Hedging Strategies Under Temporary Market Impact



Hendrik Brackmann

St. Catherine's College

University of Oxford

A thesis submitted in partial fulfillment of the MSc in

Mathematical Finance

June 21, 2013

Abstract

This thesis is concerned with the problem of hedging derivatives under temporary market impact. We are going to examine the problem of hedging derivatives in an optimal control setting similar to [14] and [10]. By considering specific combinations of volatility and liquidity specifications, we are able to obtain analytic solutions. Under the optimal control, the holdings in the stock exhibit a mean reverting behavior. While the reversion-speed is similar to the results in the literature, we are able to find a new characterization of the target portfolio as weighted average of future Δ-hedge portfolios. If the stock-price process is given by a Brownian motion, this characterization collapses to the Δ-hedge. In that case, the final hedging error is proportional to the variance of the Δ-hedge. Further, we extend the analysis by including drift using asymptotic expansions. We find that under the optimal control, the wealth process is mean-reverting towards the expectation of the derivative. The hedging portfolio adjusts more to the inclusion of drift if future hedging is less aggressive. The comparison with numerical solutions suggests that the approximations perform reasonably well.

Contents

1 Introduction **1**

2 Model Description **3**

2.1 Limit order book trading. 3

2.2 The wealth process 5

3 The Optimal Control Problem **6**

3.1 Problem formulation 6

3.2 The zero mean case 9

3.3 Hedging portfolio in the case of Brownian motion 13

3.4 Estimating the cost of the trading strategy 15

4 Asymptotic Expansions. **17**

4.1 A stylized example 17

4.2 Brownian motion with drift 18

4.2.1 Expansion of a_1, b_1, c_1 18

4.2.2 Comparison to numerical solution. 19

4.2.3 Expansion of b_2, c_2. 20

4.2.4 Interpretation of strategy adjustment 22

5 Conclusion **25**

Bibliography **26**

A Proof of theorem 3.2.1 **28**

A.1 Solution for weight function in case $n = 2$ 30

B Matlab-Code for comparison with numerical solution **31**

C Mathematica-Code for symbolical calculations **34**

List of Figures

3.1 Plot of $\gamma(t)$ and $\omega(t,u)$ for the case $n = 0, 1$. 14
3.2 Plot of $\gamma(t)$ and $\omega(t,u)$ for the case $n = 2$. 14

4.1 Plot of numerical solutions for a,b, c. 21
4.2 Plot of the error $\epsilon(t)$ for a, b, c^0, c. 21
4.3 Plot of the function β 22

Chapter 1

Introduction

Modern derivative-pricing theory is largely based on arbitrage arguments. Most models assume that each claim can be replicated by following a certain strategy, the Δ-hedge. However, in practice, the implementation of such models suffers problems. One of them is that the hedging activity has a market impact, that is, the price is going to move as a result of trading. For a derivative-trader, important questions arise from this observation. Can the market impact be used to create arbitrage? How should one hedge under these constraints? Further, liquidity is affected by economic conditions. That is, we could observe decreases in liquidity both during the LTCM collapse as well as the recent collapse of the housing market. How does one adjust the hedge to account for the decreased liquidity in these cases? The first question is based on the idea of a feedback mechanism described below. In this thesis, we are more concerned with answering the last two questions.

The feedback mechanism is considered by one part of the literature ([1, 7, 13, 15, 16]) and results from the following observation: The price of a derivative is determined by a hedging strategy in the underlying. The hedging activity, however, affects the price of the underlying, which in turn changes the payoff and therefore the price of the derivative. The models usually assume some version of supply and demand function. The price is then implicitly determined by a market clearing condition. Since the hedging strategy maps the price of the asset to a quantity of the asset, it can easily be reinterpreted as a supply/demand function. However, there are several problems with these types of market impact model. Most of them arise, because the the price depends on the the size of the of the hedging portfolio rather than the trading activity. The first problem is that this implies that round-trip trades, i.e a sequence of trades that results in the initial portfolio, do not occur at extra costs, while having large price impacts. The problems that arise from this phenomenon for derivative pricing are examined in [16]. Second, while excess demand definitely affects the price, the

empirical literature suggests that trading, i.e the change in the portfolio, is a more relevant measure for price impact. The empirical literature [9, 11, 6, 17] finds that the empirical price impact function, that is the change of price resulting from a trade, is a concave function. Explicitly, this means that the sum of the price impact from two subsequent orders is lower than the impact from one order of the sum of their sizes. Hence it is profitable to split up a bigger order into smaller ones. However, price models which are based on a market clearing condition do not distinguish between different forms of trading but are only concerned with the total change in the portfolio.

The next branch of models is not concerned with the feedback-mechanism but views impact as a form of non-linear transaction cost. For example [4, 3] measure the change of the portfolio process by its jumps and quadratic variation. This approach has the desirable feature that classical strategies, for example, the Black-Scholes hedge, are still viable, since they have finite liquidity cost. However, [4] showed that there exists a sequence of continuous, finite-variation trading strategies, i.e. strategies that don't have price impact, such that terminal wealth converges towards to the option payoff. Therefore, the option prices in this model correspond to the option prices without liquidity costs. This is clearly unrealistic. The problem can be overcome by imposing a minimum time between trades, thus forcing jumps in the portfolio process. This approach has been used to obtain option prices by super-hedging the option ([3]). However, they are not able to characterize the optimal strategy in this case.

This thesis is based on [14] and [10]. While the previous literature was concerned with finding option prices in case of liquidity costs, here the focus is more on the hedging strategy itself. To do so, the trading process is modeled directly. The liquidity cost can therefore be written as function of the trading process. Further, the focus on strategy allows for the analysis of initially mishedged portfolios In this thesis, we focus on temporary market impact. The existing literature is extended in three ways: First, we are able to obtain explicit solutions for more specifications of liquidity cost and stock-price processes. Second, we provide a novel characterization of the target portfolio. Third, we analyze the effects of hedging in an asset with drift.

The thesis is structured as follows. The first chapter presents the model. The second chapter motivates the problem of liquidity modeling in a limit order book setting. the third chapter is concerned with finding an analyzing the the solution of the corresponding optimal control problem. The fourth chapter extends the analysis to include drift. The fifth chapter concludes.

Chapter 2

Model Description

In this chapter, we discuss how to model liquidity. First, we motivate functional forms in a static limit order book setting. Then, we propose a dynamic model which is based on the static analysis.

2.1 Limit order book trading.

Assume we want to buy h shares.[1] While we would like to pay the quoted mid price $p(0)$ for the asset, there are not enough people willing to sell for that price so we have to bid more. Let $p(h)$ be the smallest amount we have to pay to buy h shares. Transferring the situation to a limit order book setting, we can express $p(h)$ implicitly by

$$h = \int_{p(0)}^{p(h)} d(\tilde{p})\,\mathrm{d}\tilde{p},$$

where $d(p)$ refers to the depth at price p. The total cost of acquiring h units can be described by

$$\int_0^h p(\tilde{h})\,\mathrm{d}\tilde{h}.$$

We assume the limit order book has instant resilience, that is, after the trade, depth returns immediately to it's previous level. The price is therefore not affected by trading in the asset. The liquidity costs from acquiring h shares are therefore given by

$$l(h) = \int_0^h p(\tilde{h}) - p(0)\,\mathrm{d}\tilde{h}$$

[1] Note that the following holds also true for a negative h, i.e. the selling of shares.

Notice that since $p(h)$ is assumed to be monotonically increasing, $l(h)$ is always positive. Assuming sufficient differentiability, we have:

$$\frac{\partial l}{\partial h}(h) = p(h) - p(0)$$
$$\frac{\partial^2 l}{\partial h^2}(h) = \frac{\partial p}{\partial h}(h)$$

Therefore, an increasing price function implies an convex liquidity costs.

Remark 1 (Empirical evidence). The empirical literature has analyzed the price impact function $\phi(|h|) = \text{sign}(h)(p(h) - p(0))$. The survey articles [8, 2] provides an overview to estimations of the specification $\phi(|h|) \propto |h|^\alpha$. The parameter α is estimated between 0.2 and 0.7. Most widely accepted is a value of 0.5. In this case, the liquidity costs would scale by $l(h) \propto h^{1.5}$. There is no evidence for instant resilience. However, most of the models incorporating this feature assume the effects of trading to last forever, which is just as unlikely.

Remark 2 (Comparison of derivation). In [14], it is assumed that the total costs are given by the relative price specification

$$p(0) \int_1^{\frac{p(h)}{p(0)}} \tilde{p}\tilde{d}(\tilde{p})\,\mathrm{d}\tilde{p}.$$

Note how this approach makes sense dimensionally. The term $\int_1^{\frac{p(h)}{p(0)}} \tilde{d}(\tilde{p})\,\mathrm{d}\tilde{p}$ refers to shares of the asset[2] and $p(0)$ is measured in $\frac{\text{money}}{\text{shares}}$. However, the following chapters indicate that this assumption might be too restrictive and the scaling relationship of the market impact with price is the basis for a rich analysis.

Remark 3 (Examples of liquidity functions). The following specifications are found in the literature

1. The constant transaction cost

   $$l(h) = K|h|$$

   is used in [5, 12].

2. In this thesis, we analyze

   $$l(h) = kp(0)^n h^2$$

   The quadratic dependence on h implies a constant depth in the limit order book. As special cases, we recover the situation in [14] with n=1 and [10] with $n = 0$.

[2] $\tilde{d}$ is measured in shares and $\tilde{p}$ is dimensionless.

2.2 The wealth process

In this section, we introduce the model dynamics. Assume a two asset economy. One risky, illiquid asset S_t and one perfectly liquid, riskless asset B_t, which we assume to be constant. Using the previous notation, we can identify S_t as the time evolution of $p(0)$. At this point, we assume the stock price follows a general Ito-process:

$$\mathrm{d}S_t = \mu(t, S_t)\,\mathrm{d}t + \sigma(t, S_t)\,\mathrm{d}W_t.$$

We assume the coefficients are nice enough to guarantee existence and uniqueness of the process. Further, the stock price process is assumed to be given exogenous, i.e. independent from the trading activity. Given a trading process h_t, the holdings are given by

$$H_t = H_0 + \int_0^t h_s\,\mathrm{d}s.$$

The wealth X_t is determined by

$$\begin{aligned} X_t &= (X_t - H_t)B_t + H_t S_t \\ &= X_0 + \int_0^t H_s\,\mathrm{d}S - \int_0^t l(h_s)\,\mathrm{d}s. \end{aligned}$$

The first two terms are well known. The third term incorporates the liquidity costs. For the purpose of derivative replication, it is going to be helpful to distinguish between the unaffected portfolio process ξ_t and the liquidity costs L_t given by:

$$\begin{aligned} \xi_t &= X_0 + \int_0^t H_u\,\mathrm{d}S_u, \\ L_t &= \int_0^t l(h_u)\,\mathrm{d}u. \end{aligned}$$

It is going to be helpful to define the generator $\mathcal{L}$ of the process by

$$\mathcal{L}f = \frac{\partial f}{\partial t} + \mu(t, S)\frac{\partial f}{\partial S} + \frac{1}{2}\sigma(t, S)^2\frac{\partial^2 f}{\partial S^2}$$

Chapter 3

The Optimal Control Problem

In this chapter, we derive the optimal strategy and analyze important properties of the solution. We start by formulating the problem in an optimal control setting. Then we restrict the analysis to specific cases and solve the problem for these cases. Finally, the terminal hedging error is estimated.

