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Nonlinear Correction to the Euler Buckling Formula for Compressible Cylinders

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Abstract Euler's celebrated buckling formula gives the critical load N for the buckling of a slender cylindrical column with radius B and length L as

$$N/(\pi^3 B^2) = (E/4)(B/L)^2,$$

where E is Young's modulus. Its derivation relies on the assumptions that linear elasticity applies to this problem, and that the slenderness (B/L) is an infinitesimal quantity. Here we ask the following question: What is the first nonlinear correction in the right hand-side of this equation when terms up to $(B/L)^4$ are kept? To answer this question, we specialize the exact solution of non-linear elasticity for the homogeneous compression of a thick cylinder with lubricated ends to the theory of third-order elasticity. In particular, we highlight the way second- and third-order constants —including Poisson's ratio— all appear in the coefficient of $(B/L)^4$.

Keywords Column buckling · Euler formula · Non-linear correction

Mathematics Subject Classification (2000) MSC 74B15 · MSC 74B10 · MSC 74B20 · MSC 74G05 · MSC 74G15 · MSC 74G60

1 Introduction

One of the first, and most important, problems to be tackled by the theory of linear elasticity is that of the buckling of a column under an axial load. Using Bernoulli's beam equations,

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Euler showed that the critical load of compression N_c leading to the buckling of a slender cylindrical column of radius B and length L is given by

$$\frac{N_c}{\pi^3 B^2} = \frac{E}{4}\left(\frac{B}{L}\right)^2, \tag{1}$$

where E is the Young modulus. The extension of this formula to the case of a thick column is a non-trivial, even sophisticated, problem of non-linear three-dimensional elasticity. In general, progress can be only made by using reductive (rod, shells, etc.) theories. However, there is a choice of boundary conditions of particular importance and significance in rod theory where one end is built-in and the other one is free to move while the direction of the rod tangent at the other end remains aligned with the original axis (See for instance Fig 2-4ab, page 49 in [1]). In such a case, the criterion (1) is also valid (this can be seen by symmetry consideration and by extending the classical Euler buckled solution for the hinged-hinged case by a half-length). In three-dimensions, these boundary conditions correspond to the case where the upper and lower faces of the cylinder remain normal to the initial axis and one end is built-in. Under such boundary conditions, the initial homogeneous deformation can be solved *exactly* in non-linear elasticity. Remarkably, the linearized equations of an incremental static deformation superimposed upon the large compression can also be solved exactly and universally (for every elastic material). In this case, the Euler formula can be extended to the case of a column with finite dimensions, for arbitrary constitutive law.

The exact incremental solution allows for an explicit derivation of Euler's formula at the *onset of non-linearity*, which combines third-order elastic constants with a term in $(B/L)^4$. In Goriely et al. [2], we showed that for an incompressible cylinder,

$$\frac{N_c}{\pi^3 B^2} = \frac{E}{4}\left(\frac{B}{L}\right)^2 - \frac{\pi^2}{96}\left(\frac{20}{3}E + 9\mathscr{A}\right)\left(\frac{B}{L}\right)^4, \tag{2}$$

where $\mathscr{A}$ is Landau's third-order elasticity constant. This formula clearly shows that *geometrical non-linearities* (term in $(B/L)^4$) are intrinsically coupled to *physical non-linearities* (term in $\mathscr{A}$) for this problem. (For the connection between Euler's theory of buckling and incremental incompressible nonlinear elasticity, see the early works of Wilkes [3], Biot [4], Fosdick and Shield [5], and the references collected in [2].)

Now, in third-order *in*compressible elasticity, there are two elastic constants [6]: the shear modulus μ $(= E/3)$ and $\mathscr{A}$. In third-order *compressible* elasticity, there are five elastic constants: λ and μ, the (second-order) Lamé constants (or equivalently, E and ν, Young's modulus and Poisson's ratio), and $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, the (third-order) Landau constants. Euler's formula at order $(B/L)^2$, equation (1), involves only one elastic constant, E. It is thus natural to ask whether Poisson's ratio, ν, plays a role in the non-linear correction to Euler formula of $(B/L)^4$, the next-order term. The final answer is found as formula (30) below, which shows that the non-linear correction involves all five elastic constants.

