Forward Performance Measurement with Applications in Indifference Valuation



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To my loving dad, who has not only earned his bread but also my bread for over twenty years.

And to my loving mum, who could never fall asleep if she has not heard my voice on the phone.

I love you!

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Abstract

In this thesis, we present basic ideas and key results for forward performance measurement. Besides, we provide an explicit construction of the optimal processes of a class of time-monotone forward performance processes. Moreover, starting with a two parameter family risk tolerance function, we construct a class of forward performance processes. By letting the parameter go to zero, we obtain the forward exponential utility. Finally, using the forward exponential utility, we solve the integrated portfolio management problem by the so-called utility-based approach and compare it with its classical backward indifference counterpart.

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Chapter 1

Introduction

Optimal asset allocation problems can be formulated as classical stochastic optimization problems. In the continuous time framework, introduced in [7], the problem consists of a time horizon, a controlled wealth process and an optimization criterion represented as the expectation of the utility of terminal wealth condition on current information. Maximizing the expectation over a given set of admissible policies gives the optimal investment strategy and the so-called value function.

Mathematically, given an investment horizon $[t, T]$, if we denote the investor's wealth at time $s \in [t, T]$ by X_s and the terminal utility function by U, then the value function is defined as

$$V(x, t; T) = \sup_{\mathcal{A}} \mathbb{E}_{\mathbb{P}} \left(U(X_T) | \mathcal{F}_t \right) = \mathbb{E}_{\mathbb{P}} \left(U(X_T^*) | \mathcal{F}_t \right),$$

where X^* represents the optimal wealth. The value function has been widely studied under rather general modelling assumptions. As a result of the dynamic programming principle, it is a martingale at the optimal and supermartingale away from it (see, for example [3]). Namely,

$$\begin{cases} V(x, t; T) = \mathbb{E}_{\mathbb{P}} \left(V(X_s^*, s; T) | \mathcal{F}_t \right), & s \in [t, T] \\ \\ V(x, t; T) \geqslant \mathbb{E}_{\mathbb{P}} \left(V(X_s^{\pi}, s; T) | \mathcal{F}_t \right), & s \in [t, T]. \end{cases}$$

From the above we can see that the value function coincides with a utility prespecified at the end of the horizon and generates itself backwards in time. The idea of assigning a datum at a future time comes from classical problems (see, for example, Merton [7], Davis and Norman [1], and Karatzas, Lehoczky, Sethi and Shreve [5]). However, it might not be an appropriate assumption under other modelling settings. Here, we mention two drawbacks of the traditional approach.

Firstly, the risk preferences are a priori specified at a future time, which do not seem to capture changes in the risk attitude accurately as the market environment

evolves. Moreover, from the indifference valuation's perspective, the traditional approach considerably constraints the set of claims that can be priced to the ones that expire before the normalization point T. This precludes opportunities related to claims arriving at a later time and maturing beyond the normalization point.

1.1 Forward Performance Measurement in the Literature

Motivated by the above considerations, an alternative approach to optimal portfolio choice based on the so-called forward performance criterion was proposed by M. Musiela and T. Zariphopoulou (see, among others, Musiela and Zariphopoulou [11] and Musiela and Zariphopoulou [12]). The idea of the forward performance criterion is to remove the drawbacks but not to move entirely away from the traditional formulation . It offers flexibility on the choices of trading horizon and risk preferences while preserving the optimality properties of the value function process. To be specific, in this approach, the investor dose not choose his risk preference at a single point in time as in the Merton model, but has the flexibility to revise them dynamically. The resulting process is called *forward performance process*.

Constructing the forward performance process as well as the associated optimal investment strategy and the optimal wealth process poses many difficulties. This is due to the fact that the implicit stochastic optimization problem is posed "forward in time" and therefore the existing stochastic optimization techniques are not directly applicable. Nevertheless, a class of time-monotone forward performance processes was recently constructed and analysed in Musiela and Zariphopoulou [12] and Berrier, Rogers and Tehranchi [2]. Explicit constructions of the optimal investment strategy and optimal wealth process also exist for a special case of the forward performance criterion (see Musiela and Zariphopoulou [13]).

1.2 Outline of the Dissertation

The detailed structure of the dissertation is as follows. In Chapter 2, I introduce the general modelling setting and review the definition of the forward performance process. Moreover, a class of time-monotone forward performance processes is studied. It can be represented as the compilation of differential and stochastic input which are specified by a solution of a non-linear partial differential equation and a triple of stochastic processes respectively. The optimal portfolio is a linear combination of the investor's optimal wealth and the time rescaled risk tolerance process. In addition, it

is proved that they solve an autonomous system of stochastic differential equations. In Musiela and Zariphopoulou [13], it is proved that for a special case of a forward performance criterion, there exists an explicit construction of the optimal investment strategy and optimal wealth process. Finally, I show that this result is also true for a more general class of forward performance processes. This chapter is based on Musiela and Zariphopoulou [12] and Musiela and Zariphopoulou [13].

As the results in Chapter 2 illustrate, it is the risk tolerance function r, and not u, that emerges as the key underlying differential quantity. This motivates us to question whether one should start with u and then define r or model r directly. In the first three sections of Chapter 3, I look at the representation of the risk tolerance function r and the differential input u, as well as the relationship between them. The last section studies a special class of forward performance processes constructed from a two parameter family risk tolerance $r(x,t;a,b) = \sqrt{a^2x^2 + b^2e^{-a^2t}}$. The differential input u is constructed by applying the results derived previously. Finally, by letting the parameter a go to 0, I show that the limiting risk tolerance function and the corresponding differential input resemble their classical exponential analogue. This chapter is based on Musiela and Zariphopoulou [13] and Zariphopoulou and Zhou [18].

Chapter 4 studies indifference valuation under forward dynamic utilities in incomplete markets. The notion of forward dynamic utilities was introduced to the field of indifference pricing by M. Musiela and T. Zariphopoulou (see Musiela and Zariphopoulou [8] and [9]). The model considered therein is a simple multi-period incomplete binomial model. This chapter extends the notion of forward dynamic utilities in a diffusion model with a correlated stochastic factor. Section 2 studies the forward indifference price and optimal investment strategy corresponding to the forward dynamic exponential utility calculated in Chapter 3. In section 3, I show that the backward and forward indifference valuations have very different characteristics by comparing their indifference values and optimal integrated investment portfolios. This chapter is based on Musiela and Zariphopoulou [10].

In the last chapter, I briefly review the main results of the forward performance process and its applications in indifference valuation. Further research directions are also discussed.

Chapter 2

The Model and Forward Performance Process

This chapter is organized as follows. Section 1 describes the general modelling setting. In Section 2, we revisit the definition of forward performance process and the main results of Musiela and Zariphopoulou [12]. The explicit construction of the optimal wealth processes and investment strategies of a class of time-monotone forward performance processes are presented in Section 3

2.1 The Model

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a fixed filtered complete probability space on which a standard $\mathcal{F}_t$-adapted d-dimensional Brownian motion $W_t = (W_t^1, ..., W_t^d)$ with $W_0 = 0$ is defined. It is assumed that $\mathcal{F}_t = \sigma\{W_s : 0 \leq s \leq t\}$, augmented by all the null sets.

The market environment consists of $k+1$ assets being traded continuously. One of the assets is a riskless bank account whose price process B_t follows

$$dB_t = r_t B_t dt,\ t \in [0, +\infty);\ B_0 = 1 \tag{2.1}$$

where the interest rate $r(.)$ is an $\mathcal{F}_t$-progressively measurable[1], bounded scalar-valued stochastic process. The other k assets are risky stocks and their prices are modelled as positive and continuous Ito processes. Namely, for $i = 1, ..., k$, the price S^i of the i^{th} risky asset solves the following equation

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ij} dW_t^j \right),\ S_0^i > 0. \tag{2.2}$$

[1]In general, a process X is progressively measurable with respect to $\mathcal{F}$ if for every $t \in [0, T]$ the map $(t, \omega) \to X(t, \omega) \equiv X_t(\omega)$ from $([0, t] \times \Omega, \mathcal{B}([0, t] \otimes \mathcal{F}_t))$ into $(R^n, \mathcal{B}(R^n))$ is measurable. Every progressively measurable process is adapted. Conversely, any adapted process with right- or left-continuous paths is progressively measurable (see Karatzas and Shreve [5], Proposition 1.13).

where μ_t^i and σ_t^{ij}, the appreciation and dispersion rates, respectively, are bounded scalar-valued $\mathcal{F}_t$-progressively measurable stochastic processes in $\mathbb{R}$ and $\mathbb{R}^d$. For brevity, we use σ_t to denote the volatility matrix, i.e. the $d \times k$ random matrix $\left(\sigma_t^{ij}\right)$, whose i^{th} column σ_t^i represents the volatility of the i^{th} risky asset. Alternatively, (2.2) can be written as

$$dS_t^i = S_t^i(\mu_t^i dt + \sigma_t^i \cdot dW_t).$$

In addition, we denote by μ_t the $k \times 1$ vector with coordinates μ_t^i and by $\mathbf{1}$ the k dimensional vector with every entry equal to 1. Then

$$\mu_t - r_t\mathbf{1} \in \text{Lin}(\sigma_t^T). \tag{2.3}$$

In consequence,

$$\sigma_t^T(\sigma_t^T)^+(\mu_t - r_t\mathbf{1}) = \mu_t - r_t\mathbf{1}, \tag{2.4}$$

where the matrix $(\sigma_t^T)^+$ is the Moore-Penrose pseudo-inverse[2] of the matrix σ_t^T.

Define λ_t, known as the market price of risk, to be a d-dimensional bounded $\mathcal{F}_t$ adapted process that satisfies the equation

$$\mu_t^i - r_t = \textstyle\sum_{j=1}^d \sigma_t^{ij}\lambda_t^j = \sigma_t^i \cdot \lambda_t,$$

for $t \geqslant 0$ and $i = 1, ..., k$. Using vector and matrix notations, the above equation can be written as

$$\sigma_t^T\lambda_t = \mu_t - r_t\mathbf{1}.$$

Hence,

$$\lambda_t = (\sigma_t^T)^+(\mu_t - r_t\mathbf{1}). \tag{2.5}$$

It is assumed that, for all $t \geqslant 0, \mathbb{E}_{\mathbb{P}}\left(\int_0^t |\sigma_s\sigma_s^+\lambda_s|^2 ds\right) < \infty$.

Starting at time $t = 0$ with initial wealth x, the investor invests his wealth in the riskless and risky assets at all future times $t > 0$. Define $\pi^i, i = 0, ..., k$ to be the present value of the weath invested in the i^{th} asset. Therefore, the present value of his total wealth is given by $X_t = \sum_{i=0}^k \pi_t^i$. Under the self-financing assumption, the discounted wealth can be written as

$$X_s = x + \sum_{i=1}^k \int_0^s \pi_u^i(\mu_u^i - r_u)du + \sum_{i=1}^k \int_0^s \pi_u^i\sigma_u^i \cdot dW_u,$$

[2]The Moore-Penrose pseudo-inverse matrix, denoted by A^+, of a $d \times k$ matrix A is the unique $k \times d$ matrix satisfying $AA^+A = A; A^+AA^+ = A^+, (AA^+)^T = AA^+ \text{and} (A^+A)^T = A^+A$ (see Penrose[15]).

for $s \geq 0$. Writing it in differential form yields the dynamics of the discounted wealth process,

$$dX_t = \sum_{i=1}^{k} \pi_t^i \sigma_t^i \cdot (\lambda_t dt + dW_t) = \sigma_t \pi_t \cdot (\lambda_t dt + dW_t), \tag{2.6}$$

where the column vector $\pi_t = (\pi_t^1, ..., \pi_t^k)$.

The set of admissible strategies, $\mathcal{A}$, is defined to be the set of all self-financing $\mathcal{F}_t$ adapted processes π_t such that $\mathbb{E}_{\mathbb{P}}\left(\int_0^s |\sigma_t \pi_t|^2 dt\right) < \infty$, for all $s \geq 0$.

Next we introduce the definition of forward performance process and the results in Musiela and Zariphopoulou [12].

2.2 Forward Performance Process

Definition 2.2.1. *An F_t-adapted process $U_t(x)$ is a forward performance if for $t \geqslant 0$ and $x \in \mathbb{R}$:*

1. *the mapping $x \longrightarrow U_t(x)$ is strictly concave and increasing,*
2. *for each self-financing strategy, π, $\mathbb{E}_{\mathbb{P}}\left(U_t(X_t^{\pi})\right)^+ < \infty$, and*

$$\mathbb{E}_{\mathbb{P}}\left(U_s(X_s^{\pi}) \middle| \mathcal{F}_t\right) \leq U_t(X_t^{\pi}), \ s \geq t, \tag{2.7}$$

3. *there exists a self-financing strategy, π^* for which*

$$\mathbb{E}_{\mathbb{P}}\left(U_s(X_s^{\pi^*}) \middle| \mathcal{F}_t\right) = U_t(X_t^{\pi^*}), \ s \geq t, \tag{2.8}$$

4. *it satisfies the initial datum $U_0(x) = u_0(x)$, $x \in \mathbb{R}$, where $u_0 : \mathbb{R} \to \mathbb{R}$ is a concave and increasing function of wealth.*

Remark 2.2.1. *Condition 4 in the definition can be relaxed. In addition, not all increasing and concave functions can serve as an initial condition, even for the class of time-monotone forward performance processes we study later (see Musiela and Zariphopoulou [13] section 4 for detailed analysis). Characterizing the set of appropriate initial conditions is a challenging question and is currently being studied by researchers in this field. Recently, a new point of view on the specification of the investor's risk preferences is introduced by M. Musiela and T. Zariphopoulou. To be specific, instead of specifying the utility function (or risk tolerance) at the beginning of the trading horizon, it is assumed that the investor is able to determine the portfolio allocations that are desirable to him at initiation. The entire analysis bypasses the need to know the risk preferences (see Musiela and Zariphopoulou [14]).*

Remark 2.2.2. *In Definition 2.2.1, in order to define the conditional expectation* $\mathbb{E}_{\mathbb{P}}\left(U_s(X_s^{\pi^*})\middle|\mathcal{F}_t\right)$*, an integrability condition is imposed on* $\mathbb{E}_{\mathbb{P}}\left(U_t(X_t^{\pi})\right)^+$*. However,this may lead to other integrability conditions which further constrain the class of forward solutions the investor may employ. Alternatively, we could avoid this by replacing conditions 2 and 3 with corresponding local-martingale statements. In consequence, the set of admissible strategies becomes*

$$\mathcal{A}^l = \{\pi_t : \textit{self-financing, } \mathcal{F}_t\textit{-adapted and } \mathbb{P}\left(\int_0^t |\sigma_s\pi_s|^2 < \infty\right) = 1;\ \textit{for } \forall t > 0\}.$$

Next, we provide a class of time-monotone forward performance process which is introduced in Musiela and Zariphopoulou [12]. It is given by the solution of a fully nonlinear PDE and a triple of stochastic processes.