3.1 Problem formulation

Assume we want to replicate an European claim with terminal payoff $G(S_T)$. Since the trading strategies are restricted to be of finite variation, almost sure replication is not possible. Thus we propose to minimize the following objective:

$$\Phi(X, H, S) = \mathbb{E}[(\xi_T - G(S_T))^2 + L_T | X_0 = X, H_0 = H, S_0 = S]$$

This objective imposes a mean-squared error on deviations from the terminal payoff while also penalizing trading. Further, it allows for reasonable tractability. To ensure the objective is finite, we assume $\mathbb{E}[G(S_T)^2] < \infty$ and restrict the trading strategies to:

$$\{h_s | \mathbb{E}[X_T^2 | X_0 = X | H_0 = H, S_0 = S] < \infty\}$$

These assumptions imply that the objective is in L^2. To minimize the objective Φ, we define the value function V by

$$V(t, X, H, S) = \mathbb{E}\left[(X_T - G(S_T))^2 + \int_t^T l(h_s)\,\mathrm{d}s | X_t = x, H_t = H, S_t = S\right].$$

A well-known argument shows that the value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$\mathcal{L}V + H\mu\frac{\partial V}{\partial X} + \frac{1}{2}H^2\sigma^2\frac{\partial^2 V}{\partial X^2} + H\sigma^2\frac{\partial^2 V}{\partial X \partial S} + \inf_h \left\{h\frac{\partial V}{\partial H} + l(h)\right\} = 0 \tag{3.1}$$

with the terminal condition

$$V(T, X, H, S) = X^2 - 2G(S)X + G(S)^2.$$

In this general form, it is hard to make progress. However, we can go a long way if we assume that $l(h) = kS^n h^2$ for some constants k, n. The motivation to do so is given in remark 3. With this specifications, the optimal control is given by

$$h^* = -\frac{1}{2kS^n}\frac{\partial V}{\partial H}. \tag{3.2}$$

Inserting this in the HJB-equation (3.1) yields

$$\mathcal{L}V + H\mu\frac{\partial V}{\partial X} + \frac{1}{2}H^2\sigma^2\frac{\partial^2 V}{\partial X^2} + H\sigma^2\frac{\partial^2 V}{\partial X \partial S} - \frac{(V_H)^2}{4kS^n} = 0, \tag{3.3}$$

which is a second-order non-linear PDE. Since all specifications so far have been at most quadratic in X and H, it seems reasonable that the value function should be of this form as well. therefore we propose the ansatz

$$V(t, X, H, S) = a(t, S)X^2 + b(t, H, S)X + c(t, H, S), \tag{3.4}$$

with

$$\begin{aligned} b(t, S, H) &= b_1(t, S)H + b_2(t, S), \\ c(t, S, H) &= c_1(t, S)H^2 + c_2(t, S)H + c_3(t, S). \end{aligned}$$

Inserting the ansatz in the PDE (3.3), we find that the solution is given by the following six coupled PDEs:

$$\mathcal{L}a - \frac{1}{4}\frac{b_1^2}{kS^n} = 0, \tag{3.5}$$

$$\mathcal{L}b_1 + 2\left(\mu a + \sigma^2\frac{\partial a}{\partial S}\right) - \frac{b_1 c_1}{kS^n} = 0, \tag{3.6}$$

$$\mathcal{L}b_2 - \frac{1}{2}\frac{b_1 c_2}{kS^n} = 0, \tag{3.7}$$

$$\mathcal{L}c_1 + \sigma^2 a + \mu b_1 + \sigma^2\frac{\partial b_1}{\partial S} - \frac{c_1^2}{kS^n} = 0, \tag{3.8}$$

$$\mathcal{L}c_2 + \mu b_2 + \sigma^2\frac{\partial b_2}{\partial S} - \frac{(c_1 c_2)}{kS^n} = 0, \tag{3.9}$$

$$\mathcal{L}c_3 - \frac{1}{4}\frac{c_2^2}{kS^n} = 0, \tag{3.10}$$

$$\tag{3.11}$$

with terminal conditions

$$\begin{aligned} a(T,S) &= 1, \\ b_1(T,S) &= 0, \\ b_2(T,S) &= -2G(S), \\ c_1(T,S) &= 0, \\ c_2(T,S) &= 0, \\ c_3(T,S) &= G(S)^2. \end{aligned}$$

This problem looks very complicated on first sight. The following sections, however, provide some guidance to find solutions. We start by making the following two observations:

- First, the functions a, b_1, c_1 have to be solved simultaneously. Given solutions to these three, b_2 and c_2 are to be solved together. At last, c_3 has to be solved
- The trio a, b_1, c_1 does not depend on the claim $G(S)$.

Again, this form is too general to obtain analytic solutions. The next section analyzes a special case. Chapter 4 explains how to extend the results to a more general setting.

Remark 4 (Option price). While the objective of this thesis is to construct hedging strategies, it is interesting to look for a notion of price. If we define the price of a claim to be its indifference value, then that price depends not only on the derivative, but also on the current holdings. This implies different prices for sellers who own different amounts of the stock. Even more interesting, the claim would have different prices when paid in cash or shares.

Remark 5 (Interpretation of the optimal control). Plugging the specification (3.4) into expression (3.2), we find that the optimal control is given by

$$h^* = \underbrace{\frac{c_1(t,S)}{kS^n}}_{\kappa(t,S)} \Bigg(\underbrace{-\frac{b_1(t,S)X + c_2(t,S)}{2c_1(t,S)}}_{=\theta(t,S,X)} - H \Bigg).$$

Rewriting the dynamics of H in this way, we find that

$$\mathrm{d}H_t = \kappa(t,S_t)(\theta(t,S_t,X_t) - H_t),$$

which describes a mean-reverting process with speed $\kappa(t,S_t)$ and target portfolio $\theta(t,S_t,X_t)$. Notice that since c_1 does not depend on the particular claim, the speed is independent of the particular contract.

3.2 The zero mean case

In this section, we analyze the case $\mu(t,S) = 0$, which is the basis for both [14] and [10]. It is easy to verify that in this case we have $a = 1, b_1 = 0$. Further, b_2 is given by the Feynman-Kac representation

$$b_2(t,S) = -2\mathbb{E}[G(S_T)|S_t = S].$$

Notice the replication portfolio in the case without transaction costs is given by $\Delta(t,S) = \frac{\partial \mathbb{E}[G(S_T)|S_t=S]}{\partial S} = -\frac{1}{2}\frac{\partial b_2}{\partial S}$.[1] Given this information, we see that the six parabolic equations (3.5) to (3.10) collapse to the three equations

$$\mathcal{L}c_1 + \sigma^2 - \frac{c_1^2}{kS^n} = 0, \tag{3.12}$$

$$\mathcal{L}c_2 - 2\sigma^2\Delta - \frac{(c_1c_2)}{kS^n} = 0, \tag{3.13}$$

$$\mathcal{L}c_3 - \frac{1}{4}\frac{c_2^2}{kS^n} = 0. \tag{3.14}$$

The case $\sigma(t,S) = \bar{\sigma}S$, and $n = 1$ is considered in [14], the case $\sigma(t,S) = \bar{\sigma}$ and $n = 0$ is a special case of the analysis in [10].

The equation (3.12) is non-linear. Given a solution (3.12), the solutions to (3.13),(3.14) are determined by their Feynman-Kac representations.

In the following, we are concerned with the scaling relationship $\sigma(t,S) = \bar{\sigma}S^{\frac{n}{2}}$ for $n \in \{0,1,2\}$. The solution can then be further separated by making the ansatz

$$c_1(t,S) = \tilde{\gamma(t)}S^n$$

Inserting into (3.12), we find that $\tilde{\gamma}(t)$ satisfies

$$\frac{\partial\tilde{\gamma}}{\partial t} + \frac{n(n-1)}{2}\bar{\sigma}^2\tilde{\gamma}(t) + \bar{\sigma}^2 - \frac{\tilde{\gamma}(t)^2}{k} = 0, \tag{3.15}$$

with terminal condition $\tilde{\gamma}(T) = 0$. The above ODE is of the Ricatti-type with constant coefficients and can therefore be solved analytically. The following theorem summarizes the results:

[1] In this thesis we make the technical assumption of finite-variation hedging strategies. The proposed delta-hedge is however of infinite variation.

Theorem 3.2.1 (Solution for the value function in case of zero mean)**.** *Assume that* $\sigma^2(t,S) = \bar{\sigma}^2 S^n$ *and* $l(h) = kS^n h^2$ *then the solutions for* (3.12) *to* (3.14) *are given by*

$$\begin{aligned} c_1(t,S) &= \gamma(t) k S^n \\ c_2(t,S) &= -2\gamma(t)\mathbb{E}\left[\int_t^T \omega(t,u)\xi_{t,u}\Delta(u,S_u)\,\mathrm{d}u | S_t = S\right] kS^n \\ c_3(t,S) &= \mathbb{E}\left[G(S_T)^2 - \frac{1}{4}\int_t^T \frac{c_2^2(u,S_u)}{kS^n}\,\mathrm{d}u | S_t = S\right] \end{aligned}$$

with

$$\gamma(t) = \begin{cases} \dfrac{\sigma}{\sqrt{k}}\tanh\left(\dfrac{\sigma}{\sqrt{k}}(T-t)\right) & \text{for } n = 1,2, \\ \dfrac{1}{\sqrt{\frac{k^2}{4}+\frac{k}{\sigma^2}}\coth\left(\sigma\sqrt{\frac{\sigma^2}{4}+\frac{1}{k}}(T-t)\right) - \frac{1}{2}k} & \text{for } n = 2, \end{cases}$$

$$\omega(t,u) = \begin{cases} \dfrac{\sigma}{\sqrt{K}}\dfrac{\cosh\left[\frac{\sigma}{\sqrt{k}}(T-u)\right]}{\cosh\left[\frac{\sigma}{\sqrt{k}}(T-t)\right]}\coth\left[\dfrac{\sigma}{\sqrt{k}}(T-t)\right] & \text{for } n = 1,2, \\ \sigma^2\dfrac{\exp\left(s(t,u)+\sigma^2(u-t)\right)}{\gamma(t)} & \text{for } n = 2, \end{cases}$$

$$\xi_{t,u} = \frac{S_u^n}{\mathbb{E}[S_u^n | S_t = S]}$$

where the function $s(t,u)$ *is defined in Appendix A.1.*

Proof. See Appendix A. □

A peak at the solution $s(t,u)$ reveals how complicated things get in the case $n = 2$. Therefore, we often have to restrict the analysis to the case $n = 0, 1$. The following lemma highlights an important property of the function ω.

Lemma 3.2.2 (Weighting property)**.** *Assume* $n = 0, 1$*, then*

$$\mathbb{E}[\int_t^T \omega(t,u)\xi_{t,u}\,\mathrm{d}u] = 1.$$

Proof. We use the fact that in this case $M_u = \xi_{t,u}$ is a martingale. An application of Fubini's theorem yields

$$\begin{aligned} \mathbb{E}[\int_t^T \omega(t,u)\xi_{t,u}\,\mathrm{d}u] &= \int_t^T \omega(t,u)\mathbb{E}[\xi_{t,u}]\,\mathrm{d}u \\ &= \int_t^T \omega(t,u)\,\mathrm{d}u = 1. \end{aligned}$$

The last line follows from direct calculation. □

We are now ready to verify that the functions γ, ω, ξ are important economic quantities. To do so, we use the interpretation introduced in remark 5. Using this interpretation, we find that the adjustment speed of the optimal strategy is given by

$$\kappa(t, S) = \gamma(t) \tag{3.16}$$

and the target portfolio is given by

$$\theta(t, S) = \mathbb{E}\left[\int_t^T \omega(t, u)\xi_{t,u}\Delta(t, S_u)\,\mathrm{d}u\right]. \tag{3.17}$$

We can interpret the target portfolio as an weighted average of the future Δ-hedge portfolios. Since $\mathbb{E}[\xi_{t,u}] = 1$, we interpret $\xi_{t,u}$ as the state weight and $\omega(t, u)$ as the time weight. The next section proves that the target portfolio coincides with the Δ-hedge for $n = 0$.

Notice that these functions only depend on σ and k through the ratio $\frac{\sigma}{\sqrt{k}}$. The idea is that both a more volatile asset and a lower transaction cost increase the incentive to hedge. Figure 3.1 show plots of γ and ω for different values of the ratio $\frac{\sigma^2}{k}$.

The adjustment speed γ is given by $\frac{\sigma}{\sqrt{k}}$ for large time to maturity $(T - t)$ and decreases towards zero as maturity approaches. The shape might be explained by the fact that there is a leverage effect in hedging early: Assume one is mis-hedged by one unit at time t and assume there is no more trading. Then the final difference to the best hedge is going to be in the order of $S_T - S_t$. Since we assumed a quadratic cost function, this implies an additional error in the order of $\int_t^T \bar{\sigma}^2 S_u^n\,\mathrm{d}u$. Since this expression is of order $(T - t)$, we conclude that mis-hedging early is a lot more expensive compared to mis-hedging late.[2] The shape of $\gamma(t)$ accounts for this. Further, an increase in $\frac{\sigma^2}{k}$ results in a higher level and a sharper decline towards end. As the incentive to hedge increases, the optimal strategy becomes more aggressively in doing so.

The expression (3.17) can be interpreted in the following way: Adjusting the portfolio is expensive. Hence there exists a trade-off between hedging perfectly today and minimizing the expected trading costs tomorrow. By averaging over future Δ-hedges, both of these objectives are acknowledged. Figure 3.1b shows the weighting function for different values of $\frac{\sigma^2}{k}$. First we note that contemporary Δ-hedges are

[2]One could also view this in the following way: The volatility of S_T at time t is of order $\bar{\sigma}\sqrt{T - t}$. Therefore, hedging early is the same as hedging a more volatile asset, therefore it adjustment becomes more important.

weighted more heavily than Δ-hedges in the far future, thus providing evidence for the first goal. Further, we see that for small $\frac{\sigma^2}{k}$, all future Δ-hedges are weighted almost equally. This might be the fact that in this case, the optimal strategy does re-hedge only slightly in the future, as indicated by the small γ. Thus the optimal strategy weights future portfolios more in advance knowledge of the low future re-hedging activities.

