The norm of a canonical isomorphism of determinant line bundles

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To everyone who has helped me
on my mathematical journey,
from my parents and teachers
to my tutors and supervisor.

Abstract

We extend an involution formula given by a canonical isomorphism of determinant line bundles in Rössler's [a] to the analytic case where the determinant of cohomology is endowed with the Quillen metric, in the case where the fixed point scheme is a Cartier divisor.

Additionally, we show the relation between Rössler's main results in [a] and the Adams-Riemann-Roch theorem; and we extend the main result of Ducrot's [b], which aims to extend Deligne's pairing to the higher relative dimensional intersection bundle, to the analytic case.

[a] Damian Rössler, *A local refinement of the Adams-Riemann-Roch theorem in degree 1*, Arithmetic L-functions and differential geometric methods, Progr. Math., vol. 338, Birkhäuser/Springer, Cham, [2021], pp. 213–246.

[b] François Ducrot, *Cube structures and intersection bundles*, J. Pure Appl. Algebra, 195 (2005), no. 1, 33–73.

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Contents

Chapter 1

Introduction

Back when we were starting this research, Rössler showed us his [23]—then titled *Canonical isomorphisms of determinant line bundles*, now published as *A local refinement of the Adams-Riemann-Roch theorem in degree 1*—and he encouraged us to try extending the results in his paper from the algebraic case to the analytic one.

In this document, we collect all the work we've done during our research. This includes three main results; firstly that the isomorphism built by Rössler in [23] is indeed a refinement of the Adams-Riemann-Roch theorem; secondly we extend Ducrot's cube structures defined in [7] from the algebraic to the analytic case; and thirdly we give a partial extension to Rössler's involution formula in [23], in the case where the fixed point scheme is a Cartier divisor.

These three main results are stated and proved in the following sections:

- Theorem 3.2, stated in §3.1.2, proven in §3.3.
- Theorem 5.2, stated in §5.2.2, proven in §5.3.
- Theorem 6.1, stated in §6.1.4, proven in §6.2.

In [6], Deligne introduced the Deligne pairing of two line bundles on a curve, together with canonical isomorphisms related to this construction, in both the algebraic and the analytic cases. He suggested that his work could be extended to higher dimensions, which has been studied by several authors.

There are two constructions of the Deligne pairing, one more geometric in nature, the other expressed in terms of the determinant of cohomology. Elkik extended Deligne's construction in [8] and [9], in both algebraic and analytic cases, by following a geometric approach. Ducrot would later provide a different algebraic construction in [7], by instead considering the determinant of cohomology.

Eriksson's [10], Franke's [11], and Rössler's [23] instead consider the canonical isomorphisms that Deligne constructed, and aimed to extend them to higher dimensions. Rössler suggested that the methods he used were chosen to be as simple as possible to make it possible to extend his results to the analytic case.

The canonical isomorphism constructed by Rössler in [23], which we call Rössler's *refinement of the Adams Riemann Roch (ARR) theorem*, is closely related to one of the isomorphisms Deligne constructs in the case of curves, but also provides isomorphisms in higher dimensions. Rössler suggested that his isomorphism was also related to the ARR theorem by following an insight from Nori's [22]. While we were familiarizing ourselves with the analytic tools we would require to extend Rössler's results, we looked into this and found, after some rather interesting combinatorics, the relationship between Rössler's isomorphism and the ARR. While not directly related to the other two results in this paper, we have decided to include it since it provides insight into Rössler's results, and why one might be interested in extending them to the analytic case.

In his construction of his canonical isomorphism, Rössler makes use of the results from Ducrot's [7]. Unlike Elkik's extension of the Deligne pairing, Ducrot's had not had an analytic extension yet, which is why we provide the extension of Ducrot's construction to the analytic case.

Rössler's refinement of the ARR is proved by using an involution formula, which forms the core of Rössler's proof. This involution formula is proved in two steps, firstly in the case where the fixed points of the involution form a Cartier divisor, and secondly when the fixed points have higher relative codimension. We provide an analytic extension to the first of these two steps, since we lack the analytic tools required to consider the blow-up used in Rössler's proof of the general case.

1.1 About this paper

1.1.1 Algebra and analysis

In this paper, there are two cases that are being considered throughout.

In the *algebraic case*, we consider schemes, vector bundles and the determinant of cohomology.

In the *analytic case*, we consider compact Hodge manifolds, hermitian vector bundles, and the determinant of cohomology endowed with the Quillen metric.

Two of the main results in this paper aim to extend results in the algebraic case to results in the analytic case. While the isomorphisms themselves can be directly obtained from the algebraic case, the analytic case has an additional metric structure, and it is important to note if the isomorphisms coming from the algebraic case are isometries in the analytic case, or if they instead have a non-trivial norm.

1.1.2 Outline

§1 Introduction We introduce the core notions and main objects that will be used throughout this paper; the algebraic and analytic cases, schemes and complex manifolds, vector bundles and hermitian vector bundles, the determinant of cohomology, canonical isomorphisms and their norms.

§2 Lambda rings, characteristic classes and Riemann-Roch algebra We introduce key concepts that require more elaboration; the Grothendieck K group, Chern characters and forms, the Adams character, and the Adams Riemann Roch (ARR) theorem. We prove the cancellation lemmas in §2.4 which form the core of the proofs in both §5 and §6. This section considers both algebraic and analytic cases, and the lemmas in §2.4 highlight the similarities between the two cases.

§3 A refinement of Adams Riemann-Roch We introduce Rössler's results from [23], which we partially extend from the algebraic to the analytic case in §6. We prove that the canonical isomorphism of determinant line bundles in Rössler's paper gives a refinement of ARR. This section focuses only on the algebraic case.

§4 $\mathbb{Z}/2$-equivariant geometry and analytic preliminaries We introduce concepts that will be used in §5 and §6; $\mathbb{Z}/2$ equivariance, the Quillen metric, Bismut's immersion formula, and Ma's branched covering formula. This section considers both algebraic and analytic tools.

§5 Cube structures and the Quillen metric We introduce Ducrot's cube structures from [7], which are used by Rössler in his refinement of ARR, and extend them from the algebraic to the analytic case.

§6 Involution formula for Cartier divisors We extend Rössler's involution formula, which is the core of his refinement to ARR, from the algebraic to the analytic case, in the case where the fixed point scheme is a Cartier divisor.

1.1.3 On ※ and †

By necessity, we must closely follow the steps in Rössler's and Ducrot's work when extending their results from the algebraic to the analytic case. This involves going over their proofs in near full detail and considering the norms of isomorphisms at each step. Furthermore, for the reader's convenience, we include in this paper a significant amount of theory that is present in other papers and books, so that this paper is mostly self-contained.

To help the reader more easily distinguish between our contributions and the results from other authors, we use the following notation;

※ Sections marked with this symbol consist of this paper's author's original work.

† Sections marked with this symbol follow closely other authors' works.

We will also use the "※" symbol for notes discussing the originality of the contribution in the section, for example when they were assisted by insights from others, or when they are fairly well-known results for which we were unable to find a major paper to quote them from.

Similarly, we will use the "†" symbol for short notes about which papers are the ones the section closely follows.

1.1.4 Suggested readings

As we do not assume previous familiarity with the area, a significant part of the content of this paper can be skimmed over by readers familiar with the concepts.

Each section can be read on its own, referring back to previous sections as necessary. There are also three main "readings" in this paper, which are closely related, but can be read independently too.

Rössler's refinement of the Adams Riemann Roch formula

> This reading explains the context behind the Adams Riemann Roch theorem, and shows why Rössler main result in [23] provides a refinement in degree 1.
>
> Start with §1.2, follow with §2 skipping over §2.3, then read all of §3.
>
> This reading focuses only on the algebraic case.

Extending Ducrot's theorem

> This reading explains Ducrot's extension of the Deligne pairing and the cube structures he constructs in [7] to define his intersection bundle; and then presents our extension to the analytic case.
>
> Start with §1.2, follow with §2 skipping over §2.5, then read all of §5, referring back to §4 as necessary.

Extending Rössler's involution formula

> This reading presents our partial extension to Rössler's involution formula to the case where the fixed point scheme is a Cartier divisor.
>
> Start with §1.2, follow with §2, read §3.1 for Rössler's results, continue with §4, read §5.2 for the statement of Ducrot's theorem and our extension, then read all of §6.
>
> This reading is closely related to the first in their study of Rössler's work, and closely related to the second in the methodology used when extending from the algebraic to the analytic case.

1.2 Preliminaries

In this section, we introduce the objects we will be working with; schemes in §1.2.1, complex manifolds in §1.2.2, vector bundles and hermitian vector bundles in §1.2.3, and the tangent, cotangent, normal and conormal bundles in §1.2.4 and §1.2.5. We also introduce the determinant of cohomology in §1.2.6, which is the algebraic construction used in all the main theorems we wish to extend to the analytic case.

We later introduce the concept of a *canonical isomorphism* in §1.2.7 and §1.2.8; this is often used implicitly throughout this paper. We also use a different notation when computing the norm of an isomorphism, which is introduced in §1.2.9.

Further preliminaries are found in §2 and §4. In particular, $\mathbb{Z}/2$-equivariant variations of schemes, complex manifolds, vector bundles and hermitian vector bundles are found in §4.1, and the analytic counterpart of the determinant of cohomology, the determinant of cohomology endowed with the Quillen metric, is introduced in §4.2.

1.2.1 Schemes †

In the algebraic case, we consider a scheme X over a base scheme S, so that we have a morphism of schemes $f : X \to S$. A scheme S is *locally Noetherian* if each $s \in S$ has an open affine neighborhood $\operatorname{Spec}(R) = U \subseteq S$ such that R is a Noetherian ring.

A scheme S over a field k is *smooth* if $S \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\bar{k})$ is regular, where $\bar{k}$ is the algebraic closure of k. Recall that a scheme S is *regular* if each $s \in S$ has an open affine neighborhood $\operatorname{Spec}(R) = U \subseteq S$ such that R is a regular Noetherian ring.

A morphism $f : X \to S$ is *proper* if it is separated, locally of finite type, quasi-compact and universally closed. Recall that f is *separated* if the diagonal of $X \times_S X$ is a closed subscheme; f is *locally of finite type* if for each $x \in X$ there exists an open affine neighborhood $U \ni x$, and an open affine $V \supseteq f(U)$, such that $f : U \to V$ is given by a morphism of finite type of rings; f is *quasi-compact* if for every quasi-compact open $V \subseteq S$, $f^{-1}(V)$ is quasi-compact; f is *universally closed* if any pullback of f is a topologically closed map (so that the image of a closed subset is closed).

A morphism $f : X \to S$ is *flat* if for each $x \in X$ the local ring $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{S,f(x)}$-module. Recall that a module M over a ring R is *flat* if the functor of R-modules mapping A to $A \otimes M$ is exact.

A morphism $f : X \to S$ is *finite* if f is affine, and if for every affine open $\mathrm{Spec}(R) = V \subset S$ with inverse image $\mathrm{Spec}(A) = f^{-1}(V) \subset X$, the associated ring map $R \to A$ is finite. Recall that f is *affine* if each inverse image of an affine open of S is an affine open in X. Recall that $R \to A$ is finite if A is finite as an R-module. Recall that f is finite if and only if it is affine and proper [26, Lemma 29.44.11].

A morphism $f : X \to S$ is *locally projective* if for each $s \in S$ there is an open affine neighborhood $U \ni s$ such that $f|_U$ factors into a closed U-immersion into $\mathbb{P}^N_U$ followed by the projection, for some N depending on U. Note that any locally projective map is proper. A morphism $f : X \to S$ is *projective* if X is isomorphic as a scheme over S to a closed subscheme of a projective bundle $\mathbb{P}(E)$ for some quasi-coherent, finite type $\mathcal{O}_S$-module E.

A morphism $f : X \to S$ is *smooth* if it is locally of finite presentation, flat, and for each geometric point $\bar{s} \to S$ the fiber $X_{\bar{s}} = X \times_S \bar{s}$ is regular. Recall that $\bar{s} \to S$ is a *geometric point* if it is a morphism into S from the spectrum $\bar{s} := \mathrm{Spec}(k)$ of an algebraically closed field k. Recall that $f : X \to S$ is locally of finite presentation if for every $x \in X$ there are open affine neighborhoods $U \ni x$ and $V \ni f(x)$ with $f(U) \subseteq V$ such that $\mathcal{O}_X(U)$ is a finitely presented algebra over $\mathcal{O}_S(V)$. Note that when S is locally Noetherian, $f : X \to S$ is locally of finite presentation if and only if it is locally of finite type.

A morphism $f : X \to S$ is of *constant relative dimension* n if at each $s \in S$, the fiber $f^{-1}(s)$ is of dimension n. A scheme X is *equidimensional* if all its irreducible components are of the same dimension.

1.2.2 Complex manifolds †

In the analytic case, we instead consider complex manifolds.

Kähler manifolds are manifolds that have compatible complex, symplectic, and Riemannian structures. We choose to consider them as complex manifolds together with a hermitian metric on the tangent bundle that satisfies certain properties, as described in §1.2.4, which will have an associated *Kähler form* ω, which is the symplectic form.

A *Hodge manifold* is a Kähler manifold whose Kähler form ω is integral, that is, its cohomology class is in the image of integral cohomology.

It is well-known from GAGA principles that one may construct the *analytification* X^{an} of a complex projective variety X, and that in this case the sheaf cohomology theories of X and X^{an} are compatible, and that furthermore X^{an} can be endowed with a Kähler structure. By Kodaira's embedding theorem [16, Theorem 4.1] all Hodge manifolds are the analytification of a complex projective variety, endowed with an appropriate Kähler structure.

We will often make use of both algebraic and analytic results together in this paper, so we will favor the use of Hodge manifolds, but it should be possible to extend our results to Kähler manifolds more generally. We will write X^{an} for a Hodge manifold, and X for the scheme it corresponds to under GAGA, while remembering that X^{an} has the additional structure of a Kähler metric on its tangent bundle.

While in the algebraic case we study a scheme X over a base scheme S, in the analytic case we will often—but not always—consider the case where $S = \operatorname{Spec}(\mathbb{C})$ is a point. This is because the norms of isomorphisms can be computed pointwise, and so it is trivial to extend results from the absolute case, where S^{an} is a point, to the relative case, where S^{an} has higher complex dimension. Note that when moving from the absolute to the relative case, we no longer make use of a Kähler metric on X, and instead require a Kähler metric on the relative tangent bundle (see §1.2.4). We will similarly assume that X^{an} is connected throughout, with the understanding that extending to the general case is a matter of applying our results to connected components.

We say $f : X^{\mathrm{an}} \to S^{\mathrm{an}}$ is a *submersion* if the differential at each $x \in X^{\mathrm{an}}$, $df_x : \mathcal{T}_{X^{\mathrm{an}},x} \to \mathcal{T}_{S^{\mathrm{an}},f(x)}$ is surjective.

We say that $f : X^{\mathrm{an}} \to S^{\mathrm{an}}$ is *proper* if for each compact $K \subseteq S^{\mathrm{an}}$, $f^{-1}(K)$ is compact. Note that this is a purely topological notion.

We say that $f : X^{\mathrm{an}} \to S^{\mathrm{an}}$ is of *constant relative dimension* n if for each $s \in S^{\mathrm{an}}$ the fiber $f^{-1}(s)$ is of complex dimension n.

In §6 we also consider *equivariant* schemes and complex manifolds, which are introduced in §4.1.1.

1.2.3 Vector bundles and hermitian vector bundles †

In the algebraic case, we consider vector bundles. A *vector bundle* E on a scheme X is a locally free coherent sheaf of $\mathcal{O}_X$-modules. The *rank* of a vector bundle E is the number $\operatorname{rk} E$ of free generators the free $O_X(U)$-modules $E(U)$ have for each open $U \subseteq X$. A *line bundle* L on a scheme X is a vector bundle of rank 1, or equivalently, an invertible sheaf under the tensor product. We let $\mathcal{O}_X$ denote the trivial bundle of X, which is a line bundle.

Suppose now that we are in the analytic case. A *holomorphic vector bundle* E on a complex manifold X is a complex vector bundle such that the total space E is a complex manifold and the projection map $\pi : E \to X$ is holomorphic. Note that the fibers $E_x \coloneqq \pi^{-1}(\{x\})$ have the structure of a complex vector space for each $x \in X$. By Serre's GAGA, the category of holomorphic vector bundles on a smooth complex projective variety X, viewed as a complex manifold, is equivalent to the category of (algebraic) vector bundles on X, which is why we call holomorphic vector bundles simply “vector bundles”.

In the analytic case, we instead wish to consider hermitian vector bundles. A *hermitian vector bundle* $\underline{E} = (E, h^E)$ on a complex manifold X consists of a (holomorphic) vector bundle E together with a smooth hermitian metric h^E on E; which is given by an inner product h^E_x on each fiber E_x, which varies smoothly with $x \in X$, and which is invariant under complex conjugation. A *hermitian line bundle* $\underline{L}$ on X is a hermitian vector bundle whose holomorphic fibers L_x are complex vector spaces of dimension 1. We let $\underline{\mathcal{O}}_X$ denote the trivial bundle on X, endowed with its canonical metric (such that the norm of the canonical section is 1), and which is a hermitian line bundle.

We can extend constructions of vector spaces to vector bundles in the algebraic and analytic cases. This includes direct sums $U \oplus V$, tensor products $U \otimes V$, tensor powers $V^{\otimes k}$, exterior powers $\bigwedge^i V$, and symmetric powers $\operatorname{Sym}^j(V)$. Recall that the tensor power is $V^{\otimes k} = \bigotimes_{i=1}^{k} V$ a tensor product of a vector space V with itself; the exterior power is the tensor power quotiented by alternating permutations of tensor factors, for example,

$$u \wedge v \wedge w = -v \wedge u \wedge w \in \bigwedge^{3} V;$$

while the symmetric power is the tensor power quotiented by permutations of the tensor factors, for example,

$$u \otimes v \otimes w = v \otimes u \otimes w \in \operatorname{Sym}^{3} V.$$

All of these constructions interact naturally with hermitian inner products, for example with the orthogonal direct sum of two inner product spaces. This allows us to extend these constructions to hermitian vector bundles.

The study of exterior powers of vector bundles $\bigwedge^{i} E$ is what motivates the construction of λ-rings, which we introduce in §2.1. The combinatorial properties of symmetric powers are relevant in §3.3, which encourages our study of complete homogeneous symmetric polynomials in §3.2.

For line bundles (in both algebraic and analytic cases), we will also consider the dual $L^{\vee}$ which is the inverse of L with respect to the tensor product $L^{\vee} \otimes L \cong L \otimes L^{\vee} \cong \mathcal{O}_X$. Once again, this interacts naturally with hermitian metrics, so we may also consider the dual $\underline{L}^{\vee}$ of a hermitian vector bundle $\underline{L}$. We then let for $k \geq 0$, $L^{\otimes k} := \bigotimes_{i=1}^{k} L$, and $L^{\otimes -k} := \bigotimes_{i=1}^{k} L^{\vee}$.

In §6 we also consider *equivariant* vector bundles and hermitian vector bundles, which are introduced in §4.1.2.

1.2.4 Cotangent and tangent bundles, Kähler forms †

Suppose that we are in the algebraic case, and consider $f : X \to S$ a separated morphism of schemes. The *sheaf of differentials*, or *cotangent sheaf*, $\Omega_{X/S}$ or Ω_f can be defined as the pullback $\Delta^*(I/I^2)$, where $\Delta : X \to X \times_S X$ is the diagonal morphism and I is the ideal associated to the closed subscheme $\Delta(X)$. Alternatively it is the sheaf of modules corresponding to the *module of derivations*; given a morphism of rings $R \to S$, the module of derivations is the S-module generated by formal elements ds for each $s \in S$ and the relations $dr = 0$ for $r \in R$,

$d(s+t) = ds + dt$ and $d(st) = sdt + tds$, together with the R-module map d from S to the derivations. The two definitions of the cotangent sheaf are canonically isomorphic, and when X is smooth over S the cotangent sheaf is a vector bundle, which we call the *cotangent bundle*. We then let the *tangent bundle* $\mathcal{T}_{X/S}$ or $\mathcal{T}_f$ be the dual of the cotangent bundle.

In the analytic case, given a complex manifold X^{an}, we have the *tangent bundle* $\mathcal{T}_X$ and its dual the *cotangent bundle* Ω_X. Moreover, given a submersion $f : X^{\mathrm{an}} \to S^{\mathrm{an}}$ of complex manifolds, we let $\mathcal{T}_{X/S}$, $\mathcal{T}_f$ denote the *relative tangent bundle*, which is the kernel of the differential $df : \mathcal{T}_X \to \mathcal{T}_S$; and we let $\Omega_{X/S}$, Ω_f denote its dual, the *relative cotangent bundle*. We will note that the relative tangent and cotangent bundles are the analytifications of the algebraic tangent and cotangent bundles, and that the tangent and cotangent bundles of X^{an} are the relative tangent and cotangent bundles of the submersion $f : X \to \operatorname{Spec} \mathbb{C}$.

Note that if $f : X \to S$ is of constant relative dimension n, then the rank of the (relative) tangent bundle $\mathcal{T}_f$ is n, in both the algebraic and the analytic cases. Moreover, in the analytic case, the rank of the tangent and cotangent bundles is equal to the dimension of the complex manifold.

In the analytic case, we say that a hermitian metric h on the tangent bundle $\mathcal{T}_X$ is *Kähler* with an *associated Kähler form* ω if the 2-form

$$\omega(u, v) := \mathrm{Re}\ h(iu, v) = \mathrm{Im}\ h(u, v),$$

is closed under the de Rham cohomology. Once we choose a Kähler metric h on $\mathcal{T}_X$, we consider the hermitian vector bundle $\underline{\mathcal{T}}_X = (\mathcal{T}_X, h)$. Since a metric on a vector bundle has an associated metric on the dual bundle, a choice of metric on $\underline{\mathcal{T}}_X$ gives us a metric h^{Ω_X} on Ω_X, so that we can consider the hermitian vector bundle $\underline{\Omega}_X = (\Omega_X, h^{\Omega_X})$.

Given a submersion of complex manifolds $f : X \to S$, we say that a hermitian metric h on the relative tangent bundle $\mathcal{T}_{X/S}$ is *Kähler* if for each fiber $X_s = f^{-1}(\{s\})$, where $s \in S$, the induced metric h_s on $\mathcal{T}_{X_s} = \mathcal{T}_{X/S}|_{X_s}$ is a Kähler metric.

1.2.5 Conormal and normal bundles, Cartier divisors †

Consider the algebraic case. Let $\iota : Z \to X$ be a closed immersion of schemes, and let $\mathcal{I} \subseteq \mathcal{O}_X$ be the associated sheaf of ideals. Since $\mathcal{I}$ annihilates the sheaf of $\mathcal{O}_X$-modules $\mathcal{I}/\mathcal{I}^2$, it can be considered as a sheaf of $\mathcal{O}_Z$-modules on Z, which we call the *conormal sheaf* $\mathcal{C}_{Z/X}$. The *normal sheaf* $\mathcal{N}_{Z/X}$ is defined to be the dual of the conormal sheaf,

$$\mathcal{N}_{Z/X} \coloneqq \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z).$$

Note that Rössler's [23] writes $N_{Z/X}$ for the conormal bundle, which agrees with EGA's [15, Def 16.1.2] notation; we instead follow the convention encouraged by the stacks project [26, §29.31], since we must consider both the conormal and the normal bundles.

We say that Z is a *Cartier divisor* of X if the ideal sheaf $\mathcal{I}$ associated to Z is a line bundle. In this case we have the dual line bundles $\mathcal{O}_X(Z) \coloneqq \mathcal{I}^\vee$, and $\mathcal{O}_X(-Z) \coloneqq \mathcal{I}$, together with a canonical section on $\mathcal{O}_X(Z)$ which corresponds to the inclusion $\mathcal{I} \to \mathcal{O}_X$, given that $\mathcal{I}^\vee = \mathrm{Hom}(\mathcal{I}, \mathcal{O}_X)$. For a vector bundle M on X, we write $M(Z) \coloneqq M \otimes \mathcal{O}_X(Z)$ and $M(-Z) \coloneqq M \otimes \mathcal{O}_X(-Z)$.

When Z is a Cartier divisor, the conormal sheaf is $\mathcal{C}_{Z/X} = \mathcal{O}_X(-Z)|_Z$, and the normal sheaf is $\mathcal{N}_{Z/X} = \mathcal{O}_X(Z)|_Z$, both of which are line bundles. More generally, if Z is a local complete intersection the conormal and normal sheaves will be locally free, and are called the *conormal and normal bundles*.

When Z is a Cartier divisor, we have two related short exact sequences

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(Z) \to \iota_*(\mathcal{N}_{Z/X}) \to 0,$$

$$0 \to \mathcal{O}_X(-Z) \to \mathcal{O}_X \to \iota_*(\mathcal{O}_Z) \to 0.$$

Both of these are due to the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0,$$

and when twisted with a line bundle M we have the following short exact sequences,

$$0 \to M \to M(Z) \to M(Z)|_Z \to 0, \tag{1.1}$$

$$0 \to M(-Z) \to M \to M|_Z \to 0. \tag{1.2}$$

While both of these short exact sequences are equivalent, they each form the core of Ducrot's [7] and Rössler's [23], respectively, both of which we aim to extend from the algebraic to the analytic case in this paper.

If we have both Z and X being smooth over a scheme S, then we have a short exact sequence in Z

$$0 \to \mathcal{C}_{Z/X} \to \iota^*\Omega_{X/S} \to \Omega_{Z/S} \to 0, \tag{1.3}$$

which gives us also a short exact sequence in terms of the tangent and normal bundles,

$$0 \to \mathcal{T}_{Z/S} \to \mathcal{T}_{X/S} \to \mathcal{N}_{Z/X} \to 0. \tag{1.4}$$

Note also that the cotangent bundle is by definition the normal bundle of the diagonal $\Delta : X \to X \times_S X$,

$$\Omega_{X/S} = \mathcal{C}_{\Delta}.$$

In the analytic case, the *normal bundle* $\mathcal{N}_{Z/X}$ is defined as the quotient of the tangent bundles $\mathcal{T}_X|_Z/\mathcal{T}_Z$, which is equivalent to the quotient of the relative tangent bundles $\mathcal{T}_{X/S}|_Z/\mathcal{T}_{Z/S}$. This is equivalent to the analytification of the algebraic normal bundle due to (1.4). The *conormal bundle* $\mathcal{C}_{Z/X}$ is defined to be the dual of the normal bundle, which coincides with the analytification of the algebraic conormal bundle.

1.2.6 Determinant of cohomology †

Consider the algebraic case. Suppose that X and S are locally Noetherian schemes, and that $f : X \to S$ is proper and flat.

A *graded line bundle* (L, α) on S consists of a line bundle L on S and some $\alpha \in \mathbb{Z}/2$, where the tensor product of graded bundles is defined by

$$(L, \alpha) \otimes (M, \beta) = (L \otimes M, \alpha + \beta),$$

and where the commutativity of the tensor product is given by an isomorphism which depends on the degree;

$$\begin{aligned}(L, \alpha) \otimes (M, \beta) &\cong (M, \beta) \otimes (L, \alpha),\\ u \otimes v &\mapsto (-1)^{\alpha+\beta} v \otimes u.\end{aligned}$$

The category of graded line bundles $\mathrm{Pic}_*(S)$ is a non-strictly commutative Picard category, which is explored in more detail in §5.1.2.

The *determinant* of a vector bundle K on S is defined to be the top exterior power of the vector bundle, together with the parity of its rank,

$$\det(K) \coloneqq \left(\bigwedge^{\mathrm{rk}(K)} K,\ \mathrm{rk}(K)\right),$$

which is a graded line bundle.

We will note that the choice of the commutative isomorphism in $\mathrm{Pic}_*(S)$ is precisely the one which makes the following diagram commute, for vector bundles E and F, due to the effect of reordering terms in the exterior power;

$$\begin{tikzcd}
\det(E) \otimes \det(F) \arrow[r] \arrow[d] & \det(E \oplus F) \arrow[d] \\
\det(F) \otimes \det(E) \arrow[r] & \det(F \oplus E)
\end{tikzcd}$$

Let $K_\bullet$ be a bounded complex of vector bundles on S. The *Knudsen-Mumford determinant* of this complex is defined to be

$$\det(K_\bullet) \coloneqq \bigotimes_{j \in \mathbb{Z}} \det(K_j)^{(-1)^j},$$

and this definition can be extended to perfect complexes, which are quasi-isomorphic to a bounded complex by definition. See [17] for more details on this construction, and note that $\det(K_\bullet)$ is also a graded line bundle.

Now let E be a vector bundle on X, and note that by the semicontinuity theorem, since f is proper and flat, then $R^\bullet f_* E$ is a perfect complex. The *determinant of cohomology* of E is defined to be

$$\lambda(E) \coloneqq \det(R^\bullet f_* E).$$

We will often write $\lambda_X(E)$ or $\lambda_{X/S}(E)$ when more schemes are involved. We will also note that there exists a *canonical isomorphism* (see §1.2.7),

$$\lambda(E \oplus E') \cong \lambda(E) \otimes \lambda(E'),$$

so we write for formal linear combinations of vector bundles with integer coefficients $n_1E_1 + \cdots + n_kE_k$,

$$\lambda(n_1E_1 + \cdots + n_kE_k) \coloneqq \lambda(E_1)^{\otimes n_1} \otimes \cdots \otimes \lambda(E_k)^{\otimes n_k}.$$

This allows us to write, for example,

$$\lambda((O_X - L) \otimes (O_X - M)) \cong \lambda(\mathcal{O}_X) \otimes \lambda(L)^\vee \otimes \lambda(M)^\vee \otimes \lambda(L \otimes M).$$

Note that $\lambda(O_X)$ and $\mathcal{O}_S$ are not isomorphic to each other in general, due to the fact that $R^\bullet f_*\mathcal{O}_X$ is not quasi-isomorphic to $\mathcal{O}_S$ in general.

The determinant of cohomology has an analogue in the analytic case; the determinant of cohomology endowed with the *Quillen metric*, which is introduced in §4.2. Both Rössler and Ducrot use the determinant of cohomology in their construction of canonical isomorphisms, which is why when extending their results to the analytic case we must consider what happens to the norms of isomorphisms involving the determinant of cohomology endowed with the Quillen metric, by using results recalled in §4.3 like Bismut's immersion formula.

1.2.7 Canonical isomorphisms †

In this paper we make ample use of the notion of "canonical isomorphisms".

We say that two objects are isomorphic to each other when there exists an isomorphism between them, but this isomorphism is not unique in general. Sometimes there exists a unique way to choose this isomorphism in a sensible way, which we call "canonical isomorphisms". For example there exists a canonical isomorphism between any object and itself; the identity morphism.

More formally, let $\mathcal{C}$ and $\mathcal{D}$ be two categories, let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be two functors, and let $\eta : F \Rightarrow G$ be an isomorphism of functors. Then η consists of a family of isomorphisms, for each object X in $\mathcal{C}$,

$$\eta_X : F(X) \to G(X).$$

When we say that there exists a *canonical isomorphism* $F(X) \cong G(X)$, we imply the categories $\mathcal{C}$, $\mathcal{D}$ as well as the functors F and G, and state that there exists an isomorphism of functors $\eta : F \Rightarrow G$.

For example, when we state that for two Abelian groups X and Y there exists a canonical isomorphism

$$X \times Y \cong Y \times X,$$

the category $\mathcal{C}$ is implied to be the category with pairs of Abelian groups (X, Y) as objects, $\mathcal{D}$ is implied to be the category of Abelian groups, the functors are implied to be

$$F((X, Y)) \coloneqq X \times Y \qquad \text{and} \qquad G((X, Y)) \coloneqq Y \times X,$$

and, in this case, the natural transformation is implied to be

$$\eta_{(X,Y)} : (x, y) \mapsto (y, x).$$

Note that there exists multiple isomorphisms between $X \times Y$ and $Y \times X$ in general, for example by considering automorphisms of X or Y, but in our case we might be more interested in the specific and unique isomorphism defined by $(x, y) \mapsto (y, x)$.

While some canonical isomorphisms are easy to understand, more complex canonical isomorphisms can be built by combining multiple canonical isomorphisms. Formally, this simply corresponds to composition of isomorphisms of functors. Informally, the composition of canonical isomorphisms is also a canonical isomorphism.

In this paper we will write "$A \cong B$" to mean "there exists a canonical isomorphism between A and B", instead of the usual "there exists an isomorphism between A and B". We will write $\alpha : A \cong B$ to mean that $\alpha : A \to B$ is the canonical isomorphism between A and B. In all of these cases, the choice of isomorphism between A and B should be either self-evident or described in the surrounding text.

We would like to note that a canonical isomorphism between A and B in this paper is a "choice of a unique isomorphism between A and B", instead of "the unique way to choose an isomorphism between A and B". For example, if V is a vector space, the map sending $v \mapsto v$ and the map sending $v \mapsto -v$ are both distinct canonical automorphisms, each formally given by a disctinct isomorphism of functors.

1.2.8 Compatibility and canonical isomorphisms †

While in general when describing canonical isomorphisms the functors F, G, the category $\mathcal{D}$ and the objects of the category $\mathcal{C}$ are easily implied, the same is not necessarily true for the morphisms in $\mathcal{C}$. When we say that a "canonical isomorphism is *compatible with* some structure", we are effectively describing the morphisms in $\mathcal{C}$.

The most common case will be when a canonical isomorphism is "compatible with base change". Here is an example; suppose that X and S are smooth schemes, $f : X \to S$ is a proper morphism, and λ is the determinant of cohomology. Take two line bundles L and M on X, then there exists a canonical isomorphism

$$\lambda(L \oplus M) \cong \lambda(L) \otimes \lambda(M),$$

compatible with base change.

Here $\mathcal{D}$ is the category of pairs of a smooth scheme S and a graded line bundle on S; the objects of $\mathcal{C}$ are tuples (X, S, f, L, M) where X and S are smooth schemes, f is a proper morphism between them, L and M are line bundles on X,

$$F(X, S, f, L, M) \coloneqq (S,\ \lambda(L \oplus M)) \qquad \text{and} \qquad G(X, S, f, L, M) \coloneqq (S,\ \lambda(L) \otimes \lambda(M)).$$

Now, what we mean by this isomorphism being compatible with base change is that the morphisms in $\mathcal{C}$, mapping $(X', S', f', L', M') \mapsto (X, S, f, L, M)$ are given by a pair of morphisms $\phi : X' \to X$ and $\psi : S' \to S$ such that the following square commutes (which is called "base change")

$$\begin{tikzcd}
X' \arrow[r, "\phi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r, "\psi"] & S
\end{tikzcd}$$

and such that

$$L' = \phi^* L \qquad \text{and} \qquad M' = \phi^* M.$$

1.2.9 Norms of isomorphisms ※

Consider the analytic case. Suppose that $\underline{L} = (L, h^L)$ and $\underline{M} = (M, h^M)$ are hermitian line bundles on a complex manifold X. Suppose that there exists a canonical isomorphism of line bundles $\sigma : L \cong M$. Then, either the metrics on L and M match, so that $\sigma^* h^M = h^L$, or they do not match.

We say that we have a *canonical isometry* of line bundles, which we write as $\underline{L} \simeq \underline{M}$, if we have a canonical isomorphism $L \cong M$ for which the metrics match. We will write $\sigma : \underline{L} \simeq \underline{M}$ to mean that $\sigma : L \cong M$ and $\sigma^* h^M = h^L$.

We now consider the case where $\sigma : L \cong M$ is not an isometry. Note that σ is given by a unique section in $L^\vee \otimes M$, and that $\underline{L}^\vee \otimes \underline{M}$ is a hermitian line bundle, so we may compute the norm $|\cdot|^2$ of its sections. The *log-norm* of σ is $\log|\sigma|^2$, where $|\sigma|^2$ is the norm of the section of $\underline{L}^\vee \otimes \underline{M}$ corresponding to $\sigma : L \cong M$.

We write

$$\underline{L} \overset{(a)}{\approx} \underline{M},$$

to mean that $\underline{L}$ and $\underline{M}$ are canonically isomorphic and the log-norm of the canonical isomorphism is (a), which we often describe separately. See §4.3 for some analytical formulae in this paper that use this notation.

We will note the following useful properties of this notation:

- $\underline{L} \simeq \underline{M}$ is equivalent to $\underline{L} \overset{0}{\approx} \underline{M}$,
- $\underline{L} \overset{(a)}{\approx} \underline{M}$ implies that $\underline{M} \overset{-(a)}{\approx} L$,
- $\underline{L} \overset{(a)}{\approx} \underline{M}$ and $\underline{M} \overset{(b)}{\approx} \underline{N}$ together imply that $\underline{L} \overset{(a)+(b)}{\approx} \underline{N}$,
- $\underline{L} \overset{(a)}{\approx} \underline{M}$ and $\underline{L}' \overset{(b)}{\approx} \underline{M}'$ together imply that $\underline{L} \otimes \underline{L}' \overset{(a)+(b)}{\approx} \underline{M} \otimes \underline{M}'$,
- $\underline{L} \overset{(a)}{\approx} \underline{M}$ implies that $\underline{L}^\vee \overset{-(a)}{\approx} \underline{M}^\vee$.

This notation allows us to more seamlessly chain canonical isometries and canonical isomorphisms with nontrivial log-norm when constructing more complex canonical isomorphisms. Throughout this paper, we refer to the log-norm as simply the *norm* of a canonical isomorphism.

Chapter 2

Lambda rings, characteristic classes and Riemann-Roch algebra

A key result used in sections §5 and §6 —to compute the norm of certain isomorphisms of line bundles—is a rather elementary observation about *Chern characters* and *Chern forms*. We take the opportunity to revise the properties of both Chern characters and forms in this section, with the aim of making the proof of the key observation above clear to any reader.

We also take the opportunity to define and recall the construction of several other mathematical objects that will be used throughout this paper. In particular, we recall the statement of the Adams Riemann Roch theorem—for which Rössler's results in [23] provide a refinement, as we showcase in §3—as well as how it relates to other results of Riemann Roch type.

Overall, this section consists mostly of content recalled from the literature, in particular; Soulé et al.'s *Lectures on Arakelov Geometry* [24] for the definition of Chern forms in §2.3; Fulton and Lang's *Riemann-Roch Algebra* [12] for the definition of Chern characters in §2.2 and for the statement of Adams Riemann Roch in §2.5; and both of the previous sources for the definition of λ-rings in §2.1. In contrast, the key results in §2.4, while rather elementary, are still part of our original research for this paper.

In §2.1 we recall the notion of a λ-ring, as well the main example of λ-rings of interest in this paper; the Grothendieck group $K(X)$ of vector bundles on a smooth scheme X. These are the types of rings on which Chern characters are defined.

In §2.3 we recall the definition of Chern characters on a general λ-ring, and give brief comments about the construction for $K(X)$ in particular. We also showcase a few other characteristic forms like the *Todd character* and, most notably, the Adams character which is considered by the Adams Riemann Roch theorem.

In §2.4 we recall the construction of Chern forms for hermitian vector bundles on complex manifolds; as well as define the forms of type (p,p) on a complex manifold X, $\bigoplus_{p\geq 0} A^{p,p}(X)$, which are used to build the Chern forms; and the $\mathrm{dd^c} : A^{p,p}(X) \to A^{p+1,p+1}(X)$ differential which is used later throughout the paper.

In §2.4 we state and prove Lemma 2.5 and Lemma 2.6, which are later used in key parts of §5 and §6. Both of these Lemmas relate to a cancellation of, respectively, Chern characters and Chern forms, when combined in a particular alternating sum.

In §2.5 we recall the notion of Riemann Roch functors, theorems of type Riemann Roch, as well as examples such as the Grothendieck Riemann Roch theorem, and—of most interest for this paper—the Adams Riemann Roch theorem.

2.1 Lambda rings †

The notion of λ-rings—which is not directly related to the determinant of cohomology—is used to describe commutative rings K where there is a family of operations $\lambda^i : K \to K$ which behave like the exterior powers $\bigwedge^i$ of vector bundles with respect to the direct sum and tensor product. For example we expect that

$$\lambda^k(x+y) = \sum_{i+j=k} \lambda^i(x)\lambda^j(y),$$

much like when we have two vector spaces V, W, the k-th exterior power of their direct sum is isomorphic to

$$\bigwedge^k(V \oplus W) \cong \bigoplus_{i+j=k} \left(\bigwedge^i(V) \otimes \bigwedge^j(W)\right).$$

The main example of λ-rings of interest to this paper are the Grothendieck group $K(X)$ of vector bundles on a smooth scheme X. These are where the Chern characters are defined, as well as where the Adams operations $\psi^k : K(X) \to K(X)$ of interest to the Adams Riemann Roch theorem are defined.

In §2.1.1 we introduce in more detail the Grothendieck group $K(X)$ of vector bundles over a smooth scheme X, the main motivation for our revision of λ-rings. In §2.1.2 we recall the fundamental theorem of symmetric polynomials, which is used for many of the technical details of the construction of λ-rings. In §2.1.3 we give a definition of λ-rings. In §2.1.4 we introduce the notion of augmentations, positive elements and line elements in a λ-ring. In §2.1.5 we recall the notion of extensions of λ-rings and state the splitting principle for a family of λ-rings, which is used in many occasions to extend constructions from line elements to all other elements in a λ-ring.

In §2.1.6 we define involutions on λ-rings. In §2.1.7 we give a brief overview of how the Grothendieck group $K(X)$ of vector bundles over a smooth scheme X has the structure of a λ-ring, and the family of such groups satisfies the splitting principle. In §2.1.8, we recall the notions of ample line elements and ample line bundles, their relation, and the fact that if there exists an ample line element then elements of the form $(1-u)$, where u is a line element, are nilpotent.

† This section summarizes definitions and statements from [12] and [24].

2.1.1 Grothendieck group †

Suppose that X is a smooth scheme. Consider vector bundles on X as in §1.2.3, and the free additive group $K'(X)$ generated by equivalence classes of vector bundles up to isomorphism. The *Grothendieck group* $K(X)$ of vector bundles on X is the group $K'(X)$ quotiented by, for each short exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0,$$

the relation

$$[E] = [E'] + [E''].$$

For example, for two vector bundles E and E' on X, the short exact sequence due to a direct sum,

$$0 \to E \to E \oplus E' \to E' \to 0,$$

implies that

$$[E \oplus E'] = [E] + [E'].$$

The Grothendieck group $K(X)$ of vector bundles on X can be given the structure of a ring, by defining multiplication to be induced by the tensor product of vector bundles; so that for two vector bundles E and E' on X we have that

$$[E] \cdot [E'] \coloneqq [E \otimes E'].$$

Let X, S be smooth locally Noetherian schemes, and $f : X \to S$ be proper and flat. By the semicontinuity theorem, for a vector bundle E on X its pushforward $R^\bullet f_* E$ is a perfect complex on S, so it is quasi-isomorphic to a bounded complex of vector bundles. We can then define the pushforward $f_* : K(X) \to K(S)$ by

$$f_*[E] \coloneqq \sum_{j \in \mathbb{Z}} (-1)^j [R^j f_* E].$$

Since the pullback of vector bundles are vector bundles, we can simply define the pullback for an element in $K(S)$ to be given by

$$f^*[E] \coloneqq [f^* E].$$

Recall from §1.2.6 that

$$\lambda(E) \coloneqq \bigotimes_{j \geq 0} \det(R^i f_* E)^{(-1)^j},$$

we can then define the determinant of cohomology on $K(X) \to K(S)$ by

$$\lambda([E]) \coloneqq [\lambda(E)],$$

which is well-defined since a canonical isomorphism of vector bundles $E \cong F$ induces a canonical isomorphism of determinant line bundles $\lambda(E) \cong \lambda(F)$, and for a short exact sequence $0 \to E' \to E \to E'' \to 0$ we have a canonical isomorphism

$$\lambda(E') \otimes \lambda(E'') \cong \lambda(E).$$

Note that in the remainder of this chapter we do not consider the determinant of cohomology.

On the other hand, we can also consider the alternating product, for each $i \geq 0$ we can take the i-th exterior power $\bigwedge^i E$ of a vector bundle E. This gives us an element $\lambda^i(E) \coloneqq \left[\bigwedge^i E\right]$ in the Grothendieck group. The various properties of these maps $\lambda^i : K(X) \to K(X)$ can be studied independently of the context of vector bundles, by only considering elements in the ring $K(X)$. We can use the λ-rings defined in §2.1.3 to codify these properties.

