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Relational Incentive Contracts with Persistent Private Information

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Abstract

This paper investigates relational incentive contracts with a continuum of privately-observed agent types that are persistent over time. For a sufficiently productive relationship, a full pooling contract exists in which all agent types continuing the relationship choose the same action. When some separation is feasible, the parties can do better than with full pooling. When future actions are optimal, however, full separation of *all* types is not possible. There is, though, an equilibrium with separation into pools each containing a non-degenerate interval of types and fully separating individual types is not generally optimal. Separation results in an increase in output.

Keywords: Relational incentive contracts, private information, ratchet effect, dynamic enforcement

JEL classification: C73, D82, D86

1 Introduction

In a variety of economic contexts, agents of different types are pooled together in groups, with those within each group persistent over time and all treated the same despite differences between them. Employees are grouped in grades, with those in a grade all paid the same. Toyota, as described by Asanuma (1989), places its suppliers into a small number of categories that receive differential treatment. In these examples, pooling of types is only partial because the different grades or categories are treated differently, and the set of agents in a particular pool is persistent over time.

Pooling of privately-observed, continuous and persistent agent types arises from the *ratchet effect* in dynamic models of procurement, see Laffont and Tirole (1993, Chapter 9). It occurs with a principal who makes "take it or leave it" offers and is legally constrained from committing to contract terms for future periods, even those conditioned on outcomes the principal can contract on when those future periods arrive. Partial pooling of privately-observed, continuous but non-persistent types arises from *dynamic enforcement* in the hidden information relational incentive contract model of Levin (2003). The present paper combines insights underlying the ratchet effect and dynamic enforcement to show that partial pooling is inherent to relational incentive contracts with privately-observed, continuous and persistent agent types when the parties cannot commit themselves to behave sub-optimally in the future. This pooling does not depend on legal constraints on committing to future contract terms that are in principle contractible, nor on the principal making "take it or leave it" offers. It depends only on efficient effort being unattainable and future payoffs if the agent's type is revealed being on the feasible Pareto frontier. When there is sufficient difference between types for some separation to occur, there are however equilibria with multiple pools each containing a non-degenerate interval of types with the set of agents in each persistent over time, as in the examples of employment and Toyota suppliers. In general, it is not optimal to fully separate individual types.

With a relational contract, parties make payments conditioned on non-contractible outcomes only if the payoff from having the relationship continue is sufficient to make that worthwhile. This imposes a constraint on the spread of rewards for performance that Levin (2003) calls the *dynamic enforcement constraint*. That constraint leads to pooling of agent types. In Levin (2003), optimal pooling takes the form of a single pool consisting of an interval of types that always includes the most productive. If that single pool does not include all types, there is an interval of less productive types each of which is separated. But it is never optimal to have separate pools each containing a non-degenerate interval of types. Moreover, because types are *iid* random draws each period, there is no systematic persistence of a particular agent in a particular pool.

With the ratchet effect, types are persistent. Pooling arises because, when the principal cannot commit to future contract terms, the principal's "take it or leave it" offers extract all future rent if the agent's type is revealed, so a more productive agent does better by pretending to be a less productive agent and receiving a future informational rent. The constraint on committing to future contract terms is appropriate for sovereign bodies that cannot commit their successors, and for regulators who are not permitted to do so. It is less appropriate for private sector principals who can bind themselves to future contract terms conditioned on outcomes that are in principle contractible. The inability to commit to future contract terms in the present paper is just as applicable to private sector principals. Because the results here also do not depend on the principal making "take it or leave it" offers, they significantly extend the set of circumstances under which persistent types are necessarily pooled.

In the model used here, the agent's type affects the cost of supplying effort to the principal and is persistent over time. It is specific to the relationship with the principal and privately observed by the agent. This framework corresponds to an extension of the classic model in Shapiro and Stiglitz (1984) to private information about the agent worker's disutility of effort, though it also allows for a continuous, not just binary, effort choice, as in MacLeod and Malcomson (1989).

In this model, provided the relationship is sufficiently productive, there always exists an equilibrium relational contract with full pooling of all agent types for which the relationship continues. With such a contract, the agent ends the relationship if the cost of effort is above a critical value but otherwise provides the same effort independent of type and the principal pays the same remuneration to all agent types who continue the relationship. So it is always an equilibrium for employers to expect the same amount of work from employees with different characteristics and to pay them the same. If further separation of agent types is feasible, such a contract is, however, dominated by one with some separation of types who continue the relationship. Even with the agent's type revealed, it is not in general possible for a relational contract to sustain the efficient level of effort that would be possible if outcomes were fully contractible. When efficient effort is not sustainable, full separation of privately-observed, continuous and persistent agent types is not feasible if future actions are optimal in the sense of attaining the feasible Pareto frontier once type is revealed. Agent types can, however, be separated into a finite number of pools each containing a non-degenerate interval of types. Furthermore, where an individual type can be fully separated but can also be pooled with a marginally lower type without detriment to other types, pooling them is better.

Dynamic enforcement has other implications. During the initial period of a relational contract with some separation of types, effort for all but the least productive is below that sustainable without private information, which is itself below the fully efficient level. So there is a cost to information being private except "at the bottom", that is, for the least productive feasible relationships. As in Watson (1999) and Watson (2002), the relationship starts out "small". As in MacLeod and Malcomson (1989) and Levin (2003), remuneration consists of two components, one that does not depend on performance and a bonus that does. Once separation has been completed, a higher fixed component goes with the agent receiving more of the gains from the relationship.

Related papers include Yang (forthcoming), who considers persistent types that are private information but allows for just two, so there is no possibility of multiple pools containing more than one type. Kennan (2001) and Battaglini (2005) also analyse revelation of two persistent types that are private information but without non-contractible effort. Athey and Bagwell (2008) analyse a model of collusion between firms in an oligopoly in which cost shocks are both private information and persistent. But collusion between firms has very different characteristics from employment or supply relationships. In particular, only one side of the market participates in the relational contract and monetary payments are not used because they make breach of antitrust rules more apparent. Finally, MacLeod and Malcomson (1988) analyse relational incentive contracts with a continuum of persistent, privately-observed agent types that are partitioned into separate pools. There, however, restrictions on rewards and punishments drive the partitioning. Here those restrictions are removed, so the result is more fundamental.

The structure of the paper is as follows. Section 2 sets out the model. Section 3 derives incentive compatibility conditions for the agent and the principal in a relational contract. Section 4 derives equilibrium conditions for relational contracts, Section 5 for relational contracts with full pooling of continuing agent types. Section 6 studies relational contracts with separation of types and shows that these exhibit partial pooling. Section 7 contains concluding remarks. Proofs of propositions are in an appendix.

2 Model

A principal uses an agent to perform a specific task each period. The relationship between the two can, in principle, continue indefinitely. The principal's payoff in period t if matched with the agent is $e_t - w_t$, where $e_t \in [0, \bar{e}]$ is the agent's effort, and w_t the payment to the agent, in period t. Effort e_t cannot be verified by third parties, so a legally enforceable agreement for performance at the task is not possible. It can be thought of as anything unverifiable the agent may do that affects the payoff to the principal. The principal's payoff for a period not matched with the agent is $\underline{v} \geq 0$.

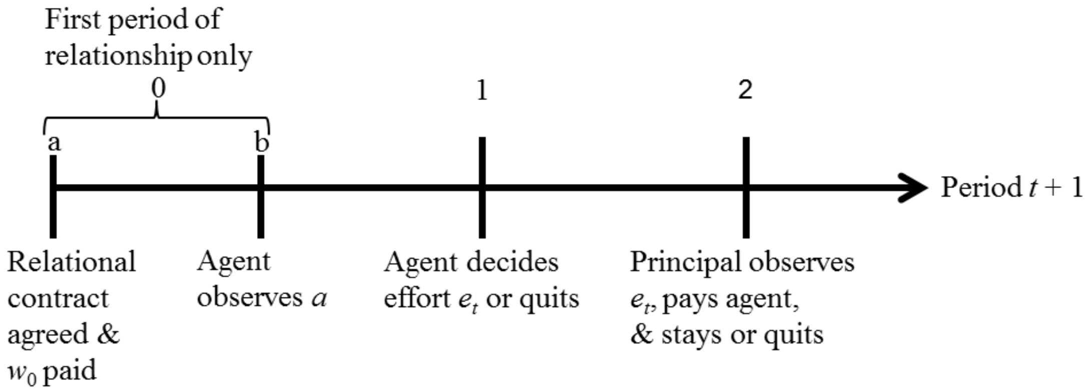

Figure 1: Timing of events in period t

The agent's payoff in period t if matched with the principal is $w_t - c(e_t, a)$, where $c(e_t, a)$ is the cost of effort e_t to agent type $a \in [\underline{a}, \overline{a}]$, with a observed privately by the agent. Agent type is distributed $F(a)$, with $dF(a) > 0$ everywhere. The agent's payoff for a period not matched with the principal is $\underline{u} \geq 0$, with $\underline{u} + \underline{v} > 0$. Principal and agent have the same discount factor δ. The function c has the following standard properties.

Assumption 1 *For all* $a \in [\underline{a}, \overline{a}]$*:* (1) $c(0, a) = 0$ *and* $c(\overline{e}, a)$ *is bounded above;* (2) *for all* $\tilde{e} \in [0, \overline{e}]$, $c(\tilde{e}, a)$ *is twice continuously differentiable, with* $c_1(\tilde{e}, a) > 0$, $c_2(\tilde{e}, a) \leq 0$ *with strict inequality for* $\tilde{e} \in (0, \overline{e}]$, $c_{11}(\tilde{e}, a) > 0$, $c_{12}(\tilde{e}, a) < 0$, *and* $c(\tilde{e}, \underline{a}) > \tilde{e} - (\underline{u} + \underline{v})$*;* (3) $c_1(0, a) < 1$ *and* $c_1(\overline{e}, a) > 1$.

The timing of events for period t is shown in Figure 1. In the first period of the relationship ($t = 1$), the parties first decide (at stage 0a) whether to agree a relational contract (to be formally defined shortly) and, if they do, make initial payment w_0. Then the agent (at stage 0b) observes a. The other stages are the same for all t. At stage 1, the agent either incurs effort e_t or ends the relationship. At stage 2, the principal observes e_t, pays the agent and decides whether to continue the relationship.

As in MacLeod and Malcomson (1989) and Levin (2003), payment has a fixed component $\underline{w}_t$ conditioned only on the relationship being continued by both parties for period t (and not on effort at t). It also has a bonus component $w_t - \underline{w}_t$ that can be conditioned on the agent's effort in period t but is not legally enforceable because effort is unverifiable. The magnitude and sign of $\underline{w}_t$ are unrestricted (negative $\underline{w}_t$ requires the agent to pay the principal) but, to avoid a decision by the agent at stage 2 of whether to accept the bonus, $w_t - \underline{w}_t$ is restricted to being non-negative. (This restriction does not restrict the set of payoffs attainable with equilibrium relational contracts.)

Let $h_t = h_{t-1} \cup (e_{t-1}, w_{t-1})$, for $t \geq 2$, with $h_1 = \{w_0\}$, denote the commonly observed history at stage 1 of period t conditional on the relationship not having ended before then. At that stage, the agent can condition actions on (a, h_t). A strategy σ^a for the agent consists of a decision rule for whether to accept w_0, a decision rule $\gamma_t(a, h_t) \in \{0, 1\}$ for each t for whether to continue the relationship at stage 1, and an effort choice $e_t(a, h_t)$ for each t conditional on continuation. At stage 2 of period t, the principal can condition actions on (h_t, e_t). A strategy σ^p for the principal consists of a decision rule for whether to pay w_0, a decision rule $\beta_t(h_t, e_t) \in \{0, 1\}$ for each t for whether to continue the relationship at stage 2, and a payment choice $w_t(h_t, e_t)$ for each t conditional on continuation. Formally, a *relational contract* is a w_0, a $\underline{w}_t(h_t)$ for each h_t and t, and a strategy pair (σ^p, σ^a).[1]

The joint payoff gain to the principal and the agent from being matched in period t conditional on a is $e_t - c(e_t, a) - (\underline{u} + \underline{v})$. *Efficient* effort $e^*(a)$ maximises this joint gain. Under Assumption 1, $e^*(a) \in (0, \bar{e})$ for all a and is uniquely determined by

$$c_1(e^*(a), a) = 1. \tag{1}$$

The natural equilibrium concept for this game is perfect Bayesian equilibrium in the strategies of the parties to the relational contract. To avoid the measurability details that can arise with mixed strategies when action spaces are continuous (see Mailath and Samuelson (2006, Remark 2.1.1)), attention is restricted to pure strategies.

3 Incentive compatibility

This section analyses the relational contracts that are incentive compatible for principal and agent. Start with the agent. Let $A_t(h_t)$ denote the set of agent types a with history h_t at t. For a best response effort, the payoff gain $U_t(a, h_t)$ to agent type $a \in A_t(h_t)$ from continuing the relationship at stage 1 of period t given history h_t is

$$U_t(a, h_t) = \max_{\tilde{e} \in [0, \bar{e}]} \Big\{ -c(\tilde{e}, a) - \underline{u} + \underline{w}_t(h_t) + \beta_t(h_t, \tilde{e}) \Big[w_t(h_t, \tilde{e}) - \underline{w}_t(h_t) + \delta \max \Big\{ 0, U_{t+1}(a, (h_t, \tilde{e}, w_t(h_t, \tilde{e}))) \Big\} \Big] \Big\}. \tag{2}$$

[1] The timing used here has each party make decisions at only one stage in each period, which simplifies the analysis by avoiding having to keep track of the parties' payoffs at other stages within a period. A party's payoff from continuing the relationship is, however, at its lowest at the stage it takes its decisions. So allowing a party to end the relationship at other stages within a period would not change the impact of the individual rationality constraints. Having the principal make the stay or quit decision simultaneously with the agent would make mutual quitting always a best response pair but would not affect the maximum sustainable effort or the set of equilibrium payoffs.

(Explicit dependence of payoff gains on the contract is suppressed to avoid cumbersome notation.) The interpretation is as follows. A type a agent continuing the relationship for period t and choosing effort $\tilde{e}$ incurs cost of effort $c(\tilde{e},a)$, forgoes utility $\underline{u}$ available if not matched with the principal, and receives payment $\underline{w}_t(h_t)$. For $\beta_t(h_t,\tilde{e})=1$, the principal continues the relationship and pays the bonus $w_t(h_t,\tilde{e})-\underline{w}_t(h_t)$. In that case, the agent receives payoff gain from the future of $U_{t+1}(a,(h_t,\tilde{e},w_t(h_t,\tilde{e})))$ if this is non-negative, so continuing is worthwhile. For $\beta_t(h_t,\tilde{e})=0$, the principal ends the relationship, in which case paying a bonus is never a best response.

All agent types in $A_t(h_t)$ gain from continuing the relationship at t if $\underline{w}_t(h_t)>\underline{u}$ because payoff gain $\underline{w}_t(h_t)-\underline{u}>0$ can then be guaranteed by setting $e_t=0$ and quitting at $t+1$. With $c_2\leq 0$, $U_t(a,h_t)$ is non-decreasing in a, so there is a lowest agent type $\alpha_t(h_t)$ that continues the relationship for period t given history h_t that satisfies

$$\begin{gathered}U_t(\alpha_t(h_t),h_t)\geq \max\left[0,\underline{w}_t(h_t)-\underline{u}\right], \text{ all } h_t,t,\\ U_t(a,h_t)\leq 0, \text{ for } a<\alpha_t(h_t), \text{ all } a\in A_t(h_t), \text{ all } h_t,t, \\ \alpha_t(h_t)=\min a\in A_t(h_t), \text{ if } \underline{w}_t(h_t)>\underline{u}.\end{gathered}\tag{3}$$

For notational convenience define, for a given relational contract,

$$A_t^+(h_t)=\{a \mid a\in A_t(h_t), a\geq \alpha_t(h_t)\}, \text{ for all } h_t,t, \tag{4}$$

$$a_t^-(a)=\alpha_t(h_t), \text{ for } a\in A_t^+(h_t), \text{ all } h_t,t. \tag{5}$$

$A_t^+(h_t)$ is the set of a with history h_t who continue the relationship at t, $a_t^-(a)$ the lowest agent type pooled with a in that set. Also define

$$\begin{aligned}\tilde{U}_t\left(a',a,h_t\right)=&-c\left(e_t\left(a',h_t\right),a\right)\\ &+\delta\beta_t\left(h_t,e_t\left(a',h_t\right)\right)\max\left\{0,U_{t+1}\left(a,\left(h_t,e_t\left(a',h_t\right),w_t\left(h_t,e_t\left(a',h_t\right)\right)\right)\right)\right\},\\ &\text{for all } a,a'\in A_t^+(h_t), \text{ all } h_t,t.\end{aligned}\tag{6}$$

$\tilde{U}_t(a',a,h_t)$ consists of the components of the maximand in (2) that depend on the agent's actual type a evaluated at the effort for type a' specified by $e_t(a',h_t)$.

Proposition 1 *Necessary conditions for decision rules for agent types $a\in A_t(h_t)$ in a relational contract to be best responses are, for all t,*

$$\gamma_t(a,h_t)=\begin{cases}1, & \text{if } a\geq \alpha_t(h_t),\\ 0, & \text{otherwise;}\end{cases}\tag{7}$$

$$\tilde{U}_t(a,a,h_t)-\tilde{U}_t\left(a,a',h_t\right)\geq U_t(a,h_t)-U_t\left(a',h_t\right)$$

$$\geq \tilde{U}_t\left(a',a,h_t\right) - \tilde{U}_t\left(a',a',h_t\right), \textit{ for all } a,a' \in A_t^+(h_t). \quad (8)$$

These conditions are also sufficient if the continuation contracts following deviation to $e_t \neq e_t(a',h_t)$ *for any* $a' \in A_t^+(h_t)$ *are the same as the continuation contract for* $e_t = e_t(\alpha_t(h_t),h_t)$ *except that (1) the principal pays no bonus at t* ($w_t(h_t,e_t) = \underline{w}_t(h_t)$) *and (2) the payment* $\underline{w}_{t+1}(h_t \cup (e_t,\underline{w}_t(h_t)))$ *is such that agent type* $\alpha_t(h_t)$ *would receive non-positive payoff gain from continuing the relationship at stage 1 of period* $t+1$ ($U_{t+1}(\alpha_t(h_t),h_t \cup (e_t,\underline{w}_t(h_t))) \leq 0$)*.*

That (7) defines a best response follows from the specification for $\alpha_t(h_t)$ in (3). The other results in Proposition 1 are related to results familiar from mechanism design for one-period models. A one-period model is equivalent to having $\delta = 0$ so $\tilde{U}_t(a',a,h_t) = -c(e_t(a',h_t),a)$ from (6). For that case, it is standard to divide all terms in (8) by $a'-a$ and take the limit as $a' \to a$ to get a condition on the derivative $c_2(e_t(a,h_t),a)$ that is used to construct the difference between the payoffs of different types and also, given $c_{12} < 0$, to establish the requirement that $e_t(a,h_t)$ is non-decreasing in a. Here the additional terms in $\tilde{U}_t(a',a,h_t)$ take account of the future consequences from $t+1$ on of agent type a choosing the effort corresponding to type a' at t. The derivative formulation is less useful here because, for relevant continuation contracts, the additional terms in $\tilde{U}_t(a',a,h_t)$ are not differentiable in a at $a' = a$.

When the agent's performance is verifiable, deviation to effort that is not on the equilibrium path for any agent type can be deterred by a sufficiently large monetary penalty. With unverifiable performance (as here), the worst penalty that can be imposed on the agent is to receive zero payoff gain following such a deviation because the agent can always quit. As in Abreu (1988), this would give the largest set of equilibria. Conditions (7) and (8) are then not only necessary for best responses but also sufficient. Ending the relationship is, however, inefficient when a mutually beneficial relationship is possible. In Levin (2003), the same penalty is achieved without the relationship ending by a continuation equilibrium following deviation that is the same as the continuation equilibrium with no deviation but with the agent paying the principal just enough to give the agent zero payoff gain from continuation. With transferable utility, there is then no efficiency loss. That approach is more complicated here because the principal may not know the agent's type and so the payment required to give the agent zero payoff gain from continuation following deviation is not common knowledge. Proposition 1, however, shows that a weaker requirement suffices to ensure that conditions (7) and (8) are sufficient, specifically that the payment following deviation at t by an agent with history h_t is such that the lowest agent type with that history who would, on the equilibrium path, continue the relationship (formally $\alpha_t(h_t)$) receives zero payoff gain from continuation. With this continuation contract, higher a would continue to receive a strictly positive payoff gain from continuation following deviation but that is not sufficient to make deviation worthwhile.