Theorem 2.2.3. *Let the process* Y *and* Z *solve*

$$dY_t = Y_t\delta_t \cdot (\lambda_t dt + dW_t) \tag{2.9}$$

and

$$dZ_t = Z_t\phi_t \cdot dW_t, \tag{2.10}$$

with $Y_0 = Z_0 = 1$*, and the coefficients* δ *and* ϕ *being* $\mathcal{F}_t$*-adapted and bounded. Moreover, we assume throughout that* δ *and* ϕ *belong to* $Lin(\sigma_t)$ *which implies that*

$$\sigma_t\sigma_t^+\delta_t = \delta_t \textit{ and } \sigma_t\sigma_t^+\phi_t = \phi_t. \tag{2.11}$$

Define the process

$$A_t = \int_0^t |\sigma_s\sigma_s^+(\lambda_s + \phi_s) - \delta_s|^2 ds,\ t \geq 0, \tag{2.12}$$

where λ_s *is given as in* (2.5)*.*

Let $u : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ *be a concave and increasing function of the spatial argument with* $u : \mathcal{C}^{3,1}(\mathbb{R} \times (0,\infty))$ *satisfying the differential constraint*

$$u_t u_{xx} = \frac{1}{2}u_x^2 \tag{2.13}$$

and the initial datum

$$u(x,0) = u_0(x), \tag{2.14}$$

with $u_0 : \mathbb{R} \to \mathbb{R}$ *be in* $\mathcal{C}^3(\mathbb{R})$*. Then the process* $U_t(x)$ *defined by*

$$U_t(x) = u\left(\frac{x}{Y_t}, A_t\right) Z_t,\ t \geq 0 \tag{2.15}$$

is a forward performance process.

In the representation (2.15), Y normalizes the wealth argument while Z appears as a multiplicative factor. One might think of Y as a benchmark (or numeraire) with regards to which the investment performance is measured. The process Z, on the other hand, can be thought as a device offering flexibility to our solutions in terms of asset returns. It refers to changes in the historical probability measure and accommodates the investor's view on future market movements. We refer to Y and Z as the benchmark and market view processes respectively. On the other hand, A is a positive non-decreasing process with zero initial value. It rescales the time argument in the differential input u.

Corollary 2.2.1. *In the special case where $\delta_t = \phi_t = 0$, $t \geq 0$, the forward performance process reduces to*

$$U_t(x) = u\left(x, \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds\right). \tag{2.16}$$

Forward solutions of the form (2.16) and its corresponding optimal policies are extensively studied in Musiela and Zariphopoulou [13].

We continue to study the optimal investment strategies and wealth processes corresponding to the time-monotone forward performance processes (2.15). For simplicity, we define the benchmarked optimal portfolio and benchmarked optimal wealth to be

$$\tilde{\pi}_t^* \equiv \frac{\pi_t^*}{Y_t} \quad \text{and} \quad \tilde{X}_t^* \equiv \frac{X_t^*}{Y_t}. \tag{2.17}$$

Given the differential input u as in (2.15), define the time-rescaled risk tolerance process to be

$$\tilde{R}_t^* = r\left(\tilde{X}_t^*, A_t\right), \tag{2.18}$$

with the local risk tolerance r given by

$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)} \text{ with initial value } r_0(x) = r(x,0) = -\frac{u_0'(x)}{u_0''(x)}, \tag{2.19}$$

and A being the time rescaling process defined in (2.12). The following theorem is proved in Musiela and Zariphopoulou [13].

Theorem 2.2.4. *The optimal benchmarked portfolio $\tilde{\pi}_t^*$, $t > 0$, is given by*

$$\tilde{\pi}_t^* = \Pi_t^*\left(X_t^*\right)$$

with

$$\Pi_t^*(x) = x\sigma^+\delta_t + r(x, A_t)\sigma_t^+(\lambda_t + \phi_t - \delta_t), \tag{2.20}$$

where A as in (2.12) *and $\tilde{X}^*_t$, $t > 0$, solves*

$$d\tilde{X}^*_t = \left(\sigma_t \tilde{\pi}^*_t - \tilde{X}^*_t \delta_t\right) \cdot ((\lambda_t - \delta_t)dt + dW_t), \tag{2.21}$$

with $\tilde{\pi}^*$ being used. Therefore,

$$\tilde{\pi}^*_t = m_t \tilde{X}^*_t + n_t \tilde{R}^*_t \tag{2.22}$$

with $\tilde{R}^*_t$ as in (2.18)*, and the portfolio weights given by*

$$m_t = \sigma_t^+ \delta_t \quad \textit{and} \quad n_t = \sigma_t^+(\lambda_t + \phi_t - \delta_t). \tag{2.23}$$

The associated optimal performance, $U^*_t = U_t\left(\tilde{X}^*_t\right)$ satisfies

$$dU^*_t = \left(Z_t u_x\left(\tilde{X}^*_t, A_t\right)\tilde{R}^*_t(\lambda_t + \phi_t - \delta_t) + U^*_t \phi_t\right)_t,$$

with $U^*_0(x) = u_0(x)$.

First we notice that the above result comes as a surprise given the non-Markovian nature of the market model. Moreover, the vector of optimal allocations can be expressed as a linear combination of two portfolios, given by

$$\tilde{\pi}^{*,X}_t = m_t \tilde{X}^*_t \quad \text{and} \quad \tilde{\pi}^{*,R}_t = n_t \tilde{R}^*_t. \tag{2.24}$$

Observe that the first component $\tilde{\pi}^{*,X}_t$ depends functionally only on the benchmarked wealth and not on the risk tolerance, while the situation is reversed for the second term $\tilde{\pi}^{*,R}_t$.

In addition, in Musiela and Zariphopoulou [13], it is proved that the processes $\tilde{X}^*_t$ and $\tilde{R}^*_t$ solves the following autonomous system of stochastic differential equations for $t > 0$.

Proposition 2.2.1. *Let r be the local risk tolerance function introduced in* (2.19)*, and A be the time rescaling process given in* (2.12)*. The, for $t > 0$, the processes $\tilde{X}^*_t$ and $\tilde{R}^*_t$, $t > 0$, representing the benchmarked optimal wealth and risk tolerance process, solve the system*

$$\begin{cases} d\tilde{X}^*_t = \tilde{R}^*_t \sigma_t n_t \cdot ((\lambda_t - \delta_t)dt + dW_t) \\ \\ \qquad d\tilde{R}^*_t = r_x\left(\tilde{X}^*_t, A_t\right) d\tilde{X}^*_t \end{cases} \tag{2.25}$$

with $\tilde{X}^*_0 = x$, $\tilde{R}^*_0 = r_0(x)$ and n_t, $t > 0$ as in (2.23)*.*

Proof. According to (2.21), we deduce that

$$d\tilde{X}_t^* = \tilde{R}_t^*(\lambda_t + \phi_t - \delta_t) \cdot ((\lambda_t - \delta_t)dt + dW_t). \tag{2.26}$$

Moreover,

$$\begin{aligned} d\tilde{R}_t^* &= dr\left(\tilde{X}_t^*, A_t\right) = r_x\left(\tilde{X}_t^*, A_t\right) d\tilde{X}_t^* + r_t\left(\tilde{X}_t^*, A_t\right) dA_t + \frac{1}{2} r_{xx}\left(\tilde{X}_t^*, A_t\right) d\langle \tilde{X}^* \rangle_t \\ &= r_x\left(\tilde{X}_t^*, A_t\right) d\tilde{X}_t^* + \left(r_t\left(\tilde{X}_t^*, A_t\right) + \frac{1}{2} r_{xx}\left(\tilde{X}_t^*, A_t\right)\left(\tilde{R}_t^*\right)^2\right) dA_t \\ &= r_x\left(\tilde{X}_t^*, A_t\right) d\tilde{X}_t^* + \left(r_t\left(\tilde{X}_t^*, A_t\right) + \frac{1}{2} r^2\left(\tilde{X}_t^*, A_t\right) r_{xx}\left(\tilde{X}_t^*, A_t\right)\right) dA_t. \end{aligned}$$

Using (2.27), we conclude that

$$d\tilde{R}_t^* = r_x\left(\tilde{X}_t^*, A_t\right) d\tilde{X}_t^*.$$

□

Notice that it is the function r, and not u, that appears in the optimal portfolio (2.20) and the system of stochastic differential equations (2.25). In Musiela and Zariphopoulou [13], it is proved that r has the following differential property.

Proposition 2.2.2. *Let $u \in \mathcal{C}^{4,1}(\mathbb{R} \times [0, +\infty))$ satisfies (2.13) and (2.14). Then the associated local risk tolerance $r(x,t)$, defined in (2.19), satisfies*

$$r_t + \frac{1}{2} r^2 r_{xx} = 0 \quad \textit{and} \quad r(x,0) = -\frac{u_0'(x)}{u_0''(x)}. \tag{2.27}$$

Proof. Differentiate the differential constraint (2.13) yields

$$u_{tx} = u_x - \frac{1}{2} u_x \left(\frac{u_x u_{xxx}}{u_{xx}^2}\right). \tag{2.28}$$

Differentiating (2.28) again with respect to x yields

$$u_{txx} = u_{xx} - \frac{1}{2} u_{xx} \left(\frac{u_x u_{xxx}}{u_{xx}^2}\right) - \frac{1}{2} u_x \left(\frac{u_x u_{xxx}}{u_{xx}^2}\right)_x. \tag{2.29}$$

Moreover, differentiating (2.19), we obtain

$$r_x = -1 + \frac{u_x u_{xxx}}{u_x^2} \quad \text{and} \quad r_{xx} = \left(\frac{u_x u_{xxx}}{u_{xx}^2}\right)_x. \tag{2.30}$$

Combining the above results, yields

$$r_t + \frac{1}{2} r^2 r_{xx} = -\frac{u_{tx}}{u_{xx}} + \frac{u_x u_{txx}}{u_{xx}^2} + \frac{1}{2}\left(\frac{u_x u_{xxx}}{u_{xx}}\right)_x = 0.$$

□

Once the model is chosen and the local risk tolerance function is obtained by solving (2.27), the processes $\left(\tilde{X}_t^*, \tilde{R}_t^*\right)$ can be fully specified by applying Proposition 2.2.1. Substituting it into (2.20) yields the optimal investment portfolio. This motivates us to question whether we should first model the differential input u and then specify r or start with r directly. This question will be further investigated in Chapter 3.

2.3 Construction of Optimal Wealth and Portfolio Processes

The explicit construction of the optimal processes corresponding to the special class of forward performance processes (2.16) is derived in Musiela and Zariphopoulou [13]. Let $h(x,t)$ be a strictly increasing function in x and solves the backward heat equation

$$h_t + \frac{1}{2}h_{xx} = 0, \tag{2.31}$$

for $(x,t) \in \mathbb{R} \times [0,+\infty)$. The function $h^{(-1)}$ stands for the spatial inverse of h. It is proved therein that the processes X_t^* and π_t^* corresponding to (2.16) are given by

$$X_t^* = h(h^{(-1)}(x,0) + A_t + M_t, A_t) \quad \text{and} \quad \pi_t^* = h_x(h^{(-1)}(X_t^*, A_t), A_t)\sigma_t^+\lambda_t, \tag{2.32}$$

with the market input processes A_t and M_t, $t > 0$ defined as

$$A_t = \int_0^t |\lambda_s|^2 ds \quad \text{and} \quad M_t = \int_0^t \lambda_s \cdot dW_s.$$

The main contribution of this section is the explicit construction of the optimal wealth processes and trading strategies corresponding to a more general class of time-monotone forward performance processes (2.15). We establish that, in analogy to the forward performance process, X_t^* and π_t^* are also given as a compilation of market input and deterministic functions of space and time. Namely, we show that

$$X_t^* = Y_t h(h^{(-1)}(x,0) + N_t + M_t, A_t)$$

and

$$\pi_t^* = \sigma_t^+\delta_t X_t^* + \sigma_t^+(\lambda_t + \phi_t - \delta_t)Y_t h_x\left(h^{(-1)}\left(\left(\frac{X_t^*}{Y_t}\right), A_t\right), A_t\right),$$

with A_t defined as in (2.12) and the market input processes (N_t, M_t) being

$$N_t = \int_0^t (\lambda_s - \delta_s)(\lambda_s + \phi_s - \delta_s)ds \quad \text{and} \quad M_t = \int_0^t \lambda_s + \phi_s - \delta_s dW_s \tag{2.33}$$

The function $h(x,t)$ is a strictly increasing solution to the equation (2.31). The function $h^{(-1)}$ stands for the spatial inverse of h.

The above formulae shows that all quantities of interest can be fully specified as long as the parameters λ_t, δ_t and ϕ_t are specified and the functions u and h are known. In the investment model we consider, the solution h represents the optimal wealth which might take arbitrary values. As a result, a more detailed study is required depending on the range of h. Here, for the reader's convenience, we state the result with the assumption that h is of full range. Other cases as well as the relation between increasing and strictly concave solutions to (2.13) and strictly increasing solutions to (2.31) will be further investigated in Chapter 3.