For example, for two vector bundles E and E' on X, we have that

$$\bigwedge^k(E \oplus E') = \bigoplus_{i+j=k} \bigwedge^i E \otimes \bigwedge^j E',$$

and this can be codified by one of the defining properties of λ-rings; that for two elements x, y in a λ-ring K we have that

$$\lambda^k(x+y) = \sum_{i+j=k} \lambda^i(x)\lambda^j(y).$$

2.1.2 Fundamental theorem of symmetric polynomials †

A polynomial $P(X_1, \ldots, X_n)$ in n variables $X_1, \ldots, X_n$ is said to be a *symmetric polynomial* if permuting the variables does not change $P(X_1, \ldots, X_n)$. For example, $X_1^2 X_2 + X_1 X_2^2$ is a symmetric polynomial in two variables, while $X_1 - X_2$ is not. We will write "$X_\bigstar$" as a shorthand for "$X_1, \ldots, X_n$", so that, for example, for a polynomial p in n variables,

$$p(X_\bigstar) \coloneqq p(X_1, X_2, \ldots, X_n),$$

and, more generally, for p as above and a polynomial expression f in one variable, we will write $p(f(X_\bigstar))$ for $p(f(X_1), \ldots, f(X_n))$, so that, for example,

$$p(1 - X_\bigstar) = p(1 - X_1, 1 - X_2, \ldots, 1 - X_n).$$

Suppose that we have n variables $U_1, \ldots, U_n$. Consider the polynomials in these n variables $\sigma_0, \ldots, \sigma_n$ which satisfy the following polynomial identity

$$\sum_{i=0}^{n} \sigma_i(U_\bigstar) t^i \coloneqq \prod_{j=1}^{n} (1 + U_j t).$$

It is clear that these polynomials are symmetric. For example,

$$\sigma_0(U_\bigstar) = 1, \qquad \sigma_1(U_\bigstar) = U_1 + \cdots + U_n,$$
$$\sigma_2(U_\bigstar) = \sum_{1 \le i < j \le n} U_i U_j, \qquad \sigma_n(U_\bigstar) = \prod_{j=1}^{n} U_j.$$

They are called the *elementary symmetric polynomials*, and the *fundamental theorem of symmetric polynomials* states that every symmetric polynomial in n variables can be uniquely expressed as a polynomial expression in the elementary symmetric polynomials. For example, the polynomial $X_1^2 X_2 + X_1 X_2^2$ can be uniquely expressed as

$$X_1^2 X_2 + X_1 X_2^2 = \sigma_1(X_1, X_2) \cdot \sigma_2(X_1, X_2).$$

The fundamental theorem of symmetric polynomials is used for many of the technical details of the construction of λ-rings since each of the λ^i operations behaves like the elementary symmetric polynomial σ_i.

2.1.3 Lambda rings †

A λ-*ring* is a commutative ring K endowed with operations $\lambda^i : K \to K$ for non-negative integers $i \ge 0$, which satisfy the following conditions;

- $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for all $x \in K$, and $\lambda^i(1) = 0$ for all $i \ge 1$.
- $\lambda^k(x + y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y)$.
- $\lambda^k(xy) = P_k(\lambda^1(x), \dots, \lambda^k(x); \lambda^1(y), \dots, \lambda^k(y))$, where P_k are the unique polynomials in $2k$ variables satisfying for $2k$ variables $U_1, \dots, U_k, W_1, \dots, W_k$ the polynomial identity

$$\sum_{i \ge 0} P_i(\sigma_1(U_\bigstar), \dots, \sigma_i(U_\bigstar); \sigma_1(W_\bigstar), \dots, \sigma_i(W_\bigstar))t^i = \prod_{1 \le i,j \le k} (1 + U_i W_j t).$$

- $\lambda^k(\lambda^\ell(x)) = P_{k,\ell}(\lambda^1(x), \dots, \lambda^{k\ell}(x))$ where $P_{k,\ell}$ are the unique polynomials satisfying for $k\ell$ variables $U_1, \dots, U_{k\ell}$ the polynomial identity

$$\sum_{i \ge 0} P_{i,\ell}(\sigma_1(U_\bigstar), \dots, \sigma_i(U_\bigstar))t^i = \prod_{1 \le i_1 < \cdots < i_\ell \le k} (1 + U_{i_1} \cdots U_{i_\ell} t).$$

These rings are called “special λ-rings” in sources such as [12], we instead follow the convention of [24]. A morphism $\alpha : K \to A$ of λ-rings is one that commutes with the λ^i maps in the two rings, so that $\lambda^i(\alpha(x)) = \alpha(\lambda^i(x))$ for all $x \in K$.

It is convenient to define the following power series for elements x in a λ-ring K;

$$\lambda_t(x) \coloneqq \sum_{i\geq 0} \lambda^i(x)t^i \qquad \in K[[t]].$$

Note that by considering $\lambda^i(x+y)$ for all $i \geq 0$ we obtain that

$$\lambda_t(x+y) = \lambda_t(x)\lambda_t(y).$$

The λ-ring structure makes $\lambda_t : (K, +) \to (1 + tK[[t]], \cdot)$ a morphism of groups, where

$$1 + tK[[t]] = \{1 + tP(t) : P \in K[[t]]\} \subseteq K[[t]],$$

is a group under multiplication.

Note that the conditions that $\lambda_t(0) = 1$ and $\lambda_t(1) = 1 + t$ induce a unique λ-ring structure on $\mathbb{Z}$, where the λ^i satisfy the polynomial equation $\sum_{i\geq 0} \lambda^i(n)t^i = (1+t)^n \in \mathbb{Z}[[t]]$. In particular, for $n \geq i \geq 0$, the λ^i coincide with the binomial coefficients $\lambda^i(n) = \binom{n}{i}$.

Another example of λ-rings are the Grothendieck groups $K(X)$ for a smooth scheme X, together with the maps induced by $\lambda^i([E]) \coloneqq [\bigwedge^i E]$.

2.1.4 Augmentations and positive elements †

A *positive structure* on a λ-ring K consists of an *augmentation* $\varepsilon : K \to \mathbb{Z}$, which is a morphism of λ-rings, and a subset $E \subset K$ of *positive elements*, satisfying the following conditions;

- $\mathbb{Z}+ \subset E$, so that the multiplicative identity 1 and the elements obtained by sums of 1 are all in E.
- $EE = E$, so that multiplication of two positive elements yields a positive element.
- $K = E - E$, that is every element in K is the difference between two positive elements.

- $\varepsilon(e) > 0$ for each $e \in E$.
- If $e \in E$ and $\varepsilon(e) = r$, then $\lambda^i(e) = 0$ for every $i > r$; and $\lambda^r(e)$ is a unit in K.

Given a positive structure on K, we define the subset of *line elements* $L \subset E$ as those elements $\ell \in E$ for which $\varepsilon(\ell) = 1$. Note that for a line element $u \in L$, we have that $\lambda_t(u) = 1 + ut$.

2.1.5 The splitting principle †

We say that a positive element e in a λ-ring K *splits* if there exists line elements $u_1, \ldots, u_m$ in $L \subset K$ such that $e = u_1 + \cdots + u_m$. Note that if $e = u_1 + \cdots + u_m$ splits, since λ_t is a group morphism from an additive group to a multiplicative one, we may write

$$\lambda_t(e) = \prod_{i=1}^{m} \lambda_t(u_i) = \prod_{i=1}^{m}(1 + u_i t).$$

An *extension* K' of a λ-ring K is a λ-ring K' which contains K and so that the operations λ^i on K' extend those in K. If K has positive elements E, then the positive elements E' of K' must contain E.

A family $\mathcal{K}$ of λ-rings satisfies the *splitting property* if for each $K \in \mathcal{K}$, and each positive element $e \in E \subset K$, there exists an extension $K' \in \mathcal{K}$ of K where e splits. Note that in this case $e = u_1 + \cdots + u_m$ for some line elements $u_1, \ldots, u_m$ in the extension K'.

Note that, if $e = u_1 + \cdots + u_m$ splits in an extension $u_1, \ldots, u_m \in K'$, by comparing the coefficients in t^k of $\lambda_t(e) = \prod_{i=1}^{m}(1 + u_i t)$, we deduce that $\lambda^i(t) = \sigma_i(u_1, \ldots, u_m)$. Now, since $\lambda^i(t) \in K$, we deduce that the elementary symmetric polynomials $\sigma_i(u_1, \ldots, u_m)$ are independent of choice of splitting of e and of the extension K' where e splits. By the fundamental theorem of symmetric polynomials, any symmetric polynomial expression of the summands of a splitting of e are in K, independent of the splitting of e, and can be computed using the $\lambda^i(e)$'s.

2.1.6 Involutions †

An *involution* on a λ-ring K is a homomorphism $x \mapsto x^\vee$ from K to itself, such that for $x \in K$ and $u \in L \subset K$ we have

$$(x^\vee)^\vee = x, \qquad \varepsilon(x^\vee) = \varepsilon(x), \qquad \text{and} \qquad u^\vee = u^{-1}.$$

Suppose that the splitting principle holds for a family of λ-rings containing K, and that the involution extends in such family in a compatible way. Then, for each $e = u_1 + \cdots + u_m \in E \subset K$, where the u_i might belong to an extension of K, we have that $e^\vee = u_1^\vee + \cdots + u_m^\vee$ and so

$$\begin{aligned}
\lambda_t(e) &= \prod_{i=1}^m (1 + u_i t) \\
&= \prod_{i=1}^m (u_i t)(1 + u_i^\vee t^{-1}) \\
&= \left(\prod_{i=1}^m u_i\right) \cdot t^m \prod_{i=1}^m (1 + u_i^\vee t^{-1}) \\
&= \lambda^m(e) \cdot \sum_{j=0}^m \lambda^j(e^\vee) t^{m-j}.
\end{aligned}$$

This implies, by comparing coefficients with $\lambda_t(e) = \sum_{i=0}^m \lambda^i(e) t^i$, that

$$\lambda^i(e) = \lambda^m(e) \lambda^{m-i}(e^\vee),$$

where $m = \varepsilon(e)$.

2.1.7 Grothendieck groups and the splitting principle †

Given a smooth scheme X, the Grothendieck group $K(X)$ defined in §2.1.1 has the structure of a λ-ring with the λ^i operations defined by, for vector bundles E on X,

$$\lambda^i([E]) := \left[\bigwedge\nolimits^i E\right].$$

This definition can be extended to all elements of $K(X)$ by using the properties of the λ^i operators with respect to addition and multiplication.

The subset $E(X) \subset K(X)$ of positive elements is the set of elements of the form $[E]$ for some vector bundle E on X. The subset $L(X) \subset K(X)$ of line elements is the set of elements of the form $[L]$ for some line bundle L on X. The augmentation $\varepsilon : K(X) \to Z$ is defined by extending the definition for each vector bundle E on X,

$$\varepsilon([E]) \coloneqq \mathrm{rk}(E),$$

to each element in $K(X)$ by taking into account ε is a morphism of rings, and $K(X) = E(X) - E(X)$.

The family of λ-rings of Grothendieck groups of vector bundles $K(X)$ for smooth schemes X satisfies the splitting property. This is proved in [12, V, §2 Theorem 2.7] by inductively considering, for a vector bundle E on a smooth scheme X, the projective scheme $f : \mathbb{P}(E) \to X$ where the tautological exact sequence

$$0 \to H \to f^*E \to \mathcal{O}(1) \to 0$$

gives $[f^*E] = [H] + [\mathcal{O}(1)]$ in the extension $K(\mathbb{P}(E))$ of $K(X)$. We will note that $f : \mathbb{P}(E) \to X$ is a projective morphism.

There is an involution on $K(X)$, defined by extending for each vector bundle E on X,

$$[E]^\vee \coloneqq [E^\vee],$$

where $E^\vee$ is the dual vector bundle of E. This involution extends to the extensions of $K(X)$ in the family of λ-rings of Grothendieck groups of vector bundles $K(Y)$ for smooth schemes Y.

2.1.8 Ample line elements and bundles †

A line element $u \in L$ is said to be *ample* if given $x \in K$ there exists an integer $n(x)$ such that for all $n \geq n(x)$,

$$u^n x = e - m,$$

for some $e \in E$ and $m \in \mathbb{Z}$.

Given a scheme X, a line bundle L on X is said to be *ample* if for any sheaf F on X there exists an integer $n(F)$ such that for all $n \geq n(F)$, $F \otimes L^{\otimes n}$ is generated by global sections, so that there exists a surjection $\bigoplus_{i=1}^{m} \mathcal{O}_X \to F \otimes L^{\otimes n}$ for some m.

These two notions are related; as is proved in [12, Lemma 3.1], one can show that if L is an ample line bundle on a scheme X, then $[L]$ is ample in the Grothendieck group $K(X)$. Take an element in $K(X)$ and write it as $[M] - [M']$ for two positive elements $[M]$ and $[M']$. For $n \geq n(M')$, there exists a surjection $\bigoplus_{i=1}^{m} O_X \to M' \otimes L^{\otimes n}$ for some m, whose kernel N is also a vector bundle by considering the fibers of the vector bundles at points. By considering the short exact sequence

$$0 \to N \to \bigoplus_{i=1}^{m} O_X \to M' \otimes L^{\otimes n} \to 0,$$

one has, in $K(X)$ by recalling $[\mathcal{O}_X] = 1$, that

$$[M'] = m - [N],$$

and so one can write

$$[L]^n([M] - [M']) = [M \otimes L^{\otimes n} \oplus N] - m.$$

We recall another relevant lemma from [12],

Lemma 2.1. *Let K be a λ-ring. If there exists an ample line element $u \in L \subset K$, then for all line elements $v \in L$, $(1 - v)$ is nilpotent.*

Reference. See [12, Lemma 1.4]. □

2.2 Characteristic Classes †

Given a λ-ring and a graded ring, one can consider certain characteristic classes, which will be relevant at different points of this paper. Firstly, the Chern and Todd characters are the algebraic analogues to the Chern and Todd forms introduced later in §2.3, and not only are they useful to better understand their analytic counterparts, they also form the core of the Grothendieck Riemann Roch theorem from §2.5.3, which we use in §3.

We also introduce the Adams character, which is used in the Adams Riemann Roch (ARR) theorem, for which Rössler's [23], which we aim to extend to the analytic case in this paper, is a local refinement in degree 1.

In §2.2.1, we introduce Chern class homomorphisms c^i from a λ-ring to a graded ring, from which characteristic classes are built. In §2.2.2 we introduce the Chow ring. In §2.2.3, we present the notion of a set of Chern roots for a positive element e, which build on the idea of splitting a positive element and are used to better understand characteristic classes.

In §2.2.4, we define the Chern character morphism of a given power series, and in §2.2.5 we introduce the Chern character, which uses the exponential power series and is a morphism of rings. In §2.2.6 we define the Todd character, which is related to the Chern character as is shown in §2.2.7.

In §2.2.8 we introduce the Adams character, which is the key construction for the Adams Riemann Roch theorem. In §2.2.9 we present the Bott-Chern cannibalistic class, which is to the Adams character like the Todd character is to the Chern character. In §2.2.10 we present some classic identities involving the Adams character and the Bott Chern cannibalistic class. In §2.2.11 we introduce a variant of the Todd character which will be useful when considering the equivariant case later in this paper.

2.2.1 Chern class homomorphisms †

Suppose that $A = \bigoplus_{i\geq 0} A^i$ is a graded ring, where A^i is the i-th graded component. We let $A^{\geq m}$ denote the ideal $\bigoplus_{i\geq m} A^i$; and $1 + A^{\geq 1} = 1 + \bigoplus_{i\geq 1} A^i$ be the set of elements of the form $1+\alpha$ for $\alpha \in A^{\geq 1}$; and $\bigwedge^\circ A$ be the multiplicative group whose elements are power series in $A[[t]]$ of the form $1 + a_1 t + a_2 t^2 + \cdots$ where $a_i \in A^i$ for all $i \geq 1$.

Given a λ-ring K and a graded ring A, we consider a homomorphism of Abelian groups c_t, from the group of K under addition, to the group of $\bigwedge^\circ A$ under multiplication. We write $c^i : K \to A^i$ for the maps that satisfy the identity

$$c_t(x) = \sum_{i\geq 0} c^i(x) t^i.$$

Note that the fact that c_t is a group homomorphism is equivalent to the identity

$$c^k(x+y) = \sum_{i+j=k} c^i(x)c^j(y).$$

The group homomorphism c_t is called a *Chern class homomorphism* with values in A if it satisfies the following three conditions, in addition to the splitting principle below.

- For each line element $u \in L$, $c_t(u) = 1 + c^1(u)t$, so that $c^i(u) = 0$ for all $i \geq 1$.
- For $u, v \in L$, we have $c^1(uv) = c^1(u) + c^1(v)$, so that $c^1 : L \to A^1$ is a homomorphism.
- For all positive elements $e \in E$ and all $i \geq 1$, $c^i(e)$ is nilpotent.

Note that due to the third condition, for a given $x \in K$ the $c^i(x)$ vanish for large enough i. We call $c_t(x)$ the *Chern polynomial* of $x \in K$, and $c^i(x)$ the *i-th Chern class* of x. The *total Chern class* of x is the sum of all the Chern classes $c(x) = \sum_{i \geq 0} c^i(x) \in 1 + A^{\geq 1}$. Note that $c : K \to A^\times$ is a group homomorphism from the group of K under addition to the multiplicative units of A.

The *splitting principle* for Chern class homomorphisms requires that given a finite set of positive elements $\{e_i\}$ in K, there exists a λ-ring extension K' of K where each e_i splits, and such that the total Chern class c extends to a homomorphism $c : K' \to A'^\times$ for some graded extension A' of A.

2.2.2 Chow ring †

Suppose that X is a smooth equidimensional scheme. An *algebraic cycle* on X is a linear combination of subvarieties of X with integer coefficients.

Suppose that W is an $(i+1)$-dimensional subvariety of X, and Z is an i-dimensional subvariety of W. For $f \in \mathcal{O}_{W,Z}$, the *order of vanishing of f along Z*, $\mathrm{ord}_Z(f)$, is the length of the $\mathcal{O}_{W,Z}$-module $\mathcal{O}_{W,Z}/(f)$. If f is a rational function on W, it may be written as $f = a/b$ for some $a, b \in \mathcal{O}_{W,Z}$, and *the order of vanishing* of f along Z is given by $\mathrm{ord}_Z(f) = \mathrm{ord}_Z(a) - \mathrm{ord}_Z(b)$.

For an $(i+1)$-dimensional subvariety W, and a rational function f on W which is not identically zero, we define the cycle

$$(f) \coloneqq \sum_{Z} \operatorname{ord}_Z(f) Z,$$

where the sum runs over the i-dimensional subvarieties Z of W.

We say that two algebraic cycles are *rationally equivalent* if they differ by an algebraic cycle generated by the cycles of the form (f) for rational functions f on subvarieties W of X. The *Chow ring* $CH(X)$ on a smooth scheme X is made of the algebraic cycles on X quotiented by rational equivalence.

A multiplication operation can be defined on the Chow ring as is shown in [13], so that for two subvarieties that intersect transversely, their product is the combination of subvarieties which form their intersection. This gives the Chow ring the structure of a ring, with 0 being the empty linear combination, and 1 being the equivalence class of X as a subvariety of X itself.

The Chow ring has a grading based on the codimension of subvarieties of X, writing $CH^q(X)$ for the equivalence classes of algebraic cycles of codimension q. Note that if two subvarieties of X intersect transversely, the codimension of their intersection can be computed by adding the codimensions of the two subvarieties, this property is what induces the grading on the Chow ring.

Given a morphism $g : Y \to X$ of smooth schemes, we may consider the pullback of elements in the Chow ring, induced by the pullback of subvarieties; for $Z \subseteq X$, $g^*Z \coloneqq Y \times_X Z$. The pullback is a morphism of graded rings

$$g^* : CH(X) \to CH(Y),$$

as such it preserves the degree of the elements, mapping $g^* : CH^q(X) \to CH^q(Y)$.

Given a proper morphism $f : X \to S$ of relative dimension n, it is possible to extend the pushforward of subvarieties of X into a pushforward in the Chow ring

$$f_* : CH(X) \to CH(S).$$

Unlike the case of the pullback, instead of preserving codimension, the pushforward of cycles preserves the dimension, and so the pushforward maps $f_* : CH^q(X) \to CH^{q-n}(S)$. For elements $\alpha \in CH(X)$, we will write $\int_{X/S} \alpha$ to mean $f_*(\alpha)$.

Following [13], it is possible to construct a Chern class homomorphism

$$c^i : K(X) \to CH(X).$$

Note that by the properties of the grading of the Chow ring, we have that for a vector bundle E on X, $c^i(E) \coloneqq c^i([E])$ is an equivalence class of cycles of codimension i.

From [13, 2.1.1] (see also [15, IV.12.6]), we will also recall the fact that $c^1 : \mathrm{Pic}(X) \to CH^1(X)$ is injective if X is normal, while noting that if X is smooth, it is then regular, which implies it is normal.

2.2.3 Chern Roots †

Let $c_t : K \to \bigwedge^\circ A$ be a Chern class homomorphism with values in A. Suppose that $e \in E \subset K$ is a positive element, and that e splits into $e = u_1 + \cdots + u_m$ in some extension K' of K. Let $a_i = c^1(u_i)$, which we say are a choice of *Chern roots* of e. Note that we can then factor the Chern polynomial of e as

$$c_t(e) = \prod_{i=1}^{m}(1 + a_i t).$$

Moreover, the i-th Chern class of e can also be expressed as $c^i(e) = \sigma_i(a_1, \ldots, a_m)$, so in particular, by the fundamental theorem of symmetric polynomials, any symmetric polynomial expression in the a_i can be expressed in terms of the $c^i(e)$, is contained in A, and is independent of the choice of splitting of e.

If $m = \varepsilon(e)$, we call $c^{\mathrm{top}}(e) \coloneqq c^m(e) = \prod_{i=1}^m a_i$ the *top Chern class* of e. Also note that the total Chern class of e can be expressed as $c(e) = \prod_{i=1}^m (1 + a_i)$.

2.2.4 Chern character morphism of a power series †

Take a power series $\varphi(t) \in \mathbb{Q}[[t]]$ with rational coefficients. To each of these power series we associate an additive homomorphism

$$\mathrm{ch}_\varphi : K \to A_\mathbb{Q},$$

where $A_\mathbb{Q}$ is $A \otimes_\mathbb{Z} \mathbb{Q}$, as follows.

For positive elements $e \in E$, we define

$$\mathrm{ch}_\varphi(e) \coloneqq \sum_{i=1}^{m} \varphi(a_i),$$

which is independent of the choice of Chern roots a_i of e by the fundamental theorem of symmetric polynomials.

If we have two positive elements $e, e' \in E$, by considering the splitting of $e + e'$ it follows that

$$\mathrm{ch}_\varphi(e + e') = \mathrm{ch}_\varphi(e) + \mathrm{ch}_\varphi(e').$$

We then define for a general element $x \in K$, by taking $x = e - e'$ for some positive elements $e, e' \in E$, since $K = E - E$,

$$\mathrm{ch}_\varphi(x) \coloneqq \mathrm{ch}_\varphi(e) - \mathrm{ch}_\varphi(e').$$

We can show that ch_φ is well-defined by considering the *associated Hirzebruch polynomials* H_j of φ. These are the polynomials in j variables satisfying the following power series identity, for variables $U_1, \ldots, U_m$ and where σ_i are the elementary polynomials;

$$\sum_{i=1}^{m} \varphi(U_i t) = \sum_{j \geq 0} H_j(\sigma_1(U_\bigstar), \ldots, \sigma_j(U_\bigstar))t^j.$$

The existence and uniqueness of the associated Hirzebruch polynomials of a power series φ follows from the fundamental theorem of symmetric polynomials. Note that given our definition of ch_φ for $x \in K$ we can write

$$\mathrm{ch}_\varphi(x) = \sum_{j \geq 0} H_j(c^1(x), \ldots, c^j(x)).$$

2.2.5 Chern character †

The *Chern character* $\mathrm{ch} : K \to A$ is one of these additive homomorphisms ch_φ where φ is set to be the exponential power series,

$$\varphi(t) = \exp(t) = \sum_{k \geq 0} \frac{t^k}{k!},$$

so that for a positive element $e \in E$ with Chern roots $a_1, \dots, a_m$ we have

$$\operatorname{ch}(e) = \sum_{i=1}^{m} \sum_{k \geq 0} \frac{a_i^k}{k!}.$$

In this case, ch is not only an additive group homomorphism, but a morphism of rings. This is because for positive elements with splitting $e = u_1 + \cdots + u_m$, $e' = v_1 + \cdots + v_n$ we have $ee' = \sum_{i,j} u_i v_j$ and so

$$\begin{aligned}
\operatorname{ch}(ee') &= \sum_{i,j} \exp(c^1(u_i v_j)) \\
&= \sum_{i,j} \exp(c^1(u_i) + c^1(v_j)) \\
&= \sum_{i,j} \exp(c^1(u_i)) \exp(c^1(v_j)) \\
&= \left(\sum_i \exp(c^1(u_i)) \right) \left(\sum_j \exp(c^1(v_j)) \right) \\
&= \operatorname{ch}(e) \operatorname{ch}(e').
\end{aligned}$$

Recall from §2.2.1 that for a line element $u \in L \subset K$ we have that u has a single Chern root $a_1 = c^1(u)$. This means that for such u we have that

$$\operatorname{ch}(u) = 1 + c^1(u) + \frac{1}{2} c^1(u)^2 + \cdots \in 1 + A^{\geq 1}.$$

Using the Chern class morphism described in §2.2.2, we may define the Chern character

$$\operatorname{ch} : K(X) \to CH(X)_{\mathbb{Q}},$$

where $CH(X)_{\mathbb{Q}}$ is the Chow ring tensored with $\mathbb{Q}$. For a vector bundle E on X, we will write $\operatorname{ch}(E) := \operatorname{ch}([E])$. For a formal sum of vector bundles with integer coefficients $k_1 E_1 + \cdots + k_m E_m$ we will let

$$\operatorname{ch}(k_1 E_1 + \cdots + k_m E_m) := k_1 \operatorname{ch}(E_1) + \cdots + k_m \operatorname{ch}(E_m),$$

which agrees with the notion of the Chern character being a morphism of rings. In a similar way, we consider the tensor product to distribute with formal sums of vector bundles in the natural way, which also agrees with the Chern character. This allows us to for example write

$$\operatorname{ch}\left(\bigotimes_{i=1}^{m} (\mathcal{O}_X - E_i) \right) = \prod_{i=1}^{m} (1 - \operatorname{ch}(E_i)).$$

2.2.6 Todd character †

Now take a power series $\varphi(t) \in 1 + t\mathbb{Q}[[t]]$. We first define the corresponding *Todd homomorphism* td_φ on positive elements by, for a positive element $e \in E$ with Chern roots a_i,

$$\mathrm{td}_\varphi(e) = \prod_{i=1}^{m} \varphi(a_i).$$

This can be extended to a homomorphism $\mathrm{td}_\varphi : K \to 1 + A^{\geq 1}$ by using the associated Hirzebruch polynomials Q_j satisfying, for variables $U_1, \ldots, U_m$,

$$\prod_{i=1}^{m} \varphi(U_i t) = \sum_{j \geq 0} Q_j(\sigma_1(U_\bigstar), \ldots, \sigma_j(U_\bigstar))t^j.$$

This is a group homomorphism from the additive group of K to the multiplicative group of units of A in $1 + A^{\geq 1}$, once again by considering the Chern roots of two positive elements. This means that for two elements $x, y \in K$ we have that

$$\mathrm{td}_\varphi(x + y) = \mathrm{td}_\varphi(x) \cdot \mathrm{td}_\varphi(y).$$

The *Todd character* $\mathrm{td} : K \to 1 + A^{\geq 1}$ is the Todd homomorphism of the power series

$$\varphi(t) = \frac{t\exp(t)}{\exp(t) - 1} = \sum_{n=0} B_n \frac{x^n}{n!} = 1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \cdots,$$

where B_n are the Bernoulli numbers with $B_1 = +\frac{1}{2}$, [27, A164555 / A027646].

2.2.7 Todd character, Chern character and involutions †

Suppose that we have a λ-ring K with an involution $x \mapsto x^\vee$, which is compatible with extensions obtained from the splitting principle.

For a line element $u \in L \subset K$, we have that

$$c^1(u^\vee) = c^1(u^{-1}) = -c^1(u).$$

From the splitting principle we deduce that for each $x \in K$ we have

$$c_i(x^\vee) = (-1)^i c_i(x),$$

which implies that

$$\operatorname{ch}(x^\vee) = -\operatorname{ch}(x).$$

In a similar way, by applying the splitting principle we deduce that for any positive element $e \in E \subset K$ we have

$$\operatorname{td}(e^\vee) = \operatorname{td}(e)\exp(-c^1(e)).$$

Now suppose that e is a positive element with Chern roots $a_1, \dots a_m$. We may then note that

$$\begin{aligned}\operatorname{ch}(\lambda_t(e^\vee)) &= \operatorname{ch}\left(\prod_{i=1}^m (1 + u_i^\vee t)\right)\\ &= \prod_{i=1}^m (1 + \operatorname{ch}(u_i^\vee)t)\\ &= \prod_{i=1}^m (1 + \exp(-a_i)t),\end{aligned}$$

So that

$$\begin{aligned}\operatorname{td}(e)\operatorname{ch}(\lambda_{-1}(e^\vee)) &= \prod_{i=1}^m \frac{a_i\exp(a_i)}{\exp(a_i)-1} \cdot \prod_{i=1}^m (1 - \exp(-a_i))\\ &= \prod_{i=1}^m \left[\frac{a_i\exp(a_i)}{\exp(a_i)-1} \cdot \frac{\exp(a_i)-1}{\exp(a_i)}\right]\\ &= \prod_{i=1}^m a_i\\ &= c^{\text{top}}(e).\end{aligned}$$

This gives us a formula for computing the Todd character based on the Chern character, which states that

$$\operatorname{td}(e) = \frac{c^{\text{top}}(e)}{\operatorname{ch}\left(\sum_{i=1}^{\varepsilon(e)} (-1)^i \lambda^i(e^\vee)\right)}. \tag{2.1}$$

2.2.8 Adams character †

Suppose that we have a λ-ring K. The j-th *Adams character* $\psi^j : K \to K$ is a ring homomorphism defined by considering the coefficients in the following expression;

$$\begin{aligned}\sum_{j\geq 0} \psi^j(x)t^j &:= \varepsilon(x) - t\frac{d}{dt}\log \lambda_{-t}(x) \\ &= \varepsilon(x) + \frac{\lambda^1(x)t - 2\lambda^2(x)t^2 + 3\lambda^3(x)t^3 - \cdots}{1 - \lambda^1(x)t + \lambda^2(x)t^2 - \cdots}.\end{aligned}$$

For line elements $u \in L$, we have that

$$\begin{aligned}\sum_{j\geq 0} \psi^j(u)t^j &= 1 + \frac{ut}{1-ut} \\ &= 1 + ut + u^2t^2 + u^3t^3 + \cdots ,\end{aligned}$$

so we have that

$$\psi^j(u) = u^j.$$

Additivity of the ψ^j follows from the fact that ε is additive, and that for $x, y \in K$ we have

$$\begin{aligned}\frac{d}{dt}\log \lambda_{-t}(x+y) &= \frac{d}{dt}\log(\lambda_{-t}(x)\lambda_{-t}(y)) \\ &= \frac{d}{dt}\log \lambda_{-t}(x) + \frac{d}{dt}\log \lambda_{-t}(y).\end{aligned}$$

We therefore have, for positive elements with splitting $e = u_1 + \cdots + u_m$,

$$\psi^j(e) = u_1^j + \cdots + u_m^j.$$

The multiplicativity of the ψ^j is deduced by considering the case where we take the product of two line elements $u, w \in L$, where

$$\psi^j(uw) = u^j w^j = \psi^j(u)\psi^j(w),$$

and extending to elements in E and then elements in K. This makes the j-th Adams character ψ^j a ring automorphism $\psi^j : K \to K$.

2.2.9 Bott-Chern cannibalistic class †

Suppose that we have a λ-ring K and that $j \geq 1$ is a unit in K. Let $\gamma^j(t)$ be the polynomial

$$\gamma^j(t) \coloneqq 1 + t + t^2 + \cdots + t^{j-1} = \frac{1 - t^j}{1 - t}.$$

We define the j-th *Bott-Chern cannibalistic class* θ^j firstly on positive elements e by considering a splitting $e = u_1 + \cdots + u_m$ and defining

$$\theta^j(e) \coloneqq \prod_{i=1}^{m} \gamma^j(u_i).$$

If we are able to find a multiplicative inverse for $\theta^j(u)$ for all line elements $u \in L \subset K$, then we are able to extend the definition of θ^j multiplicatively for all of K. Recall from §2.1.8 that if there exists an ample line element $v \in L$ (for example if there exists an ample line bundle over a scheme X for $K(X)$), then for any line element $u \in L$, $(1 - u)$ is nilpotent. We now show that $\theta^j(u)$ has a multiplicative inverse in $K[\frac{1}{j}]$ if $(1 - u)$ is nilpotent.

Let t and $s = (1 - t)$ be formal variables, we aim to consider the Taylor expansion of $\gamma^j(t)^{-1}$ around 1, so in terms of s. Write $(1 - s)^j = 1 - js + s^2 p(s)$, where p is a polynomial with integer coefficients. We then have

$$\gamma^j(t)^{-1} = \frac{1 - t}{1 - t^j} = \frac{s}{1 - (1 - s)^j} = \frac{s}{js - s^2 p(s)} = \frac{1}{j} \, \frac{1}{1 - \frac{1}{j} s p(s)},$$

so that the Taylor expansion can be computed with

$$\gamma^j(t)^{-1} = \frac{1}{j} \sum_{k \geq 0} \left(\frac{1}{j} s p(s) \right)^k.$$

In particular, this is a finite sum if t is replaced with a line element u for which $(1 - u)$ is nilpotent, and so $\theta^j(u) = \gamma^j(u)$ has an inverse in $K[\frac{1}{j}]$.

For the case $j = 2$, we have that $(1 - s)^2 = 1 - 2s + s^2$ so that $p(s) = 1$ and so

$$\theta^2(u)^{-1} = \frac{1}{2} \sum_{k \geq 0} \left(\frac{1 - u}{2} \right)^k.$$

2.2.10 Identities involving the Adams character and the Bott-Chern cannibalistic class †

We'll first define a family of morphisms of graded rings $\phi^k : A \to A$ so that for $\sum_{p\geq 0} \alpha_p$ with $\alpha_p \in A^p$,

$$\phi^k\left(\sum_{p\geq 0} \alpha_p\right) := \sum_{p\geq 0} k^p \alpha_p.$$

For example, if $\alpha \in A^1$, one has

$$\phi^k(\exp(\alpha)) = \sum_{p\geq 0} k^p \frac{\alpha^p}{p!} = \exp(k\alpha).$$

We can then check the following identity for positive elements;

$$\mathrm{ch}(\psi^k(e)) = \phi^k(\mathrm{ch}(e)), \tag{2.2}$$

which is due to, for a splitting $u_1, \dots, u_m$ of e with Chern roots $a_1, \dots, a_m$,

$$\begin{aligned}\mathrm{ch}(\psi^k(e)) &= \mathrm{ch}\left(\sum_{i=1}^m u_i^k\right) = \sum_{i=1}^m \mathrm{ch}(u_i)^k \\ &= \sum_{i=1}^m \exp(a_i)^k = \sum_{i=1}^m \exp(ka_i) = \phi^k(\mathrm{ch}(e)).\end{aligned}$$

We also have the following identity for positive elements, [12, Proposition 6.2], which is in a way the main relationship between the Bott-Chern cannibalistic class and the Adams operation;

$$\psi^k(\lambda_{-1}(e)) = \lambda_{-1}(e)\theta^k(e). \tag{2.3}$$

This is due to, for a splitting $u_1, \dots, u_m$ of e,

$$\begin{aligned}\psi^k(\lambda_{-1}(e)) &= \psi^k\left(\prod_{i=1}^m (1-u_i)\right) = \prod_{i=1}^m (1-u_i^k) \\ &= \prod_{i=1}^m (1-u_i)(1+u_i+\cdots+u_i^{k-1}) = \lambda_{-1}(e)\theta^k(e).\end{aligned}$$

We can combine these identities together in what is a well-known result in the area.

Lemma 2.2. *We have the following identity for positive elements,*

$$\mathrm{ch}(\theta^k(e^\vee)) = k^m \,\mathrm{td}(e)\phi^k(\mathrm{td}(e)^{-1}).$$

Proof. We begin with (2.3), applied to $e^\vee$;

$$\lambda_{-1}(e^\vee)\theta^k(e^\vee) = \psi^k(\lambda_{-1}(e^\vee)),$$

Then, we evaluate the Chern character on both sides;

$$\mathrm{ch}(\lambda_{-1}(e^\vee))\,\mathrm{ch}(\theta^k(e^\vee)) = \mathrm{ch}(\psi^k(\lambda_{-1}(e^\vee))),$$

We then use (2.2);

$$\mathrm{ch}(\lambda_{-1}(e^\vee))\,\mathrm{ch}(\theta^k(e^\vee)) = \phi^k(\mathrm{ch}(\lambda_{-1}(e^\vee))),$$

We then use the identity (2.1), which is $\mathrm{ch}(\lambda_{-1}(e^\vee)) = c^{\mathrm{top}}(e)\,\mathrm{td}(e)^{-1}$;

$$c^{\mathrm{top}}(e)\,\mathrm{td}(e)^{-1}\,\mathrm{ch}(\theta^k(e^\vee)) = \phi^k(c^{\mathrm{top}}(e)\,\mathrm{td}(e)^{-1}),$$

But then $c^{\mathrm{top}}(e) \in A^m$, so we have $\phi^k(c^{\mathrm{top}}(e)) = k^m c^{\mathrm{top}}(e)$;

$$c^{\mathrm{top}}(e)\,\mathrm{td}(e)^{-1}\,\mathrm{ch}(\theta^k(e^\vee)) = k^m c^{\mathrm{top}}(e)\phi^k(\mathrm{td}(e)^{-1}),$$

Now, while $c^{\mathrm{top}}(e)$ is not a unit, and in fact is frequently nilpotent, all of the expressions we've used can be uniquely expressed as polynomial expressions in the $c^i(e)$. So, while A is in general not an integral domain, $\mathbb{Q}[c^1(e), \ldots, c^{\mathrm{top}}(e)]$ is, and so we may formally divide by the common $c^{\mathrm{top}}(e)$ factor.

$$\begin{aligned}\mathrm{td}(e)^{-1}\,\mathrm{ch}(\theta^k(e^\vee)) &= k^m\phi^k(\mathrm{td}(e)^{-1}),\\ \mathrm{ch}(\theta^k(e^\vee)) &= k^m\,\mathrm{td}(e)\phi^k(\mathrm{td}(e)^{-1}).\end{aligned}$$

□

2.2.11 The td_{-1} character ※

Consider the power series

$$\varphi(t) = \frac{te^t}{e^t+1} = \sum_{n\geq 0} \frac{1}{2} E_n(1) \frac{t^{n+1}}{n!},$$

where E_n are the Euler polynomials, which satisfy

$$\frac{2e^{xt}}{e^t+1} = \sum_{n\geq 0} E_n(x) \frac{t^n}{n!}.$$

We define the -1 Todd class td_{-1} to be the Todd homomorphism td_φ of the power series φ.

The -1 Todd class satisfies the following properties.

Lemma 2.3.

$$\mathrm{td}_{-1}(e)\,\mathrm{ch}(\lambda_1(e^\vee)) = c^{\mathrm{top}}(e).$$

Proof. This proof follows the same process as (2.1). We start by noting that, for a splitting $e = u_1 + \cdots + u_m$ and Chern roots a_i,

$$\mathrm{ch}(\lambda_1(e^\vee)) = \mathrm{ch}\left(\prod_{i=1}^m (1+u_i^\vee)\right) = \prod_{i=1}^m (1+\mathrm{ch}(u_i^\vee)) = \prod_{i=1}^m (1+\exp(-a_i))$$

So that

$$\begin{aligned}
\mathrm{td}_{-1}(e)\,\mathrm{ch}(\lambda_1(e^\vee)) &= \prod_{i=1}^m \frac{a_i \exp(a_i)}{\exp(a_i)+1} \cdot \prod_{i=1}^m (1+\exp(-a_i)) \\
&= \prod_{i=1}^m \left[\frac{a_i \exp(a_i)}{\exp(a_i)+1} \cdot \frac{\exp(a_i)+1}{\exp(a_i)}\right] \\
&= \prod_{i=1}^m a_i \\
&= c^{\mathrm{top}}(e).
\end{aligned}$$

□

Lemma 2.4.

$$\mathrm{td}(e)\,\mathrm{td}_{-1}(e) = 2^{-\varepsilon(e)} c^{\mathrm{top}}(e) \phi^2(\mathrm{td}(e)).$$

Proof. Consider the $m = \varepsilon(e)$ Chern roots a_i of e.

Note that we then have that

$$\begin{aligned}\phi^2(\mathrm{td}(e)) &= \phi^2\left(\prod_{i=1}^{m}\sum_{n\geq 0} B_n\frac{a_i^n}{n!}\right)\\ &= \prod_{i=1}^{m}\sum_{n\geq 0} 2^n B_n\frac{a_i^n}{n!}\\ &= \prod_{i=1}^{m}\sum_{n\geq 0} B_n\frac{(2a_i)^n}{n!}\\ &= \prod_{i=1}^{m}\frac{2a_i\exp(2a_i)}{\exp(2a_i)-1}.\end{aligned}$$

Then we have that

$$\begin{aligned}\mathrm{td}(e)\,\mathrm{td}_{-1}(e) &= \prod_{i=1}^{m}\left(\frac{a_i\exp(a_i)}{\exp(a_i)-1}\cdot\frac{a_i\exp(a_i)}{\exp(a_i)+1}\right)\\ &= \prod_{i=1}^{m}\left(\frac{1}{2}a_i\right)\left(\frac{2a_i\exp(2a_i)}{\exp(2a_i-1)}\right)\\ &= 2^{-m}c^{\mathrm{top}}(e)\phi^2(\mathrm{td}(e)).\end{aligned}$$

□

2.3 Chern Forms †

A hermitian vector bundle $\underline{E} = (E, h)$ on a compact Kähler manifold X^{an} consists of a holomorphic vector bundle E, which corresponds to an algebraic vector bundle in $K(X)$, together with a hermitian metric h.

When computing the norms of isomorphisms related to hermitian vector bundles, one must often consider the Chern forms $\mathrm{ch}(\underline{E}) \in \bigoplus_{p\geq 0}\mathfrak{A}^{p,p}(X^{\mathrm{an}})$ and the Chern form classes $\overline{\mathrm{ch}}(\underline{E}) \in \bigoplus_{p\geq 0}\widetilde{\mathfrak{A}}^{p}(X^{\mathrm{an}})$.

While Chern characters and Chern forms are constructed in very different settings—the former in a purely algebraic one, and the latter in the case of complex geometry—they share similar properties. This similarity will make the Lemmas in §2.4, concerning both Chern characters and forms, have very similar proofs in spirit, which differ in the details.

In §2.3.1 we introduce the (p,q)-forms $\mathfrak{A}^{p,q}(X)$ on a complex manifold X, which have a subset $\bigoplus_{p\geq 0}\mathfrak{A}^{p,p}(X)$ where the Chern forms are defined. In §2.3.2 we introduce the ∂, $\bar{\partial}$ differentials, the $\mathrm{dd}^{\mathrm{c}} : \mathfrak{A}^{p,p}(X) \to \mathfrak{A}^{p+1,p+1}(X)$ differential, and differential form classes. In §2.3.3 we define connections ∇ on a holomorphic vector bundle E on X, as well as the curvature form ∇^2 of a connection. In §2.3.4 we define the unique hermitian holomorphic connection on a holomorphic bundle $\underline{E} = (E,h)$ endowed with a hermitian metric, whose curvature form is used to define Chern forms.

In §2.3.5 we give the definition of the Chern form $\mathrm{ch}(\underline{E})$ of a hermitian vector bundle $\underline{E} = (E,h)$. In §2.3.6 we give the definition of Todd forms $\mathrm{td}(\underline{E})$ of hermitian vector bundles. In §2.3.7 we introduce the secondary Chern and Todd forms of a short exact sequence of hermitian vector bundles, which vanish if and only if the short exact sequence is orthogonally split. In §2.3.8 we define the Chern form class $\overline{\mathrm{ch}}(\underline{E}) \in \bigoplus_{p\geq 0}\widetilde{\mathfrak{A}}^{p}(X^{\mathrm{an}})$.

† This section follows [24, §IV.2].

2.3.1 Differential forms of type (p,q) †

Let X be an n-dimensional complex manifold. We may think of X as a $2n$-dimensional real orientable manifold, and so we can consider the *de Rham* complex of X as a real smooth manifold

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \Omega^3(X) \xrightarrow{d} \cdots , \tag{2.4}$$

where $\Omega^0(X)$ are the smooth real functions on X as a real manifold, $\Omega^1(X)$ are the smooth real 1-forms, and so on. Recall that $\Omega^i(X) := \bigwedge^i \Omega_X$, where Ω_X is the cotangent space of X.

We define $\mathfrak{A}^i(X) := \Omega^i(X) \otimes \mathbb{C}$, the complexified i-forms on X, and consider the complexification of the de Rham exact sequence (2.4);

$$0 \to \mathfrak{A}^0(X) \xrightarrow{d} \mathfrak{A}^1(X) \xrightarrow{d} \mathfrak{A}^2(X) \xrightarrow{d} \mathfrak{A}^3(X) \xrightarrow{d} \cdots .$$

Note that $\mathfrak{A}^0(X)$ are the smooth, but not necessarily holomorphic, complex functions on X.

