



Gravity-driven draining of a thin rivulet with constant width down a slowly varying substrate

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Abstract

The locally unidirectional gravity-driven draining of a thin rivulet with constant width but slowly varying contact angle down a slowly varying substrate is considered. Specifically, the flow of a rivulet in the azimuthal direction from the top to the bottom of a large horizontal cylinder is investigated. In particular, it is shown that, despite behaving the same locally, this flow has qualitatively different global behaviour from that of a rivulet with constant contact angle but slowly varying width. For example, whereas in the case of constant contact angle there is always a rivulet that runs all the way from the top to the bottom of the cylinder, in the case of constant width this is possible only for sufficiently narrow rivulets. Wider rivulets with constant width are possible only between the top of the cylinder and a critical azimuthal angle on the lower half of the cylinder. Assuming that the contact lines de-pin at this critical angle (where the contact angle is zero) the rivulet runs from the critical angle to the bottom of the cylinder with zero contact angle, monotonically decreasing width and monotonically increasing maximum thickness. The total mass of fluid on the cylinder is found to be a monotonically increasing function of the value of the constant width.

Keywords:

rivulet flow, thin film, slowly-varying substrate, fixed width, slowly-varying contact angle,
de-pinning

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1 Introduction

The gravity-driven draining of a rivulet of fluid down an inclined substrate is a fundamental fluid mechanics problem of enduring interest, not least because of the wide range of industrial devices and processes to which it is relevant, including heat exchangers (see, for example, Vlasogiannis *et al.* [1]), trickle-bed reactors (see, for example, Maiti, Khanna and Nigam [2]), various coating processes (see, for example, Schweizer and Kistler [3]), and even the cleaning of the long and narrow tubes found in endoscopes (see the recent paper by Labib *et al.* [4]). As a result there has been considerable experimental and theoretical work on various aspects of rivulet flow (see [5]–[38] and the references therein). In particular, the pioneering studies by Towell and Rothfeld [5], Allen and Biggin [6] and Bentwich *et al.* [7] inspired a substantial body of subsequent work on unidirectional (i.e. rectilinear) rivulet flow and its stability. Duffy and Moffatt [14] used their solution for the unidirectional flow of a thin rivulet to describe the flow of a rivulet with constant contact angle but slowly varying width down a slowly varying substrate. In the present work we show how the same solution can also be used to describe a rather different situation, specifically the flow of a rivulet with constant width but slowly varying contact angle down a slowly varying substrate. In particular, we show that, despite behaving the same locally, this flow has qualitatively different global behaviour from that in the case of constant contact angle. Flow of a rivulet with constant width could be realised in the laboratory using a substrate with a strip of constant width whose surface properties are different from those of the rest of the substrate, or by engraving the substrate with parallel grooves in order to pin the contact lines. Moreover, the results obtained here may also be relevant to more complicated situations, such as the rings of fluid observed by Moffatt [39] in his pioneering experiments on the flow of liquid on the outer surface of a uniformly rotating horizontal cylinder, and the banded films of condensed ammonia-water mixtures observed by Deans and Kucuka [40] on the outer surface of a stationary horizontal cylinder.

2 Unidirectional Flow of a Thin Rivulet

Consider the steady unidirectional flow of a thin symmetric rivulet with constant semi-width a and constant volume flux $Q (> 0)$ down a planar substrate inclined at an angle α ($0 \leq \alpha \leq \pi$) to the vertical. We assume that the fluid is Newtonian with constant viscosity μ , density ρ and coefficient of surface tension γ , and choose Cartesian coordinates $Oxyz$ with the x axis down the line of greatest slope, the y axis horizontal, and the z axis normal to the substrate $z = 0$. The velocity $\mathbf{u} = u(y, z)\mathbf{i}$

and the pressure (relative to its ambient value) $p = p(y, z)$ satisfy the familiar mass-conservation and Navier–Stokes equations subject to the usual normal and tangential stress balances and the kinematic condition at the free surface $z = h(y)$, the no-slip condition at the substrate $z = 0$, and the condition of zero thickness at the contact lines (i.e. $h(\pm a) = 0$). The contact angle is denoted by $\beta = \mp h'(\pm a) (\geq 0)$, where the dash denotes differentiation with respect to argument, and the maximum thickness of the rivulet, which always occurs at $y = 0$, is denoted by $h_m = h(0)$. We non-dimensionalise y and a with ℓ , z , h and h_m with $\delta\ell$, u with $U = \delta^2 \rho g \ell^2 / \mu$, Q with $\delta \ell^2 U = \delta^3 \rho g \ell^4 / \mu$, and p with $\delta \rho g \ell$, where g is the magnitude of gravitational acceleration, $\ell = (\gamma / \rho g)^{1/2}$ is the capillary length, and δ is the transverse aspect ratio. There is some freedom regarding the definition of δ . We could define δ using the constant value of the contact angle by choosing $\delta = \beta$, corresponding to taking $\beta = 1$ without loss of generality. Alternatively, we could define δ using the prescribed value of the flux, denoted by $\bar{Q} (> 0)$, by choosing $\delta = (\mu \bar{Q} / \rho g \ell^4)^{1/3}$, corresponding to taking $\bar{Q} = 1$ without loss of generality. However, for the moment we leave δ unspecified and retain both β and $\bar{Q}$ in order to keep the subsequent presentation as general as possible.

