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RESEARCH ARTICLE

Classification of homogenized limits of diffusion problems with spatially-dependent reaction over critical-size particles

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(*Dedicated to the memory of V.V. Zhikov*)

The main goal of this paper is to characterize the change of structural behavior (i.e. the appearance of the so called "strange terms") arising in the homogenization process when applied to *distributed* microscopic chemical reactions taking place on fixed-bed nanoreactors, at the microscopic level, on the boundary of the particles of critical size. The presence of non-homogeneous distributed functions $b_\varepsilon^j(x)$ of the reaction kinetics may be originated by many different reasons. The case of quick oscillating is often due to own structure of the fixed bed reactor since the flux of the fluid acts on each particle may act in a non-homogeneous way. In some other cases, the non-homogeneous distributed functions $b_\varepsilon^j(x)$ of the reaction kinetics is artificially provoked in order to control a certain desired global effect. Our main result gives a complete classification of the strange terms according the assumed periodicity on the distributed functions $b_\varepsilon^j(x)$ of the reaction kinetic.

Keywords: homogenization, diffusion processes, periodic asymmetric particles, microscopic distributed boundary reaction, critical sizes.

AMS Subject Classifications: 35B25,35B40,35J05,35J20

Introduction

After the introduction of the basic mathematical elements of homogenization theory (see, e.g., the monographs [1] and [2]) it was found that some changes may appear in the structural modelling of the homogenized problem, for suitable "critical size" of the elements configuring the "micro-structured" material. It seems that the first result in that direction was presented in the pioneering paper by V. Marchenko and E. Hruslov [3]. A well-known different presentation of the appearance of those "strange terms" was due to D. Cioranescu and F. Murat [4]. Both articles dealt with linear equations with Neumann and Dirichlet boundary conditions, respectively. Since those dates to our days many papers have been devoted to consider different formulations: more general elliptic partial differential equations (possibly of quasilinear type), Robin and other types of boundary conditions of different nature, etc. It is impossible to mention all of them here (a few of them will be referenced in the rest of this Introduction) but the reader may imagine that the nature of this "strange term" may be completely different according to the peculiarities of the formulation in consideration (something that was already indicated at the end of the Introduction of the paper by Cioranescu

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and Murat [4]). For some recent results concerning microscopic nonlinear reactions on particles of general form we send the reader to our paper [5], where he can find many other references on this type of problems.

The main goal of this paper is to characterize the change of structural behavior (i.e. the so called "strange terms") arising in the homogenization process when applied to *distributed* microscopic chemical reactions taking place on fixed-bed nanoreactors, at the microscopic level, on the boundary of the particles. So, our main interest now is to study how the $x-$dependence in the microscopic kinetics may lead to completely different reaction-diffusion homogenized equations and to characterize the different "strange terms" which arise in terms of the different periodicity of the distributed function $b_\varepsilon^j(x)$ of the reaction kinetics. As a matter of fact, in this paper we shall only pay attention to the case of a linear kinetics in order to emphasize the effects of the distributed function $b_\varepsilon^j(x)$.

The microscopic particles G_ε^j (also identified as the microscopic perforation sets in the case of other porous media) were diffeomorphic to a ball, their diameters are a_ε, $a_\varepsilon = C_0\varepsilon^\alpha$, $\alpha = \frac{n}{n-2}$. We consider different types of coefficients depending on x: quick oscillating (in the sense that the periodicity is the size of the catalytic particle), slow oscillation (in the sense that periodicity of the position of the miscoscopic catalytic particle) and independent of small parameters. In all this cases we construct the limit problem and prove some convergence of solutions as $\varepsilon \to 0$. We mention that the presence of non-homogeneous distributed functions $b_\varepsilon^j(x)$ of the reaction kinetics may be originated by many different reasons. The case of quick oscillating is often due to own structure of the fixed bed reactor since the flux of the fluid acts on each particle may act in a non-homogeneous way (for instance due to some privilegiate directions of the flux). In some other cases the non-homogeneous distributed functions $b_\varepsilon^j(x)$ of the reaction kinetics is artificially provocated in order to control a certain desired global effect (see, e.g. the detailed exposition made in the monograph [6]). Many variants in the formulation of the model problem considered in this paper are possible but we prefer to present here only some sharp answers for a simple formulation. Similar results can be found in [7–11].

1. Statement of results

Let Ω be a bounded domain in $\mathbb{R}^n$ $n \geq 3$ with a piecewise smooth boundary $\partial\Omega$. The case $n = 2$ requires some technical modifications which will not be presented here. Let G_0 be a domain in $Y = (-\frac{1}{2}, \frac{1}{2})^n$, such that $\overline{G_0}$ is a compact set diffeomorphic to a ball. Let $C_0, \varepsilon > 0$ and set

$$a_\varepsilon = C_0\varepsilon^\alpha \quad \text{for } \alpha = \frac{n}{n-2}. \tag{1}$$

For $\delta > 0$ and B a set let $\delta B = \{x \mid \delta^{-1}x \in B\}$. Assume that ε is small enough so that $a_\varepsilon G_0 \subset \varepsilon Y$. For $j \in \mathbb{Z}^n$ we define

$$P_\varepsilon^j = \varepsilon j, \quad Y_\varepsilon^j = P_\varepsilon^j + \varepsilon Y, \quad G_\varepsilon^j = P_\varepsilon^j + a_\varepsilon G_0.$$

We define the set of admissible indexes as

$$\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : d(\varepsilon j, \partial\Omega) \geq \varepsilon\sqrt{n}\}.$$

Notice that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$ where $d > 0$ is a constant. Our problem will be set in the following domain:

$$\Omega_\varepsilon = \Omega \setminus \overline{G}_\varepsilon \text{ where } G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j.$$

Finally, let

$$\partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega, \quad S_\varepsilon = \partial G_\varepsilon.$$

The aim of this paper is to consider the asymptotic behaviour of the following problem, which models the steady-state of a diffusion-reaction process, where the reaction is taking place on the boundary of the particles G_ε^j, while the diffusion is taking place on Ω_ε:

$$\begin{cases} -\Delta u_\varepsilon = f, & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma} b_\varepsilon^j(x) u_\varepsilon = 0, & x \in \partial G_\varepsilon^j, \; j \in \Upsilon_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

Here $f \in L^2(\Omega)$, $b_\varepsilon^j \in \mathcal{C}(\partial G_\varepsilon^j)$ and

$$b_\varepsilon^j(x) \geq b_0 > 0,$$

for some constant b_0. We will consider three possible cases for b_ε^j:

- Quickly oscillating coefficients. In this case we will assume that

$$b_\varepsilon^j(x) = b\left(\frac{x - P_\varepsilon^j}{a_\varepsilon}\right) \tag{3}$$

  for a function $b \in \mathcal{C}(\partial G_0)$. This case corresponds, in the modelling, to particles which have some regions more favorable to the reaction than other, hence the reaction coefficient is not constant along the boundary. The periodicity of b_ε^j and G_ε^j is the same, and this corresponds to the presence of function $b(y)$ on G_0 driving the coefficient of the linear reaction term.
- ε-periodic coefficient case. In this case

$$b_\varepsilon^j(x) = b\left(\frac{x - P_\varepsilon^j}{\varepsilon}\right) \tag{4}$$

  for a Y-periodic function $b \in \mathcal{C}^1(\overline{Y})$.
- Independent of ε case. Here we consider

$$b_\varepsilon^j(x) = b(x) \tag{5}$$

for a function $b \in \mathcal{C}^1(\Omega)$. This case correspond, in the modelling, to a case in which the environment Ω is more favorable for the reaction in some spatial regions. In the case of fixed bed reactor with a fluid, this inhomogeneity can be caused, for example, by spatial difference of temperatures in the fluid.

In [5] the authors consider the case of boundary conditions

$$\partial_\nu u_\varepsilon + \varepsilon^{-\gamma} \sigma(u_\varepsilon) = 0$$

in which the nonlinearity σ does not depend on x. Our intention is to present similar findings when σ does depend on x, but is linear on u (i.e. $\sigma^\varepsilon(x,u) = b^\varepsilon(x)u$). This allows us to conjecture in Remark 6.4 what the expected behaviour in the case $\sigma^\varepsilon(x) = b^\varepsilon(x)\sigma(u)$.

We will make use of the auxiliary function $\widehat{w} = \widehat{w}(y;\xi,g)$ where $\xi \in \mathbb{R}, g \in \mathcal{C}(\partial G_0)$, the solution of the problem

$$\begin{cases} -\Delta \widehat{w} = 0, & y \in \mathbb{R}^n \setminus G_0 \\ \partial_\nu \widehat{w} + C_0 \xi g(y) \widehat{w} = C_0 \xi g(y), & y \in \partial G_0 \\ \widehat{w} \to 0, & |y| \to 0. \end{cases} \tag{6}$$

This auxiliary function will provide a capacity-like term defined as

$$\lambda(G_0,\xi,g) = \int_{\partial G_0} \partial_{\nu_y} \widehat{w}(y;\xi,g) d\sigma_y. \tag{7}$$

For more details on the appearance of this term we refer the reader to [5], in which the function $\widehat{w}$ is the solution of a nonlinear exterior problem. With this definition, we will show that:

1.1. *Homogenization of the linear problem*

Theorem 1.1 *Let $n \geq 3, \alpha = \gamma = \frac{n}{n-2}$, u_ε be the solution of (2) and let b_ε be of one of the above mentioned forms (3),(4),(5). Then, there exists an extension $\widetilde{u}_\varepsilon \in H_0^1(\Omega)$ such that $\widetilde{u}_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$, where u is the unique solution of*

$$\begin{cases} -\Delta u + \lambda_0(x) C_0^{n-2} u = f, & \Omega, \\ u = 0, & \partial\Omega. \end{cases} \tag{8}$$

and $\lambda_0(x)$ is given by:

- *Quickly oscillating coefficients:*

$$\lambda_0(x) = \lambda(G_0, 1, b(\cdot)). \tag{9}$$

  That is, in (6) we take $\xi = 1$ and $g(y) = b(y)$.
- *ε-oscillating coefficients:*

$$\lambda_0(x) = \lambda(G_0, 1, b(0)). \tag{10}$$

  That is, in (6) we take $\xi = 1$ and $g(y) = b(0)$.

- *Independent of ε case:*

$$\lambda_0(x) = \lambda(G_0, b(x), 1). \tag{11}$$

That is, in (6) we take $\xi = b(x)$ and $g(y) = 1$.

Remark 1.2 Notice that in (9) and (10) λ_0 does not depend on x, whereas in (11) it might.

