A Probabilistic Approach to Fractional Reaction-Diffusion Equations

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For Michael, Mom and Dad.

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Abstract

This thesis consists of two projects. In the first project, we prove a new result relating the solution interface of the scaled fractional Allen–Cahn equation to motion by mean curvature flow. More precisely, we show that, for $\alpha \in (1,2)$ and $\mathbb{d} \geq 2$, if u^ε is a solution to the scaled fractional Allen–Cahn equation

$$\partial_t u^\varepsilon = -I(\varepsilon)^{\alpha-2}(-\Delta)^{\frac{\alpha}{2}} u^\varepsilon + \varepsilon^{-2} u^\varepsilon (1-u^\varepsilon)(2u^\varepsilon - 1) \quad t > 0,\, x \in \mathbb{R}^{\mathbb{d}}$$

with a suitably chosen initial condition, then as $\varepsilon \to 0$, u^ε converges to the indicator function of a set whose boundary evolves according to motion by mean curvature flow. Here, $I(\varepsilon)$ can be chosen from a family of possible 'scaling functions', distinguishing our result from known convergence results such as [IS09]. Our proof is purely probabilistic (taking inspiration from [EFP17]) and describes solutions of the fractional Allen–Cahn equation in terms of ternary branching α-stable motions. Finally, we prove that this mean curvature flow behaviour is preserved for a stochastic analogue of the fractional Allen–Cahn equation. We do so via the Spatial Λ-Fleming-Viot process. In our second project, we prove a new result on the spreading speed of solutions to one-dimensional fractional Fisher–KPP type equations with appropriately chosen initial condition, such as the equation

$$\partial_t u = -(-\Delta)^{\frac{\alpha}{2}} u + u(1-u) \quad t > 0,\, x \in \mathbb{R}.$$

Taking inspiration from [Pen18], we use the Feynman–Kac formula to prove probabilistically that the front position of solutions to these equations grow exponentially in time, and state the precise order of this spreading speed, improving known results such as those from [CR13].

Declaration

I hereby declare that the contents of this thesis are original. The two projects featured in this thesis were both collaborations. The work in Chapters 3 to 5 was joint with Prof Alison Etheridge and Dr Ian Letter. The work in Chapter 6 was supervised by Dr Sarah Penington. This thesis is submitted for consideration for the degree of Doctor of Philosophy (Statistics) at The University of Oxford.

Contents

Chapter 1

Introduction

Reaction-diffusion equations have been used to study a wide range of phenomena within the natural sciences, and are a topic of great mathematical intrigue in their own right. They appear as models for the spread of populations [Fis37, KPP37, AW78], phase transitions [HH04, Wan02, ESS92], combustion [BNS85, BLL90], and chemical reactions [AC79, BL91]. In this thesis, we explore *fractional* reaction-diffusion equations that model populations that exhibit long range dispersal. Fractional reaction-diffusion equations are reaction-diffusion equations where the diffusive term is replaced by the generator of a pure jump process (namely, a stable process). As a result, they present new challenges that as of yet have not been fully explored in a probabilistic context.

One of the earliest applications of reaction-diffusion equations to biology appears in the 1937 work [Fis37]. Here, Fisher introduced the equation

$$\partial_t u(t,x) = \frac{m}{2}\Delta u(t,x) + su(t,x)(1-u(t,x)) \tag{1.1}$$

for $t > 0$, $x \in \mathbb{R}$ and initial condition $u_0(x) := u(0,x) \in [0,1]$. In the same year, Kolmogorov, Petrovsky and Piscounov studied (1.1) but for more general reaction terms $f(u)$ [KPP37]. Today, we know equation (1.1) as the Fisher-KPP equation. Fisher introduced this equation as a model for the spread of advantageous genes in a linear habitat, where $u(t,x)$ denotes the frequency of the gene at position x at time t, m is a diffusion constant and s is the intensity of selection in favour of the gene. The Fisher-KPP equation, which is intimately linked to branching processes and certain martingales, acts as the basis for a rich area of research explored by the likes of Skorokhod [Sko64], McKean [McK75], Bramson [Bra78, Bra83], Neveu [Nev88], Lalley and Sellke [LS87], and many others. For a detailed introduction to the Fisher-KPP equation and overview of its history, see [Fif79, Bri86].

Fisher's model assumed that individuals in the population dispersed locally. However, this assumption is erroneous for many real world populations. In this work, we consider populations that exhibit long range dispersal (i.e. the capacity for children to, on rare occasions, establish very far away from their parent). This behaviour is ubiquitous in nature and is a crucial survival mechanism for many organisms, particularly those insular species that must travel vast distances to populate new regions. Examples include the dispersal of plant seeds (which can travel by wind, water, and can be transported internally or externally by animals) [CMS00], fungi [Bus07], and small insects such as flies, moths, and bees (which have the secondary effect of facilitating the hybridisation of the flora they pollinate) [Bar96, Rea21]. The ability for certain organisms to populate islands through long range dispersal can have a profound impact on the biological composition of the land, increasing the genetic diversity of isolated regions [WKK$^+$15, Hea00, Rea21]. One example of this is the Hawaiian angiosperm flora, that cannot be attributed to a single mainland source, but instead have the genetic composition of flora from across circum-Pacific regions [Car67, WKK$^+$15]. Another recently observed instance of long range dispersal was that of a single finch that travelled over 100 km to an island in the Galápagos where it went on to produce hybrid offspring with the resident population [LHW$^+$18, Rea21].

To model populations with long range dispersal, we consider fractional reaction-diffusion equations which take the form

$$\partial_t u(t,x) = -(-\Delta)^{\frac{\alpha}{2}} u(t,x) + f(u(t,x)) \tag{1.2}$$

for all $t > 0$ and $x \in \mathbb{R}$, where $f : \mathbb{R} \to \mathbb{R}$ is a general nonlinearity (typically taken to be a quadratic or cubic, depending on the particular model) and $-(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian (i.e. the generator of an α-stable process), which will be defined in (2.1) below. Here the parameter $\alpha \in (0,2)$ determines the size and frequency of large jumps made by the stable process (as α decreases, the frequency of large jumps increases). Outside of population models, fractional reaction-diffusion equations have applications in dislocation dynamics and statistical mechanics [IS09]. Dislocation dynamics relates to the motion of defects in crystalline lattice structures, which can sometimes be modelled by non-local diffusions, as in Peierls-Nabarro models [KCnO02, GLL21]. In statistical mechanics, non-local diffusions can be derived as the mean field equation associated to stochastic Ising models [IS09]. For a comprehensive list of references see [GLL21].

In recent decades, fractional reaction-diffusion equations and reaction-diffusion equations with anomalous diffusion have surged in popularity.[1] This is in part due to their relevance to numerous physical models like those mentioned above. From a purely mathematical viewpoint, they pose technical difficulties that classical parabolic equations like the Fisher-KPP equation do not. Much of the theoretical work on these topics, to date, has been led by the PDE community [MMM11, GZ15, CR13, CS14, CS15, MRS14]. The effect of long range dispersal on various models has also been studied numerically [MVV03, del09, HF14], and applied to data from epidemics [BH13] and plant populations [CLMH03, MMC11, CMM06].

In this thesis, we study fractional reaction-diffusion equations from two distinct perspectives. In our first body of work (corresponding to Chapters 3-5) we prove a new result on the geometric motion of the solution interface generated by a scaled fractional reaction-diffusion equation with Allen–Cahn type nonlinearity. Then, in Chapter 6, we find new, tight estimates on the spreading speed of fractional reaction-diffusion equations with Fisher–KPP type nonlinearity.

1.1 The fractional Allen–Cahn equation

In this section, we provide an overview of the work in Chapters 3-5 on the fractional Allen–Cahn equation.

1.1.1 The Allen–Cahn equation

The *Allen–Cahn equation* is a reaction-diffusion equation that takes the form

$$\partial_t u^\varepsilon(t,x) = \Delta u^\varepsilon(t,x) + \frac{1}{\varepsilon^2} f(u^\varepsilon(t,x)) \tag{1.3}$$

for $\varepsilon > 0$ fixed and all $t > 0$, $x \in \mathbb{R}$. This equation can be obtained from the unscaled equation $\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))$ by defining $u^\varepsilon(t,x) := u(\varepsilon^2 t, \varepsilon x)$. Here, $f \in C^2(\mathbb{R})$ is assumed to have precisely three zeros, v_-, v_0 and v_+, such that

$$\begin{aligned}
&f > 0 \text{ on } (-\infty, v_-) \cup (v_0, v_+),\\
&f < 0 \text{ on } (v_-, v_0) \cup (v_+, \infty),\\
&f'(v_-), f'(v_+) < 0 \text{ and } f'(v_0) > 0.
\end{aligned}$$

[1] Anomalous diffusions are processes whose mean squared displacement obeys a power law. Consequently, α-stable processes (which have undefined second moment when $\alpha < 2$) are not anomalous diffusions, but their q-th moments ($q < \alpha$) can be used to define 'Lévy flights', see [MVV03, Section 2.2.4].

An example of f is shown in Figure 1.1, together with the Fisher–KPP nonlinearity.

In equation (1.3), the diffusive term Δ acts to smooth solutions (in the context of a biological model, this could be viewed as the 'mixing' of the population), while the nonlinearity f drives solutions towards one of two states, v_- or v_+ (in a population, these states might correspond to the dominance of a particular allele). These opposing effects, which are characteristic of reaction-diffusion equations, create a solution interface separating the two states. Over large spatial and temporal scales (corresponding to $\varepsilon \to 0$) this interface appears sharp and one can study its motion.

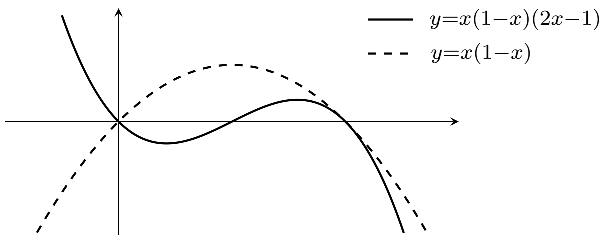


Figure 1.1: Examples of Fisher–KPP type (dashed) and Allen–Cahn type (solid) nonlinearities

The Allen–Cahn equation was originally introduced in [AC79] to model the motion of curved *antiphase boundaries* (APB) in crystalline solids. These are defective regions in the crystal lattice where atoms have the opposite configuration to that predicted by their lattice system, producing a positive excess of free energy in the system [AC79]. This is a non-equilibrium state of the lattice, resulting in the diffusive movement of the APB to minimise the total area of the boundary. The motion of the APB can then be modelled by the solution interface of the Allen–Cahn equation.[2]

It was observed by Allen and Cahn in [AC79] that the velocity of the interfacial motion described by equation (1.3) was proportional to the mean curvature of the boundary. Bronsard and Kohn [BK90, BK91] and Demottoni and Schatzman [dMS89, dMS90] provided a rigorous proof of this under a variety of dimensional and regularity restrictions, and in 1992, Chen proved this result in all dimensions under relatively

[2]In their original work [AC79], Allen and Cahn already note the relevance of long range dispersal to interfacial motion. Allen and Cahn mention that interfacial motion sometimes requires long range dispersal, citing the growth of a (solid) precipitate from a supersaturated solution and the motion of interfaces with impurities as two examples. Later, the motion of phase boundaries in particle systems with long range dispersal would be studied more rigorously, such as in the work of [GL97, GL98] for the Cahn–Hilliard equation.

weak regularity assumptions [Che92]. In this work, we prove an analogous result in the fractional setting, motivated by the motion of hybrid zones.

1.1.2 Hybrid zones

The Allen–Cahn equation can be used to model the motion of *hybrid zones* over large spatial and temporal scales. Hybrid zones are narrow geographical regions where two genetically distinct species meet and reproduce, resulting in individuals of mixed ancestry (hybrid individuals). The motion of hybrid zones was considered in Etheridge et al. [EFP17], and will be generalised in this work to populations that exhibit long range dispersal. Hybrid zones have been observed extensively in nature. Examples include the European house mouse [HS73] and North American warbler birds [TLIB18] (see [BH89] for a list of examples).

As explained in [EFP17], there are two primary mechanisms acting to maintain hybrid zones. The first is a change in environment where the two populations meet. The second, which will be considered in this work, is selection against hybrid individuals. This is the case, for example, for the European house mouse hybrid zone, where male hybrid individuals were found to be sterile [TPH+08], and for the Myrtle-Audubon warbler populations, although the selection mechanism is not known in this case [TLIB18].

To model the motion of hybrid zones in populations that exhibit long range dispersal, we consider the following special case of the fractional Allen–Cahn equation

$$\partial_t u(t,x) = -(-\Delta)^{\frac{\alpha}{2}} u(t,x) + su(t,x)(1-u(t,x))(2u(t,x)-1), \tag{1.4}$$

for all $t > 0$ and $x \in \mathbb{R}$ where s is a small selection parameter. The connection between hybrid zones and (1.4) will be justified in Chapter 3 by considering a diploid population in which the trait under selection is determined by a single genetic locus at which there are just two types (alleles) a and A. Since we consider hybrid zones maintained by selection against hybrid individuals, we assume selection favours homozygotes (aa or AA individuals) over heterozygotes (aA indivduals). We will see that the proportion of type a alleles at time t and position x will satisfy (1.4) over large spatial and temporal scales. Here, the operator $-(-\Delta)^{\frac{\alpha}{2}}$ arises from the long range dispersal of individuals (modelled as α-stable processes), and the nonlinearity will be derived as the rate of change of the proportion of type a alleles due to reproduction.

In Etheridge et al. [EFP17], populations were assumed to disperse locally. Consequently, the version of the Allen–Cahn equation considered there was the same as

(1.4), but with the fractional Laplacian replaced by the ordinary Laplacian Δ (the resulting equation is also known as the Nagumo's equation, see [NAY62, McK70]). By considering branching Brownian motions, the authors proved using purely probabilistic techniques that the (deterministic) Allen–Cahn equation converges as $\varepsilon \to 0$ to the indicator function of a set whose boundary evolves according to mean curvature flow. As we have seen, this result is due to Chen [Che92], but the proof in [EFP17] and techniques developed therein were novel. Additionally, their proof could be adapted to incorporate genetic drift, which refers to the randomness inherent in reproduction within finite populations. This was achieved via a so-called Spatial-Λ-Fleming-Viot process. It was shown in [EFP17] that, provided the genetic drift is not too strong, the limiting mean curvature flow behaviour observed in the deterministic case is preserved by the scaled stochastic model.

In view of this work, it is natural to ask: will mean curvature flow be preserved in populations that exhibit long range dispersal? The answer to this should, of course, depend on the strength of the dispersal mechanism. This is determined by the index $\alpha \in (0, 2]$. When $\alpha = 2$, the fractional Laplacian and ordinary Laplacian coincide, so in view of Chen's result [Che92], we expect mean curvature flow to be preserved for α sufficiently close to 2. As α approaches 0, the severity and frequency of large jumps increases, so for small α it seems unlikely that motion by mean curvature flow will be seen in the limit. This intuition is supported by results in the PDE literature, with the threshold between the two behaviours occurring at $\alpha = 1$. For example, in [CS10], Caffarelli and Souganidis consider a threshold dynamics type algorithm to simulate a moving front governed by the fractional heat equation. The resulting interface was shown to converge to mean curvature flow for $\alpha \geq 1$ and 'weighted' mean curvature flow for $0 < \alpha < 1$. Later, [IS09] showed that equation (1.4), when correctly scaled, converges as $\varepsilon \to 0$ to motion by mean curvature flow when $\alpha \in (1, 2)$.[3] As we will see, the scaling of (1.4) taken in [IS09] is distinct from our own, so even our deterministic convergence result is new.

1.1.3 Main results

We will prove that, under a family of possible scalings, the fractional Allen–Cahn equation with index $\alpha \in (1, 2)$ started from a suitable initial condition converges as $\varepsilon \to 0$ to the indicator function of a set whose boundary evolves according to mean curvature flow in dimension $\mathbb{d} \geq 2$. By modifying our proof to study an instance of

[3] The results of [IS09] are stated more generally for any reaction-diffusion equation with diffusive term given by a singular integral operator and reaction term given by a bistable nonlinearity.

the so-called Spatial Λ-Fleming-Viot process with selection, we show that this property is preserved in the presence of random genetic drift (provided the genetic drift is not too strong).

A rigorous definition of motion by mean curvature flow will be provided in Chapter 3. If $\mathbf{\Gamma}_0 \subset \mathbb{R}^{\mathrm{d}}$ is some initial hypersurface that evolves over time, we denote the surface at time t by $\mathbf{\Gamma}_t$. We call the collection $(\mathbf{\Gamma}_t)_{t\geq 0}$ the *mean curvature flow* of $\mathbf{\Gamma}_0$ if the normal velocity of a point in $\mathbf{\Gamma}_t$ is given by the mean curvature at that point. The resulting motion reduces the surface area of $\mathbf{\Gamma}_0$ as quickly as possible with respect to the L^2-norm over the surface (formally, mean curvature flow is the negative gradient flow for the area functional [CMP15]).

Our main deterministic result is the following (corresponding to Theorem 3.1.5 in Chapter 3). The precise assumptions on our initial condition p are omitted for the sake of brevity. These assumptions will be chosen later to ensure the existence of the mean curvature flow of the surface $\mathbf{\Gamma}_0 := \{x \in \mathbb{R}^{\mathrm{d}} : p(x) = \frac{1}{2}\}$ up to some finite time $\mathscr{T} > 0$.

Theorem 1.1.1 (The deterministic multidimensional result)**.** *Let $\varepsilon > 0$, $\mathrm{d} \geq 2$ and $\alpha \in (1,2)$. Suppose for all $t > 0$ and $x \in \mathbb{R}^{\mathrm{d}}$, $u^\varepsilon(t,x)$ solves*

$$\begin{cases} \partial_t u^\varepsilon = -\dfrac{\sigma_\alpha}{I(\varepsilon)^{2-\alpha}}(-\Delta)^{\frac{\alpha}{2}} u^\varepsilon + \dfrac{1}{\varepsilon^2} u^\varepsilon(1-u^\varepsilon)(2u^\varepsilon - 1) \\ u^\varepsilon(0,x) = p(x) \end{cases} \tag{1.5}$$

for some normalising constant $\sigma_\alpha > 0$, an appropriately chosen initial condition $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$ and any function I such that, for some $\delta > 0$, $I : (0,\delta) \to (0,\infty)$ satisfies

$$\lim_{\varepsilon\to 0} I(\varepsilon)|\log\varepsilon|^k = 0 \;\forall\, k \in \mathbb{N}, \quad \lim_{\varepsilon\to 0} \frac{\varepsilon^2}{I(\varepsilon)^2}|\log\varepsilon| = 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \frac{I(\varepsilon)^{2\alpha}}{\varepsilon^2}|\log\varepsilon|^\alpha = 0.$$

Let $\mathbf{\Gamma}_0 := \left\{x \in \mathbb{R}^{\mathrm{d}} : p(x) = \frac{1}{2}\right\}$ and denote the mean curvature flow of $\mathbf{\Gamma}_0$ at time $t \geq 0$ by $\mathbf{\Gamma}_t$. Denote the signed distance from a point $x \in \mathbb{R}^{\mathrm{d}}$ to $\mathbf{\Gamma}_t$ by $d(x,t)$. Assume that for x inside $\mathbf{\Gamma}_0$, $p(x) < \frac{1}{2}$, and for x outside $\mathbf{\Gamma}_0$, $p(x) > \frac{1}{2}$. Then $\mathbf{\Gamma}_t$ exists for all times t less than some $\mathscr{T} > 0$, and there exists $\varepsilon_{\mathrm{d}}, a_{\mathrm{d}}, c_{\mathrm{d}}, m > 0$ such that, for all $\varepsilon \in (0,\varepsilon_{\mathrm{d}})$ and $a_{\mathrm{d}}\varepsilon^2|\log\varepsilon| \leq t < \mathscr{T}$, if $x \in \mathbb{R}^{\mathrm{d}}$ satisfies $d(x,t) \geq c_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$, then

$$u^\varepsilon(t,x) \geq 1 - m\frac{\varepsilon^2}{I(\varepsilon)^2} - m\frac{I(\varepsilon)^2}{\varepsilon^{\frac{2}{\alpha}}}|\log\varepsilon| - mI(\varepsilon)^{\alpha-1} \tag{1.6}$$

and if $d(x,t) \leq -c_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$, then

$$u^\varepsilon(t,x) \leq m\frac{\varepsilon^2}{I(\varepsilon)^2} + m\frac{I(\varepsilon)^2}{\varepsilon^{\frac{2}{\alpha}}}|\log\varepsilon| + mI(\varepsilon)^{\alpha-1}. \tag{1.7}$$

Theorem 1.1.1 tracks the motion of the solution interface of equation (1.5) with width $2c_{\mathbb{d}}I(\varepsilon)|\log\varepsilon|$. On either side of this region, solutions to equation (1.5) will be close to one or zero. More precisely, they will be, at most, a distance $m\frac{\varepsilon^2}{I(\varepsilon)^2} + m\frac{I(\varepsilon)^2}{\varepsilon^{\frac{2}{\alpha}}}|\log\varepsilon| + mI(\varepsilon)^{\alpha-1}$ away from zero or one (provided enough time has passed). Since this distance (and the interface width itself) converges to zero as $\varepsilon \to 0$ (by assumption), Theorem 1.1.1 can be summarised as: solutions to equation (1.5) converge, as $\varepsilon \to 0$, to the indicator function of a set whose boundary evolves according to motion by mean curvature flow.

We can also interpret Theorem 1.1.1 in the context of hybrid zones. Recall that, to model hybrid zones maintained by selection against hybrid individuals, we consider a diploid population with selection against heterozygotes. Over large spatial and temporal scales, we can view $u^\varepsilon(t, x)$ as the proportion of type a alleles at time $t \geq 0$ and position $x \in \mathbb{R}^2$. Theorem 1.1.1 tells us that the hybrid zone, which divides the population into subsets of predominantly aa individuals (when $d(x,t) \geq c_{\mathbb{d}}I(\varepsilon)|\log\varepsilon|$) and AA individuals (when $d(x,t) \leq -c_{\mathbb{d}}I(\varepsilon)|\log\varepsilon|$), follows motion by mean curvature flow. Although the biologically relevant dimension is $\mathbb{d} = 2$, we prove Theorem 1.1.1 in all dimensions $\mathbb{d} \geq 2$.

This result is new in a number of ways. First, it is distinct from the work of [IS09], where a diffusive scaling was made.[4] This would correspond to setting $I(\varepsilon) := \varepsilon$ in equation (1.5). This scaling is not possible under our assumptions. To the authors' knowledge, this is also the first result of its kind for the fractional Allen–Cahn equation that explicitly provides an interface width as well as the estimates (1.6) and (1.7). Finally, the proof of this result is novel in that it is entirely probabilistic.

In order to prove Theorem 1.1.1, we couple solutions u^ε of the multidimensional initial value problem (1.5) to solutions of a particular initial value problem in one dimension. This coupling will enable us to prove Theorem 1.1.1 by proving an analogous result for the one-dimensional solution. The proof of our one-dimensional result in Chapter 3 constitutes the bulk of the work needed to prove Theorem 1.1.1.

The precise form of this one-dimensional result (Theorem 3.3.7) is quite technical and will take time to motivate in Chapter 3. Here, we will state a weaker version of the result that parallels Theorem 1.1.1. Since there is no notion of mean curvature flow in dimension one, in Theorem 1.1.2 we consider solutions on either side of the fixed interval of width $\mathcal{O}(I(\varepsilon)|\log\varepsilon|)$ about zero.

[4] We believe the standing example [IS09, Example 1] contains a typo, namely, the $\varepsilon^{-\alpha}\Delta^{\frac{\alpha}{2}}$ term (corresponding to the diffusive term $\varepsilon^{-2-\alpha}(-\Delta)^{\frac{\alpha}{2}}$ in their reaction-diffusion equation) should be $\varepsilon^{\alpha}\Delta^{\frac{\alpha}{2}}$ (making the diffusive term in their reaction-diffusion equation $\varepsilon^{-2+\alpha}(-\Delta)^{\frac{\alpha}{2}}$).

Theorem 1.1.2 (The deterministic one-dimensional result). *Let $\varepsilon > 0$, $\alpha \in (1,2)$ and suppose for all $t > 0$ and $x \in \mathbb{R}$ that $u^\varepsilon(t,x)$ solves*

$$\begin{cases} \partial_t u^\varepsilon = -\dfrac{\sigma_\alpha}{I(\varepsilon)^{2-\alpha}}(-\Delta)^{\frac{\alpha}{2}} u^\varepsilon + \dfrac{1}{\varepsilon^2} u^\varepsilon (1-u^\varepsilon)(2u^\varepsilon - 1) \\ u^\varepsilon(0,x) = \mathbb{1}_{\{x \geq 0\}} \end{cases} \tag{1.8}$$

for some normalising constant $\sigma_\alpha > 0$ and any function I such that, for some $\delta > 0$, $I : (0,\delta) \to (0,\infty)$ satisfies

$$\lim_{\varepsilon \to 0} I(\varepsilon) |\log \varepsilon|^k = 0 \ \forall\, k \in \mathbb{N} \quad \textit{and} \quad \lim_{\varepsilon \to 0} \frac{\varepsilon^2}{I(\varepsilon)^2} |\log \varepsilon| = 0.$$

Fix $T^* > 0$. Then there exists $\varepsilon_1 > 0$ and $c_1 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_1)$ and $t \in [0,T^*]$, if $x \in \mathbb{R}$ satisfies $x \geq c_1 I(\varepsilon)|\log \varepsilon|$, then

$$u^\varepsilon(t,x) \geq 1 - \frac{\varepsilon^2}{I(\varepsilon)^2} \tag{1.9}$$

and if $x \leq -c_1 I(\varepsilon)|\log \varepsilon|$, then

$$u^\varepsilon(t,x) \leq \frac{\varepsilon^2}{I(\varepsilon)^2}. \tag{1.10}$$

While the central ideas of our work take inspiration from [EFP17], our use of a long range dispersal mechanism throughout necessitates new, often delicate, arguments. In the Brownian setting of [EFP17], a dual to the Allen–Cahn equation was constructed by considering a ternary branching Brownian motion endowed with a 'voting system'. This is similar to the representation of solutions of the Fisher–KPP equation in terms of binary branching Brownian motions developed by Skorohod and McKean [Sko64, McK75].[5] The corresponding duality for the multidimensional and one-dimensional solutions of (1.5), (1.8) follows by almost identical arguments to [EFP17], but replacing all Brownian motions therein by α-stable processes.

Although the probabilistic dual to the fractional Allen–Cahn equation is straightforward to obtain, in practice it is difficult to work with. This is because of the rare long jumps of the α-stable process. To overcome this, in both the multidimensional and one-dimensional cases we couple a ternary branching α-stable motion to a ternary branching subordinated Brownian motion (with subordinator given by a truncated $\frac{\alpha}{2}$-stable motion). The idea here is that α-stable processes are equal in

[5] In fact, in the masters thesis of [O'D19], the author proves that a semilinear heat equation can be expressed in terms of a branching Brownian motion with a voting scheme if and only if the nonlinearity of the equation belongs to a certain (very general) family of polynomials.

distribution to Brownian motions subordinated by an (independent) $\frac{\alpha}{2}$-stable subordinator. By our choice of scaling in equations (1.5) and (1.8), the 'large' jumps of the $\frac{\alpha}{2}$-stable subordinator are sufficiently rare as to enable us to couple stable motions to Brownian motions subordinated by a *truncated* $\frac{\alpha}{2}$-stable subordinator. The resulting process has finite exponential moments, allowing us to more readily adapt the proofs in [EFP17].

Once we have proved the one-dimensional result, we go on to construct a coupling of one-dimensional and multidimensional solutions. This was relatively straightforward in the Brownian setting of [EFP17], but is a major hurdle in our own work. In short, in the Brownian setting, the motion of the interface in $\mathbb{R}^{\mathrm{d}}$ can be 'tracked' by considering the distance between a d-dimensional Brownian motion and the interface, which itself approximates a one-dimensional Brownian motion. This description is not viable in our setting because the large jumps of the stable process could see it jump over the interface, possibly repeatedly. We overcome this with a series of couplings, which will be discussed in depth in Chapter 4.

To incorporate genetic drift into our model, we mimic [EFP17] and employ the Spatial Λ-Fleming-Viot process with selection against heterozygosity (SΛFVS). The Spatial Λ-Fleming-Viot process, introduced in [BEV10, Eth08], is a measure-valued process $(w_t)_{t\geq 0}$ obtained as the limit of an individual-based model that is driven by a Poisson point process of 'events'. For $x \in \mathbb{R}^{\mathrm{d}}$, one should view $w_t(x)$ as the proportion of a alleles at time t and location x, so $w_t(\cdot)$ is the density that is defined almost everywhere.

We consider the SΛFVS with 'stable radii' to model a population with long range dispersal. The dynamics of the SΛFVS will be discussed in detail in Chapter 3. By adapting a result from [EVY20], the SΛFVS process can be seen as a (stochastic) perturbation of the fractional Allen–Cahn equation. Our main stochastic result is the following. This corresponds to Theorem 3.1.11 in Chapter 3. Once again, we omit the precise assumptions on the initial condition p, which will be provided in Assumption 3.1.4.

Theorem 1.1.3 (The stochastic multidimensional result)**.** *Let* $\mathrm{d} \geq 2$*,* $\alpha \in (1,2)$*,* $\beta \in \left(0, \frac{1}{2\alpha}\right)$*, and* $0 < \mathcal{R} \leq 1$*. Consider a sequence* $(\varepsilon_m)_{m=1}^{\infty}$ *satisfying* $\varepsilon_m \to 0$ *and* $(\log m)^{1/2}\varepsilon_m \to \infty$ *as* $m \to \infty$*. Suppose for some* $\delta > 0$ *that* $I : (0,\delta) \to (0,\infty)$ *satisfies*

$$\lim_{\varepsilon\to 0} I(\varepsilon)|\log\varepsilon|^k = 0 \;\forall\, k \in \mathbb{N}, \quad \lim_{\varepsilon\to 0} \frac{\varepsilon^2}{I(\varepsilon)^2}|\log\varepsilon| = 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \frac{I(\varepsilon)^{2\alpha}}{\varepsilon^2}|\log\varepsilon|^{\alpha} = 0.$$

Set
$$\mu(dr) := \frac{\mathbb{1}_{\{r\geq 1\}}}{r^{\mathrm{d}+\alpha+1}}dr.$$
Let $n \in \mathbb{N}$ *and set* $\mathcal{R}_n := n^{-\beta}\mathcal{R}$. *For all Borel subsets* A *of* $(0,\infty)$, *define the measure* $\mu^n(A) = \mu(n^\beta A)$. *Let* $(w_t^n)_{t\geq 0}$ *be the SΛFVS driven by* Π^n, *a Poisson point process on* $(0,\infty)\times\mathbb{R}^{\mathrm{d}}\times(\mathcal{R}_n,\infty)$ *with intensity measure* $ndt\otimes n^\beta dx\otimes\mu^n(dr)$, *with impact parameter* u_n *and selection parameter* s_n *given by*
$$u_n = \frac{u}{n^{1-\beta\alpha}I(\varepsilon_n)^{2-\alpha}}, \quad \text{and} \quad s_n = \frac{I(\varepsilon_n)^{2-\alpha}}{\varepsilon_n^2 n^{\beta\alpha}}.$$
Suppose $w_0^n(x) = p(x)$ *for a suitable choice of* $p:\mathbb{R}^{\mathrm{d}}\to[0,1]$. *Define* $\mathscr{T}$ *and* $d(x,t)$ *as in Theorem 1.1.1 and take* $T^* < \mathscr{T}$. *There exists* $n_*\in\mathbb{N}$, $a_*, c_*, m_* > 0$ *such that, for all* $n\geq n_*$ *and* $a_*\varepsilon_n^2|\log\varepsilon_n| \leq t \leq T^*$, *if* $x\in\mathbb{R}^{\mathrm{d}}$ *satisfies* $d(x,t)\geq c_* I(\varepsilon_n)|\log\varepsilon_n|$, *then*
$$\mathbb{E}\left[w_t^n(x)\right] \geq 1 - m_*\left(\frac{\varepsilon_n^2}{I(\varepsilon_n)^2} + \frac{I(\varepsilon_n)^2}{\varepsilon_n^{\frac{2}{\alpha}}}|\log\varepsilon_n| + I(\varepsilon_n)^{\alpha-1} + n^{-\beta\left(1-\frac{\alpha}{2}\right)}I(\varepsilon_n)^{\alpha-2}\right)$$
and if $d(x,t)\leq -c_* I(\varepsilon_n)|\log\varepsilon_n|$, *then*
$$\mathbb{E}\left[w_t^n(x)\right] \leq m_*\left(\frac{\varepsilon_n^2}{I(\varepsilon_n)^2} + \frac{I(\varepsilon_n)^2}{\varepsilon_n^{\frac{2}{\alpha}}}|\log\varepsilon_n| + I(\varepsilon_n)^{\alpha-1} + n^{-\beta\left(1-\frac{\alpha}{2}\right)}I(\varepsilon_n)^{\alpha-2}\right).$$

To prove Theorem 1.1.3, we consider the historical branching-coalescing dual of the SΛFVS process. We will see that, asymptotically in the scaling parameter n, under a certain parameter regime this dual is similar to a ternary branching stable motion, allowing us to apply Theorem 1.1.1. Theorem 1.1.3 tells us that, so long as genetic drift is sufficiently weak (as determined by the impact and selection parameters u_n and s_n), we will still observe an interface evolving under mean curvature flow as $\varepsilon_n\to 0$. An interesting avenue for further research would be to find the critical strength of genetic drift that determines if motion by mean curvature flow is preserved. This has been accomplished by [EGL22] in the Brownian setting.

1.2 Spreading speeds of reaction-diffusion equations

It has long been recognised by ecologists that the speed at which a population spreads is heavily dependent on the type of diffusion involved. For example, in [Ske51, Mol77] the authors note that the dispersal mechanism and, in particular, its tail distribution, changes predictions about a population's spread. In this work, we model populations that exhibit long range dispersal by a fractional Fisher–KPP type equation and calculate their spreading speed.

Numerous works have shown that Gaussian diffusion approximations can lead to dramatically inaccurate predictions for a population's spread based on historical data. The earliest of these observations came from Reid in 1899. Reid considered the northern migration of English oak trees at the beginning of the Holocene, and observed that, for these oak trees to reach their current northernmost position without 'external aid' (such as animals transporting seeds) "would take something like a million years" [Rei99, p. 25]. Later, Skellam would support Reid's claim by calculating the spread of oaks using a Gaussian diffusion approximation to show that local diffusion was not a plausible. Similar arguments have been made for the dispersal of other plant species, such as woodland herbs [CDM98, Dav76]. The discrepancy between these predictions and observations, known as "Reid's paradox", suggests that long range kernels could be more appropriate to model the spread of some populations; see [LMLN03] for an excellent historical review of these findings.

From a mathematical perspective, the spreading of a population is usually modelled by a travelling wave. Travelling waves are solutions to reaction-diffusion equations that do not change shape, that is, they appear stationary when viewed in a moving frame (at the appropriate speed). More precisely, a solution $u(t,x)$ of a one-dimensional reaction-diffusion equation is a travelling wave solution if it can be written as

$$u(t,x) = \omega(x - ct)$$

for all $t > 0$, $x \in \mathbb{R}$, some $c \geq 0$ and a sufficiently regular function ω. Here, c is called the wavespeed, which, in a population model, describes how fast the population spreads or advances. Travelling waves arise in countless other models for fluid dynamics, phase transitions, chemical reactions, epidemics, and chemotaxis [CH93, AK02, DEF$^+$07, Mur76, GBK01, HMS$^+$94].

The seminal works [Fis37, KPP37] were some of the first studies on travelling waves.[6] In [KPP37], Kolmogorov et. al. proved that solutions of 'Fisher–KPP type' equations with Heaviside initial condition approached a travelling wave as $t \to \infty$, and specified its wavespeed (this result was also conjectured in [Fis37]). Later, Aronson and Weinberger generalised this result to any dimension for compactly supported

[6] Murray [Mur93, p. 277] attributes the first analysis of travelling waves in chemical reactions to Luther [Lut06]. In this rediscovered work, Luther obtained the same wave speed as Fisher [Fis37].

initial data; part of this result is stated below.[7] Let $|\cdot|$ denote the standard Euclidean distance in $\mathbb{R}^{\mathrm{d}}$ for $\mathrm{d} \geq 1$.

Theorem 1.2.1 ([AW78])**.** *Suppose $g \in C^1([0,1])$ is concave, $g(0) = g(1) = 0$ and $g'(1) < 0 < g'(0)$. Suppose for all $t > 0$ and $x \in \mathbb{R}^{\mathrm{d}}$ that $u(t,x)$ is a solution of*

$$\begin{cases} \partial_t u = \Delta u + g(u) \\ u(0,x) = u_0(x), \quad 0 \leq u_0 \leq 1 \end{cases} \tag{1.11}$$

where $u(0,\cdot) \not\equiv 0$ and u_0 is compactly supported in $\mathbb{R}^{\mathrm{d}}$. Let $c_* := 2\sqrt{g'(0)}$. Then, for all $\varepsilon > 0$,

$$\lim_{t\to\infty} \inf_{|x| \leq (c_* - \varepsilon)t} u(t,x) = 1 \quad \textit{and} \quad \lim_{t\to\infty} \sup_{|x| \geq (c_* + \varepsilon)t} u(t,x) = 0.$$

In this work, we would like to obtain an analogous result to Theorem 1.2.1 (in one dimension) but for the fractional version of equation (1.11), to model the advancement of populations with long range dispersal. Numerous simulation studies suggest that the wave of advance in a population with long range dispersal rapidly accelerates [CSK+99, MMC11, HF14]. Mathematically rigorous studies on this topic are, however, lacking. The most accurate estimates for the spreading speed of fractional reaction-diffusion equations with Fisher–KPP type nonlinearity are, to the best of our knowledge, the following results due to [CR13].[8] These result holds for reaction-diffusion equations with diffusive term given by the generator of certain Feller semigroups, but we present it here in terms of the fractional Laplacian for simplicity.

Theorem 1.2.2 ([CR13])**.** *Suppose $g \in C^1([0,1])$ is concave, $g(0) = g(1) = 0$ and $g'(1) < 0 < g'(0)$. Let $\mathrm{d} \geq 1$ and $\sigma = \frac{g'(0)}{\mathrm{d}+\alpha}$. For all $t > 0$ and $x \in \mathbb{R}^{\mathrm{d}}$, suppose $u(t,x)$ is a solution of the initial value problem*

$$\begin{cases} \partial_t u = -(-\Delta)^{\frac{\alpha}{2}} u + g(u) \\ u(0,x) = u_0(x), \ 0 \leq u_0 \leq 1 \end{cases} \tag{1.12}$$

where u_0 is measurable, $u_0 \not\equiv 0$ and

$$u_0(x) \leq C|x|^{-\mathrm{d}-\alpha} \textit{ for all } x \in \mathbb{R}^{\mathrm{d}}$$

for some constant C. Then, for all $\varepsilon > 0$,

$$\lim_{t\to\infty} \inf_{|x| \leq e^{(\sigma-\varepsilon)t}} u(t,x) = 1 \quad \textit{and} \quad \lim_{t\to\infty} \sup_{|x| \geq e^{(\sigma+\varepsilon)t}} u(t,x) = 0.$$

[7] Of course, we have only given the definition of a one-dimensional travelling wave. In [AW78], the authors consider 'planar' travelling waves, and show that the smallest possible speed of a planar travelling wave solution to (1.11) is given by c_* from Theorem 1.2.1.

[8] Strictly speaking, a Fisher–KPP type nonlinearity is defined as any $g \in C^1([0,1])$ with $g(0) = g(1) = 0$ and $0 < g(s) < g'(0)s$ for all $s \in (0,1)$. The nonlinearities considered in Theorem 1.2.2 are a subset of this.

The following theorem considers the case when $\mathbb{d} = 1$ and the initial data is nonincreasing.

Theorem 1.2.3 ([CR13]). *Suppose $g \in C^1([0,1])$ is concave, $g(0) = g(1) = 0$ and $g'(1) < 0 < g'(0)$. Let $\sigma^* = \frac{g'(0)}{\alpha}$. For all $t > 0$ and $x \in \mathbb{R}$, suppose $u(t,x)$ is a solution of the initial value problem*

$$\begin{cases} \partial_t u = -(-\Delta)^{\frac{\alpha}{2}} u + g(u) \\ u(0,x) = u_0(x), \ 0 \le u_0 \le 1 \end{cases} \tag{1.13}$$

where u_0 is measurable and nonincreasing, $u_0 \not\equiv 0$ and

$$u_0(x) \le C x^{-\alpha} \text{ if } x > 0$$

for some constant C. Then, for all $\varepsilon > 0$,

$$\lim_{t\to\infty} \inf_{x \le e^{(\sigma^*-\varepsilon)t}} u(t,x) = 1 \quad \text{and} \quad \lim_{t\to\infty} \sup_{x \ge e^{(\sigma^*+\varepsilon)t}} u(t,x) = 0.$$

An immediate consequence of these results is that fractional reaction-diffusion equations of the forms specified above do not admit travelling waves. We therefore understand the spreading speed of solutions to equation (1.13) to be the speed of the level sets $\{u(t,x) = \lambda\}$ for $\lambda > 0$, instead of the wavespeed of a travelling wave solution.

In addition to Theorems 1.2.2 and 1.2.3, the authors tracked the speed of the level sets of equation (1.12) in a particular case.

Theorem 1.2.4 ([CR13]). *Suppose for all $t > 0$ and $x \in \mathbb{R}$ that $u(t,x)$ is a solution of the initial value problem*

$$\begin{cases} \partial_t u = -(-\Delta)^{\frac{1}{2}} u + u(1-u) \\ u(0,x) = u_0(x), \ 0 \le u_0 \le 1 \end{cases} \tag{1.14}$$

where u_0 is measurable, $u_0 \not\equiv 0$ and

$$u_0(x) \le C|x|^{-2} \text{ for all } x \in \mathbb{R}$$

for some constant C. Then, for all $\lambda \in (0,1)$ there exists a constant $C_\lambda > 1$ and a time $t_\lambda > 0$ such that, for all $t > t_\lambda$,

$$\{|x| > C_\lambda e^{\frac{t}{2}}\} \subset \{x \in \mathbb{R} : u(t,x) < \lambda\} \quad \text{and} \quad \{|x| < C_\lambda^{-1} e^{\frac{t}{2}}\} \subset \{x \in \mathbb{R} : u(t,x) > \lambda\}.$$

The reason that Theorem 1.2.4 is stated for the fractional Laplacian with parameter 1 (and not general α) is because the 1-stable process has an exact transition density, which was exploited in their proof.

Let us briefly compare Theorem 1.2.4 to Theorem 1.2.2 in the case of $\alpha = 1$ and $g(u) = u(1-u)$. Theorem 1.2.2 tells us that the spreading speed of equation (1.14) will be at most $\mathcal{O}(e^{\sigma' t})$ for $\sigma' > \sigma$. Therefore a spreading speed on the order of, say, $t^p e^{\sigma t}$ for some $p > 0$ is possible (in fact, if one linearises equation (1.14) and predicts the spreading speed using elementary techniques, they will obtain the speed of $t^{\frac{1}{1+\alpha}} e^{\sigma^* t}$, see [CR13, p. 683]). Theorem 1.2.4 eliminates this possibility in the case of $\alpha = 1$, showing that the spreading speed is precisely of order $e^{\sigma t}$, where $\sigma = \frac{g'(0)}{1+\alpha} = \frac{1}{2}$.

1.2.1 Main result

In this work, we will track the level sets of solutions to equation (1.13), taking similar initial conditions to those in Theorem 1.2.3. We will prove a similar result to Theorem 1.2.4, but for all $\alpha \in (0, 2)$ and for a general class of nonlinearities. The precise form of our result is stated below. Let $\mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$ denote the set of bounded and uniformly continuous functions $v : \mathbb{R} \to \mathbb{R}$ for which $-(-\Delta)^{\frac{\alpha}{2}} v$ is well-defined.

Theorem 1.2.5. *Suppose $g \in C^1([0, 1])$ is concave, $g(0) = g(1) = 0$ and $g'(1) < 0 < g'(0)$. Let $\sigma^* := \frac{g'(0)}{\alpha}$ and, for all $t > 0$ and $x \in \mathbb{R}$, suppose $u(t, x)$ is a solution of the initial value problem*

$$\begin{cases} \partial_t u = -(-\Delta)^{\frac{\alpha}{2}} u + g(u) \\ u(0, x) = u_0(x), \ \ 0 \le u_0 \le 1. \end{cases} \tag{1.15}$$

where $u_0 \in \mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$ satisfies

$$\widetilde{m} := \inf_{x \le 0} u_0(x) > 0 \quad \textit{and} \quad u_0(x) \le C(1 + x)^{-\alpha} \textit{ for all } x > 0$$

for some $C > 0$. For all $\varepsilon > 0$, there exists $t_\varepsilon, C_\varepsilon, C'_\varepsilon > 0$ such that, for all $t \ge t_\varepsilon$,

$$\{x \le C_\varepsilon e^{\sigma^* t}\} \subseteq \{x \in \mathbb{R} : u(t, x) > 1 - \varepsilon\} \textit{ and } \{x \ge C'_\varepsilon e^{\sigma^* t}\} \subseteq \{x \in \mathbb{R} : u(t, x) < \varepsilon\}.$$

In contrast to [CR13], which uses PDE techniques (and, in particular, the maximum principle), we will use purely probabilistic methods to prove Theorem 1.2.5. We employ a Feynman–Kac representation of solutions to (1.15), and follow broadly the work of [Pen18] to find alternating upper and lower bounds on u.

Theorem 1.2.5 tells us that the spreading speed of equation (1.15) is precisely of order $e^{\sigma^* t}$, where σ^* is the same exponent given in Cabré and Roquejoffre's result, Theorem 1.2.3. This parallels Theorems 1.2.2 and 1.2.4 for equation (1.14), since the former result predicted a spreading speed of order at most $e^{\sigma' t}$ for $\sigma' > \sigma$, and the latter result showed the spreading speed was of precisely order $e^{\sigma t}$ (in one specific case).

1.3 Outline

Chapters 2: Here we provide background on α-stable processes, α-stable subordinators, and travelling waves.

Chapter 3-5: This work is joint with Prof Alison Etheridge and Dr Ian Letter. In Chapter 3 we begin by stating the precise forms of Theorems 1.1.1, 1.1.2 and 1.1.3. We then describe the probabilistic dual to the fractional Allen–Cahn equation, and go on to prove the main one-dimensional result. To conclude this chapter, we present two results that will be needed in the multidimensional case (these are the 'slope of the interface' and coupling results). In Chapter 4, we prove our main deterministic multidimensional result. Finally, in 5 we prove the stochastic multidimensional result.

Chapter 6: This chapter consists of joint work with Dr Sarah Penington. We prove the spreading speed result (Theorem 1.2.5) in several stages. First, we present some standard results on the Feynman–Kac formula and stable processes that will be needed. We go on to prove some general properties on the growth and spread of solutions to our fractional reaction-diffusion equation, and use this to prove the main theorem.

Appendices A and B: In Appendix A and Appendix B we provide supplementary material for Chapters 3-5 and Chapter 6, respectively.

Chapter 2

Background

In this chapter we present some general theory on α-stable processes (Section 2.1) and reaction-diffusion equations (Section 2.2). In the first section, we are primarily interested in stable processes and stable subordinators. However, we will state results for the more general class of Lévy processes when convenient to do so. In particular, results like the Lévy-Itô decomposition will provide us with an intuitive description of the paths of Lévy processes (and hence stable processes). To conclude Section 2.1, we present two results that relate stable processes to Brownian motion. These results will be fundamental to the forthcoming chapters. In Section 2.2, we provide background material for Chapter 6. To begin, we recall several classical results on the speed of travelling wave solutions to the Fisher-KPP equation. We then explore how these results change when the ordinary Laplacian in the Fisher-KPP equation is replaced by the generator of an α-stable process.

2.1 Stable processes

In this section, we introduce α-stable processes and α-stable subordinators, along with several useful tools in their analysis. These processes are part of a large class of processes known as Lévy processes. There are many excellent references on these topics, see [Wat13, Sat99, Lal07, Pap08].

2.1.1 First definitions

To begin, we define Lévy processes and subordinators. We work on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ throughout.

Definition 2.1.1 (Lévy process, Lévy subordinator)**.** A continuous-time process $(X_t)_{t\geq 0}$ taking values in $\mathbb{R}^d$ is a *Lévy process* if it is càdlàg and enjoys both stationary and independent increments, i.e.

(1) for $0 \le s \le t$, $X_t - X_s$ is equal in distribution to $X_{t-s} - X_0$;

(2) for all $0 = t_0 \le t_1 \le \cdots \le t_k$, the increments $X_{t_i} - X_{t_{i-1}}$ are independent.

A *Lévy subordinator* is a Lévy process on $\mathbb{R}$ with non-decreasing sample paths.

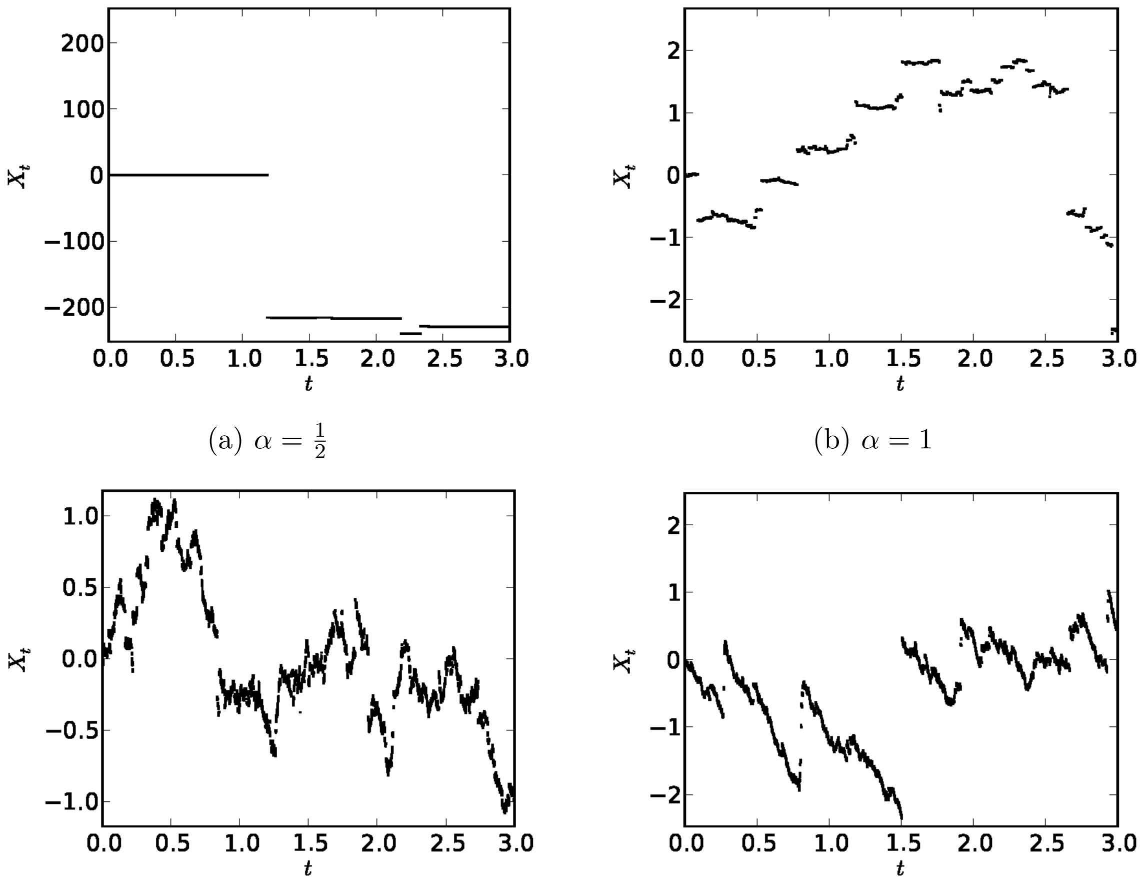
We refer to a Lévy process defined on $\mathbb{R}^{\mathrm{d}}$ as a $\mathbb{d}$-dimensional Lévy process. The prototypical examples of a Lévy process and a Lévy subordinator are a standard Brownian motion and a Poisson process, respectively. Indeed, a standard Brownian motion $(W_t)_{t\ge 0}$ can be defined as a Lévy process with the additional 'self-similarity property', namely, W_t is equal in distribution to $\sqrt{t}W_1$ for all $t > 0$. By generalising the self-similarity property of Brownian motion to indices $\alpha < 2$, we arrive at the definition of an α-stable process and α-stable subordinator. Let $\overset{D}{=}$ denote equality in distribution.

Definition 2.1.2 (Stable process, stable subordinator)**.** Fix $\alpha \in (0, 2]$. An *α-stable process* is a Lévy process $(X_t)_{t\ge 0}$ such that $X_t \overset{D}{=} t^{\frac{1}{\alpha}} X_1$ for all $t > 0$. An *α-stable subordinator* is an α-stable process on $\mathbb{R}$ with $0 < \alpha < 1$ that is also a subordinator.

Assumption 2.1.3. We will assume throughout this thesis that all α-stable processes (that are not subordinators) are *symmetric*, namely, $X_t \overset{D}{=} -X_t$ for all $t \ge 0$. In dimension $\mathbb{d} \ge 2$, we assume all stable processes are *rotation invariant*, i.e. $X_t \overset{D}{=} UX_t$ for every orthogonal matrix U (this is equivalent to symmetry in dimension one).

Lévy processes satisfying the scaling property from Definition 2.1.2 are sometimes referred to in the literature as *strictly* α-stable processes. This terminology is used to distinguish between stable processes in the sense of Definition 2.1.2 and a slightly more general definition that will not appear in this work - the interested reader should see [Kyp06].

As we have seen, a 2-stable process is equal in distribution to a standard Brownian motion. To exclude this case, we henceforth assume that all stable processes referred to in this text have index $\alpha < 2$. Unlike Brownian motion, α-stable processes are almost surely discontinuous, and experience an infinite number of discontinuities (or 'jumps') in any bounded time interval. Notably, α-stable processes are heavy tailed, in the sense that they only have moments of order strictly less than α [Kyp06]. In particular, since $\alpha < 2$, all α-stable processes have infinite variance, and when $\alpha < 1$, the expected value is not well-defined. This alludes to a behaviour that we will explore



(a) $\alpha = \frac{1}{2}$ (b) $\alpha = 1$

(c) $\alpha = \frac{3}{2}$ (d) $\alpha = \frac{3}{2}$

Figure 2.1: Simulations of symmetric α-stable processes with the given value of α, courtesy of Dr Alexander Watson [Wat13]

more rigorously below: as α decreases, the 'large' jumps of an α-stable process become more prevalent (see Figure 2.1).

Outside of the cases when $\alpha = 2$ (corresponding to Brownian motion), $\alpha = 1$ (the Cauchy distribution) and $\alpha = \frac{1}{2}$ (the Lévy distribution) there are not closed form expressions of the densities of α-stable processes. Instead, the density of a stable process can be written as a convergent power series, see [Sat99]. In practice, it is usually more convenient to use estimates for the stable density. It was shown by Lévy in 1925 [Lév25] that the tails of an α-stable distribution obey a Pareto law. That is, if X is an α-stable random variable centred about zero, then, as $x \to \infty$,

$$\mathbb{P}(X > x) \sim Cx^{-\alpha}$$

for a fixed constant C. Much later, [Kol00] proved the following estimates for the transition density p_{X_t} of a $\mathbb{d}$-dimensional α-stable process $(X_t)_{t\geq 0}$. Here, we state the version found in [CR13]. Let $|\cdot|$ denote the standard Euclidean norm in $\mathbb{R}^{\mathrm{d}}$.