To this end, we introduce the set of measures $\mathcal{B}^+(\mathbb{R})$ defined by

$$\mathcal{B}^+(\mathbb{R}) = \{\nu \in \mathcal{B}(\mathbb{R}) : \forall B \in \mathcal{B}, \nu(B) \geq 0 \text{ and } \int_{\mathbb{R}} e^{yx}\nu(dy) < \infty, x \in \mathbb{R}\}, \tag{2.34}$$

where $\mathcal{B}(\mathbb{R})$ is the set of Borel measures in $\mathbb{R}$.

Proposition 2.3.1. *i) Let $\nu \in \mathcal{B}^+(\mathbb{R})$. Then the function h defined, for $(x,t) \in [0,+\infty)$, by*

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx-\frac{1}{2}y^2t} - 1}{y}\nu(dy), \tag{2.35}$$

is a strictly increasing solution to (2.31).

ii) Assuming that h above is of full range, for each $t \geq 0$ and let $h^{(-1)} : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ be its spatial inverse. Then the function u defined for $(x,t) \in \mathbb{R} \times [0,+\infty)$ and given by

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x\left(h^{(-1)}(x,s),s\right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz,$$

is an increasing and strictly concave solution of (2.13).

We proceed with the main theorem in which we provide closed form expressions for optimal wealth processes and investment strategies. For simplicity, the following theorem is stated without specifying the range of h and u.

Theorem 2.3.1. *1) Let h be a strictly increasing solution to (2.31), for $(x,t) \in \mathbb{R} \times [0,+\infty)$, and assume that the associated measure ν satisfies*

$$\int_{\mathbb{R}} e^{yx+\frac{1}{2}y^2t}\nu(dy) < \infty, \quad \left|\int_{\mathbb{R}} \frac{e^{yx+\frac{1}{2}y^2t}}{y}\nu(dy)\right| < \infty \text{ and } \left|\int_{\mathbb{R}} \frac{\nu(dy)}{y}\right| < \infty \tag{2.36}$$

Let also N_t and M_t be as in (2.33) *and A_t be as in* (2.12) *for $t \geq 0$. Define the processes X_t^* and π_t^* by*

$$X_t^* = Y_t h(h^{(-1)}(x,0) + N_t + M_t, A_t) \tag{2.37}$$

and

$$\pi_t^* = \sigma_t^+ \delta_t X_t^* + \sigma_t^+ (\lambda_t + \phi_t - \delta_t) Y_t h_x \left(h^{(-1)} \left(\left(\frac{X_t^*}{Y_t} \right), A_t \right), A_t \right), \tag{2.38}$$

$t \geq 0$, $x \in \mathbb{R}$ with h as above and $h^{(-1)}$ standing for its spatial inverse. Then, the benchmarked portfolio $\tilde{\pi}^*$ is admissible and the portfolio π_t^* generates X_t^*, i.e.,

$$X_t^* = x + \int_0^t \sigma_s \pi_s^* \cdot (\lambda_s ds + dW_s). \tag{2.39}$$

ii) Let u be the associated with h increasing and strictly concave solution to (2.13). *Then, the process $u(X_t^*, A_t)$, $t \geq 0$, satisfies*

$$\begin{aligned} &d(u(\tilde{X}_t^*, A_t) Z_t) \\ &= [u_x(\tilde{X}_t^*, A_t) h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t)(\lambda_t + \phi_t - \delta_t) + u(\tilde{X}_t^*, A_t)\phi_t] Z_t dW_t, \end{aligned} \tag{2.40}$$

with X_t^* and π_t^* as in (2.37) *and* (2.38).

iii) Let $U_t(x)$, $t > 0$, $x \in \mathbb{R}$ be given by

$$U_t(x) = u(\frac{x}{Y_t}, A_t) Z_t. \tag{2.41}$$

Then, $U_t(x)$ is a forward performance process and the processes X_t^* and π_t^* are optimal.

Proof. i) We provide the proof under the assumption that h is of infinite range. Other semi-infinite range cases can be proved by similar arguments. For simplicity, we take $C = 0$ in (3.7) and h becomes

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2}y^2 t} - 1}{y} \nu(dy), \tag{2.42}$$

for $(x, y) \in \mathbb{R} \times [0, +\infty)$.

The benchmarked optimal wealth and investment strategy are given by

$$\tilde{X}_t^* = h(h^{(-1)}(x,0) + N_t + M_t, A_t) \tag{2.43}$$

and

$$\tilde{\pi}_t^* = \sigma_t^+ \delta_t \tilde{X}_t^* + \sigma_t^+ (\lambda_t + \phi_t - \delta_t) h_x \left(h^{(-1)} \left(\left(\tilde{X}_t^* \right), A_t \right), A_t \right), \tag{2.44}$$

To establish that $\tilde{\pi}_t^* \in \mathcal{A}$, it suffices to show that the integrability condition $\mathbb{E}_{\mathbb{P}}\left(\int_0^s |\sigma_t \tilde{\pi}_t^*|^2 dt\right) < \infty$ holds.

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(\int_0^t |\sigma_s \tilde{\pi}_s^*|^2 ds\right) &= \mathbb{E}_{\mathbb{P}}\left(\int_0^t |\delta_s \tilde{X}_s^* + (\lambda_s + \phi_s - \delta_s) h_x(h^{(-1)}((\tilde{X}_s^*, A_s), A_s)|^2 ds\right) \\
&\leq \mathbb{E}_{\mathbb{P}}\left(\int_0^t 2|\delta_s \tilde{X}_s^*|^2 + 2|(\lambda_s + \phi_s - \delta_s) h_x(h^{(-1)}((\tilde{X}_s^*, A_s), A_s)|^2 ds\right) \\
&= 2\mathbb{E}_{\mathbb{P}}\left(\int_0^t |\delta_s \tilde{X}_s^*|^2 ds\right) + 2\mathbb{E}_{\mathbb{P}}\left(\int_0^t |h_x(h^{(-1)}((\tilde{X}_s^*, A_s), A_s)|^2 dA_s\right)
\end{aligned} \tag{2.45}$$

Firstly, we show that the second part in (2.45) is finite. Using that

$$h_x(x,t) = \int_{\mathbb{R}} e^{yx - \frac{1}{2}y^2 t} \nu(dy)$$

and (2.37), we have

$$(h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t))^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{(y_1+y_2)(h^{(-1)}(x,0)+N_t+M_t) - \frac{1}{2}(y_1^2+y_2^2)A_t} \nu(dy_1)\nu(dy_2).$$

From Fubini's theorem, we duduce that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left(\int_0^t |h_x(h^{(-1)}((\tilde{X}_s^*, A_s), A_s)|^2 dA_s\right) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}\left(\int_0^t e^{(y_1+y_2)(h^{(-1)}(x,0)+N_s+M_s) - \frac{1}{2}(y_1^2+y_2^2)A_s} dA_s\right) \nu(dy_1)\nu(dy_2).
\end{aligned}$$

Using that M_s, $s > 0$ is $N(0, A_s)$ we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left(\int_0^t |h_x(h^{(-1)}((\tilde{X}_s^*, A_s), A_s)|^2 dA_s\right) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^t e^{(y_1+y_2)(h^{(-1)}(x,0)+N_s) + y_1 y_2 A_s} dA_s \nu(dy_1)\nu(dy_2) \\
&= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} e^{(y_1+y_2)(h^{(-1)}(x,0)+N_s) + y_1 y_2 A_s} \nu(dy_1)\nu(dy_2) dA_s \\
&\leq \int_0^t \left(\int_{\mathbb{R}} e^{y(h^{(-1)}(x,0)+N_s) + \frac{1}{2}y^2 A_s} \nu(dy)\right)^2 dA_s.
\end{aligned}$$

Recall that the parameters λ_s, ϕ_s, and δ_s are bounded, which implies that N_s and A_s are finite for all $s \geq 0$. Combining with the first inequality in (2.36), yields

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^t |h_x(h^{(-1)}((\tilde{X}_s^*, A_s), A_s)|^2 dA_s\right) < \infty.$$

Next, we show that the first part in (2.45) is finite. Since the parameter δ_s is bounded for $s \geq 0$, we only need to show $\mathbb{E}_{\mathbb{P}}\left(\int_0^t |\tilde{X}_s^*|^2 ds\right) < \infty$. According to (2.43)

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^t |\tilde{X}_s^*|^2 ds\right) = \mathbb{E}_{\mathbb{P}}\left(\int_0^t (h(h^{(-1)}(\tilde{X}_s^*, A_s), A_s))^2 ds\right)$$

Using (2.42), we have

$$\begin{aligned}
&(h(h^{(-1)}(\tilde{X}_t^*, A_t), A_t))^2 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{(y_1+y_2)(h^{(-1)}(x,0)+N_t+M_t)-\frac{1}{2}(y_1^2+y_2^2)A_t}+1}{y_1 y_2}\nu(dy_1)\nu(dy_2) \\
&- \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{y_1(h^{(-1)}(x,0)+N_t+M_t)-\frac{1}{2}y_1^2 A_t}+e^{y_2(h^{(-1)}(x,0)+N_t+M_t)-\frac{1}{2}y_2^2 A_t}}{y_1 y_2}\nu(dy_1)\nu(dy_2).
\end{aligned}$$

From Fubini's theorem, we deduce that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left(\int_0^t |h(h^{(-1)}((\tilde{X}_s^*, A_s), A_s)|^2 ds\right) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}\left(\int_0^t \frac{e^{(y_1+y_2)(h^{(-1)}(x,0)+N_s+M_s)-\frac{1}{2}(y_1^2+y_2^2)A_s}}{y_1 y_2} ds\right)\nu(dy_1)\nu(dy_2) \qquad (2.46) \\
&- \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}\left(\int_0^t \frac{e^{y_1(h^{(-1)}(x,0)+N_s+M_s)-\frac{1}{2}y_1^2 A_s}+e^{y_2(h^{(-1)}(x,0)+N_s+M_s)-\frac{1}{2}y_2^2 A_s}}{y_1 y_2} ds\right)\nu(dy_1)\nu(dy_2) \qquad (2.47) \\
&+ \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}\left(\int_0^t \frac{1}{y_1 y_2} ds\right)\nu(dy_1)\nu(dy_2). \qquad (2.48)
\end{aligned}$$

Next, we will show that (2.46), (2.47) and (2.48) are all finite. Since M_s, $s > 0$ is $N(0, A_s)$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}\left(\int_0^t \frac{e^{(y_1+y_2)(h^{(-1)}(x,0)+N_s+M_s)-\frac{1}{2}(y_1^2+y_2^2)A_s}}{y_1 y_2} ds\right)\nu(dy_1)\nu(dy_2) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^t \frac{e^{(y_1+y_2)(h^{(-1)}(x,0)+N_s)+y_1 y_2 A_s}}{y_1 y_2} ds\nu(dy_1)\nu(dy_2) \\
&= \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{(y_1+y_2)(h^{(-1)}(x,0)+N_s)+y_1 y_2 A_s}}{y_1 y_2}\nu(dy_1)\nu(dy_2) ds \\
&\leq \int_0^t \left(\int_{\mathbb{R}} \frac{e^{y(h^{(-1)}(x,0)+N_s)+\frac{1}{2}y^2 A_s}}{y}\nu(dy)\right)^2 ds.
\end{aligned}$$

Using that N_s and A_s are both bounded and combining with the second inequality in (2.36), we conclude that (2.46) is finite.

To show (2.47) is finite, we only need to show that

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}} \left(\int_0^t \frac{e^{y_1(h^{(-1)}(x,0)+N_s+M_s)-\frac{1}{2}y_1^2 A_s}}{y_1 y_2} ds \right) \nu(dy_1)\nu(dy_2) \right| < \infty. \tag{2.49}$$

By (2.36), we have

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}} \left(\int_0^t \frac{e^{y_1(h^{(-1)}(x,0)+N_s+M_s)-\frac{1}{2}y_1^2 A_s}}{y_1 y_2} ds \right) \nu(dy_1)\nu(dy_2) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^t \frac{e^{y_1(h^{(-1)}(x,0)+N_s)}}{y_1 y_2} ds \nu(dy_1)\nu(dy_2) \\
&= \int_{\mathbb{R}} \frac{\nu(dy_2)}{y_2} \int_0^t \int_{\mathbb{R}} \frac{e^{y_1(h^{(-1)}(x,0)+N_s)}}{y_1} \nu(dy_1) ds
\end{aligned}$$

which is finite by (2.36).

Finally, one can easily see that (2.48) is finite by the third constraint in (2.36). Hence, we conclude that the benchmarked optimal portfolio $\tilde{\pi}_t^*$ is admissible.