Now, consider complex local coordinates $z_1, \dots, z_n$ on X as an n-dimensional complex manifold, where we consider the real and imaginary parts $z_j = x_j + iy_j$, which give us a set of real local coordinates $x_1, y_1,\ x_2, y_2,\ \dots,\ x_n, y_n$ for X as a $2n$-dimensional real manifold. A basis for $\mathfrak{A}^1(X)$ as a complex bundle will be given by $\{dx_1, dy_1, \dots, dx_n, dy_n\}$, and by change of basis, we may consider instead the basis $\{dz_1, d\bar{z}_1, \dots, dz_n, d\bar{z}_n\}$ where $dz_j := dx_j + idy_j$ and $d\bar{z}_j := dx_j - idy_j$. We then define the *holomorphic cotangent bundle* $\mathfrak{A}^{1,0}(X)$ as the span of $dz_1, dz_2, \dots, dz_n$, and the *antiholomorphic cotangent bundle* $\mathfrak{A}^{0,1}(X)$ as the span of $d\bar{z}_1, d\bar{z}_2, \dots d\bar{z}_n$. It is immediate from the definition that

$$\mathfrak{A}^1(X) = \mathfrak{A}^{1,0}(X) \oplus \mathfrak{A}^{0,1}(X).$$

The *differential forms of type* (p, q) on X are the differential $(p+q)$-forms in

$$\mathfrak{A}^{p,q}(X) := \bigwedge^{p} \mathfrak{A}^{1,0}(X) \wedge \bigwedge^{q} \mathfrak{A}^{0,1}(X).$$

Note that since $\mathfrak{A}^k(X) = \bigwedge^k (\mathfrak{A}^{1,0}(X) \oplus \mathfrak{A}^{0,1}(X))$, by considering the expansion of the exterior product, we have that

$$\mathfrak{A}^k(X) = \bigoplus_{p+q=k} \mathfrak{A}^{p,q}(X).$$

2.3.2 The ∂, $\bar{\partial}$ and $\mathrm{dd^c}$ differentials †

By considering $\mathfrak{A}^1(X) = \mathfrak{A}^{1,0}(X) \oplus \mathfrak{A}^{0,1}(X)$, we may define

$$\partial : \mathfrak{A}^0(X) \to \mathfrak{A}^{1,0}(X) \qquad \text{and} \qquad \bar{\partial} : \mathfrak{A}^0(X) \to \mathfrak{A}^{0,1}(X),$$

such that the de Rham differential $d : \mathfrak{A}^0(X) \to \mathfrak{A}^1(X)$ splits as $d = \partial + \bar{\partial}$. These naturally extend to differentials

$$\partial : \mathfrak{A}^{p,q}(X) \to \mathfrak{A}^{p+1,q}(X) \qquad \text{and} \qquad \bar{\partial} : \mathfrak{A}^{p,q}(X) \to \mathfrak{A}^{p,q+1}(X),$$

as can be seen in Figure 2.1.

Since $d^2 = 0$, we deduce that $\partial^2 = 0$, $\bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. We define $d^c := \frac{1}{4\pi i}(\partial - \bar{\partial})$, so that $\mathrm{dd^c}$ is the operator $\mathrm{dd^c} : \mathfrak{A}^{p,q}(X) \to \mathfrak{A}^{p+1,q+1}(X)$ given by

$$\mathrm{dd^c} = -\frac{1}{4\pi i}\partial\bar{\partial} = \frac{1}{4\pi i}\bar{\partial}\partial.$$

$$\begin{tikzcd}
 & & & & \cdots \\
 & & & \mathfrak{A}^{3,0}(X) \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"] & \\
 & & \mathfrak{A}^{2,0}(X) \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"] & & \cdots \\
 & \mathfrak{A}^{1,0}(X) \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"] & & \mathfrak{A}^{2,1}(X) \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"] & \\
\mathfrak{A}^{0,0}(X) \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"] \arrow[rr, "\mathrm{dd^c}"] & & \mathfrak{A}^{1,1}(X) \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"] \arrow[rr, "\mathrm{dd^c}"] & & \mathfrak{A}^{2,2}(X)\cdots \\
 & \mathfrak{A}^{0,1}(X) \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"] & & \mathfrak{A}^{1,2}(X) \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"] & \\
 & & \mathfrak{A}^{0,2}(X) \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"] & & \cdots \\
 & & & \mathfrak{A}^{0,3}(X) \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"] & \\
 & & & & \cdots
\end{tikzcd}$$

Figure 2.1: Splitting of the differential $d : \mathfrak{A}^k(X) \to \mathfrak{A}^{k+1}(X)$ into $\partial + \bar{\partial}$. Note that $\partial\bar{\partial} = -\bar{\partial}\partial$. The $\mathrm{dd^c}$ differential for $\mathfrak{A}^{p,p}(X) \to \mathfrak{A}^{p+1,p+1}(X)$.

Note that we have a complex

$$0 \to \mathfrak{A}^{0,0}(X) \xrightarrow{\mathrm{dd^c}} \mathfrak{A}^{1,1}(X) \xrightarrow{\mathrm{dd^c}} \mathfrak{A}^{2,2}(X) \xrightarrow{\mathrm{dd^c}} \cdots \xrightarrow{\mathrm{dd^c}} \mathfrak{A}^{p,p}(X) \xrightarrow{\mathrm{dd^c}} \cdots .$$

The set of forms $\bigoplus_{p\geq 0} \mathfrak{A}^{p,p}(X)$ can be given the structure of a graded ring where multiplication is given by the exterior product $\wedge$. Note that this multiplication is well defined since if $\alpha \in \mathfrak{A}^{p,p}(X)$ and $\alpha \in \mathfrak{A}^{q,q}(X)$ then

$$\alpha\beta \in \mathfrak{A}^{p+q,p+q}(X) = \bigwedge^{p+q} \mathfrak{A}^{1,0}(X) \wedge \bigwedge^{p+q} \mathfrak{A}^{0,1}(X);$$

and that this multiplication commutative since all the forms in $\mathfrak{A}^{p,p}(X)$ are differential forms in even degree $2p$, and the exterior product of two forms in even degree is commutative.

Given a map of complex manifolds $g : Y \to X$ we may consider the pullback of k-forms which induces a pullback

$$g^* : \mathfrak{A}^{p,p}(X) \to \mathfrak{A}^{p,p}(Y).$$

Given a proper submersion of complex manifolds $f : X \to S$, of relative dimension n, the *integration along fibers* induces a pushforward

$$f_* : \mathfrak{A}^{p,p}(X) \to \mathfrak{A}^{p-n,p-n}(S).$$

We will write $\int_{X/S} \alpha$ for $f_*\alpha$. Note that f being a proper submersion implies by Ehresmann's Theorem that the fibers of f are locally trivial, which allows one to integrate along the fibers.

We define the p-th differential form classes to be the quotient

$$\widetilde{\mathfrak{A}}^p(X) := \frac{\mathfrak{A}^{p,p}(X)}{\operatorname{Im}\partial + \operatorname{Im}\bar{\partial}},$$

and note that we may extend the quotient map to

$$\bigoplus_{p\geq 0} \mathfrak{A}^{p,p}(X) \to \bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^p(X).$$

2.3.3 Connections and curvature forms †

Suppose now that X is a complex manifold, and consider a holomorphic vector bundle E on X of rank r. Let $\mathfrak{A}^n(X, E)$ be the smooth sections of $\bigwedge^n \Omega_X \otimes E$, where Ω_X is the cotangent bundle of X. Note that $\mathfrak{A}^n(X, E)$ is an $\mathfrak{A}^0(X)$-module.

A *connection* on E is a $\mathbb{C}$-linear map

$$\nabla : \mathfrak{A}^0(X, E) \to \mathfrak{A}^1(X, E),$$

satisfying

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s),$$

for all $f \in \mathfrak{A}^0(X)$ and $s \in \mathfrak{A}^0(X, E)$.

A connection $\nabla : \mathfrak{A}^0(X, E) \to \mathfrak{A}^1(X, E)$ on E induces a map

$$\nabla : \mathfrak{A}^1(X, E) \to \mathfrak{A}^2(X, E),$$

where

$$\nabla(w \otimes s) := dw \otimes s - w \otimes \nabla(s),$$

for $w \in \mathfrak{A}^1(X)$ and $s \in \mathfrak{A}^0(X, E)$. A simple computation then implies that for $f \in \mathfrak{A}^0(X)$ and $s \in \mathfrak{A}^0(X, E)$ we have

$$\nabla^2(f \cdot s) = f \cdot \nabla^2(s),$$

which makes $\nabla^2 : \mathfrak{A}^0(X, E) \to \mathfrak{A}^2(X, E)$ an $\mathfrak{A}^0(X)$-linear map, which we call the *curvature form* of ∇. We often consider ∇^2 as an element of $\mathfrak{A}^2(X, \mathrm{End}(E))$, or a matrix of forms in $\mathfrak{A}^2(X, E)$.

2.3.4 Unitary connections and the hermitian holomorphic connection †

First, note that since the elements of $\mathfrak{A}^1(X, E)$ are the linear combinations of elements which are the tensor product of an element in $\mathfrak{A}^1(X) = \mathfrak{A}^{1,0}(X) \oplus \mathfrak{A}^{0,1}(X)$ and one in $\mathfrak{A}^0(X, E)$, we may consider the splitting of $\nabla : \mathfrak{A}^0(X, E) \to \mathfrak{A}^1(X, E)$ into $\nabla = \nabla^{1,0} + \nabla^{0,1}$ in a similar way that $d = \partial + \bar{\partial}$.

Now, suppose that E is equipped with a hermitian metric, which is given by a hermitian inner product $\langle \cdot, \cdot \rangle$ on each fiber of E, varying smoothly depending on points on E. This hermitian metric induces, for two sections $s, t \in \mathfrak{A}^0(X, E)$, a pairing $\langle s, t\rangle \in \mathfrak{A}^0(X, E)$ so that

$$\langle s, t\rangle(x) := \langle s(x), t(x)\rangle;$$

together with pairings $\mathfrak{A}^0(X, E) \otimes \mathfrak{A}^1(X, E) \to \mathfrak{A}^1(X)$ and $\mathfrak{A}^1(X, E) \otimes \mathfrak{A}^0(X, E) \to \mathfrak{A}^1(X)$ such that for $s, t \in \mathfrak{A}^0(X, E)$ and $w \in \mathfrak{A}^1(X)$ we have

$$\langle s \otimes w, t\rangle := \langle s, t\rangle \cdot w,$$

$$\langle s, t \otimes w\rangle := \langle s, t\rangle \cdot w.$$

A connection $\nabla : \mathfrak{A}^0(X, E) \to \mathfrak{A}^1(X, E)$ is said to be *unitary* if it satisfies the following equation for all $s, t \in \mathfrak{A}^0(X, E)$:

$$d\langle s, t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle.$$

The *hermitian holomorphic connection* of a holomorphic bundle E endowed with a hermitian metric h is the unique unitary connection $\nabla : \mathfrak{A}^0(X, E) \to \mathfrak{A}^1(X, E)$ which satisfies $\nabla^{0,1} = \bar{\partial}_E$, where $\bar{\partial}_E$ is map $\mathfrak{A}^0(X, E) \to \mathfrak{A}^1(X, E)$ induced by $\bar{\partial} : \mathfrak{A}^0(X) \to \mathfrak{A}^{0,1}(X)$. See [24, Lemma 1 §IV.2.4] for a proof of the existence and uniqueness of the hermitian holomorphic connection.

2.3.5 Chern forms †

Let X be a complex manifold, $\underline{E} = (E, h)$ a holomorphic vector bundle on X endowed with a hermitian metric, and ∇ be its hermitian holomorphic connection. The *Chern form* of $\underline{E}$, which depends on both the holomorphic vector bundle E and the hermitian metric h, is given by

$$\mathrm{ch}(\underline{E}) := \mathrm{tr}_E \exp\left(\frac{-1}{2\pi i}\nabla^2\right) \in \bigoplus_{p \geq 0} \mathfrak{A}^{p,p}(X),$$

where tr_E is the trace in $\mathrm{End}(E)$.

One may also define a Chern class homomorphism c^k, with $c^k(\underline{E}) \in \mathfrak{A}^{k,k}(X)$, where

$$\sum_{k \geq 0} c^k(\underline{E}) t^k = \det\left(I + \frac{-1}{2\pi i}\nabla^2 t\right),$$

where $\mathrm{ch}(\underline{E})$ can then be defined in terms of the $c^i(\underline{E})$ as in §2.2.5.

By considering the exponential power series and the trace of the identity matrix, it follows that $\mathrm{ch}(\underline{E}) = \mathrm{rk}(E) + \chi$ for some $\chi \in \bigoplus_{p\geq 1} \mathfrak{A}^{p,p}(X)$. In particular, for line bundles endowed with a hermitian metric $\underline{L}$ we have that

$$\mathrm{ch}(\underline{L}) \in 1 + \bigoplus_{p\geq 1} \mathfrak{A}^{p,p}(X).$$

Chern forms are known to be d and d^c closed, and satisfy the following properties;

- $g^* \mathrm{ch}(\underline{E}) = \mathrm{ch}(g^*\underline{E})$ for holomorphic maps $g : Y \to X$ of complex manifolds.
- $\mathrm{ch}(\underline{E} \oplus \underline{F}) = \mathrm{ch}(\underline{E}) + \mathrm{ch}(\underline{F})$ for all hermitian vector bundles $\underline{E}, \underline{F}$.
- $\mathrm{ch}(\underline{E} \otimes \underline{F}) = \mathrm{ch}(\underline{E}) \cdot \mathrm{ch}(\underline{F})$ for all hermitian vector bundles $\underline{E}, \underline{F}$.
- If $\underline{\mathcal{O}}_X$ is the trivial bundle on X together with the trivial metric, then the curvature of the hermitian holomorphic connection is zero and so $\mathrm{ch}(\underline{\mathcal{O}}_X) = 1$.

For holomorphic vector bundles $\underline{E}_1, \dots, \underline{E}_m$ on X, and integer coefficients $k_1, \dots, k_m \in \mathbb{Z}$, we define the Chern form of the formal sum so that

$$\mathrm{ch}(k_1\underline{E}_1 + \cdots + k_m\underline{E}_m) := k_1 \mathrm{ch}(\underline{E}_1) + \cdots + k_m \mathrm{ch}(\underline{E}_m).$$

In a similar way, we consider the tensor product to distribute with formal sums of holomorphic vector bundles in the natural way, which also agrees with Chern forms. This allows us to for example write

$$\mathrm{ch}\left(\bigotimes_{i=1}^{m}(\underline{\mathcal{O}}_X - \underline{E}_i)\right) = \prod_{i=1}^{m}(1 - \mathrm{ch}(\underline{E}_i)).$$

2.3.6 Todd forms †

The *Todd form* $\mathrm{td}(\underline{E})$ of $\underline{E}$ is defined to be

$$\mathrm{td}(\underline{E}) = \frac{c^{\mathrm{rk}\,E}(\underline{E})}{\mathrm{ch}\left(\sum_{j=1}^{\mathrm{rk}\,E}(-1)^j \bigwedge^j \underline{E}^\vee\right)} \in \bigoplus_{p\geq 0} \mathfrak{A}^{p,p}(X).$$

Note that this is taking §2.2.7, a formula in the case of abstract λ-rings, and using it to construct an analogous definition in the case of complex geometry. One may define the Todd form in terms of the $c^i(\underline{E})$ directly following §2.2.6 instead, if preferred.

We will note the following property of the Todd forms; for orthogonal sums of vector bundles we have

$$\mathrm{td}(\underline{E} \oplus \underline{F}) = \mathrm{td}(\underline{E})\, \mathrm{td}(\underline{F}).$$

2.3.7 Secondary Chern and Todd characteristic forms †

In the algebraic case, given a short exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0,$$

we have the following identities in terms of Chern and Todd classes;

$$\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E''),$$

$$\mathrm{td}(E) = \mathrm{td}(E')\, \mathrm{td}(E'').$$

In the analytic case, if we have a short exact sequence of hermitian vector bundles

$$0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0,$$

we no longer have such identities in general unless the metric on $\underline{E}$ agrees with the metric on the orthogonal direct sum $\underline{E}' \oplus \underline{E}''$, in which case we say the short exact sequence is *orthogonally split*.

Instead there exists, following [24, §IV.3], *secondary Chern and Todd characteristic forms* $\widetilde{\mathrm{ch}}(0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0), \widetilde{\mathrm{td}}(0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0) \in \bigoplus_{p \geq 0} \widetilde{A}^p(X)$ satisfying the following identities;

$$\mathrm{dd^c}\, \widetilde{\mathrm{ch}}(0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0) = \mathrm{ch}(\underline{E}') - \mathrm{ch}(\underline{E}) + \mathrm{ch}(\underline{E}''),$$

$$\mathrm{dd^c}\, \widetilde{\mathrm{td}}(0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0) = \mathrm{td}(\underline{E}')\, \mathrm{td}(\underline{E}'') - \mathrm{td}(\underline{E}).$$

In particular, if $\underline{E} = (E, h_1)$ and $\underline{E}' = (E, h_2)$ are two hermitian vector bundles with the same underlying holomorphic vector bundle, but different metrics, then the difference between $\mathrm{ch}(\underline{E})$ and $\mathrm{ch}(\underline{E}')$ is determined by

$$\mathrm{ch}(\underline{E}') - \mathrm{ch}(\underline{E}) = \mathrm{dd^c}\widetilde{\mathrm{ch}}(0 \to \underline{E}' \to \underline{E} \to 0 \to 0).$$

For such cases, we write

$$\widetilde{\mathrm{ch}}(0 \to \underline{E}' \to \underline{E} \to 0) := \widetilde{\mathrm{ch}}(0 \to \underline{E}' \to \underline{E} \to 0 \to 0),$$

$$\widetilde{\mathrm{td}}(0 \to \underline{E}' \to \underline{E} \to 0) := \widetilde{\mathrm{td}}(0 \to \underline{E}' \to \underline{E} \to 0 \to 0).$$

2.3.8 Chern form classes †

Given a hermitian vector bundle $\underline{E}$, we may consider the differential form class of its Chern form $\mathrm{ch}(\underline{E})$ by considering the quotient $\bigoplus_{p\geq 0} \mathfrak{A}^{p,p}(X) \to \bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^{p}(X)$. We call this the Chern form class

$$\overline{\mathrm{ch}}(\underline{E}) \in \bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^{p}(X).$$

While the Chern form of $\underline{E} = (E, h^E)$ does depend on the hermitian metric h^E on E, from our note in §2.3.7, the difference between two hermitian bundles with the same underlying holomorphic vector bundle but with a different hermitian metric is the image of a form under the $\mathrm{dd^c}$ map. This difference is therefore in the image of ∂ and $\bar{\partial}$ and so it vanishes in $\bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^{p}(X)$. This means that $\overline{\mathrm{ch}}(\underline{E})$ is independent of the hermitian metric h^E on E, which is why most authors write "$\mathrm{ch}(E)$" for the Chern form class of $\underline{E}$. We will use a different notation to avoid confusion with the Chern character in the algebraic case.

We will similarly write $\overline{\mathrm{td}}(\underline{E})$ for the Todd form class, which are analogously given by the image of the Todd form $\mathrm{td}(\underline{E})$ under the quotient $\bigoplus_{p\geq 0} \mathfrak{A}^{p,p}(X) \to \bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^{p}(X)$. Note that Todd form classes are also independent of the hermitian metric, for the same reason as the Chern forms classes.

2.4 A cancellation of Chern characters and forms ※

Bismut's immersion formula (see §4.3.4) and Ma's branched covering formula (see §4.3.8) are both used to compute the norms of isomorphisms involving the determinant of cohomology endowed with the Quillen metric (see §4.2), in different geometric situations.

Both of these formulas involve the Chern form of a hermitian vector bundle $\underline{E}$ over a complex manifold X, $\operatorname{ch}(\underline{E}) \in \bigoplus_{p\geq 0} \mathfrak{A}^{p,p}(X)$, as well as its Chern form class, $\overline{\operatorname{ch}}(\underline{E}) \in \bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^{p}(X)$.

Bismut's immersion formula is used in §5 and both formulas are used in §6. The constructions used in these two sections involve an alternating product of hermitian line bundles $\underline{L}_1, \ldots, \underline{L}_m$, which can be informally expressed as either

$$\text{“}\bigoplus_{J\subseteq\{1,\ldots,m\}} (-1)^{|J|} \bigotimes_{i\in J} \underline{L}_i\text{”}, \qquad \text{or} \qquad \text{“}\bigotimes_{1\leq i\leq m} (\underline{\mathcal{O}}_X - \underline{L}_i)\text{”}.$$

When computing the norms of isomorphisms involving these constructions, as well as in similar algebraic situations such as the one considered in §3.3.7, we often observe a cancellation of Chern characters, forms, and form classes, purely due to the properties of the Chern character and Chern form, the grading of the Chow ring and the ring of (p,p)-forms, and the behavior of the pushforward with respect to this grading. These cancellations form the core of this paper, and are presented in Lemmas 2.5, 2.6 and 2.7. While these Lemmas relate to different constructions, their proof makes use of the same core ideas.

2.4.1 Cancellation of Chern characters ※

Lemma 2.5. *Let X and S be smooth schemes, $f : X \to S$ be a proper morphism of relative dimension n, $\ell \geq 0$, and $m \geq n+\ell+1$. Suppose that $L_1, \ldots, L_m$ are m line bundles on X. Let $P \in CH(X)_{\mathbb{Q}}$, and let $[\cdot]^\ell$ denote the degree ℓ part of an element of $CH(S)$. Then*

$$\left[\int_{X/S} P \operatorname{ch}\left(\bigotimes_{i=1}^{m} (\mathcal{O}_X - L_i)\right)\right]^\ell = 0.$$

Proof. Recall from §2.2.5 that for line bundles we have that

$$\mathrm{ch}(L_i) = 1 + c^1(L_i) + \frac{1}{2}c^1(L_i)^2 + \cdots \in 1 + CH(X)_{\mathbb{Q}}^{\geq 1}.$$

Therefore, by recalling that the Chern character is a ring homomorphism and $\mathcal{O}_X$ is the multiplicative unit in $K(X)$, it follows that

$$\mathrm{ch}(\mathcal{O}_X - L_i) = -c_1(L_i) - \frac{1}{2}c_1(L_i)^2 - \cdots \in CH(X)_{\mathbb{Q}}^{\geq 1}.$$

By taking into account the degrees of the terms, we deduce that

$$P\,\mathrm{ch}\left(\bigotimes_{i=1}^{m}(\mathcal{O}_X - L_i)\right) = P\prod_{i=1}^{m}\mathrm{ch}(\mathcal{O}_X - L_i) \in CH(X)_{\mathbb{Q}}^{\geq m}.$$

Recall from §2.2.2 that the pushforward $f_* = \int_{X/S}(\cdot) : CH(X)_{\mathbb{Q}} \to CH(S)_{\mathbb{Q}}$ preserves the dimension of cycles, while the Chow rings are graded according to the codimension, so that the pushforward maps elements in $CH^k(X)$ to elements in $CH^{k-n}(S)$. This implies that the pushforward maps elements in $CH(X)_{\mathbb{Q}}^{\geq m}$ to elements in $CH(S)_{\mathbb{Q}}^{\geq m-n} \subseteq CH(S)_{\mathbb{Q}}^{\geq \ell+1}$, since $m \geq n + \ell + 1$.

We therefore conclude that

$$\left[\int_{X/S} P\,\mathrm{ch}\left(\bigotimes_{i=1}^{m}(\mathcal{O}_X - L_i)\right)\right]^{\ell} = 0.$$

□

2.4.2 Cancellation of Chern forms ※

Lemma 2.6. *Let X be a complex manifold of dimension n, and $m \geq n + 1$. Suppose that $\underline{L}_1, \ldots, \underline{L}_m$ are hermitian line bundles on X. Let $P \in \bigoplus_{p\geq 0} \widetilde{A}^p(X)$. Then*

$$\int_{X/S} P\,\mathrm{ch}\left(\bigotimes_{i=1}^{m}(\underline{\mathcal{O}}_X - \underline{L}_i)\right) = 0.$$

Proof. Recall from §2.3.5 that for each hermitian vector bundle $\underline{L}_i$ we have that

$$\mathrm{ch}(\underline{L}_i) \in 1 + \bigoplus_{p\geq 1} A^{p,p}(X),$$

and that $\mathrm{ch}(\underline{\mathcal{O}}_X) = 1$, so that by our definition of the Chern form for a linear combination of hermitian vector bundles with integer coefficients we have that

$$\mathrm{ch}(\underline{\mathcal{O}}_X - \underline{L}_i) = 1 - \mathrm{ch}(\underline{L}_i) \in \bigoplus_{p\geq 1} A^{p,p}(X).$$

By recalling the graded ring structure of $\bigoplus_{p\geq 0} A^{p,p}(X)$ described in §2.3.2, we deduce that

$$\mathrm{ch}\left(\bigotimes_{i=1}^{m}(\underline{\mathcal{O}}_X - \underline{L}_i)\right) = \prod_{i=1}^{m} \mathrm{ch}(\underline{\mathcal{O}}_X - \underline{L}_i) \in \bigoplus_{p\geq m} A^{p,p}(X).$$

However, since $m \geq n+1 > \dim(X)$, it follows that $A^{p,p}(X) = 0$ for $p \geq m$. Therefore

$$\mathrm{ch}\left(\bigotimes_{i=1}^{m}(\underline{\mathcal{O}}_X - \underline{L}_i)\right) = 0,$$

and so

$$\int_{X/S} P\,\mathrm{ch}\left(\bigotimes_{i=1}^{m}(\underline{\mathcal{O}}_X - \underline{L}_i)\right) = 0.$$

□

Lemma 2.7. *Let X be a complex manifold of dimension n, and $m \geq n+1$. Suppose that $\underline{L}_1, \ldots, \underline{L}_m$ are hermitian line bundles on X. Let $P \in \bigoplus_{p\geq 0} \widetilde{A}^p(X)$. Then*

$$\int_{X/S} P\,\overline{\mathrm{ch}}\left(\bigotimes_{i=1}^{m}(\underline{\mathcal{O}}_X - \underline{L}_i)\right) = 0.$$

Proof. The same proof as for Lemma 2.6 applies to the case where we consider Chern form classes instead of Chern forms. □

2.5 Riemann Roch algebra †

Our research aims to extend Rössler's [23], which is titled *A refinement of the Adams Riemann Roch theorem in degree 1.*

In this section, we give a brief introduction of the Adams Riemann Roch theorem in §2.5.4, which describes the behavior of the Adams character ψ^k when combined with pushforward of vector bundles; as well as the Grothendieck Riemann Roch theorem in §2.5.3, which describes the behavior of the Chern character ch when combined with the pushforward.

Both of these theorems can be understood to form part of a family of results called Riemann Roch theorems, which we describe in §2.5.2; and in §2.5.1 we recall the notion of Riemann Roch functors, such as the Grothendieck K group or the Chow ring $CH_{\mathbb{Q}}$.

† This section summarizes definitions and statements from [12].

2.5.1 Riemann Roch functors †

Let $\mathcal{CR}ing$ denote the category of commutative rings, and $\mathcal{A}b\mathcal{G}rp$ denote the category of Abelian groups, and note that any commutative ring is an Abelian group under addition.

A *Riemann Roch functor* H on a category $\mathcal{C}$ consists of a mapping $H : X \mapsto H(X) \in \mathcal{CR}ing$ for each object in $\mathcal{C}$; which forms both a contravariant functor $\mathcal{C} \to \mathcal{CR}ing$ with pullback $f \mapsto f^*$, and a covariant functor $\mathcal{C} \to \mathcal{A}b\mathcal{G}rp$ with pushforward $f \mapsto f_*$, for each morphism f in the category $\mathcal{C}$; and so that the *projection formula* applies, for all $f : X \to S$ in $\mathcal{C}$ and all $x \in X$, $s \in S$, we have that

$$f_*(f^*s \cdot x) = s \cdot f_*(x).$$

We will sometimes write f^H and f_H for the pullback and pushforward morphisms in H when more than one Riemann Roch functors are used. For a morphism $f : X \to S$ we will often write $\int_{X/S} x$ to mean $f_*(x)$. For example, the projection formula states that for $x \in X$ and $s \in S$ we have that

$$\int_{X/S} f^*s \cdot x = s \cdot \int_{X/S} x.$$

2.5.2 Riemann Roch theorems †

Suppose that K and A are two Riemann Roch functors on a category $\mathcal{C}$. We may consider a natural transformation $\eta : K \Rightarrow A$ as contravariant functors into $\mathcal{CR}ing$. Such natural transformation will, by definition, consist of morphisms of rings $\eta_X : K(X) \to A(X)$ for each object $X \in \mathcal{C}$, which commutes with the pullback; for each $f : X \to S$ the following diagram commutes;

$$\begin{tikzcd}
K(S) \arrow[r, "\eta_S"] \arrow[d, "f^K"] & A(S) \arrow[d, "f^A"] \\
K(X) \arrow[r, "\eta_X"] & A(X).
\end{tikzcd}$$

We might hope that the morphisms η_X also commute with the pushforward, but that is not often the case. Instead, there can be a "correction term" that describes the way η_X interacts with the pushforward.

Given (K, η, A) as above, we say that *Riemann Roch holds* for $f : X \to S$ *with multiplier* $\tau_f \in A(X)$ if for all $x \in K(X)$ the following identity is satisfied

$$\eta_S(f_K(x)) = f_A(\eta_X(x) \cdot \tau_f),$$

or, using integral notation,

$$\eta_S\left(\int_{X/S} x\right) = \int_{X/S} \eta_X(x) \cdot \tau_f.$$

2.5.3 Grothendieck Riemann Roch theorem †

Recall from §2.1.1 that the Grothendieck groups $K(X)$ for smooth schemes X form a family of λ-ring which satisfy the splitting property. Recall from §2.2.2 that it is possible to construct a Chern class homomorphisms c_t on $K(X)$ with values in $CH(X)$, which in turn allows one to construct a Chern character

$$\mathrm{ch} : K(X) \to CH(X)_{\mathbb{Q}}.$$

Both K and CH can be shown to be Riemann Roch functors, and the *Grothendieck Riemann Roch theorem* (GRR) states that for proper morphisms $f : X \to S$ of non-singular reduced irreducible schemes, Riemann Roch holds for $(K, \mathrm{ch}, CH_{\mathbb{Q}})$ with multiplier $\mathrm{td}(\mathcal{T}_X)$, the Todd class of the tangent bundle of X. This means that for a vector bundle E on X we have that

$$\mathrm{ch}(f_*E) = \int_{X/S} \mathrm{td}(\mathcal{T}_X)\,\mathrm{ch}(E).$$

2.5.4 Adams Riemann Roch theorem †

The *Adams Riemann Roch theorem* (ARR) states that for $k \geq 1$, Riemann Roch holds for f with respect to $(K, \psi^k, \mathbb{Z}[\frac{1}{k}] \otimes K)$, where K is the Grothendieck group and ψ^k are the Adams operators, with multiplier $\theta^k(\Omega_f)^{-1}$, the multiplicative inverse in $\mathbb{Z}[\frac{1}{k}] \otimes K(X)$ of the Bott-Chern cannibalistic class of the cotangent bundle of f. Note that this inverse does not exist in general in $K(X)$ but does in $\mathbb{Z}[\frac{1}{k}] \otimes K(X)$.

This means that for a line bundle L, for which

$$\psi^k([L]) = [L]^k,$$

we have that

$$(f_*[L])^k = f_*\left([L]^k \cdot \theta^k([\Omega_f])^{-1}\right).$$

Rössler's canonical isomorphism of determinant line bundles, found in [23]—and which we aim to extend to the analytic case in this paper—is a refinement of the Adams Riemann Roch theorem in degree one. This is proved and explained in more detail in §3, below.

Chapter 3

A refinement of Adams Riemann-Roch

In [6], Deligne introduced various canonical isomorphisms in the case where we have a morphism of schemes $X \to S$ of relative dimension 1, and computed the norms of these isomorphisms in the analytic case. Since then, multiple authors have aimed to extend Deligne's results to higher relative dimensions.

His results are mainly stated in terms of the *Deligne pairing*, which can be written in terms of the determinant of cohomology. We will look into Ducrot's extension of the Deligne pairing in higher relative dimensions, the *intersection bundle* constructed using *cube structures* [7], in §5 where we extend Ducrot's results to the analytic case. Elkik provides a different construction of the intersection bundle in [8] and later considered the analytic case in [9].

Other authors have instead searched for canonical isomorphisms analogous to those constructed by Deligne in higher relative dimensions, at least in the algebraic case. This includes the works of Franke [11], Eriksson [10] and now Rössler in [23]. Rössler's construction is of particular interest since the methods used are much more suitable for extending his results to the analytic case.

When we were starting this research, Rössler showed us an early version of his [23], where a canonical isomorphism of determinant line bundles was the main result. Rössler mentioned to us that this isomorphism must be related to the Adams Riemann Roch theorem (ARR) from §2.5.4. While we were getting familiar with the analytic tools required to extend his results to the analytic case, we worked on finding the relationship between Rössler's ARR refinement and the ARR theorem.

This required some fairly interesting combinatorics, and, while not directly related to the rest of our research, the possibility exists that these ideas might come up once again when dealing with the full extension of Rössler's results to the analytic case. While a previous version of parts of this chapter can be found in our Transfer of Status, we have added the details required for a robust proof.

3.1 Canonical isomorphisms of determinant line bundles †

In this section, we recall the statement of Rössler's refinement of the Adams Riemann Roch (ARR) theorem in degree 1 [23, Theorem 1.1], and give a brief summary of its method of proof.

In §3.1.1 we recall the main result, which we call Rössler's ARR refinement. In §3.1.2 we give an explanation of the relationship between Rössler's result and the ARR theorem. In §3.1.3, we briefly recall the statement of [23, Theorem 6.1], which we call Rössler's involution formula; note that this theorem uses definitions from §4, and is looked at in more detail in §6.1. In §3.1.4 we give a brief overview of Rössler's proof of their ARR refinement. In §3.1.5 and §3.1.6 we recall some results that are related to Rössler's refinement of the ARR.

3.1.1 Rössler's ARR refinement †

The main result in [23] is Rössler's local refinement of the Adams Riemann Roch theorem in degree 1, [23, Theorem 1.1], which we'll call Rössler's ARR refinement. This theorem states the following;

Theorem 3.1. *Let X and S be locally Noetherian $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$-schemes, so that 2 is invertible on X and S. Let $f : X \to S$ be a locally projective and smooth morphism of constant relative dimension d. Let L be a line bundle on X and assume L is cohomologically flat over S, so that $R^i f_* L$ is locally free for $i \geq 0$.*

Then there exists a canonical isomorphism

$$\lambda(L)^{\otimes 2^{2d+2}} \cong \bigotimes_{j=0}^{2d} \lambda(L^{\otimes 2} \otimes \mathrm{Sym}^j(\Omega_f))^{\otimes(-1)^j \sum_{i=0}^{2d-j} \binom{2d+1}{i}},$$

compatible with base change to any locally Noetherian scheme.

Here and henceforth, λ denotes once again the determinant of cohomology introduced in §1.2.6, instead of the λ-ring operations. Note that in [23] a more formal notation is used for the statement of this theorem through the use of category theory, instead of the more informal notion of "canonical isomorphisms" which we introduced in §1.2.7. Recall that $\Omega_f = \Omega_{X/S}$ is the cotangent bundle.

3.1.2 A local refinement of Adams Riemann Roch in degree 1 †

Rössler's [23] is titled *A local refinement of the Adams-Riemann-Roch theorem in degree 1*. The canonical isomorphism in Rössler's ARR refinement from §3.1.1 is related to the equation in the Adam Riemann Roch theorem from §2.5.4, for the second Adams operator ψ^2,

$$(f_*[L])^2 = f_*\left([L]^2 \cdot \theta^2([\Omega_f])^{-1}\right) \in K(S)[\tfrac{1}{2}], \tag{3.1}$$

in the following way.

We first take determinants on either side, defining $\det([E]) := [\det(E)]$, and using the canonical isomorphisms $\det(E^{\otimes 2}) \cong \det(E)^{\otimes 2}$ as well as

$$\det(f_*[E]) = \det\left(\sum_{j\in\mathbb{Z}} (-1)^j [R^j f_* E]\right) = \bigotimes_{j\in\mathbb{Z}} \det([R^j f_* E])^{(-1)^j} = [\lambda(E)],$$

we obtain from (3.1) the following equality in the Picard group $\mathrm{Pic}(S)[\frac{1}{2}] \subset K(S)[\frac{1}{2}]$ of line bundles up to isomorphism

$$\lambda([L])^2 = \lambda([L]^2 \cdot \theta^2([\Omega_f])^{-1}).$$

Since we are working in $\mathrm{Pic}(S)[\frac{1}{2}]$ instead of only in $\mathrm{Pic}(S)$, taking powers of 2 is and invertible process, and so the above equation is equivalent to

$$\lambda([L])^{2^{2d+2}} = \lambda([L]^2 \cdot 2^{2d+1}\theta^2([\Omega_f])^{-1}),$$

by taking $(\cdot)^{2^{2d+1}}$ on both sides, and recalling that the determinant of cohomology sends sums to products.

We then use the following theorem, which is proved later in this section in §3.3,

Theorem 3.2. *Let X and S locally Noetherian* $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$*-schemes, $f : X \to S$ be a locally projective and smooth morphism of constant relative dimension d, and suppose that X has an ample line bundle. Let L be a line bundle on X which is cohomologically flat over S. Then we have an equality in* $\operatorname{Pic}(S)[\frac{1}{2}]$*,*

$$\lambda([L]^2 \cdot 2^{2d+1}\theta^2([\Omega_f])^{-1}) = \left[\bigotimes_{j=0}^{2d} \lambda(L^{\otimes 2} \otimes \operatorname{Sym}^j(\Omega_f))^{\otimes (-1)^j \sum_{i=0}^{2d-j} \binom{2d+1}{i}}\right].$$

This finally gives us an equation in $\operatorname{Pic}(S)[\frac{1}{2}]$ which implies, in terms of line bundles instead of elements of $\operatorname{Pic}(S)[\frac{1}{2}]$, that

$$\lambda(L)^{\otimes 2^{2d+2}} \quad \text{and} \quad \bigotimes_{j=0}^{2d} \lambda(L^{\otimes 2} \otimes \operatorname{Sym}^j(\Omega_f))^{\otimes (-1)^j \sum_{i=0}^{2d-j} \binom{2d+1}{i}}$$

are isomorphic to each other up to an indeterminate 2^∞-torsion line bundle, which is due to working over $\operatorname{Pic}(S)[\frac{1}{2}]$ instead of $\operatorname{Pic}(S)$.

Rössler's ARR refinement is a refinement of this last fact since it not only implies the existence of an isomorphism, but constructs a canonical isomorphism, compatible with base change. Rössler's ARR refinement furthermore removes the indeterminate 2^∞-torsion line bundle altogether.

3.1.3 Rössler's involution formula †

The core to Rössler's ARR refinement is a canonical isomorphism which we call Rössler's involution formula in this paper. It is [23, Theorem 6.1], and its statement uses equivariant constructions and definitions which we will recall later in §4; in particular λ^{eq}, the *equivariant determinant of cohomology*, is defined in §4.2.6.

Theorem 3.3 (Rössler's involution formula)**.** *Let $G = \mathbb{Z}/2$, and S and X be locally Noetherian G-equivariant schemes where 2 is invertible, where the action of G on S is trivial. Let $f : X \to S$ be a smooth, locally projective, separated, G-equivariant morphism of constant relative dimension d and of finite type. Suppose that the* *orbit of every point in X is contained in an affine subscheme. Let $\iota : Z \to X$ be the fixed point subscheme $Z = X_G$, and suppose the induced $Z \to S$ is flat, which implies Z is regularly immersed in X. Let $\mathcal{C}$ be the conormal bundle on Z of the immersion $Z \to X$, endowed with its canonical G-equivariant structure, and let M be a G-equivariant line bundle on X.*

We then have a canonical isomorphism

$$\lambda_X^{\mathrm{eq}}(M)^{\otimes 2^{d+1}} \cong \bigotimes_{j=0}^{d} \lambda_Z^{\mathrm{eq}}(\iota^*(M) \otimes \mathrm{Sym}^j(\mathcal{C}))^{\otimes \sum_{i=0}^{d} \binom{d+1}{i}},$$

compatible with base change.

The original goal of this paper was to extend Rössler's involution formula from the algebraic to the analytic case, which would then provide an extension to Rössler's ARR refinement in the analytic case. In §6, we give a partial extension of Theorem 3.3, for the case where Z is a Cartier divisor of X, by following Rössler's proof of Theorem 3.3 and considering the norms of canonical isomorphisms at each step.

If one were to follow Rössler's proof for the general case, where Z has a higher codimension than 1, one would need to study the blow-up of Z, and how Quillen metrics interact with this construction. We describe Rössler's proof of Theorem 3.3 in §6.1.

3.1.4 Rössler's method of proof for Theorem 3.1 †

To prove Theorem 3.1 for the map $g : Y \to S$ of relative dimension d, Rössler applies Theorem 3.3 to the case where $X = Y \times Y$, and the action of G on X swaps factors. This strategy uses the fact that the fixed point formula for an involution recovers Adams Riemann Roch for ψ^2, which was first noted in Nori's [22].

In this situation, $\iota : Z \to X$ is the diagonal of $Y \times Y$. In particular, the codimension of Z is equal to the dimension of Y, and thus is a Cartier divisor only when Y is a curve. This means that while our extension of Theorem 3.3 is applicable to X of any relative dimension over S, we would only be able to obtain an extension of Theorem 3.1 in the case where Y is of relative dimension 1 over S, which was studied by Deligne as we recall in §3.1.6.

3.1.5 Franke's and Eriksson's work †

In [10], Eriksson constructs a refinement of the Adams Riemann Roch formula using the homotopy theory of schemes. It gives an analogous result to Rössler's ARR refinement but includes an undetermined 2^∞ torsion line bundle in the isomorphism, and the resulting linear combination in the symmetric powers of Ω_f would a priori depend on the dimension of the total space.

In [11], Franke gives a refinement of the Adams Riemann Roch formula using higher K-theory. In a similar way to Eriksson's work, the result is analogous to Rössler's ARR refinement, but in Franke's work's case an undetermined torsion bundle (not necessarily 2^∞) is included.

However, while the results of Rössler's ARR refinement have no undetermined line bundles, our main interest is in Rössler's method of proof for Theorems 3.1 and 3.3, which involves neither higher K-theory nor the homotopy theory of schemes (both requiring a vast categorical apparatus), and instead relies on elementary and explicitly constructed canonical isomorphisms at each step. This makes it feasible to compute the norm of the canonical isomorphism in Rössler's ARR refinement in the analytic case, where both sides are endowed with the Quillen metric.

3.1.6 Deligne's *Le détérminant de la cohomologie* †

In [6], Deligne presents us with results for the case where the relative dimension of X over S is 1, and program encouraging us to extend his results to higher relative dimensions. Franke's, Eriksson's and Rössler's work all are related to this program, in their goal of refining Adams Riemann Roch in higher relative dimension.

In [6] Deligne also constructs and presents the Deligne pairing, which we describe in the introduction to §5. Deligne once again proposed that this construction could be extended to higher relative dimensions as the *intersection bundle*, and at least two different authors have provided such constructions; Elkik in [8], and Ducrot in [7].

It is worth noting that in [6], Deligne also considers the analytic case, where one endows the determinant of cohomology with the Quillen metric. Of the authors we have mentioned in this section, only Elkik has provided a similar analogue for her work in [9], and one of Rössler's motivations for relying on elementary methods in his proof of Theorem 3.1 was to make the extension of his results to the analytic case feasible.

Interestingly, Rössler's proof of Theorem 3.3 makes use of Ducrot's intersection bundle, which does have a different construction from Elkik's more geometrical approach. In this paper, we provide an analogue to Ducrot's construction in the analytic case in §5, and a partial analogue to Rössler's involution formula in §6.

3.2 Combinatorial identities ※

In this section we collect all the combinatorial results that will be used throughout this paper. Most of them are of immediate use to §3.3, though the polynomial identity of §3.2.2 will instead be used in §6.

In §3.2.1 we present a combinatorial result, which was our first contribution to this research; a simplification which made its way into Rössler's paper as [23, Lemma 6.3]. In §3.2.2 we introduce a polynomial identity related to this simplified combinatorial expression.

Rössler's canonical isomorphism involves $\mathrm{Sym}^j(\Omega_f)$ terms, whereas the Adams Riemann Roch (ARR) theorem has instead a $\theta^2(\Omega_f)^{-1}$ term. If the cotangent bundle Ω_f splits into line bundles, we can write both of these in terms of symmetric polynomials of the line bundles, which means through λ-ring methods that these can ultimately be expressed as polynomial expressions of the $\bigwedge^i \Omega_f$, even in the case where Ω_f does not split. Either way, we're ultimately working with symmetric polynomials.