Lemma 2.1.4. *There exists $B > 1$ such that transition density $p_{X_t}(x)$ of $(X_s)_{s\geq 0}$ at time t started at $X_0 = 0$ satisfies*

$$\frac{B^{-1}}{t^{\frac{\mathrm{d}}{\alpha}} + t^{-1}|x|^{\mathrm{d}+\alpha}} \leq p_{X_t}(x) \leq \frac{B}{t^{\frac{\mathrm{d}}{\alpha}} + t^{-1}|x|^{\mathrm{d}+\alpha}}$$

for all $t > 0$ and $x \in \mathbb{R}^{\mathrm{d}}$.

To conclude this section, we describe the generator of an α-stable process on $\mathbb{R}^{\mathrm{d}}$ for $\mathrm{d} \geq 1$, called the *fractional Laplacian* and denoted $-(-\Delta)^{\frac{\alpha}{2}}$. We restrict our attention to the case of $\alpha \in (1, 2)$ and note that a similar identity holds for $\alpha \leq 1$. Let $\mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$ be the domain of $-(-\Delta)^{\frac{\alpha}{2}}$ and $C_\infty(\mathbb{R}^{\mathrm{d}})$ denote the Banach space of continuous functions $f : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$ for which $\lim_{|x|\to\infty} f(x) = 0$ under the uniform norm. Denote by $C^2_\infty(\mathbb{R}^{\mathrm{d}})$ the subspace of functions $f \in C_\infty(\mathbb{R}^{\mathrm{d}})$ that are twice-differentiable with first and second derivatives belonging to $C_\infty(\mathbb{R}^{\mathrm{d}})$. Then, for functions $u \in C^2_\infty(\mathbb{R}^{\mathrm{d}}) \subset \mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$, the fractional Laplacian is given by

$$-(-\Delta)^{\frac{\alpha}{2}} u(x) := C_\alpha \lim_{\delta\to 0} \int_{\mathbb{R}^{\mathrm{d}} \setminus B_\delta(x)} \frac{u(y) - u(x)}{|y - x|^{\mathrm{d}+\alpha}} dy, \tag{2.1}$$

where $C_\alpha := \frac{\alpha 2^{\alpha-1}\Gamma\left(\frac{\alpha+\mathrm{d}}{2}\right)}{\pi^{\frac{\mathrm{d}}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)}$ and $B_\delta(x) \subset \mathbb{R}^{\mathrm{d}}$ is the δ-sphere about x. For more on the fractional Laplacian, see [Sat99, Example 32.7] or [BSW14, Example 1.25].

2.1.2 The Lévy-Itô decomposition

A powerful tool in the study of Lévy processes is their characteristic exponent. For any Lévy process $(X_t)_{t\geq 0}$, the characteristic exponent $\Psi : \mathbb{R} \to \mathbb{R}$ defined by

$$e^{-t\Psi(\theta)} = \mathbb{E}\left[\exp\left(i\theta X_t\right)\right]$$

for all $t \geq 0$ takes on a particularly nice form, described by the Lévy-Khintchine formula. We will see that, in addition to being useful in calculations, this result provides us with deep insights into the path structure of a Lévy process. Although this result holds for d-dimensional Lévy process, for convenience we restrict our attention to one dimension.

Theorem 2.1.5 (Lévy-Khintchine formula)**.** *Let $(X_t)_{t\geq 0}$ be a one-dimensional Lévy process. Then there exists $a \in \mathbb{R}$, $\sigma \geq 0$ and a measure ν with no atom at 0 satisfying $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$ such that the characteristic exponent Ψ of $(X_t)_{t\geq 0}$ satisfies*

$$\Psi(\theta) = ia\theta + \tfrac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x \mathbb{1}_{\{|x|<1\}}\right) \nu(dx) \tag{2.2}$$

for all $\theta \in \mathbb{R}$.

For reasons that will become clear, the constants a and σ from (2.2) are referred to as the *centre* and *Brownian coefficient*, respectively. The measure ν is called the *Lévy measure* of the process $(X_t)_{t\geq 0}$. Informally, the Lévy measure describes the expected number of jumps of a certain size made by $(X_t)_{t\geq 0}$ in a unit time interval [Pap08]. The condition that $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$ ensures ν has bounded mass away from 0, so $(X_t)_{t\geq 0}$ will only make a finite number of 'large' jumps (jumps of size greater than one, say) in a finite time interval. For α-stable processes, $\nu(-\varepsilon, \varepsilon) = \infty$ for all $\varepsilon > 0$, so stable processes have infinitely many 'small' jumps.

Examples 2.1.6. (1) The Lévy measure of a Lévy process $(X_t)_{t\geq 0}$ is identically zero if and only $(X_t)_{t\geq 0}$ is Gaussian [Sat99].

(2) The characteristic exponent of a one-dimensional (symmetric, strictly) α-stable process is given by

$$\Psi(\theta) = \begin{cases} c|\theta|^\alpha & \text{if } \alpha \in (0,1) \cup (1,2) \\ c|\theta| & \text{if } \alpha = 1 \end{cases} \tag{2.3}$$

for some $c > 0$ [Kyp06]. It can be shown that Ψ satisfies (2.2) by setting $\sigma = 0$, $a = 0$ and $\nu(dx) := c'|x|^{-1-\alpha}$ for some $c' > 0$.

(3) For $\alpha \in (0, 2)$, the characteristic exponent of an $\frac{\alpha}{2}$-stable subordinator is given (up to a multiplicative constant) by

$$\Psi(-\theta) = -\theta^{\frac{\alpha}{2}} = \frac{\alpha}{2\Gamma\left(1 - \frac{\alpha}{2}\right)} \int_0^\infty \left(e^{-i\theta x} - 1\right) x^{-1-\frac{\alpha}{2}}\, dx$$

for $\theta \geq 0$ [Sat99, Example 32.6].

(4) The notion of a *truncated* stable process will be crucial to our later work. Given a stable process $(X_t)_{t\geq 0}$ with Lévy measure ν, we define the truncated stable process at level $l > 0$ (or the *l-truncated stable process*) to be the process with Lévy measure $\nu(dx)\mathbb{1}_{\{|x|\leq l\}}$. The truncated stable process has finite exponential moments [Sat99]. See [KS07] for an introduction to truncated stable processes. Truncated stable subordinators can be defined similarly.

Remark 2.1.7. Truncated stable processes in the sense of Example 2.1.6 (4) are not to be confused with *tempered stable processes*, also known as *truncated Lévy flights* in the physics literature. Similar to truncation, the tempering of a stable process is a technique used to approximate stable distributions by a process with finite variance. It is obtained by multiplying the α-stable Lévy measure with an exponential, and results in both Gaussian and stable trends (seen in the tail and bulk of the distribution, respectively). The interested reader should see [Ros07] for an introduction to tempered stable processes.

The Lévy-Khintchine formula can be used to understand the path structure of any Lévy process in terms of three simpler, independent Lévy processes. This discussion is called the Lévy-Itô decomposition. Here, we follow the exposition of [Pap08]. Let $(X_t)_{t\geq 0}$ be a one-dimensional Lévy process with characteristic exponent given by (2.2). We may decompose Ψ as the sum $\Psi = \Psi^1 + \Psi^2 + \Psi^3 + \Psi^4$ where

$$\begin{aligned}
&\Psi^1(\theta) = ia\theta, \quad \Psi^2(\theta) = \tfrac{1}{2}\sigma^2\theta^2,\\
&\Psi^3(\theta) = \int_{|x|\geq 1}(1-e^{i\theta x})\nu(dx),\\
&\Psi^4(\theta) = \int_{|x|<1}(1-e^{i\theta x}+i\theta x)\nu(dx).
\end{aligned}$$

Here, the choice of one in the integral bounds $\{|x|\geq 1\}$ and $\{|x|<1\}$ is arbitrary: what is important is that we have distinguished between the 'small' and 'large' jumps of $(X_t)_{t\geq 0}$. Each of Ψ^1, Ψ^2, Ψ^3 and Ψ^4 are the characteristic exponents of independent Lévy processes X_t^1, X_t^2, X_t^3 and X_t^4 respectively, that satisfy $X_t \stackrel{D}{=} X_t^1+X_t^2+X_t^3+X_t^4$. The first three of these processes are straightforward to describe: X^1 is a deterministic drift term, X^2 is equal in law to a Brownian motion with coefficient σ, and X^3 is a compound Poisson process with Poisson parameter $\nu(\mathbb{R}\backslash(-1,1))$ and compounding distribution $F(dx) = \frac{\nu(dx)}{\nu(\mathbb{R}\backslash(-1,1))}\mathbb{1}_{|x|\geq 1}$. That is, if $\{\zeta_i\}_{i\geq 1}$ is a collection of i.i.d. random variables with distribution F that are independent of a Poisson process $(N_t)_{t\geq 0}$ with parameter $\nu(\mathbb{R}\backslash(-1,1))$, then

$$X_t^3 = \sum_{i=0}^{N_t}\zeta_i.$$

For a review of compound Poisson processes, see [Kyp06]. The final process X^4 reflects the 'small jumps' of $(X_t)_{t\geq 0}$, of which there can be infinitely many. For this reason, extra care must be taken when describing X^4. Let $\Delta X_t^4 = X_t^4 - X_{t-}^4$ and define, for all $\varepsilon > 0$, the compensated compound Poisson process

$$X_t^{(4,\varepsilon)} := \sum_{0\leq s\leq t}\Delta X_s^4 \mathbb{1}_{\{\varepsilon<|\Delta X_s^4|<1\}} - t\int_{\varepsilon<|x|<1} x\nu(dx).$$

One can show that the characteristic exponent of $X^{(4,\varepsilon)}$ is

$$\Psi^{(4,\varepsilon)}(\theta) = \int_{\varepsilon<|x|<1}\left(1-e^{i\theta x}-i\theta x\right)\nu(dx).$$

Taking $\varepsilon \to 0$, standard martingale arguments imply $X^{(4,\varepsilon)} \to X^4$ uniformly on a closed time interval with probability one and $\Psi^{(4,\varepsilon)} \to \Psi^4$.

One consequence of the Lévy-Itô decomposition is that 'large' jumps of a stable process can be described as a compound Poisson process. In particular, we have the following (see, for instance, [Lal07]).

Lemma 2.1.8. *Fix $y > 0$ and let $(X_s)_{s\geq 0}$ be a one-dimensional α-stable process. Let N_t denote the number of jumps of size at least y made by $(X_s)_{s\geq 0}$ up to time t. Then $(N_t)_{t\geq 0}$ is a Poisson process with mean $c_\alpha t y^{-\alpha}$ for some $c_\alpha > 0$.*

If we replace the stable process in Lemma 2.1.8 by an $\frac{\alpha}{2}$-stable subordinator, an almost identical result holds, but for a Poisson process with mean $\mathcal{O}(ty^{-\frac{\alpha}{2}})$. We will frequently make use of Lemma 2.1.8 to estimate the first time that a stable process (or stable subordinator) made a 'large' jump.

2.1.3 Connections to Brownian motion

Any α-stable process can be described as a *subordinated* Brownian motion, and a Brownian motion can be obtained as the weak limit of a stable process with 'small jumps'. These two perspectives link stable processes to Brownian motions, and will be crucial to our later work. We explore these facts more rigorously below.

To begin, we consider stable processes as subordinated Brownian motions. This description is consistent with the simulations from Figure 2.1. The following result can be found in [Sat99, Example 30.6]. Recall that all stable processes in this text are assumed to be rotation invariant (without this, the following proposition would not be true).

Proposition 2.1.9. *Let $(W_t)_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}^d$ and $(R_t)_{t\geq 0}$ be an independent $\frac{\alpha}{2}$-stable subordinator for $\alpha \in (0,2)$. Then $(W(R_t))_{t\geq 0}$ is an α-stable process on $\mathbb{R}^d$. Conversely, all α-stable processes can be obtained in this way.*

Proposition 2.1.9 will be exploited throughout this thesis. By conditioning on the behaviour of the stable subordinator and using Proposition 2.1.9, we will be able to describe the motion of a stable processes in terms of more tractable Brownian motions.

The second connection we draw between stable processes and Brownian motions involves the 'small jumps' of a stable process. In short, if we remove all jumps of size larger than $\varepsilon > 0$ from the path of a stable process and take $\varepsilon \to 0$, we can approximate a Brownian motion. The following proposition is a consequence of a more general theorem, [AR01, Theorem 2.1]. This result also holds for d-dimensional stable processes, see, for instance [CR07, Proposition 3.2].

Proposition 2.1.10 ([AR01])**.** *Let $\varepsilon > 0$, $\alpha \in (0,2)$ and $X = (X_t)_{t\geq 0}$ be an α-stable process on $\mathbb{R}$ with Lévy measure $\nu(dx) = c|x|^{-1-\alpha}$ for $c > 0$. Let $\Delta X_s := X_s - X_{s-}$ and consider the 'small jump process' of $(X_t)_{t\geq 0}$ defined by*

$$X_t^\varepsilon := X_t - \sum_{0\leq s\leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|\geq \varepsilon\}}.$$

Define

$$\sigma(\varepsilon)^2 := \int_{|x|\leq\varepsilon} x^2 \nu(dx) = \frac{c}{2-\alpha}\varepsilon^{2-\alpha}.$$

Then $\sigma(\varepsilon)^{-1} X_t^\varepsilon$ converges weakly to a standard Brownian motion as $\varepsilon \to 0$.

This decomposition, which is often exploited in simulations of stable processes, appears, for example, in [KHT10, Dia13]. Although we do not apply this result directly in our work, it does inspire much of our approach to Chapters 3-5. Notably, Proposition 2.1.10 tells us that the 'small jump' stable process must be *sped up* in order to approximate a Brownian motion. This will be reflected in the scaling of the fractional Allen–Cahn equation that we choose in Chapter 3.

2.2 Reaction-diffusion equations

In this section, we provide background for Chapter 6. Recall the Fisher–KPP equation is given by

$$\partial_t u(t,x) = \tfrac{m}{2}\Delta u(t,x) + su(t,x)(1-u(t,x)) \;\forall\; t>0,\, x\in\mathbb{R} \tag{2.4}$$

with initial condition $u_0(x) := u(0,x) \in [0,1]$. The Fisher-KPP equation is a classical example of a partial differential equation that admits *travelling wave solutions*, namely, solutions of the form

$$u(t,x) = \omega(x-\mu t) \;\forall\; t>0,\, x\in\mathbb{R}$$

for some monotone, twice continuously differentiable function $\omega : \mathbb{R} \to [0,1]$ satisfying $\omega(-\infty) = 0$, $\omega(\infty) = 1$. This function ω is called the *wave profile* and the constant μ is called the *wave speed.* It was shown in [Uch78, Bra83] that any solution to the Fisher-KPP equation with sufficient decay converges to the travelling wave solution with minimal wavespeed given by $\sqrt{2ms_0}$ and wave profile of order xe^{-x}.

It is natural to ask how (2.4) would change if the underlying model for the spread of genes in a diploid population experienced selection against heterozygotes. In this setting, the governing equation for the frequency of the gene at position x and time t is given by

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x)) \quad \forall\ t>0,\ x\in\mathbb{R} \tag{2.5}$$

where

$$f(x) := x(1-x)(2x-1+\gamma) \tag{2.6}$$

for some $\gamma > 0$. It can be shown that equation (2.5) admits travelling wave solutions for all γ (for suitable initial conditions), but the behaviour of these solutions depends greatly on the choice of γ. For $\gamma \in (0,2)$, there is an exact travelling wave solution with wave profile of order e^{-x}. When $\gamma > 2$, the travelling wave solution with minimal wave speed has profile of order xe^{-x}. This change in behaviour between $\gamma \in (0,2)$ to $\gamma > 2$ corresponds to a transition from *pushed* to *pulled* waves. Waves are said to be pushed when the per-capita growth rate of individuals is highest in the bulk of the wave (this might occur in populations that benefit from cooperation), and pulled when individuals at the front of the wave experience a higher growth rate (e.g. due to a lack of competition). See [GGHR12, EP22] for a comprehensive review of these results.

We now turn our attention to travelling wave solutions of reaction-diffusion equations with fractional Laplacian:

$$\partial_t u(t,x) = -(-\Delta)^{\frac{\alpha}{2}} u(t,x) + f(u(t,x)) \ \forall\ t>0,\ x\in\mathbb{R}. \tag{2.7}$$

When f is defined as in (2.4), we call (2.7) the fractional Fisher-KPP equation, and when f is given by (2.6), we call it the fractional Allen–Cahn equation. We refer the reader to [Imb05, DI06] for proofs that solutions to these equations exist. As we shall see, the existence (or lack thereof) of travelling wave solutions to the Fisher-KPP and Allen–Cahn equations changes drastically when the ordinary Laplacian is replaced by the fractional Laplacian.

In [CR13], Cabré and Roquejoffre proved that the fractional Fisher-KPP equation does not admit travelling wave solutions. More generally, they showed that for any nonlinearity f satisfying

$$f \in C^1([0,1]) \text{ is concave, } f(0)=f(1)=0 \text{ and } f'(1)<0<f'(0), \tag{2.8}$$

the front position of solutions to (2.7) varies exponentially in time, and consequently there does not exist a travelling wave solution. An interesting generalisation of this result, proved by Gui and Huan [GH15] using PDE techniques, illustrates the delicate balance between the index α of the fractional Laplacian, and the order of the polynomial f.

Theorem 2.2.1 ([GH15])**.** *Assume there exists some $\theta \in (0,1)$, $A_1, A_2 > 0$ and $0 < p < \infty$ such that*

$$\begin{cases} f(u) > 0 = f(0) = f(1) \quad \forall\, u \in (0,1), \;\; f'(1) < 0 \\ A_1 u^p \leq f(u) \leq A_2 u^p \quad \forall\, u \in [0,\theta], \\ f'(u) \geq A_1 u^{p-1} \quad \forall\, u \in (0,\theta). \end{cases} \tag{2.9}$$

Then (2.7) has a travelling wave solution if and only if $p > 2$ and $\alpha \geq \frac{p}{p-1}$.

Unlike the fractional Fisher-KPP equation, the fractional Allen–Cahn equation does admit travelling wave solutions (in some cases). In [GZ15], the authors prove existence and uniqueness of travelling wave solutions to equation (2.7) for all $\alpha \in (0,2)$ if $f \in C^2(\mathbb{R})$ is a bistable potential, corresponding to the case when $\gamma \in (0,1)$ in equation (2.6). By Theorem 2.2.1, when $\gamma \geq 1$ equation (2.5) does not admit a travelling wave solution. This is consistent with our intuition of pulled and pushed waves: when γ is small, the behaviour of the wave is dominated by those individuals in the bulk, where their long-range dispersal will have less of an effect. Conversely, for large γ, the wave is 'pulled' along by those individuals in the front, which should increase the speed of the advancing wave in the presence of long-range dispersal. In fact, when $\gamma \geq 3$, the Allen–Cahn type nonlinearity (2.6) satisfies (2.8), so [CR13] tells us that the position of the front is exponential in time in this case. In Chapter 6, we will review the result of [CR13] in more detail, and prove a stronger version of their result using probabilistic techniques.

Chapter 3

Majority voting in one dimension

To begin this chapter, we formally state our main results in Section 3.1. Then, in Section 3.2, we go on to construct a probabilistic representation of solutions to the fractional Allen–Cahn equation. Finally, in Section 3.3 we prove a one-dimensional analogue of our main deterministic result which will be needed to prove the multi-dimensional result in Chapter 4. To conclude this chapter, we will prove two other results required in the multidimensional setting (namely the 'slope of the interface' and 'coupling' results in Subsections 3.3.3 and 3.3.4).

3.1 Main Results

To begin, we recall the definition of the mean curvature at a point on a hypersurface $M \subset \mathbb{R}^{\mathrm{d}}$. Let $n : M \to \mathbb{R}^{\mathrm{d}}$ be the Gauss map, i.e. the map that assigns to each point $p \in M$ the outward facing unit vector $n(p)$ orthogonal to the tangent space of M at p, denoted T_pM. By choosing an appropriate orthonormal basis of the tangent space T_pM for all $p \in M$, we can define the shape operator $\mathbb{S}_p$ at p (locally) as the negative Jacobian of n at p. It can be shown that there exists an inner product on T_pM (called the first fundamental form) and $\mathbb{S}_p$ can be diagonalised, $\mathbb{S}_p = \mathrm{diag}(\kappa_1(p), ..., \kappa_{\mathrm{d}-1}(p))$. The *mean curvature at* p is then

$$\kappa(p) := \frac{1}{\mathrm{d}-1} \sum_{i=1}^{\mathrm{d}-1} \kappa_i(p).$$

Definition 3.1.1 (Motion by mean curvature flow)**.** Fix $\mathscr{T} > 0$. Let $S^{\mathrm{d}-1} \subset \mathbb{R}^{\mathrm{d}}$ be the unit sphere and $(\mathbf{\Gamma}_t)_{0 \le t < \mathscr{T}}$ be a family of smooth embeddings $S^{\mathrm{d}-1} \to \mathbb{R}^{\mathrm{d}}$. Let $\mathbf{n} = \mathbf{n}_t(\phi)$ be the unit inward normal vector to $\mathbf{\Gamma}_t$ at ϕ and let $\kappa = \kappa_t(\phi)$ be the mean curvature of $\mathbf{\Gamma}_t$ at ϕ. Then $(\mathbf{\Gamma}_t)_{0 \le t < \mathscr{T}}$ is a *mean curvature flow* if, for all t and ϕ,

$$\frac{\partial \mathbf{\Gamma}_t(\phi)}{\partial t} = \kappa_t(\phi)\mathbf{n}_t(\phi). \tag{3.1}$$

It can be shown that mean curvature flow in dimension $\mathbb{d} = 2$ (also called curve shortening flow) terminates after a finite time $\mathscr{T}$, and by the theorems of Gage and Hamilton (1986) and Grayson (1987), any smoothly embedded closed curve shrinks to a point as $t \uparrow \mathscr{T}$. When $\mathbb{d} > 2$, this does not always hold as singularities may develop. Following Chen [Che92], we shall impose sufficient regularity conditions to ensure the existence of a finite time $\mathscr{T}$ before which the mean curvature flow exists and does not develop a singularity. For an overview of existence results for mean curvature flow see [EFP17].

3.1.1 The deterministic result

Let $\mathbb{d} \geq 1$ and denote the standard Euclidean distance in $\mathbb{R}^{\mathbb{d}}$ by $|\cdot|$. Recall from Chapter 2 that the fractional Laplacian is defined on functions $u : \mathbb{R}^{\mathbb{d}} \to \mathbb{R}$ with sufficient decay by

$$-(-\Delta)^{\frac{\alpha}{2}} u(x) := C_\alpha \lim_{r \to 0} \int_{\mathbb{R}^{\mathbb{d}} \setminus B_r(x)} \frac{u(y) - u(x)}{|y - x|^{\mathbb{d}+\alpha}} dy,$$

where $C_\alpha := \frac{\alpha 2^{\alpha-1} \Gamma\left(\frac{\alpha+\mathbb{d}}{2}\right)}{\pi^{\frac{\mathbb{d}}{2}} \Gamma\left(1-\frac{\alpha}{2}\right)}$ and $B_r(x) \subset \mathbb{R}^{\mathbb{d}}$ is the sphere of radius r about x. Our first objective is to show that in dimension $\mathbb{d} \geq 2$, for suitable initial conditions, the solution of the scaled fractional Allen–Cahn equation

$$\begin{cases} \partial_t u^\varepsilon = -\sigma_\alpha I(\varepsilon)^{\alpha-2} (-\Delta)^{\frac{\alpha}{2}} u^\varepsilon + \varepsilon^{-2} u^\varepsilon (1 - u^\varepsilon)(2u^\varepsilon - 1), \quad t \geq 0,\ x \in \mathbb{R}^{\mathbb{d}} \\ u^\varepsilon(0, x) = p(x), \quad x \in \mathbb{R}^{\mathbb{d}} \end{cases} \tag{3.2}$$

converges as $\varepsilon \to 0$ to the indicator function of a set whose boundary evolves according to mean curvature flow. Here,

$$\sigma_\alpha := \left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} \Gamma\left(1 - \tfrac{\alpha}{2}\right) \tag{3.3}$$

is a normalising constant that will simplify our later calculations, and I can be any function satisfying the following.

Assumptions 3.1.2. For some $\delta > 0$, assume that $I : (0, \delta) \to (0, \infty)$ satisfies

(A) $\lim_{\varepsilon \to 0} I(\varepsilon) |\log \varepsilon|^k = 0 \quad \forall\, k \in \mathbb{N}$,

(B) $\lim_{\varepsilon \to 0} \frac{\varepsilon^2}{I(\varepsilon)^2} |\log \varepsilon| = 0$,

(C) $\lim_{\varepsilon \to 0} \frac{I(\varepsilon)^{2\alpha}}{\varepsilon^2} |\log \varepsilon|^\alpha = 0$.

The rate of convergence and width of the 'solution interface' in our convergence result will ultimately depend on this choice of I. Note that Assumption 3.1.2 (B) and Assumption 3.1.2 (C) are incompatible as soon as $\alpha \leq 1$. When $\alpha > 1$ a standing example that fulfills all the conditions is $I(\varepsilon) = \varepsilon|\log \varepsilon|$.

Remark 3.1.3. Equation (3.2) can be obtained from the unscaled fractional Allen–Cahn equation by scaling time and space by

$$t \mapsto \varepsilon^2 t \text{ and } x \mapsto \left(\sigma_\alpha I(\varepsilon)^{\alpha-2}\varepsilon^2\right)^{\frac{1}{\alpha}} x.$$

Using the definition of σ_α from (3.3), as $\alpha \to 2^-$, $(\sigma_\alpha I(\varepsilon)^{\alpha-2}\varepsilon^2)^{\frac{1}{\alpha}} x \to \varepsilon x$ which is consistent with the diffusive scaling considered in the Brownian setting of [EFP17]. We discuss the origin of our chosen scaling more in Section 3.2.1.

Our assumptions on the initial condition p in (3.2) mirror those in [EFP17]. First, p is assumed to take values in $[0, 1]$. Set

$$\mathbf{\Gamma}_0 := \left\{x \in \mathbb{R}^{\mathrm{d}} : p(x) = \tfrac{1}{2}\right\}. \tag{3.4}$$

Assume $\mathbf{\Gamma}_0$ is a smooth hypersurface and is the boundary of an open set homeomorphic to a sphere. Further suppose the following.

Assumptions 3.1.4. Let $p : \mathbb{R}^{\mathrm{d}} \to [0, 1]$ and $\mathbf{\Gamma}_0$ be as in (3.4). Denote the shortest Euclidean distance between a point $x \in \mathbb{R}^{\mathrm{d}}$ and the surface $\mathbf{\Gamma}_0$ by $\mathrm{dist}(x, \mathbf{\Gamma}_0)$.

(A) $\mathbf{\Gamma}_0$ is C^a for some $a > 3$.

(B) For x inside $\mathbf{\Gamma}_0$, $p(x) < \frac{1}{2}$, and for x outside $\mathbf{\Gamma}_0$, $p(x) > \frac{1}{2}$.

(C) There exist $r, \gamma > 0$ such that, for all $x \in \mathbb{R}^{\mathrm{d}}$, $\left|p(x) - \frac{1}{2}\right| \geq \gamma(\mathrm{dist}(x, \mathbf{\Gamma}_0) \wedge r)$.

Assumption 3.1.4 (B) establishes a sign convention, and Assumption 3.1.4 (C) ensures $p(x)$ is bounded away from $\frac{1}{2}$ when x is away from the interface. Together, these conditions ensure that the mean curvature flow started from $\mathbf{\Gamma}_0$, $(\mathbf{\Gamma}_t(\cdot))_{t\geq 0}$, exists up until some finite time $\mathscr{T}$.

Following [EFP17], we let $d(x, t)$ be the signed distance from $\mathbf{\Gamma}_t$ to x, which we choose to be negative inside $\mathbf{\Gamma}_t$ and positive outside $\mathbf{\Gamma}_t$. Then, as sets,

$$\mathbf{\Gamma}_t = \left\{x \in \mathbb{R}^{\mathrm{d}} : d(x, t) = 0\right\}.$$

Lastly, define $F(\varepsilon)$ by

$$F(\varepsilon) = \frac{I(\varepsilon)^2}{\varepsilon^{\frac{2}{\alpha}}}|\log \varepsilon| + I(\varepsilon)^{\alpha-1}. \tag{3.5}$$

Note that, for any $\alpha \in (1,2)$ and function $I : (0,\delta) \to (0,\infty)$ satisfying Assumptions 3.1.2, $F(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The scaling function I and parameter α will be fixed throughout this work. Just as we have done for F, when defining new functions, we will typically not make explicit their dependence on the choice of I and choice of α. Our main (deterministic) theorem is the following.

Theorem 3.1.5. *Let $\alpha \in (1,2)$, $\mathbb{d} \geq 2$ and fix a function I satisfying Assumptions 3.1.2. Suppose u^ε solves equation (3.2) with initial condition p satisfying Assumptions 3.1.4. Let $\mathscr{T}$ and $d(x,t)$ be as above, F be as in (3.5) and fix $T^* \in (0,\mathscr{T})$. Then there exists $\varepsilon_{\mathbb{d}}(\alpha), a_{\mathbb{d}}(\alpha), c_{\mathbb{d}}(\alpha), m > 0$ such that, for $\varepsilon \in (0,\varepsilon_{\mathbb{d}})$ and $a_{\mathbb{d}}\varepsilon^2|\log\varepsilon| \leq t \leq T^*$,*

(1) for x with $d(x,t) \geq c_{\mathbb{d}} I(\varepsilon)|\log\varepsilon|$, we have $u^\varepsilon(t,x) \geq 1 - m\frac{\varepsilon^2}{I(\varepsilon)^2} - mF(\varepsilon)$,

(2) for x with $d(x,t) \leq -c_{\mathbb{d}} I(\varepsilon)|\log\varepsilon|$, we have $u^\varepsilon(t,x) \leq m\frac{\varepsilon^2}{I(\varepsilon)^2} + mF(\varepsilon)$.

Of course, we could have stated Theorem 3.1.5 in terms of an error function $F'(\varepsilon) := F(\varepsilon) + \frac{\varepsilon^2}{I(\varepsilon)^2}$. We choose to make the $\frac{\varepsilon^2}{I(\varepsilon)^2}$ term explicit since it will also appear in the one-dimensional analogue of Theorem 3.1.5.

Throughout this work, we often discuss the *solution interface* and its corresponding width. The term solution interface refers to the spatial region outside of which the solution $u^\varepsilon(t,\cdot)$ is very close to zero or one. Explicitly, in Theorem 3.1.5 the solution interface at time t is the set of $x \in \mathbb{R}^{\mathbb{d}}$ for which $|d(x,t)| \leq c_{\mathbb{d}} I(\varepsilon)|\log\varepsilon|$, and we call $2c_{\mathbb{d}} I(\varepsilon)|\log\varepsilon|$ the *interface width*. We will refer to the error bounds on u^ε in Theorem 3.1.5 as the *sharpness* of the interface.

Examples 3.1.6. (1) It is easy to verify that $I(\varepsilon) = \varepsilon|\log\varepsilon|$ satisfies Assumptions 3.1.2, so for this choice of I the interface width is $\mathcal{O}\left(\varepsilon|\log\varepsilon|^2\right)$. One can also check that $F(\varepsilon) = \mathcal{O}(\varepsilon^{\alpha-1}|\log\varepsilon|^{\alpha-1})$, which is dominated by $\frac{\varepsilon^2}{I(\varepsilon)^2}$, so the sharpness of the interface in Theorem 3.1.5 is $\mathcal{O}\left(\varepsilon^2 I(\varepsilon)^{-2}\right) = \mathcal{O}\left(|\log\varepsilon|^{-2}\right)$.

(2) We now provide an example in which $F(\varepsilon)$ dominates $\varepsilon^2 I(\varepsilon)^{-2}$. Set $I(\varepsilon) = \varepsilon^q$ where $q = \frac{3\alpha+1}{2\alpha(1+\alpha)}$. This choice of I satisfies Assumptions 3.1.2, and the resulting interface width and sharpness given by Theorem 3.1.5 are $\mathcal{O}\left(\varepsilon^q|\log\varepsilon|\right)$ and $\mathcal{O}\left(\varepsilon^{\frac{\alpha-1}{\alpha+1}}|\log\varepsilon|^\alpha\right)$, respectively.

We often reference the Brownian analogue of Theorem 3.1.5 proved using probabilistic techniques in [EFP17]. This result, originally due to Chen [Che92], is given as follows (here we state the version found in [EFP17, Theorem 1.3]).

Theorem 3.1.7 ([Che92]). *Let u^ε solve*

$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-2} u^\varepsilon (1-u^\varepsilon)(2u^\varepsilon - 1), & t \geq 0,\ x \in \mathbb{R}^{\mathrm{d}} \\ u^\varepsilon(0,x) = p(x), & x \in \mathbb{R}^{\mathrm{d}} \end{cases} \tag{3.6}$$

with initial condition p satisfying Assumptions 3.1.4. Let $\mathscr{T}$ and $d(x,t)$ be as above. Fix $T^* \in (0, \mathscr{T})$ and $k \in \mathbb{N}$. Then there exists $\varepsilon_{\mathrm{d}}(k), a_{\mathrm{d}}(k), c_{\mathrm{d}}(k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$ and t with $a_{\mathrm{d}}\varepsilon^2 |\log \varepsilon| \leq t \leq T^*$,

(1) for x such that $d(x,t) \geq c_{\mathrm{d}}\varepsilon|\log\varepsilon|$, we have $u^\varepsilon(t,x) \geq 1 - \varepsilon^k$,

(2) for x such that $d(x,t) \leq -c_{\mathrm{d}}\varepsilon|\log\varepsilon|$, we have $u^\varepsilon(t,x) \leq \varepsilon^k$.

Remark 3.1.8. The interface width $\mathcal{O}(\varepsilon|\log\varepsilon|)$ in Theorem 3.1.7 is not achievable in Theorem 3.1.5, but we may approximate it by choosing $I(\varepsilon) = \varepsilon^\delta$ where $\frac{1}{\alpha} < \delta < 1$, and considering $\delta \to 1$. The interface width and sharpness in this case are $\mathcal{O}(\varepsilon^\delta|\log\varepsilon|)$ and $\varepsilon^{2-2\delta}$, respectively.

We note some key differences between our theorem, Theorem 3.1.5, and Theorem 3.1.7. Firstly, in Theorem 3.1.7, the width of the solution interface is $\mathcal{O}(\varepsilon|\log\varepsilon|)$, compared to the strictly larger width of $\mathcal{O}(I(\varepsilon)|\log\varepsilon|)$ in the fractional setting. Secondly, the sharpness of the interface in Theorem 3.1.7, ε^k, is much better than the sharpness in Theorem 3.1.5. Both of these differences are consistent with our intuition by considering the (soon to be formalised) probabilistic representation of solutions to the ordinary and fractional Allen–Cahn equations in terms of Brownian and α-stable motions, respectively. In the stable case, the rare large jumps of the α-stable motion act to 'fatten' the interface compared to the Brownian case. While the effect of these large jumps is small enough that we still observe mean curvature flow (as $\alpha > 1$), it does require much longer for solutions to converge in the fractional setting.

3.1.2 Hybrid zones and the stochastic result

As explained in [EFP17], the connection between the hybrid zones and the (deterministic) Allen–Cahn equation can be motivated as follows. Consider a diploid population that is in Hardy-Weinberg proportions: if the proportion of a-alleles in the parental population is w, then the proportions of parents of type aa, aA and AA are $w^2, 2w(1-w)$ and $(1-w)^2$, respectively. To reflect the selection against heterozygotes taking place in the hybrid zone, we assign to each of the allele pairs aa, aA and AA the relative fitnesses $1, 1-s$ and 1, for $s > 0$ a small selection parameter. These fitnesses refer to the relative proportion of germ cells produced by heterozygotes and homozygotes during reproduction. It follows that if w is the proportion of type a alleles before reproduction, then the proportion of type a alleles after reproduction is

$$w^* = \frac{w^2 + w(1-w)(1-s)}{1-2sw(1-w)} = w + sw(1-w)(2w-1) + \mathcal{O}(s^2).$$

Taking $s = \frac{1}{N}$ and measuring time in units of N generations, the above calculation implies that, as $N \to \infty$, $\frac{dw}{dt} = w(1-w)(2w-1)$. Adding long range dispersal gives

$$\frac{\partial w}{\partial t} = -(-\Delta)^{\frac{\alpha}{2}} w + w(1-w)(2w-1).$$

Finally, to recover equation (3.2) we scale time and space by

$$t \mapsto \varepsilon^2 t, x \mapsto \left(\sigma_\alpha I(\varepsilon)^{\alpha-2}\varepsilon^2\right)^{\frac{1}{\alpha}} x.$$

In [EFP17] this is extended to the stochastic setting, proving that hybrid zones maintained by selection, even in the presence of random genetic drift, evolve according to mean curvature flow (provided the drift is weak enough). The biological model used to incorporate this random genetic drift was the Spatial Λ-Fleming-Viot model with selection against heterozygosity. Following [EFP17], we consider a population with a and A alleles, and define, up to a Lebesgue null set, the random function $\{w_t(x), x \in \mathbb{R}^{\mathrm{d}}\}$ by

$$w_t(x) := \text{ the proportion of type } a \text{ alleles at time } t \text{ and location } x. \tag{3.7}$$

We arbitrarily impose that $w_t(x) = 0$ on the null set for which (3.7) does not specify w_t, so that w_t is defined on all of $\mathbb{R}$. The appropriate state space for the process $x \mapsto w_t(x)$ was derived in [VW15], and shown to be in one-to-one correspondence with the space $\mathcal{M}_\lambda$ of measures on $\mathbb{R}^{\mathrm{d}} \times \{a, A\}$ with 'spatial marginal' Lebesgue measure, endowed with the topology of vague convergence. We abuse notation and denote the state space of $(w_t)_{t\in\mathbb{R}}$ by $\mathcal{M}_\lambda$ throughout. The following definition can be found in [EFP17].

Definition 3.1.9 (Spatial Λ-Fleming-Viot with selection against heterozygosity)**.** Fix $u \in (0, 1]$. Let μ be a finite measure on $(0, \infty)$ which we assume satisfies

$$\int_0^\infty r^{\mathrm{d}} \mu(dr) < \infty.$$

Further, let Π be a Poisson point process on $(0, \infty) \times \mathbb{R}^{\mathrm{d}} \times (0, \infty)$ with intensity measure

$$dt \otimes dx \otimes \mu(dr). \tag{3.8}$$

The Spatial Λ-Fleming-Viot process with selection driven by Π is the $\mathcal{M}_\lambda$-valued process $(w_t)_{t\geq 0}$ with dynamics given as follows.

If $(t,x,r)\in\Pi$, a reproduction event occurs at time t within the closed ball $B_r(x)$ of radius r centred at x. With probability $1-s$ the event is *neutral*, in which case:

(1) Choose a parental location z uniformly at random within $B_r(x)$, and a parental type, k_0, according to $w_{t-}(z)$, that is $k_0=a$ with probability $w_{t-}(z)$ and $k_0=A$ with probability $1-w_{t-}(z)$.

(2) For every $y\in B_r(x)$, set $w_t(y)=(1-u)w_{t-}(y)+u\mathbb{1}_{k_0=a}$.

With probability s the event is *selective*, in which case:

(1) Choose three 'potential' parental locations z_1,z_2,z_3 independently and uniformly at random within $B_r(x)$, and at each of these sites sample 'potential' parental types k_1,k_2,k_3 according to $w_{t-}(z_1),w_{t-}(z_2),w_{t-}(z_3)$ respectively. Let $\widehat{k}$ denote the most common allelic type in k_1,k_2,k_3.

(2) For each $y\in B_r(x)$, set $w_t(y)=(1-u)w_{t-}(y)+u\mathbb{1}_{\hat{k}=a}$.

To reflect the long range dispersal present in our population, we consider the SΛFVS with stable radii. That is, for $\alpha\in(1,2)$ we consider the SΛFVS from Definition 3.1.9 driven by the point process Π with intensity measure $dt\otimes dx\otimes\mu(dr)$ where

$$\mu(dr)=\frac{\mathbb{1}_{\{r\geq 1\}}}{r^{\mathrm{d}+\alpha+1}}dr. \tag{3.9}$$

At the n-th stage of rescaling (for $n\in\mathbb{N}$), the SΛFVS with stable radii is assumed to have impact parameter u_n and selection parameter s_n satisfying

$$u_n=\frac{u}{n^{1-\beta\alpha}},\quad\text{and}\quad s_n=\frac{\rho}{n^{\beta\alpha}}$$

for some scaling parameter $\beta\in\left(0,\frac{1}{\alpha}\right)$ and $\rho>0$ independent of n. Define the averaged process

$$\overline{w}^n_t(x)=\frac{1}{V_1}\int_{B_1(n^\beta x)}w_{nt}(y)dy \tag{3.10}$$

where V_1 is the volume of the ball of radius 1 in $\mathbb{R}^\mathrm{d}$. Denote $\mathcal{M}_\lambda(\mathbb{R}^\mathrm{d}\times\{a,A\})$ by $\mathcal{M}$ and the set of càdlàg paths with values in $\mathcal{M}$ by $D_{\mathcal{M}}[0,\infty)$. Let $C^\infty_\mathrm{c}(\mathbb{R}^\mathrm{d})$ be the set of smooth compactly supported functions on $\mathbb{R}^\mathrm{d}$. Define

$$\langle w,f\rangle=\int_{\mathbb{R}^\mathrm{d}}w(x)f(x)dx.$$

Then, by a straightforward adaptation of [EVY20, Theorem 1.14], we have the following convergence result.

Theorem 3.1.10 (Adaptation of [EVY20, Theorem 1.14]). *Let $\alpha \in (0,2)$, $\mathrm{d} \geq 2$ and $0 < \beta < \frac{1}{2\alpha-1}$. Assume that $\overline{w}_0^n$ converges weakly to some $w^0 \in \mathcal{M}$. Then as $n \to \infty$, the process $(\overline{w}_t^n)_{t\geq 0}$ converges weakly in $D_{\mathcal{M}}[0,\infty)$ to a process $(w_t^\infty)_{t\geq 0}$ with initial value $w_0^\infty = w_0$. Furthermore, $(w_t^\infty)_{t\geq 0}$ is the unique deterministic process for which, for every $f \in C_c^\infty(\mathbb{R}^{\mathrm{d}})$ and $t \geq 0$,*

$$\langle w_t^\infty, f\rangle = \langle w_0^\infty, f\rangle + \int_0^t \left\{ \langle w_s^\infty, -u(-\Delta^{\frac{\alpha}{2}})f\rangle + \frac{u\rho V_1}{\alpha}\langle w_s^\infty(1-w_s^\infty)(2w_s^\infty - 1), f\rangle \right\} ds.$$

In view of this result, it is natural to ask if the scaling of s_n can be chosen in such a way that $s_n n^{\beta\alpha} \to \infty$ as $n \to \infty$ and the asymptotic behaviour from the deterministic result is maintained. That is, we would like to obtain an analogous convergence statement to that of Theorem 3.1.5 even in the presence of genetic drift.

Fix $0 < \mathcal{R} \leq 1$, $n \in \mathbb{N}$ and set $\mathcal{R}_n = n^{-\beta}\mathcal{R}$. Define $\mu^n(A) := \mu(n^\beta A)$ for all Borel subsets $A \subset (0,\infty)$. Let Π^n be the driving Poisson point process on $(0,\infty) \times \mathbb{R}^{\mathrm{d}} \times (\mathcal{R}_n, \infty)$ in the scaled SΛFVS with stable radii having intensity measure $ndt \otimes n^\beta dx \otimes \mu^n(dr)$. Let

$$u_n = \frac{u}{n^{1-\beta\alpha} I(\varepsilon_n)^{2-\alpha}}, \quad \text{and} \quad s_n = \frac{I(\varepsilon_n)^{2-\alpha}}{\varepsilon_n^2 n^{\beta\alpha}}. \tag{3.11}$$

Let $(\overline{w}_t^n)_{t\geq 0}$ be the averaged process from (3.10), where $w_{nt}(\cdot)$ is defined as in Definition 3.1.9 but for the Poisson point process Π^n and impact and selection parameters given by u_n and s_n above.

Given a sequence $(\varepsilon_m)_{m\geq 0}$, set

$$G(n) = n^{-\beta\left(1-\frac{\alpha}{2}\right)} I(\varepsilon_n)^{\alpha-2} + \frac{\varepsilon_n^2}{I(\varepsilon_n)^2} + n^{-\frac{\beta}{2\alpha}\left(1-\frac{\alpha}{2}\right)}\varepsilon_n^{-\frac{2}{\alpha}} + I(\varepsilon_n)^{\alpha-1}. \tag{3.12}$$

Our main result in the stochastic setting is stated as follows.

Theorem 3.1.11. *Let $\alpha \in (1,2)$, $\beta \in \left(0, \frac{1}{2\alpha}\right)$, $\mathrm{d} \geq 2$ and consider a sequence $(\varepsilon_n)_{n=1}^\infty$ satisfying $\varepsilon_n \to 0$ and $(\log n)^{\frac{1}{2}}\varepsilon_n \to \infty$ as $n \to \infty$. Fix I satisfying Assumptions 3.1.2. Let G be as above and $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$ satisfy Assumptions 3.1.4. Let $(w_t^n)_{t\geq 0}$ be the SΛFVS driven by Π^n with u_n, s_n given by (3.11) and initial condition $w_0^n(x) = p(x)$. Define $\mathscr{T}$ and $d(x,t)$ as in Theorem 3.1.5 and take $T^* < \mathscr{T}$. There exists $n_*(\alpha) \in \mathbb{N}$ and $a_*(\alpha), c_*(\alpha), m_* \in (0,\infty)$ such that, for all $n \geq n_*$ and $a_*\varepsilon_n^2|\log \varepsilon_n| \leq t \leq T^*$,*

(1) for x with $d(x,t) \geq c_* I(\varepsilon_n)|\log \varepsilon_n|$, we have $\mathbb{E}[w_t^n(x)] \geq 1 - m_* G(n)$,

(2) for x with $d(x,t) \leq -c_* I(\varepsilon_n)|\log \varepsilon_n|$, we have $\mathbb{E}[w_t^n(x)] \leq m_* G(n)$.

To prove Theorem 3.1.11 in Chapter 5, we will approximate the SΛFVS dual by a subordinated Brownian motion, which then allows us to apply Theorem 3.1.5. The condition that $(\log n)^{\frac{1}{2}}\varepsilon_n \to \infty$ ensures that, with high probability, the SΛFVS dual (which has branching rate ε_n^{-2}) does not have too many lineages by a fixed time T^*.

3.2 A probabilistic dual

In this section, we define a probabilistic dual to the scaled fractional Allen–Cahn equation (3.2), following closely the work of [EFP17]. To do this, we consider a ternary-branching α-stable motion in which each individual, independently, follows an α-stable motion, until the end of its exponentially distributed lifetime (with mean ε^2) at which point it splits into three particles. Let $Y(t)$ denote a $\mathbb{d}$-dimensional α-stable process and $\boldsymbol{Y}(t)$ denote a $\mathbb{d}$-dimensional historical ternary branching α-stable motion. That is, $\boldsymbol{Y}(t)$ traces out the space-time trees that record the position of all particles alive at time s for $s \in [0,t]$. Throughout this work, we adopt the following convention. Recall that $\sigma_\alpha := \left(\frac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} \Gamma\left(1-\frac{\alpha}{2}\right)$.

Assumption 3.2.1. All α-stable motions have generator $-\sigma_\alpha I(\varepsilon)^{\alpha-2}(-\Delta)^{\frac{\alpha}{2}}$ for a fixed $\alpha \in (1,2)$.

To record the genealogy of the process we employ the Ulam-Harris notation to label individuals by elements of $\mathcal{U} = \bigcup_{m=0}^{\infty}\{1,2,3\}^m$, where $\{1,2,3\}^0 = \emptyset$ by convention. For example, $(3,1)$ represents the particle which is the first child of the third child of the initial ancestor $\emptyset$. Let $N(t) \subset \mathcal{U}$ denote the set of individuals alive at time t.

We call $\mathcal{T}$ a *time-labelled ternary tree* if $\mathcal{T}$ is a finite subset of $\mathcal{U}$ with each internal vertex v labelled with a time $t_v > 0$, where t_v is strictly greater than the label of the parent vertex of v. Ignoring the spatial position of individuals, we see that $\boldsymbol{Y}(t)$ traces out a time-labelled ternary tree which associates to each branch point the time of the branching event. Denote the time-labelled ternary tree traced out by $\boldsymbol{Y}(t)$ by $\mathcal{T}(\boldsymbol{Y}(t))$.

Definition 3.2.2 ($\mathbb{V}_p$)**.** Fix $p : \mathbb{R}^{\mathbb{d}} \to [0,1]$ and define the *majority voting procedure* on $\mathcal{T}(\boldsymbol{Y}(t))$ as follows.

(1) each leaf i of $\mathcal{T}(\boldsymbol{Y}(t))$ independently votes 1 with probability $p(Y_i(t))$ and otherwise votes 0;

(2) at each branching event in $\mathcal{T}(\boldsymbol{Y}(t))$, the vote of the parent particle j is given by the majority vote of its offspring $(j,1),(j,2),(j,3)$.

This voting procedure runs inward from the leaves of $\mathcal{T}(\boldsymbol{Y}(t))$ to the root $\emptyset$. Under this voting procedure, define $\mathbb{V}_p(\boldsymbol{Y}(t))$ to be the vote associated to the root $\emptyset$ of the ternary branching stable process.

For $x \in \mathbb{R}^{\mathrm{d}}$, we shall write $\mathbb{P}_x$ and $\mathbb{E}_x$ for the probability measure and expectation associated to the law of a stable motion starting at x. Write $\mathbb{P}^{\varepsilon}_x$ for the probability measure under which $(\boldsymbol{Y}(t), t \geq 0)$ has the law of the historical process of a ternary branching α-stable motion in $\mathbb{R}^{\mathrm{d}}$ with branching rate ε^{-2} started from a single particle at location x at time 0. We write $\mathbb{E}^{\varepsilon}_x$ for the corresponding expectation. We emphasise that the variable ε used to define the speed of the stable process, $I(\varepsilon)^{\alpha-2}$, is the same variable ε that defines the branching rate, ε^{-2}. Then the root vote $\mathbb{V}_p(\boldsymbol{Y}(t))$ provides us with a dual to equation (3.2) in the following sense.

Theorem 3.2.3. *Let $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$. Then*

$$u^{\varepsilon}(t,x) := \mathbb{P}^{\varepsilon}_x[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \tag{3.13}$$

is a solution to equation (3.2) with initial condition $u^{\varepsilon}(0,x) = p(x)$.

Sketch of proof of Theorem 3.2.3. This proof follows closely that of [EFP17, Theorem 2.2]. In this proof we neglect the superscript ε on $\mathbb{P}^{\varepsilon}_x, \mathbb{E}^{\varepsilon}_x$ and u^{ε}. Let u be as in (3.13) and consider $u(t+\delta t, x)$. If τ is the time of the first branching event, we have

$$\begin{aligned} u(t+\delta t, x) = \mathbb{P}_x[\mathbb{V}_p(\boldsymbol{Y}(t+\delta t)) = 1 \mid \tau \leq \delta t]\, \mathbb{P}[\tau \leq \delta t] \\ + \mathbb{P}_x[\mathbb{V}_p(\boldsymbol{Y}(t+\delta t)) = 1 \mid \tau > \delta t]\ (1 - \mathbb{P}[\tau \leq \delta t])\,. \end{aligned} \tag{3.14}$$

We will consider each of the terms in (3.14) separately. Let V_1, V_2, and V_3 be the votes of the three offspring created at time τ. Conditional on $\tau \leq \delta t$, the probability of a second branching event before time δt is $\mathcal{O}(\delta t)$, so for $s \leq \delta t$

$$\mathbb{E}_x\left[V_1 \mid (\tau, Y_\tau) = (s,y)\right] = \mathbb{E}_y\left[u(t, Y_{\delta t - s})\right] + \mathcal{O}(\delta t).$$

We can then approximate

$$\mathbb{E}_x[V_1 \mid \tau \leq \delta t] \approx u(t,x) + \mathcal{O}(\delta t).$$

Since the root vote of $\mathcal{T}(\boldsymbol{Y}(t))$ will equal one if and only if at most one of V_1, V_2, and V_3 is zero, it follows by conditional independence of each V_i given (τ, Y_τ) that

$$\begin{aligned}\mathbb{P}_x[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1 \mid \tau \le \delta t] &= u(t,x)^3 + 3u(t,x)^2(1-u(t,x)) + o(1) \\ &= u(t,x)(1-u(t,x))(2u(t,x)-1) + u(t,x) + o(1).\end{aligned}$$

For the second term in (3.14) note that

$$\mathbb{P}_x[\mathbb{V}_p(\boldsymbol{Y}(t+\delta t)) = 1 \mid \tau > \delta t] = \mathbb{E}_x[\mathbb{P}_{Y_{\delta t}}(\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] = \mathbb{E}_x[u(t, Y_{\delta t})]$$

by the Markov property of $\boldsymbol{Y}$ at time δt. Since $\mathbb{P}[\tau \le \delta t] \approx \delta t \varepsilon^{-2} + \mathcal{O}(\delta t^2)$, putting the above estimates into (3.14) gives

$$\begin{aligned}&\lim_{\delta t \to 0} \frac{u(t+\delta t, x) - u(t,x)}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\mathbb{E}_x[u(t, Y_{\delta t})] - u(t,x)}{\delta t} + \varepsilon^{-2} u(t,x)(1-u(t,x))(2u(t,x)-1) \\ &= -\sigma_\alpha I(\varepsilon)^{\alpha-2}(-\Delta)^{\frac{\alpha}{2}} u(t,x) + \varepsilon^{-2} u(t,x)(1-u(t,x))(2u(t,x)-1).\end{aligned}$$

□

Remark 3.2.4. To prove Theorem 3.2.3 more formally, one can adapt the proof of the analogous Brownian result that appears in this thesis [O'D19, Theorem 3.2.2, Proposition 4.2.3]. There, the author expresses (3.13) (with $\boldsymbol{Y}(t)$ replaced by a branching Brownian motion) as an explicit integral in terms of the heat kernel. The analytic arguments that follow remain true when the heat kernel is replaced by the fractional heat semigroup. We note that, using this expression, it is immediate that u^ε is in the domain of the fractional Laplacian.

With this in mind, we can restate Theorem 3.1.5 as follows.

Theorem 3.2.5. *Fix a function I satisfying Assumptions 3.1.2. Suppose the initial condition p satisfies Assumptions 3.1.4. Let $\mathscr{T}$ and $d(x,t)$ be as in Section 3.1, F be as in (3.5) and fix $T^* \in (0, \mathscr{T})$. Then there exist $\varepsilon_{\mathrm{d}}(\alpha)$, $a_{\mathrm{d}}(\alpha)$, $c_{\mathrm{d}}(\alpha)$, $m > 0$ such that, for $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$ and $a_{\mathrm{d}}\varepsilon^2 |\log \varepsilon| \le t \le T^*$,*

(1) for x with $d(x,t) \ge c_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$, $\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \ge 1 - m\dfrac{\varepsilon^2}{I(\varepsilon)^2} - mF(\varepsilon)$,

(2) for x with $d(x,t) \le -c_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$, $\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \le m\dfrac{\varepsilon^2}{I(\varepsilon)^2} + mF(\varepsilon)$.