Duffy and Moffatt [14] (see also [5], [6], [17], [19], [21] and [22]) showed that when $\beta > 0$ at leading order in the limit of small transverse aspect ratio $\delta \rightarrow 0$ (i.e. for a thin rivulet) the governing equations are readily solved to yield the velocity $u = \sin \alpha (2h - z)z/2$, the pressure $p = \cos \alpha (h - z) - h''$, the free surface shape

$$h(y) = \beta \times \begin{cases} \frac{\cosh ma - \cosh my}{m \sinh ma} & \text{for } 0 \leq \alpha < \frac{\pi}{2}, \\ \frac{a^2 - y^2}{2a} & \text{for } \alpha = \frac{\pi}{2}, \\ \frac{\cos my - \cos ma}{m \sin ma} & \text{for } \frac{\pi}{2} < \alpha \leq \pi, \end{cases} \quad (1)$$

the maximum thickness of the rivulet

$$h_m = \beta \times \begin{cases} \frac{1}{m} \tanh \frac{ma}{2} & \text{for } 0 \leq \alpha < \frac{\pi}{2}, \\ \frac{a}{2} & \text{for } \alpha = \frac{\pi}{2}, \\ \frac{1}{m} \tan \frac{ma}{2} & \text{for } \frac{\pi}{2} < \alpha \leq \pi, \end{cases} \quad (2)$$

and the flux

$$Q = \frac{\beta^3 \sin \alpha}{9m^4} f(ma), \quad (3)$$

where $m = |\cos \alpha|^{1/2}$. The function $f = f(ma)$ appearing in (3) is given by

$$f(ma) = \begin{cases} 15ma \coth^3 ma - 15 \coth^2 ma - 9ma \coth ma + 4 & \text{for } 0 \leq \alpha < \frac{\pi}{2}, \\ \frac{12}{35} (ma)^4 & \text{for } \alpha = \frac{\pi}{2}, \\ -15ma \cot^3 ma + 15 \cot^2 ma - 9ma \cot ma + 4 & \text{for } \frac{\pi}{2} < \alpha \leq \pi, \end{cases} \quad (4)$$

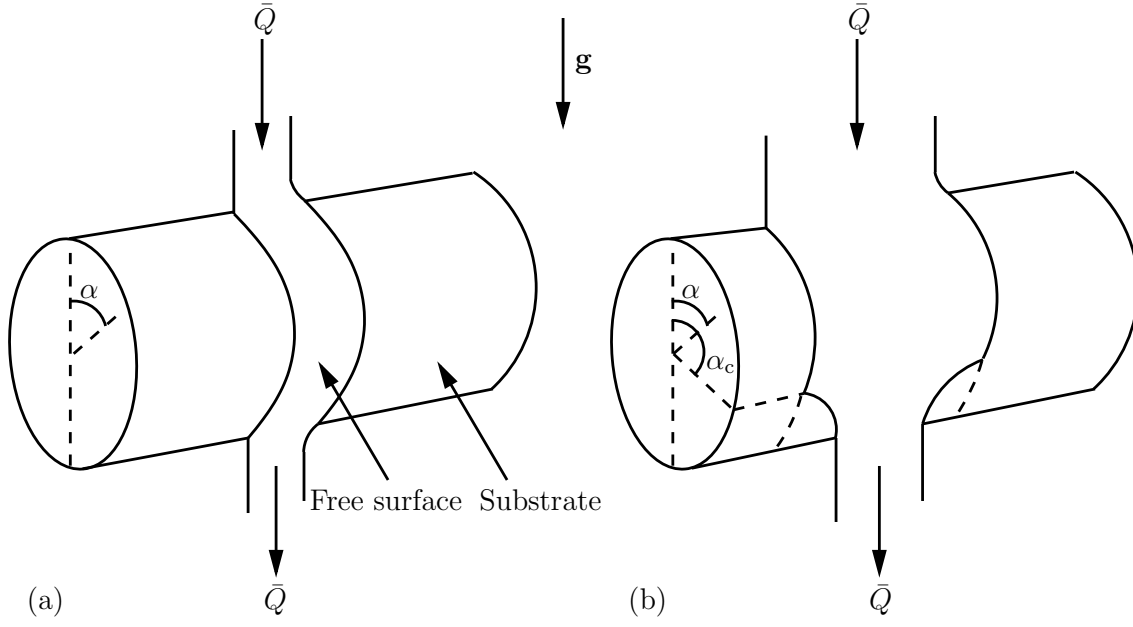


Figure 1: Sketches of a slowly varying rivulet with prescribed flux $\bar{Q}$ that runs from the top $\alpha = 0$ to the bottom $\alpha = \pi$ of a large horizontal cylinder. (a) A narrow rivulet with constant semi-width $a = \bar{a} < \pi$ and slowly varying contact angle β for all $0 \leq \alpha \leq \pi$, and (b) a wide rivulet with constant semi-width $a = \bar{a} > \pi$ and slowly varying contact angle β for $0 \leq \alpha \leq \alpha_c$ and slowly varying semi-width $a = \pi/m$ ($\pi \leq a \leq \bar{a}$) and zero contact angle $\beta = 0$ for $\alpha_c \leq \alpha \leq \pi$.

and satisfies $f \sim 12(ma)^4/35 \rightarrow 0$ as $ma \rightarrow 0$, $f \sim 6ma - 11 \rightarrow \infty$ as $ma \rightarrow \infty$ for $0 \leq \alpha < \pi/2$, and $f \sim 15\pi(\pi - ma)^{-3} \rightarrow \infty$ as $ma \rightarrow \pi$ for $\pi/2 < \alpha \leq \pi$.