For the principal, let $P_t(a,(h_t,e_t))$ denote the payoff gain from continuing the relational contract with agent type a at stage 2 of period t given history (h_t,e_t), conditional on paying the bonus $w_t(h_t,e_t)-\underline{w}_t(h_t)$.

Proposition 2 *Suppose the continuation contracts following the principal's deviation to $w_t \neq w_t(h_t,e_t)$ are the same as that for $w_t = w_t(h_t,e_t)$ except that the payment $\underline{w}_{t+1}(h_t \cup (e_t,w_t))$ is such that the principal receives non-positive payoff gain from continuing the relationship at stage 2 of period t when paying $w_t = \underline{w}_t(h_t)$. Then best response decision rules for the principal are, for all h_t, e_t and t,*

$$\beta_t(h_t,e_t) = \begin{cases} 1, & \text{if } E_{a|h_t,e_t}\left[P_t(a,(h_t,e_t))\right] \geq 0, \\ 0, & \text{otherwise;} \end{cases} \tag{9}$$

if $w_t(h_t,e_t)-\underline{w}_t(h_t) > 0$, pay $w_t(h_t,e_t)$ if and only if $\beta_t(h_t,e_t)=1$; otherwise, pay $w_t(h_t,e_t)=\underline{w}_t(h_t)$.

Most of this result follows directly from the definition of $P_t(a,(h_t,e_t))$. The principal does not deviate to a bonus smaller than specified in the relational contract when continuing the relationship because that would trigger a continuation equilibrium the same as with no deviation but with the principal paying the agent just enough to give the principal zero payoff gain from continuation. Because the principal's type is common knowledge, the payment required for this is also common knowledge.

For stage 0a of the first period of the relationship, neither party has information about the agent's type beyond its initial distribution. The agent starts a relational contract only if the initial payoff gain U_0 satisfies

$$U_0 \equiv w_0 + \int_{\alpha_1(h_1)}^{\bar{a}} U_1(\tilde{a},h_1)\,dF(\tilde{a}) \geq 0. \tag{10}$$

The principal starts a relational contract only if the expected payoff gain from starting the relationship given the initial distribution of a, denoted P_0, satisfies $P_0 \geq 0$.

4 Equilibrium relational contracts

The previous section derived conditions for decisions in a relational contract to be best responses. This section is concerned with the effort functions $e_t(a,h_t)$, and the functions $\alpha_t(h_t)$ specifying the lowest agent type who continues the relationship, that can be sustained as equilibria. In a Bayesian equilibrium, the principal's beliefs about the agent's type when an event occurs that is on the equilibrium path for some type are defined by Bayes' rule. For an event at t that is not on the equilibrium path for *any* type with history h_t, the continuation contracts are taken to be those specified in Propositions 1 and 2. The former is consistent with the principal believing the agent to be the lowest type in $A_t^+(h_t)$. A relational contract that is a perfect Bayesian equilibrium satisfying these conditions is referred to as an *equilibrium relational contract*. Such contracts are also referred to as *self-enforcing*. In describing equilibria, the history argument is omitted for simplicity where that does not result in ambiguity; for pure strategy equilibria, h_t at each t of a continuing relationship is fully determined by the relational contract and the agent's type.

To be an equilibrium, the parties' payoffs have to be consistent with the total output produced. The joint gain to the principal and the agent (also called the *surplus*) from continuing the relationship can be measured at stages 1 and 2 in each period. Let $S_t^i(a)$ denote the joint gain from continuing the relationship at stage i of period t for given a for a given relational contract. These two measures can be defined recursively as

$$S_t^1(a) = e_t(a) - c(e_t(a), a) - \underline{u} - \underline{v} + \beta_t(e_t(a))\, S_t^2(a), \quad \text{all } a,t; \tag{11}$$

$$S_t^2(a) = \delta\gamma_{t+1}(a)\, S_{t+1}^1(a), \quad \text{all } a,t. \tag{12}$$

These joint gains depend only on agent type and effort, not the division between parties. The joint gain to starting a relational contract is

$$S_0 = \int_{\alpha_1}^{\overline{a}} S_1^1(a)\, dF(a). \tag{13}$$

A necessary condition for a relational contract to start is that $S_0 \geq 0$. Moreover, provided $S_0 \geq 0$, there is always a w_0 such that the agent's and the principal's initial payoff gains U_0, given by (10), and P_0 are both non-negative. Equilibrium requires that the agent receives that part of the joint gain not received by the principal. It follows from (2) that

$$U_t(a) = -c(e_t(a), a) - \underline{u} + \underline{w}_t + \beta_t(e_t(a)) \left[S_t^2(a) - P_t(a)\right], \quad \text{all } a,t. \tag{14}$$

This condition is the *budget balance constraint* from which the dynamic enforcement constraint in Levin (2003) is derived.

A *continuation contract for* h_τ is the part of a relational contract applying from period τ on to relationships with history h_τ. Suppose type a is the only type with history h_τ and so is fully separated at τ from other types in $[\underline{a}, \overline{a}]$. In that case, both parties know the joint gain $S_\tau^1(a)$ from a given continuation contract for h_τ. Because joint gains can be distributed in any proportions at stage 1 of period τ, it is optimal for them to select a continuation contract that maximises $S_\tau^1(a)$ subject to the feasibility and incentive constraints, called here an *optimal continuation contract for* h_τ. The set of optimal continuation contracts corresponds to the feasible Pareto frontier at stage 1 of period τ. It allows for *any* division between the parties of the joint continuation gains and so is independent of what determines that division.

Proposition 3 *Suppose agent type a is the only agent type with history h_τ at τ.*

1. *There exists a continuation contract for h_τ for which continuation of the relationship is an equilibrium if*

$$\max_{\tilde{e}\in[0,\bar{e}]}\left[\delta\tilde{e}-c(\tilde{e},a)\right]\geq\delta\left(\underline{u}+\underline{v}\right). \tag{15}$$

2. *For a satisfying (15), an optimal continuation contract for h_τ has, for all $t\geq\tau$, stationary effort $e_t(a)=e(a)$ that satisfies*

$$\delta e(a)-c(e(a),a)-\delta\left(\underline{u}+\underline{v}\right)\geq 0. \tag{16}$$

   Moreover, for any continuation payoff gains $P_t(a)\geq 0$ and $U_t(a)\geq 0$ for $t\geq\tau$ consistent with the budget balance constraint (14) and independent of t, there exists an optimal continuation contract for h_τ with $w_t(e(a))$ and $\underline{w}_t$ independent of t that has those continuation payoff gains.

3. *If (15) is satisfied for type a but efficient effort $e^*(a)$ does not satisfy (16), an optimal continuation contract for h_τ has effort $e(a)$ the highest that satisfies (16) with equality, $P_t(a)=0$, $U_t(a)=\underline{w}_t-\underline{u}\geq 0$ and*

$$c(e(a),a)=S_t^2(a),\qquad \textit{for all } t\geq\tau. \tag{17}$$

Part 1 of Proposition 3 gives a condition for continuation of the relational contract to be an equilibrium. By an argument in Levin (2003, Theorem 2), if an optimal contract exists, there are stationary contracts that are optimal. For a satisfying (15), there is certainly a stationary effort $e(a)$ that satisfies (16). Part 2 of Proposition 3 shows that effort in an optimal continuation contract must satisfy (16) because of the budget balance constraint (14). When the principal and agent type a continue the relationship at each date along an equilibrium path, $\gamma_t(a)=\beta_t(e(a))=1$ for all $t\geq\tau$ from (7) and (9). Then, from (11) and (12),

$$S_t^2(a)=\frac{\delta}{1-\delta}\left[e(a)-c(e(a),a)-\underline{u}-\underline{v}\right],\quad \text{for all } t\geq\tau. \tag{18}$$

Combined with the budget balance constraint (14), this gives

$$\delta e(a) - c(e(a), a) - \delta(\underline{u} + \underline{v}) = (1-\delta)\left[U_t(a) + \underline{u} - \underline{w}_t + P_t(a)\right], \quad \text{for all } t \geq \tau. \tag{19}$$

With the agent's type revealed to be a, continuation of the relationship requires $U_t(a) \geq \max[0, \underline{w}_t - \underline{u}]$ and $P_t(a) \geq 0$ (from (3) and (9)), so the right-hand side of (19) must be non-negative. Thus (16) is necessary. Part 2 of Proposition 3 also establishes that, for any stationary effort $e(a)$ that satisfies (16), there exist equilibrium continuation contracts with that stationary effort. It further establishes that there exist payments that distribute the joint gain in any way consistent with individual rationality. The reason can be seen from (19), which $w_t(e(a))$ enters only through the payoff gains $U_t(a)$ and $P_t(a)$ and cancels out in their sum. By changing $w_t(e(a))$, these payoff gains can, for given $e(a)$, range from $U_t(a) = \underline{w}_t - \underline{u}$ to $P_t(a) = 0$ without changing the value of the square bracket on the right-hand side. Moreover, $\underline{w}_t$ can be set equal to $\underline{u}$, so $U_t(a) = 0$ is also possible.[2]

Efficient effort for a is $e^*(a)$ defined by (1). If this satisfies (16), it is optimal because it maximizes the joint gain to be distributed between the parties. If it does not satisfy (16), the parties jointly gain by choosing $e(a)$ at the highest level that does, in which case (16) holds with equality, as specified in Part 3 of Proposition 3. Denote by $\hat{\alpha}$ the lowest a for which (15) is satisfied. It follows that, for any $a \geq \hat{\alpha}$ the only type with its history, effort in an optimal continuation contract is

$$\hat{e}(a) = \begin{cases} e^*(a), \text{ if } e^*(a) \text{ satisfies (16);} \\ \max e(a) \text{ that satisfies (16) with equality, otherwise;} \end{cases} \quad \text{for } a \in [\hat{\alpha}, \overline{a}]. \tag{20}$$

Part 3 of Proposition 3 also establishes that, when efficient effort is not attainable, the bonus is set to make $P_t(a) = 0$. A higher bonus makes it possible to induce higher effort. So, when efficient effort is unattainable, it is optimal to have the bonus at the

[2] To see why Proposition 3 is robust to the changes in timing discussed in footnote 1, let $P_t(a)$ be measured at stage 1 of period t. Then the budget balance constraint (14) becomes, for $\beta_t(e(a)) = \gamma_{t+1}(a) = 1$,

$$U_t(a) = -c(e_t(a), a) - \underline{u} + \underline{w}_t + \left[S_t^2(a) + w_t(e(a)) - \underline{w}_t - \delta P_{t+1}(a)\right].$$

For $e_t(a) = e(a)$ and with (18), which is unaffected by the change in timing, this changes (19) to

$$\delta e(a) - c(e(a), a) - \delta(\underline{u} + \underline{v}) = (1-\delta)\left[U_t(a) + \underline{u} - \underline{w}_t + \delta P_{t+1}(a) - (w_t(e(a)) - \underline{w}_t)\right].$$

With this timing, continuation of the relationship requires $U_t(a) \geq \max[0, \underline{w}_t - \underline{u}]$ and also $\delta P_{t+1}(a) \geq w_t(e(a)) - \underline{w}_t$ because otherwise the principal will not pay the bonus $w_t(e(a)) - \underline{w}_t$. So, by the same argument as for the timing in the text, (15) and (16) apply to the revised timing. The only change to the proposition under the revised timing is to Part 3, for which $P_{t+1}(a) = [w_t(e(a)) - \underline{w}_t]/\delta$. This change does not affect the results that follow.

highest level consistent with the principal continuing the relationship. That requires the principal's future payoff gain from continuing the relationship by paying the bonus to be zero. The agent's payoff gain is $U_t(a) = \underline{w}_t - \underline{u}$. This is the lowest payoff gain consistent with the agent incurring the required effort because the agent can guarantee payoff gain of at least $\underline{w}_t - \underline{u}$ by putting in no effort at t and ending the relationship in period $t+1$ even when the principal pays no bonus. The shares of the joint gain are determined by $\underline{w}_t$. For $\underline{w}_t - \underline{u} = S_t^1(a)$, $U_t(a) = S_t^1(a)$, so the agent receives all the joint gain at stage 1 of period t. For lower $\underline{w}_t$, the principal receives some of the joint gain at that stage (even though $P_t(a)$, which is measured at stage 2 of period t, is zero). For $\underline{w}_t = \underline{u}$, the principal receives all of the joint gain. Because the joint gain can be shared in any proportions in this way, it is in the interests of both parties to choose a continuation contract for h_τ that is optimal, independently of how the additional joint gain is divided between them (and hence of any question of relative bargaining powers).

The best response criteria used in the proof of Proposition 3 are those in Propositions 1 and 2. In particular, with the agent's type fully revealed, it is sufficient punishment for defection that the defecting party is required to make a monetary payment that leaves it zero payoff gain from continuing the relationship but in other respects the parties follow the original continuation equilibrium. Then for the optimal continuation equilibria in Proposition 3, the continuation equilibrium following defection is also an optimal continuation equilibrium, just one with a different division of the joint gains. Thus these optimal continuation equilibria are strongly optimal in the sense of Levin (2003, p. 841). Indeed, as shown by Goldlücke and Kranz (2013, Section 4.3), they are also strong perfect (and hence also strong renegotiation-proof in the sense of Farrell and Maskin (1989)). No other continuation contracts are.[3]

Proposition 3 applies to continuation contracts for an agent type fully separated from other types in $[\underline{a}, \overline{a}]$, that is, for a the only type with its history. Separation of agent types may, however, be only partial. The next result applies to that case.

Proposition 4 *Suppose, for $a \in A_\tau^+(h_\tau)$, (15) is satisfied but efficient effort $e^*(a)$ does not satisfy (16) for $a_\tau^-(a)$, so $\hat{e}(a_\tau^-(a)) < e^*(a_\tau^-(a))$. Conditional on types $a \in A_\tau^+(h_\tau)$ all choosing the same effort $e(a)$ for all $t \geq \tau$, a continuation contract for h_τ satisfying Proposition 3 that is optimal for type $a_\tau^-(a)$ is optimal for all $a \in A_\tau^+(h_\tau)$. Any continuation contract for h_τ with effort $\hat{e}(a_\tau^-(a))$ for all $a \in A_\tau^+(h_\tau)$ for all $t \geq \tau$ has $U_t(a_\tau^-(a)) = \underline{w}_t - \underline{u} \geq 0$, and*

[3]If the parties were able to commit to a sub-optimal continuation contract for h_τ, that would in general affect the extent of separation possible in previous periods. But such commitment seems inappropriate for parties who, as here, cannot commit not to renegotiate.

$$U_t(a) = U_t\left(a_\tau^-(a)\right) + \frac{1}{1-\delta}\left[c\left(\hat{e}\left(a_\tau^-(a)\right), a_\tau^-(a)\right) - c\left(\hat{e}\left(a_\tau^-(a)\right), a\right)\right],$$

$$\textit{for } a \in A_\tau^+(h_\tau),\ t \geq \tau. \quad (21)$$

Efficient effort is increasing in a. So if two agent types $a'' > a'$ with the same history h_τ are to be pooled with stationary effort from τ on and efficient effort for a' is not achievable, effort for a'' is below its efficient level. Thus what is optimal at τ for type a'' must be optimal for type a'. So, as Proposition 4 shows, if some types are to be pooled indefinitely, an optimal continuation contract has the characteristics shown in Proposition 3 to be optimal for the lowest type in the pool. Types $a > a_\tau^-(a)$ necessarily receive the strictly positive payoff gain in (21) because of their lower cost of effort.

5 Contracts with full pooling of continuing types

This section gives necessary and sufficient conditions for there to exist an equilibrium relational contract for the whole relationship in which all agent types who continue the relationship are pooled, choosing the same effort and being paid the same.

Proposition 5 *There exists an equilibrium pooling relational contract for the whole relationship in which all agent types $a \in [a', \overline{a}]$ continue the relationship and choose effort $e(a')$ for all t, and all types $a \in [\underline{a}, a')$ end it in the first period, if and only if*

$$\delta e\left(a'\right) - c\left(e\left(a'\right), a'\right) - \delta\left(\underline{u} + \underline{v}\right) \geq 0. \quad (22)$$

Proposition 5 follows from the budget balance condition (14). For stationary effort, this can be written as (19). With all $a \in [a', \overline{a}]$ choosing the same effort and being paid the same, $P_t(a) = P_t(a')$ for all $a \in [a', \overline{a}]$, so the principal continues the relationship only if $P_t(a') \geq 0$. Agent type a' does so only if $U_t(a') \geq \max\left[0, \underline{w}_t - \underline{u}\right]$. Substitution of these into the right-hand side of (19) establishes that (22) is necessary. Moreover, $U_t(a)$ is strictly increasing in a for a given effort sequence, so continuing the relationship and choosing $e_t = e(a')$ for all $t \geq 1$ is better for all $a \in (a', \overline{a}]$ than having it end if it is for a'. The only effort choice permitted by the contract is $e(a')$ and that is a best response for all $a \in [a', \overline{a}]$ if $U_t(a') \geq 0$. Furthermore, the contract can be chosen such that $U_t(a') = 0$, so types $a \in [\underline{a}, a')$ prefer to end the relationship in the first period. Finally, there is always some w_0 such that the initial participation conditions U_0, given by (10), and P_0 are non-negative because $U_t(a), P_t(a) \geq 0$ for all $a \in [a', \overline{a}]$, $\alpha_1(h_1) = a'$, and the payoff gains for $a \in [\underline{a}, a')$ are zero. Indeed, w_0 can be chosen to give $U_0 = 0$, $P_0 = 0$ or any convex combination of these.

Clearly, (22) can be satisfied for some a' if and only if (15) is satisfied for the highest agent type $\overline{a}$. Moreover, Proposition 4 applies to a pooling contract that satisfies the conditions of Proposition 5. Thus, if efficient effort $e^*(a')$ does not satisfy (22), it is optimal to set $e(a')$ at the level that satisfies (22) with equality.

The pooling contracts in Proposition 5 are inefficient in not tailoring $e_t(a)$ to each type a that continues the relationship, which is what full efficiency would do. The next section explores equilibria with separation of types that continue the relationship.

6 Contracts with separation of continuing types

A natural question is whether there exist equilibrium contracts that fully separate all agent types when the continuation equilibria for separated types are optimal. The following proposition gives conditions under which there do not.

Proposition 6 *Consider period t of an equilibrium relational contract with $[\underline{a}_t, \overline{a}_t] \subseteq A_t^+(h_t)$. Suppose, under that contract, all types $a \in [\underline{a}_t, \overline{a}_t] \cup \{a \mid a_{t+1}^-(a) = \underline{a}_t\}$ continue the relationship with effort $e_\tau(a, h_\tau) = \hat{e}\big(a_{t+1}^-(a)\big) < e^*\big(a_{t+1}^-(a)\big)$ for all $\tau > t$. Then*

1. *for $a, a' \in [\underline{a}_t, \overline{a}_t]$ with $a > a'$ and separated from a' at t, $e_t(a, h_t) - e_t(a', h_t)$ is bounded below by some $\varepsilon > 0$;*
2. *the equilibrium relational contract cannot separate all $a \in [\underline{a}_t, \overline{a}_t]$ at t;*
3. *$e_t(a, h_t) < \hat{e}(a)$ for $a \in (\underline{a}_t, \overline{a}_t]$ and $e_t(\underline{a}_t, h_t) \leq \hat{e}(\underline{a}_t)$;*
4. *for $\delta \geq 1/2$ and $a' \in [\underline{a}_t, \overline{a}_t]$ for which $a' = a_{t+1}^-(a')$, there exists $a'' \in (a', \overline{a}_t]$ such that $a \in (a', a'']$ is not separated from a'.*

Proposition 6 considers equilibrium relational contracts with agent types that have the same history at t (including $t = 1$ when all agent types necessarily have the same history) and that, from $t + 1$ on, choose the effort level that is optimal for the lowest type with which they still have the same history at $t + 1$. (That is, $e_\tau(a, h_\tau) = \hat{e}\big(a_{t+1}^-(a)\big)$ for $\tau > t$.) For type a fully separated from all other types at t, this implies $e_\tau(a, h_\tau) = \hat{e}(a)$ for $\tau > t$, which has been shown in Proposition 3 to be optimal for a continuation contract. Thus Proposition 6 covers as a special case full separation of a type with an optimal continuation contract. The proposition considers a for which efficient effort is not attained in the continuation equilibrium, so $\hat{e}\big(a_{t+1}^-(a)\big) < e^*\big(a_{t+1}^-(a)\big)$.