2 Finite compression and buckling

In this section, we recall the equations governing the homogeneous compression of a cylinder in the theory of exact (finite) elasticity. We also recall the form of some incremental solutions that is, of some small-amplitude static deformations which may be superimposed upon the large compression, and which indicate the onset of instability for the cylinder.

2.1 Large deformation

We take a cylinder made of an hyperelastic, compressible, isotropic solid with strain-energy density W say, with radius B and length L in its undeformed configuration. We denote by (R, Θ, Z) the coordinates of a particle in the cylinder at rest, in a cylindrical system.

Then we consider that the cylinder is subject to the following deformation,

$$r = \lambda_1 R, \qquad \theta = \Theta, \qquad z = \lambda_3 Z, \tag{3}$$

where (r, θ, z) are the current coordinates of the particle initially at (R, Θ, Z), λ_1 is the radial stretch ratio and λ_3 is the axial stretch ratio. Explicitly, $\lambda_1 = b/B$ and $\lambda_3 = l/L$, where b and l are the radius and length of the deformed cylinder, respectively. The physical components of F, the corresponding deformation gradient, are: $\mathsf{F} = \mathrm{Diag}\,(\lambda_1, \lambda_1, \lambda_3)$, showing that the deformation is equi-biaxial and homogeneous (and thus, universal).

The (constant) Cauchy stresses required to maintain the large homogeneous compression are (see, for instance, Ogden [7]),

$$\sigma_i = (\det \mathsf{F})^{-1}\, \lambda_i W_i, \qquad i = 1,2,3 \quad \text{(no sum)}, \tag{4}$$

where $W_i \equiv \partial W/\partial \lambda_i$. In our case, $\sigma_1 = \sigma_2$ because the deformation is equi-biaxial, and $\sigma_1 = \sigma_2 = 0$ because the outer face of the cylinder is free of traction. Hence

$$\sigma_1 = \sigma_2 = \lambda_1^{-1}\lambda_3^{-1} W_1 = 0, \qquad \sigma_3 = \lambda_1^{-2} W_3. \tag{5}$$

Note that we may use the first equality to express one principal stretch in terms of the other (provided, of course, that inverses can be performed).

2.2 Incremental equations

Now we recall the equations governing the equilibrium of incremental solutions, in the neighborhood of the finite compression. They read in general as [7]

$$\mathrm{div}\, \mathsf{s} = \mathbf{0}, \tag{6}$$

where s is the incremental nominal stress tensor. It is related to $\mathbf{u}$, the incremental mechanical displacement, through the incremental constitutive law,

$$\mathbf{s} = \mathsf{B}\,(\mathrm{Grad}\, \mathbf{u})^T, \tag{7}$$

where B is the fourth-order tensor of incremental elastic moduli and the gradient is computed in the current cylindrical coordinates [8]. The non-zero components of B, in a coordinate system aligned with the principal axes associated with the deformation (3), are given in general by [7]

$$\begin{aligned}
& B_{iijj} = \lambda_i \lambda_j W_{ij}/(\lambda_1\lambda_2\lambda_3), && \\
& B_{ijij} = (\lambda_i W_i - \lambda_j W_j)\lambda_i^2/(\lambda_i^2 - \lambda_j^2), && i \neq j,\ \lambda_i \neq \lambda_j, \\
& B_{ijji} = (\lambda_j W_i - \lambda_i W_j)\lambda_i \lambda_j/(\lambda_i^2 - \lambda_j^2), && i \neq j,\ \lambda_i \neq \lambda_j, \\
& B_{ijij} = (B_{iiii} - B_{iijj} + \lambda_i W_i)/2, && i \neq j,\ \lambda_i = \lambda_j, \\
& B_{ijji} = B_{jiij} = B_{ijij} - \lambda_i W_i, && i \neq j,\ \lambda_i = \lambda_j,
\end{aligned} \tag{8}$$

