



Hypergraph colouring and the Lovász Local Lemma

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Received 7 July 1995

Abstract

The Lovász Local Lemma yields sufficient conditions for a hypergraph to be 2-colourable, that is, to have a colouring of the points blue or red so that no edge is monochromatic. The method yields a general theorem, which shows for example that, if H is a hypergraph in which each edge contains at least 9 points and each point is contained in at most 11 edges, then H is 2-colourable.

Here we see that this approach can go a little further: we use the ‘lopsided’ version of the Local Lemma to give an improved version of the theorem on hypergraph 2-colouring, from which it follows for example that the numbers 9, 11 above may be replaced by 8, 12.

1. Introduction

A hypergraph $H = (V, E)$ consists of a collection E of subsets of a (finite) set V : the members of V are the *points* and the members of E are the *edges*. A 2-colouring of a hypergraph is a colouring of the points blue or red so that no edge is monochromatic.

The original application of the Local Lemma presented in 1973 concerns hypergraph colouring (see e.g. [3, Theorem 2.1, Ch. 5] and yields the following result.

Theorem 1 (Erdős and Lovász [6]). *Let H be a hypergraph in which every edge contains at least k points and meets at most d other edges. If $e(d+1) \leq 2^{(k-1)}$ then H has a 2-colouring.*

A hypergraph is *k-uniform* if each edge contains exactly k points, and the *degree* of a point is the number of edges containing it. For which values of k and l is it true that each k -uniform hypergraph in which all points have degree at most l must be

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2-colourable? Call such a pair (k, l) *good*. Theorem 1 shows that $(9, 11)$ is good (take $d = 90$ and note that $91e \leq 2^8$).

Seymour [10] showed that any minimal non-2-colourable hypergraph must have at least as many edges as points (in edges). Thus any pair (k, l) with $k > l$ is good; and the pair (k, k) is good if and only if each k -uniform and k -regular (that is, each degree equals k) hypergraph is 2-colourable. Note also that if (k, l) is good then so is $(k + 1, l + 1)$. For $k \leq l$ this follows from the fact that in any bipartite graph there is matching covering all the vertices of maximum degree.

The special case when $k = l$ has received much attention. We have just seen from the Local Lemma approach [6] that $(9, 11)$ is good, and thus so is $(9, 9)$. Alon and Bregman [2] in 1988 showed that $(8, 8)$ is good, by a quite different method using the theorem asserting the truth of the Van der Waerden conjecture. Then in 1992 Thomassen [12] showed from his work on the even cycle problem for directed graphs that in fact $(4, 4)$ is good. Note that $(3, 3)$ is not good, since the hypergraph formed by the points and lines of the Fano plane is 3-uniform and 3-regular but not 2-colourable.

Here, we see that the approach to showing the existence of 2-colourings by using the Local Lemma goes a little further than had previously been apparent, and we obtain a slight improvement on Theorem 1. In the next section we shall prove the following result, using the ‘lopsided’ version of the Lovász Local Lemma.

Theorem 2. *Let H be a hypergraph in which every edge contains at least k points and meets at most d other edges. If $e(d + 2) \leq 2^k$ then H has a 2-colouring.*

It follows from Theorem 2 that $(7, 7)$ is good (take $d = 42$ and note that $44e \leq 2^7$); but this is weaker than Thomassen’s result noted above. However, we also find for example that $(8, 12)$ is good (take $d = 88$ and note that $90e \leq 2^8$). Indeed, more generally, if we could deduce from Theorem 1 that (k, l) is good, where $k < l$, then we can deduce from Theorem 2 that $(k - 1, l + 1)$ is good (since $(k - 1)l + 2 \leq k(l - 1) + 1$).

Much interest has been focussed recently on the Lovász Local Lemma and hypergraph 2-colouring, following the presentation by Beck [4] (see also [1, 3]) of an efficient method to find 2-colourings if edges are sufficiently large and degrees sufficiently small. Using Theorem 2 these conditions may be relaxed very slightly. Finally, let us note here that if we use Theorem 2 rather than its predecessor Theorem 1, we may obtain slightly improved conditions for lower bounds on the diagonal Ramsey numbers (see, e.g. [5, Ch. 12, Theorem 5] or [8, Ch. 4] or [3, Ch. 5] though there is no asymptotic improvement.