In the limit $\beta \rightarrow 0^+$ there is no solution for $0 \leq \alpha \leq \pi/2$, but for $\pi/2 < \alpha \leq \pi$ we recover the solution in the special case of a perfectly wetting fluid $\beta = 0$ described by Wilson and Duffy [28] (see also [15], [23], [29] and [33]), namely

$$a = \frac{\pi}{m}, \quad h = \frac{h_m}{2}(1 + \cos my), \quad Q = \frac{5\pi \sin \alpha h_m^3}{24m}. \quad (5)$$

3 Case of Constant Contact Angle

Duffy and Moffatt [14] used the solution given by (1)–(4) to describe the locally unidirectional flow down a slowly varying substrate, specifically the flow in the azimuthal direction from the top $\alpha = 0$ to the bottom $\alpha = \pi$ of a large horizontal cylinder, of a rivulet with constant contact angle β (taken to be $\beta = 1$ without loss of generality) but slowly varying semi-width $a = a(\alpha)$. The key results in this case are summarised here both for completeness and to highlight the qualitative differences from those in the case of constant semi-width a given subsequently in Section 4. Note that both here

and in Section 4 “slowly varying” means that the longitudinal aspect ratio $\epsilon = \ell/R$, where R is the radius of the cylinder, satisfies $\epsilon \ll \delta$ so that $\epsilon/\delta \rightarrow 0$ in the limit $\epsilon \rightarrow 0$. Imposing the condition of prescribed flux, $Q = \bar{Q}$ with Q given by (3), yields a non-linear algebraic equation for the semi-width a which can, in general, be solved only numerically or asymptotically. In particular, for all values of $\bar{Q}$ there is a slowly varying rivulet that runs all the way from $\alpha = 0$ [where $a = O(\alpha^{-1}) \rightarrow \infty$ and $h_m \rightarrow 1^+$ as $\alpha \rightarrow 0^+$] to $\alpha = \pi$ [where $a \rightarrow \pi^-$ and $h_m = O(\pi - \alpha)^{-1/3} \rightarrow \infty$ as $\alpha \rightarrow \pi^-$]. The rivulet does not have top-to-bottom symmetry; its semi-width a has a single minimum on the lower half of the cylinder (i.e. for $\pi/2 < \alpha \leq \pi$), and its maximum thickness h_m either increases monotonically or has a single maximum and a single minimum on the upper half of the cylinder (i.e. for $0 \leq \alpha < \pi/2$). Furthermore, in the limit of small flux, $\bar{Q} \rightarrow 0^+$, the rivulet satisfies $a = O(\bar{Q}^{1/4})$ and $h_m = O(\bar{Q}^{1/4})$ while in the limit of large flux, $\bar{Q} \rightarrow \infty$, it satisfies $a = O(\bar{Q})$ and $h_m = O(1)$ on the upper half of the cylinder, $a = O(\bar{Q}^{1/4})$ and $h_m = O(\bar{Q}^{1/4})$ at $\alpha = \pi/2$, and $a = O(1)$ and $h_m = O(\bar{Q}^{1/2})$ on the lower half of the cylinder.

Wilson and Duffy [28] interpreted the solution in the special case $\beta = 0$ given by (5) in a similar manner. Specifically, imposing the condition of prescribed flux, $Q = \bar{Q}$ with Q given by (5), immediately yields

$$h_m = \left(\frac{24\bar{Q}m}{5\pi \sin \alpha} \right)^{\frac{1}{3}}, \quad (6)$$

and so for all values of $\bar{Q}$ there is a slowly varying rivulet on the lower half of the cylinder with monotonically decreasing semi-width a and monotonically increasing maximum thickness h_m that runs from $\alpha = \pi/2^+$ [where $a = O(\alpha - \pi/2)^{-1/2} \rightarrow \infty$ and $h_m = O(\alpha - \pi/2)^{1/6} \rightarrow 0^+$ as $\alpha \rightarrow \pi/2^+$] to $\alpha = \pi$ [where $a \rightarrow \pi^+$ and $h_m = O(\pi - \alpha)^{-1/3} \rightarrow \infty$ as $\alpha \rightarrow \pi^-$].

4 Case of Constant Semi-Width

The purpose of the present work is to use the solution given by (1)–(5) to describe the locally unidirectional flow down a slowly varying substrate, specifically the flow in the azimuthal direction from the top $\alpha = 0$ to the bottom $\alpha = \pi$ of a large horizontal cylinder of a rivulet with constant semi-width $a = \bar{a}$ but slowly varying contact angle $\beta = \beta(\alpha) (\geq 0)$. Imposing the conditions of prescribed flux, $Q = \bar{Q} (> 0)$ with Q given by (3), and of constant semi-width, $a = \bar{a} (> 0)$, yields an explicit solution for the contact angle β ,

$$\beta = \left(\frac{9\bar{Q}m^4}{f(m\bar{a}) \sin \alpha} \right)^{\frac{1}{3}} = \left(\frac{9\bar{Q} \cos^2 \alpha}{f(|\cos \alpha|^{1/2} \bar{a}) \sin \alpha} \right)^{\frac{1}{3}}. \quad (7)$$