Applying Ito's formula on X_t^* and using (2.43), we obtain

$$\begin{aligned}
dX_t^* &= d(\tilde{X}_t^* Y_y) \\
&= Y_t h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t)(\lambda_t + \sigma_t - \delta_t)((\lambda_t - \delta_t)dt + dW_t) \\
&\quad + \tilde{X}_t^* Y_t \delta_t(\lambda_t dt + dW_t) + h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t)(\lambda_t + \phi_t - \delta_t) Y_t \delta_t dt \\
&= Y_t h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t)(\lambda_t + \phi_t - \delta_t)(\lambda_t dt + dW_t) + X_t^* \delta_t(\lambda_t dt + dW_t) \\
&= [\delta_t X_t^* + (\lambda_t + \phi_t - \delta_t) Y_t h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t)](\lambda_t dt + dW_t) \\
&= \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t).
\end{aligned}$$

ii) To show (2.40), we apply Ito's formula to $u(\tilde{X}_t^*)Z_t,\ t \geq 0$,

$$\begin{aligned}
d(u(\tilde{X}_t^*, A_t)Z_t) &= Z_t d(u(\tilde{X}_t^*, A_t)) + u(\tilde{X}_t^*, A_t) dZ_t + d\langle u(\tilde{X}^*, A), Z\rangle_t \\
&= Z_t[u_x(\tilde{X}^*, A) h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t)(\lambda_t + \phi_t - \delta_t)((\lambda_t - \delta_t)dt + dW_t) \\
&\quad + u_t(\tilde{X}_t^*, A_t) dA_t + \frac{1}{2} u_{xx}(\tilde{X}^*, A) d\langle \tilde{X}^* \rangle_t] + u(\tilde{X}_t^*, A_t) Z_t \phi_t dW_t \\
&\quad + u_x(\tilde{X}^*, A) Z_t \phi_t d\langle \tilde{X}^*, W \rangle_t \\
&= Z_t[u_x(\tilde{X}^*, A) h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t) dA_t \\
&\quad + u_x(\tilde{X}^*, A) h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t)(\lambda_t + \phi_t - \delta_t)(-\phi_t dt + dW_t) \\
&\quad + u_t(\tilde{X}_t^*, A_t) dA_t + \frac{1}{2} u_{xx}(\tilde{X}_t^*, A_t) d\langle \tilde{X} \rangle_t] \\
&\quad + u(\tilde{X}_t^*, A_t) Z_t \phi_t dW_t + u_x(\tilde{X}_t^*, A_t) Z_t \phi_t d\langle \tilde{X}^*, W \rangle_t.
\end{aligned}$$

In the second section of Chapter 3, we deduce that

$$-\frac{u_x(\tilde{X}_t^*, A_t)}{u_{xx}(\tilde{X}_t^*, A_t)} = h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t), \tag{2.50}$$

Combining (2.50) with the above results yields,

$$\begin{aligned} d(u(\tilde{X}_t^*, A_t)Z_t) = Z_t[&u_x(\tilde{X}^*, A)h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t)(\lambda_t + \phi_t - \delta_t)(-\phi_t dt + dW_t) \\ &- \frac{u_x(\tilde{X}_t^*, A_t)^2}{u_{xx}(\tilde{X}_t^*, A_t)}dA_t + u_t(\tilde{X}_t^*, A_t)dA_t + \frac{1}{2}\frac{u_x(\tilde{X}_t^*, A_t)^2}{u_{xx}(\tilde{X}_t^*, A_t)}dA_t \\ &+ u(\tilde{X}_t^*, A_t)Z_t\phi_t dW_t + u_x(\tilde{X}_t^*, A_t)Z_t\phi_t d\langle \tilde{X}^*, W\rangle_t. \end{aligned}$$

Using (2.13), the middle terms cancel with each other and we obtain

$$\begin{aligned} d(u(\tilde{X}_t^*, A_t)Z_t) &= Z_t[u_x(\tilde{X}^*, A)h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t)(\lambda_t + \phi_t - \delta_t)(-\phi_t dt + dW_t) \\ &\quad + u(\tilde{X}_t^*, A_t)Z_t\phi_t dW_t \\ &\quad + u_x(\tilde{X}_t^*, A_t)Z_t\phi_t h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t)(\lambda_t + \phi_t - \delta_t)dt] \\ &= [u(\tilde{X}_t^*, A_t)\phi_t + u_x(\tilde{X}^*, A)h_x(h^{(-1)}(\tilde{X}_t^*, A_t), A_t)(\lambda_t + \phi_t - \delta_t)]Z_t dW_t, \end{aligned}$$

which coincides with (2.40).

iii) We need to establish that $U_t(x)$ satisfies all conditions in Definition 2.2.1. The facts that $u(\frac{x}{Y_t}, A_t)Z_t$ is $\mathcal{F}_t$-adapted and that the mapping $x \to u(\frac{x}{Y_t}, A_t)Z_t$ is increasing and strictly concave follow trivially from the properties of u and the processes (Y_t, Z_t, A_t). Next, we will show that the integrability condition $\mathbb{E}_{\mathbb{P}}\left(U_t(X_t^\pi)\right)^+ < \infty$ also holds.

First, we observe that the strict concavity of u together with (2.13) yields $u_t < 0$ and therefore $u(x,t) < u(x,0) < ax^+ + b$ for some positive constant a and b. Moreover, applying Ito's formula to $\tilde{X}_T$ we obtain that

$$d(\tilde{X}_t) = (\sigma_t\tilde{\pi}_t - \delta_t\tilde{X}_t) \cdot ((\lambda_t - \delta_t)dt + dW_t). \tag{2.51}$$

which implies that

$$\begin{aligned} (\tilde{X}_t^\pi)^+ &\leq x^+ + \int_0^t |\sigma_s\tilde{\pi}_s - \delta_s\tilde{X}_s||\lambda_s - \delta_s|ds \\ &\quad + \left|\int_0^t \delta_s\tilde{X}_s dW_s\right| + \left|\int_0^t \sigma_s\tilde{\pi}_s dW_s\right| \\ &\leq x^+ + \frac{1}{2}\int_0^t |\sigma_s\tilde{\pi}_s - \delta_s\tilde{X}_s|^2 ds + \frac{1}{2}\int_0^t |\lambda_s - \delta_s|^2 ds \\ &\quad + \left|\int_0^t \delta_s\tilde{X}_s dW_s\right| + \left|\int_0^t \sigma_s\tilde{\pi}_s dW_s\right| \end{aligned}$$

$$\leq x^+ + \int_0^t |\sigma_s\tilde{\pi}_s|^2 ds + \int_0^t |\delta_s\tilde{X}_s|^2 ds + \frac{1}{2}\int_0^t |\lambda_s - \delta_s|^2 ds$$
$$+ \left|\int_0^t \delta_s\tilde{X}_s dW_s\right| + \left|\int_0^t \sigma_s\tilde{\pi}_s dW_s\right|.$$

Notice that the processes λ_s and δ_s are taken to be bounded for all $s \geq 0$ and using the results we proved in i) we conclude that the integrability condition is satisfied.

To show (2.7), we apply Ito's formula to $U_t(x)$,

$$dU_t(X_t) = d(u(\tilde{X}_t, A_t)Z_t)$$
$$= (du(\tilde{X}_t, A_t))Z_t + u(\tilde{X}_t, A_t)dZ_t + d\langle u(\tilde{X}, A), Z\rangle_t. \tag{2.52}$$

Applying Ito's formula to $u(\tilde{X}_t, A_t)$, we get

$$\begin{aligned} d(u(\tilde{X}_t, A_t)) &= u_x(\tilde{X}_t, A_t)d\tilde{X}_t + u_t(\tilde{X}_t, A_t)dA_t + \frac{1}{2}u_{xx}(\tilde{X}_t, A_t)d\langle\tilde{X}\rangle_t \\ &= u_x(\tilde{X}_t, A_t)((\sigma_t\tilde{\pi}_t - \delta_t\tilde{X}_t)\cdot((\lambda_t - \delta_t)dt + dW_t)) \\ &\quad + u_t(\tilde{X}_t, A_t)dA_t + \frac{1}{2}u_{xx}(\tilde{X}_t, A_t)(\sigma_t\tilde{\pi}_t - \delta_t\tilde{X}_t)^2 dt. \end{aligned}$$

Therefore,

$$\begin{aligned} d(u(\tilde{X}_t, A_t))Z_t &= Z_t u_x(\tilde{X}_t, A_t)(\sigma_t\tilde{\pi}_t - \delta_t\tilde{X}_t)\cdot((\lambda_t - \delta_t)dt + dW_t)) \\ &\quad + Z_t u_t(\tilde{X}_t, A_t)dA_t + \frac{1}{2}Z_t u_{xx}(\tilde{X}_t, A_t)(\sigma_t\tilde{\pi}_t - \delta_t\tilde{X}_t)^2 dt. \end{aligned} \tag{2.53}$$

Moreover,

$$d\langle u(\tilde{X}, A), Z\rangle_t = u_x(\tilde{X}_t, A_t)d\langle\tilde{X}, Z\rangle_t = u_x(\tilde{X}_t, A_t)(\sigma_t\tilde{\pi}_t - \delta_t\tilde{X}_t)Z_t\phi_t dt \tag{2.54}$$

Substituting (2.54) and (2.53) into (2.52), yields

$$\begin{aligned} dU_t(x) &= (u_x(\tilde{X}_t, A_t)(Z_t\sigma_t\tilde{\pi}_t - Z_t\delta_t\tilde{X}_t) + \phi_t U_t(x))\cdot dW_t \\ &\quad + u_x(\tilde{X}_t, A_t)(Z_t\sigma_t\tilde{\pi}_t - Z_t\delta_t\tilde{X}_t)(\lambda_t + \phi_t - \delta_t)dt \\ &\quad + u_t(\tilde{X}_t, A_t)Z_t(\lambda_t + \phi_t - \delta_t)^2 dt + \frac{1}{2}u_{xx}(\tilde{X}_t, A_t)Z_t(\sigma_t\tilde{\pi}_t - \delta_t\tilde{X}_t)^2 dt \\ &= (u_x(\tilde{X}_t, A_t)(Z_t\sigma_t\tilde{\pi}_t - Z_t\delta_t\tilde{X}_t) + \phi_t U_t(x))\cdot dW_t \\ &\quad + \frac{1}{2}u_{xx}(\tilde{X}_t, A_t)Z_t\left(\sigma_t\tilde{\pi}_t - \left(\delta_t\tilde{X}_t - (\lambda_t + \phi_t - \delta_t)\frac{u_x(\tilde{X}_t, A_t)}{u_{xx}(\tilde{X}_t, A_t)}\right)\right)^2 \\ &\quad + \left(u_t(\tilde{X}_t, A_t) - \frac{1}{2}u_{xx}(\tilde{X}_t, A_t)\left(\frac{u_x(\tilde{X}_t, A_t)}{u_{xx}(\tilde{X}_t, A_t)}\right)^2\right)(\lambda_t + \phi_t - \delta_t)^2 dt \end{aligned}$$

$$
\begin{aligned}
&= (u_x(\tilde{X}_t, A_t)(Z_t\sigma_t\tilde{\pi}_t - Z_t\delta_t\tilde{X}_t) + \phi_t U_t(x)) \cdot dW_t \\
&\quad + \frac{1}{2}u_{xx}(\tilde{X}_t, A_t)Z_t \left(\sigma_t\tilde{\pi}_t - \left(\delta_t\tilde{X}_t - (\lambda_t + \phi_t - \delta_t)\frac{u_x(\tilde{X}_t, A_t)}{u_{xx}(\tilde{X}_t, A_t)}\right)\right)^2 \\
&\quad + \left(u_t(\tilde{X}_t, A_t) - \frac{1}{2}\frac{u_x(\tilde{X}_t, A_t)^2}{u_{xx}(\tilde{X}_t, A_t)}\right)(\lambda_t + \phi_t - \delta_t)^2 dt.
\end{aligned}
$$

According to (2.13), we have

$$
\begin{aligned}
dU_t(x) &= (u_x(\tilde{X}_t, A_t)Z_t(\sigma_t\tilde{\pi}_t - \delta_t\tilde{X}_t) + \phi_t U_t(x)) \cdot dW_t \\
&\quad + \frac{1}{2}u_{xx}(\tilde{X}_t, A_t)Z_t \left(\sigma_t\tilde{\pi}_t - \left(\delta_t\tilde{X}_t - (\lambda_t + \phi_t - \delta_t)\frac{u_x(\tilde{X}_t, A_t)}{u_{xx}(\tilde{X}_t, A_t)}\right)\right)^2 dt. \qquad (2.55)
\end{aligned}
$$

Since u is concave and the process $Z_t > 0$ for all $t \geq 0$, the drift part is non-positive. Therefore, we conclude that $U_t(X_t^\pi)$ is a supermartingale. Moreover, we observe $U_t(X_t^*)$ is a martingale by substituting (2.38) into (2.55).

□

Chapter 3

Representation of the Differential Input and the Local Risk Tolerance Function

From the results in Chapter 2, especially Theorem 2.2.4, we can see that it is the risk tolerance process r, not the differential input u, that emerges as the key quantity in constructing the optimal allocations. This observation motivates us to question whether one should start with the differential input and then define the risk tolerance function r or model r directly. In other words, given the differential input u, one could specify r by directly applying its definition. On the other hand, given a risk tolerance function r, are we always able to construct the corresponding differential input u? Moreover, as it is proved in Theorem 2.3.1, the functions h plays an important role in the construction of optimal wealth and portfolio processes. What is the relationship among the functions h, u and r?

We try to answer the above questions in the first three sections of this chapter. In the last section, we specify the function h and derive the corresponding r and u. Moreover, we show that the limiting risk tolerance and differential input resemble the classical exponential utility case.

3.1 The Relationship Between Functions h and u

Recall the definition of differential input is given by

Definition 3.1.1. *The function $u \in \mathcal{C}^{3,1}(\mathbb{R} \times (0, \infty))$ is a differential input if it is concave and increasing in the spatial argument and satisfies the differential constraint*

$$u_t u_{xx} = \frac{1}{2} u_x^2. \tag{3.1}$$

Let the function $h : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ be strictly increasing in x and solves the backward heat equation

$$h_t + \frac{1}{2} h_{xx} = 0. \tag{3.2}$$

In Musiela and Zariphopoulou [13], it is proved that there is a one-to-one correspondence (modulo normalization constants) between u and h. In this section, we provide an overview of the results proved therein. The key idea is to represent h in terms of the bilateral Laplace transform of the positive finite Borel measure, denoted by ν, where $\nu \in \mathcal{B}^+(\mathbb{R})$ for $\mathcal{B}^+(\mathbb{R})$ defined in (2.34)[1]. The above representation originates from the classical result of Widder (see, Widder [16]).