(no sums), where $W_{ij} \equiv \partial^2 W/(\partial\lambda_i \partial\lambda_j)$. Note that here, some of these components are not independent one from another because $\lambda_1 = \lambda_2$ and $\sigma_1 = \sigma_2 = 0$. In particular, we find that

$$\begin{aligned} &B_{1212} = B_{2121} = B_{1221}, \qquad B_{2323} = B_{1313} = B_{1331} = B_{2332},\\ &B_{2222} = B_{1111}, \qquad B_{2233} = B_{1133}, \qquad B_{3232} = B_{3131}, \qquad B_{1122} = B_{1111} - 2B_{1212}. \end{aligned} \tag{9}$$

2.3 Incremental solutions

We look for incremental static solutions that are periodic along the circumferential and axial directions, and have yet unknown radial variations. Thus our ansatz for the components of the mechanical displacement is the same as Wilkes's [3]:

$$u_r = U_r(r)\cos n\theta \cos kz, \qquad u_\theta = U_\theta(r)\sin n\theta \cos kz, \qquad u_z = U_z(r)\cos n\theta \sin kz, \tag{10}$$

where $n = 0, 1, 2, \dots$ is the *circumferential number*; k is the *axial wavenumber*; the subscripts (r, θ, z) refer to components within the cylindrical coordinates (r, θ, z); and all upper-case functions are functions of r alone.

Dorfmann and Haughton [8] show that the following displacements $\mathbf{U}^{(1)}$, $\mathbf{U}^{(2)}$, and $\mathbf{U}^{(3)}$ are solutions to the incremental equations (6),

$$\mathbf{U}^{(1)}(r), \mathbf{U}^{(2)}(r) = \left[I_n'(qkr), -\frac{n}{qkr} I_n(qkr), -\frac{(B_{1111}q^2 - B_{3131})}{q(B_{1313} + B_{1133})} I_n(qkr) \right]^T, \tag{11}$$

and

$$\mathbf{U}^{(3)}(r) = \left[\frac{1}{r} I_n(q_3 kr), -\frac{q_3 k}{n} I_n'(q_3 kr), 0 \right]^T, \tag{12}$$

where $q = q_1, q_2$ and I_n is the modified Bessel function of order n. Here q_1, q_2, and q_3 are the square roots of the roots q_1^2, q_2^2 of the following quadratic in q^2:

$$B_{1313}B_{1111}q^4 + [(B_{1133} + B_{1313})^2 - B_{1313}B_{3131} - B_{3333}B_{1111}]q^2 + B_{3333}B_{3131} = 0, \tag{13}$$

and of the root of the following linear equation in q^2

$$B_{1212}q^2 - B_{3131} = 0, \tag{14}$$

respectively.

From (7) we find that the incremental traction on planes normal to the axial direction has components of the same form as that of the displacements, namely

$$s_{rr} = S_{rr}(r)\cos n\theta \cos kz, \qquad s_{r\theta} = S_{r\theta}(r)\sin n\theta \cos kz, \qquad s_{rz} = S_{rz}(r)\cos n\theta \sin kz, \tag{15}$$

say, where again all upper-case functions are functions of r alone. Then we find that the traction solutions corresponding to the solutions (11)-(12) are given by

$$\begin{aligned} &r\mathbf{S}^{(1)}(r), \;\; r\mathbf{S}^{(2)}(r) = \\ &\Bigg[2B_{1212}I_n'(qkr) - \left(2B_{1212}\frac{n^2}{qkr} + qkrB_{1111} - \frac{krB_{1133}\left(B_{1111}q^2 - B_{3131}\right)}{q\left(B_{1313} + B_{1133}\right)} \right) I_n(qkr), \\ &\qquad 2nB_{1212}\left(\frac{I_n(qkr)}{qkr} - I_n'(qkr) \right), -krB_{1313}\left(\frac{B_{1111}q^2 - B_{3131}}{B_{1313} + B_{1133}} + 1 \right) I_n'(qkr) \Bigg]^T, \end{aligned} \tag{16}$$