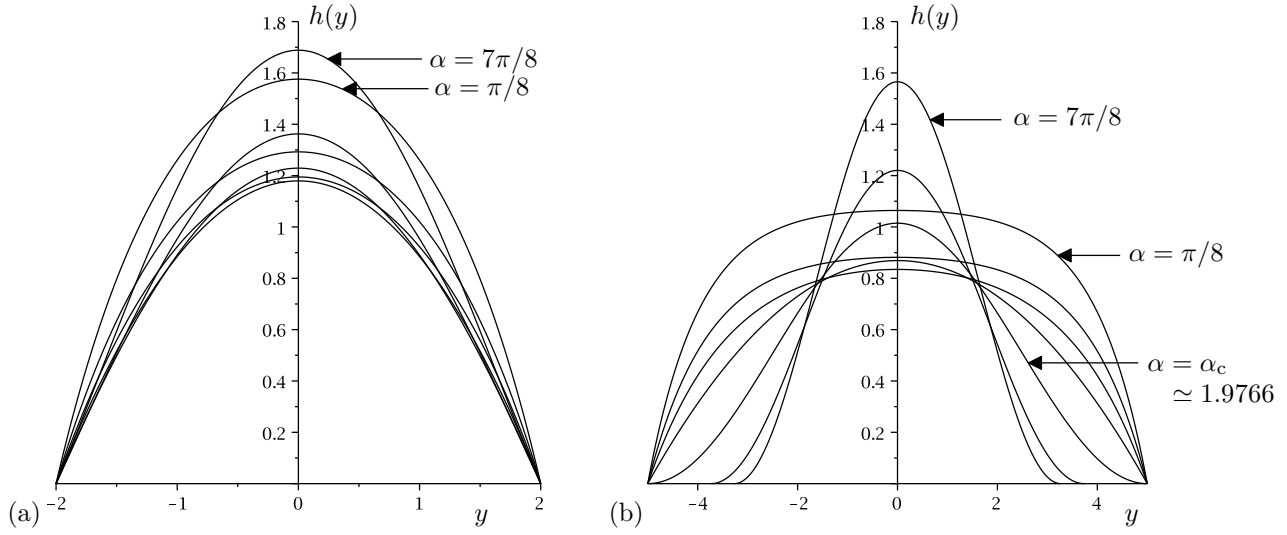


Figure 2: Cross-sectional free-surface profiles, $z = h(y)$, with constant semi-width $a = \bar{a}$ and flux $Q = \bar{Q}$ of (a) a narrow rivulet with $\bar{a} = 2 (< \pi)$ and $\bar{Q} = 1$ for $\alpha = \pi/8, \pi/4, 3\pi/8, \pi/2, 5\pi/8, 3\pi/4$ and $7\pi/8$, and (b) a wide rivulet with $\bar{a} = 5 (> \pi)$ and $\bar{Q} = 1$ for $\alpha = \pi/8, \pi/4, 3\pi/8, \pi/2, \alpha_c = \cos^{-1}(-\pi^2/25) \simeq 1.9766, 3\pi/4$ and $7\pi/8$. Note that while parts (a) and (b) use the same vertical range, for clarity they use different horizontal ranges (namely from $y = -\bar{a}$ to $y = +\bar{a}$ in each case).

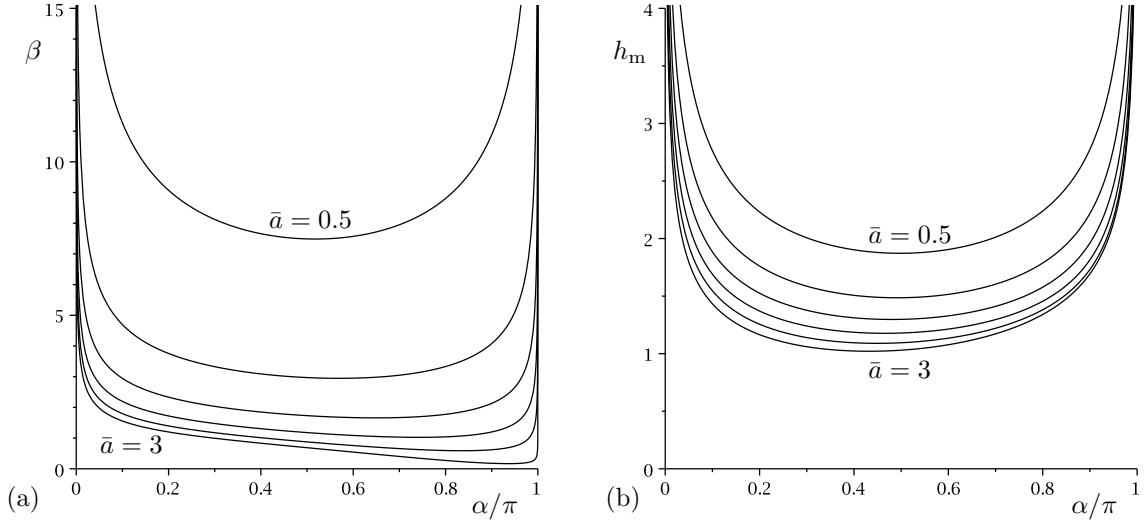


Figure 3: Plots of (a) the contact angle β and (b) the maximum thickness h_m of a narrow rivulet with flux $\bar{Q} = 1$ as functions of α/π for $\bar{a} = 0.5, 1, 1.5, 2, 2.5$ and 3 .

As in the case of constant β , the rivulet does not have top-to-bottom symmetry. The solution (7) reveals that, unlike in the case of constant β in which the dependence of the rivulet on $\bar{Q}$ is not straightforward, for all values of $\bar{a}$ and α , β is simply proportional to $\bar{Q}^{1/3}$. Inspection of the solution (7) also reveals that, unlike in the case of constant β in which the behaviour is qualitatively the same for all values of β , the behaviour of the rivulet is qualitatively different for “narrow” rivulets with $\bar{a} < \pi$, the special case $\bar{a} = \pi$, and “wide” rivulets with $\bar{a} > \pi$. We shall therefore describe the behaviour of the rivulet in each of these three cases separately.