The symmetric power $\mathrm{Sym}^j(\bigoplus_{i=1}^m L_i)$ can be written in terms of the complete homogeneous symmetric polynomial of degree j in m variables, which is the sum of all monomials of degree j in m variables. We introduce these polynomials in §3.2.4. We study these complete homogeneous symmetric polynomials by using non-strictly increasing sequences of finite length, each of which corresponds to a different monomial. We introduce these increasing sequences in §3.2.3.

Finally, in §3.2.5, we consider a combinatorial identity which is the key to going from Rössler's ARR refinement to the ARR theorem.

※ We are unfamiliar with the literature surrounding combinatorics, as such it is possible that previous authors have found these results, and that we are unaware of them.

3.2.1 A combinatorial simplification ※

We could say that this result is our first contribution to our study of Rössler's [23], which was included in his work as [23, Lemma 6.3].

Lemma 3.1. *Suppose that* $0 \leq j \leq n$. *Then*

$$\sum_{k=0}^{n-j} 2^{n-j-k} \binom{j+k}{j} = \sum_{k=0}^{n-j} \binom{n+1}{k}.$$

Proof. We use induction on $n \geq j$. In the base case, where $n = j$, k may only be 0, and so the statement follows from the identity

$$2^0 \binom{j}{j} = 1 = \binom{n+1}{0}.$$

Now suppose for induction that the statement holds for $n-1$. Then starting with the right hand side we have

$$\begin{aligned}
\sum_{k=0}^{n-j} \binom{n+1}{k} &= 1 + \sum_{k=1}^{n-j} \left[\binom{n}{k-1} + \binom{n}{k} \right] \\
&= \left[\sum_{(k-1)=0}^{(n-1)-j} \binom{(n-1)+1}{(k-1)} \right] + \left[1 + \sum_{k=1}^{(n-1)-j} \binom{(n-1)+1}{k} \right] + \binom{n}{n-j} \\
&= 2 \left[\sum_{k=0}^{(n-1)-j} \binom{(n-1)+1}{k} \right] + \binom{n}{j} \\
&= 2 \left[\sum_{k=0}^{(n-1)-j} 2^{(n-1)-j-k} \binom{j+k}{j} \right] + 2^{n-j-(n-j)} \binom{j+(n-j)}{j} \\
&= \sum_{k=0}^{n-j} 2^{n-j-k} \binom{j+k}{j}.
\end{aligned}$$

□

In essence, this simplification involves two steps when looking at Pascal's triangle; writing down the binomial coefficients of the form

$$\binom{n+1}{0}, \binom{n+1}{1}, \dots, \binom{n+1}{k}, \dots, \binom{n+1}{n-j},$$

in terms of binomial coefficients of the form

$$\binom{j}{0}, \binom{j+1}{1}, \dots, \binom{j+k}{k}, \dots, \binom{n}{n-j},$$

by using expansions of the form $\binom{a+1}{b} = \binom{a}{b-1} + \binom{a}{b}$—which introduces the factors of 2^{n-j-k}—and finally using the symmetry $\binom{a}{b} = \binom{a}{a-b}$ to replace the terms above with binomial coefficients of the form

$$\binom{j}{j}, \binom{j+1}{j}, \dots, \binom{j+k}{j}, \dots, \binom{n}{j}.$$

3.2.2 Polynomial with combinatorial sum coefficients ※

Consider the family of polynomials whose coefficients are the combinatorial sums presented in §3.2.1,

$$p_n(x) := \sum_{j=0}^{n} \left(\sum_{k=0}^{n-j} \binom{n+1}{k} \right) x^j.$$

The following result allows us to easily compute the combinatorial sums for multiple j, by using computer algebra systems or otherwise. We will also use it in our computation of the norm of the canonical isomorphism due to Rössler's involution formula in §6.2.7.

Lemma 3.2.

$$p_n(x) = \frac{2^{n+1} - (1+x)^{n+1}}{1-x}.$$

Proof. When we multiply $p_n(x)$ with $(1-x)$, we are left with a polynomial whose coefficients are the difference between two partial sums, of which one has one more term than the other. This effectively leaves us with the terms of the partial sums as the new coefficients, which coincide with the binomial coefficients.

We compute;

$$
\begin{aligned}
(1-x)\,p_n(x) &= (1-x)\left(\sum_{j=0}^{n}\left(\sum_{k=0}^{n-j}\binom{n+1}{k}\right)x^j\right)\\
&= \left[\sum_{j=1}^{n}\left(\sum_{k=0}^{n-j}\binom{n+1}{k} - \sum_{k=0}^{n-(j-1)}\binom{n+1}{k}\right)x^j\right]\\
&\quad + \left(\sum_{k=0}^{n}\binom{n+1}{k}\right)x^0 - x^{n+1}\\
&= -\left[\sum_{j=1}^{n}\binom{n+1}{n-(j-1)}x^j\right] + (2^{n+1}-1) - x^{n+1}\\
&= 2^{n+1} - \left(1 + \left[\sum_{j=1}^{n}\binom{n+1}{j}x^j\right] + x^{n+1}\right)\\
&= 2^{n+1} - \sum_{j=0}^{n+1}\binom{n+1}{j}x^j\\
&= 2^{n+1} - (1+x)^{n+1}.
\end{aligned}
$$

□

For the proof of [23, Theorem 6.1], Rössler introduces the family of polynomials

$$P_n(t) := \sum_{i=0}^{n} 2^{n-i}(2-t)^i,$$

and proves the identity

$$tP_n(t) = 2^{n+1} - (2-t)^{n+1}.$$

Lemma 3.3. *The relationship between P_n and p_n is that*

$$p_n(x) = P_n(1-x).$$

Proof. We make use of Lemma 3.1, and compute, starting on the right hand side, and using the substitution $i = j + k$,

$$\begin{aligned}
P_n(1-x) &= \sum_{i=0}^{n} 2^{n-i}(2-(1-x))^i \\
&= \sum_{i=0}^{n} 2^{n-i}(1+x)^i \\
&= \sum_{i=0}^{n} 2^{n-i} \sum_{j=0}^{i} \binom{i}{j} x^j \\
&= \sum_{j=0}^{n} \sum_{i=j}^{n} 2^{n-i} \binom{i}{j} x^j \\
&= \sum_{j=0}^{n} \left[\sum_{k=0}^{n-j} 2^{n-k-j} \binom{k+j}{j} \right] x^j \\
&= \sum_{j=0}^{n} \left[\sum_{k=0}^{n-j} \binom{n+1}{k} \right] x^j \\
&= p_n(x).
\end{aligned}$$

□

3.2.3 Increasing sequences ※

If we wish to sum over all monomials of a given degree k, in m variables $X_1, \ldots, X_m$, we can do so by considering the sum of the monomial $X_{i_1} X_{i_2} \cdots X_{i_k}$ over all $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq m$. We thus study the "increasing sequences" $[i_1, \ldots, i_k]$.

Let $[n] = \{1, 2, \ldots, n\}$ denote the ordered set of integers from 1 to n. We now consider a category $\mathcal{IS}eq_m$ of *increasing sequences up to* m. The objects of $\mathcal{IS}eq_m$ are non-strictly increasing morphisms of ordered sets $\alpha : [\ell(\alpha)] \to [m]$, for some integer $\ell(\alpha) \geq 0$ called the *length* of the increasing sequence α. We let $\mathcal{IS}eq_{m,k}$ be the subset of increasing sequences up to m of length k.

We write

$$[\alpha(1), \alpha(2), \ldots, \alpha(\ell(\alpha))],$$

to represent an increasing sequence in $\mathcal{IS}eq_m$. For example, $[\,]$, $[1, 2, 3]$, $[1, 1, 1, 2]$ and $[1, 1, 3, 3]$ are all objects in $\mathcal{IS}eq_3$, while $[2, 1]$, $[2, 3, 4]$ aren't.

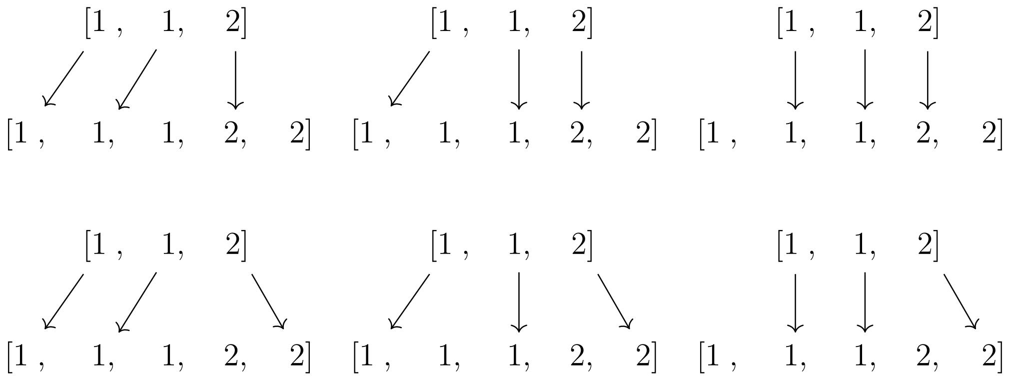

Figure 3.1: The six embeddings of $[1, 1, 2]$ into $[1, 1, 1, 2, 2]$.

Given a monomial of degree k in m variables $X_1, \dots, X_m$, there exists a unique increasing sequence up to m of length k, $\alpha \in \mathcal{ISeq}_{m,k}$, such that the monomial can be expressed as

$$X_{\alpha(1)} X_{\alpha(2)} \cdots X_{\alpha(k)}.$$

For example, the monomial $X_1^3 X_2 X_4^2$ corresponds to the increasing sequence $[1, 1, 1, 2, 4, 4]$.

The morphisms in $\mathcal{ISeq}_m$ between an increasing sequence β of length j and an increasing sequence α of length k are the *embeddings* $\eta : \beta \to \alpha$, consisting of *strictly* increasing morphisms $\eta : [j] \to [k]$ such that the following diagram commutes:

$$\begin{tikzcd}
{[j]} \arrow[rr, "\eta"] \arrow[dr, "\beta"'] & & {[k]} \arrow[dl, "\alpha"] \\
& {[m]} &
\end{tikzcd}$$

Note that there exists no morphisms $\beta \to \alpha$ if the length of β is longer than the length of α. As an example, there exists 6 embeddings of $[1, 1, 2]$ into $[1, 1, 1, 2, 2]$, which are described in Figure 3.1.

Lemma 3.4. *Fix an increasing sequence $\beta \in \mathcal{ISeq}_m$ of length j. Then the number of embeddings from β into any increasing sequence of length k depends only on m, k and j and is given by*

$$|\{\beta \to \alpha \ s.t.\ \alpha \in \mathcal{ISeq}_{m,k}\}| = \binom{m+k-1}{k-j}.$$

Proof. We adapt a strategy that is often used to count the number of distinct monomials of a fixed degree. We find a correspondence between increasing sequences up to m of length k and sequences of k *stars* "$*$" and $(m-1)$ *bars* "$|$". We interpret these sequence of stars and bars as "instructions on how to build an increasing sequence", with a star indicating that one should add another copy of a number to the increasing sequence, and a bar indicating that we must move to the next number. For example, $[1,1,1,2,2] \in \mathcal{IS}eq_{3,5}$ corresponds to the sequence of stars and bars $(*\,*\,*|\,*\,*|\,)$, and $[2,4,4,5] \in \mathcal{IS}eq_{5,4}$ corresponds to $(\,|\,*\,|\,|\,*\,*\,|\,*)$. The number of these sequences of stars and bars can be found by simply choosing k of the $m+k-1$ symbols to be stars instead of bars, and as such, by using this correspondence, we have that

$$|\,\mathcal{IS}eq_{m,k}\,| = \binom{m+k-1}{k}.$$

While the above strategy allows one to count the number of distinct monomials of degree k in m variables by using the correspondence between such monomials and objects in $\mathcal{IS}eq_{m,k}$, a small alteration allows us to count the size of the set of embeddings from a fixed $\beta \in \mathcal{IS}eq_{m,j}$ to an increasing sequence of length k.

Fix a $\beta \in \mathcal{IS}eq_{m,j}$, and consider its stars and bars representation, where there are j stars and $(m-1)$ bars. We then add $(k-j)$ *balls* "$\bullet$" to the sequence, and we note that each choice of placing of the balls corresponds to a unique embedding of β into an increasing sequence $\alpha \in \mathcal{IS}eq_{m,k}$, whose representation is given by forgetting the distinction between stars and balls. The balls will correspond to the elements in α not contained in the image of the embedding.

For example, suppose that $m=3$, $k=5$, $j=3$, $\beta=[1,1,2]$ and $\alpha=[1,1,1,2,2]$. The representation of β is $(*\,*\,|\,*\,|\,)$, and α's is $(*\,*\,*|\,*\,*|\,)$. Each embedding $\beta \to \alpha$ in Figure 3.1 corresponds respectively to each of the following representations using stars, bars and balls;

$$(*\,*\,\bullet|\,*\,\bullet|\,) \quad (*\,\bullet\,*|\,*\,\bullet|\,) \quad (\bullet\,*\,*|\,*\,\bullet|\,)$$

$$(*\,*\,\bullet|\,\bullet\,*|\,) \quad (*\,\bullet\,*|\,\bullet\,*|\,) \quad (\bullet\,*\,*|\,\bullet\,*|\,)$$

while, for example, $(*\,*\,|\,*\,|\,\bullet\,\bullet)$ corresponds to the unique embedding of β into $[1,1,2,3,3]$.

By a similar argument as for the stars and bars case, the number of embeddings from β to a increasing sequences of length k is equal to the number of ways of choosing $(k-j)$ balls among $m+k-1$ symbols that could be stars, bars or balls. Note that the placement of the stars and bars between each other is determined by β, but does not affect the number of embeddings of β into elements in $\mathcal{IS}eq_{m,k}$, which is

$$|\{\beta \to \alpha \text{ s.t. } \alpha \in \mathcal{IS}eq_{m,k}\}| = \binom{m+k-1}{k-j}.$$

□

For an example of Lemma 3.4 in practice; with $m = k = 3$, $[1,1]$ can be embedded thrice into $[1,1,1]$, and once into $[1,1,2]$ and $[1,1,3]$ each, for a total of 5 embeddings; while $[1,2]$ can be embedded twice into $[1,1,2]$ and $[1,2,2]$ each, as well as once into $[1,2,3]$, for another total of 5 embeddings, but obtained by adding $2+2+1$ instead of $3+1+1$.

3.2.4 Complete homogeneous symmetric polynomials ※

The *k-th complete homogeneous symmetric polynomial* in m variables $h_{m,k}$, often written h_k if the number of variables is clear, is the sum of all monomials of degree k in m variables;

$$h_{m,k}(X_\bigstar) := \sum_{1\le i_1\le\cdots\le i_k\le m} X_{i_1}\cdots X_{i_k}.$$

For example

$$h_2(x,y,z) = x^2 + xy + xz + y^2 + yz + z^2,$$

$$h_k(x,y) = x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k.$$

A standard result in the theory of symmetric polynomials is that one might construct a power series whose coefficients are complete homogeneous symmetric polynomials by taking the product of the geometric series in each variable;

$$\sum_{k\ge 0} h_{m,k}(X_\bigstar)t^k = \prod_{i=1}^{m}\sum_{k=0} X_i^k t^k = \prod_{i=1}^{m}(1-X_i t)^{-1}. \tag{3.2}$$

While the complete homogeneous symmetric polynomials do not generate all the symmetric polynomials like the elementary symmetric polynomials do, we are able to write the expression $h_{m,k}(1-X_\bigstar)$ in terms of the j-th complete homogeneous symmetric polynomials for $0 \leq j \leq k$.

Lemma 3.5.

$$h_{m,k}(1-X_\bigstar) = \sum_{j=0}^{k} (-1)^j \binom{m+k-1}{k-j} h_{m,j}(X_\bigstar).$$

Proof. Note that $(-1)^j h_{m,j}(X_\bigstar) = h_{m,j}(-X_\bigstar)$ since the $h_{m,j}$ are homogeneous of degree j. By replacing $X_\bigstar$ with $-X_\bigstar$, it follows that the statement to prove is equivalent to proving

$$h_{m,k}(1+X_\bigstar) = \sum_{j=0}^{k} \binom{m+k-1}{k-j} h_{m,j}(X_\bigstar).$$

By the correspondence between monomials and increasing sequences described in §3.2.3, since the complete symmetric homogeneous polynomial in degree k and m variables $X_1, \ldots, X_m$ is given by adding all monomials of degree k in m variables, and each such monomial corresponds to an increasing sequence up to m of length k, we have that

$$h_{m,k}(X_\bigstar) = \sum_{\alpha \in \mathcal{IS}eq_{m,k}} X_{\alpha(1)} \cdots X_{\alpha(k)}. \tag{3.3}$$

Now, fix an increasing sequence $\alpha \in \mathcal{IS}eq_m$, and consider the product

$$(1+X_{\alpha(1)})(1+X_{\alpha(2)}) \cdots (1+X_{\alpha(\ell(\alpha))}).$$

When expanding, we add over monomials obtained by making a choice for each factor $(1+X_{\alpha(i)})$ whether we pick the 1 or the $X_{\alpha(i)}$ term. Each of such choices corresponds to an immersion $\beta \to \alpha$ for some $\beta \in \mathcal{IS}eq_m$. This allows us to write

$$\prod_{i=1}^{\ell(\alpha)} (1+X_{\alpha(i)}) = \sum_{\beta \to \alpha} X_{\beta(1)} \cdots X_{\beta(\ell(\beta))}, \tag{3.4}$$

where the sum is over all embeddings $\beta \to \alpha$ for some $\beta \in \mathcal{IS}eq_m$, not over all the $\beta \in \mathcal{IS}eq_m$.

For example, the immersion

$$\begin{tikzcd}
 & {[1\,,} \arrow[dl] & {1,} \arrow[d] & {2]} \arrow[dr] & \\
{[1\,,} & {1,} & {1,} & {2,} & {2]}
\end{tikzcd}$$

corresponds to the choice of taking the first and third X_1's, as well as the second X_2, and 1's in the other factors, when expanding

$$(1+X_1)(1+X_1)(1+X_1)(1+X_2)(1+X_2),$$

which gives us an $X_1^2X_2$ summand. Note that we then may find the coefficient of the $X_1^2X_2$ term in the expansion of the product above by counting the number of distinct embeddings from $[1,1,2]$ to $[1,1,1,2,2]$, which we know to be 6 from Figure 3.1.

We now compute, using (3.3),

$$h_{m,k}(1+X_\bigstar) = \sum_{\alpha\in\mathcal{IS}eq_{m,k}} (1+X_{\alpha(1)})\cdots(1+X_{\alpha(k)}),$$

Then we use (3.4),

$$= \sum_{\alpha\in\mathcal{IS}eq_{m,k}} \sum_{\beta\to\alpha} X_{\beta(1)}\cdots X_{\beta(\ell(\beta))},$$

We then double count the morphisms $\beta\to\alpha$, above we count the α first, below we instead count the β first;

$$= \sum_{j=0}^{k} \sum_{\beta\in\mathcal{IS}eq_{m,j}} \sum_{\beta\to\alpha}^{\alpha\in\mathcal{IS}eq_{m,k}} X_{\beta(1)}\cdots X_{\beta(\ell(\beta))},$$

Then, noting that the summand depends only on β, we can sum over all α,

$$= \sum_{j=0}^{k} \sum_{\beta\in\mathcal{IS}eq_{m,j}} |\{\beta\to\alpha \text{ s.t. } \alpha\in\mathcal{IS}eq_{m,k}\}|\, X_{\beta(1)}\cdots X_{\beta(\ell(\beta))},$$

We may then invoke Lemma 3.4,

$$= \sum_{j=0}^{k} \sum_{\beta\in\mathcal{IS}eq_{m,j}} \binom{m+k-1}{k-j} X_{\beta(1)}\cdots X_{\beta(\ell(\beta))},$$

And finally rearrange and use (3.3) once more,

$$= \sum_{j=0}^{k} \binom{m+k-1}{k-j} \sum_{\beta \in \mathcal{IS}eq_{m,j}} X_{\beta(1)} \cdots X_{\beta(\ell(\beta))},$$
$$= \sum_{j=0}^{k} \binom{m+k-1}{k-j} h_{m,j}(X_\bigstar).$$

□

3.2.5 An alternating sum of a product of binomial coefficients ※

Consider the following alternating combinatorial sum (this is [27, A239473]);

$$T(n,i) := \sum_{j=i}^{n} (-1)^{i+j} \binom{j}{i}.$$

Note that, setting $T(n,i) = 0$ for $i < 0$ or $i > n$, we have that, for $n \geq 0$ and $0 \leq i \leq n$:

$$T(n,n) = 1; \qquad T(n,0) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases};$$
$$\text{and} \quad T(n+1,i) = T(n,i-1) - T(n,i).$$

This makes $T(n,i)$ generate a “Triangle of Pascal” where the right side is made of 1’s, the left side alternates 1’s and 0’s, and instead of adding two values to get the one beneath, we take the difference. See Figure 3.2.

We now consider the following alternating sum of products of binomial coefficients, which can be expressed in terms of powers of 2 or the $T(n,i)$.

Lemma 3.6. *Let $d \geq 1$ and $0 \leq i \leq 2d$. We then have*

$$\sum_{\ell=0}^{2d-i} \sum_{k=0}^{2d-i-\ell} (-1)^{\ell} \binom{d+i+\ell-1}{\ell} \binom{2d+1}{k} = \begin{cases} 2^{2+1-i} & \text{if } 0 \leq i \leq d+1 \\ T(d-2, i-d-2) & \text{if } d+2 \leq i \leq 2d \end{cases}$$

Proof. The sum in the left hand side can be rearranged by “adding along the diagonal” of a right triangle instead of along the sides, as is shown in Figure 3.3.

$$\sum_{\ell=0}^{2d-i} \sum_{k=0}^{2d-i-\ell} F(k,\ell) = \sum_{p=0}^{2d-i} \sum_{k,\ell \geq 0}^{k+\ell=p} F(k,\ell).$$

n							
0	1						
1	0	1					
2	1	-1	1				
3	0	2	-2	1			
4	1	-2	4	-3	1		
5	0	3	-6	7	-4	1	
6	1	-3	9	-13	11	-5	1
	0	1	2	3	4	5	6
				i			

Figure 3.2: $T(n,i)$ for $0 \leq i \leq n \leq 6$.

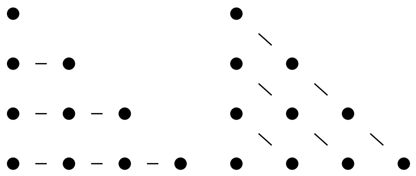

Figure 3.3: $\sum_{\ell=0}^{N}\sum_{k=0}^{N-\ell}F(k,\ell)=\sum_{p=0}^{N}\sum_{k,\ell\geq 0}^{k+\ell=p}F(k,\ell)$. In this picture, ℓ is the column from left to right, and k is the row from bottom to top.

Recall the following two standard binomial expansions, for $n, m \geq 0$,

$$\begin{aligned}(1+t)^n &= \sum_{k=0}^{\infty}\binom{n}{k}t^k\\ (1+t)^{-m} &= \sum_{\ell=0}^{\infty}(-1)^\ell\binom{m+\ell-1}{\ell}t^\ell\end{aligned}$$

Therefore, by considering the product

$$(1+t)^n(1+t)^{-m} = (1+t)^{n-m},$$

we deduce the following identity by considering the p-th coefficient of a product of power series;

$$\sum_{k,\ell\geq 0}^{k+\ell=p}(-1)^\ell\binom{m+\ell-1}{\ell}\binom{n}{k} = \begin{cases}\binom{n-m}{p} & \text{if } n \geq m\\ (-1)^p\binom{m-n+p-1}{p} & \text{if } n < m\end{cases}$$

Now let $n = 2d+1$ and $m = d+i$, so that $n-m = d-i+1$ and $m-n = i-d-1$, and, starting by the left hand side, we have

$$\begin{aligned}
&\sum_{\ell=0}^{2d-i}\sum_{k=0}^{2d-i-\ell}(-1)^{\ell}\binom{d+i+\ell-1}{\ell}\binom{2d+1}{k}\\
&\qquad=\sum_{p=0}^{2d-i}\sum_{k,\ell\geq 0}^{k+\ell=p}(-1)^{\ell}\binom{[d+i]+\ell-1}{\ell}\binom{[2d+1]}{k}\\
&\qquad=\sum_{p=0}^{2d-i}\begin{cases}\binom{[d-i+1]}{p} & \text{if } \ d+1\geq i\\ (-1)^{p}\binom{[i-d-1]+p-1}{p} & \text{if } \ d+1<i\end{cases}
\end{aligned}$$

We first consider the case where $i \leq d+1$;

$$\begin{aligned}
\sum_{p=0}^{2d-i}\binom{d-i+1}{p} &= \sum_{p=0}^{d-i+1}\binom{d-i+1}{p}\\
&= 2^{d-i+1}.
\end{aligned}$$

Now, in the case where $i > d+1$, or $i \geq d+2$, we have that, using the substitution $q = i-d+p-2$,

$$\begin{aligned}
\sum_{p=0}^{2d-i}(-1)^{p}\binom{[i-d-1]+p-1}{p} &= \sum_{p=0}^{2d-i}(-1)^{p}\binom{i-d+p-2}{p}\\
&= \sum_{p=0}^{2d-i}(-1)^{p}\binom{i-d+p-2}{i-d-2}\\
&= \sum_{q=[i-d-2]}^{[d-2]}(-1)^{q-[i-d-2]}\binom{q}{[i-d-2]}\\
&= T(d-2, i-d-2).
\end{aligned}$$

□

3.3 Proof of Theorem 3.2 ※

Recall Theorem 3.2, where we let X and S locally Noetherian $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$-schemes, $f : X \to S$ be a locally projective and smooth morphism of constant relative dimension d, and suppose that X has an ample line bundle. We then let L be a line bundle on X which is cohomologically flat over S. And then we have an equality in $\mathrm{Pic}(S)[\frac{1}{2}]$,

$$\lambda([L]^2 \cdot 2^{2d+1}\theta^2([\Omega_f])^{-1}) = \left[\bigotimes_{j=0}^{2d} \lambda(L^{\otimes 2} \otimes \mathrm{Sym}^j(\Omega_f))^{\otimes(-1)^j \sum_{i=0}^{2d-j} \binom{2d+1}{i}}\right].$$

In §3.3.1 we define an equivalence relation on $K(X)[\frac{1}{2}]$ that simplifies the statement of the theorem. In §3.3.2 we consider a splitting cotangent bundle. In §3.3.3 we write the inverse of the Bott-Chern cannibalistic class of the cotangent bundle in terms of its splitting, and in §3.3.4 we write the symmetric powers of the cotangent bundle in terms of its splitting.

The main part of the proof is found in §3.3.5, where we make use of the Lemmas in §3.2. This proof makes use of a Lemma which is proved in §3.3.6 when the cotangent bundle splits in $K(X)$ and in §3.3.7 when we must consider an extension of $K(X)$ where the cotangent bundle splits. While the second proof implies the first, we provide both since the first showcases methods that will be used in the later sections, and the second hints at the difficulties one may face when considering cases of higher codimension in §6.

3.3.1 A relation in $K(X)[\frac{1}{2}]$ ※

Consider the relation $\sim$ on $K(X)[\frac{1}{2}]$ where

$$\alpha \sim \beta \quad \text{if and only if} \quad \lambda([L]^2 \cdot \alpha) = \lambda([L]^2 \cdot \beta) \in K(S)[\tfrac{1}{2}].$$

This is an equivalence relation.

Our aim in this section is to prove Theorem 3.2, which states that

$$2^{2d+1}\theta^2([\Omega_f])^{-1} \sim \sum_{j=0}^{2d}(-1)^j \sum_{i=0}^{2d-j} \binom{2d+1}{i} \mathrm{Sym}^j([\Omega_f]).$$

3.3.2 A splitting of $[\Omega_f]$ ※

Recall from §2.1.7 that the family of λ-rings of Grothendieck groups of vector bundles $K(X)$ for smooth schemes X satisfies the splitting property. This means that there exists a λ-ring extension $K(X)'$ where $[\Omega_f]$ splits as

$$[\Omega_f] = M_1 + \cdots + M_d,$$

where $d = \mathrm{rk}[\Omega_f]$ is the constant relative dimension of f, and M_i are elements of $K(X)'$.

Note that from the splitting principle, we know that there exists some smooth scheme X' over X such that $K(X)' = K(X')$. By the construction of the extension described in §2.1.7, we note that $\pi : X' \to X$ is projective since it is the composition of projective morphisms of schemes. Since X has an ample line bundle and π is projective, by [26, Lemma 37.50.1], X' too has an ample line bundle. By using Lemma 2.1 from §2.1.8, we have that for any line element M_i in $K(X)'$, $(1 - M_i)$ is nilpotent.

Finally, recall from §2.1.5 that any symmetric polynomial expression of the M_i's is in $K(X)$, is independent of the choice of splitting, and can be computed using the $\lambda^i([\Omega_f]) = [\bigwedge^i \Omega_f] \in K(X)$. We can thus consider symmetric polynomial expressions of the M_i as elements of $K(X)[\frac{1}{2}]$ without need to move to $K(X)'[\frac{1}{2}]$.

3.3.3 The inverse of the Bott-Chern cannibalistic class $\theta^2([\Omega_f])^{-1}$ ※

From §2.2.9, for each M_i we have that $\theta^2(M_i)^{-1}$ in $K(X)'[\frac{1}{2}]$ is given by

$$\theta^2(M_i)^{-1} = \frac{1}{2} \sum_{k \geq 0} \left(\frac{1 - M_i}{2} \right)^k,$$

which is well defined since for each M_i, $(1 - M_i)$ is nilpotent.

Now, given that θ^2 is defined multiplicatively, $\theta^2([\Omega_f])^{-1}$ in $K(X)[\frac{1}{2}]$ is given by

$$\begin{aligned} \theta^2([\Omega_f])^{-1} &= \prod_{i=1}^{d} \frac{1}{2} \sum_{k \geq 0} \left(\frac{1 - M_i}{2} \right)^k \\ &= \frac{1}{2^d} \sum_{k \geq 0} h_{d,k} \left(\frac{1 - M_\bigstar}{2} \right) \\ &= \frac{1}{2^d} \sum_{k \geq 0} \frac{1}{2^k} h_{d,k} \left(1 - M_\bigstar \right) \end{aligned}$$

where $h_{d,k}$ are the complete homogeneous polynomials of degree k in d variables introduced in §3.2.4. Here we have used (3.2) from §3.2.4.

3.3.4 The symmetric powers $\mathrm{Sym}^j([\Omega_f])$ ※

For an n-dimensional k-vector space V with basis $e_1, \dots, e_n$, we may decompose $V = k_1 \oplus \cdots \oplus k_n$ as a direct sum where each k_i is a copy of k corresponding to each basis vector e_i.

A basis of the j-th symmetric power of V, $\mathrm{Sym}^j(V)$, is then given by

$$\{e_{\alpha(1)} \otimes e_{\alpha(2)} \otimes \cdots e_{\alpha(j)} \text{ s.t. } 1 \leq \alpha(1) \leq \cdots \leq \alpha(j) \leq n\},$$

in other words, where α is a sequence up to n of length j. We may then decompose $\mathrm{Sym}^j(V)$ as a direct sum

$$\mathrm{Sym}^j(V) = \bigoplus_{\alpha \in \mathcal{ISeq}_{n,j}} k_{\alpha(1)} \otimes \cdots \otimes k_{\alpha(j)} = h_{n,j}(k_\bigstar)$$

This can be extended to vector bundles over X with a splitting in some extension of $K(X)$, which implies that

$$\mathrm{Sym}^j([\Omega_f]) = h_{d,j}(M_\bigstar).$$

For another example of this, for a pair of line bundles L and M,

$$\mathrm{Sym}^2(L \oplus M) \cong L^{\otimes 2} \oplus (L \otimes M) \oplus M^{\otimes 2} \cong h_{2,2}(L, M).$$

3.3.5 Computation ※

We now only require one last lemma to show that

$$2^{2d+1}\theta^2([\Omega_f])^{-1} \sim \sum_{j=0}^{2d} \sum_{i=0}^{2d-j} (-1)^j \binom{2d+1}{i} \mathrm{Sym}^j([\Omega_f]),$$

which is the following statement.

Lemma 3.7. *For $k \geq d+2$,*

$$h_{d,k}(1 - M_\bigstar) \sim 0$$

Deferred proof. We prove this in two different cases; if $[\Omega_f]$ splits in $K(X)$, which we prove in §3.3.6; and if $[\Omega_f]$ doesn't split in $K(X)$, which we prove in §3.3.7.

Note that the second proof is a more general case and implies the result of first case. However, in the first case, we are able to give a stronger result by constructing a canonical isomorphism, and using Ducrot's theorem; in other words, this case uses methods similar to the ones used in §5 and §6. Rössler suggests a similar proof should exist in the second case due to the splitting principle; we instead give a proof which makes use of one of our cancellation lemmas, and of the Grothendieck Riemann Roch theorem, to showcase how these results can be used in these types of situations.

□

We begin with the right hand side, and note that from §3.3.4 we have that

$$\begin{aligned}
&\sum_{j=0}^{2d}\sum_{k=0}^{2d-j}(-1)^j\binom{2d+1}{k}\operatorname{Sym}^j([\Omega_f])\\
&\qquad=\sum_{j=0}^{2d}\sum_{k=0}^{2d-j}(-1)^j\binom{2d+1}{k}h_{d,j}(M_\bigstar)\\
&\qquad=\sum_{j=0}^{2d}\sum_{k=0}^{2d-j}(-1)^j\binom{2d+1}{k}h_{d,j}(1-(1-M_\bigstar)),
\end{aligned}$$

and we can then use Lemma 3.5 from §3.2.4 to compute $h_{d,j}(1-(1-M_\bigstar))$ in terms of $h_{d,j}(1-M_\bigstar)$,

$$\begin{aligned}
&=\sum_{j=0}^{2d}\sum_{k=0}^{2d-j}(-1)^j\binom{2d+1}{k}\sum_{i=0}^{j}(-1)^i\binom{d+j-1}{j-i}h_{d,i}(1-M_\bigstar)\\
&=\sum_{j=0}^{2d}\sum_{k=0}^{2d-j}\sum_{i=0}^{j}(-1)^{j-i}\binom{d+j-1}{j-i}\binom{2d+1}{k}h_{d,i}(1-M_\bigstar)\\
&=\sum_{i=0}^{2d}\sum_{j=i}^{2d}\sum_{k=0}^{2d-j}(-1)^{j-i}\binom{d+j-1}{j-i}\binom{2d+1}{k}h_{d,i}(1-M_\bigstar),
\end{aligned}$$

we then let $\ell = j-i$,

$$=\sum_{i=0}^{2d}\sum_{\ell=0}^{2d-i}\sum_{k=0}^{2d-i-\ell}(-1)^\ell\binom{d+i+\ell-1}{\ell}\binom{2d+1}{k}h_{d,i}(1-M_\bigstar),$$

and use Lemma 3.6 from §3.2.5,

$$= \sum_{i=0}^{d+1} 2^{d+1-i} h_{d,i}(1 - M_\bigstar) + \sum_{i=d+2}^{2d} T(d-2, i-d-2) h_{d,i}(1 - M_\bigstar),$$

we now use Lemma 3.7 to remove the parts where $i \geq d+2$,

$$\sim \sum_{i=0}^{d+1} 2^{d+1-i} \, h_{d,i}(1 - M_\bigstar)$$

and we use Lemma 3.7 once again to complete the power series, noting that we can do so because $h_{d,i}(1 - M_\bigstar)$ vanishes for large enough i since the $(1 - M_k)$ are nilpotent, as is noted in §3.3.3,

$$\begin{aligned} &\sim \sum_{i \geq 0} 2^{d+1-i} \, h_{d,i}(1 - M_\bigstar) \\ &= 2^{2d+1} \, \frac{1}{2^d} \sum_{i \geq 0} \frac{1}{2^i} \, h_{d,i}(1 - M_\bigstar) \end{aligned}$$

and we conclude by using §3.3.3, which gives us the left hand side,

$$= 2^{2d+1} \theta^2([\Omega_f])^{-1}.$$

3.3.6 Proof of lemma 3.7 if $[\Omega_f]$ splits in $K(X)$ ※

Suppose that $[\Omega_f] = M_1 + \cdots + M_d$ splits in $K(X)$, so that $M_i = [L_i]$ for some line bundles L_i on X, and let $k \geq d+2$, where d is the relative dimension of X over S. Our aim is then to show that

$$h_{d,k}(1 - M_\bigstar) \sim 0,$$

or, equivalently, that

$$\lambda([L]^2 \cdot h_{d,k}(1 - M_\bigstar)) = 1.$$

Instead, we are able to prove a stronger statement, that there exists a canonical isomorphism

$$\lambda(L^{\otimes 2} \otimes h_{d,k}(\mathcal{O}_X - L_\bigstar)) \cong \mathcal{O}_S.$$

Recall that the complete homogeneous symmetric polynomial $h_{d,k}$ is the sum of all homogeneous monomials of degree k, and we may consider them all by considering the increasing sequences up to d of length k, $\alpha \in \mathcal{IS}eq_{d,k}$. Then, using the canonical isomorphisms $\lambda(E+E') \cong \lambda(E) \otimes \lambda(E')$, we have that

$$\lambda(L^{\otimes 2} \otimes h_{d,k}(\mathcal{O}_X - L_\bigstar)) \cong \bigotimes_{\alpha \in \mathcal{IS}eq_{d,k}} \lambda\left(L^{\otimes 2} \otimes \bigotimes_{i=1}^{k} (\mathcal{O}_X - L_{\alpha(i)})\right).$$

We now use Ducrot's theorem, §5.2 Theorem 5.1, and which is studied in great detail in §5. In particular, Lemma 5.1 implies that there exists a canonical trivialization for each $\alpha \in \mathcal{IS}eq_{d,k}$,

$$\lambda\left(L^{\otimes 2} \otimes \bigotimes_{i=1}^{k} (\mathcal{O}_X - L_{\alpha(i)})\right) \cong \mathcal{O}_S.$$

Combining these canonical trivializations gives us a canonical trivialization for $\lambda(L^{\otimes 2} \otimes h_{d,k}(\mathcal{O}_X - L_\bigstar))$, which in turn implies that

$$h_{d,k}(1 - M_\bigstar) \sim 0.$$

3.3.7 Proof of lemma 3.7 if $[\Omega_f]$ doesn't split in $K(X)$ ※

Recall from §2.2.2 that $c^1 : \mathrm{Pic}(S) \to CH^1(S)$ is injective, since S is smooth. We can then prove that $\alpha \sim 0$ by finding an equation in $CH^1(S)$ giving us

$$c^1(\lambda([L]^2 \cdot \alpha)) = 0.$$

Let $e = u_1 + \cdots + u_m$ an element of $K(S)$ which splits for some u_i in an extension of $K(S)$. Then

$$c^1(\det(e)) = c^1(u_1 \cdots u_m) = c^1(u_1) + \cdots + c^1(u_m) = c^1(e).$$

This means that for $\alpha \in K(X)$, $\alpha \sim 0$ if and only if

$$0 = c^1(\lambda([L]^2 \cdot \alpha)) = c^1(\det f_*([L]^2 \cdot \alpha)) = c^1(f_*([L]^2 \cdot \alpha)).$$

We may now use the Grothendieck Riemann Roch theorem from §2.5.3, and write

$$c^1(f_*([L]^2 \cdot \alpha)) = [\mathrm{ch}(f_*([L]^2 \cdot \alpha))]^1 = \left[\int_{X/S} \mathrm{td}([\mathcal{T}_f])\,\mathrm{ch}([L]^2 \cdot \alpha)\right]^1,$$

where $[\star]^1 : CH(S) \to CH^1(S)$ is taking the degree 1 part in the graded Chow ring.

Now, we let $\alpha = h_{d,k}(1 - M_{\bigstar})$ and $P = \operatorname{td}([\mathcal{T}_f])\operatorname{ch}([L]^2)$, and recall that we're considering the case where $k \geq d + 2$. Our aim is then to prove that

$$\left[\int_{X/S} \operatorname{td}([\mathcal{T}_f])\operatorname{ch}([L]^2 \cdot \alpha)\right]^1 = \left[\int_{X/S} P\operatorname{ch}(h_{d,k}(1 - M_{\bigstar}))\right]^1 \text{ vanishes.}$$

We know from §2.2.1 that there is a graded extension of $CH(X)$, $CH(X)'$, for which we can extend the Chern character to an extension $K(X)'$ of $K(X)$ where $[\mathcal{T}_f]$ splits as $[\mathcal{T}_f] = M_1 + \cdots + M_d$. We now consider $h_{d,k}(1 - M_{\bigstar})$ to be written in terms of the M_i instead of the exterior powers $\lambda^i([\Omega_f])$, and, since the Chern character is a morphism of rings, we have that

$$\operatorname{ch}(h_{d,k}(1 - M_{\bigstar})) = h_{d,k}(\operatorname{ch}(1 - M_{\bigstar})).$$

Now, each $\operatorname{ch}(1 - M_i)$ satisfies

$$\operatorname{ch}(1 - M_i) = -c^1(M_i) - \frac{1}{2}c^2(M_i) - \cdots \quad \in CH(X)'^{\geq 1}_{\mathbb{Q}},$$

so, since $h_{d,k}$ is a combination of monomials of degree k, by considering the degree in the graded ring $CH(X)'$ we have that

$$h_{d,k}(\operatorname{ch}(1 - M_{\bigstar})) \in CH(X)'^{\geq k}_{\mathbb{Q}}.$$

By taking into account that $h_{d,k}(1 - M_{\bigstar}) \in K(X)$ and that $CH(X)'$ is an extension of graded rings, we then have that

$$P\operatorname{ch}(h_{d,k}(1 - M_{\bigstar})) \in CH(X)^{\geq k}_{\mathbb{Q}}.$$

Recall from §2.2.2 that the pushforward of cycles preserves the dimension of cycles, and maps elements in $CH(X)^m$ to elements in $CH(S)^{m-d}$, since the Chow ring is graded according to co-dimension.

It then follows that

$$\int_{X/S} P\operatorname{ch}(h_{d,k}(1 - M_{\bigstar})) \in CH(S)^{k-d}_{\mathbb{Q}} \subseteq CH(S)^2_{\mathbb{Q}},$$

so that

$$\left[\int_{X/S} P\,\mathrm{ch}(h_{d,k}(1 - M_\bigstar))\right]^1 = 0,$$

and so

$$h_{d,k}(1 - M_\bigstar) \sim 0.$$

Note that this later argument closely follows the proof of one of our cancellation lemmas, Lemma 2.5, which we would have been able to use directly if the M_i's were in $K(X)$, as in §3.3.6.

Chapter 4

$\mathbb{Z}/2$-equivariant geometry and analytic preliminaries

In this section we introduce the remaining tools that will be necessary for §5 and §6.

In §4.1, we introduce $\mathbb{Z}/2$ equivariant schemes, complex manifolds, vector bundles and hermitian vector bundles. The equivariant case is studied in the literature more generally for cyclic groups acting on schemes and complex manifolds, but for §6, where we study an involution formula, it is enough to consider the $\mathbb{Z}/2$-equivariant case.

In §4.2, we introduce the Quillen metric, which we endow to the determinant of cohomology when considering the analytic case.

In 4.3 we introduce the analytic formulae used in §5 and §6; Bismut's immersion formula, which determines the norm of the isomorphism of determinant line bundles obtained when comparing a vector bundle with a resolution of its pushforward along a closed immersion; the equivariant Bismut's immersion formula, which is its equivariant analogue; and Ma's branched covering formula, which determines the norm of the isomorphism of determinant line bundles obtained when comparing a line bundle with its pushforward along a branched covering.

In §4.4 we provide a few results related to the determinant of cohomology endowed with the Quillen metric.

4.1 $\mathbb{Z}/2$-equivariant geometry ※†

In this section we recall definitions and propositions related to G-equivariant geometry in both the algebraic and the analytic cases, restricted to the case where the cyclic group G is $\mathbb{Z}/2$. Throughout this section, we let G be the group $\mathbb{Z}/2$, with generator g.

In §4.1.1 we recall the definitions of G-equivariant schemes and complex manifolds. In §4.1.2 we recall the definitions of G-equivariant vector bundles and hermitian vector bundles. In §4.1.3 we introduce the notion of the G-equivariant decomposition of a vector bundle or a hermitian vector bundle, which expresses the (hermitian) vector bundle restricted to the fixed points as an (orthogonal) direct sum of two (hermitian) vector bundles;

$$E|_{X_G} = E_0 \oplus E_1 \qquad \text{or} \qquad \underline{E}|_{X_G} = \underline{E}_0 \oplus \underline{E}_1.$$

In §4.1.4 we introduce the $\{-1\}$ G-equivariant vector bundle and the $\underline{\{-1\}}$ G-equivariant hermitian vector bundle, which coincide with $\mathcal{O}_X$ and $\underline{\mathcal{O}}_X$ when the equivariant structure is removed.

In §4.1.5 we study the immersion of the fixed points, together with the natural G-equivariant structure on the normal and conormal bundles, in the case where the fixed points form a Cartier divisor. In §4.1.6 we recall the definition of the quotient scheme and study an analogue in the analytic case.

