\mathrm{d}s^2_{M_6}\,|_{\xi=\frac{\pi}{2},\,\text{string}} = X^{-2}\mathrm{d}s^2_6\,, \tag{6.5.7}$$

where $\mathrm{d}s^2_6$ is the Romans supergravity metric. The B-field then uplifts to the type IIA B-field with curvature $F_{(3)} = H = \mathrm{d}B$ via (6.2.1) at the north pole $\xi = \frac{\pi}{2}$. In section 6.2 we have set most of the physical scaling parameters to specific numerical values – for example the Romans mass is set to $m_{\text{IIA}} = \frac{\sqrt{2}}{3}$, while the correctly normalized value for the supergravity dual to the $USp(2N)$ gauge theories is $(8 - N_f)/(2\pi\ell_s)$ where ℓ_s is the string length. In particular restoring the AdS radius to its physical value (as in [164])

$$L^4 = \frac{8\pi^2 N}{9(8-N_f)}\ell_s^4\,, \tag{6.5.8}$$

the string frame action is

$$S = \frac{\sqrt{2}\,N^{1/2}}{3\sqrt{(8-N_f)}}\int_{\Sigma_2} X^{-2}\sqrt{\det\gamma}\,\mathrm{d}^2x + \mathrm{i}B\,, \tag{6.5.9}$$

where γ_{ab} is the metric induced on Σ_2 via its embedding into the Romans metric $\mathrm{d}s^2_6$ on M_6, and we have included the usual Wess-Zumino coupling to the ten-dimensional B-field. More precisely, (6.5.9) is divergent, and as usual one may regularize it by cutting off the r

integral at some $r=\Lambda$, and including a boundary counterterm given by the length of the boundary $S^1 \subset S^5$ at $r=\Lambda$. Thus the regularized action reads

$$S_{\text{string}} = \frac{\sqrt{2}\,N^{1/2}}{3\sqrt{(8-N_f)}}\left[\int_{\Sigma_2}\left(X^{-2}\sqrt{\det\gamma}\,\mathrm{d}^2x + \mathrm{i}B\right) - \frac{3}{\sqrt{2}}\text{length}(\partial\Sigma_2)\right], \qquad (6.5.10)$$

where this is understood to mean the limit as one takes the cut-off $\Lambda\to\infty$. We now compute this for our various solutions.

1/4 BPS background

We begin with the 1/4 BPS background, as in this case the supersymmetric Killing vector bilinear is simply $K=\partial_\tau$. Via the $SU(3)$ symmetry of the background all orbits of K are equivalent, and thus there is effectively only one Wilson loop to compute. This wraps the τ and r directions at, say, $\sigma=0$ (which is a point on the base $\mathbb{CP}^2$ of $S^1_{\text{Hopf}} \hookrightarrow S^5 \to \mathbb{CP}^2$, all points being equivalent under $SU(3)$). The regularized string action (6.5.10) is

$$S_{\text{string}} = \lim_{\Lambda\to\infty}\frac{N^{1/2}2\sqrt{2}\pi}{3\sqrt{(8-N_f)}}\left[\int_{r=\frac{1}{\sqrt{6}}}^{\Lambda}\left[X^{-2}(r)\alpha(r)\gamma(r) + \mathrm{i}\,p(r)\right]\mathrm{d}r - \frac{3}{\sqrt{2}}\gamma(\Lambda)\right], \quad (6.5.11)$$

where we have used that τ has period 2π. Evaluating this for the two-parameter family of 1/4 BPS solutions, as a series in the parameter δ, we find

$$-S_{\text{string}} = \frac{3\sqrt{2}\pi}{\sqrt{8-N_f}}N^{1/2} + \mathcal{O}(\delta^5), \qquad (6.5.12)$$

which agrees precisely with the large N field theory result (6.5.4) since $K=\partial_\tau=\partial_{\varphi_1}+\partial_{\varphi_2}+\partial_{\varphi_3}$ so that $b_1=b_2=b_3=1$.

3/4 BPS background

For the 3/4 BPS solution recall that the supersymmetric Killing vector K has $b_1 = 1+\sqrt{1-s^2}$, $b_2=b_3=1-\sqrt{1-s^2}$. For generic values of the squashing parameter s the generic orbit of K will be open. However, the orbits always close over the circles S^1_i defined in section 6.5.1, which have lengths $\mathscr{L} = 2\pi/b_i$. Since $b_2 = b_3$ these circles give rise to two distinct Wilson loop VEVs:

$$\frac{\log\langle W_{\mathrm{fund},\, S^1_i}\rangle}{\log\langle W_{\mathrm{fund}}\rangle_{\mathrm{round}}} = \begin{cases} \dfrac{3-\sqrt{1-s^2}}{3(1+\sqrt{1-s^2})}\,, & i = 1\,, \\[2ex] \dfrac{3-\sqrt{1-s^2}}{3(1-\sqrt{1-s^2})}\,, & i = 2,3\,. \end{cases} \tag{6.5.13}$$

We may then compare these results to the regularized string action (6.5.10), where for S^1_i the fundamental string wraps the circle φ_i together with the r direction. More precisely, S^1_1 is located at $\sigma = 0$ in the coordinates (6.1.1), while S^1_2 is located at $\{\sigma = \frac{\pi}{2},\, \theta = 0\}$, as one sees from (6.1.22). The result for S^1_3 is the same as that for S^1_2 due to the $SU(2) \subset SU(3)$ symmetry preserved by the bosonic solution and supersymmetric Killing vector. On the other hand, due to the signs in (6.1.23) the relevant string actions to compute are then

$$\frac{N^{1/2}2\sqrt{2}\pi}{3\sqrt{(8-N_f)}}\left[\int_{r=\frac{1}{\sqrt{6}}}^{\Lambda}\left[X^{-2}(r)\alpha(r)\gamma(r) \pm \mathrm{i}\, p(r)\right]\mathrm{d}r - \frac{3}{\sqrt{2}}\gamma(\Lambda)\right]\,, \tag{6.5.14}$$

respectively. Evaluating this for the one-parameter family of 3/4 BPS solutions, as a series in the parameter δ up to sixth order where $\delta^2 = \frac{1}{s} - 1$, we find that they agree precisely with the series expansions of (6.5.13) computed in field theory.

6.6 Two conjectures

Given any supersymmetric supergravity solution one can construct the Killing vector $K^\mu = \varepsilon^{IJ}\epsilon_I^T\mathcal{C}\gamma^\mu\epsilon_J$, where ϵ_I, $I = 1, 2$, is the $SU(2)_R$ doublet of Killing spinors. For our solutions the free energy takes the form

$$\mathcal{F} = \frac{(|b_1|+|b_2|+|b_3|)^3}{27|b_1b_2b_3|}\mathcal{F}_{\mathrm{AdS}_6}\,, \tag{6.6.1}$$

where we write the supersymmetric Killing vector as $K = \sum_{i=1}^{3} b_i \partial_{\varphi_i}$, and ∂_{φ_i} are standard generators of $U(1)^3 \subset SU(3) \times U(1)$ acting on $S^5 \subset \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$. Given the corresponding $4d/3d$ results of [89, 91], it is then natural to conjecture that (6.6.1) holds for *any* supersymmetric supergravity solution with the topology of a six-ball and for which the supersymmetric Killing vector K may be written as $K = \sum_{i=1}^{3} b_i \partial_{\varphi_i}$. In the present chapter we chose orientation conventions so that $b_i > 0$ for $i = 1, 2, 3$. More generally we expect the orientations of ∂_{φ_i} to be fixed as in [91], leading to the modulus signs in (6.6.1). We also conjecture that any supersymmetric gauge theory, with finite N, defined on the conformal boundary of such a supergravity solution depends only on b_1, b_2, b_3.

We find that one can write the Wilson loop VEV as

$$\log \langle W \rangle \; = \; \frac{|b_1| + |b_2| + |b_3|}{3|b_i|} \log \langle W \rangle_{\mathrm{AdS}_6} \, , \tag{6.6.2}$$

where the Wilson loop wraps the φ_i circle. Again, it is natural to conjecture that (6.6.2) holds for general supergravity backgrounds with $U(1)^3$ symmetry and the topology of a six-ball. In a forthcoming paper we will prove this conjecture from the supergravity point of view in the general setting of any supergravity solution of six-ball topology [8].

Chapter 7

Supersymmetric gauge theories on five-manifolds

In this chapter we construct rigid supersymmetric gauge theories on Riemannian five-manifolds. We follow a holographic approach, realizing the manifold as the conformal boundary of the six-dimensional bulk supergravity solution discussed in Chapter 6. This leads to a systematic classification of five-dimensional supersymmetric backgrounds with gravity duals. We show that the background metric is furnished with a conformal Killing vector, which generates a transversely holomorphic foliation with a transverse Hermitian structure. Moreover, we prove that any such metric defines a supersymmetric background.[1]

7.1 Rigid supersymmetry from holography

In this section we determine the form of the Euclidean Romans supersymmetry conditions – as given in section 6.2, equations (6.2.6) and (6.2.7) – at the five-dimensional conformal boundary. We employ the notation $M, N, \ldots \in \{0,1,2,3,4,5\}$ and $\mu, \nu, \ldots \in \{1,\ldots,5\}$ in the following. We again use the Fefferman-Graham coordinates outlined in section 6.4.2, with a change of coordinates $z \to 1/r$, so that the conformal boundary is now at $r = \infty$.

[1] In [6], we also explicitly construct supersymmetric Lagrangians for all these background. However due to space limitations, we will not discuss this here.

The metric hence admits an expansion of the form

$$\mathrm{d}s^2 = \frac{9}{2}\frac{\mathrm{d}r^2}{r^2} + r^2\left[g_{\mu\nu} + \frac{1}{r^2}g^{(2)}_{\mu\nu} + \cdots\right]\mathrm{d}x^\mu\mathrm{d}x^\nu\,. \tag{7.1.1}$$

Here the conformal boundary has metric $g = (g_{\mu\nu})$. Notice that the particular form of the metric in (7.1.1) is not reparametrization invariant under $r \to \Lambda r$, where $\Lambda = \Lambda(x^\mu)$. However, the correction terms under such a transformation are subleading in the $1/r$ expansion. This will play an important role in the next section however.

Let us introduce six-dimensional vielbeins $\{E^M\}_{M=0}^5$ such that

$$\mathrm{d}s_6^2 = E^M E^M = E^0E^0 + E^\mu E^\mu\,. \tag{7.1.2}$$

If we denote by $\{e^\mu\}_{\mu=1}^5$ the five-dimensional vielbeins for $g_{\mu\nu}$, then the six-dimensional frame components may be written as

$$E^0 = \frac{3}{\sqrt{2}}\frac{\mathrm{d}r}{r}\,, \qquad E^\mu = re^\mu + \cdots\,, \tag{7.1.3}$$

where the ellipsis denotes subleading powers of r. The six-dimensional spin connection $\Omega_A{}^{BC}$ then expands as

$$\Omega_0{}^{\mu\nu} = 0 = \Omega_0{}^{0\mu}\,, \quad \Omega_\mu{}^{0\nu} = -\frac{\sqrt{2}}{3}\delta_\mu{}^\nu + \frac{1}{r^2}\omega_\mu{}^\nu + \cdots\,, \quad \Omega_\mu{}^{\nu\rho} = \frac{1}{r}\omega^{(5d)\nu\rho}_\mu + \ldots\,, \tag{7.1.4}$$

where $\omega^{(5d)\nu\rho}_\mu$ is the spin connection associated with the five dimensional boundary metric $(g_{\mu\nu})$ and $\omega_\mu{}^\nu$ is a higher order correction.

As detailed in section 6.4.2, the asymptotic bulk field expansions in the local six-dimensional coordinates are

$$\begin{aligned} X &= 1 + \frac{1}{r^2}X_2 + \cdots\,, & B &= \frac{2r}{3}b - \frac{2}{3r^2}\mathrm{d}r\wedge A^{(0)} + \cdots\,, \\ F^3 &= f + \cdots\,, & A &= a + \cdots\,. \end{aligned} \tag{7.1.5}$$

For simplicity we have assumed Abelian solutions in which $A^1 = A^2 = 0$, and $A^3 \equiv A$, with field strength $F^1 = F^2 = 0$ and $F^3 \equiv \mathrm{d}A$, such that $f \equiv \mathrm{d}a$.

We start by decomposing the six-dimensional gamma matrices and spinors. We take our coordinate-independent Cliff$(6,0)$ gamma matrices to be

$$\Gamma_0 = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix}, \qquad \Gamma_\mu = \begin{pmatrix} 0 & \mathrm{i}\gamma_\mu \\ -\mathrm{i}\gamma_\mu & 0 \end{pmatrix}, \qquad \Gamma_7 = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4 \end{pmatrix}, \tag{7.1.6}$$

where γ_μ are a Hermitian basis of Cliff$(5,0)$. Then the six-dimensional spinor ϵ_I is decomposed as

$$\epsilon_I = \begin{pmatrix} \epsilon_I^+ \\ \epsilon_I^- \end{pmatrix}, \tag{7.1.7}$$

where $\epsilon_I^\pm$ are four-component spinors.