4.1 Narrow Rivulets with $\bar{a} < \pi$

For narrow rivulets with $\bar{a} < \pi$ the solution with β given by (7) corresponds to a rivulet with constant semi-width but varying contact angle $\beta > 0$ that runs all the way from the top $\alpha = 0$ to the bottom $\alpha = \pi$ of the cylinder. This scenario is sketched in Figure 1(a). The contact angle β has a single minimum on the lower half of the cylinder and the maximum thickness h_m has a single minimum on the upper half of the cylinder. In particular, the rivulet becomes deep near the top and the bottom of the cylinder according to

$$\beta \sim \left(\frac{9\bar{Q}}{f(\bar{a})\alpha} \right)^{\frac{1}{3}} \rightarrow \infty \quad \text{and} \quad h_m \sim \left(\frac{9\bar{Q}}{f(\bar{a})\alpha} \right)^{\frac{1}{3}} \tanh\left(\frac{\bar{a}}{2}\right) \rightarrow \infty \quad (8)$$

as $\alpha \rightarrow 0^+$ and

$$\beta \sim \left(\frac{9\bar{Q}}{f(\bar{a})(\pi - \alpha)} \right)^{\frac{1}{3}} \rightarrow \infty \quad \text{and} \quad h_m \sim \left(\frac{9\bar{Q}}{f(\bar{a})(\pi - \alpha)} \right)^{\frac{1}{3}} \tan\left(\frac{\bar{a}}{2}\right) \rightarrow \infty \quad (9)$$

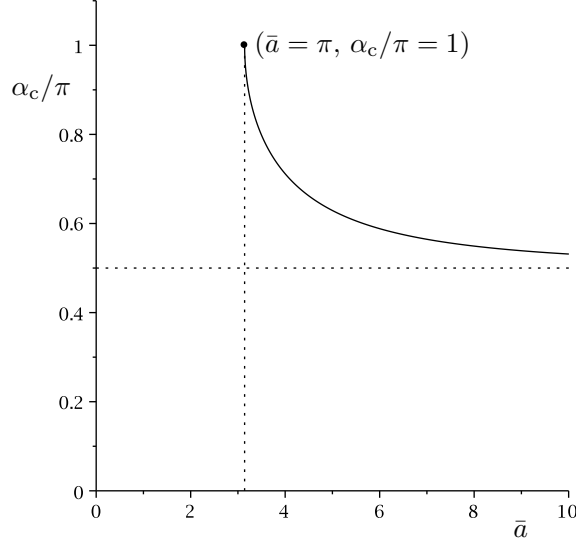


Figure 4: The scaled critical azimuthal angle, α_c/π , plotted as a function of $\bar{a}$.

as $\alpha \rightarrow \pi^-$ (so that the thin-film approximation ultimately fails in these limits); also β and h_m take the $O(1)$ values

$$\beta = \left(\frac{105\bar{Q}}{4\bar{a}^4} \right)^{\frac{1}{3}} \quad \text{and} \quad h_m = \left(\frac{105\bar{Q}}{32\bar{a}} \right)^{\frac{1}{3}} \quad (10)$$

at $\alpha = \pi/2$.

Figure 2(a) shows typical cross-sectional free-surface profiles of the rivulet in the case $\bar{a} = 2 (< \pi)$, and Figure 3 shows plots of β and h_m as functions of α/π for a range of values of $\bar{a}$ satisfying $\bar{a} < \pi$. Figure 3 also shows that, in order to maintain the correct flux, in the limit of a very narrow rivulet, $\bar{a} \rightarrow 0^+$, the rivulet becomes narrow and deep everywhere according to

$$\beta \sim \left(\frac{105\bar{Q}}{4\bar{a}^4 \sin \alpha} \right)^{\frac{1}{3}} \rightarrow \infty \quad \text{and} \quad h_m \sim \left(\frac{105\bar{Q}}{32\bar{a} \sin \alpha} \right)^{\frac{1}{3}} \rightarrow \infty. \quad (11)$$

4.2 Wide Rivulets with $\bar{a} > \pi$

For wide rivulets with $\bar{a} > \pi$ the solution for β given by (7) first attains the value $\beta = 0$ at the critical azimuthal angle $\alpha = \alpha_c$, where $\alpha_c = \alpha_c(\bar{a})$ ($\pi/2 < \alpha_c \leq \pi$) is a monotonically decreasing function of $\bar{a}$ given by

$$\alpha_c = \cos^{-1} \left(-\frac{\pi^2}{\bar{a}^2} \right) \quad \text{for} \quad \bar{a} \geq \pi, \quad (12)$$

and satisfying $\alpha_c \sim \pi + O(\bar{a} - \pi)^{1/2} \rightarrow \pi^-$ as $\bar{a} \rightarrow \pi^+$ and $\alpha_c \sim \pi/2 + O(\bar{a}^{-2}) \rightarrow \pi/2^+$ as $\bar{a} \rightarrow \infty$.