Theorem 3.1.1. *(Widder). Let $g(x,t), (x,t) \in \mathbb{R} \times [0, +\infty)$, be a positive solution of (3.2). Then, there exists $\mu \in \mathcal{B}^+(\mathbb{R})$ such that g is represented as*

$$g(x,t) = \int_{\mathbb{R}} e^{yx - \frac{1}{2}y^2 t} \mu(dy). \tag{3.3}$$

Widder's theorem can't be applied therein since the solutions h we consider herein may not be positive. Indeed, in (2.37), the function h represents the optimal wealth which is not always positive or above a given threshold. However, due to the assumed monotonic property of h, Widder's theorem can be applied to its spatial derivative h_x which is always positive and solves (3.2).

To this end, we introduce the following subsets of $\mathcal{B}^+(\mathbb{R})$.

$$\mathcal{B}_0^+(\mathbb{R}) = \{\nu \in \mathcal{B}^+(\mathbb{R}) \text{ and } \nu(\{0\}) = 0\}, \tag{3.4}$$

$$\mathcal{B}_+^+(\mathbb{R}) = \{\nu \in \mathcal{B}_0^+(\mathbb{R}) : \nu((-\infty, 0)) = 0\}, \tag{3.5}$$

$$\mathcal{B}_-^+(\mathbb{R}) = \{\nu \in \mathcal{B}_0^+(\mathbb{R}) : \nu((0, +\infty)) = 0\}. \tag{3.6}$$

It is assumed that the case $\nu(\mathbb{R}) = 0$ is excluded.

The following proposition proved in Musiela and Zariphopoulou [13] shows that the range of h depends on the support and integrability properties of the positive finite Borel measure ν.

Proposition 3.1.1. *i) Let $\nu \in \mathcal{B}^+(\mathbb{R})$. Then, the function h defined for $(x,t) \in \mathbb{R} \times [0, +\infty)$, by*

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2}y^2 t} - 1}{y} \nu(dy) + C, \tag{3.7}$$

is a strictly increasing solution to (3.2).

[1]Notice that this condition is sufficient to ensure u and h to be finite. However, for forward performance processes of the form (2.15), we need stronger conditions, namely (2.36), to guarantee the admissibility of the optimal investment strategies.

Moreover, if $\nu(\{0\}) > 0$, *or* $\nu \in \mathcal{B}_0^+(\mathbb{R})$, *or* $\nu \in \mathcal{B}_+^+(\mathbb{R})$ *and* $\int_{0^+}^{+\infty} \frac{\nu(dy)}{y} = +\infty$, *or* $\nu \in \mathcal{B}_-^+(\mathbb{R})$ *and* $\int_{-\infty}^{0^-} \frac{\nu(dy)}{y} = -\infty$, *then* $Range(h) = (-\infty, +\infty)$, *for* $t \geq 0$.

On the other hand, if $\nu \in \mathcal{B}_+^+(\mathbb{R})$ *with* $\int_{0^+}^{+\infty} \frac{\nu(dy)}{y} < +\infty$ *(resp.* $\nu \in \mathcal{B}_-^+(\mathbb{R})$ *with* $\int_{-\infty}^{0^-} \frac{\nu(dy)}{y} > -\infty$*), then* $Range(h) = \left(C - \int_{0^+}^{+\infty} \frac{\nu(dy)}{y}, +\infty\right)$ *(resp.* $Range(h) = \left(-\infty, C - \int_{-\infty}^{0^-} \frac{\nu(dy)}{y}\right)$*), for* $t \geq 0$.

ii) Conversely, let $h : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ *be a strictly increasing solution to* (3.2). *Then there exists* $\nu \in \mathcal{B}^+(\mathbb{R})$ *such that* h *is given by* (3.7).

Moreover, if $Range(h) = (-\infty, +\infty), t \geq 0$, *then it must be either that* $\nu(\{0\}) > 0$, *or* $\nu \in \mathcal{B}_0^+(\mathbb{R})$, *or* $\nu \in \mathcal{B}_+^+(\mathbb{R})$ *and* $\int_{0^+}^{+\infty} \frac{\nu(dy)}{y} = +\infty$, *or* $\nu \in \mathcal{B}_-^+(\mathbb{R})$ *and* $\int_{-\infty}^{0^-} \frac{\nu(dy)}{y} = -\infty$.

On the other hand, if $Range(h) = (x_0, +\infty)$ *(resp.* $Range(h) = (-\infty, x_0)$*),* $t \geq 0$ *and* $x_0 \in \mathbb{R}$, *then it must be that* $\nu \in \mathcal{B}_+^+(\mathbb{R})$ *with* $\int_{0^+}^{+\infty} \frac{\nu(dy)}{y} < +\infty$ *(resp.* $\nu \in \mathcal{B}_-^+(\mathbb{R})$ *with* $\int_{-\infty}^{0^-} \frac{\nu(dy)}{y} > -\infty$*).*

Moreover, the following result of Widder (see Widder [16]) shows that if the range is finite, then h is a constant. Therefore, it violates our monotonicity assumption.

Proposition 3.1.2. *Let* h *be a solution for* (3.2) *such that for* $(x, t) \in \mathbb{R} \times [0, +\infty)$, $-M \leq h(x, t) \leq M$, *for some constant* M. *Then,* $h(x, t)$ *is constant.*

In Musiela and Zariphopoulou [13], depending on the range of h, the one-to-one correspondence (modulo normalization constant) between u and h is proved in three cases, namely when $Range(h) = (-\infty, +\infty), (x_0, +\infty)$ and $(-\infty, x_0)$.

Case I: $Range(h) = (-\infty, +\infty)$

Here the constant C is taken to be 0 and thus $h(0, 0) = 0$. Without loss of generality, assume

$$u(0, 0) = 0 \quad \text{and} \quad u_x(0, 0) = 1. \tag{3.8}$$

Proposition 3.1.3. *i) Let* $\nu \in \mathcal{B}^+(\mathbb{R})$ *andh* $: \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ *be as in* (3.7) *with the measure* ν *being used. Assume that* h *is of full range for each* $t \geq 0$, *and let* $h^{(-1)} : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ *be its spatial inverse. The the function* u *defined for* $(x, t) \in [0, +\infty)$ *and given by*

$$u(x, t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x,s) + \frac{s}{2}} h_x(h^{(-1)}(x, s), s) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz, \tag{3.9}$$

is an increasing and strictly concave solution of (3.1) *satisfying* (3.8).

Moreover, for $t \geq 0$, the Inada conditions,

$$\lim_{x\to-\infty} u_x(x,t) = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} u_x(x,t) = 0, \tag{3.10}$$

are satisfied.

ii) Conversely, let u be an increasing and strictly concave function satisfying, for $(x,t) \in \mathbb{R}\times[0,+\infty)$, (3.1) and (3.8), and the Inada conditions (3.10), for $t \geq 0$. Then there exists $\nu \in \mathcal{B}^+(\mathbb{R})$, such that u admits representation (3.9) with h given by (3.7), for $(x,t) \in \mathbb{R}\times[0,+\infty)$. Moreover, h is of full range, for each $t \geq 0$, and $h(0,0) = 0$.

Case II: *Range*$(h) = (x_0, +\infty),\ x_0 \in \mathbb{R}$

In consequence of Proposition 3.1.1, h is the same as in (3.7) where ν satisfies

$$\nu \in \mathcal{B}^+_+(\mathbb{R}) \quad \text{and} \quad \int_{0^+}^{+\infty} \frac{\nu(dy)}{y} < +\infty, \tag{3.11}$$

with $\mathcal{B}^+_+(\mathbb{R})$ given in (3.5), and *Range*$(h) = (C - \int_{0^+}^{+\infty} \frac{\nu(dy)}{y}, +\infty)$. By setting $C = \int_{0^+}^{+\infty} \frac{\nu(dy)}{y}$, we obtain that

$$h(x,t) = \int_{0^+}^{+\infty} \frac{e^{yx-\frac{1}{2}y^2t}}{y}\nu(dy). \tag{3.12}$$

and *Range*$(h) = (0, +\infty)$. In consequence, the function is defined on the positive half line and considerations need to be given to $\lim_{x\to0} u(x,t), t \geq 0$. The result is proved in two cases, depending on the limit being finite or infinite.

Proposition 3.1.4. *i) Let ν satisfies (3.11), $\nu((0,1]) = 0$ and $\int_{1^+}^{+\infty} \frac{\nu(dy)}{y-1} < +\infty$. Let $h : \mathbb{R}\times[0,+\infty) \to (0,+\infty)$ be as in (3.12) and $h^{(-1)} : (0,+\infty)\times[0,+\infty) \to \mathbb{R}$ be its spatial inverse. Then the function u defined for $(x,t) \in (0,+\infty)\times[0,+\infty)$, by*

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x(h^{(-1)}(x,s),s)ds + \int_0^x e^{-h^{(-1)}(z,0)}dz, \tag{3.13}$$

is an increasing and strictly concave solution of (3.1) with

$$\lim_{x\to0} u_x(x,t) = 0, \quad \textit{for} \quad t \geq 0. \tag{3.14}$$

Moreover, for $t \geq 0$, the Inada conditions,

$$\lim_{x\to0} u_x(x,t) = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} u_x(x,t) = 0, \tag{3.15}$$

are satisfied.

ii) Conversely, let u, defined for $(x,t) \in (0,+\infty) \times [0,+\infty)$, be an increasing and strictly concave function satisfying (3.1), (3.14) *and the Inada conditions* (3.15). *Then there exists $\nu \in \mathcal{B}^+(\mathbb{R})$ satisfying* (3.11), *$\nu((0,1]) = 0$ and $\int_{1^+}^{+\infty} \frac{\nu(dy)}{y-1} < +\infty$, such that u admits representation* (3.13) *with h given by* (3.12) *for $(x,t) \in \mathbb{R} \times [0,+\infty)$.*

Proposition 3.1.5. *i) Let ν satisfies* (3.11) *and either $\nu((0,1]) > 0$ or $\nu((0,1]) = 0$ and $\int_{1^+}^{+\infty} \frac{\nu(dy)}{y-1} < +\infty$. Let $h : \mathbb{R} \times [0,+\infty) \to (0,+\infty)$ be as in* (3.12) *and $h^{(-1)} : (0,+\infty) \times [0,+\infty) \to \mathbb{R}$ be its spatial inverse. Then the function u defined for $(x,t) \in (0,+\infty) \times [0,+\infty)$, by*

$$u(x,t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x(h^{(-1)}(x,s),s)ds + \int_0^x e^{-h^{(-1)}(z,0)}dz, \tag{3.16}$$

is an increasing and strictly concave solution of (3.1) *with*

$$\lim_{x \to 0} u_x(x,t) = -\infty, \quad \textit{for} \quad t \geq 0. \tag{3.17}$$

Moreover, for $t \geq 0$, the Inada conditions,

$$\lim_{x \to 0} u_x(x,t) = +\infty \quad \textit{and} \quad \lim_{x \to +\infty} u_x(x,t) = 0, \tag{3.18}$$

are satisfied.

ii) Conversely, let u, defined for $(x,t) \in (0,+\infty) \times [0,+\infty)$, be an increasing and strictly concave function satisfying (3.1), (3.17) *and the Inada conditions* (3.18). *Then there exists $\nu \in \mathcal{B}^+(\mathbb{R})$ satisfying* (3.11) *and $\nu((0,1]) > 0$ or $\nu((0,1]) = 0$ and $\int_{1^+}^{+\infty} \frac{\nu(dy)}{y-1} < +\infty$, such that u admits representation* (3.13) *with h given by* (3.12) *for $(x,t) \in \mathbb{R} \times [0,+\infty)$.*

Case III: *$Range(h) = (-\infty, x_0)$, $x_0 \in \mathbb{R}$*

In consequence of Proposition 3.1.1, h is the same as in (3.7) where ν satisfies

$$\nu \in \mathcal{B}_-^+(\mathbb{R}) \quad \text{and} \quad \int_{-\infty}^{0^-} \frac{\nu(dy)}{y} > -\infty, \tag{3.19}$$

with $\mathcal{B}_-^+(\mathbb{R})$ given in (3.6), and $Range(h) = (-\infty, C - \int_{-\infty}^{0^-} \frac{\nu(dy)}{y})$. By setting $C = \int_{-\infty}^{0^-} \frac{\nu(dy)}{y}$, we obtain that

$$h(x,t) = \int_{-\infty}^{0^-} \frac{e^{yx-\frac{1}{2}y^2t}}{y} \nu(dy). \tag{3.20}$$

and $Range(h) = (-\infty, 0)$.

Proposition 3.1.6. *i) Let ν satisfies* (3.19)*. Let, also, $h : \mathbb{R} \times [0, +\infty) \to (-\infty, 0)$ be as in* (3.20) *and $h^{(-1)} : (-\infty, 0) \times [0, +\infty) \to \mathbb{R}$ be its spatial inverse. Then the function u defined for $(x, t) \in (-\infty, 0) \times [0, +\infty)$, by*

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x(h^{(-1)}(x,s),s)ds - \int_x^0 e^{-h^{(-1)}(z,0)}dz, \tag{3.21}$$

is an increasing and strictly concave solution of (3.1) *with*

$$\lim_{x\to 0} u_x(x,t) = 0, \quad \textit{for} \quad t \geq 0. \tag{3.22}$$

Moreover, for $t \geq 0$, the Inada conditions,

$$\lim_{x\to-\infty} u_x(x,t) = +\infty \quad \textit{and} \quad \lim_{x\to 0} u_x(x,t) = 0, \tag{3.23}$$

are satisfied.

ii) Conversely, let u, defined for $(x, t) \in (-\infty, 0) \times [0, +\infty)$, be an increasing and strictly concave function satisfying (3.1)*,* (3.22) *and the Inada conditions* (3.23)*. Then there exists ν as in* (3.19) *such that u admits representation* (3.21) *with h given by* (3.20)*.*

3.2 The Relationship Between Functions u and r

In this section, we will investigate the relationship between the differential input u and local risk tolerance function r. Specifically, given u satisfying (3.1) with a concave and increasing initial datum, could one construct the corresponding r from it or vice versa?