Remark 1.3 Even though, in this case, $\alpha = \gamma$ we have preserved both constants for consistency in the notation the non-critical cases $\alpha < \frac{n}{n-2}$ and critical cases when $-\Delta$ is replaced by $-\Delta_p$.

1.2. *Homogenization of the associated spectral problem*

The spectral problem corresponding to the boundary value problem (2) can be considered as in [12].

THEOREM 1.4 *Let $n \geq 3$, $\alpha = \gamma = \frac{n}{n-2}$ and let $\{\lambda_\varepsilon^m\}$ be the nondecreasing sequence of eigenvalues of the eigenvalue problem*

$$\begin{cases} -\Delta u_\varepsilon^m = \lambda_\varepsilon^m u_\varepsilon^m, & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon^m + \varepsilon^{-\gamma} b_\varepsilon^j(x) u_\varepsilon^m = 0, & x \in \partial G_\varepsilon^j,\ j \in \Upsilon_\varepsilon, \\ u_\varepsilon^m = 0, & x \in \partial\Omega, \end{cases}$$

where $b_\varepsilon^j(x)$ is of one of the mentioned forms (3), (4) or (5). Let $\{\lambda^m\}$ be the nondecreasing sequence of eigenvalues of the eigenvalue problem

$$\begin{cases} -\Delta u^m + \lambda_0(x) C_0^{n-2} u^m = \lambda^m u^m, & x \in \Omega, \\ u^m = 0, & x \in \partial\Omega, \end{cases}$$

and

- $\lambda_0(x) = \lambda(G_0, 1, b)$ *if $b_\varepsilon^j(x)$ is given by (3),*
- $\lambda_0(x) = \lambda(G_0, 1, b(0))$ *if $b_\varepsilon^j(x)$ is given by (4),*
- $\lambda_0(x) = \lambda(G_0, b(x), 1)$ *if $b_\varepsilon^j(x)$ is given by (5).*

Then $\lambda_\varepsilon^m \to \lambda^m$, as $\varepsilon \to 0$.

Results of similar nature can be found in [13].

2. Existence, bounds and asymptotic behaviour of u_ε assuming $b_\varepsilon^j \in \mathcal{C}(\partial G_\varepsilon^j)$

Let us consider the weak solution of the problem (2), i.e. the function

$$u_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega) = \overline{\{u \in C^\infty(\Omega_\varepsilon) : \text{dist}(\text{supp } u, \partial\Omega) > 0\}}^{H^1(\Omega_\varepsilon)}$$

which satisfies the integral identity

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi dx + \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x) u_\varepsilon \varphi ds = \int\limits_{\Omega_\varepsilon} f \varphi dx \tag{12}$$

for all $\varphi \in H^1(\Omega_\varepsilon, \partial\Omega)$.

Taking $\varphi = u_\varepsilon$ in (12) we deduce that

$$\|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq K, \qquad \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x) u_\varepsilon^2 ds \leq K\varepsilon^\gamma. \tag{13}$$

Here and in what follows K will be some constant which does not depend on ε.

2.1. *Extension of solutions*

We consider the family of extension operators

$$P_\varepsilon : H^1(\Omega_\varepsilon, \partial\Omega) \to H_0^1(\Omega), \tag{14}$$

such that $P_\varepsilon v = v$ a.e. in Ω_ε and

$$\|\nabla P_\varepsilon v\|_{L^2} \leq \|\nabla v\|_{L^2} \qquad \forall v \in H^1(\Omega_\varepsilon, \partial\Omega).$$

For the details of the construction see [1, 14].

2.2. *Existence of a limit*

Due to the previous statement the sequence $\widetilde{u}_\varepsilon = P_\varepsilon u_\varepsilon$ is a bounded sequence in $H_0^1(\Omega)$. Therefore, it is weakly convergent in $H_0^1(\Omega)$. There exists $u \in H_0^1(\Omega)$ such that

$$\tilde{u}_\varepsilon \rightharpoonup u \text{ in } H_0^1(\Omega) \text{ as } \varepsilon \to 0.$$

From here on, we will simply use u_ε instead of $\widetilde{u}_\varepsilon$.

2.3. *From integrals on ∂G_ε^j to integrals on $\partial T_{\frac{\varepsilon}{4}}$. The auxiliary functions w_ε^j*

Let us study the asymptotic behavior of the solution u_ε as $\varepsilon \to 0$. To study the surface integral in the left part of (12) we need some auxiliary functions. To do this, we introduce the function w_ε^j as the solution of the problem, for sufficiently small ε,

$$\begin{cases} \Delta w_\varepsilon^j = 0, & \text{if } x \in T_{\frac{\varepsilon}{4}}^j \setminus \overline{G_\varepsilon^j}, \\ \partial_{\nu_x} w_\varepsilon^j + \varepsilon^{-\gamma} b_\varepsilon^j(x) w_\varepsilon^j = \varepsilon^{-\gamma} b_\varepsilon^j(x), & \text{if } x \in \partial G_\varepsilon^j, \\ w_\varepsilon^j = 0, & \text{if } x \in \partial T_{\frac{\varepsilon}{4}}^j, \end{cases} \tag{15}$$

where $T_r^j = \{x \in \mathbb{R}^n : |x - P_\varepsilon^j| \leq r\}$. This function is harmonic, and it is clear that $\underline{w}_\varepsilon^j = 0$ is a subsolution and $\overline{w}_\varepsilon^j = 1$ is a supersolution of problem (15). Therefore

$$0 \leq w_\varepsilon^j \leq 1. \tag{16}$$

We set

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j(x) & x \in T_{\frac{\varepsilon}{4}}^j \setminus \overline{G_\varepsilon^j}, j \in \Upsilon_\varepsilon, \\ 0 & x \in \mathbb{R}^n \setminus \bigcup\limits_{j \in \Upsilon_\varepsilon} \overline{T_{\frac{\varepsilon}{4}}^j}. \end{cases} \tag{17}$$

Let us prove some properties of W_ε. If one takes w_ε^j as a test functions in the integral identity for problem (15):

$$\int\limits_{T_{\frac{\varepsilon}{4}}^j \setminus G_\varepsilon^j} | \nabla w_\varepsilon^j |^2 \, dx + \varepsilon^{-\gamma} \int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x) \mid w_\varepsilon^j \mid^2 \, ds = \varepsilon^{-\gamma} \int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x) w_\varepsilon^j ds.$$

From the properties of b_ε^j and Cauchy's inequality we derive

$$\|\nabla w_\varepsilon^j\|^2_{L^2(T_{\frac{\varepsilon}{4}}^j \setminus G_\varepsilon^j)} + \varepsilon^{-\gamma} b_0 \int\limits_{\partial G_\varepsilon^j} | w_\varepsilon^j |^2 \, ds \leq \varepsilon^{-\gamma} \left(C_{b_0} \mid \partial G_\varepsilon^j \mid + \frac{b_0}{2} \|w_\varepsilon^j\|^2_{L^2(\partial G_\varepsilon^j)} \right).$$

Thus,

$$\|\nabla w_\varepsilon^j\|^2_{L^2(T_{\frac{\varepsilon}{4}}^j \setminus G_\varepsilon^j)} + \varepsilon^{-\gamma} \frac{b_0}{2} \|w_\varepsilon^j\|^2_{L^2(\partial G_\varepsilon^j)} \leq K a_\varepsilon^{n-2} = K \varepsilon^n,$$

and

$$\|\nabla W_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma} \frac{b_0}{2} \|W_\varepsilon\|^2_{L^2(\partial S_\varepsilon)} \leq K.$$

Using Friedrich's inequality we obtain that

$$\|w_\varepsilon^j\|^2_{L^2(T_{\frac{\varepsilon}{4}}^j \setminus G_\varepsilon^j)} \leq \varepsilon^2 K \|\nabla w_\varepsilon^j\|^2_{L^2(T_{\frac{\varepsilon}{4}}^j \setminus G_\varepsilon^j)}.$$

Hence, $\|W_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq K\varepsilon^2$ and

$$\widetilde{W}_\varepsilon \rightharpoonup 0 \text{ as } \varepsilon \to 0 \text{ in } H^1(\Omega), \tag{18}$$

$$\widetilde{W}_\varepsilon \to 0 \text{ as } \varepsilon \to 0 \text{ in } L^2(\Omega), \tag{19}$$

where $\widetilde{W}_\varepsilon = P_\varepsilon W_\varepsilon$. Let $\widetilde{W}_\varepsilon \varphi$ be the test function in (12), then:

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla (\widetilde{W}_\varepsilon \varphi) dx + \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x) u_\varepsilon \widetilde{W}_\varepsilon \varphi ds = \int\limits_{\Omega_\varepsilon} f \widetilde{W}_\varepsilon \varphi dx. \tag{20}$$

We can rewrite this in the form:

$$\int\limits_{\Omega_\varepsilon} \nabla \widetilde{W}_\varepsilon \nabla (u_\varepsilon \varphi) dx - \int\limits_{\Omega_\varepsilon} u_\varepsilon \nabla \widetilde{W}_\varepsilon \nabla \varphi dx + \int\limits_{\Omega_\varepsilon} \widetilde{W}_\varepsilon \nabla \varphi \nabla u_\varepsilon dx + \\ + \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x) u_\varepsilon w_\varepsilon^j \varphi ds = \int\limits_{\Omega_\varepsilon} f \widetilde{W}_\varepsilon \varphi dx.$$

From the Green's formula we obtain

$$\sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} (\partial_\nu w_\varepsilon^j) u_\varepsilon \varphi ds + \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} (\partial_\nu w_\varepsilon^j) u_\varepsilon \varphi ds - \int\limits_{\Omega_\varepsilon} u_\varepsilon \nabla \widetilde{W}_\varepsilon \nabla \varphi dx + \int\limits_{\Omega_\varepsilon} \widetilde{W}_\varepsilon \nabla \varphi \nabla u_\varepsilon dx + \\ + \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x) u_\varepsilon w_\varepsilon^j \varphi ds = \int\limits_{\Omega_\varepsilon} f \widetilde{W}_\varepsilon \varphi dx.$$

Using the boundary conditions on ∂G_ε^j we have

$$\sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} (\partial_\nu w_\varepsilon^j) u_\varepsilon \varphi ds + \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x) u_\varepsilon \varphi ds = \int\limits_{\Omega_\varepsilon} f \widetilde{W}_\varepsilon \varphi dx \\ + \int\limits_{\Omega_\varepsilon} u_\varepsilon \nabla \widetilde{W}_\varepsilon \nabla \varphi dx - \int\limits_{\Omega_\varepsilon} \widetilde{W}_\varepsilon \nabla \varphi \nabla u_\varepsilon dx$$

or,

$$\varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x) u_\varepsilon \varphi ds = - \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} (\partial_\nu w_\varepsilon^j u_\varepsilon) \varphi ds + \int\limits_{\Omega_\varepsilon} f \widetilde{W}_\varepsilon \varphi dx + \\ + \int\limits_{\Omega_\varepsilon} u_\varepsilon \nabla \widetilde{W}_\varepsilon \nabla \varphi dx - \int\limits_{\Omega_\varepsilon} \widetilde{W}_\varepsilon \nabla \varphi \nabla u_\varepsilon dx. \tag{21}$$

From (18), (19) we obtain that

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} f \widetilde{W}_\varepsilon \varphi dx = 0, \\ \lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} u_\varepsilon \nabla \widetilde{W}_\varepsilon \nabla \varphi dx = 0, \\ \lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} \widetilde{W}_\varepsilon \nabla \varphi \nabla u_\varepsilon dx = 0. \tag{22}$$

The aim of the next sections will be to characterize the limit of

$$\sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\frac{\varepsilon}{4}}} (\partial_\nu w^j_\varepsilon) u_\varepsilon \varphi ds$$

in the different cases.