In §4.1.7 we recall the G-equivariant analogues of Chern and Todd forms, and in §4.1.8 we briefly consider G-equivariant Todd forms in the particular case where $G = \mathbb{Z}/2$.

† We collect definitions and theorems used in [23] and [25] where the general G-equivariant case, where G is a cyclic group, is studied.

※ Some observations specialized to the specific $\mathbb{Z}/2$-equivariant case are not taken from the literature.

4.1.1 Equivariant schemes and complex manifolds †

Consider the algebraic case, and let S and X be a schemes. A *G-equivariant structure* on X, making X a *G-equivariant scheme*, is given by a group homomorphism $G \to \mathrm{Aut}(X)$, which in the case of $G = \mathbb{Z}/2$ is determined by an involution

$g : X \to X$ such that $g \circ g = \mathrm{Id}_X$. Note that $g : X \to X$ is the image of the generator $g \in G$ through the group homomorphism $G \to \mathrm{Aut}(X)$.

Consider the analytic case, and let S and X be complex manifolds. A *G-equivariant structure* on X, making X a *G-equivariant complex manifold*, is given by a group homomorphism $G \to \mathrm{Aut}(X)$, which in the case of $G = \mathbb{Z}/2$ is determined by a holomorphism $g : X \to X$ which is an involution, so that $g \circ g = \mathrm{Id}_X$. If X is a Kähler manifold, we will require that the hermitian metric on the tangent bundle makes $\underline{\mathcal{T}}_X$ a G-equivariant hermitian vector bundle, as defined in §4.1.2.

In both algebraic and analytic cases, we say that the action of G is trivial if the image of the group homomorphism is trivial, that is, the involution is the identity. Throughout this section, we will assume that both X and S are G-equivariant, and that the action of G on S is trivial.

4.1.2 Equivariant vector bundles †

Consider the algebraic case, let $G = \mathbb{Z}/2$ and X be a G-equivariant scheme, with involution $g : X \to X$. A *G-equivariant vector bundle* E on X is a vector bundle E endowed with an isomorphism of vector bundles $\alpha_E : E \to g_*(E)$ such that $g_*(\alpha_E) \circ \alpha_E = \mathrm{Id}_E$.

Consider the analytic case, let $G = \mathbb{Z}/2$ and X be a G-equivariant complex manifold, with involution $g : X \to X$. A *G-equivariant hermitian vector bundle* $\underline{E}$ on X is a hermitian vector bundle $\underline{E}$ endowed with an isometry of hermitian vector bundles $\alpha_{\underline{E}} : \underline{E} \to g_*(\underline{E})$ such that $g_*(\alpha_{\underline{E}}) \circ \alpha_{\underline{E}} = \mathrm{Id}_{\underline{E}}$.

4.1.3 Equivariant decomposition †

Consider the algebraic case, let $G = \mathbb{Z}/2$, let S be a G-equivariant scheme with a trivial G-equivariant structure, and let E be a G-equivariant vector bundle on S.

As the involution $g : S \to S$ is the identity, at each point $s \in S$ we may identify the fibers $(g_*E)_s = E_s$, and so we may consider $\alpha_E : E \to g_*E$ as a (possibly non-trivial) automorphism of E. As $g_*(\alpha_E) \circ \alpha_E = \mathrm{Id}_E$, in this case we have that at each fiber E_s, $(\alpha_E|_{E_s})^2 = \mathrm{Id}_{E_s}$, and so we may decompose E into the vector subbundles $E = E_0 \oplus E_1$, where α_E restricted to E_0 is the identity, and α_E restricted to E_1 is given by multiplication by -1.

Now consider the analytic case, let $G = \mathbb{Z}/2$, let S be a G-equivariant complex manifold with a trivial G-equivariant structure, and let $\underline{E} = (E, h^E)$ be a G-equivariant hermitian vector bundle on S.

We consider the decomposition of the holomorphic vector bundle $E = E_0 \oplus E_1$ in the same way as for the algebraic case. We then restrict the hermitian metric h^E on E to the vector subbundles E_0 and E_1, giving us the hermitian vector bundles $\underline{E}_0$ and $\underline{E}_1$.

Since $\alpha_{\underline{E}}$ is an isometry, if e_0 and e_1 are sections of E_0 and E_1 respectively, we have that

$$\langle e_0, e_1 \rangle_{h^E} = \langle \alpha_E(e_0), \alpha_E(e_1) \rangle_{h^E} = \langle e_0, -e_1 \rangle_{h^E} = -\langle e_0, e_1 \rangle_{h^E},$$

and so the sections of E_0 and E_1 are orthogonal to each other, so the direct sum decomposition $E = E_0 \oplus E_1$ extends to an orthogonal decomposition $\underline{E} = \underline{E}_0 \oplus \underline{E}_1$ of hermitian vector bundles.

Note that when the action of G is trivial, in both algebraic and analytic cases we may consider vector bundles E and hermitian vector bundles $\underline{E}$ as G-equivariant (hermitian) vector bundles "with trivial G action" by choosing $\alpha_E = \mathrm{Id}_E$ (resp. $\alpha_{\underline{E}} = \mathrm{Id}_{\underline{E}}$). This allows us to consider, given a G-equivariant vector bundle E, the vector bundles E_0 and E_1 as G-equivariant vector bundles with the trivial G action, and similarly in the analytic case. However, note that in this case in general $E \neq E_0 \oplus E_1$ as G-equivariant vector bundles, since they might not have the same G-equivariant structure (see §4.1.4 for the right decomposition), and the same applies in the analytic case.

When the action of G is non-trivial, we instead consider the restriction of the (hermitian) vector bundle E (resp. $\underline{E}$) over a scheme (resp. complex manifold) X to the fixed point scheme (resp. complex manifold) X_G, so that $E|_{X_G} = E_0 \oplus E_1$ as vector bundles in the algebraic case, and $\underline{E}|_{X_G} = \underline{E}_0 \oplus \underline{E}_1$ as hermitian vector bundles in the analytic case.

We will also note that both $(\cdot)_0$ and $(\cdot)_1$ are exact functors, in both the algebraic case and the analytic case.

4.1.4 The $\{-1\}$ G-equivariant line bundle †

Consider both algebraic and analytic cases; let X be a G-equivariant scheme or complex manifold.

While X may not have a trivial G-equivariant structure, the existence of the canonical section 1 of $\mathcal{O}_X$ or $\underline{\mathcal{O}}_X$ allows us to identify, for each $x \in X$, the fibers of $\mathcal{O}_X$ and $g_*\mathcal{O}_X$ at x. We let the G-equivariant vector bundle $\mathcal{O}_X$ denote the trivial bundle $\mathcal{O}_X$ endowed with the isomorphism $\alpha_{\mathcal{O}_X} : \mathcal{O}_X \to g_*\mathcal{O}_X$ which preserves the canonical section 1, and the G-equivariant vector bundle $\{-1\}$ denote the trivial bundle $\mathcal{O}_X$ endowed with the isomorphism $\alpha_{\{-1\}} : \mathcal{O}_X \to g_*\mathcal{O}_X$ which maps the canonical section 1 to its negative -1. We define the G-equivariant hermitian vector bundles $\underline{\mathcal{O}}_X$ and $\underline{\{-1\}}$ analogously.

We then let $E\{-1\} = E \otimes \{-1\}$ for G-equivariant vector bundles and $\underline{E}\{-1\} = \underline{E} \otimes \underline{\{-1\}}$ for G-equivariant hermitian vector bundles. Note that, in the case of vector bundles over a scheme where the G-action is trivial, while $E|_{X_G} = E_0 \oplus E_1$ as vector bundles, we have that $E|_{X_G} = E_0 \oplus E_1\{-1\}$ as G-equivariant vector bundles, where both E_0 and E_1 are endowed with the trivial G-equivariant action; and analogously in the analytic case.

4.1.5 The fixed point scheme and the normal and conormal bundles †※

Consider the algebraic case, with X a G-equivariant scheme over S, which has a trivial G-equivariant structure. We let $Z = X_G$ be the fixed point subscheme of X over S, which exists if X is separated over S by [23, Proposition 2.3], and let $\iota : Z \to X$ be the immersion.

Provided that Z is a local complete intersection, for example if Z is a Cartier divisor of X, we have a short exact sequence relating the tangent bundles of X and Z, as well as the normal bundle; this is (1.4).

$$0 \to \mathcal{T}_{Z/S} \to \mathcal{T}_{X/S}|_Z \to \mathcal{N}_{Z/X} \to 0. \tag{1.4}$$

Furthermore, if Z is a Cartier divisor, by [23, Proposition 2.5.(2)] we have that the conormal bundle $\mathcal{C}_{Z/X}$, which carries a natural action of G, satisfies $(\mathcal{C}_{Z/X})_0 = 0$.

By duality, we have that the normal bundle too satisfies $(\mathcal{N}_{Z/X})_0 = 0$. Moreover, note that since the G-action on Z is trivial, then the natural G-equivariant structure on $\mathcal{T}_{Z/S}$ is trivial, so that $(\mathcal{T}_{Z/S})_1 = 0$. By applying the exact functors $(\cdot)_0$ and $(\cdot)_1$ to (1.4), we obtain two canonical isomorphisms of vector bundles $\mathcal{T}_{Z/S} \cong (\mathcal{T}_{X/S}|_Z)_0 = (\mathcal{T}_{X/S})_0$ and $\mathcal{N}_{Z/X} \cong (\mathcal{T}_{X/S}|_Z)_1 = (\mathcal{T}_{X/S})_1$.

By the equivariant decomposition of an equivariant vector bundle from §4.1.3, we have a canonical isomorphism

$$\mathcal{T}_{X/S}|_Z \cong \mathcal{T}_{Z/S} \oplus \mathcal{N}_{Z/X},$$

and so we may consider both $\mathcal{T}_{Z/S}$ and $\mathcal{N}_{Z/X}$ as vector subbundles of $\mathcal{T}_{X/S}|_Z$.

In the analytic case, if $\mathcal{T}_X$ is endowed with a Kähler metric, then both the tangent bundle $\mathcal{T}_Z$ and the normal bundle $\mathcal{N}_{Z/X}$ on Z^{an} inherit a metric from the one on $\mathcal{T}_X$ as vector subbundles of $\mathcal{T}_X|_Z$, and one can verify that the metric on $\mathcal{T}_Z$ is Kähler. The metric on the normal bundle then induces a metric on its dual, the conormal bundle $\mathcal{C}_{Z/X}$.

Finally, by §4.1.3, we have the following orthogonal decomposition;

$$\underline{\mathcal{T}}_X|_Z \simeq \underline{\mathcal{T}}_Z \oplus \underline{\mathcal{N}}_{Z/X}.$$

4.1.6 The quotient scheme †※

In the algebraic case, a *categorical quotient* of a G-equivariant scheme X, if it exists, is the scheme up to unique isomorphism X/G, together with a morphism $q : X \to X/G$; such that the action of G on X/G is trivial; and, whenever the action of G on a G-equivariant scheme Y' is trivial, there exists, for each $q' : X \to Y'$, a unique morphism $h : X/G \to Y'$ such that $h \circ q = q'$. By [23, Proposition 2.1], if X is a G-equivariant scheme and the orbit of every point in X is contained in an affine open subscheme, then the quotient X/G exists and $q : X \to X/G$ is integral and surjective. We let $Y = X/G$ be the quotient of X.

Suppose that X is a G-equivariant scheme, that the orbit of every point in X is contained in an affine open subscheme, that $\iota : Z \to X$ is the immersion of the fixed point subscheme, and that $q : X \to Y$ is the quotient map. We recall some useful facts from [23, Proposition 2.5],

Lemma 4.1. *If Z is a Cartier divisor of X, then q is flat.*

Proof. This is [23, Proposition 2.5.(1)] in the case where $G = \mathbb{Z}/2$. $\square$

Lemma 4.2. *The composition $q \circ \iota : Z \to Y$ is a closed immersion, we have set theoretic equality $q^{-1}(q(Z)) = Z$ when considering Z a subset of X, and we have a canonical isomorphism*

$$Y\backslash Z \cong (X\backslash Z)/G.$$

Proof. This is [23, Proposition 2.5.(3)], identifying Z with closed subschemes in X and Y due to the closed immersion. $\square$

Lemma 4.3. *Let M be a G-equivariant vector bundle on X. Suppose that ι^*M has the trivial action and that q is flat. Then the natural morphism $q^*(q_*M)_0 \to M$ is an isomorphism.*

Proof. This is [23, Proposition 2.5.(5)]. $\square$

Lemma 4.4. *If X is smooth, $Z = X_G$ is a Cartier divisor in X, and $Z \to S$ is flat, then $Y \to S$ is also smooth, where $Y = X/G$.*

Proof. This is [23, Proposition 2.5.(7)]. $\square$

In the analytic case we can use these results to prove the following lemma.

Lemma 4.5. *Let $f : X^{\mathrm{an}} \to S^{\mathrm{an}}$ be a proper submersion of G-equivariant Hodge manifolds, where $G = \mathbb{Z}/2$, and such that S^{an} has a trivial G-equivariant structure. Let $Z^{\mathrm{an}} = X_G^{\mathrm{an}}$ be the fixed points of X^{an}, and suppose that Z^{an} is a complex submanifold, such that $\iota : Z^{\mathrm{an}} \to X^{\mathrm{an}}$ is a closed immersion and such that $f \circ \iota : Z^{\mathrm{an}} \to S^{\mathrm{an}}$ is a submersion.*

If Z^{an} is of codimension 1, then quotient $Y^{\mathrm{an}} = X^{\mathrm{an}}/G$ is a complex manifold with a submersion to S^{an}, and if $q : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is the quotient map, then $q \circ \iota : Z^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a closed immersion, and q is a branched covering whose branch points are Z^{an}.

Proof. We consider X and S the schemes corresponding to X^{an} and S^{an}. The fact that $f : X^{\mathrm{an}} \to S^{\mathrm{an}}$ is a proper submersion implies that $f : X \to S$ is a proper morphism of schemes. The fact that the orbit of every point x in X is contained in an affine open subscheme follows from the fact that each orbit $\mathrm{orb}(x)$ is contained in the fiber $X_{f(x)}$, so we may reduce to the case where $S = \mathrm{Spec}(\mathbb{C})$ is a point, where then f being proper implies that X is a complex projective variety. For such varieties, if we have two points, we may find a homogeneous polynomial H in the homogeneous coordinate ring, such that H vanishes at neither point, and so both points will be contained in the basic open set defined by H, which is an open affine.

This means that $Y = X/G$ exists, and that the quotient $q : X \to Y$ is integral and surjective.

Now, if Z^{an} is a complex submanifold of codimension 1 in X, then Z is a Cartier divisor of X. If $Z^{\mathrm{an}} \to S^{\mathrm{an}}$ is a submersion, then it $Z \to S$ is smooth, and so $Z \to S$ is flat. By Lemma 4.4, $Y \to S$ is smooth, which implies that the analytification of Y, Y^{an}, is a complex manifold and $Y^{\mathrm{an}} \to S^{\mathrm{an}}$ is a submersion.

By Lemma 4.1, q is flat, and by Lemma 4.2, $q \circ \iota$ is a closed immersion of Z in Y, and $Y \backslash Z \cong (X \backslash Z)/G$. This means that $q|_{X \backslash Z}$ is a covering map, and so since q is 2-1 away from Z and 1-1 on Z, q is a branched covering with branching points Z. □

Lemma 4.4 implies that, in the analytic case, if $Z^{\mathrm{an}} := X_G^{\mathrm{an}}$ is a complex submanifold of codimension 1, then $Y^{\mathrm{an}} = X^{\mathrm{an}}/G$ is a complex manifold. Note that in the analytic case we will mostly work over a point $S = \mathrm{Spec}(\mathbb{C})$.

4.1.7 Equivariant forms †

Consider the G-equivariant analytic case, and let X be a G equivariant complex manifold. For a G-equivariant hermitian vector bundle $\underline{E}$ on X, we let

$$\mathrm{ch}_g(\underline{E}) := \mathrm{ch}(\underline{E}_0) - \mathrm{ch}(\underline{E}_1) \in \bigoplus_{p \geq 0} \mathfrak{A}^{p,p}(X_G),$$

be its *equivariant Chern form*, which is calculated on

$$\underline{E}|_{X_G} = \underline{E}_0 \oplus \underline{E}_1.$$

We define the *equivariant Chern form class* of $\underline{E}$ analogously, as

$$\overline{\mathrm{ch}}_g(\underline{E}) := \overline{\mathrm{ch}}(\underline{E}_0) - \overline{\mathrm{ch}}(\underline{E}_1) \in \bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^p(X_G),$$

which is also simply taking the Chern form in $\bigoplus_{p\geq 0} \mathfrak{A}^{p,p}(X_G)$ and considering its image in $\bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^p(X_G)$.

The *equivariant Todd form* of E is given by the formula

$$\mathrm{td}_g(\underline{E}) = \frac{c^{\mathrm{rk}\, E}(\underline{E}|_{X_G})}{\mathrm{ch}_g\left(\sum_{j=0}^{\mathrm{rk}\, E}(-1)^j \bigwedge^j \underline{E}^\vee\right)} \in \bigoplus_{p\geq 0} \mathfrak{A}^{p,p}(X_G),$$

and the *equivariant Todd form class* $\overline{\mathrm{td}}_g(\underline{E})$ is its image in $\bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^p(X_G)$.

As for the non-equivariant case in §2.3.7, following [18, §3], given a short exact sequence of equivariant hermitian vector bundles,

$$0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0,$$

there exists equivariant secondary Chern and Todd characteristic classes $\widetilde{\mathrm{ch}}_g(0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0), \widetilde{\mathrm{td}}_g(0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0) \in \bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^p(X_G)$ satisfying

$$\mathrm{dd^c}\, \widetilde{\mathrm{ch}}_g(0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0) = \mathrm{ch}_g(\underline{E}') - \mathrm{ch}_g(\underline{E}) + \mathrm{ch}_g(\underline{E}''),$$

$$\mathrm{dd^c}\, \widetilde{\mathrm{td}}_g(0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0) = \mathrm{td}_g(\underline{E}')\, \mathrm{td}_g(\underline{E}'') - \mathrm{td}_g(\underline{E}).$$

Finally, we write

$$\widetilde{\mathrm{ch}}_g(0 \to \underline{E}' \to \underline{E} \to 0) := \widetilde{\mathrm{ch}}_g(0 \to \underline{E}' \to \underline{E} \to 0 \to 0),$$

$$\widetilde{\mathrm{td}}_g(0 \to \underline{E}' \to \underline{E} \to 0) := \widetilde{\mathrm{td}}_g(0 \to \underline{E}' \to \underline{E} \to 0 \to 0).$$

4.1.8 Equivariant Todd form decomposition ※

Suppose that $\underline{E}$ has a trivial G-action, then, as one might expect, we have that

$$\mathrm{td}_g(\underline{E}) = \mathrm{td}(\underline{E}).$$

We then define

$$\mathrm{td}_{-1}(\underline{E}) := \mathrm{td}_g(\underline{E}\{-1\}).$$

For any equivariant hermitian vector bundle E, from equivariant decomposition we will then have, from the multiplicativity of Todd forms,

$$\mathrm{td}_g(\underline{E}) = \mathrm{td}(\underline{E}_0)\,\mathrm{td}_{-1}(\underline{E}_1).$$

For an hermitian line bundle $\underline{L}$ with a trivial G-action, we can compute

$$\begin{aligned}\mathrm{td}_{-1}(\underline{L}) &= \mathrm{td}_g(\underline{L}\{-1\})\\ &= \frac{c^1(\underline{L})}{\mathrm{ch}_g(\mathcal{O}_X - \underline{L}^\vee\{-1\})}\\ &= \frac{c^1(\underline{L})}{1+\mathrm{ch}(\underline{L}^\vee)}\\ &= \frac{c^1(\underline{L})\,\mathrm{ch}(\underline{L})}{\mathrm{ch}(\underline{L})+1}.\end{aligned}$$

Which means that td_{-1} can be considered to be the Todd homomorphism td_ϕ of the power series

$$\phi(t) = \frac{t\exp(t)}{\exp(t)+1} = \sum_{n\geq 0}\frac{1}{2}E_n(1)\frac{t^{n+1}}{n!},$$

where E_n are the Euler polynomials, which satisfy

$$\frac{2e^{xt}}{e^t+1} = \sum_{n\geq 0}E_n(x)\frac{t^n}{n!}.$$

Recall that we have studied this td_{-1} in §2.2.11.

4.2 Quillen metric †

When moving into the analytic case, given a hermitian vector bundle $\underline{E}$ and a Kähler metric on the relative tangent bundle with Kähler form ω, it is possible to construct an L_2 metric on the determinant of cohomology $\lambda(E)$ of E, as is described in §4.2.1. However, this metric is not smooth, but, if we multiply it by the analytic torsion introduced in §4.2.2, we obtain the Quillen metric, which is smooth, and is formally introduced in §4.2.5. For this reason, when considering the analytic case, one often endows the determinant of cohomology with the Quillen metric.

The analytic torsion, and so also the Quillen metric, depend both on the hermitian metric on $\underline{E}$, as well as the Kähler metric on the relative tangent bundle. How the analytic torsion changes if these metrics are changed is determined by the anomaly formula, introduced in §4.2.3. There also exists equivariant analogues to the analytic torsion, which we recall in §4.2.4. We use Rössler's definition of the $\mathbb{Z}/2$-equivariant determinant of cohomology, which differs from the usual definition by making use of the dual of line bundles, which may only be used in the case where $G = \mathbb{Z}/2$. For this reason, the equivariant determinant of cohomology with Quillen metric which we use in this paper, which we introduce in §4.2.6, differs from the standard definition.

† We follow [24, §VI.3] for the definition of the Quillen metric. We follow the construction of the $\mathbb{Z}/2$-equivariant determinant of cohomology from [23, §5].

4.2.1 L_2 metric on $\lambda(E)$ †

We are in the analytic case, and consider a submersion $f : X \to S$ of constant relative dimension n of Hodge manifolds, and endow the relative tangent bundle $\mathcal{T}_{X/S}$ with a Kähler metric with associated Kähler form ω. This corresponds to a proper and flat morphism, so by the semicontinuity theorem, for a Hermitian vector bundle $\underline{E} = (E, h^E)$, the pushforward $R^\bullet f_* E$ is a perfect complex.

Since hermitian metrics are defined locally, it is enough to define the L_2 metric on the fibers. Following [24, §VI.3.1], we note that for each point $s \in S$, the fiber of the determinant of cohomology is given by;

$$\lambda(E)_s = \bigotimes_{q \geq 0} (\det H^q(X_s, E))^{(-1)^q}.$$

By the Hodge theorem, there exists a canonical isomorphism between $H^q(X_s, E)$ and the kernel of the Laplace operator acting on $\mathfrak{A}^{0,q}(X_s, E)$. The forms in $\mathfrak{A}^{0,q}(X_s, E)$ have an L_2 metric given by the formula

$$\langle \sigma, \tau \rangle_{L_2} = \int_{X_s} \langle \sigma(x), \tau(x) \rangle \frac{\omega^n}{n!}.$$

Using the isomorphism between $H^q(X_s, E)$ and the kernel of the Laplace operator, we give each $H^q(X_s, E)$ a hermitian metric.

Now, recall that the exterior product interacts naturally with hermitian metrics. In particular, for the top exterior product, the determinant, we have

$$\langle v_1 \wedge \cdots \wedge v_n,\ w_1 \wedge \cdots \wedge w_n\rangle = \det((\langle v_i, w_j\rangle)_{i,j}).$$

This allows us to extend the L_2 metric from each $H^q(X_s, E)$ to each fiber

$$\lambda(E)_s = \bigotimes_{q\geq 0} (\det H^q(X_s, E))^{(-1)^q},$$

and combining the metric on each fiber gives us the *determinant of cohomology endowed with the L_2 metric.*

4.2.2 Analytic torsion †

Consider the analytic case, let $f : X \to S$ be a submersion of complex manifolds, and endow the relative tangent bundle $\mathcal{T}_f$ with a Kähler metric, with associated Kähler form ω. This gives us the hermitian vector bundle $\underline{\mathcal{T}}_X$.

The *analytic torsion form* $\mathfrak{T}_f(\underline{E})$ of a hermitian vector bundle $\underline{E} = (E, h^E)$ on X, as constructed in [4] (where it's written as $T(\omega, h^E)$), is a sum of forms of type (p, p) in $\bigoplus_{p\geq 0} \mathfrak{A}^{p,p}(S)$ and satisfies the following equation;

$$\mathrm{dd^c}\, \mathfrak{T}_f(\underline{E}) = -\int_{X/S} \mathrm{td}(\underline{\mathcal{T}}_f)\,\mathrm{ch}(\underline{E}).$$

This form depends on the hermitian metric on $\underline{E}$; as well as the Kähler form ω—or equivalently—the metric on $\underline{\mathcal{T}}_f$.

4.2.3 Anomaly formula †

The *anomaly formula* [4, 3.10] describes the behavior of the analytic torsion $\mathfrak{T}_f(\underline{E})$ when either the hermitian metric on the vector bundle or the Kähler metric on the tangent bundle (and thus the Kähler form) are changed. Suppose that $\underline{E} = (E, h^E)$ and $\underline{E}' = (E, h'^E)$ are two hermitian vector bundles with the same underlying holomorphic vector bundle, but different metrics; and that $f' : X \to S$ is the same morphism as f, but endowed with a different Kähler form ω' giving rise to a different metric on the tangent bundle $\underline{\mathcal{T}_f}' = (\mathcal{T}_f, h'^{\mathcal{T}_f})$. Then, the anomaly formula states that

$$\begin{aligned}\mathfrak{A}\mathfrak{T}_{f'}(\underline{E}') - \mathfrak{A}\mathfrak{T}_f(\underline{E}) = {} & \widetilde{\mathrm{ch}}(0 \to R^\bullet f_* \underline{E} \to R^\bullet f_* \underline{E}' \to 0)\\ & - \int_{X/S} \widetilde{\mathrm{td}}(0 \to \underline{\mathcal{T}_f} \to \underline{\mathcal{T}_f}' \to 0)\,\mathrm{ch}(\underline{E})\\ & - \int_{X/S} \mathrm{td}(\underline{\mathcal{T}_f}')\widetilde{\mathrm{ch}}(0 \to \underline{E} \to \underline{E}' \to 0),\end{aligned}$$

in $\bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^p(S)$.

4.2.4 Equivariant analytic torsion †

In the equivariant case, we similarly have the *equivariant analytic torsion form* $\mathfrak{A}\mathfrak{T}_{f,g}(\underline{E})$ of an equivariant hermitian vector bundle $\underline{E}$ on X. Its construction can be found in [19] and it is a sum of forms (p,p) over S, and is such that

$$\mathrm{dd^c}\, \mathfrak{A}\mathfrak{T}_{f,g}(\underline{E}) = -\int_{X_G} \mathrm{td}_g(\underline{\mathcal{T}_f})\,\mathrm{ch}_g(\underline{E}).$$

The anomaly formula in the equivariant case [19, Thm. 2.13], states that

$$\begin{aligned}\mathfrak{A}\mathfrak{T}_{f',g}(\underline{E}') - \mathfrak{A}\mathfrak{T}_{f,g}(\underline{E}) = {} & \widetilde{\mathrm{ch}}_g(0 \to R^\bullet f_* \underline{E} \to R^\bullet f_* \underline{E}' \to 0)\\ & - \int_{X_G/S} \widetilde{\mathrm{td}}_g(0 \to \underline{\mathcal{T}_f} \to \underline{\mathcal{T}_f}' \to 0)\,\mathrm{ch}_g(\underline{E})\\ & - \int_{X_G/S} \mathrm{td}_g(\underline{\mathcal{T}_f}')\widetilde{\mathrm{ch}}_g(0 \to \underline{E} \to \underline{E}' \to 0),\end{aligned}$$

in $\bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^p(S)$.

4.2.5 The Quillen metric †

Following [24, §VI Definition 3], the *Quillen metric* on the determinant of cohomology $\lambda(E)$ is the metric obtained by taking the product of the L_2 metric on $\lambda(E)$ and the exponential of the analytic torsion $\mathfrak{A}\mathfrak{T}_f(\underline{E})$. We write $\lambda(\underline{E})$ for the determinant of cohomology of E endowed with the Quillen metric.

4.2.6 $\mathbb{Z}/2$-equivariant determinant of cohomology †

Now suppose that E is an equivariant vector bundle. Then $R^\bullet f_*^{\mathrm{eq}} E$ consists of equivariant vector bundles on S, where the G-action is trivial. Following [23, §5], and by noting that the functors $F \mapsto F_0$ and $F \mapsto F_1$ are exact, we then define

$$\lambda^{\mathrm{eq}}(E) = \det((R^\bullet f_*^{\mathrm{eq}} E)_0) \ \otimes \ \det((R^\bullet f_*^{\mathrm{eq}} E)_1)^\vee. \tag{4.1}$$

Now suppose that $\underline{E}$ is an equivariant hermitian vector bundle. Since $(R^\bullet f_*^{\mathrm{eq}} E)_0$ and $(R^\bullet f_*^{\mathrm{eq}} E)_1$ are made of subbundles of the vector bundles that make $(R^\bullet f_*^{\mathrm{eq}} E)$, they inherit the L_2 metric constructed before. We may thus construct an L_2 metric on $\lambda^{\mathrm{eq}}(E)$. The Quillen metric on $\lambda^{\mathrm{eq}}(E)$ is obtained by multiplying the L_2 metric by the equivariant analytic torsion form, and we let $\lambda^{\mathrm{eq}}(\underline{E})$ denote the line bundle $\lambda^{\mathrm{eq}}(E)$ endowed with the Quillen metric.

Now, in [3], the equivariant determinant of cohomology is instead defined as a direct sum $\det((R^\bullet f_*^{\mathrm{eq}} E)_0) \oplus \det((R^\bullet f_*^{\mathrm{eq}} E)_1)$ instead of a tensor product inverting the second term $\det((R^\bullet f_*^{\mathrm{eq}} E)_0) \otimes \det((R^\bullet f_*^{\mathrm{eq}} E)_1)^\vee$. This is because the latter definition cannot be extended to the cases where $G = \mathbb{Z}/n\mathbb{Z}$ for larger n, whereas the former can be used to explore a more general equivariant theory by considering $\bigoplus_{\xi^n=1} \det((R^\bullet f_*^{\mathrm{eq}} E)_\xi)$, where ξ goes over the n-th roots of unity.

However, when computing the norms of isomorphisms, both definitions will agree. This is because in the case of using direct sums, the norms of the isomorphisms are weighted by ξ for each root of unity, and in the case where $G = \mathbb{Z}/2\mathbb{Z}$ this will be weights of 1 and -1, which is the same effect as considering the tensor product with the second term being inverted.

4.3 Analytic formulae †

Bismut, Ma, and other authors have studied the behavior of the Quillen metric on the determinant of cohomology in various geometric circumstances. In this section we recall those that will be used in §5 and §6; Bismut's immersion formulae in the classic (§4.3.4) and equivariant (§4.3.6) cases, as well as Ma's branched covering formula (§4.3.8).

We will also examine more closely the case where we consider the immersion of a Cartier divisor; we do this in both the classic (§4.3.5) and equivariant cases (§4.3.7).

In §4.3.1 we introduce Bismut's assumption (A) on the metrics involved to use Bismut's immersion formulae. In §4.3.2 we introduce the Bott Chern singular current, and in §4.3.3 we introduce Bismut's R-genus, both of which are used in Bismut's immersion formulae.

† We collect results from [3], [5], and [20].

4.3.1 Bismut's assumption (A) †

Suppose that $\iota : Z \to X$ is an embedding of compact complex manifolds. Take a vector bundle η on Z, and a finite complex of vector bundles $\xi_\bullet$ on X which is a resolution of $\iota_* \eta$. Then by [17], there is a canonical isomorphism of line bundles $\lambda(\xi_\bullet) \cong \lambda(\eta)$.

We may endow both sides of this isomorphism with the Quillen metric, once hermitian metrics are chosen on η, the vector bundles in the resolution $\xi_\bullet$, and Kähler metrics are chosen on $\mathcal{T}_X$ and $\mathcal{T}_Z$. Bismut's immersion formula then gives us the norm of this canonical isomorphism, provided that the metrics chosen are compatible in the following ways.

- the metric on $\mathcal{T}_X$ is Kähler.
- the metric on $\mathcal{T}_Z$ is the one induced by the metric on $\mathcal{T}_X$.
- we choose a metric on $\mathcal{N}_{Z/X}$ to be the one induced by the metric on $\mathcal{T}_X$.
- the metrics on $\xi_\bullet$ are compatible with the ones on $\mathcal{N}_{Z/X}$ and η, satisfying Bismut's assumption (A) from [2, Definition 1.5].

Bismut's assumption (A) is satisfied when the hermitian metrics on $\xi_\bullet$, when restricted to Z, are isometrically related to the resolution $\bigwedge^\bullet \underline{\mathcal{C}}_{Z/X} \otimes \underline{\eta}$ of $\underline{\eta}$. Note that if $\xi_\bullet$ is a resolution of $\iota_* \eta$, then by [2, Proposition 1.6] there exists hermitian metrics on the $\xi_\bullet$ which satisfy Bismut's assumption (A).

4.3.2 Bott-Chern singular current †

Suppose that $\underline{\xi}_\bullet$ is a bounded complex of hermitian vector bundles, such that for all $i \in \mathbb{Z}$ the homology sheaves H_i are vector bundles, which are endowed with hermitian metrics h^{H_i}. We say that $\underline{\xi}_\bullet$ is *homologically split* if the short exact sequences

$$0 \to \underline{\mathrm{Im}} \to \underline{\mathrm{Ker}} \to \underline{H}_i \to 0$$

and

$$0 \to \underline{\mathrm{Ker}} \to \underline{\xi}_i \to \underline{\mathrm{Im}} \to 0,$$

are all orthogonally split, where kernel and image of each differential are endowed with the hermitian metrics induced by the metrics on the $\underline{\xi}_i$.

If we have a bounded complex $\underline{\xi}_\bullet$ which resolves $\iota_* \eta$ as in §4.3.1 and satisfies Bismut's Assumption (A), then we have the Bott-Chern singular current $\mathfrak{BC}(\underline{\xi}_\bullet)$ defined by Bismut, Gillet and Soulé in [1] which generalizes the Chern secondary characteristic class to the case where one considers the resolutions $\underline{\xi}_\bullet$ of hermitian vector bundles $\iota_* \underline{\eta}$ associated to closed immersions. They satisfy the following differential equation [1, (2.8)],

$$\mathrm{dd^c}\, \mathfrak{BC}(\underline{\xi}_\bullet) = \iota_* \operatorname{ch}(\underline{\eta}) \operatorname{td}^{-1}(\underline{N}) - \operatorname{ch}(\underline{\xi}_\bullet).$$

There is an equivariant analogue, the equivariant Bott-Chern singular current $\mathfrak{BC}_g(\underline{\xi}_\bullet)$. A good introduction is found in [25, §4.2].

An useful fact of the Bott-Chern singular current is that it vanishes if $\underline{\xi}_\bullet$ is homologically split, see [25, Theorem 5.7]. In the equivariant case, the equivariant Bott-Chern singular current vanishes if $\underline{\xi}_\bullet|_{X_g}$ is homologically split, this is in [25, Theorem 5.9].

Furthermore, the behavior when considering a tensor product would be, by considering the above together with the projection formula, that

$$\mathfrak{BC}(\underline{\xi}_\bullet \otimes \underline{\mu}) = \mathfrak{BC}(\underline{\xi}_\bullet) \operatorname{ch}(\underline{\mu}),$$

where $\underline{\xi}_\bullet \otimes \underline{\mu}$ is the resolution of $\iota_*(\underline{\eta} \otimes \iota^* \underline{\mu})$ obtained by tensoring the chain complex $\underline{\xi}_\bullet$ by a hermitian vector bundle $\underline{\mu}$. This is because

$$\iota_* \operatorname{ch}(\underline{\eta} \otimes \iota^* \underline{\mu}) \operatorname{td}^{-1}(\underline{N}) - \operatorname{ch}(\underline{\xi}_\bullet \otimes \underline{\mu}) = (\iota_* \operatorname{ch}(\underline{\eta}) \operatorname{td}^{-1}(\underline{N}) - \operatorname{ch}(\underline{\xi}_\bullet)) \operatorname{ch}(\underline{\mu}).$$

4.3.3 Bismut's R-genus †

The *polylogarithm* $\mathrm{Li}_s(z)$ is defined by the power series

$$\mathrm{Li}_s(z) \coloneqq \sum_{k \geq 1} \frac{z^k}{k^s},$$

which is extended by analytic continuation. One has that the *Riemann zeta function* is given by $\zeta(s) = \mathrm{Li}_s(1)$, and similarly the *Dirichlet eta function* is given by $\eta(s) = -\mathrm{Li}_s(-1)$. We also let $\mathcal{H}_k \coloneqq 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ denote the partial sums of the harmonic series.

We then define two power series,

$$R(x) \coloneqq \sum_{k \geq 0} \left(2\zeta'(-k) + \mathcal{H}_k \zeta(-k)\right) \frac{x^k}{k!},$$

$$R(z, x) \coloneqq \sum_{k \geq 0} \left(2 \frac{\partial}{\partial s} \mathrm{Li}_{-k}(z) + \mathcal{H}_k \mathrm{Li}_{-k}(z)\right) \frac{x^k}{k!}.$$

Given a hermitian vector bundle $\underline{E}$, there is a class called the Bismut R-genus $R(\underline{E})$ which does not depend on the metric on E, and is furthermore additive, as defined in [3]. This form $R(\underline{E})$ is given by the Chern character morphism of the power series R defined above.

If we have an equivariant hermitian vector bundle $\underline{E}$, there is a class called the equivariant R-genus $R_g(\underline{E})$ which is similarly independent on the metric on E, as defined in [18, Definition 3.5]. This form is given by

$$R_g(\underline{E}) = R(1, \underline{E}_0) + R(-1, \underline{E}_1),$$

in the case where $G = \mathbb{Z}/2$.

For a hermitian vector bundle with a trivial G-action, we can see that $R_g(\underline{E}) = R(1, \underline{E}) = R(\underline{E})$. We let $R_{-1}(\underline{E}) \coloneqq R_g(\underline{E}\{-1\}) = R(-1, \underline{E})$. Equivariant decomposition and additivity will then mean that

$$R_g(\underline{E}) = R(\underline{E}_0) + R_{-1}(\underline{E}_1).$$

We will note that while R is given by the Riemann zeta function, R_{-1} is instead given by the Dirichlet eta function;

$$R_{-1}(x) = -\sum_{k \geq 0} \left(2\eta'(-k) + \mathcal{H}_k \eta(-k)\right) \frac{x^k}{k!}.$$

We will also note that it is possible to write R_{-1} in terms of the usual R-genus and the Todd character, by using the standard results that $\eta(k) = (1 - 2^{1-k})\zeta(k)$ and that $\zeta(-n) = (-1)^n B_{n+1}/(n+1)$. This means that $\eta(-k) = (1 - 2^{1+k})\zeta(-k)$ and $\eta'(-k) = (1 - 2^{1+k})\zeta(-k) + 2^{1+k}\log(2)\zeta(-k)$.

$$\begin{aligned} R_{-1}(x) = -\sum_{k\geq 0}\left[(1-2^{1+k})(2\zeta'(-k) + \mathcal{H}_k\zeta(-k))\frac{x^k}{k!}\right] \\ - \log(2)\sum_{k\geq 0}\left[(-1)^k 2^{1+k} B_{k+1}\frac{x^k}{(k+1)!}\right]. \end{aligned}$$

Which gives that

$$R_{-1}(\underline{E}) = -R(\underline{E}) + 2\phi^2(R(\underline{E})) - \log(2)\overline{c}^{\mathrm{top}}(\underline{E}^\vee)^{-1}\phi^2(\overline{\mathrm{td}}(\underline{E}^\vee)),$$

where ϕ^2 is the operator defined in §2.2.10.

4.3.4 Bismut's immersion formula †

Bismut's immersion formula [3, Theorem 0.1] gives us that the norm of the canonical isomorphism described in §4.3.1, $\lambda(\underline{\xi}_\bullet) \overset{(\alpha)}{\approx} \lambda(\underline{\eta})$, is given by

$$\begin{aligned} (\alpha) = &-\int_X \mathrm{td}(\underline{\mathcal{T}}_X)\mathfrak{B}\mathcal{C}(\underline{\xi}_\bullet) \\ &+ \int_Z \mathrm{td}^{-1}(\underline{\mathcal{N}}_{Z/X})\widetilde{\mathrm{td}}(0 \to \underline{\mathcal{T}}_Z \to \underline{\mathcal{T}}_X|_Z \to \underline{\mathcal{N}}_{Z/X} \to 0)\,\mathrm{ch}(\underline{\eta}) \\ &- \int_X \overline{\mathrm{td}}(\underline{\mathcal{T}}_X)R(\underline{\mathcal{T}}_X)\overline{\mathrm{ch}}(\underline{\xi}_\bullet) \\ &+ \int_Z \overline{\mathrm{td}}(\underline{\mathcal{T}}_Z)R(\underline{\mathcal{T}}_Z)\overline{\mathrm{ch}}(\underline{\eta}). \end{aligned}$$

Note that in Bismut's paper [3], λ refers to the inverse of the determinant of cohomology instead of the determinant of cohomology. However, the paper describes the norm of the canonical section on $\lambda(\eta)^\vee \otimes \lambda(\xi_\bullet)$, whereas (α) is the log-norm of the canonical section on $\lambda(\xi_\bullet)^\vee \otimes \lambda(\eta)$. Overall, we negate twice to obtain the right sign.

4.3.5 Bismut's immersion formula for Cartier divisors ※

We apply Bismut's immersion formula to the case where the immersion is that of a Cartier divisor. In this case, we have the following short exact sequence,

$$0 \to \underline{L} \to \underline{\mathcal{O}}_X \to \iota_* \underline{\mathcal{O}}_Z \to 0,$$

where $L \cong \mathcal{O}_X(-Z)$ is endowed with the metric that makes $0 \to \underline{L} \to \underline{\mathcal{O}}_X \to 0$ satisfy Bismut's assumption (A). In particular, the compatibility with the norm of $\underline{\mathcal{N}}_{Z/X}$ implies that we have a canonical isometry $\iota^* \underline{L} \simeq \underline{\mathcal{C}}_{Z/X} = \underline{\mathcal{N}}_{Z/X}^{\vee}$. We may then tensor this by a hermitian line bundle $\underline{M}$, so we have

$$0 \to \underline{L} \otimes \underline{M} \to \underline{M} \to \iota_*(\iota^* \underline{M}) \to 0.$$

So we have a canonical isomorphism of line bundles

$$\lambda_X(\underline{M} \otimes (\underline{\mathcal{O}}_X - \underline{L})) \overset{(a)}{\approx} \lambda_Z(\iota^* \underline{M}),$$

with norm given by

$$\begin{aligned}(a) = &- \int_X \mathrm{td}(\underline{\mathcal{T}}_X) \mathfrak{B}\mathcal{C}(0 \to \underline{L} \to \underline{\mathcal{O}}_X \to 0)\, \mathrm{ch}(\underline{M}) \\ &+ \int_Z \mathrm{td}^{-1}(\underline{\mathcal{N}}_{Z/X}) \widetilde{\mathrm{td}}(0 \to \underline{\mathcal{T}}_Z \to \underline{\mathcal{T}}_X|_Z \to \underline{\mathcal{N}}_{Z/X} \to 0)\, \mathrm{ch}(\iota^* \underline{M}) \\ &- \int_X \overline{\mathrm{td}}(\underline{\mathcal{T}}_X) R(\underline{\mathcal{T}}_X)(\overline{\mathrm{ch}}(\underline{\mathcal{O}}_X) - \overline{\mathrm{ch}}(\underline{L}))\overline{\mathrm{ch}}(\underline{M}) \\ &+ \int_Z \overline{\mathrm{td}}(\underline{\mathcal{T}}_Z) R(\underline{\mathcal{T}}_Z) \overline{\mathrm{ch}}(\iota^* \underline{M}).\end{aligned}$$

4.3.6 The equivariant Bismut immersion formula †

Bismut's immersion formula has been generalized to the equivariant case. In its most general version, we have an immersion $\iota : Z \to X$ as in the non-equivariant case, but now assume that both X and Z are G-equivariant for some compact Lie group G and that the action preserves Z in X, but could act non-trivially on Z.

In our case, we have a much more specific situation; G is $\mathbb{Z}/2\mathbb{Z}$, Z will be the fixed point subscheme of X, and so the action on Z will be trivial. We will furthermore consider the case when Z is a Cartier divisor.

Recall that our construction of the equivariant determinant of cohomology endowed with the Quillen metric is different from the one used in Bismut's work since it is specific to the case where $G = \mathbb{Z}/2$, but the norms of the isomorphisms agree whether we use either definition, as noted in §4.2.6.