For the sake of completeness, we mention the Brownian analogue of Theorem 3.2.3 and Theorem 3.2.5 from [EFP17]. There, the authors considered a historical ternary branching d-dimensional Brownian motion $(\boldsymbol{W}(t), t \ge 0)$ with branching rate ε^{-2}.

Let $\mathbb{P}_x^\varepsilon$ and $\mathbb{V}_p$ be defined as above but for the process $(\boldsymbol{W}(t), t \geq 0)$. Then, by [EFP17, Theorem 2.2], given $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$,

$$u^\varepsilon(t,x) := \mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{W}(t)) = 1] \tag{3.15}$$

is a solution to equation (3.6), and Theorem 3.1.7 can be restated in terms of $\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{W}(t)) = 1]$. We will refer to the restatement of Theorem 3.1.7 in terms of $\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{W}(t)) = 1]$ as the 'probabilistic version of Theorem 3.1.7'.

Notation 3.2.6. It will be convenient to distinguish between one-dimensional and multidimensional α-stable processes. We adopt the convention that $X(t)$ will denote the one-dimensional α-stable process, with the corresponding historical branching stable process denoted by $\boldsymbol{X}(t)$. When $\mathrm{d} > 1$, we denote the α-stable process by $Y(t)$ and denote the corresponding historical branching stable process by $\boldsymbol{Y}(t)$.

3.2.1 Remarks on the choice of scaling

Now that we have described the probabilistic dual to the fractional Allen–Cahn equation, we can explain the origin of the scaling taken in equation (3.2). As we will see, this choice of scaling is intimately linked to our strategy of proof for Theorem 3.2.5. To explain this, we will need to consider stable processes run at varying speeds. For this reason, in this section (and this section only) we do *not* adopt Assumption 3.2.1, so the speeds of all stable process, stable subordinators, and historical stable trees will be made explicit.

Recall from Proposition 2.1.10 that if $(Y_t^\varepsilon)_{t\geq 0}$ is a d-dimensional ε-truncated α-stable process with $\alpha \in (1,2)$, then for some $c_\alpha > 0$, we have

$$c_\alpha \varepsilon^{\frac{\alpha}{2}-1} Y_t^\varepsilon \xrightarrow{w} W_t \quad \text{as } \varepsilon \to 0 \tag{3.16}$$

where $(W_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion and $\xrightarrow{w}$ denotes weak convergence of the one-dimensional distributions. Note that

$$\varepsilon^{\frac{\alpha-2}{\alpha}} Y_t^{\varepsilon^{2/\alpha}} \overset{D}{=} Y_{\varepsilon^{\alpha-2}t}^\varepsilon, \tag{3.17}$$

where $(Y_t^{\varepsilon^{2/\alpha}})_{t\geq 0}$ denotes the $\varepsilon^{2/\alpha}$-truncated α-stable process.[1] Therefore, by replacing ε by $\varepsilon^{2/\alpha}$ in (3.16), and applying (3.17), we see that

$$c_\alpha Y_{\varepsilon^{\alpha-2}t}^\varepsilon \xrightarrow{w} W_t \quad \text{as } \varepsilon \to 0. \tag{3.18}$$

[1]To see (3.17), let ψ^ε and $\psi^{\varepsilon^{2/\alpha}}$ be the characteristic exponents of $(Y_t^\varepsilon)_{t\geq 0}$ and $(Y_t^{\varepsilon^{2/\alpha}})_{t\geq 0}$ respectively. By definition of the characteristic exponent, (3.17) will hold if and only if, for all $\theta \in \mathbb{R}$, $\psi^{\varepsilon^{2/\alpha}}(\theta\varepsilon^{1-\frac{2}{\alpha}}) = \psi^\varepsilon(\theta)\varepsilon^{\alpha-2}$, which is straightforward to verify using the Lévy-Khintchine formula (Theorem 2.1.5).

More generally, one could replace ε in (3.18) by a function $I(\varepsilon)$, satisfying $I(\varepsilon) \to 0$ as $\varepsilon \to 0$. The $I(\varepsilon)$-truncated stable process $Y^{I(\varepsilon)}_{I(\varepsilon)^{\alpha-2}t}$ will approximate a Brownian motion, so it is reasonable to expect that the probabilistic version of Theorem 3.1.7 holds when the branching Brownian motion is replaced by a branching $I(\varepsilon)$-truncated stable process, run at speed $I(\varepsilon)^{\alpha-2}$. Therefore to prove Theorem 3.2.5, it should suffice to show:

Step 1: the probabilistic version of Theorem 3.1.7 holds when $\boldsymbol{W}(t)$ is replaced by $\boldsymbol{Y}^{I(\varepsilon)}(I(\varepsilon)^{\alpha-2}t)$, a $\mathbb{d}$-dimensional ternary branching $I(\varepsilon)$-truncated stable tree run at speed $I(\varepsilon)^{\alpha-2}$;

Step 2: there exists a coupling of the root votes of $\boldsymbol{Y}(I(\varepsilon)^{\alpha-2}t)$ and $\boldsymbol{Y}^{I(\varepsilon)}(I(\varepsilon)^{\alpha-2}t)$ in such a way that Step 1 implies Theorem 3.2.5.

The purpose of this two-step approach is that, by using a truncated stable process (which is not heavy tailed) we can more readily adapt proofs from the Brownian case of [EFP17].

The discussion above explains why the fractional Laplacian in equation (3.2) is sped up by a factor of $I(\varepsilon)^{\alpha-2}$. However, we have not addressed our choice of $I(\varepsilon)$ as outlined by Assumptions 3.1.2. In particular, we require $\lim_{\varepsilon\to 0} \frac{\varepsilon}{I(\varepsilon)} = 0$, so $I(\varepsilon) \neq \varepsilon$, which might be unexpected in view of the limit (3.18). This is because we must carefully balance two opposing effects: the truncation level must be small enough that the truncated motion is 'similar' to a Brownian motion (to prove Step 1), but it must be large enough so that the truncated motion and original stable motion are themselves 'similar' (to prove Step 2). Although Step 1 will hold when $I(\varepsilon) = \varepsilon$, Step 2 does not.

More concretely, consider Assumption 3.1.2 (B): $\lim_{\varepsilon\to 0} \frac{\varepsilon^2}{I(\varepsilon)^2} |\log \varepsilon| = 0$. Recall that the ternary branching stable motion branches at rate ε^{-2}. Suppose $\tau \sim exp(\varepsilon^{-2})$ is the time of one such branching event. Then, for $k \in \mathbb{N}$, conditional on $\tau \leq k\varepsilon^2 |\log \varepsilon|$ (which happens with probability $1 - \varepsilon^k$), an individual in the tree $\boldsymbol{Y}(I(\varepsilon)^{\alpha-2}t)$ is expected to make, at most,

$$k\frac{\varepsilon^2}{I(\varepsilon)^2}|\log \varepsilon| \tag{3.19}$$

jumps larger than $I(\varepsilon)$ in its lifetime, because the arrival rate of jumps larger then $I(\varepsilon)$ made by a stable process run at speed $I(\varepsilon)^{\alpha-2}$ is[2]

$$\mathcal{O}\left(I(\varepsilon)^{\alpha-2}\int_{I(\varepsilon)}^{\infty} x^{-\alpha-1}dx\right) = \mathcal{O}\left(I(\varepsilon)^{-2}\right). \tag{3.20}$$

To prove Step 2, each individual stable motion in $\boldsymbol{Y}(I(\varepsilon)^{\alpha-2}t)$ should approximate (asymptotically in ε) an $I(\varepsilon)$-truncated process, so the quantity (3.19) should converge to zero with ε, which is precisely Assumption 3.1.2 (B). The remaining assumptions on $I(\varepsilon)$ from Assumptions 3.1.2 arise from several technical lemmas needed to prove Theorem 3.2.5.

To the best of our knowledge, Theorem 3.2.5 is the first result on the solution interface of equation (3.2) with our chosen scaling. Other related works, such as [CS10, IS09], suggest that our result should hold even when $I(\varepsilon) = \varepsilon$. However, it does not seem likely that we can achieve the $I(\varepsilon) = \varepsilon$ scaling using our current method of proof.

The two-step proof described above motivates our choice of scaling. However, in reality, we opt to work with $W\left(R^{I(\varepsilon)^2}_{I(\varepsilon)^{\alpha-2}t}\right)$, a Brownian motion subordinated by an $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator run at speed $I(\varepsilon)^{\alpha-2}$, instead of the $I(\varepsilon)$-truncated stable process $Y^{I(\varepsilon)}_{I(\varepsilon)^{\alpha-2}t}$. The central idea of our proof remains the same as before, however, we found the subordinated process easier to work with. Like the truncated stable process, the subordinated Brownian motion is not heavy tailed, and its explicit description in terms of a Brownian motion allows us to more elegantly adapt the Brownian proof of [EFP17].[3] Moreover, the error made by approximating the stable process $Y_{I(\varepsilon)^{\alpha-2}t}$ by $W\left(R^{I(\varepsilon)^2}_{I(\varepsilon)^{\alpha-2}t}\right)$ is roughly equal to the error we would obtain using $Y^{I(\varepsilon)}_{I(\varepsilon)^{\alpha-2}}$. Recall that $Y_{I(\varepsilon)^{\alpha-2}t} \stackrel{D}{=} W(R_{I(\varepsilon)^{\alpha-2}t})$, a Brownian motion subordinated by an $\frac{\alpha}{2}$-stable subordinator. The arrival rate of jumps larger that $I(\varepsilon)^2$ made by the $\frac{\alpha}{2}$-stable subordinator is

$$\mathcal{O}\left(I(\varepsilon)^{\alpha-2}\int_{I(\varepsilon)^2}^{\infty} x^{-\frac{\alpha}{2}-1}dx\right) = \mathcal{O}\left(I(\varepsilon)^{-2}\right),$$

[2]To obtain (3.20), we use that the arrival rate of jumps larger than $I(\varepsilon)$ made by the $\mathbb{d}$-dimensional process $Y_{I(\varepsilon)^{\alpha-2}t}$ is proportional to $I(\varepsilon)^{\alpha-2}\int_{x\in\mathbb{R}^{\mathbb{d}}:|x|>I(\varepsilon)}\nu(dx)$ where $\nu(dx) = |x|^{-\alpha-\mathbb{d}}dx$ is the Lévy measure of $(Y_t)_{t\geq 0}$. Equation (3.20) then follows using spherical coordinates.

[3]In the proof of the one-dimensional result (described in Section 3.3) our use of subordinated Brownian motions instead of truncated stable motions is largely superficial (indeed, we have verified that the one-dimensional result holds for truncated stable motions). However, the bottleneck of our multidimensional proof, the coupling result (Section 3.3.4) relies heavily on our use of subordinated Brownian motions. It is unclear how the proofs in the section could be adapted for truncated stable motions.

which is the same order as the arrival rate of jumps larger than $I(\varepsilon)$ made by a stable process from (3.20).

3.3 Majority voting in one dimension

For the remainder of this chapter, we prove a one-dimensional analogue of Theorem 3.2.5, which will enable us to prove Theorem 3.2.5 in Chapter 4. This parallels the structure of proof for the Brownian result, Theorem 3.1.7.

For each $x \in \mathbb{R}$, let $\mathbb{P}_x$ be the law of a one-dimensional α-stable process $X(t)$ satisfying Assumption 3.2.1 started at x, with corresponding expectation $\mathbb{E}_x$. Write $\mathbb{P}_x^\varepsilon$ for the probability measure under which $(\boldsymbol{X}(t), t \geq 0)$ has the law of a one-dimensional historical ternary branching α-stable motion, with branching rate ε^{-2} started from a single particle at location x at time 0. Write $\mathbb{E}_x^\varepsilon$ for the corresponding expectation. In accordance with Assumption 3.2.1, each particle in $(\boldsymbol{X}(t), t \geq 0)$ is assumed to run at speed $\sigma_\alpha I(\varepsilon)^{\alpha-2}$ for σ_α given in (3.3).

Define

$$p_0(x) = \mathbb{1}_{\{x \geq 0\}}. \tag{3.21}$$

With this choice of initial condition, under majority voting (Definition 3.2.2), the leaves of $\mathcal{T}(\boldsymbol{X}(t))$ will vote one if and only if they are on the right half line. Denote the root vote of $\mathcal{T}(\boldsymbol{X}(t))$ under majority voting with initial condition (3.21) by $\mathbb{V}(\boldsymbol{X}(t)) := \mathbb{V}_{p_0}(\boldsymbol{X}(t))$. The following result is the natural analogue of Theorem 3.2.5 in one dimension.

Theorem 3.3.1. *Let $T^* \in (0, \infty)$. Suppose I satisfies Assumptions 3.1.2 (A)-(B). Then there exist $c_1(\alpha), \varepsilon_1(\alpha) > 0$ such that, for all $t \in [0, T^*]$ and all $\varepsilon \in (0, \varepsilon_1)$,*

(1) for $x \geq c_1 I(\varepsilon)|\log \varepsilon|$, we have $\mathbb{P}_x^\varepsilon[\mathbb{V}(\boldsymbol{X}(t)) = 1] \geq 1 - \frac{\varepsilon^2}{I(\varepsilon)^2}$,

(2) for $x \leq -c_1 I(\varepsilon)|\log \varepsilon|$, we have $\mathbb{P}_x^\varepsilon[\mathbb{V}(\boldsymbol{X}(t)) = 1] \leq \frac{\varepsilon^2}{I(\varepsilon)^2}$.

This result tells us that, for positive x, 'typical' leaves of the tree $\mathcal{T}(\boldsymbol{X}(t))$ based at x are more likely to vote 1 than 0. As we mentioned in our introduction, Theorem 3.3.1 is weaker than the actual one-dimensional result that will be used to prove Theorem 3.2.5. This stronger result (Theorem 3.3.7) will be developed in Section 3.3.1, and shown to imply Theorem 3.3.1.

Remark 3.3.2. There is some evidence in the literature that the interface width in Theorem 3.3.1 could be improved. Let f be any bistable nonlinearity and consider the one-dimensional equation

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}u(y) = f(u(y)) \quad \forall\, y \in \mathbb{R}, \\ \lim_{y\to\infty} u(y) = 1, \quad \lim_{y\to-\infty} u(y) = 0.\end{cases} \tag{3.22}$$

Then, by [GZ15, Proposition 3.2], a solution $u \in C^2(\mathbb{R})$ to (3.22) satisfies

$$\frac{A}{y^\alpha} \leq 1 - u(y) \leq \frac{B}{y^\alpha}$$

for all $y > 1$ and some $A, B > 0$ (with a similar inequality holding if $y \leq -1$). We rewrite this equation under our scaling by setting $z = \varepsilon^{\frac{2}{\alpha}} I(\varepsilon)^{1-\frac{2}{\alpha}} y$, and obtain

$$\frac{A\varepsilon^2 I(\varepsilon)^{\alpha-2}}{z^\alpha} \leq 1 - u(z) \leq \frac{B\varepsilon^2 I(\varepsilon)^{\alpha-2}}{z^\alpha}.$$

Therefore if $z \geq I(\varepsilon)$, $1 - u(z)$ is of order $\frac{\varepsilon^2}{I(\varepsilon)^2}$, the interface sharpness from Theorem 3.3.1. This suggests that Theorem 3.3.1 might hold for an interface of width $I(\varepsilon)$, and that this would be the narrowest width possible for the given interface sharpness. However, we were only able to prove Theorem 3.3.1 for an interface width of order $I(\varepsilon)|\log \varepsilon|$.

The choice of initial condition $p_0(x) = \mathbb{1}_{\{x\geq 0\}}$ affords us several useful inequalities. First, for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$, we have

$$\mathbb{P}^\varepsilon_{x_1}[\mathbb{V}(\boldsymbol{X}(t)) = 1] \leq \mathbb{P}^\varepsilon_{x_2}[\mathbb{V}(\boldsymbol{X}(t)) = 1]. \tag{3.23}$$

For any time-labelled ternary tree $\mathcal{T}$, write

$$\mathbb{P}^t_x(\mathcal{T}) := \mathbb{P}^\varepsilon_x[\mathbb{V}(\boldsymbol{X}(t)) = 1 \mid \mathcal{T}(\boldsymbol{X}(t)) = \mathcal{T}].$$

This conditioning should be understood formally by first determining the time labelled ternary tree, then determining the stable motions. It then follows by symmetry of α-stable motions and the definition of $p_0(x)$ that, for any $x \in \mathbb{R}$ and $t > 0$,

$$\mathbb{P}^t_x(\mathcal{T}) = 1 - \mathbb{P}^t_{-x}(\mathcal{T}). \tag{3.24}$$

Setting $x = 0$ in equation (3.24) yields $\mathbb{P}^t_0(\mathcal{T}) = \frac{1}{2}$, so by monotonicity (3.23)

$$\mathbb{P}^t_x(\mathcal{T}) \geq \tfrac{1}{2} \text{ for } x > 0, \text{ and } \mathbb{P}^t_x(\mathcal{T}) \leq \tfrac{1}{2} \text{ for } x < 0.$$

It will be convenient in our later calculations to introduce notation for the majority voting system. Mimicking [EFP17], define the function $g : [0,1]^3 \to [0,1]$ by

$$g(p_1, p_2, p_3) = p_1 p_2 p_3 + p_1 p_2 (1 - p_3) + p_2 p_3 (1 - p_1) + p_3 p_1 (1 - p_2). \tag{3.25}$$

This is the probability that a majority vote gives the result 1, in the special case when the three voters are independent and have probabilities p_1, p_2 and p_3 of voting 1. We will abuse notation slightly and write $g(q) := g(q, q, q)$. Note that, for all $q \in [0,1]$,

$$g(q) = 1 - g(1 - q). \tag{3.26}$$

3.3.1 A coupling of voting systems

In this section, we will couple the root vote of $\mathcal{T}(\boldsymbol{X}(t))$ under majority voting to the root vote of another ternary branching process under a different voting system. This other branching process will be a ternary branching subordinated Brownian motion, with subordinator given by a truncated $\frac{\alpha}{2}$-stable subordinator. We endow this process with a voting procedure that we call 'marked majority voting'. Once we have achieved this coupling of root votes, we state a more general theorem in terms of this new branching process that will imply Theorem 3.3.1.

The intuition behind marked majority voting, which we denote by $\mathbb{V}^\times$, is straightforward. However, formally proving a coupling of the majority and marked majority systems $\mathbb{V}$ and $\mathbb{V}^\times$ is more challenging. To aid us with this, we introduce an intermediate voting system $\widehat{\mathbb{V}}$ that can be readily compared to both voting systems. We call this the 'exponentially marked' voting procedure. The definition of the exponentially marked voting procedure, together with a proof that it couples with the ordinary majority voting system in the appropriate sense (Theorem 3.3.4) make up the content of Section 3.3.1.1. After this, in Section 3.3.1.2, we define the marked majority voting procedure that will be carried through the one-dimensional proof, and prove (using the intermediate voting system) that it can be coupled to majority voting. Theorem 3.3.1 can then be restated in terms of the marked voting system on the subordinated Brownian tree (Theorem 3.3.7), and will be proved in Section 3.3.2.

3.3.1.1 Exponentially marked voting

In this section we will couple the majority voting system on $\mathcal{T}(\boldsymbol{X}(t))$ with the exponentially marked voting system defined on a ternary branching subordinated

Brownian motion. Fix $\varepsilon > 0$ throughout. We will consider an $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator denoted R_t^ε. Assumption 3.2.1 will also be adopted for all stable subordinators. To be precise, if

$$\nu(dx) := \frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)} x^{-1-\frac{\alpha}{2}} dx$$

is the Lévy measure of the $\frac{\alpha}{2}$-stable subordinator, then the Lévy measure of R_t^ε is given by

$$\sigma_\alpha I(\varepsilon)^{\alpha-2}\nu(dx)\mathbb{1}_{0\le x\le \frac{2-\alpha}{\alpha}I(\varepsilon)^2}.$$

Although we truncate at level $\frac{2-\alpha}{\alpha}I(\varepsilon)^2$, will shall refer to this as the $I(\varepsilon)^2$-truncated stable subordinator. Let $\boldsymbol{B}_{R^\varepsilon}(t)$ denote the historical process of a ternary branching R_t^ε-subordinated Brownian motion with branching rate ε^{-2}. Unless stated otherwise, all subordinators in this work will be zero at time zero.

Let us now make precise the form of the coupling that we desire. As before, let $\mathbb{V}(\boldsymbol{X}(t))$ denote the root vote of $\mathcal{T}(\boldsymbol{X}(t))$ under majority voting (Definition 3.2.2). We will define a voting system on $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ with root vote $\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t))$ satisfying

$$\mathbb{P}_x^\varepsilon\left[\mathbb{V}(\boldsymbol{X}(t))=1\right] \ge \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t))=1\right] \quad \text{for all} \quad x \ge 0 \tag{3.27}$$

with the reverse inequality holding for $x < 0$. Having obtained this, it will suffice to prove the analogue of Theorem 3.3.1 with $\mathbb{V}(\boldsymbol{X})$ replaced by $\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon})$. In this way, we will have incorporated the problematic 'large jumps' of the α-stable process $\boldsymbol{X}(t)$ into the voting system $\widehat{\mathbb{V}}$.

Before defining a voting system on $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, it will be useful to note the following. Let R_t be an $\frac{\alpha}{2}$-stable subordinator, and let

$$\tau^\times := \inf\{t \ge 0 : |R(t) - R(t-)| > \tfrac{2-\alpha}{\alpha} I(\varepsilon)^2\}$$

be the first time that the subordinator makes a jump of size greater than $\frac{2-\alpha}{\alpha}I(\varepsilon)^2$. Then $\tau^\times$ is exponentially distributed with parameter

$$\begin{aligned}
&\int_{\frac{2-\alpha}{\alpha}I(\varepsilon)^2}^\infty \sigma_\alpha I(\varepsilon)^{\alpha-2}\nu(dx)\mathbb{1}_{0\le x\le \frac{2-\alpha}{\alpha}I(\varepsilon)^2}\\
&= \tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I(\varepsilon)^{\alpha-2}\int_{\frac{2-\alpha}{\alpha}I(\varepsilon)^2}^\infty x^{-\frac{\alpha}{2}-1}dx\\
&= I(\varepsilon)^{-2}
\end{aligned}$$

using the definition of σ_α from (3.3) (indeed, we chose σ_α so that the above equality would hold). With this in mind, we can define the exponentially marked voting system on $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$.

Definition 3.3.3 ($\widehat{\mathbb{V}}_p$)**.** Fix $\varepsilon > 0$. For $t \geq 0$, let $\boldsymbol{X}(t)$ by a ternary branching α-stable process, which is equal in distribution to a ternary branching subordinated Brownian motion $\boldsymbol{B}_R(t)$, where the individual with label i travels according to a Brownian motion subordinated by an $\frac{\alpha}{2}$-stable subordinator, $B_i(R_i)$. Let $M(t)$ be the set of all individuals that have ever been alive in $\mathcal{T}(\boldsymbol{B}_R(t))$, and for each $i \in M(t)$ let

$$\tau_i^{\times} = \inf\{t \geq 0 : |R_i(t) - R_i(t-)| > \tfrac{2-\alpha}{\alpha} I(\varepsilon)^2\}$$

and let τ_i be the time that the individual with label i branches into three. Let $\boldsymbol{B}_{R^\varepsilon}(t)$ be the process obtained from $\boldsymbol{B}_R(t)$ by removing all jumps of size larger than $\frac{2-\alpha}{\alpha} I(\varepsilon)^2$ in the path of each stable subordinator R_i. Then the individual with label i in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ travels according to $B_i(R_i^\varepsilon)$ for R_i^ε an $I(\varepsilon)^2$-truncated stable subordinator. For $p : \mathbb{R} \to [0,1]$, define the *exponentially marked voting procedure* on $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ as follows.

(1) Each individual $B_i(R_i^\varepsilon)$ is said to be *marked* if $\tau_i^{\times} < \tau_i$. Each marked individual votes 1 with probability $\frac{1}{2}$ and otherwise votes 0.

(2) Each unmarked leaf i of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, independently, votes 1 with probability $p(B_i(R_i^\varepsilon(t)))$ and otherwise votes 0.

(3) At each branch point in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, if the parent particle k is unmarked, she votes according to the majority vote of her three offspring $(k,1)$, $(k,2)$ and $(k,3)$.

Under this voting procedure, define $\widehat{\mathbb{V}}_p(\boldsymbol{B}_{R^\varepsilon}(t))$ to be the vote associated to the root $\emptyset$ of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$.

When an individual in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ is marked, its vote is independent of the votes of its ancestors. Therefore if at least two individuals born at the same branching event are marked, the vote of their parent is independent of all of its ancestors, making it effectively random. Reassuringly, this scenario is very unlikely, since down a 'typical' line of descent in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, two individuals will not be marked at the same branching event.

We now describe the intuition behind the exponentially marked voting procedure. Consider a tree $\mathcal{T}(\boldsymbol{B}_R(t))$ rooted at $x > 0$. Suppose the tree $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, also rooted at x, has been obtained from $\mathcal{T}(\boldsymbol{B}_R(t))$ as described in Definition 3.3.3. Then we can consider how the vote of each individual $B_i(R_i)$ in $\mathcal{T}(\boldsymbol{B}_R(t))$ compares to the vote of the corresponding individual $B_i(R_i^\varepsilon)$ in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$. If the subordinator R_i has not made a jump bigger than $\frac{2-\alpha}{\alpha} I(\varepsilon)^2$ in its lifetime ($\tau_i^{\times} \geq \tau_i$), then $B_i(R_i^\varepsilon)$ votes in the same way as $B_i(R_i)$ according to majority voting. However, if R_i does make a large jump ($\tau_i^\times < \tau_i$), then, since $x > 0$, $B_i(R_i)$ is more likely to jump into right-half line than the left. Therefore the vote of $B_i(R_i)$ should be one with probability strictly greater than $1/2$. In contrast, when $\tau_i^\times < \tau_i$, $B_i(R_i^\varepsilon)$ votes one with probability exactly $1/2$, which reduces the probability that the root vote of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ will equal one, so we expect (3.27) to hold.

Now, instead of considering the initial condition $p_0(x) = \mathbb{1}_{\{x\geq 0\}}$ as we did for majority voting, we will use

$$\widehat{p}_0(x) = u_+\mathbb{1}_{\{x\geq 0\}} + u_-\mathbb{1}_{\{x\leq 0\}}, \tag{3.28}$$

where $0 < u_- < u_+ < 1$ satisfy $1-u_+ = u_-$. We will choose u_- and u_+ later in (3.44) and (3.45) to be two of the fixed points of the 'marked majority voting' function, which will be defined in (3.42). For this choice of initial condition, we write $\widehat{\mathbb{V}} := \widehat{\mathbb{V}}_{\widehat{p}_0}$. Noting that $\widehat{p}_0$ is increasing, for any $x_1 \leq x_2 \in \mathbb{R}$,

$$\mathbb{P}^\varepsilon_{x_1}\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] \leq \mathbb{P}^\varepsilon_{x_2}\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right]. \tag{3.29}$$

Let $\mathcal{T}$ be a time-labelled ternary tree and define

$$\widehat{\mathbb{P}}^\varepsilon_x(\mathcal{T}) := \mathbb{P}^\varepsilon_x\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \mathcal{T} = \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))\right].$$

Then, since 0 and 1 are exchangeable in the exponentially marked voting system,

$$\widehat{\mathbb{P}}^t_x(\mathcal{T}) = 1 - \widehat{\mathbb{P}}^t_{-x}(\mathcal{T}) \tag{3.30}$$

for all $x \in \mathbb{R}$ and $t \geq 0$. Setting $x = 0$ in equation (3.30) shows that $\widehat{\mathbb{P}}^t_0(\mathcal{T}) = \frac{1}{2}$ for all $t > 0$, and together with monotonicity (3.29), this gives

$$\widehat{\mathbb{P}}^\varepsilon_x(\mathcal{T}) \geq \tfrac{1}{2} \ \text{ for } \ x > 0, \quad \widehat{\mathbb{P}}^\varepsilon_x(\mathcal{T}) \leq \tfrac{1}{2} \ \text{ for } \ x < 0.$$

To conclude this section, we prove the claimed coupling of voting systems.

Theorem 3.3.4. *Let $\varepsilon > 0$ and $t \geq 0$. Let $\boldsymbol{X}(t)$ be the α-stable ternary branching process used to define $\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t))$ as described in Definition 3.3.3. Then*

(1) for all $x \geq 0$, $\mathbb{P}^\varepsilon_x[\mathbb{V}(\boldsymbol{X}(t)) = 1] \geq \mathbb{P}^\varepsilon_x\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right]$,

(2) for all $x \leq 0$, $\mathbb{P}^\varepsilon_x[\mathbb{V}(\boldsymbol{X}(t)) = 1] \leq \mathbb{P}^\varepsilon_x\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right]$.

Proof. We only prove the first inequality, since the second inequality will follow by the symmetry relations (3.24) and (3.30). Recall the initial conditions for $\mathbb{V}(\boldsymbol{X}(t))$ and $\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t))$ are given by

$$p_0(x) = \mathbb{1}_{\{x \geq 0\}} \quad \text{and} \quad \widehat{p}_0(x) = u_+ \mathbb{1}_{\{x \geq 0\}} + u_- \mathbb{1}_{\{x \leq 0\}}$$

respectively. To ease notation, let $p \equiv p_0$ and $\widehat{p} \equiv \widehat{p}_0$ for the remainder of this proof.

Consider $\boldsymbol{B}_R(t) \overset{D}{=} \boldsymbol{X}(t)$ and denote the time of the first branching event in $\mathcal{T}(\boldsymbol{B}_R(t)) = \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}$ by $\tau_\emptyset$. Let $\tau_\emptyset^\times \sim exp(I(\varepsilon)^{-2})$ be the first time that the subordinator associated to the ancestral particle in $\boldsymbol{B}_R(t)$ makes a jump of size larger than $\frac{2-\alpha}{\alpha} I(\varepsilon)^2$. To ease notation let $\tau^\times = \tau_\emptyset^\times$ and $\tau = \tau_\emptyset$. We proceed by induction on the number of branching events in $\mathcal{T}$.

To prove the base case, let $\mathcal{T}_0$ denote the tree with a root and a single leaf. Conditional on $\{\mathcal{T}_0 = \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))\}$, let $B(R_t^\varepsilon)$ be the position of the single individual at time t where $(B_s)_{s\geq 0}$ is a standard Brownian motion and $(R_s^\varepsilon)_{s \geq 0}$ is an $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator. Under the exponentially marked voting procedure, this individual votes 1 with probability $\widehat{p}_0(B(R_t^\varepsilon))$ if she is unmarked (i.e. $\tau^\times \geq \tau$), or she votes 1 with probability $\frac{1}{2}$ if she is marked ($\tau^\times < \tau$). Since the event $\{\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}_0\}$ is equivalent to $\{\tau > t\}$, we have, for all $x \geq 0$,

$$\begin{aligned}
\widehat{\mathbb{P}}_x^t(\mathcal{T}_0) &= \mathbb{E}_x^\varepsilon\left[\widehat{p}\left(B(R_t^\varepsilon)\right) \mathbb{1}_{\tau^\times \geq \tau} \mid \tau > t\right] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times < \tau \mid \tau > t] \\
&= \mathbb{E}_x^\varepsilon\left[\widehat{p}\left(B(R_t^\varepsilon)\right) \mid \tau > t\right] \mathbb{P}_x^\varepsilon[\tau^\times \geq \tau \mid \tau > t] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times < \tau \mid \tau > t] \\
&\leq \mathbb{E}_x^\varepsilon\left[\widehat{p}\left(B(R_t^\varepsilon)\right) \mid \tau > t\right] \mathbb{P}_x^\varepsilon[\tau^\times \geq t] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times < t]
\end{aligned} \tag{3.31}$$

where in the second line we have used that, conditional on the event $\{\tau > t\}$, the events $\{B(R_t^\varepsilon) > 0\}$ and $\{\tau^\times \geq \tau\}$ are independent. It is straightforward to verify the final inequality holds using that $\mathbb{P}_x^\varepsilon[\tau^\times \geq \tau \mid \tau > t] \leq \mathbb{P}_x^\varepsilon[\tau^\times \geq t]$ and that $\mathbb{E}_x^\varepsilon\left[\widehat{p}\left(B(R_t^\varepsilon)\right) \mid \tau > t\right] \geq \frac{1}{2}$ since $x \geq 0$, which follows by similar arguments to those used to obtain (3.24).

Continuing the proof of the base case, conditional on $\{\mathcal{T}(\boldsymbol{B}_R(t)) = \mathcal{T}_0\}$, denote the single individual in $\mathcal{T}(\boldsymbol{B}_R(t))$ at time t by $B(R_t)$. Then, since $\tau^\times$ is the first time that R_t makes a jump of size larger than $\frac{2-\alpha}{\alpha} I(\varepsilon)^2$, we have

$$\begin{aligned}
&\mathbb{P}_x^t(\mathcal{T}_0) \\
&= \mathbb{E}_x^\varepsilon[p\left(B(R_t)\right) \mid \tau^\times \geq t, \tau > t]\mathbb{P}_x^\varepsilon[\tau^\times \geq t] + \mathbb{E}_x^\varepsilon[p(B(R_t)) \mid \tau^\times < t < \tau]\mathbb{P}_x^\varepsilon[\tau^\times < t] \\
&\geq \mathbb{E}_x^\varepsilon[p(B(R_t^\varepsilon)) \mid \tau > t]\, \mathbb{P}_x^\varepsilon[\tau^\times \geq t] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times < t]
\end{aligned} \tag{3.32}$$

using that, conditional on $\{\tau^\times > t\}$, $B(R_t) \overset{D}{=} B(R_t^\varepsilon)$, and

$$\mathbb{E}_x^\varepsilon[p(B(R_t)) \mid \tau^\times < t < \tau] \geq \tfrac{1}{2}$$

since $x \geq 0$, which can be shown using a similar identity to (3.24). It is straightforward to verify that $\mathbb{E}_x^\varepsilon[p(B(R_t)) \mid \tau > t] \geq \mathbb{E}_x^\varepsilon[\widehat{p}(B(R_t)) \mid \tau > t]$ for all $x \geq 0$, which, together with (3.31) and (3.32) gives us

$$\mathbb{P}_x^t(\mathcal{T}_0) \geq \widehat{\mathbb{P}}_x^t(\mathcal{T}_0)$$

for $x \geq 0$, proving the base case.

Now suppose that, for all trees $\mathcal{T}^{n-1}$ with at most $n-1 > 1$ branching events,

$$\mathbb{P}_x^t(\mathcal{T}^{n-1}) \geq \widehat{\mathbb{P}}_x^t(\mathcal{T}^{n-1})$$

for $x \geq 0$. Let $\mathcal{T}^n$ be a tree with n branching events. Define the first three trees of descent, denoted $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$, to be the three subtrees of $\mathcal{T}^n$ generated at time τ. Note that $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ have strictly less than n branching events. Recall that g from (3.25) is the majority voting function associated to $\mathbb{V}$. Write

$$g\left(\mathbb{P}^{t-\tau}_{B(R_\tau)}(\mathcal{T}\star)\right) := g\left(\mathbb{P}^{t-\tau}_{B(R_\tau)}(\mathcal{T}_1), \mathbb{P}^{t-\tau}_{B(R_\tau)}(\mathcal{T}_2), \mathbb{P}^{t-\tau}_{B(R_\tau)}(\mathcal{T}_3)\right) \tag{3.33}$$

and define $g\left(\widehat{\mathbb{P}}^{t-\tau}_{B(R_\tau^\varepsilon)}(\mathcal{T}\star)\right)$ similarly. By (3.24), (3.30) and (3.26)

$$\begin{aligned} g\left(\mathbb{P}^{t-\tau}_{B(R_\tau)}(\mathcal{T}\star)\right) &= 1 - g\left(\mathbb{P}^{t-\tau}_{-B(R_\tau)}(\mathcal{T}\star)\right) \\ \text{and } g\left(\widehat{\mathbb{P}}^{t-\tau}_{B(R_\tau^\varepsilon)}(\mathcal{T}\star)\right) &= 1 - g\left(\widehat{\mathbb{P}}^{t-\tau}_{-B(R_\tau^\varepsilon)}(\mathcal{T}\star)\right). \end{aligned} \tag{3.34}$$

Let $E_n := \{\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}(\boldsymbol{X}(t)) = \mathcal{T}^n\}$. By almost identical arguments to those used to obtain (3.32), but now conditioning on the event $\{\tau^\times > \tau\}$, we have

$$\begin{aligned} &\mathbb{P}_x^t(\mathcal{T}^n) \\ &= \mathbb{E}_x^\varepsilon\left[g\left(\mathbb{P}^{t-\tau}_{B(R_\tau^\varepsilon)}(\mathcal{T}\star)\right)\mathbb{1}_{\tau^\times > \tau} \mid E_n\right] + \mathbb{E}_x^\varepsilon\left[g\left(\mathbb{P}^{t-\tau}_{B(R_\tau)}(\mathcal{T}\star)\right) \mid \tau^\times \leq \tau, E_n\right]\mathbb{P}_x^\varepsilon[\tau^\times \leq \tau \mid E_n] \\ &\geq \mathbb{E}_x^\varepsilon\left[g\left(\mathbb{P}^{t-\tau}_{B(R_\tau^\varepsilon)}(\mathcal{T}\star)\right)\mathbb{1}_{\tau^\times > \tau} \mid E_n\right] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times \leq \tau \mid E_n], \end{aligned} \tag{3.35}$$

using that, for all $x \geq 0$, $\mathbb{E}_x^\varepsilon\left[g\left(\mathbb{P}^{t-\tau}_{B(R_\tau)}(\mathcal{T}\star)\right) \mid \tau^\times \leq \tau, E_n\right] \geq \frac{1}{2}$ by a similar symmetry relation to (3.30). By definition of the exponentially marked voting system, we also have

$$\widehat{\mathbb{P}}_x^t(\mathcal{T}^n) = \mathbb{E}_x^\varepsilon\left[g\left(\widehat{\mathbb{P}}^{t-\tau}_{B(R_\tau^\varepsilon)}(\mathcal{T}\star)\right)\mathbb{1}_{\tau^\times > \tau} \mid E_n\right] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times \leq \tau \mid E_n]. \tag{3.36}$$

Therefore by (3.35) and (3.36), it suffices to show that

$$\mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\tau^\times > \tau} \mid E_n \right] \geq \mathbb{E}_x^\varepsilon \left[g\left(\widehat{\mathbb{P}}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\tau^\times > \tau} \mid E_n \right]. \tag{3.37}$$

Now, for all $x \geq 0$,

$$\begin{aligned}
&\mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\tau^\times > \tau} \mid E_n \right] \\
&= \mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\tau^\times > \tau} \mid B(R^\varepsilon_\tau) > 0, E_n \right] \mathbb{P}_x^\varepsilon \left[B(R^\varepsilon_\tau) > 0 \mid E_n \right] \\
&\quad + \mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\tau^\times > \tau} \mid B(R^\varepsilon_\tau) \leq 0, E_n \right] \mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) \leq 0 \mid E_n] \\
&= \mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\tau^\times > \tau} \mid B(R^\varepsilon_\tau) > 0, E_n \right] \mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) > 0 \mid E_n] \\
&\quad + \mathbb{P}_x^\varepsilon [\tau^\times > \tau \mid E_n] \mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) \leq 0 \mid E_n] \\
&\quad - \mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{-B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\tau^\times > \tau} \mid B(R^\varepsilon_\tau) \leq 0, E_n \right] \mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) \leq 0 \mid E_n] \\
&= \mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\tau^\times > \tau} \mid B(R^\varepsilon_S) > 0, E_n \right] \Big(\mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) > 0 \mid E_n] \\
&\quad - \mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) \leq 0 \mid E_n] \Big) + \mathbb{P}_x^\varepsilon [\tau^\times > \tau \mid E_n] \mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) \leq 0 \mid E_n] \\
&\geq \mathbb{E}_x^\varepsilon \left[g\left(\widehat{\mathbb{P}}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\tau^\times > \tau} \mid B(R^\varepsilon_\tau) > 0, E_n \right] \Big(\mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) > 0 \mid E_n] \\
&\quad - \mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) \leq 0 \mid E_n] \Big) + \mathbb{P}_x^\varepsilon [\tau^\times > \tau \mid E_n] \mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) \leq 0 \mid E_n] \\
&= \mathbb{E}_x^\varepsilon \left[g\left(\widehat{\mathbb{P}}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\tau^\times > \tau} \mid E_n \right]
\end{aligned}$$

where, in the second equality, we have applied the symmetry (3.34), and in the second to last line, we used monotonicity of g together with our inductive hypothesis, and that, given $x \geq 0$, the difference $\mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) > 0 \mid E_n] - \mathbb{P}_x^\varepsilon [B(R^\varepsilon_\tau) \leq 0 \mid E_n]$ is non-negative. The final equality follows by reversing the arguments used above but for $\widehat{\mathbb{P}}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star)$. We conclude that (3.37) holds, proving our inductive step. □

3.3.1.2 Marked majority voting

In this section, we describe what will be the final voting system in one dimension, denoted $\mathbb{V}^\times$, defined on $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$. This voting system will be carried throughout the one-dimensional proof. In spirit, $\mathbb{V}^\times$ is very similar to $\widehat{\mathbb{V}}$, but no longer relies on knowing the lifetime of particles in order to mark them. Instead, particles are marked (independently) when they are born. In Theorem 3.3.6, it will be shown that our new voting system $\mathbb{V}^\times$ can be coupled to the exponentially marked voting system $\widehat{\mathbb{V}}$, so by Theorem 3.3.4, it can also be coupled to the original majority voting system $\mathbb{V}$.

Under the marked majority voting system, particles will be marked (independently) with probability b_ε, defined as follows. Recall that, for $i \in M(t)$, $\tau_i^\times \sim exp(I(\varepsilon)^{-2})$ is the first time the subordinator $(R_i(s))_{s\geq 0}$ makes a jump of size larger than $\frac{2-\alpha}{\alpha} I(\varepsilon)^2$. Further recall that $\tau_i \sim exp(\varepsilon^{-2})$, where τ_i is the time that the particle with label i in $\mathcal{T}(\boldsymbol{X}(t))$ branches into three. Define

$$b_\varepsilon := \mathbb{P}[\tau_i^\times < \tau_i] = \frac{I(\varepsilon)^{-2}}{I(\varepsilon)^{-2} + \varepsilon^{-2}} \sim \frac{\varepsilon^2}{I(\varepsilon)^2} \tag{3.38}$$

where $x \sim y$ for some x, y depending on ε means that there exists constants $c, d > 0$ independent of ε such that $cy < x < dy$. The quantity b_ε is the probability that the subordinator associated to individual i makes a large jump in its lifetime (if individual i is a leaf, b_ε gives an upper bound on this probability). By Assumption 3.1.2 (B), $b_\varepsilon \to 0$ as $\varepsilon \to 0$.

Definition 3.3.5 ($\mathbb{V}_p^\times$)**.** Let $\varepsilon > 0$. For $p : \mathbb{R} \to [0,1]$, we define a *marked majority* voting procedure on $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ as follows.

(1) At each branch point in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, the parent particle j marks each of her three offspring $(j,1), (j,2)$ and $(j,3)$ independently with probability b_ε. Each marked particle (independently) votes 1 with probability $\frac{1}{2}$ and otherwise votes 0.

(2) Each unmarked leaf i of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, independently, votes 1 with probability $p(B_i(R_i^\varepsilon)))$ and otherwise votes 0.

(3) At each branch point in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, if the parent particle k is unmarked, she votes according to the majority vote of her three offspring $(k,1), (k,2)$ and $(k,3)$.

Observe that the initial ancestor of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ is never marked under this procedure. With the marked majority voting procedure described above, define $\mathbb{V}_p^\times$ to be the vote associated to the root $\emptyset$ of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$.

The marked majority voting procedure makes it more difficult for the root of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ (rooted at $x > 0$) to vote 1 compared to majority voting. Consider, for example, a particle in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ that is not the root or a leaf. Then even if all three offspring of this individual vote 1, under marked majority voting the individual can still vote 0 with positive probability, thereby reducing the probability that the root vote equals one. This can be viewed as the penalty one must pay for truncating the underlying spatial motion (where this truncation really takes place on the subordinator). In other words, to couple $\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))$ to $\mathbb{V}(\boldsymbol{X}(t))$, the voting procedure $\mathbb{V}^\times$ should make it more difficult for individuals to vote 1, to compensate for the new underlying spatial motion, which makes it easier for individuals to vote 1 (since, for a tree rooted at $x > 0$, $\boldsymbol{B}_{R^\varepsilon}(t)$ is more likely to remain on the right-half line than $\boldsymbol{X}(t)$).

Here, we will use the same initial condition as $\widehat{p}_0$ (3.28) that we did for exponentially marked voting, and write $\mathbb{V}^\times := \mathbb{V}^\times_{\widehat{p}_0}$. With this choice of initial condition, the marked majority voting system, $\mathbb{V}^\times$, retains many of the symmetry relations exploited in [EFP17], that we have already used here for $\mathbb{V}$ and $\widehat{\mathbb{V}}$. Namely, for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$,

$$\mathbb{P}^\varepsilon_{x_1}\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] \leq \mathbb{P}^\varepsilon_{x_2}\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right], \tag{3.39}$$

and, for any time-labelled tree $\mathcal{T}$, if we set

$$\overset{\times}{\mathbb{P}}{}^t_x(\mathcal{T}) = \mathbb{P}^\varepsilon_x\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}\right],$$

then by symmetry of the historical stable process and exchangeability of 0 and 1 in the marked voting procedure,

$$\overset{\times}{\mathbb{P}}{}^t_x(\mathcal{T}) = 1 - \overset{\times}{\mathbb{P}}{}^t_{-x}(\mathcal{T}) \tag{3.40}$$

for all $\mathcal{T}$, $x \in \mathbb{R}$, and $t \geq 0$. Setting $x = 0$ in (3.40) gives $\overset{\times}{\mathbb{P}}{}^t_0(\mathcal{T}) = \frac{1}{2}$, so by monotonicity (3.39), for any time-labelled ternary tree $\mathcal{T}$,

$$\overset{\times}{\mathbb{P}}{}^t_x(\mathcal{T}) \geq \tfrac{1}{2} \text{ for } x > 0, \text{ and } \overset{\times}{\mathbb{P}}{}^t_x(\mathcal{T}) \leq \tfrac{1}{2} \text{ for } x < 0. \tag{3.41}$$

We next introduce notation for our marked majority voting procedure. Define the *marked majority voting function* $g_\times : [0,1]^3 \to [0,1]$ by

$$g_\times(p_1, p_2, p_3) := g\left((1-b_\varepsilon)p_1 + \tfrac{b_\varepsilon}{2}, (1-b_\varepsilon)p_2 + \tfrac{b_\varepsilon}{2}, (1-b_\varepsilon)p_3 + \tfrac{b_\varepsilon}{2}\right). \tag{3.42}$$

This is the probability that an unmarked parent particle votes 1 under $\mathbb{V}^\times$, in the special case when the three offspring are independent and have probabilities p_1, p_2 and p_3 of voting 1 if they are unmarked. We abuse notation and write $g_\times(q) := g_\times(q, q, q)$. It is easy to check using symmetry of the majority voting function (3.26) that, for all $q \in [0, 1]$,

$$g_\times(q) = 1 - g_\times(1-q). \tag{3.43}$$

Let $\{u_-, \frac{1}{2}, u_+\}$ be the three solutions to $g_\times(q) = q$, satisfying $0 < u_- < \frac{1}{2} < u_+ < 1$. The exact derivation of these fixed points can be found in the Proposition A.1.1.

It will be useful later to note that these fixed points can be approximated by Taylor expansion as

$$u_- = \frac{1}{2} - \frac{\sqrt{(1-b_\varepsilon)^3(1-3b_\varepsilon)}}{2(1-b_\varepsilon)^3} = \frac{3}{4}b_\varepsilon^2 + \mathcal{O}(b_\varepsilon^3) \tag{3.44}$$

$$u_+ = \frac{1}{2} + \frac{\sqrt{(1-b_\varepsilon)^3(1-3b_\varepsilon)}}{2(1-b_\varepsilon)^3} = 1 - \frac{3}{4}b_\varepsilon^2 + \mathcal{O}(b_\varepsilon^3). \tag{3.45}$$

Henceforth, these fixed points u_+ and u_- will be used to define the initial condition

$$\widehat{p}_0(x) = u_+ \mathbb{1}_{\{x\geq 0\}} + u_- \mathbb{1}_{\{x\geq 0\}}.$$

We now return to the coupling of $\mathbb{V}^\times$ and the original voting system $\mathbb{V}$ via the intermediate system $\widehat{\mathbb{V}}$.

Theorem 3.3.6. *Let $\widehat{\mathbb{V}}$ be the exponentially marked majority voting system (Definition 3.3.3) and $\mathbb{V}^\times$ be the marked majority voting system (Definition 3.3.5), both with initial condition $\widehat{p}_0(x) = u_+ \mathbb{1}_{\{x\geq 0\}} + u_- \mathbb{1}_{\{x\leq 0\}}$. Then, for all $x \in \mathbb{R}$ and $t \geq 0$,*

$$\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] = \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right](1-b_\varepsilon) + \frac{b_\varepsilon}{2}. \tag{3.46}$$

Proof. Recall that, under the voting system $\widehat{\mathbb{V}}$, each particle i is marked if $\tau_i^\times < \tau_i$. By definition of b_ε, all particles in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ are marked with probability b_ε under both $\widehat{\mathbb{V}}$ and $\mathbb{V}^\times$, except for ancestral particle, which remains unmarked under $\mathbb{V}^\times$ by definition. Conditioning on the marking of the ancestral particle in $\widehat{\mathbb{V}}$, we obtain

$$\begin{aligned}\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] &= \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \tau^\times > \tau_0\right]\mathbb{P}_x^\varepsilon[\tau^\times > \tau_0] \\ &\quad + \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \tau^\times \leq \tau_0\right]\mathbb{P}_x^\varepsilon[\tau^\times \leq \tau_0] \\ &= \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right](1-b_\varepsilon) + \frac{b_\varepsilon}{2},\end{aligned}$$

where the last line follows by definition of b_ε. □

Finally, by Theorem 3.3.4 and Theorem 3.3.6, to prove our main one-dimensional result, Theorem 3.3.1, it suffices to show the following.

Theorem 3.3.7. *Suppose I satisfies Assumptions 3.1.2 (A)-(B). Fix $k \in \mathbb{N}$ and $T^* \in (0,\infty)$. Let u_+, u_- be as in (3.44) and (3.45). Then there exist $c_1(\alpha,k), \varepsilon_1(\alpha,k) > 0$ such that, for all $t \in [0,T^*]$ and all $\varepsilon \in (0,\varepsilon_1(k))$,*

(1) for $x \geq c_1(k)I(\varepsilon)|\log\varepsilon|$, we have $\mathbb{P}_x^\varepsilon[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] \geq u_+ - \varepsilon^k$,

(2) for $x \leq -c_1(k)I(\varepsilon)|\log \varepsilon|$, we have $\mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] \leq u_- + \varepsilon^k$.

Observe that the sharpness of the interface in Theorem 3.3.7 is of the same order as the sharpness from the Brownian result, Theorem 3.1.7. This is a result of the truncated subordinated Brownian motion behaving similarly to a Brownian motion (as discussed in Section 3.2.1).

Remark 3.3.8. By a similar proof to that of Theorem 3.2.3, we see that

$$u^\varepsilon(t,x) = \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right]$$

is a solution to the equation

$$\begin{aligned}\partial_t u^\varepsilon &= \mathcal{L}^\varepsilon u^\varepsilon + \varepsilon^{-2}(g_\times(u^\varepsilon) - u^\varepsilon) \\ &= \mathcal{L}^\varepsilon u^\varepsilon + \mathcal{O}\left(\varepsilon^2 I(\varepsilon)^{-4}\right)\left(\tfrac{1}{2} - u^\varepsilon\right)^3 + \mathcal{O}\left(\varepsilon^{-2}\right)u^\varepsilon(2u^\varepsilon - 1)(1 - u^\varepsilon) \qquad (3.47)\end{aligned}$$

with initial condition $u^\varepsilon(0,x) = \widehat{p}_0(x)$, where $\mathcal{L}^\varepsilon$ denotes the infinitesimal generator of $(B(R_t^\varepsilon))_{t\geq 0}$. Rather remarkably, the work from this section tells us that solutions to (3.47) and (3.2) are related. More precisely, the couplings from Theorem 3.3.4 and Theorem 3.3.6 tell us that solutions to equation (3.47) (after transformation by the function $v \mapsto (1 - b_\varepsilon)v + \frac{b_\varepsilon}{2}$) are lower and upper bounds to solutions of the scaled fractional Allen–Cahn equation (3.2) restricted to $x \geq 0$ and $x \leq 0$, respectively. It would be interesting to see if this relationship provides any insights into a PDE-theoretic proof of our main result.

3.3.2 Proof of Theorem 3.3.7

We now prove Theorem 3.3.7. To do so, we will adapt ideas from both [EFP17, HD21]. In [EFP17], the majority voting function g was used throughout, while [HD21] builds upon this work and considers more general voting functions. This makes [HD21] useful when proving results about the marked majority voting function $g_\times$.

Throughout this section, we take the initial condition

$$\widehat{p}_0(x) = u_+\mathbb{1}_{\{x\geq 0\}} + u_-\mathbb{1}_{\{x\leq 0\}}.$$

Our next result verifies that the marked majority voting procedure cannot reduce the positive voting bias on the leaves when the root, x, is non-negative. By the symmetry (3.40), the same result holds for the negative voting bias on the leaves when $x < 0$. In view of this symmetry, we often state results only for positive x when convenient to do so. Recall that

$$\check{\mathbb{P}}_x^t(\mathcal{T}) := \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}\right].$$

Lemma 3.3.9. *For any time-labelled ternary tree $\mathcal{T}$, time $t > 0$, and any $x \geq 0$,*

$$\check{\mathbb{P}}_x^t(\mathcal{T}) \geq u_+ \mathbb{P}_x[B(R_t^\varepsilon) \geq 0] + u_- \mathbb{P}_x[B(R_t^\varepsilon) \leq 0].$$

Proof. This proof follows exactly [HD21, Lemma 3.1] with $g_\times$ instead of g therein. □

Lemma 3.3.9 partly motivated our definitions of $\widehat{\mathbb{V}}$ and $\mathbb{V}^\times$. Recall that marked individuals in $\mathbb{V}^\times$ and $\widehat{\mathbb{V}}$ vote 1 or 0 with equal probability. However, the proof of Theorem 3.3.4 would have simplified greatly if we had asked marked individuals under $\widehat{\mathbb{V}}$ to vote 0 with probability 1.[4] Unlike $g_\times$, which satisfies (3.43), the voting function corresponding to this other system would not be symmetric. As a result, the proof of [HD21, Lemma 3.1] would no longer apply, and we are unsure if Lemma 3.3.9 would hold at all. The symmetry of $g_\times$ will be used in several other proofs throughout this section as well.

We now show that the iterative voting procedure amplifies a small positive bias at the leaves to a large positive voting bias at the root. To do this, we define $g_\times^{(n)}(q)$ inductively by

$$g_\times^{(1)}(q) = g_\times(q), \; g_\times^{(n+1)}(q) = g_\times^{(n)}(g_\times(q)).$$

Noting that the ancestral particle is never marked under $\mathbb{V}^\times$, we see that $g_\times^{(n)}(q)$ is the probability of voting 1 at the root of an n-level regular ternary tree under $\mathbb{V}^\times$ if the votes of the *unmarked* leaves are i.i.d. Bernoulli(q). In the following result, we consider the rate of convergence of $g_\times$ to its fixed points. Let I be a scaling function satisfying Assumptions 3.1.2 (A)-(B) throughout.