Let us focus on the expansion of the $M = 0$ direction in the bulk Killing spinor equation (6.2.6). To do so we substitute in the expansions of the fields (7.1.5). At first order this leads to the simple differential equation

$$\begin{pmatrix} \partial_r \epsilon_I^+ \\ \partial_r \epsilon_I^- \end{pmatrix} = \frac{\mathrm{i}}{2\,r} \begin{pmatrix} \epsilon_I^- \\ -\epsilon_I^+ \end{pmatrix} \tag{7.1.8}$$

with solution

$$\epsilon_I = \begin{pmatrix} \epsilon_I^+ \\ \epsilon_I^- \end{pmatrix} = \sqrt{r} \begin{pmatrix} \chi_I \\ -\mathrm{i}\chi_I \end{pmatrix} + \frac{1}{\sqrt{r}} \begin{pmatrix} \varphi_I \\ \mathrm{i}\varphi_I \end{pmatrix} + \cdots. \tag{7.1.9}$$

As in the Chapter 6 we consider a "real" class of solutions for which ϵ_I satisfies the symplectic Majorana condition $\varepsilon_I{}^J \epsilon_J = \mathcal{C}_6 \epsilon_I^* \equiv \epsilon_I^c$, where $\mathcal{C}_6$ denotes the charge conjugation matrix, satisfying $\Gamma_M^T = \mathcal{C}_6^{-1} \Gamma_M \mathcal{C}_6$. The bosonic fields are all taken to be real, with the exception of the B-field which is purely imaginary. With these reality properties, one can show that the Killing spinor equation (6.2.6) and dilatino equation (6.2.7) for ϵ_2 are simply the charge conjugates of the corresponding equations for ϵ_1. In this way we effectively reduce to a single Killing spinor $\epsilon \equiv \epsilon_1$, with $SU(2)_R$ doublet (ϵ, ϵ^c). We then note the following large r expansions of bilinears:

$$\begin{aligned} \epsilon^\dagger \Gamma_7 \epsilon &= 4\alpha S + \cdots \,, \\ \mathrm{i}\epsilon^\dagger \Gamma_7 \Gamma_{(1)} \epsilon &= 2 S r K_2 - 3\sqrt{2} S \mathrm{d}r + \cdots \,. \end{aligned} \tag{7.1.10}$$

Here we have defined $\Gamma_{(1)} \equiv \Gamma_M E^M$ and $S \equiv \chi^\dagger \chi$. We also note that the bilinear $\epsilon^\dagger \Gamma_{(1)} \epsilon$ is a Killing one-form in the bulk – see Chapter 6. This will hence restrict to a conformal Killing vector on the boundary at $r = \infty$.

Using the expansion for the Killing spinors (7.1.9) and analyzing the large r expansion of the remaining components of the Killing spinor equation (6.2.6), we find

$$\left(\nabla_\mu + \frac{\mathrm{i}}{2} a_\mu\right) \chi = -\frac{\sqrt{2}}{3} \mathrm{i} \gamma_\mu \varphi - \frac{\mathrm{i}}{12\sqrt{2}} b_{\nu\sigma} \gamma_\mu{}^{\nu\sigma} \chi + \frac{\mathrm{i}}{3\sqrt{2}} b_{\mu\nu} \gamma^\nu \chi \,, \tag{7.1.11}$$

$$\begin{aligned} \left(\nabla_\mu + \frac{\mathrm{i}}{2} a_\mu\right) \varphi &= -\frac{\mathrm{i}}{6\sqrt{2}} b_{\mu\nu} \gamma^\nu \varphi + \frac{1}{16\sqrt{2}} f_{\nu\sigma} \gamma_\mu{}^{\nu\sigma} \chi - \frac{3}{8\sqrt{2}} f_{\mu\nu} \gamma^\nu \chi \\ &\quad + \frac{1}{48} (\mathrm{d}b)_{\nu\rho\sigma} \gamma^{\nu\rho\sigma} \gamma_\mu \chi - \frac{1}{36} A^{(0)}_\nu \gamma_\mu{}^\nu \chi + \frac{1}{12} A^{(0)}_\mu \chi + \frac{\mathrm{i}}{2} \omega_\mu{}^\nu \gamma_\nu \chi \,, \end{aligned} \tag{7.1.12}$$

to first and second order respectively. Here ∇ denotes the Levi-Civita spin connection for the boundary metric $(g_{\mu\nu})$.

A similar analysis for the bulk dilatino equation (6.2.7) gives the algebraic five dimensional boundary condition

$$-\frac{1}{6\sqrt{2}} b_{\mu\nu} \gamma^{\mu\nu} \varphi - \frac{\sqrt{2}}{3} X_2 \chi + \frac{\mathrm{i}}{8\sqrt{2}} f_{\mu\nu} \gamma^{\mu\nu} \chi + \frac{\mathrm{i}}{24} (\mathrm{d}b)_{\mu\nu\sigma} \gamma^{\mu\nu\sigma} \chi - \frac{\mathrm{i}}{18} A^{(0)}_\mu \gamma^\mu \chi = 0 \,. \tag{7.1.13}$$

We note here that equation (7.1.11) may be rewritten in the form of a charged conformal Killing spinor equation, with additional b-field couplings [5]. Setting $b = 0$ one obtains the standard charged conformal Killing spinor equation, whose solutions (twistor spinors) have been studied in the holographic context for three-manifolds and four-manifolds in [104, 124, 165–167]. On the other hand, previous work on rigid supersymmetry in five dimensions [119–121] has used Killing spinor equations of a different form, without the coupling to φ in (7.1.11). We may make closer contact with this work by noting that supersymmetry in the bulk also implies the algebraic relation

$$\varphi \;=\; -\alpha\chi - \frac{\mathrm{i}}{2}(K_2)_\nu\gamma^\nu\chi\,. \tag{7.1.14}$$

This follows from the bilinear expansions (7.1.10).

In the remainder of this chapter we shall take equations (7.1.11), (7.1.12), (7.1.13), and (7.1.14) as our starting point for a purely five-dimensional analysis.

7.2 Background geometry

In this section we start with a Riemannian five-manifold (M_5, g), on which we would like to define rigid supersymmetric gauge theories. The gauge/gravity correspondence implies that this should be possible, provided the spinor equations derived in the previous section hold.

Let us summarize the background data. In addition to the real metric g, we have two generalized Killing spinors χ, φ. Globally these are spin^c spinors, being sections of the spin bundle of M_5 tensored with $L^{-1/2}$, $\chi,\varphi\in\Gamma[\mathrm{Spin}(M_5)\otimes L^{-1/2}]$, where L is the complex line bundle for which the real gauge field a is a connection. This Abelian gauge field is a background field for $U(1)_R\subset SU(2)_R$, with (χ,χ^c), (φ,φ^c) forming $SU(2)_R$ doublets, where $\chi^c\equiv\mathcal{C}\chi^*$ with $\mathcal{C}$ the five-dimensional charge conjugation matrix. The spinors χ, φ then satisfy the coupled Killing spinor equations (7.1.11), (7.1.12), where the background b-field is taken to be a purely imaginary two-form, $A^{(0)}$ is a purely imaginary one-form, while $\omega_{\mu\nu} = g_{\nu\sigma}\omega_\mu{}^\sigma$ is real and symmetric. Furthermore, χ and φ are related algebraically by (7.1.14), which introduces the additional background fields α and K_2, which are respectively a real function and real one-form. Finally we have the dilatino equation (7.1.13), which introduces the real background function X_2.

In the remainder of this section we shall analyse the geometric constraints that these equations impose on (M_5, g). Although the background data and equations (7.1.11)–(7.1.14) appear *a priori* complicated, in fact we shall see that the geometry they are equivalent to is rather simple.

7.2.1 Differential constraints

In the analysis that follows it is convenient to assume that the spin^c spinor χ is nowhere zero. More generally χ could vanish along some locus $Z \subset M_5$, and the local geometry we shall derive below is valid on $M_5 \setminus Z$. If Z is non-empty one would need to impose suitable boundary conditions, although we shall not consider this further here. A nowhere zero spin^c spinor equips (M_5, g) with a local $SU(2)$ structure. Specifically, we may define the bilinears

$$\begin{aligned} S &\equiv \chi^\dagger \chi \,, & K_1 &\equiv \frac{1}{S}\chi^\dagger \gamma_{(1)} \chi \,, \\ J &\equiv -\frac{\mathrm{i}}{S}\chi^\dagger \gamma_{(2)} \chi \,, & \Omega &\equiv -\frac{1}{S}(\chi^c)^\dagger \gamma_{(2)} \chi \,. \end{aligned} \tag{7.2.1}$$

Here we have introduced the notation $\gamma_{(n)} \equiv \frac{1}{n!}\gamma_{\mu_1 \cdots \mu_n} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$, where x^μ, $\mu = 1, \ldots, 5$, are local coordinates on M_5. Since χ is nowhere zero the scalar function S is strictly positive, and it makes sense to normalize the bilinears as in (7.2.1). We note that K_1 is a real unit length one-form, while J is a real two-form with square length $\|J\|^2 = 2$. Here the square norm of a p-form ϕ is defined via $\|\phi\|^2 \, \mathrm{vol}_5 = \phi \wedge *\phi$, where $*$ denotes the Hodge duality operator on (M_5, g) and vol_5 denotes the Riemannian volume form. The complex bilinear Ω is globally a two-form valued in the line bundle L^{-1}.

That χ, or equivalently the bilinears (7.2.1), defines a local $SU(2)$ structure follows from some simple group theory. The spin group is $\mathrm{Spin}(5) \cong Sp(2) \subset U(4)$, with the latter acting in the fundamental representation on the spinor space $\mathbb{C}^4$. The stabilizer of a non-zero spinor is then $Sp(1) \cong SU(2)$. When M_5 is spin and L is trivial, so that $\chi \in \Gamma[\mathrm{Spin}(M_5)]$, this defines a global $SU(2)$ structure. However, more generally we require only that M_5 is spin^c, and in this case the *global* stabilizer group is enlarged to $U(2)$: the additional $U(1)$ factor rotates the spinor by a phase, which may be undone by a $U(1)$ gauge transformation. To see this in more detail we introduce a local orthonormal frame

e^a, $a = 1, \ldots, 5$, so that

$$K_1 \;=\; e^5\,, \qquad J \;=\; e^1\wedge e^2 + e^3\wedge e^4\,, \qquad \Omega \;=\; (e^1+\mathrm{i}e^2)\wedge(e^3+\mathrm{i}e^4)\,, \tag{7.2.2}$$

where the metric is $g = \sum_{a=1}^5 (e^a)^2$. The $U(2) = SU(2)\times_{\mathbb{Z}_2} U(1)$ structure group acts in the obvious way on the $\mathbb{C}^2$ spanned by $e^1+\mathrm{i}e^2, e^3+\mathrm{i}e^4$. This leaves K_1, J and the metric g invariant, but rotates Ω by the determinant of the $U(2)$ transformation. In order for this to be undone by a gauge transformation, this identifies the line bundle as $L = \Lambda^{2,0}$. The latter is the space of Hodge type $(2,0)$-forms for the four-dimensional vector bundle spanned by e^1, e^2, e^3, e^4, and with almost complex structure I for which $e^1+\mathrm{i}e^2$ and $e^3+\mathrm{i}e^4$ are $(1,0)$-forms. Thus our rigid supersymmetric geometry will in general be equipped with a global $U(2)$ structure on M_5 (or more precisely on $M_5\setminus Z$).

The one-form $SK_1 = \chi^\dagger\gamma_{(1)}\chi$ arises simply from the restriction of the bulk Killing one-form $\epsilon^\dagger\Gamma_{(1)}\epsilon$ to the conformal boundary, and thus defines a conformal Killing one-form on (M_5, g). This is easily confirmed from the Killing spinor equation (7.1.11) for χ, which implies

$$\nabla_{(\mu}(SK_1)_{\nu)} \;=\; \mathcal{L}_\xi(\log S)g_{\mu\nu}\,, \tag{7.2.3}$$

where we have introduced the dual vector field ξ, defined by $g(\xi,\cdot) = SK_1$, and $\mathcal{L}$ denotes the Lie derivative.

One finds that the spinor equations (7.1.11)–(7.1.14) imply the following differential constraints:

$$\mathrm{d}S \;=\; -\frac{\sqrt{2}}{3}\left(SK_2 + \mathrm{i}i_\xi b\right)\,, \qquad \mathrm{d}(S\alpha) \;=\; -\frac{1}{2\sqrt{2}}i_\xi \mathrm{d}a\,, \tag{7.2.4}$$

$$\mathrm{d}(SK_1) \;=\; \frac{2\sqrt{2}}{3}\left[2\alpha SJ + SK_1\wedge K_2 + \mathrm{i}Sb - \frac{\mathrm{i}}{2}i_\xi(*b)\right]\,, \tag{7.2.5}$$

$$\mathrm{d}(SK_2) \;=\; \mathrm{i}i_\xi \mathrm{d}b - \mathrm{i}\mathcal{L}_\xi(\log S)b\,, \tag{7.2.6}$$

$$\mathrm{d}(SJ) \;=\; -\sqrt{2}K_2\wedge(SJ)\,, \tag{7.2.7}$$

$$\mathrm{d}(S\Omega) \;=\; -\mathrm{i}\left(a - 2\sqrt{2}\alpha K_1 - \mathrm{i}\sqrt{2}K_2\right)\wedge(S\Omega)\,. \tag{7.2.8}$$

Here $(i_V\phi)_{a_1\cdots a_{p-1}} = V^b\phi_{ba_1\cdots a_{p-1}}$ defines the interior contraction of a vector V into a p-form ϕ. Notice that the background data X_2, $A^{(0)}$ and $\omega_{\mu\nu}$ in (7.1.11)–(7.1.14) does not enter equations (7.2.4)–(7.2.8): they simply drop out (one only needs to use the reality properties we specified, together with the fact that $\omega_{\mu\nu} = \omega_{\nu\mu}$ is symmetric).

It is straightforward to verify that (7.2.4)–(7.2.8) are invariant under the Weyl transformations

$$\begin{aligned} \alpha &\to \Lambda^{-1}\alpha\,, & a &\to a\,, & K_2 &\to K_2 - \frac{3}{\sqrt{2}}\mathrm{d}\log\Lambda\,, \\ S &\to \Lambda S\,, & K_1 &\to \Lambda K_1\,, & b &\to \Lambda b\,, \\ g &\to \Lambda^2 g\,, & J &\to \Lambda^2 J\,, & \Omega &\to \Lambda^2\Omega\,. \end{aligned} \tag{7.2.9}$$

This symmetry is of course inherited from invariance under the change of radial variable $r \to \Lambda r$ in the bulk. If S is nowhere zero notice that one might use this symmetry to set $S \equiv 1$.