The equivariant Bismut immersion formula [5, Theorem 0.1] states that the norm of the canonical isomorphism $\lambda^{\mathrm{eq}}(\underline{\xi}_{\bullet}) \overset{(\alpha)}{\approx} \lambda^{\mathrm{eq}}(\underline{\eta})$ is given by

$$\begin{aligned}(\alpha) = &-\int_{X_g} \mathrm{td}_g(\underline{\mathcal{T}}_X)\mathfrak{B}\mathcal{C}_g(\underline{\xi}_{\bullet}) \\ &+\int_{Z_g} \mathrm{td}_g^{-1}(\underline{\mathcal{N}}_{Z/X})\widetilde{\mathrm{td}}_g(0 \to \underline{\mathcal{T}}_Z \to \underline{\mathcal{T}}_X|_Z \to \underline{\mathcal{N}}_{Z/X} \to 0)\,\mathrm{ch}_g(\underline{\eta}) \\ &-\int_{X_g} \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X)R_g(\underline{\mathcal{T}}_X)\overline{\mathrm{ch}}_g(\underline{\xi}_{\bullet}) \\ &+\int_{Z_g} \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_Z)R_g(\underline{\mathcal{T}}_Z)\overline{\mathrm{ch}}_g(\underline{\eta}).\end{aligned}$$

4.3.7 Equivariant Bismut immersion formula for Cartier divisors ※

We now apply the equivariant Bismut immersion formula to the case where the immersion is that of a Cartier divisor Z. As in the non-equivariant case, we have the following short exact sequence,

$$0 \to \underline{L} \to \underline{\mathcal{O}}_X \to \iota_*\underline{\mathcal{O}}_Z \to 0,$$

where once again $L \cong \mathcal{O}_X(-Z)$ is endowed with the metric that makes $0 \to \underline{L} \to \underline{\mathcal{O}}_X \to 0$ satisfy Bismut's assumption (A). Tensoring by a hermitian line bundle $\underline{M}$, we have

$$0 \to \underline{L} \otimes \underline{M} \to \underline{M} \to \iota_*(\iota^*\underline{M}),$$

which gives us a canonical isomorphism of line bundles

$$\lambda_X^{\mathrm{eq}}(\underline{M} \otimes (\underline{\mathcal{O}}_X - \underline{L})) \overset{(a)}{\approx} \lambda_Z^{\mathrm{eq}}(\iota^*\underline{M}),$$

with the norm given by

$$
\begin{aligned}
(a) = &- \int_{X_g} \mathrm{td}_g(\underline{\mathcal{T}}_X)\mathfrak{BC}_g(0 \to \underline{L} \to \underline{\mathcal{O}}_X \to 0)\, \mathrm{ch}_g(\underline{M}) \\
&+ \int_{Z_g} \mathrm{td}_g^{-1}(\underline{\mathcal{N}}_{Z/X})\widetilde{\mathrm{td}}_g(0 \to \underline{\mathcal{T}}_Z \to \underline{\mathcal{T}}_X|_Z \to \underline{\mathcal{N}}_{Z/X} \to 0)\, \mathrm{ch}_g(\iota^* \underline{M}) \\
&- \int_{X_g} \overline{\mathrm{td}}(\underline{\mathcal{T}}_X) R_g(\underline{\mathcal{T}}_X)(\overline{\mathrm{ch}}_g(\underline{\mathcal{O}}_X) - \overline{\mathrm{ch}}_g(\underline{L}))\overline{\mathrm{ch}}_g(\underline{M}) \\
&+ \int_{Z_g} \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_Z) R_g(\underline{\mathcal{T}}_Z)\overline{\mathrm{ch}}_g(\iota^* \underline{M}).
\end{aligned}
$$

4.3.8 Ma's branched covering formula †

In [20], Ma studies the norm of a canonical isomorphism of determinant line bundles when considering a branched covering. When we consider the quotient $q : X \to Y = X/G$, in particular when the fixed point scheme $Z = X_G$ is a Cartier divisor, we find ourselves in the situation Ma studies in [20, §4]. Here we recall Ma's statement in the case where $G = \mathbb{Z}/2$.

We consider a submersion of complex compact Kähler manifolds Y^{an} over S^{an}, a holomorphic line bundle F on Y^{an}, and a section α of $F^{\otimes 2}$. We then let $P^{\mathrm{an}} = \mathbb{P}(F \oplus 1)$ and identify Y with $\{(y, [0, 1]) \in P^{\mathrm{an}} \text{ s.t. } y \in Y\}$, with $\pi : P^{\mathrm{an}} \to Y^{\mathrm{an}}$ being the projection. We then consider

$$X^{\mathrm{an}} := \{(y, t) \in F \text{ s.t. } t^2 + \alpha(y) = 0\} \subset F,$$

which can be identified with a complex submanifold of P^{an} due to the canonical isomorphism between F and the complement of $\mathbb{P}(F)$ in P^{an}. The projection $\pi : P^{\mathrm{an}} \to Y^{\mathrm{an}}$ then induces a branched covering $q : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ of degree 2.

For $G = \mathbb{Z}/2\mathbb{Z}$, we suppose Y^{an} to be endowed with the trivial G-action, and X^{an} with the natural G-action sending $(y, t) \mapsto (y, -t)$. We can then identify $Y^{\mathrm{an}} \cong X^{\mathrm{an}}/G$, and so we also let $Z^{\mathrm{an}} = X_G^{\mathrm{an}}$ be the fixed point set, which can be considered as a submanifold of Y^{an} determined by the zero set of the section α, and is thus a Cartier divisor.

Then one chooses a Kähler metric on P^{an}, which then induces Kähler metrics on X^{an}, Y^{an}, and Z^{an}.

We then take a hermitian vector bundle $\underline{\widetilde{M}}$ on Y, consider $\underline{M} \simeq q^*\underline{\widetilde{M}}$ on X and the pushforward $R^\bullet q_*\underline{M}$. By [20, Theorem 4.1], we then have a canonical isomorphism of determinant line bundles

$$\lambda^{\mathrm{eq}}(\underline{M}) \overset{(a)}{\approx} \lambda^{\mathrm{eq}}(R^\bullet q_*\underline{M}),$$

whose norm is given by

$$\begin{aligned}
(a) = &\int_Y \mathrm{td}(\underline{\mathcal{T}}_Y)\,\mathrm{td}^{-1}([X], h^{[X]})\,\mathrm{td}_g(\underline{\mathcal{N}}_{Y/P})\log\|\alpha\|^2_{h^{[X]}}\,\mathrm{ch}(\underline{\widetilde{M}}) \\
&- \int_Z \mathrm{td}^{-1}(\underline{\mathcal{N}}_{X/P})\,\mathrm{td}_g(\underline{\mathcal{N}}_{Y/P})\widetilde{\mathrm{td}}(0 \to \underline{\mathcal{T}}_Z \to \underline{\mathcal{T}}_Y \to \underline{\mathcal{N}}_{Z/Y} \to 0)\,\mathrm{ch}(\underline{\widetilde{M}}) \\
&+ \int_Z \mathrm{td}(\underline{\mathcal{T}}_Y)\,\mathrm{td}_g(\underline{\mathcal{N}}_{Y/P})\widetilde{\mathrm{td}}(0 \to [d\underline{Y}] \to \underline{\mathcal{N}}_{X/P} \to 0)^{-1}\,\mathrm{ch}(\underline{\widetilde{M}}) \\
&+ \int_Y \overline{\mathrm{td}}(\underline{\mathcal{T}}_Y)R(\underline{\mathcal{T}}_Y)\overline{\mathrm{ch}}_g(R^\bullet q_*\underline{\mathcal{O}}_X)\overline{\mathrm{ch}}(\underline{\widetilde{M}}) \\
&- \int_Z \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X)R_g(\underline{\mathcal{T}}_X)\overline{\mathrm{ch}}(\underline{\widetilde{M}}).
\end{aligned}$$

4.4 Properties of the analytic determinant of cohomology ※

In this section, we provide a couple of standard results in the area about the determinant of cohomology endowed with the Quillen metric.

In §4.4.1 we note a canonical isometry of determinant line bundles due to the orthogonal sum of hermitian vector bundles. In §4.4.2 we note an analytic consequence of the projection formula.

※ While these results should be well known, we were unable to locate a source from which we may quote them.

4.4.1 Direct sums †

A consequence of Bismut's immersion formula, when applied to the case where the immersion is the trivial immersion of either the empty subset or the full manifold, allows us to compute the norm of the isomorphisms between determinants of cohomology due to short exact sequences.

Theorem 4.1. *Let* $0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0$ *be a short exact sequence of vector bundles, equipped with metrics. Then we have that* $\lambda(E) \cong \lambda(E') \otimes \lambda(E'')$ *and the norm of this isomorphism is given by*

$$\widetilde{\mathrm{ch}}(0 \to \lambda(\underline{E}) \to \lambda(\underline{E}') \otimes \lambda(\underline{E}'') \to 0) = -\int_X \mathrm{td}(\underline{\mathcal{T}_X})\widetilde{\mathrm{ch}}(0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0).$$

Notes. This is a direct consequence of Bismut's immersion formula, applied to the case where the immersion is the identity. See [18, Theorem 3.7] for the more general equivariant version, which reduces to this in the case where the group acting on X is the trivial group. □

Theorem 4.2. *Suppose that* $\underline{E}$ *and* $\underline{E}'$ *are two hermitian vector bundles, then we have a canonical isometry*

$$\lambda(\underline{E} \oplus \underline{E}') \simeq \lambda(\underline{E}) \otimes \lambda(\underline{E}')$$

Proof. We apply Theorem 4.1 to the short exact sequence $0 \to \underline{E} \to \underline{E} \oplus \underline{E}' \to \underline{E}' \to 0$, and recall that $\widetilde{\mathrm{ch}}(0 \to \underline{E} \to \underline{F} \to 0) = 0$ if and only if $\underline{E} \to \underline{F}$ is an isometry, and that $\widetilde{\mathrm{ch}}(0 \to \underline{E}' \to \underline{E} \to \underline{E}'' \to 0) = 0$ if $E \cong E' \oplus E''$ and E is endowed with the orthogonal direct sum metric. □

4.4.2 Projection formula ※

Recall from §2.5 that projection formulae are results of the form

$$f_*(x \cdot f^*s) = s \cdot f_*(x),$$

which are applicable to many circumstances where there is a pushforward and a pullback. While the determinant of cohomology endowed with the Quillen metric is not a pushforward, it is still built using the pushforward and so has an analogue to the projection formula.

We let

$$\chi_{X/S}(E) \coloneqq \sum_{i \geq 0} (-1)^i \,\mathrm{rk}(R^i f_*(E)),$$

and we note that in the case where S is a point, this coincides with the Euler characteristic, so we may use the Hirzebruch-Riemann-Roch theorem, which states that

$$\chi_{X/S}(E) = \int_X \mathrm{td}(\underline{\mathcal{T}_X})\,\mathrm{ch}(\underline{E}).$$

Lemma 4.6. *Let X and S be Hodge manifolds, $f : X \to S$ be of constant relative dimension d, and endow the relative tangent bundle with a Kähler metric. Let $\underline{E}$ be a hermitian vector bundle on X and $\underline{L}$ be a hermitian line bundle on S. Then the canonical isomorphism from the projection formula gives a canonical isometry*

$$\lambda(\underline{E} \otimes f^*\underline{L}) \simeq \lambda(\underline{E}) \otimes \underline{L}^{\chi_{X/S}(E)}.$$

Proof. By the projection formula, we have a canonical isomorphism (see [7, 3.3.2]),

$$\lambda(E \otimes f^*L) \cong \lambda(E) \otimes L^{\chi_{X/S}(E)}.$$

This is because

$$\begin{aligned}
\lambda(E \otimes f^*L) &\cong \det(R^\bullet f_*(E \otimes f^*L)) \\
&\cong \det(R^\bullet f_*(E) \otimes L) \\
&\cong \lambda(E)^{\operatorname{rk} L} \otimes L^{\sum_{i\geq 0}(-1)^i \operatorname{rk}(R^i f_*(E))} \\
&\cong \lambda(E) \otimes L^{\chi_{X/S}(E)}.
\end{aligned}$$

We thus have a canonical isomorphism

$$\lambda(\underline{E} \otimes f^*\underline{L}) \cong \lambda(\underline{E}) \otimes \underline{L}^{\chi_{X/S}(E)},$$

which is an isometry provided the norm of the section associated to the isomorphism is 1. This norm can be determined locally, and, since the projection formula commutes with base change, it suffices to show that the norm is 1 in the case when S is a point.

In this case, $\underline{L}$ is the trivial bundle $\mathcal{O}_S$, endowed with some non-trivial metric, which is a positive constant $\langle 1_S, 1_S \rangle = e^c$ given that S is a point. The norm of the isomorphism can be determined by using the anomaly formula from §4.2.3, which gives us that

$$\lambda(\underline{E} \otimes f^*\underline{L}) \overset{(a)}{\approx} \lambda(\underline{E}),$$

with norm

$$\begin{aligned}(a) &= \int_{X/S} \operatorname{td}(\underline{\mathcal{T}_X})\widetilde{\operatorname{ch}}(0 \to \underline{E} \to \underline{E} \otimes f^*\underline{L} \to 0)\\ &= \int_{X/S} \operatorname{td}(\underline{\mathcal{T}_X})\widetilde{\operatorname{ch}}(0 \to \underline{\mathcal{O}_X} \to f^*\underline{L} \to 0)\operatorname{ch}(\underline{E})\\ &= \int_{X/S} f^*(\widetilde{\operatorname{ch}}(0 \to \underline{\mathcal{O}_S} \to \underline{L} \to 0))\operatorname{td}(\underline{\mathcal{T}_X})\operatorname{ch}(\underline{E})\\ &= \widetilde{\operatorname{ch}}(0 \to \underline{\mathcal{O}_S} \to \underline{L} \to 0)\int_{X/S} \operatorname{td}(\underline{\mathcal{T}_X})\operatorname{ch}(\underline{E})\\ &= -c\,\chi_{X/S}(E).\end{aligned}$$

Since the norm of the isomorphism $\underline{\mathcal{O}_S} \overset{-(a)}{\approx} \underline{L}^{\chi_{X/S}(E)}$ is

$$-(a) = c\,\chi_{X/S}(E),$$

which can be computed by considering that the norm of the isomorphism $\underline{\mathcal{O}_S} \cong \underline{L}$ is c, we can tensor both isomorphisms to obtain an isometry

$$\lambda(\underline{E} \otimes f^*\underline{L}) \simeq \lambda(\underline{E}) \otimes \underline{L}^{\chi_{X/S}(E)}.$$

So the norm is 1, and we have an isometry. $\square$

Chapter 5

Cube structures and the Quillen metric

In [6], Deligne introduced the *Deligne pairing* of two line bundles. We're in the situation where X is an algebraic curve over S, of relative dimension $n = 1$, and given two line bundles L_1, L_2 on X, their Deligne pairing is a line bundle $\langle L_1, L_2\rangle$ on S.

This Deligne pairing is stable under base change $T \to S$, and also has a *multiadditive property*; for all line bundles L, M, N on X we have canonical isomorphisms

$$\langle L \otimes M, N\rangle \cong \langle L, N\rangle \otimes \langle M, N\rangle, \qquad \text{and} \qquad \langle L, M \otimes N\rangle \cong \langle L, M\rangle \otimes \langle L, N\rangle.$$

Deligne's first construction of the Deligne pairing was geometrical in nature, but he would later provide a different point of view that considers the Deligne pairing as

$$\langle L, M\rangle \cong \lambda(L \otimes M) \otimes \lambda(L)^\vee \otimes \lambda(M)^\vee \otimes \lambda(\mathcal{O}_X), \tag{5.1}$$

where $\lambda(L) = \det(R^\bullet f_* L)$ is the determinant of cohomology.

Deligne proposed that this construction could be extended to higher relative dimensions n of X/S, which we call the *intersection bundle* $I_{X/S}(L_1, \ldots, L_{n+1})$ of $n+1$ line bundles $L_1, \ldots, L_{n+1}$ on X. However, Deligne noted that such extensions would require particular care with signs. Several authors would later construct such extensions by using different approaches, most notably Elkik in [8] where she provides a geometric construction following Deligne's first construction, and more recently Ducrot in [7] where he introduces a construction using the determinant of cohomology.

Similarly to the Deligne pairing, we expect that a construction of the intersection bundle $I_{X/S}$ of $n+1$ line bundles over X to be both stable under base change $T \to S$ and satisfy a multiadditive property; given bundles $L_1, \dots, L_{n+1}, M$ on X and some $1 \le i \le n+1$, we have a canonical isomorphism

$$\begin{aligned} & I_{X/S}(L_1, \dots, L_i \otimes M, \dots L_{n+1}) \cong \\ & I_{X/S}(L_1, \dots, L_{i-1}, L_i, L_{i+1}, \dots L_{n+1}) \otimes I_{X/S}(L_1, \dots, L_{i-1}, M, L_{i+1}, \dots L_{n+1}). \end{aligned} \tag{5.2}$$

The construction of the family of multiadditive isomorphisms is the key to a definition of the intersection bundle.

In Deligne's original construction for the case where the relative dimension is $n = 1$, he considers not only the algebraic case, but also the analytic one. In a similar way, Elkik extends her construction to the analytic case in [9]. We provide an extension of Ducrot's construction to the analytic case in §5.3.

In [23], Rössler uses the isomorphism coming from the multiadditive property of Ducrot's construction of the intersection bundle. As we aim to extend Rössler's work to the analytic case, we require the extension of Ducrot's construction.

In this section, we will first recall in §5.1 the machinery introduced by Ducrot in [7] to construct his version of the intersection bundle.

Then, in §5.2, we recall a statement of Ducrot's main theorem, which establishes the construction of the multiadditive isomorphisms for his line bundle. In §5.2 we also state the theorem we aim to prove to extend Ducrot's work, and give a brief sketch of Ducrot's proof of his theorem.

Finally, in §5.3 we give a proof of our extension of Ducrot's work, by following Ducrot's construction and considering the norms of isomorphisms at each step, once line bundles are given hermitian metrics and the determinant of cohomology is endowed with the Quillen metric.

5.1 Ducrot's cube structures †

In this section, we recall the constructions used by Ducrot to define his version of the intersection line bundles. How this intersection bundle is related to the cube structures defined by Ducrot is briefly explained in §5.1.1. In §5.1.2 to §5.1.7 we go over the objects and constructions Ducrot uses to define cube structures, and which can be found in [7, §1]. Finally, from §5.1.8 to §5.1.12 we review some of the remaining tools that will be required in our extension of Ducrot's results, and which can be found in [7, §3].

† This section recalls Ducrot's definitions from [7, §1, §3].

5.1.1 Ducrot's intersection bundle †

Let $f : X \to S$ be a flat n-dimensional projective morphism of varieties, and λ is the determinant of cohomology.

Given $L_1, \ldots, L_{n+1}$ line bundles on X, Ducrot defines the intersection bundle following the point of view of (5.1), as

$$I_{X/S}(L_1, \ldots, L_{n+1}) = \bigotimes_{J \subseteq \{1,\ldots,n+1\}} \lambda \left(\bigotimes_{i \in J} L_i \right)^{(-1)^{|J|}}. \tag{5.3}$$

For example, in the case where $n = 2$ this expression is

$$\begin{aligned} I_{X/S}(L, M, N) = \quad & \lambda(L \otimes M \otimes N)^{\vee} \\ & \otimes \lambda(L \otimes M) \otimes \lambda(M \otimes N) \otimes \lambda(N \otimes L) \\ & \otimes \lambda(L)^{\vee} \otimes \lambda(M)^{\vee} \otimes \lambda(N)^{\vee} \\ & \otimes \lambda(\mathcal{O}_X). \end{aligned}$$

Note that this agrees with (5.1) in the case where the relative dimension is $n = 1$.

However, it remains to define the isomorphisms from the multiadditive property. Now, note that, for example, in the case where $n = 1$ we have that the isomorphism

$$I_{X/S}(L \otimes M, N) \cong I_{X/S}(L, N) \otimes I_{X/S}(M, N),$$

is equivalent to having an isomorphism

$$\begin{aligned}\mathcal{O}_S &\cong I_{X/S}(L \otimes M, N)^\vee \otimes I_{X/S}(L, N) \otimes I_{X/S}(M, N)\\ &\cong \lambda(L \otimes M \otimes N)^\vee \otimes \lambda(L \otimes M) \otimes \lambda(N) \otimes \lambda(\mathcal{O}_X)^\vee\\ &\quad \otimes \lambda(L \otimes N) \otimes \lambda(L)^\vee \otimes \lambda(N)^\vee \otimes \lambda(\mathcal{O}_X)\\ &\quad \otimes \lambda(M \otimes N) \otimes \lambda(M)^\vee \otimes \lambda(N)^\vee \otimes \lambda(\mathcal{O}_X)\\ &\cong \lambda(L \otimes M \otimes N)^\vee\\ &\quad \otimes \lambda(L \otimes M) \otimes \lambda(M \otimes N) \otimes \lambda(N \otimes L)\\ &\quad \otimes \lambda(L)^\vee \otimes \lambda(M)^\vee \otimes \lambda(N)^\vee\\ &\quad \otimes \lambda(\mathcal{O}_X).\end{aligned}$$

Note the similarity with the definition of the intersection bundle in the case where $n = 2$. This argument can be replicated for larger relative dimension n, and this implies that the construction of the isomorphisms from the multiadditive property (5.2) are equivalent to the construction of *trivialization isomorphisms*

$$\mathcal{O}_S \cong \bigotimes_{J \subseteq \{1,\dots,n+2\}} \lambda\left(\bigotimes_{i \in J} L_i\right)^{(-1)^{|J|}}, \tag{5.4}$$

where we now have $n + 2$ line bundles $L_1, \dots L_{n+2}$ on X instead of the $n + 1$ used for the definition of the intersection bundle.

In essence, the cube structures that Ducrot defines in [7], and that we review in the remainder of this section, are families of such trivialization isomorphisms.

5.1.2 Picard categories †

A *commutative Picard category* is a category $\mathcal{D}$ endowed with a product operation $\otimes$ and a unit 1, where each object $L \in \mathcal{D}$ has an inverse $L^\vee$ such that $L \otimes L^\vee \cong L^\vee \otimes L \cong 1$, together with isomorphisms $\psi_{L,M} : L \otimes M \to M \otimes L$ giving a notion of commutativity. A commutative Picard category is *strictly commutative* if the expression $L^\vee \otimes L \otimes L^\vee$ can be uniquely be reduced to $L^\vee$, in other words that the following diagram commutes:

$$\begin{tikzcd}
L^\vee \otimes L \otimes L^\vee \arrow[r] \arrow[d] & L^\vee \otimes 1 \arrow[d] \\
1 \otimes L^\vee \arrow[r] & L^\vee
\end{tikzcd}$$

Let $\mathcal{C}$ and $\mathcal{D}$ be commutative Picard categories, with $\mathcal{C}$ being strictly commutative.

For a commutative Picard category $\mathcal{D}$, we let $\pi_0(\mathcal{D})$ be the group of isomorphism classes under tensor product, and $\pi_1(\mathcal{D})$ be the automorphism group of the identity object 1. For an object $M \in \mathcal{D}$, the translation functor $X \mapsto X \otimes M$, $f \mapsto (f \otimes \mathrm{id}_M)$, canonically identifies $\pi_1(\mathcal{D}) = \mathrm{Aut}_{\mathcal{D}}(1)$ with $\mathrm{Aut}_{\mathcal{D}}(M)$, and for $L \in \mathcal{D}$, choosing $M = L \otimes L$, the canonical isomorphism $\psi_{L,L} \in \mathrm{Aut}_{\mathcal{D}}(L \otimes L)$ from the commutative property of the commutative Picard category determines a canonical element $\varepsilon(L) \in \pi_1(\mathcal{D})$. This defines a morphism of groups $\varepsilon : \pi_0(\mathcal{D}) \to \pi_1(\mathcal{D})$. Recall that a commutative Picard category is strictly commutative if and only if $\varepsilon(L) = 1$ for all objects in the category, and that $\varepsilon(L)^2 = 1$ for all objects in a commutative Picard category, as is noted in [6, §4.1.1].

5.1.3 Line bundles and graded line bundles †

We now give two examples of Picard categories used by Ducrot in [7]. Firstly is the category of line bundles over an algebraic variety S, $\mathrm{Pic}(S)$, which is a strictly commutative Picard category.

Secondly is the non-strictly commutative Picard category $\mathrm{Pic}_*(S)$ of *graded line bundles* (recall §1.2.6), which Ducrot defines in the following way. The objects are pairs (L, d) where L is a line bundle on S and d is a locally constant map $S \to \mathbb{Z}$. There are arrows between (L, d) and (M, e) only if $d = e$, and in this case they are isomorphisms of line bundles $L \cong M$. The product is defined by $(L, d) \otimes (M, e) = (L \otimes M, d+e)$, with the standard associativity morphisms, however when defining the commutativity morphisms there is a twist; the commutativity morphism $\psi : (L, d) \otimes (M, e) \to (M, e) \otimes (L, d)$ is defined by the morphisms of sheaves

$$L \otimes M \to M \otimes L; \qquad l \otimes m \mapsto (-1)^{de} m \otimes l.$$

In $\mathrm{Pic}_*(S)$, the identity object is $(\mathcal{O}_S, 0)$, and the inverse of (L, d) is $(L^\vee, -d)$. Note that if the reduction morphism $(L, d) \otimes (L, d)^\vee \to (\mathcal{O}_S, 0)$ corresponds to the

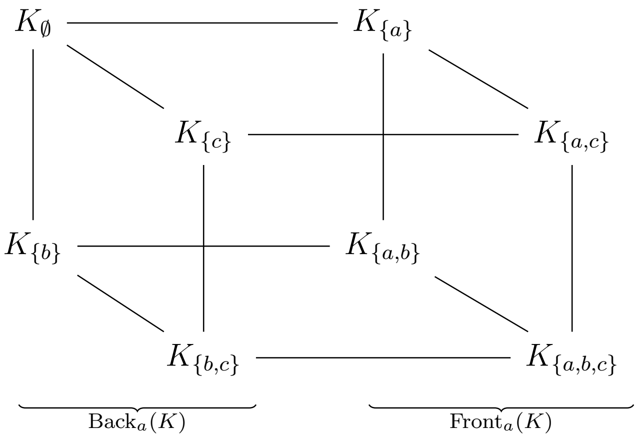

Figure 5.1: An I-cube for the set $I = \{a, b, c\}$. $\mathrm{Back}_a(K)$ and $\mathrm{Front}_a(K)$.

contraction map $c_L : L \otimes L^\vee \to \mathcal{O}_S$, then the reduction morphism of $(L, d)^\vee \otimes (L, d) \to (\mathcal{O}_S, 0)$ corresponds to $(-1)^d c_L$.

5.1.4 Cubes †

For a finite set I, the category $I\text{-Cube}(\mathcal{C})$ of I-cubes has as objects families $K = (K_J)_{J \subseteq I}$ where K_J are objects of $\mathcal{C}$, with morphisms given term-wise by $\mathcal{C}$-morphisms. If the size of I is p, we call K a p-cube. The category $p\text{-Cube}(\mathcal{C})$ of p-cubes has as morphisms $(\phi, \varphi) : K \to K'$, with K an I-cube and K' an I'-cube, where $\phi : I \to I'$ and $\varphi : K \to \phi^* K'$ is a morphism of I-cubes.

Given disjoint subsets $J, I' \subseteq I$, $J \cap I' = \emptyset$, with the size of $I' = m$, the m-face of K associated to J, I' is the I'-cube $\mathcal{F}_{J,I'}K$ with $(\mathcal{F}_{J,I'}K)_{J'} = K_{J \cup J'}$ for each $J' \subseteq I'$. Given $i \in I$, we define the back and front faces of K with respect to i as the $I \backslash \{i\}$-cubes $\mathrm{Back}_i(K) = \mathcal{F}_{\emptyset, I\backslash\{i\}}K$ and $\mathrm{Front}_i(K) = \mathcal{F}_{\{i\}, I\backslash\{i\}}K$. For example, if $I = \{a, b, c\}$, $\mathrm{Back}_a(K)$ has as vertices $K_\emptyset$, $K_{\{b\}}$, $K_{\{c\}}$ and $K_{\{b,c\}}$; whereas $\mathrm{Front}_a(K)$ has as vertices $K_{\{a\}}$, $K_{\{a,b\}}$, $K_{\{a,c\}}$ and $K_{\{a,b,c\}}$ (see Fig. 5.1). For an I-cube K, we write $K = A - B$ to mean that $A = \mathrm{Back}_i(K)$ and $B = \mathrm{Front}_i(K)$, for some $i \in I$.

5.1.5 Alternating product †

When we are working in a strict commutative Picard category, given an I-cube K, the *alternating product* of K is $\Sigma K = \bigotimes_{J\subseteq I} K_J^{(-1)^{|J|}}$. More generally, to define ΣK in a (possibly non-strict) commutative Picard category we let, for a finite set I and two total orders $<, \prec$ on I, and $J \subseteq I$,

$$N_{J,<,\prec} = |\{\{i,j\} \subset I \backslash J \text{ s.t. } i < j \text{ and } j \prec i\}|,$$

we define an automorphism $\eta_{J,<,\prec}$ on K_J sending $u \mapsto \varepsilon(K_J)^{N_{J,<,\prec}} u$, we let sK be the I-cube defined so that $(sK)_J$ is the inductive limit of the system of all $\eta_{J,<,\prec}$, and we let

$$\Sigma K \coloneqq \bigotimes_{J\subseteq I} (sK)_J^{(-1)^{|J|}}.$$

Note that an isomorphism $\alpha : \Sigma K \to L$ is given by a family of isomorphisms $\alpha_< : \bigotimes_{J\subseteq I} K^{(-1)^{|J|}} \to L$ indexed by all total orders of I such that $\alpha_< = \alpha_\prec \otimes \bigotimes_{J\subseteq I} \varepsilon(K_J)^{N_{J,<,\prec}}$.

For each n-cube $K = A - B$, we have the following canonical isomorphism (see [7, §1.4.3]),

$$\Sigma(A - B) \cong \Sigma(B) \otimes \Sigma(A)^\vee, \tag{5.5}$$

which is deduced from expanding the terms. In a similar way we can deduce that given two n-cubes $K = A - B$ and $K' = C - D$, together with an isomorphisms $u : B \cong C$, we have a canonical induced gluing isomorphism (see [7, §1.4.4])

$$\rho_u : \Sigma(A - B) \otimes \Sigma(C - D) \cong \Sigma(A - D). \tag{5.6}$$

5.1.6 Decorated cubes †

A p-cube K is *decorated* if each of its 2-faces F is endowed with an isomorphism $m_F : \mathcal{O} \cong \Sigma(F)$, in such a way that for each 3-face C of K, and every pair of indices i, j, the decorations of the 2-faces $\mathrm{Front}_i(C)$, $\mathrm{Back}_i(C)$, $\mathrm{Front}_j(C)$ and $\mathrm{Back}_j(C)$ are compatible with the canonical isomorphism

$$\Sigma(\mathrm{Front}_i(C)) \otimes \Sigma(\mathrm{Back}_i(C))^\vee \cong \Sigma(C) \cong \Sigma(\mathrm{Front}_j(C)) \otimes \Sigma(\mathrm{Back}_j(C))^\vee. \tag{5.7}$$

Morphisms of decorated p-cubes are morphisms of p-cubes which preserve the decoration. This forms the category $p\text{-}\mathrm{Cube}_d(\mathcal{C})$ of decorated p-cubes. From [7,

$$\begin{tikzcd}
& L_0 \otimes L_2 \otimes L_3 \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & L_0 \otimes L_1 \otimes L_2 \otimes L_3 \arrow[dl, dash] \arrow[dd, dash] \\
L_0 \otimes L_2 \arrow[rr, dash] \arrow[dd, dash] & & L_0 \otimes L_1 \otimes L_2 \arrow[dd, dash] & \\
& L_0 \otimes L_3 \arrow[rr, dash] \arrow[dl, dash] & & L_0 \otimes L_1 \otimes L_3 \arrow[dl, dash] \\
L_0 \arrow[rr, dash] & & L_0 \otimes L_1 &
\end{tikzcd}$$

Figure 5.2: The decorated 3-cube $K_{L_0}(L_1, L_2, L_3)$.

§1.5] we know that all decorated p-cubes are isomorphic to an I-cube of the form $K_J = L_0 \otimes \bigotimes_{i \in J} L_i$ for all $J \subseteq I$, where L_0 and L_i, $i \in I$ are in $\mathcal{C}$, with its natural decoration which is explained in the next paragraph. We call such p-cube $K_{L_0}(L_1, \dots, L_p)$. For example, $K_{L_0}(L_1, L_2, L_3)$ is shown in Figure 5.2.

Note that giving a decoration to an I-cube K means that for each $J \subsetneq I$ with $|J| + 2 \leq |I|$ and distinct $i, j \notin J$, we have an isomorphism

$$\mathcal{O}_X \cong K_J \otimes K_{J \cup \{i\}}^\vee \otimes K_{J \cup \{j\}}^\vee \otimes K_{J \cup \{i,j\}},$$

which is equivalent to having a family of isomorphisms

$$K_{J \cup \{i\}} \otimes K_{J \cup \{j\}} \cong K_J \otimes K_{J \cup \{i,j\}}.$$

In the case where $K = K_{L_0}(L_1, \dots, L_p)$, with J, i, j as above, and recalling $K_J := L_0 \otimes \bigotimes_{k \in J} L_k$, the natural decoration on K is given by the canonical isomorphisms of the form

$$(K_J) \otimes (K_J \otimes L_i \otimes L_j) \cong (K_J \otimes L_i) \otimes (K_J \otimes L_j).$$

5.1.7 Cube structures †

Let $\mathcal{C}$ and $\mathcal{D}$ be two commutative Picard categories, and suppose that $\mathcal{C}$ is strictly commutative. We let $i_{p-1} : (p-1)\text{-}\operatorname{Cube}_d(\mathcal{C}) \to p\text{-}\operatorname{Cube}_d(\mathcal{C})$ be the functor sending a $(p-1)$-cube A to the p-cube $A - A$, and we let $\underline{1} : p\text{-}\operatorname{Cube}_d(\mathcal{C}) \to \mathcal{D}$ be the constant functor mapping all decorated p-cubes K of $\mathcal{C}$ objects to the identity object of $\mathcal{D}$.

Suppose now that we have a functor $\delta : \mathcal{C} \to \mathcal{D}$. A *$p$-cube structure over δ* is an isomorphism of functors $s : \underline{1} \to \Sigma\delta$, compatible with the gluing isomorphisms (5.6) as well as the canonical trivialization of $(\Sigma\delta) \circ i_{p-1}$. A p-cube structure on $\delta : \mathcal{C} \to \mathcal{D}$ therefore consists of a family of canonical trivializations $s_K : \mathcal{O}_D \cong \Sigma\delta K$ for each decorated p-cube K.

Recall that in order to prove that Ducrot's construction of the intersection line bundle is multiadditive we require families of trivialization isomorphisms (5.4). Cube structures allow us to consider these families of trivializations.

5.1.8 Determinant of cohomology †

Let $D^b(S)$ be the derived category of bounded complexes of $\mathcal{O}_S$-modules, and let $D^{\text{is}}_{\text{pf}}(S)$ be the subcategory of perfect complexes where the arrows are the isomorphisms in $D^b(S)$. The determinant of cohomology for a projective flat morphism $f : X \to S$ defined by Knudsen and Mumford in [17], and recalled in §1.2.6, is a functor

$$\lambda_X : D^{\text{is}}_{\text{pf}}(X) \to \text{Pic}_*(S), \qquad K^\bullet \mapsto \det(R^\bullet f_* K^\bullet),$$

where $\text{Pic}_*(S)$ is the non-strictly commutative Picard category of graded line bundles introduced in §5.1.3.

5.1.9 Cubic complexes †

Given a finite set I, an I-complex is an I-cube K together with a set of morphisms $u_{J,i} : K_J \to K_{J\cup\{i\}}$ for each $J \subseteq I$ and each $i \in I\backslash J$, such that for all $J \subseteq I$ and all $i, j \in I\backslash J$ the following squares commute

$$\begin{tikzcd}
K_J \arrow[r, "u_{J,i}"] \arrow[d, "u_{J,j}"] & K_{J\cup\{i\}} \arrow[d, "u_{J\cup\{i\},j}"] \\
K_{J\cup\{j\}} \arrow[r, "u_{J\cup\{j\},i}"] & K_{J\cup\{i,j\}}.
\end{tikzcd}$$

A morphism of I-complexes $\alpha : K \to K'$ is a set of morphisms $\alpha_J : K_J \to K'_J$ for each $J \subseteq I$, such that the resulting diagram commutes.

We can interpret an I-complex as an I-multicomplex by associating to each $J \subseteq I$ the function $k_J : I \to \{-1, 0\}$ so that

$$k_J(x) = \begin{cases} -1 \text{ if } x \notin J \\ \ \ 0 \text{ if } x \in J \end{cases},$$

and letting $K^\bullet = (K^k)_{k \in \mathbb{Z}^I}$, where for each $J \subseteq I$ we have that $K_J = K^{k_J}$, be the I-multicomplex whose degrees are concentrated in $\{-1, 0\}^I$ and whose differentials d_i^k commute and so that the maps $u_{J,i}$ corresponds to the differential $d_i^{k_J}$.

5.1.10 The simple complex associated to a cubic complex †

Let $\mathcal{A}$ be an Abelian category and $\mathcal{B}$ be the category of simple complexes in $\mathcal{A}$. Set $\widetilde{I} = I \cup \{i_0\}$, and consider for each cubic I-complexes in $\mathcal{B}$, the associated I-multicomplex as an $\widetilde{I}$-multicomplex $K^\bullet$ in $\mathcal{A}$, concentrated in $\{-1, 0\}^I \times \mathbb{Z}^{\{i_0\}}$, and with commuting differentials. Note that we are adding the i_0 term to allow us to consider simple complexes in $\mathcal{A}$ instead of only considering objects in $\mathcal{A}$.

We extend any order $<$ on I to $\widetilde{I}$ by setting $I < i_0$, and we let $K^\bullet_<$ be the $\widetilde{I}$-multicomplex with the same vertices as $K^\bullet$, but such that the differentials satisfy

$$(d_i^k)_< = (-1)^{\sum_{j<i} k(j)} d_i^k.$$

The simple complex associated to $K^\bullet_<$ is

$$C(K^\bullet_<) = \cdots \to \bigoplus_{p = \sum_{i \in I} k(i)} K^k \xrightarrow{\sum (d_i^k)_<} \bigoplus_{p+1 = \sum_{i \in I} k(i)} K^k \to \cdots .$$

Given two orders $<$ and $\prec$, there is a way to construct an isomorphism $K^\bullet_< \cong K^\bullet_\prec$ given by multiplication by $(-1)^{\sum_{i<j \prec i} k(i)k(j)}$ on each K^k. This induces an isomorphism of simple complexes $C(K^\bullet_<) \cong C(K^\bullet_\prec)$ which forms a family of isomorphisms that commute in the natural way when comparing three different orders. We define $C(K^\bullet)$, the *simple complex associated to* $K^\bullet$, to be the inductive limit of the system of $C(K^\bullet_<)$ relative to these isomorphisms.

5.1.11 Determinant of a cubic complex †

Now let $\mathcal{A}$ be the category of coherent sheaves on X and $\mathcal{B}$ the category of simple complexes over $\mathcal{A}$, where $f : X \to S$ is a projective flat morphism. For any cubic I-complex $K^\bullet$ in $\mathcal{B}$, where each $K_J \in \mathcal{B}$ is a perfect complex for each $J \subseteq I$, we let $\lambda(K^\bullet)$ be the I-cube in $\mathrm{Pic}_*(S)$ whose vertices are $\lambda(K^\bullet)_J = \lambda(K_J)$, the determinant of cohomology of the perfect complex K_J of coherent sheaves on X, for each $J \subseteq I$.

For each order $<$ on I with minimal element i_1, the inclusion $\mathrm{Front}_{i_1}(K) \to K$ and the projection $K \to \mathrm{Back}_{i_1}(K)$ induce morphisms of simple complexes that form an exact sequence

$$0 \to C(\mathrm{Front}_{i_1}(K^\bullet)_<) \to C(K^\bullet_<) \to C(\mathrm{Back}_{i_1}(K^\bullet)_<)[1] \to 0,$$

which in turn induces an isomorphism

$$\lambda(C(K^\bullet_<)) \cong \lambda(C(\mathrm{Front}_{i_1}(K^\bullet)_<)) \otimes \lambda(C(\mathrm{Back}_{i_1}(K^\bullet)_<))^\vee. \tag{5.8}$$

Note that this would not be possible with other, non-minimal, $i \in I$.

We can form an isomorphism

$$\alpha_{K,<} : \lambda(C(K^\bullet_<)) \cong \bigotimes_{J \subseteq I} \lambda(K_J)^{(-1)^{|I \setminus J|}} = \Sigma\lambda K,$$

by continuing the expansion of $\lambda(C(K^\bullet_<))$ as in (5.8); first expanding $\lambda(C(K^\bullet_<))$ for the I-cube K along i_1, then doing the same expansion for the $(I\setminus\{i_1\})$-cubes $\mathrm{Front}_{i_1}(K)$ and $\mathrm{Back}_{i_1}(K)$ along i_2, the minimal element in $I\setminus\{i_1\}$, and so on.

Given two orders $<$ and $\prec$ of I, it is shown in [7, §3.5.1] that $\alpha_{K,<}$ and $\alpha_{K,\prec}$ are compatible with the isomorphism between $C(K^\bullet_<)$ and $C(K^\bullet_\prec)$, and so the system of the $\alpha_{K,<}$ define an isomorphism $\alpha_K : \lambda(C(K^\bullet)) \cong \Sigma\lambda K$.

5.1.12 Reduction isomorphism †

Take an exact sequence of cubic $(I\setminus\{i\})$-complexes in $\mathcal{B}$

$$\mathcal{E} : 0 \to X^\bullet \xrightarrow{\phi} Y^\bullet \xrightarrow{\psi} Z^\bullet \to 0,$$

where all $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ are made of perfect complexes of coherent sheaves, and let $A^\bullet$ be the cubic I-complex defined by $\phi : X^\bullet \to Y^\bullet$.

If we take an order $<$ on I, an easy computation shows that ψ defines a morphism of simple complexes $C(A^\bullet_<) \to C(Z^\bullet_<)$. The system of such morphisms for each order $<$ on I are compatible with each other and induce a morphism $C(A^\bullet) \to C(Z^\bullet)$, which can be shown to be a quasi-isomorphism. This in turn induces an isomorphism $\lambda(C(A^\bullet)) \cong \lambda(C(Z^\bullet))$, which by applying α gives us a canonical *reduction isomorphism* $r : \Sigma\lambda A^\bullet \cong \Sigma\lambda Z^\bullet$.

For example, by taking an I-cube K in $\mathrm{Pic}(X)$, $K^\bullet$ any cubic complex over K, and any effective relative Cartier divisor D on X, we can consider the short exact sequence

$$0 \to K^\bullet \to K^\bullet(D) \to K^\bullet(D)|_D \to 0.$$

This induces the reduction isomorphism

$$r_D : \Sigma\lambda(K - K(D)) \cong \Sigma\lambda(K(D)|_D).$$

Note that this reduction isomorphism is completely independent of the choice of cube complex on K, and depends only on the I-cube K itself.

5.2 Ducrot's theorem †

Now, in order to construct his intersection bundle, Ducrot makes use of the cube structures he defines in [7, §1] and which we recall in §5.1. In particular, he proves the existence of a family of cube structures for each flat projective morphism $f : X \to S$ over the determinant of cohomology functor λ, which is compatible with base change $T \to S$ and the reduction isomorphism when taking Cartier divisors. This is [7, Theorem 4.2], which states:

Theorem 5.1 (Ducrot)**.** *There exists a unique correspondence which associates to each integer n and each flat n-dimensional projective morphism of schemes $f : X \to S$ an $(n+2)$-cube structure $\mathcal{S}^{n+2}_{X/S}$ over the determinant of cohomology $\lambda_{X/S}$ such that*

(BC) the construction of $\mathcal{S}^{n+2}_{X/S}$ commutes with base changes $T \to S$.

(R) For any X/S of dimension n, any decorated cube $K \in (n+1)-\mathrm{Cub}_d(\mathrm{Pic}(X))$, and any relative Cartier divisor D, the restriction isomorphism

$$\sum \lambda_{X/S}(K - K(D)) \cong \sum \lambda_{D/S}(K(D)|_D)$$

identifies the trivializations of $\sum \lambda_{X/S}(K - K(D))$ and $\sum \lambda_{D/S}(K(D)|_D)$ induced by $\mathcal{S}^{n+2}_{X/S}$ and $\mathcal{S}^{n+1}_{D/S}$.

Where for a Cartier divisor D and a decorated $(n+1)$-cube $K = K_{L_0}(L_1, \dots, L_{n+1})$, $K(D)$ is the decorated $(n+1)$-cube $K_{L_0 \otimes \mathcal{O}_X(D)}(L_1, \dots, L_{n+1})$. In other words, for all $J \subseteq \{1, \dots n+1\}$, we have that

$$K(D)_J = K_J \otimes \mathcal{O}_X(D).$$

5.2.1 Ducrot's theorem in terms of canonical isomorphisms ※

Note that cube structures give canonical trivializations in the sense of §1.2.7, and so Ducrot's theorem states the existence of a canonical isomorphism. We now give a corollary of Ducrot's theorem that is stated in terms of canonical isomorphisms.