3. The capacity term. The auxiliary function $\widehat{w}$

3.1. *The auxiliary function* $\widehat{w}$

Since $b > 0$ we will always consider that $\xi, g(y) > 0$. Existence and uniqueness of $\widehat{w}$ follow in a straightforward way. The first thing we must point out is that $\widehat{w}$ is a classical solution of the problem, and hence a harmonic function. Again, the constant functions 0 and 1 are a sub and supersolution respectively, therefore

$$0 \le \widehat{w} \le 1.$$

We have that

$$\partial_\nu \widehat{w} = C_0 \xi g(y)(1-\widehat{w}) \ge 0$$

in ∂G_0 since $\xi, g(y) > 0$. Therefore

$$\lambda(G_0, \xi, g) > 0. \tag{23}$$

Let us define

$$K_0 = \max_{z\in\partial G_0} |y|^{n-2}. \tag{24}$$

Since $\widehat{w} \le 1$ we have that

$$0 \le \widehat{w} \le \frac{K_0}{|y|^{n-2}} \quad \text{in } \partial G_0.$$

By the maximum principle

$$0 \le \widehat{w} \le \frac{K_0}{|y|^{n-2}} \quad \text{in } \mathbb{R}^n \setminus G_0. \tag{25}$$

Remark 3.1 Let us note that $\widehat{w}(y;\xi,g)$ is continuous with respect to ξ. Let us consider the function

$$V(y;\xi_1,\xi_2) = \widehat{w}(y;\xi_1,1) - \widehat{w}(y;\xi_2,1). \tag{26}$$

Due to the maximum principle and the Robin type boundary condition (see, e.g., [15, Theorem 29]) it holds that

$$|V(y;\xi_1,\xi_2)| \le \frac{1}{|\xi_1|} \max_{y\in\partial G_0} |(\xi_1-\xi_2)(1-\widehat{w}(y;\xi_2,1))|.$$

Since $0 \le \widehat{w} \le 1$ then

$$|\widehat{w}(y;\xi_1,1) - \widehat{w}(y;\xi_2,1)| \le \frac{1}{|\xi_1|}|\xi_1 - \xi_2|. \tag{27}$$

3.2. *The auxiliary function $\widehat{w}_\varepsilon^j$*

We will use use the following estimate on

$$\widehat{w}_\varepsilon^j(x;\xi_\varepsilon^j,g) = \widehat{w}\left(\frac{x-P_\varepsilon^j}{a_\varepsilon};\xi_\varepsilon^j,g\right) \tag{28}$$

It is clear that this function satisfies

$$\begin{cases} -\Delta \widehat{w}_\varepsilon^j = 0, & x \in \mathbb{R}^n \setminus G_\varepsilon^j, \\ \partial_\nu \widehat{w}_\varepsilon^j + \xi_\varepsilon^j g_\varepsilon^j(x)\widehat{w}_\varepsilon^j = \xi_\varepsilon^j g_\varepsilon^j(x), & x \in \partial G_\varepsilon^j, \\ \widehat{w}_\varepsilon^j \to 0, & |x| \to +\infty, \end{cases}$$

where $\xi_\varepsilon^j \in \mathbb{R}$ and

$$g_\varepsilon^j(x) = g\left(\frac{x-P_\varepsilon^j}{a_\varepsilon}\right).$$

Due to the properties of $\widehat{w}$ we have that

$$0 \le \widehat{w}_\varepsilon^j(x) \le \frac{K}{\left|\frac{x-P_\varepsilon^j}{a_\varepsilon}\right|^{n-2}} \le K\frac{a_\varepsilon^{n-2}}{|x-P_\varepsilon^j|^{n-2}} \le K\frac{\varepsilon^n}{|x-P_\varepsilon^j|^{n-2}}. \tag{29}$$

We also point out that

$$\max_{\partial T^j_{\frac{\varepsilon}{4}}} \left|\partial_{\nu_x}\widehat{w}_\varepsilon^j(x)\right| \le K\frac{a_\varepsilon^{-1}}{|\frac{x}{a_\varepsilon}|^{n-1}} = Ka_\varepsilon^{n-2}\varepsilon^{1-n} \le K\varepsilon.$$

3.3. *The auxiliary function $v_\varepsilon^j = \widehat{w}_\varepsilon^j - w_\varepsilon^j$*

We will make use of the auxiliary function

$$v_\varepsilon^j(x;\xi_\varepsilon^j,g_\varepsilon^j) = \widehat{w}_\varepsilon^j(x;\xi_\varepsilon^j,g_\varepsilon^j) - w_\varepsilon^j.$$

We have that $v_\varepsilon^j(x;u,g)$ is the solution of the following problem:

$$\begin{cases} -\Delta v_\varepsilon^j = 0, & x \in T^j_{\frac{\varepsilon}{4}} \setminus \overline{G_\varepsilon^j}, \\ \partial_\nu v_\varepsilon^j + \varepsilon^{-\gamma} b_\varepsilon^j(x) v_\varepsilon^j & \\ \quad = -\varepsilon^{-\gamma}(b_\varepsilon^j(x) - \xi_\varepsilon^j g_\varepsilon^j)(1-\widehat{w}_\varepsilon^j), & x \in \partial G_\varepsilon^j, \\ v_\varepsilon^j = \widehat{w}_\varepsilon^j, & x \in \partial T^j_{\frac{\varepsilon}{4}}. \end{cases} \tag{30}$$

We will consider a different choice of ξ_ε^j and g_ε^j in each of the following cases:

- Quickly oscillating coefficients: $\xi_\varepsilon^j g_\varepsilon^j(x) = b_\varepsilon^j(x)$
- ε-periodic coefficients: $\xi_\varepsilon^j g_\varepsilon^j(x) = b_\varepsilon^j(0)$
- Coefficients independent of ε: $\xi_\varepsilon^j g_\varepsilon^j(x) = b(P_\varepsilon^j)$.

In all the previous cases we have that

$$|b_\varepsilon^j - \xi_\varepsilon^j g_\varepsilon^j| \le K|x - P_\varepsilon^j| \le K\varepsilon, \qquad x \in T^j_{\frac{\varepsilon}{4}} \setminus G_\varepsilon^j.$$

It is easy to check that $\widehat{w}_\varepsilon^j$ is a supersolution of this problem. Hence, via the comparison principle

$$0 \le v_\varepsilon^j(x) \le \widehat{w}_\varepsilon^j(x). \tag{31}$$

We have that

$$|v_\varepsilon^j(x)| \le |\widehat{w}_\varepsilon^j(x)| \le \frac{K}{\left|\frac{x-P_\varepsilon^j}{a_\varepsilon}\right|} = \frac{K a_\varepsilon^{n-2}}{|x-P_\varepsilon^j|^{n-2}} \le K\frac{a_\varepsilon^{n-2}}{\varepsilon^n} \le K\varepsilon^2, \qquad \forall x \in T^j_{\frac{\varepsilon}{4}} \setminus T^j_{\frac{\varepsilon}{8}}.$$

Applying the estimates of derivatives of harmonic function and the maximum principle we get for $x_0 \in \partial T_{\frac{\varepsilon}{8}}$ and $T_r^{x_0}$ a ball of radius r centered at x_0, that:

$$\begin{aligned}\left|\partial_{x_i} v_\varepsilon^j(x_0)\right| &= \frac{1}{\left|T^{x_0}_{\frac{\varepsilon}{16}}\right|}\left|\int_{T^{x_0}_{\frac{\varepsilon}{16}}} \frac{\partial v_\varepsilon^j}{\partial x_i}dx\right| = \frac{K}{\varepsilon^n}\left|\int_{\partial T^{x_0}_{\frac{\varepsilon}{16}}} v_\varepsilon^j \nu_i ds\right| \\ &\le K\varepsilon.\end{aligned} \tag{32}$$

Finally let us indicate how to obtain a L^2 estimate for v_ε^j from L^2 estimates on ∇v_ε^j. We start by recalling the following estimate:

Lemma 3.2 *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that, for all $f \in H^1(\mathcal{O})$,*

$$\int_{\mathcal{O}} |f|^2 dy \le C\left(\int_{\mathcal{O}} |\nabla f|^2 dy + \int_{\partial\mathcal{O}} |f|^2 ds\right). \tag{33}$$

Since many similar results hold, for the convenience of the reader and for the sake of completeness, we provide an indication of the proof.