Part 1 of Proposition 6 establishes that, for a contract to separate type a from type $a' < a$ in period t when both have the same history at t, the effort of type a at t must be discretely greater than that of a'. It follows that, as established in Part 2, it is not possible for a contract to separate at t all types in an interval $[\underline{a}_t, \overline{a}_t]$ who have the same history h_t because a monotone function defined on an interval cannot have a continuum of jumps. To understand why separation requires a discrete jump in effort, consider the case with the cost of effort multiplicatively separable in effort and agent type. Agent type can then be normalized to write $c(\tilde{e}, a) = \hat{c}(\tilde{e})/a$ for some increasing function $\hat{c}(\tilde{e})$. (The proof in the Appendix does not rely on this special case.) Consider the lowest type $a > a'$ to be separated from a' at t. Under the conditions of the proposition, $e_\tau(a, h_\tau) = \hat{e}(a) < e^*(a)$ for $\tau > t$ if a separates from a'. By Proposition 4, this implies payoff gain $U_\tau(a) = \underline{w}_\tau - \underline{u} \geq 0$ for all $\tau > t$. But any type less than a taking the action for a in period t can guarantee the same payoff gain $\underline{w}_{t+1} - \underline{u}$ at $t+1$ as a by continuing the relationship for $t+1$ (so forgoing outside opportunity with payoff $\underline{u}$) and collecting the fixed wage $\underline{w}_{t+1}$, but delivering no effort (so receiving no bonus at $t+1$) and quitting for $t+2$. Moreover, by choosing the effort for a' from t on, a can obtain an additional payoff gain over a' given by (21) for $a_\tau^-(a) = a'$, amounting to $\hat{c}(\hat{e}(a'))\left(\frac{1}{a'} - \frac{1}{a}\right)$ for each period from $t+1$ on. From (8) in Proposition 1, for a to choose separation, efforts in period t must therefore satisfy

$$
\begin{aligned}
&-\hat{c}(e_t(a, h_t))\left(\frac{1}{a} - \frac{1}{a'}\right) \\
&\qquad \geq U_t(a, h_t) - U_t\left(a', h_t\right) \\
&\qquad\qquad \geq -\hat{c}\left(e_t\left(a', h_t\right)\right)\left(\frac{1}{a} - \frac{1}{a'}\right) - \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a'\right)\right)\left(\frac{1}{a} - \frac{1}{a'}\right). \qquad (23)
\end{aligned}
$$

(When the principal continues the relationship, $\beta_t(.) = 1$.) For $a > a'$, (23) requires

$$
\hat{c}(e_t(a, h_t)) \geq \hat{c}\left(e_t\left(a', h_t\right)\right) + \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a'\right)\right). \qquad (24)
$$

The second term on the right-hand side of (24) is strictly positive. Thus, (24) implies $e_t(a, h_t)$ must be greater than $e_t(a', h_t)$ by a discrete amount. This applies no matter how close a is to a' because the right-hand side of (24) is independent of a. Indeed, (24) implies the same jump in effort no matter what a is to be separated from a' but this specific property applies only to the case $c(\tilde{e}, a) = \hat{c}(\tilde{e})/a$.

The intuition behind (23) and (24) is related to that for the ratchet effect in the dynamic procurement model in Laffont and Tirole (1993, Chapter 9). For full separation, there would have to be a different effort for each type. Suppose there was only one period. With no future to consider, that corresponds to $\delta = 0$. In that case, (24) implies the standard incentive compatibility condition that $e_t(a, h_t)$ is non-decreasing in a.

(More generally, this conclusion follows when $c_{12}(\tilde{e},a) < 0$, which is always satisfied when $c(\tilde{e},a) = \hat{c}(\tilde{e})/a$.) With the ratchet effect model, current output is contractible so the principal can commit to rewarding current effort. With just one period, the optimal contract then has an effort function that is continuous non-decreasing, which satisfies (24) when $\delta = 0$. Payment increases with effort sufficiently to compensate higher agent types for higher effort. With two periods, because the principal does not commit to the second-period contract before the agent chooses first-period effort and makes "take it or leave it" contract offers, agent type a that is fully revealed receives no rent in the second period. But the same second-period payoff can be obtained by type $a' < a$ that takes the action in the first period intended for a by quitting for the second period. So, if choosing the first-period effort for a, the only difference in payoff for a and a' would be the first-period one. Money is equally valuable to all types, so that payoff difference is just the difference in the cost of effort, which corresponds to the term on the left-hand side of the first inequality in (23). But, if choosing the first-period effort for a', the difference in payoff for a and a' is not just the difference in the cost of effort (corresponding to the first term on the right-hand side of the second inequality in (23)) but also the difference in the informational rent that a can obtain in the second period because a has a lower cost of effort than a'. That difference in rent corresponds to the one-period equivalent of the second term on the right-hand side of the second inequality in (23). If effort was a continuous function of type, the difference in the additional cost of effort from choosing the first-period effort for a over that for a' (that is, $\hat{c}(e_t(a,h_t)) - \hat{c}(e_t(a',h_t))$) would go to zero as a approaches a'. Then the gain in future informational rent to a from choosing the first-period effort for a' would always make that choice advantageous. So a continuous effort function cannot induce full separation when $\delta > 0$. An implication is that, as noted in Laffont and Tirole (1993, p. 382), the revelation principle does not apply to repeated relationships in the absence of commitment because truthful revelation of types in one period would result in full separation in subsequent periods.

The relational contract model is, however, crucially different from the ratchet effect model in what determines the payoff gain to an agent type that has been fully revealed. With the ratchet effect, a fully revealed agent type receives no payoff gain in period 2 because the principal cannot commit in advance to a continuation contract, even on outcomes that are contractible in future periods, and extracts all the continuation rent by making a "take it or leave it" continuation contract offer. A lower type imitating a higher type in period 1 can obtain that same payoff for period 2 by quitting. So it is because the principal chooses the point on the feasible Pareto frontier following revelation of the agent's type that there is no difference in future payoff between a fully revealed type and a lower type imitating that type in the first period. With the relational contract when efficient effort is not attainable, agent type a receives future payoff gain $U_{t+1}(a) = \underline{w}_{t+1} - \underline{u}$ at $t+1$ following full revelation of type at t for *every* optimal continuation contract, no matter how the joint gain from adopting such a contract is divided between the parties. The reason is that optimal effort from $t+1$ on is the highest effort consistent with dynamic enforcement and, if $U_{t+1}(a) > \underline{w}_{t+1} - \underline{u}$, effort could be increased without the agent preferring to choose zero effort and end the relationship next period. But a lower type $a' < a$ taking the action for a at t can also guarantee payoff gain $\underline{w}_{t+1} - \underline{u}$ at $t+1$ by continuing the relationship for $t+1$ (so forgoing outside opportunity with payoff $\underline{u}$) and collecting the fixed wage $\underline{w}_{t+1}$, but delivering no effort (so receiving no bonus at $t+1$) and quitting for $t+2$.[4] Thus type a separating fully at t receives no higher payoff from $t+1$ on than $a' < a$ would by imitating a at t. That does not preclude type a receiving some of the joint gain from continuation of the contract because $\underline{w}_{t+1}$ can be greater than $\underline{u}$. But the characteristic that prevents full separation is not that type a receives no payoff gain once fully revealed but that there is no *difference* in future payoff between type a fully revealed at t and $a' < a$ that chooses the effort at t intended for a. As long as that is the case, the argument in the previous paragraph applies.

This rationale for partial pooling when efficient effort is not attainable differs from that for partial pooling in the hidden information model of Levin (2003). In that model, types are *iid* draws each period, so all types are pooled at the start of each period and revelation of type does not affect future payoffs. Thus, unlike here, full separation is possible. But full separation is not optimal when efficient effort is unattainable because the budget balance constraint restricts the spread of bonuses that are incentive compatible. That, in turn, restricts the spread of incentive compatible efforts that are available for separating types and pooling the most productive types is the optimal way to limit the spread of efforts. This implication of the budget balance constraint plays no role in the derivation of (24) and so is not the reason full separation is not achievable. But it still limits how much separation is achievable. To see why in the case $c(\tilde{e}, a) = \hat{c}(\tilde{e})/a$, use the right-hand inequality in (23) to substitute for $U_t(a)$ in the budget balance constraint (14), note that $\beta_t(e_t(a)) = 1$ for $e_t(a)$ the effort specified in the contract for some a with history h_t, use the value of $S_t^2(a)$ given in (17), and re-arrange to get

$$\frac{\hat{c}(e_t(a,h_t))}{a} + \left[\hat{c}\big(e_t(a',h_t)\big) + \frac{\delta}{1-\delta}\hat{c}\big(\hat{e}(a')\big)\right]\left(\frac{1}{a'} - \frac{1}{a}\right) \leq \frac{\hat{c}(\hat{e}(a))}{a} - P_t(a) - U_t\big(a'\big) - \underline{u} + \underline{w}_t. \qquad (25)$$

[4]Allowing negative bonuses would not alter this conclusion because type $a' < a$ would not pay a negative bonus in period $t+1$ if intending to quit for $t+2$.

The second term on the left-hand side of (25) is positive for $a > a'$. With the conditions $P_t(a) \geq 0$ and $U_t(a) \geq \max[0, \underline{w}_t - \underline{u}]$, (25) thus places an upper bound on $\hat{c}(e_t(a,h_t))/a$ in addition to the lower bound placed by (24). In particular, it implies $\hat{c}(e_t(a,h_t)) < \hat{c}(\hat{e}(a))$, and hence $e_t(a,h_t) < \hat{e}(a)$, for $a > a'$, which is the result in Part 3 of Proposition 6. It also restricts the set of types that can be separated from a' to those types a for which the upper bound on $\hat{c}(e_t(a,h_t))$ implied by (25) is at least as great as the lower bound required to satisfy (24). Thus, Proposition 6 combines the insight of Laffont and Tirole (1993) that separation requires a discrete jump in effort between types (with the implication that full separation is unattainable) with the insight of Levin (2003) that dynamic enforcement restricts the spread of incentive compatible efforts (which limits the extent to which partial separation can be attained). Moreover, by Assumption 1, $e_t(a',h_t) \geq 0$, $\hat{c}(\tilde{e})$ is continuous and $\hat{c}(0) = 0$. So, with $e_t(a,h_t) < \hat{e}(a)$, (24) can never be satisfied for a sufficiently close to a' for $\delta/(1-\delta) \geq 1$, that is $\delta \geq 1/2$, which is the result in Part 4 of Proposition 6. In that case, the form of separation Levin (2003) finds optimal for non-persistent types (a single pool consisting of an interval of the most productive types, with other types fully separated) is not feasible for persistent types.

It follows from (24) and (25) that separation is more easily achieved with lower $e_t(a',h_t)$ and with lower $\hat{e}(a')$. The former illustrates the benefits of starting a relationship "small", as in Watson (1999) and Watson (2002). The latter illustrates the limitations that arise from the parties being unable to commit themselves to inefficient actions in the future. If the parties could commit to sub-optimal effort for type a' in period $t+1$, separation of types at t would be easier to achieve.

Proposition 6 applies to an interval of pooled agent types and hence to all agent types in the first period of a relationship, so not all types can be separated in that period. The next result extends Proposition 6 to the whole relationship.

Proposition 7 *If there is more than one agent type $a \in [\underline{a}, \overline{a}]$ for which a mutually beneficial relational contract is possible, there exists no equilibrium relational contract with optimal continuation that continues the relationship for all those types and fully separates them.*

When there is more than one agent type for which a mutually beneficial relational contract is possible, Assumption 1 ensures that there is an interval of agent types a for which efficient effort does not satisfy (16), so $\hat{e}(a) < e^*(a)$. By Proposition 6, it is not possible to separate in one period all such agent types with the same history. That applies for any number of periods as long as the contract retains an interval of types with the same history. Such contracts are not, however, the only possible continuation contracts exhibiting partial separation. Laffont and Tirole (1993, p. 383) describe, in the context of a two-period procurement model, continuation equilibria that exhibit infinite reswitching in which actions that generate the same outcome are chosen by different types, but never by neighbouring types. That is, for any two types choosing the same action, there is always some intermediate type that chooses an action that generates a different outcome. Sun (2011) shows that, in the two-period procurement model, such continuation contracts are not optimal. In the relational contract model used here, contracts with infinite reswitching are no more effective at achieving full separation with optimal continuation than are contracts with intervals of types that are pooled. So, as stated in Proposition 7, not all agent types for which a mutually beneficial relational contract is possible can be separated. The only restriction on continuation contracts used to derive this result is that, conditional on full revelation of type a, effort for that type is $\hat{e}(a)$ thereafter. No restriction is imposed on effort in continuation contracts for types that are still pooled.

Assumption 1 does not rule out efficient continuation effort being attainable for some agent types that are fully revealed. For such types, effort in an optimal continuation contract is efficient. Moreover, it may not require the agent's payoff gain in periods τ following revelation to equal $\underline{w}_\tau - \underline{u}$, so the argument used to establish Proposition 7 does not go through. But this is never the case for all types for which a mutually beneficial relational contract is possible.[5]

Where, though, separation of types is possible, it is better than having these types remain pooled indefinitely, as the next proposition shows.

Proposition 8 *Consider period t of an equilibrium relational contract in which, for all $a \in [\underline{a}_t, \bar{a}_t] \subseteq A_t^+(h_t)$, $e_t(a,h_t) = e_t(\underline{a}_t, h_t)$ and $e_\tau(a,h_t) = \hat{e}(\underline{a}_t) < e^*(\underline{a}_t)$ for all $\tau > t$. If there exist $a' \in (\underline{a}_t, \bar{a}_t]$ and $e_t(a',h_t) < \hat{e}(a')$ that satisfy the conditions for a' to be separated from $\underline{a}_t$ at t given $e_t(\underline{a}_t, h_t)$, the joint gain $S_t^1(a)$ can be increased for $a \in [a', \bar{a}_t]$ without being reduced for $a \in [\underline{a}_t, a')$ by further partitioning $[\underline{a}_t, \bar{a}_t]$ in period t.*

The intuition for Proposition 8 is that the higher types in the further separation can deliver higher, and therefore closer to efficient, effort without reducing the effort of the lower types. This applies, in particular, to the case in which the original equilibrium relational contract pools all agent types who continue the relationship, as in Proposition 5. So some separation, if feasible, always dominates full pooling of types that continue the relationship. Moreover, Proposition 8 implies that, once all separation that is going

[5] For agent types a for which efficient effort is attainable following separation, as long as the difference in future payoff between $a > a'$ being fully revealed at t and a' choosing the effort for a at t is less than $\frac{\delta}{1-\delta}\hat{c}(\hat{e}(a'))\left(\frac{1}{a} - \frac{1}{a'}\right)$, the additional term on the left-hand side of (24) is insufficient to avoid the need for a jump in effort to achieve separation. This certainly applies if the principal receives all the joint gain, so a's payoff gain following separation at t is $U_{t+1}(a) = 0$.

to occur under an optimal contract has occurred, it is not possible to separate types further given the history.

If some separation is feasible, it is always possible to fully separate some types. For agent type distributions with no mass points, however, it is never optimal to fully separate one type if a marginally lower type could be pooled with it without detriment to other types, as the next result shows.

Proposition 9 *Consider period t of an equilibrium relational contract with optimal continuation for which $[a', a''] \subseteq A_t^+(h_t)$ for a' sufficiently close to a'', a'' is fully separated at t from all $a \in A_t^+(h_t)$, $e_\tau\left(a_{t+1}^-(a'), h_\tau\right) = \hat{e}\left(a_{t+1}^-(a')\right)$ for $\tau \geq t+1$, and $\hat{e}(a'') < e^*(a'')$. If $F(a)$ is continuous at a'' and it is feasible to separate $a \in [a', a'')$ from all $a \in A_t^+(h_t)$ with $a < a'$ at t without changing $e_t\left(a_{t+1}^-(a'), h_t\right)$, the overall joint gain at t for $a \in [a', a'']$, $\int_{a'}^{a''} S_t^1(a)\, dF(a)$, can be increased by pooling $a \in [a', a'']$ without reducing $S_t^1(a)$ for $a \in A_t^+(h_t) - [a', a'']$.*

While the statement of Proposition 9 is somewhat technical, the basic point is intuitive. As shown in Proposition 6, separating a type from some lower type involves a discrete jump in effort. Pooling a type with a marginally higher type requires at worst only a marginal reduction in effort, now and in the future, for that higher type. So, when the higher type is fully separated, the distribution of types is continuous and effort is below the efficient level, the discrete jump in effort for the additionally separated type increases the joint gain more than the marginal reduction in effort for the marginally higher type reduces it. This applies even when the alternative is merely delaying the additional separation for one period in order to achieve full separation of both types without reducing the effort of the marginally higher type a''. That is because the reduction in the discrete gain for $a \in [a', a'')$ resulting from discounting outweighs the marginal gain from higher effort for the marginally higher type. The proposition applies to fully separated types anywhere in the distribution of types. It may, of course, be optimal to have some types fully separated, for example when a marginally lower type cannot be induced to incur the discrete jump in effort required for separation. But when marginally lower types can be induced to incur that effort, the joint payoff gain is increased by having them do so. Propositions 8 and 9 motivate the following definitions.

Definition 1 *A* one-period partition contract *is a relational contract with agent types partitioned by $\underline{a} < a^1 < \ldots < a^n < \bar{a}$ and the characteristics that, for given a^1 and $e_1\left(a^1, h_1\right)$:*

1. *all agent types $a \in [\underline{a}, a^1)$ end the relationship in period 1;*

2. *all agent types $a \in [a^i, a^{i+1})$ for $i = 1, \dots n$, with a^{n+1} defined as $a^{n+1} = \overline{a}$, choose the same effort in period 1 and effort $\hat{e}(a^i)$ defined by (20) in subsequent periods.*

Definition 2 *A* finest one-period partition contract *is a one-period partition contract for which a^i, for $i = 2, \dots n$, is the lowest type that can be separated from a^{i-1}, with n given by the highest integer such that $a^n < \overline{a}$ when a^i is defined in this way.*

A one-period partition contract satisfies the assumptions of Proposition 6, so the results in that proposition apply. Part 3 of Proposition 6 implies that effort in period 1 for all types $a > a^1$ is below $\hat{e}(a)$. There is thus a cost to information being private except possibly "at the bottom", that is, for the least productive relationships. Other characteristics of one-period partition contracts are given in the next proposition for the separable case $c(\tilde{e}, a) = \hat{c}(\tilde{e})/a$.

Proposition 10 *Consider $c(\tilde{e}, a) = \hat{c}(\tilde{e})/a$ with $\underline{a} > 0$ and $\hat{e}(a) < e^*(a)$ for all $a \in [\underline{a}, \overline{a}]$. For given $a^1 = \alpha_1(h_1)$ that satisfies (15) and $e_1(a^1, h_1) \le \hat{e}(a^1)$, the following hold.*

1. *Necessary and sufficient conditions for a one-period partition contract to be sustainable as an equilibrium contract are*

$$\frac{\hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)}{a^{i+1}} - \sum_{j=1}^{i} \left[\hat{c}\left(e_1\left(a^j, h_1\right)\right) + \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a^j\right)\right)\right] \left[\frac{1}{a^j} - \frac{1}{a^{j+1}}\right]$$
$$\ge \frac{\hat{c}\left(e_1\left(a^{i+1}, h_1\right)\right)}{a^{i+1}}$$
$$\ge \left[\hat{c}\left(e_1\left(a^i, h_1\right)\right) + \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a^i\right)\right)\right] \frac{1}{a^{i+1}}, \quad i = 1, \dots, n-1. \tag{26}$$

   For any such contract that maximizes S_0, the first inequality holds with equality for $i = n-1$.