The answer to the first question is affirmative and the proof is trivial. Given u, by definition, $r : \mathbb{R} \times [0, +\infty) \to \mathbb{R}^+$ is given by

$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}. \tag{3.24}$$

In this section, we focus on constructing u from r, which is more complicated. To be specific, in light of Proposition 2.2.2, the problem we are considering herein is constructing the function u satisfying (3.1) and (3.24), where r solves

$$r_t + \frac{1}{2}r^2 r_{xx} = 0, \tag{3.25}$$

with initial datum $r(x, 0) = r_0(x)$.

One way to solve this problem is by successive integration, provided certain time dependent quantities are correctly specified (see Zariphopoulou and Zhou [18] for a

two parameter family example). Here, we provide an alternative way of constructing u from r.

At $t = 0$, by (3.24), we have

$$-\frac{u_0'(x)}{u_0''(x)} = r_0(x). \tag{3.26}$$

and therefore, the initial condition $u_0(x)$ can be obtained by integrating (3.26).

$$\begin{aligned}
(\log u_0'(x))_x &= -\frac{1}{r_0(x)} \\
\log u_0'(x) &= \int_{-\infty}^{x} -\frac{1}{r_0(y)} dy \\
u_0(x)' &= \exp\left(\int_{-\infty}^{x} -\frac{1}{r_0(y)} dy\right) \\
u_0(x) &= \int_{-\infty}^{x} \exp\left(\int_{-\infty}^{y} -\frac{1}{r_0(z)} dz\right) dy
\end{aligned}$$

Next, using the definition of r, we can rewrite (3.1) as a first order PDE, i.e.

$$\begin{aligned}
u_t - \frac{1}{2}\left(\frac{u_x}{u_{xx}}\right) u_x &= 0, \\
u_t + \frac{1}{2} r(x,t) u_x &= 0,
\end{aligned} \tag{3.27}$$

which is a transport equation. Next, we solve the equation (3.27) using method of characteristics:

If we assume that $u = u(x(t), t)$, then differentiating u yields

$$\frac{d}{dt} u(x(t), t) = \frac{\partial u}{\partial t}(x(t), t) + \frac{\partial u}{\partial x}(x(t), t) \frac{dx(t)}{dt}. \tag{3.28}$$

Let compare (3.27) and (3.28). If we set

$$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t) \text{ with } x(0) = x_0,$$

then the rate of change of u w.r.t t is zero, i.e.,

$$\frac{du}{dt} = 0.$$

This implies

$$u(x(t), t) = C = u_0(x_0).$$

Therefore, the problem becomes

$$u(x(t), t) = u_0(x_0), \tag{3.29}$$

where $x(t)$ solves the IVP

$$\frac{dx(t)}{dt} = \frac{1}{2}r(x(t), t) \quad \text{with} \quad x(0) = x_0. \tag{3.30}$$

Hence, if we can find a solution to (3.30) then the differential input can be obtained by writing x_0 as a function of (x, t) and substituting it into the RHS of (3.29).

The existence of local solutions of the IVP (3.30) is an immediate consequence of *Peano existence theorem* as well as *Carathéodory's existence theorem*. In addition, the uniqueness of local solutions of the IVP (3.30) can also be established by applying *Picard-Lindelöf theorem*, provided $r(x, t)$ is Lipschitz continuous[2].

Global solutions to the IVP (3.30) exists provided the above ordinary differential equation is of separable, linear or homogeneous type. However, in general, characterizing the existence of global solutions of the IVP (3.30) is a challenging question and is left for further studies.

3.3 The Relationship Between Functions r and h

In this section, we investigate the relationship between local risk tolerance functions r and the bilateral Laplace transform of positive finite Borel measures. Specifically, given function h as in (3.7), could one construct the corresponding local risk tolerance function r satisfying (3.25) or vice versa?

The answer is affirmative. The idea is simple. Recall that in Section 1, it is stated that there is a one-to-one correspondence between u and h. Given h, one could construct the differential input u and, thus, obtain r using the definition and vice versa.

Proposition 3.3.1. *i) Given a positive finite Borel measure $\nu \in \mathcal{B}^+(\mathbb{R})$ and the corresponding bilateral Laplace transform h as defined in* (3.7) *with Range$(h) = \mathbb{D}$, then there exists a function $r : \mathbb{D} \times [0, +\infty) \to (0, +\infty)$ satisfying*

$$r(x, t) = -\frac{u_x(x, t)}{u_{xx}(x, t)}, \tag{3.31}$$

where u is given by

$$u(x, t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x(h^{(-1)}(x, s), s)ds + \int_0^x e^{-h^{(-1)}(z,0)}dz, \tag{3.32}$$

[2]Given two metric spaces (X, d_X) and (Y, d_Y), where d_X denotes the metric on the set X and d_Y is the metric on set Y, a function $f : X \to Y$ is called Lipschitz continuous if there exists a real constant $K \geq 0$ such that, for all $x1$ and $x2$ in X, $d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$.

Moreover, $r(x,t)$ is given by

$$r(x,t) = h_x(h^{(-1)}(x,t),t). \tag{3.33}$$

ii) Conversely, given function $r : \mathbb{D} \times [0,+\infty) \to (0,+\infty)$ satisfying

$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}, \tag{3.34}$$

then there exists a corresponding bilateral Laplace transform h as defined in (3.7) *with* $Range(h) = \mathbb{D}$ *and satisfies* (3.32) *and* (3.33)*.*

Here, we will give the proof of part i), and the proof of part ii) follows along similar arguments.

Proof of part i). In Musiela and Zariphopoulou [13], it is proved that

$$h(x,t) = (u_x)^{(-1)}\left(e^{-x+\frac{t}{2}},t\right), \tag{3.35}$$

with u as in (3.32).

Notice that (3.35) yields

$$u_x(h(x,t),t) = e^{-x+\frac{t}{2}}. \tag{3.36}$$

Differentiating (3.35), we obtain

$$h_x(x,t) = -\frac{e^{-x+\frac{t}{2}}}{u_{xx}(h(x,t),t)}. \tag{3.37}$$

Substituting (3.36) into (3.37), yields

$$h_x(x,t) = -\frac{u_x(h(x,t),t)}{u_{xx}(h(x,t),t)}. \tag{3.38}$$

Hence, by (3.33)

$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}. \tag{3.39}$$

□

3.4 Forward Utilities via Dirac Measures

In this section, we solve a concrete (and very involved) example to demonstrate the general results obtained in previous chapters. We start from a Borel measure ν such that

$$\nu(dy) = \frac{b}{2}(\delta_a + \delta_{-a}),$$

where $\delta_{\pm a}$ are Dirac measures at $\pm a$ with $a, b > 0$. Its bilateral Laplace transform h, which is defined as

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx-\frac{1}{2}y^2t}-1}{y}\nu(dy),$$

is given by

$$h(x,t;a,b) = \frac{b}{a}e^{-\frac{1}{2}a^2t}\sinh(ax). \tag{3.40}$$

Moreover, its spatial inverse $h^{(-1)}(x,t)$ is

$$h^{(-1)}(x,t;a,b) = \frac{1}{a}\ln\left(\frac{a}{b}xe^{\frac{1}{2}a^2t} + \sqrt{\frac{a^2}{b^2}x^2e^{-\frac{1}{2}a^2t}+1}\right). \tag{3.41}$$

Using Proposition 3.3.1 we obtain the corresponding r

$$r(x,t;a,b) = \sqrt{a^2x^2 + b^2e^{-a^2t}}. \tag{3.42}$$

In the rest of this section, we construct the corresponding optimal wealth process and portfolio strategy using results derived in Chapter 2. In addition, the corresponding differential input is also constructed by applying results in Chapter 3. Finally, by letting the parameter a go to zero, we show that the limiting risk tolerance and differential input assemble the classical exponential case.

Remark 3.4.1. *The same class of local risk tolerance functions were studied in Zariphopoulou and Zhou [18] as well. However, the approaches are different. For instance, the optimal process therein is constructed by solving a system of stochastic differential equations, whereas here we apply Theorem 2.3.1 directly. Moreover, the differential input therein is constructed by successive integration, whereas, here we use propositions in Chapter 3. Nevertheless, our results coincide.*

3.4.1 Construction of Optimal Processes

Firstly, we compute the benchmarked optimal process which are defined as in (2.17). According to Theorem 2.3.1,

$$\tilde{X}_t^* = h(h^{(-1)}(x,0) + N_t + M_t, A_t), \tag{3.43}$$

for A_t, N_t and M_t defined as in (2.12) and (2.33). Substituting (3.40) and (3.41) into (3.43), yields

$$\begin{aligned}
\tilde{X}_t^* &= \frac{b}{a} e^{-\frac{1}{2}a^2 A_t} \sinh\left(\ln\left(\frac{ax}{b} + \sqrt{\frac{a^2x^2}{b^2}+1}\right) + aM_t + aN_t\right) \\
&= \frac{1}{2}\frac{b}{a} e^{-\frac{1}{2}a^2 A_t}\left[\left(\frac{ax}{b} + \sqrt{\frac{a^2x^2}{b^2}+1}\right) e^{aN_t+aM_t} - \left(-\frac{ax}{b} + \sqrt{\frac{a^2x^2}{b^2}+1}\right) e^{-aN_t-aM_t}\right] \\
&= \frac{1}{2}\frac{b}{a} e^{-\frac{1}{2}a^2 A_t}\left[\frac{ax}{b}\left(e^{aN_t+aM_t} + e^{-aN_t-aM_t}\right) + \sqrt{\frac{a^2x^2}{b^2}+1}\left(e^{aN_t+aM_t} - e^{-aN_t-aM_t}\right)\right] \\
&= e^{-\frac{1}{2}a^2 A_t}\left[x\cosh(aN_t+aM_t)) + \sqrt{x^2+\frac{b^2}{a^2}}\sinh(aN_t+aM_t)\right]
\end{aligned}$$

Substituting into to (3.42) and replacing t with A_t, yields

$$\begin{aligned}
\tilde{R}_t^* &= \sqrt{a^2(\tilde{X}_t^*)^2 + b^2 e^{-a^2 A_t}} \\
&= \sqrt{b^2 e^{-a^2 A_t}\sinh\left(\ln\left(\frac{ax}{b} + \sqrt{\frac{a^2x^2}{b^2}+1}\right) + aM_t + aN_t\right)^2 + b^2 e^{-a^2 A_t}} \\
&= \sqrt{b^2 e^{-a^2 A_t}\cosh\left(\ln\left(\frac{ax}{b} + \sqrt{\frac{a^2x^2}{b^2}+1}\right) + aM_t + aN_t\right)^2} \\
&= be^{-\frac{1}{2}a^2 A_t}\cosh\left(\ln\left(\frac{ax}{b} + \sqrt{\frac{a^2x^2}{b^2}+1}\right) + aM_t + aN_t\right)
\end{aligned}$$

Following similar arguments as in the derivation of $\tilde{X}_t^*$, we obtain

$$\begin{aligned}
\tilde{R}_t^* &= be^{-\frac{1}{2}a^2 A_t}\cosh\left(\ln\left(\frac{ax}{b} + \sqrt{\frac{a^2x^2}{b^2}+1}\right) + aM_t + aN_t\right) \\
&= \frac{b}{2} e^{-\frac{1}{2}a^2 A_t}\left[\left(\frac{ax}{b} + \sqrt{\frac{a^2x^2}{b^2}+1}\right) e^{aN_t+aM_t} + \left(-\frac{ax}{b} + \sqrt{\frac{a^2x^2}{b^2}+1}\right) e^{-aN_t-aM_t}\right] \\
&= \frac{b}{2} e^{-\frac{1}{2}a^2 A_t}\left[\frac{ax}{b}\left(e^{aN_t+aM_t} - e^{-aN_t-aM_t}\right) + \sqrt{\frac{a^2x^2}{b^2}+1}\left(e^{aN_t+aM_t} + e^{-aN_t-aM_t}\right)\right] \\
&= e^{-\frac{1}{2}a^2 A_t}\left[ax\sinh(aN_t+aM_t)) + \sqrt{a^2x^2+b^2}\cosh(aN_t+aM_t)\right]
\end{aligned}$$

Therefore, according to Theorem 2.3.1 and Proposition 3.3.1

$$\tilde{\pi}_t^* = \sigma_t^+ \delta_t \tilde{X}_t^* + \sigma_t^+(\lambda_t + \phi_t - \delta_t)\tilde{R}_t^*. \tag{3.44}$$

Hence, the optimal wealth and portfolio processes are given by

$$X_t^* = Y_t \tilde{X}_t^* \quad \text{and} \quad \pi_t^* = Y_t \tilde{\pi}_t^*,$$

where

$$Y_t = \exp\left(\int_0^t \left(\delta_s \cdot \lambda_s - \frac{1}{2}\delta_s^2\right) ds + \int_0^t \delta_s \cdot dW_s\right).$$

3.4.2 Construction of Differential Input

Since h is given, and $Range(h) = (-\infty, +\infty)$, one could construct the corresponding differential input using Proposition 3.1.3. Using this approach, the following result is derived in Musiela and Zariphopoulou [13].

If $a = 1$, then

$$u(x,t;a,b) = \frac{1}{2}\left(\ln\left(x + \sqrt{x^2 + b^2 e^{-t}}\right) - \frac{e^t}{b^2} x\left(x - \sqrt{x^2 + b^2 e^{-t}}\right) - \frac{t}{2}\right) - \frac{1}{2}\ln b,$$

whereas, if $a \neq 1$, then

$$u(x,t;a,b) = \frac{(\sqrt{a})^a}{a^2 - 1} e^{\frac{1-a}{2}t} \frac{b^2 e^{-a^2 t} + a(1+a)\left(ax^2 + x\sqrt{a^2x^2 + b^2 e^{-a^2 t}}\right)}{\left(ax + \sqrt{a^2x^2 + b^2 e^{-a^2 t}}\right)^{1+\frac{1}{a}}} - \frac{(\sqrt{a})^a}{a^2 - 1} b^{1-\frac{1}{a}}. \tag{3.45}$$

3.4.3 Special Case: Forward Exponential Utility

Here, we look at the behaviour of the risk tolerance function and differential input when the parameter a vanish.