Lemma 3.3.10. *Fix $k \in \mathbb{N}$. There exists $A(k) < \infty$ and $\varepsilon_1(k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1)$ and $n \geq A(k)|\log \varepsilon|$,*

$$g_\times^{(n)}\left(\tfrac{1}{2} + \varepsilon\right) \geq u_+ - \varepsilon^k \quad \textit{and} \quad g_\times^{(n)}\left(\tfrac{1}{2} - \varepsilon\right) \leq u_- + \varepsilon^k.$$

Proof. We follow the proof of [HD21, Lemma 3.2], with some slight changes. Recall that

$$g_\times(p) = g((1 - b_\varepsilon)p + \tfrac{b_\varepsilon}{2})$$

where $b_\varepsilon = \mathcal{O}\left(\frac{\varepsilon^2}{I(\varepsilon)^2}\right)$. We prove only the first inequality since the second follows by completely symmetric arguments. First, we show that there exists $C_k > 0$ and some fixed $q_0 > 0$ such that, after $n \geq C_k |\log \varepsilon|$ iterations,

$$g_\times^{(n)}(u_+ - q) \geq u_+ - \varepsilon^k \tag{3.48}$$

[4]Technically, this version of $\widehat{\mathbb{V}}$ would only satisfy part (1) of Theorem 3.3.4. To obtain part (2) of Theorem 3.3.4, one would need to define $\widehat{\mathbb{V}}$ so that marked individuals vote 1 with probability 1. In fact, we will explore these voting systems more in Chapter 4.

for all $q \leq q_0$. We then show that there exists $D > 0$ such that, after $n \geq D|\log \varepsilon|$ iterations,

$$g_\times^{(n)}\left(\tfrac{1}{2}+\varepsilon\right) \geq u_+ - q_0. \tag{3.49}$$

Combining (3.48) and (3.49) then gives the result. To prove (3.48), choose ε_1 sufficiently small so that $b_\varepsilon \leq \frac{1}{6}$ for all $\varepsilon \in (0, \varepsilon_1)$. Then

$$g'_\times\left(\tfrac{1}{2}\right) = (1-b_\varepsilon)g'\left(\tfrac{1}{2}\right) = \tfrac{3}{2}(1-b_\varepsilon) \geq \tfrac{5}{4} > 1. \tag{3.50}$$

Next, using that $g'(0) = 0$ and g' is continuous, together with the estimate (3.44), for ε_1 sufficiently small we have

$$g'\left(u_-(1-b_\varepsilon) + \tfrac{b_\varepsilon}{2}\right) < \tfrac{1}{4}$$

for all $\varepsilon \in (0, \varepsilon_1)$. It follows that, for this choice of ε_1,

$$g'_\times(u_-) = (1-b_\varepsilon)g'\left(u_-(1-b_\varepsilon) + \tfrac{b_\varepsilon}{2}\right) < \tfrac{1}{4}.$$

Since $g'_\times\left(\frac{1}{2}\right) > 1$ and $g'_\times$ is continuous,

$$q_0 := \inf\left\{q \geq 0 : g'_\times(u_- + q) \geq \tfrac{1}{2}\right\} > 0.$$

By the Mean Value Theorem and definition of q_0, together with the symmetry of the marked voting function (3.43), for all $q < q_0$

$$u_+ - g_\times(u_+ - q) = g_\times(u_- + q) - u_- \leq q\, g'_\times(u_- + q_0) = \tfrac{q}{2}.$$

Iterating this yields

$$u_+ - g_\times^{(n)}(u_+ - q) \leq \tfrac{1}{2^n}\left(u_+ - q\right) \leq \tfrac{1}{2^n}.$$

It follows that there exists $C_k > 0$ such that if $n \geq C_k|\log \varepsilon|$ and $q \leq q_0$ then

$$g_\times^{(n)}(u_+ - q) \geq u_+ - \varepsilon^k,$$

thereby proving (3.48). We now prove (3.49). By equation (3.50), ε_1 is sufficiently small so that, for all $\varepsilon \in (0, \varepsilon_1)$, $g'_\times(\frac{1}{2}) > 1$. Since $g_\times$ is increasing and $u_-, \frac{1}{2}, u_+$ are the only fixed points of $g_\times$, we have

$$g_\times(q) > q \text{ for all } q \in \left(\tfrac{1}{2}, u_+\right). \tag{3.51}$$

By definition of q_0 and since $g'_\times$ is increasing on $(0, \frac{1}{2})$, $u_+ - q_0 - \frac{1}{2} = \frac{1}{2} - q_0 - u_- > 0$. Therefore

$$q_1 := \inf_{\varepsilon \le q \le u_+ - q_0 - \frac{1}{2}} \frac{g_\times\left(\frac{1}{2}+q\right) - \left(\frac{1}{2}+q\right)}{q} \ge 0,$$

and so for $q \in \left[\varepsilon, u_+ - q_0 - \frac{1}{2}\right]$, by definition of q_1 we have

$$g_\times\left(\tfrac{1}{2}+q\right) - \tfrac{1}{2} > (1+q_1)q. \tag{3.52}$$

Now, if $g_\times\left(\frac{1}{2}+\varepsilon\right) \ge u_+ - q_0$, we are done. If not, we can apply (3.52) twice to obtain

$$\begin{aligned} g_\times^{(2)}\left(\tfrac{1}{2}+\varepsilon\right) &= g_\times\left(\tfrac{1}{2} + \left(g_\times(\tfrac{1}{2}+\varepsilon) - \tfrac{1}{2}\right)\right) \\ &\ge (1+q_1)\left[g_\times(\tfrac{1}{2}+\varepsilon) - \tfrac{1}{2}\right] + \tfrac{1}{2} \\ &\ge (1+q_1)^2\varepsilon + \tfrac{1}{2}. \end{aligned}$$

Repeating this argument $n-1$ times, we obtain

$$g_\times^{(n)}\left(\tfrac{1}{2}+\varepsilon\right) \ge (1+q_1)^n\varepsilon + \tfrac{1}{2}.$$

It follows that, for $D := \frac{1}{\log(1+q_1)}$, after $n > D|\log\varepsilon|$ iterations,

$$g_\times^{(n)}\left(\tfrac{1}{2}+\varepsilon\right) \ge u_+ - q_0.$$

Setting $A = C_k + D$ proves the result. □

We now briefly note a useful inequality for $g_\times$, that will be used in the proof of Theorem 3.3.7.

Lemma 3.3.11. *If $p_1, p_2, p_3 \ge \frac{1}{2}$ then,*

$$g_\times(p_1, p_2, p_3) \ge \min(p_1, p_2, p_3, u_+).$$

If $p_1, p_2, p_3 \le \frac{1}{2}$ then,

$$g_\times(p_1, p_2, p_3) \le \max(p_1, p_2, p_3, u_-).$$

Proof. We prove only the first inequality, since the second one follows by a symmetric argument. Denote

$$p_{\min} = \min\{p_1, p_2, p_3, u_+\}.$$

If $p_1, p_2, p_3 \ge \frac{1}{2}$, it follows that $\frac{1}{2} \le p_{\min} \le u_+$. Since $g_\times$ is increasing in each variable,

$$g_\times(p_1, p_2, p_3) \ge g_\times(p_{\min}).$$

Therefore it suffices to show that $g_\times(p_{\min}) \geq p_{\min}$. For this recall that $\{u_-, \frac{1}{2}, u_+\}$ are the only fixed points of $g_\times$. Factorising $g_\times(p_{\min}) - p_{\min}$ yields

$$g_\times(p_{\min}) - p_{\min} = (p_{\min} - u_-)(2p_{\min} - 1)(u_+ - p_{\min}).$$

Since $\frac{1}{2} \leq p_{\min} \leq u_+$, $g_\times(p_{\min}) - p_{\min} \geq 0$, as required. □

We next show that, with high probability, by time $t \geq a\varepsilon^2|\log\varepsilon|$, each ancestral line of descent in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ contains at least $\mathcal{O}(|\log\varepsilon|)$ branching events. Let

$$\mathcal{T}_n^{reg} = \cup_{k\leq n}\{1,2,3\}^k \subset \mathcal{U}$$

denote the n-level regular ternary tree, and for $l \in \mathbb{R}$, let $\mathcal{T}_l^{reg} = \mathcal{T}_{\lceil l\rceil}^{reg}$. For $\mathcal{T}$ a time-labelled ternary tree, we use $\mathcal{T}_l^{reg} \subseteq \mathcal{T}$ to mean that, as subtrees of $\mathcal{U}$, $\mathcal{T}_l^{reg}$ is contained inside $\mathcal{T}$ (ignoring time labels).

Lemma 3.3.12. *Let $k \in \mathbb{N}$ and $A(k)$ be as in Lemma 3.3.10. There exists $a_1(\alpha, k) > 0$ and $\varepsilon_1(\alpha, k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1)$ and $t \geq a_1(k)\varepsilon^2|\log\varepsilon|$,*

$$\mathbb{P}^\varepsilon\left[\mathcal{T}\left(\boldsymbol{B}_{R^\varepsilon}(t)\right) \supseteq \mathcal{T}_{A(k)|\log\varepsilon|}^{reg}\right] \geq 1 - \varepsilon^k. \tag{3.53}$$

Proof. This proof proceeds exactly as that of [EFP17, Lemma 2.10]. □

Next, we control the displacement of leaves in $\boldsymbol{B}_{R^\varepsilon}(t)$.

Lemma 3.3.13. *Fix $k \in \mathbb{N}$ and let $a_1(k)$ be as in Lemma 3.3.12. There exists $\varepsilon_1(\alpha, k) > 0$ and $l_1(\alpha, k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1)$ and $s \leq a_1(k)\varepsilon^2|\log\varepsilon|$,*

$$\mathbb{P}_x^\varepsilon\left[\exists i \in N(s) : |B_i(R_i^\varepsilon(s)) - x| \geq l_1(k)I(\varepsilon)|\log\varepsilon|\right] \leq \varepsilon^k.$$

Lemma 3.3.13 highlights the importance of working with a truncated subordinated Brownian motion instead of the original stable process. In the proof of Lemma 3.3.13, we control the position of the leaves in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ using a many-to-one lemma. If we were to use the same approach for the stable tree (without any truncation), we could not obtain the polynomial error ε^k in Lemma 3.3.13, which is crucial to our later proofs.

Proof of Lemma 3.3.13. First note that for any $m > 0$

$$\begin{aligned}
&\mathbb{P}_x^\varepsilon\left[\exists i \in N(s) : |B_i(R_i^\varepsilon(s)) - x| \geq mI(\varepsilon)|\log\varepsilon|\right] \\
&\quad\leq \mathbb{E}_x^\varepsilon[N(s)]\mathbb{P}_0\left[|B(R_s^\varepsilon)| \geq mI(\varepsilon)|\log\varepsilon|\right] \\
&\quad= e^{\frac{2s}{\varepsilon^2}}\mathbb{P}_0\left[|B(R_s^\varepsilon)| \geq mI(\varepsilon)|\log\varepsilon|\right] \\
&\quad\leq \varepsilon^{-2a_1}\mathbb{P}_0\left[|B(R_s^\varepsilon)| \geq mI(\varepsilon)|\log\varepsilon|\right].
\end{aligned} \tag{3.54}$$

Denote the density of R_s^ε by f_ε. Then for any $h \geq 0$, partitioning over the event $\{R_s^\varepsilon \leq hI(\varepsilon)^2|\log\varepsilon|\}$ and its complement, we obtain

$$\begin{aligned}
&\mathbb{P}_0\left[|B(R_s^\varepsilon)| \geq mI(\varepsilon)|\log\varepsilon|\right] \\
&\leq \int_0^{hI(\varepsilon)^2|\log\varepsilon|} \mathbb{P}_0\left[|B_t| \geq mI(\varepsilon)|\log\varepsilon| \mid R_s^\varepsilon = t\right] f_\varepsilon(t)dt + \mathbb{P}_0[R_s^\varepsilon \geq hI(\varepsilon)^2|\log\varepsilon|] \\
&\leq \sup_{0\leq t\leq hI(\varepsilon)^2|\log\varepsilon|} \mathbb{P}_0[|B_t| \geq mI(\varepsilon)|\log\varepsilon|] + \mathbb{P}_0[R_s^\varepsilon \geq hI(\varepsilon)^2|\log\varepsilon|].
\end{aligned}$$

We bound each term separately. First, by a Chernoff bound, for all $t \leq hI(\varepsilon)^2|\log\varepsilon|$,

$$\begin{aligned}
\mathbb{P}_0\left[|B_t| \geq mI(\varepsilon)|\log\varepsilon|\right] &= \mathbb{P}_0\left[\sqrt{2t}|Z| \geq mI(\varepsilon)|\log\varepsilon|\right] \\
&\leq \mathbb{P}_0\left[\sqrt{2h}|Z| \geq m|\log\varepsilon|^{\frac{1}{2}}\right] \\
&\leq \exp\left(-\tfrac{1}{4}\tfrac{m^2}{h}|\log\varepsilon|\right) \\
&= \varepsilon^{\frac{m^2}{4h}}.
\end{aligned}$$

Fix $h := k + 2a_1(k) + 1$. By Lemma A.4.2, there exists $\varepsilon_1(k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1)$

$$\mathbb{P}_0\left[R_s^\varepsilon \geq hI(\varepsilon)^2|\log\varepsilon|\right] \leq \varepsilon^{k+2a_1(k)}.$$

Putting this together, we obtain

$$\mathbb{P}_x^\varepsilon\left[\exists i \in N(s) : |B_i(R_i^\varepsilon(s)) - x| \geq mI(\varepsilon)|\log\varepsilon|\right] \leq \varepsilon^k + \varepsilon^{\frac{m^2}{4h} - 2a_1(k)}$$

so the result holds by choosing $m = l_1(k)$ sufficiently large. □

In the following proof of Theorem 3.3.7, we suppose $x \geq 2l_1(k)I(\varepsilon)|\log\varepsilon|$, $s^* := a_1(k)\varepsilon^2|\log\varepsilon|$ and consider the cases $t \leq s^*$ and $t \geq s^*$ separately. First, if $t \leq s^*$, by Lemma 3.3.13, with high probability none of the particles in $\mathcal{T}(\boldsymbol{X}(t))$ have moved a distance further than $l_1(k)I(\varepsilon)|\log\varepsilon|$, and the result follows easily. When $t \geq s^*$, we use Lemma 3.3.9 to show that the leaves of $\boldsymbol{B}_{R^\varepsilon}(t)$ have a positive voting bias, which, by Lemma 3.3.12 is magnified by $\mathcal{O}(|\log\varepsilon|)$ rounds of voting, so Lemma 3.3.10 applies and gives the result.

Proof of Theorem 3.3.7. We follow closely the proof of [EFP17, Theorem 2.6], with some slight differences. We suppress the superscript ε of $\mathbb{P}_x^\varepsilon$ throughout. Fix $k \in \mathbb{N}$ and $T^* \in (0,\infty)$. Let $\varepsilon < \frac{1}{2}$ and define z_ε implicitly by the relation

$$\mathbb{P}_{z_\varepsilon}[B(R_{T^*}^\varepsilon) \geq 0] = \tfrac{1}{2} + (u_+ - u_-)^{-1}\varepsilon. \tag{3.55}$$

By Lemma A.4.4 we may choose ε_1 sufficiently small so that, for all $\varepsilon \in (0, \varepsilon_1)$, $z_\varepsilon \le 8\sqrt{2\pi(T^*+2)}\,\varepsilon$. Further suppose $\varepsilon_1(k) < \frac{1}{2}$ is sufficiently small so that Lemmas 3.3.12 and 3.3.13 hold for all $\varepsilon \in (0, \varepsilon_1)$. Let $c_1(k) = 2l_1(k)$, for $l_1(k)$ as in Lemma 3.3.13. By Assumption 3.1.2 (B), $\varepsilon I(\varepsilon)^{-1} \to 0$ as $\varepsilon \to 0$, so we can choose ε_1 sufficiently small so that

$$l_1(k)I(\varepsilon)|\log\varepsilon| + z_\varepsilon \le c_1(k)I(\varepsilon)|\log\varepsilon|$$

for all $\varepsilon \in (0, \varepsilon_1)$. Let a_1 be as in Lemma 3.3.12 and define

$$s^*(\varepsilon) = a_1(k)\varepsilon^2|\log\varepsilon|.$$

Let $t \in (0, s^*)$ and $z \ge c_1(k)I(\varepsilon)|\log\varepsilon|$. Note that $g_\times(u_+) = u_+$, so if the initial condition is constant with $p(x) \equiv u_+$, then

$$\mathbb{P}_z\left[\mathbb{V}^\times_{u_+}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] = u_+ \text{ for all } t > 0, z \in \mathbb{R}. \tag{3.56}$$

Recall that the initial condition for marked majority voting is chosen to be $\widehat{p}_0 = u_+\mathbb{1}_{\{x\ge 0\}} + u_-\mathbb{1}_{\{x\le 0\}}$, so

$$\begin{aligned}
\mathbb{P}_z\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 0\right] &\le \mathbb{P}_z\left[\exists\, i \in N(t) : |B_i(R_i^\varepsilon(t)) - z| \ge c_1 I(\varepsilon)|\log\varepsilon|\right] \\
&\quad + \mathbb{P}_z\left[\{\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 0\} \cap \{\nexists\, i \in N(t) : |B_i(R_i^\varepsilon(t)) - z| \ge c_1 I(\varepsilon)|\log\varepsilon|\}\right] \\
&\le \varepsilon^k + 1 - u_+ \\
&= \varepsilon^k + u_-
\end{aligned}$$

where we have used (3.56) in the second inequality. This proves the result when $t < s^*$. Now suppose $t \in [s^*, T^*]$ and $z \ge c_1(k)I(\varepsilon)|\log\varepsilon|$. Let $\mathcal{T}_{s^*} = \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(s^*))$ and define

$$q_{t-s^*}(z) = \mathbb{P}_z[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t-s^*)) = 1]$$

for all $z \in \mathbb{R}$. Write $\{\boldsymbol{B}_{R^\varepsilon}(s^*) > z_\varepsilon\}$ for the event $B_i(R_i^\varepsilon(s^*)) > z_\varepsilon$ for all $i \in N(s^*)$. Then

$$\begin{aligned}
\mathbb{P}_z[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] &= \mathbb{P}_z\left[\mathbb{V}^\times_{q_{t-s^*}(\cdot)}(\boldsymbol{B}_{R^\varepsilon}(s^*)) = 1\right] \\
&\ge \mathbb{P}_z\left[\left\{\mathbb{V}^\times_{q_{t-s^*}(z_\varepsilon)}(\boldsymbol{B}_{R^\varepsilon}(s^*)) = 1\right\} \cap \{\boldsymbol{B}_{R^\varepsilon}(s^*) > z_\varepsilon\}\right] \\
&\ge \mathbb{P}_z\left[\mathbb{V}^\times_{q_{t-s^*}(z_\varepsilon)}(\boldsymbol{B}_{R^\varepsilon}(s^*)) = 1\right] - \varepsilon^k,
\end{aligned} \tag{3.57}$$

where the first line follows by the Markov property of $\boldsymbol{B}_{R^\varepsilon}$ at time s^*, the second line follows by monotonicity (3.39), and the third line follows by the Lemma 3.3.13

and our assumption $z \geq c_1 I(\varepsilon)|\log \varepsilon|$. Now, by Lemma 3.3.9 and the definition of z_ε (3.55) (noting that $t - s^* \leq T^*$), we have

$$\begin{aligned} q_{t-s^*}(z_\varepsilon) &\geq \mathbb{P}_{z_\varepsilon}[B(R^\varepsilon_{t-s^*}) \geq 0]u_+ + \mathbb{P}_{z_\varepsilon}[B(R^\varepsilon_{t-s^*}) \leq 0]u_- \\ &\geq u_+ \left(\tfrac{1}{2} + (u_+ - u_-)^{-1}\varepsilon\right) + u_- \left(\tfrac{1}{2} - (u_+ - u_-)^{-1}\varepsilon\right) \\ &= \tfrac{1}{2} + \varepsilon. \end{aligned} \tag{3.58}$$

Substituting (3.58) into (3.57), we obtain

$$\mathbb{P}_z \left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] \geq \mathbb{P}_z \left[\mathbb{V}^\times_{\frac{1}{2}+\varepsilon}(\boldsymbol{B}_{R^\varepsilon}(s^*)) = 1\right] - \varepsilon^k. \tag{3.59}$$

Note that, if $p_i \geq \frac{1}{2} + \varepsilon$ for $i = 1, 2, 3$, then $g_\times(p_1, p_2, p_3) \geq \min(p_1, p_2, p_3, u_+)$ from Lemma 3.3.11. Therefore, if each leaf of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(s^*))$ votes 1 independently with probability greater than $\frac{1}{2}+\varepsilon$, and $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(s^*)) \supseteq \mathcal{T}^{reg}_{A|\log\varepsilon|}$, then each leaf in $\mathcal{T}^{reg}_{A|\log\varepsilon|}$ also votes 1 with probability greater than $\frac{1}{2}+\varepsilon$. By Lemma 3.3.12, $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(s^*)) \supseteq \mathcal{T}^{reg}_{A|\log\varepsilon|}$ with probability at least $1 - \varepsilon^k$, so by Lemma 3.3.10

$$\begin{aligned} \mathbb{P}_z \left[\mathbb{V}^\times_{\frac{1}{2}+\varepsilon}(\boldsymbol{B}_{R^\varepsilon}(s^*)) = 1\right] &\geq (1 - \varepsilon^k) g_\times^{(\lceil A|\log\varepsilon|\rceil)}\left(\tfrac{1}{2} + \varepsilon\right) \\ &\geq (1 - \varepsilon^k)(u_+ - \varepsilon^k) \\ &\geq u_+ - 2\varepsilon^k. \end{aligned}$$

Substituting this into (3.59) yields

$$\mathbb{P}_z \left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] \geq u_+ - 3\varepsilon^k, \tag{3.60}$$

thereby proving part (1) of Theorem 3.3.7. Part (2) of Theorem 3.3.7 follows by completely symmetric arguments. □

Remark 3.3.14. Observe that (3.60) contains a coefficient in front of the polynomial error term, ε^k, that is not mentioned in the statement of Theorem 3.3.7. Our convention here and in the following chapters is that coefficients of polynomial error terms will not be stated in theorems and propositions (this convention was also used in [EFP17]).

3.3.3 Slope of the interface

Just as in [EFP17], to prove the multidimensional result, Theorem 3.2.5, we make use of a lower bound on the 'slope' of the interface in dimension $\mathrm{d} = 1$. For a time-labelled ternary tree $\mathcal{T}$, recall that

$$\overset{\times}{\mathbb{P}}{}^t_x(\mathcal{T}) := \mathbb{P}^\varepsilon_x[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1) \mid \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}].$$

Proposition 3.3.15. *Suppose $x \geq 0$ and $\eta > 0$. Then for any time-labelled ternary tree $\mathcal{T}$ and any time t,*

$$\check{\mathbb{P}}^t_x(\mathcal{T}) - \check{\mathbb{P}}^t_{x-\eta}(\mathcal{T}) \geq \check{\mathbb{P}}^t_{x+\eta}(\mathcal{T}) - \check{\mathbb{P}}^t_x(\mathcal{T}). \tag{3.61}$$

Proof. This proofs follows [EFP17] with some minor differences, because of our different choice of voting function $g_\times$. We proceed by induction on the number of branching events. Let $\mathcal{T}_0$ denote a time-labelled tree with a root and a single leaf. Recall that, under $\mathbb{V}^\times$, the initial ancestor is never marked. Denote the transition density of $B(R^\varepsilon_t)$ started at $z \in \mathbb{R}$ by $p_{z,t}(\cdot)$. Then for $x \geq 0$ and $\eta > 0$,

$$\check{\mathbb{P}}^t_x(\mathcal{T}_0) - \check{\mathbb{P}}^t_{x-\eta}(\mathcal{T}_0) = \int_{x-\eta}^{x} p_{0,t}(z)dz \geq \int_{x}^{x+\eta} p_{0,t}(z)dz = \check{\mathbb{P}}^t_{x+\eta}(\mathcal{T}_0) - \check{\mathbb{P}}^t_x(\mathcal{T}_0).$$

Now assume the inequality holds for all time-labelled ternary trees with at most n branch points. Let $\mathcal{T}$ be a time-labelled ternary tree with $n+1$ internal vertices, and denote the time of the first branching event in $\mathcal{T}$ by τ. Let $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ denote the three trees of descent from the first branching event in $\mathcal{T}$. Write

$$g_\times\left(\check{\mathbb{P}}^{t-\tau}_x(\mathcal{T}\star)\right) := g_\times\left(\check{\mathbb{P}}^{t-\tau}_x(\mathcal{T}_1), \check{\mathbb{P}}^{t-\tau}_x(\mathcal{T}_2), \check{\mathbb{P}}^{t-\tau}_x(\mathcal{T}_3)\right).$$

Then

$$\begin{aligned}
\check{\mathbb{P}}^t_x(\mathcal{T}) - \check{\mathbb{P}}^t_{x-\eta}(\mathcal{T}) &= \mathbb{E}^\varepsilon_x\left[g_\times\left(\check{\mathbb{P}}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star)\right)\right] - \mathbb{E}^\varepsilon_{x-\eta}\left[g_\times\left(\check{\mathbb{P}}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star)\right)\right] \\
&= \int_{-\infty}^{\infty}\left[g_\times\left(\check{\mathbb{P}}^{t-\tau}_y(\mathcal{T}\star)\right) - g_\times\left(\check{\mathbb{P}}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right)\right] p_{x,\tau}(y)dy \\
&= \int_{0}^{\infty}\left[g_\times(\check{\mathbb{P}}^{t-\tau}_y(\mathcal{T}\star)) - g_\times\left(\check{\mathbb{P}}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right)\right](p_{x,\tau}(y) - p_{x,\tau}(-y))dy,
\end{aligned}$$

where the final line follows from the symmetry relations (3.40) and (3.43). By identical arguments applied to the right hand side of (3.61), we obtain

$$\begin{aligned}
&\left(\check{\mathbb{P}}^t_x(\mathcal{T}) - \check{\mathbb{P}}^t_{x-\eta}(\mathcal{T})\right) - \left(\check{\mathbb{P}}^t_{x+\eta}(\mathcal{T}) - \check{\mathbb{P}}^t_{x-\eta}(\mathcal{T})\right) \\
&= \int_0^\infty \left(g_\times\left(\check{\mathbb{P}}^{t-\tau}_y(\mathcal{T}\star)\right) - g_\times\left(\check{\mathbb{P}}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right)\right)(p_{x,\tau}(y) - p_{x,\tau}(-y))dy \\
&\quad - \int_0^\infty \left(g_\times\left(\check{\mathbb{P}}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right) - g_\times\left(\check{\mathbb{P}}^{t-\tau}_y(\mathcal{T}\star)\right)\right)(p_{x,\tau}(y) - p_{x,\tau}(-y))dy.
\end{aligned}$$

Since $x \geq 0$, $p_{x,\tau}(y) - p_{x,\tau}(-y) \geq 0$ for $y \geq 0$, and it suffices to show that, for $y \geq 0$,

$$\left(g_\times\left(\check{\mathbb{P}}^{t-\tau}_y(\mathcal{T}\star)\right) - g_\times\left(\check{\mathbb{P}}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right)\right) - \left(g_\times\left(\check{\mathbb{P}}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right) - g_\times\left(\check{\mathbb{P}}^{t-\tau}_y(\mathcal{T}\star)\right)\right) \geq 0. \tag{3.62}$$

By the inductive hypothesis, for each $i = 1, 2, 3$

$$\left(\check{\mathbb{P}}^{t-\tau}_y(\mathcal{T}_i) - \check{\mathbb{P}}^{t-\tau}_{y-\eta}(\mathcal{T}_i)\right) - \left(\check{\mathbb{P}}^{t-\tau}_{y+\eta}(\mathcal{T}_i) - \check{\mathbb{P}}^{t-\tau}_y(\mathcal{T}_i)\right) \geq 0 \tag{3.63}$$

for $y \geq 0$. By monotonicity of $g_\times$ and (3.63)

$$g_\times\left(\check{\mathbb{P}}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right) \leq g_\times\left(2\check{\mathbb{P}}^{t-\tau}_{y}(\mathcal{T}\star) + \check{\mathbb{P}}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right). \tag{3.64}$$

Substituting (3.64) into (3.62), it suffices to show

$$g_\times\left(\check{\mathbb{P}}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right) - 2g_\times\left(\check{\mathbb{P}}^{t-\tau}_{y}(\mathcal{T}\star)\right) + g_\times\left(2\check{\mathbb{P}}^{t-\tau}_{y}(\mathcal{T}\star) - \check{\mathbb{P}}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right) \leq 0. \tag{3.65}$$

By definition of $g_\times$, (3.65) is equivalent to

$$\begin{aligned} g\left((1-b_\varepsilon)\check{\mathbb{P}}^{t-\tau}_{y+\eta}(\mathcal{T}\star) + \tfrac{b_\varepsilon}{2}\right) - 2g\left((1-b_\varepsilon)\check{\mathbb{P}}^{t-\tau}_{y}(\mathcal{T}\star) + \tfrac{b_\varepsilon}{2}\right) \\ +g\left((1-b_\varepsilon)(2\check{\mathbb{P}}^{t-\tau}_{y}(\mathcal{T}\star) - \check{\mathbb{P}}^{t-\tau}_{y+\eta}(\mathcal{T}\star)) + \tfrac{b_\varepsilon}{2}\right) \leq 0. \end{aligned} \tag{3.66}$$

To see that (3.66) holds, note that

$$\begin{aligned} &g(p_1+q_1, p_2+q_2, p_3+q_3) - 2g(p_1,p_2,p_3) + g(p_1-q_1, p_2-q_2, p_3-q_3) \\ &\quad = 2q_1q_2(1-2p_3) + 2q_2q_3(1-2p_1) + 2q_3q_1(1-2p_2). \end{aligned}$$

Setting $p_i = (1-b_\varepsilon)\check{\mathbb{P}}^{t-\tau}_{y}(\mathcal{T}_i) + \frac{b_\varepsilon}{2}$ and $q_i = (1-b_\varepsilon)\left(\check{\mathbb{P}}^{t-\tau}_{y+\eta}(\mathcal{T}_i) - \check{\mathbb{P}}^{t-\tau}_{y}(\mathcal{T}_i)\right)$, we see that (3.66) will hold if $p_i \geq \frac{1}{2}$, or equivalently, $\check{\mathbb{P}}^{t-\tau}_{y}(\mathcal{T}_i) \geq \frac{1}{2}$, which holds by (3.41) since $y \geq 0$. □

With this, we can prove the slope of the interface result.

Corollary 3.3.16. *Let $\varepsilon_1(\alpha)$ and $c_1(\alpha)$ be as in Theorem 3.3.7. Let $\varepsilon < \min(\varepsilon_1, \frac{1}{24})$. Suppose that for some $t \in [0, T^*]$ and $z \in \mathbb{R}$,*

$$\left|\mathbb{P}^\varepsilon_z[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] - \tfrac{1}{2}\right| \leq \tfrac{5}{12},$$

and let $w \in \mathbb{R}$ with $|z-w| \leq c_1(\alpha)I(\varepsilon)|\log\varepsilon|$. Then

$$\left|\mathbb{P}^\varepsilon_z[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] - \mathbb{P}^\varepsilon_w[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1]\right| \geq \frac{|z-w|}{48c_1(\alpha)I(\varepsilon)|\log\varepsilon|}.$$

Proof. This follows exactly that of [EFP17, Corollary 2.12], replacing the interface width $\varepsilon|\log\varepsilon|$ with $I(\varepsilon)|\log\varepsilon|$. □

3.3.4 Coupling one-dimensional and $\mathbb{d}$-dimensional processes

In this section, we will construct a coupling of the the one-dimensional and multidimensional voting systems, so that the results of Section 3.3 can be used to prove our multidimensional result in the next chapter. To accomplish this, we require the following regularity properties also used in [EFP17], which follow from Assumptions 3.1.4 by [Che92]. Recall that the sets $(\boldsymbol{\Gamma}_t)_{0\leq t<\mathscr{T}}$ denote the mean curvature flow of $\boldsymbol{\Gamma}_0$ defined in (3.4).

(1) There exists $c_0 > 0$ such that for all $t \in [0, T^*]$ and $x \in \{y : |d(y,t)| \leq c_0\}$

$$|\nabla d(x,t)| = 1. \tag{3.67}$$

Moreover, d is a $C^{a,\frac{a}{2}}$ function in $\{(x,t) : |d(x,t)| \leq c_0, t \leq T^*\}$ for a as in Assumption 3.1.4 (A).

(2) Viewing $\mathbf{n} := \nabla d$ as the positive normal direction, for $x \in \mathbf{\Gamma}_t$, the normal velocity of $\mathbf{\Gamma}_t$ at x is $-d(x,t)$, and the curvature of $\mathbf{\Gamma}_t$ at x is $-\Delta d(x,t)$. Thus, the equation defining mean curvature flow, equation (3.1), becomes

$$\dot{d}(x,t) = \Delta d(x,t) \tag{3.68}$$

for all x such that $d(x,t) = 0$.

(3) There exists $C_0 > 0$ such that for all $t \in [0, T^*]$ and x such that $|d(x,t)| \leq c_0$ for c_0 as in the first assumption,

$$\left|\nabla \left(\dot{d}(x,t) - \Delta d(x,t)\right)\right| \leq C_0. \tag{3.69}$$

(4) There exists $v_0, V_0 > 0$ such that for all $t \in [0, T^* - v_0]$ and all $s \in [t, t + v_0]$,

$$|d(x,t) - d(x,s)| \leq V_0(s-t). \tag{3.70}$$

Before explaining our result, let us briefly recall the coupling in [EFP17] that compares $d(W_s, t-s)$, the distance from a $\mathbb{d}$-dimensional Brownian motion to $\mathbf{\Gamma}_{t-s}$, to a one-dimensional Brownian motion.

Proposition 3.3.17 ([EFP17])**.** *Let $(W_s)_{s\geq 0}$ denote a $\mathbb{d}$-dimensional Brownian motion started at $x \in \mathbb{R}^{\mathrm{d}}$. Suppose that $t \leq T^*, \beta \leq c_0$ and let*

$$T_\beta = \inf(\{s \in [0,t) : |d(W_s, t-s)| \geq \beta\} \cup \{t\}).$$

Then we can couple $(W_s)_{s\geq 0}$ with a one-dimensional Brownian motion $(B_s)_{s\geq 0}$ started from $z = d(x,t)$ in such a way that for $s \leq T_\beta$,

$$B_s - C_0\beta s \leq d(W_s, t-s) \leq B_s + C_0\beta s. \tag{3.71}$$

This result was a key ingredient in the proofs of [EFP17, Proposition 2.17, Lemma 2.18] that gave a comparison between the multidimensional and one-dimensional results. It turns out that [EFP17, Proposition 2.17, Lemma 2.18] are extremely sensitive to any change in the coupling (3.71). Indeed, if there is any additional error term in the left and right bounds of (3.71) (that is *not* of the form $f(\varepsilon)s$ for some f satisfying $\lim_{\varepsilon\to 0} f(\varepsilon)=0$), then this error propagates, and the strategy of proof in [EFP17] no longer works. This provides us with a major hurdle, since, if we mimic the proof of the Brownian coupling but with subordinated Brownian motions, the error term in (3.71) changes drastically. To overcome this, we employ not one but *two* coupling results. The first, Theorem 3.3.18, is a straightforward adaptation of Proposition 3.3.17 to our setting. We use this result to show that if we replace the multidimensional subordinated Brownian motion by one that is *shifted* along an appropriately chosen outward facing normal vector of $\mathbf{\Gamma}_{t-s}$, then the coupling has the desired form (this is the content of Theorem 3.3.20).

Theorem 3.3.18. *Let $k\in\mathbb{N}$. Let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion started at $x\in\mathbb{R}^{\mathrm{d}}$, and $(R^{\varepsilon}_t)_{t\geq 0}$ be an $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator satisfying Assumption 3.2.1. Fix $t\leq T^*$ and $\beta<c_0$ for c_0 as in (3.69). Define the stopping time*

$$T_\beta=\inf(\{s\in[0,(k+1)\varepsilon^2|\log\varepsilon|):|d(W_s,t-s)|>\beta\}\cup\{t\}).$$

Fix $s\geq 0$. If $R^{\varepsilon}_s<T_\beta\wedge t$, then there exists a one-dimensional standard Brownian motion $(B_t)_{t\geq 0}$ started at $d(x,t)$, constants $C_0,D_0>0$ and $\varepsilon_1(k)>0$ such that, for all $\varepsilon\in(0,\varepsilon_1(k))$, with probability at least $1-\varepsilon^{k+1}$,

$$|d(W(R^{\varepsilon}_s),t-s)-B(R^{\varepsilon}_s)|\leq C_0\beta s+D_0(k+2)I(\varepsilon)^2|\log\varepsilon|. \tag{3.72}$$

Proof. We first rewrite $d(W(R^{\varepsilon}_s),t-s)$ as

$$d(W(R^{\varepsilon}_s),t-R^{\varepsilon}_s)+\left[d(W(R^{\varepsilon}_s),t-s)-d(W(R^{\varepsilon}_s),t-R^{\varepsilon}_s)\right]. \tag{3.73}$$

By the Proposition 3.3.17, since $R^{\varepsilon}_s\leq T_\beta\wedge t$ by assumption, there exists a one-dimensional Brownian motion $(B_t)_{t\geq 0}$ started from $d(x,t)$ and $C_0>0$ such that

$$B(R^{\varepsilon}_s)-C_0\beta R^{\varepsilon}_s\leq d(W(R^{\varepsilon}_s),t-R^{\varepsilon}_s)\leq B(R^{\varepsilon}_s)+C_0\beta R^{\varepsilon}_s. \tag{3.74}$$

By (3.70), the second term in (3.73) is bounded as

$$|d(W(R^{\varepsilon}_s),t-s)-d(W(R^{\varepsilon}_s),t-R^{\varepsilon}_s)|\leq V_0|R^{\varepsilon}_s-s|. \tag{3.75}$$

Combining (3.73), (3.74) and (3.75),

$$\begin{aligned}|d(W(R^{\varepsilon}_s),t-s)-B(R^{\varepsilon}_s)|&\leq C_0\beta R^{\varepsilon}_s+V_0|R^{\varepsilon}_s-s|\\&\leq C_0\beta s+D_0|R^{\varepsilon}_s-s|\end{aligned}$$

where $D_0 := V_0 + C_0 c_0$ for c_0 as in (3.67). By Lemma A.4.2, there exists $\varepsilon_1(k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1(k))$,

$$\mathbb{P}(|R^\varepsilon_s - s| > (k+2)I(\varepsilon)^2|\log\varepsilon|) \leq \varepsilon^{k+1}$$

and the result follows. □

We now define the shifted subordinated Brownian motions that will ultimately move the unwanted error term in (3.72) into the multidimensional spatial motion.

Definition 3.3.19 (Z^+_s, Z^-_s)**.** Let $(W(R^\varepsilon_t))_{t\geq 0}$ be a d-dimensional subordinated Brownian motion. Fix $0 < t, T < \mathscr{T}$, $l > 0$ and $\beta < c_0$ for c_0 as in (3.67). Let $x_s \in \mathbf{\Gamma}_{t-s}$ be the unique point on $\mathbf{\Gamma}_{t-s}$ that is the shortest distance from $W(R^\varepsilon_s)$, and $\mathbf{v}_s$ be the outward facing unit vector perpendicular to the tangent hypersurface of $\mathbf{\Gamma}_{t-s}$ at x_s. Then we define the processes $(Z^+_s)_{0\leq s\leq T}$ and $(Z^-_s)_{0\leq s\leq T}$ by

$$Z^+_s = \begin{cases} W(R^\varepsilon_s) + lI(\varepsilon)^2|\log\varepsilon|\mathbf{v}_s & \text{if } |d(W(R^\varepsilon_s), t-s)| \leq \beta \\ W(R^\varepsilon_s) & \text{otherwise.} \end{cases} \tag{3.76}$$

$$Z^-_s = \begin{cases} W(R^\varepsilon_s) - lI(\varepsilon)^2|\log\varepsilon|\mathbf{v}_s & \text{if } |d(W(R^\varepsilon_s), t-s)| \leq \beta \\ W(R^\varepsilon_s) & \text{otherwise.} \end{cases} \tag{3.77}$$

Observe that we may choose ε sufficiently small so that any point x on the line segment between $W(R^\varepsilon_s)$ and Z^+_s (or Z^-_s) satisfies $d(x, t-s) \leq c_0$. Then by (3.67) $|\nabla d(x, t-s)| = 1$ so

$$d(Z^+_s, t-s) = d(W(R^\varepsilon_s), t-s) + lI(\varepsilon)^2|\log\varepsilon| \tag{3.78}$$

$$d(Z^-_s, t-s) = d(W(R^\varepsilon_s), t-s) - lI(\varepsilon)^2|\log\varepsilon|. \tag{3.79}$$

Consequently, we obtain the following important restatement of Theorem 3.3.18.

Theorem 3.3.20. *Let $k \in \mathbb{N}$. For $\alpha \in (1,2)$ and D_0 as in Theorem 3.3.18, let Z^+_t and Z^-_t be as in Definition 3.3.19 for $l := D_0(k+2)$, started at $x \in \mathbb{R}^{\mathrm{d}}$. Let $(R^\varepsilon_t)_{t\geq 0}$ be an $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator satisfying Assumption 3.2.1. Fix $t \leq T^*$, $\beta < c_0$ for c_0 as in (3.69). Define the stopping time*

$$T_\beta = \inf(\{s \in [0, (k+1)\varepsilon^2|\log\varepsilon|) : |d(W_s, t-s)| > \beta\} \cup \{t\}).$$

Fix $s \geq 0$. If $R^\varepsilon_s < T_\beta \wedge t$, then there exists a one-dimensional standard Brownian motion $(B_t)_{t\geq 0}$ started at $d(x,t)$ and $C_0 > 0$ such that, with probability at least $1-\varepsilon^{k+1}$,

$$d(Z^+_s, t-s) \geq B(R^\varepsilon_s) - C_0\beta s \tag{3.80}$$

$$d(Z^-_s, t-s) \leq B(R^\varepsilon_s) + C_0\beta s. \tag{3.81}$$

Proof. This follows from Theorem 3.3.18 and equations (3.78), (3.79). □

Notation 3.3.21. As we have seen in Theorem 3.3.20, Z^+ and Z^- satisfy (3.80) and (3.81) when $l := D_0(k+2)$ in (3.76) and (3.77). For the remainder of this work, we shall take $l := D_0(k+2)$ in the definition of Z^+ and Z^-, where the choice of k will be clear in the given context.

Equipped with Theorem 3.3.20, we will be able to use our one-dimensional result, but for the processes Z^+, Z^- instead of a d-dimensional stable process. To translate this back to a result in terms of stable processes, we use the following comparison between root votes.

Let $\boldsymbol{Z}^+$ be the ternary branching process (with branching rate ε^{-2}) in which individuals independently travel according to $(Z_s^+)_{0\le s\le \mathscr{T}}$. Define $\boldsymbol{Z}^-$ similarly. Denote the historical ternary branching process associated to the R_t^ε-subordinated d-dimensional Brownian motion by $\boldsymbol{W}_{R^\varepsilon}$.

Proposition 3.3.22. *Let $0 < t < \mathscr{T}$, $0 \le \beta < c_0$, $k \in \mathbb{N}$, $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$ and F be as in (3.5). Let $\boldsymbol{Z}^+(t)$ and $\boldsymbol{Z}^-(t)$ be the historical path of branching processes defined above. Then, for any $x \in \mathbb{R}^{\mathrm{d}}$, there exists $m_1, m_2 > 0$ such that*

$$\left|\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1)] - \mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1)]\right| \le m_1 e^{-\frac{t}{\varepsilon^2}} + m_2 F(\varepsilon) \tag{3.82}$$

$$\left|\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t)) = 1)] - \mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1)]\right| \le m_1 e^{-\frac{t}{\varepsilon^2}} + m_2 F(\varepsilon). \tag{3.83}$$

Proposition 3.3.22 is integral to the proof of the main multidimensional result. While intuitively the root votes in (3.82) and (3.83) should be close (since the spatial motions are) to obtain the precise bound above requires lengthy calculations which are not illuminating. So as to not disrupt our flow, we defer the proof of Proposition 3.3.22 to Section A.2 of the appendix.

Remark 3.3.23. Observe that Proposition 3.3.22 marks the first appearance of the term $F(\varepsilon)$ that will later contribute to the sharpness of the interface in Theorem 3.1.5. It is also the first time we require that $\alpha \in (1,2)$ and that Assumption 3.1.2 (C) holds, to ensure that $F(\varepsilon) \to 0$.

Chapter 4

Majority voting in dimension $\mathbb{d} \geq 2$

In this chapter, we will use the one-dimensional result to prove the main multidimensional result, Theorem 3.2.5. To begin, in Section 4.1, we will prove a series of couplings. This will allow us to restate Theorem 3.2.5 in terms of the processes $\boldsymbol{Z}^+(t)$ and $\boldsymbol{Z}^-(t)$ in Theorem 4.1.6. We go on to prove Theorem 4.1.6 in Section 4.2 and Section 4.3 following similar arguments to those in [EFP17]. The proof of a technical lemma will make up the content of Section 4.4.

Let us briefly recall the notation introduced in Chapter 3. We write X_t for the one-dimensional α-stable process (with associated historical ternary branching process $\boldsymbol{X}(t)$), and Y_t for the $\mathbb{d}$-dimensional α-stable process (with associated historical ternary branching process $\boldsymbol{Y}(t)$). The one-dimensional R_t^ε-subordinated Brownian motion is denoted $B(R_t^\varepsilon)$ (with associated historical ternary branching process $\boldsymbol{B}_{R^\varepsilon}(t)$), and the $\mathbb{d}$-dimensional R_t^ε-subordinated Brownian motion is denoted $W(R_t^\varepsilon)$ (with associated historical ternary branching process $\boldsymbol{W}_{R^\varepsilon}(t)$). Finally, let $Z^+(t)$ and $Z^-(t)$ be the processes from Definition 3.3.19 (with associated historical ternary branching processes $\boldsymbol{Z}^+(t)$ and $\boldsymbol{Z}^-(t)$). As ever, all stable processes and subordinators are assumed to satisfy Assumption 3.2.1. Finally, for convenience, we restate Theorem 3.2.5.

Theorem 3.2.5. *Fix a function I satisfying Assumptions 3.1.2. Suppose the initial condition p satisfies Assumptions 3.1.4. Let $\mathscr{T}$ and $d(x,t)$ be as in Section 3.1, F be as in (3.5) and fix $T^* \in (0,\mathscr{T})$. Then there exist $\varepsilon_{\mathbb{d}}(\alpha)$, $a_{\mathbb{d}}(\alpha)$, $c_{\mathbb{d}}(\alpha)$, $m > 0$ such that, for $\varepsilon \in (0,\varepsilon_{\mathbb{d}})$ and $a_{\mathbb{d}}\varepsilon^2|\log\varepsilon| \leq t \leq T^*$,*

(1) for x with $d(x,t) \geq c_{\mathbb{d}} I(\varepsilon)|\log\varepsilon|$, $\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1\right] \geq 1 - m\dfrac{\varepsilon^2}{I(\varepsilon)^2} - mF(\varepsilon)$,

(2) for x with $d(x,t) \leq -c_{\mathbb{d}} I(\varepsilon)|\log\varepsilon|$, $\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1\right] \leq m\dfrac{\varepsilon^2}{I(\varepsilon)^2} + mF(\varepsilon)$.

4.1 A coupling of voting systems in higher dimensions

Recall that $Z^+(t)$ and $Z^-(t)$ satisfy the coupling result Theorem 3.3.20. This is almost identical to the coupling result from the Brownian setting, Proposition 3.3.17. Using this and our one-dimensional result (Theorem 3.3.7) it will be straightforward to prove an analogue of Theorem 3.2.5 for the processes $\boldsymbol{Z}^+(t)$ and $\boldsymbol{Z}^-(t)$ by adapting the techniques of [EFP17]. In this section, we will show that this analogue of Theorem 3.2.5 for the processes $\boldsymbol{Z}^+(t)$ and $\boldsymbol{Z}^-(t)$ (stated in Theorem 4.1.6) will imply Theorem 3.2.5. To do this, we construct the following couplings:

$$\begin{aligned}\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p(\boldsymbol{Y}(t))=1\right] &\overset{Prop\ 4.1.2}{\approx} \mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t))=1\right]\\ &\overset{Prop\ 4.1.4}{\approx} \mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t))=1\right]\\ &\overset{Prop\ 3.3.22}{\approx} \mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t))=1\right]\end{aligned}$$

where the voting system $\mathbb{V}_p^+$ will be defined in Definition 4.1.1.[1] Note that the final coupling of $\mathbb{V}_p^\times(\boldsymbol{Z}^-(t))$ to $\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t))$ has already been developed in Chapter 3.

We now proceed to construct a coupling of $\mathbb{V}_p(\boldsymbol{Y}(t))$ to $\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t))$. To begin, we introduce the positively and negatively biased asymmetric marked voting procedures.

Definition 4.1.1 **($\mathbb{V}_p^+, \mathbb{V}_p^-$).** Let $\varepsilon > 0$ and $t \geq 0$. Let b_ε be as in (3.38). For a fixed function $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$, we define the *positively biased* (resp. *negatively biased*) asymmetric marked voting procedures on $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t))$ as follows.

(1) At each branch point in $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t))$, the parent particle j marks each of their three offspring $(j,1),(j,2)$ and $(j,3)$ independently with probability b_ε. All marked particles vote 1 with probability 1 (resp. 0 for the negatively biased procedure).

(2) Each unmarked leaf i of $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t))$, independently, votes 1 with probability $p(W_i(R_i^\varepsilon(t))$ and otherwise votes 0.

(3) At each branch point in $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t))$, if the parent particle k is unmarked, she votes according to the majority vote of her three offspring $(k,1),(k,2)$ and $(k,3)$.

Define $\mathbb{V}_p^+$ (resp. $\mathbb{V}_p^-$) to be the vote associated to the root $\emptyset$ of the ternary branching truncated stable tree under the positively biased (negatively biased) asymmetric marked voting procedure described above.

[1] As we will see, a similar series of couplings also relate $\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p(\boldsymbol{Y}(t))=1\right]$ to $\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t))=1\right]$.

We can now prove the first coupling result.

Proposition 4.1.2. *For all $\varepsilon > 0$, $x \in \mathbb{R}^{\mathrm{d}}$, $t \geq 0$ and $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$ we have*

$$(1-b_\varepsilon)\mathbb{P}_x^\varepsilon[\mathbb{V}_p^-(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] \leq \mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \leq (1-b_\varepsilon)\mathbb{P}_x^\varepsilon[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] + b_\varepsilon.$$

Proof. This proof proceeds almost identically to the proof of the one-dimensional coupling of voting systems, Theorems 3.3.4 and 3.3.6 using an intermediate asymmetric exponentially marked voting system. More specifically, define the negatively biased exponential marked voting procedure $\widehat{\mathbb{V}}_p^-$ like the exponential marked voting procedure from Definition 3.3.3, except that marked individuals vote 1 with probability 0. Similarly define the positively biased exponential marked voting procedure $\widehat{\mathbb{V}}_p^+$ where marked individuals vote 1 with probability 1. Then, mimicking the proof of Theorem 3.3.4, we can show that, for all $x \in \mathbb{R}^{\mathrm{d}}$,

$$\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^-(\boldsymbol{W}_{R^\varepsilon}(t)) = 1\right] \leq \mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \leq \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1\right]. \tag{4.1}$$

Finally, by conditioning on whether or not the initial ancestor in $\boldsymbol{W}_{R^\varepsilon}(t)$ is marked (just as we did in the proof of Theorem 3.3.6) we obtain

$$\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^-(\boldsymbol{W}_{R^\varepsilon}(t)) = 1\right] \geq (1-b_\varepsilon)\mathbb{P}_x^\varepsilon[\mathbb{V}_p^-(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]$$

and

$$\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1\right] \leq (1-b_\varepsilon)\mathbb{P}_x^\varepsilon[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] + b_\varepsilon,$$

proving the result. □

In the one-dimensional analogue of Proposition 4.1.2 (Theorems 3.3.4, 3.3.6), we were able to couple $\mathbb{V}(\boldsymbol{X}(t))$ and $\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))$ using the (symmetric) exponentially marked voting procedure $\widehat{\mathbb{V}}$. However, we could not adapt this proof to couple $\mathbb{V}_p(\boldsymbol{Y}(t))$ to $\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t))$ (having instead to use the auxiliary voting procedures $\mathbb{V}_p^+$ and $\mathbb{V}_p^-$). This is because the initial condition p is no longer assumed to be symmetric.

To better understand this, let us revisit the proof of Theorem 3.3.4, where we showed that, for $x \geq 0$,

$$\mathbb{P}_x^\varepsilon[\mathbb{V}(\boldsymbol{X}(t)) = 1] \geq \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] \tag{4.2}$$

with the reverse inequality holding when $x < 0$. We then obtained a coupling of $\mathbb{V}(\boldsymbol{X}(t))$ to $\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))$ by showing in Theorem 3.3.6 that

$$\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] = (1-b_\varepsilon)\mathbb{P}_x^\varepsilon[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] + \frac{b_\varepsilon}{2}.$$

To prove Theorem 3.3.4, we used an inductive argument on the number of branching events in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ and $\mathcal{T}(\boldsymbol{X}(t))$. In the base case, when $\mathcal{T}(\boldsymbol{X}(t)) = \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}_0$, the tree with a single leaf, we considered the single individual in $\mathcal{T}(\boldsymbol{X}(t))$, $X(t) \overset{D}{=} B(R_t)$ for $B(t)$ a standard one-dimensional Brownian motion and $R(t)$ an $\frac{\alpha}{2}$-stable subordinator. We saw in equation (3.32) that, if $\tau^\times$ was the first time that R_t made a large jump, and τ was the time of the first branching event in $\mathcal{T}(\boldsymbol{X}(t))$, then if $x \geq 0$ and $p_0(x) = \mathbb{1}_{\{x \geq 0\}}$

$$\mathbb{E}_x^\varepsilon[p_0(B(R_t)) \,|\, \tau^\times < t < \tau] \geq \tfrac{1}{2}, \tag{4.3}$$

from which it followed that

$$\begin{aligned}
&\mathbb{P}_x^t(\mathcal{T}_0)\\
&= \mathbb{E}_x^\varepsilon[p_0\left(B(R_t)\right) \,|\, \tau^\times \geq t, \tau > t]\mathbb{P}_x^\varepsilon[\tau^\times \geq t] + \mathbb{E}_x^\varepsilon[p_0(B(R_t)) \,|\, \tau^\times < t < \tau]\mathbb{P}_x^\varepsilon[\tau^\times < t]\\
&\geq \mathbb{E}_x^\varepsilon[p_0(B(R_t^\varepsilon)) \,|\, \tau > t]\, \mathbb{P}_x^\varepsilon[\tau^\times \geq t] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times < t]. \qquad (4.4)
\end{aligned}$$

The quantity in (4.4) is an upper bound for $\widehat{\mathbb{P}}_x^t(\mathcal{T}_0)$, so by induction we obtained (4.2).