Using equations (7.2.4)–(7.2.8) one can show that the conformal Killing vector ξ preserves all of the background geometric structure, provided one rescales the fields by appropriate powers of S according to their Weyl weights in (7.2.9). For instance, contracting ξ into the second equation in (7.2.4) shows that $\mathcal{L}_\xi(S\alpha) = 0$. On the other hand, taking the exterior derivative of the same equation one finds $\mathcal{L}_\xi \mathrm{d}a = 0$. One can hence locally choose a gauge in which a is invariant under $\mathcal{L}_\xi$, so that the second equation in (7.2.4) is solved by

$$S\alpha = \frac{1}{2\sqrt{2}} i_\xi a\,. \tag{7.2.10}$$

In a similar way, one can show that also $S^{-1}b$ and $S^{-2}J$ are invariant under $\mathcal{L}_\xi$, while $S^{-2}\Omega$ is invariant under $\mathcal{L}_\xi$ in the gauge choice for which (7.2.10) holds. Notice that the first equation in (7.2.4) implies that $i_\xi K_2 = -\frac{3}{\sqrt{2}}\mathcal{L}_\xi(\log S)$.

Without loss of generality it is convenient to henceforth impose $\mathcal{L}_\xi S = 0$.[2] In terms of the bulk expansion in section 7.1 this means choosing the radial coordinate r to be

[2]An exception being the $S^1 \times S^4$ geometry discussed in section 7.2.3.

independent of the bulk Killing vector. This is a natural choice, which in turn implies that $\mathcal{L}_\xi S = 0$ and SK_1 is Killing, and we shall make this convenient (partial) conformal gauge choice in the following. We may then introduce a local coordinate ψ so that

$$\xi \;=\; \partial_\psi \,. \tag{7.2.11}$$

The condition $\mathcal{L}_\xi S = 0$ is then equivalent to S being independent of ψ.

7.2.2 Geometric structure

The Killing vector ξ has norm S, and the dual one-form K_1 may be written locally as

$$K_1 \;=\; S(\mathrm{d}\psi + \rho) \;\equiv\; S\eta \,, \tag{7.2.12}$$

where $i_\xi \rho = 0$. Notice that η has Weyl weight zero and norm $1/S$. The local frame e_1, e_2, e_3, e_4 provide a basis for $D = \ker \eta$, and D inherits an almost complex structure from J. One then defines an endomorphism Φ of the tangent bundle of M_5 by

$$\Phi \,|_D = \; I \,, \qquad \Phi \,|_\xi = \; 0 \,, \tag{7.2.13}$$

where I is the almost complex structure. One easily verifies that $\Phi^2 = -1 + \xi \otimes \eta$, which is a defining relation of an *almost contact structure*. Moreover, the five-dimensional metric takes the form

$$\mathrm{d}s^2_{M_5} \;=\; S^2\eta^2 + \mathrm{d}s^2_4 \,, \tag{7.2.14}$$

where $\mathrm{d}s^2_4$ is (almost) Hermitian with respect to I. Although ξ is Killing, this structure is in general *not* a K-contact structure, which is a stronger condition. In particular the latter requires [168] that $\mathrm{d}\eta$ is the fundamental $(1,1)$-form J associated to the transverse almost complex structure (which in general is not the case here), which in turn implies that η is a contact form, *i.e.* that $\eta \wedge \mathrm{d}\eta \wedge \mathrm{d}\eta$ is a volume form (which in general is also not the case here). Notice that since ξ is nowhere zero, its orbits define a foliation of M_5.

Let us now turn to the differential constraints (7.2.4)–(7.2.8). The two equations (7.2.4) allow us to write

$$b \;=\; \mathrm{i}S\eta\wedge\left(K_2+\frac{3}{\sqrt{2}}\mathrm{d}\log S\right)+b_\perp\,, \qquad a \;=\; 2\sqrt{2}S\alpha\eta+a_\perp\,, \tag{7.2.15}$$

where $b_\perp$ and $a_\perp$ are *basic* forms for the foliation defined by ξ; that is, they are invariant under, and have zero interior contraction with, ξ. Recall that in writing the gauge field in the form in (7.2.15) we have made a (partial) gauge choice, as in (7.2.10). This leaves a residual gauge freedom $a_\perp \to a_\perp + \mathrm{d}\lambda$, where λ is a basic (ξ-invariant) function. The equation (7.2.6) is simply equivalent to b being invariant under ξ.

The differential constraint (7.2.5) reduces to

$$\mathrm{d}\rho \;=\; \frac{\sqrt{2}}{3S}\left(-\mathrm{i}*_4 b_\perp+2\mathrm{i}b_\perp+4\alpha J\right)\,. \tag{7.2.16}$$

Here $*_4$ is the Hodge dual with respect to the transverse four-dimensional metric $\mathrm{d}s^2_4$, with volume form $e^1\wedge e^2\wedge e^3\wedge e^4$. It is then convenient to introduce

$$b_\perp \;=\; b^+ + b^-\,, \tag{7.2.17}$$

decomposing into the transversely self-dual and anti-self-dual parts. Equation (7.2.16) is then equivalent to

$$b^+ \;=\; \mathrm{i}\left(4\alpha J-\frac{3}{\sqrt{2}}S\mathrm{d}\rho^+\right)\,, \qquad b^- \;=\; -\frac{\mathrm{i}}{\sqrt{2}}S\mathrm{d}\rho^-\,. \tag{7.2.18}$$

The constraint (7.2.7) simply identifies

$$\theta \;\equiv\; J\lrcorner\mathrm{d}J \;=\; -\sqrt{2}K_2-\mathrm{d}\log S\,, \tag{7.2.19}$$

with the *Lee form* θ of the transverse four-dimensional Hermitian structure. That is, every four-dimensional Hermitian structure with fundamental two-form J satisfies $\mathrm{d}J=\theta\wedge J$.

Finally, the differential constraint (7.2.8) now reads

$$\mathrm{d}\Omega \;=\; (\theta - \mathrm{i}a_\perp)\wedge\Omega\,. \tag{7.2.20}$$

This implies that the almost complex structure I is *integrable*, thus defining a *transversely holomorphic foliation* of M_5. In particular this means that the almost Hermitian structure of the transverse space is actually Hermitian. We may introduce local coordinates ψ, z_1, z_2 adapted to the foliation, where the transition functions between the z_1, z_2 coordinates are holomorphic.

Notice that we may rewrite (7.2.20) as

$$\mathrm{d}\Omega \;=\; -\mathrm{i}a_{\mathrm{Chern}}\wedge\Omega\,, \tag{7.2.21}$$

where we have defined

$$a_{\mathrm{Chern}} \;\equiv\; a_\perp - I(\theta)\,, \tag{7.2.22}$$

and $I(\theta) \equiv -i_{\theta^\#}J$, where $\theta^\#$ is the vector field dual to θ. To obtain an explicit expression for the Chern connection a_{Chern}, we begin by noting that $\Omega\wedge\bar\Omega = 2J\wedge J$. Using local coordinates z^α, $\alpha = 1, 2$, for the transverse space we may write

$$\Omega \;=\; f\,\mathrm{d}z^1\wedge\mathrm{d}z^2\,, \qquad J \;=\; \frac{\mathrm{i}}{2}g^{(4)}_{\alpha\bar\beta}\mathrm{d}z^\alpha\wedge\mathrm{d}\bar z^{\bar\beta}\,, \tag{7.2.23}$$

which implies that $|f| = \sqrt{\det g^{(4)}}$. Notice that globally f is a section of L^{-1}, where $L \cong \Lambda^{2,0} \equiv \mathcal{K}$ is the canonical bundle. Writing $f = |f|\mathrm{e}^{\mathrm{i}\phi}$ we then have

$$\mathrm{d}\Omega \;=\; \mathrm{d}\log f\wedge\Omega \;=\; \mathrm{i}\left(\frac{1}{2}\mathrm{d}^c\log\det g^{(4)} + \mathrm{d}\phi\right)\wedge\Omega\,, \tag{7.2.24}$$

where $\mathrm{d}^c \equiv I\circ\mathrm{d}$. We thus recognize (up to gauge)

$$a_{\mathrm{Chern}} \;=\; -\frac{1}{2}\mathrm{d}^c\log\det g^{(4)} \tag{7.2.25}$$

as the Chern connection on the canonical bundle.

The geometric content of the differential constraints (7.2.4)–(7.2.8) may hence be summarized as follows. M_5 is equipped with a transversely Hermitian structure, so that the metric takes the form

$$\mathrm{d}s^2_{M_5} \;=\; S^2(\mathrm{d}\psi+\rho)^2+\mathrm{d}s^2_4\,. \tag{7.2.26}$$

Here the Killing vector is $\xi=\partial_\psi$, which generates a transversely holomorphic foliation. The almost contact form is $\eta=(\mathrm{d}\psi+\rho)$, and $\mathrm{d}s^2_4$ is a transverse Hermitian metric. One is also free to specify the functions α and S. Given this data, the remaining background fields a and b that enter (7.2.4)–(7.2.8) are determined via

$$\begin{aligned} a &= 2\sqrt{2}S\alpha\eta + a_{\mathrm{Chern}} + I(\theta)\,,\\ b &= -\frac{\mathrm{i}}{\sqrt{2}}S\eta\wedge(\theta-2\,\mathrm{d}\log S)+4\mathrm{i}\alpha J-\frac{\mathrm{i}}{\sqrt{2}}S(3\mathrm{d}\rho^+ +\mathrm{d}\rho^-)\,. \end{aligned} \tag{7.2.27}$$

In particular the choice of a transverse Hermitian metric $g^{(4)}$ fixes the two-form J, and hence the Lee form θ, while the Hodge type $(2,0)$-form Ω and Chern connection a_{Chern} in (7.2.25) are also determined up to gauge. Notice that the terms $S\alpha\eta$ and $I(\theta)$ entering the formula for a in (7.2.27) are both global one-forms on M_5, implying that globally a is a connection on $L=\Lambda^{2,0}$.

We shall furthermore show in section 7.2.4 that *any* choice of transversely Hermitian structure on M_5 of the above form gives a supersymmetric background. In particular the remaining background fields X_2, $A^{(0)}$, and $\omega_{\mu\nu}$ appearing in the spinor equations (7.1.11)–(7.1.14) are also determined by the above geometric data.

7.2.3 Examples

In this section we shall present some explicit examples of the above construction. These include all explicit examples appearing in the literature (within the Abelian truncation on which we are mostly focusing), including examples with six-dimensional gravity duals, plus large families of new solutions.

General families

We begin by noting some special families of backgrounds:

- ◇ Setting $\rho = 0$ and $S \equiv 1$ gives a product metric $M_5 = \mathbb{R} \times M_4$ or $M_5 = S^1 \times M_4$, where M_4 is any Hermitian four-manifold. Notice this four-manifold geometry is the same as the rigid supersymmetric geometry one finds in four dimensions [104, 123]. The first reference here follows a similar holographic approach to the present chapter, while the second takes a rigid limit of "new minimal" supergravity in four dimensions.
- ◇ If $\mathrm{d}\theta = 0$ then the transverse Hermitian metric is locally conformally Kähler.
  - ⋆ If furthermore $\theta = 0$ then the transverse four-metric is Kähler.
  - ⋆ If $\theta = 0$ and $\mathrm{d}\rho$ is a positive constant multiple of J then the five-metric is locally conformally Sasakian. Supersymmetric gauge theories on Sasaki-Einstein manifolds, for which furthermore $S \equiv 1$ and g is a positively curved Einstein metric, were defined in [155], and further studied in [169–172].
- ◇ We may take any circle bundle over a product of Riemann surfaces $S^1 \hookrightarrow \Sigma_1 \times \Sigma_2$. The Hermitian metric may be taken to be simply a product of two metrics on the Riemann surfaces, while ρ is the connection one-form for the fibration. One can generalize this further by allowing S^1 orbibundles over a product of orbifold Riemann surfaces.
  - ⋆ If we only fibre over Σ_1, this leads to direct product $M_3 \times \Sigma_2$ solutions, where M_3 is a Seifert fibred three-manifold. Notice this three-manifold geometry is the same as the rigid supersymmetric geometry in three dimensions [122]. Maximally supersymmetric Yang-Mills theory has been studied on similar backgrounds in [42, 111–114], including the direct products $S^3 \times \Sigma_2$ and $M_3 \times S^2$. Here the spheres are equipped with round metrics and the associated canonical spinors, while the spinors on Σ_2 and M_3 are constructed by topologically twisting with the $SU(2)_R$ symmetry.

◇ Finally, if $\mathrm{d}\rho$ has Hodge type $(1,1)$ the transversely holomorphic foliation admits a complexification [173], *i.e.* adding a radial direction to ξ we naturally have a complex six-manifold M_6, with a transversely holomorphic foliation. Notice that Sasakian geometry and the direct product $S^1 \times M_4$ are special cases. When the orbits of ξ all close, M_5 fibres over a Hermitian four-orbifold M_4, and the associated $U(1)$ orbibundle is the unit circle in a Hermitian holomorphic line orbibundle over M_4. The corresponding complex M_6 is then simply the total space of the associated $\mathbb{C}^*$ bundle over M_4.