2. *For $\delta > 1/2$, there exists a unique partition that satisfies the conditions for an equilibrium finest one-period partition contract.*[6] *It has a finite number of sub-intervals, with a^i for $i = 2, \dots, n$ given by*

$$\frac{\hat{e}\left(a^2\right) - (\underline{u} + \underline{v})}{\hat{e}(a^1) - (\underline{u} + \underline{v})} = \frac{1}{1-\delta} - \left[1 - \frac{\hat{c}\left(e_1\left(a^1, h_1\right)\right)}{\hat{c}(\hat{e}(a^1))}\right] \tag{27}$$

$$\frac{\hat{e}\left(a^{i+1}\right) - (\underline{u} + \underline{v})}{\hat{e}(a^i) - (\underline{u} + \underline{v})} = \frac{1}{1-\delta}, \quad i = 2, \dots, n-1, \tag{28}$$

[6] The condition $\delta > 1/2$ ensures that, by Part 4 of Proposition 6, the lowest sub-interval of the partition is not degenerate. If that is not the case, (27) takes a different form for some $e_1\left(a^1, h_1\right)$.

and $e_1\left(a^i,h_1\right)$ *for* $i=2,\ldots,n$ *by*

$$\hat{c}\left(e_1\left(a^{i+1},h_1\right)\right)=\hat{c}\left(e_1\left(a^i,h_1\right)\right)+\frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a^i\right)\right),\quad i=1,\ldots,n-1. \tag{29}$$

3. *There exists no continuation contract for histories* h_2 *generated by the contract in Part 2 that separates* $a\in\left(a^i,a^{i+1}\right)$ *from* a^i *for* $i=1,\ldots,n-1$ *when* $e_t(a,h_t)=\hat{e}\left(a_t^-(a)\right)$ *for* $t>2$.

Part 1 of Proposition 10 uses (24) and (25) to put bounds on $\hat{c}\left(e_1\left(a^{i+1},h_1\right)\right)/a^{i+1}$ in an equilibrium one-period partition contract. The right-hand inequality in (26) is just (24) for $t=1$, $a'=a^i$ and $a=a^{i+1}$. The left-hand inequality corresponds to (25), again for $t=1$, $a'=a^i$ and $a=a^{i+1}$, with the value of $U_1\left(a^i\right)$ determined by the efforts for lower elements in the partition, $\underline{w}_1$ at the highest value (specifically $\underline{u}$) with types below a^1 ending the relationship, and $P_1\left(a^{i+1}\right)$ at the lowest value for which the principal will continue the relationship, that is, zero. From (26), increasing $e_1\left(a^j,h_1\right)$ for $j\le i$ reduces the left-hand side of the first inequality. It thus has consequences for higher sub-intervals in the partition because it reduces the upper bound on $e_1\left(a^{i+1},h_1\right)$ further below $e^*\left(a^{i+1}\right)$. For $i=n-1$, however, there is no higher sub-interval in the partition, so it is always optimal to raise $e_1(a^n,h_1)$ to make the upper bound in (26) tight.

Even though full separation of agent types is not feasible when the parties cannot commit to sub-optimal actions in the future, from Part 1 there exists an equilibrium relational contract with some separation if there exist a^1 that satisfies (15), and $e_1\left(a^1,h_1\right)\le\hat{e}\left(a^1\right)$ and $e_1\left(a^2,h_1\right)$ that satisfy (26) for $a^2\le\bar{a}$. The minimum gap between $\hat{c}\left(e_1\left(a^{i+1},h_1\right)\right)$ and $\hat{c}\left(e_1\left(a^i,h_1\right)\right)$ is given by the right-hand inequality in this condition. Because $\hat{c}\left(\hat{e}\left(a^i\right)\right)$ increases with a^i, this minimum gap is greater for sub-intervals of the partition with more productive types. That contrasts with a partition equilibrium for the standard ratchet effect, for which the minimum gap for the special case that corresponds to $\hat{c}(\tilde{e})=b\tilde{e}/2$ with b a positive constant (the only case for which Laffont and Tirole (1993) derive the gap) is δ/b, which is independent of type.

An equilibrium *finest* one-period partition contract has a^{i+1} the lowest type that can be separated from a^i and thus for which the upper bound on $\hat{c}\left(e_1\left(a^{i+1},h_1\right)\right)/a^{i+1}$ in (26) exceeds the lower bound. Because $\hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)/a^{i+1}$ is increasing in a^{i+1} and the other terms in these bounds involving a^{i+1} are the same, the upper and lower bounds are equal. There is thus a unique equilibrium finest one-period partition contract for given a^1 and $e_1\left(a^1,h_1\right)$. Part 2 of Proposition 10 characterizes that. Effort $\hat{e}\left(a^1\right)$ is the highest effort sustainable by a^1. For $e_1\left(a^1,h_1\right)=\hat{e}\left(a^1\right)$, (27) has the same form as (28).

Part 3 of Proposition 10 establishes that, with separation of agent types in the first period the finest achievable, no further separation with optimal continuation is possible in subsequent periods. The reason is that the continuation contract has $e_t\left(a^i, h_t\right) = \hat{e}\left(a^i\right)$ for $t \geq 2$. With a^i taking the place of a' in (25), that increases the left-hand side because in the finest one-period partition contract $e_1\left(a^i, h_1\right) < \hat{e}\left(a^i\right)$ for $i \geq 2$, which makes the requirement (25) for separation more stringent. As a result, no further separation is possible in subsequent periods. Moreover, Proposition 4 implies that $\hat{e}\left(a^i\right)$ is also optimal from period 2 on for all $a \in [a^i, a^{i+1})$ given that no further separation is possible.

The following example illustrates the requirements in Proposition 10 that (15) is satisfied and that $\hat{e}(a) < e^*(a)$.

Example 1 *Consider $\hat{c}(\tilde{e}) = \tilde{e}^2/2$. Efficient effort defined by (1) is then $e^*(a) = a$, so this specification identifies an agent type with its efficient effort. By (20), for $\hat{e}(a) < e^*(a)$,*

$$\hat{e}(a) = \delta a + \sqrt{(\delta a)^2 - 2\delta a\,(\underline{u} + \underline{v})}.$$

For a^1 to satisfy (15), this must have a real root, which requires

$$a^1 \geq \frac{2\,(\underline{u} + \underline{v})}{\delta}.$$

Moreover, $\hat{e}(a) < e^*(a)$ for all $a \in [\underline{a}, \overline{a}]$ whenever

$$\frac{\underline{u} + \underline{v}}{\overline{a}} > 1 - \frac{1}{2\delta}.$$

Proposition 10 further establishes that, with a finest one-period partition contract, types are separated into only a finite number of sub-intervals despite being continuous. Even with continuous types, employees will be placed in a finite number of grades that include different abilities, a characteristic of many employment situations, and suppliers will be grouped into a finite number of bands, as Asanuma (1989) explains for Toyota. A natural question for empirical application is how an equilibrium finest one-period partition contract changes with the parameters of the model.

Corollary 1 *For the conditions of Proposition 10, given a^1 and $e_1\left(a^1, h_1\right) \leq \hat{e}\left(a^1\right)$,*

1. *an increase in δ increases a^i for $i = 2, \ldots, n$ such that $\hat{e}\left(a^{i+1}\right) - (\underline{u} + \underline{v})$ is increased relative to $\hat{e}\left(a^i\right) - (\underline{u} + \underline{v})$ for $i = 2, \ldots, n-1$;*

2. *a decrease in $\underline{u} + \underline{v}$ increases a^i for $i = 2, \ldots, n$ such that $\hat{e}\left(a^i\right) - (\underline{u} + \underline{v})$ is increased in the same proportion for all $i = 2, \ldots, n$;*

3. *n is non-increasing in δ;*

4. *n is non-decreasing in $\overline{a}$.*

Thus, for example, where the employees of a firm have less good outside opportunities (lower $\underline{u}$), the difference between the output of workers in a sub-interval of the finest partition and the joint value of separation $(\underline{u}+\underline{v})$ increases in the same proportion for all sub-intervals. With higher δ (which could be because the match has a lower probability of ending for exogenous reasons), that difference increases more for sub-intervals with more productive workers. The latter also (weakly) reduces the number of sub-intervals, so workers are less finely separated. The reason is that an increase in δ increases the value of the future rent from imitating a less productive type. This, as shown by (24), increases the difference in effort required to separate types which, with a limited spread of feasible efforts, increases the size of the sub-intervals in the partition and may thus reduce the number of such sub-intervals. These characteristics have the potential for empirical investigation.

MacLeod and Malcomson (1988) also derive an equilibrium relational contract with a finite number of ranks that corresponds to a partition of continuous, privately-observed agent types that are persistent. The reason for the finite partition is, however, different. MacLeod and Malcomson (1988) follow the efficiency wage model of Shapiro and Stiglitz (1984) in having no bonus component to pay. With no bonus, the only way for the principal to induce agent effort is by the threat that the agent's payoff will in future be reduced to the level $\underline{u}$ per period that can be obtained outside the relationship. Moreover, in MacLeod and Malcomson (1988), agent types are not specific to the relationship, but equally valuable to competing principals, who can observe the payment made to an agent. This corresponds to the payoff $\underline{u}$ being a function of information that is revealed. Specifically, an agent dismissed for not complying with the relational contract in one rank is believed by other principals to be appropriate for the rank below. The discrete difference in payoff between ranks that results in discrete partitioning of types thus corresponds to the difference in payoff between the employed and the unemployed required to induce effort in Shapiro and Stiglitz (1984). These differences are required to maintain incentive compatibility even after types have been partitioned into ranks. In the model used here, bonuses are permitted and types are relationship-specific. The discrete differences in payoff between ranks are not then necessary to maintain incentive compatibility once types have been partitioned. They arise from the impossibility of getting agents to reveal finer information about types in the first place.

In the relational contracts in Proposition 10, different partitions result from different a^1 and $e_1(a^1,h_1)$ and different values of these will in general result in different joint gains S_0 from the contract. Lowering a^1, where that is feasible, increases the set of types that continue the relationship and, from (27), also lowers a^2. That, in turn, lowers a^i for $i>2$ from (28). But it may be feasible only if $e_1(a^1,h_1)$ is also lowered. From

(27), that lowers a^2, and so on, but it reduces effort for all $a \in [a^1, a^2)$. So which finest one-period partition contract is optimal depends on the distribution of agent types.

There is also a question of when a finest one-period partition contract for some a^1 and $e_1(a^1, h_1)$ is optimal. To give insight into this, consider the case in which the distribution of types $F(a)$ has mass points at a' and $\overline{a} > a'$, with all other types having negligible probability weights, and where $a' \geq \hat{\alpha}$ so that a continued relationship is feasible for type a'. In this case, any gain from delaying separation until after the first period is negligible, so candidates for optimality are:

1. set $a^1 = \overline{a}$, discontinuing the relationship for $a < \overline{a}$;
2. set $a^1 = a'$ and pool $\overline{a}$ with a';
3. set $a^1 = a'$ and $a^2 = \overline{a}$.

The first of these is a (degenerate) finest one-period partition contract for which it is optimal to set effort in every period at the highest feasible level for $\overline{a}$, $\hat{e}(\overline{a})$. For the second, it is optimal to set effort in every period at the highest feasible level for a', $\hat{e}(a')$. This is a finest one-period partition contract if and only if there is no $a^2 \leq \overline{a}$ that satisfies (27) for $a^1 = a'$ and $e_1(a^1, h_1) = \hat{e}(a^1)$. If there is such an a^2, it increases the joint gain to separate $\overline{a}$ from a' by setting $e_1(a^1, h_1) = \hat{e}(a^1)$ and setting $a^2 = \overline{a}$ because this gives higher effort for $\overline{a}$ in every period without reducing that for a' in any period and, with effort below the efficient level, an increase in effort increases the joint gain. This outcome is a finest one-period partition contract unless the a^2 that satisfies (27) with $e_1(a^1, h_1) = \hat{e}(a^1)$ is strictly less than $\overline{a}$. If there is no $a^2 \leq \overline{a}$ that satisfies (27) for $a^1 = a'$ and $e_1(a^1, h_1) = \hat{e}(a^1)$, it may still be optimal to separate $\overline{a}$ from a' by setting $e_1(a^1, h_1)$ such that (27) is satisfied for $a^2 = \overline{a}$ (which is optimal conditional on separation because, from (29), $e_1(a^2, h_1)$ is increasing in $e_1(a^1, h_1)$). That is also a finest one-period partition contract. So the only case in which a finest one-period partition contract is not optimal is when there exists $a^2 < \overline{a}$ that satisfies (27) for $a^1 = a'$ and $e_1(a^1, h_1) = \hat{e}(a^1)$.

Now suppose there is $a'' \in (a', \overline{a})$ with some probability mass for which (26) is satisfied for $a^1 = a'$ and $a^2 = a''$ for some $e_1(a^1, h_1) \leq \hat{e}(a')$ but for which there is no $a^3 \leq \overline{a}$ that satisfies (28) for $a^2 = a''$. Then a', a'' and $\overline{a}$ cannot all be separated in a single period. Suppose though it is possible to separate $\overline{a} = a^2$ from $a' = a^1$ in period 1 for some $e_1(a^1, h_1)$. It might then, depending on the magnitudes of their probability weights, increase the joint gain to pool a'' with a' in period 1 in order to separate $\overline{a}$ from a'' and then separate a'' from a' in period 2. So it is not always optimal to do all separation in one period.

Even if not all separation takes place in the first period, however, Proposition 9 gives conditions under which the optimal separation of types is into pools of sub-intervals. Moreover, Proposition 8 implies bounds on the coarseness of the sub-intervals independent of the distribution of agent types.

Proposition 11 *Consider $c(\tilde{e},a)=\hat{c}(\tilde{e})/a$ and $\hat{e}(a)<e^{*}(a)$ for all $a\in[\underline{a},\overline{a}]$ with $\underline{a}>0$. If, in an equilibrium partition contract with optimal continuation, all separation that is going to occur has occurred and there remain n sub-intervals of agent types continuing the relationship, those sub-intervals are no coarser than implied by*

$$\frac{\hat{e}\left(a^{i+1}\right)-(\underline{u}+\underline{v})}{\hat{e}(a^{i})-(\underline{u}+\underline{v})}=\frac{1}{1-\delta},\quad i=1,\ldots,n.$$

7 Conclusion

This paper combines insights from the ratchet effect in the literature on procurement and dynamic enforcement in the literature on relational incentive contracts to analyse relational incentive contracts when the agent's type is privately known by the agent and is persistent over time. It differs from the models of Levin (2003) and MacLeod (2003), in which the agent's type is an *iid* random draw each period. Applied to employment, it generalizes the models of Shapiro and Stiglitz (1984) and MacLeod and Malcomson (1989) to private information about workers' disutility of effort. Provided the relationship is sufficiently productive, there always exists a pooling contract in which the agent ends the relationship if the disutility of effort is too high but otherwise, whatever the agent's type, the agent provides the same effort and the principal pays the same remuneration. Some separation between agent types who continue the relationship may be feasible — necessary and sufficient conditions for this are derived. If such additional separation of agent types is feasible, an optimal pooling contract is always dominated by a contract with some separation of agent types who continue the relationship.

With relational contracts for which future actions are optimal once the agent's type is fully revealed (that is, are at any point on the Pareto frontier following full revelation of a type), it may not be possible to achieve any separation of types that continue the relational contract and full separation of such types is not feasible. Thus, the ratchet effect result that some pooling of agent types is unavoidable applies even though the parties are not legally constrained from committing to future contract terms and the principal does not have the power to make "take it or leave it" contract offers. These results significantly extend the set of circumstances under which persistent types are pooled beyond the traditional ratchet effect.

The essential reason behind this result is the following. In a relational contract with the agent's type fully revealed and efficient effort unattainable, optimal effort is the highest effort consistent with dynamic enforcement, that is, consistent with the agent being willing to choose that effort over choosing zero effort. That makes the agent's payoff following revelation of type the same as from choosing zero effort. Thus a less productive type that chooses the effort for a more productive type in the period they are to be separated can, by choosing zero effort in the following period, attain the same future payoff as the more productive type. So there is no difference in *future* payoff that makes the effort for the more productive type attractive to that type and not to the less productive type. But, as with the ratchet effect, there is a difference in future informational rent that makes it attractive for the more productive type to imitate the less productive type. Because monetary rewards are equally attractive to both types, the only way to make the effort for the more productive type attractive to that type but not to the less productive type is to accompany monetary reward by sufficiently higher effort to be unattractive to less productive types because of their higher disutility of effort. But effort that is continuous in type is not sufficient for this because the cost of the additional effort intended for the more productive type goes to zero for less productive types sufficiently close to the more productive type. It is thus outweighed by the difference in informational rent from the more productive type imitating the less productive. To overcome that, there has to be a discrete jump in the effort for the more productive type no matter how close the types are. But then not all types can be fully separated because, with continuous types, it is not possible to have a discrete jump in effort between each of them. Moreover, where an individual type can be fully separated but can also be pooled with a marginally lower type without detriment to other types, pooling them is better. This is because the jump in effort to separate the marginally lower type from yet lower types more than outweighs the marginal reduction in effort for the higher type.

When some separation is feasible, agent types (though continuous) can be partitioned, with all types within a sub-interval of the partition delivering the same performance, but the maximum number of sub-intervals is finite. This is similar in spirit to a result in MacLeod and Malcomson (1988), also for relational contracts with persistent private information about a continuum of agent types. But there restrictions on rewards and punishments drive the partitioning. Here those restrictions are removed, so the result is more fundamental.

Where it is possible to separate agent types who continue the relationship, an additional cost to inducing separation is that effort in the period in which types are revealed is, for all but the least productive relationships, necessarily below the level that could be sustained without private information. As in MacLeod and Malcomson (1989) and Levin (2003), remuneration consists of two components, one that does not depend on performance and a bonus that does. Once separation has been completed, a higher fixed component goes with the agent receiving more of the gains from the relationship.

Appendix: Proofs

Proof of Proposition 1. That (7) defines a best response continuation rule follows from the specification of $\alpha_t(h_t)$ in (3).

Effort function $e_t(a,h_t)$ may not be a best response at t because agent type a prefers to deviate to either (1) $\tilde{e} = e_t(a',h_t) \neq e_t(a,h_t)$ for some $a' \in A_t^+(h_t)$ or (2) $\tilde{e} \neq e_t(a',h_t)$ for any $a' \in A_t^+(h_t)$. Given a relational contract, let $\check{U}(a',a,h_t)$ denote the maximand in (2) for agent type a choosing $\tilde{e} = e_t(a',h_t)$. Incentive compatibility to the first type of deviation corresponds to $U_t(a,h_t) = \check{U}(a,a,h_t)$. That in turn corresponds to

$$U_t(a,h_t) \geq \check{U}\left(a',a,h_t\right) = U_t\left(a',h_t\right) + \check{U}\left(a',a,h_t\right) - \check{U}\left(a',a',h_t\right), \quad \forall a,a' \in A_t^+(h_t),$$

and, with the roles of a and a' interchanged,

$$U_t\left(a',h_t\right) \geq \check{U}\left(a,a',h_t\right) = U_t(a,h_t) + \check{U}\left(a,a',h_t\right) - \check{U}(a,a,h_t), \quad \forall a,a' \in A_t^+(h_t).$$

These two conditions imply

$$\begin{aligned}\check{U}(a,a,h_t) - \check{U}\left(a,a',h_t\right) &\geq U_t(a,h_t) - U_t\left(a',h_t\right) \\ &\geq \check{U}\left(a',a,h_t\right) - \check{U}\left(a',a',h_t\right), \quad \forall a,a' \in A_t^+(h_t). \quad \text{(A.1)}\end{aligned}$$

For $\tilde{U}_t$ defined in (6), $\check{U}(a,a,h_t) - \check{U}(a,a',h_t) = \tilde{U}_t(a,a,h_t) - \tilde{U}_t(a,a',h_t)$ for $a,a' \in A_t^+(h_t)$. The same holds with a and a' interchanged. So (A.1) implies (8) is necessary. It also implies (8) is sufficient to deter deviation to $\tilde{e} = e_t(a',h_t) \neq e_t(a,h_t)$ for $a' \in A_t^+(h_t)$.