In the multidimensional setting, for a general initial condition p, this argument does not hold. More specifically, (4.3) need not hold since p may not be symmetric; instead, we only have the trivial inequality $\mathbb{E}_x^\varepsilon[p(W(R_t)) \,|\, \tau^\times < t < \tau] \geq 0$. Using this, the multidimensional analogue of (4.4) becomes

$$\mathbb{P}_x^t(\mathcal{T}_0) \geq \mathbb{E}_x^\varepsilon[p(W(R_t^\varepsilon)) \,|\, \tau > t]\, \mathbb{P}_x^\varepsilon[\tau^\times \geq t],$$

where the right hand side is an upper bound for the probability that, conditional on $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t)) = \mathcal{T}_0$, the single individual votes 1 under the negatively biased exponentially marked voting procedure $\widehat{\mathbb{V}}_p^-$ defined in the proof of Proposition 4.1.2. Similarly, we can use the trivial bound $\mathbb{E}_x^\varepsilon[p(W(R_t)) \,|\, \tau^\times < t < \tau] \leq 1$ to obtain

$$\mathbb{P}_x^t(\mathcal{T}_0) \leq \mathbb{E}_x^\varepsilon[p(W(R_t^\varepsilon)) \,|\, \tau > t]\, \mathbb{P}_x^\varepsilon[\tau^\times \geq t] + \mathbb{P}_x^\varepsilon[\tau^\times \leq t]$$

where the right hand side is equal to the probability that, conditional on $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t)) = \mathcal{T}_0$, the single individual in votes 1 under the positively biased exponentially marked voting procedure, $\widehat{\mathbb{V}}_p^+$. These bounds, together with an inductive argument, can be used to prove equation (4.1) from the proof of Proposition 4.1.2.

Remark 4.1.3. Just as in Remark 3.3.8, we can write down the partial differential equation solved by $\mathbb{P}_x^\varepsilon[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]$ and $\mathbb{P}_x^\varepsilon[\mathbb{V}_p^-(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]$. Denote the infinitesimal generator of $(W(R_t^\varepsilon))_{t \geq 0}$ by $\mathcal{L}^\varepsilon$. Then it is straightforward to verify, using similar arguments to those in the proof of Theorem 3.2.3, that $v_+^\varepsilon(t,x) := \mathbb{P}_x^\varepsilon[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]$ solves

$$\partial_t v_+^\varepsilon = \mathcal{L}^\varepsilon v_+^\varepsilon + \varepsilon^{-2} f_+(v_+^\varepsilon), \quad v_+^\varepsilon(0,x) = p(x)$$

and $v_-^\varepsilon(t,x) := \mathbb{P}_x^\varepsilon[\mathbb{V}_p^-(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]$ solves

$$\partial_t v_-^\varepsilon = \mathcal{L}^\varepsilon v_+^\varepsilon + \varepsilon^{-2} f_-(v_-^\varepsilon), \quad v_-^\varepsilon(0,x) = p(x)$$

where the nonlinearities f_+ and f_- are given by

$$f_+(x) := x(1-x)(2x-1) - 2b_\varepsilon^3(1-x)^3 - 3b_\varepsilon^2(1-x)^2(2x-1) + 6b_\varepsilon x(1-x)^2$$

and

$$f_-(x) := x(1-x)(2x-1) + 2b_\varepsilon^3 x^3 - 3b_\varepsilon^2 x^2(2x-1) - 6b_\varepsilon x^2(1-x).$$

Proposition 4.1.2 relates solutions to these equations to the original (scaled) fractional Allen–Cahn equation, equation (3.2).

The positively and negatively biased voting systems can be compared to our (symmetric) marked system as follows. Combining Propositions 4.1.2 and 4.1.4 will give us the desired comparison between $\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1]$ and $\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]$.

Proposition 4.1.4. *There exists $C > 0$ such that, for all $\varepsilon > 0$, $x \in \mathbb{R}^{\mathrm{d}}$, $t \geq 0$ and $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$,*

$$\sup_{x \in \mathbb{R}^{\mathrm{d}}} \left(\mathbb{P}_x^\varepsilon[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] - \mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] \right) \leq C b_\varepsilon \tag{4.5}$$

$$\sup_{x \in \mathbb{R}^{\mathrm{d}}} \left(\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] - \mathbb{P}_x^\varepsilon[\mathbb{V}_p^-(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] \right) \leq C b_\varepsilon. \tag{4.6}$$

Proof. We prove only (4.5), noting that (4.6) follows by symmetric arguments. Define $g_+ : [0,1] \to [0,1]$ by $g_+(q) = g((1-b_\varepsilon)q + b_\varepsilon)$ where g is the ordinary majority voting function. This is the probability that an unmarked parent particle votes 1 under $\mathbb{V}_p^+$, in the special case when the three offspring are independent and each have probability q of voting 1 if they are unmarked. Write τ for the time of the first branching event in $\boldsymbol{W}_{R^\varepsilon}(\cdot)$. To ease notation, set

$$u_\times^\varepsilon(t,x) = \mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] \quad \text{and} \quad u_+^\varepsilon(t,x) = \mathbb{P}_x^\varepsilon[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1].$$

Then, by the Markov property at time $t \wedge \tau$ and definition of $\mathbb{V}_p^\times$ and $\mathbb{V}_p^+$ (noting that the initial ancestor is never marked in both schemes) we have

$$\begin{aligned} u_\times^\varepsilon(t,x) &= \mathbb{E}_x^\varepsilon \left[g_\times(u_\times^\varepsilon(t-\tau, W(R_\tau^\varepsilon))) \mathbb{1}_{\tau \leq t} \right] + \mathbb{E}_x^\varepsilon \left[p(W(R_t^\varepsilon)) \mathbb{1}_{\tau > t} \right] \\ u_+^\varepsilon(t,x) &= \mathbb{E}_x^\varepsilon \left[g_+ \left(u_+^\varepsilon(t-\tau, W(R_\tau^\varepsilon)) \right) \mathbb{1}_{\tau \leq t} \right] + \mathbb{E}_x^\varepsilon [p(W(R_t^\varepsilon)) \mathbb{1}_{\tau > t} \right]. \end{aligned}$$

It follows that

$$|u^\varepsilon_\times(t,x) - u^\varepsilon_+(t,x)| \le \mathbb{E}^\varepsilon_x\left[\left|g_\times(u^\varepsilon_\times(t-\tau, W(R^\varepsilon_\tau))) - g_+(u^\varepsilon_+(t-\tau, W(R^\varepsilon_\tau)))\right| \mathbb{1}_{\tau\le t}\right].$$

By definition of $g_\times$ and g_+, and since g is Lipschitz with constant $\frac{3}{2}$, we have

$$\begin{aligned}
&|u^\varepsilon_\times(t,x) - u^\varepsilon_+(t,x)| \\
&\le \tfrac{3}{2}\mathbb{E}^\varepsilon_x\left[\left|(1-b_\varepsilon)(u^\varepsilon_\times(t-\tau, W(R^\varepsilon_\tau)) - u^\varepsilon_+(t-\tau, W(R^\varepsilon_\tau))) - \tfrac{b_\varepsilon}{2}\right| \mathbb{1}_{\{\tau\le t\}}\right] \\
&\le \tfrac{3}{4}b_\varepsilon + \tfrac{3}{2}(1-b_\varepsilon)\mathbb{E}^\varepsilon_x\left[\left|u^\varepsilon_\times(t-\tau, W(R^\varepsilon_\tau)) - u^\varepsilon_+(t-\tau, W(R^\varepsilon_\tau))\right| \mathbb{1}_{\{\tau\le t\}}\right] \\
&= \tfrac{3}{4}b_\varepsilon + \tfrac{3}{2}(1-b_\varepsilon)\int_0^t \frac{e^{-\rho\varepsilon^{-2}}}{\varepsilon^2}\mathbb{E}^\varepsilon_x\left[|u^\varepsilon_\times(t-\rho, W(R^\varepsilon_u)) - u^\varepsilon_+(t-\rho, W(R^\varepsilon_u))|\right] d\rho \\
&\le \tfrac{3}{4}b_\varepsilon + \tfrac{3}{2}(1-b_\varepsilon)\int_0^t \frac{e^{-\rho\varepsilon^{-2}}}{\varepsilon^2}||u^\varepsilon_\times(t-\rho,\cdot) - u^\varepsilon_+(t-\rho,\cdot)||_\infty d\rho,
\end{aligned}$$

where $||\cdot||_\infty$ denotes the uniform norm, and we have used that $\tau \sim exp(\varepsilon^{-2})$ is independent of the spatial motion. Noting that the above inequality holds for all $x \in \mathbb{R}^{\mathrm{d}}$, and applying the change of variables $\rho \mapsto t-\rho$, we obtain

$$||u^\varepsilon_\times(t,\cdot) - u^\varepsilon_+(t,\cdot)||_\infty \le \tfrac{3}{4}b_\varepsilon + \tfrac{3}{2}e^{-t\varepsilon^{-2}}\int_0^t e^{\rho\varepsilon^{-2}}\varepsilon^{-2}||u^\varepsilon_\times(\rho,\cdot) - u^\varepsilon_+(\rho,\cdot)||_\infty d\rho.$$

By an adaptation of Grönwall's inequality, available, for instance, in [Dra03, Theorem 15],

$$\begin{aligned}
||u^\varepsilon_\times(t,\cdot) - u^\varepsilon_+(t,\cdot)||_\infty &\le \tfrac{3}{4}b_\varepsilon \exp\left(\tfrac{3}{2}\left(\int_0^t \exp\left(-s\varepsilon^{-2}\right)\varepsilon^{-2}\right) ds\right) \\
&= \tfrac{3}{4}b_\varepsilon \exp\left(\tfrac{3}{2}\mathbb{P}[\tau\le t]\right) \\
&\le \tfrac{3}{4}b_\varepsilon \exp\left(\tfrac{3}{2}\right).
\end{aligned}$$

Setting $C := \frac{3}{4}\exp\left(\frac{3}{2}\right)$ gives the result. □

Now, using the coupling result Proposition 3.3.22 from Chapter 3, we obtain our main coupling result of this section.

Corollary 4.1.5. *Let $\varepsilon \in (0,1)$, $x \in \mathbb{R}^{\mathrm{d}}$ and $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$. Let F be as in (3.5). Then there exists $a_{\mathrm{d}}(\alpha) > 0$ and $m > 0$ such that, for all $t \ge a_{\mathrm{d}}\varepsilon^2|\log\varepsilon|$,*

$$\mathbb{P}^\varepsilon_x[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \le \mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^-(t)) = 1] + mF(\varepsilon) + mb_\varepsilon$$

and

$$\mathbb{P}^\varepsilon_x[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \ge (1-b_\varepsilon)\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^+(t)) = 1] - mF(\varepsilon) - mb_\varepsilon.$$

Proof. We prove only the first inequality, noting that the second follows by similar arguments. By Propositions 4.1.2, 4.1.4, and 3.3.22 there exists $m_1, m_2, C > 0$ such that

$$\begin{aligned}\mathbb{P}^\varepsilon_x[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] &\leq (1-b_\varepsilon)\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] + (C+1)b_\varepsilon \\ &\leq (1-b_\varepsilon)\left(\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^-(t)) = 1] + m_1 e^{-\frac{t}{\varepsilon^2}} + m_2 F(\varepsilon)\right) + (C+1)b_\varepsilon.\end{aligned}$$

Choose a_{d} sufficiently large so that, for $t \geq a_{\mathrm{d}}\varepsilon^2|\log\varepsilon|$, $e^{-\frac{t}{\varepsilon^2}} \leq F(\varepsilon)$. Choosing m sufficiently large then gives the upper bound. □

Next, we will state our main theorem for $\boldsymbol{Z}^+(t)$ and $\boldsymbol{Z}^-(t)$ and show using Corollary 4.1.5 that it implies Theorem 3.1.5. Recall that $u_- = \frac{3}{4}b_\varepsilon^2 + \mathcal{O}(b_\varepsilon^3)$ and $u_+ = 1 - \frac{3}{4}b_\varepsilon^2 + \mathcal{O}(b_\varepsilon^3)$.

Theorem 4.1.6. *Fix I satisfying Assumptions 3.1.2 and $k \in \mathbb{N}$. Suppose the initial condition p satisfies Assumptions 3.1.4. Let $\mathscr{T}$ and $d(x,t)$ be as in Section 3.1, F be as in (3.5) and fix $T^* \in (0, \mathscr{T})$. Let u_+, u_- be as in (3.44) and (3.45). Then there exists $\varepsilon_{\mathrm{d}}(\alpha,k), a_{\mathrm{d}}(\alpha,k), c_{\mathrm{d}}(\alpha,k) > 0$ such that, for $\varepsilon \in (0,\varepsilon_{\mathrm{d}})$ and $a_{\mathrm{d}}\varepsilon^2|\log\varepsilon| \leq t \leq T^*$,*

(1) for x with $d(x,t) \geq c_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$, $\mathbb{P}^\varepsilon_x\left[\mathbb{V}^\times_p(\boldsymbol{Z}^+(t)) = 1\right] \geq u_+ - \varepsilon^k$,

(2) for x with $d(x,t) \leq -c_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$, $\mathbb{P}^\varepsilon_x\left[\mathbb{V}^\times_p(\boldsymbol{Z}^-(t)) = 1\right] \leq u_- + \varepsilon^k$.

To see that this implies Theorem 3.1.5, let $k \in \mathbb{N}$ and suppose

$$\mathbb{P}^\varepsilon_x\left[\mathbb{V}^\times_p(\boldsymbol{Z}^+(t)) = 1\right] \geq u_+ - \varepsilon^k.$$

By Corollary 4.1.5, this implies

$$\mathbb{P}^\varepsilon_x[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \geq (1-b_\varepsilon)(u_+ - \varepsilon^k) - mF(\varepsilon) - mb_\varepsilon$$

for some $m > 0$. Since $u_+ \geq 1 - b_\varepsilon$, it is straightforward to see that, for $\varepsilon > 0$ sufficiently small, we may increase m as necessary so that

$$\mathbb{P}^\varepsilon_x[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \geq 1 - mF(\varepsilon) - m\frac{\varepsilon^2}{I(\varepsilon)^2}.$$

Similar arguments using Theorem 4.1.6 (2) prove the lower bound in Theorem 3.1.5.

4.2 Generation of interface

We now show that in a time $\mathcal{O}(\varepsilon^2|\log\varepsilon|)$, an interface of width $\mathcal{O}(I(\varepsilon)|\log\varepsilon|)$ is created.[2] We will make use of the following one-dimensional result, where we recall that $\mathbb{V}^\times = \mathbb{V}^\times_{\widehat{p}_0}$ is the marked majority voting system with initial condition

$$\widehat{p}_0(x) = u_+\mathbb{1}_{\{x\geq 0\}} + u_-\mathbb{1}_{\{x\leq 0\}}.$$

Proposition 4.2.1. *Let a_1 be as in Lemma 3.3.12 and fix $k \in \mathbb{N}$. Then there exists $\varepsilon_{\mathrm{d}}(k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $t \geq a_1(k)\varepsilon^2|\log\varepsilon|$ and $x \in \mathbb{R}$,*

$$u_- - \varepsilon^k \leq \mathbb{P}^\varepsilon_x[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] \leq u_+ + \varepsilon^k. \tag{4.7}$$

Proof. We prove the right hand inequality in (4.7) and note that the left hand inequality follows by very similar arguments. It is easy to verify that $\delta := g'_\times(u_+) = \mathcal{O}(b_\varepsilon) < 1$, and by the Mean Value Theorem, since $g'_\times$ is decreasing on $[u_+, 1]$, for all $q \in (0, 1 - u_+]$,

$$g_\times(u_+ + q) - g_\times(u_+) \leq \delta q.$$

Since u_+ is a fixed point of $g_\times$, for ε sufficiently small, $g_\times(q) < q$ for $q \in (u_+, 1]$ so iterating the above inequality as in the proof of Lemma 3.3.10 gives us

$$g^{(n)}_\times(u_+ + q) - u_+ \leq \delta^n q$$

for all $q \in (0, 1 - u_+]$. In particular,

$$g^{(n)}_\times(u_+ + (1 - u_+)) - u_+ \leq \delta^n(1 - u_+) \leq \varepsilon^k$$

after $n \geq C(k)|\log\varepsilon|$ iterations, for some $C(k) > 0$. That is, $g^{(n)}_\times(1) \leq u_+ + \varepsilon^k$ if $n \geq C(k)|\log\varepsilon|$. We note that, since $g_\times$ is increasing on $[0, 1]$, the largest value of the iterates of $g_\times(x)$ will be when $x = 1$. Finally, by Lemma 3.3.12, for $t \geq a_1(k)\varepsilon^2|\log\varepsilon|$,

$$\mathbb{P}^\varepsilon_x\left[\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) \supset \mathcal{T}^{reg}_{A(k)|\log\varepsilon|}\right] \geq 1 - \varepsilon^k.$$

Therefore when $t \geq a_1(k)\varepsilon^2|\log\varepsilon|, \mathbb{P}^\varepsilon_x[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] \leq u_+ + 2\varepsilon^k$. □

It is straightforward to adapt the proof of Lemma 3.3.12 to show that, for all $k \in \mathbb{N}$ and $A(k)$ as in Lemma 3.3.10, there exists $\rho^+_{\mathrm{d}}(k) > 0$ and $\varepsilon_{\mathrm{d}} > 0$ such that, for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $x \in \mathbb{R}^{\mathrm{d}}$ and $t \geq \rho^+_{\mathrm{d}}(k)\varepsilon^2|\log\varepsilon|$,

$$\mathbb{P}^\varepsilon_x\left[\mathcal{T}(\boldsymbol{Z}^+(t)) \supset \mathcal{T}^{reg}_{A(k)|\log\varepsilon|}\right] \geq 1 - \varepsilon^k.$$

[2]Here, we refer to the solution interface associated to the partial differential equation solved by $\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^-(t)) = 1]$ with initial condition p.

Similarly, there exists $\rho_{\mathrm{d}}^-(k)$ and $\varepsilon'_{\mathrm{d}} > 0$ such that, for all $\varepsilon \in (0, \varepsilon'_{\mathrm{d}})$ and $t \geq \rho_{\mathrm{d}}^- \varepsilon^2 |\log \varepsilon|$,

$$\mathbb{P}_x^{\varepsilon}\left[\mathcal{T}(\boldsymbol{Z}^-(t)) \supset \mathcal{T}^{reg}_{A(k)|\log\varepsilon|}\right] \geq 1 - \varepsilon^k.$$

With this, we can adapt the proof of Proposition 4.2.1 to show that, for any $k \in \mathbb{N}$ and ε sufficiently small, if $t \geq \rho_{\mathrm{d}}^+(k)\varepsilon^2|\log\varepsilon|$,

$$u_- - \varepsilon^k \leq \mathbb{P}_x^{\varepsilon}[\mathbb{V}_p^{\times}(\boldsymbol{Z}^+(t)) = 1] \leq u_+ + \varepsilon^k \tag{4.8}$$

for any initial condition p, and if $t \geq \rho_{\mathrm{d}}^-(k)\varepsilon^2|\log\varepsilon|$

$$u_- - \varepsilon^k \leq \mathbb{P}_x^{\varepsilon}[\mathbb{V}_p^{\times}(\boldsymbol{Z}^-(t)) = 1] \leq u_+ + \varepsilon^k. \tag{4.9}$$

In our later proofs, we will want (4.7), (4.8) and (4.9) to hold simultaneously, so it will be useful to define

$$\rho_{\mathrm{d}}(k) := \rho_{\mathrm{d}}^-(k) \vee \rho_{\mathrm{d}}^+(k) \vee a_1(k). \tag{4.10}$$

Proposition 4.2.2. *Let $k \in \mathbb{N}$ and $\rho_{\mathrm{d}}(k)$ be as in equation (4.10). Fix I satisfying Assumptions 3.1.2 and let u_+, u_- be as in (3.44), (3.45). Then there exists $\varepsilon_{\mathrm{d}}(\alpha, k), b_{\mathrm{d}}(\alpha, k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, if*

$$t_{\mathrm{d}}(k,\varepsilon) := \rho_{\mathrm{d}}(k)\varepsilon^2|\log\varepsilon|,$$
$$t'_{\mathrm{d}}(k,\varepsilon) := (2\rho_{\mathrm{d}}(k) + k + 1)\varepsilon^2|\log\varepsilon|,$$

then for $t \in [t_{\mathrm{d}}, t'_{\mathrm{d}}]$,

(1) for $d(x,t) \geq b_{\mathrm{d}}(k)I(\varepsilon)|\log\varepsilon|$, we have $\mathbb{P}_x^{\varepsilon}[\mathbb{V}_p^{\times}(\boldsymbol{Z}^-(t)) = 1] \geq u_+ - \varepsilon^k$,

(2) for $d(x,t) \leq -b_{\mathrm{d}}(k)I(\varepsilon)|\log\varepsilon|$, we have $\mathbb{P}_x^{\varepsilon}[\mathbb{V}_p^{\times}(\boldsymbol{Z}^-(t)) = 1] \leq u_- + \varepsilon^k$.

Remark 4.2.3. By almost identical arguments, Proposition 4.2.2 holds when $\boldsymbol{Z}^-$ is replaced with $\boldsymbol{Z}^+$. Note that our choice of t_{d} and t'_{d} are stricter than needed for this result alone, but it will be convenient to define them in this way for use in later proofs.

Proof. We follow the proof of [EFP17, Proposition 2.16] closely, and consider the multidimensional analogues of Lemmas 3.3.12 and 3.3.13. First, by choice of ρ_{d}^-, there exists $\varepsilon_{\mathrm{d}}(\alpha, k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $x \in \mathbb{R}^{\mathrm{d}}$ and $t \geq \rho_{\mathrm{d}}^-(k)\varepsilon^2|\log\varepsilon|$,

$$\mathbb{P}_x^{\varepsilon}\left[\mathcal{T}(\boldsymbol{Z}^-(t)) \supseteq \mathcal{T}^{\mathrm{reg}}_{A(k)|\log\varepsilon|}\right] \geq 1 - \varepsilon^k \tag{4.11}$$

for $A(k)$ as in Lemma 3.3.10. By standard estimates for the multidimensional standard normal variable, the proof of Lemma 3.3.13 can be adapted to show that there exists $h_{\mathrm{d}}(k) > 0$ and $\varepsilon_{\mathrm{d}}(k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$ and $t \in [t_{\mathrm{d}}, t'_{\mathrm{d}}]$,

$$\mathbb{P}_x^\varepsilon \left[\exists i \in N(s) : |W_i(R_i^\varepsilon(s)) - x| \geq h_{\mathrm{d}}(k) I(\varepsilon)|\log \varepsilon|\right] \leq \varepsilon^k. \tag{4.12}$$

By definition of Z_s^-, $|Z_s^- - W(R_s^\varepsilon)| \leq D_0(k+2)I(\varepsilon)^2|\log\varepsilon|$, for D_0 the constant from Theorem 3.3.18. Therefore (4.12) implies that, for ε sufficiently small, there exists $l_{\mathrm{d}}(k) > h_{\mathrm{d}}(k)$ for which

$$\mathbb{P}_x^\varepsilon \left[\exists i \in N(s) : |Z_i^-(s) - x| \geq l_{\mathrm{d}}(k) I(\varepsilon)|\log \varepsilon|\right] \leq \varepsilon^k. \tag{4.13}$$

Set $b_{\mathrm{d}} = 2l_{\mathrm{d}}$. Recall that $d(x,t)$ is the signed distance between $x \in \mathbb{R}^{\mathrm{d}}$ and $\mathbf{\Gamma}_t$. By the regularity assumption on $\mathbf{\Gamma}_t$ (3.70), there exist $v_0, V_0 > 0$ such that, for $t \leq v_0$ and $x \in \mathbb{R}^{\mathrm{d}}$, $|d(x,0) - d(x,t)| \leq V_0 t$. Reduce ε_{d} if necessary so that $t'_{\mathrm{d}} \leq v_0$ for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$. Let $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $t \in [t_{\mathrm{d}}, t'_{\mathrm{d}}]$ and x be such that $d(x,t) \geq b_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$ and $|Z_i^-(t) - x| \leq l_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$. It follows by the triangle inequality and Lipschitz continuity of $d(\cdot, t)$ that

$$\begin{aligned}
d(Z_i^-(t), 0) &\geq d(x,t) - \left|d(x,t) - d(Z_i^-(t), t)\right| - \left|d(Z_i^-(t), t) - d(Z_i^-(t), 0)\right| \\
&\geq b_{\mathrm{d}} I(\varepsilon)|\log\varepsilon| - l_{\mathrm{d}} I(\varepsilon)|\log\varepsilon| - V_0 t'_{\mathrm{d}} \\
&= \tfrac{1}{2} b_{\mathrm{d}} I(\varepsilon)|\log\varepsilon| - V_0(2\rho_{\mathrm{d}} + k + 1)\varepsilon^2|\log\varepsilon|.
\end{aligned}$$

By Assumption 3.1.2 (B) we may reduce ε_{d} if necessary so that

$$d(Z_i^-(t), 0) \geq \tfrac{1}{4} b_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$$

for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$. Finally, by Assumptions 3.1.4 (B)-(C), and reducing ε_{d} if necessary,

$$\begin{aligned}
p(Z_i^-(t)) &\geq \tfrac{1}{2} + \gamma\left(\tfrac{1}{4} b_{\mathrm{d}} I(\varepsilon)|\log\varepsilon| \wedge r\right) \\
&\geq \tfrac{1}{2} + \varepsilon
\end{aligned} \tag{4.14}$$

for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$. We then combine (4.11), (4.12) and (4.14) exactly as the proof of Theorem 3.3.7, to obtain that, for $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $t \in [t_{\mathrm{d}}, t'_{\mathrm{d}}]$ and x such that $d(x,t) \geq b_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$,

$$\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] \geq u_+ - 3\varepsilon^k.$$

The upper bound is obtained using the same approach. $\square$

4.3 Propagation of the interface

In this section, we will compare $\mathbb{V}_p^\times(\boldsymbol{Z}^-(t))$ to $\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))$, and use this to show that the interface propagates. Throughout this section, define

$$\gamma(t) := K_1 e^{K_2 t} I(\varepsilon)|\log \varepsilon| \tag{4.15}$$

where the choice of K_1, K_2 and ε will be clear in the given context.

Proposition 4.3.1. *Let $l \in \mathbb{N}$ with $l \geq 4$ and fix I satisfying Assumptions 3.1.2. Let t_{d} be as in Proposition 4.2.2. There exist $K_1, K_2 > 0$ such that for $\gamma(\cdot)$ as in (4.15), $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$ and $t \in [t_{\mathrm{d}}(l), T^*]$ we have*

$$\sup_{x \in \mathbb{R}^{\mathrm{d}}} \left(\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] - \mathbb{P}^\varepsilon_{d(x,t)+\gamma(t)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] \right) \leq \varepsilon^l \tag{4.16}$$

and

$$\sup_{x \in \mathbb{R}^{\mathrm{d}}} \left(\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t)) = 0] - \mathbb{P}^\varepsilon_{d(x,t)-\gamma(t)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 0] \right) \leq \varepsilon^l. \tag{4.17}$$

Throughout this section, we will extend the domain of $g_\times : [0,1] \to [0,1]$ to all of $\mathbb{R}$. Namely, we set

$$g_\times(p) = \begin{cases} g_\times(0) & \text{if } p < 0 \\ g_\times(p) & \text{if } p \in [0,1] \\ g_\times(1) & \text{if } p > 1. \end{cases}$$

Key to the proof of Proposition 4.3.1 will be Lemma 4.3.2, which parallels [EFP17, Lemma 2.18]. The proof of Theorem 4.1.6 will then follow easily. We defer the lengthy proof of Lemma 4.3.2 to Section 4.4.

Lemma 4.3.2. *Let $K_1 > 0$, $l \in \mathbb{N}$ with $l \geq 4$, and fix I satisfying Assumptions 3.1.2. Let t'_{d} be as in Proposition 4.2.2. Then there exists $K_2 = K_2(K_1, l) > 0$ and $\varepsilon_{\mathrm{d}}(K_1, K_2, l) > 0$ such that, for $\gamma(\cdot)$ as in (4.15), $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $x \in \mathbb{R}^{\mathrm{d}}$, $s \in [0, (l+1)\varepsilon^2|\log\varepsilon|]$ and $t \in [t'_{\mathrm{d}}(l), T^*]$,*

$$\begin{aligned} &\mathbb{E}_x\left[g_\times\left(\mathbb{P}^\varepsilon_{d(Z_s^-, t-s)+\gamma(t-s)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t-s)) = 1] + \varepsilon^l \right) \right] \\ &\quad \leq \frac{3}{4}\varepsilon^l + \mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{P}^\varepsilon_{B(R_s^\varepsilon)+\gamma(t)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t-s)) = 1] \right) \right] + \mathbb{1}_{\{s \leq \varepsilon^3\}}\varepsilon^l \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} &\mathbb{E}_x\left[g_\times\left(\mathbb{P}^\varepsilon_{d(Z_s^+, t-s)-\gamma(t-s)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t-s)) = 0] + \varepsilon^l \right) \right] \\ &\quad \leq \frac{3}{4}\varepsilon^l + \mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{P}^\varepsilon_{B(R_s^\varepsilon)-\gamma(t)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t-s)) = 0] \right) \right] + \mathbb{1}_{\{s \leq \varepsilon^3\}}\varepsilon^l. \end{aligned} \tag{4.19}$$

Proof of Proposition 4.3.1. We only prove (4.16), since (4.17) follows by completely symmetric arguments. Set $K_1 = b_{\mathrm{d}}(l) + c_1(l)$ for b_{d} as in Proposition 4.2.2 and c_1 as in Theorem 3.3.7. Take $\varepsilon_{\mathrm{d}} > 0$ sufficiently small so that Theorem 3.3.7, Proposition 4.2.2, Proposition 4.2.1 and Lemma 4.3.2 hold for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$. We first observe that, for $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $t \in [t_{\mathrm{d}}(l), t'_{\mathrm{d}}(l)]$ (for t_{d} and t'_{d} as in Proposition 4.2.2) and $x \in \mathbb{R}^{\mathrm{d}}$,

$$\mathbb{P}^{\varepsilon}_{x}[\mathbb{V}^{\times}_{p}(\boldsymbol{Z}^{-}(t)) = 1] - \mathbb{P}^{\varepsilon}_{d(x,t)+\gamma(t)}[\mathbb{V}^{\times}(\boldsymbol{B}_{R^{\varepsilon}}(t)) = 1] \leq \varepsilon^{l}. \tag{4.20}$$

To see this, first suppose that $d(x,t) \leq -b_{\mathrm{d}}(l)I(\varepsilon)|\log \varepsilon|$. Now, reducing ε_{d} if necessary, by Proposition 4.2.2

$$\mathbb{P}^{\varepsilon}_{x}[\mathbb{V}^{\times}_{p}(\boldsymbol{Z}^{-}(t)) = 1] \leq u_{-} + \varepsilon^{l}.$$

Also, by Proposition 4.2.1,

$$\mathbb{P}^{\varepsilon}_{d(x,t)+\gamma(t)}[\mathbb{V}^{\times}(\boldsymbol{B}_{R^{\varepsilon}}(t)) = 1] \geq u_{-} - \varepsilon^{l},$$

hence (4.20) holds. Here, we continue to ignore coefficients in front of polynomial error terms following Remark 3.3.14. If we added a coefficient to the error term in (4.20), it would appear in all polynomial error terms that follow, but would not affect our proof.

Now suppose $d(x,t) \geq -b_{\mathrm{d}}(l)I(\varepsilon)|\log \varepsilon|$. Then, reducing ε if necessary,

$$d(x,t) + \gamma(t) \geq c_1(l)I(\varepsilon)|\log \varepsilon|,$$

so by Theorem 3.3.7, $\mathbb{P}^{\varepsilon}_{d(x,t)+\gamma(t)}[\mathbb{V}^{\times}(\boldsymbol{B}_{R^{\varepsilon}}(t)) = 1] \geq u_{+} - \varepsilon^{l}$. By (4.9),

$$\mathbb{P}^{\varepsilon}_{x}[\mathbb{V}^{\times}_{p}(\boldsymbol{Z}^{-}(t)) = 1] \leq u_{+} + \varepsilon^{l},$$

and (4.20) holds.

It remains to verify (4.20) for $t \in [t'_{\mathrm{d}}, T^{*}]$. Assume for the purpose of a contradiction that there exists $t \in [t'_{\mathrm{d}}, T^{*}]$ such that, for some $x \in \mathbb{R}^{\mathrm{d}}$,

$$\mathbb{P}^{\varepsilon}_{x}[\mathbb{V}^{\times}_{p}(\boldsymbol{Z}^{-}(t)) = 1] - \mathbb{P}^{\varepsilon}_{d(x,t)+\gamma(t)}[\mathbb{V}^{\times}(\boldsymbol{B}_{R^{\varepsilon}}(t)) = 1] > \varepsilon^{l}.$$

Let T' be the infimum of the set of such t, and choose

$$T \in [T', \min(T' + \varepsilon^{l+3}, T^{*})],$$

which is in the set of such t. So there exists some $x \in \mathbb{R}^{\mathrm{d}}$ such that

$$\mathbb{P}^{\varepsilon}_{x}[\mathbb{V}^{\times}_{p}(\boldsymbol{Z}^{-}(T)) = 1] - \mathbb{P}^{\varepsilon}_{d(x,T)+\gamma(T)}[\mathbb{V}^{\times}(\boldsymbol{B}_{R^{\varepsilon}}(T)) = 1] > \varepsilon^{l}. \tag{4.21}$$

We will show

$$\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^-(T))=1] \le \tfrac{7}{8}\varepsilon^l + \mathbb{P}^\varepsilon_{d(x,T)+\gamma(T)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T))=1]. \tag{4.22}$$

Let τ be the time of the first branching event in $\boldsymbol{Z}^-(T)$ and Z^-_τ be the position of the initial ancestor particle at that time. Then, by the Strong Markov Property at time $\tau \wedge (T-t_{\mathbb{d}})$,

$$\begin{aligned}\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^-(t))=1] = \mathbb{E}^\varepsilon_x\left[g_\times(\mathbb{P}^\varepsilon_{Z^-_\tau}[\mathbb{V}^\times_p(\boldsymbol{Z}^-(T-\tau))=1])\mathbb{1}_{\tau\le T-t_{\mathbb{d}}}\right] \\ +\mathbb{E}^\varepsilon_x\left[\mathbb{P}^\varepsilon_{Z^-_{T-t_{\mathbb{d}}}}[\mathbb{V}^\times_p(\boldsymbol{Z}^-(t_{\mathbb{d}}))=1]\mathbb{1}_{\tau\ge T-t_{\mathbb{d}}}\right].\end{aligned} \tag{4.23}$$

Since $T-t_{\mathbb{d}} \ge t'_{\mathbb{d}} - t_{\mathbb{d}} > (l+1)\varepsilon^2|\log\varepsilon|$ and $\tau \sim exp(\varepsilon^{-2})$, the second term on the right side of (4.23) is bounded by

$$\begin{aligned}\mathbb{E}^\varepsilon_x\left[\mathbb{P}^\varepsilon_{Z^-_{T-t_{\mathbb{d}}}}[\mathbb{V}_p(\boldsymbol{Z}^-(t_{\mathbb{d}}))=1]\mathbb{1}_{\tau\ge T-t_{\mathbb{d}}}\right] &\le \mathbb{P}\left[\tau \ge (l+1)\varepsilon^2|\log\varepsilon|\right] \\ &= \varepsilon^{l+1}.\end{aligned} \tag{4.24}$$

To bound the first term on the right hand side of (4.23), we partition over the event $\{\tau \le \varepsilon^{3+l}\}$ (which has probability $\le \varepsilon^{l+1}$) and its complement to obtain

$$\begin{aligned}&\mathbb{E}^\varepsilon_x\left[g_\times(\mathbb{P}^\varepsilon_{Z^-_\tau}[\mathbb{V}^\times_p(\boldsymbol{Z}^-(T-\tau))=1])\mathbb{1}_{\tau\le T-t_{\mathbb{d}}}\right] \\ &\le \mathbb{E}^\varepsilon_x\left[g_\times(\mathbb{P}^\varepsilon_{Z^-_\tau}[\mathbb{V}^\times_p(\boldsymbol{Z}^-(T-\tau))=1])\mathbb{1}_{\varepsilon^{l+1}\le\tau\le T-t_{\mathbb{d}}}\right] + \varepsilon^{l+1} \\ &\le \mathbb{E}^\varepsilon_x\left[g_\times\left(\mathbb{P}^\varepsilon_{d(Z^-_\tau,T-\tau)+\gamma(T-\tau)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T-\tau))=1]+\varepsilon^l\right)\mathbb{1}_{\tau\le T-t_{\mathbb{d}}}\right] + \varepsilon^{l+1},\end{aligned} \tag{4.25}$$

where the last line follows by minimality of T', since $\varepsilon^{l+3} \le \tau \le T-t_{\mathbb{d}}$, so $T-\tau \in [t_{\mathbb{d}}, T')$, and by monotoncity of $g_\times$. Then, conditioning on the value of τ, and noting that the path of the ancestral particle $(B(R^\varepsilon_\cdot))$ is independent of τ,

$$\begin{aligned}&\mathbb{E}^\varepsilon_x\left[g_\times\left(\mathbb{P}^\varepsilon_{d(Z^-_\tau,T-\tau)+\gamma(T-\tau)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T-\tau))=1]+\varepsilon^l\right)\mathbb{1}_{\tau\le T-t_{\mathbb{d}}}\right] \\ &\le \int_0^{(l+1)\varepsilon^2|\log\varepsilon|}\frac{e^{-\varepsilon^{-2}s}}{\varepsilon^2}\mathbb{E}_x\left[g_\times\left(\mathbb{P}^\varepsilon_{d(Z^-_s,T-s)+\gamma(T-s)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T-s))=1]+\varepsilon^l\right)\right]ds \\ &\quad + \mathbb{P}[\tau \ge (l+1)\varepsilon^2|\log\varepsilon|] \\ &\le \int_0^{(l+1)\varepsilon^2|\log\varepsilon|}\frac{e^{-\varepsilon^{-2}s}}{\varepsilon^2}\mathbb{E}_{d(x,T)}\left[g_\times\left(\mathbb{P}^\varepsilon_{B(R^\varepsilon_s)+\gamma(T)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T-s))=1]\right)\right]ds \\ &\quad + \varepsilon^{l+1} + \varepsilon^l\left(\tfrac{3}{4}+\mathbb{P}[\tau\le\varepsilon^3]\right) \\ &\le \mathbb{E}^\varepsilon_{d(x,T)}\left[g_\times\left(\mathbb{P}^\varepsilon_{B(R^\varepsilon_{\tau'})+\gamma(T)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T-\tau'))=1]\right)\mathbb{1}_{\tau'\le T-t_{\mathbb{d}}}\right] \\ &\quad + \tfrac{3}{4}\varepsilon^l + 2\varepsilon^{l+1},\end{aligned} \tag{4.26}$$

where the second inequality follows by Lemma 4.3.2. Here τ' denotes the time of the first branching event in $\boldsymbol{B}_{R^\varepsilon}$, which has the same distribution as τ. The final inequality holds since $T \geq t'_{\mathrm{d}}$, so $T - t_{\mathrm{d}} \geq (l+1)\varepsilon^2|\log\varepsilon|$. Putting (4.24), (4.25) and (4.26) into (4.23), we obtain

$$\begin{aligned}\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^-(T))=1] &\leq \mathbb{E}^\varepsilon_{d(x,T)}\left[g_\times\left(\mathbb{P}^\varepsilon_{B(R^\varepsilon_{\tau'})+\gamma(T)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T-\tau'))=1]\right)\mathbb{1}_{\tau'\leq T-t_{\mathrm{d}}}\right]\\ &\quad+\tfrac{3}{4}\varepsilon^l+4\varepsilon^{l+1}\\ &\leq \mathbb{P}^\varepsilon_{d(x,T)+\gamma(T)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T))=1]+\tfrac{3}{4}\varepsilon^l+4\varepsilon^{l+1},\end{aligned}$$

where the last line follows by the Strong Markov Property for $(\boldsymbol{B}_{R^\varepsilon}(\cdot))$ at time $\tau' \wedge (T-t_{\mathrm{d}})$. We can reduce ε_{d} if necessary so that $4\varepsilon^{l+1}+\frac{3}{4}\varepsilon^l \leq \frac{7}{8}\varepsilon^l$ for all $\varepsilon \in (0,\varepsilon_{\mathrm{d}})$. This gives (4.22), thereby proving (4.16). The inequality (4.17) follows by a similar argument, using (4.19). □

With this, we can now prove Theorem 4.1.6.

Proof of Theorem 4.1.6. Set $c_{\mathrm{d}}(l) := c_1(l) + K_1 e^{K_2 T^*}$. Then, for any $x \in \mathbb{R}^{\mathrm{d}}$ and $t \in [t_{\mathrm{d}}, T^*]$ such that $d(x,t) \leq -c_{\mathrm{d}}(l)I(\varepsilon)|\log\varepsilon|$, we have

$$d(x,t) + K_1 e^{K_2 t} I(\varepsilon)|\log\varepsilon| \leq -c_1(l)I(\varepsilon)|\log\varepsilon|.$$

Then, by Theorem 3.3.7, reducing ε_{d} if necessary so that $\varepsilon_{\mathrm{d}} < \varepsilon_1(l)$,

$$\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^-(t))=1)] \leq u_- + 2\varepsilon^l.$$

Similarly, for x and t such that $d(x,t) \geq c_{\mathrm{d}}(l)I(\varepsilon)|\log\varepsilon|$, by Theorem 3.3.7 and (4.17), $\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^+(t))=1)] \geq u_+ - 2\varepsilon^l$. Theorem 4.1.6 then holds by setting $a_{\mathrm{d}} := \rho_{\mathrm{d}}$. □

Proof of Theorem 3.1.5. This follows immediately by combining Theorem 4.1.6 and Corollary 4.1.5. □

4.4 Proof of Lemma 4.3.2

To prove Lemma 4.3.2, we follow the proof of [EFP17, Lemma 2.18] and consider separately the cases $|d(x,t)| \leq DI(\varepsilon)|\log\varepsilon|$ and $|d(x,t)| \geq DI(\varepsilon)|\log\varepsilon|$, for some large $D > 0$. Since neither the one-dimensional process $B(R^\varepsilon_s)$ nor the multidimensional process Z^+ (or Z^-) travel further than a distance $\mathcal{O}(I(\varepsilon)|\log\varepsilon|)$ in time $s = \mathcal{O}(\varepsilon^2|\log(\varepsilon)|)$ with high probability, if D is sufficiently large and $|d(x,t)| \leq DI(\varepsilon)|\log\varepsilon|$, we will see that the result follows from the main one-dimensional result for $\mathbb{V}_p^\times(\boldsymbol{B}_{R^\varepsilon})$, Theorem 3.3.7. When $|d(x,t)| \le DI(\varepsilon)|\log\varepsilon|$, we apply Theorem 3.3.20 so that, with probability at least $1-\varepsilon^{l+1}$,

$$d(Z_s^-, t-s) \le B(R_s^\varepsilon) + \mathcal{O}(I(\varepsilon)|\log\varepsilon|)s.$$

Using this and monotonicity of $g_\times$ we can bound the left hand side of (4.18) by

$$\mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{P}^\varepsilon_{B(R_s^\varepsilon)+\gamma(t-s)+\mathcal{O}(s)I(\varepsilon)|\log\varepsilon|}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))=1]+\varepsilon^l\right)\right]+\mathcal{O}(\varepsilon^{l+1}). \tag{4.27}$$

We then control (4.27) by considering two cases: when the argument of $g_\times$ in (4.27) is bounded away from $\frac{1}{2}$, and when it is close to $\frac{1}{2}$. In the former case, we use that $|g_\times'(y)| < \frac{2}{3}$ when y is bounded far enough away from $\frac{1}{2}$, together with monotonicty of $g_\times$, to obtain (4.18). In the second case, we apply the slope of the interface result, Corollary 3.3.16, to bound the difference between the two expectations appearing in the inequality (4.18) directly.

Proof of Lemma 4.3.2. Fix $l \ge 4$. For all $u \ge 0$ and $z \in \mathbb{R}$, let

$$\mathbb{Q}_z^{\varepsilon,u} = \mathbb{P}_z^\varepsilon[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(u))=1].$$

Let C_0 be as in (3.69) and c_1 be defined as in Theorem 3.3.7. Let

$$R := 2c_1(l) + 4(l+1)\mathbb{d} + 1 \tag{4.28}$$

and fix K_2 such that

$$K_1(K_2 - C_0) - C_0R - C_1 = c_1(1). \tag{4.29}$$

To start we let $\varepsilon_{\mathbb{d}}(l) = \varepsilon_1(l)$ where $\varepsilon_1(l)$ is defined in Theorem 3.3.7.

Following the proof of [EFP17, Lemma 2.18], we begin by estimating the probability that a $\mathbb{d}$-dimensional subordinated Brownian motion moves further than a distance $I(\varepsilon)|\log\varepsilon|$ in time $s \le (l+1)\varepsilon^2|\log\varepsilon|$. Define the event

$$A_x = \left\{\sup_{u\in[0,R_s^\varepsilon]}|W_u - x| \le 2(l+1)I(\varepsilon)|\log\varepsilon|\right\}.$$

Then, bounding $|W_u - x|$ by the sum of the moduli of $\mathbb{d}$ one-dimensional Brownian motions, and by Lemma A.4.2, which bounds the displacement of the subordinator R_s^ε for small times, we obtain

$$\begin{aligned}\mathbb{P}_x[A_x^c] &\le 2\mathbb{d}\mathbb{P}_0\left[\sup_{u\in[0,R_s^\varepsilon]}B_u > 2(l+1)I(\varepsilon)|\log\varepsilon|\right]\\ &\le 2\mathbb{d}\mathbb{P}_0\left[\sup_{u\in[0,(l+2)I(\varepsilon)^2|\log\varepsilon|]}B_u > 2(l+1)I(\varepsilon)|\log\varepsilon|\right] + 2\mathbb{d}\varepsilon^{l+1}\\ &\le 4\mathbb{d}\mathbb{P}_0\left[B_1 > 2((l+1)|\log\varepsilon|)^{1/2}\right] + 2\mathbb{d}\varepsilon^{l+1}\\ &\le 6\mathbb{d}\varepsilon^{l+1}\end{aligned} \tag{4.30}$$

where the second inequality follows by the reflection principle, and the final inequality follows by identical arguments to those in the proof of Lemma 3.3.13. Now consider the cases

(i) $d(x,t) \le -(2c_1(l) + 2(l+1)\mathbb{d} + K_1 e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|$

(ii) $d(x,t) \ge (2c_1(l) + 2(l+1)\mathbb{d} + K_1 e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|$

(iii) $|d(x,t)| \le (2c_1(l) + 2(l+1)\mathbb{d} + K_1 e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|$.

Case (i): By (3.69), there exist $v_0, V_0 > 0$ such that, if $s \le v_0$ and $x \in \mathbb{R}^{\mathrm{d}}$, then

$$|d(x,t) - d(x,t-s)| \le V_0 s.$$

Reduce $\varepsilon_{\mathbb{d}}$ if necessary to ensure that, for all $\varepsilon \in (0, \varepsilon_{\mathbb{d}})$, $(l+1)\varepsilon^2|\log\varepsilon| \le v_0$. Then if A_x occurs,

$$\begin{aligned}
&d(W(R_s^\varepsilon), t-s) + K_1 e^{K_2(t-s)} I(\varepsilon)|\log\varepsilon| && (4.31)\\
&\le -(2c_1(l) + 2(l+1)\mathbb{d})I(\varepsilon)|\log\varepsilon| + |d(W(R_s^\varepsilon), t-s) - d(x,t)| \\
&\le -(2c_1(l) + 2(l+1)\mathbb{d})I(\varepsilon)|\log\varepsilon| + |d(x,t) - d(x,t-s)| + |W(R_s^\varepsilon) - x| \\
&\le -2c_1(l)I(\varepsilon)|\log\varepsilon| + V_0(l+1)\varepsilon^2|\log\varepsilon|. && (4.32)
\end{aligned}$$

By Assumption 3.1.2 (B), we may reduce $\varepsilon_{\mathbb{d}}$ if necessary so that

$$d(W(R_s^\varepsilon), t-s) + K_1 e^{K_2(t-s)} I(\varepsilon)|\log\varepsilon| \le -c_1(l)I(\varepsilon)|\log\varepsilon|,$$

for all $\varepsilon \in (0, \varepsilon_{\mathbb{d}})$. Then, since $d(Z_s^-, t-s) \le d(W(R_s^\varepsilon), t-s)$,

$$d(Z_s^-, t-s) + K_1 e^{K_2(t-s)} I(\varepsilon)|\log\varepsilon| \le -c_1(l)I(\varepsilon)|\log\varepsilon|.$$

So, by Theorem 3.3.7 and definition of $g_\times$,

$$\begin{aligned}
\mathbb{E}_x\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{d(Z_s^-,t-s)+\gamma(t-s)} + \varepsilon^l\right)\right] &\le \mathbb{E}_x[g_\times(u_- + 2\varepsilon^l)\mathbb{1}_{A_x}] + \mathbb{P}_x\left[A_x^c\right] \\
&\le u_- + 6\mathbb{d}\varepsilon^{l+1} + 12\varepsilon^l b_\varepsilon
\end{aligned}$$

where the last line follows by calculating $g_\times(u_- + 2\varepsilon^l)$ explicitly and reducing $\varepsilon_{\mathbb{d}}$ if necessary.

Next, recall that $g_\times(y) = g((1-b_\varepsilon)y + \frac{b_\varepsilon}{2})$ for $y \in [0,1]$. So

$$g_\times'(y) = 6(1-b_\varepsilon)\left((1-b_\varepsilon)y + \tfrac{b_\varepsilon}{2}\right)\left(1 - \left((1-b_\varepsilon)y + \tfrac{b_\varepsilon}{2}\right)\right).$$

Hence, if

$$(1-b_\varepsilon)(y+\delta)+\tfrac{b_\varepsilon}{2}\le\tfrac{1}{9}\quad\text{or}\quad(1-b_\varepsilon)y+\tfrac{b_\varepsilon}{2}\ge\tfrac{8}{9} \tag{4.33}$$

then

$$g_\times(y+\delta)\le g_\times(y)+\tfrac{2}{3}\delta. \tag{4.34}$$

From Proposition 4.2.1 (since $t-s\ge\rho_{\mathbb{d}}(l)\varepsilon^2|\log\varepsilon|$) and (4.34), reducing ε if necessary,

$$\mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+\gamma(t)}\right)\right]\ge u_- - \tfrac{2}{3}\varepsilon^l.$$

By choosing ε small enough so that $6\mathbb{d}\varepsilon^{l+1}+12\varepsilon^l b_\varepsilon+\frac{2}{3}\varepsilon^l\le\frac{3}{4}\varepsilon^l$ (4.18) holds in this case.

Case (ii): Suppose now that $d(x,t)\ge(2c_1(l)+2(l+1)\mathbb{d}+K_1e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|$. Using this, together with a similar argument to that used to obtain (4.30), we have

$$\mathbb{P}_{d(x,t)}[|B(R^\varepsilon_s)|\ge c_1(l)I(\varepsilon)|\log\varepsilon|]\le\varepsilon^{l+1}. \tag{4.35}$$

It follows that

$$\begin{aligned}\mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+\gamma(t)}\right)\right]&\ge\mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+\gamma(t)}\right)\mathbb{1}_{\{B(R^\varepsilon_s)\ge c_1(l)I(\varepsilon)|\log\varepsilon|\}}\right]\\&\ge g_\times(u_+-\varepsilon^l)-\varepsilon^{l+1}\\&\ge u_+-\varepsilon^{l+1}-12\varepsilon^l b_\varepsilon\end{aligned}$$

where the second inequality follows by Theorem 3.3.7, and in the third inequality we expand $g_\times(u_+-\varepsilon^l)$ and reduced $\varepsilon_{\mathbb{d}}$ if necessary. From Proposition 4.2.1 we can get that, for ε small enough,

$$\mathbb{E}_x\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{d(Z^-_s,t-s)+\gamma(t-s)}+\varepsilon^l\right)\right]\le u_++\tfrac{2}{3}\varepsilon^l.$$

Hence, reducing ε if necessary (4.18) holds in this case.