Squashed Sasaki-Einstein

We have already noted that a Sasakian five-manifold is a particular case of a supersymmetric background. Recall that Sasakian metrics take the form

$$\mathrm{d}s_5^2 \;=\; \eta^2 + \mathrm{d}s_4^2\,, \tag{7.2.28}$$

where η defines a contact structure on M_5, with Reeb Killing vector field ξ, and $\mathrm{d}s_4^2$ is a transverse Kähler metric. Moreover $\mathrm{d}\eta = \mathrm{d}\rho = 2J$. If the transverse Kähler metric $g^{(4)}$ is Einstein, then the metric (7.2.28) is said to be a *squashed* Sasaki-Einstein metric.[3] For a given choice of transverse Kähler-Einstein metric, we obtain a two-parameter family of backgrounds, parametrized by the constants c_1, c_2:

$$\begin{aligned} S \;&\equiv\; 1\,, \qquad & K_2 \;&=\; -\frac{1}{\sqrt{2}}\theta \;\equiv\; 0\,, \qquad & \alpha \;&=\; c_1\,, \\ a \;&=\; c_2\eta\,, \qquad & b \;&=\; \mathrm{i}(4c_1 - 3\sqrt{2})J\,. \end{aligned} \tag{7.2.29}$$

The Kähler-Einstein metric $g^{(4)}$ satisfies the Einstein equation $\mathrm{Ric}^{(4)} = 2(2\sqrt{2}c_1 - c_2)g^{(4)}$. Notice that we have presented the solution (7.2.29) in a different gauge choice to (7.2.10). We may impose the latter gauge choice by simply transforming $a \to a + (2\sqrt{2}c_1 - c_2)\mathrm{d}\psi$, although the form of a in (7.2.29) makes it clear that we may take a to be a global one-form

[3] In the mathematical literature [168] these are called η-Sasaki-Einstein metrics.

on M_5 for this particular class of solutions.

When g_4 is taken to be the standard metric on $\mathbb{CP}^2$, the above geometry is a squashed five-sphere. This corresponds to the conformal boundary of the 1/4 BPS bulk Romans supergravity solutions constructed in the previous chapter.

Black hole boundary

In this section we consider the conformal boundary of the 1/2 BPS topological black hole solutions constructed in [174]. We begin with the following product metric on $S^1 \times \mathbb{H}^4$, where $\mathbb{H}^4$ is hyperbolic four-space:

$$\mathrm{d}s_5^2 \;=\; \mathrm{d}\tau^2 + \frac{1}{q^2+1}\mathrm{d}q^2 + q^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi_1^2 + \cos^2\vartheta\mathrm{d}\varphi_2^2)\,. \tag{7.2.30}$$

Here τ is a periodic coordinate on S^1, q is a radial coordinate with $q \in [0,\infty)$, $\vartheta \in [0, \frac{\pi}{2}]$ while φ_1, φ_2 have period 2π. The metric in brackets is simply the round metric on S^3. For this solution b vanishes identically, while a is gauge-equivalent to zero. The Killing spinors for this background in general depend on four integration constants, but for simplicity here we present only the "toric" solution discussed in [174]. The remaining fields are then

$$S \;=\; \sqrt{q^2+1}\,, \qquad \alpha \;=\; -\frac{3}{2\sqrt{2}\sqrt{q^2+1}}\,, \qquad K_2 \;=\; -\frac{3}{2\sqrt{2}}\,\mathrm{d}\log(q^2+1)\,, \tag{7.2.31}$$

while in a gauge[4] in which $a=0$ the $U(2)$ structure is given by

$$\begin{aligned}
K_1 \;&=\; \frac{1}{\sqrt{q^2+1}}\left[\mathrm{d}\tau + q^2(\cos^2\vartheta\mathrm{d}\varphi_2 - \sin^2\vartheta\mathrm{d}\varphi_1)\right]\,,\\
J \;&=\; \frac{q^2}{2}\,\sin 2\vartheta\,\mathrm{d}\vartheta\wedge(\mathrm{d}\varphi_1+\mathrm{d}\varphi_2) + \frac{q}{(q^2+1)}\mathrm{d}q\wedge\left[\mathrm{d}\tau + \sin^2\vartheta\mathrm{d}\varphi_1 - \cos^2\vartheta\mathrm{d}\varphi_2\right]\,,\\
\Omega \;&=\; -\frac{q\,\mathrm{e}^{\mathrm{i}(\varphi_1-\tau-\varphi_2)}}{2\sqrt{q^2+1}}\Big[\sin 2\vartheta\,(q\,\mathrm{d}\tau - \mathrm{i}\mathrm{d}q)\wedge(\mathrm{d}\varphi_1+\mathrm{d}\varphi_2) + q\,\sin 2\vartheta\,\mathrm{d}\varphi_1\wedge\mathrm{d}\varphi_2\\
&\quad +2\mathrm{i}\,q\,\mathrm{d}\vartheta\wedge(\mathrm{d}\tau + \sin^2\vartheta\mathrm{d}\varphi_1 - \cos^2\vartheta\mathrm{d}\varphi_2) - 2\,\mathrm{d}q\wedge\mathrm{d}\vartheta\Big]\,.
\end{aligned} \tag{7.2.32}$$

[4] This is different to the gauge choice (7.2.10), where instead $a = -3\mathrm{d}\tau$ for this solution.

The supersymmetric Killing vector is

$$\xi \;=\; g(SK_1,\,\cdot\,) \;=\; \partial_\tau + \partial_{\varphi_2} - \partial_{\varphi_1}\,. \qquad (7.2.33)$$

Furthermore, notice that rescaling J by $1/(q^2+1)$ leads to a closed two-form, hence showing that the Hermitian metric transverse to ξ is conformal to a Kähler metric. Moreover, one can also check that the almost contact form $\eta = K_1/S$ is in fact a contact form in this case, *i.e.* that $\eta \wedge \mathrm{d}\eta \wedge \mathrm{d}\eta$ is a volume form.

Conformally flat $S^1 \times S^4$

In this section we consider the conformally flat metric on $S^1 \times S^4$, which we may write as

$$\mathrm{d}s^2_5 \;=\; \mathrm{d}\tau^2 + \mathrm{d}s^2_{S^4}\,, \qquad (7.2.34)$$

where

$$\mathrm{d}s^2_{S^4} \;=\; \mathrm{d}\beta^2 + \sin^2\beta(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi_1^2 + \cos^2\vartheta\mathrm{d}\varphi_2^2)\,. \qquad (7.2.35)$$

Here τ is a periodic coordinate on S^1, while the metric in brackets in (7.2.35) is simply the round metric on a sphere of unit radius, as in the previous black hole boundary example. The polar coordinate $\beta \in [0,\pi]$. The metric (7.2.34) of course arises as the conformal boundary of Euclidean AdS in global coordinates, and as such the background fields a and b vanish, $a = 0 = b$. There are many Killing spinors in this case, and here we simply choose one so as to present simple expressions for the remaining background data. We find

$$S \;=\; \mathrm{e}^{-\tau}\,, \qquad \alpha \;=\; 0\,, \qquad K_2 \;=\; \frac{3}{\sqrt{2}}\mathrm{d}\tau\,. \qquad (7.2.36)$$

The $U(2)$ structure is given by

$$
\begin{aligned}
K_1 &= \sin\beta\,\mathrm{d}\beta - \cos\beta\,\mathrm{d}\tau\,,\\
J &= \sin^2\beta\sin(\varphi_1+\varphi_2)\Big\{\cot(\varphi_1+\varphi_2)\,(\mathrm{d}\vartheta\wedge\mathrm{d}\tau - \cot\beta\,\mathrm{d}\beta\wedge\mathrm{d}\vartheta) - \sin^2\vartheta\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi_1\\
&\quad -\cos^2\vartheta\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi_2 + \sin\vartheta\cos\vartheta\Big[(\cot\beta\,\mathrm{d}\beta+\mathrm{d}\tau)\wedge(\mathrm{d}\varphi_1-\mathrm{d}\varphi_2)\\
&\quad -\cot(\varphi_1+\varphi_2)\,\mathrm{d}\varphi_1\wedge\mathrm{d}\varphi_2\Big]\Big\}\,,\\
\Omega &= \mathrm{i}\sin^2\beta\sin(\varphi_1+\varphi_2)\Big[\cot\beta\,\mathrm{d}\beta\wedge\mathrm{d}\vartheta - \mathrm{d}\vartheta\wedge\mathrm{d}\tau + \sin\vartheta\cos\vartheta\,\mathrm{d}\varphi_1\wedge\mathrm{d}\varphi_2\Big]\\
&\quad +\sin^2\beta\sin\vartheta\Big[\sin\vartheta + \mathrm{i}\cos\vartheta\cos(\varphi_1+\varphi_2)\Big]\Big(\cot\beta\,\mathrm{d}\beta\wedge\mathrm{d}\varphi_1 - \cot\vartheta\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi_2\\
&\quad +\mathrm{d}\tau\wedge\mathrm{d}\varphi_1\Big) + \sin^2\beta\cos\vartheta\Big[\cos\vartheta - \mathrm{i}\sin\vartheta\cos(\varphi_1+\varphi_2)\Big]\Big(\cot\beta\,\mathrm{d}\beta\wedge\mathrm{d}\varphi_2\\
&\quad +\tan\vartheta\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi_1 + \mathrm{d}\tau\wedge\mathrm{d}\varphi_2\Big)\,.
\end{aligned}
\tag{7.2.37}
$$

Notice that in this example we obtain a *conformal* Killing vector from the Killing spinor bilinear, but not a Killing vector. As described at the end of section 7.2.1, we may always make a Weyl transformation of the background to obtain a Killing vector. In the case at hand this corresponds to the Weyl factor $\Lambda = \mathrm{e}^{\tau}$, and the corresponding Weyl-transformed metric is then (locally) flat, with the Weyl-transformed J and Ω both closed and hence defining a transverse hyperKähler structure. Nevertheless, the fact that the metric (7.2.34) leads to a conformal Killing vector explains why this background is missing from the rigid supersymmetric geometry in [119, 120]: in the latter references the corresponding bilinear is necessarily a Killing vector. This also suggests that the conjecture made in [120] is likely to be correct: that is, to obtain the $S^1 \times S^4$ background from a rigid limit of supergravity, one should begin with *conformal* supergravity in five dimensions, rather than Poincaré supergravity.

Squashed S^5

We consider the squashed five-sphere metric

$$\begin{aligned} \mathrm{d}s_5^2 &= \frac{1}{s^2}(\mathrm{d}\tau + C)^2 + \mathrm{d}\sigma^2 + \frac{1}{4}\sin^2\sigma(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2) \\ &\quad + \frac{1}{4}\cos^2\sigma\sin^2\sigma(\mathrm{d}\beta + \cos\vartheta\mathrm{d}\varphi)^2 , \end{aligned}$$

where $s \in (0, 1]$ is the squashing parameter and

$$C \equiv -\frac{1}{2}\sin^2\sigma(\mathrm{d}\beta + \cos\vartheta\mathrm{d}\varphi) . \tag{7.2.38}$$

The coordinates $\sigma, \beta, \vartheta, \varphi$ are coordinates on the base $\mathbb{CP}^2$, with β having period 4π, φ having period 2π, while $\sigma \in [0, \frac{\pi}{2}]$, $\vartheta \in [0, \pi]$, and $\frac{1}{2}\mathrm{d}C$ is the Kähler two-form on $\mathbb{CP}^2$. For the "toric" family discussed in Chapter 6 we find

$$S = \frac{\cos^2\sigma}{b_2} + \frac{\sin^2\sigma}{b_1} , \tag{7.2.39}$$

where

$$b_1 = 1 + \sqrt{1 - s^2} , \qquad b_2 = 1 - \sqrt{1 - s^2} . \tag{7.2.40}$$

The other background fields are, in an appropriate gauge (*i.e.* not that in (7.2.10)),

$$\begin{aligned} \alpha &= \frac{b_1(b_1 + b_2)(b_1 - 7b_2 + (b_1 - b_2)\cos 2\sigma)}{4\sqrt{2}(b_1\cos^2\sigma + b_2\sin^2\sigma)} , \\ a &= \frac{b_1 - b_2}{2b_2}(\mathrm{d}\tau + C) , \\ b &= -\frac{\mathrm{i}(b_1 - b_2)}{2\sqrt{2}b_1 b_2(b_1 + b_2)}\mathrm{d}C , \\ K_2 &= \frac{\sqrt{2}\,(b_1 - b_2)\,\sin 2\sigma}{b_1\cos^2\sigma + b_2\sin^2\sigma}\,\mathrm{d}\sigma = -\sqrt{2}\,\mathrm{d}\log(b_1\cos^2\sigma + b_2\sin^2\sigma) . \end{aligned} \tag{7.2.41}$$