Now consider deviation to $\tilde{e} \neq e_t(a',h_t)$ for any $a' \in A_t^+(h_t)$. To simplify notation in the proof, let h'_{t+1} denote the history at $t+1$ conditional on the agent choosing effort $e_t(\alpha_t(h_t),h_t)$ and the principal paying the corresponding bonus. Formally

$$h'_{t+1} = h_t \cup (e_t(\alpha_t(h_t),h_t), w_t(h_t, e_t(\alpha_t(h_t),h_t))).$$

With the specified continuation contracts, $\beta_t(h_t,\tilde{e}) = 1$, $w_t(h_t,\tilde{e}) = \underline{w}_t(h_t)$ and

$$\underline{w}_{t+1}(h_t \cup (\tilde{e}, \underline{w}_t(h_t))) \leq -\left[\tilde{U}_{t+1}\left(\alpha_t(h_t), \alpha_t(h_t), h'_{t+1}\right) - \underline{u} + \underline{w}_{t+1}\left(h'_{t+1}\right)\right]. \quad \text{(A.2)}$$

The payoff gain at stage 1 of period t to $a \geq \alpha_t(h_t)$ continuing the relationship while deviating to $\tilde{e}$ would, given (A.2), be

$$\begin{aligned}
&-c(\tilde{e},a)-\underline{u}+\underline{w}_t(h_t)\\
&\qquad+\delta\left[\tilde{U}_{t+1}\left(\alpha_t(h_t),a,h'_{t+1}\right)-\underline{u}+\underline{w}_{t+1}\left(h'_{t+1}\right)+\underline{w}_{t+1}(h_t\cup(\tilde{e},\underline{w}_t(h_t)))\right]\\
&\quad\leq -c(\tilde{e},a)-\underline{u}+\underline{w}_t(h_t)+\delta\left[\tilde{U}_{t+1}\left(\alpha_t(h_t),a,h'_{t+1}\right)-\tilde{U}_{t+1}\left(\alpha_t(h_t),\alpha_t(h_t),h'_{t+1}\right)\right].
\end{aligned}$$

Choice of $\tilde{e}$ affects only the first term, so this payoff gain cannot be greater than for $\tilde{e}=0$ so that $c(\tilde{e},a)=0$. Thus, not deviating to $\tilde{e}$ is a best response if

$$U_t(a,h_t)\geq -\underline{u}+\underline{w}_t(h_t)+\delta\left[\tilde{U}_{t+1}\left(\alpha_t(h_t),a,h'_{t+1}\right)-\tilde{U}_{t+1}\left(\alpha_t(h_t),\alpha_t(h_t),h'_{t+1}\right)\right],\quad \text{for all } a\in A_t^+(h_t). \tag{A.3}$$

For $a'=\alpha_t(h_t)$, (8) implies

$$U_t(a,h_t)\geq \tilde{U}_t(\alpha_t(h_t),a,h_t)-\tilde{U}_t(\alpha_t(h_t),\alpha_t(h_t),h_t)+U_t(\alpha_t(h_t),h_t),\quad \text{for all } a\in A_t^+(h_t). \tag{A.4}$$

So, if the right-hand side of (A.4) is greater than that of (A.3), (8) is sufficient to deter $a\in A_t^+(h_t)$ from deviating to $\tilde{e}\neq e_t(a',h_t)$ for any $a'\in A_t^+(h_t)$.

Type $a\geq\alpha_t(h_t)$ imitating $\alpha_t(h_t)$ at t cannot receive a lower payoff than from continuing to imitate $\alpha_t(h_t)$ at $t+1$. So, from (2) and (6),

$$\begin{aligned}
\tilde{U}_t(\alpha_t(h_t),a,h_t)\geq{}& -c(e_t(\alpha_t(h_t),h_t),a)+\delta\left[w_t(h_t,e_t(\alpha_t(h_t),h_t))-\underline{w}_{t+1}\left(h'_{t+1}\right)\right.\\
&\left.-\underline{u}+\underline{w}_{t+1}\left(h'_{t+1}\right)+\tilde{U}_{t+1}\left(\alpha_t(h_t),a,h'_{t+1}\right)\right],\ \text{for all } a\geq\alpha_t(h_t),
\end{aligned}$$

and this holds with equality for $a=\alpha_t(h_t)$. Thus

$$\begin{aligned}
&\tilde{U}_t(\alpha_t(h_t),a,h_t)-\tilde{U}_t(\alpha_t(h_t),\alpha_t(h_t),h_t)\\
&\quad\geq -c(e_t(\alpha_t(h_t),h_t),a)+c(e_t(\alpha_t(h_t),h_t),\alpha_t(h_t))\\
&\qquad+\delta\left[\tilde{U}_{t+1}\left(\alpha_t(h_t),a,h'_{t+1}\right)-\tilde{U}_{t+1}\left(\alpha_t(h_t),\alpha_t(h_t),h'_{t+1}\right)\right],\ \text{for all } a\in A_t^+(h_t).
\end{aligned}$$

With $c_2(\tilde{e},a)<0$ for $\tilde{e}\in(0,\overline{e}]$, (A.4) therefore implies

$$\begin{aligned}
U_t(a,h_t)\geq{}&\delta\left[\tilde{U}_{t+1}\left(\alpha_t(h_t),a,h'_{t+1}\right)-\tilde{U}_{t+1}\left(\alpha_t(h_t),\alpha_t(h_t),h'_{t+1}\right)\right]\\
&+U_t(\alpha_t(h_t),h_t),\quad \text{for all } a\in A_t^+(h_t),
\end{aligned}$$

and so, because (3) requires $U_t(\alpha_t(h_t),h_t)\geq\underline{w}_t(h_t)-\underline{u}$, (A.4) implies (A.3). ■

Lemma 1 *Suppose a relational contract specifies* $e_\tau(a'',h''_\tau)\geq e_\tau(a',h'_\tau)$ *for all* $\tau\geq t$*,* *where $a', a'' \in A_t(h_t)$ and h'_τ, h''_τ are the histories at $\tau \geq t$ from choosing the effort paths for a' and a'' respectively from t on when the parties adhere to the relational contract. If choosing $e_\tau(a'', h''_\tau)$ for all $\tau \geq t$ yields as high a payoff gain to agent type a'' at t as choosing $e_\tau(a', h'_\tau)$, then it does so for all agent types $a \in A_t(h_t)$ with $a \geq a''$.*

Proof. Let $\check{U}(a', a, h_t)$ denote the maximand in (2) for agent type a choosing $\tilde{e} = e_t(a', h_t)$ given the relational contract. Then the proposition certainly holds if

$$\check{U}\left(a'', a, h_t\right) - \check{U}\left(a', a, h_t\right) \geq \check{U}\left(a'', a'', h_t\right) - \check{U}\left(a', a'', h_t\right),\ \forall a \in A_t(h_t), a \geq a'',$$

or, re-arranging, if

$$\check{U}\left(a'', a, h_t\right) - \check{U}\left(a'', a'', h_t\right) \geq \check{U}\left(a', a, h_t\right) - \check{U}\left(a', a'', h_t\right),\ \forall a \in A_t(h_t), a \geq a''.$$

For $\tilde{U}_t$ defined in (6), $\check{U}(\tilde{a}, a, h_t) - \check{U}(\tilde{a}, a'', h_t) = \tilde{U}_t(\tilde{a}, a, h_t) - \tilde{U}_t(\tilde{a}, a'', h_t)$ for any $a, \tilde{a} \in A_t(h_t)$, so this condition is equivalent to

$$\tilde{U}_t\left(a'', a, h_t\right) - \tilde{U}_t\left(a'', a'', h_t\right) \geq \tilde{U}_t\left(a', a, h_t\right) - \tilde{U}_t\left(a', a'', h_t\right),\ \forall a \in A_t(h_t), a \geq a''. \quad \text{(A.5)}$$

For $\tilde{a} \in \{a', a''\}$ generating history $\tilde{h}_t$,

$$\tilde{U}_t(\tilde{a}, a, h_t) - \tilde{U}_t\left(\tilde{a}, a'', h_t\right) = \sum_{\tau=t}^{\infty} \delta^{\tau-t}\left[c\left(e_\tau\left(\tilde{a}, \tilde{h}_\tau\right), a''\right) - c\left(e_\tau\left(a, \tilde{h}_\tau\right), a\right)\right].$$

By Assumption 1, $c_{12}(\tilde{e}, a) < 0$ for all $(\tilde{e}, a)$. Hence, (A.5) holds when $e_\tau(a'', h''_\tau) \geq e_\tau(a', h'_\tau)$ for all $\tau \geq t$. ■

Proof of Proposition 2. From the definition of $P_t(a, (h_t, e_t))$, the principal does at least as well by continuing the relationship ($\beta_t(h_t, e_t) = 1$) and paying the bonus $w_t(h_t, e_t) - \underline{w}_t(h_t)$ as by ending it if $E_{a|h_{t}, e_t}\left[P_t(a, (h_t, e_t))\right] \geq 0$. Moreover, if ending the relationship at t ($\beta_t(h_t, e_t) = 0$), the principal clearly cannot gain by paying a bonus at t. Suppose the principal were to continue the relationship at t ($\beta_t(h_t, e_t) = 1$) but pay $w_t \neq w_t(h_t, e_t)$ when $w_t(h_t, e_t) - \underline{w}_t(h_t) > 0$. Under the specified continuation contract, the principal would receive non-positive payoff gain from continuation and so, given (9), would not make a greater payoff gain than from paying $w_t(h_t, e_t)$. ■

Definition 3 *A* stationary pooling continuation contract for h_τ *has the continuation contracts following deviation in Propositions 1 and 2 and, for $a \in A_t^+(h_t)$:*

1. *every agent type a choose the same effort $e_t = e(a_\tau^-(a))$ at $t \geq \tau$;*

2. *the principal continue the relationship at $t \geq \tau$ if $e_t = e(a_\tau^-(a))$.*

Lemma 2 *The following apply to stationary pooling continuation contracts for h_τ:*

1. *There exists a stationary pooling continuation contract for h_τ that is a continuation equilibrium at τ if and only if*

$$\delta e\big(a_\tau^-(a)\big) - c\big(e\big(a_\tau^-(a)\big), a\big) - \delta\left(\underline{u}+\underline{v}\right) \geq 0, \quad \textit{for all } a \in A_\tau^+(h_\tau), \tag{A.6}$$

or, equivalently,

$$S_{t-1}^2\big(a_\tau^-(a)\big) \geq c\big(e\big(a_\tau^-(a)\big), a_\tau^-(a)\big), \quad \textit{for all } a \in A_\tau^+(h_\tau) \textit{ and } t \geq \tau. \tag{A.7}$$

2. *Consider continuation payoff gains $P_t(a_\tau^-(a)) \geq 0$ and $U_t(a_\tau^-(a)) \geq 0$ consistent with (14) and independent of $t \geq \tau$. If $e(a_\tau^-(a))$ satisfies (A.6) and either $A_\tau^+(h_\tau) = A_\tau(h_\tau)$ or $U_t(a_\tau^-(a)) = 0$, there exists a stationary pooling continuation contract for h_τ with $w_t(e(a_\tau^-(a)))$ and $\underline{w}_t$ independent of $t \geq \tau$ that is a continuation equilibrium for h_τ with those continuation payoff gains.*

3. *To be a continuation equilibrium for h_τ, any stationary pooling continuation contract for h_τ for which $e(a_\tau^-(a))$ satisfies (A.6) with equality for $a = a_\tau^-(a)$ has $P_t(a_\tau^-(a)) = 0$, $U_t(a_\tau^-(a)) = \underline{w}_t - \underline{u} \geq 0$ and*

$$c\big(e\big(a_\tau^-(a)\big), a_\tau^-(a)\big) = S_{t-1}^2\big(a_\tau^-(a)\big), \textit{ for all } t \geq \tau. \tag{A.8}$$

Proof. Part 1: Necessity. For stationary pooling continuation contracts for h_τ, $S_{t-1}^2(a)$ is stationary, and $\gamma_t(a) = \beta_t(e(a)) = 1$, for all $a \in A_\tau^+(h_\tau)$ and all $t \geq \tau$. Then, from (11) and (12),

$$S_{t-1}^2(a) = \frac{\delta}{1-\delta}\left[e(a) - c(e(a), a) - \underline{u} - \underline{v}\right], \text{ for all } a \in A_\tau^+(h_\tau) \text{ and } t \geq \tau. \tag{A.9}$$

Combined with the budget balance constraint (14), this gives

$$\begin{aligned} &\delta e(a) - c(e(a), a) - \delta\left(\underline{u}+\underline{v}\right) \\ &\qquad = (1-\delta)\Big[U_t(a) + \underline{u} - \underline{w}_t + P_t(a)\Big], \quad \text{for all } a \in A_\tau^+(h_\tau) \text{ and } t \geq \tau. \end{aligned} \tag{A.10}$$

All $a \in A_\tau^+(h_\tau)$ have the same history for $t \geq \tau$, so $P_t(a)$ is the same. Thus, from (9), continuation of the relationship requires $P_t(a) \geq 0$ for $a \in A_\tau^+(h_\tau)$ and $t \geq \tau$. Moreover, from (3) and Proposition 1, continuation by agent types $a \in A_\tau^+(h_\tau)$ implies

$$U_t(a) \geq \max\left[0, \underline{w}_t - \underline{u}\right], \quad \text{for all } a \in A_\tau^+(h_\tau) \text{ and } t \geq \tau. \tag{A.11}$$

Together with (A.10), these imply that, for $e(a_\tau^-(a))$ to be a continuation equilibrium at τ for all $a \in A_\tau^+(h_\tau)$, it must satisfy (A.6). (A.9) for $a = a_\tau^-(a)$ implies that (A.7) is equivalent to (A.6) for $a = a_\tau^-(a)$. That and $c_2(\tilde{e}, a) < 0$ for $\tilde{e} > 0$ imply (A.6) is satisfied for all $a \geq a_\tau^-(a)$ if it is satisfied for $a_\tau^-(a)$, so (A.7) and (A.6) are equivalent.

Sufficiency. Suppose $e(a_\tau^-(a))$ satisfies (A.6) and consider the stationary pooling continuation contract for h_τ that has $e_t(a) = e(a_\tau^-(a))$, $\gamma_t(a) = \beta_t(e(a_\tau^-(a))) = 1$, $w_t(e(a_\tau^-(a))) = w$, and $\underline{w}_t = \underline{w}$ for all $a \in A_\tau^+(h_\tau)$ and $t \geq \tau$ for some w and $\underline{w}$ with $w \geq \underline{w}$. Under this continuation contract, payoff gains are stationary. It follows from (2) for the agent, and a corresponding calculation for the principal, that

$$U_t(a) = \frac{-c(e(a_\tau^-(a)), a) - \underline{u} + \underline{w} + (w - \underline{w})}{1-\delta}, \quad \text{for all } a \in A_\tau^+(h_\tau), \, t \geq \tau, \tag{A.12}$$

$$P_t(a) = \frac{-(w - \underline{w}) + \delta\left[e(a_\tau^-(a)) - \underline{v} - \underline{w}\right]}{1-\delta}, \quad \text{for all } a \in A_\tau^+(h_\tau), \, t \geq \tau. \tag{A.13}$$

With (A.6) satisfied, there certainly exist $w \geq \underline{w}$ such that $U(a) \geq \max[0, \underline{w} - u]$ and $P(a) \geq 0$ for all $a \in A_\tau^+(h_\tau)$, specifically when

$$c\big(e\big(a_\tau^-(a)\big), a_\tau^-(a)\big) + \underline{u} - \underline{w} \leq w - \underline{w} \leq \delta\left[e\big(a_\tau^-(a)\big) - \underline{v} - \underline{w}\right] \tag{A.14}$$

because, with $c_2(\tilde{e}, a) < 0$ for $\tilde{e} > 0$, $c(e, a)$ is decreasing in a. With the continuation contracts for deviation to $\tilde{e} \neq e(a_\tau^-(a))$ in Proposition 1, the only conditions for agent types $a_\tau^-(a)$ to continue the relationship for $t \geq \tau$ are those in (A.11). With $U_t(a)$ necessarily non-decreasing in a, this is sufficient to ensure that (A.11) is satisfied for all $a \in A_\tau^+(h_\tau)$ and, by Proposition 1, do not deviate to $\tilde{e} \neq e(a_\tau^-(a))$. That and the condition in (9) for it to be a best response for the principal to continue the relationship are thus satisfied for the specified stationary pooling continuation contract. Moreover, with the continuation contracts for deviation to $w_t \neq w$ in Proposition 2, it is a best response for the principal to pay w. Thus the specified stationary pooling continuation contract for h_τ is a continuation equilibrium for h_τ.

Part 2. The stationary pooling continuation contract specified in the proof of sufficiency for Part 1 has $w_t(e(a_\tau^-(a)))$ and $\underline{w}_t$ independent of $t \geq \tau$. Consider $\underline{w} = \underline{u}$. Then w satisfies (A.14) if

$$\underline{u} + c\big(e\big(a_\tau^-(a)\big), a_\tau^-(a)\big) \leq w \leq \underline{u} + \delta\left[e\big(a_\tau^-(a)\big) - \underline{v} - \underline{u}\right].$$

By choosing w appropriately between the upper and lower bounds in this, $U_t(a_\tau^-(a))$ and $P_t(a_\tau^-(a))$ specified by (A.12) and (A.13) can take on any non-negative values independent of $t \geq \tau$ that are consistent with (A.10) and thus (A.6). That establishes the result for $A_\tau^+(h_\tau) = A_\tau(h_\tau)$. For $A_\tau^+(h_\tau) \subset A_\tau(h_\tau)$, it may need to be that $U_t(a_\tau^-(a)) = 0$ for $a \in A_\tau^+(h_\tau)$ for $a \notin A_\tau^+(h_\tau)$ not to continue the relationship.

Part 3. For (A.6) to hold with equality for $a = a_\tau^-(a)$, the right-hand side of (A.10) must be zero for $a = a_\tau^-(a)$. That is consistent with $P_t(a) \geq 0$ and (A.11) only if $P_t(a_\tau^-(a)) = 0$ and $U_t(a_\tau^-(a)) = \underline{w}_t - \underline{u} \geq 0$. (A.8) follows from the equivalence of (A.6) and (A.7). ■

Lemma 3 *Consider a for which (15) is satisfied and let*

$$\hat{e}(a) = \arg\max_{\tilde{e}} [\tilde{e} - c(\tilde{e}, a)] \text{ subject to (16).}$$

1. *Either $\hat{e}(a) = e^*(a)$ or $\hat{e}(a)$ satisfies (16) with equality.*

2. *If $\hat{e}(a)$ satisfies (16) with equality, $\hat{e}(a)$ increases with δ.*

Proof. If (16) is not a binding constraint, $\hat{e}(a) = e^*(a)$ by the definition of $e^*(a)$. That establishes Part 1. Kuhn-Tucker necessary conditions for the maximization problem, with λ a multiplier attached to (16), are

$$\begin{aligned} 1 - c_1(\hat{e}(a), a) + \lambda\, [\delta - c_1(\hat{e}(a), a)] &= 0 \\ \lambda\, [\delta \hat{e}(a) - c(\hat{e}(a), a) - \delta\, (\underline{u} + \underline{v})] &= 0 \\ \lambda \geq 0,\ \text{(16) satisfied.} & \end{aligned}$$

From these,

$$\lambda = -\frac{1 - c_1(\hat{e}(a), a)}{\delta - c_1(\hat{e}(a), a)},$$

so $\lambda \geq 0$ implies $c_1(\hat{e}(a), a) \in (\delta, 1]$. For $\hat{e}(a)$ satisfying (16) with equality, differentiation yields

$$\frac{\partial \hat{e}(a)}{\partial \delta} = -\frac{\hat{e}(a) - (\underline{u} + \underline{v})}{\delta - c_1(\hat{e}(a), a)},$$

which must be strictly positive because for (16) to hold requires $\hat{e}(a) > \underline{u} + \underline{v}$. ■

Proof of Proposition 3. The proof proceeds in two steps. Step 1 shows that, if (15) is satisfied for a, there exists an equilibrium stationary pooling continuation contract for h_τ for a and establishes the proposition for continuation contracts restricted to the class of stationary pooling continuation contracts, see Definition 3. Step 2 establishes that no continuation contract with non-stationary effort can do as well, so *any* optimal continuation contract must have stationary effort.