Case (iii): Finally, suppose $|d(x,t)|\le(2c_1(l)+2(l+1)\mathbb{d}+K_1e^{K_2(t-s)})I(\varepsilon)|\log(\varepsilon)|$. If A_x occurs and $u\in[0,(l+2)I(\varepsilon)^2|\log\varepsilon|]$,

$$\begin{aligned}&|d(W(R^\varepsilon_u),t-u)|\\&\le|W(R^\varepsilon_u)-x|+|d(x,t)|+|d(x,t)-d(x,t-u)|\\&\le(2c_1(l)+4(l+1)\mathbb{d}+K_1e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|+V_0(l+2)I(\varepsilon)^2|\log(\varepsilon)|.\end{aligned}$$

Therefore, reducing ε if necessary, with probability at least $1 - \varepsilon^{l+1}$,

$$|d(W(R_s^\varepsilon), t-s)| \leq (R + K_1 e^{K_2(t-s)}) I(\varepsilon) |\log \varepsilon|$$

for R as in (4.28). Now set

$$\beta = (R + K_1 e^{K_2(t-s)}) I(\varepsilon) |\log \varepsilon|. \tag{4.36}$$

Reduce ε_{d} if necessary so that, for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $\beta \leq c_0/2$, for c_0 as in (3.67). Recall that

$$T_\beta = \inf \left(\{ s \in [0, (l+1)\varepsilon^2 |\log \varepsilon|) : |d(W_s, t-s)| > \beta \} \cup \{t\} \right).$$

Note that $\mathbb{P}[R_s^\varepsilon > T_\beta] \leq 2\varepsilon^{l+1}$: by the above calculation, if A_x occurs, then $T_\beta > R_s^\varepsilon$ with probability at least $\geq 1 - \varepsilon^{l+1}$. Therefore, by Theorem 3.3.20, and reducing ε if necessary so that $T_\beta < t$,

$$d(Z_s^-, t-s) \leq B(R_s^\varepsilon) + C_0 \beta s \tag{4.37}$$

with probability at least $1 - 2\varepsilon^{l+1}$. Then, by monotonicity of $g_\times$ and (4.37), partitioning over $\{R_s^\varepsilon > T_\beta\}$ and A_x, we obtain

$$\begin{aligned} &\mathbb{E}_x \left[g_\times \left(\mathbb{Q}^{\varepsilon, t-s}_{d(Z_s^-, t-s) + \gamma(t-s)} + \varepsilon^l \right) \right] \\ &\leq \mathbb{E}_{d(x,t)} \left[g_\times \left(\mathbb{Q}^{\varepsilon, t-s}_{B(R_s^\varepsilon) + C_0\beta s + \gamma(t-s)} + \varepsilon^l \right) \right] + (2 + 6\mathrm{d})\varepsilon^{l+1}. \end{aligned} \tag{4.38}$$

Let

$$D := \left\{ \left| \mathbb{Q}^{\varepsilon, t-s}_{B(R_s^\varepsilon) + C_0\beta s + \gamma(t-s)} - \tfrac{1}{2} \right| \leq \tfrac{5}{12} \right\}.$$

We consider D and D^c separately to bound the right hand side of (4.38). First suppose the event D occurs. Then, by definition of β (4.36),

$$\begin{aligned} &K_1 e^{K_2 t} I(\varepsilon)|\log \varepsilon| - \left(C_0 \beta s + K_1 e^{K_2(t-s)} I(\varepsilon) |\log \varepsilon| \right) \\ &= \left(K_1 e^{K_2(t-s)} \left(e^{K_2 s} - 1 - C_0 s \right) - C_0 R s \right) I(\varepsilon) |\log \varepsilon| \\ &\geq (K_1(K_2 - C_0) - C_0 R) \, s I(\varepsilon) |\log \varepsilon| \\ &= c_1(1) s I(\varepsilon) |\log \varepsilon| \end{aligned} \tag{4.39}$$

where the final equality follows by (4.29). Reducing ε_{d} if necessary so that $\varepsilon_{\mathrm{d}} < \min(\varepsilon_1(1), \frac{1}{24})$, for $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$ we may apply Corollary 3.3.16 with

$$z = B(R_s^\varepsilon) + C_0 \beta s + K_1 e^{K_2(t-s)} I(\varepsilon) |\log \varepsilon|$$

and

$$w = z + c_1(1) s I(\varepsilon) |\log \varepsilon| \leq B(R_s^\varepsilon) + \gamma(t)$$

to give

$$\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+C_0\beta s+\gamma(t-s)}\mathbb{1}_D \le \left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+\gamma(t)} - \tfrac{1}{48}s\right)\mathbb{1}_D. \tag{4.40}$$

Now suppose the event D^c occurs. Reduce $\varepsilon_{\mathbb{d}}$ if necessary so that

$$\tfrac{1}{12} < \tfrac{1}{9} - \varepsilon^l(1-b_\varepsilon) - \tfrac{b_\varepsilon}{2},$$

which implies (4.33) for $\delta = \varepsilon^l$. Thus, for $\varepsilon \in (0,\varepsilon_{\mathbb{d}})$, we have

$$\begin{aligned} g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+C_0\beta s+\gamma(t-s)} + \varepsilon^l\right)\mathbb{1}_{D^c} &\le \left(g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+C_0\beta s+\gamma(t-s)}\right) + \tfrac{2}{3}\varepsilon^l\right)\mathbb{1}_{D^c} \\ &\le \left(g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+\gamma(t)}\right) + \tfrac{2}{3}\varepsilon^l\right)\mathbb{1}_{D^c} \end{aligned} \tag{4.41}$$

where the first inequality follows by (4.34) and the second inequality by (4.39) and monotonicity of $g_\times$. Putting (4.40) and (4.41) into (4.38), and since $2+6\mathbb{d} \le 8\mathbb{d}$ we obtain

$$\begin{aligned} &\mathbb{E}_x\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{d(Z^-_s,t-s)+\gamma(t-s)} + \varepsilon^l\right)\right] \\ &\le \mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+\gamma(t)} - \tfrac{1}{48}s + \varepsilon^l\right)\mathbb{1}_D\right] + \mathbb{E}_{d(x,t)}\left[\left(g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+\gamma(t)}\right) + \tfrac{2}{3}\varepsilon^l\right)\mathbb{1}_{D^c}\right] \\ &\quad + 8\mathbb{d}\varepsilon^{l+1} \\ &\le \mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+\gamma(t)}\right)\right] + \tfrac{2}{3}\varepsilon^l + \varepsilon^l\mathbb{1}_{\{\frac{1}{48}s\le\varepsilon^l\}} + 8\mathbb{d}\varepsilon^{l+1} \end{aligned}$$

where in the final inequality, we use that $g'_\times(y) \le \frac{3}{2}$ for all $y \in [0,1]$. Further reducing $\varepsilon_{\mathbb{d}}$ if necessary so that $8\mathbb{d}\varepsilon^{l+1} \le \frac{1}{12}\varepsilon^l$ and $48\varepsilon^l \le \varepsilon^3$ for all $\varepsilon \in (0,\varepsilon_{\mathbb{d}})$ gives the result. □

Chapter 5

The stochastic fractional Allen–Cahn equation

In this chapter, we prove that after rescaling the SΛFVS with stable radii started from appropriately chosen initial conditions converges to a set whose boundary evolves as mean curvature flow (this is Theorem 3.1.11). Crucially, the branching stable trees used throughout the deterministic setting are now the historical SΛFVS dual, which exhibits both branching and coalescing events. Following [EFP17], we show that, as our scaling parameter $n \to \infty$, this historical dual process is close to a branching pure jump process. We then diverge from the format of proof in [EFP17], and use a similar proof to that of our coupling from Proposition 3.3.22 to compare the root vote under majority voting of the (approximately) stable branching jump process to the root vote under marked majority voting of the truncated subordinated Brownian tree. We then apply the results from the previous section to conclude the proof.

5.1 A branching-coalescing dual for the SΛFVS

Throughout this chapter, fix $\alpha \in (1, 2)$ and a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} \varepsilon_m = 0 \quad \text{and} \quad \lim_{m\to\infty} (\log m)^{\frac{1}{2}} \varepsilon_m = \infty.$$

Fix $0 < \beta < \frac{1}{2\alpha}$. Let $n \in \mathbb{N}$ and recall that, at the n-th stage of rescaling, the SΛFVS with stable radii $(w_t^n)_{t\geq 0}$ (Definition 3.1.9) has impact parameter u_n and selection parameter s_n given by

$$u_n = \frac{u}{n^{1-\beta\alpha} I(\varepsilon_n)^{2-\alpha}} \quad \text{and} \quad s_n = \frac{I(\varepsilon_n)^{2-\alpha}}{\varepsilon_n^2 n^{\beta\alpha}}, \tag{5.1}$$

where I is any function satisfying Assumptions 3.1.2. Fix $0 < \mathcal{R} \leq 1$ and set $\mathcal{R}_n := n^{-\beta}\mathcal{R}$. Then the driving Poisson Point Process of the scaled SΛFVS, denoted Π^n, is defined on $[0,\infty)\times\mathbb{R}^{\mathrm{d}}\times(\mathcal{R}_n,\infty)$ with intensity measure $ndt\otimes n^{\beta}dx\otimes\mu^n(dr)$ where

$$\mu^n(dr)=\frac{\mathbb{1}_{\{r\geq n^{-\beta}\}}}{n^{\beta\alpha+\mathrm{d}\beta}r^{\alpha+\mathrm{d}+1}}. \tag{5.2}$$

Let $B_r(x)$ denote the d-ball centred at x of radius r throughout. The dual process $(\mathcal{P}_t^n)_{t\geq 0}$ to the SΛFVS is then defined as follows.

Definition 5.1.1 (SΛFVS dual, [EFP17, Definition 3.1])**.** For $n\in\mathbb{N}$, the process $(\mathcal{P}_t^n)_{t\geq 0}$ is the $\bigcup_{l\geq 1}(\mathbb{R}^{\mathrm{d}})^l$-valued Markov process with dynamics defined as follows. The process is started with a single individual $\mathcal{P}_0^n=z$, and for $t\geq 0$, $\mathcal{P}_t^n=(\xi_1^n(t),...,\xi_{N(t)}^n(t))$ for some $N(t)\in\mathbb{N}$. At each event $(t,x,r)\in\Pi^n$, independently of all else, the event is said to be *neutral* with probability $1-s_n$. In this case:

(1) For each $\xi_i^n(t-)\in B_r(x)$, independently tag the corresponding individual with probability u_n;

(2) if at least one individual is tagged, all tagged individuals coalesce into a single offspring individual, whose location is drawn uniformly at random from within $B_r(x)$.

With the complementary probability s_n, the event is said to be *selective*, in which case:

(1) For each $\xi_i^n(t-)\in B_r(x)$, independently tag the corresponding individual with probability u_n;

(2) if at least one individual is tagged, all of the tagged individuals are replaced by three offspring individuals, whose locations are drawn independently and uniformly from within $B_r(x)$.

In both cases, if no individual is tagged, then nothing happens.

Of course, it would be more biologically accurate to replace all instances of the term ‘offspring’ in Definition 5.1.1 by ‘parents’ or ‘potential parents’. However, for simplicity and consistency with the deterministic setting, we will continue to use the term offspring.

Before making precise the duality relation between $(\mathcal{P}_t^n)_{t\geq 0}$ and $(w_t^n)_{t\geq 0}$, we require some more definitions. Define the *historical process* of branching and coalescing lineages by

$$\Xi^n(t):=(\mathcal{P}_s^n)_{0\leq s\leq t}.$$

We write $\mathbb{P}_x$ for the law of Ξ^n when $\mathcal{P}_0^n = x$ and $\mathbb{E}_x$ for the corresponding expectation. For all $\mathbf{i} = (i_1, i_2, ...) \in \{1,2,3\}^{\mathbb{N}}$, let $(\xi_{\mathbf{i}}^n(s), 0 \le s \le t) \subset \Xi(t)$ denote the $\mathbb{R}^{\mathrm{d}}$-valued path that jumps to the location of the i_k-th offspring when the individual in $\mathcal{P}_s^n$ at its location is affected by the k^{th} selective event. We refer to the path $(\xi_{\mathbf{i}}^n(s), 0 \le s \le t)$ as an *ancestral lineage*. We define a majority voting procedure on $\Xi^n(t)$ as follows.

Definition 5.1.2 ($\mathbb{V}_p(\Xi(t))$)**.** For $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$ and all $0 \le j \le N(t)$, the individual $\xi_j^n(t)$ votes 1 with probability $p(\xi_j^n(t))$ and otherwise votes 0. The votes of different individuals are independent, and as we trace backwards in time through $\Xi(t)$,

(1) at each neutral event in Π^n, all individuals that are tagged in the event adopt the vote of the offspring individual of the event;

(2) at each selective event in Π^n, all individuals that are tagged in the event adopt the majority vote of the three offspring individuals of the event.

With the voting procedure defined above, set $\mathbb{V}_p(\Xi(t))$ to be the vote associated to the root $\emptyset$.

The duality relation between Ξ and the SΛFVS is then identical to that of [EFP17, Theorem 3.4]. Since the SΛFVS is only defined Lebesgue a.e., in order to state this we must consider the integral of $w_t^n(\cdot)$ against a test function.

Theorem 5.1.3. *The SΛFVS driven by Π^n, $(w_t^n(x), x \in \mathbb{R}^{\mathrm{d}})_{t\ge 0}$, is dual to the historical process $(\Xi^n(t), t \ge 0)$ in the sense that, for every $\psi \in C(\mathbb{R}^{\mathrm{d}}) \cap L^1(\mathbb{R}^{\mathrm{d}})$, we have*

$$\begin{aligned}\mathbb{E}_p\left[\int_{\mathbb{R}^{\mathrm{d}}} \psi(x) w_t^n(x) dx\right] &= \int_{\mathbb{R}^{\mathrm{d}}} \psi(x) \mathbb{E}_x\left[\mathbb{V}_p\left(\Xi^n(t)\right)\right] dx \\ &= \int_{\mathbb{R}^{\mathrm{d}}} \psi(x) \mathbb{P}_x\left[\mathbb{V}_p\left(\Xi^n(t)\right) = 1\right] dx.\end{aligned}$$

Here we have abused notation and written expectations with respect to different measures on either side of the above equation. In both cases, the subscript of the expectation denotes the initial value of both processes.

We omit a formal proof of this result as it requires lengthy generator calculations which are a straightforward adaptation of those developed in [EVY20]. To see why it should be true, recall that, up to a Lebesgue null set,

$$w_t^n(x) = \text{the proportion of type } a \text{ alleles at time } t \text{ and location } x$$

at the n-th stage of rescaling. To determine the type of any individual that is at location x at time t, we look to the most recent time in the past when an event both covered x and may have affected that individual's type (which happens with probability u_n). For a neutral event, the individual will adopt the type of its parent, whose location is chosen uniformly at random from within a ball containing x. For a selective event, the individual adopts the majority type of three potential parents whose locations are independently and uniformly chosen from within a ball containing x. We continue this procedure to find the type of each parent, terminating once we reach time 0. In this way, we have constructed a branching-coalescing system, with root x, of 'potential ancestors' of any individual at x at time t, corresponding precisely to $\boldsymbol{\Xi}(t)$ started at x. The probability that our given individual is of type a then corresponds to the probability that the root vote of $\boldsymbol{\Xi}(t)$ equals one under the voting procedure described above.

Equipped with this duality relation, we can now restate our main result, Theorem 3.1.11. Recall from (3.12) that, given a sequence $(\varepsilon_m)_{m\geq 0}$,

$$G(n) = n^{-\beta\left(1-\frac{\alpha}{2}\right)} I(\varepsilon_n)^{\alpha-2} + \frac{\varepsilon_n^2}{I(\varepsilon_n)^2} + n^{-\frac{\beta}{2\alpha}\left(1-\frac{\alpha}{2}\right)}\varepsilon_n^{-\frac{2}{\alpha}} + I(\varepsilon_n)^{\alpha-1}. \tag{5.3}$$

Theorem 5.1.4. *Let $\beta \in \left(0, \frac{1}{2\alpha}\right)$ and consider a sequence $(\varepsilon_n)_{n=1}^{\infty}$ satisfying $\varepsilon_n \to 0$ and $(\log n)^{\frac{1}{2}}\varepsilon_n \to \infty$ as $n \to \infty$. Fix I satisfying Assumptions 3.1.2. Let G be as above and $p : \mathbb{R}^d \to [0,1]$ satisfy Assumptions 3.1.4. Let $(w_t^n)_{t\geq 0}$ be the SΛFVS driven by Π^n with u_n, s_n given by (5.1) and initial condition $w_0^n(x) = p(x)$. Define $\mathscr{T}$ and $d(x,t)$ as in Theorem 3.1.5 and take $T^* < \mathscr{T}$. There exists $n_*(\alpha) \in \mathbb{N}$ and $a_*(\alpha), c_*(\alpha), m_* \in (0,\infty)$ such that, for all $n \geq n_*$ and t satisfying $a_*\varepsilon_n^2|\log\varepsilon_n| \leq t \leq T^*$,*

(1) for x with $d(x,t) \geq c_* I(\varepsilon_n)|\log\varepsilon_n|$, we have $\mathbb{P}_x\left[\mathbb{V}_p(\boldsymbol{\Xi}^n(t)) = 1\right] \geq 1 - m_* G(n)$,

(2) for x with $d(x,t) \leq -c_* I(\varepsilon_n)|\log\varepsilon_n|$, we have $\mathbb{P}_x\left[\mathbb{V}_p(\boldsymbol{\Xi}^n(t)) = 1\right] \leq m_* G(n)$.

To eliminate the complication of coalescing lineages, we will couple $\boldsymbol{\Xi}^n$ to a pure jump branching process (asymptotically in n). As we work towards this goal, it will be useful to keep in mind the following quantities. Let $V_r(x,y)$ denote the volume of $B_r(x) \cap B_r(y)$, and V_r denote the volume of $B_r(0)$. The rate at which lineages jump from y to $y+z$ in $\boldsymbol{\Xi}^n$ is given by

$$\begin{aligned} m_n(dz) &= n u_n n^{\mathrm{d}\beta} \int_{\mathcal{R}_n}^{\infty} \frac{V_r(0,z)}{V_r}\mu^n(dr)dz \\ &= \frac{u}{I(\varepsilon_n)^{2-\alpha}} \int_{\mathcal{R}_n}^{\infty} \frac{V_r(0,z)}{V_r}\frac{1}{r^{1+\alpha+\mathrm{d}}}drdz. \end{aligned} \tag{5.4}$$

To see this, first note that, by spatial homogeneity, we may equivalently consider jumps from 0 to z. For a lineage to make this jump, it must be impacted by an event (which happens with probability u_n) whose centre is contained in the set $V_{n^{-\beta}r}(0,z)$, with r chosen according to $\mu(n^\beta dr) = \mu^n(dr)$. Incorporating our temporal scaling, this implies that the rate at which such a centre, x, is chosen is given by $nu_n V_{n^{-\beta}r}(0,z)\mu^n(dr) = nu_n n^{\mathrm{d}\beta} V_r(0, n^\beta z)\mu^n(dr)$. The location of the offspring is chosen uniformly at random from $B_r(x)$, so the probability that the offspring is at location z is dz/V_r. Putting this together and incorporating our spatial scaling yields (5.4).

It can be shown after some straightforward calculations (mimicking [EFP17, Equation 3.3]) that a lineage is affected by a selective event at rate

$$\left(\frac{un^{\beta\alpha}}{I(\varepsilon_n)^{2-\alpha}} V_1 \int_{\mathcal{R}}^\infty r^{\mathrm{d}} \mu(dr)\right) s_n = \eta \varepsilon_n^{-2} \tag{5.5}$$

where $\eta = uV_1 \int_{\mathcal{R}}^\infty r^{\mathrm{d}}\mu(dr) = \frac{uV_1}{\alpha}$. Since selective events correspond to a ternary 'branching' event, $\eta\varepsilon_n^{-2}$ will be the 'branching' rate.

By almost identical arguments to those in [EFP17], we see that, for some fixed $T^* < \mathscr{T}$ and large enough n, there should be no coalescing of lineages before time T^*: first, by bounding the total number of individuals alive at time T^* by the total number alive in a process whose lineages independently branch at rate $\eta\varepsilon_n^{-2} = \mathcal{O}(\log n)$, we see that for all $\delta > 0$, there are $o(n^\delta)$ lineages in $\Xi^n(T^*)$ with high probability. Since lineages can only coalesce if both are impacted by an event, and pairs of lineages are impacted by at most $\mathcal{O}(n)$ events by time T^*, the probability of any coalescence is $o(nu_n^2 n^\delta)$. Note that $nu_n^2 = \mathcal{O}(n^{2\beta\alpha-1} I(\varepsilon_n)^{2\alpha-4})$ and $\beta < \frac{1}{2\alpha}$ by assumption. Choose δ sufficiently small so that $\delta + 2\beta\alpha - 1 =: q < 0$. Then by Assumption 3.1.2 (B) and our choice of ε_n, for some constant c (that we allow to change from line to line)

$$\begin{aligned} nu_n^2 n^\delta &\leq \frac{c}{n^q I(\varepsilon_n)^2} \\ &\leq \frac{c}{n^q \varepsilon_n^2 |\log \varepsilon_n|} \\ &\leq \frac{c|\log n|}{n^q}, \end{aligned}$$

therefore with high probability lineages will not coalesce by time T^*.

5.2 Majority voting in the SΛFVS

In this section, we mimic [EFP17, Section 3.2.2] to couple $\Xi^n(t)$ asymptotically to a process $\boldsymbol{\Psi}^n(t)$ in which lineages evolve independently after branching, hence do not coalesce.

5.2.1 Independence after branching

Throughout this section we consider a ternary branching jump process defined as follows.

Definition 5.2.1 (Branching jump process)**.** For $n \in \mathbb{N}$ and starting point $x \in \mathbb{R}^{\mathrm{d}}$, let $(\boldsymbol{\Psi}^n(t))_{t\geq 0}$ be the historical process of the branching random walk defined as follows.

(1) The lifetime of individuals are independently exponentially distributed with parameter $\eta\varepsilon_n^{-2}$.

(2) During it's lifetime, each individual independently evolves according to a pure jump process with jump intensity $(1-s_n)m_n(dz)$ for m_n as in (5.4).

(3) At the end of its lifetime an individual branches into three offspring.

(4) For each branching event, independently pick $r \in (\mathcal{R}_n, \infty)$ according to

$$\frac{r^{\mathrm{d}-1}\mu^n(dr)}{\int_{\mathcal{R}_n}^{\infty} s^{\mathrm{d}-1}\mu^n(ds)}.$$

If the parent is at location $z \in \mathbb{R}^{\mathrm{d}}$, then each of the three offspring, independently, samples its location uniformly from $B_r(z)$.

What follows is the analogue of [EFP17, Lemma 3.12] in the stable radii case.

Lemma 5.2.2. *Let $T^* \in (0,\infty)$, $k \in \mathbb{N}$ and $z \in \mathbb{R}^{\mathrm{d}}$. There exists $n_* \in \mathbb{N}$ such that, for all $n \geq n_*$, there is a coupling of $\boldsymbol{\Xi}^n$ started from z and $\boldsymbol{\Psi}^n$ started from z such that, with probability $1-\varepsilon_n^k$,*

$$\boldsymbol{\Xi}^n(T^*) = \boldsymbol{\Psi}^n(T^*).$$

We prove Lemma 5.2.2 following closely the proof of [EFP17, Lemma 3.12], noting a few differences. First, we redefine the SΛFVS dual to give us a process $\boldsymbol{\Phi}^n(t)$, that is equal in law to $\boldsymbol{\Xi}^n(t)$, but has a different 'tagging' procedure that will allow us to more easily capture when two lineages are moving independently. We then describe how likely it is that two distinct lineages are not both tagged at the same time (and are thus moving independently) in Lemma 5.2.4. We use this result to prove Lemma 5.2.2 at the end of this section.

Definition 5.2.3 (Pre-emptive SΛFVS Dual [EFP17, Definition 3.13])**.** For $n \in \mathbb{N}$, the process $(\widetilde{\mathcal{P}}_t^n)_{t\geq 0}$ is a $\bigcup_{l\geq 1}(\mathbb{R}^d)^l$-valued process of individuals, each of which may be tagged. The dynamics are described as follows.

The process is started with a single individual at the point x and we write

$$(\xi_1^n(t), ..., \xi_{N(t)}^n(t))$$

for the locations of the random number $N(t)$ of individuals alive at time t. At time zero, independently of all else, the individual $\xi_1^n(0)$ is tagged with probability u_n. At each event $(t, x, r) \in \Pi^n$, independently, the event is said to be *neutral* with probability $1 - s_n$. In this case:

(1) if at least one individual $\xi_i^n(t-) \in B_r(x)$ is tagged, then all tagged individuals in $B_r(x)$ are replaced by a single offspring individual, whose location is drawn uniformly at random from within $B_r(x)$;

(2) for each $\xi_i^n(t) \in B_r(x)$, including the offspring individual if any, independently tag the corresponding individual with probability u_n and untag it otherwise.

With the complementary probability s_n, the event is said to be *selective*, in which case:

(1) if at least one individual $\xi_i^n(t-) \in B_r(x)$ is tagged, the collection of tagged individuals in $B_r(x)$ is replaced by three offspring individuals, whose locations are drawn independently and uniformly from within $B_r(x)$;

(2) for each $\xi_i^n(t) \in B_r(x)$, including the offspring individuals if any, independently tag the corresponding individual with probability u_n, and untag it otherwise.

In between events in Π^n, nothing happens. In particular, once tagged, an individual remains tagged until it is in the region covered by an event, and, during events, all individuals in the affected region (whether they were marked before the event or not) sample afresh from independent Bernoulli random variables to decide whether they are tagged immediately after the event.

We denote the historical pre-emptive process defined above by $\mathbf{\Phi}^n$ (ignoring tags). Then $\mathbf{\Phi}^n$ and $\mathbf{\Xi}^n$ are equal in law. The key difference is that, in the pre-emptive dual, the tagging of individuals is decided in the previous time step, rather than at the time of the event. In the following lemma we bound the probability that two lineages are tagged at the same time. Our proof is almost identical to that of [EFP17, Lemma 3.14], but we choose to reproduce it in this case to emphasise that we require the scaling parameter β to satisfy $\beta < \frac{1}{2\alpha}$ (as opposed to $\beta < \frac{1}{4}$ in the fixed radius case of [EFP17]).

Lemma 5.2.4. *Let $T^* \in (0, \infty)$. There exists $p > 0$ such that*

$$\begin{aligned}\mathbb{P}\,[\exists \xi_{\mathbf{i}}^n \neq \xi_{\mathbf{j}} \subset \mathbf{\Phi}^n(T^*), t \in [0, T^*] \text{ such that} \\ \xi_{\mathbf{i}}^n \text{ and } \xi_{\mathbf{j}}^n \text{ are both tagged at time } t\Big] = \mathcal{O}(n^{-p}).\end{aligned}$$

Proof. We follow closely [EFP17, Lemma 3.14]. Denote the genealogy of $\mathbf{\Phi}^n(t)$ by $\mathcal{T}(\mathbf{\Phi}^n(t))$. We first note that for any $b > 0$, with high probability,

$$\mathcal{T}(\mathbf{\Phi}^n(T^*)) \subset \mathcal{T}^{\text{reg}}_{b \log n}.$$

This follows by identical arguments to those in the proof of [EFP17, Lemma 3.14], which shows that, if U^n is Poisson distributed with mean $T^* \eta \varepsilon_n^{-2}$, by a Chernoff bound (and taking n sufficiently large)

$$\mathbb{P}[U^n > b \log n] \leq 3^{-2b \log n}.$$

Since selective events affect lineages at rate $\eta \varepsilon_n^{-2} = o(\log n)$, it follows using a union bound on all lineages that

$$\mathbb{P}\left[\mathcal{T}(\mathbf{\Phi}^n(T^*)) \not\subseteq \mathcal{T}^{\text{reg}}_{b \log n}\right] \leq 3^{b \log n} \mathbb{P}\left[U^n > b \log n\right] \leq 3^{-b \log n}.$$

Next consider the first reproduction event that results in two distinct lineages $\xi_{\mathbf{i}}^n, \xi_{\mathbf{j}}^n \subset \mathbf{\Phi}^n(t)$ being tagged. If this event occurs in time $[0, T^*]$, then at least one of $\xi_{\mathbf{i}}^n$ or $\xi_{\mathbf{j}}^n$ must have been in the region effected by the event. Now, the probability that both lineages are tagged is u_n^2, and the number of reproduction events before time T^* whose region contained $\xi_{\mathbf{i}}^n$ is Poisson with mean $\Theta(n)$ since, for any $x \in \mathbb{R}^{\mathrm{d}}$,

$$\begin{aligned}
&nT^* n^{\mathrm{d}\beta} \int_{\mathbb{R}^{\mathrm{d}}} \int_{\mathcal{R}_n}^{\infty} \frac{V_r(x,z)}{V_r} \mu^n(dr) dz \\
&\quad = nT^* n^{\mathrm{d}\beta} \int_{\mathcal{R}_n}^{\infty} \frac{1}{V_r} \int_{\mathbb{R}^{\mathrm{d}}} \int_{\mathbb{R}^{\mathrm{d}}} \mathbb{1}_{|y-x|<r} \mathbb{1}_{|y-z|<r} dz dy \mu^n(dr) \\
&\quad = nT^* n^{\mathrm{d}\beta} \int_{\mathcal{R}_n}^{\infty} V_r \frac{1}{n^{\beta\alpha + \mathrm{d}\beta}} \frac{1}{r^{\alpha + \mathrm{d} + 1}} dr \\
&\quad = nT^* n^{\mathrm{d}\beta} \int_{\mathcal{R}}^{\infty} V_{n^{-\beta} r} \frac{1}{n^{\beta\alpha + \mathrm{d}\beta}} \frac{1}{(n^{-\beta} r)^{\alpha + \mathrm{d} + 1}} n^{-\beta} dr \\
&\quad = nT^* n^{\mathrm{d}\beta} \int_{\mathcal{R}}^{\infty} \frac{r^{\mathrm{d}}}{n^{\mathrm{d}\beta}} V_1 \frac{1}{n^{\beta\alpha + \mathrm{d}\beta}} \frac{1}{(n^{-\beta} r)^{\alpha + \mathrm{d} + 1}} n^{-\beta} dr \\
&\quad = nT^* V_1 \int_{\mathcal{R}}^{\infty} \frac{1}{r^{1+\alpha}} dr,
\end{aligned}$$

where in the third equality, we use substitution and that $\mathcal{R}_n = n^{-\beta}\mathcal{R}$. It follows that the probability that $\xi^n_{\mathbf{i}}, \xi^n_{\mathbf{j}}$ are both marked at time $t \in [0, T^*]$ is $\mathcal{O}(nu_n^2) = \mathcal{O}\left(n^{2\beta\alpha-1}I(\varepsilon_n)^{2\alpha-4}\right)$. Then, conditioning on the event $\{\mathcal{T}(\mathbf{\Phi}^n(T^*)) \subset \mathcal{T}^{\mathrm{reg}}_{b\log n}\}$ and applying a union bound over all pairs of lineages, we obtain

$$\begin{aligned}
&\mathbb{P}\left[\exists \xi^n_{\mathbf{i}} \neq \xi^n_{\mathbf{j}} \subset \mathbf{\Phi}^n(T^*) \text{ and } t \in [0, T^*] : \xi^n_{\mathbf{i}} \text{ and } \xi^n_{\mathbf{j}} \text{ are both marked at time } t\right] \\
&\quad \leq 3^{-b\log n} + 3^{2b\log n}\mathcal{O}\left(n^{2\beta\alpha-1}I(\varepsilon_n)^{4-2\alpha}\right) \\
&\quad \leq 3^{-b\log n} + 3^{2b\log n}\mathcal{O}\left(n^{2\beta\alpha-1}\right) \\
&\quad \leq 3^{-b\log n} + \mathcal{O}\left(\exp(2b(\log 3)(\log n) + (2\beta\alpha - 1)\log n)\right).
\end{aligned}$$

Since $2\beta\alpha - 1 < 0$ we can choose b so that $2b(\log 3) + (2\alpha\beta - 1) < 0$ and we obtain the result. □

At this point, we can prove the main duality result of this section, Lemma 5.2.2.

Proof of Lemma 5.2.2. Again, we follow the proof of [EFP17, Lemma 3.12]. Set

$$\tau' = \inf\{t \geq 0 : \exists \xi^n_{\mathbf{i}} \neq \xi^n_{\mathbf{j}} \subset \mathbf{\Phi}^n(T^*) \text{ such that } \xi^n_{\mathbf{i}} \text{ and } \xi^n_{\mathbf{j}} \text{ are both tagged at time } t\}.$$

For any $p > 0$, $n^{-p} = o((\log n)^{-\frac{k}{2}}) = o(\varepsilon_n^k)$, so by Lemma 5.2.4, $\mathbb{P}[\tau' \geq T^*] \geq 1 - \varepsilon_n^k$. When lineages are not both marked, they evolve independently, so can be coupled with $\Xi^n(t)$ until time τ', and the result follows. □

5.2.2 Proof of Theorem 3.1.11

Let $(\xi^n(t))_{t\geq 0}$ be a pure jump process started at $x \in \mathbb{R}^{\mathrm{d}}$ with rate of jumps from y to $y + z$ given by $m_n(dz)$ where we recall that

$$m_n(dz) = \frac{u}{I(\varepsilon_n)^{2-\alpha}} \int_{\mathcal{R}_n}^{\infty} \frac{V_r(0, z)}{V_r} \frac{1}{r^{1+\alpha+\mathrm{d}}} dr dz.$$

Let $(W_t)_{t\geq 0}$ be a d-dimensional Brownian motion started at x, $(R_t)_{t\geq 0}$ be an $\frac{\alpha}{2}$-stable subordinator and $(R^{\varepsilon_n}_t)_{t\geq 0}$ be an $I(\varepsilon_n)^2$-truncated $\frac{\alpha}{2}$-stable subordinator. All subordinators are implicitly assumed to run at speed $uI(\varepsilon_n)^{\alpha-2}$.

Our first aim is to provide a coupling of $W(R^{\varepsilon_n}_t)$ to $\xi^n(t)$. We begin by showing in Lemma 5.2.5 that, with high probability, the distance $|\xi^n(t) - W(R_t)|$ is small for any $t > 0$. Then, in Corollary 5.2.6, we use this to show that an individual $\xi^n_i(\tau)$ created at the first branching event of Ξ^n is close to $W(R^{\varepsilon_n}_\tau)$. This will allow us to prove that the root vote of $\Xi^n(t)$ under majority voting is close to the root vote of $W(R^{\varepsilon_n}_t)$ under marked majority voting in Proposition 5.2.7. We may then apply the work from Chapter 4 to prove the main result.

Lemma 5.2.5. *For $t > 0$ fixed, there exists a coupling of $W(R_t)$ and $\xi^n(t)$ such that*

$$\mathbb{P}\left[|\xi^n(t) - W(R_t)| \geq n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)}\right] = \mathcal{O}\left((t \vee 1) n^{-\beta\left(1-\frac{\alpha}{2}\right)} I(\varepsilon_n)^{\alpha-2}\right).$$

Proof. Define another pure jump process, $(\upsilon^n(t))_{t\geq 0}$ independent of ξ^n with jump measure given by the integral

$$\widehat{m}_n(dz) = \frac{u}{I(\varepsilon_n)^{2-\alpha}} \int_0^{\mathcal{R}_n} \frac{V_r(0,z)}{V_r} \frac{1}{r^{1+\alpha+\mathbb{d}}} dr dz.$$

Then $\upsilon^n + \xi^n$ is a pure jump process with jump measure given by the integral

$$\overline{m}_n(dz) = \frac{u}{I(\varepsilon_n)^{2-\alpha}} \int_0^{\infty} \frac{V_r(0,z)}{V_r} \frac{1}{r^{1+\alpha+\mathbb{d}}} dr dz.$$

By [EVY20, Lemma 6.1], $\overline{m}_n(dz)$ is the Lévy measure associated to a symmetric α-stable process (run at speed $uI(\varepsilon_n)^{\alpha-2}$), which we denote by $W(R_t)$. It follows that

$$|\xi^n(t) - W(R_t)| \stackrel{D}{=} |\upsilon^n(t)|.$$

Since υ^n is a mean-zero martingale, by Doob's inequality and the Burkholder-Davis-Gundy inequality,

$$\mathbb{P}\left[\sup_{0\leq s\leq t} |\upsilon^n_s| \geq \mathcal{R}_n^{\frac{2-\alpha}{4}}\right] \leq \mathbb{E}\left[|\upsilon^n_t|^2\right] \mathcal{R}_n^{\frac{\alpha}{2}-1} \leq C \mathcal{R}_n^{\frac{\alpha}{2}-1} \mathbb{E}[\langle|\upsilon^n|\rangle_t] \tag{5.6}$$

for some constant $C > 0$. Since υ^n is a pure jump process,

$$\begin{aligned}
\mathbb{E}[\langle|\upsilon^n|\rangle_t] &= \mathbb{E}\left[\sum_{0\leq s\leq t} (|\eta^n_s| - |\eta^n_{s-}|)^2\right] \\
&= \frac{u}{I(\varepsilon_n)^{2-\alpha}} \int_0^t \int_{\mathbb{R}^{\mathbb{d}}} \int_0^{\mathcal{R}_n} \frac{V_r(0,z)}{V_r} \frac{|z|^2}{r^{1+\alpha+\mathbb{d}}} dr dz ds \\
&\leq \frac{u}{I(\varepsilon_n)^{2-\alpha}} t \int_0^{\mathcal{R}_n} \frac{1}{r^{1+\alpha+2\mathbb{d}}} \int_{\mathbb{R}^{\mathbb{d}}} \int_{\mathbb{R}^{\mathbb{d}}} \mathbb{1}_{|x|<r} \mathbb{1}_{|z|<r-|x|} |z|^2 dz dx dr \\
&= \frac{u}{I(\varepsilon_n)^{2-\alpha}} D t \int_0^{\mathcal{R}_n} \frac{r^{2\mathbb{d}+2}}{r^{1+\alpha+2\mathbb{d}}} dr \\
&\leq D u t \left(\frac{\mathcal{R}_n}{I(\varepsilon_n)}\right)^{2-\alpha}
\end{aligned}$$

for some $D > 0$. Substituting this into (5.6) and using that $\mathcal{R}_n := n^{-\beta}\mathcal{R}$ gives the result. □

Recall η from equation (5.5) is given by $\eta = \frac{uV_1}{\alpha}$.

Corollary 5.2.6. *Let τ be the first branch time of Ξ^n. There exists a coupling of $\Xi^n(t)$ and $W(R_t^{\varepsilon_n})$ under which $\tau \sim exp(\eta\varepsilon_n^{-2})$ and for $i = 1, 2, 3$*

$$\mathbb{P}\left[|\xi_i^n(\tau) - W(R_\tau^{\varepsilon_n})| \geq 2n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)}\right] = \mathcal{O}\left(n^{-\beta\left(1-\frac{\alpha}{2}\right)} I(\varepsilon_n)^{\alpha-2} + \varepsilon_n^2 I(\varepsilon_n)^{-2}\right) \qquad (5.7)$$

for all n larger than some $n^* > 0$.

Proof. The distribution of τ follows from (5.5). To prove (5.7), it suffices to bound the difference $\xi_i^n(\tau) - W(R_\tau)$, since, with probability $\mathcal{O}(1 - b_{\varepsilon_n})$, $W(R_\tau) \stackrel{D}{=} W(R_\tau^{\varepsilon_n})$. Consider a lineage $\xi^n(\cdot) \in \Xi^n$. Since τ is the time of the first branching event, for $t < \tau$, $\xi^n(\cdot)$ evolves as a pure jump process with jump intensity $(1 - s_n)m_n(dz)$. By Lemma 5.2.5 (replacing m_n therein by $(1 - s_n)m_n$), there is a time changed $\frac{\alpha}{2}$-stable subordinated Brownian motion $(W(R_t))_{t\geq 0}$ such that, for $t > 0$,

$$\mathbb{P}\left[|\xi^n(t) - W(R_t)| \geq n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)} \;\middle|\; t < \tau\right] \leq \mathcal{O}\left((t \vee 1) n^{-\beta\left(1-\frac{\alpha}{2}\right)} I(\varepsilon_n)^{\alpha-2}\right).$$

In particular,

$$\mathbb{P}\left[|\xi^n(\tau-) - W(R_{\tau-})| \geq n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)} \;\middle|\; \tau\right] \leq \mathcal{O}\left((\tau \vee 1) n^{-\beta\left(1-\frac{\alpha}{2}\right)} I(\varepsilon_n)^{\alpha-2}\right).$$

We next bound (with high probability) the distance of any offspring at time τ from the initial ancestor at time $\tau-$. Recall that the event radius $R > \mathcal{R}_n$ is picked according to

$$\frac{r^{\mathbb{d}-1}\mu^n(dr)}{\int_{\mathcal{R}_n}^\infty s^{\mathbb{d}-1}\mu^n(ds)}.$$

Therefore

$$\begin{aligned}\mathbb{P}[R \leq r \,|\, r > \mathcal{R}_n] &= \frac{\int_{\mathcal{R}_n}^r w^{\mathbb{d}-1}\mu^n(dw)}{\int_{\mathcal{R}_n}^\infty s^{\mathbb{d}-1}\mu^n(ds)} \\ &= 1 - c_\alpha n^{-\beta\alpha-\mathbb{d}\beta} r^{-\alpha-1}\end{aligned}$$

for $c_\alpha := (\alpha + 1)^{-1}$. Suppose n is sufficiently large so that $\mathcal{R}_n < n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)}$. Then setting $r = n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)}$ in the previous computation gives

$$\mathbb{P}\left[R \leq n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)}\right] = 1 - c_\alpha n^{-\frac{3\beta\alpha}{4}-\mathbb{d}\beta+\frac{\beta}{2}-\frac{\beta\alpha^2}{4}}.$$

That is, for $i = 1, 2, 3$,

$$\mathbb{P}\left[|\xi_i^n(\tau) - \xi_1^n(\tau-)| \geq n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)} \;\middle|\; \tau\right] \leq c_\alpha n^{-\frac{3\beta\alpha}{4}-\mathbb{d}\beta+\frac{\beta}{2}-\frac{\beta\alpha^2}{4}}.$$

Finally, we note that $\mathbb{E}[\tau] = \Theta(\varepsilon_n^2) = o(1)$. It follows from the triangle inequality, Assumption 3.1.2 (B) and since $n^{-1} = o(\varepsilon_n)$ that, for n sufficiently large,

$$n^{-\beta\left(1-\frac{\alpha}{2}\right)} I(\varepsilon_n)^{\alpha-2} > c_\alpha n^{-\frac{3\beta\alpha}{4}-\mathbb{d}\beta+\frac{\beta}{2}-\frac{\beta\alpha^2}{4}}$$

and

$$\mathbb{P}\left[|\xi_i^n(\tau) - W(R_\tau)| \geq n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)}\right] \leq \mathcal{O}\left(n^{-\beta\left(1-\frac{\alpha}{2}\right)} I(\varepsilon_n)^{\alpha-2}\right),$$

and the result follows. □

Denote the $\mathbb{d}$-dimensional historical branching truncated subordinated Brownian motion, with branching rate $\eta\varepsilon_n^{-2}$, by $\boldsymbol{W}_{R^{\varepsilon_n}}$.

Proposition 5.2.7. *Let* $0 < t < \mathscr{T}$ *and* G *be as in (5.3). Then, for any* $x \in \mathbb{R}^{\mathbb{d}}$*, there exists* $m_1, m_2 > 0$ *such that, for* n *sufficiently large,*

$$\Big|\mathbb{P}_x[\mathbb{V}_p(\boldsymbol{\Xi}^n(t)) = 1)] - \mathbb{P}_x[\mathbb{V}_p^{\times}(\boldsymbol{W}_{R^{\varepsilon_n}}(t)) = 1)]\Big| \leq m_1 e^{-\frac{t}{\varepsilon^2}} + m_2 G(n) \tag{5.8}$$

The proof of Proposition 5.2.7 follows a similar format to the proof of Proposition 3.3.22, and is deferred to Appendix A. From this the proof of Theorem 5.1.4 is immediate.

Proof of Theorem 5.1.4. This follows from Proposition 5.2.7 and the multidimensional deterministic result, Theorem 3.2.5. □

Chapter 6

The spreading speed of solutions to fractional reaction-diffusion equations

In this chapter, we find precise estimates for the spreading speed of solutions to a family of fractional reaction-diffusion equations. Our result improves that of [CR13], in that we find the precise order of the spreading speed. In contrast to [CR13], our proofs use purely probabilistic techniques, relying heavily on the Feynman–Kac formula. This work takes inspiration from [Pen18], where similar probabilistic arguments were used to find the spreading speed of solutions to the Fisher–KPP equation with non-local nonlinearity.

To begin, in Section 6.1 we state our main results and compare them to those of [CR13]. In Section 6.2 we collect several key results that will be used throughout this work, including the Feynman–Kac formula. Our two main theorems (Theorem 6.1.2 and Theorem 6.1.3) will then proved in the following three sections. First, in Section 6.3, we prove several general results on the growth and spreading properties of solutions to fractional reaction-diffusion equations. We then go on to prove Theorem 6.1.2 and Theorem 6.1.3 in Sections 6.4 and 6.5, respectively.

6.1 Main results

We consider solutions $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ of the initial value problem

$$\begin{cases} \partial_t u = -(-\Delta)^{\frac{\alpha}{2}} u + g(u), \ t \geq 0, \ x \in \mathbb{R} \\ u(0, x) = u_0(x), \ x \in \mathbb{R}, \ 0 \leq u_0 \leq 1 \end{cases} \tag{6.1}$$

for all $\alpha \in (0, 2)$. Before stating our results, let us briefly review the work of [CR13]. There, reaction-diffusion equations of the form (6.1) were studied, but with the fractional Laplacian replaced by the generator of a more general Feller semigroup. In the case of equation (6.1), [CR13, Theorem 1.5] amounts to the following. Throughout this chapter, if $A, B \subseteq \mathbb{R}$ let $C^1(A)$ denote the space of continuously differentiable functions $A \to \mathbb{R}$ and $C^1(A, B)$ denote the space of continuously differentiable functions $A \to B$.

Theorem 6.1.1 ([CR13]). *Suppose $g \in C^1([0,1])$ is concave, $g(0) = g(1) = 0$ and $g'(1) < 0 < g'(0)$. Let $\sigma^* = \frac{g'(0)}{\alpha}$ and u be a solution of (6.1) where u_0 is measurable and nonincreasing, $u_0 \not\equiv 0$ and*

$$u_0(x) \le Cx^{-\alpha} \text{ if } x > 0$$

for some constant C. Then, for all $\varepsilon > 0$,

$$\lim_{t\to\infty} \inf_{x \le e^{(\sigma^*-\varepsilon)t}} u(t,x) = 1 \quad \text{and} \quad \lim_{t\to\infty} \sup_{x \ge e^{(\sigma^*+\varepsilon)t}} u(t,x) = 0.$$

Before the work of [CR13], our understanding of equations of the form (6.1) was largely heuristic. For example, in [dCL03], numerical simulations of the fractional Fisher-KPP equation were used to argue that the spreading speed of the front is exponential in time. In [MVV03], the authors came to the same conclusion, using both numerical simulations and a linear analysis of the front that relied on the Feynman–Kac formula and the maximum principle.

While Theorem 6.1.1 is the most mathematically rigorous result in the literature for the spreading speed of solutions to (6.1), it does not specify the precise order of this speed. For instance, it could be the case that solutions spread at speed $t^p e^{\sigma^* t}$ for some $p > 0$, as suggested by a linearisation argument of the front in [CR13]. In our work, probabilistic arguments enable us to show that the speed of the front is exactly order $e^{\sigma^* t}$.

Let us now state the precise form of our result. For $u \in [0,1]$, set

$$f(u) := \begin{cases} \frac{g(u)}{u} & \text{if } u \neq 0 \\ g'(0) & \text{if } u = 0. \end{cases} \tag{6.2}$$

We assume that

$$f \in C^1([0,1],[0,\infty)) \text{ is decreasing}, f(0) > 0 \text{ and } f(1) = 0 \tag{6.3}$$

and there exists $C_0 > 0$ such that

$$u_0 \in \mathcal{D}((-\Delta)^{\frac{\alpha}{2}}),\ \widetilde{m} := \inf_{x\le 0} u_0(x) > 0 \text{ and } u_0(x) \le C_0(1+x)^{-\alpha} \text{ for all } x > 0 \tag{6.4}$$

where $\mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$ denotes the set of bounded and uniformly continuous functions $u : \mathbb{R} \to \mathbb{R}$ for which $-(-\Delta)^{\frac{\alpha}{2}} u$ is well-defined. We note that our assumptions on f in (6.3) amount to the same assumptions for the function g stated in Theorem 6.1.1 (this follows easily by recalling that, since g is concave, it is bounded above by its first order Taylor expansion). Unlike [CR13], we need not assume u_0 is nonincreasing. However, we do ask that $u_0 \in \mathcal{D}((-\Delta)^{\frac{\alpha}{2}}) \subset C(\mathbb{R})$, which arises from our use of the Feynman–Kac formula. While it should be possible to generalise our result to discontinuous initial data by considering mild solutions to (6.1) expressed using the semigroup associated to $(-\Delta)^{\frac{\alpha}{2}}$, this is beyond the scope of our project.

Due to the structure of our proof, it will be convenient to split the statement of our main result into the following two theorems.

Theorem 6.1.2. *Suppose u is a solution to (6.1) and f is defined as in (6.2). Assume f and u_0 satisfy (6.3) and (6.4), respectively. Define $\sigma^* := \frac{f(0)}{\alpha}$. For all $\varepsilon > 0$, there exists $T_\varepsilon, T'_\varepsilon, C_\varepsilon, C'_\varepsilon > 0$ and $m_\varepsilon \in (0, 1)$ such that*

(a) *for all $t \geq T_\varepsilon$, $\{x \leq C_\varepsilon e^{\sigma^* t}\} \subseteq \{x \in \mathbb{R} : u(t, x) > m_\varepsilon\}$;*

(b) *for all $t \geq T'_\varepsilon$, $\{x \geq C'_\varepsilon e^{\sigma^* t}\} \subseteq \{x \in \mathbb{R} : u(t, x) < \varepsilon\}$.*

We can strengthen part (a) of Theorem 6.1.2 to obtain the following.

Theorem 6.1.3. *Suppose u is a solution to (6.1) and f is defined as in (6.2). Assume f and u_0 satisfy (6.3) and (6.4), respectively. Let $\sigma^* := \frac{f(0)}{\alpha}$ and $\varepsilon > 0$. There exists $T_\varepsilon > 0$ and $C_\varepsilon > 0$ such that, for all $t \geq T_\varepsilon$,*

$$\{x \leq C_\varepsilon e^{\sigma^* t}\} \subseteq \{x \in \mathbb{R} : u(t, x) > 1 - \varepsilon\}.$$

As we shall see, part (a) of Theorem 6.1.2 enables us to prove Theorem 6.1.3. A similar strategy is employed in [CR13]: the authors first show that solutions $u(t, x)$ of equation (6.1) are bounded below by some positive constant for all t sufficiently large and $x < e^{(\sigma - \varepsilon)t}$. Then, using this result, they are able to prove that the first of the two limits in Theorem 6.1.1 holds.

Example 6.1.4. An example of an equation of the form (6.1) is the unbalanced fractional Allen–Cahn equation given by

$$\partial_t u = -(-\Delta)^{\frac{\alpha}{2}} u + u(1 - u)(2u - 1 + \gamma). \tag{6.5}$$

It is straightforward to verify that this equation satisfies (6.3) if and only if $\gamma > 3$ (see Figure 6.1). In this case, Theorem 6.1.2 tells us that solutions u of (6.5), with appropriate initial condition, spread at speed $\mathcal{O}\left(e^{\frac{\gamma-1}{\alpha} t}\right)$.

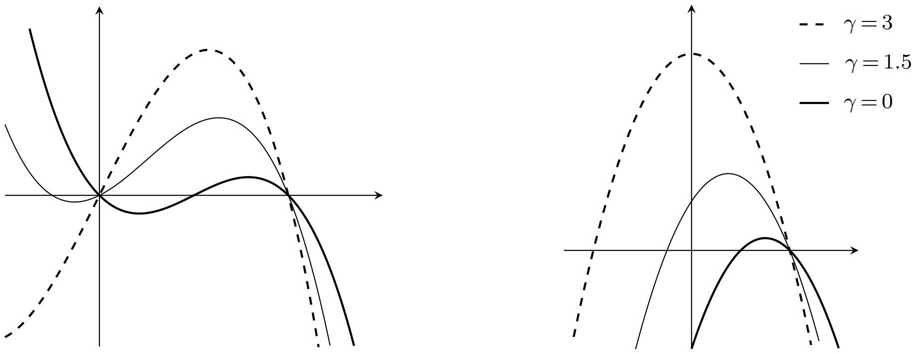


Figure 6.1: Graphs of $g(x) = x(1-x)(2x-1+\gamma)$ (left) and $f(x) = \frac{g(x)}{x}$ (right) for varying values of γ

6.2 Preliminary results

We now record some key results that will be used throughout this chapter. In Section 6.2.1, we will state the Feynman–Kac formula for fractional reaction-diffusion equations and verify it applies in our setting. Then in Section 6.2.2 we will state several useful results for α-stable processes.

Let $\alpha \in (0,2)$ and for $x \in \mathbb{R}$ let $\mathbb{P}_x$ denote the probability measure under which $(X_t)_{t\geq 0}$ is one-dimensional symmetric α-stable process started at x with corresponding expectation $\mathbb{E}_x$.

6.2.1 The Feynman–Kac formula

The main tool in this work is the Feynman–Kac formula. The traditional statement of this formula provides an integral representation of solutions to heat equations with dissipation, i.e. equations of the form

$$\partial_t u(t,x) = \tfrac{1}{2}\Delta u(t,x) + f(t,x)u(t,x) \text{ on } (0,\infty)\times\mathbb{R}. \tag{6.6}$$

An excellent introduction to the Feynman–Kac formula and its applications can be found in [Bra82]. Here, we state a different version of the Feynman–Kac formula for fractional heat equations with dissipation.

Theorem 6.2.1 (The Feynman–Kac formula)**.** *Take $u_0 \in \mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$ and suppose $f \in C([0,\infty)\times\mathbb{R},[0,\infty))$ is differentiable in the first variable with measurable derivative. For $x \in \mathbb{R}$ and $t > 0$ consider the equation*

$$\partial_t u(t,x) = -(-\Delta)^{\frac{\alpha}{2}} u(t,x) + f(t,x)u(t,x), \quad u(0,x) = u_0(x). \tag{6.7}$$

Then a unique solution u to (6.7) exists with $u(t,\cdot) \in \mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$ for each $t \geq 0$. Moreover, u satisfies

$$u(t,x) = \mathbb{E}_x \left[\exp\left(\int_0^t f(t-s, X_s) ds \right) u_0(X_t) \right]. \tag{6.8}$$

We note that Theorem 6.2.1 can be stated more generally by replacing the fractional Laplacian by the generator of any Feller process. It will be useful to note that the Feynman–Kac formula (6.8) can also be written as

$$u(t,x) = \mathbb{E}_x \left[\exp\left(\int_0^r f(t-s, X_s) ds \right) u(t-r, X_r) \right] \tag{6.9}$$

for any $r \in [0,t]$. A proof of this fact, along with the proof of Theorem 6.2.1, will be provided in Appendix B.

We note that Theorem 6.2.1 applies to equation (6.1) by assumptions (6.3) and (6.4). This follows from [CR13, Remark 2.6] (which itself follows from [Paz12]), which states that if $u_0 \in \mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$, then $u(t,\cdot) \in \mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$ for all $t \geq 0$ and $u \in C^1([0,\infty),\mathbb{R})$. It follows that $f(t,x) \equiv f(u(t,x))$ is differentiable in time with measurable derivative since $f \in C^1([0,1])$ by assumption.

6.2.2 Stable processes

In this section, we state several results for stable processes. To begin, we recall the following result from Chapter 2 that can be attributed to [Kol00]. We state here the one-dimensional version of this result.

Lemma 6.2.2. *There exists $B > 1$ such that transition density $p_{X_t}(x)$ of $(X_s)_{s\geq 0}$ at time t started at $X_0 = 0$ satisfies*

$$\frac{B^{-1}}{t^{\frac{1}{\alpha}} + t^{-1}|x|^{1+\alpha}} \leq p_{X_t}(x) \leq \frac{B}{t^{\frac{1}{\alpha}} + t^{-1}|x|^{1+\alpha}}$$

for all $t > 0$ and $x \in \mathbb{R}$.

The following result describes the maximum displacement of a stable process by time t. The proof of this result relies on well-known bounds for displacement of a standard Brownian motion together with the description of a stable process as a subordinated Brownian motion. We defer the proof of this result to the appendix.

Lemma 6.2.3. *We have the following two bounds for the displacement of X_s.*

(a) *There exists $\gamma > 0$ such that, for any $c \geq 0, t \geq 0$ and $w > 0$,*

$$\mathbb{P}_0 \left[\sup_{s\in[0,t]} |X_s| \geq c \right] \leq 4e^{-\frac{c^2}{2w}} + \gamma(1 \wedge tw^{-\frac{\alpha}{2}}).$$

(b) *If $t \geq 1$ and $0 \leq x_0 \leq 1$, there exist constants $\eta_1(\alpha), \eta_2(\alpha) > 0$ such that*

$$\mathbb{P}_0\left[|X_s| \leq 1\ \forall\, s \leq t,\ |X_t| \leq x_0\right] \geq \eta_1 \min(x_0, \tfrac{2}{3}) e^{-\eta_2 t}.$$

6.3 Growth and spread of solutions

We now prove several results that will be key to our proofs in later sections. We continue with our earlier notation for the probability measure $\mathbb{P}_x$ and expectation $\mathbb{E}_x$ of an α-stable process $(X_s)_{s\geq 0}$ started at x. To begin, we present a comparison principle for equations of the form (6.1).

Proposition 6.3.1 ([CR13])**.** *Suppose $u_1, u_2 : \mathbb{R} \to \mathbb{R}$ are bounded and uniformly continuous in $\mathbb{R}$. Assume $g_1, g_2 \in C^1([0,1])$ are concave with $g_i(0) = g_i(1) = 0$ and $g_i'(1) < 0 < g_i'(0)$ for $i \in \{1,2\}$. For $i \in \{1,2\}$ let u_i be the solution of (6.1) with g replaced by g_i and u_0 replaced by u_i. If $g_1 \leq g_2$ and $u_1(0,\cdot) \leq u_2(0,\cdot)$ in $\mathbb{R}$, then $u_1(t,\cdot) \leq u_2(t,\cdot)$ for all $t \geq 0$.*

In this work, we will only utilise Proposition 6.3.1 to obtain the following result. This follows immediately since $u_0 \in [0,1]$ and 0 and 1 are steady state solutions of (6.1).