The limit of (3.42) as a approaches zero is

$$\lim_{a \to 0} r(x,t;a,b) = b, \quad x \in \mathbb{R}. \tag{3.46}$$

Recall the exponential utility in the traditional setup

$$u^e(x;T) = -e^{-\gamma x}, \quad x \in \mathbb{R} \quad \text{and} \quad \gamma > 0, \tag{3.47}$$

and its corresponding local risk tolerance function

$$r^e(x;T) = \frac{1}{\gamma}, \quad x \in \mathbb{R}. \tag{3.48}$$

Notice that (3.46) is of the same form as (3.48). Therefore, we expect that by letting the parameter a go to zero, the corresponding differential input will also assemble the classical exponential case.

Note that solutions of (3.1) is invariant under affine transformations. Hence, here we consider differential inputs of the form

$$\bar{u} = Mu + N,$$

where M, N are constants and u is defined as in (3.45). Without loss of generality, we consider

$$\bar{u} = Me^{\frac{1-a}{2}t}\frac{b^2e^{-a^2t} + a(1+a)\left(ax^2 + x\sqrt{a^2x^2 + b^2e^{-a^2t}}\right)}{\left(ax + \sqrt{a^2x^2 + b^2e^{-a^2t}}\right)^{1+\frac{1}{a}}}.$$

Hence, its limit as a approaches zero is

$$\begin{aligned}\lim_{a\to 0}\bar{u}(x,t;a,b) &= \lim_{a\to 0} Me^{\frac{1-a}{2}t}\left(b^2e^{-a^2t} + a(1+a)\left(ax^2 + x\sqrt{a^2x^2 + b^2e^{-a^2t}}\right)\right)\\ &\quad\times \lim_{a\to 0}\left(ax + \sqrt{a^2x^2 + b^2e^{-a^2t}}\right)^{-1-\frac{1}{a}}\end{aligned}$$

$$\begin{aligned}&= \lim_{a\to 0} Me^{\frac{1-a}{2}t}\left(b^2e^{-a^2t} + a(1+a)\left(ax^2 + x\sqrt{a^2x^2 + b^2e^{-a^2t}}\right)\right)\\ &\quad\times \lim_{a\to 0}\left(\frac{a}{b}x + \sqrt{\frac{a^2}{b^2}x^2 + e^{-a^2t}}\right)^{-1-\frac{1}{a}} \times b^{-1-\frac{1}{a}}\end{aligned}$$

Using the fact that

$$\lim_{a\to 0}\left(\frac{a}{b}x + \sqrt{\frac{a^2}{b^2}x^2 + 1}\right)^{-1-\frac{1}{a}} = e^{-\frac{x}{b}},$$

we obtain

$$\begin{aligned}\lim_{a\to 0}\bar{u}(x,t;a,b) &= \lim_{a\to 0} Me^{\frac{1-a}{2}t}b^2e^{-\frac{x}{b}}b^{-1-\frac{1}{a}}\\ &= \lim_{a\to 0} Mb^{1-\frac{1}{a}}e^{-\frac{x}{b}+\frac{1}{2}t}\end{aligned}$$

For simplicity, take $M = -b^{-1+\frac{1}{a}}$ yields

$$u^e(x,t) = \lim_{a\to 0}\bar{u}(x,t;a,b) = -e^{-\frac{x}{b}+\frac{t}{2}}, \quad x \in \mathbb{R}. \tag{3.49}$$

In consequence, the corresponding forward exponential utility is given by

$$U_t^e(x) = -\exp\left(-\frac{1}{b}\frac{x}{Y_t} + \frac{A_t}{2}\right)Z_t. \tag{3.50}$$

Following similar arguments, we obtain the optimal benchmarked wealth and benchmarked portfolio processes

$$\tilde{X}_t^{*,e} = x + b\left(N_t + M_t\right) \text{ and } \tilde{\pi}_t^{*,e} = \sigma_t^+\delta_t\left(x + b(N_t + M_t)\right) + \sigma_t^+(\lambda_t + \phi_t - \delta_t)b. \tag{3.51}$$

In the next chapter, we show the application of forward utilities similar to (3.50) in derivative pricing in incomplete markets, namely forward indifference valuation.

Chapter 4

Forward Indifference Valuation in a Stochastic Factor Model with Exponential Preferences

In this chapter, we look at the application of forward performance measures in integrated portfolio management problems in a stochastic factor model. It is assumed that the investor trades between a riskless bond and a risky asset, and may incorporate in his portfolio derivatives and liabilities. The corresponding optimal investment problem can be solved by the so-called utility-based pricing approach.

This chapter is organized as follows. In Section 1, we introduce the stochastic factor model. Section 2 contains the derivation of the forward indifference value and its associated optimal investment strategy. In Section 3, we compare the forward indifference valuation to its backward counterpart. This chapter is based on Musiela and Zariphopoulou [10].

4.1 The Model

The market we consider there is the same as in Musiela and Zariphopoulou [10]. We assume a dynamic market environment with two assets, namely a riskless bond and a risky stock. It is assumed that the bond offers zero interest rate and the price of the stock solves

$$dS_s = \mu(Y_s)S_s ds + \sigma(Y_s)S_s dW_s^1, \tag{4.1}$$

with $s \geq 0$ and $S_0 = S > 0$ and the stochastic factor Y satisfies

$$dY_s = b(Y_s)ds + a(Y_s)dW_s, \tag{4.2}$$

for $s \geq 0$ and $Y_0 = y \in \mathbb{R}$.

The processes W^1 and W are standard Brownian motions with correlation coefficient $\rho \in (-1, 1)$ on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})$ with $\mathcal{F}_s$ being the augmented σ-algebra. Hence, we may write

$$dW_s = \rho dW_s^1 + \sqrt{1-\rho^2} dW_s^{1,\perp}, \tag{4.3}$$

where $W^{1,\perp}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})$ orthogonal to W^1.

The following assumption is imposed throughout the chapter.

Assumption 4.1.1. *The market coefficients μ, σ, a and b are $C^2(\mathbb{R})$ functions that satisfy $|f(y)| \leq C(1+|y|)$ for $f = \mu$, σ, a and b, and are such that (4.1) and (4.2) have a unique strong solution satisfying $S_s > 0$ a.e. for $s \geq 0$. There also exists $\varepsilon > 0$ such that $\sigma(y) > \varepsilon$, for $y \in \mathbb{R}$.*

Given an arbitrary trading horizon $[0, T]$, we consider an agent who, starting at time $t_0 \in [0, T]$, trades between bond and stock with initial wealth x. His current wealth can be represented by $X_s = \pi_s^0 + \pi_s$, where the self-financing portfolios π_s^0 and π_s represent the amount invested in the bond and the stock respectively. In addition, the dynamics of his wealth process can be written as

$$dX_s = \mu(Y_s)\pi_s ds + \sigma(Y_s)\pi_s dW_s^1, \tag{4.4}$$

with $X_{t_0} = x$. The set of admissible strategies is defined as

$$\mathcal{A} = \{\pi : \pi \text{ is } \mathcal{F}_s - \text{measurable. self-financing and } \mathbb{E}\left(\int_0^T \sigma^2(Y_s)\pi_s^2 ds\right) < +\infty\}.$$

Recall in Chapter 3 Section 3, we constructed a class of forward dynamic exponential utilities of the form

$$U_t^e(x) = -\exp\left(-\frac{1}{b}\frac{x}{Y_t} + \frac{A_t}{2}\right) Z_t, \tag{4.5}$$

where the processes (A_t, Y_t, Z_t) are defined as in (2.12), (2.9) and (2.10). Here, for simplicity, we take $Y_t = Z_t = 1$. Assuming an initial exponential preferene at time $s \geq 0$, the corresponding forward exponential utility is given by

$$U_t^F(x; s) = U_t^e(x) = -\exp\left(-\gamma x + \frac{A_t}{2}\right), \tag{4.6}$$

where $\gamma = \frac{1}{b}$ and $A_t = \int_s^t \lambda(Y_u)^2 du$.

4.2 Investment and Valuation Under Forward Exponential Utilities

In this section, we derive the forward indifference value and integrated optimal portfolio corresponding to the forward exponential utility (4.6).

4.2.1 Forward Indifference Price

The following concept of forward indifference value is introduced in Musiela and Zariphopoulou [10].

Definition 4.2.1. *Let $s \geq 0$ be the forward normalization point and consider a claim $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$, written at $t_0 \geq s$ and maturing at T. For $t \in [t_0, T]$, the forward indifference value process $v_t^F(C_{\bar{T}}; s)$ is defined as the amount that satisfies the pricing condition*

$$U_t^F(x - v_t^F(C_{\bar{T}}; s); s) = \sup_{\mathcal{A}} \mathbb{E}_{\mathbb{P}}(U_T^F(X_T - C_{\bar{T}}; s)|\mathcal{F}_t) \tag{4.7}$$

for all $x \in \mathbb{R}$, $X_t = x$.

In light of the above definition, forward indifference value and its corresponding integrated optimal portfolio are derived in Musiela and Zariphopoulou [10].

Proposition 4.2.1. *Let $s \geq 0$ be the forward normalization point and consider a European claim written at $t_0 > s$ and maturing at time T yielding payoff $C_T = C(S_T, Y_T)$. For $t \in [t_0, T]$, its forward indifference value $v_t^F(C_T; s)$ is given by*

$$v_t^F(C_T; s) = p^F(S_t, Y_t, t),$$

where S and Y solve (4.1) and (4.2), and $p^F : \mathbb{R}^+ \times \mathbb{R} \times [0, T] \to \mathbb{R}$ satisfies

$$\begin{cases} p_t^F + \mathcal{L}^{(S,T),mm} p^F + \frac{1}{2}\gamma(1-\rho^2)a^2(y)(p_y^F)^2 = 0 \\ \\ p^F(S, y, T) = C(S, y), \end{cases} \tag{4.8}$$

with

$$\begin{aligned} \mathcal{L}^{(S,Y),mm} =& \frac{1}{2}\sigma^2(y)S^2\frac{\partial^2}{\partial S^2} + \rho\sigma(y)Sa(y)\frac{\partial^2}{\partial S \partial y} + \frac{1}{2}a^2(y)\frac{\partial^2}{\partial y^2} \\ &+ (b(y) - \rho\lambda(y)a(y))\frac{\partial}{\partial y}. \end{aligned}$$

Proof. According to the representation (4.6),

$$\begin{aligned} \sup_{\mathcal{A}} \mathbb{E}_{\mathbb{P}}(U_T^F(X_T - C_T; s)|\mathcal{F}_t) =& \sup_{\mathcal{A}} \mathbb{E}_{\mathbb{P}}\left(-e^{-\gamma(X_T - C_T) + \int_s^T \frac{1}{2}\lambda^2(Y_u)du} \middle| \mathcal{F}_t\right) \\ =& e^{\int_s^t \frac{1}{2}\lambda^2(Y_u)du} \sup_{\mathcal{A}} \mathbb{E}_{\mathbb{P}}\left(-e^{-\gamma(X_T - C_T) + \int_t^T \frac{1}{2}\lambda^2(Y_u)du} \middle| \mathcal{F}_t\right) \end{aligned} \tag{4.9}$$

Define $u^C : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times [0, T] \to \mathbb{R}^-$

$$u^C(x, S, y, t) = \sup_{\mathcal{A}} \mathbb{E}_{\mathbb{P}}\left(-e^{-\gamma(X_T - C_T) + \int_t^T \frac{1}{2}\lambda^2(Y_u)du} \middle| X_t = x, S_t = S, Y_t = y\right). \tag{4.10}$$

It solves the Hamilton-Jacobi-Bellman equation

$$u_t^C + \max_{\pi}\Big(\frac{1}{2}\sigma^2(y)\pi^2 u_{xx}^C + \pi(\sigma^2(y)Su_{xS}^C + \rho a(y)\sigma(y)u_{xy}^C + \mu(y)u_x^C) \tag{4.11}$$
$$+ \mathcal{L}^{(S,Y)}u^C + \frac{\lambda^2(y)}{2}u^C = 0$$

with

$$u^C(x,S,y,T) = -e^{-\gamma(x-p^F(S,y,t))}$$

and

$$\mathcal{L}^{(S,Y)} = \frac{1}{2}\sigma^2(y)S^2\frac{\partial^2}{\partial S^2} + \rho\sigma(y)Sa(y)\frac{\partial^2}{\partial S\partial y} + \frac{1}{2}a^2(y)\frac{\partial^2}{\partial y^2}$$
$$+ \mu(y)S\frac{\partial}{\partial S} + b(y)\frac{\partial}{\partial y}.$$

Verification results yield that the optimal policy $\pi_S^{B,*}$ is given in the feedback form

$$\pi_s^{B,*} = \pi^{B,*}(X_s^{B,*}, S_s, Y_s, s)$$

with

$$\pi^{B,*}(X_s^{B,*}, S_s, Y_s, s) = -\frac{\sigma^2(y)Su_{xS}^C + \rho a(y)\sigma(y)u_{xy}^C + \mu(y)u_x^C}{\sigma^2(y)u_{xx}^C}. \tag{4.12}$$

Hence, (4.11) can be written as

$$u_t^C - \frac{1}{2}\frac{(\sigma^2(y)Su_{xS}^C + \rho a(y)\sigma(y)u_{xy}^C + \mu(y)u_x^C)^2}{\sigma^2(y)u_{xx}^C} + \mathcal{L}^{(S,Y)}u^C + \frac{\lambda^2(y)}{2}u^C = 0. \tag{4.13}$$

Using the transformation

$$u^C(x,S,y,t) = -e^{-\gamma(x-p^F(S,y,t))}$$

and substituting it into (4.13), yields

$$\gamma p_t^F u - \frac{1}{2}\frac{(-\gamma^2\sigma^2(y)Sp_y^F u - \gamma^2\rho a(y)\sigma(y)p_y^F u - \gamma\mu(y)u)^2}{\sigma^2(y)\gamma^2 u} \tag{4.14}$$
$$+\frac{1}{2}\sigma^2(y)S^2(\gamma p_{SS}^F + \gamma^2 p_S^F)^2))u + \rho\sigma(y)Sa(y)(\gamma p_{Sy}^F + \gamma^2 p_S^F p_y^F)u$$
$$+\frac{1}{2}a^2(y)(\gamma^2(p_y^F)^2 + \gamma p_{yy}^F)u + \mu(y)S\gamma p_S^F u + b(y)\gamma p_y^F u + \frac{\lambda^2(y)}{2}u = 0$$

Expand the second term in (4.14) and factor out γ and u yields

$$p_t^F - \frac{1}{2}[\gamma\sigma^2(y)S^2p_S^F + \gamma\rho^2a^2(y)(p_y^F)^2 + \frac{\lambda^2(y)}{\gamma} + 2\gamma S\rho a(y)\sigma(y)p_S^F p_y^F + 2S\mu(y)p_S^F$$

$$+2\rho a(y)\lambda(y)p_y^F] + \frac{1}{2}\sigma^2(y)S^2(p_{SS}^F + \gamma(p_S^F)^2) + \rho\sigma(y)Sa(y)(p_{Sy}^F + \gamma p_S^F p_y^F)$$
$$+\frac{1}{2}a^2(y)(\gamma(p_y^F)^2 + p_{yy}^F) + \mu(y)Sp_S^F + b(y)p_y^F + \frac{\lambda^2(y)}{2\gamma} = 0$$

After some cancellations, we obtain (4.8).