The only difference between the cases $n = 0$ and $n = 1$ is in the factor $\xi_{t,u}$. This might be due to the fact that in this case, the liquidity cost is stochastic. The factor $\xi_{t,u}$ can be viewed as a normalized version of the stochastic component of the liquidity cost. Therefore, Δ-hedges in states of low liquidity costs are weighted less compared to states with high liquidity costs. Therefore, the portfolio is skewed towards Δ-hedges in states with high liquidity cost. This has the benefit of reducing the need to trade in states where liquidity costs are high and increases the need to trade in states where liquidity costs are high. Therefore, such a portfolio choice exploits the stochastic liquidity imbalance

Remark 6 (The case $n = 2$.). Figure 3.2 shows plots of γ and ω. Note that since these functions depend on σ and k separately, we had to fix the value $\sigma = 0.1$. In this case, the adjustment speed is essentially time independent. Consequently, the portfolio weights ω exhibit an exponential decay. It is worth pointing out that the level of γ and ω is similar to the one in the cases $n = 0, 1$. The main difference to the previous cases seems to be that the adjustment speed decays later and sharper. It is hard to interpret this difference, since it hinges on a very specific formulation of the stochastic process and liquidity cost. However, it might be in relation with with the fact that $l(h)$ is not a martingale in this case.

While we cannot verify the condition $\int_t^T \omega(t,u)\,\mathrm{d}u = 1$ by means of symbolic computations, attempts to evaluate the integral numerically result in values close to 1 for a range of parameters. However, the deviations are likely to to be the result of the quadrature rule. Despite the appearance of Figure 3.2a γ is a continuous function with $\lim_{t\to T}\gamma(t) = 0$. Therefore, the computational evaluation is likely inaccurate.

Remark 7 (Definition of a new measure). In view of lemma 3.2.2 we might interpret $\omega(t,u)\xi_{t,u}$ as a change of measure. The target portfolio could therefore be expressed as an expectation under a measure $\mathcal{Q}$ on $[t,T]\times\Omega$. This is very interesting, because it raises the question of whether there exists an analogy to the fundamental theorem of asset pricing for the case the case of stochastic liquidity costs.

Remark 8 (Limiting cases). First, we note that in the case $\frac{\sigma}{\sqrt{k}} \to \infty$, we have that $\gamma(u) \to \infty$ and $\omega(t,u) \to 0$ for all $T > u > t$. Further, from the condition $\int_t^T \omega(t,u)\,\mathrm{d}u = 1$ we can conclude that $\omega(t,u) \to \delta_t(u)$, the Dirac-delta function. Therefore, $\theta(t,S) \to \Delta(t,S)$. We recover the classical hedging theory in the sense that the optimal holding is mean-reverting to the classical hedge with infinite speed. In the case $\frac{\sigma}{\sqrt{k}} \to 0$, we obtain $\gamma(t) \to 0$, i.e, trading is stopped completely. While the previous limit can only be sensibly interpreted as the limit $k \to 0$, here this might be the result of $\sigma \to 0$ or $k \to \infty$. In the latter case, the interpretation is strait forward. In the former case, we have to note that in the case $\sigma = 0$ there are two assets in the market with zero drift and volatility. Trading allows us to shift money between these two. However, since the assets are identical trading does not change the wealth except for the costs suffered from trading. Hence it is optimal to not trade at all.

3.3 Hedging portfolio in the case of Brownian motion

In this section, we proof that the target portfolio coincides with the Δ-hedge in case $n = 0$

Theorem 3.3.1. *In the case $n = 0$, we have*

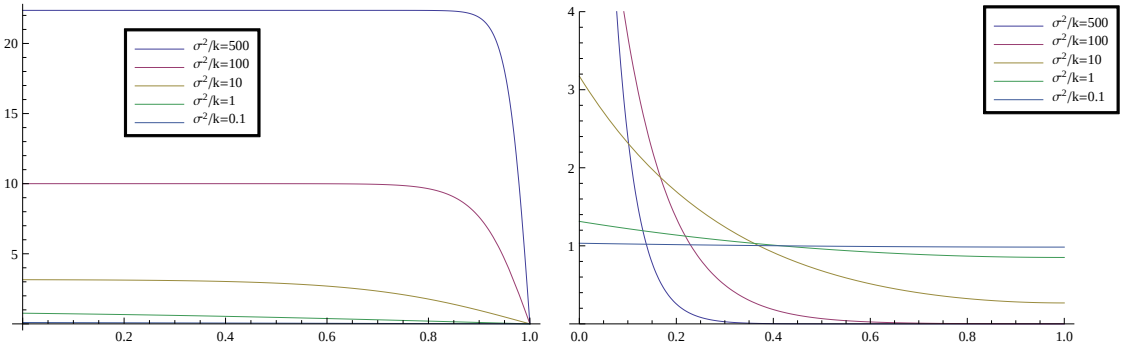
$$\theta(t,S) = \Delta(t,S)$$

Proof. The important observation is that in this case, $\Delta(t,S)\xi_{t,u}$ is a martingale. For any $r > t$, we have:

$$\begin{aligned}
\mathbb{E}\left[\Delta(r,S_r)\xi_{t,r}|\mathcal{F}_t\right] &= \mathbb{E}\left[\frac{\partial}{\partial S_r}\mathbb{E}\left[G(S_T)|\mathcal{F}_r\right]\frac{S_r^n}{\mathbb{E}[S_r^n|\mathcal{F}_r]}|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[G'(S_T)\frac{\partial S_T}{\partial S_r}|\mathcal{F}_r\right]\frac{S_r^n}{\mathbb{E}[S_r^n|\mathcal{F}_r]}|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[G'(S_T)\frac{\partial S_T}{\partial S_r}\frac{\partial S_r}{\partial S_t}|\mathcal{F}_r\right]\frac{S_r^n}{\mathbb{E}[S_r^n|\mathcal{F}_r]}|\mathcal{F}_t\right].
\end{aligned}$$

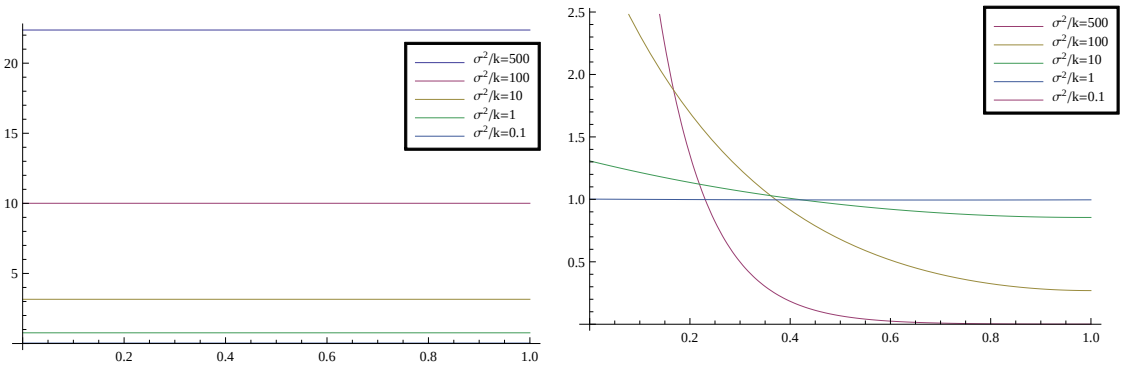
Now, for $n = 0$, we have

$$\frac{\frac{S_r^n}{\mathbb{E}[S_r^n|\mathcal{F}_r]}}{\frac{\partial S_r}{\partial S_t}} = \left(\frac{\partial}{\partial S_t}\left(S_t + \int_t^r \sigma\,\mathrm{d}W_t\right)\right)^{-1} = 1.$$



(a) $\gamma(\cdot)$ (b) $\omega(0,\cdot)$

Figure 3.1: Plot of $\gamma(t)$ and $\omega(t,u)$ for the case $n = 0, 1$.
Parameters are set to $t = 0, T = 1$.



(a) $\gamma(\cdot)$ (b) $\omega(0,\cdot)$

Figure 3.2: Plot of $\gamma(t)$ and $\omega(t,u)$ for the case $n = 2$.
Parameters are set to $t = 0, T = 1, \sigma = 0.1$.

Therefore, the conditional expectation simplifies to

$$\mathbb{E}\left[\mathbb{E}\left[G'(S_T)\frac{\partial S_T}{\partial S_t}|\mathcal{F}_r\right]|\mathcal{F}_t\right] = \mathbb{E}\left[G'(S_T)\frac{\partial S_T}{\partial S_t}|\mathcal{F}_t\right] = \frac{\partial}{\partial S_t}\mathbb{E}\left[G(S_T)\right] = \Delta(t,S).$$

Since the assumptions on integrability are given, we have proofed the martingale part. The proof is concluded by

$$\begin{aligned}\theta(t,S) &= \mathbb{E}[\int_0^t \omega(t,u)\Delta(u,S_u)\,\mathrm{d}u|S_t = S] \\ &= \int_0^t \omega(t,u)\mathbb{E}[\Delta(u,S_u)|S_t = S]\,\mathrm{d}u \\ &= \int_0^t \omega(t,u)\Delta(t,S)\,\mathrm{d}u = \Delta(t,S).\end{aligned}$$

□

Remark 9 (Comparison with existing literature). Both [10] and [14] analyze the case of a small transaction cost k and find that their optimal holdings are mean reverting towards the Δ-hedge. Since [10] assumes a Brownian motion, the result is in agreement with their strategy. The combination of stochastic process and liquidity cost considered by [14] is not part of this dissertation but can be recovered as limiting case.. Further, our expression for the mean reversion speed is very similar to the one obtained in [10].

3.4 Estimating the cost of the trading strategy

In this section, we are going to examine the cost associated with applying the trading strategy given in the last section. We do so by comparing the value function with the case of no transaction costs. It should be noted, however, that the case of zero transaction costs is not the one considered in classical theory, since we still impose the restriction of finite variation strategies. Therefore, we look at the difference of the value functions for different transaction costs and take a limit from that expression.

$$\begin{aligned}V_{k_1} - V_{k_2} = \gamma_1(t)k_1S^n(H^2 - 2\theta_1(t,S)H) - \gamma_2(t)k_2S^n(H^2 - 2\theta_2(t,S)_1H) \\ + \mathbb{E}\left[\int_t^T k_2S^n\gamma_2(u)^2\theta_2^2(u,S_u) - k_1S^n\gamma_1(u)^2\theta_1^2(u,S_u)\,\mathrm{d}u\right].\end{aligned}$$

We have $\gamma(t)k \xrightarrow{k\to 0} 0$ and $\gamma(t)^2 k \xrightarrow{k\to 0} \sigma^2$ for all $t < T$. Remark 8 indicates the relation $\theta_k(t,S) \xrightarrow{k\to 0} \Delta(t,S)$. Further, from $\gamma(T) = 0$, we have the relations

$$-\gamma(t)k = \int_t^T k\gamma'(u)\,\mathrm{d}u = -\int_t^T \sigma^2(1-\lambda(u))\,\mathrm{d}u. \tag{3.18}$$

where we defined $\lambda(u) = \tanh(\frac{\sigma}{\sqrt{k}}(T-t))^2$. We can think of λ as a normalized metric of the adjustment speed γ. In the limiting case $k_2 \to 0$, we therefore find the error to be

$$\begin{aligned}
\mathcal{E} &= \gamma(t)kS^n(H-\theta(t,S))^2 \\
&\quad + \mathbb{E}\left[\int_t^T \sigma^2(S_u^n\Delta(u,S_u)^2 - \left[(1-\lambda(u))S_t^n\theta(t,S_t)^2 + \lambda(u)S_u^n\theta(u,S_u)^2\right])\,\mathrm{d}u | S_t = S\right] \\
&= \gamma(t)kS^n(H-\theta(t,S))^2 \\
&\quad + \mathbb{E}\left[\int_t^T \Delta(u,S_u)^2 - \left[(1-\lambda(u))\frac{S_t^n}{S_u^n}\theta(t,S_t)^2 + \lambda(u)\theta(u,S_u)^2\right]\,\mathrm{d}\langle S_u\rangle\,|S_t = S\right].
\end{aligned}$$

The first term in the above expression represents the error due to holding the wrong portfolio at time t. Holding the target portfolio does not prevent one from the need to re-balance in the future, which is described by the second term. It can be written as the difference of the expected quadratic variation of of the perfectly hedged portfolio and the expected quadratic variation of holding a weighted average of today's portfolio and the target portfolio. Therefore, the error increases if the Δ-hedge varies widely. In such a case, further rebalancing is required, thus increasing the error.