Proposition 6.3.2. *Suppose u is a solution to (6.1) with f defined as in (6.2). Assume f and u_0 satisfy (6.3) and (6.4), respectively. Then $0 \leq u(t,x) \leq 1$ for all $t \geq 0$ and $x \in \mathbb{R}$.*

In the remainder of this section, let f be defined as in (6.2) and u be a solution to (6.1) for f and u_0 satisfying (6.3) and (6.4), respectively. The following lemma gives us a quantitative uniform continuity result for u.

Lemma 6.3.3. *Let $t \geq 1$. There exists $b > 0$ such that, for all $\varepsilon \in (0,b)$, if $|x-y| < \varepsilon^{2+\frac{2}{\alpha}}$ then $|u(t,x) - u(t,y)| < \varepsilon$.*

Proof. This proof follows closely that of [Pen18, Lemma 2.2]. Let $\varepsilon < \frac{1}{e^{f(0)}(B+f(0))} =: b$ for B be as in Lemma 6.2.2. Suppose that $t \geq 1$ and $|x-y| < \varepsilon^{2+\frac{2}{\alpha}}$. By the Feynman–Kac formula given in (6.9), and since f is decreasing by (6.3), for all $z \in \mathbb{R}$

$$\mathbb{E}_z[u(t-\varepsilon^2, X_{\varepsilon^2})] \leq u(t,z) \leq e^{f(0)\varepsilon^2}\mathbb{E}_z[u(t-\varepsilon^2, X_{\varepsilon^2})].$$

This, together with Proposition 6.3.2, gives us

$$
\begin{aligned}
&u(t,x)-u(t,y)\\
&\le e^{f(0)\varepsilon^2}\mathbb{E}_x\left[u(t-\varepsilon^2,X_{\varepsilon^2})\right]-\mathbb{E}_y\left[u(t-\varepsilon^2,X_{\varepsilon^2})\right]\\
&=e^{f(0)\varepsilon^2}\left(\mathbb{E}_x\left[u(t-\varepsilon^2,X_{\varepsilon^2})\right]-\mathbb{E}_y\left[u(t-\varepsilon^2,X_{\varepsilon^2})\right]\right)+(e^{f(0)\varepsilon^2}-1)\mathbb{E}_y\left[u(t-\varepsilon^2,X_{\varepsilon^2})\right]\\
&\le e^{f(0)\varepsilon^2}\left(\mathbb{E}_x\left[u(t-\varepsilon^2,X_{\varepsilon^2})\right]-\mathbb{E}_y\left[u(t-\varepsilon^2,X_{\varepsilon^2})\right]\right)+(e^{f(0)\varepsilon^2}-1).
\end{aligned}
\tag{6.10}
$$

Denote the transition density of the α-stable process at time ε^2 starting at x by $p_{X_{\varepsilon^2}}(x,\cdot)$. Then for $x\le y$, $p_{X_{\varepsilon^2}}(x,z)-p_{X_{\varepsilon^2}}(y,z)\ge 0$ if and only if $z\le\frac{x+y}{2}$, so by Proposition 6.3.2

$$
\begin{aligned}
&\mathbb{E}_x\left[u(t-\varepsilon^2,X_{\varepsilon^2})\right]-\mathbb{E}_y\left[u(t-\varepsilon^2,X_{\varepsilon^2})\right]\\
&\le\int_{-\infty}^{\frac{1}{2}(x+y)}p_{X_{\varepsilon^2}}(x,z)-p_{X_{\varepsilon^2}}(y,z)\,dz\\
&=\mathbb{P}_0\left[X_{\varepsilon^2}\le\tfrac{1}{2}(y-x)\right]-\mathbb{P}_y\left[X_{\varepsilon^2}\le\tfrac{1}{2}(x-y)\right]\\
&=\mathbb{P}_0\left[|X_{\varepsilon^2}|\le\tfrac{1}{2}|x-y|\right]\\
&\le 2\int_0^{\frac{1}{2}|x-y|}\frac{B}{\varepsilon^{\frac{2}{\alpha}}+\varepsilon^{-2}z^{\alpha+1}}dz\\
&\le\frac{B}{\varepsilon^{\frac{2}{\alpha}}}|x-y|
\end{aligned}
$$

where the second to last inequality follows from Lemma 6.2.2. Substituting this into (6.10) and recalling that $|x-y|<\varepsilon^{2+\frac{2}{\alpha}}$ and $\varepsilon<\frac{1}{e^{f(0)}(B+f(0))}=b$, we obtain

$$
\begin{aligned}
u(t,x)-u(t,y)&\le e^{f(0)\varepsilon^2}\left(B\varepsilon^{-\frac{2}{\alpha}}|x-y|+1-e^{-f(0)\varepsilon^2}\right)\\
&\le\varepsilon^2e^{f(0)}\left(B+f(0)\right)\\
&\le\varepsilon be^{f(0)}\left(B+f(0)\right)\\
&=\varepsilon.
\end{aligned}
$$

By identical arguments this also holds for $y>x$, proving the result. □

We now consider the spread and growth of solutions u. First, in Lemma 6.3.4, we show that when u is small it grows exponentially. Then, in Lemma 6.3.5, we show that solutions spread, i.e. if $u(t,x)>m$ for some $m>0$, then there exists $m'>0$ such that $u(t+s,x)>m'$ for all $s>0$ sufficiently large. These results will often be used in conjunction with one another: if $u(t,x)$ is bounded below by a function that goes to zero as $t\to\infty$, we can first apply Lemma 6.3.4 to grow u to obtain a constant order lower bound, then we apply Lemma 6.3.5 to show that u is bounded below by some constant for all (large enough) future times. We begin with the growth result.

Lemma 6.3.4. *Let $\delta \in (0,1)$. There exists $C = C(\delta), R = R(\delta)$ and $z_0 = z_0(\delta)$ such that for $z \in (0, z_0)$, if $t \geq 1$ and $u(t,x) > z$ then there exists $s \in [t, t + C|\log z|]$ and $y \in [x - R, x + R]$ such that $u(s,y) \geq 1 - \delta$.*

Proof. Fix $\delta \in (0,1)$ and suppose $z_0 < (1-\delta) \wedge b$ for b as in Lemma 6.3.3. For reasons that will become clear later in this proof, we choose $R > 0$ and $C > 0$ sufficiently large so that

$$R^\alpha \geq \eta_2(\lambda\delta)^{-1} \text{ and } C \geq (3 + \tfrac{2}{\alpha}) \left(\lambda\delta - \eta_2 R^{-\alpha}\right)^{-1} \tag{6.11}$$

for η_2 as in Lemma 6.2.3 (b) and $\lambda > 0$ satisfying $f(u) \geq \lambda(1-u)$. We note that such a $\lambda > 0$ exists by the Mean Value Theorem, since, by assumption (6.3), f is decreasing and $f(1) = 0$, so

$$-\frac{f(u)}{1-u} = \frac{f(1) - f(u)}{1-u} = f'(v) < 0$$

for some $v \in (u, 1)$. Decrease z_0 if necessary so that

$$C|\log(z)|R^{-\alpha} \geq 1 \quad \text{and } R^{-1} \left(\tfrac{z}{2}\right)^{2+\frac{2}{\alpha}} \leq \tfrac{2}{3} \tag{6.12}$$

for all $z \in (0, z_0)$. Let $z \in (0, z_0)$ with $u(t,x) > z$ and assume for the purpose of a contradiction that for all $s \in [t, t + C|\log z|]$ and $y \in [x - R, x + R]$ we have $u(s,y) < 1 - \delta$. Therefore if $|X_s - x| \leq R$ for all $s \in [0, C|\log z|]$, then

$$\int_0^{C|\log z|} f(u(t + C|\log z| - s, X_s))ds > C|\log z|\lambda\delta.$$

Since $z_0 < b$, $z < z_0$ and $u(t,x) > z$, by Lemma 6.3.3 we have

$$u(t,y) > \tfrac{1}{2}z \quad \forall\, y \in \left[x - \left(\tfrac{1}{2}z\right)^{2+\frac{2}{\alpha}}, x + \left(\tfrac{1}{2}z\right)^{2+\frac{2}{\alpha}}\right].$$

By the Feynman–Kac formula (6.9)

$$\begin{aligned}
&u(t + C|\log z|, x) \\
&= \mathbb{E}_x\left[\exp\left(\int_0^{C|\log z|} f(u(t + C|\log z| - s, X_s))ds\right) u(t, X_{C|\log z|})\right] \\
&\geq \tfrac{1}{2}ze^{C\lambda\delta|\log z|}\mathbb{P}_x\left[|X_s - x| \leq R\; \forall\, s \in [0, C|\log z|],\, |X_{C|\log z|} - x| \leq \left(\tfrac{z}{2}\right)^{2+\frac{2}{\alpha}}\right]. \quad (6.13)
\end{aligned}$$

By equation (6.13) and Lemma 6.2.3 (b), there exists $\eta_1, \eta_2 > 0$ such that

$$u(t + C|\log z|, x) \geq \frac{\eta_1 R^{-1}}{2^{3+\frac{2}{\alpha}}} z^{3+\frac{2}{\alpha} - C\lambda\delta + \eta_2 C R^{-\alpha}}.$$

By our choice of C and R in (6.11), $p := 3 + \frac{2}{\alpha} + C\left(\eta_2 R^{-\alpha} - \lambda\delta\right) < 0$, so if

$$z \le \left(\frac{(1-\delta)2^{3+\frac{2}{\alpha}}R}{\eta_1}\right)^{1/p}$$

we have $u(t + C|\log z|, x) > 1 - \delta$ which contradicts our assumption. $\square$

We conclude this section by describing the spread of solutions u.

Lemma 6.3.5. *For all $m > 0$, there exists $T = T(m) > 0$ such that if $t_0 \ge 1$, $x_0 \in \mathbb{R}$ and $u(t_0, x_0) \ge m$, there exists $m' = m'(T)$ such that $u(s, x_0) \ge m'$ for all $s \ge t_0 + T$.*

Proof. Fix $t_0 \ge 1$, $x_0 \in \mathbb{R}$ and $m > 0$ with $u(t_0, x_0) > m$. By Lemma 6.3.3, we may fix $\varepsilon > 0$ such that, if $s \ge 1$ and $x, y \in \mathbb{R}$ with $|x - y| \le \varepsilon$, then $|u(s, x) - u(s, y)| \le \frac{1}{2}m$. Choose $T, R > 1$ sufficiently large so that

$$\frac{\varepsilon B^{-1}}{1 + (\varepsilon + 2R)^{1+\alpha}} m\eta_1 e^{(T-2)f(m) - \eta_2\frac{T-2}{R^\alpha}} > \frac{3}{2} \tag{6.14}$$

for η_1, η_2 the constants from Lemma 6.2.3 (b). This proof will be straightforward once we have proved the following:

$$\text{for all } t \ge t_0, \sup_{\substack{|y - x_0| \le R \\ s \in [t, t+T]}} u(s, y) \ge m. \tag{6.15}$$

Suppose for the purpose of a contradiction that (6.15) does not hold. Define

$$\hat{t} := \inf\left\{ r \ge t_0 : \sup_{\substack{|y - x_0| \le R \\ s \in [r, r+T]}} u(s, y) < m \right\}.$$

Note that there exists $x_1 \in [x_0 - R, x_0 + R]$ such that $u(\hat{t}, x_1) \ge m$. To see this, suppose there did not exist any such x_1. Then by continuity of u, for some small $\delta > 0$, u would be bounded above by m in the space-time box $[\hat{t} - \delta, \hat{t} - \delta + T] \times [x_0 - R, x_0 + R]$, but this would contradict our choice of $\hat{t}$.

By our choice of ε, and since $u(\hat{t}, x_1) \ge m$,

$$u(\hat{t}, x) \ge \tfrac{1}{2}m \;\forall\, x \in [x_1 - \varepsilon, x_1 + \varepsilon]. \tag{6.16}$$

Therefore by the Feynman–Kac formula, definition of $\hat{t}$, and since f is decreasing by (6.3),

$$\begin{aligned}
&u(\hat{t} + T - 1, x_0) \\
&\ge \mathbb{E}_z\left[\exp\left(\int_0^{T-2} f(u(\hat{t} + T - 1 - s, X_s))ds\right) u(\hat{t}, X_{T-1})\right] \\
&\ge \tfrac{1}{2}m e^{(T-2)f(m)} \mathbb{P}_{x_0}\left[|X_s - x_0| \le R \;\forall\, s \le T-2,\; |X_{T-1} - x_1| \le \varepsilon\right].
\end{aligned} \tag{6.17}$$

Recall that $|x_0 - x_1| \leq 2R$. If $x_0 - x_1 \geq 0$, then by monotonicity of the stable density,

$$\begin{aligned}\mathbb{P}_0\left[|X_1 + x_0 - x_1| \leq \varepsilon\right] &= \mathbb{P}_0\left[-\varepsilon + x_1 - x_0 \leq X_1 \leq \varepsilon + x_1 - x_0\right] \\ &\geq \mathbb{P}_0\left[|X_1 - 2R| \leq \varepsilon\right].\end{aligned}$$

The identical inequality holds when $x_0 - x_1 < 0$ by symmetry of the stable process. Using this, and conditioning on the value of X_{T-2}, we obtain

$$\begin{aligned}&\mathbb{P}_{x_0}\left[|X_s - x_0| \leq R\ \forall\, s \leq T-2,\ |X_{T-1} - x_1| \leq \varepsilon\right] \\ &\quad \geq \mathbb{P}_0\left[|X_s| \leq R\ \forall\, s \leq T-2\right] \mathbb{P}_0\left[|X_1 - 2R| \leq \varepsilon\right].\end{aligned}$$

By Lemma 6.2.3 (b),

$$\mathbb{P}_0\left[|X_s| \leq R\ \forall\, s \leq T-2\right] \geq \tfrac{2}{3}\eta_1 e^{-\eta_2 \frac{T-2}{R^\alpha}}$$

and by Lemma 6.2.2

$$\mathbb{P}_0\left[|X_1 - 2R| \leq \varepsilon\right] \geq \frac{2\varepsilon B^{-1}}{1 + (\varepsilon + 2R)^{1+\alpha}}.$$

Therefore (6.17) becomes

$$\begin{aligned}u(\hat{t} + T - 1, x_0) &\geq \frac{2}{3}\frac{\varepsilon B^{-1}}{1 + (\varepsilon + 2R)^{1+\alpha}} m\eta_1 e^{(T-2)f(m) - \eta_2 \frac{T-2}{R^\alpha}} \\ &> 1\end{aligned}$$

by our choice of R and T from (6.14), contradicting Proposition 6.3.2, so (6.15) holds.

Now fix $s \geq t_0$. By (6.15), there exists $x \in [x_0 - R, x_0 + R]$ and $r \in [s, s+T]$ for which $u(r, x) \geq m$. So by the Feynman–Kac formula and Lemma 6.2.2,

$$\begin{aligned}u(s + T, x_0) &\geq \tfrac{1}{2} m\, \mathbb{P}_{x_0}\left[|X_{s+T-r} - x| \leq \varepsilon\right] \\ &\geq \tfrac{1}{2} m \inf_{l \in [0,T]} \mathbb{P}_0\left[X_l \in [R - \varepsilon, R + \varepsilon]\right] \\ &\geq \tfrac{1}{2} m \frac{2B^{-1}(R + \varepsilon)}{T^{\frac{1}{\alpha}} + T^{-1}(R + \varepsilon)^{1+\alpha}} \\ &=: m'\end{aligned}$$

proving the result. $\square$

6.4 Proof of Theorem 6.1.2

Now that we have shown solutions to equation (6.1) grow and spread, we can proceed with the main proof. Our strategy follows that of [Pen18], where upper and lower bounds for u are proved alternately. To begin, we calculate one such upper bound, which results in a weakened version of the second limit in our main theorem, Theorem 6.1.2. We will see that this result already improves the upper bound found in [CR13], stated in Theorem 6.1.1 part (a).

Proposition 6.4.1. *Suppose u is a solution to (6.1) and f is defined as in (6.2). Assume f and u_0 satisfy (6.3) and (6.4), respectively. Define $\sigma^* := \frac{f(0)}{\alpha}$, $x_t^\varepsilon := t^{\frac{1}{\alpha}+\varepsilon}e^{\sigma^* t}$ and $x_t := t^{\frac{1}{\alpha}}e^{\sigma^* t}$. There exist constants $A_1, A_2 > 0$ such that, for all $\varepsilon > 0$, $t > 1$, $s \in [0,t)$ and $y \geq 0$,*

$$u(t-s, x_t^\varepsilon + y) \leq A_1 \frac{(t-s)e^{f(0)(t-s)}}{(x_t^\varepsilon - x_t)^\alpha} + \frac{A_2}{t}.$$

One consequence of Proposition 6.4.1 is that

$$\lim_{t\to\infty} \sup_{x \geq t^{\frac{1}{\alpha}+\varepsilon}e^{\sigma^* t}} u(t,x) = 0. \tag{6.18}$$

This follows by the Mean Value Theorem since $(x_t^\varepsilon - x_t)^{-\alpha} \leq (t\varepsilon^\alpha e^{f(0)t}(\log t)^\alpha)^{-1}$ where we recall that $t > 1$ and $\alpha\sigma^* = f(0)$. Then, by Proposition 6.4.1, for any $x \geq t^{\frac{1}{\alpha}+\varepsilon}e^{\sigma^* t}$,

$$u(t,x) \leq \frac{A_1}{\varepsilon^\alpha (\log t)^\alpha} + \frac{A_2}{t}$$

and (6.18) is immediate. Notably, this is already an improvement of the analogous result in [CR13], and is proved using straightforward probabilistic techniques.

Proof of Proposition 6.4.1. Let $y \geq 0$, $t > 1$, $s \in [0,t)$ and $\varepsilon > 0$. By assumptions (6.3) and (6.4), $u_0 \in [0,1]$ and $f \geq 0$ is decreasing. It follows by the Feynman–Kac formula (Theorem 6.2.1) that $u \geq 0$ and

$$\begin{aligned} u(t-s, x_t^\varepsilon + y) &\leq e^{f(0)(t-s)}\mathbb{E}_{x_t^\varepsilon + y}[u_0(X_{t-s})] \\ &\leq e^{f(0)(t-s)}\left(\mathbb{P}_{x_t^\varepsilon+y}\left[X_{t-s} < x_t\right] + \mathbb{E}_{x_t^\varepsilon+y}\left[u_0(X_{t-s})\mathbb{1}_{X_{t-s}\geq x_t}\right]\right). \end{aligned} \tag{6.19}$$

We consider each of these terms separately. First, by Lemma 6.2.2 there exists $B > 0$ for which

$$\mathbb{P}_{x_t^\varepsilon+y}\left[X_{t-s} < x_t\right] \leq \int_{-\infty}^{x_t - x_t^\varepsilon} \frac{B}{(t-s)^{\frac{1}{\alpha}} + (t-s)^{-1}|z|^{1+\alpha}} dz$$

$$\leq B(t-s)\int_{x_t^\varepsilon - x_t}^{\infty} z^{-1-\alpha}dz$$
$$= \frac{B(t-s)}{\alpha\left(x_t^\varepsilon - x_t\right)^\alpha}. \tag{6.20}$$

For the second term, by Lemma 6.2.2 and since $u_0(x) \leq C_0(1+x)^{-\alpha}$ for all $x > 0$ by assumption (6.4), we have

$$\begin{aligned}
&\mathbb{E}_{x_t^\varepsilon + y}\left[u_0(X_{t-s})\mathbb{1}_{X_{t-s}\geq x_t}\right] \\
&\leq \int_{x_t - x_t^\varepsilon - y}^{\infty} \frac{Bu_0(z + x_t^\varepsilon + y)}{(t-s)^{\frac{1}{\alpha}} + (t-s)^{-1}|z|^{\alpha+1}}dz \\
&\leq \int_{x_t^\varepsilon + y - x_t}^{\infty} \frac{B(t-s)}{z^{1+\alpha}}dz + \int_{x_t - x_t^\varepsilon - y}^{x_t^\varepsilon + y - x_t} \frac{BC_0(1+z+x_t^\varepsilon+y)^{-\alpha}}{(t-s)^{\frac{1}{\alpha}} + (t-s)^{-1}|z|^{\alpha+1}}dz \\
&\leq \frac{B(t-s)}{\alpha\left(x_t^\varepsilon - x_t\right)^\alpha} + \int_{x_t - x_t^\varepsilon - y}^{x_t^\varepsilon + y - x_t} \frac{BC_0(1+x_t)^{-\alpha}}{(t-s)^{\frac{1}{\alpha}} + (t-s)^{-1}|z|^{\alpha+1}}dz \\
&\leq \frac{B(t-s)}{\alpha\left(x_t^\varepsilon - x_t\right)^\alpha} + \frac{2BC_0}{(1+x_t)^\alpha}\left(\int_{(t-s)^{\frac{1}{\alpha}}}^{x_t^\varepsilon + y - x_t} \frac{t-s}{z^{\alpha+1}}dz + \int_0^{(t-s)^{\frac{1}{\alpha}}} (t-s)^{-\frac{1}{\alpha}}dz\right) \\
&\leq \frac{B(t-s)}{\alpha\left(x_t^\varepsilon - x_t\right)^\alpha} + \frac{2BC_0(1+\frac{1}{\alpha})}{x_t^\alpha}.
\end{aligned} \tag{6.21}$$

Substituting (6.20) and (6.21) back into (6.19) gives us

$$u(t-s, x_t^\varepsilon + y) \leq \frac{2B(t-s)e^{f(0)(t-s)}}{\alpha(x_t^\varepsilon - x_t)^\alpha} + \frac{2BC_0(1+\frac{1}{\alpha})e^{f(0)(t-s)}}{x_t^\alpha}.$$

Since $e^{f(0)(t-s)}x_t^{-\alpha} \leq e^{f(0)t}x_t^{-\alpha} = t^{-1}$ this proves the result. □

Using Proposition 6.4.1, we prove the following lower bound for u at the point $t^{\frac{1}{\alpha}+2\varepsilon}e^{\sigma^* t}$. We will then use the growing and spreading results (Lemmas 6.3.4, 6.3.5) to obtain a lower bound on $u(t,x)$ over the region $x \leq t^{-K}e^{\sigma^* t}$ for some $K > 0$ and large enough t (this is the content of Proposition 6.4.3).

Lemma 6.4.2. *Let $\varepsilon, \delta > 0$. There exists $T = T(\varepsilon,\delta) > 0$ such that for all $t \geq T$*

$$u(t, t^{\frac{1}{\alpha}+2\varepsilon}e^{\sigma^* t}) \geq t^{-1-\delta-3\varepsilon\alpha}.$$

Proof. Let $\varepsilon > 0$ and $t > 1$. Continuing with our earlier notation, set $x_t^\varepsilon := t^{\frac{1}{\alpha}+\varepsilon}e^{\sigma^* t}$ and $x_t := t^{\frac{1}{\alpha}}e^{\sigma^* t}$. Using the Feynman–Kac formula and that $\widetilde{m} := \inf_{x\leq 0} u_0(x) > 0$ by assumption (6.4),

$$\begin{aligned}
u(t, x_t^{2\varepsilon}) &\geq \widetilde{m}\,\mathbb{E}_{x_t^{2\varepsilon}}\left[\exp\left(\int_0^t f(u(t-s, X_s))ds\right)\mathbb{1}_{X_t<0}\mathbb{1}_{X_s \geq x_t^\varepsilon\ \forall s\leq t-1}\right] \\
&\geq \widetilde{m}\,\mathbb{E}_{x_t^{2\varepsilon}}\left[\exp\left(\int_0^{t-1} f(u(t-s, X_s))ds\right)\middle|\, X_s \geq x_t^\varepsilon\ \forall\, s \leq t-1\right] \\
&\quad \cdot \mathbb{P}_{x_t^{2\varepsilon}}\left[X_t < 0,\, X_s \geq x_t^\varepsilon\ \forall\, s \leq t-1\right].
\end{aligned} \tag{6.22}$$

Here, we have used that $\int_0^t f(u(t-s,X_s))ds \geq \int_0^{t-1} f(u(t-s,X_s))ds$ since f is non-negative, which allows us to remove the event $\{X_t < 0\}$ from the conditioning. The idea here is to condition on possible paths of $(X_s)_{0\leq s\leq t-1}$ for which the Feynman–Kac growth term, $f(u(t-s,X_s))$, is large, which will enable us to obtain a suitable lower bound on $u(t,x_t^{2\varepsilon})$. By Proposition 6.4.1, conditional on $\{X_s \geq x_t^\varepsilon \ \forall\, s \leq t-1\}$, $u(t-s,X_s)$ will be small for all $s \leq t-1$, and since f is decreasing, this gives us a lower bound on $f(u(t-s,X_s))$.

We begin by bounding the second term in (6.22). Denote the transition density of $(X_s)_{s\geq 0}$ started at x at time t by $p_{X_t}(x,\cdot)$. Now,

$$\begin{aligned}
&\mathbb{P}_{x_t^{2\varepsilon}}\left[X_t < 0,\, X_s \geq x_t^\varepsilon \ \forall\, s \leq t-1\right] \\
&\geq \int_{x_t^\varepsilon}^{x_t^{3\varepsilon}} \mathbb{P}_{x_t^{2\varepsilon}}\left[X_t < 0,\, X_s \geq x_t^\varepsilon \ \forall\, s \leq t-1 \,|\, X_{t-1} = z\right] p_{X_{t-1}}(x_t^{2\varepsilon}, z) dz \\
&\geq \mathbb{P}_{x_t^{3\varepsilon}}\left[X_1 < 0\right] \mathbb{P}_{x_t^{2\varepsilon}}\left[X_s \geq x_t^\varepsilon \ \forall\, s \leq t-1,\, X_{t-1} \leq x_t^{3\varepsilon}\right].
\end{aligned}$$

We bound each of these terms separately. First, noting that $x_t^{3\varepsilon} - x_t^{2\varepsilon} \geq x_t^{2\varepsilon} - x_t^\varepsilon$, and by the Mean Value Theorem $x_t^{2\varepsilon} - x_t^\varepsilon \geq \varepsilon \log t\, x_t^\varepsilon$,

$$\begin{aligned}
&\mathbb{P}_{x_t^{2\varepsilon}}\left[X_s \geq x_t^\varepsilon \ \forall\, s \leq t-1,\, X_{t-1} \leq x_t^{3\varepsilon}\right] \\
&\geq \mathbb{P}_0\left[x_t^\varepsilon - x_t^{2\varepsilon} \leq X_s \leq x_t^{3\varepsilon} - x_t^{2\varepsilon} \ \forall\, s \leq t-1\right] \\
&\geq \mathbb{P}_0\left[|X_s| \leq x_t^{2\varepsilon} - x_t^\varepsilon \ \forall\, s \leq t-1\right] \\
&\geq \tfrac{2}{3}\eta_1 \exp\left(-\frac{\eta_2(t-1)}{(x_t^{2\varepsilon} - x_t^\varepsilon)^\alpha}\right) \\
&\geq \tfrac{2}{3}\eta_1 \exp\left(-\frac{\eta_2(t-1)}{\varepsilon^\alpha t^{1+\varepsilon\alpha}(\log t)^\alpha e^{f(0)t}}\right)
\end{aligned} \tag{6.23}$$

where the second to last inequality follows by Lemma 6.2.3 and since $\sigma^*\alpha = f(0)$. By Lemma 6.2.2,

$$\mathbb{P}_{x_t^{3\varepsilon}}\left[X_1 < 0\right] \geq \frac{B^{-1}}{2\alpha}(x_t^{3\varepsilon})^{-\alpha}. \tag{6.24}$$

Combining (6.23) and (6.24) gives

$$\begin{aligned}
&\mathbb{P}_{x_t^{2\varepsilon}}\left[X_t < 0,\, X_s \geq x_t^\varepsilon \ \forall\, s \leq t-1\right] \\
&\geq \frac{\eta_1 B^{-1}}{3\alpha} t^{-1-3\varepsilon\alpha} \exp\left(-f(0)t - \frac{\eta_2(t-1)}{\varepsilon^\alpha t^{1+\varepsilon\alpha}(\log t)^\alpha e^{f(0)t}}\right) \\
&\geq \overline{C} t^{-1-3\varepsilon\alpha} \exp\left(-f(0)t - \tfrac{1}{2}\right)
\end{aligned} \tag{6.25}$$

where $\overline{C} := \frac{\eta_1 B^{-1}}{3\alpha}$ and the final line holds by increasing T if necessary, thereby bounding the second term of (6.22).

Next, we find a lower bound for

$$\mathbb{E}_{x_t^{2\varepsilon}}\left[\exp\left(\int_0^{t-1} f(u(t-s, X_s))ds\right)\middle| X_s \geq x_t^\varepsilon \ \forall\, s \leq t-1\right].$$

By Proposition 6.4.1, there exists constants $A_1, A_2 > 0$ such that, for any $y \geq 0$ and $s \in [0, t)$, if

$$D(t, s) := A_1 \frac{(t-s)e^{f(0)(t-s)}}{(x_t^\varepsilon - x_t)^\alpha} + \frac{A_2}{t}$$

then $u(t-s, x_t^\varepsilon + y) \leq D(t, s)$. Note that, by assumption (6.3) and the Mean Value Theorem, there exists $\kappa > 0$ such that $f(u) \geq f(0) - \kappa u$ for all $u \in [0, 1]$. Therefore for T sufficiently large and $t \geq T$

$$\begin{aligned}
&\mathbb{E}_{x_t^{2\varepsilon}}\left[\exp\left(\int_0^{t-1} f(u(t-s, X_s))ds\right)\middle| X_s \geq x_t^\varepsilon \ \forall\, s \leq t-1\right] \\
&\geq \exp\left(\int_0^{t-1} f(0) - \kappa D(t, s) ds\right) \\
&= \exp\left((t-1)f(0) - \kappa \int_0^{t-1} f(0)\left(A_1 \frac{(t-s)e^{f(0)(t-s)}}{(x_t^\varepsilon - x_t)^\alpha} + \frac{A_2}{t}\right) ds\right) \\
&\geq \exp\left((t-1-\kappa A_2)f(0) + \frac{\kappa A_1}{f(0)(x_t^\varepsilon - x_t)^\alpha}\left[e^{f(0)}(f(0)-1) + e^{tf(0)}(1-tf(0))\right]\right) \\
&\geq \exp\left((t-1-\kappa A_2)f(0) - \tfrac{1}{2}\right) \qquad (6.26)
\end{aligned}$$

using that $(x_t^\varepsilon - x_t)^\alpha \geq \varepsilon^\alpha t(\log t)^\alpha e^{f(0)t}$ so the last line holds by increasing T as necessary. Substituting (6.25) and (6.26) into (6.22) gives us

$$u(t, x_t^{2\varepsilon}) \geq \widetilde{m}\,\overline{C} e^{-(1+\kappa A_2)f(0)-1}\, t^{-1-3\varepsilon\alpha}.$$

Clearly for any $\delta > 0$ and $T(\delta)$ sufficiently large the latter is bounded below by $t^{-\delta-1-3\varepsilon\alpha}$ for all $t \geq T$, proving the result. ◻

Equipped with Lemma 6.4.2, we can prove our first lower bound for u over a large region.

Proposition 6.4.3. *There exists $K, T', m^* > 0$ such that, for all $t \geq T'$, $u(t, x) > m^*$ for all $x \leq t^{-K} e^{\sigma^* t}$.*

Proof. Fix $\varepsilon, \delta > 0$. By Lemma 6.4.2, there exists $T = T(\varepsilon, \delta) > 0$ such that $u(t, t^{\frac{1}{\alpha}+2\varepsilon} e^{\sigma^* t}) \geq t^{-q}$ for all $t \geq T$, where $q \equiv q(\varepsilon, \delta) := 1 + \delta + 3\varepsilon\alpha$. Increase T

if necessary so that $T > 1$ and $T^{-q} < z_0(1/2)$ for z_0 as in Lemma 6.3.4. Then by Lemma 6.3.4, there exists $C, R > 0$, $s_0 \in [0, Cq \log t]$ and $y_0 \in [-R, R]$ such that $u(t+s_0, t^{\frac{1}{\alpha}+2\varepsilon}e^{\sigma^* t}+y_0) \geq \frac{1}{2}$ for all $t \geq T$. By Lemma 6.3.5, there exists $\widehat{T}, m > 0$ such that

$$u(t + s_0 + r, t^{\frac{1}{\alpha}+2\varepsilon}e^{\sigma^* t} + y_0) \geq m$$

for all $r \geq \widehat{T}$. That is, for all $s \geq Cq \log t + \widehat{T}$ and $t \geq T$,

$$u(t + s, t^{\frac{1}{\alpha}+2\varepsilon}e^{\sigma^* t} + y_0) \geq m.$$

By Lemma 6.3.3, there exists $\varepsilon_0 > 0$ for which $|u(t,x) - u(t,z)| < \frac{m}{2}$ if $|x - z| < \varepsilon_0$ and $t \geq 1$. Therefore

$$u(t + s, t^{\frac{1}{\alpha}+2\varepsilon}e^{\sigma^* t} + y) \geq \frac{m}{2}$$

for all $y \in [y_0 - \varepsilon_0, y_0 + \varepsilon_0]$. By the Feynman–Kac formula and Lemma 6.2.2

$$\begin{aligned} u(t + s + 1, t^{\frac{1}{\alpha}+2\varepsilon}e^{\sigma^* t}) &\geq \frac{m}{2}\mathbb{P}_0\left[X_1 \in [y_0 - \varepsilon_0, y_0 + \varepsilon_0]\right] \\ &= \frac{m}{2}\frac{2B^{-1}\varepsilon_0}{1 + (|y_0| + \varepsilon_0)^{1+\alpha}} \\ &=: m' \end{aligned}$$

for all $s \geq Cq \log t + \widehat{T}$ and $t \geq T$. Choose $N > 0$ sufficiently large so that $\sigma^*(Cq + N) > \frac{1}{\alpha}$. By increasing T appropriately,

$$u(t + (Cq + N)\log t, t^{\frac{1}{\alpha}+2\varepsilon}e^{\sigma^* t}) \geq m' \tag{6.27}$$

for all $t \geq T$.

Let $t \geq T$ and define $t' := t + c' \log t$ where $c' := Cq + N$. Increase T if necessary so that, if $t \geq T$, $c' \log t < t'/2$. Then for all $t \geq T$

$$\begin{aligned} t^{\frac{1}{\alpha}+2\varepsilon}e^{\sigma^* t} &= (t' - c' \log t)^{\frac{1}{\alpha}+2\varepsilon}e^{\sigma^*(t' - c' \log t)} \\ &\geq 2^{-\frac{1}{\alpha}-2\varepsilon}(t')^{\frac{1}{\alpha}+2\varepsilon-c'\sigma^*}e^{\sigma^* t'} \\ &\geq (t')^{-K}e^{\sigma^* t'} \end{aligned}$$

for some $K > 0$, where, by our choice of N, we may assume ε is sufficiently small so that $\frac{1}{\alpha} + 2\varepsilon - c'\sigma^* < 0$, so the final line holds by increasing T if necessary. Define $y_{t'} := (t' - c' \log t)^{\frac{1}{\alpha}+2\varepsilon}e^{\sigma^*(t' - c' \log t)}$. Then by (6.27) there exists $T' > 0$ such that

$$u(t', y_{t'}) \geq m'$$

for all $t' \geq T'$ where $y_{t'} \geq (t')^{-K}e^{\sigma^* t'}$.

Let $\overline{T} > 0$ and suppose $s \geq T' + \overline{T}$. Take $y \in [y_{T'}, s^{-K}e^{\sigma^* s}]$. Since $y_{t'}$ is continuous and increasing in t', there exists $T' \leq r \leq s$ such that $y = y_r$, so $u(r, y) \geq m'$. Increasing $\overline{T}$ as needed, by Lemma 6.3.5 there exists $m'' > 0$ such that $u(s, y) \geq m''$, proving the result for large enough y.

Now suppose $y < y_{T'}$. Recall by assumption (6.4) that $\widetilde{m} := \inf_{x \leq 0} u_0(x) > 0$. Therefore by the Feynman–Kac formula and Lemma 6.2.2

$$\begin{aligned} u(1, y) &\geq \mathbb{E}_y \left[u_0(X_1) \right] \\ &\geq \widetilde{m} \mathbb{P}_0 \left[X_1 \leq -y_{T'} \right] \\ &\geq \widetilde{m} \frac{B^{-1}}{2\alpha} y_{T'}^{-\alpha}. \end{aligned}$$

It follows from Lemma 6.3.5 that there exists $T'', \overline{m} > 0$ such that $u(r, y) \geq \overline{m}$ for all $r \geq 1 + T''$. In particular, if s is sufficiently large, $u(s, y) \geq \overline{m}$. Combining this with the above result, we conclude that there exists $T > 0$ and $m^* > 0$ such that, for all $s \geq T$, $u(s, y) \geq m^*$ for all $y \leq s^{-K} e^{\sigma^* s}$. □

We now prove our final upper bound for u, which we state as a corollary to the following proposition.

Proposition 6.4.4. *Fix $q > \frac{4}{f(0)}$ and define $z_t := \frac{1}{2}(q \log t)^{-K} t^{q\sigma^*}$ for K as in Proposition 6.4.3. For all $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ and $M = M(\varepsilon) > 0$ such that, for all $t \geq T$, $y > 0$ and $x \geq M y e^{\sigma^* t}$,*

$$\mathbb{E}_x \left[e^{\int_0^t f(u(t-s, X_s)) ds} \mathbb{1}_{X_t \leq \frac{z_t}{2}} \right] < \varepsilon y^{-\alpha}.$$

Using Proposition 6.4.4, it is straightforward to prove the following strengthened version of Theorem 6.1.2 part (b).

Corollary 6.4.5. *For all $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ and $M = M(\varepsilon) > 0$ such that, for all $t \geq T$, $y > 0$ and $x \geq M y e^{\sigma^* t}$, $u(t, x) < \varepsilon y^{-\alpha}$.*

We begin with the proof of Proposition 6.4.4.

Proof of Proposition 6.4.4. Fix $\varepsilon, y, T > 0$. Suppose $t \geq T$ and $x \geq M y e^{\sigma^* t}$ for some $M > 0$ that will be determined in the course of this proof. We begin by defining several constants that will be used throughout. First, by Proposition 6.4.3, there exists $T', K, m > 0$ such that, for all $t \geq T'$, $u(t, z) > m$ for all $z \leq t^{-K} e^{\sigma^* t}$. Let $D := D_1 + D_2$ where $D_1 > D_2$ and

$$D_1 > \frac{1}{f(0) - f(m)}, \quad D_2 > \frac{4}{f(0)}. \tag{6.28}$$

It will be useful to rewrite z_t as

$$z_t := \frac{1}{2}(D_2 \log t)^{-K} t^{D_2\sigma^*}. \tag{6.29}$$

Finally, define

$$h(s) = \begin{cases} z_t \text{ if } s \le t - D\log t \\ \frac{1}{2}\left(\frac{D_2}{D}(t-s)\right)^{-K} e^{D_2\sigma^*(t-s)/D} \text{ if } s > t - D\log t. \end{cases}$$

Choose $C \in (0, D\log t)$ sufficiently large so that $h(s)$ is decreasing over $[0, t-C]$. To bound above the quantity $\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{X_t\le\frac{z_t}{2}}\right]$, we condition on the following three events. Define

$$\begin{aligned} E &:= \left\{\exists\, s \in (0, t-C) : X_s \in [h(s), x - z_t],\ X_t \le \tfrac{z_t}{2}\right\} \\ F &:= \left\{\exists\, s \in (0, t-C) :\ X_s < h(s),\ X_r \notin [h(r), x - z_t]\,\forall\, r \le s,\ X_t \le \tfrac{z_t}{2}\right\} \\ G &:= \left\{X_s > x - z_t\ \forall\, s \in [0, t-C],\ X_t \le \tfrac{z_t}{2}\right\}. \end{aligned}$$

Clearly $\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{X_t\le\frac{z_t}{2}}\right] \le \mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\left(\mathbb{1}_E + \mathbb{1}_F + \mathbb{1}_G\right)\right]$, so it suffices to bound each of these terms separately. To better visualise these events see Figure 6.2.

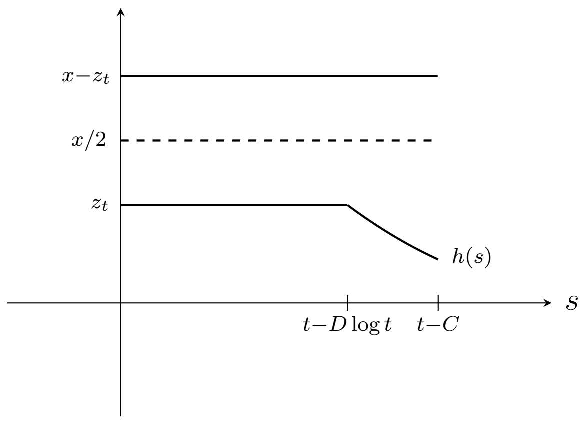


Figure 6.2: The graphs $y = x - z_t$, $y = x/2$ and $y = h(s)$

We will see that, conditional on each of the events E, F, and G, either the Feynman–Kac growth term $\int_0^t f(u(t-s,X_s))ds$ will be small, or the probability of the event we are conditioning on will be small; in both situations, we obtain an upper bound for $\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{X_t\le\frac{z_t}{2}}\right]$. We begin by considering the event F. Note that

$F \subseteq F_1 \cup F_2$ where

$$F_1 := \left\{\exists\, s \in (0, t - D\log t] : X_s < z_t, X_t \leq \tfrac{z_t}{2}\right\}$$
$$F_2 := \left\{\exists\, s \in (t - D\log t, t - C) : X_s < h(s),\ X_t \leq \tfrac{z_t}{2}\right\}$$
$$\cap \left\{X_r \geq h(r)\ \forall\, r \in (0, t - D\log t),\ X_v \notin [h(v), x - z_t]\ \forall\, v \leq s\right\}.$$

We first consider F_1.

Claim 1. There exists $T = T(\varepsilon) > 0$ such that, for all $t \geq T$,

$$\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{F_1}\right] < \frac{\varepsilon}{8y^\alpha}.$$

Proof of Claim 1. We will show that when F_1 occurs, X_s spends a substantial amount of time (of order $\log t$) in a region for which $f(u(t-s, X_s))$ is small, which will give us the upper bound in Claim 1. To do this, we first show that, once $X_s < z_t$ for some $s \in (0, t - D\log t]$, then with high probability, for a further $\mathcal{O}(\log t)$ time, the process stays below $2z_t$. By our choice of z_t and Proposition 6.4.3, it will follow that, with high probability, $u(t-s, X_s)$ will be large for all s in an interval of length $\mathcal{O}(\log t)$, so $f(u(t-s, X_s))$ will be small over this interval.

Let T be as above and fix $t \geq T$. Define $\tau_1 := \inf\{s \leq t : X_s < h(s)\}$. Decompose F_1 as $F_1 = F_1^1 \cup F_1^2$ where

$$F_1^1 := \left\{\tau_1 \leq t - D\log t,\ \exists\, s\ \in [\tau_1, \tau_1 + D_1 \log t] : X_s > 2z_t,\ X_t \leq \tfrac{z_t}{2}\right\}$$
$$F_1^2 := \left\{\tau_1 \leq t - D\log t,\ \forall\, s\ \in [\tau_1, \tau_1 + D_1 \log t],\ X_s \leq 2z_t,\ X_t \leq \tfrac{z_t}{2}\right\}.$$

We show that F_1^1 occurs with low probability. By the Strong Markov Property,

$$\begin{aligned}\mathbb{P}_x\left[F_1^1\right] &= \mathbb{E}_x\left[\mathbb{P}_{X_{\tau_1}}\left[\exists\, s \leq D_1\log t : X_s > 2z_t,\ X_{t-\tau_1} \leq \tfrac{z_t}{2}\right]\mathbb{1}_{\{\tau_1 \leq t - D\log t\}}\right] \\ &\leq \mathbb{E}_x\left[\mathbb{P}_{X_{\tau_1}}\left[\exists\, s \leq D_1\log t : X_s > 2z_t\right]\mathbb{1}_{\{\tau_1 \leq t - D\log t\}}\right] \\ &\leq \mathbb{P}_0\left[\sup_{s\in(0,D_1\log t)} |X_s| \geq z_t\right]\mathbb{E}_x\left[\mathbb{1}_{\{\tau_1 \leq t - D\log t\}}\right] \end{aligned} \tag{6.30}$$

where the last line follows since $X_{\tau_1} < z_t$ by definition of τ_1 and h. By Lemma 6.2.3 part (a) and definition of z_t (6.29), and recalling that $\sigma^*\alpha = f(0)$, there exists $\gamma > 0$ such that

$$\begin{aligned}\mathbb{P}_0\left[\sup_{s\in(0,D_1\log t)} |X_s| \geq z_t\right] &\leq 4e^{-\frac{z_t}{2}} + \gamma D_1 z_t^{-\frac{\alpha}{2}}\log t \\ &= 4\exp\left(-\frac{t^{\sigma^* D_2}}{4(D_2\log t)^K}\right) + 2^{\frac{\alpha}{2}}\gamma D_1 D_2^{\frac{K\alpha}{2}}(\log t)^{\frac{K\alpha}{2}+1}t^{-\frac{f(0)D_2}{2}} \\ &\leq B_1(\log t)^{\frac{K\alpha}{2}+1}t^{-\frac{f(0)D_2}{2}} \end{aligned} \tag{6.31}$$

for some $B_1 > 0$ where we increase T if necessary for the last inequality to hold for all $t \geq T$. Also,

$$\begin{aligned}
\mathbb{E}_x\left[\mathbb{1}_{\{\tau_1 \leq t - D\log t\}}\right] &\leq \mathbb{P}_x\left[\exists\, s \in (0,t) : X_s < z_t\right] \\
&\leq \sum_{j=0}^{\lfloor t \rfloor}\left(\mathbb{P}_x\left[X_j \leq \tfrac{x}{2}\right] + \mathbb{P}_x\left[\sup_{s\in[j,j+1]}|X_j - X_s| \geq \tfrac{x}{2} - z_t\right]\right) \\
&\leq \sum_{j=0}^{\lfloor t \rfloor}\left(\frac{2^\alpha B}{\alpha(My)^\alpha}\frac{j}{e^{f(0)t}} + \mathbb{P}_0\left[\sup_{s\in[0,1]}|X_s| \geq \tfrac{My}{2}e^{\sigma^* t} - z_t\right]\right) \\
&\leq \frac{2^\alpha B}{\alpha(My)^\alpha}\frac{t^2}{e^{f(0)t}} + t\mathbb{P}_0\left[\sup_{s\in[0,1]}|X_s| \geq \tfrac{My}{2}e^{\sigma^* t} - z_t\right] \qquad (6.32)
\end{aligned}$$

where we have used in the second inequality that $x > Mye^{\sigma^* t}$. Increase T if necessary so that, for all $t \geq T$, $\frac{My}{2}e^{\sigma^* t} - z_t > \frac{My}{4}e^{\sigma^* t}$. Then by Lemma 6.2.3 part (a), where we set $w = \frac{M^2y^2e^{2\sigma^* t}}{32f(0)t}$, we have

$$\begin{aligned}
\mathbb{P}_0\left[\sup_{s\in[0,1]}|X_s| \geq \tfrac{My}{2}e^{\sigma^* t} - z_t\right] &\leq \mathbb{P}_0\left[\sup_{s\in[0,1]}|X_s| \geq \tfrac{My}{4}e^{\sigma^* t}\right] \\
&\leq 4e^{-f(0)t} + \frac{\gamma}{(My)^\alpha}(32f(0)t)^{\frac{\alpha}{2}}e^{-f(0)t} \\
&\leq B_2 t^{\frac{\alpha}{2}}e^{-f(0)t}y^{-\alpha} \qquad (6.33)
\end{aligned}$$

for some $B_2 > 0$ where the last line follows by increasing T if necessary. Combining (6.32) and (6.33), there exists $T > 0$ and $B_3 > 0$ such that, for all $t \geq T$,

$$\mathbb{E}_x\left[\mathbb{1}_{\{\tau_1 \leq t - D\log t\}}\right] \leq B_3 t^2 e^{-f(0)t}y^{-\alpha}. \qquad (6.34)$$

Substituting (6.31) and (6.34) into (6.30) yields

$$\mathbb{P}_x\left[F_1^1\right] \leq B_1B_3(\log t)^{\frac{K\alpha}{2}+1}t^{2-\frac{f(0)D_2}{2}}e^{-f(0)t}y^{-\alpha} \qquad (6.35)$$

for all $t \geq T$.

We now turn to the event F_1^2. Increase T if necessary so that for all $t \geq T$, $D_2\log t \geq T'$ for T' as in Proposition 6.4.3. Then by Proposition 6.4.3 and our choice of z_t, if F_1^2 occurs, $u(t-r, X_r) > m$ for all $r \in [\tau_1, \tau_1 + D_1\log t]$. To see this, note that since $\tau_1 \in [0, t - D\log t]$ and $D = D_1 + D_2$, $r \in [0, t - D_2\log t]$. Increase T if necessary to ensure that, for $t \geq T$, $(t-r)^{-K}e^{\sigma^*(t-r)}$ is decreasing in r for $r \in [0, t - D_2\log t]$. Then

$$(t-r)^{-K}e^{\sigma^*(t-r)} \geq (D_2\log t)^{-K}e^{\sigma^* D_2\log t} = 2z_t.$$

In particular, for any $r \in [\tau_1, \tau_1 + D_1 \log t]$, $X_r \leq 2z_t$ implies $X_r \leq (t-r)^{-K} e^{\sigma^*(t-r)}$ and Proposition 6.4.3 gives us that $u(t-r, X_r) > m$. Since f is decreasing by assumption (6.3), $f(u(t-r, X_r)) \leq f(m)$ for all such r. Therefore by (6.35)

$$\begin{aligned}
&\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds} \mathbb{1}_{F_1}\right] \\
&\leq e^{f(0)t}\mathbb{P}_x\left[F_1^1\right] + e^{f(0)(t-D_1\log t)+f(m)D_1\log t}\mathbb{P}_x[X_t \leq \tfrac{z_t}{2}] \\
&\leq B_1 B_3 (\log t)^{\frac{K\alpha}{2}+1} t^{2-\frac{f(0)D_2}{2}} y^{-\alpha} + \frac{B2^\alpha}{\alpha(My)^\alpha} t^{1+D_1(f(m)-f(0))} \\
&< \frac{\varepsilon}{8y^\alpha}
\end{aligned}$$

where in the second inequality we increase T if necessary and use Lemma 6.2.2 and that $x \geq Me^{\sigma^* t}$. The final inequality follows by our choice of D_1 and D_2 in (6.28) where we increase $T = T(\varepsilon)$ if necessary. ■

We now consider the event F_2, recalling that

$$\begin{aligned}
F_2 := &\left\{\exists\, s \in (t - D\log t, t - C) : X_s < h(s),\, X_t \leq \tfrac{z_t}{2}\right\} \\
&\cap \left\{X_r \geq h(r)\ \forall\, r \in (0, t - D\log t),\, X_v \notin [h(v), x - z_t]\ \forall\, v \leq s\right\}.
\end{aligned}$$

Claim 2. There exists $T = T(\varepsilon) > 0$ and $M = M(\varepsilon) > 0$ such that, for all $t \geq T$ and $x \geq Mye^{\sigma^* t}$

$$\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds} \mathbb{1}_{F_2}\right] < \frac{\varepsilon}{8y^\alpha}.$$

Proof of Claim 2. First, note that $\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds} \mathbb{1}_{F_2}\right] \leq \sum_{j=C}^{\lfloor D\log t\rfloor} A_j$ where

$$A_j := \mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds} \mathbb{1}_{\exists s\in[t-j-1,t-j]:X_s<h(s)} \mathbb{1}_{X_t \leq \frac{z_t}{2}} \mathbb{1}_{X_{t-j-1}\geq x - z_t} \mathbb{1}_{X_r \notin [h(r), x-z_t]\,\forall r \leq s}\right].$$

Conditional on the event A_j, the first time the stable process satisfies $X_s < h(s)$ is at a time $s \in [t-j-1, t-j]$, and to arrive at this location, the stable process had to jump over the region between $y = h(s)$ and $y = x - z_t$.

Let $\tau_j := \inf\{s \geq t-j-1 : X_s \leq h(s)\}$. By the Strong Markov Property at time τ_j,

$$\begin{aligned}
A_j \leq \mathbb{E}_x\Big[&e^{(t-j)f(0)} \mathbb{1}_{X_{t-j-1}\geq x-z_t} \mathbb{1}_{\tau_j \leq t-j} \mathbb{1}_{X_s \notin [h(s), x-z_t]\ \forall s \leq \tau_j} \\
&\cdot \mathbb{E}_{X_{\tau_j}}\left[e^{\int_0^{t-\tau_j} f(u(t-\tau_j-s,X_s))ds} \mathbb{1}_{X_{t-\tau_j} \leq \frac{z_t}{2}}\right]\Big].
\end{aligned} \tag{6.36}$$

To control the inner expectation in (6.36), we will show that, with high probability, once the stable process has gone below the curve $h(s)$, it remains for a period of time in a region for which $f(u(t-\tau_j-s,X_s))$ is strictly smaller than its maximum value. To this end, let $l\in[j,j+1]$, $y\le h(t-j-1)$ and consider

$$\mathbb{E}_y\left[e^{\int_0^l f(u(l-s,X_s))ds}\mathbb{1}_{X_l\le\frac{z_t}{2}}\right]\le\mathbb{E}_y\left[e^{\int_0^l f(u(l-s,X_s))ds}\mathbb{1}_{X_l\le\frac{z_t}{2}}\mathbb{1}_{X_s\le y+\exp\left(\frac{\sigma^*j}{4}\right)\ \forall s\le\frac{l}{2}}\right] \\ +e^{lf(0)}\mathbb{P}_0\left[\sup_{0\le s\le\frac{j+1}{2}}|X_s|>\exp\left(\frac{\sigma^*j}{4}\right)\right]. \tag{6.37}$$

We have introduced the (deterministic) variables l and y to simplify the coming calculations. Later, we will use $t-\tau_j$ is place of l and X_{τ_j} in place of y. We will bound the first term on the right hand side of (6.37) by showing that, if $X_s\le y+e^{\sigma^*j/4}$ for all $s\le l/2$, then $X_s\le(l-s)^{-K}e^{\sigma^*(l-s)}$ for all $s\le l/2$ and Proposition 6.4.3 applies. This provides us with an upper bound on the Feynman–Kac growth term. We will see that the second term on the right hand side of (6.37) occurs with low probability. The resulting bound for (6.37) will then be substituted back into (6.36) to give us an upper bound that is summable in j.

Before bounding the terms in (6.37), we show that for all integers $C\le j\le\lfloor D\log t\rfloor$, as long as C is large enough,

$$2h(t-j-1)\le\frac{e^{\sigma^*j/2}}{(j+1)^K}. \tag{6.38}$$

It will then follow immediately that, as long as C is large enough, for all $s\le l/2$ and all integers $C\le j\le\lfloor D\log t\rfloor$,

$$h(t-j-1)+e^{\frac{\sigma^*}{4}j}\le(l-s)^{-K}e^{\sigma^*(l-s)}. \tag{6.39}$$

First consider the case when $C\le j\le\lfloor D\log t\rfloor-1$. Then, by definition of f, (6.38) will hold if

$$(j+1)^{-K}\,e^{\sigma^*j/2}\ge\left(\tfrac{D_2}{D}(j+1)\right)^{-K}\,e^{\frac{D_2}{D}\sigma^*(j+1)}.$$

Since $D_2<D_1$, $\frac{D_2}{D}<\frac{1}{2}$ so $\sigma^*j\left(\frac{1}{2}-\frac{D_2}{D}\right)>0$ if C is large enough. Therefore the previous inequality holds by increasing C as necessary. This proves (6.38) in the case of $j\le\lfloor D\log t\rfloor-1$. Now suppose $j=\lfloor D\log t\rfloor$. Then (6.38) holds if

$$(D_2\log t)^{-K}t^{D_2\sigma^*}\le\frac{e^{-\sigma^*/2}t^{\frac{\sigma^*D}{2}}}{(D\log t+1)^K}$$

which clearly holds for T sufficiently large since $\frac{D}{2}>D_2$. So (6.38), and hence (6.39) hold by choosing C and T sufficiently large.