and

$$r\mathbf{S}^{(3)}(r) = \Bigg[2B_{1212}\left(\frac{I_n(q_3kr)}{r} - q_3kI_n'(q_3kr)\right),$$
$$B_{1212}\left(\frac{2q_3k}{n}I_n'(q_3kr) - \left(\frac{2n}{r} + \frac{q_3^2k^2r}{n}\right)I_n(q_3kr)\right), -kB_{1313}I_n(q_3kr)\Bigg]^T. \quad (17)$$

The general solution to the incremental equations of equilibrium is thus of the form

$$r\mathbf{S}(r) = \left[r\mathbf{S}^{(1)}(r)\middle|r\mathbf{S}^{(2)}(r)\middle|r\mathbf{S}^{(3)}(r)\right]\mathbf{c}, \quad (18)$$

where $\mathbf{S} \equiv [S_{rr}, S_{r\theta}, S_{rz}]^T$ and $\mathbf{c}$ is a constant three-component vector. Note that we use the quantity $r\mathbf{S}$ for the traction (instead of $\mathbf{S}$), because it is the Hamiltonian conjugate to the displacement in cylindrical coordinates [9].

Now when the cylinder is compressed (by platens say), its end faces should stay in full contact with the platens so that the first incremental boundary condition is $u_z = 0$ on $z = 0$, l, which leads to

$$k = m\pi/l, \quad (19)$$

for some integer m (we take $m = 1$ without loss of generality, see [8,2]).

The other boundary condition is that the cylindrical face is free of incremental traction: $\mathbf{S}(b) = \mathbf{0}$. This gives

$$\Delta \equiv \det\left[b\mathbf{S}^{(1)}(b)\middle|b\mathbf{S}^{(2)}(b)\middle|b\mathbf{S}^{(3)}(b)\right] = 0. \quad (20)$$

3 Euler buckling

3.1 Asymptotic expansions

We now specialize the analysis to the asymmetric buckling mode $n = 1$, corresponding to the Euler buckling, in the limit where the axial compressive stretch λ_3 is close to 1 (the mode $n = 0$ corresponds to the barreling mode and is not unstable for long enough cylinders). To this end, we only need to consider the so-called *third-order elasticity* expansion of the strain energy density, for example that of Landau and Lifshitz [10],

$$W = \frac{\lambda}{2}(\mathrm{tr}\mathsf{E})^2 + \mu\,\mathrm{tr}(\mathsf{E}^2) + \frac{\mathscr{A}}{3}\mathrm{tr}(\mathsf{E}^3) + \mathscr{B}(\mathrm{tr}\mathsf{E})\,\mathrm{tr}(\mathsf{E}^2) + \frac{\mathscr{C}}{3}(\mathrm{tr}\mathsf{E})^3, \quad (21)$$

where $\mathsf{E} = \mathsf{E}^T$ is the Lagrange, or Green, strain tensor defined as $\mathsf{E} = \left(\mathsf{F}^T\mathsf{F} - \mathsf{I}\right)/2$, λ and μ are the Lamé moduli, and $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are the Landau third-order elastic constants (Note that there are other, equivalent, expansions based on other invariants, such as the ones proposed by Murnaghan [11], Toupin and Bernstein [12], Bland [13], or Eringen and Suhubi [14], see Norris [15] for the connections.)