Further, in the case $n = 0$, further simplifications can be made by using theorem 3.3:

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T \Delta(u,S_u)^2\,\mathrm{d}\langle S_u\rangle\right] - \mathbb{E}\left[\int_t^T (1-\lambda(u))\theta(t,S_t)^2 + \lambda(u)\theta(u,S_u)^2\,\mathrm{d}\langle S_u\rangle\,|S_t = S\right] \\
&= \mathbb{E}\left[\int_t^T (1-\lambda(u))(\Delta(u,S_u)^2 - \theta(t,S_t)^2)\,\mathrm{d}\langle S_u\rangle\,|S_t = S\right] \\
&= \int_t^T (1-\lambda(u))\mathbb{E}\left[(\Delta(u,S_u)^2|S_t = S\right] - \Delta(t,S_t)^2)\,\mathrm{d}\langle S_u\rangle \\
&= \int_t^T (1-\lambda(u))\mathbb{V}\left[\Delta(u,S_u)|S_t = S\right]\,\mathrm{d}\langle S_u\rangle
\end{aligned}$$

The factor $(1-\lambda(\cdot))$ is inversely related to the trading intensity. Therefore, the error is proportional to the variance of future Δ-hedges, weighted more heavily when the optimal strategy indicates to trade less aggressively.

Chapter 4

Asymptotic Expansions.

In this chapter, we discuss how to extend the results derived in the last chapter to more general cases by using asymptotic expansions. First, we present a stylized example, which shows the idea behind asymptotic expansions in this case. Second, we apply the methodology to include drift in the case $n = 0$. Third, we compare the expansions to numerical results. Finally, we examine how the optimal portfolio adjusts to the inclusion of drift.

4.1 A stylized example

Assume we are given the PDE

$$\mathcal{L}_\varepsilon f - f^2 = 0 \tag{4.1}$$

with some parametrized generator $\mathcal{L}_\varepsilon$ of a stochastic process S_t^ε and terminal condition $f(t,s) = G(s)$. The main problem in solving the above expression is that the Feynman-Kac-representation, given by

$$f(t,s) = \mathbb{E}[\exp(-\int_t^T f(u, S_u^\varepsilon)\,\mathrm{d}u)G(S_T^\varepsilon)|S_t = s],$$

is self-referential. However, assume we know the solution for some parametrization $\varepsilon = 0$. Expanding the solution by

$$f(t,s) = f_0(t,S) + \varepsilon f_1(t,S) + \varepsilon^2 f_2(t,S) + ...$$

and inserting this in the above equation , we find

$$\begin{aligned} 0 &= \mathcal{L}_\varepsilon f - f^2 \\ &= \left[\mathcal{L}_0 f_0 - f_0^2\right] + \left[(\mathcal{L}_\varepsilon - \mathcal{L}_0)f_0 + \epsilon(\mathcal{L}_0 f_1 - 2f_1 f_0)\right] \\ &\quad + \left[\epsilon(\mathcal{L}_\varepsilon - \mathcal{L}_0)f_1 + \epsilon^2(\mathcal{L}_0 f_2 - 2f_2 f_0 - f_1^2)\right] + \mathcal{O}(\varepsilon^3) \end{aligned}$$

We can make the natural assumption that $(\mathcal{L}_\varepsilon - \mathcal{L}_0)f$ is of order ε to obtain the Feynman-Kac representations

$$f_1(t,s) = \mathbb{E}\left[\int_t^T \exp(-2\int_t^u f_0(r,S_r^0)\,\mathrm{d}r)\left((\mathcal{L}_\varepsilon - \mathcal{L}_0)f_0(u,S_u^0)\right)\,\mathrm{d}u|S_t = s\right],$$
$$f_2(t,s) = \mathbb{E}\left[\int_t^t \exp(-2\int_t^u f_0(r,S_r^0)\,\mathrm{d}r)\left((\mathcal{L}_\varepsilon - \mathcal{L}_0)f_1(u,S_u^0) - f_1(u,S_u^0)^2\right)\,\mathrm{d}u|S_t = s\right].$$

This representation is not self referential anymore and can therefore be solved by conventional methods. Before we proceed, a word of caution: The n-th expression of the above representation is given by

$$\mathbb{E}\left[\int_t^t e^{-2\int_t^u f_0(r,S_r^0)\,\mathrm{d}r}\left((\mathcal{L}_\varepsilon - \mathcal{L}_0)f_{n-1}(u,S_u^0) - \sum_{k=1}^{n-1} f_k(u,S_u)f_{n-k}(u,S_u)\right)\,\mathrm{d}u|S_t = s\right].$$

For the errors in the approximation to vanish, we would like to bound f_n. However, the summation of the lower order terms makes it hard to proof such a result. The numerical results indicate however, that the approximations perform pretty well.

4.2 Brownian motion with drift

4.2.1 Expansion of a_1, b_1, c_1

We are now ready to extend the results of the previous chapter by using the method described in the section 4.1. To obtain solutions, we restrict the analysis to the case of a Brownian motion with drift, that is, we assume $\mu(t,S) = \mu$ and $\sigma(t,S)^2 = \bar{\sigma}^2$. To obtain solutions for this case, we expand around around the case of zero mean, defined in theorem 3.2.1. Since a_1, b_1 and c_1 are independent from the other functions, we can consider these first. It is easy to check that a_1, b_1, c_1 do not depend on S. Therefore, we obtain three coupled non-linear ODEs for $a(t) = a(t,S), b(t) = b_1(t,S), c(t) = c_1(t,S)$:

$$\begin{aligned}\frac{\partial a}{\partial t} - \frac{1}{4}\frac{b^2}{k} &= 0,\\ \frac{\partial b}{\partial t} + 2\mu a - \frac{cb}{k} &= 0,\\ \frac{\partial c}{\partial t} + \sigma^2 a + \mu b - \frac{c^2}{k} &= 0.\end{aligned}$$

In the asymptotic analysis, we now consider the expansions $\phi^0(t,S) + \phi^1(t,S)\mu + \phi^2\mu^2 + ...$ for $\phi \in \{a,b,c\}$. Table 4.1 shows the expansions for the first two orders. The numerical results in section 4.2.2 indicate that the second order expansions approximate the true solutions reasonably well. Before we proceed, we make the following observation. The proof illustrates the recursive definition.

Lemma 4.2.1. *All even terms in the expansion of b and all uneven terms in the expansions of a, c vanish.*

Proof. Proof by induction. Table 4.1 shows this is true for the orders $0, 1, 2$. For higher orders, the recursion is given by

$$a^n(t) = -\frac{1}{4k}\int_t^T \sum_{l=0}^{n} b^l(u)b^{n-l}(u)\,\mathrm{d}u$$
$$b^n(t) = \int_t^T \exp(-\int_t^u \frac{c^0(r)}{k}\,\mathrm{d}r)\left(2a^{n-1} - \frac{\sum_{l=1}^{n} c^l(u)b^{n-l}(u)}{k}\right)\mathrm{d}u$$
$$c^n(t) = \int_t^T \exp(-2\int_t^u \frac{c^0(r)}{k}\,\mathrm{d}r)\left(\sigma^2 a^n(u) + b^{n-1}(u) - \frac{\sum_{l=1}^{n-1} c^l(u)c^{n-l}(u)}{k}\right)\mathrm{d}u$$

The theorem therefore follows by noting that if n is even, then $n - k \mod 2 = k \mod 2$, while this is not true for the reverse case. □

Function	Order 0	Order 1	Order 2
a_1	1	0	$-\frac{k}{\sigma^4}\int_t^T \gamma(u)^2\,\mathrm{d}u$
b_1	0	$\frac{2k}{\sigma^2}\gamma(t)$	0
c_1	$\gamma(t)k$	0	$\frac{k^3}{\sigma^6}\gamma(t)^2\int_t^T \omega(t,u)^2(2\gamma(u) - \int_u^T \gamma(r)^2\,\mathrm{d}r)\,\mathrm{d}u$

Table 4.1: Asymptotic expansions for the drift term.
Notice that $\gamma(t) \propto \frac{\sigma}{\sqrt{k}}$, therefore the leading order terms are somewhat misleading.

4.2.2 Comparison to numerical solution.

In this section, we compare the asymptotic expansions with values obtained by numerical estimations. To do so, we use the ODE-solver *ODE45* in Matlab, which implements an explicit Runge-Kutta scheme. We restrict the maximum time step to $1E-4$. Using this time step, we obtain similar solution for all tested solvers. Figure 4.1 shows a plot of expansions and the numerical solutions. The expansions seem to be close to the numerical values and seem to capture the functional form of the numerical solutions quite well.

To gain a clearer picture of the quality of the expansions, we compute the relative error as

$$\epsilon(t) = \frac{\phi^{\text{Expansion}} - \phi^{\text{Numerical}}(t)}{\phi^{\text{Numerical}}(t)}$$

Figure 4.2 shows plots of $\epsilon(t)$ for different parametrizations. The first thing to realize is that the error is very small. The largest relative error is in the region of 3%.

Further, it seems that all corrections overshoot in the sense that the error has a different sign compared to the correction term. This might be an indication for the correction terms to have alternating signs. In all cases, increasing the volatility leads to a reduction in error. Further, it seems that the errors exhibit very different scaling relationships. For example, while the relative error of b seems to scale with $\frac{\mu}{\sigma}$, this is clearly not the case for a. For example, compare the case $\mu = 0.4, \sigma = 2$ with the case $\mu = 0.1, \sigma = 0.1$. The error for b seems to be to be essentially the same, while there is a big difference for a and c. For b_1, we see that the errors from the first order expansions are already close to the solution. Analyzing c, we see that the zero order expansion performs already pretty good and the error can further be reduced by the first correction term. We conclude that the expansion-technique explained so far works fairly well, even for first order approximations.

4.2.3 Expansion of b_2, c_2.

Having found the asymptotic expressions for both a_1, b_1 and c_1, we can now proceed to analyze how b_1 and c_1 react to the introduction of drift. Since these functions clearly depend on S, we have to solve the PDEs given by

$$\begin{aligned} \frac{\partial b_2}{\partial t} + \mu \frac{\partial b_2}{\partial S} + \frac{\sigma^2}{2} \frac{\partial^2 b_2}{\partial S^2} - \frac{1}{2} \frac{b_1 c_2}{k} = 0 \\ \frac{\partial c_2}{\partial t} + \mu \frac{\partial c_2}{\partial S} + \frac{\sigma^2}{2} \frac{\partial^2 c_2}{\partial S^2} + \mu b_2 + \sigma^2 \frac{\partial b_2}{\partial S} - \frac{c_1 c_2}{k} = 0 \end{aligned}$$

The recursive procedure can be written using Feynman-Kac-repesentations:

$$\begin{aligned} b_2^n(t,S) &= \mathbb{E}^0 \left[\int_t^T \frac{\partial b_2^{n-1}}{\partial S} - \frac{1}{2k} \sum_{k=0}^{n} b_1^k c_2^{n-k} \, \mathrm{d}u \right] \\ c_2^n(t,S) &= \mathbb{E}^0 \left[\int_t^T e^{-\int_t^u \frac{c_0^0}{k}(r) \, \mathrm{d}r} \left(\frac{\partial c^{n-1}}{\partial S} + b_2^{n-1} + \sigma^2 \frac{\partial b_2^n}{\partial S} - \frac{1}{k} \sum_{k=0}^{n} c_1^k c_2^{n-k} \right) \, \mathrm{d}u \right] \\ &= k\gamma(t) \mathbb{E}^0 \left[\int_t^T \omega(t,u) \left(\frac{\partial b_2^n}{\partial S} + \frac{1}{\sigma^2} \left(\frac{\partial b^{n-1}}{\partial S} + b_2^{n-1} - \frac{1}{k} \sum_{k=0}^{n} c_1^k c_2^{n-k} \right) \right) \, \mathrm{d}u \right] \end{aligned}$$

where all functions are evaluated at (u) or (u, S_u) unless indicated otherwise. $\mathbb{E}^0$ indicates that the expectations are taken under a measure such that S_t is a martingale.