The $U(2)$ structure is

$$
\begin{aligned}
K_1 &= \frac{1}{4b_1b_2(b_1+b_2)\big(b_1\cos^2\sigma+b_2\sin^2\sigma\big)}\Bigg[(b_1+b_2)(b_1-b_2+(b_1+b_2)\cos 2\sigma)\mathrm{d}\tau \\
&\quad -\frac{1}{2}\sin^2\sigma\left((b_1-b_2)^2\cos 2\sigma+b_1^2-4b_1b_2-b_2^2\right)(\mathrm{d}\beta+\cos\vartheta\mathrm{d}\varphi)\Bigg]\,, \\
J &= \frac{\sin\sigma}{8\,b_1b_2\,(b_1+b_2)^2\,\left(b_1\cos^2\sigma+b_2\sin^2\sigma\right)}\Bigg[4\cos\sigma\Big(\,2(b_1+b_2)\,\mathrm{d}\sigma\wedge\mathrm{d}\tau \\
&\quad -b_1\mathrm{d}\sigma\wedge(\mathrm{d}\beta+\cos\vartheta\mathrm{d}\varphi)\Big)+2\sin\vartheta\sin\sigma\,(b_1\cos^2\sigma+b_2\sin^2\sigma)\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\Bigg]\,, \\
\Omega &= \frac{\sin\sigma\,\mathrm{e}^{\mathrm{i}(\tau-\beta)}}{8b_1b_2(b_1+b_2)^2\left(b_1\cos^2\sigma+b_2\sin^2\sigma\right)}\Bigg[-\sin 2\sigma\Big(\mathrm{i}\sin\vartheta\,(\,b_1\mathrm{d}\varphi\wedge\mathrm{d}\beta \\
&\quad +2(b_1+b_2)\,\mathrm{d}\tau\wedge\mathrm{d}\varphi)-2(b_1+b_2)\,\mathrm{d}\vartheta\wedge\mathrm{d}\tau+b_1\,\mathrm{d}\vartheta\wedge(\mathrm{d}\beta+\cos\vartheta\mathrm{d}\varphi)\Big) \\
&\quad -4\,(b_1\cos^2\sigma+b_2\sin^2\sigma)\,(\sin\vartheta\;\mathrm{d}\sigma\wedge\mathrm{d}\varphi+\mathrm{i}\,\mathrm{d}\vartheta\wedge\mathrm{d}\sigma)\Bigg]\,.
\end{aligned}
\tag{7.2.42}
$$

The supersymmetric Killing vector is

$$
\xi \;=\; b_1\partial_\tau+2(b_1+b_2)\partial_\beta\,. \tag{7.2.43}
$$

One also computes

$$
\begin{aligned}
\eta\wedge\mathrm{d}\eta\wedge\mathrm{d}\eta \;&=\; \frac{b_1^3b_2^3\;(b_1+b_2)^2}{2\left(b_1\cos^2\sigma+b_2\sin^2\sigma\right)^5}\left[(b_1-b_2)^2\cos 2\sigma+b_1^2-4b_1b_2-b_2^2\right] \\
&\quad\times\left[\left(b_1^2-b_2^2\right)\cos 2\sigma+b_1^2-6b_1b_2+b_2^2\right]\mathrm{vol}_5\,,
\end{aligned}
\tag{7.2.44}
$$

where vol_5 denotes the Riemannian volume form and $\eta = K_1/S$ is the almost contact form. The right hand side of (7.2.44) can have non-trivial zeros, and we thus see that in general η *does not* define a contact structure. These backgrounds arise as the conformal boundary of the 3/4 BPS solutions of Romans supergravity constructed in Chapter 6.

7.2.4 From geometry to supersymmetry

In this section we will show that *any* choice of transversely Hermitian structure on M_5 defines a supersymmetric background. The background $U(1)_R$ gauge field a and the b-field are given in terms of the geometry by (7.2.27). It then remains to show that the geometry also determines the fields X_2, $A^{(0)}$ and $\omega_{\mu\nu}$, in such a way that the original spinor equations (7.1.11)–(7.1.14) are satisfied.

We first examine the Killing spinor equation (7.1.11) for χ. In order to proceed it is convenient to choose a set of projection conditions (see for example [175])

$$\gamma_{12}\chi \; = \; \gamma_{34}\chi \; = \; \mathrm{i}\chi \, , \qquad \gamma_5\chi \; = \; \chi \, . \tag{7.2.45}$$

These allow one to substitute for the fields b and K_2 in terms of the geometry, via (7.2.27) and (7.2.19), into the right hand side of equation (7.1.11). In doing this calculation it is also convenient to write $\Omega = J_2 + \mathrm{i}J_1$, $J = J_3$ so that

$$J_1 \; = \; e_{14} + e_{23} \, , \qquad J_2 \; = \; e_{13} - e_{24} \, , \qquad J_3 \; = \; e_{12} + e_{34} \, . \tag{7.2.46}$$

Notice that J_i, $i = 1,2,3$ span the transverse self-dual two-forms, and hence may be used as a basis thereof. One can furthermore make use of various identities that easily follow from (7.2.45), such as $\mathrm{i}\gamma_m\chi = J_{mn}\gamma^n\chi$, where $m,n = 1,\dots,4$, and $(\beta^-)_{mn}\gamma^{mn}\chi = 0$ for any transverse anti-self-dual two-form β^-.

In this way it is straightforward to show that the $\mu = 5$ (the ψ direction) component of (7.1.11) simply imposes $\partial_\psi\chi = 0$.[5] Thus χ is independent of ψ. Taking instead $\mu = m$, $m = 1,2,3,4$, one finds (7.1.11) is equivalent to

$$\nabla^{(4)}_m\chi \; = \; \frac{1}{4}\theta^n\gamma_{mn}\chi - \frac{\mathrm{i}}{2}(a_\perp)_m\chi + \frac{1}{2}(\partial_m \log S)\chi \, , \tag{7.2.47}$$

[5] Without loss of generality we take the four-dimensional frame $e^1,\dots,e^4$ to be independent of the Killing vector $\xi = \partial_\psi$.

where $\nabla^{(4)}$ denotes the Levi-Civita spin connection for the transverse four-dimensional metric. Recall that the latter metric is Hermitian. It is then more natural to express equation (7.2.47) in terms of an appropriate Hermitian connection, which preserves both the metric and the two-form J. The Chern connection is such a connection, defined by

$$\begin{aligned} \nabla_m^{\mathrm{Chern}}\chi &= \partial_m\chi + \frac{1}{4}(\omega_m^{\mathrm{Chern}})_{pq}\gamma^{pq}\chi\,, \\ \text{where} \qquad (\omega_m^{\mathrm{Chern}})_{pq} &\equiv (\omega_m^{(4)})_{pq} + \frac{1}{2}J_m{}^n(\mathrm{d}J)_{npq}\,. \end{aligned} \tag{7.2.48}$$

This coincides with the Levi-Civita connection if and only if $\mathrm{d}J = 0$ (equivalently $\theta = 0$), so that the metric is Kähler.

Next, let us notice that under the Weyl transformation (7.2.9) we have $\chi \to \Lambda^{1/2}\chi$, so that it is also natural to introduce

$$\tilde{\chi} \equiv S^{-1/2}\chi\,, \tag{7.2.49}$$

so that $\tilde{\chi}$ is Weyl invariant. In this notation (7.2.47) becomes

$$\nabla_m^{\mathrm{Chern}}\tilde{\chi} + \frac{\mathrm{i}}{2}a_{\mathrm{Chern}}\tilde{\chi} = 0\,, \tag{7.2.50}$$

where recall that $a_{\mathrm{Chern}} = a_\perp - I(\theta)$ is the Chern connection for the canonical bundle $\mathcal{K} \equiv \Lambda^{2,0}$, given explicitly by (7.2.25). It is then a standard fact – and is straightforward to show – that any Hermitian space admits a canonical solution $\tilde{\chi}$ to (7.2.50). Specifically, any Hermitian space admits a canonical spin^c structure, with twisted spin bundles $\mathrm{Spin}^c = \mathrm{Spin} \otimes \mathcal{K}^{-1/2}$. In four dimensions this is isomorphic to

$$\mathrm{Spin}^c \cong \left(\Lambda^{0,0} \oplus \Lambda^{0,2}\right) \oplus \Lambda^{0,1}\,, \tag{7.2.51}$$

where $\Lambda^{p,q}$ denotes the bundle of forms of Hodge type (p,q). In the case at hand, these are defined transversely to the foliation generated by the Killing vector ξ. Under (7.2.51) the Killing spinor $\tilde{\chi} = S^{-1/2}\chi$ is a section of the trivial line bundle $\Lambda^{0,0}$. Moreover, the Chern connection restricted to this summand is flat, with the induced connection $-\frac{1}{2}a_{\mathrm{Chern}}$ on the twist factor $\mathcal{K}^{-1/2}$ effectively cancelling that coming from the spin bundle. Concretely, in terms of local complex coordinates z^α, $\alpha = 1, 2$, we have $(\omega^{\mathrm{Chern}})_\alpha{}^\beta = (\partial g^{(4)})_{\alpha\bar{\gamma}}(g^{(4)})^{\bar{\gamma}\beta}$, and using the projection conditions (7.2.45) one can show this precisely cancels the contribution from (7.2.25). The spin^c spinor $\tilde{\chi}$ is simply a constant length section of this flat line bundle. Put simply, the rescaled Killing spinor $\tilde{\chi} = S^{-1/2}\chi$ is constant.

Next we turn to the dilatino equation (7.1.13). Substituting for φ in terms of χ, using (7.1.14), after a somewhat lengthy computation one finds the dilatino equation holds provided

$$A^{(0)} = -\frac{9}{4} * \left(\mathrm{d} * b - \frac{\mathrm{i}\sqrt{2}}{3} b \wedge b\right) , \tag{7.2.52}$$

and

$$X_2 = -4\alpha^2 - \frac{1}{4}\langle K_2, K_2\rangle - \frac{\mathrm{i}}{6\sqrt{2}} S\langle \eta, A^{(0)}\rangle - \frac{3}{16}\langle \mathrm{d}a_\perp, J\rangle - \frac{3}{4\sqrt{2}}\langle K_2, \mathrm{d}\log S\rangle . \tag{7.2.53}$$

Here we have introduced the notation $\phi_1 \wedge *\phi_2 = \frac{1}{p!}\langle \phi_1, \phi_2\rangle\, \mathrm{vol}_5$ for the inner product between two p-forms ϕ_1, ϕ_2. Notice that under the Weyl scaling (7.2.9) we have

$$A^{(0)} \to \frac{1}{\Lambda}\left(A^{(0)} + \frac{9}{2} i_{\mathrm{d}\log\Lambda^\#} b\right) , \qquad X_2 \to \frac{1}{\Lambda^2} X_2 . \tag{7.2.54}$$

The fact that X_2 has Weyl weight -2 is clearly consistent with the bulk expansion (7.1.5), but the "anomalous" transformation of $A^{(0)}$ in (7.2.54) naively appears to contradict (7.1.5), for which $A^{(0)}$ has Weyl weight -1. However, this is where the comment above equation (7.1.5) is relevant: the reparametrization $r \to \Lambda r$ does *not* preserve the *subleading* terms in the metric (7.1.1). It is therefore not a strict symmetry of the system we have defined. However, the *leading order* terms in the expansions (7.1.1), (7.1.5) *are* invariant. This explains why the differential constraints (7.2.4)–(7.2.8) have the Weyl symmetry (7.2.9), while the higher order term $A^{(0)}$ arising in the expansion of the B-field does not. One could restore the full Weyl symmetry by adding a cross term $9\frac{\mathrm{d}r}{r}\mathscr{C}_\mu \mathrm{d}x^\mu$ into the metric (7.1.1), so that

$$\mathscr{C} \;\to\; \mathscr{C} - \mathrm{d}\log\Lambda\,, \tag{7.2.55}$$

under $r \to \Lambda r$ preserves the form of the metric. Then $\mathscr{C}$ is a new background field on M_5, and one finds

$$A^{(0)} \;=\; -\frac{9}{4} * \left[(\mathrm{d} + 2\mathscr{C}\wedge) * b - \frac{\mathrm{i}\sqrt{2}}{3} b\wedge b\right]\,. \tag{7.2.56}$$

This now has Weyl weight -1, as expected, and the anomalous variation in (7.2.54) arises simply because we have made the gauge choice $\mathscr{C} = 0$ in our original expansion. In general notice that a field of Weyl weight w will couple to a Weyl covariant derivative $\mathscr{D}_\mu \equiv \partial_\mu + w\,\mathscr{C}_\mu$, with $w = 2$ for $*b$.

It remains to show that the background geometry implies the φ Killing spinor equation (7.1.12). At this point notice that everything is fixed uniquely in terms of the free functions α and S, and the transversely Hermitian structure on M_5, apart from the higher order spin connection term $\omega_{\mu\nu}$ which appears in (7.1.12). After a lengthy computation, in our orthonormal frame one finds the expression

$$\begin{aligned}
\omega_{55} \;&=\; -\,6\sqrt{2}\alpha^2 - \frac{1}{3\sqrt{2}}\langle K_2, K_2\rangle - \sqrt{2}X_2 - \frac{1}{2\sqrt{2}}\langle \mathrm{d}a_\perp, J\rangle - \langle K_2, \mathrm{d}\log S\rangle\,,\\
\omega_{5m} \;=\; \omega_{m5} \;&=\; \left[-\frac{\mathrm{i}}{3\sqrt{2}} i_{K_2^\#} b_\perp + i_{\mathrm{d}\log S\#}\left(2\alpha J + \frac{1}{\sqrt{2}} S\mathrm{d}\rho^-\right)\right]_m\,,\\
\omega_{mn} \;&=\; \frac{\sqrt{2}}{3}(K_2)_m (K_2)_n - \nabla^{(4)}{}_{(m}(K_2)_{n)} - \Big(\frac{4}{3}S\alpha\,\mathrm{d}\rho^- + \frac{1}{\sqrt{2}}\mathrm{d}a_\perp^-\Big)_{mp} J^p{}_n\\
&\quad + \left(2\sqrt{2}\alpha^2 + \frac{\sqrt{2}}{3}X_2 - \frac{1}{3\sqrt{2}}\langle K_2, K_2\rangle + \frac{1}{4\sqrt{2}}\langle \mathrm{d}\alpha_\perp, J\rangle\right)\delta_{mn}\,.
\end{aligned} \tag{7.2.57}$$

This is manifestly real and symmetric, apart from the last term in the penultimate line. However, it is straightforward to show that $(\beta^-)_{mp} J^p{}_n$ is symmetric for any transverse anti-self-dual two-form β^-. Thus (7.1.12) is satisfied provided $\omega_{\mu\nu}$ is given by (7.2.57).