Lemma 5.1. *Let $X \to S$ be a flat d-dimensional projective morphism of schemes, let $L_1, \dots, L_k$ be line bundles on X for some $k \geq d+2$, and let M be a linear combination of line bundles on X with integer coefficients.*

Then we have a canonical trivialization

$$\lambda \left(M \otimes \bigotimes_{i=1}^{k} (\mathcal{O}_X - L_i) \right) \cong \mathcal{O}_S,$$

compatible with base change, taking Cartier Divisors as in (R), the canonical trivialization when one of the line bundles is $\mathcal{O}_X$, and with the canonical isomorphisms from replacing L_i with $L_i' \otimes L_i''$.

Proof. The first two compatibility conditions directly address the compatibility conditions of Theorem 5.1; while the other two compatibility conditions are related to the definition of cube structures.

Now, without loss of generality, we may let $k = d + 2$, as we can absorb $\bigotimes_{i=d+3}^{k}(\mathcal{O}_X - L_i)$ into M otherwise. We may similarly reduce to the case where M is a line bundle, since if $M = \sum_{j=1}^{\ell} n_j M_j$ then

$$\lambda\left(M \otimes \bigotimes_{i=1}^{k}(\mathcal{O}_X - L_i)\right) \cong \bigotimes_{j=1}^{\ell} \lambda\left(M_j \otimes \bigotimes_{i=1}^{k}(\mathcal{O}_X - L_i)\right)^{\otimes n_j},$$

and we can combine canonical trivializations with the tensor product.

Once we reduce to $k = d+2$ and M being a line bundle, it is enough to consider the trivialization isomorphism s_K for the $(d+2)$-cube

$$K = K_M(L_1, \dots, L_{d+2}),$$

given by the cube structure defined by Theorem 5.1.

□

5.2.2 Extending Ducrot's theorem to the analytic case ※

Recall that a cube structure $\mathcal{S}_{X/S}^{n+2}$ gives us a family of trivializations $s_K : \mathcal{O}_S \cong \sum \lambda K$ for each decorated cube K of line bundles over X. This trivialization is used by Rössler in [23] when he constructs his canonical isomorphism of line bundles. Since we aim to compute the norm of Rössler's isomorphism when the determinant of cohomology is endowed with the Quillen metric, we must extend Ducrot's theorem to the analytic case. This extension is Theorem 5.2.

Theorem 5.2. *Let $f : X^{\mathrm{an}} \to S^{\mathrm{an}}$ be a proper submersion of constant relative dimension of Hodge manifolds, and let X and S be the complex projective varieties whose analytifications are X^{an} and S^{an}.*

Take $\underline{L}_0, \dots, \underline{L}_{n+2}$ hermitian line bundles on X^{an}, where $\underline{L}_i = (L_i, h^{L_i})$, and consider the decorated $(n+2)$-cube $\underline{K} = K_{\underline{L}_0}(\underline{L}_1, \dots, \underline{L}_{n+2})$ of hermitian line bundles, and $K = K_{L_0}(L_1, \dots, L_{n+2})$ of line bundles on X. Let λ be the determinant of cohomology endowed with the Quillen metric.

Then the isomorphism $s_K : \mathcal{O}_S \cong \Sigma\lambda(K)$ from the cube structure $\mathcal{S}_{X/S}^{n+2}$ constructed in Theorem 5.1 extends to an isometry $s_{\underline{K}} : \underline{\mathcal{O}}_S \simeq \Sigma\lambda(\underline{K})$.

We give a proof of Theorem 5.2 in §5.3 by following Ducrot's construction of the cube structure and considering the norms at each step, once the determinant of cohomology is endowed with the Quillen metric.

We rewrite the result of Theorem 5.2 in the terms that we will use in §6.

Lemma 5.2. *Suppose that Y is a Hodge manifold, that $S = \operatorname{Spec}\mathbb{C}$, and let $\underline{M}$ and $\underline{J}$ be hermitian line bundles on Y. Then we have the following canonical isometry*

$$\lambda_Y(\underline{M} \otimes (\underline{\mathcal{O}}_Y - \underline{J})^{\otimes d+2}) \simeq \underline{\mathcal{O}}_S.$$

Proof. This immediately follows from Theorem 5.2, with $\underline{L}_0 = \underline{M}$ and $\underline{L}_i = \underline{J}$ for $1 \leq i \leq d+2$, in the case where $S = \operatorname{Spec}\mathbb{C}$. □

We will note that in §6 we only look at the absolute case, where $S = \operatorname{Spec}\mathbb{C}$, whereas in this section we still consider the relative case where S might have non-trivial dimension. This is because when computing the norms of isomorphisms, it is enough to consider the absolute case, since norms of isomorphisms are determined point-wise. However, in the proof of Theorem 5.2, we must consider S of higher dimensions when constructing the algebraic trivializations inductively, as such a more general relative statement is required for induction.

5.2.3 Ducrot's proof of Theorem 5.1 †

We now present a sketch of the proof of Theorem 5.1 that Ducrot gave in [7, §4]. At its core, it is a proof by induction on n, the relative dimension of X over S. However, the induction step is split into three different cases of increasing generality, leaving us with the following four parts:

1. The base case $n = 0$. See §5.3.1.
2. The induction step when K has one edge which is a Cartier divisor. See §5.3.2.
3. The case when K has two edges which are "sufficiently positive". See §5.3.3.
4. The general case. See §5.3.4.

At each part of the proof, Ducrot focuses on two objectives; firstly, he constructs a canonical trivialization $s_K : \mathcal{O}_S \to \sum \lambda K$ for cubes K of increasing generality; secondly he proves that the trivialization s_K is independent on any choice made during its construction, and depends only on the morphism $f : X \to S$ as well as the decorated cube K.

In the extension to the analytic case that we provide, we already have the isomorphisms s_K, and we only must prove that they are isometries once we endow the determinant of cohomology with the Quillen metric. In particular, we do not need to concern ourselves about the uniqueness of s_K, and can solely focus on Ducrot's first objective, the construction of s_K; in fact we make use of the freedom of choice we have when following Ducrot's construction in the first part.

Ducrot considers the base case separately in [7, Lemma 4.1.1], which states the following:

Lemma 5.3. *There exists an unique correspondence $\mathcal{S}$ which associates to each finite flat morphism $X \to S$ a square structure $\mathcal{S}^2_{X/S}$ on the determinant of cohomology functor $\lambda_{X/S} : \mathrm{Pic}(X) \to \mathrm{Pic}_*(S)$, and which is compatible with base changes $T \to S$.*

In this situation, given a decorated cube $K = K_{L_0}(L_1, L_2)$, he constructs a trivialization $s_K : \mathcal{O}_S \to \sum \lambda K$ locally by considering open neighborhoods U of S where $L_i|_U \cong \mathcal{O}_U$ for all $i \in \{0, 1, 2\}$. This is recalled in more detail in §5.3.1.

In his inductive argument, Ducrot first considers the case where one of the line bundles in the $(n+2)$-cube on X has a regular section. In this case, the line bundle is obtained from a Cartier divisor D, and so requirements of the condition (R) allows him to construct the trivialization isomorphism of the $(n+2)$-cube on X by combining the reduction isomorphism from §5.1.12 with the trivialization of an $(n+1)$-cube on D. This is detailed in §5.3.2.

Now, a line bundle L on X/S is said to be *sufficiently positive* if L is very ample relatively to $f : X \to S$, and if for each $k > 0$ we have that $R^k f_* L = 0$. In order to remove the assumption that one of the line bundles has a regular section, Ducrot first considers the case where at least two of the line bundles L_i, L_j in the $(n+2)$-cube $K = K_{L_0}(L_1, \ldots, L_{n+2})$, where $i, j \in \{1, \ldots, n+2\}$, are sufficiently positive.

In this case, Ducrot considers the base change to $P = \mathbb{P}(f_*L_1) \times_S \mathbb{P}(f_*L_2)$, and there are two open sets of P where line bundles on $X_P = X \times_S P$ have a regular section. In each open set there is a trivialization isomorphism, Ducrot then shows that these two isomorphisms match in the intersection of the two open sets, and can be extended uniquely and continuously to all of P. He finally shows how the trivialization isomorphism on X_P induces one on X. This is explained in full detail in §5.3.3.

To prove the general case, Ducrot uses the fact that for any line bundle L on X there exists a sufficiently positive line bundle M such that $L \otimes M$ is also sufficiently positive. This allows him to compare cubes where not enough sides are sufficiently positive with cubes where there are enough. This is reviewed in §5.3.4.

5.3 Proof of Theorem 5.2 ※

Let $f : X^{\mathrm{an}} \to S^{\mathrm{an}}$ be a proper submersion of Hodge manifolds of constant relative dimension n, and let X and S be the complex projective varieties whose analytifications are X^{an} and S^{an}. Take a Kähler metric on the relative tangent bundle $\mathcal{T}_f$ with associated Kähler form ω.

Let $\underline{L}_i = (L_i, h^{L_i})$, where $0 \leq i \leq n+2$, be $n+3$ hermitian line bundles on X. Let $\underline{A} = K_{\underline{L}_0}(\underline{L}_1, \dots, \underline{L}_{n+2})$ be an $(n+2)$-cube of hermitian line bundles on X^{an}, and $A = K_{L_0}(L_1, \dots, L_{n+2})$ be an $(n+2)$-cube of line bundles on X obtained by forgetting the hermitian metrics on $\underline{A}$.

Now, by Theorem 5.1, we have a trivialization isomorphism of line bundles $s_A : \mathcal{O}_S \to \sum \lambda A$, given by the $(n+2)$-cube structure on the determinant of cohomology functor λ. We then consider the analytic case, where we endow the determinant of cohomology with the Quillen metric. The trivialization isomorphism s_A naturally extends to an isomorphism $s_{\underline{A}} : \underline{\mathcal{O}}_S \to \sum \lambda \underline{A}$, and our goal is to prove that the norm of the isomorphism $s_{\underline{A}}$ vanishes, so that $s_{\underline{A}}$ is an isometry.

※ This proof closely follows [7, §4], extending it from the algebraic to the analytic case.

5.3.1 Base case ※

† Compare with [7, §4.1].

We begin with the case where $n = 0$ so that $f : X \to S$ is finite and flat; and we take a decorated square $A = K_{L_0}(L_1, L_2)$ in $\mathrm{Pic}(X)$. Ducrot in [7, Lemma 4.1.1] builds the trivialization isomorphism s_A locally on small enough open neighborhoods U of S, so that the line bundles are trivial, and thus isomorphic to each other. We therefore have four isomorphisms $u_{J,i} : A_J \cong A_{J\cup\{i\}}$, for $J \subseteq \{1, 2\}$, $0 \leq |J| \leq 1$ and $i \in \{1, 2\}\backslash J$, such that

$$u_{\emptyset,1} \otimes u^{\vee}_{\{2\},1} : A_{\emptyset} \otimes A_{\{1,2\}} \cong A_{\{1\}} \otimes A_{\{2\}}, \tag{5.9}$$

gives the decoration of A.

While the choice of $u_{J,i}$ is not unique, in [7, Lemma 4.1.1] it is proven that the isomorphism s_A constructed from these maps does not depend on this choice. We can thus scale $u_{\emptyset,1}$ and $u_{\{2\},1}$ by the same non-zero section, so that the decoration of A in (5.9) is not changed; and similarly with $u_{\emptyset,2}$ and $u_{\{1\},2}$.

Moving to the analytic case, using this scaling, we can choose without loss of generality that all $u_{J,i}$ are isometries; this is because $u_{\emptyset,1}$ and $u_{\{2\},1}$ will be isometries precisely when they both correspond to an isometry $u_1 : \underline{\mathcal{O}}_X \cong \underline{L}_1$, in that $u_{\emptyset,1} : \underline{L}_0 \cong \underline{L}_0 \otimes \underline{L}_1$ and $u_{\{2\},1} : \underline{L}_0 \otimes \underline{L}_2 \cong \underline{L}_0 \otimes \underline{L}_1 \otimes \underline{L}_2$, and similarly for $u_{\emptyset,2}$ and $u_{\{1\},2}$. We let $\underline{A}^{\bullet}$ be the 2-complex over $\underline{A}$ with these morphisms.

$$\begin{tikzcd}
 & & \underline{A}^{(1,0)} = \underline{L}_0 \otimes \underline{L}_1 \arrow[dr, "u_{\{1\},2}"] & & \\
0 \arrow[r] & \underline{A}^{(0,0)} = \underline{L}_0 \arrow[ur, "u_{\emptyset,1}"] \arrow[dr, "u_{\emptyset,2}"'] & & \underline{A}^{(1,1)} = \underline{L}_0 \otimes \underline{L}_1 \otimes \underline{L}_2 \arrow[r] & 0. \\
 & & \underline{A}^{(0,1)} = \underline{L}_0 \otimes \underline{L}_2 \arrow[ur, "u_{\{2\},1}"'] & &
\end{tikzcd}$$

Note that this means that the decoration of $\underline{A}$ is an isometry, and this is the case even if we didn't rescale the $u_{J,i}$, since in any case the norm of $u_{\emptyset,1}$ and $u_{\{2\},1}$ are the same, so that the norm of the decoration morphism $u_{\emptyset,1} \otimes u^{\vee}_{\{2\},1}$ is trivial.

Now, the complex $C(\underline{A}^{\bullet}) = 0 \to \underline{A}^{(0,0)} \to \underline{A}^{(1,0)} \oplus \underline{A}^{(0,1)} \to \underline{A}^{(1,1)} \to 0$ obtained by collapsing the 2-complex $\underline{A}^{\bullet}$ is a short exact sequence, since it corresponds to an orthogonal direct sum via the isometries $u_{J,i}$. By Theorem 4.2, we have an isometry

$$\lambda(\underline{A}^{(1,0)} \oplus \underline{A}^{(0,1)}) \simeq \lambda(\underline{A}^{(0,0)}) \otimes \lambda(\underline{A}^{(1,1)}),$$

which implies that the canonical trivialization $\lambda(0) : \underline{\mathcal{O}}_S \cong \lambda(C(\underline{A}^{\bullet}))$ is an isometry.

Furthermore, the morphism $\alpha_A : \lambda(C(\underline{A}^{\bullet})) \to \Sigma\lambda(\underline{A})$ obtained by extending the morphism α_A from §5.1.11 to the case of hermitian line bundles is clearly an isometry. This is because α_A is constructed as the inductive limit of morphisms $\alpha_{A,<}$ for each of the two orders of the set $\{0, 1\}$, and each of these can be verified to be an isometry since all the involved maps in the cube $\{0, 1\}$-complex $\underline{A}^{\bullet}$ are isometries.

Therefore, the trivialization $s_{A^{\bullet}} = \alpha_A \circ \lambda(0) : \underline{\mathcal{O}}_S \to \lambda(C(\underline{A}^{\bullet})) \to \Sigma\lambda(\underline{A})$ is an isometry. The isomorphism s_A is locally constructed in such way, and, since isomorphisms are isometries if and only if they are locally isometries, we conclude that in the case where $n = 0$ the isomorphism s_A is an isometry once we give the line bundles a hermitian metric.

5.3.2 $\underline{A}$ has at least one edge with a regular section ※

† Compare with [7, §4.3].

Recall that in Ducrot's proof of Theorem 5.1, he splits the induction step into three cases of increasing generality, the first being when one of the edges has a regular section. We now follow his construction of the trivialization isomorphism s_A and show that it induces an isometry $s_{\underline{A}}$ when extended to hermitian line bundles.

Suppose that the decorated $(n+2)$-cube $\underline{A} = K_{\underline{L}_0}(\underline{L}_1, \ldots, \underline{L}_{n+2})$ has an edge L_i with an f-regular section σ. This section defines an isomorphism $L_i \cong \mathcal{O}_X(D)$, where $D = \mathrm{div}(\sigma)$. Take a hermitian metric on $\mathcal{O}_X(D)$ such that the above isomorphism is an isometry. We can then write $\underline{A} \simeq (\underline{K} - \underline{K}(D))$, where $\underline{K} = Back_i(\underline{A}) = K_{\underline{L}_0}(\underline{L}_1, \ldots, \hat{\underline{L}}_i, \ldots, \underline{L}_{n+2})$ and $\underline{K}(D)_J \simeq \underline{K}_J \otimes \underline{L}_i$ for each $J \subseteq I = \{1, \ldots, n+2\} \backslash \{i\}$.

In the algebraic setting, Ducrot constructs the trivialization $s_{A,\sigma}$ as the composition of the reduction isomorphism $r_D : \sum \lambda_{X/S}(A) \to \sum \lambda_{D/S}(K(D)|_D)$ and the

$$\begin{tikzcd}
K_{\emptyset} \arrow[rr] \arrow[dr] \arrow[dd] & & K(D)_{\emptyset} \arrow[rr, dashed] \arrow[dr] \arrow[dd] & & (K(D)|_D)_{\emptyset} \arrow[dr] \arrow[dd] & \\
& K_{\{1\}} \arrow[rr] \arrow[dd] & & K(D)_{\{1\}} \arrow[rr, dashed] \arrow[dd] & & (K(D)|_D)_{\{1\}} \arrow[dd] \\
K_{\{2\}} \arrow[rr] \arrow[dr] & & K(D)_{\{2\}} \arrow[rr, dashed] \arrow[dr] & & (K(D)|_D)_{\{2\}} \arrow[dr] & \\
& K_{\{1,2\}} \arrow[rr] & & K(D)_{\{1,2\}} \arrow[rr, dashed] & & (K(D)|_D)_{\{1,2\}}
\end{tikzcd}$$

Figure 5.3: The cube $(n+2)$-cube $A = K - K(D)$ and the $(n+1)$-cube $K(D)|_D$, whose rows form short exact sequences $0 \to K_J \to K(D)_J \to (K(D)|_D)_J \to 0$ for each $J \subseteq \{1, \ldots, n+2\} \backslash \{i\}$.

trivialization $s_{K(D)|_D} : \sum \lambda_{D/S}(K(D)|_D) \to \mathcal{O}_S$ from the $(n+1)$-cube structure of $\lambda_{D/S}$. By induction on n, we assume that $s_{K(D)|_D}$ is an isometry. It is therefore enough to prove that r_D is an isometry to deduce that $s_{A,\sigma}$ is an isometry.

Now, the reduction isomorphism r_D is constructed by combining the isomorphisms of the form

$$\lambda_{X/S}(K_J)^{\vee} \otimes \lambda_{X/S}(K(D)_J) \cong \lambda_{D/S}((K(D)|_D)_J),$$

arising from the short exact sequences due to Cartier Divisors

$$0 \to K_J \to K(D)_J \to (K(D)|_D)_J \to 0, \tag{5.10}$$

for each subset $J \subseteq I = \{1, \ldots, n+2\} \backslash \{i\}$, in the alternating sum defined by $\sum \lambda_{X/S}(A)$ (see Figure 5.3). Note that the sequence (5.10) is the standard short exact sequence (1.1) from §1.2.5.

We can therefore use Bismut's immersion formula for Cartier divisors, as described in §4.3.5, to compute the norm of the reduction isomorphism r_D. Note that since the norm of the isomorphism is computed pointwise on S, it is enough to prove the norm vanishes for the case where $S = \operatorname{Spec} \mathbb{C}$; alternatively we might instead use a relative version of Bismut's immersion formula. We have isomorphisms

$$\lambda_{X/S}(\underline{K}_J)^{\vee} \otimes \lambda_{X/S}(\underline{K}(D)_J) \overset{(a_J)}{\approx} \lambda_{D/S}((\underline{K}(D)|_D)_J),$$

whose norms are given by Bismut's immersion formula to be

$$(a_J) = -\int_{X/S} P_1 \operatorname{ch}(\underline{K}_J) + \int_{D/S} P_2 \iota^* \operatorname{ch}(\underline{K}_J) - \int_{X/S} P_3 \overline{\operatorname{ch}}(\underline{K}_J) + \int_{D/S} P_4 \iota^* \overline{\operatorname{ch}}(\underline{K}_J), \quad (5.11)$$

where

$$\begin{aligned}
P_1 &= \operatorname{td}(\underline{\mathcal{T}}_X) \mathfrak{BC}(\underline{L} \to \underline{\mathcal{O}}_X) \operatorname{ch}(\underline{\mathcal{O}}_X(D)),\\
P_2 &= \operatorname{td}_g^{-1}(\underline{N}_{X/D}) \widetilde{\operatorname{td}}(\underline{\mathcal{T}}_D, \underline{\mathcal{T}}_X|_D, \underline{N}_{X/D}) \operatorname{ch}(\iota^* \underline{\mathcal{O}}_X(D)),\\
P_3 &= \overline{\operatorname{td}}(\underline{\mathcal{T}}_X) R(\underline{\mathcal{T}}_X) (\overline{\operatorname{ch}}(\underline{\mathcal{O}}_X) - \overline{\operatorname{ch}}(L)) \overline{\operatorname{ch}}(\underline{\mathcal{O}}_X(D)),\\
P_4 &= \overline{\operatorname{td}}(\underline{\mathcal{T}}_D) R(\underline{\mathcal{T}}_D) \overline{\operatorname{ch}}(\iota^* \underline{\mathcal{O}}_X(D)),
\end{aligned}$$

are constants that depend only on the immersion of D in X, not on J.

Now, note that

$$\sum_{J \subseteq I} (-1)^{|J|} \int_{X/S} P_1 \operatorname{ch}(\underline{K}_J) = \int_{X/S} \left[P_1 \operatorname{ch}(\underline{L}_0) \operatorname{ch}\left(\bigotimes_{j \in I} (\underline{\mathcal{O}}_X - \underline{L}_j) \right) \right] = 0,$$

which vanishes by our cancellation Lemma 2.6, since $|I| = n+1$. By the same argument we have that $\sum_{J \subseteq I} (-1)^{|J|} \int_{D/S} P_2 \operatorname{ch}(\iota^* \underline{K}_J)$ vanishes. Using a similar argument, and by Lemma 2.7, both $\sum_{J \subseteq I} (-1)^{|J|} \int_{X/S} P_3 \overline{\operatorname{ch}}(\underline{K}_J)$ and $\sum_{J \subseteq I} (-1)^{|J|} \int_{D/S} P_4 \overline{\operatorname{ch}}(\iota^* \underline{K}_J)$ vanish. By combining the terms of (5.11), we deduce that $\sum_{J \subseteq I} (-1)^{|J|} (a_J)$ vanishes.

Therefore, the norm of the reduction isomorphism r_D, given by

$$\sum \lambda_{X/S}(\underline{A}) \overset{(b)}{\approx} \sum \lambda_{D/S}(\underline{K}(D)|_D),$$

where $(b) = \sum_{J \subseteq I} (-1)^{|J|} (a_J)$ vanishes, and so r_D is an isometry, so that $s_{A,\sigma}$ is an isometry.

5.3.3 $\underline{A}$ has at least two sufficiently positive edges ※

† Compare with [7, §4.4].

Recall that a line bundle L on X is said to be sufficiently positive if it is both very ample relatively to $f : X \to S$, and for each $i > 0$ we have $R^i f_* L = 0$. Note that for every line bundle L there exists a sufficiently positive line bundle M such that $L \otimes M$ is sufficiently positive.

Following Ducrot's proof of Theorem 5.1, we now suppose that $\underline{A}$ has two edges, $\underline{L}_i$ and $\underline{L}_j$, such that L_i and L_j are sufficiently positive, and show that the trivialization isomorphism $s_A^{i,j}$ of $\sum \lambda(A)$ constructed in [7][§4.4] gives an isometric trivialization of $\sum \lambda(\underline{A})$.

Let $P_i = \mathbb{P}(f_* L_i)$ and $P_j = \mathbb{P}(f_* L_j)$. Then let $P = P_i \times_S P_j$ and consider the base change

$$\begin{tikzcd}
X_P \arrow[r, "g"] \arrow[d, "f"] & X \arrow[d, "f"] \\
P \arrow[r, "h"] & S
\end{tikzcd}$$

We then let $L'_k = g^* L_k \otimes f^* \mathcal{O}_{P_k}(1)$, for $k = i, j$, which have canonical sections σ_i, σ_j respectively. We then let $U_i \subset P_i$ (respectively $U_j \subset P_j$) be the open subset subset of P_i where σ_i is f-regular, and let $U = (U_i \times P_j) \cup (P_i \times U_j) \subset P$.

The norm on $\underline{L}_k$ together with the Kähler form ω induce an L_2 metric on $f_* L_k$, which gives us a hermitian vector bundle $f_* \underline{L}_k$. This hermitian metric induces hermitian metrics on each 1-dimensional subbundle of $f_* \underline{L}_k$, and therefore on each fiber of the tautological bundle $\mathcal{O}_{P_k}(-1)$, which gives us a natural metric to define the hermitian vector bundle $\underline{\mathcal{O}}_{P_k}(-1)$, from which we can define a hermitian metric on $\mathcal{O}_{P_k}(1)$, giving $\underline{\mathcal{O}}_{P_k}(1)$. Let $\underline{L}'_k := g^* \underline{L}_k \otimes f^* \underline{\mathcal{O}}_{P_k}(1)$.

Writing

$$\underline{A} = \begin{tikzcd}
\underline{K} \otimes \underline{L}_i \arrow[r, dash] \arrow[d, dash] & \underline{K} \otimes \underline{L}_i \otimes \underline{L}_j \arrow[d, dash] \\
\underline{K} \arrow[r, dash] & \underline{K} \otimes \underline{L}_j
\end{tikzcd}, \text{ and } \underline{A}' = \begin{tikzcd}
g^* \underline{K} \otimes \underline{L}'_i \arrow[r, dash] \arrow[d, dash] & g^* \underline{K} \otimes \underline{L}'_i \otimes \underline{L}'_j \arrow[d, dash] \\
g^* \underline{K} \arrow[r, dash] & g^* \underline{K} \otimes \underline{L}'_j
\end{tikzcd},$$

we can construct, following §5.3.2, a trivialization s_i of $\sum \lambda_{U_i \times P_j / S}(\underline{A}')$ over $U_i \times P_j$ by using the regular section σ_i, as well as a trivialization s_j constructed analogously for $\sum \lambda_{P_i \times U_j / S}(\underline{A}')$.

Both s_i and s_j are isometries, and, as in [7][Lemma 4.4.3], we know that there exists a unique trivialization s' of $\sum \lambda_{X_P/S}(\underline{A}')$ over P extending s_i and s_j, which is continuous. It follows that s' is an isometry, since it is given by a continuous section whose norm is 1 on a dense open subset.

Then, following [7, §4.4.4], we have the following sequence of canonical isometries, where we start by expanding the square representation of $\underline{A}'$,

$$\sum \lambda_{X_P/P}(\underline{A}')$$

$$\simeq \bigotimes_{J\subseteq\{i,j\}} \left(\sum \lambda_{X_P/P}\left(g^*\underline{K}\otimes \bigotimes_{k\in J} \underline{L}'_k\right)\right)^{\otimes(-1)^{|J|}},$$

Then, we recall that $\underline{L}'_k := g^*\underline{L}_k \otimes f^*\underline{\mathcal{O}}_{P_k}(1)$,

$$\simeq \bigotimes_{J\subseteq\{i,j\}} \left(\sum \lambda_{X_P/P}\left(g^*\underline{K}\otimes \bigotimes_{k\in J} g^*\underline{L}_k \otimes f^*\underline{\mathcal{O}}_{P_k}(1)\right)\right)^{\otimes(-1)^{|J|}},$$

We then let $\chi = \sum \chi_{X/S}(K\otimes L_i\otimes L_j) = \sum \chi_{X/S}(K\otimes L_i) = \sum \chi_{X/S}(K\otimes L_j)$, the equality due to $\chi_{X/S}$ itself being endowed with an $(n+1)$-cube structure as in [7, §1.7.2], and use the projection formula from Lemma 4.6,

$$\simeq \bigotimes_{J\subseteq\{i,j\}} \left(\sum \lambda_{X_P/P}\left(g^*\underline{K}\otimes \bigotimes_{k\in J} g^*\underline{L}_k\right) \otimes \bigotimes_{k\in J} \underline{\mathcal{O}}_{P_k}(1)^{\chi}\right)^{\otimes(-1)^{|J|}}$$

Then, we use the cancellation $(\underline{\mathcal{O}}_{P_i}(1)^{\chi}\otimes\underline{\mathcal{O}}_{P_j}(1)^{\chi})\otimes(\underline{\mathcal{O}}_{P_i}(1)^{\chi})^{\vee}\otimes(\underline{\mathcal{O}}_{P_j}(1)^{\chi})^{\vee}\simeq\underline{\mathcal{O}}$,

$$\simeq \bigotimes_{J\subseteq\{i,j\}} \left(\sum \lambda_{X_P/P}\left(g^*\underline{K}\otimes \bigotimes_{k\in J} g^*\underline{L}_k\right)\right)^{\otimes(-1)^{|J|}}$$

We then use base change,

$$\simeq \bigotimes_{J\subseteq\{i,j\}} h^*\left(\sum \lambda_{X/S}\left(\underline{K}\otimes \bigotimes_{k\in J} \underline{L}_k\right)\right)^{\otimes(-1)^{|J|}}$$

And we end by noting that we have the square representation of $\underline{A}$.

$$\simeq h^*\sum \lambda_{X/S}(\underline{A})$$

Overall, we are left with an isometry

$$\sum \lambda_{X_P/P}(\underline{A}') \simeq h^*\sum \lambda_{X/S}(\underline{A}) \tag{5.12}$$

By composing this isometry (5.12) with s, we obtain an isometric trivialization of $h^*\sum\lambda_{X/S}(\underline{A})\simeq\underline{\mathcal{O}}_P$, which will correspond to an isometric trivialization $\sum\lambda_{X/S}(\underline{A})\simeq\underline{\mathcal{O}}_S$, since $h_*\underline{\mathcal{O}}_P\simeq\underline{\mathcal{O}}_S$ (see [7, §4.4.4]) and by using h^*-h_* adjunction.

5.3.4 Removing the positivity hypothesis on $\underline{A}$ ※

In the algebraic case, to generalize to the case where A is any cube of line bundles, Ducrot constructs the trivialization isomorphism s_A by using the trivializations of cubes which have one more side which is sufficiently positive. By doing this step twice, we can ensure we are working with cubes with at least two sides which are sufficiently positive, and so we may use the previous case.

Suppose that A has an edge L_i which is not sufficiently positive. Then there exists a sufficiently positive line bundle M such that $L_i \otimes M$ is also sufficiently positive. We let B be $K_{L_0 \otimes L_i}(L_1, \ldots, L_{i-1}, M, L_{i+1}, \ldots, L_{n+2})$, so we have an isomorphism $u : Front_i(A) \cong Back_i(B)$. We also consider the gluing $A \star_u B := K_{L_0}(L_1, \ldots, L_{i-1}, L_i \otimes M, L_{i+1}, \ldots, L_{n+2})$. See Figure 5.4.

$$\begin{tikzcd}
K_\emptyset \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & K_\emptyset \otimes L_i \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & K_\emptyset \otimes L_i \otimes M \arrow[dr, dash] \arrow[dd, dash] & \\
& K_{\{1\}} \arrow[rr, dash] \arrow[dd, dash] & & K_{\{1\}} \otimes L_i \arrow[rr, dash] \arrow[dd, dash] & & K_{\{1\}} \otimes L_i \otimes M \arrow[dd, dash] \\
K_{\{2\}} \arrow[rr, dash] \arrow[dr, dash] & & K_{\{2\}} \otimes L_i \arrow[rr, dash] \arrow[dr, dash] & & K_{\{2\}} \otimes L_i \otimes M \arrow[dr, dash] & \\
& K_{\{1,2\}} \arrow[rr, dash] & & K_{\{1,2\}} \otimes L_i \arrow[rr, dash] & & K_{\{1,2\}} \otimes L_i \otimes M \\
& & \underbrace{\hphantom{K_{\{1,2\}} \otimes L_i \otimes M \otimes L_i \otimes M}}_{A} & & & \\
& & & \underbrace{\hphantom{K_{\{1,2\}} \otimes L_i \otimes M \otimes L_i \otimes M}}_{B} & & \\
& & & \underbrace{\hphantom{K_{\{1,2\}} \otimes L_i \otimes M \otimes L_i \otimes M \otimes L_i \otimes M}}_{A \star_u B} & &
\end{tikzcd}$$

Figure 5.4: $A \star_u B$ when $n = 1$. For higher n, the picture is similar but higher dimensional.

Then both B and $A \star_u B$ will have an additional edge which is sufficiently positive, given by M and $L_i \otimes M$ respectively both being sufficiently positive. This allows us to consider a short induction on the number of sides which are not sufficiently positive, noting that the case where at least two sides are sufficiently positive is considered in §5.3.3. Now, given trivializations s_B and $s_{A \star_u B}$ by induction, there exists a unique trivialization s of A such that $\rho_u \circ (s \otimes s_B) = s_{A \star_u B}$, which we call $s_{A \star_u B} \otimes s_B^\vee$, where ρ_u is the canonical gluing isomorphism given in (5.6). In this way Ducrot constructs a trivialization for A, which he then proves in [7, Lemma 4.5.1] is independent of the choices of i, B and u.

We now look at the case where we endow the line bundles with hermitian metrics. We can assume that $u : Front_i(A) \cong Back_i(B)$ is an isometry, by endowing $K_J \otimes L_i$ with the same metrics in both $Front_i(A)$ and $Back_i(B)$. Then ρ_u is an isometry since it only expresses cancellations arising from comparing $Front_i(A)$ and $Back_i(B)$.

Finally, supposing that s_B and $s_{A \star_u B}$ are isometries by induction, we can compute the norm of s pointwise by using the equation $\rho_u \circ (s \otimes s_B) = s_{A \star_u B}$, giving us norm 1 at each point in S, so that s is also an isometric trivialization.

Chapter 6

Involution formula for Cartier divisors

In [23], Rössler gives a refinement of the Adams Riemann Roch theorem in degree 1. In his method of proof, he follows an insight from Nori's [22], where he noted that one can recover the Adams Riemann Roch theorem for ψ^2 when studying the $\mathbb{Z}/2$-equivariant scheme $X \times X$ whose action swaps the coordinates in the Cartesian product.

The core of Rössler's proof is an involution formula, [23, Theorem 6.1], whose statement is recalled in §3.1.3 and in §6.1.1. As detailed in §6.1, Rössler's proof of this involution formula is split into two cases; firstly when the fixed point subscheme of the involution is a Cartier divisor, and secondly when the fixed point subscheme is of higher codimension; where he considers the blow-up to make use of the previous case.

Our aim is to extend Rössler's involution formula to the analytic case, in the case where the fixed point subscheme is a Cartier divisor; and we do so in §6.2. To extend Rössler's involution formula to the general case, and therefore be able to extend Rössler's refinement of the Adams Riemann Roch formula to the analytic case, one would need to study the behavior of the equivariant Quillen metric when considering blow-ups, which is beyond the scope of this paper.

6.1 Rössler's involution formula †

In order to prove his refinement of the Adams Riemann Roch theorem in degree 1, Theorem 3.1, Rössler makes use of an involution formula, Theorem 3.3. In this section, we will examine the proof of Rössler's involution formula in detail. In §6.2, we will then retrace some of the steps in this proof to extend it to the analytic case, but only in the case where the fixed point scheme is a Cartier divisor.

In §6.1.1 we briefly recall the statement of Theorem 3.3. In §6.1.2 we recall Rössler's proof in the case where the fixed point subscheme is a Cartier divisor, which we extend to the analytic case in §6.2. In §6.1.3 we recall Rössler's proof of the general case, where the fixed point subscheme is of higher codimension.

† In this section, we recall the proof of [23, Theorem 6.1].

6.1.1 Outline of Rössler's involution formula †

We are in the algebraic case. Let $G = \mathbb{Z}/2$, and S and X be locally Noetherian G-equivariant schemes where 2 is invertible, and so that the action of G on S is trivial. Let $f : X \to S$ be a smooth, locally projective, separated, G-equivariant morphism of constant relative dimension d and of finite type. Suppose that the G-orbit of every point in X is contained in an affine subscheme.

We let $Z = X_G$ be the fixed point scheme, and $\iota : Z \to X$ be the immersion. Suppose that the induced $Z \to S$ is flat, which implies Z is regularly immersed in X. Let $\mathcal{C}$ be the conormal bundle on Z of the immersion $Z \to X$, endowed with its natural G-equivariant structure (see §4.1.5). We also let $Y = X/G$ be the quotient scheme and $q : X \to Y$ be the quotient morphism.

Rössler's involution formula, Theorem 3.3, states that, for each G-equivariant line bundle M on X, we have a canonical isomorphism

$$\lambda_X^{\mathrm{eq}}(M)^{\otimes 2^{d+1}} \cong \bigotimes_{j=0}^{d} \lambda_Z^{\mathrm{eq}}(\iota^*(M) \otimes \mathrm{Sym}^j(\mathcal{C}))^{\otimes \sum_{i=0}^{d} \binom{d+1}{i}},$$

compatible with base change.

Rössler proves this involution formula in two steps; first starting by considering the case where Z is a Cartier divisor, and then continuing by considering the general case.

6.1.2 The case where Z is a Cartier divisor †

In the case where Z is a Cartier divisor, we have two additional properties to work with. Firstly, as noted in §4.1.5, [23, Proposition 2.5.(1)] implies that the quotient morphism $q : X \to Y$ is flat, which means for a vector bundle E, $R^\bullet q_* E$ is a perfect complex.

Secondly, letting $L \coloneqq \mathcal{O}(-Z)$, we have the canonical short exact sequence (1.2) from §1.2.5,

$$0 \to L \otimes M \to M \to \iota_* \iota^* M \to 0.$$

Rössler writes *"The existence of this sequence, unspectacular as it may seem, is the linchpin of the proof"*, which in our case applies both here and in §5.3.2, where we examined Ducrot's theorem.

Rössler starts his proof by defining a family of polynomials with integer coefficients

$$P_k(t) \coloneqq \sum_{j=0}^{k} 2^{k-j}(2-t)^j \in \mathbb{Z}[t],$$

and noting the polynomial identity

$$tP_k(t) = 2^{k+1} - (2-t)^{k+1}. \tag{6.1}$$

Following one of the referees of [23], who provided a simplification, one can prove this identity by setting $2q = 2 - t$, so that

$$\begin{aligned} tP_k(t) &= 2(1-q) \cdot 2^k(1 + q + q^q + \cdots + q^k) \\ &= 2^{k+1}(1 - q^{k+1}) \\ &= 2^{k+1} - (2-t)^{k+1}. \end{aligned}$$

Then, Rössler notes that, by definition or adjunction formula, $\iota^* L \cong \mathcal{C}$, and defines $J \coloneqq (q_* L\{-1\})_0$. He further notes that G acts on $\mathcal{C}$ by -1, and that $q^* J \cong L\{-1\}$. Note that Rössler defines L as a vector bundle, and so implicitly endows it with the trivial equivariant structure.

He then combines a chain of canonical isomorphisms to prove that there is a canonical isomorphism

$$\lambda_Z^{\mathrm{eq}}(\iota^* M \otimes P_k(\mathcal{O}_Z - \mathcal{C})) \cong \lambda_X^{\mathrm{eq}}(M)^{\otimes 2^{k+1}} \otimes \lambda_Y^{\mathrm{eq}}(q_* M \otimes (\mathcal{O}_Y - J)^{\otimes k+1})^\vee.$$

Our main computation in §6.2.7 closely follows this chain of canonical isomorphisms, with the exception that we use the polynomial

$$p_n(x) \coloneqq \sum_{j=0}^{n} \left(\sum_{k=0}^{n-j} \binom{n+1}{k} \right) x^j = P_n(1-x),$$

from §3.2.2, instead of P_n.

Next, Rössler uses Ducrot's theorem, together with a few more algebraic manipulations, to prove that when $k = d$, the relative dimension of X over S, there is a canonical trivialization

$$\lambda_Y^{\mathrm{eq}}(q_* M \otimes (\mathcal{O}_Y - J)^{\otimes k+1}) \cong O_S.$$

Our work in §5 will allow us to move from the algebraic to the analytic case in this step.

This therefore leaves us with a canonical isomorphism

$$\lambda_X^{\mathrm{eq}}(M)^{\otimes 2^{k+1}} \cong \lambda_Z^{\mathrm{eq}}(\iota^* M \otimes P_k(\mathcal{O}_Z - \mathcal{C})). \tag{6.2}$$

From here, Rössler considers the case where Z is not a Cartier divisor, but we'll note that it is simple to get from this canonical isomorphism to the one we wish to prove,

$$\lambda_X^{\mathrm{eq}}(M)^{\otimes 2^{d+1}} \cong \bigotimes_{j=0}^{d} \lambda_Z^{\mathrm{eq}}(\iota^*(M) \otimes \mathrm{Sym}^j(\mathcal{C}))^{\otimes \sum_{i=0}^{d} \binom{d+1}{i}},$$

by using the fact that when Z is a Cartier divisor, $\mathcal{C}$ is a line bundle and so $\mathrm{Sym}^j(\mathcal{C}) \cong \mathcal{C}^{\otimes j}$; together with the combinatorial simplification in Lemma 3.1, which we communicated to Rössler, and is found in [23, Lemma 6.3].

6.1.3 The case where Z is not a Cartier divisor †

In the general case, where the fixed point subscheme Z is not a Cartier divisor, Rössler considers the blow-up $b : \widetilde{X} \to X$ of X along Z, and notes that the exceptional divisor E is the fixed point subscheme of $\widetilde{X}$, a Cartier divisor on $\widetilde{X}$, and isomorphic to the projectivisation of the conormal bundle on Z, $\mathbb{P}(\mathcal{C})$.

Note that then the previous case gives us a canonical isomorphism analogous to (6.2) working with $\widetilde{X}$ and E instead of X and Z. Rössler shows that one can move between $\widetilde{X}$ and X, or between E and Z, by making use of the projection formula, giving us the following canonical isomorphisms, where $\mu : E \to \widetilde{X}$ is the immersion, $p : E \to Z$ is the projection, and $\widetilde{\mathcal{C}}$ is the conormal bundle of E in $\widetilde{X}$,

$$\lambda_X^{\mathrm{eq}}(M)^{\otimes 2^{k+1}} \cong \lambda_{\widetilde{X}}^{\mathrm{eq}}(b^*M)^{\otimes 2^{k+1}},$$

$$\lambda_E^{\mathrm{eq}}(\mu^*b^*M \otimes P_k(\mathcal{O}_E - \widetilde{\mathcal{C}})) \cong \lambda_Z^{\mathrm{eq}}(\iota^*M \otimes R^\bullet p_* P_k(\mathcal{O}_E - \mathcal{C}_{E/\widetilde{X}})).$$

With the final addition that there is a canonical isomorphism

$$R^\bullet p_*(\widetilde{\mathcal{C}}^{\otimes j}) \cong \mathrm{Sym}^j(\mathcal{C}),$$

we can construct the canonical isomorphism required by going up from X to $\widetilde{X}$ using the projection formula, then using the previous case to go to E, then use the projection formula again to go back down to Z.

We will note that, if one were to consider the analytic case, one would be able to use the equivariant Bismut submersion formula to consider the norm of the isomorphism involving Z and E; however when considering the isomorphism involving $\widetilde{X}$ and X, one would need a formula which considers the behavior of the determinant of cohomology, endowed with the Quillen metric, when considering a blow-up, in the equivariant case. We were unable to find such formula in the literature. We conjecture that it is possible to go around this requirement by considering a different construction, but this would require changes to the algebraic case in addition to the consideration of the analytic one.

6.1.4 Analytic involution formula for Cartier divisors ※

Suppose that X^{an} is a Hodge manifold of dimension d, with Kähler metric with an associated Kähler form ω, which gives us a Kähler metric on the tangent bundle $\underline{\mathcal{T}}_X = (\mathcal{T}_X, h^{\mathcal{T}_X})$. Let $G = \mathbb{Z}/2\mathbb{Z}$ and suppose that X^{an} has a G-equivariant structure. Let $S = \operatorname{Spec}\mathbb{C}$ be endowed with the trivial G-action, and let $f : X^{\mathrm{an}} \to S^{\mathrm{an}}$ be the canonical submersion.

Let $Z^{\mathrm{an}} := X_G^{\mathrm{an}}$ be the fixed points of X, where $\iota : Z^{\mathrm{an}} \to X^{\mathrm{an}}$ is the immersion, and suppose that Z^{an} is a complex submanifold of codimension one, and that $f \circ \iota : Z^{\mathrm{an}} \to S^{\mathrm{an}}$ is a submersion. As in §4.1.5, we can consider the tangent bundle $\mathcal{T}_Z$ as a vector subbundle of $\mathcal{T}_X|_Z$, and so the hermitian metric on $\mathcal{T}_X$

induces a hermitian metric $h^{\mathcal{T}_Z}$ on $\mathcal{T}_Z$, giving us $\underline{\mathcal{T}}_Z = (\mathcal{T}_Z, h^{\mathcal{T}_Z})$. It is easy to check that $h^{\mathcal{T}_Z}$ is a Kähler metric.