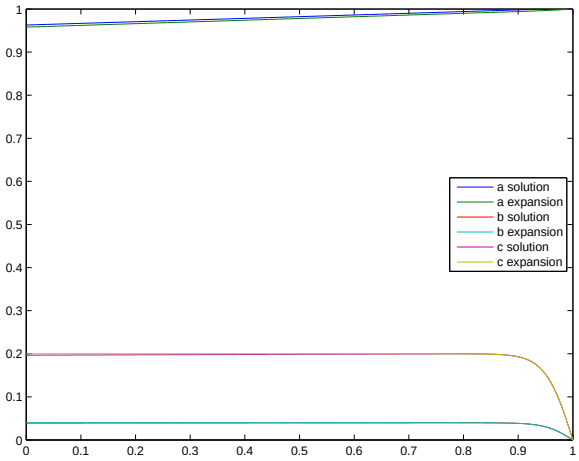
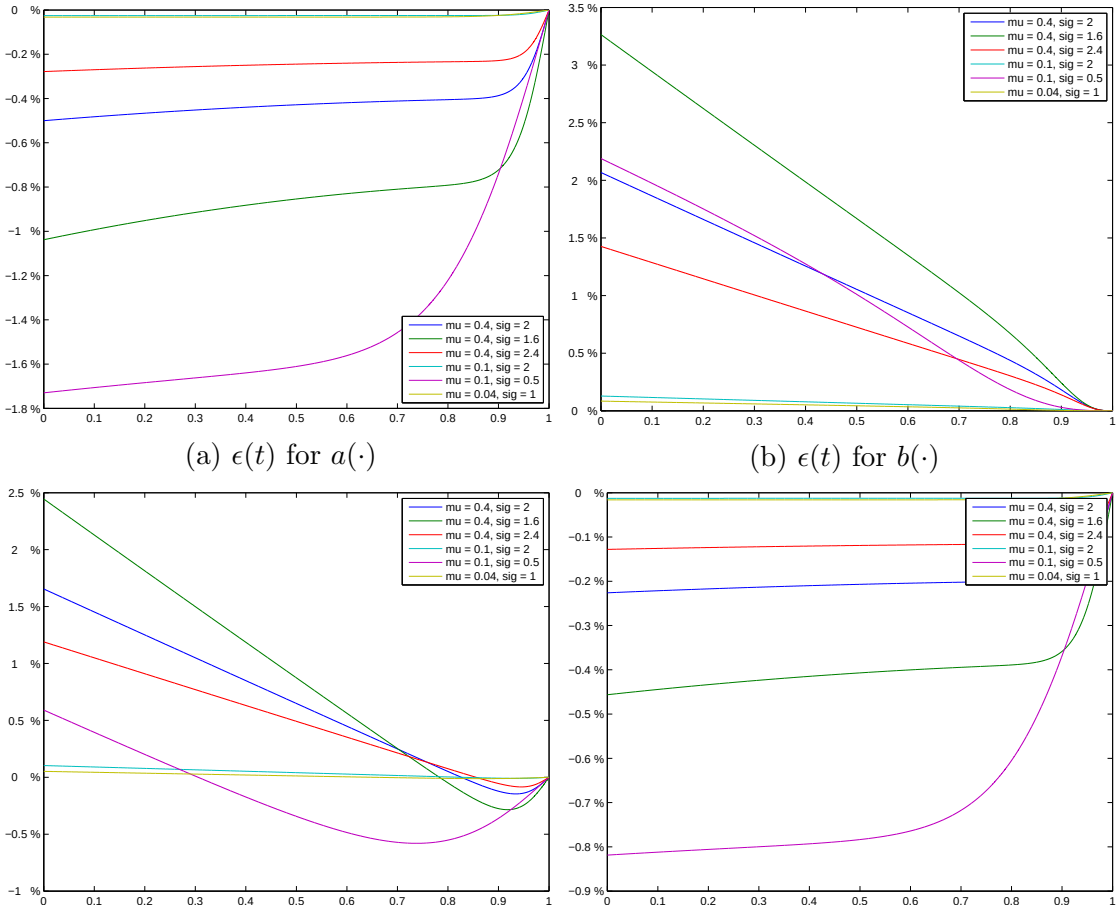


Figure 4.1: Plot of numerical solutions for a,b, c.
The parameters are given by $\mu = 0.4, \sigma = 2, k = 0.01, t = 0, T = 1$



(a) $\epsilon(t)$ for $a(\cdot)$ (b) $\epsilon(t)$ for $b(\cdot)$

(c) $\epsilon(t)$ for $c^0(\cdot)$ (d) $\epsilon(t)$ for $c(\cdot)$

Figure 4.2: Plot of the error $\epsilon(t)$ for a, b, c^0, c.
The parameters are given by $k = 0.01, t = 0, T = 1$

For $n = 1$, we find in view of theorem 3.3:

$$\begin{aligned}
b_2^1(t,S) &= \mathbb{E}^0\left[\int_t^T -2\Delta(u,S_u) + 2\gamma(u)^2\theta^0(u,S_u)\frac{k}{\sigma^2}\,\mathrm{d}u\right] \\
&= \Delta(t,S_t)2\int_t^T (2(\lambda(u)-1))\,\mathrm{d}u = -\frac{2k}{\sigma^2}\gamma(u) \\
c_2^1(t,S) &= k\gamma(t)\mathbb{E}^0\left[\int_t^T \omega(t,u)\left(\frac{\partial b_2^1}{\partial S}(u,S_u) + \frac{1}{\sigma^2}\left(\frac{\partial c^0}{\partial S} + b_2^0\right)\right)\,\mathrm{d}u\right] \\
&= k\gamma(t)\mathbb{E}^0\left[\int_t^T \omega(t,u)\left(-\frac{4k}{\sigma^2}\gamma(u)\right)\frac{\partial \Delta}{\partial S}(u,S_u) - \frac{2\mathbb{E}^0[G(S)|\mathcal{F}_u]}{\sigma^2}\,\mathrm{d}u\right] \\
&= -2k\gamma(t)\left[\beta(t)\frac{\partial \Delta}{\partial S}(t,S_t) + \frac{2\mathbb{E}^0[G(S_T)|\mathcal{F}_u]}{\sigma^2}\right]
\end{aligned}$$

with

$$\beta(t) = 2\frac{\sqrt{k}}{\sigma}\left(1 - \operatorname{sech}\left(\frac{\sigma}{\sqrt{k}}(T-t)\right)\right)\coth\left(\frac{\sigma}{\sqrt{k}}(T-t)\right).$$

Figure 4.3 shows the plot for β. It seems that β tends towards zero for $\frac{\sigma^2}{k} \to \infty$ and towards $(T-t)$ for $\frac{\sigma^2}{k} \to 0$. We can verify this relationships by using L'Hopital's rule. The next section illustrates why this indicates a sensible behavior.

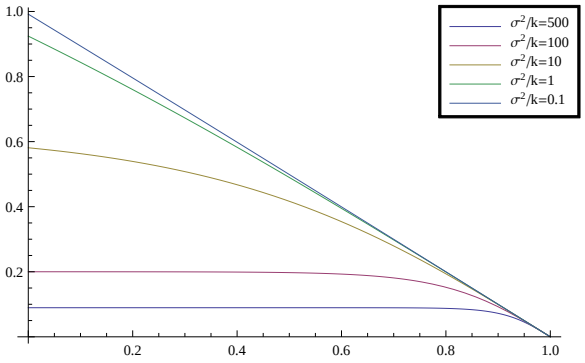


Figure 4.3: Plot of the function β
The parameters are given by $t = 0, T = 1$

4.2.4 Interpretation of strategy adjustment

We are now ready to put together all the parts of this chapter. First, the numerical results in section 4.2.2 indicate that the adjustment speed is already well described by it's zero order approximation. Further, lemma 4.2.1 indicates that the adjustment speed is symmetric in μ. This is consistent with the interpretation of speed, which

does not have a natural interpretation of direction. Further, the optimal portfolio is given by:

$$\theta(t,S) = \Delta(t,S) + \mu\beta(t)\frac{\partial\Delta}{\partial S} + \frac{\mu}{\sigma^2}(\mathbb{E}^0[G(S_T)|S_t = S] - X)$$

In a classical framework, the corresponding expansion would result in $\Delta(t,S) + \frac{\partial\Delta}{\partial S}\mu(T-t)$.[1]

The last term introduces a mean reversion towards the expectation of G into the wealth process. To understand the relationship, we note the following lower bound for the value function for $\mu = 0$:

$$\begin{aligned} V(t,X,H,S) &\geq \mathbb{E}[(\xi_T - G(S_T))^2] \geq \mathbb{E}[\xi_T - G(S_T)]^2 \\ &= (X_0 - \mathbb{E}[G(S_T)])^2 \end{aligned}$$

The bound arises because in the case $\mu = 0$, the wealth-process is a martingale. Therefore, no strategy can adjust the mean of the wealth towards the expectation of G. After the introduction of drift, there exists the possibility to affect the mean of the wealth by trading in the asset. If the wealth is too low to replicate the claim, then the drift of the asset can used to adjust the mean of the wealth process. However, it also implies that if the initial wealth is too high, then the optimal strategy suggests to loose expected wealth by trading in the asset. Obviously, this is in contradiction to the idea of replication. It arises, because the penalty function penalizes positive and negative deviations of terminal wealth equally. Economically, only negative deviations should be penalized. Therefore, the suggested quadratic penalty does not seem appropriate to model the hedging problem considered here.

Putting these economic objections aside, it is interesting to see how the target portfolio reacts to the incentives of the problem. First, the speed of the mean-adjustment term is given by $\frac{\mu}{\sigma^2}$. Specifically, the speed does not depend on the size of the transaction cost but is purely determined by properties of the stock-price process. Therefore, the ratio $\frac{\mu}{\sigma^2}$ describes the role of the asset. If the mean is big relative the volatility, then the asset is primarily used to adjust the mean of the wealth. If the mean is small compared to the volatility, then the asset is mainly used to hedge the stochastic component of terminal wealth.

Further, $\beta(t)$ depends on the ratio $\frac{\sigma^2}{k}$. If the volatility is low or the liquidity cost is high, then the adjustment term becomes $\mu(T-t)\frac{\partial\Delta}{\partial S}(t,S)$, the adjustment that would occur in a classical Δ-hedging setting. On the other hand, if the volatility

[1] It is worth pointing out, that the differential asymptotics computed here correspond to moment expansions in a stochastic framework.

is large or the liquidity cost is low, then the adjustment-term vanishes. Since the optimal adjustment-speed scales with $\frac{\sigma^2}{k}$, we can interpret this in the following way: If the optimal portfolio is going to be adjusted very often in the future, then today's portfolio should focus more heavily on events that are in the close future. Now, since the drift adjustment is of order $(T-t)$ whereas the volatility is of order $\sqrt{T-t}$, the volatility adjustments play a higher role than the drift adjustments as the time horizon shortens. Therefore, they should play a bigger role as the portfolio is rebalanced more frequently.

Another way to look at this is that including drift adds a finite-variation process to the stock price. Usually the Δ-hedge is used to get rid of the quadratic variation of a portfolio that is being short one option. Therefore, we could interpret the zero order expansion of Δ to be mainly concerned with quadratic variation whereas the first order expansion is a finite variation adjustment. The above formulation suggests it is better to adjust by applying mean-reversion rather than using a rather static portfolio adjustment.

Remark 10 (CIR-process). The case $n = 1$ is interesting because the liquidity cost scales with S, which seems most natural. However, the process defined by the SDE

$$\mathrm{d}S_t = \sigma\sqrt{S_t}\,\mathrm{d}W_t$$

is a poor model for the stock price process. Notice that since the Lipschitz-continuity of the volatility is missing, it might not even be uniquely defined. The solutions defined in this thesis can be used find expressions to a full CIR-process with mean-reversion speed κ and level θ. In this case, a two dimensional expansion of the form

$$\phi(t,S) = \phi(t,S)^{0,0} + \kappa\phi(t,S)^{1,0} + \theta\phi(t,S)^{0,1} + \kappa^2\phi(t,S)^{2,0} + \theta^2\phi(t,S)^{0,2} + \kappa\theta\phi^{1,1} + ...$$

is necessary. However, introducing drift might lead to problems regarding the sensibility of the derived strategy, as explained above.

Remark 11 (Volatility expansion). The case $\sigma(t,S) = \sigma S$ and liquidity cost $l(h) = kSh^2$ is considered in [14]. In this case, we could try to expand around $\sigma(t,S) = 0$ using the methodology described in this chapter. However, since the results do not involve expectations, it is harder to interpret the results in this case.

Chapter 5

Conclusion

In this thesis, we have investigated the problem of hedging derivatives under liquidity costs. At first, we motivated the formulation of liquidity by considering a limit order book. Then we formulated the hedging problem in an optimal control setting. Restricting the stochastic processes, we were able to solve the problem. We find that under the optimal strategy the holdings in the stock are mean-reverting towards a weighted average of future Δ-hedge portfolios. In the case of Brownian motion and deterministic liquidity cost, the target portfolio coincides with the Δ-hedge portfolio. The error of hedging the derivative is proportional to the variance of the Δ-hedge in this case. These results could be extended to include mean by using asymptotic analysis. We find that the adjustment-speed reacts symmetrically to the inclusion of mean. Numerical results indicate that even low order expansions describe the solutions reasonably well. Finally, the effects on the optimal portfolio are analyzed. We find that the optimal portfolio leads to a mean-reverting wealth process. However this might be the result of a questionable problem formulation, since both positive and negative deviations of terminal wealth are penalized. Last, if the optimal strategy recommends to hedge more aggressively in the future, then the Δ-hedge reacts less strongly to the inclusion of drift.