To measure how close λ_3 is to 1, we introduce ε, a small parameter proportional to the slenderness of the deformed cylinder,

$$\varepsilon = kb = \pi b/l. \quad (22)$$

Then we expand the radial stretch λ_1 and the critical buckling stretch λ_3 in terms of ε up to order M,

$$\lambda_1 = \lambda_1(\varepsilon) = 1 + \sum_{p=1}^{M} \alpha_p \varepsilon^p + \mathscr{O}(\varepsilon^{M+1}), \qquad \lambda_3 = \lambda_3(\varepsilon) = 1 + \sum_{p=1}^{M} \beta_p \varepsilon^p + \mathscr{O}(\varepsilon^{M+1}), \quad (23)$$

say, where the α's and β's are to be determined shortly. Similarly, we expand Δ in powers of ε,

$$\Delta = \sum_{p=1}^{M_d} d_p \varepsilon^p + \mathscr{O}(\varepsilon^{M_d+1}),$$

say, and solve each order $d_p = 0$ for the coefficients α_p and β_p, making use of the condition $\sigma_1 = 0$. We find that α_p and β_p vanish identically for all odd values of p, and that λ_1 and λ_3, up to the fourth-order in ε, are given by

$$\lambda_1 = 1 + \alpha_2 \varepsilon^2 + \alpha_4 \varepsilon^4 + O(\varepsilon^6), \qquad \lambda_3 = 1 + \beta_2 \varepsilon^2 + \beta_4 \varepsilon^4 + O(\varepsilon^6), \quad (24)$$

with α_2 and α_4 given by

$$\begin{aligned} \alpha_2 &= \frac{\nu}{4}, \\ \alpha_4 &= -\frac{\nu(1+\nu)}{32} - \frac{(1+\nu)(1-2\nu)}{16E}\left[\nu^2\mathscr{A} + (1-2\nu+6\nu^2)\mathscr{B} + (1-2\nu)^2\mathscr{C}\right] - \nu\beta_4, \end{aligned} \quad (25)$$

wherein

$$\begin{aligned} \beta_2 &= -\frac{1}{4}, \\ \beta_4 &= \frac{29+39\nu+8\nu^2}{96(1+\nu)} + \frac{1}{16E}\left[(1-2\nu^3)\mathscr{A} + 3(1-2\nu)(1+2\nu^2)\mathscr{B} + (1-2\nu)^3\mathscr{C}\right]. \end{aligned} \quad (26)$$

Note that we switched from Lamé constants to Poisson's ratio and Young's modulus for these expressions, using the connections $\nu = \lambda/(2\lambda + 2\mu)$ and $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$.

3.2 Onset of nonlinear Euler buckling

The analytical results presented above are formulated in terms of the *current* geometrical parameter ε, defined in (22). In order to relate these results to the classical form of Euler buckling, we introduce the *initial* geometric slenderness B/L. Recalling that $\varepsilon = \pi b/l$, $\lambda_3 = l/L$, and $b = \lambda_1 B$, we find that

$$\varepsilon\lambda_3 = \pi\lambda_1(B/L). \quad (27)$$

We expand ε in powers of B/L, and solve (27) to obtain

$$\begin{aligned} \varepsilon &= \pi(B/L) + (\alpha_2 - \beta_2)\pi^3(B/L)^3 + \mathscr{O}\left((B/L)^4\right) \\ &= \pi(B/L) + (1+\nu)(\pi^3/4)(B/L)^3 + \mathscr{O}\left((B/L)^4\right). \end{aligned} \quad (28)$$

Second, we wish to relate the axial compression to the current axial load N. To do so, we integrate the axial stress over the faces of the cylinder,

$$N = -2\pi\int_0^b r\sigma_3 \mathrm{d}r = -\pi b^2\sigma_3 = -\pi\lambda_1^2 B^2\sigma_3, \quad (29)$$

because σ_3 is constant, given by $(5)_2$.