Step 1. If (15) is satisfied for a, there certainly exists $e(a)$ that satisfies (16) and thus satisfies (A.6) for $a_\tau^-(a) = a$. By Part 1 of Lemma 2, (A.6) is sufficient for there to exist a stationary pooling continuation contract for h_τ that is a continuation equilibrium for a (in this case with only one type in the pool). That establishes Part 1 of the proposition.

By Part 1 of Lemma 2, (A.6) is also necessary for there to exist a stationary pooling continuation contract for h_τ that is a continuation equilibrium for a. Thus an optimal continuation contract from that class must satisfy (16), as specified in Part 2 of the proposition. It must, moreover, maximise $S^1_\tau(a)$ subject to (16). For stationary pooling continuation contracts for h_τ with $\gamma_\tau(a) = 1$, maximizing $S^1_\tau(a)$ corresponds to maximizing $S^2_{\tau-1}(a)$ from (12). For such contracts, $S^2_t(a)$ for $t \geq \tau - 1$, given by (A.9), is a continuous function of $e(a)$ to be maximised by choice of $e(a)$ from the compact set defined by (16), so an optimal $e(a)$ certainly exists. It is, moreover, independent of $w_t(e(a))$ and $\underline{w}_t$. Because $A^+_\tau(h_\tau) = A_\tau(h_\tau)$ for a the only type with history h_τ, Part 2 of Lemma 2 suffices to complete the proof of Part 2 of the proposition for the class of stationary pooling continuation contracts for h_τ.

If efficient effort $e^*(a)$ satisfies (16), then clearly that maximizes $S^2_{\tau-1}(a)$ subject to (16). If not, $S^2_{\tau-1}(a)$ is maximized by effort $\hat{e}(a)$ defined in Lemma 3 that satisfies (16) with equality. It then follows from Part 3 of Lemma 2 with $a^-_\tau(a) = a$ that an optimal stationary pooling continuation contract for h_τ has $P_t(a) = 0$ and $U_t(a) = \underline{w}_t - \underline{u} \geq 0$ for all $t \geq \tau$, and has effort $\hat{e}(a)$ that satisfies (17) for all $t \geq \tau$. That establishes Part 3 of the proposition for the class of stationary pooling continuation contracts for h_τ.

Step 2. Now consider whether it is possible to achieve as high or higher $S^2_{\tau-1}(a)$ with a continuation contract that has non-stationary effort. That cannot be the case if efficient effort $e^*(a)$ is attainable because efficient effort is stationary, non-stationary effort must depart from efficient effort for some $t \geq \tau$ and that must lower $S^2_{\tau-1}(a)$. Consider, therefore, an optimal contract among the class of those with stationary effort $\hat{e}(a) < e^*(a)$ and let $\hat{S}^2(a)$ denote the joint gain $S^2_{\tau-1}(a)$ with such a contract from τ on. From Levin (2003, Theorem 2), no non-stationary continuation equilibrium contract can achieve a higher joint gain than $\hat{S}^2(a)$ at $\tau - 1$. The same applies to any $t \geq \tau$ so, for any optimal continuation contract, $S^2_t(a) = \hat{S}^2(a)$. Thus any optimal continuation contract must satisfy the budget balance constraint (14) with $S^2_t(a) = \hat{S}^2(a)$. For any continuation contract with non-stationary effort to do as well as one with stationary effort $\hat{e}(a) < e^*(a)$, it must have $e_t(a) > \hat{e}(a)$ for some $t \geq \tau$. By step 1, an optimal contract with stationary effort $\hat{e}(a) < e^*(a)$ has $P_t(a) = 0$ and $U_t(a) = \underline{w}_t - \underline{u}$ for all $t \geq \tau$. It is thus not possible to satisfy (14) with $S^2_t(a) = \hat{S}^2(a), P_t(a) \geq 0, U_t(a) \geq \underline{w}_t - \underline{u}$ (as required by (3)) and $\beta_t(\hat{e}(a)) = 1$ for all $t \geq \tau$ with $e_t(a) > \hat{e}(a)$ for some $t \geq \tau$. Thus, any optimal continuation contract for h_τ must have $e_t(a) = \hat{e}(a)$ for all $t \geq \tau$. ■

Proof of Proposition 4. With (15) satisfied for $a^-_\tau(a)$, a continuation contract for h_τ optimal for agent type $a^-_\tau(a)$ satisfies Proposition 3 with $a^-_\tau(a)$ substituted for a. Such a contract has stationary effort $\hat{e}(a^-_\tau(a))$ and satisfies (16) for $a = a^-_\tau(a)$. With

$c_2 \leq 0$, it also satisfies (A.6). It is, therefore, a stationary pooling contract for h_τ that satisfies (A.6) and so, by Lemma 2, is a continuation equilibrium for all $a \in A_\tau^+(h_\tau)$. For $\hat{e}(a_\tau^-(a)) < e^*(a_\tau^-(a))$, $\hat{e}(a_\tau^-(a))$ is the highest stationary effort sustainable for $a_\tau^-(a)$. Thus it is the highest stationary effort sustainable subject to the best response constraints for all $a \in A_\tau^+(h_\tau)$. Because efficient effort $e^*(a)$ is non-decreasing, $\hat{e}(a_\tau^-(a)) < e^*(a_\tau^-(a))$ implies $\hat{e}(a_\tau^-(a)) < e^*(a)$ for $a \in A_\tau^+(h_\tau)$. Thus, under the conditions of the proposition, $\hat{e}(a_\tau^-(a))$ is optimal for all $a \in A_\tau^+(h_\tau)$. Then, by Part 3 of Lemma 2, $U_t(a_\tau^-(a)) = \underline{w}_t - \underline{u}$ for all $t \geq \tau$ and (21) follows from (2). ■

Proof of Proposition 5. It follows from Lemma 2 that, provided $a < a'$ end the relationship at $\tau = 1$, there exists a relational contract with the properties specified if and only if (A.6) with a' substituted for $a_\tau^-(a)$ is satisfied for $A_\tau^+(h_\tau) = [a', \overline{a}]$. With $c_2 \leq 0$, condition (22) is necessary and sufficient for that. Lemma 2 also ensures there exists such a contract for which $U_t(a') = 0$ for all t so, because $U_t(a)$ is necessarily strictly increasing in a for a given strictly positive effort sequence, that contract ensures the best response condition in (7) for $a < a'$ to end the relationship in the first period is satisfied. Thus all the conditions are satisfied for the contract specified to be a continuation equilibrium in period 1. Moreover, from the definition of $S_t^1(a)$ in (11), that (22) is satisfied implies that $S_t^1(a) \geq 0$ for all $a \geq a'$, and $S_t^1(a) > 0$ for all $a > a'$, for all t and hence, from (13), that $S_0 > 0$ for $\alpha_1 = a'$. There thus exists a w_0 for which both parties gain from starting the relational contract, establishing the proposition. ■

Lemma 4 *Consider $[a', a''] \subseteq A_t^+(h_t)$ and continuation contracts for h_t, with continuation histories denoted by*

$$\hat{h}_{t+1}(a) = h_t \cup (e_t(a(h_t), h_t), w_t(h_t, e_t(a(h_t), h_t))), \qquad \textit{for } a \in [a', a''],$$

for which, for $\tilde{a} \in (a', a'']$ separated from all $a \in [a', \tilde{a})$ at t, $U_{t+1}(a, \hat{h}_{t+1}(a'))$ is differentiable with respect to a and $U_{t+1}(a, \hat{h}_{t+1}(\tilde{a})) = U_{t+1}(\tilde{a}, \hat{h}_{t+1}(\tilde{a}))$, for $a \in [a', \tilde{a}]$. Necessary conditions for there to exist such a continuation contract for h_t that is a continuation equilibrium for all $a \in [a', a'']$ and that separates agent type a'' from agent type a' at t for given $e_t(a', h_t) > 0$ are that there exists an $\tilde{a} \in (a', a'']$ and $\tilde{e}$ such that

$$c_2(\tilde{e}, \tilde{a}) \leq c_2(e_t(a', h_t), \tilde{a}) - \delta \frac{\partial}{\partial a} \left[U_{t+1}(a, \hat{h}_{t+1}(a')) \right]_{a=\tilde{a}} \tag{A.15}$$

and

$$c(\tilde{e}, \tilde{a}) + \Big\{ c(e_t(a', h_t), a') - c(e_t(a', h_t), \tilde{a}) + \delta \left[U_{t+1}(\tilde{a}, \hat{h}_{t+1}(a')) \right.$$

$$-U_{t+1}\left(a',\hat{h}_{t+1}\left(a'\right)\right)\Big]\Bigg\} \leq S_t^2(\tilde{a}) - P_t(\tilde{a}) - U_t\left(a',h_t\right) - \underline{u} + \underline{w}_t(h_t)\,. \quad \text{(A.16)}$$

For a' satisfying (15) and $\hat{e}(a) < e^*(a)$ for all $a \in [a',a'']$, (A.16) and

$$c_2(\tilde{e},a) \leq c_2\left(e_t\left(a',h_t\right),a\right) - \delta\frac{\partial U_{t+1}\left(a,\hat{h}_{t+1}(a')\right)}{\partial a}, \quad \textit{all } a \in \left[a',\tilde{a}\right], \qquad \text{(A.17)}$$

satisfied for some $\tilde{e}$, $P_t(a') \geq 0$, $U_t(a',h_t) \geq \max\left[0,\underline{w}_t(h_t) - \underline{u}\right]$, and $\tilde{a} \in (a',a'']$ are sufficient for there to exist continuation contracts for h_t with the characteristics specified that are continuation equilibria for all agent types in $[a',a'']$ for all $\tau \geq t$ and separate agent type a'' from type a' at t for given $e_t(a',h_t) \leq \hat{e}(a')$. In particular, there exist equilibrium continuation contracts for h_t with effort functions conditional on the relationship continuing

$$e_t(a,h_t) = \begin{cases} e_t(a',h_t)\,, & \textit{for } a \in [a',\tilde{a})\,, \\ \tilde{e}, & \textit{for } a \in [\tilde{a},a''] \cup \left\{a \mid a^-_{t+1}(a) = \tilde{a}\right\}, \end{cases} \qquad \text{(A.18)}$$

$$e_\tau(a,h_\tau) = \begin{cases} \hat{e}(a')\,, & \textit{for } a \in [a',\tilde{a})\,, \\ \hat{e}(\tilde{a})\,, & \textit{for } a \in [\tilde{a},a''] \cup \left\{a \mid a^-_{t+1}(a) = \tilde{a}\right\}, \end{cases} \quad \textit{for all } \tau > t, \qquad \text{(A.19)}$$

with $\tilde{e}$ such as to satisfy (A.16) and (A.17).

Proof. For a continuation contract for h_τ to separate a'' from a' at t, either a'' is the lowest type in $(a',a'']$ separated or there exists some lower type $\tilde{a}$ separated from all $a \in [a',\tilde{a})$. For $\tilde{a}$ the lowest type separated from a' at t, $e_t(a,h_t) = e_t(a',h_t)$ for $a \in [a',\tilde{a})$. Moreover, under the conditions specified in the lemma, $\beta_t(h_t,e_t(a,h_t)) = 1$ and $U_{t+1}\left(a,\hat{h}_{t+1}(\tilde{a})\right) = U_{t+1}\left(\tilde{a},\hat{h}_{t+1}(\tilde{a})\right)$ for $a \in [a',\tilde{a})$. So condition (8) for $\tilde{a}$ to choose $e_t(\tilde{a},h_t)$ and $a \in [a',\tilde{a})$ to choose $e_t(a,h_t) = e_t(a',h_t)$ at t, corresponds to

$$\begin{aligned} &-c\left(e_t\left(a',h_t\right),a\right) + \delta U_{t+1}\left(a,\hat{h}_{t+1}\left(a'\right)\right) + c\left(e_t\left(a',h_t\right),\tilde{a}\right) - \delta U_{t+1}\left(\tilde{a},\hat{h}_{t+1}\left(a'\right)\right) \\ &\qquad \geq U_t(a,h_t) - U_t(\tilde{a},h_t) \\ &\qquad\qquad \geq -c(e_t(\tilde{a},h_t),a) + c(e_t(\tilde{a},h_t),\tilde{a}), \quad \text{all } a \in \left[a',\tilde{a}\right), \end{aligned}$$

and, with re-arrangement,

$$\begin{aligned} &c(e_t(\tilde{a},h_t),a) - c(e_t(\tilde{a},h_t),\tilde{a}) \\ &\quad \geq U_t(\tilde{a},h_t) - U_t(a,h_t) \\ &\qquad \geq c\left(e_t\left(a',h_t\right),a\right) - c\left(e_t\left(a',h_t\right),\tilde{a}\right) \\ &\qquad\quad + \delta\left[U_{t+1}\left(\tilde{a},\hat{h}_{t+1}\left(a'\right)\right) - U_{t+1}\left(a,\hat{h}_{t+1}\left(a'\right)\right)\right], \quad \text{all } a \in \left[a',\tilde{a}\right). \quad \text{(A.20)} \end{aligned}$$

By Assumption 1, $c(\tilde{e}, a)$ is differentiable, and hence continuous, in a and, as specified in the statement of the lemma, so is $U_{t+1}\left(a, \hat{h}_{t+1}(a')\right)$, so to satisfy (A.20) requires $U_t(\tilde{a}, h_t) = \lim_{a \to \tilde{a}-} U_t(a, h_t)$.

Necessity. By Proposition 1, (8) is necessary for best response efforts and, hence, (A.20) with $e_t(\tilde{a}, h_t) \neq e_t(a', h_t)$ is necessary for separation of $\tilde{a}$ from $a \in [a', \tilde{a})$ under the conditions of the lemma. For (A.20) to hold with $U_t(\tilde{a}, h_t) = \lim_{a \to \tilde{a}-} U_t(a, h_t)$, the derivative of the left-hand side with respect to a must be no greater than the derivative of the right-hand side when both are evaluated at $a = \tilde{a}$. This implies (A.15) is necessary for some $\tilde{e} = e_t(\tilde{a}, h_t)$. Use of the right-hand inequality in (A.20) for $a = a'$ to substitute for $U_t(\tilde{a}, h_t)$ given by (14) and of $\beta_t(e_t(\tilde{a}, h_t)) = 1$ implies

$$U_t\left(a', h_t\right) + c\left(e_t\left(a', h_t\right), a'\right) - c\left(e_t\left(a', h_t\right), \tilde{a}\right) + \delta\left[U_{t+1}\left(\tilde{a}, \hat{h}_{t+1}\left(a'\right)\right) - U_{t+1}\left(a', \hat{h}_{t+1}\left(a'\right)\right)\right] \leq -c(e_t(\tilde{a}, h_t), \tilde{a}) - \underline{u} + \underline{w}_t + S_t^2(\tilde{a}) - P_t(\tilde{a}).$$

Re-arrangement yields the necessary condition

$$\begin{aligned} c(e_t(\tilde{a}, h_t), \tilde{a}) + c\left(e_t\left(a', h_t\right), a'\right) - c\left(e_t\left(a', h_t\right), \tilde{a}\right) \\ + \delta\left[U_{t+1}\left(\tilde{a}, \hat{h}_{t+1}\left(a'\right)\right) - U_{t+1}\left(a', \hat{h}_{t+1}\left(a'\right)\right)\right] \\ \leq S_t^2(\tilde{a}) - P_t(\tilde{a}) - U_t\left(a', h_t\right) - \underline{u} + \underline{w}_t(h_t). \end{aligned} \tag{A.21}$$

To satisfy (A.21), it is necessary that $\tilde{e} = e_t(\tilde{a}, h_t)$ satisfies (A.16).

Sufficiency. Consider $\tau > t$ for continuation contracts for h_t with effort functions satisfying (A.18) and (A.19). Continuation contracts for h_τ with the efforts in (A.19) satisfy Proposition 4, so $U_{t+1}\left(\tilde{a}, \hat{h}_{t+1}(\tilde{a})\right) = \underline{w}_{t+1}\left(, \hat{h}_{t+1}(a')\right) - \underline{u} \geq 0$ and

$$U_{t+1}\left(a, \hat{h}_{t+1}\left(a'\right)\right) = U_t\left(a', \hat{h}_{t+1}\left(a'\right)\right) + \frac{1}{1-\delta}\left[c\left(\hat{e}\left(a'\right), a'\right) - c\left(\hat{e}\left(a'\right), a\right)\right], \quad \text{for } a \geq a', a \in A_t^+(h_t), \tag{A.22}$$

which is differentiable with respect to a. Moreover, type $a \in [a', \tilde{a})$ taking the action for $\tilde{a}$ in period t can achieve payoff gain $U_{t+1}\left(a, \hat{h}_{t+1}(\tilde{a})\right) = \underline{w}_{t+1}\left(\hat{h}_{t+1}(a')\right) - \underline{u}$ at $t+1$ by continuing the relationship for $t+1$, collecting the fixed wage $\underline{w}_{t+1}\left(\hat{h}_{t+1}(a')\right)$, setting $e_{t+1} = 0$ and quitting for $t+2$. With $U_{t+1}\left(a, \hat{h}_{t+1}(\tilde{a})\right)$ necessarily non-decreasing in a, type a could clearly not do better than type $\tilde{a}$ at $t+1$ from taking the action for $\tilde{a}$ at t. So $U_{t+1}\left(a, \hat{h}_{t+1}(\tilde{a})\right) = \underline{w}_{t+1}\left(\hat{h}_{t+1}(a')\right) - \underline{u} = U_{t+1}\left(\tilde{a}, \hat{h}_{t+1}(\tilde{a})\right)$. Thus the contract has the characteristics specified for the lemma. By the same argument, the stationary pooling continuation contract for $a \in [a', \tilde{a})$ is not a continuation equilibrium for h_τ for $a < a'$, so no $a < a'$ can have the same history as a' for all $\tau > t$. For stationary effort, $S_\tau^2(a)$ is given by (18). Because $\hat{e}(a)$ satisfies (16) with equality, and hence (17), for $a = a', \tilde{a}$,

$$S_\tau^2(a) = c\left(\hat{e}\left(a'\right), a'\right) + \frac{\delta}{1-\delta}\left[c\left(\hat{e}\left(a'\right), a'\right) - c\left(\hat{e}\left(a'\right), a\right)\right], \quad \text{for } a \in \left[a', \tilde{a}\right), \tau > t,$$
$$S_\tau^2(\tilde{a}) = c(\hat{e}(\tilde{a}), \tilde{a}), \quad \text{for } \tau > t.$$

Now consider period t conditional on these continuation equilibria for $\tau > t$. The efforts specified in (A.18) and (A.19) are the same for all $a \in [a', \tilde{a})$ for all $\tau \geq t$, so it must be that $P_t(a) = P_t(a')$ for $a \in [a', \tilde{a})$. Moreover, for $e_t(a', h_t) \leq \hat{e}(a')$, there exist equilibrium payments such that $P_t(a') \geq 0$. Thus, with the efforts in (A.18) and (A.19), the expressions just given for $S_\tau^2(\tilde{a})$ and $S_\tau^2(a)$, and $\beta_t(e_t(\tilde{a})) = 1$, (14) can be written for $a \in [a', \tilde{a})$ and $\tilde{a}$ respectively as

$$U_t(a, h_t) = -c\left(e_t\left(a', h_t\right), a\right) - \underline{u} + \underline{w}_t + c\left(\hat{e}\left(a'\right), a'\right)$$
$$+ \frac{\delta}{1-\delta}\left[c\left(\hat{e}\left(a'\right), a'\right) - c\left(\hat{e}\left(a'\right), a\right)\right] - P_t\left(a'\right), \text{ for } a \in \left[a', \tilde{a}\right), \quad \text{(A.23)}$$
$$U_t(\tilde{a}, h_t) = -c(\tilde{e}, \tilde{a}) - \underline{u} + \underline{w}_t + c(\hat{e}(\tilde{a}), \tilde{a}) - P_t(\tilde{a}). \quad \text{(A.24)}$$

Moreover, from (A.23) for $a = a'$,

$$P_t\left(a'\right) = c\left(\hat{e}\left(a'\right), a'\right) - c\left(e_t\left(a', h_t\right), a'\right) - U_t\left(a', h_t\right) - \underline{u} + \underline{w}_t(h_t), \quad \text{(A.25)}$$

so

$$U_t(a, h_t) = -c\left(e_t\left(a', h_t\right), a\right) + c\left(e_t\left(a', h_t\right), a'\right)$$
$$+ \frac{\delta}{1-\delta}\left[c\left(\hat{e}\left(a'\right), a'\right) - c\left(\hat{e}\left(a'\right), a\right)\right] + U_t\left(a', h_t\right), \text{ for } a \in \left[a', \tilde{a}\right). \quad \text{(A.26)}$$

From (A.24) and (A.26), for $a \in [a', \tilde{a})$,

$$U_t(\tilde{a}, h_t) - U_t(a, h_t) = -\left[c(\tilde{e}, \tilde{a}) - c\left(e_t\left(a', h_t\right), a\right)\right] + \left[c(\hat{e}(\tilde{a}), \tilde{a}) - c\left(e_t\left(a', h_t\right), a'\right)\right]$$
$$- \frac{\delta}{1-\delta}\left[c\left(\hat{e}\left(a'\right), a'\right) - c\left(\hat{e}\left(a'\right), a\right)\right] - P_t(\tilde{a}) - U_t\left(a', h_t\right) - \underline{u} + \underline{w}_t(h_t). \quad \text{(A.27)}$$

Provided $U_t(a, h_t) \geq 0$, (8) is sufficient for $e_t(a, h_t)$ to be a best response for a by Proposition 1. For the efforts specified in (A.18) and (A.19), that implies (A.20) with $e_t(\tilde{a}, h_t) = \tilde{e}$ is sufficient for best response efforts by $\tilde{a}$ and $a \in [a', \tilde{a})$. From (A.27),

$$\frac{\partial}{\partial a}\left[U_t(\tilde{a}, h_t) - U_t(a, h_t)\right] = c_2\left(e_t\left(a', h_t\right), a\right) + \frac{\delta}{1-\delta} c_2\left(\hat{e}\left(a'\right), a\right), \quad \text{for } a \in \left[a', \tilde{a}\right),$$

which is just the derivative of the right-most term in (A.20) for the efforts in (A.18) and (A.19). So, for $U_t(\tilde{a}, h_t) = \lim_{a \to \tilde{a}-} U_t(a, h_t)$, the right-hand inequality in (A.20) is satisfied with equality for all $a \in [a', \tilde{a})$. With (A.17) satisfied, integration of both sides of (A.17) with respect to a from $a^0 \in [a', \tilde{a})$ to $\tilde{a}$, with $e_t(\tilde{a}, h_t)$ set equal to $\tilde{e} > e_t(a', h_t)$ and the efforts in (A.18) and (A.19), ensures that the left-most term in (A.20) is no less than the right-most term for $a = a^0 \in [a', \tilde{a})$. Thus the efforts in (A.18) and (A.19) for $\tilde{a}$ and for $a \in [a', \tilde{a})$ satisfy (A.20), and hence (8), and so are best responses.