Figure 4 shows the scaled critical angle, α_c/π , plotted as a function of $\bar{a}$. The solution with β given by (7) corresponds to a rivulet with constant semi-width but varying contact angle $\beta > 0$ that runs

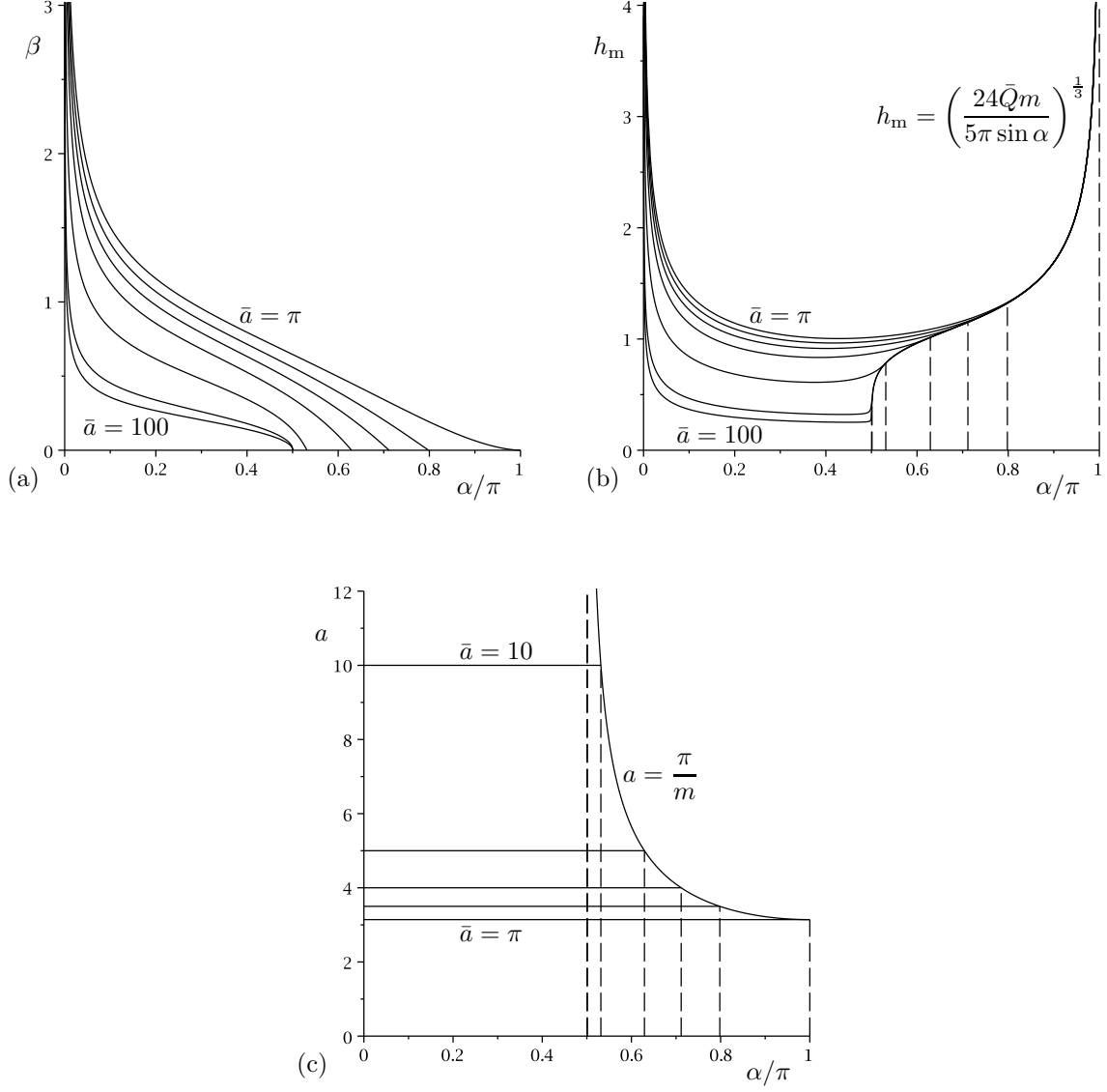


Figure 5: Plots of (a) the contact angle β , (b) the maximum thickness h_m , and (c) the semi-width a of a wide rivulet with flux $\bar{Q} = 1$ as functions of α/π for $\bar{a} = \pi, 3.5, 4, 5, 10, 50$ and 100 . In parts (b) and (c) the vertical dashed lines indicate the corresponding values of α_c/π .

from the top $\alpha = 0$ of the cylinder to $\alpha = \alpha_c$ on the lower half of the cylinder. In particular, the rivulet again becomes deep near the top of the cylinder according to (8) and again β and h_m take the $O(1)$ values given by (10) at $\alpha = \pi/2$. At $\alpha = \alpha_c$ the rivulet has zero contact angle $\beta = 0$, semi-width $a = \bar{a}$, and maximum thickness $h_m = h_{mc}$, where

$$h_{mc} = \left(\frac{24\bar{Q}}{5\bar{a} \sin \alpha_c} \right)^{\frac{1}{3}} = \left(\frac{24\bar{a}\bar{Q}}{5\sqrt{\bar{a}^4 - \pi^4}} \right)^{\frac{1}{3}}. \quad (13)$$

However, beyond $\alpha = \alpha_c$ the solution with β given by (7) is no longer physically realisable because it always predicts that $h < 0$ somewhere in the interval $y = -a$ to $y = +a$, and so an alternative description of the behaviour beyond $\alpha = \alpha_c$ is required. Physically it is possible that the rivulet simply falls off the cylinder at $\alpha = \alpha_c$ or that the flow becomes unsteady beyond $\alpha = \alpha_c$. However, an alternative possibility in which steady rivulet flow still occurs is that the contact lines de-pin at $\alpha = \alpha_c$, and that the rivulet runs from $\alpha = \alpha_c$ to the bottom of the cylinder $\alpha = \pi$ with zero contact angle according to the solution in the case $\beta = 0$ given by (5) and (6), with monotonically decreasing semi-width $a = \pi/m$ ($\pi \leq a \leq \bar{a}$) and monotonically increasing maximum thickness $h_m (\geq h_{mc})$. This scenario is sketched in Figure 1(b).