We then easily see that

$$\sup_{\mathcal{A}} \mathbb{E}_{\mathbb{P}}\left(U_T^F(X_T - C_T; s|\mathcal{F}_t)\right) = -e^{\int_s^t \frac{1}{2}\lambda^2(Y_u)du} u^C(x, S_t, Y_t, t),$$

and using (4.6) and Definition 4.2.1 we conclude. □

4.2.2 Forward Optimal Integrated Portfolio

Moreover, the associated optimal integrated portfolio can also be constructed explicitly.

Proposition 4.2.2. *Let $s \geq 0$ be the forward normalization point and consider an investor endowed with the forward dynamic exponential utility U^F and facing a liability $C_T = C(S_T, Y_T)$. The processes $\pi_s^{F,*}$ and $\pi_s^{F,0,*}$, representing the optimal investment in the risky and riskless asset in the integrated portfolio choice problem, are given, for $u \in [t, T]$, by*

$$\pi_u^{F,*} = \pi^{F,*}(X_u^{F,*}, S_u, Y_u, u) = \frac{\mu(Y_u)}{\gamma\sigma^2(Y_u)} \tag{4.15}$$
$$+ S_u p_S^F(S_u, Y_u, u) + \rho\frac{a(Y_u)}{\sigma(Y_u)}p_y^F(S_u, Y_u, u) \tag{4.16}$$

and

$$\pi_u^{F,0,*} = \pi^{F,0,*}(X_u^*, S_u, Y_u, u) = X_u^{F,*} - \pi_u^{F,*}. \tag{4.17}$$

Herein, $X_s^{F,*}$ solves (4.4) with $\pi_s^{F,*}$ being used and p^F satisfies (4.8).

Proof. Recall the optimal policy $\pi_u^{F,*}$ is given in (4.12). Let

$$u^C(x, S, y, t) = -e^{-\gamma(x - p^F(S,y,t))}, \tag{4.18}$$

where p^F satisfies (4.8). Substituting (4.18) into (4.12) yields the result. □

4.3 Forward and Backward Indifference Valuation

In this section, we review the classic backward indifference valuation and compare the backward indifference value and optimal investment strategies with the newly introduced forward counterpart.

4.3.1 Indifference Value: Backward vs Forward

First, we revisit the definition of the classic backward dynamic exponential utility.

Definition 4.3.1. *Let $T > 0$. An $\mathcal{F}_t$-measurable stochastic process $U_t^B(x,t)$ is called a backward dynamic exponential utility (BDU), normalized at T, if for all $t, \bar{T}$, it satisfies the stochastic optimality criterion*

$$U_t^B(x,t) = \begin{cases} -e^{-\gamma x}, & t = T \\ \sup_{\mathcal{A}} \mathbb{E}(U_{\bar{T}}^B(X_{\bar{T}};T)|\mathcal{F}_t), & 0 \le t \le \bar{T} \le T, \end{cases}$$

with X given by (4.4) and $X_t = x \in \mathbb{R}$.

To this end, we introduce two equivalent martingale measures, namely, the minimal martingale measure and the minimal entropy measure, denoted respectively by $\mathbb{Q}^{mm}$ and $\mathbb{Q}^{me}$. They are defined as the minimizer of the entropic functionals:

$$\mathcal{H}^0(\mathbb{Q}^{mm}|\mathbb{P}) = \min_{\mathcal{Q}\in\mathcal{Q}_e} \mathbb{E}_{\mathbb{P}}\left(-\ln\frac{dQ}{d\mathbb{P}}\right),$$
$$\mathcal{H}(\mathbb{Q}^{me}|\mathbb{P}) = \min_{\mathcal{Q}\in\mathcal{Q}_e} \mathbb{E}_{\mathbb{P}}\left(\frac{dQ}{d\mathbb{P}}\ln\frac{dQ}{d\mathbb{P}}\right),$$

where $\mathcal{Q}_e$ stands for the set of equivalent martingale measures. When the market becomes complete, the two measures, $\mathbb{Q}^{mm}$ and $\mathbb{Q}^{me}$, coincide with the unique risk neutral measure. Next, we introduce the following two quantities which will facilitate our analysis. For $0 \le t \le \tilde{T} \le T$, we define the aggregate relative entropy process

$$H\left(t,\tilde{T}\right) = \mathbb{E}_{\mathbb{Q}_{me}}\left(\int_t^{\tilde{T}} \frac{1}{2}(\lambda^2(Y_s) + \hat{\lambda}^2(Y_s,s;T))ds \middle| \mathcal{F}_t\right), \tag{4.19}$$

and the function

$$\tilde{H}\left(y,t;\tilde{T}\right) = \mathbb{E}_{\mathbb{Q}_{me}}\left(\int_t^{\tilde{T}} \frac{1}{2}(\lambda^2(Y_s) + \hat{\lambda}^2(Y_s,s;T))ds \middle| Y_t = y\right) \tag{4.20}$$

where λ is the Sharpe ratio process and $\hat{\lambda} : \mathbb{R}\times[0,T] \to \mathbb{R}^+$ is defined as

$$\hat{\lambda}(y,t;T) = -\frac{1}{\sqrt{1-\rho^2}}a(y)\frac{f_y(y,t;T)}{f(y,t,T)}, \tag{4.21}$$

where $f : \mathbb{R}\times[0,T] \to \mathbb{R}^+$ is the unique $C^{1,2}(\mathbb{R}\times[0,T])$ solution of the terminal wealth problem

$$\begin{cases} f_t + \frac{1}{2}a^2(y)f_{yy} + (b(y) - \rho\lambda(y)a(y))f_y = \frac{1}{2}(1-\rho^2)\lambda^2(y)f \\ \\ f(y,T) = 1. \end{cases} \tag{4.22}$$

Next, we provide an analytic representation of the backward dynamic exponential utility as in Definition 4.3.1 and its associated indifference value (see Musiela and Zariphopoulou [10] for detailed explanation).

Proposition 4.3.1. *Let $\mathbb{Q}_{me}$ be the minimal relative entropy martingale measure and $H(t,T)$ the aggregate relative entropy process as in (4.19). Then, for $x \in \mathbb{R}$, $t \in [0,T]$, the process $U_t^B \in \mathcal{F}_t$, given by*

$$U_t^B(x;T) = -e^{-\gamma x - H(t,T)} \tag{4.23}$$

is a backward dynamic exponential utility.

Proposition 4.3.2. *Let T be the backward normalization point and consider a European claim written at $t_0 > 0$ and maturing at $\bar{T} < T$, yielding payoff $C_{\bar{T}} = C(S_{\bar{T}};T)$. For $t \in [t_0, \bar{T}]$, its backward indifference value process $v_t^B(C_{\bar{T}};T)$ is given by*

$$v_t^B(C_{\bar{T}};T) = p^B(S_t, Y_t, t)$$

where S and Y solve (4.1) and (4.2) and $p^F : \mathbb{R}^+ \times \mathbb{R} \times [0,\bar{T}] \to \mathbb{R}$ satisfies

$$\begin{cases} p_t^B + \mathcal{L}^{(S,Y),me} p^B + \frac{1}{2}\gamma(1-\rho^2)a^2(y)(p_y^B)^2 = 0 \\ \\ p^B(S,y,\bar{T}) = C(S,y). \end{cases} \tag{4.24}$$

Herein,

$$\begin{aligned} \mathcal{L}^{(S,Y),me} =& \frac{1}{2}\sigma^2(y)S^2\frac{\partial^2}{\partial S^2} + \rho\sigma(y)Sa(y)\frac{\partial^2}{\partial S \partial y} + \frac{1}{2}a^2(y)\frac{\partial^2}{\partial y^2} \\ &+ \left(b(y) - \rho\lambda(y)a(y)a^2(y)\frac{f_y(y,t;T)}{f(y,t;T)}\right)\frac{\partial}{\partial y}, \end{aligned} \tag{4.25}$$

and f solved (4.22).

Notice that the backward and forward indifference values, $v_t^B(C_{\bar{T}};T)$ and $v_t^F(C_{\bar{T}};s)$, $0 \le s \le t \le \bar{T} \le T$, have very distinct characteristics. They are both represented as solutions of similar quasilinear partial differential equations. While the nonlinearities in the pricing partial differential equations are of the same type, the associated linear operators differ. The linear operator associate with the backward indifference value corresponds to the minimal relative entropy measure, while the one associate to the forward indifference value corresponds to the minimal martingale measure. Moreover, the forward indifference value is independent on the forward normalization point s, whereas, the backward indifference value depends on the backward normalization point T.

The two prices do not coincide unless the market is complete. This is a direct consequence of the fact that the internal market incompleteness is processed by the backward and forward dynamic utilities in a different way. The backward dynamic utility complies changes in the market in an aggregate manner, whereas the forward utility does it in a much finer way.

4.3.2 Optimal Integrated Portfolio: Backward vs Forward

Now, we compare the backward and forward optimal investment strategies. The following result is derived in Musiela and Zariphopoulou [10].

Proposition 4.3.3. *Let T be the backward normalization point and consider tan investor endowed with the backward dynamic exponential utility U^B and facing a liability $C_{\bar{T}} = C(S_{\bar{T}}, Y_{\bar{T}})$. The processes, $\pi_s^{B,*}$ and $\pi_s^{B,0,*}$,representing the optimal investments in the risky asset and riskless asset, are given, for $s \in [t, \bar{T}]$, by*

$$\pi_s^{B,*} = \pi^{B,*}(X_s^{B,*}, S_s, Y_s, s) = \frac{\mu(Y_s)}{\gamma\sigma^2(Y_s)} - \rho\frac{a(Y_s)}{\sigma(Y_s)}\tilde{H}(Y_s, s; T) + S_s p_S^B(S_s, Y_s, s) + \rho\frac{a(Y_s)}{\sigma(Y_s)}p_y^B(S_s, Y_s, s) \tag{4.26}$$

and

$$\pi_s^{B,0,*} = \pi^{B,0,*}(X_s^*, S_s, Y_s, s) = X_s^{B,*} - \pi_s^{B,*}.$$

Herein, $X_s^{B,*}$ solves (4.4) *with $\pi_s^{B,*}$ being used, $\tilde{H}$ as in* (4.20) *and p^B solves* (4.24).

Notice that the optimal portfolios corresponding to the backward and forward dynamic utilities have very different characteristics. The optimal backward investment consists of the myopic portfolio, the backward indifference risk monitoring strategies and the excess risky demand. The forward optimal investment has the same structure as their backward counterpart but do not include the excess risk demand. Moreover, the backward optimal strategy depends on the backward normalization point T, whereas its forward counterpart is independent on the forward normalization point s.

Chapter 5

Conclusions and Final Remarks

In this thesis, we studied the forward performance measurement and its application in indifference valuation.

We extended the results in Musiela and Zariphopoulou [13] and provided an explicit construction of the optimal investment and wealth processes corresponding to a time-monotone forward performance process introduced in Musiela and Zariphopoulou [12].

Motivated by the form of the optimal processes, we then investigated the representation of the differential input and the corresponding risk tolerance function. Our result showed that instead of specifying the differential input u and then defining r, one could start with the risk tolerance process r directly. To demonstrate this idea, we started with a two parameter family risk tolerance and constructed its corresponding differential input and optimal processes. By letting the parameter go to zero, we obtained a special forward performance process, namely, the forward exponential utility.

Finally, using the forward exponential utility, we solved the integrated portfolio management problem by the so-called utility-based approach and compared it with its classical backward counterpart.

Our results showed that, in general, the backward and forward indifference values and investment strategies do not coincide. This is because the market incompleteness is processed by the backward and forward exponential utilities in a very distinct manner. Characterizing the market conditions for which the two approaches coincide is an open question.

Another point of interest with forward indifference valuation would be to investigate the characteristics of forward indifference value and optimal strategies when the Markovian structure of the model is relaxed.

Recently, T. Leung, R. Sircar and T. Zariphopoulou discussed the forward indif-

ference valuation for American options in a stochastic factor model. Future research could also be done in other applications, for example, credit derivatives, volatility derivatives and order book modelling.

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