Sketch of proof. Assume, towards a contradiction, that the inequality is false. Let $f_n \in H^1(\mathcal{O})$ be such that the inequality is reversed and n appears instead of C. Let $v_n = \frac{u_n}{\|u_n\|_{L^2}}$. One shows that $v_n \rightharpoonup 0$ in $H^1(\mathcal{O})$ and $\|v_n\|_{L^2} = 1$. This is a contradiction due to the compact embedding of H^1 in L^2. ■

Lemma 3.3 *Let $f \in H^1(T^j_{\frac{\varepsilon}{4}} \setminus G_\varepsilon^j)$. Then*

$$\|f\|^2_{L^2(T_{\frac{\varepsilon}{4}} \setminus G_\varepsilon^j)} \le C\left(\varepsilon^2 \|\nabla f\|^2_{L^2(T_{\frac{\varepsilon}{4}} \setminus G_\varepsilon^j)} + \varepsilon \|f\|^2_{L^2(\partial T^j_{\frac{\varepsilon}{4}})}\right). \tag{34}$$

In particular,

$$\|v_\varepsilon^j\|^2_{L^2(T_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j)} \le C\left(\varepsilon^2\|\nabla v_\varepsilon^j\|^2_{L^2(T_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j)} + \varepsilon^{n+4}\right). \tag{35}$$

Proof. Let $\widetilde{f} \in H^1(T^j_{\frac{\varepsilon}{4}})$ be an extension of f as in [14]. Let $\widehat{f}(y) = \widetilde{f}(\varepsilon j + \varepsilon y)$ for $y \in \mathcal{O} = T^0_{\frac{1}{4}}$. Applying the previous lemma and the change in variable $x = \varepsilon j + \varepsilon y$ we have that

$$\int_{T^j_{\frac{\varepsilon}{4}}} \left|\widetilde{f}\right|^2 dx \le C\left(\varepsilon^2 \int_{T^j_{\frac{\varepsilon}{4}}} \left|\nabla\widetilde{f}\right|^2 dx + \varepsilon\int_{\partial T^j_{\frac{\varepsilon}{4}}} \left|\widetilde{f}\right|^2 ds\right). \tag{36}$$

Therefore

$$\int_{T^j_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j} |f|^2\, dx \le \int_{T^j_{\frac{\varepsilon}{4}}} \left|\widetilde{f}\right|^2 dx \le C\left(\varepsilon^2 \int_{T^j_{\frac{\varepsilon}{4}}} \left|\nabla\widetilde{f}\right|^2 dx + \varepsilon\int_{\partial T^j_{\frac{\varepsilon}{4}}} \left|\widetilde{f}\right|^2 ds\right)$$
$$\le C\left(\varepsilon^2 \int_{T^j_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j} |\nabla f|^2\, dx + \varepsilon\int_{\partial T^j_{\frac{\varepsilon}{4}}} |f|^2\, ds\right).$$

Since $|v_\varepsilon^j| \le |\widehat{w}_\varepsilon^j| \le K\varepsilon^2$ we have that

$$\int_{\partial T^j_{\frac{\varepsilon}{4}}} |v_\varepsilon^j|^2 ds \le \varepsilon^4\varepsilon^{n-1} = \varepsilon^{n+3}. \tag{37}$$

This completes the proof. ■

4. Case of quickly oscillating coefficients

In this case we set $\xi_\varepsilon^j = 1$ and $g(y) = b(y)$. Hence, in this section we will write $\widehat{w}(y) = \widehat{w}(y; 1, b(\cdot))$.

LEMMA 4.1 *The following estimate holds:*

$$\sum_{j\in\Upsilon_\varepsilon} \left\|w_\varepsilon^j - \widehat{w}_\varepsilon^j\right\|^2_{H^1(T^j_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j)} \le K\varepsilon^2.$$

Proof. We use v_ε^j as the test function:

$$\int\limits_{T^j_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j} |\nabla v_\varepsilon^j|^2\, dx + \varepsilon^{-\gamma} \int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x) \mid v_\varepsilon^j \mid^2 ds = -\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} \partial_\nu v_\varepsilon^j \widehat{w}_\varepsilon^j(x) ds.$$

Therefore

$$\|\nabla v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j)} + \varepsilon^{-\gamma}\int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x) \mid v_\varepsilon^j \mid^2 ds = -\int\limits_{T^j_{\frac{\varepsilon}{4}}\setminus T^j_{\frac{\varepsilon}{8}}} \nabla v_\varepsilon^j \nabla\widehat{w} dx + \int\limits_{\partial T^j_{\frac{\varepsilon}{8}}} \partial_\nu v_\varepsilon^j \widehat{w} ds. \tag{38}$$

Then

$$\left|\int\limits_{\partial T^j_{\frac{\varepsilon}{8}}} \partial_\nu v^j_\varepsilon \widehat{w}^j ds\right| \leq K\varepsilon \|\widehat{w}\|_{L^\infty(\partial T^j_{\frac{\varepsilon}{8}})} \mid \partial T^j_{\frac{\varepsilon}{8}} \mid = K\varepsilon^{n+2}.$$

From (38) we have

$$\|\nabla v^j_\varepsilon\|^2_{L^2(T^j_{\frac{\varepsilon}{4}} \setminus G^j_\varepsilon)} + \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} b^j_\varepsilon(x) \mid v^j_\varepsilon \mid^2 ds \leq K\varepsilon^{n+2}.$$

Due to Lemma 3.3

$$\|v^j_\varepsilon\|^2_{L^2(T^j_{\frac{\varepsilon}{4}} \setminus G^j_\varepsilon)} \leq K\varepsilon^{n+4}.$$

Then, adding over $j \in \Upsilon_\varepsilon$ we obtain

$$\sum_{j\in\Upsilon_\varepsilon} \|\nabla v^j_\varepsilon\|^2_{L^2(T^j_{\frac{\varepsilon}{4}} \setminus G^j_\varepsilon)} + \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial G^j_\varepsilon} b^j_\varepsilon(x) \mid v^j_\varepsilon \mid^2 ds \leq K\varepsilon^2,$$

and

$$\sum_{j\in\Upsilon_\varepsilon} \|v^j_\varepsilon\|^2_{L^2(T^j_{\frac{\varepsilon}{4}} \setminus G^j_\varepsilon)} \leq K\varepsilon^4. \tag{39}$$

These estimates complete the proof. ■

To obtain the homogenized problem we need the following lemma:

LEMMA 4.2 *Let λ_0 be given by (9) and let $h_\varepsilon, h \in H^1_0(\Omega)$ such that $h_\varepsilon \rightharpoonup h$ in $H^1_0(\Omega)$. Then*

$$\left|\sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} (\partial_{\nu_x} \widehat{w}^j_\varepsilon) h_\varepsilon ds + C_0^{n-2} \lambda_0 \int\limits_\Omega h dx\right| \to 0$$

as $\varepsilon \to 0$, where ν is the unit outward normal vector to $\partial T^j_{\frac{\varepsilon}{4}}$.

Remark 4.3 Notice that, in this case, λ_0 does not depend on j or x.

Proof of Lemma 4.2. Let us consider the auxiliary problem

$$\begin{cases} -\Delta \theta^j_\varepsilon = \mu_\varepsilon, & \text{in } \widehat{Y}^j_\varepsilon = Y^j_\varepsilon \setminus \overline{T^j_{\frac{\varepsilon}{4}}}, \\ -\partial_\nu \theta^j_\varepsilon = \partial_\nu \widehat{w}^j_\varepsilon, & \text{on } \partial T^j_{\frac{\varepsilon}{4}}, \\ \partial_\nu \theta^j_\varepsilon = 0, & \text{on } \partial Y^j_\varepsilon \setminus \partial T^j_{\frac{\varepsilon}{4}}, \\ \langle \theta^j_\varepsilon \rangle_{\widehat{Y}^j_\varepsilon} = 0. \end{cases} \tag{40}$$

The constant μ_ε is defined from the solvability conditions of problem (40):

$$\mu_\varepsilon \varepsilon^n (1 - 2^{-2n}\omega(n)) = - \int_{\partial T^j_{\frac{\varepsilon}{4}}} \partial_\nu \widehat{w}^j_\varepsilon ds_x = \int_{\partial G^j_\varepsilon} \partial_\nu \widehat{w}^j_\varepsilon ds_x = a_\varepsilon^{n-2} \int_{\partial G_0} \partial_{\nu_y} \widehat{w}(y) ds_y,$$

therefore

$$\mu_\varepsilon = \frac{a_\varepsilon^{n-2}\lambda_0}{(1 - 2^{-2n}\omega(n))\varepsilon^n} = \frac{C_0^{n-2}\lambda_0}{1 - 2^{-2n}\omega(n)},$$

where $\omega(n)$ is the area of the unit sphere surface. From the integral identity for the problem (40) we obtain

$$\int_{\widehat{Y}^j_\varepsilon} \left|\nabla \theta^j_\varepsilon\right|^2 dx = \mu_\varepsilon \int_{\widehat{Y}_\varepsilon} \theta^j_\varepsilon dx + \int_{\partial T^j_{\frac{\varepsilon}{4}}} (\partial_\nu \widehat{w}^j_\varepsilon)\theta^j_\varepsilon ds.$$

From some estimates proved in [16] we have

$$\int_{\widehat{Y}^j_\varepsilon} \mid \theta^j_\varepsilon \mid dx \le K\varepsilon^{\frac{n}{2}} \|\theta^j_\varepsilon\|_{L^2(\widehat{Y}^j_\varepsilon)} \le K\varepsilon^{\frac{n}{2}+1} \|\nabla\theta^j_\varepsilon\|_{L^2(\widehat{Y}^j_\varepsilon)}.$$

Using the estimates proved in [16] we deduce

$$\begin{aligned}
\int_{\partial T^j_{\frac{\varepsilon}{4}}} \left|\partial_{\nu_x} \widehat{w}^j_\varepsilon \theta^j_\varepsilon\right| ds &\le K\varepsilon \int_{\partial T^j_{\frac{\varepsilon}{4}}} \mid \theta^j_\varepsilon \mid ds \\
&\le K\varepsilon^{\frac{n-1}{2}+1} \|\theta^j_\varepsilon\|_{L^2\left(\partial T^j_{\frac{\varepsilon}{4}}\right)} \\
&\le K\varepsilon^{\frac{n+1}{2}} \left\{ \varepsilon^{-\frac{1}{2}} \|\theta^j_\varepsilon\|_{L^2\left(\widehat{Y}^j_\varepsilon\right)} + \varepsilon^{\frac{1}{2}} \|\nabla\theta^j_\varepsilon\|_{L^2\left(\widehat{Y}^j_\varepsilon\right)} \right\} \\
&\le K\varepsilon^{\frac{n+2}{2}} \|\nabla\theta^j_\varepsilon\|_{L^2\left(\widehat{Y}^j_\varepsilon\right)}.
\end{aligned}$$

Thus, we have

$$\|\nabla\theta^j_\varepsilon\|^2_{L^2\left(\widehat{Y}^j_\varepsilon\right)} \le K\varepsilon^{n+2}.$$

Using this estimate we obtain that

$$\sum_{j\in\Upsilon_\varepsilon} \int_{\widehat{Y}^j_\varepsilon} \mid \nabla\theta^j_\varepsilon \mid^2 dx \le K\varepsilon^2.$$

Due to the definition of θ^j_ε we have that

$$\left| \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\frac{\varepsilon}{4}}} (\partial_\nu \widehat{w}^j_\varepsilon) h_\varepsilon ds + \mu_\varepsilon \sum_{j\in\Upsilon_\varepsilon} \int_{\widehat{Y}^j_\varepsilon} h_\varepsilon dx \right| = \left| \sum_{j\in\Upsilon_\varepsilon} \int_{\widehat{Y}^j_\varepsilon} \nabla\theta^j_\varepsilon \nabla_\varepsilon h dx \right| \le K\varepsilon \|h_\varepsilon\|_{H_1(\Omega)}.$$