Future research should be concerned with extending the results to more general processes and stochastic liquidity formulations. The asymptotic analysis of the mean term also indicates that the problem setup needs to be adjusted to account for super-hedging if drift is included.

References

[1] Peter Bank and Dietmar Baum. Hedging and portfolio optimization in financial markets with a large trader. *Mathematical Finance*, **14**(1):1–18, 2004.

[2] Jean-Philippe Bouchaud, J Farmer, and Fabrizio Lillo. How markets slowly digest changes in supply and demand. *Fabrizio, How Markets Slowly Digest Changes in Supply and Demand (September 11, 2008)*, 2008.

[3] Umut Cetin, Robert Jarrow, Philip Protter, and Mitch Warachka. Pricing options in an extended black scholes economy with illiquidity: Theory and empirical evidence. *Review of Financial Studies*, **19**(2):493–529, 2006.

[4] Umut Cetin, Robert A Jarrow, and Philip Protter. Liquidity risk and arbitrage pricing theory. *Finance and Stochastics*, **8**(3):311–341, 2004.

[5] Mark HA Davis and Andrew R Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, **15**(4):676–713, 1990.

[6] J Doyne Farmer, Paolo Patelli, and Ilija I Zovko. The predictive power of zero intelligence in financial markets. *Proceedings of the National Academy of Sciences of the United States of America*, **102**(6):2254–2259, 2005.

[7] Rüdiger Frey and Alexander Stremme. Market volatility and feedback effects from dynamic hedging. *Mathematical finance*, **7**(4):351–374, 1997.

[8] Martin Gould, Mason Porter, Stacy Williams, Mark McDonald, Daniel Fenn, and Sam Howison. Limit order books. *Available at SSRN 1970185*, 2012.

[9] Joel Hasbrouck. Measuring the information content of stock trades. *The Journal of Finance*, **46**(1):179–207, 1991.

[10] Tianhui Michael Li and Robert Almgren. A fully-dynamic closed-form solution for δ-hedging with market impact. *working paper*, 2011.

[11] Fabrizio Lillo, J Doyne Farmer, and Rosario N Mantegna. Econophysics: Master curve for price-impact function. *Nature*, **421**(6919):129–130, 2003.

[12] Michael JP Magill and George M Constantinides. Portfolio selection with transaction costs. *Journal of Economic Theory*, **13**(2):245–263, 1976.

[13] Eckhard Platen and Martin Schweizer. On feedback effects from hedging derivatives. *Mathematical Finance*, **8**(1):67–84, 1998.

[14] Leonard CG Rogers and Surbjeet Singh. The cost of illiquidity and its effects on hedging. *Mathematical Finance*, **20**(4):597–615, 2010.

[15] K Ronnie Sircar and George Papanicolaou. General black-scholes models accounting for increased market volatility from hedging strategies. *Applied Mathematical Finance*, **5**(1):45–82, 1998.

[16] Paul Wilmott and Philipp J Schönbucher. The feedback effect of hedging in illiquid markets. *SIAM Journal on Applied Mathematics*, **61**(1):232–272, 2000.

[17] Wei-Xing Zhou. Universal price impact functions of individual trades in an order-driven market. *Quantitative Finance*, **12**(8):1253–1263, 2012.

Appendix A

Proof of theorem 3.2.1

Define $A = \frac{n(n-1)}{2}\sigma^2 = \mathbb{1}_{n=2}\sigma^2, B = \sigma^2$. Now substitute

$$\tilde{\gamma}(t) = -k\frac{u'(t)}{u(t)}$$

into the ODE (3.15) and divide by $-k$ to arrive at

$$\begin{aligned}\frac{u''u - u'^2}{u^2} + A\frac{u'}{u} - \frac{1}{k}B + \frac{u'^2}{u^2} &= 0,\\ u'' + Au' - \frac{1}{k}Bu &= 0.\end{aligned}$$

This is a constant coefficient second order ODE. The solution can be obtained by first solving for the roots of the characteristic equation

$$\begin{aligned}0 &= r^2 + Ar - \frac{1}{k}B\\ r_{1,2} &= -\frac{A}{2} \pm \sqrt{\frac{A^2}{4} + \frac{B}{k}}.\end{aligned}$$

Since $\frac{A^2}{4} + \frac{B}{k} > 0$, the general solution is given by

$$u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

for some constants C_1, C_2. This implies that

$$\begin{aligned}\tilde{\gamma}(t) &= -k\frac{C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t}}{C_1 e^{r_1 t} + C_2 e^{r_2 t}}\\ &= -k\frac{\frac{C_1}{C_2} r_1 e^{r_1 t} + r_2 e^{r_2 t}}{\frac{C_1}{C_2} e^{r_1 t} + e^{r_2 t}}.\end{aligned}$$

From the terminal condition $\gamma(T) = 0$, we find $\frac{C_1}{C_2} = -\frac{r_2 e^{r_2 T}}{r_1 e^{r_1 T}}$. Inserting this into the equation, we find

$$\tilde{\gamma}(t) = -k \frac{e^{-r_1(T-t)} - e^{-r_2(T-t)}}{\frac{1}{r_1} e^{-r_1(T-t)} - \frac{1}{r_2} e^{-r_2(T-t)}}.$$

Rewriting the last equation in terms of A, B leads to

$$\tilde{\gamma}(t) = \frac{B}{\sqrt{\frac{A^2}{4} + \frac{B}{k}} \coth\left(\sqrt{\frac{A^2}{4} + \frac{B}{k}}(T-t)\right) - \frac{A}{2}}.$$

Reinserting the the expressions for A, B and noticing $\gamma(t) = \frac{\tilde{\gamma(t)}}{k}$

$$\gamma(t) = \frac{1}{k} \frac{1}{\sqrt{\frac{\mathbb{1}_{n=2}}{4} + \frac{1}{k\sigma^2}} \coth\left(\sigma\sqrt{\frac{\sigma^2}{4}\mathbb{1}_{n=2} + \frac{1}{k}}(T-t)\right) - \frac{\mathbb{1}_{n=2}}{2}}.$$

To obtain the expression for ω, we note that the Feynman-Kac representation of c_2 is given by

$$\begin{aligned} c_2(t,S) &= \mathbb{E}\left[-\int_t^T \exp\left(-\int_t^u \frac{c_1(r,S_r)}{kS_r^n}\,\mathrm{d}r\right) 2\sigma^2 S_u^n \theta_\Delta(u,S_u)\,\mathrm{d}u\right] \\ &= \mathbb{E}\left[-\int_t^T \exp\left(-\int_t^u \gamma(r)\,\mathrm{d}r\right) 2\sigma^2 \xi_{t,u}\theta_\Delta(u,S_u)\,\mathrm{d}u\right] S^n. \end{aligned}$$

Therefore, we have to choose $\omega(t,u) = \sigma^2 \frac{\exp\left(-\int_t^u \gamma(u)\,\mathrm{d}u\right)}{\gamma(t)k} \frac{\mathbb{E}[S_u^n | S_t = S]}{S}$. Evaluating this expression and noting $\frac{\mathbb{E}[S_u^n|S_t=S]}{S} = 1$ of $n = 0, 1$ and $\exp(\sigma^2(u-t))$ for $n = 2$, we find:

$$\omega(t,u) = \sigma \frac{\cosh\left[\frac{\sigma}{\sqrt{k}}(T-u)\right]}{\cosh\left[\frac{\sigma}{\sqrt{k}}(T-t)\right]} \frac{\coth\left[\frac{\sigma}{\sqrt{k}}(T-t)\right]}{\sqrt{k}}.$$

in the case of $n = 0, 1$ and

$$\omega(t,u) = \sigma^2 \frac{\exp\left(s(t,u) + \sigma^2(u-t)\right)}{\tilde{\gamma}(t)}.$$

in the case $n = 2$. The function $s(t,u)$ is given on the following page.

A.1 Solution for weight function in case $n = 2$

$$
\begin{aligned}
s(t,u) = \frac{1}{k\left(\frac{1}{k}+\frac{\sigma}{4}\right)^{0.5}\sigma}\Bigg(&\frac{\mathrm{Log}\left[1.-\coth\left[(T-t)\left(\frac{1}{k}+0.25\sigma\right)^{0.5}\sigma\right]\right]}{-1.+2.\left(0.25+\frac{1}{k\sigma^2}\right)^{0.5}}\\
&-\frac{0.5\mathrm{Log}\left[1.-\coth\left[(T-1.u)\left(\frac{1}{k}+\frac{\sigma}{4}\right)^{0.5}\sigma\right]\right]}{-0.5+\left(0.25+\frac{1}{k\sigma^2}\right)^{0.5}}\\
&+\frac{0.5\mathrm{Log}\left[1.+\coth\left[(T-t)\left(\frac{1}{k}+\frac{\sigma}{4}\right)^{0.5}\sigma\right]\right]-0.5\mathrm{Log}\left[1.+\coth\left[(T-u)\left(\frac{1}{k}+\frac{\sigma}{4}\right)^{0.5}\sigma\right]\right]}{0.5+\left(0.25+\frac{1}{k\sigma^2}\right)^{0.5}}\\
&+1\Big/\left(-0.25+\left(0.25+\frac{1}{k\sigma^2}\right)\right)\left(0.25+\frac{1}{k\sigma^2}\right)^{0.5}\left(-\mathrm{Log}\left[-1.+2.\left(0.25+\frac{1}{k\sigma^2}\right)^{0.5}\coth\left[(T-t)\left(\frac{1}{k}+\frac{\sigma}{4}\right)^{0.5}\sigma\right]\right]\right.\\
&\left.+\mathrm{Log}\left[-1+2\left(0.25+\frac{1}{k\sigma^2}\right)^{0.5}\coth\left[(T-u)\left(\frac{1}{k}+\frac{\sigma}{4}\right)^{0.5}\sigma\right]\right]\right)\Bigg).
\end{aligned}
$$