2. Proof of Theorem 2

Let A_1, \dots, A_n be events in a probability space, and let G be a graph on the vertex set $V = \{1, \dots, n\}$. Following [7] we say that G is a *lopsidedependency graph* for the

events if

$$P\left(A_i \mid \bigcap_{j \in J} \overline{A_j}\right) \leq P(A_i)$$

for each $i \in V$ and each non-empty $J \subseteq V$ such that no $j \in J$ is adjacent to i (and $P(\bigcap_{j \in J} \overline{A_j}) > 0$).

We shall use the following ‘symmetric lopsided’ version of the Local Lemma, which follows immediately from the usual proof of the Local Lemma [6,7,3].

Lemma 1 (Local Lemma: symmetric lopsided version). *Let A_1, \dots, A_n be events in a probability space Ω , with a lopsidedependency graph G . Suppose that each event A_i has probability at most p and that each vertex degree in G is at most d .*

If $ep(d+1) \leq 1$ then $P(\bigcap_{i=1}^n \overline{A_i}) > 0$.

Proof of Theorem 2. Colour the points of the hypergraph $H=(V,E)$ blue or red independently and with equal probability; that is, pick a colouring uniformly from the set Ω of the $2^{|V|}$ possible blue-red colourings. For each edge $f \in E$ define two events B_f and R_f by letting B_f [R_f] be the set of $\omega \in \Omega$ such that ω colours f all blue [red]. Thus $P(B_f) = P(R_f) = 2^{-|f|} \leq 2^{-k}$ for each $f \in E$.

Let G be the bipartite graph with parts $\{B_f: f \in E\}$ and $\{R_f: f \in E\}$ and with vertices B_f and R_g adjacent if and only if $f \cap g \neq \emptyset$. Note that each vertex in G has degree at most $d+1$.

Claim. G is a lopsidedependency graph for the family of events B_f, R_f .

Once this claim is established, the theorem will follow immediately from Lemma 1.

Let us then prove the claim. Fix some edge $g \in E$, and consider the corresponding event B_g . (An entirely analogous argument will work for R_g .) Let J be any non-empty collection of events B_f or R_f not adjacent in G to B_g , and suppose that the event $D = \bigcap_{A \in J} \overline{A}$ is non-empty. We must show that

$$P(B_g | D) \leq P(B_g). \quad (1)$$

Let x be a partial colouring of the form $\omega|_{V \setminus g}$ for some $\omega \in D$. Let C_x be the set of colourings $\omega \in \Omega$ such that $\omega|_{V \setminus g} = x$. To prove inequality (1) it suffices to show that

$$P(B_g | C_x \cap D) \leq P(B_g). \quad (2)$$

Note that for any event B_f in J with $f \cap g = \emptyset$ there must be a point in f coloured red by x , and for any event R_f in J (necessarily $f \cap g = \emptyset$) there must be a point in f coloured blue by x . If there is an edge $f \in E$ such that $B_f \in J$, $f \cap g \neq \emptyset$ and $f \setminus g$ is all blue under x , then $P(B_g | C_x \cap D) = 0 \leq P(B_g)$; and if there is no such edge then $C_x \subseteq D$ so $P(B_g | C_x \cap D) = P(B_g)$. Thus the inequality (2) holds in either case, and the proof is complete. \square

3. Further remarks

3.1. Hypergraph t -colouring

The following result on hypergraph t -colouring extends Theorem 2, and may be proved along exactly the same lines — see also Lemma 2 below.

Theorem 3. *As in Theorem 2, let H be a hypergraph in which every edge contains at least k points and meets at most d other edges. Let $t \geq 2$ be an integer. If $e((d+1)(t-1)+1)(1-1/t)^k \leq 1$ then H has a t -colouring in which each colour appears on each edge.*

This result yields a slight improvement in a result from [6] — see for example Theorem 2.2 of chapter 5 of [3]. We may see that for each $k \geq 19$ every k -uniform k -regular hypergraph can be 3-coloured so that each colour appears on each edge.