Finally, from [1, Corollary 1.7] and the fact, that $\mu_\varepsilon \left| Y \setminus T_{\frac{1}{4}} \right| = C_0^{n-2}\lambda_0$, we derive that

$$\left| \mu_\varepsilon \sum_{j\in\Upsilon_\varepsilon} \int_{\widehat{Y}_\varepsilon^j} h_\varepsilon dx - C_0^{n-2}\lambda_0 \int_{\Omega} h dx \right| \to 0,$$

as $\varepsilon \to 0$. ■

4.1. *Proof of Theorem 1.1*

From (21) we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} b_\varepsilon^j(x) u_\varepsilon \varphi ds = -\lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T_{\frac{\varepsilon}{4}}^j} (\partial_\nu w_\varepsilon^j) u_\varepsilon \varphi ds.$$

By Lemma 4.1, (39) and the Green's formula we obtain

$$\begin{aligned}
\sum_{j\in\Upsilon_\varepsilon} \left| \int_{\partial T_{\frac{\varepsilon}{4}}^j} (\partial_\nu \widehat{w}_\varepsilon^j - \partial_\nu w_\varepsilon^j) h_\varepsilon ds \right| &= \sum_{j\in\Upsilon_\varepsilon} \left| \int_{\partial T_{\frac{\varepsilon}{4}}^j} (\partial_\nu v_\varepsilon^j) h_\varepsilon ds \right| \\
&= \sum_{j\in\Upsilon_\varepsilon} \left| \int_{T_{\frac{\varepsilon}{4}}^j \setminus G_\varepsilon^j} \nabla v_\varepsilon^j \nabla h_\varepsilon dx - \int_{\partial G_\varepsilon^j} (\partial_\nu v_\varepsilon^j) h_\varepsilon ds \right| \\
&= \sum_{j\in\Upsilon_\varepsilon} \left| \int_{T_{\frac{\varepsilon}{4}}^j \setminus G_\varepsilon^j} \nabla v_\varepsilon^j \nabla h_\varepsilon dx \right| + \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \left| \int_{\partial G_\varepsilon^j} b_\varepsilon^j v_\varepsilon^j h_\varepsilon ds \right| \\
&\leq K\varepsilon \|h_\varepsilon\|_{H^1(\Omega_\varepsilon)} + K\varepsilon^{\frac{n+2}{2}} \sum_{j\in\Upsilon_\varepsilon} \left(\varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} b_\varepsilon^j(x) h_\varepsilon^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Taking into account that $h_\varepsilon = u_\varepsilon \varphi$ and using estimates (13), we have:

$$\begin{aligned}
\sum_{j\in\Upsilon_\varepsilon} \left| \int_{\partial T_{\frac{\varepsilon}{4}}^j} (\partial_\nu \widehat{w}_\varepsilon^j - \partial_\nu w_\varepsilon^j) u_\varepsilon \varphi ds \right| &\leq K \left(\varepsilon + \varepsilon^{\frac{n+2}{2} - \frac{n}{2}} \left(\sum_{j\in\Upsilon_\varepsilon} \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} b_\varepsilon^j(x) h_\varepsilon^2 ds \right)^{\frac{1}{2}} \right) \\
&\leq K\varepsilon.
\end{aligned}$$

Hence, from Lemma 4.2 we derive that

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial G_\varepsilon^j} b_\varepsilon^j(x)u_\varepsilon\varphi ds = -\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T_{\frac{\varepsilon}{4}}^j}(\partial_\nu w_\varepsilon^j)u_\varepsilon\varphi ds$$

$$= -\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T_{\frac{\varepsilon}{4}}^j}(\partial_\nu \widehat{w}_\varepsilon^j)u_\varepsilon\varphi ds$$

$$= C_0^{n-2}\lambda_0\int\limits_\Omega u\varphi dx. \qquad (41)$$

Applying Lemmas 4.1 and 4.2 , (22) and (41) we obtain that u satisfies the identity

$$\int\limits_\Omega \nabla\phi\nabla u dx + C_0^{n-2}\lambda_0\int\limits_\Omega u\phi dx = \int\limits_\Omega f\phi dx$$

for all $\phi\in H_0^1(\Omega)$.

5. Case of ε - periodic coefficients

Now we set $\xi_\varepsilon^j = b(0)$ and $g_\varepsilon^j \equiv 1$.

Lemma 5.1 *The following estimate holds*

$$\sum_{j\in\Upsilon_\varepsilon}\|w_\varepsilon^j(x)-\widehat{w}_\varepsilon^j(x)\|^2_{H^1(T_{\frac{\varepsilon}{4}}^j\setminus G_\varepsilon^j)} \le K\max\{\varepsilon^2, \varepsilon^{\frac{4}{n-2}}\}.$$

Proof. In this case we have that

$$\partial_\nu v_\varepsilon^j = -\varepsilon^{-\gamma}b(0)v_\varepsilon^j - \varepsilon^{-\gamma}(b_\varepsilon^j(x)-b(0))w_\varepsilon^j + \varepsilon^{-\gamma}(b_\varepsilon^j(x)-b(0)) \quad \text{if } x\in\partial G_\varepsilon^j. \quad (42)$$

We use v_ε^j as the test function in the integral identity associated to (42):

$$\int\limits_{T_{\frac{\varepsilon}{4}}^j\setminus G_\varepsilon^j} |\nabla v_\varepsilon^j|^2\, dx + \varepsilon^{-\gamma}b(0)\int\limits_{\partial G_\varepsilon^j}|v_\varepsilon^j|^2\, ds = \varepsilon^{-\gamma}\int\limits_{\partial G_\varepsilon^j}(b(0)-b_\varepsilon(x))w_\varepsilon^j v_\varepsilon^j ds +$$

$$+\varepsilon^{-\gamma}\int\limits_{\partial G_\varepsilon^j}(b_\varepsilon(x)-b(0))v_\varepsilon^j ds + \int\limits_{\partial T_{\frac{\varepsilon}{4}}^j}\partial_\nu v_\varepsilon^j\widehat{w}_\varepsilon^j(x)ds,$$

so

$$\|\nabla v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon)} + \varepsilon^{-\gamma} b(0) \int\limits_{\partial G^j_\varepsilon} | v_\varepsilon^j |^2 \, ds$$

$$= \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} (b(0) - b_\varepsilon(x)) w_\varepsilon^j v_\varepsilon^j ds + \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} (b_\varepsilon(x) - b(0)) v_\varepsilon^j ds$$

$$+ \int\limits_{T^j_{\frac{\varepsilon}{4}}\setminus T^j_{\frac{\varepsilon}{8}}} \nabla v_\varepsilon^j \nabla \widehat{w} dx - \int\limits_{\partial T^j_{\frac{\varepsilon}{8}}} \partial_\nu v_\varepsilon^j \widehat{w} ds. \tag{43}$$

We have that

$$\left| \int\limits_{\partial T^j_{\frac{\varepsilon}{8}}} \partial_\nu v_\varepsilon^j \widehat{w} ds \right| \leq K\varepsilon \|\widehat{w}\|_{L^\infty(\partial T^j_{\frac{\varepsilon}{8}})} \mid \partial T^j_{\frac{\varepsilon}{8}} \mid = K\varepsilon^{n+2}.$$

Using the smoothness of function b we have that $|b_\varepsilon(x) - b(0)| \leq K \left| \frac{x}{\varepsilon} \right|$ and thus we deduce

$$\varepsilon^{-\gamma} \left| \int\limits_{\partial G^j_\varepsilon} (b_\varepsilon(x) - b(0)) w_\varepsilon^j v_\varepsilon^j ds \right| \leq \varepsilon^{-\gamma} \left(a_\varepsilon^2 \varepsilon^{-2} C_{b(0)/4} \int\limits_{\partial G^j_\varepsilon} | w_\varepsilon^j |^2 \, ds + \frac{b(0)}{4} \int\limits_{\partial G^j_\varepsilon} | v_\varepsilon^j |^2 \, ds \right)$$

$$= \varepsilon^{-\gamma} \frac{b(0)}{4} \int\limits_{\partial G^j_\varepsilon} | v_\varepsilon^j |^2 \, ds + \varepsilon^{-2} a_\varepsilon C_{b(0)/4} \int\limits_{\partial G^j_\varepsilon} | w_\varepsilon^j |^2 \, ds,$$

and

$$\varepsilon^{-\gamma} \left| \int\limits_{\partial G^j_\varepsilon} (b_\varepsilon(x) - b(0)) v_\varepsilon^j ds \right| \leq \varepsilon^{-\gamma} \left(C_{b(0)/4} a_\varepsilon^2 \varepsilon^{-2} \mid \partial G^j_\varepsilon \mid + \frac{b(0)}{4} \int\limits_{\partial G^j_\varepsilon} | v_\varepsilon^j |^2 \, ds \right)$$

$$\leq C_{b(0)/4} \varepsilon^{-2} a_\varepsilon^n + \varepsilon^{-\gamma} \frac{b(0)}{4} \int\limits_{\partial G^j_\varepsilon} | v_\varepsilon^j |^2 \, ds.$$

From (43) we have

$$\|\nabla v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon)} + \varepsilon^{-\gamma} b(0) \int\limits_{\partial G^j_\varepsilon} | v_\varepsilon^j |^2 \, ds$$

$$\leq K \left(\varepsilon^{n+2} + \varepsilon^{-2} a_\varepsilon^n + \varepsilon^{-2} a_\varepsilon C_{b(0)/4} \int\limits_{\partial G^j_\varepsilon} | w_\varepsilon^j |^2 \, ds \right)$$

$$+ \varepsilon^{-\gamma} \frac{b(0)}{2} \int\limits_{\partial G^j_\varepsilon} | v_\varepsilon^j |^2 \, ds.$$

Therefore

$$\|\nabla v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon)} + \varepsilon^{-\gamma}\frac{b(0)}{2}\int\limits_{\partial G^j_\varepsilon} \mid v_\varepsilon^j \mid^2 ds$$

$$\leq K\left(\varepsilon^{n+2} + \varepsilon^{-2}a_\varepsilon^n + \varepsilon^{-2}a_\varepsilon C_{b(0)/4}\int\limits_{\partial G^j_\varepsilon} \mid w_\varepsilon^j \mid^2 ds\right).$$

Due to Lemma 3.3

$$\|v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon)} \leq K\varepsilon^{n+4}.$$

Then, adding over all $j \in \Upsilon_\varepsilon$ we obtain

$$\sum_{j\in\Upsilon_\varepsilon} \|\nabla v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon)} + \varepsilon^{-\gamma}\frac{b(0)}{2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial G^j_\varepsilon} \mid v_\varepsilon^j \mid^2 ds \leq K\max\{\varepsilon^2, \varepsilon^{\frac{4}{n-2}}\},$$

and

$$\sum_{j\in\Upsilon_\varepsilon} \|v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon)} \leq K\varepsilon^4. \tag{44}$$

This estimates complete the proof. ■

To identify the homogenized problem we need the following lemma, the proof of this result follows similarly to proof of Lemma 4.2.