Furthermore, from (A.27), $U_t(\tilde{a}, h_t) = \lim_{a \to \tilde{a}-} U_t(a, h_t)$ requires

$$c(\tilde{e}, \tilde{a}) + c\left(e_t\left(a', h_t\right), a'\right) - c\left(e_t\left(a', h_t\right), \tilde{a}\right) + \frac{\delta}{1-\delta}\left[c\left(\hat{e}\left(a'\right), a'\right) - c\left(\hat{e}\left(a'\right), \tilde{a}\right)\right]$$
$$= c\left(\hat{e}(\tilde{a}), \tilde{a}\right) - P_t(\tilde{a}) - U_t\left(a', h_t\right) - \underline{u} + \underline{w}_t(h_t).$$

That $\tilde{e}$ satisfies (A.16) for $P_t(\tilde{a}) \geq 0$ and $P_t(a') \geq 0$ imply that the principal's best response condition in (9) for continuing at t is satisfied for all $a \in [a', \tilde{a})$ and for all $a \in [\tilde{a}, a'']$. Moreover, with $c_2(\tilde{e}, a) \leq 0$, $U_t(a, h_t)$ is non-decreasing in a for $a \in [a', \tilde{a})$ from (A.26). Thus, with $U_t(\tilde{a}, h_t) = \lim_{a \to \tilde{a}-} U_t(a, h_t)$, the agent's condition in (7) for continuing the relationship at t is satisfied for all $a \in [a', \tilde{a}]$.

For any $a \in (\tilde{a}, a''] \cup \left\{a \mid a^-_{t+1}(a) = \tilde{a}\right\}$, Lemma 1 ensures $e_t(\tilde{a}, h_t) = \tilde{e}$ is preferred to $e_t(a', h_t)$ given that it is preferred by $\tilde{a}$. Proposition 1 ensures that, for the continuation contracts for deviation specified there, $e_t(\tilde{a}, h_t) = \tilde{e}$ is also preferred to any other effort. Thus the continuation contract for h_t specified is also a continuation equilibrium for $a \in (\tilde{a}, a''] \cup \left\{a \mid a^-_{t+1}(a) = \tilde{a}\right\}$. Under the conditions specified, therefore, there exists an equilibrium continuation contract for h_t satisfying the specifications of the lemma that separates $a'' \geq \tilde{a}$ from a'. ■

Lemma 5 *Effort function $\hat{e}(a)$, defined in (20), is strictly increasing for all $a \in [\hat{\alpha}, \overline{a}]$ and $c(\hat{e}(a), a)$ is strictly increasing in a for all $a \in [\hat{\alpha}, \overline{a}]$ such that $\hat{e}(a) < e^*(a)$.*

Proof. For $\hat{e}(a) = e^*(a)$, the first result follows immediately from $e^*(a)$ strictly increasing. For $\hat{e}(a) < e^*(a)$, from (16) and (20),

$$\delta \hat{e}(a) - c(\hat{e}(a), a) - \delta\left(\underline{u} + \underline{v}\right) = 0, \quad \text{for } a \in [\hat{\alpha}, \overline{a}], \tag{A.28}$$

so, differentiating with respect to a,

$$\left[\delta - c_1(\hat{e}(a), a)\right] \frac{\partial}{\partial a}\hat{e}(a) - c_2(\hat{e}(a), a) = 0, \quad \text{for } a \in [\hat{\alpha}, \overline{a}]. \tag{A.29}$$

Define $e^0(a)$ by

$$e^0(a) = \arg \max_{\tilde{e} \in [0, \overline{e}]} \left[\delta \tilde{e} - c(\tilde{e}, a)\right], \quad \text{for } a \in [\hat{\alpha}, \overline{a}]. \tag{A.30}$$

Because, by definition, $\hat{\alpha}$ is the lowest value of a for which (15) is satisfied, $\hat{e}(\hat{\alpha}) = e^0(\hat{\alpha})$. Because $c_2(\tilde{e},a) < 0$ for $\tilde{e} > 0$, the maximand in (A.30) is increasing in a. So $\hat{e}(a) > e^0(a)$ for all $a > \hat{\alpha}$ because $\hat{e}(a)$ is the maximum $e(a)$ that satisfies (16). But then the square bracket in (A.29) is certainly negative for $a > \hat{\alpha}$. Hence, with $c_2(\tilde{e},a) < 0$ for $\tilde{e} > 0$, it follows from (A.29) that $\partial\hat{e}(a)/\partial a$ is strictly positive for $a > \hat{\alpha}$. With $\hat{e}(a)$ strictly increasing in a, it follows from (A.28) that $c(\hat{e}(a),a)$ is too for all $a \in [\hat{\alpha},\overline{a}]$ such that $\hat{e}(a) < e^*(a)$. ■

Proof of Proposition 6. Part 1. The continuation efforts for agent type a for $\tau > t$ specified in the proposition satisfy Proposition 4, which implies that the continuation payoff gains of type a for $\tau > t$ satisfy the conditions of Lemma 4 and also

$$\frac{dU_{t+1}(a)}{da}|_{a^-_{t+1}(a)} = -\frac{1}{1-\delta}c_2\left(\hat{e}\left(a^-_{t+1}(a)\right),a\right).$$

By Lemma 4, a necessary condition for a to be separated from a' is that there exists $\tilde{a} \in (a',a]$ and $\tilde{e}$ satisfying (A.15). For the specification in the proposition, this becomes

$$c_2(\tilde{e},\tilde{a}) \leq c_2\left(e_t\left(a',h_t\right),\tilde{a}\right) + \frac{\delta}{1-\delta}c_2\left(\hat{e}\left(a^-_{t+1}(a')\right),\tilde{a}\right). \tag{A.31}$$

By Assumption 1, $c_2(\tilde{e},a) < 0$ for $\tilde{e} \in (0,\overline{e}]$ and $c_{12}(\tilde{e},a) < 0$. It follows from (A.31) that $\tilde{e} - e_t(a',h_t)$ is bounded below by some $\varepsilon > 0$. By definition of $a^-_{t+1}(a)$, $e_t(a,h_t) = e_t\left(a^-_{t+1}(a),h_t\right)$ for all a. If $a^-_{t+1}(a) = \tilde{a}$, then $e_t(a,h_t) = e_t(\tilde{a},h_t) = \tilde{e} \geq e_t(a',h_t) + \varepsilon$. If $a^-_{t+1}(a) > \tilde{a}$, then $a^-_{t+1}(a)$ is separated from $\tilde{a}$ and hence also from $a^-_{t+1}(\tilde{a})$, so (A.31) must be satisfied when $a^-_{t+1}(a)$ is substituted for $\tilde{a}$ and $a^-_{t+1}(\tilde{a})$ for a'. That requires $e_t\left(a^-_{t+1}(a),h_t\right) > e_t\left(a^-_{t+1}(\tilde{a}),h_t\right)$. Thus $e_t(a,h_t) = e_t\left(a^-_{t+1}(a),h_t\right) > e_t\left(a^-_{t+1}(\tilde{a}),h_t\right) = e_t(\tilde{a},h_t) = \tilde{e} \geq e_t(a',h_t) + \varepsilon$.

Part 2. From Part 1, for full separation $e_t(a,h_t)$ must have an upward jump discontinuity at every $a \in [\underline{a}_t,\overline{a}_t]$. It must, therefore, be monotone. But for a monotone function defined on an interval, the set of jump continuities is at most countable, which results in a contradiction because the set of $a \in [\underline{a}_t,\overline{a}_t]$ is uncountable.

Part 3. Consider first $\underline{a}_t$. Then $e_t(\underline{a}_t,h_t) \leq \hat{e}(\underline{a}_t)$ because, by Proposition 4, $\hat{e}(\underline{a}_t)$ is the highest sustainable effort for $\underline{a}_t$ in an equilibrium continuation contract for h_t with $\hat{e}(\underline{a}_t) < e^*(\underline{a}_t)$. Now consider $a \in (\underline{a}_t,\overline{a}_t]$. One possibility is that a is not separated from $\underline{a}_t$ at t, in which case $e_t(a,h_t) = e_t(\underline{a}_t,h_t) \leq \hat{e}(\underline{a}_t) < \hat{e}(a)$, the final inequality following because, by Lemma 5, $\hat{e}(a)$ is strictly increasing in a. The other possibility is that a is separated from $\underline{a}_t$ at t, in which case $a^-_{t+1}(a)$ must be separated from $\underline{a}_t$ at t because $e_t\left(a^-_{t+1}(a),h_t\right) = e_t(a,h_t)$ by the definition of $a^-_{t+1}(a)$ having the same history at $t+1$ as a. That $e_t\left(a^-_{t+1}(a),h_t\right) < \hat{e}\left(a^-_{t+1}(a)\right)$ follows from (A.16). The term in braces on its left-hand side is necessarily positive for $\tilde{a} > a'$. Also, $U_t(\underline{a}_t) \geq \max[0, \underline{w}_t - \underline{u}]$, $P_t\left(a^-_{t+1}(a)\right) \geq 0$ and, by Proposition 4, $S^2_t\left(a^-_{t+1}(a)\right)$ is given by (17) for $a = a^-_{t+1}(a)$, so the right-hand side cannot be greater than $c\left(\hat{e}\left(a^-_{t+1}(a)\right), a^-_{t+1}(a)\right)$. But then $e_t(a,h_t) = e_t\left(a^-_{t+1}(a), h_t\right) < \hat{e}\left(a^-_{t+1}(a)\right) < \hat{e}(a)$, again with the final inequality following from $\hat{e}(a)$ strictly increasing in a.

Part 4. By Part 1, a necessary condition for a to be separated from a' under the conditions of the proposition is that there exists $\tilde{a} \in (a', a]$ and $\tilde{e}$ satisfying (A.31). By Part 3, $\tilde{e} < \hat{e}(\tilde{a})$. By Assumption 1, $c_2(\tilde{e}, a) < 0$ for $\tilde{e} \in (0, \overline{e}]$ and $c_{12}(\tilde{e}, a) < 0$, so there can exist an $\tilde{a} \in (a', a]$ and an $\tilde{e}$ satisfying (A.31) only if

$$c_2(\hat{e}(\tilde{a}), \tilde{a}) < \frac{\delta}{1-\delta} c_2\left(\hat{e}\left(a^-_{t+1}(a')\right), \tilde{a}\right).$$

Also by Assumption 1, c is twice continuously differentiable, so c_2 is continuous and so is $\hat{e}(a)$. Thus, with $c_2(\tilde{e}, a) < 0$ for $\tilde{e} \in (0, \overline{e}]$, there can exist an $\tilde{a} \in (a', a]$ that satisfies this condition as a approaches a' only if $\delta/(1-\delta) < 1$; that is $\delta < 1/2$. ■

Proof of Proposition 7. Under Assumption 1, if there is more than one agent type for which a mutually beneficial relational contract is possible, there exists a non-degenerate interval of such types $\left[a^0, a''\right]$ with $a'' > a^0$ for which $\hat{e}(a) < e^*(a)$ for all $a \in \left[a^0, a''\right]$. Suppose it were possible to separate all types for which a mutually beneficial relational contract is possible. Then it must be possible to separate all $a \in \left[a^0, a''\right]$.

Consider an equilibrium with a non-degenerate interval $[\underline{a}_t(h_t), \overline{a}_t(h_t)] \subseteq \left[a^0, a''\right]$ of types a with the same history h_t for which $\hat{e}(a) < e^*(a)$, where t is the period in which full separation of $a \in \left[a^0, a''\right]$ first occurs. If all types $a \in [\underline{a}_t(h_t), \overline{a}_t(h_t)]$ were fully separated at t, optimal continuation contracts would, by Proposition 3, result in $e_\tau(a, h_\tau) = \hat{e}(a)$ for all $\tau > t$, so the assumptions of Proposition 6 would be satisfied. But then Part 2 of Proposition 6 would imply that full separation of $a \in [\underline{a}_t(h_t), \overline{a}_t(h_t)]$ is not feasible, a contradiction that establishes the proposition for this case.

The alternative to an equilibrium in which full separation follows directly from the non-degenerate interval $[\underline{a}_t(h_t), \overline{a}_t(h_t)] \subseteq \left[a^0, a''\right]$ of types a with the same history h_t is one in which $a \in [\underline{a}_t(h_t), \overline{a}_t(h_t)]$ are separated at t into pools that do not include an interval of types, so every pooled type is separated from its immediate neighbours, with full separation occurring only in some later period. Consider such an equilibrium, with $a, a' \in [\underline{a}_t(h_t), \overline{a}_t(h_t)]$ with $a' > a$ to be separated into different pools at t. A necessary condition for this separation is that (8) is satisfied for some $e_t(a', h_t) \neq e_t(a, h_t)$. Let $k(a)$ denote the number of periods after t for which a has payoff gain strictly greater than $\underline{w}_\tau - \underline{u}$ for $t < \tau \leq k(a)$, which must be finite if a is eventually to be fully separated with optimal continuation because, by Part 3 of Proposition 3, the payoff gain to a once fully separated at τ is $\underline{w}_\tau - \underline{u}$. Also, let $\hat{k}(a,a')$ denote the number of periods after t for which it is optimal for a to continue imitating a' conditional on deviating by doing so at t. Certainly $\hat{k}(a,a') \leq k(a')$ because, if a' has payoff gain $\underline{w}_\tau - \underline{u}$ from continuing the relationship at $\tau > t$ with strictly positive effort, $a < a'$ has payoff gain strictly less than $\underline{w}_\tau - \underline{u}$ from continuing to imitate a at τ but can get payoff gain $\underline{w}_\tau - \underline{u}$ by choosing zero effort. Under these conditions, (8) corresponds to

$$\begin{aligned}
-\sum_{i=0}^{k(a)} \delta^i \left[c(e_{t+i}(a),a) - c\left(e_{t+i}(a),a'\right)\right] - \frac{\delta^{k(a)+1}}{1-\delta}\left[c(\hat{e}(a),a) - c\left(\hat{e}(a),a'\right)\right] \\
\geq U_t(a,h_t) - U_t\left(a',h_t\right) \\
\geq -\sum_{i=0}^{\hat{k}(a,a')} \delta^i \left[c\left(e_{t+i}\left(a'\right),a\right) - c\left(e_{t+i}\left(a'\right),a'\right)\right] + \sum_{i=\hat{k}(a,a')+1}^{k(a')} \delta^i c\left(e_{t+i}\left(a'\right),a'\right), \\
\text{for } a,a' \in [\underline{a}_t(h_t),\overline{a}_t(h_t)], a < a'. \qquad \text{(A.32)}
\end{aligned}$$

By Assumption 1, $c(\tilde{e},a)$ is continuous and differentiable in a. So, by the Mean Value Theorem, there exist $a_{t+i}, a'_{t+i}, a'' \in (a,a')$ such that

$$\begin{aligned}
c\left(e_{t+i}(a),a'\right) - c(e_{t+i}(a),a) &= c_2(e_{t+i}(a),a_{t+i})\left(a'-a\right) \\
c\left(e_{t+i}\left(a'\right),a'\right) - c\left(e_{t+i}\left(a'\right),a\right) &= c_2\left(e_{t+i}\left(a'\right),a'_{t+i}\right)\left(a'-a\right) \\
c\left(\hat{e}(a),a'\right) - c(\hat{e}(a),a) &= c_2\left(\hat{e}(a),a''\right)\left(a'-a\right).
\end{aligned}$$

Use of these in (A.32), division by $a'-a > 0$, and re-arrangement gives the requirement

$$\begin{aligned}
\sum_{i=0}^{k(a)} \delta^i c_2(e_{t+i}(a),a_{t+i}) - \sum_{i=0}^{\hat{k}(a,a')} \delta^i c_2\left(e_{t+i}\left(a'\right),a'_{t+i}\right) \\
\geq -\frac{\delta^{k(a)+1}}{1-\delta} c_2\left(\hat{e}(a),a''\right) + \sum_{i=\hat{k}(a,a')+1}^{k(a')} \delta^i \frac{c\left(e_{t+i}\left(a'\right),a'\right)}{a'-a}, \\
\text{for } a,a' \in [\underline{a}_t(h_t),\overline{a}_t(h_t)], a < a'. \qquad \text{(A.33)}
\end{aligned}$$

For a' to be separated from its immediate neighbours, (A.33) must hold in the limit as $a \to a'$. As a gets close a', $\hat{k}(a,a') = k(a')$ because as long as a' has payoff gain strictly greater than $\underline{w}_\tau - \underline{u}$, a sufficiently close to a' also does from imitating a', so the final term on the right-hand side of (A.33) is zero. Thus, with $a_{t+i}, a'_{t+i}, a'' \in (a,a')$, (A.33) can, in the limit as $a \to a'$, be written

$$\lim_{a \to a'} \sum_{i=0}^{k(a)} \delta^i c_2(e_{t+i}(a),a) - \sum_{i=0}^{k(a')} \delta^i c_2\left(e_{t+i}\left(a'\right),a'\right) \geq -\lim_{a \to a'} \frac{\delta^{k(a)+1}}{1-\delta} c_2(\hat{e}(a),a). \qquad \text{(A.34)}$$

Define

$$h(a) = -\sum_{i=0}^{k(a)} \delta^i c_2(e_{t+i}(a), a).$$

Then (A.34) implies

$$h(a') - \lim_{a \to a'} h(a) \geq -\lim_{a \to a'} \frac{\delta^{k(a)+1}}{1-\delta} c_2(\hat{e}(a), a). \tag{A.35}$$

Because $c_2(\tilde{e}, a) < 0$ for $\tilde{e} > 0$, the right-hand side of (A.35) is strictly positive for $k(a)$ finite, so $h(a)$ must have an upward jump discontinuity at a'. But for all agent types to be separated from their immediate neighbours, (A.35) must hold for all $a' \in (\underline{a}_t(h_t), \overline{a}_t(h_t)]$. Thus, for full separation to occur, $h(a)$ must have an upward jump discontinuity at every $a \in (\underline{a}_t(h_t), \overline{a}_t(h_t)]$. Such a function is certainly monotone, so the set of such continuities is at most countable, a contradiction because the set of a in $(\underline{a}_t(h_t), \overline{a}_t(h_t)]$ is uncountable. ■

Proof of Proposition 8. Consider a contract that is identical to the equilibrium relational contract in the proposition except that it separates $a \in [a', \overline{a}_t]$ from $a \in [\underline{a}_t, a')$ in period t but pools within these intervals, with continuation efforts for $\tau > t$ of $e_\tau(a, h_\tau) = \hat{e}(\underline{a}_t)$ for $a \in [\underline{a}_t, a')$ and $e_\tau(a, h_\tau) = \hat{e}(a')$ for $a \in [a', \overline{a}_t]$. By the definition of $\hat{e}(a)$ in (20), (A.6) is satisfied for $a_\tau^-(a) = \underline{a}_t$ with effort $\hat{e}(\underline{a}_t)$ and for $a_\tau^-(a) = a'$ with effort $\hat{e}(a')$, so Lemma 2 ensures that there exist equilibrium continuation contracts for $\tau > t$ with these continuation efforts. For all $a \in [\underline{a}_t, a')$, $e_\tau(a, h_\tau)$ for $\tau > t$ is no further from the efficient level $e^*(a)$. By Lemma 5, the effort function $\hat{e}(a)$, defined by (20), is strictly increasing for $a \in [\hat{\alpha}, \overline{a}_t]$, so $\hat{e}(a') > \hat{e}(\underline{a}_t)$. Thus, for all $a \in [a', \overline{a}_t]$, $e_\tau(a, h_\tau)$ for $\tau > t$ is strictly closer to the efficient level $e^*(a)$. So the joint gain for $\tau > t$ is strictly greater for $a \in [a', \overline{a}_t]$, and no less for $a \in [\underline{a}_t, a')$, with this relational contract than with the original equilibrium contract.