7.2.5 Summary

A supersymmetric asymptotically locally AdS solution to six-dimensional Romans supergravity leads to the coupled spinor equations (7.1.11)–(7.1.14) on the conformal boundary M_5. These are a rather complicated looking equations for the spin^c spinors χ and φ, depending on the large number of background fields $g, X_2, a, A^{(0)}, b$ and $\omega_{\mu\nu}$ on M_5, with φ and χ related to each other by the further background fields α and K_2 via (7.1.14). However, we have shown these equations are completely equivalent to a very simple geometric structure:

(i) The five-manifold M_5 is equipped with a transversely holomorphic foliation, with the one-dimensional leaves generated by the (conformal) Killing vector field $\xi = \partial_\psi$. This structure is a natural odd-dimensional cousin of a complex manifold, and means we may cover M_5 locally with coordinates ψ, z_1, z_2, where the transition functions between the z_1, z_2 coordinates are holomorphic. More formally we have an open cover $\{U_i\}$ and submersions $f_i : U_i \to \mathbb{C}^2$ with one-dimensional fibres, such that on overlaps $U_i \cap U_j$ we have $f_j = g_{ji} \circ f_i$ where g_{ji} are biholomorphisms of open sets in $\mathbb{C}^2$.

(ii) This foliation is compatible with an almost contact form $\eta = \mathrm{d}\psi + \rho$. Choose a particular $\rho = \rho_0$, which is defined only locally in the foliation patches, gluing together to give the global η. Then for fixed foliation any other choice of ρ is related to this by $\rho = \rho_0 + \nu$, where ν is a *global basic one-form*. That is, ν is a global one-form on M_5 satisfying $\mathcal{L}_\xi \nu = 0 = i_\xi \nu$.

(iii) One can choose an *arbitrary* transverse Hermitian metric $\mathrm{d}s^2_4$, invariant under ξ and compatible with the foliation.

(iv) Finally, one is free to choose the ξ-invariant real functions α and S (with S nowhere zero).

We have shown that any choice of the data (i)–(iv) determines a supersymmetric background, solving the spinor equations (7.1.11)–(7.1.14), and conversely any such solution determines a choice of the above geometric data. Furthermore, solving (7.1.11)–(7.1.14) is equivalent to finding a supersymmetric asymptotically locally AdS solution to Romans supergravity, to the first few orders in an expansion around the conformal boundary M_5. Of course whether or not this extends to a complete, non-singular supergravity solution, as some of the explicit examples in section 7.2.3 do, is another matter.

7.3 Discussion

In this chapter we have constructed rigid supersymmetric gauge theories with matter on a general class of five-manifold backgrounds. By construction these are the most general backgrounds that arise as conformal boundaries of six-dimensional Romans supergravity solutions. We find that (M_5, g) is equipped with a conformal Killing vector which generates a transversely holomorphic foliation. In particular the transverse metric $g^{(4)}$ is an arbitrary Hermitian metric with respect to the transverse complex structure. This is a natural hybrid/generalization of the rigid supersymmetric geometries in three and four dimensions constructed in [122–124], and includes many previous constructions as special cases.

Let us mention here, that an entirely different approach to defining supersymmetry on M_5 is followed in [105, 172]. In [105] a twisted version of $\mathcal{N} = 1$ super-Yang-Mills theory is defined on *contact* five-manifolds (M_5, η). Here η is a contact one-form, meaning that $\eta \wedge \mathrm{d}\eta \wedge \mathrm{d}\eta$ is a volume form. On a Sasaki-Einstein five-manifold [176] one can construct $\mathcal{N} = 1$ supersymmetric Yang-Mills coupled to matter [155]. This is essentially because the two Killing spinors on a Sasaki-Einstein manifold satisfy the same Killing spinor equations as those on the round sphere. For the special class of toric Sasaki-Einstein manifolds of [177] the localized perturbative partition function has been computed in [169–171], with the last reference also giving a conjectured formula for the full partition function. The authors of [172] furthermore show that one can define a twisted version of $\mathcal{N} = 2$ super-Yang-Mills theory on any *K-contact* five-manifold.

It is interesting to compare the geometry we find to the rigid limit of Poincaré supergravity [119, 120] and the twisting of [172]. In the former case the backgrounds naively appear to be more general, as there is no almost complex structure singled out, nor integrability condition. However, they do not include the $S^1 \times S^4$ geometry relevant for the supersymmetric index, which as we show in section 7.2.3 *is* included in our backgrounds. In fact the singling out of the almost complex structure is related to the fact that in section 7.2 we focus on the case where we turn on only an Abelian $U(1)_R \subset SU(2)_R$. This was motivated in part for simplicity, and in part because the known solutions to Romans supergravity discussed in Chapter 6 also have this property. It should be straightforward to analyse the geometric constraints in the more general case with arbitrary $SU(2)_R$ gauge field. Indeed, this is certainly necessary, and presumably sufficient, to reproduce the partially topologically twisted backgrounds $S^2 \times M_3$ of [112–114], since the $SU(2)$ spin connection of M_3 is twisted by $SU(2)_R$. On the other hand recall that the twisting in [172] requires that M_5 be a K-contact manifold. This shares many features with our geometry, with one important difference: for a K-contact manifold the transverse two-form is closed, so the corresponding foliation is *transversely symplectic*; however, our case is in some sense precisely the opposite, namely transversely holomorphic. These intersect precisely for Sasakian manifolds. It is interesting that these various approaches generally seem to lead to different supersymmetric geometries, with varying degrees of overlap.

Given the geometry we find and the results of [90], it is natural to conjecture that the partition function and other BPS observables depend only on the transversely holomorphic foliation, *i.e.* for fixed such foliation they are independent of the choice of the remaining background data (functions S, α, the one-form ν defined in section 7.2.5, and the transverse Hermitian metric $g^{(4)}$). It will be interesting to verify that this is indeed the case, and to compute these quantities using localization methods. Notice that *locally* a transversely holomorphic foliation always looks like $\mathbb{R} \times \mathbb{C}^2$, which perhaps also explains why in [120] the authors found that *locally* all deformations of their backgrounds were Q-exact.

Appendix A

Macdonald polynomials and the refined S-matrix

A.1 Group theory

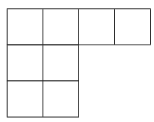
The finite dimensional irreducible representations of A_N are in one-to-one correspondence with dominant integral weights,

$$\lambda \;=\; \sum_{i=1}^{N-1} \lambda_i \, \omega_i$$

whose Dynkin labels $(\lambda_1, \lambda_2, ..., \lambda_{N-1})$ are nonnegative integers. Equivalently, irreducible representations are labeled by partitions $(\ell_1, \ell_2, \ldots, \ell_N)$ where $\ell_1 \geq \ell_2 \geq ... \geq \ell_N = 0$, such that

$$\ell_i \;=\; \lambda_i + \lambda_{i+1} + ... + \lambda_{N-1} \, . \tag{A.1}$$

Each partition is associated to a Young diagram whose i-th row has length ℓ_i. For instance, the following diagram



in $SU(4)$ corresponds to the partition $(4,2,2,0)$. The partition labels $(\ell_1, \ell_2, \ldots, \ell_N)$ are related to the components of the weight in the orthogonal basis

$$\omega_i \;=\; \epsilon_1 + \cdots + \epsilon_i - \frac{\mathrm{i}}{N}\sum_{j=1}^{N}\epsilon_j \tag{A.2}$$

where

$$\lambda \;=\; \sum_{i=1}^{N}\kappa_i\epsilon_i\,, \qquad \kappa_i \;=\; \ell_i - \frac{1}{N}\sum_{j=1}^{N-1} j(\ell_j - \ell_{j+1})\,. \tag{A.3}$$

The states in a given irreducible representation are in one-to-one correspondence with *semi-standard Young tableaux*. They are obtained by filling the boxes of a Young diagram with the numbers $\{1,\ldots,N\}$, such that the numbers are non-decreasing from left to right and strictly increasing from top to bottom. Finally, to each semi-standard Young tableau, we attach the labels $(n_1, ..., n_N)$, where n_i denotes the number of times that i appears in the semi-standard tableau. As an example below we include a few semi-standard tableaux for the adjoint **8** representation of $SU(3)$ with their corresponding labels.

$$\begin{array}{cccc} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \\ (2,1,0) & (1,1,1) & (1,1,1) & (0,2,1) \end{array}$$

A.2 Schur polynomials and the modular S-matrix

Let us introduce coordinates a_j, for $j = 1, ..., N$, obeying $\prod_{i=1}^{N} a_i = 1$. For $a_j = e^{i\theta_j}$ they are coordinates on the maximal torus of $SU(N)$. The Schur polynomials form a basis of symmetric functions in the variables $\{a_1, \ldots, a_N\}$ labeled by irreducible representations. The Schur polynomial of the irreducible representation of highest weight λ labeled by the partition $(\ell_1, \ldots, \ell_N)$ is given by the determinant formula

$$\chi_\lambda(a) \;=\; \frac{\det a_j^{\ell_i+N-i}}{\det a_j^{N-i}}\,. \tag{A.1}$$

An important property of the Schur polynomials is that they are orthonormal with respect to the inner product on the space of symmetric functions

$$\langle f, g\rangle \;\equiv\; \int \Delta(a) f(a) g(a^{-1}) \tag{A.2}$$

where $\Delta(a)$ is the Haar measure and the integration is over the maximal torus of $SU(N)$. Products of Schur polynomials decompose according to the tensor product of the irreducible representations

$$\chi_{\lambda_1}(a)\,\chi_{\lambda_1}(a) \;=\; \sum_{\mu} N_{\lambda_1,\lambda_2}{}^{\mu}\,\chi_{\mu}(a)\,, \tag{A.3}$$

where $N_{\lambda_1,\lambda_2}{}^{\mu}$ are the Littlewood-Richardson numbers.

In order to construct the modular S-matrix we introduce the Weyl weight ρ, which is the highest weight of the adjoint representation of $SU(N)$. Its components in the Dynkin basis are $\rho = (1,1,\ldots,1)$. In the orthogonal basis $\{\epsilon_i\}$, in which the simple roots are written as $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $i = 1,\ldots,N-1$, the components of ρ are $\rho_j = (N-2j+1)/2$. Now consider two irreducible representations λ and λ' with components κ_i and κ'_i in the orthogonal basis. Then the modular S-matrix is given by

$$S_{\lambda\lambda'} \;=\; S_{00}\,\chi_{\lambda}(q^{\rho_1},...,q^{\rho_N})\chi_{\bar{\lambda}'}(q^{\rho_1+\kappa_1},...,q^{\rho_N+\kappa_N})\,, \tag{A.4}$$

where $\bar{\lambda}'$ denotes the complex conjugate representation of λ'.

A.3 Macdonald polynomials and the refined S-matrix

The Macdonald polynomials are symmetric polynomials in the variables $\{a_1,\ldots,a_N\}$ that depend on two additional complex parameters q and t. They are labeled by irreducible representations λ of $SU(N)$ and reduce to the corresponding Schur polynomials when $q = t$.

The Macdonald polynomial labeled by the irreducible representation λ is

$$P_\lambda(a,q,t) \;=\; \chi_\lambda(a) + \sum_{\mu<\lambda} c_{\lambda,\mu}(q,t)\,\chi_\mu(a) \tag{A.1}$$

where $c_{\lambda,\mu}(q,t)$ are rational functions of q and t that are uniquely determined by ensuring $P_\lambda(a,q,t)$ is a simultaneous eigenfunctions of the difference operators

$$G_r \;=\; t^{r(1-N)} \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=r}} \prod_{i\in I, j\notin I} \frac{ta_i - a_j}{a_i - a_j}\, T_I\,, \qquad\qquad r = 1,\ldots,N-1\,, \tag{A.2}$$

where

$$T_I\,:\; a_i \;\to\; \begin{cases} q^{1-1/N} a_i\,, & \text{if } i\in I \\ q^{-1/N} a_i\,, & \text{if } i\notin I\,. \end{cases} \tag{A.3}$$

Here we have included a background shift by $q^{-1/N}$ compared to the standard Macdonald difference operators in order to preserve the condition $\prod_i^N a_i = 1$ relevant for $SU(N)$. The difference operators are self-adjoint with respect to the inner product

$$\langle f,g\rangle \;=\; \int \Delta^{(n)}_{q,t}(a) f(a) g(a^{-1})\,, \qquad \Delta^{(n)}_{q,t}(a) \;=\; \frac{1}{N!}\prod_{i=1}^{N-1}\frac{\mathrm{d}a_i}{2\pi\mathrm{i}a_i}\prod_{i\neq j}\frac{(a_i/a_j;\,q)}{(ta_i/a_j;\,q)}\,, \tag{A.4}$$

where $(a;\,q) = \prod_{i=0}^\infty(1-q^i a)$ is the q-Pochhammer symbol. The Macdonald polynomials are non-degenerate and orthogonal with respect to the same measure.

The product of Macdonald polynomials decomposes according to the tensor product of irreducible representations

$$P_{\lambda_1}(a,q,t)P_{\lambda_2}(a,q,t) \;=\; \sum_\mu \mathcal{N}_{\lambda_1,\lambda_2}{}^{\mu}(q,t) P_\mu(a,q,t) \tag{A.5}$$

where the $\mathcal{N}_{\lambda_1,\lambda_2}{}^{\mu}(q,t)$ are rational functions in q and t.