LEMMA 5.2 *Let λ_0 be given by (10) (notice that λ_0 is independent of j) and let $h_\varepsilon, h \in H_0^1(\Omega)$ be such that $h_\varepsilon \rightharpoonup h$ in $H_0^1(\Omega)$. Then*

$$\left|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} \left(\partial_{\nu_x}\widehat{w}_\varepsilon^j\right) h_\varepsilon ds + C_0^{n-2}\lambda_0\int\limits_\Omega h dx\right| \to 0$$

as $\varepsilon \to 0$, where ν is the unit outward normal vector to $\partial T^j_{\frac{\varepsilon}{4}}$.

5.1. *Proof of Theorem 1.1*

From (21) we have

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon} b_\varepsilon(x)u_\varepsilon\varphi ds = -\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} (\partial_\nu w_\varepsilon^j)u_\varepsilon\varphi ds.$$

By Lemma 3, (44) and the Green's formula we obtain

$$\sum_{j\in\Upsilon_\varepsilon}\left|\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}}(\partial_\nu\widehat{w}^j_\varepsilon(x)-\partial_\nu w^j_\varepsilon)u_\varepsilon\varphi ds\right| = \sum_{j\in\Upsilon_\varepsilon}\left|\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}}\partial_\nu v^j_\varepsilon u_\varepsilon\varphi ds\right| =$$

$$= \sum_{j\in\Upsilon_\varepsilon}\left|\int\limits_{T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon}\nabla v^j_\varepsilon\nabla(u_\varepsilon\varphi)dx - \int\limits_{\partial G^j_\varepsilon}\partial_\nu v^j_\varepsilon u_\varepsilon\varphi ds\right|$$

$$= \sum_{j\in\Upsilon_\varepsilon}\left|\int\limits_{T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon}\nabla v^j_\varepsilon\nabla(u_\varepsilon\varphi)dx + +\varepsilon^{-\gamma}b(0)\int\limits_{\partial G^j_\varepsilon}v^j_\varepsilon u_\varepsilon\varphi ds\right.$$

$$\left. +\varepsilon^{-\gamma}\int\limits_{\partial G^j_\varepsilon}(b_\varepsilon(x)-b(0))w^j_\varepsilon u_\varepsilon\varphi ds - \varepsilon^{-\gamma}\int\limits_{\partial G^j_\varepsilon}(b_\varepsilon(x)-b(0))u_\varepsilon\varphi ds\right|.$$

From Cauchy's inequality, for smooth φ we have

$$\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\left|\int\limits_{\partial G^j_\varepsilon}(b_\varepsilon(x)-b(0))w^j_\varepsilon u_\varepsilon\varphi ds\right|$$

$$\leq Ka_\varepsilon\varepsilon^{-1}\sum_{j\in\Upsilon_\varepsilon}\left(\varepsilon^{-\gamma}\int\limits_{\partial G^j_\varepsilon}|\, w^j_\varepsilon\,|^2\, ds\right)^{\frac{1}{2}}\left(\varepsilon^{-\gamma}\int\limits_{\partial G^j_\varepsilon}|\, u_\varepsilon\varphi\,|^2\, ds\right)^{\frac{1}{2}}$$

$$\leq Ka_\varepsilon\varepsilon^{-1}\varepsilon^{\frac{n}{2}}\sum_{j\in\Upsilon_\varepsilon}\left(\varepsilon^{-\gamma}\int\limits_{\partial G^j_\varepsilon}|\, u_\varepsilon\varphi\,|^2\, ds\right)^{\frac{1}{2}}$$

$$\leq Ka_\varepsilon\varepsilon^{-1}\left(\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}|\, u_\varepsilon\,|^2\, ds\right)^{\frac{1}{2}} \leq K\varepsilon^{\frac{2}{n-2}}.$$

Also

$$\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\left|\int\limits_{\partial G^j_\varepsilon}(b_\varepsilon(x)-b(0))u_\varepsilon\varphi ds\right| \leq K\varepsilon^{-\gamma}a_\varepsilon\varepsilon^{-1}\sum_{j\in\Upsilon_\varepsilon}\sqrt{|\,\partial G^j_\varepsilon\,|}\left(\int\limits_{\partial G^j_\varepsilon}|\, u_\varepsilon\varphi\,|^2\, ds\right)^{\frac{1}{2}}$$

$$\leq K\varepsilon^{-\gamma}a_\varepsilon^{\frac{n+1}{2}}\varepsilon^{-1}\varepsilon^{-\frac{n}{2}}\left(\int\limits_{S_\varepsilon}|\, u_\varepsilon\,|^2\, ds\right)^{\frac{1}{2}} \leq K\varepsilon^{\frac{2}{n-2}}.$$

Therefore

$$\left|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}}(\partial_\nu\widehat{w}^j_\varepsilon-\partial_\nu w^j_\varepsilon)u_\varepsilon\varphi ds\right|\le K(\varepsilon\|u_\varepsilon\varphi\|_{H^1(\Omega_\varepsilon)}+\varepsilon^{\frac{2}{n-2}}).$$

Taking into account estimates (13), we have that

$$\left|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}}(\partial_\nu\widehat{w}^j_\varepsilon-\partial_\nu w^j_\varepsilon)u_\varepsilon\varphi ds\right|\le K\max\{\varepsilon,\varepsilon^{\frac{2}{n-2}}\}.$$

So, from Lemma 5.2 we deduce that

$$\lim_{\varepsilon\to0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}b_\varepsilon u_\varepsilon\varphi ds=-\lim_{\varepsilon\to0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}}(\partial_\nu w^j_\varepsilon)u_\varepsilon\varphi ds=-\lim_{\varepsilon\to0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}}(\partial_\nu\widehat{w}^j_\varepsilon)u_\varepsilon\varphi ds$$
$$=C_0^{n-2}\lambda\int\limits_\Omega u_0\varphi dx. \tag{45}$$

Applying the previous lemmas and (45), we obtain that u satisfies the identity

$$\int\limits_\Omega\nabla\phi\nabla u dx+C_0^{n-2}\lambda_0\int\limits_\Omega u\phi dx=\int\limits_\Omega f\phi dx$$

for all $\phi\in H^1_0(\Omega)$. Hence $u\in H^1_0(\Omega)$ is a weak solution of the problem

$$\begin{cases}-\Delta u_0+C_0^{n-2}\lambda_0u_0=f & x\in\Omega,\\ u_0=0 & x\in\partial\Omega.\end{cases} \tag{46}$$

6. Case of coefficients independent of ε

In this case $g(y)=1$, and we will change the value of ξ. Hence in this section we will use the simplified notation $\widehat{w}=\widehat{w}(y;\xi)$ and

$$\lambda(\xi)=\lambda(G_0,\xi,y).$$

Lemma 6.1 *The estimate*

$$\sum_{j\in\Upsilon_\varepsilon}\|w^j_\varepsilon(x)-\widehat{w}^j_\varepsilon(x;b(P^j_\varepsilon))\|^2_{H^1(T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon)}\le K\varepsilon^{\frac{2(n-1)}{n-2}}$$

holds.

Proof. Let us note that $0\le v^j_\varepsilon(x)\le\widehat{w}^j_\varepsilon(x;b(P^j_\varepsilon))$. For the rest of the proof

$$v^j_\varepsilon(x)=\widehat{w}^j_\varepsilon(x;b(P^j_\varepsilon))-w^j_\varepsilon(x).$$

We use v_ε^j as the test function in the integral identity of the weak formulation of its problem:

$$\int\limits_{T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon} \mid \nabla v_\varepsilon^j \mid^2 dx + \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} b(x) \mid v_\varepsilon^j \mid^2 ds = \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} (b(P^j_\varepsilon) - b(x))\widehat{w}^j_\varepsilon(x) v^j_\varepsilon ds$$

$$+ \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} (b(x) - b(P^j_\varepsilon)) v^j_\varepsilon ds + \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} \partial_\nu v^j_\varepsilon \widehat{w} ds,$$

so

$$\|\nabla v^j_\varepsilon\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon)} + \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} b(x) \mid v_\varepsilon^j \mid^2 ds = \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} (b(P^j_\varepsilon) - b(x))\widehat{w}^j_\varepsilon(x) v^j_\varepsilon ds$$

$$+ \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} (b(x) - b(P^j_\varepsilon)) v^j_\varepsilon ds + \int\limits_{T^j_{\frac{\varepsilon}{4}}\setminus T^j_{\frac{\varepsilon}{8}}} \nabla v^j_\varepsilon \nabla \widehat{w} dx + \int\limits_{\partial T^j_{\frac{\varepsilon}{8}}} \partial_\nu v^j_\varepsilon \widehat{w} ds. \tag{47}$$

We have that $\mid v^j_\varepsilon \mid \leq \mid \widehat{w}^j_\varepsilon \mid \leq K\varepsilon^2$. Applying the estimates on derivatives of harmonic function and the maximum principle we get that, if $x_0 \in \partial T^j_{\frac{\varepsilon}{8}}$, then

$$\mid \partial_{x_i} v^j_\varepsilon(x_0) \mid = \frac{1}{\mid T^{x_0}_{\frac{\varepsilon}{16}} \mid} \left| \int\limits_{T^{x_0}_{\frac{\varepsilon}{16}}} \frac{\partial v^j_\varepsilon}{\partial x_i} dx \right|$$

$$= \frac{K}{\varepsilon^n} \left| \int\limits_{\partial T^{x_0}_{\frac{\varepsilon}{16}}} v^j_\varepsilon \nu_i ds \right| \leq K\varepsilon,$$

where T^x_τ is the ball of radius τ centered at x. Hence, $\mid \nabla v^j_\varepsilon \mid \leq K\varepsilon$ on $\partial T^j_{\frac{\varepsilon}{8}}$. Then

$$\left| \int\limits_{\partial T^j_{\frac{\varepsilon}{8}}} (\partial_\nu v^j_\varepsilon) \widehat{w}^j_\varepsilon ds \right| \leq K\varepsilon \|\widehat{w}\|_{L^\infty(\partial T^j_{\frac{\varepsilon}{8}})} \mid \partial T^j_{\frac{\varepsilon}{8}} \mid = K\varepsilon^{n+2}.$$