Now consider period t. Under the conditions of the proposition, $a' \in (\underline{a}_t, \overline{a}_t]$ and $\tilde{e} = e_t(a', h_t)$ satisfy the sufficient conditions in Lemma 4 for a' to be separated from $\underline{a}_t$ given $e_t(\underline{a}_t, h_t)$. From Part 1 of Proposition 6 with a' substituted for a and $\underline{a}_t$ for a', $e_t(a', h_t) > e_t(\underline{a}_t, h_t)$. Moreover, by Lemma 4, there then exists an equilibrium continuation contract for h_t with pooling in period t of all $a \in [a', \overline{a}_t]$ with effort $e_t(a', h_t)$, and of all types $a \in [\underline{a}_t, a')$ with effort $e_t(\underline{a}_t, h_t)$, and the equilibrium continuation contract specified above for $\tau > t$. Then, for all $a \in [\underline{a}_t, a')$, $e_t(a, h_t)$ is no further from the efficient level $e^*(a)$ and, for all $a \in [a', \overline{a}_t]$, $e_t(a, h_t)$ is strictly closer to the efficient level $e^*(a)$ than with the original equilibrium continuation contract. Thus $S_t^1(a)$ is increased for $a \in [a', \overline{a}_t]$ without being reduced for $a \in [\underline{a}_t, a')$ by the additional separation. ■

Proof of Proposition 9. Denote by $S_t^s(a)$ the joint gain for a from t on for the contract in the proposition with a'' fully separated and by $S_t^p(a)$ that for the alternative contract with a' pooled with a''. The overall joint gains for all $a \in [a', a'']$ from t on are

$$JG^i(a', a'') \equiv \int_{a'}^{a''} S_t^i(a)\, dF(a), \quad \text{for } i \in \{s, p\}.$$

For $S_t^i(a)$ Riemann integrable and $F(a)$ continuous, $JG^i(a', a'')$ is continuous in a' by a standard result (see Apostol (1957, Theorem 9-31, p. 214)), so $\lim_{a' \to a''} JG^i(a', a'') = 0$. Thus $JG^p(a', a'') > JG^s(a', a'')$ as $a' \to a''$ if the left derivative of the former with respect to a' is less than the left derivative of the latter as $a' \to a''$. That is

$$\lim_{a' \to a''} \left\{ \left[-S_t^p(a') + S_t^s(a') \right] dF(a') \right\} < 0.$$

Let a^0 be the lowest type with which a' is pooled at t under the contract that fully separates a''. (Formally, $a^0 = \left(a_{t+1}^-(a'), h_t \right)$ given that contract.) $S_t^s(a')$ cannot be greater than if the continuation contract were to separate $a \in [a', a'')$ from a^0 in the following period (at $t+1$) without changing the continuation contract for $a \in A_t^+(h_t) - [a', a'']$. (That may not be an equilibrium continuation contract but a continuation equilibrium could not give higher joint gain than if it were.) $S_t^p(a')$ cannot be less than if $a \in [a', a'']$ are pooled from t on with effort $e_t(a', h_t)$ at t and effort $\hat{e}(a')$ thereafter, which is feasible under the conditions of the proposition. (With $e_t\left(a^0, h_t\right)$ unchanged from the contract that fully separates a'', (A.15) and (A.16) with a' replaced by $a_{t+1}^-(\tilde{a})$ are unchanged for $\tilde{a} \in [\alpha_t(h_t), a')$, and lowering $e_t(a'', h_t)$ and reducing $e_\tau(a'', h_\tau)$ from $\hat{e}(a'')$ to $\hat{e}(a')$ for $\tau > t$ relaxes (A.15) and (A.16) with a' replaced by a'' for $\tilde{a} > a''$.) Thus

$$\begin{aligned}
&\left[-S_t^p(a') + S_t^s(a') \right] dF(a') \\
&\quad \leq \Bigg\{ \left[e_t\left(a^0, h_t\right) - c\left(e_t\left(a^0, h_t\right), a'\right) \right] - \left[e_t\left(a', h_t\right) - c\left(e_t\left(a', h_t\right), a'\right) \right] \\
&\qquad + \delta \Big[\left[e_{t+1}\left(a', h_{t+1}\right) - c\left(e_{t+1}\left(a', h_{t+1}\right), a'\right) \right] \; - \; \left[\hat{e}(a') - c\left(\hat{e}(a'), a'\right) \right] \Big] \Bigg\} dF(a').
\end{aligned} \tag{A.36}$$

By Proposition 6, $e_t(a', h_t)$ is discretely greater than $e_t\left(a^0, h_t\right)$ for a' to separate from a^0 at t and $\hat{e}(a') > e_{t+1}(a', h_{t+1})$ when, as specified in the proposition, $e_\tau\left(a^0, h_\tau\right) = \hat{e}\left(a^0\right)$ for $\tau \geq t+1$. Moreover, because all these efforts are strictly less than efficient effort $e^*(a')$, $\tilde{e} - c(\tilde{e}, a)$ is strictly increasing in $\tilde{e}$ for $a \in [a', a'']$. Thus the expression on the right-hand side of (A.36) is strictly negative and there is, therefore, a strictly positive overall joint gain from pooling a' with a'' as $a' \to a''$. ∎

Proof of Proposition 10. Part 1. Efforts in a one-period partition contract satisfy (A.18)–(A.19) in Lemma 4 for $t=1$, $a''=a^i$ and $a'=a^{i-1}$ for all $i\geq 2$. With the continuity of $U_t(a,h_t)$ established there, (A.26) must therefore hold for $t=1$, $a''=a^i$ and $a'=a^{i-1}$ for all $i\geq 2$. With $c(\tilde{e},a)=\hat{c}(\tilde{e})/a$, (A.26) for $t=1$ then becomes

$$U_1\big(a^i,h_1\big)=\left[\hat{c}\big(e_1\big(a^{i-1},h_1\big)\big)+\frac{\delta}{1-\delta}\hat{c}\big(\hat{e}\big(a^{i-1}\big)\big)\right]\left[\frac{1}{a^{i-1}}-\frac{1}{a^i}\right] \\ +U_1\big(a^{i-1},h_1\big),\quad i=1,\ldots,n.$$

Recursive substitution for $U_1\big(a^{i-1},h_1\big)$ yields

$$U_1\big(a^i,h_1\big)=\sum_{j=1}^{i-1}\left[\hat{c}\big(e_1\big(a^j,h_1\big)\big)+\frac{\delta}{1-\delta}\hat{c}\big(\hat{e}\big(a^j\big)\big)\right]\left[\frac{1}{a^j}-\frac{1}{a^{j+1}}\right] \\ +U_1\big(a^1,h_1\big),\quad i=1,\ldots,n. \quad \text{(A.37)}$$

For $a<a^1$ to end the relationship at $t=1$ but a^1 to continue it requires $U_1\big(a^1,h_1\big)=0$ which, from (3), implies $\underline{w}_1\leq\underline{u}$ because only $a^1=\alpha_1(h_1)>\underline{a}$ can satisfy (15), as specified in the proposition. For $c(\tilde{e},a)=\hat{c}(\tilde{e})/a$ with $\underline{a}>0$ and the specified continuation efforts, (A.15) and (A.17) correspond to (24), and (A.16) to (25), for $t=1$ and $\tilde{e}=e_1(a,h_1)$ so, by Lemma 4, (24) and (25) with a replaced by a^{i+1} and a' by a^i are necessary and sufficient for a one-period partition contract to be an equilibrium contract. Condition (26) follows from these by substitution for $U_1\big(a^i,h_1\big)$ from (A.37) and use of $U_1\big(a^1,h_1\big)=0$, $\underline{w}_1\leq\underline{u}$ and $P_1\big(a^i\big)\geq 0$ for all $i=1,\ldots,n$. Since $e_1(a^n,h_1)<e^*(a^n)$, and increasing it for given a^n has, in (26), no knock-on effect on a^i for $i<n$ or the efforts for $a<a^n$, the joint gain S_0 is maximized by maximizing $e_1(a^n,h_1)$. That makes the left-hand inequality in (26) hold with equality for $i=n-1$.

Part 2. For there to exist $e_1\big(a^{i+1},h_1\big)$ that satisfies (26), the left-hand side of the first inequality must be no less than the right-hand side of the second. That requirement can be rearranged as

$$\frac{\hat{c}\big(\hat{e}\big(a^{i+1}\big)\big)}{a^{i+1}}\geq\left[\hat{c}\big(e_1\big(a^i,h_1\big)\big)+\frac{\delta}{1-\delta}\hat{c}\big(\hat{e}\big(a^i\big)\big)\right]\frac{1}{a^i} \\ +\sum_{j=1}^{i-1}\left[\hat{c}\big(e_1\big(a^j,h_1\big)\big)+\frac{\delta}{1-\delta}\hat{c}\big(\hat{e}\big(a^j\big)\big)\right]\left[\frac{1}{a^j}-\frac{1}{a^{j+1}}\right], \\ i=1,\ldots,n-1, \quad \text{(A.38)}$$

with the summation term zero for $i=1$. In a finest one-period partition contract, a^i is the lowest type that can be separated from a^{i-1}. With $\hat{e}(a)<e^*(a)$ for all $a\in[\underline{a},\overline{a}]$, it

follows from Lemma 5 that $c(\hat{e}(a), a)$, and hence $\hat{c}(\hat{e}(a))/a$ in the specification in the proposition, is strictly increasing in a for all a for which the relationship is continued. Thus, the a^{i+1} closest to a^i consistent with (A.38) has (A.38) hold with equality, provided this yields $a^{i+1} > a^i$. Moreover, the a^{i+1} that satisfies this requirement is unique. It also implies $\hat{c}\left(e_1\left(a^{i+1}, h_1\right)\right)/a^{i+1}$ is given by equality in the right-hand inequality in (26) and hence by (29).

By (16) and (20), for $\hat{e}(a) < e^*(a)$ as in the proposition and $c(\tilde{e}, a) = \hat{c}(\tilde{e})/a$,

$$\frac{\hat{c}(\hat{e}(a))}{a} = \delta\left[\hat{e}(a) - (\underline{u} + \underline{v})\right], \quad \text{for } a \in \left[a^1, \overline{a}\right]. \tag{A.39}$$

For $i = 1$, the summation term in (A.38) is zero. Thus, for $i = 1$, division of (A.38) holding with equality by $\hat{c}\left(\hat{e}\left(a^1\right)\right)/a^1$ and use of (A.39) yields (27), which certainly implies $a^2 > a^1$ for $\delta > 1/2$ because $\hat{c}\left(e_1\left(a^1, h_1\right)\right) \geq 0$ necessarily. For $i \geq 2$, subtraction of (A.38) for $i-1$ from (A.38) for i when both hold with equality yields

$$\begin{aligned}
&\frac{\hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)}{a^{i+1}} - \frac{\hat{c}\left(\hat{e}\left(a^{i}\right)\right)}{a^{i}} \\
&= \left[\hat{c}\left(e_1\left(a^i, h_1\right)\right) + \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a^i\right)\right)\right]\frac{1}{a^i} \\
&\quad + \sum_{j=1}^{i-1}\left[\hat{c}\left(e_1\left(a^j, h_1\right)\right) + \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a^j\right)\right)\right]\left[\frac{1}{a^j} - \frac{1}{a^{j+1}}\right] \\
&\quad - \left[\hat{c}\left(e_1\left(a^{i-1}, h_1\right)\right) + \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a^{i-1}\right)\right)\right]\frac{1}{a^{i-1}} \\
&\quad - \sum_{j=1}^{i-2}\left[\hat{c}\left(e_1\left(a^j, h_1\right)\right) + \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a^j\right)\right)\right]\left[\frac{1}{a^j} - \frac{1}{a^{j+1}}\right], \quad i = 2, \ldots, n-1,
\end{aligned}$$

or, cancelling terms under the summation signs,

$$\begin{aligned}
&\frac{\hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)}{a^{i+1}} - \frac{\hat{c}\left(\hat{e}\left(a^{i}\right)\right)}{a^{i}} \\
&\quad = \left[\hat{c}\left(e_1\left(a^i, h_1\right)\right) + \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a^i\right)\right)\right]\frac{1}{a^i} \\
&\qquad - \left[\hat{c}\left(e_1\left(a^{i-1}, h_1\right)\right) + \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a^{i-1}\right)\right)\right]\frac{1}{a^i}, \quad i = 2, \ldots, n-1.
\end{aligned}$$

Use of (29) to substitute for $\hat{c}\left(e_1\left(a^i, h_1\right)\right)$ in this yields

$$\begin{aligned}
&\frac{\hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)}{a^{i+1}} - \frac{\hat{c}\left(\hat{e}\left(a^{i}\right)\right)}{a^{i}} \\
&\quad = \left[\hat{c}\left(e_1\left(a^{i-1}, h_1\right)\right) + \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a^{i-1}\right)\right) + \frac{\delta}{1-\delta}\hat{c}\left(\hat{e}\left(a^i\right)\right)\right]\frac{1}{a^i}
\end{aligned}$$

$$-\left[\hat{c}\big(e_1\big(a^{i-1},h_1\big)\big)+\frac{\delta}{1-\delta}\hat{c}\big(\hat{e}\big(a^{i-1}\big)\big)\right]\frac{1}{a^i}$$

$$=\frac{\delta}{1-\delta}\frac{\hat{c}\big(\hat{e}\big(a^i\big)\big)}{a^i},\quad i=2,\ldots,n-1. \tag{A.40}$$

Division by $\hat{c}\big(\hat{e}\big(a^i\big)\big)/a^i$ and use of (A.39) yields (28), which certainly implies $a^{i+1} > a^i$. It also implies $\hat{e}\big(a^{i+1}\big) > \hat{e}\big(a^i\big)$ by an amount bounded away from zero. With $\hat{e}(\overline{a})$ bounded above by $e^*(\overline{a})$, that is sufficient to ensure a finite number of partitions.

Part 3. For further separation with the continuation effort specified given history h_2 generated by a finest one-period partition contract, there must exist an $a \in \big(a^i,a^{i+1}\big)$, for some $i=1,\ldots,n-1$, that can be separated from a^i when $e_t(a,h_t)=\hat{e}(a)$ for $t>2$. From Definition 1, the continuation contracts for h_2 are stationary pooling continuation contracts with $a^i=a_t^-(a)$ for $a\in[a^i,a^{i+1})$ and $e_t\big(a^i,h_t\big)=\hat{e}\big(a^i\big)$ for $i=1,\ldots,n$. So, from Lemma 2 with $a_\tau^-(a)=a^i$, $U_t\big(a^i,h_t\big)=\underline{w}_t(h_t)-\underline{u}$ for $t\geq 2$. It follows from (24) and (25) with a' replaced by a^i and from $P_t(a)\geq 0$ that, to be feasible to separate $a\in\big(a^i,a^{i+1}\big)$ from a^i at $t\geq 2$, it must be that

$$\left[\hat{c}\big(\hat{e}\big(a^i\big)\big)+\frac{\delta}{1-\delta}\hat{c}\big(\hat{e}\big(a^i\big)\big)\right]\frac{1}{a^i}\leq\frac{\hat{c}(\hat{e}(a))}{a} \tag{A.41}$$

or

$$\frac{1}{1-\delta}\frac{\hat{c}\big(\hat{e}\big(a^i\big)\big)}{a^i}\leq\frac{\hat{c}(\hat{e}(a))}{a}. \tag{A.42}$$

By Lemma 5 and $\hat{e}(a)<e^*(a)$ for all $a\in[\underline{a},\overline{a}]$, $\hat{c}(\hat{e}(a))/a$ is strictly increasing for all a for which the relationship will be continued, so that can hold only if $\hat{c}(\hat{e}(a))/a < \hat{c}\big(\hat{e}\big(a^{i+1}\big)\big)/a^{i+1}$. But (A.40) and (A.42) imply $\hat{c}(\hat{e}(a))/a\geq\hat{c}\big(\hat{e}\big(a^{i+1}\big)\big)/a^{i+1}$ for $i\geq 2$. For $i=1$, the summation term in (A.38) is zero and necessarily $\hat{c}\big(e_1\big(a^i,h_1\big)\big)\leq\hat{c}\big(\hat{e}\big(a^1\big)\big)$, so (A.38) and (A.41) imply $\hat{c}(\hat{e}(a))/a\geq\hat{c}\big(\hat{e}\big(a^1\big)\big)/a^1$. Thus there can exist no $a\in\big(a^i,a^{i+1}\big)$ for any $i=1,\ldots,n$ that can be separated from a^i at $t\geq 2$. ∎

Proof of Proposition 11. Let t be a date at which all separation that is going to occur under the equilibrium partition contract has occurred. Suppose, contrary to the claim in the proposition,

$$\frac{\hat{e}\big(a^{i+1}\big)-(\underline{u}+\underline{v})}{\hat{e}(a^i)-(\underline{u}+\underline{v})}>\frac{1}{1-\delta},\quad\text{for some } i\in\{1,\ldots,n\}.$$

By (20), for $\hat{e}(a)<e^*(a)$, it then follows from (16) that

$$\frac{1}{1-\delta}\frac{\hat{c}\big(\hat{e}\big(a^i\big)\big)}{a^i}<\frac{\hat{c}\big(\hat{e}\big(a^{i+1}\big)\big)}{a^{i+1}},\quad\text{for some } i\in\{1,\ldots,n\}.$$

By Proposition 4, optimal continuation implies $e_t\big(a^i,h_t\big)=\hat{e}\big(a^i\big)$ and also $U_t\big(a^i,h_t\big)=$

$\underline{w}_t(h_t) - \underline{u}$. By Lemma 5 and $\hat{e}(a) < e^*(a)$ for all $a \in [\underline{a}, \overline{a}]$, $\hat{c}(\hat{e}(a))/a$ is strictly increasing for all $a \in \left[a^i, a^{i+1}\right]$. So there then exists $a \in \left(a^i, a^{i+1}\right)$ that satisfies (24) and (25) for $a' = a^i$ and $P_t(a) = 0$ and so can be separated from a^i with effort $\hat{e}\left(a^i\right)$. Furthermore, by Proposition 8, separation of a from a^i is optimal, contradicting optimal continuation in the equilibrium partition contract. ■

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