Figure 2(b) shows typical cross-sectional free-surface profiles of the rivulet in the case $\bar{a} = 5 (> \pi)$, and Figure 5 shows plots of β , h_m and a as functions of α/π for a range of values of $\bar{a}$ satisfying $\bar{a} \geq \pi$. Figure 5 shows that when $\bar{a} > \pi$ the solutions for $\beta \geq 0$ (valid for $0 \leq \alpha \leq \alpha_c$) and for $\beta = 0$ (valid for $\alpha_c \leq \alpha \leq \pi$) join continuously at $\alpha = \alpha_c$, where $\beta \rightarrow 0^+$ as $\alpha \rightarrow \alpha_c^-$ according to

$$\beta = \left(\frac{3(\bar{a}^4 - \pi^4)\bar{Q}}{40\bar{a}^2} \right)^{\frac{1}{3}} (\alpha_c - \alpha) + O(\alpha_c - \alpha)^2, \quad (14)$$

$h_m \rightarrow h_{mc}$ as $\alpha \rightarrow \alpha_c$ according to

$$h_m = h_{mc} + \frac{(\bar{a}^4 + \pi^4)h_{mc}}{6\pi^2\sqrt{\bar{a}^4 - \pi^4}}(\alpha - \alpha_c) + O(\alpha - \alpha_c)^2, \quad (15)$$

and $a \rightarrow \bar{a}^-$ as $\alpha \rightarrow \alpha_c^+$ according to

$$a = \bar{a} - \frac{\bar{a}\sqrt{\bar{a}^4 - \pi^4}}{2\pi^2}(\alpha - \alpha_c) + O(\alpha - \alpha_c)^2. \quad (16)$$

Figure 5 also shows that near the bottom of the cylinder $\alpha = \pi$ the rivulet becomes deep with finite semi-width π according to

$$a = \pi + \frac{\pi}{4}(\pi - \alpha)^2 + O(\pi - \alpha)^4 \rightarrow \pi^+ \quad \text{and} \quad h_m \sim \left(\frac{24\bar{Q}}{5\pi(\pi - \alpha)} \right)^{\frac{1}{3}} \rightarrow \infty \quad (17)$$

as $\alpha \rightarrow \pi^-$. Figure 5 also shows that, in order to maintain the correct flux, in the limit of a very wide rivulet on the upper half of the cylinder, $\bar{a} \rightarrow \infty$, (in which $\alpha_c \rightarrow \pi/2^+$) the rivulet becomes

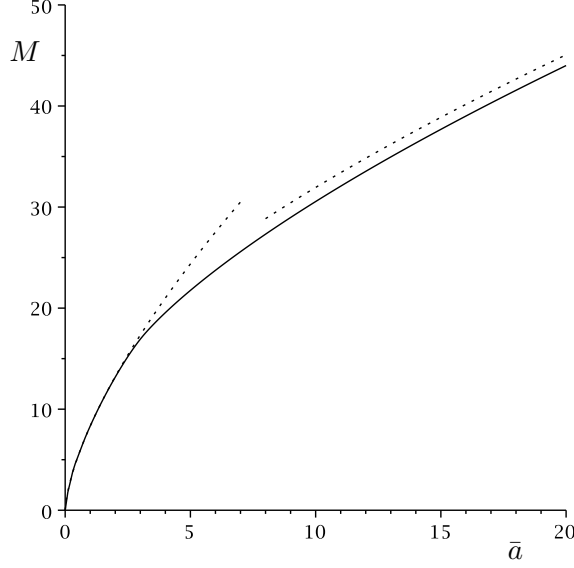


Figure 6: The total mass of fluid on the cylinder, M , for a rivulet with flux $\bar{Q} = 1$ plotted as a function of $\bar{a}$. The asymptotic solutions for M in the limits $\bar{a} \rightarrow 0^+$ and $\bar{a} \rightarrow \infty$, given by (22) and (23), respectively, are shown with dashed lines.

wide and flat on the upper half of the cylinder according to

$$\beta \sim \left(\frac{3\bar{Q}m^3}{2\bar{a}\sin\alpha} \right)^{\frac{1}{3}} \rightarrow 0^+ \quad \text{and} \quad h_m \sim \left(\frac{3\bar{Q}}{2\bar{a}\sin\alpha} \right)^{\frac{1}{3}} \rightarrow 0^+ \quad (18)$$

and is given by the solution in the case $\beta = 0$ given by (5) and (6) on the lower half of the cylinder.

4.3 The Special Case $\bar{a} = \pi$

In the special case $\bar{a} = \pi$ the solution behaves qualitatively as in the case of a narrow rivulet with $\bar{a} < \pi$ except that, since in this case $\beta = 0$ at $\alpha = \pi$, instead of satisfying (9) the rivulet becomes deep with zero contact angle near the bottom of the cylinder according to

$$\beta \sim \left(\frac{3\pi^2\bar{Q}(\pi-\alpha)^5}{320} \right)^{\frac{1}{3}} \rightarrow 0^+ \quad \text{and} \quad h_m \sim \left(\frac{24\bar{Q}}{5\pi(\pi-\alpha)} \right)^{\frac{1}{3}} \rightarrow \infty \quad (19)$$

as $\alpha \rightarrow \pi^-$. Plots of β and h_m as functions of α/π in the special case $\bar{a} = \pi$ are included in Figure 5.