3.2. Asymmetric 2-colouring

Our next result also extends Theorem 1, and may be proved along the same lines, except that we must start from the natural ‘general lopsided’ version of the Local Lemma — see [3].

Theorem 4. *Let $\mathcal{S} = (S_i: i \in I)$ and $\mathcal{T} = (T_j: j \in J)$ be two families of subsets of a set V . Suppose that there exist $0 < p < 1$, $0 < x_i < 1$ ($\forall i \in I$) and $0 < y_j < 1$ ($\forall j \in J$) such that*

$$p^{|S_i|} \leq x_i \prod_{j: S_i \cap T_j \neq \emptyset} (1 - y_j) \quad (\forall i \in I)$$

and

$$(1 - p)^{|T_j|} \leq y_j \prod_{i: S_i \cap T_j \neq \emptyset} (1 - x_i) \quad (\forall j \in J).$$

Then there is a blue-red colouring of V such that no set S_i in \mathcal{S} is all blue and no set T_j in \mathcal{T} is all red.

To see that this result contains Theorem 2, let $\mathcal{S} = \mathcal{T} = E$, let $p = \frac{1}{2}$ and let $x_i = y_j = 1/(d+2)$ for each i, j ; and recall that $(1 - (1/(d+2)))^{d+1} > \frac{1}{e}$ — see [3]. We may use Theorem 4 to obtain for example a slight improvement in previous lower bounds on the off-diagonal Ramsey number $R(k, 3)$ — see [5, Ch. 12] or [3, Ch. 5] — but not an asymptotic improvement. We find that $R(k, 3) > n$ if there exist $0 < p < 1$, $0 < x < 1$ and $0 < y < 1$ such that $p^3 \leq x(1-y)\binom{n}{3}$ and $(1-p)\binom{n}{3} \leq y(1-x)\binom{n}{3}$.

3.3. A general lemma

The main step in the proof of Theorem 1 (and of the other theorems above) is noting that a certain bipartite or multipartite graph is a lopsidedependency graph. The following lemma gives a natural generalisation of this result. As we shall see, it can be used for example if we are interested in t -colouring a hypergraph so that each colour appears at least twice on each edge. The lemma may be proved as in the proof of Theorem 2, except that at the last step we use the correlation inequality of Harris [9] (generalised by the FKG inequality, etc. — see for example [3]) in a natural way.

Let n and t be positive integers and let P be a product measure on $\Omega = \{1, \dots, t\}^n$. For each (colour) $j = 1, \dots, t$ let $\mathcal{S}^{(j)}$ be a collection of subsets of $V = \{1, \dots, n\}$; and for each set f in $\mathcal{S}^{(j)}$, let $D_f^{(j)}$ be a hereditary collection (downset) of subsets of f and let $A_f^{(j)}$ be the event

$$A_f^{(j)} = \{\omega \in \Omega : \{i \in f : \omega_i = j\} \in D_f^{(j)}\}.$$

Lemma 2. *Let G be the t -partite graph with parts $\{A_f^{(j)}\}$ for each colour $j = 1, \dots, t$, and with vertices $A_f^{(j)}$ and $A_{f'}^{(j')}$ adjacent if and only if $j \neq j'$ and $f \cap f' \neq \emptyset$. Then G is a lopsidedependency graph for the events $A_f^{(j)}$.*

To see how to apply this lemma, let us consider 2-colourings of a hypergraph such that each edge contains at least 2 points of each colour. Suppose as earlier that each edge contains at least k points and meets at most d other edges. For each edge f , let $D_f^{(1)}$ and $D_f^{(2)}$ both be the set of all subsets of f of size at most 1. Then from Lemmas 1 and 2, we find that, if $e(k+1)2^{-k}(d+2) \leq 1$ then H has a colouring as required. In particular, if say $k = 12$ and each degree is at most 10, then H has such a colouring.

Acknowledgements

I would like to thank Noga Alon and Bruce Reed for helpful comments.

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