Finally, in order to write the nonlinear buckling formula, we expand λ_1 and λ_3 in (29), using (24), and then expand ε in powers of the slenderness (B/L), using (28). It gives the desired expression for the first *non-linear correction to the Euler formula*,

$$\frac{N_c}{\pi^3 B^2} = \frac{E}{4}\left(\frac{B}{L}\right)^2 - \frac{\pi^2}{96}\delta_{\text{NL}}\left(\frac{B}{L}\right)^4, \tag{30}$$

where

$$\delta_{\text{NL}} = 2\frac{13+12\nu-2\nu^2}{(1+\nu)}E + 12\left[(1-2\nu^3)\mathscr{A} + 3(1-2\nu)(1+2\nu^2)\mathscr{B} + (1-2\nu)^3\mathscr{C}\right]. \tag{31}$$

We now check this equation against its incompressible counterpart (2). Theoretical considerations and experimental measurements [6], [17], [18], [19], show that in the incompressible limit, E and $\mathscr{A}$ remain finite, $\nu \to 1/2$, $(1-2\nu)\mathscr{B} \to -E/3$, and $(1-2\nu)^3\mathscr{C} \to 0$. It is then a simple exercise to verify that (30) is indeed consistent with (2) in those limits.

3.3 Examples

Table 1 Lamé constants and Landau third-order elastic moduli for five solids (10^{-9} N$\cdot$ m^{-2})

material	λ	μ	$\mathscr{A}$	$\mathscr{B}$	$\mathscr{C}$
Polystyrene	1.71	0.95	-10	-8.3	-10.6
Steel Hecla 37	111	82.1	-358	-282	-177
Aluminium 2S	57	27.6	-228	-197	-102
Pyrex glass	13.5	27.5	420	-118	132
SiO_2 melted	15.9	31.3	-44	93	36

To evaluate the importance of the non-linear correction, we computed the critical axial stretch ratio of column buckling for two solids. In Table 1, we list the second- and third-order elastic constants of five compressible solids, as collected by Porubov [16] (in the Table we converted the "Murnaghan constants" given by Porubov to the Landau constants $\mathscr{A}, \mathscr{B}, \mathscr{C}$). Figure 1 shows the variations of λ_3 with the squared slenderness $(B/L)^2$, for pyrex and silica (two last lines of Table 1).

4 Conclusions

The present analysis provides an asymptotic formula for the critical value of the load for the Euler buckling problem. This formula was checked both in the incompressible limit and on particular cases against the exact value of the buckling obtained from the exact solutions. Not surprisingly it reinforces the universal and generic nature of the Euler buckling formula as the correction is small for most systems even when nonlinear elastic effects and nonlinear geometric effects are taken into account. It would be of great interest to see if these effects could be observed experimentally.

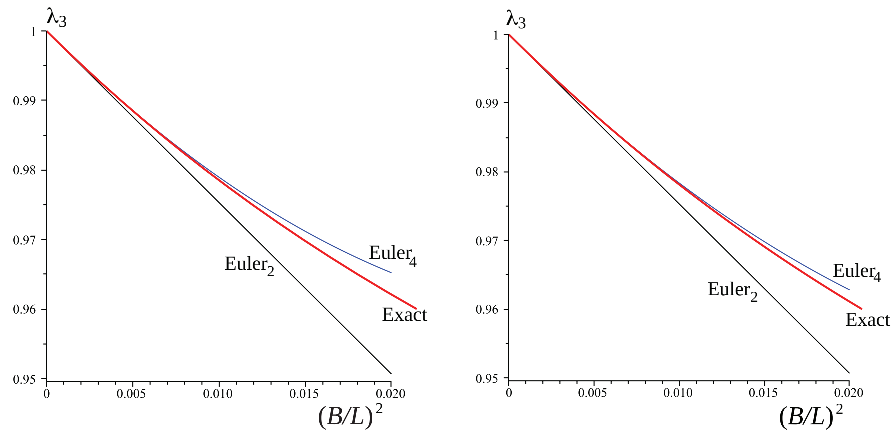

Fig. 1 Comparison of the different Euler formulas obtained by expanding the exact solution to order 2 (classical Euler buckling formula, plot labeled "Euler$_2$") and to order 4 (plot labeled "Euler$_4$"), for pyrex (figure on the left) and for silica (figure on the right).

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