Using that $|b(x) - b(P^j_\varepsilon)| \leq K a_\varepsilon$ for any $x \in \partial G^j_\varepsilon$, and that $0 \leq \widehat{w} \leq 1$ we deduce

$$\varepsilon^{-\gamma} \left| \int\limits_{\partial G^j_\varepsilon} (b(P^j_\varepsilon) - b(x))\widehat{w}^j_\varepsilon(x) v^j_\varepsilon ds \right| \leq \varepsilon^{-\gamma} \left(C_{b_0/4} \int\limits_{\partial G^j_\varepsilon} \mid P^j_\varepsilon - x \mid^2 ds + \frac{b_0}{4} \int\limits_{\partial G^j_\varepsilon} \mid v^j_\varepsilon \mid^2 ds \right)$$

$$\leq \varepsilon^{-\gamma} \frac{b_0}{4} \int\limits_{\partial G^j_\varepsilon} \mid v^j_\varepsilon \mid^2 ds + a_\varepsilon C_{b_0/4} \mid \partial G^j_\varepsilon \mid,$$

and

$$\varepsilon^{-\gamma}\left|\int\limits_{\partial G_\varepsilon^j}(b(x)-b(P_\varepsilon^j))v_\varepsilon^j ds\right| \le \varepsilon^{-\gamma}\left(a_\varepsilon^2 C_{b_0/4}\mid \partial G_\varepsilon^j\mid + \frac{b_0}{4}\int\limits_{\partial G_\varepsilon^j}\mid v_\varepsilon^j\mid^2 ds\right)$$
$$\le C_{b_0/4}a_\varepsilon^n + \varepsilon^{-\gamma}\frac{b_0}{4}\int\limits_{\partial G_\varepsilon^j}\mid v_\varepsilon^j\mid^2 ds.$$

From (47) we have

$$\|\nabla v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j)} + \varepsilon^{-\gamma}b_0\int\limits_{\partial G_\varepsilon^j}\mid v_\varepsilon^j\mid^2 ds \le K(a_\varepsilon^n + \varepsilon a_\varepsilon^{n-1}) + \varepsilon^{-\gamma}\frac{b_0}{2}\int\limits_{\partial G_\varepsilon^j}\mid v_\varepsilon^j\mid^2 ds.$$

Hence

$$\|\nabla v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j)} + \varepsilon^{-\gamma}\frac{b_0}{2}\int\limits_{\partial G_\varepsilon^j}\mid v_\varepsilon^j\mid^2 ds \le K(a_\varepsilon^n + \varepsilon a_\varepsilon^{n-1}) \le K\varepsilon a_\varepsilon^{n-1}. \tag{48}$$

Due to Lemma 3.3 we have

$$\|v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j)} \le K\varepsilon^{n+4}.$$

Then, adding over $j\in\Upsilon_\varepsilon$ we obtain

$$\sum_{j\in\Upsilon_\varepsilon}\|\nabla v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j)} + \varepsilon^{-\gamma}\frac{b_0}{2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial G_\varepsilon^j}\mid v_\varepsilon^j\mid^2 ds \le K\varepsilon^{\frac{2(n-1)}{n-2}},$$

and

$$\sum_{j\in\Upsilon_\varepsilon}\|v_\varepsilon^j\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G_\varepsilon^j)} \le K\varepsilon^4. \tag{49}$$

This estimates complete the proof. ■

To identify the homogenized problem we need the following lemma.

LEMMA 6.2 *Let $\lambda_0(x)$ be given by (11) and $h_\varepsilon, h\in H_0^1(\Omega)$ be such that $h_\varepsilon \rightharpoonup h$ in $H_0^1(\Omega)$. Then*

$$\left|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}}\big(\partial_{\nu_x}\widehat{w}_\varepsilon^j(x;b(P_\varepsilon^j))\big)h ds + C_0^{n-2}\int\limits_\Omega\lambda_0(x)h dx\right| \to 0$$

as $\varepsilon\to 0$, where ν is an unit outward normal vector to $\partial T^j_{\frac{\varepsilon}{4}}$.

Proof. Let us consider the auxiliary problem

$$\begin{cases} -\Delta\theta_\varepsilon^j = \mu_\varepsilon^j, & \widehat{Y}_\varepsilon^j, \\ -\partial_\nu\theta_\varepsilon^j = \partial_\nu \widehat{w}_\varepsilon^j(x; b(P_\varepsilon^j)), & \partial T^j_{\frac{\varepsilon}{4}}, \\ \partial_\nu\theta_\varepsilon^j = 0 & \partial Y_\varepsilon^j, \\ \langle\theta_\varepsilon\rangle_{Y_\varepsilon^j} = 0. \end{cases} \tag{50}$$

The constant μ_ε^j is defined from the solvability conditions of problem (50)

$$\begin{aligned} \mu_\varepsilon^j \varepsilon^n(1 - 2^{-2n}\omega(n)) &= -\int\limits_{\partial T^0_{\frac{\varepsilon}{4}}} \partial_\nu \widehat{w}_\varepsilon^j(x; b(P_\varepsilon^j)) ds_x \\ &= \int\limits_{\partial G_\varepsilon} \partial_\nu \widehat{w}_\varepsilon^j(x; b(P_\varepsilon^j)) ds_x \\ &= a_\varepsilon^{n-2} \int\limits_{\partial G_0} \partial_{\nu_y} \widehat{w}(b(P_\varepsilon^j), y) ds_y. \end{aligned}$$

Hence

$$\mu_\varepsilon^j = \frac{a_\varepsilon^{n-2}\lambda(b(P_\varepsilon^j))}{(1 - 2^{-2n}\omega(n))\varepsilon^n} = \frac{C_0^{n-2}\lambda(b(P_\varepsilon^j))}{1 - 2^{-2n}\omega(n)}.$$

From the integral identity for the problem (50) we obtain

$$\int\limits_{\widehat{Y}_\varepsilon^j} |\nabla\theta_\varepsilon^j|^2 \, dx = \mu_\varepsilon^j \int\limits_{\widehat{Y}_\varepsilon^j} \theta_\varepsilon^j dx + \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} \big(\partial_\nu \widehat{w}_\varepsilon^j(x; b(P_\varepsilon^j))\big)\theta_\varepsilon^j ds.$$

From some estimates proved in [16] we have

$$\int\limits_{Y_\varepsilon^j} |\theta_\varepsilon^j| \, dx \le K\varepsilon^{\frac{n}{2}} \|\theta_\varepsilon^j\|_{L^2(\widehat{Y}_\varepsilon^j)} \le K\varepsilon^{\frac{n}{2}+1} \|\nabla\theta_\varepsilon^j\|_{L^2(\widehat{Y}_\varepsilon^j)}.$$

Taking into account that $\max_{\partial T^j_{\frac{\varepsilon}{4}}} |\partial_{\nu_x} \widehat{w}_\varepsilon^j(x; b(P_\varepsilon^j))| \le K\varepsilon$, and using the estimates proved in [16] we deduce

$$\begin{aligned} \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} |\partial_{\nu_x} \widehat{w}_\varepsilon^j(x; b(P_\varepsilon^j))\theta_\varepsilon^j| \, ds &\le K\varepsilon \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} |\theta_\varepsilon^j| \, ds \le K\varepsilon^{\frac{n-1}{2}+1} \|\theta_\varepsilon^j\|_{L^2(\partial T^j_{\frac{\varepsilon}{4}})} \\ &\le K\varepsilon^{\frac{n+1}{2}} \{\varepsilon^{-\frac{1}{2}} \|\theta_\varepsilon^j\|_{L^2(\widehat{Y}_\varepsilon^j)} + \sqrt{\varepsilon} \|\nabla\theta_\varepsilon^j\|_{L^2(\widehat{Y}_\varepsilon^j)}\} \\ &\le K\varepsilon^{\frac{n+2}{2}} \|\nabla\theta_\varepsilon^j\|_{L^2(\widehat{Y}_\varepsilon^j)}. \end{aligned}$$

Thus, we have

$$\|\nabla\theta_\varepsilon^j\|^2_{L^2(\widehat{Y}_\varepsilon^j)} \le K\varepsilon^{n+2}.$$

Using this estimate we obtain

$$\sum_{j\in\Upsilon_\varepsilon}\int_{\widehat{Y}_\varepsilon^j} |\nabla\theta_\varepsilon^j|^2\,dx \le K\varepsilon^2.$$

According to the definition of θ_ε^j we obtain

$$\left|\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{\frac{\varepsilon}{4}}} \partial_\nu\Big(\widehat{w}_\varepsilon^j(x;b(P_\varepsilon^j))\Big)hds + \sum_{j\in\Upsilon_\varepsilon}\int_{\widehat{Y}_\varepsilon^j}\mu_\varepsilon^j hdx\right| = \left|\sum_{j\in\Upsilon_\varepsilon}\int_{\widehat{Y}_\varepsilon^j}\nabla\theta_\varepsilon^j\nabla hdx\right| \le K\varepsilon\|h\|_{H_1(\Omega)}.$$

To identify

$$\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int_{\widehat{Y}_\varepsilon^j}\mu_\varepsilon^j hdx$$

we need some additional analysis; from the expression for μ_ε we deduce

$$\begin{aligned}&\left|\sum_{j\in\Upsilon_\varepsilon}\int_{\widehat{Y}_\varepsilon^j}\mu_\varepsilon^j hdx - \frac{C_0^{n-2}}{|\,Y\setminus T^0_{\frac{1}{4}}\,|}\sum_{j\in\Upsilon_\varepsilon}\int_{\cup\widehat{Y}_\varepsilon^j}\lambda(b(x))hdx\right|\\ &\qquad\le\left|\frac{C_0^{n-2}}{|\,Y\setminus T^0_{\frac{1}{4}}\,|}\sum_{j\in\Upsilon_\varepsilon}\int_{\widehat{Y}_\varepsilon^j}(\lambda(b(P_\varepsilon^j))-\lambda(b(x)))hdx\right|.\end{aligned}$$