Appendix B

Matlab-Code for comparison with numerical solution

```
%% Solve for mean reversion speed by nonlinear ODE solver:
%%  Set up Parameters

mu = [0.4 0.4 0.4 0.1, 0.1, 0.04];
sigma = [2 1.6 2.4 2 0.5 1];
k = 0.01;

%% bind parameters to ODE
for j = 1:6
diff{j} = @(t,x)(DifferenceMapping(t,x,k,sigma(j),mu(j)));
end

%% Solve ODE
options = odeset('RelTol',1e-8,'AbsTol',[1e-8 1e-8 1e-8],'Stats','on', 'MaxStep',1E
    -4,'Refine',0.5);

for j = 1:6
[T{j},Y{j}] = ode45(diff{j},[0 1],[1 0 0],options);
T{j}= T{j}(end:-1:1);
end


%% Define Approximations
for j = 1:6
lambda{j}=@(t)(tanh(sigma(j)/sqrt(k)*(1-t)));
gamma{j}= @(t)(sigma(j)/sqrt(k)*lambda{j}(t));
a0{j} =@(t)(ones(length(t),1));
a2{j} =@(t)(- k/sigma(j)^4*(gamma{j}(t) + sigma(j)^2/k*(1-t)));
b1{j} =@(t)(2*k/sigma(j)^2 *gamma{j}(t));
c0{j} =@(t)(k*gamma{j}(t));
funToInt{j} = @(t,u)(sigma(j)^2/k (cosh(sigma(j)/sqrt(k)*(1-u))./cosh(sigma(j)/sqrt(k
    )*(1-t)).*coth(sigma(j)/sqrt(k)*(1-u))).^2.*(2*gamma(u) +sigma^4/k* a2{j}(u)));
c2{j} =@(t)(k^3/sigma{j}^6*gamma{j}(t)^2 *quad(funToInt,t,1));


a{j}=@(t)(a0{j}(t)+ mu(j)^2*a2{j}(t));
b{j}=@(t)(mu(j)*b1{j}(t));
c{j}=@(t)(c0{j}(t) + mu(j)^2*c2{j}(t));
end
%% Plots:
plot(T{1},Y{1}(:,1),'-',T{1},a{1}(T{1}),'-',T{1},Y{1}(:,2),'-',T{1},b{1}(T{1}),'-',T
    {1},Y{1}(:,3),'-',T{1},c0{1}(T{1}),'-');
legend('a solution','a expansion','b solution','b expansion','c solution','c
    expansion','Location','East');
```

```
41 print('-depsc','C:\Users\Hendrik\Google Drive\Master Thesis\LaTeX\Figures\
       AsymptoticPlot.eps')
42 %% Relative differences a,
43 for j = 1:6
44   relDiff{j} =100 * (a{j}(T{j})./Y{j}(:,1)-1);
45 end
46
47 plot(T{1},relDiff{1},'-',T{2},relDiff{2},'-',T{3},relDiff{3},'-',T{4},relDiff{4},'-',
       T{5},relDiff{5},'-',T{6},relDiff{6},'-');
48 legend(['mu = '  num2str(mu(1)) ', sig = '  num2str(sigma(1))],['mu = ' num2str(mu(2)
       ) ', sig = ' num2str(sigma(2))],['mu = ' num2str(mu(3)) ', sig = '  num2str(sigma
       (3))], ['mu = ' num2str(mu(4)) ', sig = '  num2str(sigma(4))],['mu = ' num2str(mu
       (5)) ', sig = ' num2str(sigma(5))],['mu = ' num2str(mu(6)) ', sig = ' num2str(
       sigma(6))],'Location','SouthEast');
49 h=gca;
50 labels=get(h,'yticklabel');
51 clear labels_modif
52 for i=1:size(labels,1)
53    labels_modif(i,:)=[labels(i,:) ' %'];
54 end
55 set(h,'yticklabel',labels_modif);
56 print('-depsc','C:\Users\Hendrik\Google Drive\Master Thesis\LaTeX\Figures\
       AsymptoticErrorA.eps')
57 %% b
58 for j = 1:6
59   relDiff{j} =100 * (b{j}(T{j})./Y{j}(:,2)-1);
60 end
61
62 plot(T{1},relDiff{1},'-',T{2},relDiff{2},'-',T{3},relDiff{3},'-',T{4},relDiff{4},'-',
       T{5},relDiff{5},'-',T{6},relDiff{6},'-');
63
64 legend(['mu = '  num2str(mu(1)) ', sig = '  num2str(sigma(1))],['mu = ' num2str(mu(2)
       ) ', sig = ' num2str(sigma(2))],['mu = ' num2str(mu(3)) ', sig = '  num2str(sigma
       (3))], ['mu = ' num2str(mu(4)) ', sig = '  num2str(sigma(4))],['mu = ' num2str(mu
       (5)) ', sig = ' num2str(sigma(5))],['mu = ' num2str(mu(6)) ', sig = ' num2str(
       sigma(6))]);
65 h=gca;
66 labels=get(h,'yticklabel');
67 clear labels_modif
68 for i=1:size(labels,1)
69    labels_modif(i,:)=[labels(i,:) ' %'];
70 end
71 set(h,'yticklabel',labels_modif);
72 print('-depsc','C:\Users\Hendrik\Google Drive\Master Thesis\LaTeX\Figures\
       AsymptoticErrorB.eps')
73 %% c
74 for j = 1:6
75   relDiff{j} =100 * (c{j}(T{j})./Y{j}(:,3)-1);
76 end
77
78 plot(T{1},relDiff{1},'-',T{2},relDiff{2},'-',T{3},relDiff{3},'-',T{4},relDiff{4},'-',
       T{5},relDiff{5},'-',T{6},relDiff{6},'-');
79
80 legend(['mu = '  num2str(mu(1)) ', sig = '  num2str(sigma(1))],['mu = ' num2str(mu(2)
       ) ', sig = ' num2str(sigma(2))],['mu = ' num2str(mu(3)) ', sig = '  num2str(sigma
       (3))], ['mu = ' num2str(mu(4)) ', sig = '  num2str(sigma(4))],['mu = ' num2str(mu
       (5)) ', sig = ' num2str(sigma(5))],['mu = ' num2str(mu(6)) ', sig = ' num2str(
       sigma(6))]);
81 h=gca;
82 labels=get(h,'yticklabel');
83 clear labels_modif
84 for i=1:size(labels,1)
85    labels_modif(i,:)=[labels(i,:) ' %'];
86 end
87 set(h,'yticklabel',labels_modif);
88 print('-depsc','C:\Users\Hendrik\Google Drive\Master Thesis\LaTeX\Figures\
       AsymptoticErrorC.eps')
```

Listing B.1: Comparison

```
function [dx_dt]= DifferenceMapping(t,x, k, sigma, mu)
%a function which returns a rate of change vector
dx_dt(1) = - 0.25* x(2)^2/k;
dx_dt(2) =   2 * mu * x(1) - x(2)*x(3)/k;
dx_dt(3) =   sigma^2 * x(1)  + mu * x(2) - x(3)^2/k;
%transpose dx_dt so it is a column vector
dx_dt = dx_dt';
return
```