Analogous to the modular S-matrix, the refined S-matrix is given by

$$S_{\lambda\lambda'} \;=\; S_{00}\, P_\lambda(t^{\rho_1}, ..., t^{\rho_N})\, P_{\bar{\lambda}'}(t^{\rho_1} q^{\kappa_1}, ..., t^{\rho_N} q^{\kappa_N})\,. \tag{A.6}$$

It is then an easy exercise to check that the ratios $S_{R,S}/S_{0,S}$ are indeed the eigenvalues of the difference operators G_R in the Macdonald limit, namely

$$G_R \cdot P_S(a_i, q, t) \;=\; \frac{S_{R,S}}{S_{0,S}}\, P_S(a_i, q, t)\,. \tag{A.7}$$

Appendix B

S-duality kernel

It is natural to expect that the mass-deformed $\mathcal{T}(SU(N))$ theory encodes the field theory degrees of freedom on a so-called S-duality domain wall in the $\mathcal{N} = 2^*$ theory [144, 178]. Such a domain wall is defined so that the four-dimensional theories on either side are related by the transformation $S : (\tau, m) \to (-1/\tau, -m)$. In this appendix we will verify that this is indeed the case if we assume that $\mathcal{G}_b$ is the partition function on the half-sphere with Dirichlet boundary conditions.

Before introducing the S-duality domain wall, let us briefly consider the ellipsoid partition function $\mathcal{Z}_{S^4_b}$ of the $\mathcal{N} = 2^*$ theory with gauge group $SU(N)$. The AGT correspondence relates this to a Toda correlator on the once-punctured torus. We thus expect that the ellipsoid partition function transforms as a modular form [179]

$$\mathcal{Z}_{S^4_b}(-m; -1/\tau) \;=\; |\tau|^{2\Delta(m)} \mathcal{Z}_{S^4_b}(m; \tau)\,, \tag{B.1}$$

with modular weight

$$\Delta(m) \;=\; \frac{N(N-1)}{2}\left(\frac{Q^2}{4} + m^2\right). \tag{B.2}$$

This modular property of the ellipsoid partition function is guaranteed if the half-sphere partition function $\mathcal{G}_b$ transforms as

$$\mathcal{G}_b(-m,a;-1/\tau) \;=\; (-\mathrm{i}\tau)^{\Delta(m)} \int \mathrm{d}a'\, \mu_b(a')\, \mathcal{Z}_b(a_i,a',m)\, \mathcal{G}_b(m,a';\tau)\,, \tag{B.3}$$

where we integrate over a real slice of the Coulomb branch. The integration kernel $\mathcal{Z}_b(a,a',m)$ must obey two important properties:

(i) It must obey the symmetry $\mathcal{Z}_b(a,a',m) = \mathcal{Z}_b(a',a,-m)$.

(ii) It must be unitary with respect to the measure $\mu_b(a)\,\mathrm{d}a$, in the sense that

$$\int \mathrm{d}a\, \mu_b(a)\, \overline{\mathcal{Z}_b(a',a,m)}\, \mathcal{Z}_b(a,a'',-m) \;=\; \mu_b(a')\,\delta(a',a'')\,. \tag{B.4}$$

Now consider the ellipsoid partition function with the insertion of an S-duality domain wall. Assuming that $\mathcal{G}_b$ is the half-sphere partition function of the $\mathcal{N}=2^*$ theory with Dirichlet boundary conditions, the S-duality partition function on the squashed four-sphere should be given by

$$\int \mathrm{d}a\, \mu(a)\, \overline{\mathcal{G}_b(m,a;\tau)}\, \mathcal{G}_b(-m,a;-1/\tau) = \\ = \int \mathrm{d}a\, \mu(a) \int \mathrm{d}a'\, \mu(a')\, \overline{\mathcal{G}_b(m,a;\tau)}\, \mathcal{Z}_b(a,a',m)\, \mathcal{G}_b(m,a';\tau)\,. \tag{B.5}$$

Consequently, $\mathcal{Z}_b(a,a',m)$ should encode the gauge degrees of freedom localized on the domain wall. Specifically, we expect that $\mathcal{Z}_b(a,a',m)$ is the partition function of the mass-deformed $\mathcal{T}(SU(N))$ theory on a squashed three-sphere.

In this context, the symmetry $\mathcal{Z}_b(a,a',m) = \mathcal{Z}_b(a',a,-m)$ is equivalent to three-dimensional mirror symmetry. The unitarity property (B.4) follows because the partition function $\mathcal{Z}_b(a,a',m)$ is an eigenfunction of the self-adjoint operator $G_R^{(3\mathrm{d})}$ with respect to the measure $\mu_b(a)\,\mathrm{d}a$ (see equation (5.1.9)). Indeed, let us denote the integral (B.4) by

$$\mathcal{I}(a',a'') \;=\; \int \mathrm{d}a\, \mu_b(a)\, \overline{\mathcal{Z}_b(a',a,m)}\, \mathcal{Z}_b(a,a'',-m)\,. \tag{B.6}$$

The self-adjoint operator $G_R^{(3d)}(a')$ can act inside this integrand in either direction, which must lead to the same answer. Consequently we find

$$(W_R(a) - W_R(a'')) \, \mathcal{I}(a, a'') \;=\; 0\,, \tag{B.7}$$

where $W_R(a)$ is the expectation value of a Wilson loop in the representation R. This implies that the integral vanishes if $a \neq a''$ modulo Weyl transformations.

B.1 Example: $\mathcal{N} = 4$ theory on S^4

Let us check the above transformation properties of the half-sphere partition function $\mathcal{G}_b$ on the round four-sphere, when $b = 1$, and in the $\mathcal{N} = 4$ limit, when $m \to 0$.

First, we compute the explicit expression for $\mathcal{G}_{b=1}(\tau, m, a_i)$ for gauge group $SU(N)$. Its one-loop contribution (5.2.6) simplifies to the formula

$$\mathcal{G}_{1-\text{loop}}(m, a_i) \;=\; \frac{1}{\sqrt{2\pi}} \prod_{i<j} \frac{\pi a_{ij}}{\sinh(\pi a_{ij})}\,, \tag{B.1}$$

where $a_{ij} = a_i - a_j$ with the constraint that $\sum_{i=1}^N a_i = 0$. Its classical contribution times its instanton contribution is given by

$$\mathcal{G}_{\text{cl}}(a_i;\tau)\,\mathcal{G}_{\text{inst}}(a_i;\tau) \;=\; e^{-\pi i \tau \left(\sum_{i=1}^N a_i^2\right)}\,\eta(\tau)^{1-N}\,, \tag{B.2}$$

where $\eta(\tau)$ is the Dedekind η-function. This can be argued as follows. If the gauge group would be $U(N)$, the instanton contribution would be $\mathcal{G}_{\text{inst}} = 1$ [180]. For gauge group $SU(N)$, however, one must first divide by the $U(1)$ factor. We can find this $U(1)$ factor by comparing with the $q = \exp(\tau)$-expansion of the Toda conformal block

$$\mathcal{F}(a_i;\tau) \;=\; q^{\Delta(a_i) - \frac{c}{24}} \sum_k q^k F_k\,. \tag{B.3}$$

In particular, using the known expressions for the Toda central charge c and the momentum

$\Delta(a_i)$, we can verify the classical contribution to $\mathcal{G}_{\rm cl}\,\mathcal{G}_{\rm inst}$ for any N. Furthermore, we can match the full expressions in an expansion of the instanton parameter q for $N = 2, 3$.

Putting the pieces together, we have

$$\mathcal{G}_{b=1}(a_i;\tau) \;=\; e^{-\pi i \tau\left(\sum_{i=1}^N a_i^2\right)} \frac{1}{\sqrt{2\pi}} \prod_{i<j} \frac{\pi a_{ij}}{\sinh(\pi a_{ij})} \eta(\tau)^{1-N}\,. \tag{B.4}$$

After performing $(N-1)$ Gaussian integrals we find the partition function

$$\mathcal{Z}_{S^4_{b=1}}(\tau) \;\propto\; \frac{1}{|\eta(\tau)|^{2(N-1)}\,\mathrm{Im}(\tau)^{(N^2-1)/2}}\,, \tag{B.5}$$

which transforms as expected under S-duality

$$\mathcal{Z}_{S^4_{b=1}}\left(-\frac{1}{\tau}\right) \;=\; |\tau|^{N(N-1)}\,\mathcal{Z}_{S^4_{b=1}}(\tau)\,. \tag{B.6}$$

We have indeed verified this for $N = 2, 3$. In the above, we have used $\mu_{b=1}(a) = \prod_{i<j} 4\sinh(\pi a_{ij})^2$ and $\Delta(0) = \frac{N(N-1)}{2}$.

We can also check that the three-sphere partition function

$$\mathcal{Z}_{b=1}(a_i, a_i') \;=\; \frac{\sum_{\rho\in S_N}(-1)^\rho e^{2\pi\sum_{j=1}^N a_{\rho(j)}a_j'}}{\prod_{i<j} 2\sinh\pi(a_{ij})\,2\sinh\pi(a_{ij}')}\,, \tag{B.7}$$

is the S-duality kernel for the half-sphere partition function $\mathcal{G}_{b=1}(\tau, a_i)$. This is again a matter of performing Gaussian integrals and using the modular property of the η-function. In particular, for $N = 2, 3$ we explicitly verified that

$$\int \mathrm{d}a_i'\,\mu_{b=1}(a_i')\,\mathcal{Z}_{b=1}(a_i, a_i')\,\mathcal{G}_{b=1}(a_i';\tau) \;\sim\; (-i\tau)^{\frac{-N(N-1)}{2}}\mathcal{G}_{b=1}(a_i;-1/\tau)\,.$$

This completes the argument and gives some evidence that $\mathcal{G}_b$ is indeed the half-sphere partition function with Dirichlet boundary conditions.

Appendix C

Factorization of Toda 3-point function

Let us briefly review some properties of special functions we need in order to manipulate one-loop contributions. As in the main text, $b \in \mathbb{R}_{>0}$ is a real parameter and we define $q \equiv b+b^{-1}$. The double gamma function $\Gamma_b(x)$ is a meromorphic function of x characterized by the functional equation

$$\Gamma_b(x+b) \;=\; \sqrt{2\pi}\, b^{bx-\frac{1}{2}}\Gamma_b(x)/\Gamma(bx)\,, \tag{C.1}$$

where $\Gamma(x)$ is the Euler gamma function and its value $\Gamma_b(q/2) = 1$. We will also need the double sine function, which is a meromorphic function that can be defined in terms of the double gamma function by the formula $S_b(x) \equiv \Gamma_b(x)/\Gamma_b(q-x)$. The double sine function is characterized by the functional equation

$$S_b(x+b) \;=\; 2\sin(\pi bx)S_b(x)\,. \tag{C.2}$$

We will furthermore need the function $\Upsilon_b(x)^{-1} = \Gamma_b(x)\Gamma_b(q-x)$, which is entirely analytic. A more complete discussion of the properties of these functions can be found, for example, in [181].

Let us begin by considering the three-point function $C(z, 2Q-z, \nu)$ in A_{N-1} Toda theory corresponding to the trivalent vertex in the pants decomposition of a torus with a simple puncture. The momentum in the internal channel $z = Q + \mathrm{i}a$, with $a \in \mathbb{R}$, is non-degenerate and describes a delta-function normalizable state, while the momentum $\nu = N(q/2 + \mathrm{i}m)\omega_{N-1}$, with $m \in \mathbb{R}$, is semi-degenerate. Substituting these momenta into the more general result of [182, 183] we find that

$$C(z, 2Q - z, \nu) \;=\; f(m)\, \frac{\prod_{i<j}^{N} \Upsilon_b(\mathrm{i}a_{ij})\, \Upsilon_b(-\mathrm{i}a_{ij})}{\prod_{i,j=1}^{N} \Upsilon_b\left(\frac{q}{2} + \mathrm{i}a_{ij} + \mathrm{i}m\right)}\,, \tag{C.3}$$

where $a_{ij} = a_i - a_j$. The proportionality factor $f(m)$ is independent of the internal parameter a. We will not need to know the details of $f(m)$ and so it will be omitted whenever convenient in what follows.

The complete correlation function on a torus with simple puncture is

$$\int \mathrm{d}a\, C(z, 2Q - z, \mu)\overline{\mathcal{F}(z, \mu; \tau)}\, \mathcal{F}(z, \mu; \tau)\,, \tag{C.4}$$

where $\mathcal{F}(z, \mu; \tau)$ are the W_N-algebra conformal blocks. This correlation function computes the ellipsoid partition function of the four-dimensional $\mathcal{N} = 2^*$ theory on an ellipsoid, with the parameters identified as in the main text.

We now consider two different ways of factorizing the three-point function and absorbing it into the W_N-algebra conformal blocks. The first way is chosen to maximally simplify the expressions for the Verlinde operators and we expect that this corresponds to a half-sphere partition function of $\mathcal{N} = 2^*$ theory with Dirichlet boundary conditions for the vectormultiplet. The second way corresponds to computing the Nekrasov partition function of the $\mathcal{N} = 2^*$ theory with deformation parameters $\epsilon_1 = b$ and $\epsilon_2 = b^{-1}$.

C.1 Normalized conformal blocks

Let us express the Toda three-point function in terms of double gamma functions and manipulate the answer into a convenient factorized form. For the hypermultiplet contribution, we have

$$\prod_{i,j=1}^{N} \Upsilon_b\left(\frac{q}{2} + \mathrm{i}a_{ij} + \mathrm{i}m\right)^{-1} = \left|\prod_{i,j=1}^{N} \Gamma_b\left(\frac{q}{2} + \mathrm{i}a_{ij} + \mathrm{i}m\right)\right|^2 . \tag{C.1}$$

For the vectormultiplet contribution

$$\prod_{i<j}^{N} \Upsilon_b(\mathrm{i}a_{ij})\,\Upsilon_b(-\mathrm{i}a_{ij}) = \mu(a)\left|\prod_{i<j}^{N} \frac{1}{\Gamma_b(q+\mathrm{i}a_{ij})\,\Gamma_b(q-\mathrm{i}a_{ij})}\right|^2 , \tag{C.2}$$

where $\mu(a)$ is the $3d$ partition function of an $\mathcal{N}=2$ vectormultiplet on a squashed three-sphere [184], which is identified here with the equator $\{x_0 = 0\}$.