From estimate (27) we have

$$\begin{aligned}&\left|\sum_{j\in\Upsilon_\varepsilon}\int_{\widehat{Y}_\varepsilon^j}(\lambda(b(P_\varepsilon^j))-\lambda(b(x)))hdx\right|\\
&\le\|h\|_{L^1(\Omega)}\max_j\left|\int_{\partial G_0}\partial_{\nu_y}\widehat{w}(y;b(P_\varepsilon^j))-\partial_{\nu_y}\widehat{w}(y;b(x))ds_y\right|\\
&=\|h\|_{L^1(\Omega)}\\
&\quad\times\max_{j\in\Upsilon_\varepsilon}\left|C_0\int_{\partial G_0}\Big(b(x)\widehat{w}(y;b(x))-b(P_\varepsilon^j)\widehat{w}_\varepsilon^j(y;b(P_\varepsilon^j))+b(P_\varepsilon^j)-b(x)\Big)ds_y\right|\\
&\le K\max_{j\in\Upsilon_\varepsilon}\left\{|b(x)-b(P_\varepsilon^j)|+b(P_\varepsilon^j)|\widehat{w}(b(x),y)-\widehat{w}(b(P_\varepsilon^j),y)|\right\}\\
&\le Ka_\varepsilon\to 0\end{aligned}$$

as $\varepsilon \to 0$. From Corollary 1.7 in [1] we deduce that

$$\left| \frac{C_0^{n-2}}{\mid Y \setminus T^0_{\frac{1}{4}} \mid} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\widehat{Y}^j_\varepsilon} \lambda(b(x)) h dx - C_0^{n-2} \int\limits_{\Omega} \lambda(b(x)) h dx \right| \to 0,$$

as $\varepsilon \to 0$. This completes the proof. ■

6.1. *Proof of Theorem 1.1*

From (21) we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} b(x) u_\varepsilon \varphi ds = - \lim_{\varepsilon \to 0} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} (\partial_\nu w^j_\varepsilon) u_\varepsilon \varphi ds.$$

From Lemma 6.1 and Green's formula we obtain

$$\sum_{j \in \Upsilon_\varepsilon} \left| \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} (\partial_\nu \widehat{w}^j_\varepsilon(x; b(P^j_\varepsilon)) - \partial_\nu w^j_\varepsilon) u_\varepsilon \varphi ds \right| = \sum_{j \in \Upsilon_\varepsilon} \left| \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} \partial_\nu v^j_\varepsilon u_\varepsilon \varphi ds \right|$$

$$= \sum_{j \in \Upsilon_\varepsilon} \left| \int\limits_{T^j_{\frac{\varepsilon}{4}} \setminus G^j_\varepsilon} \nabla v^j_\varepsilon \nabla (u_\varepsilon \varphi) dx - \int\limits_{\partial G^j_\varepsilon} \partial_\nu v^j_\varepsilon u_\varepsilon \varphi ds \right|,$$

so we have

$$\sum_{j \in \Upsilon_\varepsilon} \left| \int\limits_{\partial T^j_{\frac{\varepsilon}{4}}} (\partial_\nu \widehat{w}^j_\varepsilon(x; b(P^j_\varepsilon)) - \partial_\nu w^j_\varepsilon) u_\varepsilon \varphi ds \right|$$

$$\leq \sum_{j \in \Upsilon_\varepsilon} \left| \int\limits_{T^j_{\frac{\varepsilon}{4}} \setminus G^j_\varepsilon} \nabla v^j_\varepsilon \nabla (u_\varepsilon \varphi) dx \right|$$

$$+ \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \left| \int\limits_{\partial G^j_\varepsilon} b(x) v^j_\varepsilon u_\varepsilon \varphi ds \right|$$

$$+ \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \left| \int\limits_{\partial G^j_\varepsilon} (b(P^j_\varepsilon) - b(x)) \widehat{w}^j_\varepsilon(x; b(P^j_\varepsilon)) u_\varepsilon \varphi ds \right|$$

$$+ \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \left| \int\limits_{\partial G^j_\varepsilon} (b(x) - b(P^j_\varepsilon)) u_\varepsilon \varphi ds \right|.$$

From Cauchy's inequality and the properties of v_ε^j, for smooth φ we have

$$\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\left|\int\limits_{\partial G_\varepsilon^j}(b(P_\varepsilon^j)-b(x))\widehat{w}_\varepsilon^j(x)u_\varepsilon\varphi ds\right|\le K\sqrt{|\,S_\varepsilon\,|}\left(\int\limits_{S_\varepsilon}|\,u_\varepsilon\varphi\,|^2\,ds\right)^{\frac{1}{2}}$$
$$\le K\varepsilon^{\frac{n}{n-2}},$$

also

$$\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\left|\int\limits_{\partial G_\varepsilon^j}(b(x)-b(P_\varepsilon^j))u_\varepsilon\varphi ds\right|\le K\varepsilon^{-\gamma}a_\varepsilon\left(\int\limits_{S_\varepsilon}|\,u_\varepsilon\varphi\,|^2\,ds\right)^{\frac{1}{2}}\sqrt{|\,S_\varepsilon\,|}$$
$$\le K\varepsilon^{\frac{n}{n-2}},$$

and, finally, considering (13) and (48), we have that

$$\begin{aligned}\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\left|\int\limits_{\partial G_\varepsilon^j}b(x)v_\varepsilon^j u_\varepsilon\varphi ds\right| &\le \varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\left(\int\limits_{\partial G_\varepsilon^j}b(x)(v_\varepsilon^j)^2ds\right)^{\frac{1}{2}}\left(\int\limits_{\partial G_\varepsilon^j}b(x)u_\varepsilon^2ds\right)^{\frac{1}{2}}\\ &\le K\sqrt{\varepsilon a_\varepsilon^{n-1}}\sum_{j\in\Upsilon_\varepsilon}\left(\varepsilon^{-\gamma/2}\int\limits_{\partial G_\varepsilon^j}b(x)u_\varepsilon^2ds\right)^{\frac{1}{2}}\\ &\le K\sqrt{\varepsilon a_\varepsilon^{n-1}}\varepsilon^{-\frac{n}{2}}\\ &\le K\varepsilon^{\frac{n-1}{n-2}}.\end{aligned}$$

From these inequalities we deduce that

$$\sum_{j\in\Upsilon_\varepsilon}\left|\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}}(\partial_\nu\widehat{w}_\varepsilon^j(x;b(P_\varepsilon^j))-\partial_\nu w_\varepsilon^j)u_\varepsilon\varphi ds\right|\le K\left(\varepsilon\|u_\varepsilon\varphi\|_{H^1(\Omega_\varepsilon)}+\varepsilon^{\frac{n-1}{n-2}}+\varepsilon^{\frac{n}{n-2}}\right).$$

Taking into account that the estimates (13) are valid, we have:

$$\sum_{j\in\Upsilon_\varepsilon}\left|\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}}(\partial_\nu\widehat{w}_\varepsilon^j(x;b(P_\varepsilon^j))-\partial_\nu w_\varepsilon^j)u_\varepsilon\varphi ds\right|\le K\varepsilon.$$

Hence, from Lemma 6.2 we derive

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon} b(x)u_\varepsilon\varphi ds = -\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}}\partial_\nu w^j_\varepsilon u_\varepsilon\varphi ds$$

$$= -\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\frac{\varepsilon}{4}}}(\partial_\nu \widehat{w}^j_\varepsilon(x;b(P^j_\varepsilon)))u_\varepsilon\varphi ds$$

$$= C_0^{n-2}\int\limits_\Omega \lambda(b(x))u_0\varphi dx. \tag{51}$$

Applying Lemmas 6.1, 6.2 and the previous estimates, we obtain that u satisfies the identity

$$\int\limits_\Omega \nabla\phi\nabla u dx + C_0^{n-2}\int\limits_\Omega \lambda(b(x))u\phi dx = \int\limits_\Omega f\phi dx$$

for all $\phi\in H_0^1(\Omega)$. Hence $u\in H_0^1(\Omega)$ is the (unique) weak solution of the problem

$$\begin{cases} -\Delta u + C_0^{n-2}\lambda(b(x))u = f, & \Omega,\\ u=0, & \partial\Omega. \end{cases} \tag{52}$$

■

Example 6.3 Let us apply techniques above to the particular case, where G_0 is a ball of radius 1 and coefficient $b(x)$ does not depend on ε. This example is well known (it has been proven with various simpler techniques), and hence we check that our findings are consistent with previous results. In this case we consider the auxiliary problem

$$\begin{cases} \Delta_y\widehat{w}=0, & y\in\mathbb{R}^n\setminus\overline{G_0},\\ \partial_{\nu_y}\widehat{w} = -C_0 b(x)\widehat{w} + C_0 b(x), & y\in\partial G_0,\\ \widehat{w}\to 0, & \mid y\mid\to\infty. \end{cases} \tag{53}$$

This problem has the (unique) solution $\widehat{w}=\frac{C}{r^{n-2}}$, where C may be obtained from the boundary conditions. From direct calculations we have that $C=\frac{b(x)}{b(x)+\frac{n-2}{C_0}}$. Using the result of Section 3 we have that

$$\lambda_0(x) = \int\limits_{\partial G_0}\partial_\nu\widehat{w}(y;b(x))ds = C(n-2)\omega_n = \frac{(n-2)\omega_n b(x)}{b(x)+\frac{n-2}{C_0}},$$

where ω_n is the square of the surface of the unit sphere. Hence, the limit problem distributed reaction on the surface of particles given by balls is:

$$\begin{cases} -\Delta u + C_1\frac{b(x)}{b(x)+\frac{n-2}{C_0}}u = f, & \Omega,\\ u=0, & \partial\Omega, \end{cases} \tag{54}$$

where $C_1 = (n-2)C_0^{n-2}\omega_n$.

Remark 6.4 As mentioned at the introduction, it seems possible to adapt the techniques of proof of this paper with other developed to the study of microscopical nonlinear reactions (see, e.g. [5] for the case of a Hólder continuous increasing kinetics and particles of arbitrary shape and [17] for the case of a kinetics given by a general maximal monotone graph and particles given by balls). So, we conjecture (and it will be developed in some separated work) that the analysis of the case in which the microscopical reaction is given by

$$\begin{cases} -\Delta_p u_\varepsilon = f(x), & x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma} b_\varepsilon^j(x)\sigma(u_\varepsilon), & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases}$$

would lead (under suitable conditions on $b_\varepsilon^j(x)$) to the global homogenization problem

$$\begin{cases} -\Delta u_0 + H_{G_0}(x, u_0) = f, & \text{in } \Omega, \\ u_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

for some Lipschitz continuous nondecreasing function $H_{G_0}(x, u_0)$ of u_0, once we assume the critical relation $\gamma = \alpha = n/(n-2)$, and thus the "strange term" $H_{G_0}(x, u_0)$ would present both, a $x-$dependence distribution absorption coefficient and a nonlinear kinetic, different to the case of the microscopic problem.

7. Proof of Theorem 1.4

As in [12] the result follows from applying Lemma 1.6 in Chapter III of [1] about the spectrum of a sequence of singularly perturbed operators together with Theorem 1.1. ∎

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