Listing B.2: ODE-system

Appendix C

Mathematica-Code for symbolical calculations

```
In[1]:= h[t_,a_] = 1/((a/4+1/k/  ^2)^0.5*Coth[  *((  /4*a+1/k)^0.5(T-t))]-a/2)
Out[1]= 1/(-(a/2)+(a/4+1/(k   ^2))^0.5 Coth[(-t+T)    (1/k+(a   )/4)^0.5])
In[2]:= o[r_,a_] =Integrate[h[r,a],r]
Out[2]= -(1/(   (1/k+0.25 a   )^0.5))2 (-((0.25 Log[1. -1. Coth[(-1. r+T)    (1/k
    +0.25 a   )^0.5]])/(-0.5 a+1. (0.25 a+1/(k   ^2))^0.5))-(0.5 Log[1. +Coth[(-1. r+
    T)    (1/k+0.25 a   )^0.5]])/(a+2. (0.25 a+1/(k   ^2))^0.5)+(0.5 (0.25 a+1/(k
    ^2))^0.5 Log[-1. a+2. (0.25 a+1/(k   ^2))^0.5 Coth[(-1. r+T)    (1/k+0.25 a   )
    ^0.5]])/(-0.25 a^2+1. (0.25 a+1/(k   ^2))^1.))
In[3]:= l[ _ ,k_,t_,T_]=   ^2*FullSimplify[Exp[(-o[u,0]+o[t,0])/k]/h[t,0],(k|  |t| u|
    T)  R e a l s && k>0&&  >0&& T>u && u>t]
Out[3]= (1.   ^1. Cosh[((T-1. u)   )/k^0.5] Coth[((-t+T)   )/k^0.5] Sech[((-1. t+T)
      )/k^0.5])/k^0.5
In[4]:= l2[ _ ,k_,t_,T_] =FullSimplify[h[t,0]/k,(k|  |t|T)  R e a l s && k>0&&  >0&& T>t
    ]
Out[4]= (1.   ^1. Tanh[((-t+T)   )/k^0.5])/k^0.5
l3[ _ ,k_,t_,T_]=FullSimplify[h[t,1]/k,(k|  |t|T)  R e a l s && k>0&&  >0&& T>t ]
1/(-(k/2)+k (1/4+1/(k   ^2))^0.5 Coth[(-t+T) (1/k+  /4)^0.5   ])
l4[ _ ,k_,t_,T_]=  ^2*FullSimplify[Exp[(-o[u,1]+o[t,1])/k +1*  ^2*(u-t)]/h[t,1],(k|s|
    t| u|T)  R e a l s && k>0&&  >0&& T>u && u>t]
E^((-t+u)   ^2+((0.5 Log[1. -1. Coth[(-1. t+T) (1/k+0.25   )^0.5   ]]-0.5 Log[1. -1.
    Coth[(T-1. u) (1/k+0.25   )^0.5   ]])/(-0.5+1. (0.25 +1/(k   ^2))^0.5)+(0.5 Log
    [1. +Coth[(-1. t+T) (1/k+0.25   )^0.5   ]]-0.5 Log[1. +Coth[(T-1. u) (1/k+0.25
    )^0.5   ]])/(0.5 +1. (0.25 +1/(k   ^2))^0.5)+((0.25 +1/(k   ^2))^0.5 (-1. Log
    [-1.+2. (0.25 +1/(k   ^2))^0.5 Coth[(-1. t+T) (1/k+0.25   )^0.5   ]]+1. Log
    [-1.+2. (0.25 +1/(k   ^2))^0.5 Coth[(T-1. u) (1/k+0.25   )^0.5   ]]))/(-0.25+1.
    (0.25 +1/(k   ^2))^1.))/(k (1/k+0.25   )^0.5   ))   ^2 (-(1/2)+(1/4+1/(k   ^2))
    ^0.5 Coth[(-t+T) (1/k+  /4)^0.5   ])
FullSimplify[1/(k (1/k+0.25‘   )^0.5‘   ) (Log[1.‘ -1.‘ Coth[(-1.‘ t+T) (1/k+0.25‘
    )^0.5‘   ]]/(-1.‘+2.‘ (0.25‘ +1/(k   ^2))^0.5‘)-(0.5‘ Log[1.‘ -1.‘ Coth[(T-1.‘ u)
     (1/k+0.25‘   )^0.5‘   ]])/(-0.5‘+1.‘ (0.25‘ +1/(k   ^2))^0.5‘)+(0.5‘ Log[1.‘ +
    Coth[(-1.‘ t+T) (1/k+0.25‘   )^0.5‘   ]]-0.5‘ Log[1.‘ +Coth[(T-1.‘ u) (1/k+0.25‘
      )^0.5‘   ]])/(0.5‘ +1.‘ (0.25‘ +1/(k   ^2))^0.5‘)+1/(-0.25‘+1.‘ (0.25‘ +1/(k
    ^2))^1.‘)(0.25‘ +1/(k   ^2))^0.5‘ (-1.‘ Log[-1.‘+2.‘ (0.25‘ +1/(k   ^2))^0.5‘
    Coth[(-1.‘ t+T) (1/k+0.25‘   )^0.5‘   ]]+1.‘ Log[-1.‘+2.‘ (0.25‘ +1/(k   ^2))
    ^0.5‘ Coth[(T-1.‘ u) (1/k+0.25‘   )^0.5‘   ]])),(k|s|t| u|T)  R e a l s && k>0&&s>0&&
     T>u && u>t]
1/(k (1/k+0.25   )^0.5   ) (Log[1. -1. Coth[(-1. t+T) (1/k+0.25   )^0.5   ]]/(-1.+2.
    (0.25 +1/(k   ^2))^0.5)-(0.5 Log[1. -1. Coth[(T-1. u) (1/k+0.25   )^0.5   ]])
    /(-0.5+1. (0.25 +1/(k   ^2))^0.5)+(0.5 Log[1. +Coth[(-1. t+T) (1/k+0.25   )^0.5
      ]]-0.5 Log[1. +Coth[(T-1. u) (1/k+0.25   )^0.5   ]])/(0.5 +1. (0.25 +1/(k   ^2)
    )^0.5)+1/(-0.25+1. (0.25 +1/(k   ^2))^1.) (0.25 +1/(k   ^2))^0.5 (-1. Log[-1.+2.
    (0.25 +1/(k   ^2))^0.5 Coth[(-1. t+T) (1/k+0.25   )^0.5   ]]+1. Log[-1.+2. (0.25
    +1/(k   ^2))^0.5 Coth[(T-1. u) (1/k+0.25   )^0.5   ]]))
Integrate[l[k,  ,t,T],{u,t,T}]
1.
```

```
Integrate[l3[k,  ,t,T],{u,t,T}]

 SubsuperscriptBox[        ,    t    ,    T   ]
 SuperscriptBox[   E    ,
FractionBox[
 FractionBox[   0 .5‘\ Log[1.‘  - 1.‘\ Coth[k\    (   -1.‘   \ t + T)   \
 SuperscriptBox[   (0.25‘\ k +
 FractionBox[   1    ,         ])   ,    0 .5‘   ]]] - 0.5‘\ Log[1.‘  - 1.‘\ Coth[k\
      (T - 1.‘\ u)   \
 SuperscriptBox[   (0.25‘\ k +
 FractionBox[   1    ,         ])   ,    0 .5‘   ]]]   ,       -0.5‘    + 1.‘\
 SuperscriptBox[   (0.25‘  +
 FractionBox[   1    ,
 SuperscriptBox[   k    ,    2   ]\      ])   ,    0 .5‘   ]   ] +
 FractionBox[   0 .5‘\ Log[1.‘  + Coth[k\    (   -1.‘   \ t + T)   \
 SuperscriptBox[   (0.25‘\ k +
 FractionBox[   1    ,         ])   ,    0 .5‘   ]]] - 0.5‘\ Log[1.‘  + Coth[k\    (T
     - 1.‘\ u)   \
 SuperscriptBox[   (0.25‘\ k +
 FractionBox[   1    ,         ])   ,    0 .5‘   ]]]   ,    0 .5‘  + 1.‘\
 SuperscriptBox[   (0.25‘  +
 FractionBox[   1    ,
 SuperscriptBox[   k    ,    2   ]\      ])   ,    0 .5‘   ]   ] +
 FractionBox[
 SuperscriptBox[   (0.25‘  +
 FractionBox[   1    ,
 SuperscriptBox[   k    ,    2   ]\      ])   ,    0 .5‘   ]\    (   -1.‘   \ Log[
    -1.‘    + 2.‘\
 SuperscriptBox[   (0.25‘  +
 FractionBox[   1    ,
 SuperscriptBox[   k    ,    2   ]\      ])   ,    0 .5‘   ]\ Coth[k\    (   -1.‘   \
     t + T)   \
 SuperscriptBox[   (0.25‘\ k +
 FractionBox[   1    ,         ])   ,    0 .5‘   ]]] + 1.‘\ Log[   -1.‘    + 2.‘\
 SuperscriptBox[   (0.25‘  +
 FractionBox[   1    ,
 SuperscriptBox[   k    ,    2   ]\      ])   ,    0 .5‘   ]\ Coth[k\    (T - 1.‘\ u)
       \
 SuperscriptBox[   (0.25‘\ k +
 FractionBox[   1    ,         ])   ,    0 .5‘   ]]])       ,       -0.25‘    + 1.‘\
 SuperscriptBox[   (0.25‘  +
 FractionBox[   1    ,
 SuperscriptBox[   k    ,    2   ]\      ])   ,    1 .‘   ]   ]   ,    k \
 SuperscriptBox[   (0.25‘\ k +
 FractionBox[   1    ,         ])   ,    0 .5‘   ]\      ]]\    (   -
 FractionBox[   1    ,    2   ]    +
 SuperscriptBox[   (
 FractionBox[   1    ,    4   ] +
 FractionBox[   1    ,
 SuperscriptBox[   k    ,    2   ]\      ])   ,    0 .5‘   ]\ Coth[k\    (   - t    +
    T)   \
 SuperscriptBox[   (
 FractionBox[   k    ,    4   ] +
 FractionBox[   1    ,         ])   ,    0 .5‘   ]])       u

Integrate[l4[k,  ,t,T],{u,t,T}]

 SubsuperscriptBox[        ,    t    ,    T   ]
 SuperscriptBox[   E    ,       - t    + u +
 FractionBox[
 FractionBox[   0 .5‘\ Log[1.‘  - 1.‘\ Coth[   (   -1.‘   \ t + T)   \
 SuperscriptBox[   (0.25‘  +
 FractionBox[   1    ,         ])   ,    0 .5‘   ]]] - 0.5‘\ Log[1.‘  - 1.‘\ Coth[
    (T - 1.‘\ u)   \
 SuperscriptBox[   (0.25‘  +
 FractionBox[   1    ,         ])   ,    0 .5‘   ]]]   ,       -0.5‘    + 1.‘\
 SuperscriptBox[   (0.25‘  +
 FractionBox[   1    ,         ])   ,    0 .5‘   ]   ] +
```

```
 FractionBox[  0.5`\ Log[1.`  + Coth[   (   -1.`   \ t + T)   \
 SuperscriptBox[   (0.25`  +
 FractionBox[  1   ,         ])   ,   0.5`   ]]] - 0.5`\ Log[1.`  + Coth[   (T -
    1.`\ u)   \
 SuperscriptBox[   (0.25`  +
 FractionBox[  1   ,         ])   ,   0.5`   ]]]   ,   0.5`  + 1.`\
 SuperscriptBox[   (0.25`  +
 FractionBox[  1   ,         ])   ,   0.5`   ]   ] +
 FractionBox[
 SuperscriptBox[   (0.25`  +
 FractionBox[  1   ,         ])   ,   0.5`   ]\    (   -1.`   \ Log[   -1.`    +
    2.`\
 SuperscriptBox[   (0.25`  +
 FractionBox[  1   ,         ])   ,   0.5`   ]\ Coth[   (   -1.`   \ t + T)   \
 SuperscriptBox[   (0.25`  +
 FractionBox[  1   ,         ])   ,   0.5`   ]]] + 1.`\ Log[   -1.`    + 2.`\
 SuperscriptBox[   (0.25`  +
 FractionBox[  1   ,         ])   ,   0.5`   ]\ Coth[   (T - 1.`\ u)   \
 SuperscriptBox[   (0.25`  +
 FractionBox[  1   ,         ])   ,   0.5`   ]]])      ,       -0.25`    + 1.`\
 SuperscriptBox[   (0.25`  +
 FractionBox[  1   ,         ])   ,   1.`   ]   ]   ,
 SuperscriptBox[   (0.25`  +
 FractionBox[  1   ,         ])   ,   0.5`   ]\      ]   ]\    (   -
 FractionBox[  1   ,    2   ]    +
 SuperscriptBox[   (
 FractionBox[  1   ,    4   ] +
 FractionBox[  1   ,         ])   ,   0.5`   ]\ Coth[   (   - t    + T)   \
 SuperscriptBox[   (
 FractionBox[  1   ,    4   ] +
 FractionBox[  1   ,         ])   ,   0.5`   ])       u
NIntegrate[l4[0.00001,0.2,0,3],{u,0,3},MinRecursion->4]
1. +0. I

In[152]:= options[legendMaker]=Join[FilterRules[Options[Framed],Except[{ImageSize,
    FrameStyle,Background,RoundingRadius,ImageMargins}]],{FrameStyle->None,Background
    ->Directive[Opacity[.7],LightGray],RoundingRadius->10,ImageMargins->0,PlotStyle->
    Automatic,PlotMarkers->None,"LegendLineWidth"->35,"LegendLineAspectRatio"->.3,"
    LegendMarkerSize"->8,"LegendGridOptions"->{Alignment->Left,Spacings->{.4,.1}}}];

legendMaker[textLabels_,opts:OptionsPattern[]]:=Module[{f,lineDirectives,
    markerSymbols,n=Length[textLabels],x},lineDirectives=((PlotStyle/.{opts})/.
    PlotStyle|Automatic:>Map[ColorData[1],Range[n]])/.None->{None};
markerSymbols=Replace[((PlotMarkers/.{opts})/.Automatic:>(Drop[ListPlot[Transpose[{
    Range[n]}],PlotMarkers->Automatic][[1,2]][[1]],-1]/.Inset[x_,i__]:>x)[[All,-1]])
    /.{Graphics[gr__],sc_}:>Graphics[gr,ImageSize->("LegendMarkerSize"/.{opts}/.
    Options[legendMaker,"LegendMarkerSize"]/.{"LegendMarkerSize"->8})],PlotMarkers|
    None:>Map[Style["",Opacity[0]]&,textLabels]]/.None|{}->Style["",Opacity[0]];
lineDirectives=PadRight[lineDirectives,n,lineDirectives];
markerSymbols=PadRight[markerSymbols,n,markerSymbols];
f=Grid[MapThread[{Graphics[{#1/.None->{},If[#1==={None}||(PlotStyle/.{opts})===None
    ,{},Line[{{-.1,0},{.1,0}}]],Inset[#2,{0,0},Background->None]},AspectRatio->("
    LegendLineAspectRatio"/.{opts}/.Options[legendMaker,"LegendLineAspectRatio"]/.{"
    LegendLineAspectRatio"->.2}),ImageSize->("LegendLineWidth"/.{opts}/.Options[
    legendMaker,"LegendLineWidth"]/.{"LegendLineWidth"->35}),ImagePadding
    ->{{1,1},{0,0}}],Text[#3,FormatType->TraditionalForm]}&,{lineDirectives,
    markerSymbols,textLabels}],Sequence@Evaluate[("LegendGridOptions"/.{opts}/.
    Options[legendMaker,"LegendGridOptions"]/.{"LegendGridOptions"->{Alignment->Left,
    Spacings->{.4,.1}}})]];
Framed[f,FilterRules[{Sequence[opts,Options[legendMaker]]},FilterRules[Options[Framed
    ],Except[ImageSize]]]];


extractStyles[plot_]:=Module[{lines,markers,extract=First[Normal[plot]]},(*In a plot,
    the list of lines contains no insets,so I use this to find it:*)lines=Select[
    extract,FreeQ[#1,Inset]&];
(*Most plot markers are inside Inset,except for Point in list plots:*)markers=Select[
    extract,!FreeQ[#1,Inset]&];
```

```
(*The function returns a list of lists:*){(*The first return value is the list of
    line plot styles:*)Replace[Cases[lines,{c__,Line[__],___}:>Flatten[
    Directive@@Cases[{c},Except[_Line]]],Infinity],{}->None],(*Second return value:
    marker symbols*)Replace[Join[Cases[markers,{c__,Inset[s_,pos_,d___],e___}:>If[(*
    markers "s" can be strings or graphics*)Head[s]===Graphics,(*Append scale factor
    in case it's needed later;
default 0.01*){s,Last[{.01,d}]/.Scaled[f_]:>First[f]},If[(*For strings,add line color
     if no color specified via text styles:*)FreeQ[s,CMYKColor|RGBColor|GrayLevel|Hue
    ],Style[s,c],s]],Infinity],(*"Point" counts as marker but appears in "lines".
    Filter out Pointsize-legends don't need it:*)Cases[lines,{c___,PointSize[_],d___,
    Point[pt__],___}:>{Graphics[{c,d,Point[{0,0}]}],.01},Infinity]],{}->None]}]

Options[autoLegend]={Alignment->{Right,Top},Background->White,AspectRatio->Automatic
    };
autoLegend[plot_Graphics,labels_,opts:OptionsPattern[]]:=Module[{lines,markers,align=
    OptionValue[Alignment]},{lines,markers}=extractStyles[plot];
Graphics[{Inset[plot,{-1,-1},{Left,Bottom},Scaled[1]],Inset[legendMaker[labels,
    PlotStyle->lines,PlotMarkers->markers,opts],align,Map[If[NumericQ[#],Center,#]&,
    align]]},PlotRange->{{-1,1},{-1,1}},AspectRatio->(OptionValue[AspectRatio]/.
    Automatic:>(AspectRatio/.Options[plot,AspectRatio])/.Automatic:>(AspectRatio/.
    AbsoluteOptions[plot,AspectRatio]))]]
plot1 =Plot[{l[Sqrt[500],1,0,1],l[10,1,0,1],l[Sqrt[10],1,0,1],l[1,1,0,1],l[1,10,0,1]
    },{u,0,1},PlotRange->{0.0,4.0}]
autoLegend[plot1,{ "  ^2/k=500","  ^2/k=100","  ^2/k=10","  ^2/k=1","  ^2/k=0.1"}]

plot1 =Plot[{l2[Sqrt[500],1,t,1],l2[10,1,t,1],l2[Sqrt[10],1,t,1],l2[1,1,t,1],l2[1,10,
    t,1] },{t,0,1}]
autoLegend[plot1,{ "  ^2/k=500","  ^2/k=100","  ^2/k=10","  ^2/k=1","  ^2/k=0.1"}]
plot1 =Plot[{l3[0.1,0.00002,0,1],l3[0.1,0.0001,0,1],l3[0.1,0.001,0,1],l3
    [0.1,0.01,0,1],l3[0.1,Sqrt[0.1],0,1] },{u,0,1}]
autoLegend[plot1,{ "  ^2/k=500","  ^2/k=100","  ^2/k=10","  ^2/k=1","  ^2/k=0.1"}]
plot1 =Plot[{l4[0.1,0.00002,0,1],l4[0.1,0.0001,0,1],l4[0.1,0.001,0,1],l4
    [0.1,0.01,0,1],l4[0.1,Sqrt[0.1],0,1]},{u,0,1}]
autoLegend[plot1,{ "  ^2/k=500","  ^2/k=100","  ^2/k=10","  ^2/k=1","  ^2/k=0.1"}]
In[6]:= a2[t_] = Integrate[l2[  ,k,u,T]^2,{u,t,T}]
Out[6]= -((1.   ^1. (1. t   -1. T   -1. k^0.5 Tanh[(1. t   -1. T   )/k^0.5]))/k^1.)
In[7]:= Integrate[l[  ,k,t,T]^2*(2*l2[  ,k,u,T] -a2[u] ),u]
Out[5]= (1.   ^2. Coth[((-t+T)   )/k^0.5]^2 (  C o s h [((T-1. u)   )/k^0.5]^2 (-a2[u
    ]+(2.   ^1. Tanh[((T-u)   )/k^0.5])/k^0.5)  u ) Sech[((-1. t+T)   )/k^0.5]^2)/k
    ^1.
In[159]:= g[ _ ,k_,t_,T_]= -Integrate[l[  ,k,t,T]*(-2*k/  ^2)* l2[  ,k,u,T],{u,t,T}]
Out[159]= (2. k^0.5 (-1.+1. Cosh[(1. t   -1. T   )/k^0.5]) Coth[((-t+T)   )/k^0.5]
    Sech[((-1. t+T)   )/k^0.5])/
In[160]:= plot1 =Plot[{g[1,0.002,t,1],g[1,0.01,t,1],g[1,0.1,t,1],g[1,1,t,1],g[1,10,t
    ,1]},{t,0,1}]
autoLegend[plot1,{ "  ^2/k=500","  ^2/k=100","  ^2/k=10","  ^2/k=1","  ^2/k=0.1"}]
```

Listing C.1: Comparision