As described in the main text, we can now absorb the three-point function into the W_N-algebra conformal blocks, by defining new renormalized blocks

$$\mathcal{G}(a,m;\tau) = \frac{\prod_{i,j=1}^{N} \Gamma_b\left(\frac{q}{2} + \mathrm{i}a_{ij} + \mathrm{i}m\right)}{\prod_{i<j}^{N} \Gamma_b(q+\mathrm{i}a_{ij})\,\Gamma_b(q-\mathrm{i}a_{ij})}\,\mathcal{F}(a,m;\tau) \tag{C.3}$$

such that the correlation function becomes

$$\int \mathrm{d}a\,\mu(a)|\,\mathcal{G}(a,m;\tau)\,|^2 . \tag{C.4}$$

In order to transform between Verlinde operators acting on $\mathcal{F}(a,m;\tau)$ and those acting on $\mathcal{G}(a,m;\tau)$ we have to conjugate by the factor in front of $\mathcal{F}(a,m;\tau)$ in equation (C.3).

Let us concentrate on the Verlinde operator corresponding to the fundamental 't Hooft loop. Acting on the unnormalized conformal blocks, the difference operator has been computed in [61]. The result is given by

$$\sum_{j=1}^{N}\left[\prod_{k\neq j}^{N} \frac{\Gamma(\mathrm{i}ba_{kj})}{\Gamma\left(\frac{bq}{2} + \mathrm{i}ba_{kj} - \mathrm{i}bm\right)} \frac{\Gamma(bq + \mathrm{i}ba_{kj})}{\Gamma\left(\frac{bq}{2} + \mathrm{i}ba_{kj} + \mathrm{i}bm\right)}\right]\Delta_j , \tag{C.5}$$

where we have introduced the notation $\Delta_j : a \to a + \mathrm{i}bh_j$. Now, by patient and repeated application of the functional equation for the double gamma function, we find that the conjugated operators are given by

$$\sum_{j=1}^{N} \left[\prod_{k\neq j}^{N} \frac{\sin \pi b \left(\frac{q}{2} + \mathrm{i}a_{kj} - \mathrm{i}m\right)}{\sin \pi b \left(\mathrm{i}a_{kj}\right)} \right] \Delta_j \,, \tag{C.6}$$

as claimed in the main text. With patient bookkeeping, the same computation can be performed for the difference operators in any other completely antisymmetric tensor representation.

C.2 Nekrasov partition function

For comparison with the exact computation of an 't Hooft loop on the four-sphere in [58], it is necessary to consider another factorization of the Toda three-point function. In this factorization the difference operators act on the Nekrasov partition function $\mathcal{Z}(a, m; \tau)$, with $\epsilon_1 = b$ and $\epsilon_2 = b^{-1}$.

Thus we now express the three-point function as

$$C(z, 2Q - z, \mu) \;=\; f(m) \,|\, \mathcal{Z}^{1-\mathrm{loop}}(a, m; \tau) \,|^2 \,, \tag{C.1}$$

where

$$\mathcal{Z}^{1-\mathrm{loop}}(a, m; \tau) \;=\; \left[\frac{\prod_{i<j}^{N} \Upsilon_b \left(\mathrm{i}a_{ij}\right) \Upsilon_b \left(-\mathrm{i}a_{ij}\right)}{\prod_{i,j=1}^{N} \Upsilon_b \left(\frac{q}{2} + \mathrm{i}a_{ij} + \mathrm{i}m\right)} \right]^{1/2} \tag{C.2}$$

is the one-loop contributions to the Nekrasov partition function and $f(m)$ is independent of the internal momentum a as before. The classical and instanton contributions to the Nekrasov partition function are encoded in the W_N-algebra conformal blocks. Thus, up to the factor $f(m)$, the complete Toda correlator can be expressed as

$$\int \mathrm{d}a \,|\, \mathcal{Z}(a, m; \tau) \,|^2 \,, \tag{C.3}$$

in agreement with the exact computation of the partition function of the $\mathcal{N} = 2^*$ theory on an ellipsoid in [149].

To obtain difference operators acting on the Nekrasov partition function, it is easier at this stage to start from the relationship to the normalized W_N-algebra conformal blocks. In fact, from the relationship between the double gamma, the double sine functions and upsilon functions, we find that

$$\mathcal{Z}(a, m; \tau) \; = \; \left[\frac{\prod\limits_{i<j}^{N} S_b(q + \mathrm{i}a_{ij}) S_b(q - \mathrm{i}a_{ij})}{\prod\limits_{i,j=1}^{N} S_b(\frac{q}{2} + \mathrm{i}a_{ij} + \mathrm{i}m)} \right]^{1/2} \mathcal{G}(a, m; \tau)\,. \tag{C.4}$$

Now, using the functional equation for the double sine function, we conjugate Δ_j with the prefactor in (C.4) and we obtain

$$\left[\prod_{k \neq j}^{N} \frac{\sin \pi b \left(\frac{q}{2} + \mathrm{i}a_{kj} + \mathrm{i}m\right) \sin \pi b(\mathrm{i}a_{kj})}{\sin \pi b \left(\frac{q}{2} + \mathrm{i}a_{kj} - \mathrm{i}m\right) \sin \pi b(q + \mathrm{i}a_{kj})} \right]^{1/2} . \tag{C.5}$$

Thus, combining with equation (C.6), we conjecture that the fundamental 't Hooft loop operator acting on the Nekrasov partition function with $\epsilon_1 = b$ and $\epsilon_2 = b^{-1}$ has the general form

$$\sum_{j=1}^{k} \left[\prod_{k \neq j}^{N} \frac{\sin \pi b \left(\frac{q}{2} + \mathrm{i}a_{kj} + \mathrm{i}m\right) \sin \pi b \left(\frac{q}{2} + \mathrm{i}a_{kj} - \mathrm{i}m\right)}{\sin \pi b \left(\mathrm{i}a_{kj}\right) \sin \pi b(q + \mathrm{i}a_{kj})} \right]^{1/2} \Delta_j\,. \tag{C.6}$$

This agrees with the exact computation of the fundamental 't Hooft loop operator in the case of a round four-sphere $b = 1$ [58]. Again, with patient bookkeeping the same conclusion can be reached for 't Hooft loops labeled by any antisymmetric tensor representation.

Appendix D

Supergravity Killing spinors

Having found the supersymmetric solutions for the gravity dual of a squashed five-sphere in Romans $F(4)$ gauged supergravity (as presented in the main text), we now proceed to solve the dilatino (6.2.7) and Killing spinor equation (6.2.6) for the Killing spinors ϵ_I, $I = 1, 2$.

1/4 **BPS solution**

For the 1/4 BPS solution we find

$$\epsilon_1 \;=\; c_+ \,\mathrm{e}^{-\frac{3\mathrm{i}\tau}{2}} \begin{pmatrix} 0 \\ k_2(r) \\ 0 \\ 0 \\ 0 \\ -\mathrm{i}\, k_1(r) \\ 0 \\ 0 \end{pmatrix}, \qquad \epsilon_2 \;=\; -c_- \,\mathrm{e}^{\frac{3\mathrm{i}\tau}{2}} \begin{pmatrix} k_1(r) \\ 0 \\ 0 \\ 0 \\ -\mathrm{i}\, k_2(r) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{D.1}$$

where

$$k_1(r) \;=\; \sqrt{r} + \frac{(f_0-3)s}{6\sqrt{6}}\frac{1}{\sqrt{r}} + \frac{5(f_0-3)^2 s^2 + 6(4f_0-9)}{432}\left(\frac{1}{r}\right)^{3/2} + \dots\,,$$

$$k_2(r) = \sqrt{r} - \frac{(f_0-3)s}{6\sqrt{6}}\frac{1}{\sqrt{r}} + \frac{5(f_0-3)^2s^2+6(4f_0-9)}{432}\left(\frac{1}{r}\right)^{3/2} + \dots , \qquad \text{(D.2)}$$

as expansions around the conformal boundary at $r \to \infty$. The Killing spinors contain two constants of integration $c_\pm$. These constants are generically complex, but imposing the symplectic Majorana condition enforces certain reality conditions. The functions $k_i(r)$ depend only on r and can also be expanded around Euclidean AdS.

Supersymmetry gets enhanced for the case $f_0 = 0$ (or $\omega = 0$). In this limit the gauge field vanishes and the two Killing spinors ϵ_I for $I = 1, 2$ decouple and have the same structure. They read

$$\epsilon_I = \begin{pmatrix} c_I^{(2)}\, k_1(r) e^{\frac{3i\tau}{2}} \\ c_I^{(1)}\, k_2(r) e^{-\frac{3i\tau}{2}} \\ 0 \\ 0 \\ -i\, c_I^{(2)}\, k_2(r) e^{\frac{3i\tau}{2}} \\ -i\, c_I^{(1)}\, k_1(r) e^{-\frac{3i\tau}{2}} \\ 0 \\ 0 \end{pmatrix}, \qquad \text{(D.3)}$$

where $c_I^{(j)}$ for $j = 1, 2$ are the integration constants and $k_i(r)$ are as in (D.2) with $f_0 = 0$. This solution may be referred to as the 1/2 BPS solution.

3/4 **BPS solution**

For the 3/4 BPS solution we find

$$\epsilon_1 = a_+^{(1)}\, e^{i\frac{\tau}{2}} \begin{pmatrix} k_2(r)\left[\cos\sigma + i\lambda_+(s) e^{i\frac{\psi}{2}} S_+^{(1)} \sin\sigma\right] \\ 0 \\ i k_3(r)\left[\sin\sigma - i\lambda_+(s) e^{i\frac{\psi}{2}} S_+^{(1)} \cos\sigma\right] \\ i k_3(r)\lambda_+(s)\, e^{-i\frac{\psi}{2}} S_+^{(2)} \\ -i k_4(r)\left[\cos\sigma + i\lambda_+(s) e^{i\frac{\psi}{2}} S_+^{(1)} \sin\sigma\right] \\ 0 \\ k_1(r)\left[\sin\sigma - i\lambda_+(s) e^{i\frac{\psi}{2}} S_+^{(1)} \cos\sigma\right] \\ k_1(r)\, \lambda_+(s) e^{-i\frac{\psi}{2}} S_+^{(2)} \end{pmatrix}, \qquad \text{(D.4)}$$

$$\epsilon_2 = a_-^{(1)}\,\mathrm{e}^{-\mathrm{i}\frac{\tau}{2}} \begin{pmatrix} 0 \\ \mathrm{i}k_4(r)\left[\cos\sigma - \mathrm{i}\lambda_-(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}}S_-^{(1)}\sin\sigma\right] \\ -k_1(r)\lambda_-(s)\,\mathrm{e}^{\mathrm{i}\frac{\psi}{2}}S_-^{(2)} \\ k_1(r)\left[\sin\sigma + \mathrm{i}\lambda_-(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}}S_-^{(1)}\cos\sigma\right] \\ 0 \\ k_2(r)\left[\cos\sigma - \mathrm{i}\lambda_(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}}S_-^{(1)}\sin\sigma\right] \\ \mathrm{i}k_3(r)\,\lambda_-(s)\mathrm{e}^{\mathrm{i}\frac{\psi}{2}}S_-^{(2)} \\ -\mathrm{i}k_3(r)\left[\sin\sigma + \mathrm{i}\lambda_-(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}}S_-^{(1)}\cos\sigma\right] \end{pmatrix}, \qquad \text{(D.5)}$$

where we have introduced

$$\begin{aligned} S_\pm^{(1)} &= S_\pm^{(1)}(\theta,\varphi) = a_\pm^{(3)}\mathrm{e}^{\pm\mathrm{i}\frac{\varphi}{2}}\cos\frac{\theta}{2} - a_\pm^{(2)}\mathrm{e}^{\mp\mathrm{i}\frac{\varphi}{2}}\sin\frac{\theta}{2}\,, \\ S_\pm^{(2)} &= S_\pm^{(2)}(\theta,\varphi) = a_\pm^{(2)}\mathrm{e}^{\mp\mathrm{i}\frac{\varphi}{2}}\cos\frac{\theta}{2} + a_\pm^{(3)}\mathrm{e}^{\pm\mathrm{i}\frac{\varphi}{2}}\sin\frac{\theta}{2}\,, \\ \lambda_\pm(s) &= \frac{\pm 1 + \sqrt{1-s^2}}{s}\,. \end{aligned} \qquad \text{(D.6)}$$

The Killing spinors now contain in total six constants of integration $a_\pm^{(i)}$, $i = 1,2,3$. Imposing the symplectic Majorana condition enforces certain reality conditions. The functions $k_i(r)$ expanded around the conformal boundary are

$$\begin{aligned} k_1(r) &= \frac{-1+\sqrt{1-s^2}}{s}\sqrt{r} + \frac{1}{2\sqrt{6}}\frac{1}{\sqrt{r}} + \dots\,, \\ k_2(r) &= \sqrt{r} - \frac{5\sqrt{1-s^2}-3}{6\sqrt{6}s}\frac{1}{\sqrt{r}} + \dots\,, \\ k_3(r) &= \frac{-1+\sqrt{1-s^2}}{s}\sqrt{r} - \frac{1}{2\sqrt{6}}\frac{1}{\sqrt{r}} + \dots\,, \\ k_4(r) &= \sqrt{r} + \frac{5\sqrt{1-s^2}-3}{6\sqrt{6}s}\frac{1}{\sqrt{r}} + \dots\,. \end{aligned} \qquad \text{(D.7)}$$

From the expansion (D.2) and (D.7) the corresponding Killing spinors at the boundary $r \to \infty$ can be read off. They precisely agree with their corresponding $5d$ counterparts (6.1.13) and (6.1.14).

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