Let $\mathcal{C} \coloneqq \mathcal{C}_{Z/X}$ and $\mathcal{N} \coloneqq \mathcal{N}_{Z/X}$ be the conormal and normal bundles of $\iota : Z \to X$, and recall that by [23, Proposition 2.5.(2)] the natural G-equivariant structure on $\mathcal{C}$ and $\mathcal{N}$ satisfies $\mathcal{C}_0 = 0$ and $\mathcal{N}_0 = 0$. Let $\underline{\mathcal{C}} = (\mathcal{C}, h^{\mathcal{C}})$ and $\underline{\mathcal{N}} = (\mathcal{N}, h^{\mathcal{N}})$ be the conormal and normal bundles endowed with the metrics induced by the metric on the tangent bundle $\underline{\mathcal{T}}_X$, as described in §4.1.5. Note that Rössler uses N to denote the conormal bundle $\mathcal{C}$ in [23, §6], following the convention from EGA [15, Def 16.1.2]; while in Bismut's and Ma's formulae N denotes the normal bundle $\mathcal{N}$. We choose follow the convention from [26, §29.31].

Theorem 6.1. *Let $\underline{M}$ be a G-equivariant hermitian line bundle on X, such that $(\underline{M}|_Z)_1 = 0$. Then there exists a canonical isomorphism*

$$\lambda_X^{\mathrm{eq}}(\underline{M})^{\otimes 2^{d+1}} \overset{(n)}{\approx} \bigotimes_{j=0}^{d} \lambda_Z^{\mathrm{eq}}(\iota^*\underline{M} \otimes \underline{\mathcal{C}}_{Z/X}^{\otimes j})^{\otimes \sum_{i=0}^{d-j}\binom{d+1}{i}},$$

whose norm is given by

$$(n) = \int_Z 2^{d+1}\, \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X)\overline{\mathrm{ch}}(\iota^*\underline{M})\left(R(\underline{\mathcal{T}}_Z)\overline{c}^1(\underline{\mathcal{N}})^{-1} - R_g(\underline{\mathcal{T}}_X)\right)$$

Here $\overline{\mathrm{ch}}(\cdot)$ denotes the Chern form class recalled in §2.3.8; $\overline{\mathrm{td}}_g(\cdot)$ denotes the equivariant Todd form class recalled in §4.1.7; $R(\cdot)$ denotes the R-genus, and $R_g(\cdot)$ denotes the equivariant R-genus, both recalled in §4.3.3. The expression $\overline{c}^1(\underline{\mathcal{N}})^{-1}$ is formal division by $\overline{c}^1(\underline{\mathcal{N}})$, the class of $c^1(\underline{\mathcal{N}})$ in $\bigoplus_{p\geq 0} \widetilde{\mathfrak{A}}^p(X)$, which is a factor of $\overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X)$ by §4.1.8 and Lemma 2.3.

Note that the norm (n) does not depend on the choice of hermitian metric on M, or on the choice of Kähler metric on the tangent bundle $\mathcal{T}_X$, though the Quillen metrics on the determinants of cohomology do depend on the metric on $\mathcal{T}_X$.

6.2 Proof of Theorem 6.1 ※

Our proof of Theorem 6.1 is mainly concentrated in §6.2.7, where we construct a chain of canonical isomorphisms and consider the norms at each step. The norm obtained after the chain of canonical isomorphisms is then simplified in §6.2.8.

In the first three sections, we introduce the geometry we are working with; such as the line bundle $L \coloneqq \mathcal{O}_X(-Z)$ in §6.2.1; the quotient $q : X \to Y$ and the equivariant decomposition of $q_*\mathcal{O}_X$ in §6.2.2; and how the quotient is a branched covering in §6.2.3.

Before we do our final computation, we consider the norms of the isomorphisms that might not be isometries; first by using Bismut's immersion formula in §6.2.4; then the anomaly formula in §6.2.5; and then Ma's branched covering formula in §6.2.6.

※ This proof follows [23, Theorem 6.1], extending it from the algebraic to the analytic case.

6.2.1 The line bundle $L \coloneqq \mathcal{O}_X(-Z)$ ※

We recall the short exact sequence due to a Cartier divisor (1.2), which means that, using $L \coloneqq \mathcal{O}_X(-Z)$, we have a short exact sequence of vector bundles

$$0 \to M \otimes L \to M \to \iota_*\iota^* M \to 0. \tag{6.3}$$

Note that since L is constructed as a sheaf of ideals of $\mathcal{O}_X$, the immersion $L \to \mathcal{O}_X$ induces a natural G-equivariant structure on L, and that (6.3) is also a short exact sequence of G-equivariant vector bundles.

Recall from §1.2.5 that

$$\iota^* L = \mathcal{O}_X(-Z)|_Z = \mathcal{C}_{Z/X} = \mathcal{C},$$

And in particular, since $\mathcal{C}_0 = 0$ (see §4.1.5), then $\iota^*(L\{-1\})$ has the trivial action. Furthermore, we choose a metric h_B on L, with $\underline{L} \coloneqq (L, h_B)$, which satisfies Bismut's assumption (A) for the resolution $0 \to \underline{L} \to \underline{\mathcal{O}}_X \to 0$ of $\iota_*\underline{\mathcal{O}}_Z$. Note that in particular, since metrics that satisfy Bismut's assumption (A) are compatible with the metric on the normal bundle $\underline{\mathcal{N}}$, we have a canonical isometry

$$\iota^*\underline{L} \simeq \underline{\mathcal{C}}.$$

We will note that then

$$\mathrm{ch}_g(\underline{L}) = \mathrm{ch}((\underline{L}|_Z)_0) - \mathrm{ch}((\underline{L}|_Z)_1) = \mathrm{ch}(\underline{\mathcal{C}}_0) - \mathrm{ch}(\underline{\mathcal{C}}_1) = -\,\mathrm{ch}(\underline{\mathcal{C}}).$$

6.2.2 The decomposition $q_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus F$ ※

By Lemma 4.5, we have the quotient $Y^{\mathrm{an}} = X^{\mathrm{an}}/G$, which is a complex manifold with a submersion $Y^{\mathrm{an}} \to S^{\mathrm{an}}$. Furthermore, we let $q : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the quotient map, then $q \circ \iota : Z^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a closed immersion, and $q : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a branched cover whose branching points are Z^{an}.

By [23, Corollary 2.2], q is finite. Since q is finite and flat, and Y is locally Noetherian, then by [26, Lemma 29.48.2] q is *finite locally free*, which means that $q_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module, so $q_*\mathcal{O}_X$ is a vector bundle. Note that $\iota^*\mathcal{O}_X \cong \mathcal{O}_Z$ has the trivial G-action, and so by Lemma 4.3 we have a canonical isomorphism

$$q^*(q_*\mathcal{O}_X)_0 \cong \mathcal{O}_X.$$

Now, since q is flat and surjective, it is faithfully flat, and so the pullback q^* is faithful. This means that, given the canonical isomorphism

$$q^*\mathcal{O}_Y \cong \mathcal{O}_X \cong q^*(q_*\mathcal{O}_X)_0,$$

we then have a canonical isomorphism

$$\mathcal{O}_Y \cong (q_*\mathcal{O}_X)_0.$$

Now let $F \coloneqq (q_*\mathcal{O}_X)_1\{-1\}$ so that $q_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus F$ as an equivariant vector bundle, using the equivariant decomposition described in §4.1.3. By taking the pushforward of (6.3), with $M = \mathcal{O}_X$, and by left exactness of the pushforward, we have the following exact sequence:

$$0 \to q_*L \to q_*\mathcal{O}_X \to (q\iota)_*\mathcal{O}_Z. \tag{6.4}$$

Now $q\iota : Z \to Y$ is a closed immersion of G-equivariant varieties with trivial action, so $(q\iota)_*\mathcal{O}_Z \cong ((q\iota)_*\mathcal{O}_Z)_0$ and $((q\iota)_*\mathcal{O}_Z)_1 = 0$. Since $(\bullet)_1$ is exact (see [23, §4]), by applying $(\bullet)_1$ to (6.4) we obtain a canonical isomorphism

$$0 \to (q_*L)_1 \to (q_*\mathcal{O}_X)_1 \to 0.$$

It follows that $(q_*L\{-1\})_0 \cong F\{-1\}$. Using Lemma 4.3, since $\iota^*(L\{-1\})$ has the trivial G-action then we have a canonical isomorphism

$$L\{-1\} \cong q^*((q_*L\{-1\})_0) \cong q^*(F\{-1\}),$$

which then implies that $L \cong q^*F$.

We let $J \cong (q_*L\{-1\})_0 \cong F\{-1\}$ as in [23], to follow Rössler's notation in the main computation, and note $q^*J \cong L\{-1\}$.

We now consider (6.3) with $M = \mathcal{O}_X$ again. Note that since q is finite, it is then affine, and so by [26, Lemma 68.8.2] we have that $R^1q_*(L) = 0$. Taking the pushforward of (6.3) again, and using the projection formula to get

$$q_*q^*F \cong F \otimes q_*\mathcal{O}_X \cong F \otimes (\mathcal{O}_Y \oplus F) \cong F^{\otimes 2} \oplus F,$$

we obtain the following exact sequence:

$$0 \to F^{\otimes 2} \oplus F \to \mathcal{O}_Y \oplus F \to (q\iota)_*\mathcal{O}_Z \to 0.$$

Taking the $(\cdot)_0$ functor, which is exact (see [23, §4]), we obtain the following short exact sequence:

$$0 \to F^{\otimes 2} \xrightarrow{\alpha} \mathcal{O}_Y \to (q\iota)_*\mathcal{O}_Z \to 0, \tag{6.5}$$

and so $F^{\otimes 2}$ is isomorphic to $\mathcal{O}_Y(-Z)$, and there is a canonical section $\alpha^{-1}(1_{\mathcal{O}_Y}) \in F^{\otimes 2}$.

We will fix a hermitian metric h^F on F, giving us $\underline{F} = (F, h^F)$; this metric is induced by a choice of metric in Ma's branched covering formula, this choice will ultimately not affect the final norm. This induces a hermitian metric q^*h^F on L, and we let $\underline{L}' \coloneqq (L, q^*h^F)$. Note that $\underline{L} \not\simeq \underline{L}'$, but we will be able to use the anomaly formula to compute the difference between determinants of cohomology involving $\underline{L}$ and those involving $\underline{L}'$. We thus have

$$\underline{L}' \simeq q^*\underline{F}.$$

Restricting to Z, we define $\underline{\mathcal{C}}' \coloneqq (\mathcal{C}, \iota^*q^*h^F) = \iota^*\underline{L}'$, and we note that

$$\mathrm{ch}_g(\underline{L}') = -\,\mathrm{ch}(\underline{\mathcal{C}}').$$

We similarly have an induced hermitian metric on $J = F\{-1\}$,

$$\underline{J} \coloneqq \underline{F}\{-1\}.$$

Now, since $(M|_Z)_1 = 0$ and q is flat, by Lemma 4.3, $M \cong q^*((q_*M)_0)$. We let $\widetilde{M} \coloneqq (q_*M)_0$, and choose a hermitian metric $h^{\widetilde{M}}$ on $\widetilde{M}$, with $\underline{\widetilde{M}} \coloneqq (\widetilde{M}, h^{\widetilde{M}})$. This induces a hermitian metric $q^*h^{\widetilde{M}}$ on M, and we let $\underline{M}' \coloneqq (M, q^*h^{\widetilde{M}})$. We thus have

$$\underline{M}' \simeq q^*\underline{\widetilde{M}}.$$

6.2.3 $\mathcal{O}_Y$-algebras ※

Now let $\mathcal{A} = \mathcal{O}_Y \oplus F = q_*\mathcal{O}_X$ be the $\mathcal{O}_Y$-algebra with algebra structure from $\mathcal{O}_X$, note that due to the equivariance we have that the algebra structure on $\mathcal{A}$ is given by a map $\alpha : F^{\otimes 2} \to \mathcal{O}_Y$, which agrees with the map in the exact sequence (6.5). Note that in this situation $X = \mathrm{Spec}_Y(\mathcal{A})$.

If we let $\mathcal{B} = \bigoplus_{i=0}^{\infty} F^{\otimes i}$ be the graded $\mathcal{O}_Y$-algebra with multiplication given by the natural maps $F^{\otimes n} \times F^{\otimes m} \to F^{\otimes(n+m)}$, note that the vector bundle F, as a scheme, is given by $F = \mathrm{Spec}_Y(\mathcal{B})$. We can therefore define a morphism of schemes over Y from $X \to F$ by giving a morphism of $\mathcal{O}_Y$-algebras $\mathcal{B} \to \mathcal{A}$, which we define to be the natural morphism which is the identity on $\mathcal{O}_Y \oplus F$ in the zeroth and first degrees of $\mathcal{B}$ and is given in higher degrees by the map $F^{\otimes 2} \to \mathcal{O}_Y$ which defines the algebra structure of $\mathcal{A}$.

By noting that the kernel of the map $\mathcal{B} \to \mathcal{A}$ we have defined is the ideal generated by $\{r - \alpha(r) : r \in F^{\otimes 2}\}$, which in turn can be generated by $\alpha^{-1}(1_{\mathcal{O}_Y}) - 1_{\mathcal{O}_Y} \in \mathcal{B}$, we deduce that

$$X \cong \{(y,t) \in F \text{ s.t. } t^{\otimes 2} - \alpha^{-1}(1_{\mathcal{O}_Y})(y) = 0\}.$$

Let $P = \mathbb{P}_Y(\mathcal{O}_Y \oplus F)$ and, following §4.3.8, we can identify X^{an} and Y^{an} as complex submanifolds of P^{an}, so that the restriction of the projection $\pi : P^{\mathrm{an}} \to Y^{\mathrm{an}}$ to X^{an} is the branched covering $q : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. We then consider a Kähler metric on P^{an}, which will induce Kähler metrics on X^{an}, Y^{an} and Z^{an}. Note that the new Kähler metrics on X^{an} and Z^{an} are different from the ones we start with, but we can work with them by making use of the anomaly formulae from §4.2.3. We will write λ_X and λ_Z for the determinants of cohomology endowed with the Quillen metric with the original Kähler metric, and $\lambda_{X'}$ and $\lambda_{Z'}$ for the determinants of cohomology endowed with the Quillen metric with the new Kähler metrics, induced by the Kähler metric on P^{an}.

This is the situation described in [20, §4], and which we recalled in §4.3.8, so we may make use of Ma's branched covering formula.

6.2.4 The immersion $Z \to X$ ※

Lemma 6.1. *Let $\underline{\eta}$ be a hermitian vector bundle on X. Then we have that*

$$\lambda_Z^{\mathrm{eq}}(\iota^*\underline{\eta}) \overset{(a)}{\approx} \lambda_X^{\mathrm{eq}}(\underline{\eta} \otimes (\underline{\mathcal{O}}_X - \underline{L})) \tag{6.6}$$

with norm

$$\begin{aligned}(a) = &+\int_{X_g} \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X) R_g(\underline{\mathcal{T}}_X)(\overline{\mathrm{ch}}_g(\underline{\mathcal{O}}_X) - \overline{\mathrm{ch}}_g(\underline{L}))\overline{\mathrm{ch}}_g(\underline{\eta}) \\ &- \int_{Z_g} \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_Z) R_g(\underline{\mathcal{T}}_Z)\overline{\mathrm{ch}}_g(\iota^*\underline{\eta})\end{aligned}$$

Proof. We are in the situation considered in §4.3.7. First note that we have a resolution for $\iota_*(\iota^*\underline{\eta})$ given by the Cartier divisor,

$$0 \to \underline{\eta} \otimes \underline{L} \to \underline{\eta} \to \iota_*(\iota^*\underline{\eta}) \to 0. \tag{6.7}$$

The equivariant Bismut immersion formula then gives us the isomorphism (6.6), as well as its norm which is

$$\begin{aligned}(a) = &\int_{X_g} \mathrm{td}_g(\underline{\mathcal{T}}_X)\mathfrak{BC}_g(\underline{L} \to \underline{\mathcal{O}}_X)\,\mathrm{ch}_g(\underline{\eta}) \\ &- \int_{Z_g} \mathrm{td}_g^{-1}(\underline{N})\widetilde{\mathrm{td}}_g(\underline{\mathcal{T}}_Z, \underline{\mathcal{T}}_X|_Z, \underline{N})\,\mathrm{ch}_g(\iota^*\underline{\eta}) \\ &+ \int_{X_g} \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X) R_g(\underline{\mathcal{T}}_X)(\overline{\mathrm{ch}}_g(\underline{\mathcal{O}}_X) - \overline{\mathrm{ch}}_g(\underline{L}))\overline{\mathrm{ch}}_g(\underline{\eta}) \\ &- \int_{Z_g} \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_Z) R_g(\underline{\mathcal{T}}_Z)\overline{\mathrm{ch}}_g(\iota^*\underline{\eta})\end{aligned}$$

Let $\underline{\xi}_\bullet = 0 \to \underline{L} \to \underline{\mathcal{O}}_X \to 0$, with $\underline{\mathcal{O}}_X$ in degree 0. Note that $\underline{\xi}|_{Z,\bullet} = 0 \to \underline{\mathcal{C}} \to \underline{\mathcal{O}}_Z \to 0$, but since $\underline{\mathcal{C}}_0 = 0$ and $(\underline{\mathcal{O}}_Z)_1 = 0$, then the map $\underline{\mathcal{C}} \to \underline{\mathcal{O}}_Z$ must be the zero map, by applying the $(\cdot)_0$ and $(\cdot)_1$ functors to the exact sequence $\underline{\xi}|_{Z,\bullet}$. Now, if the differential is zero, then $\underline{\mathrm{Im}} = 0$ and $\underline{\mathrm{Ker}} \simeq \underline{\xi}|_{Z,*} \simeq \underline{H}|_{Z,*}$, where the kernel, image, and homology vector bundles are endowed with the hermitian metrics induced from $\underline{\xi}|_{Z,\bullet}$. This means that the exact sequences

$$0 \to \underline{\mathrm{Im}} \to \underline{\mathrm{Ker}} \to \underline{H}|_{Z,*} \to 0,$$

and

$$0 \to \underline{\mathrm{Ker}} \to \underline{\xi}|_{Z,*} \to \underline{\mathrm{Im}} \to 0,$$

are trivially orthogonally split, and so $\underline{\xi}_{\bullet}|_Z$ is homologically split. From §4.3.2, this means that $\mathfrak{B}\mathcal{C}_g(\underline{L} \to \underline{\mathcal{O}}_X) = 0$, and so the first term in the expression of (a) vanishes.

Secondly, we have that $\underline{\mathcal{N}}_0 = 0$ and $(\underline{\mathcal{T}_Z})_1 = 0$, so we can conclude that, by applying the $(\cdot)_0$ and $(\cdot)_1$ functors to the exact sequence

$$0 \to \underline{\mathcal{T}}_Z \to \underline{\mathcal{T}}_X|_Z \to \underline{\mathcal{N}} \to 0 \tag{6.8}$$

That $(\underline{\mathcal{T}}_X|_Z)_0 \simeq \underline{\mathcal{T}}_Z$ and $(\underline{\mathcal{T}}_X|_Z)_1 \simeq \underline{\mathcal{N}}_1$, and so that $\underline{\mathcal{T}}_X|_Z \simeq \underline{\mathcal{T}}_Z \oplus \underline{\mathcal{N}}$, so that the equivariant Todd class $\widetilde{\mathrm{td}}_g(\underline{\mathcal{T}}_Z, \underline{\mathcal{T}}_X|_Z, \underline{\mathcal{N}}) = 0$ associated to the short exact sequence vanishes.

We are thus left with the remaining terms, which contribute to the norm of the isomorphism. ☐

6.2.5 A change of metrics on X ※

Lemma 6.2. *We have the following isometry*

$$\lambda_X^{\mathrm{eq}}\left(\underline{M} \otimes (\underline{\mathcal{O}}_X + \underline{L})^{\otimes(d+1)}\right) \simeq \lambda_{X'}^{\mathrm{eq}}\left(\underline{M'} \otimes (\underline{\mathcal{O}}_X + \underline{L}')^{\otimes(d+1)}\right) \tag{6.9}$$

Proof. We prove this isomorphism is an isometry by splitting it into a chain of isometries, where we only change one metric at a time. We begin by changing the metric on $\mathcal{T}_X$, then the metric on M, then the metric on each L in the tensor power.

We let

$$\underline{\xi}_{-1} := \underline{M} \otimes (\underline{\mathcal{O}}_X + \underline{L})^{\otimes(d+1)},$$

and for $0 \leq j \leq (d+1)$,

$$\underline{\xi}_j := \underline{M'} \otimes (\underline{\mathcal{O}}_X + \underline{L})^{\otimes(d+1-j)} \otimes (\underline{\mathcal{O}}_X + \underline{L}')^{\otimes j}.$$

We begin by considering the norm (b_{-1}) of the isomorphism

$$\lambda_X^{\mathrm{eq}}(\underline{\xi}_{-1}) \overset{(b_{-1})}{\approx} \lambda_{X'}^{\mathrm{eq}}(\underline{\xi}_{-1}),$$

which is given by the anomaly formula from §4.2.4 to be equal to

$$(b_{-1}) = -\int_Z \widetilde{\mathrm{td}}_g(0 \to \underline{\mathcal{T}}_X \to \underline{\mathcal{T}}'_X \to 0)\,\mathrm{ch}_g(\underline{\xi}_{-1}).$$

Now,

$$\mathrm{ch}_g(\underline{\xi}_{-1}) = \mathrm{ch}_g(\underline{M})\,\mathrm{ch}_g((\underline{\mathcal{O}}_X + \underline{L})^{\otimes(d+1)}) = \mathrm{ch}_g(\underline{M})\,\mathrm{ch}((\underline{\mathcal{O}}_Z - \underline{\mathcal{C}})^{\otimes(d+1)}).$$

By using Lemma 2.6, since $d+1$ is greater than the relative dimension of Z, the norm $(b_{-1}) = 0$ vanishes.

We now consider the norms (b_j) for $0 \leq j \leq d+1$, of the isomorphisms

$$\lambda^{\mathrm{eq}}_{X'}(\underline{\xi}_{j-1}) \overset{(b_j)}{\approx} \lambda^{\mathrm{eq}}_{X'}(\underline{\xi}_j),$$

which are given by the anomaly formula to be equal to

$$(b_j) = -\int_Z \mathrm{td}_g(\underline{\mathcal{T}}'_X)\widetilde{\mathrm{ch}}_g(0 \to \underline{\xi}_{j-1} \to \underline{\xi}_j \to 0).$$

We see that ξ_{-1} and ξ_0 have a common tensor factor, so by the multiplicativity of secondary Chern forms we have that

$$\begin{aligned}\widetilde{\mathrm{ch}}_g(0 \to \underline{\xi}_{-1} \to \underline{\xi}_0 \to 0) &= \widetilde{\mathrm{ch}}(0 \to \underline{M} \to \underline{M}' \to 0)\,\mathrm{ch}_g((\underline{\mathcal{O}}_X + \underline{L})^{\otimes(d+1)}) \\ &= \widetilde{\mathrm{ch}}(0 \to \underline{M} \to \underline{M}' \to 0)\,\mathrm{ch}((\underline{\mathcal{O}}_Z - \underline{\mathcal{C}})^{\otimes(d+1)}),\end{aligned}$$

which will once again vanish by Lemma 2.6, so that $(b_0) = 0$.

For $1 \leq j \leq d+1$, ξ_{j-1} and ξ_j also have a common tensor factor of

$$\underline{M}' \otimes (\underline{\mathcal{O}}_X + \underline{L})^{\otimes d+1-j} \otimes (\underline{\mathcal{O}}_X + \underline{L}')^{\otimes j-1},$$

so that

$$\begin{aligned}&\widetilde{\mathrm{ch}}_g(0 \to \underline{\xi}_{j-1} \to \underline{\xi}_j \to 0) \\ &\quad = \widetilde{\mathrm{ch}}(0 \to \underline{L} \to \underline{L}' \to 0)\,\mathrm{ch}_g(\underline{M}')\,\mathrm{ch}_g((\underline{\mathcal{O}}_X + \underline{L})^{\otimes(d+1-j)} \otimes (\underline{\mathcal{O}}_X + \underline{L}')^{\otimes(j-1)}) \\ &\quad = \widetilde{\mathrm{ch}}(0 \to \underline{L} \to \underline{L}' \to 0)\,\mathrm{ch}_g(\underline{M}')\,\mathrm{ch}((\underline{\mathcal{O}}_Z - \underline{\mathcal{C}})^{\otimes(d+1-j)} \otimes (\underline{\mathcal{O}}_Z - \underline{\mathcal{C}}')^{\otimes(j-1)}),\end{aligned}$$

which will once more vanish by Lemma 2.6, giving us $(b_j) = 0$.

It follows then that by combining these isometries, we obtain an isometry

$$\lambda^{\mathrm{eq}}_X(\underline{\xi}_{-1}) \simeq \lambda^{\mathrm{eq}}_{X'}(\underline{\xi}_{d+1}).$$

□

6.2.6 The quotient $X \to Y$ ※

Lemma 6.3. *We have the following isometry*

$$\lambda^{\mathrm{eq}}_{X'}\left(q^*\left(\underline{\widetilde{M}} \otimes (\underline{\mathcal{O}}_Y - \underline{J})^{\otimes(d+1)}\right)\right) \simeq \lambda^{\mathrm{eq}}_{Y}\left(\underline{\widetilde{M}} \otimes (\underline{\mathcal{O}}_Y - \underline{J})^{\otimes d+1} \otimes (\underline{\mathcal{O}}_Y + \underline{F})\right) \tag{6.10}$$

Proof. Let $\underline{\xi} = \underline{\widetilde{M}} \otimes (\underline{\mathcal{O}}_Y - \underline{J})^{\otimes d+1}$. By noting that $q_*\mathcal{O}_X = \mathcal{O}_Y + F$, and that the norm chosen on F is the one that works for Ma's branched covering formula, we have that the above isomorphism is precisely

$$\lambda^{\mathrm{eq}}_{X'}(q^*\underline{\xi}) \overset{(b)}{\approx} \lambda^{\mathrm{eq}}_{Y}(R^\bullet q_* q^* \underline{\xi}),$$

whose norm (b) is given by Ma's branched covering formula. We use Ma's branched covering formula [20, Theorem 4.1], which gives us that the two bundles are canonically isomorphic and that the norm of the isomorphism is given by

$$\begin{aligned}
&\int_Y \mathrm{td}(\underline{\mathcal{T}}_Y)\,\mathrm{td}^{-1}([\underline{X}])\,\mathrm{td}_g(\underline{\mathcal{N}}_{Y/P})\log\|\alpha\|^2_{h[X]}\,\mathrm{ch}(\underline{\xi})\\
&-\int_Z \mathrm{td}^{-1}(\underline{\mathcal{N}}_{X/P})\,\mathrm{td}_g(\underline{\mathcal{N}}_{Y/P})\widetilde{\mathrm{td}}(\underline{\mathcal{T}}_Z, \underline{\mathcal{T}}_Y)\,\mathrm{ch}(\underline{\xi})\\
&+\int_Z \mathrm{td}(\underline{\mathcal{T}}_Y)\,\mathrm{td}_g(\underline{\mathcal{N}}_{Y/P})\widetilde{\mathrm{td}}([d\underline{Y}], \underline{\mathcal{N}}_{X/P})\,\mathrm{ch}(\underline{\xi})\\
&+\int_Y \overline{\mathrm{td}}(\underline{\mathcal{T}}_Y)R(\underline{\mathcal{T}}_Y)\overline{\mathrm{ch}}_g(R^\bullet q_*\underline{\mathcal{O}}_X)\overline{\mathrm{ch}}(\underline{\xi})\\
&-\int_Z \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X)R_g(\underline{\mathcal{T}}_X)\overline{\mathrm{ch}}(\underline{\xi})
\end{aligned}$$

Note that

$$\begin{aligned}
\mathrm{ch}(\underline{\xi}) &= \mathrm{ch}(\underline{\widetilde{M}})\,\mathrm{ch}((\underline{\mathcal{O}}_Y - \underline{J})^{\otimes(d+1)}),\\
\overline{\mathrm{ch}}(\underline{\xi}) &= \overline{\mathrm{ch}}(\underline{\widetilde{M}})\overline{\mathrm{ch}}((\underline{\mathcal{O}}_Y - \underline{J})^{\otimes(d+1)}),
\end{aligned}$$

and that the relative dimensions of Y and Z over S are d and $d-1$ respectively. We can then apply Lemmas 2.6 and 2.7 to each of the terms, which implies that the norm of the isomorphism vanishes, and so we are left with an isometry.

□

6.2.7 Main computation ※

We aim to compute the norm (n) of the isomorphism

$$\lambda_X^{\mathrm{eq}}(\underline{M})^{\otimes 2^{d+1}} \overset{(n)}{\approx} \bigotimes_{j=0}^{d} \lambda_Z^{\mathrm{eq}}(\iota^*\underline{M} \otimes \underline{\mathcal{C}}_{Z/X}^{\otimes j})^{\otimes \sum_{i=0}^{d-j}\binom{d+1}{i}}, \tag{6.11}$$

obtained by following the isomorphism constructed in [23, Theorem 6.1] in the case where $Z = X_G$ is a Cartier divisor.

We begin with the right hand side, and use the polynomial $p_d(x) = \sum_{j=0}^{d}\sum_{i=0}^{d-j}\binom{d+1}{j}x^j$, from §3.2.2.

$$\bigotimes_{j=0}^{d} \lambda_Z^{\mathrm{eq}}(\iota^*\underline{M} \otimes \underline{\mathcal{C}}^{\otimes j})^{\otimes \sum_{i=0}^{d-j}\binom{d+1}{i}}$$

$$\begin{aligned}
&\simeq \lambda_Z^{\mathrm{eq}}\left(\iota^*\underline{M} \otimes \sum_{j=0}^{d}\sum_{i=0}^{d-j}\binom{d+1}{i}\underline{\mathcal{C}}^{\otimes j}\right)\\
&\simeq \lambda_Z^{\mathrm{eq}}\left(\iota^*\underline{M} \otimes p_d(\underline{\mathcal{C}})\right),
\end{aligned}$$

Then, we recall that $\underline{\mathcal{C}} \simeq \iota^*\underline{L}$;

$$\simeq \lambda_Z^{\mathrm{eq}}\left(\iota^*(\underline{M} \otimes p_d(\underline{L}))\right),$$

We can then use the equivariant Bismut immersion formula, as in Lemma 6.1, which gives us an isomorphism with non-zero norm (a), which we recall at the end;

$$\overset{(a)}{\approx} \lambda_X^{\mathrm{eq}}\left(\underline{M} \otimes p_d(\underline{L}) \otimes (\underline{\mathcal{O}}_X - \underline{L})\right),$$

We may then use the polynomial identity $(1-x)p_d(x) = 2^{d+1} - (1+x)^{d+1}$, which is Lemma 3.2;

$$\begin{aligned}
&\simeq \lambda_X^{\mathrm{eq}}\left(\underline{M} \otimes \left(\underline{\mathcal{O}}_X^{\oplus 2^{d+1}} - (\underline{\mathcal{O}}_X + \underline{L})^{\otimes(d+1)}\right)\right)\\
&\simeq \lambda_X^{\mathrm{eq}}\left(\underline{M} \otimes \underline{\mathcal{O}}_X^{\oplus 2^{d+1}}\right) \otimes \lambda_X^{\mathrm{eq}}\left(\underline{M} \otimes (\underline{\mathcal{O}}_X + \underline{L})^{\otimes(d+1)}\right)^{\vee},
\end{aligned}$$

We may then recall that $\lambda^{\mathrm{eq}}(\underline{E} \oplus \underline{E}') \simeq \lambda^{\mathrm{eq}}(\underline{E}) \otimes \lambda^{\mathrm{eq}}(\underline{E}')$, and apply this to the left term;

$$\simeq \lambda_X^{\mathrm{eq}}\left(\underline{M}\right)^{\otimes 2^{d+1}} \otimes \lambda_X^{\mathrm{eq}}\left(\underline{M} \otimes (\underline{\mathcal{O}}_X + \underline{L})^{\otimes(d+1)}\right)^{\vee},$$

It remains to compute that the right term is isomorphically trivial, and, in fact, it is isometrically trivial. We begin by invoking the anomaly formula, as in Lemma 6.2, to change the hermitian metrics on various bundles to those coming from the quotient;

$$\simeq \lambda_X^{\mathrm{eq}}\left(\underline{M}\right)^{\otimes 2^{d+1}} \otimes \lambda_{X'}^{\mathrm{eq}}\left(\underline{M}' \otimes (\underline{\mathcal{O}}_X + \underline{L}')^{\otimes(d+1)}\right)^{\vee},$$

We then use the isometries $\underline{\mathcal{O}}_X \simeq q^*\underline{\mathcal{O}}_Y$, $\underline{M}' \simeq q^*\underline{\widetilde{M}}$, and $\underline{L}' \simeq q^*\underline{F}$;

$$\simeq \lambda_X^{\mathrm{eq}}\left(\underline{M}\right)^{\otimes 2^{d+1}} \otimes \lambda_{X'}^{\mathrm{eq}}\left(q^*\left(\underline{\widetilde{M}} \otimes (\underline{\mathcal{O}}_Y + \underline{F})^{\otimes(d+1)}\right)\right)^{\vee},$$

Then we use $\underline{J} = \underline{F}\{-1\}$;

$$\simeq \lambda_X^{\mathrm{eq}}\left(\underline{M}\right)^{\otimes 2^{d+1}} \otimes \lambda_{X'}^{\mathrm{eq}}\left(q^*\left(\underline{\widetilde{M}} \otimes (\underline{\mathcal{O}}_Y - \underline{J})^{\otimes(d+1)}\right)\right)^{\vee},$$

And then, we invoke Ma's branched covering formula, following Lemma 6.3,

$$\begin{aligned}
&\simeq \lambda_X^{\mathrm{eq}}\left(\underline{M}\right)^{\otimes 2^{d+1}} \otimes \lambda_Y^{\mathrm{eq}}\left(\underline{\widetilde{M}} \otimes (\underline{\mathcal{O}}_Y - \underline{J})^{\otimes(d+1)} \otimes (\underline{\mathcal{O}}_Y + \underline{F})\right)^{\vee} \\
&\simeq \lambda_X^{\mathrm{eq}}\left(\underline{M}\right)^{\otimes 2^{d+1}} \otimes \lambda_Y^{\mathrm{eq}}\left(\underline{\widetilde{M}} \otimes (\underline{\mathcal{O}}_Y - \underline{J})^{\otimes(d+2)}\right)^{\vee},
\end{aligned}$$

But, we know $\underline{\widetilde{M}}_1 = 0$, $\underline{\mathcal{O}}_{Y,1} = 0$ and $\underline{J}_1 = 0$, so we may write the equivariant determinant of cohomology in terms of the determinant of cohomology;

$$\simeq \lambda_X^{\mathrm{eq}}\left(\underline{M}\right)^{\otimes 2^{d+1}} \otimes \lambda_Y\left(\underline{\widetilde{M}} \otimes (\underline{\mathcal{O}}_Y - \underline{J})^{\otimes(d+2)}\right)^{\vee},$$

We may then invoke the analytic Ducrot's cube structure theorem, as in Lemma 5.2;

$$\begin{aligned}
&\simeq \lambda_X^{\mathrm{eq}}\left(\underline{M}\right)^{\otimes 2^{d+1}} \otimes \underline{\mathcal{O}}_S \\
&\simeq \lambda_X^{\mathrm{eq}}\left(\underline{M}\right)^{\otimes 2^{d+1}}.
\end{aligned}$$

Rearranging, we have a canonical isomorphism

$$\lambda_X^{\mathrm{eq}}(\underline{M})^{\otimes 2^{d+1}} \overset{(n)}{\approx} \bigotimes_{j=0}^{d} \lambda_Z^{\mathrm{eq}}(\iota^*\underline{M} \otimes \underline{\mathcal{C}}_{Z/X}^{\otimes j})^{\otimes \sum_{i=0}^{d-j}\binom{d+1}{i}},$$

whose norm (n) is $-(a)$, where (a) is the norm given in Lemma 6.1, with $\underline{\eta} = \underline{M} \otimes p_d(\underline{L})$;

$$\begin{aligned}(n) = &-\int_{X_g} \overline{\mathrm{td}}_g(\underline{\mathcal{T}_X}) R_g(\underline{\mathcal{T}_X})(\overline{\mathrm{ch}}_g(\underline{\mathcal{O}_X}) - \overline{\mathrm{ch}}_g(\underline{L}))\overline{\mathrm{ch}}_g(\underline{\eta}) \\ &+ \int_{Z_g} \overline{\mathrm{td}}_g(\underline{\mathcal{T}_Z}) R_g(\underline{\mathcal{T}_Z})\overline{\mathrm{ch}}_g(\iota^*\underline{\eta}).\end{aligned}$$

6.2.8 The norm of a canonical isomorphism of determinant line bundles ※

We aim to give a simpler expression for the norm (n).

$$\begin{aligned}(n) = &-\int_{X_g} \overline{\mathrm{td}}_g(\underline{\mathcal{T}_X}) R_g(\underline{\mathcal{T}_X})(\overline{\mathrm{ch}}_g(\underline{\mathcal{O}_X}) - \overline{\mathrm{ch}}_g(\underline{L}))\overline{\mathrm{ch}}_g(\underline{\eta}) \\ &+ \int_{Z_g} \overline{\mathrm{td}}_g(\underline{\mathcal{T}_Z}) R_g(\underline{\mathcal{T}_Z})\overline{\mathrm{ch}}_g(\iota^*\underline{\eta}),\end{aligned}$$

We now use the fact that $X_g = Z_g = Z$;

$$\begin{aligned}= &-\int_{Z} \overline{\mathrm{td}}_g(\underline{\mathcal{T}_X}) R_g(\underline{\mathcal{T}_X})(\overline{\mathrm{ch}}_g(\underline{\mathcal{O}_X}) - \overline{\mathrm{ch}}_g(\underline{L}))\overline{\mathrm{ch}}_g(\underline{\eta}) \\ &+ \int_{Z} \overline{\mathrm{td}}_g(\underline{\mathcal{T}_Z}) R_g(\underline{\mathcal{T}_Z})\overline{\mathrm{ch}}_g(\iota^*\underline{\eta}),\end{aligned}$$

Now, we have that $\mathrm{ch}_g(\underline{\mathcal{O}_X}) = \mathrm{ch}(\underline{\mathcal{O}_Z})$, $\mathrm{ch}_g(\underline{M}) = \mathrm{ch}_g(\iota^*\underline{M}) = \mathrm{ch}(\iota^*\underline{M})$ since $(\iota^*\underline{M})_1 = 0$, $\mathrm{ch}_g(\underline{L}) = -\mathrm{ch}(\underline{\mathcal{C}})$. We can then write $\mathrm{ch}_g(\underline{\eta}) = \mathrm{ch}_g(\iota^*\underline{\eta}) = \mathrm{ch}(\iota^*\underline{M})\,\mathrm{ch}(p_d(-\underline{\mathcal{C}}))$, and the same for Chern form classes;

$$\begin{aligned}= &-\int_{Z} \overline{\mathrm{td}}_g(\underline{\mathcal{T}_X}) R_g(\underline{\mathcal{T}_X})\overline{\mathrm{ch}}(\underline{\mathcal{O}_Z} + \underline{\mathcal{C}})\overline{\mathrm{ch}}(\iota^*\underline{M})\overline{\mathrm{ch}}(p_d(-\underline{\mathcal{C}})) \\ &+ \int_{Z} \overline{\mathrm{td}}_g(\underline{\mathcal{T}_Z}) R_g(\underline{\mathcal{T}_Z})\overline{\mathrm{ch}}(\iota^*\underline{M})\overline{\mathrm{ch}}(p_d(-\underline{\mathcal{C}})),\end{aligned}$$

Now recall Lemma 2.3, which implies that $\mathrm{td}_{-1}(\underline{\mathcal{N}})\,\mathrm{ch}(\underline{\mathcal{O}_Z} + \underline{\mathcal{C}}) = c^1(\underline{\mathcal{N}})$ since $\underline{\mathcal{N}}$ is a hermitian line bundle. We may then write $1 = \overline{\mathrm{td}}_{-1}(\underline{\mathcal{N}})\overline{\mathrm{ch}}(\underline{\mathcal{O}_Z} + \underline{\mathcal{C}})\overline{c}^1(\underline{\mathcal{N}})^{-1}$.

$$\begin{aligned}= &-\int_{Z} \overline{\mathrm{td}}_g(\underline{\mathcal{T}_X}) R_g(\underline{\mathcal{T}_X})\overline{\mathrm{ch}}(\underline{\mathcal{O}_Z} + \underline{\mathcal{C}})\overline{\mathrm{ch}}(\iota^*\underline{M})\overline{\mathrm{ch}}(p_d(-\underline{\mathcal{C}})) \\ &+ \int_{Z} \overline{\mathrm{td}}_g(\underline{\mathcal{T}_Z})\overline{\mathrm{td}}_{-1}(\underline{\mathcal{N}}) R_g(\underline{\mathcal{T}_Z})\overline{c}^1(\underline{\mathcal{N}})^{-1}\overline{\mathrm{ch}}(\underline{\mathcal{O}_Z} + \underline{\mathcal{C}})\overline{\mathrm{ch}}(\iota^*\underline{M})\overline{\mathrm{ch}}(p_d(-\underline{\mathcal{C}})),\end{aligned}$$

Now, recall that due to the orthogonal splitting of $\underline{\mathcal{T}}_X|_Z = \underline{\mathcal{T}}_Z \oplus \underline{\mathcal{N}}$, by the equivariant Todd decomposition in §4.1.8 we have that $\mathrm{td}_g(\underline{\mathcal{T}}_X) = \mathrm{td}(\underline{\mathcal{T}}_Z)\,\mathrm{td}_{-1}(\underline{\mathcal{N}})$.

$$
\begin{aligned}
&= -\int_Z \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X) R_g(\underline{\mathcal{T}}_X) \overline{\mathrm{ch}}(\underline{\mathcal{O}}_Z + \underline{\mathcal{C}}) \overline{\mathrm{ch}}(\iota^* \underline{M}) \overline{\mathrm{ch}}(p_d(-\underline{\mathcal{C}})) \\
&\quad + \int_Z \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X) R_g(\underline{\mathcal{T}}_Z) \overline{c}^1(\underline{\mathcal{N}})^{-1} \overline{\mathrm{ch}}(\underline{\mathcal{O}}_Z + \underline{\mathcal{C}}) \overline{\mathrm{ch}}(\iota^* \underline{M}) \overline{\mathrm{ch}}(p_d(-\underline{\mathcal{C}})),
\end{aligned}
$$

Grouping these terms, rearranging, and using $R_g(\underline{\mathcal{T}}_Z) = R(\underline{\mathcal{T}}_Z)$ since $\underline{\mathcal{T}}_Z$ has the trivial equivariant action, we have

$$
\begin{aligned}
= \int_Z \Big[& \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X) \left(R(\underline{\mathcal{T}}_Z) \overline{c}^1(\underline{\mathcal{N}})^{-1} - R_g(\underline{\mathcal{T}}_X) \right) \overline{\mathrm{ch}}(\iota^* \underline{M}) \\
& \qquad \overline{\mathrm{ch}}((\underline{\mathcal{O}}_Z + \underline{\mathcal{C}}) p_d(-\underline{\mathcal{C}})) \Big]
\end{aligned}
$$

We now once again make use of Lemma 3.2, that $(1-x)p_d(x) = 2^{d+1} - (1+x)^{d+1}$, and, in particular, changing the sign of x, $(1+x)p_d(-x) = 2^{d+1} - (1-x)^{d+1}$;

$$
\begin{aligned}
= \int_Z \Big[& \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X) \left(R(\underline{\mathcal{T}}_Z) \overline{c}^1(\underline{\mathcal{N}})^{-1} - R_g(\underline{\mathcal{T}}_X) \right) \overline{\mathrm{ch}}(\iota^* \underline{M}) \\
& \qquad \left(2^{d+1} - \overline{\mathrm{ch}}((\mathcal{O}_Z - \underline{\mathcal{C}})^{\otimes(d+1)}) \right) \Big]
\end{aligned}
$$

And we finish by making use of our cancellation lemmas, Lemma 2.7, one last time;

$$
\begin{aligned}
&= \int_Z \Big[\overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X) \left(R(\underline{\mathcal{T}}_Z) \overline{c}^1(\underline{\mathcal{N}})^{-1} - R_g(\underline{\mathcal{T}}_X) \right) \overline{\mathrm{ch}}(\iota^* \underline{M}) \\
&\qquad\qquad \left(2^{d+1} \right) \Big] \\
&= \int_Z 2^{d+1}\, \overline{\mathrm{td}}_g(\underline{\mathcal{T}}_X) \overline{\mathrm{ch}}(\iota^* \underline{M}) \left(R(\underline{\mathcal{T}}_Z) \overline{c}^1(\underline{\mathcal{N}})^{-1} - R_g(\underline{\mathcal{T}}_X) \right).
\end{aligned}
$$

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