4.4 Total Mass of Fluid

A quantity of some interest is the (finite) total mass of fluid on the cylinder (non-dimensionalised with $\delta\rho\ell^2R$) denoted by M , and given by $M = M_{\text{upper}} + M_{\text{lower}}$, where

$$M_{\text{upper}} = \int_0^{\frac{\pi}{2}} \int_{-a}^{+a} h \, dy \, d\alpha = \int_0^{\frac{\pi}{2}} \frac{2\beta(m\bar{a}\coth m\bar{a} - 1)}{m^2} \, d\alpha \quad (20)$$

and

$$M_{\text{lower}} = \int_{\frac{\pi}{2}}^{\pi} \int_{-a}^{+a} h \, dy \, d\alpha = \begin{cases} \int_{\frac{\pi}{2}}^{\pi} \frac{2\beta(1 - m\bar{a} \cot m\bar{a})}{m^2} d\alpha & \text{for } \bar{a} < \pi, \\ \int_{\frac{\pi}{2}}^{\alpha_c} \frac{2\beta(1 - m\bar{a} \cot m\bar{a})}{m^2} d\alpha + \int_{\alpha_c}^{\pi} \left(\frac{24\pi^2 \bar{Q}}{5m^2 \sin \alpha} \right)^{\frac{1}{3}} d\alpha & \text{for } \bar{a} > \pi, \end{cases} \quad (21)$$

are the masses of the fluid on the upper and lower halves of the cylinder, respectively. Figure 6 shows M plotted as a function of $\bar{a}$, and shows that M is a monotonically increasing function of $\bar{a}$. Figure 6 also shows that in the limit of a very narrow rivulet, $\bar{a} \rightarrow 0^+$, $M \rightarrow 0^+$ according to

$$M \sim C \left(\frac{70\bar{a}^2 \bar{Q}}{9} \right)^{\frac{1}{3}}, \quad (22)$$

while in the limit of a very wide rivulet on the upper half of the cylinder, $\bar{a} \rightarrow \infty$, $M \rightarrow \infty$ according to

$$M = C \bar{Q}^{\frac{1}{3}} \left[\left(\frac{3\bar{a}^2}{2} \right)^{\frac{1}{3}} + \left(\frac{6\pi^2}{5} \right)^{\frac{1}{3}} + O\left(\frac{1}{\bar{a}^{\frac{1}{3}}} \right) \right], \quad (23)$$

where the constant C is given by

$$C = \int_0^{\pi} \frac{d\alpha}{(\sin \alpha)^{\frac{1}{3}}} = \frac{\sqrt{\pi} \Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)} \simeq 4.2065. \quad (24)$$

5 Conclusions

In the present work we considered the locally unidirectional gravity-driven draining of a thin rivulet with constant semi-width a but slowly varying contact angle β down a slowly varying substrate. Specifically, we investigated the flow of a rivulet in the azimuthal direction from the top $\alpha = 0$ to the bottom $\alpha = \pi$ of a large horizontal cylinder. In particular, we showed that, despite behaving the same locally, this flow has qualitatively different global behaviour from that in the case of constant β . For example, whereas in the case of constant β the rivulet becomes flat and wide near the top of the cylinder, in the case of constant a it becomes deep according to (8), and, whereas in the case of constant β the dependence on the prescribed flux $\bar{Q}$ is not straightforward, in the case of constant a the contact angle β (and hence u , p , h and h_m) is simply proportional to $\bar{Q}^{1/3}$. More dramatically, whereas in the case of constant β there is always a rivulet that runs all the way from $\alpha = 0$ to $\alpha = \pi$, in the case of constant a this is possible only for sufficiently narrow rivulets satisfying $a = \bar{a} \leq \pi$. Wider rivulets satisfying $a = \bar{a} > \pi$ are possible only between $\alpha = 0$ and $\alpha = \alpha_c$, where $\alpha_c = \alpha_c(\bar{a})$ ($\pi/2 < \alpha_c \leq \pi$) is given by (12). Assuming that the contact lines de-pin at $\alpha = \alpha_c$ (where $\beta = 0$) the rivulet runs from $\alpha = \alpha_c$ to $\alpha = \pi$ with zero contact angle according to the solution in the case

$\beta = 0$ given by (5) and (6). The total mass of fluid on the cylinder was found to be a monotonically increasing function of $\bar{a}$. If required, the present analysis could be generalised to the situation in which de-pinning occurs at a non-zero value of β .

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Authors’ Short Vitae

Colin Paterson is a postgraduate student in the Department of Mathematics and Statistics at the University of Strathclyde and is currently undertaking theoretical research into various situations in which a thin layer of fluid is subject to an external air flow. The present study arose directly from his ongoing research into rivulet flow in the presence of an external air flow.

Professor Stephen Wilson is Professor of Applied Mathematics at the University of Strathclyde and has research interests in the application of mathematical techniques to a wide range of real-world problems in fluid mechanics. He is Joint Editor-in-Chief (with Professor Tom Witelski, Duke University, USA) of the *Journal of Engineering Mathematics* and is a Fellow of the Institute for Mathematics and its Applications (IMA).

Dr Brian Duffy is a Reader in Mathematics at the University of Strathclyde and has a long-standing interest in the mathematical analysis of many problems in fluid mechanics, notably thin-film flows and flows of liquid crystals. Together with Professor Wilson and Dr Khellil Sefiane (University of Edinburgh) he won the 2009 Institute of Physics Printing and Graphics Science Group Prize for his work on “a fundamental study of droplet evaporation”.

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