Increase C if necessary so that $C \geq 2T'$ for T' as in Proposition 6.4.3. Then $l - s \geq T'$ so by (6.39) and Proposition 6.4.3, there exists $m \in (0,1]$ such that, for $\delta < (f(0) - f(m)) \wedge \frac{f(0)}{4}$,

$$\begin{aligned}
\mathbb{E}_y\left[e^{\int_0^l f(u(l-s,X_s))ds}\mathbb{1}_{X_l \leq \frac{z_t}{2}}\right] &\leq \mathbb{E}_y\left[e^{\int_0^l f(u(l-s,X_s))ds}\mathbb{1}_{X_l \leq \frac{z_t}{2}}\mathbb{1}_{X_s \leq y+\exp\left(\frac{\sigma^* j}{4}\right)\ \forall s \leq \frac{l}{2}}\right] \\
&\quad + e^{lf(0)}\mathbb{P}_0\left[\sup_{0 \leq s \leq \frac{j+1}{2}} |X_s| > \exp\left(\tfrac{\sigma^* j}{4}\right)\right] \\
&\leq e^{\frac{l}{2}f(0) + \frac{l}{2}(f(0)-\delta)} + e^{lf(0)}\left(4e^{-\frac{1}{2}e^{\sigma^* j/4}} + \tfrac{\gamma}{2}(j+1)e^{-f(0)j/8}\right) \\
&\leq B_4 e^{j(f(0)-\delta/2)}
\end{aligned}$$

for some $B_4 > 0$, where in the second inequality we apply Lemma 6.2.3 part (a) (which gives the stated bound by increasing C if necessary so that $\frac{j+1}{2}e^{-f(0)j/8} < 1$ for all $j \geq C$). The last inequality follows by our choice of δ and increasing C if necessary. Substituting this into (6.36), we obtain

$$\begin{aligned}
A_j &\leq B_4 e^{tf(0)-j\delta/2}\mathbb{P}_x\left[X_{t-j-1} \geq x - z_t,\ \tau_j \leq t-j,\ X_s \notin [h(s), x-z_t]\,\forall\, s \leq \tau_j\right] \\
&\leq B_4 e^{tf(0)-j\delta/2}\mathbb{P}_x\left[\exists s \in [t-j-1, t-j] : |X_s - X_{s-}| \geq x - z_t - h(t-j-1)\right] \\
&\leq B_5 e^{tf(0)-j\delta/2}x^{-\alpha} \\
&\leq B_5 (My)^{-\alpha}e^{-j\delta/2}
\end{aligned}$$

for some $B_5 > 0$, where we use Lemma 2.1.8 and increase T if necessary. Therefore

$$\begin{aligned}
\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{F_2}\right] &\leq B_5(My)^{-\alpha}\sum_{j=C}^{\lfloor D\log t\rfloor} e^{-\delta j/2} \\
&\leq (My)^{-\alpha}\frac{B_5 e^{-\delta C/2}}{1-e^{-\delta/2}} \\
&< \frac{\varepsilon}{8y^\alpha}
\end{aligned}$$

by increasing $M(\varepsilon)$ if necessary. ■

Combining Claim 1 and Claim 2, and increasing T and M if necessary, we conclude that for all $t \geq T$ and $x \geq Mye^{\sigma^* t}$

$$\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_F\right] < \frac{\varepsilon}{4y^\alpha}. \tag{6.40}$$

We now turn our attention to the event

$$E = \left\{\exists\, s \in (0, t-C) : X_s \in [h(s), x-z_t], X_t \leq \tfrac{z_t}{2}\right\}.$$

Let $\tau_2 := \inf\{s \geq 0 : X_s \in [h(s), x - z_t]\}$. We consider separately the events

$$E_1 := \left\{\tau_2 \leq t - D\log t, X_t \leq \tfrac{z_t}{2}\right\}$$
$$E_2 := \left\{\tau_2 \in (t - D\log t, t - C), X_t \leq \tfrac{z_t}{2}\right\}$$

and note that $E = E_1 \cup E_2$. We start by considering the event E_1 and a subset of E_2 in Claim 3.

Claim 3. There exists $T = T(\varepsilon) > 0$ such that, for all $t \geq T$ and $x \geq Mye^{\sigma^* t}$

$$\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{E_1}\right] < \frac{\varepsilon}{8y^\alpha} \quad \text{and} \quad \mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{E_2\cap\{X_{\tau_2}\geq\frac{x}{2}\}}\right] < \frac{\varepsilon}{8y^\alpha}.$$

Proof of Claim 3. We show that E_1 occurs with low probability because the stable process starting at x is unlikely to reach the negative half line via an intermediate jump into the region $[h(s), x - z_t]$. First note that, by the Strong Markov Property at time τ_2,

$$\begin{aligned}
\mathbb{P}_x[E_1] &= \mathbb{E}_x\left[\mathbb{P}_{X_{\tau_2}}\left[X_{t-\tau_2} \leq \tfrac{z_t}{2}\right]\mathbb{1}_{\tau_2\leq t-D\log t}\right] \\
&\leq \mathbb{E}_x\left[\mathbb{P}_{X_{\tau_2}}\left[X_{t-\tau_2} \leq \tfrac{z_t}{2}\right]\mathbb{1}_{\tau_2\leq t-D\log t}\mathbb{1}_{X_{\tau_2}\in\left[z_t,\frac{x}{2}\right]}\right] \\
&\quad + \mathbb{E}_x\left[\mathbb{P}_{X_{\tau_2}}\left[X_{t-\tau_2} \leq \tfrac{z_t}{2}\right]\mathbb{1}_{\tau_2\leq t-D\log t}\mathbb{1}_{X_{\tau_2}\in\left[\frac{x}{2},x-z_t\right]}\right]. \qquad (6.41)
\end{aligned}$$

We bound each of these terms separately. By Lemma 6.2.2, Lemma 6.2.3 and since $x > Mye^{\sigma^* t}$,

$$\begin{aligned}
&\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_2}}\left[X_{t-\tau_2} \leq \tfrac{z_t}{2}\right]\mathbb{1}_{\tau_2\leq t-D\log t}\mathbb{1}_{X_{\tau_2}\in\left[z_t,\frac{x}{2}\right]}\right] \\
&\quad\leq \mathbb{E}_x\left[\mathbb{P}_{z_t}\left[X_{t-\tau_2} \leq \tfrac{z_t}{2}\right]\mathbb{1}_{\tau_2\leq t}\mathbb{1}_{X_{\tau_2}\in\left[z_t,\frac{x}{2}\right]}\right] \\
&\quad\leq \frac{2^\alpha B}{\alpha}\frac{t}{z_t^\alpha}\mathbb{P}_x\left[\exists\, s \in [0,t] : X_s \leq \tfrac{x}{2}\right] \\
&\quad\leq \frac{2^\alpha B}{\alpha}\frac{t}{z_t^\alpha}\mathbb{P}_0\left[\sup_{s\in(0,t)}|X_s| \geq \tfrac{x}{2}\right] \\
&\quad\leq \frac{2^\alpha B}{\alpha}\frac{te^{-f(0)t}}{z_t^\alpha}\left(4 + \gamma t(8f(0)t)^{\frac{\alpha}{2}}\right) \\
&\quad\leq B_6\frac{t^{2+\frac{\alpha}{2}}}{z_t^\alpha e^{f(0)t}} \\
&\quad= B_6\frac{2^\alpha(D_2\log t)^{K\alpha}}{t^{D_2f(0)-2-\frac{\alpha}{2}}e^{f(0)t}} \\
&\quad\leq B_6 y^{-\alpha}\frac{2^\alpha(D_2\log t)^{K\alpha}}{t^{D_2f(0)-3-\frac{\alpha}{2}}e^{f(0)t}} \qquad (6.42)
\end{aligned}$$

for some $B_6 > 0$ provided T is sufficiently large. By similar arguments,

$$\begin{aligned}\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_2}}\left[X_{t-\tau_2} \le \tfrac{z_t}{2}\right]\mathbb{1}_{\tau_2 \le t - D\log t}\mathbb{1}_{X_{\tau_2}\in\left[\frac{x}{2}, x-z_t\right]}\right] &\le \frac{2^\alpha B}{\alpha}\frac{t}{(x-z_t)^\alpha}\left(4e^{-\frac{z_t}{2}} + \gamma t z_t^{-\frac{\alpha}{2}}\right)\\ &\le B_7 y^{-\alpha}\frac{1}{e^{f(0)t}}\frac{(D_2\log t)^{K\frac{\alpha}{2}}}{t^{\frac{f(0)D_2}{2}-2}} \qquad (6.43)\end{aligned}$$

for some $B_7 > 0$ provided T is sufficiently large. Substituting (6.42) and (6.43) back into (6.41), and using that $f(u(t,z)) \le f(0)$ for all $t \ge 0, z \in \mathbb{R}$, we have that there exists $B_8 > 0$ such that for any $\varepsilon > 0$

$$\begin{aligned}\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{E_1}\right] &\le B_8 y^{-\alpha}\left(\frac{(\log t)^{K\alpha}}{t^{D_2 f(0)-3-\frac{\alpha}{2}}} + \frac{(\log t)^{K\frac{\alpha}{2}}}{t^{\frac{f(0)D_2}{2}-2}}\right)\\ &< \frac{\varepsilon}{8y^\alpha}\end{aligned}$$

for all $t \ge T$ for $T = T(\varepsilon)$ sufficiently large, using our choice of D_2 from (6.28). This proves the first part of the claim. By similar arguments as those used to obtain (6.43), and increasing T if necessary,

$$\begin{aligned}\mathbb{P}_x\left[E_2 \cap \{X_{\tau_2} \ge \tfrac{x}{2}\}\right] &\le \mathbb{E}_x\left[\mathbb{P}_{X_{\tau_2}}\left[X_{t-\tau_2} \le \tfrac{z_t}{2}\right]\mathbb{1}_{\tau_2\in(t-D\log t, t-C)}\mathbb{1}_{X_{\tau_2}\in\left[\frac{x}{2}, x-z_t\right]}\right]\\ &\le B_8 y^{-\alpha}e^{-f(0)t}\frac{(\log t)^{1+\frac{K\alpha}{2}}}{t^{\frac{f(0)D_2}{2}-1}}\\ &< \frac{\varepsilon}{8y^\alpha}\end{aligned}$$

by our choice of D_2, so the second part of the claim holds. ■

It remains to consider the event

$$E_2 \cap \{X_{\tau_2} < \tfrac{x}{2}\} = \left\{\tau_2 \in (t - D\log t, t - C), X_t \le \tfrac{z_t}{2}\right\} \cap \{X_{\tau_2} < \tfrac{x}{2}\}$$

where $\tau_2 := \inf\{s \ge 0 : X_s \in [h(s), x - z_t]\}$.

Claim 4. There exists $T = T(\varepsilon) > 0$ and $M = M(\varepsilon) > 0$ such that, for all $t \ge T$ and $x \ge Me^{\sigma^* t}$

$$\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{E_2\cap\{X_{\tau_2}<\frac{x}{2}\}}\right] < \frac{3\varepsilon}{8y^\alpha}.$$

Proof of Claim 4. We decompose $E_2 \cap \{X_{\tau_2} < \frac{x}{2}\}$ into the union E_2^1 and E_2^2 where

$$\begin{aligned}E_2^1 &:= E_2 \cap \{X_{\tau_2} < \tfrac{x}{2}\} \cap \{X_s > x - z_t \forall\, s \le \tau_2\}\\ E_2^2 &:= E_2 \cap \{X_{\tau_2} < \tfrac{x}{2}\} \cap \{\exists s < \tau : X_s < h(s)\}.\end{aligned}$$

Note that $E_2^2 \subset F$, so by (6.40) and increasing T and M as necessary,

$$\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{E_2^2}\right] < \frac{\varepsilon}{4y^\alpha}. \tag{6.44}$$

It remains to consider the event E_2^1. First, note that

$$\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{E_2^1}\right] \leq \sum_{j=C}^{\lfloor D\log t\rfloor} \hat{A}_j$$

where

$$\hat{A}_j := e^{f(0)t}\mathbb{E}_x\left[\mathbb{1}_{\tau_2\in[t-j-1,t-j]}\mathbb{1}_{X_{\tau_2}\in\left[h(\tau_2),\frac{x}{2}\right]}\mathbb{1}_{\{X_s>x-z_t\forall\, s\leq\tau_2\}}\mathbb{1}_{X_t<\frac{z_t}{2}}\right].$$

By Lemmas 6.2.2 and 2.1.8, and recalling that $\sigma^*\alpha = f(0)$,

$$\begin{aligned}
\hat{A}_j &\leq e^{f(0)t}\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_2}}\left[X_{t-\tau_2} < \tfrac{z_t}{2}\right]\mathbb{1}_{\tau_2\in[t-j-1,t-j]}\mathbb{1}_{X_{\tau_2}\in\left[h(\tau_2),\frac{x}{2}\right]}\mathbb{1}_{\{X_s>x-z_t\forall\, s\leq\tau_2\}}\right]\\
&\leq \frac{B}{\alpha}e^{f(0)t}\frac{j+1}{(h(t-j)-\frac{z_t}{2})^\alpha}\mathbb{P}\left[\exists s\leq 1 : |X_s - X_{s-}| > \tfrac{x}{2} - z_t\right]\\
&\leq B_9 e^{f(0)t}\frac{j+1}{h(t-j)^\alpha}\frac{1}{x^\alpha}\\
&\leq B_9 2^\alpha\left(\tfrac{D_2}{D}\right)^{K\alpha} e^{f(0)t}j^{K\alpha}\frac{j+1}{e^{\frac{D_2}{D}f(0)j}}\frac{1}{(My)^\alpha e^{f(0)t}}
\end{aligned}$$

for some $B_9 > 0$ using that $h(s) - \frac{z_t}{2} \geq \frac{h(s)}{2}$ for all $s \leq t$. Set $B_{10} := B_9 2^\alpha\left(\frac{D_2}{D}\right)^{K\alpha}$. Then

$$\begin{aligned}
\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{E_2^1}\right] &\leq B_{10}(My)^{-\alpha}\sum_{j=C}^{\lfloor D\log t\rfloor}\frac{(j+1)j^{K\alpha}}{e^{\frac{D_2}{D}f(0)j}}\\
&\leq B_{10}(My)^{-\alpha}\sum_{j=C}^{\lfloor D\log t\rfloor} e^{-\frac{D_2}{2D}f(0)j}\\
&\leq B_{10}(My)^{-\alpha}\frac{e^{-\frac{D_2}{2D}f(0)C}}{1-e^{-\frac{D_2}{2D}f(0)}}\\
&< \frac{\varepsilon}{8y^\alpha}
\end{aligned}$$

where the second inequality follows by increasing C if necessary and the final line follows by choosing $M = M(\varepsilon)$ sufficiently large. This and (6.44) prove the claim. ■

Finally, we consider $\mathbb{P}_x[G]$ where we recall that

$$G := \left\{X_s > x - z_t\ \forall\, s\in[0,t-C],\ X_t \leq \tfrac{z_t}{2}\right\}.$$

Increase T if necessary so that $\frac{x}{2} > \frac{3z_t}{2}$ for all $t \geq T$. Then by the Markov Property at time $t - C$,

$$\begin{aligned} e^{f(0)t}\mathbb{P}_x[G] &\leq e^{f(0)t}\mathbb{P}_{x-z_t}[X_C \leq \tfrac{z_t}{2}] \\ &\leq e^{f(0)t}\frac{B}{\alpha}\frac{C}{\left(\frac{1}{2}x\right)^\alpha} \\ &\leq \frac{2^\alpha BC}{\alpha(My)^\alpha} \\ &< \frac{\varepsilon}{8y^\alpha} \end{aligned}$$

where the last line follows by choosing M sufficiently large.

In summary, we have shown that

$$\begin{aligned} &\mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{X_t \leq \frac{z_t}{2}}\right] \\ &\leq \mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\left(\mathbb{1}_{F_1} + \mathbb{1}_{F_2} + \mathbb{1}_{E_1} + \mathbb{1}_{E_2\cap\{X_{\tau_2}\geq\frac{x}{2}\}} + \mathbb{1}_{E_2\cap\{X_{\tau_2}<\frac{x}{2}\}} + \mathbb{1}_G\right)\right] \\ &< \frac{\varepsilon}{y^\alpha}. \end{aligned}$$

□

Finally, we prove Corollary 6.4.5, thereby proving part (b) of Theorem 6.1.2.

Proof of Corollary 6.4.5. Fix $\varepsilon > 0$ and $y > 0$. Let $T = T(\varepsilon/2)$ and $M = M(\varepsilon/2)$ be as in the statement of Proposition 6.4.4. Suppose $t \geq T$ and $x \geq Mye^{\sigma^* t}$. Recall that $z_t := \frac{1}{2}(q\log t)^{-K}t^{q\sigma^*}$ for $q > \frac{4}{f(0)}$. Then, by the Feynman–Kac formula, Proposition 6.4.4 and assumptions (6.3) and (6.4)

$$\begin{aligned} u(t,x) &\leq \mathbb{E}_x\left[e^{\int_0^t f(u(t-s,X_s))ds}\mathbb{1}_{X_t\leq\frac{z_t}{2}}\right] + e^{f(0)t}\,\mathbb{E}_x\left[\frac{C_0}{(1+X_t)^\alpha}\left(\mathbb{1}_{X_t\in\left(\frac{z_t}{2},\frac{x}{2}-1\right)} + \mathbb{1}_{X_t\geq\frac{x}{2}-1}\right)\right] \\ &< \frac{\varepsilon}{2y^\alpha} + \frac{C_0e^{f(0)t}}{(\frac{z_t}{2}+1)^\alpha}\frac{2^\alpha Bt}{\alpha x^\alpha} + \frac{2^\alpha C_0 e^{f(0)t}}{x^\alpha} \end{aligned}$$

using that $\mathbb{P}_x\left[X_t \in \left(\frac{z_t}{2}, \frac{x}{2}-1\right)\right] \leq \mathbb{P}_0\left[X_t < -\frac{x}{2} - 1\right] \leq \frac{2^\alpha Bt}{\alpha x^\alpha}$ by Lemma 6.2.2. By definition of z_t we may increase T if necessary so that $(\frac{z_t}{2}+1)^{-\alpha} \leq t^{1-qf(0)}$ for all $t \geq T$, where we recall that $\sigma^*\alpha = f(0)$. Therefore, since $x \geq Mye^{\sigma^* t}$, for all $t \geq T$

$$u(t,x) \leq \frac{\varepsilon}{2y^\alpha} + \frac{2^\alpha C_0 B}{\alpha(My)^\alpha}\frac{1}{t^{qf(0)-2}} + \frac{2^\alpha C_0}{(My)^\alpha}.$$

By assumption, $qf(0) - 2 > 0$, and the result follows by increasing T and M as necessary. □

It remains to prove part (a) of Theorem 6.1.2. We mimic our earlier proof of the lower bound for u given in Proposition 6.4.3, which relied on first proving that u was bounded below at a particular point (this was Lemma 6.4.2).

Lemma 6.4.6. *For all $\varepsilon > 0$ here exists $M = M(\varepsilon), T = T(\varepsilon), \rho = \rho(\varepsilon) > 0$ such that, for all $t \geq T$*

$$u(t, 2Me^{\sigma^* t}) \geq \rho.$$

Proof. Let $\varepsilon > 0$ and $M = M(\varepsilon)$ be as in Corollary 6.4.5. Using the Feynman–Kac formula and assumption (6.4),

$$\begin{aligned} u(t, 2Me^{\sigma^* t}) \geq \widetilde{m}\,\mathbb{E}_{2Me^{\sigma^* t}} &\left[\exp\left(\int_0^{t-1} f(u(t-s, X_s))ds\right)\middle| X_s \geq Me^{\sigma^* t}\ \forall\, s \leq t-1\right] \\ &\cdot \mathbb{P}_{2Me^{\sigma^* t}}\left[X_t < 0,\, X_s \geq Me^{\sigma^* t}\ \forall\, s \leq t-1\right]. \end{aligned} \tag{6.45}$$

We begin by bounding the latter probability. Following the proof of Lemma 6.4.2,

$$\begin{aligned} &\mathbb{P}_{2Me^{\sigma^* t}}\left[X_t < 0,\, X_s \geq Me^{\sigma^* t}\ \forall\, s \leq t-1\right] \\ &\geq \mathbb{P}_{3Me^{\sigma^* t}}\left[X_1 < 0\right] \mathbb{P}_{2Me^{\sigma^* t}}\left[X_s \geq Me^{\sigma^* t}\ \forall\, s \leq t-1,\, X_{t-1} \leq 3Me^{\sigma^* t}\right]. \end{aligned}$$

We bound each of these terms separately. First, recalling that $\sigma^* \alpha = f(0)$,

$$\begin{aligned} \mathbb{P}_{2Me^{\sigma^* t}}&\left[X_s \geq Me^{\sigma^* t}\ \forall\, s \leq t-1,\, X_{t-1} \leq 3Me^{\sigma^* t}\right] \\ &\geq \mathbb{P}_0\left[|X_s| \leq Me^{\sigma^* t}\ \forall\, s \leq t-1\right] \\ &\geq \tfrac{2}{3}\eta_1 \exp\left(-\frac{\eta_2(t-1)}{M^\alpha e^{f(0)t}}\right) \end{aligned} \tag{6.46}$$

where the second to last inequality follows by Lemma 6.2.3. By Lemma 6.2.2,

$$\mathbb{P}_{3Me^{\sigma^* t}}\left[X_1 < 0\right] \geq \frac{B^{-1}}{2\alpha}(3Me^{\sigma^* t})^{-\alpha}. \tag{6.47}$$

Combining (6.46) and (6.47) gives us

$$\mathbb{P}_{2Me^{\sigma^* t}}\left[X_t < 0,\, X_s \geq Me^{\sigma^* t}\ \forall\, s \leq t-1\right] \geq B_6 \exp\left(-f(0)t - \frac{\eta_2(t-1)}{M^\alpha e^{f(0)t}}\right)$$

where $B_0 := \frac{\eta_1 B^{-1}}{3\alpha}(3M)^{-\alpha}$.

It remains to bound the expected value from (6.45). By Proposition 6.4.4 and our choice of M, there exists $T = T(\varepsilon) > 0$ such that, for all $t \geq T$ and $x \geq Mye^{\sigma^* t}$, $u(t, x) < \varepsilon y^{-\alpha}$. In particular, if $x \geq Mye^{\sigma^* t}$ then for all $s \leq t$, $u(t-s, x) < \varepsilon e^{-f(0)s}$.

Recall that, by assumption (6.3) and the Mean Value Theorem, there exists $\kappa > 0$ such that $f(u) \geq f(0) - \kappa u$ for all $u \in [0,1]$. Therefore for all $t \geq T$

$$\begin{aligned}
&\mathbb{E}_{2Me^{\sigma^* t}}\left[\exp\left(\int_0^{t-1} f(u(t-s,X_s))ds\right)\middle| X_s \geq Me^{\sigma^* t}\ \forall\, s \leq t-1\right] \\
&\geq \mathbb{E}_{2Me^{\sigma^* t}}\left[\exp\left(\int_0^{t-1} f(0) - \kappa\varepsilon e^{-f(0)s}\right) ds\middle| X_s \geq Me^{\sigma^* t}\ \forall\, s \leq t-1\right] \\
&\geq \exp\left(f(0)(t-1) + \kappa\varepsilon\left(e^{-f(0)(t-1)} - 1\right)\right).
\end{aligned}$$

Combining this with our earlier bound, (6.45) becomes

$$u(t, 2Me^{\sigma^* t}) \geq B_0 \exp\left(-f(0) - \frac{\eta_2(t-1)}{M^\alpha e^{f(0)t}} + \kappa\varepsilon\left(e^{-f(0)(t-1)} - 1\right)\right)$$

for all $t \geq T$, so the result follows from some $\rho(\varepsilon) > 0$ by increasing T if necessary. □

Finally, we can prove part (a) of Theorem 6.1.2.

Proposition 6.4.7. *For all $\varepsilon > 0$ there exists $T = T(\varepsilon), N = N(\varepsilon) > 0$ and $m^* = m^*(\varepsilon) > 0$ such that, for all $t \geq T$, $u(t,x) > m^*$ for all $x \leq Ne^{\sigma^* t}$.*

Proof. By Lemma 6.4.2, there exists $T = T(\varepsilon)$, $M = M(\varepsilon)$ and $\rho = \rho(\varepsilon)$ for which $u(t, 2Me^{\sigma^* t}) \geq \rho$ for all $t \geq T$. Let $\overline{T} > 0$ and suppose $s \geq T + \overline{T}$. Take $y \in [2Me^{\sigma^* T}, 2Me^{\sigma^* s}]$. Then there exists $T \leq r \leq s$ such that $y = 2Me^{\sigma^* r}$, so $u(r,y) \geq \rho$. Increasing $\overline{T}$ as needed, by Lemma 6.3.5 there exists $m = m(\varepsilon) > 0$ such that $u(s,y) \geq m$ for all $s \geq T + \overline{T}$ proving the result for large enough y.

Now suppose $y < 2Me^{\sigma^* T}$. By the Feynman–Kac formula and Lemma 6.2.2,

$$\begin{aligned}
u(1,y) &\geq \mathbb{E}_y\left[u_0(X_1)\right] \\
&\geq \widetilde{m}\mathbb{P}_0\left[X_1 \leq -2Me^{\sigma^* T}\right] \\
&\geq \widetilde{m}\frac{B^{-1}}{2\alpha}(2Me^{\sigma^* T'})^{-\alpha}.
\end{aligned}$$

Therefore by Lemma 6.3.5, there exists $T', \overline{m} > 0$ such that $u(r,y) \geq \overline{m}$ for all $r \geq T'$. In particular, if s is sufficiently large, $u(s,y) \geq \overline{m}$. Combining this with the above case gives us the result. □

6.5 Proof of Theorem 6.1.3

We now proceed with the proof of Theorem 6.1.3, restated here for convenience.

Theorem 6.1.3. *Suppose u is a solution to (6.1) and f is defined as in (6.2). Assume*

f and u_0 satisfy (6.3) and (6.4), respectively. Let $\sigma^* := \frac{f(0)}{\alpha}$ and $\varepsilon > 0$. There exists $T_\varepsilon > 0$ and $C_\varepsilon > 0$ such that, for all $t \geq T_\varepsilon$,

$$\{x \leq C_\varepsilon e^{\sigma^* t}\} \subseteq \{x \in \mathbb{R} : u(t,x) > 1 - \varepsilon\}.$$

To begin, we will describe the growth of solutions from time t to time $t+1$. A straightforward corollary of this result, Corollary 6.5.2, will enable us to prove Theorem 6.1.3.

Proposition 6.5.1. *Fix $k > 1$, $\delta \in (0,1)$, $m \in (0, 1-2\delta)$, and $t \geq 0$. Fix $x_0 \in \mathbb{R}$. There exists $R_0 = R_0(\delta) > 0$ and $C_\delta > 1$ such that, for any $R \geq R_0$, if $u(t,x) \geq m$ for all $x \in [x_0 - Rk, x_0 + Rk]$, then $u(t+1, z) \geq C_\delta m$ for all $z \in [x_0 - Rk + R, x_0 + Rk - R]$.*

Proof. Fix $\delta \in (0,1)$ and $m \in (0, 1-2\delta)$. Define

$$\varepsilon := \min\left(\delta, \inf\{f(u) : u \in [0, 1-\delta]\}\right) > 0.$$

For reasons that will become clear in the course of this proof, we fix $R_0 > 2$ sufficiently large so that

$$(1 + ||f||_\infty) \frac{2B}{\alpha R_0^\alpha} < \frac{\varepsilon}{24B} \quad \text{and} \quad \left(1 - \frac{2^{\alpha+1} B}{\alpha R_0^\alpha}\right)^2 > 1 - \frac{\varepsilon}{24B} \tag{6.48}$$

where B is as in Lemma 6.2.2 and we understand $||f||_\infty$ to mean the supremum norm of $f \in C([0,1])$.

Choose $R \geq R_0$. Suppose that $u(t,x) \geq m$ for all $x \in [x_0 - Rk, x_0 + Rk]$ for some $k > 1$. Take $z \in [x_0 - Rk + R, x_0 + Rk - R]$. By the Feynman–Kac formula,

$$u(t+1, z) = \mathbb{E}_z\left[e^{\int_0^1 f(u(t+1-s, X_s))ds} u(t, X_1)\right]. \tag{6.49}$$

We will find a lower bound for (6.49) by considering the following two cases:

Case (1): For all $r \in [\frac{1}{2}, \frac{3}{4}]$, $\displaystyle\int_{-1}^{1} \mathbb{1}_{u(t+r, y+z) > 1-\delta}\, dy < 1$;

Case (2): There exists $t^* \in [\frac{1}{2}, \frac{3}{4}]$ such that $\displaystyle\int_{-1}^{1} \mathbb{1}_{u(t+t^*, y+z) > 1-\delta}\, dy \geq 1$.

We begin with Case (1). Equation (6.49) becomes

$$\begin{aligned}
u(t+1,z) &\geq m\,\mathbb{E}_z\left[e^{\int_0^1 f(u(t+1-s,X_s))ds}\mathbb{1}_{|X_1-x_0|\leq Rk}\right] \\
&\geq m\,\mathbb{E}_z\left[\left(1+\int_0^1 f(u(t+1-s,X_s))ds\right)\mathbb{1}_{|X_1-x_0|\leq Rk}\right] \\
&= m\,\mathbb{P}_z\left[|X_1-x_0|\leq Rk\right] + m\,\mathbb{E}_z\left[\int_0^1 f(u(t+1-s,X_s))ds\right] \\
&\quad - m\,\mathbb{E}_z\left[\left(1-\mathbb{1}_{|X_1-x_0|\leq Rk}\right)\int_0^1 f(u(t+1-s,X_s))ds\right] \\
&\geq m\,\mathbb{P}_z\left[|X_1-x_0|\leq Rk\right] + m\int_0^1 \mathbb{E}_z\left[f(u(t+1-s,X_s))\right]ds \\
&\quad - m\,\mathbb{E}_z\left[\mathbb{1}_{|X_1-x_0|>Rk}\,||f||_\infty\right]. \qquad (6.50)
\end{aligned}$$

Now, for $s\in[0,1]$, by definition of ε

$$\mathbb{E}_z\left[f(u(t+1-s,X_s))\right]\geq\varepsilon\,\mathbb{P}_z\left[u(t+1-s,X_s)\leq 1-\delta\right],$$

so (6.50) becomes

$$\begin{aligned}
u(t+1,z) &\geq m\left(1-(1+||f||_\infty)\,\mathbb{P}_z\left[|X_1-x_0|>Rk\right]\right) \\
&\quad + \varepsilon m\int_0^1 \mathbb{P}_z\left[u(t+1-s,X_s)\leq 1-\delta\right]ds. \qquad (6.51)
\end{aligned}$$

Now, for $s\in[\frac{1}{4},\frac{1}{2}]$, by Lemma 6.2.2 and our assumption from Case (1),

$$\begin{aligned}
\mathbb{P}_z\left[u(t+1-s,X_s)\leq 1-\delta\right] &\geq \int_{-\infty}^{\infty}\frac{B^{-1}}{s^{\frac{1}{\alpha}}+s^{-1}|y|^{1+\alpha}}\mathbb{1}_{u(t+1-s,y+z)\leq 1-\delta}\,dy \\
&\geq \frac{B^{-1}}{3}\int_{-1}^{1}\mathbb{1}_{u(t+1-s,y+z)\leq 1-\delta}\,dy \\
&\geq \frac{B^{-1}}{3}.
\end{aligned}$$

By Lemma 6.2.2 again, and since $z\in[x_0-Rk+R,x_0+Rk-R]$,

$$\begin{aligned}
\mathbb{P}_z\left[|X_1-x_0|>Rk\right] &\leq \mathbb{P}_0\left[|X_1|>R\right] \\
&\leq \frac{2B}{\alpha}R^{-\alpha}.
\end{aligned}$$

Therefore (6.51) becomes

$$u(t+1,z)\geq m\left(1-(1+||f||_\infty)\frac{2B}{\alpha R^\alpha}\right)+\varepsilon m\frac{1}{4}\frac{B^{-1}}{3},$$

which is our lower bound for Case (1).

Now we consider Case (2), where $t^*\in[\frac{1}{2},\frac{3}{4}]$ satisfies $\int_{-1}^{1}\mathbb{1}_{u(t+t^*,y+z)>1-\delta}\,dy\geq 1.$

Note that, for all y with $|y+\frac{z-x_0}{2}| \le \frac{Rk}{2}$, by the Feynman–Kac formula, and recalling that $|z-x_0| \le Rk - R$

$$
\begin{aligned}
u(t+t^*, y+z) &\ge \mathbb{E}_{y+z}\left[u(t, X_{t^*})\right] \\
&\ge m\,\mathbb{P}_0\left[|X_{t^*}+y+z-x_0| \le Rk\right] \\
&\ge m\mathbb{P}_0\left[|X_{t^*}| \le Rk - \tfrac{|z-x_0|}{2} - |y+\tfrac{z-x_0}{2}|\right] \\
&\ge m\mathbb{P}_0\left[|X_{t^*}| \le \tfrac{Rk}{2} - \tfrac{|z-x_0|}{2}\right] \\
&\ge m\mathbb{P}_0\left[|X_{t^*}| \le \tfrac{R}{2}\right] \\
&\ge m\left(1-\frac{2^{\alpha+1}B}{\alpha R^\alpha}\right) \\
&=: m_R
\end{aligned}
$$

where in the final inequality we apple Lemma 6.2.2. By assumption, $1-\delta > m_R$. Note that, if $|y| \le \frac{R}{2}$, then $|y+\frac{z-x_0}{2}| \le \frac{Rk}{2}$. Therefore by the Feynman–Kac formula

$$
\begin{aligned}
u(t+1,z) &\ge \mathbb{E}_z\left[u(t+t^*, X_{1-t^*})\right] \\
&\ge \int_{-\frac{R}{2}}^{\frac{R}{2}} p_{X_{1-t^*}}(y)\left((1-\delta)\mathbb{1}_{u(t+t^*,y+z)>1-\delta} + m_R\mathbb{1}_{u(t+t^*,y+z)\le 1-\delta}\right)dy \\
&\ge \int_{-1}^{1} p_{X_{1-t^*}}(y)\left((1-\delta)\mathbb{1}_{u(t+t^*,y+z)>1-\delta} + m_R\left(1-\mathbb{1}_{u(t+t^*,y+z)>1-\delta}\right)\right)dy \\
&\quad + \int_{[-\frac{R}{2},\frac{R}{2}]\setminus[-1,1]} m_R\, p_{X_{1-t^*}}(y)dy \\
&= \int_{-1}^{1} p_{X_{1-t^*}}(y)\left((1-\delta-m_R)\mathbb{1}_{u(t+t^*,y+z)>1-\delta} + m_R\right)dy \\
&\quad + \int_{[-\frac{R}{2},\frac{R}{2}]\setminus[-1,1]} m_R\, p_{X_{1-t^*}}(y)dy \\
&= (1-\delta-m_R)\int_{-1}^{1} p_{X_{1-t^*}}(y)\mathbb{1}_{u(t+t^*,y+z)>1-\delta}\,dy + m_R\mathbb{P}_0\left[|X_{1-t^*}| \le \tfrac{R}{2}\right] \\
&\ge (1-\delta-m_R)\frac{B^{-1}}{5}\int_{-1}^{1}\mathbb{1}_{u(t+t^*,y+z)>1-\delta}\,dy + m_R\left(1-\frac{2^{\alpha+1}B}{\alpha R^\alpha}\right) \\
&\ge \frac{1-\delta-m}{5B} + m\left(1-\frac{2^{\alpha+1}B}{\alpha R^\alpha}\right)^2
\end{aligned}
$$

where in the second to last inequality, we apply Lemma 6.2.2 and our assumption from Case (2). The final inequality follows by definition of m_R and since $m > m_R$. Therefore in Case (1) and Case (2) respectively, we have

$$
u(t+1,z) \ge m\left(1-(1+||f||_\infty)\frac{2B}{\alpha R^\alpha}\right) + \frac{\varepsilon m}{12B} \tag{6.52}
$$

and

$$u(t+1,z) \geq \frac{1-\delta-m}{5B} + m\left(1 - \frac{2^{\alpha+1}B}{\alpha R^\alpha}\right)^2. \tag{6.53}$$

Recall that, by assumption, $m \leq 1-2\delta$ and $\varepsilon \leq \delta$, therefore

$$\frac{1-\delta-m}{5B} \geq \frac{\delta}{5B} > \frac{\varepsilon}{12B}.$$

This, together with (6.52), (6.53) and our choice of $R > R_0$ in (6.48) gives us

$$\begin{aligned} u(t+1,z) &\geq \min\left(m\left(1+\frac{\varepsilon}{24B}\right), \frac{1-\delta-m}{5B} + m\left(1-\frac{\varepsilon}{24B}\right)\right) \\ &= m\left(1+\frac{\varepsilon}{24B}\right) \\ &=: mC_\delta. \end{aligned}$$

□

Using Proposition 6.5.1, we can grow u to be arbitrarily close to one.

Corollary 6.5.2. *Fix $\delta \in (0,1)$, $m \in (0, 1-2\delta)$, $x_0 \in \mathbb{R}$ and $t \geq 0$. There exists $R_0 = R_0(\delta) > 0$, $k^* = k^*(\delta, m) > 1$ such that, for any $R \geq R_0$, if $u(t,x) \geq m$ for all $x \in [x_0 - Rk^*, x_0 + Rk^*]$, then $u(t+k^*-1, z) > 1-2\delta$ for all $z \in [x_0 - R, x_0 + R]$.*

Proof. This follows immediately by setting $k^* = \min\left\{k \in \mathbb{N} : mC_\delta^{k-1} > 1-2\delta\right\}$ in Proposition 6.5.1. □

We are now able to prove Theorem 6.1.3.

Proof of Theorem 6.1.3. Recall that, by Proposition 6.4.7, there exists $T, N > 0$ and $m^* > 0$ such that, for all $t \geq T$, $u(t,x) > m^*$ for all $x \leq Ne^{\sigma^* t}$. Choose $\delta > 0$ sufficiently small so that $m^* < 1-2\delta$. Let $R_0 = R_0(\delta)$ and $k^* = k^*(\delta, m^*)$ be as in Corollary 6.5.2. Increase T if necessary so that, for all $t \geq T$, $\frac{N}{k^*}e^{\sigma^* t} \geq R_0$. Then, by Corollary 6.5.2, for all $t \geq T$ and $x \in \left[-\frac{N}{k^*}e^{\sigma^* t}, \frac{N}{k^*}e^{\sigma^* t}\right]$, $u(t+k^*-1, x) \geq 1-2\delta$. That is, for all $t \geq T + k^* - 1$ and $x \in \left[-\frac{Ne^{1-k^*}}{k^*}e^{\sigma^* t}, \frac{Ne^{1-k^*}}{k^*}e^{\sigma^* t}\right]$, $u(t,x) > 1-2\delta$.

Now suppose $x < -\frac{Ne^{1-k^*}}{k^*}e^{\sigma^* t}$. Let $R_0 = R_0(\delta)$ be as above and define $k_0 := k^*\left(\delta, \frac{\widetilde{m}}{2}\right) > 1$. Increase T if necessary so that, for all $t \geq T + k^* - 1$,

$$-\frac{Ne^{1-k^*}}{k^*}e^{\sigma^* t} + \left(\frac{2Bt}{\alpha}\right)^{\frac{1}{\alpha}} + R_0 k_0 < 0.$$

By assumption (6.4) and the Feynman–Kac formula, for all $z \in [x - R_0 k_0, x + R_0 k_0]$, since $|x + R_0 k_0| \geq \left(\frac{2Bt}{\alpha}\right)^{\frac{1}{\alpha}}$,

$$u(t,z) \geq \widetilde{m}\mathbb{P}_z\left[X_t \leq 0\right] \geq \widetilde{m}\left(1 - \frac{Bt}{\alpha|x+R_0k_0|^\alpha}\right) \geq \frac{\widetilde{m}}{2}.$$

Therefore by our choice of k_0, Corollary 6.5.2 implies that $u(t + k_0 - 1, x) \geq 1 - 2\delta$ for all t sufficiently large. In particular, we may increase T so that, for all $t \geq T$, $u(t, x) > 1 - 2\delta$, proving the result for all $x \leq \frac{Ne^{1-k^*}}{k^*} e^{\sigma^* t}$. $\square$

Chapter 7

Conclusion

In this thesis, we have used probabilistic techniques to obtain new results for fractional reaction-diffusion equations. In doing so, we have improved results from the PDE literature and demonstrated the effectiveness of our probabilistic approach. We also believe our methods of proof to be valuable in of themselves, providing several avenues for further research.

Our work on the fractional Allen–Cahn equation contributes to the literature in a number of ways. In the deterministic setting, our main result, Theorem 3.1.5, states that solutions of the fractional Allen–Cahn equation converges to mean curvature flow under a family of possible scalings.[1] To the best of our knowledge, this family of scalings has not been considered before, and our result appears to be the first of its kind in the fractional setting that explicitly states the speed of convergence and interface width. The analogue of this result for the SΛFVS process, Theorem 3.1.11, appears to be the first result to consider motion by mean curvature flow for the fractional Allen–Cahn equation in the stochastic setting.

To prove Theorem 3.1.5, we represented stable processes as stable subordinated Brownian motions. The central idea was to truncate these stable subordinators so that we could more readily adapt work from the Brownian setting of [EFP17]. In doing so, we have demonstrated how truncation can be used to translate results from Gaussian to stable settings, in the context of voting systems for branching processes.

A natural extension of this work would be to find a more general class of reaction-diffusion equations for which our results hold. For example, it should be straightforward to change the Allen–Cahn nonlinearity to another bistable function using [O'D19], which tells us which nonlinearities enable a voting representation. Changing the fractional Laplacian to the generator of another Feller process would, however, be

[1] Of course, we really mean 'convergence as $\varepsilon \to 0$ of solutions of the scaled fractional Allen–Cahn equation to the indicator function of a set whose boundary follows motion by mean curvature flow'.

more challenging. It may be possible to prove our result for a general class of subordinated Brownian motions. One ambitious venture would be to unify the Brownian and stable results into a single theorem, stated for subordinated Brownian motions, where the interface width and speed of convergence would, presumably, depend on the tail distribution of the diffusive term. A result of this form would include processes with mixed Gaussian and stable behaviours, such as Lévy flights. Other pursuits include proving our result in the case of $I(\varepsilon) = \varepsilon$ or taking $\alpha \in (0, 1]$; our method of proof is not applicable in either case. In the stochastic setting, future work could be to find the critical amount of noise for which mean curvature flow is preserved in the limit, which has been accomplished in the Brownian setting [EGL22]. Additionally, one might consider other models for the stochastic fractional Allen–Cahn equation other than the SΛFVS process (or a variant of the SΛFVS process itself).

In our second body of work, we have used probabilistic techniques to find the spreading speed of solutions to the fractional Fisher–KPP equation. This improves the result of [CR13], which uses PDE techniques and the maximum principle. In contrast, the majority of our proof does not assume a maximum principle holds. In view of this, and [Pen18],[2] it should be possible to extend our work to reaction-diffusion equations that do not satisfy a maximum principle, such as the fractional Fisher–KPP equation with non-local nonlinearity. Future research might also include replacing the fractional Laplacian in the fractional Fisher–KPP equation by a more general generator,[3] or weakening the assumptions on our nonlinearity and initial condition. To obtain a more biologically relevant result, one could also study the spreading speed of a stochastic fractional Fisher–KPP equation. To date, rigorous results on this subject are lacking, but simulation studies such as [HF14] suggest that the presence of noise would drastically change our results.

[2]In [Pen18], the author estimates the spreading speed of a Fisher–KPP equation with non-local nonlinearity using a Feynman–Kac representation of solutions.

[3]This is accomplished in [CR13], for a family of generators whose transition kernel satisfies a power decay assumption.

Appendix A

Chapters 3-5 supplementary material

A.1 Fixed points of $g_\times$

Proposition A.1.1. *The function $g_\times$ has fixed points u_-, $\frac{1}{2}$, and u_+, where*

$$u_- = \tfrac{1}{2} - \tfrac{\sqrt{(1-b_\varepsilon)^3(1-3b_\varepsilon)}}{2(1-b_\varepsilon)^3}, \quad u_+ = \tfrac{1}{2} + \tfrac{\sqrt{(1-b_\varepsilon)^3(1-3b_\varepsilon)}}{2(1-b_\varepsilon)^3}.$$

Proof. We aim to find a such that $g_\times\left(\frac{1}{2}+a\right) = \frac{1}{2}+a$. Now,

$$\begin{aligned} g_\times\left(\tfrac{1}{2}+a\right) &= 3\left((1-b_\varepsilon)(\tfrac{1}{2}+a)+\tfrac{b_\varepsilon}{2}\right)^2 - 2\left((1-b_\varepsilon)(\tfrac{1}{2}+a)+\tfrac{b_\varepsilon}{2}\right)^3 \\ &= 2(b_\varepsilon-1)^3a^3 + \tfrac{3}{2}(1-b_\varepsilon)a + \tfrac{1}{2}. \end{aligned}$$

Setting this equal to $\frac{1}{2}+a$ we obtain the quadratic equation

$$2(1-b_\varepsilon)^3a^2 + \tfrac{3}{2}(1-b_\varepsilon) = 0,$$

for which $u_- - \frac{1}{2}$ and $u_+ - \frac{1}{2}$ are clearly solutions. To see that $g_\times(\frac{1}{2}) = \frac{1}{2}$, note that

$$g_\times\left(\tfrac{1}{2}\right) = g\left((1-b_\varepsilon)\tfrac{1}{2}+\tfrac{b_\varepsilon}{2}\right) = g\left(\tfrac{1}{2}\right) = \tfrac{1}{2}. \qquad \square$$

A.2 Proof of Proposition 3.3.22

Before proving Proposition 3.3.22, we will need the following result.

Proposition A.2.1. *Let $h_t(x,\cdot)$ denote the transition density of a $\mathbb{d}$-dimensional Brownian motion W_t started at x. There exists a constant $C>0$ such that, for all $r \geq 0$,*

$$|h_r(x,y) - h_r(x,y+z)| \leq Cr^{-\frac{\mathbb{d}+1}{2}}|z|$$

for all $x,y,z \in \mathbb{R}^{\mathbb{d}}$.

Proof. Fix $r \geq 0$. Since

$$h_r(x,y) = \frac{1}{(4\pi r)^{\frac{\mathrm{d}}{2}}} \exp\left(-\tfrac{|x-y|^2}{4r}\right),$$

by the Mean Value Theorem

$$|h_r(x,y) - h_r(x,y+z)| \leq \frac{1}{(4\pi r)^{\frac{\mathrm{d}}{2}}} |z| \left|\nabla \exp\left(-\tfrac{|\xi|^2}{4r}\right)\right|$$

for some ξ on the line segment between $y-x$ and $y+z-x$. Now,

$$\left|\nabla \exp\left(-\tfrac{|\xi|^2}{4r}\right)\right| = \frac{|\xi|}{2r} \exp\left(-\tfrac{|\xi|^2}{4r}\right) \leq r^{-\frac{1}{2}}$$

since $x\exp(-x^2) \leq 1$ for all $x \in \mathbb{R}$. The result follows by setting $C := (4\pi)^{-\frac{\mathrm{d}}{2}}$. $\square$

We now prove Proposition 3.3.22, restated here for convenience.

Proposition 3.3.22. *Let $\alpha \in (1,2)$, $0 < t < \mathscr{T}$, $0 \leq \beta < c_0$, $k \in \mathbb{N}$ and F be as in (3.5). Then, for any $x \in \mathbb{R}^{\mathrm{d}}$, there exists $m_1, m_2 > 0$ such that*

$$\left|\mathbb{P}_x[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1)] - \mathbb{P}_x[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1)]\right| \leq m_1 e^{-\frac{t}{\varepsilon^2}} + m_2 F(\varepsilon) \tag{A.1}$$

$$\left|\mathbb{P}_x[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t)) = 1)] - \mathbb{P}_x[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1)]\right| \leq m_1 e^{-\frac{t}{\varepsilon^2}} + m_2 F(\varepsilon). \tag{A.2}$$

Proof. We prove (A.2) and note that (A.1) follows by identical arguments. Denote the standard Euclidean distance in $\mathbb{R}^{\mathrm{d}}$ as $|\cdot|$ throughout. To begin, we will construct a coupling of $\boldsymbol{Z}^+(t)$ to a historical ternary branching subordinated Brownian motion, $\boldsymbol{W}_{R^\varepsilon}(t)$. Define

$$u(t,x) := \mathbb{P}_x[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t)) = 1] \quad \text{and} \quad v(t,x) := \mathbb{P}_x[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1].$$

Abuse notation and let τ denote the time of the first branching event in both $\boldsymbol{Z}^+(t)$ and $\boldsymbol{W}_{R^\varepsilon}(t)$. By the Markov property at time τ, u and v can be written as

$$u(t,x) = \mathbb{E}_x[g_\times(u(t-\tau, Z_\tau^+))\mathbb{1}_{\tau \leq t}] + \mathbb{E}_x[p(Z_t^+)\mathbb{1}_{\tau > t}] \tag{A.3}$$

$$v(t,x) = \mathbb{E}_x[g_\times(v(t-\tau, W(R_\tau^\varepsilon)))\mathbb{1}_{\tau \leq t}] + \mathbb{E}_x[p(W(R_t^\varepsilon))\mathbb{1}_{\tau > t}]. \tag{A.4}$$

To bound the difference of u and v, we control the difference of the first terms and second terms in (A.3) and (A.4) separately. First, since $\tau \sim exp(\varepsilon^{-2})$,

$$\mathbb{E}_x[p(Z_t^+)\mathbb{1}_{\tau > t}] - \mathbb{E}_x[p(W(R_t^\varepsilon))\mathbb{1}_{\tau > t}] \leq \mathbb{P}[t \leq \tau] \leq e^{-\frac{t}{\varepsilon^2}}. \tag{A.5}$$

To bound the difference of the first terms in (A.3) and (A.4), set $t^* := k\varepsilon^2|\log(\varepsilon)|$ and $\delta_\varepsilon := D_0(k+2)I^2(\varepsilon)|\log(\varepsilon)|$ for D_0 as in Theorem 3.3.18. Denote the transition density of $W(R^\varepsilon_t)$ started at x by $f_t(x,\cdot)$. Then, since $g_\times$ is bounded above by one and, by definition of Z^+_t,

$$\begin{aligned}
&\mathbb{E}_x\left[g_\times(u(t-\tau,Z^+_\tau))\mathbb{1}_{\tau\le t}\right]\\
&\le \mathbb{E}_x\left[g_\times(u(t-\tau,Z^+_\tau))\mathbb{1}_{\tau\le t\wedge t^*}\right]+\mathbb{P}[\tau>t^*]\\
&\le \sup_{|w|\le\delta_\varepsilon}\mathbb{E}_x\left[g_\times(u(t-\tau,W(R^\varepsilon_\tau)+w))\mathbb{1}_{\tau\le t\wedge t^*}\right]+\varepsilon^k\\
&=\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}} g_\times(u(t-\tau,z+w))f_\tau(x,z)dz\mathbb{1}_{\tau\le t\wedge t^*}\right]+\varepsilon^k\\
&=\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}} g_\times(u(t-\tau,z))f_\tau(x,z-w)dz\mathbb{1}_{\tau\le t\wedge t^*}\right]+\varepsilon^k\\
&\le \mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}} g_\times(u(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\le t\wedge t^*}\right]+\varepsilon^k\\
&\quad+\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}} g_\times(u(t-\tau,z))[f_\tau(x,z-w)-f_\tau(x,z)]\mathbb{1}_{\tau\le t\wedge t^*}dz\right].
\end{aligned}\tag{A.6}$$

Consider the d-dimensional ball $B_x\left(\tau^{\frac{1}{\alpha}}+\delta_\varepsilon\right):=\left\{z\in\mathbb{R}^{\mathrm{d}}:|z-x|\le\tau^{\frac{1}{\alpha}}+\delta_\varepsilon\right\}$. To ease notation, let $B_x:=B_x\left(\tau^{\frac{1}{\alpha}}+\delta_\varepsilon\right)$. Here, B_x has been chosen so that, for $z\in B_x$, the difference $|f_\tau(x,z-w)-f_\tau(x,z)|$ is sufficiently small, and for $z\in B_x^{\mathrm{c}}$ (the complement of B_x in $\mathbb{R}^{\mathrm{d}}$), the probability of the subordinated Brownian motion jumping from x to z by time τ is sufficiently small. Then, to bound the last term in (A.6) we use that $g(u(t-\tau,z))$ is bounded by 1 to get

$$\begin{aligned}
&\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}} g(u(t-\tau,z))[f_\tau(x,z-w)-f_\tau(x,z)]\mathbb{1}_{\tau\le t\wedge t^*}dz\right]\\
&\le \sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}}|f_\tau(x,z-w)-f_\tau(x,z)|\mathbb{1}_{\tau\le t\wedge t^*}dz\right]\\
&\le \sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{B_x}|f_\tau(x,z-w)-f_\tau(x,z)|dz\mathbb{1}_{\tau\le t\wedge t^*}\right]\\
&\quad+\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{B_x^{\mathrm{c}}}|f_\tau(x,z-w)-f_\tau(x,z)|dz\mathbb{1}_{\tau\le t\wedge t^*}\right].
\end{aligned}\tag{A.7}$$

To bound the first term of (A.7), first note that

$$\mathbb{P}[\tau<\delta_\varepsilon^\alpha]\le 1-\exp\left(-\delta_\varepsilon^\alpha\varepsilon^{-2}\right)\le I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log(\varepsilon)|^\alpha.$$

By Proposition A.2.1 (noting that $f_\tau(x,\cdot)\equiv h_{R^\varepsilon_\tau}(x,\cdot)$ for h the transition density of a d-dimensional Brownian motion) using that the first integral is bounded above by two, and allowing the constant C to change from line to line,

$$\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{B_x}|f_\tau(x,z-w)-f_\tau(x,z)|dz\mathbb{1}_{\tau\le t\wedge t^*}\right]$$

$$
\begin{aligned}
&\le \sup_{|w|\le\delta_\varepsilon} C\mathbb{E}\left[\int_{B_x} |w|(R^\varepsilon_\tau)^{-\frac{\mathrm{d}+1}{2}} dz \mathbb{1}_{\tau\le t\wedge t^*}\mathbb{1}_{\tau\ge\delta_\varepsilon^\alpha}\right] + 2I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log(\varepsilon)|^\alpha \\
&\le C\delta_\varepsilon \mathbb{E}\left[V(B_x)(R^\varepsilon_\tau)^{-\frac{\mathrm{d}+1}{2}}\mathbb{1}_{\tau\ge\delta_\varepsilon^\alpha}\right] + 2I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log(\varepsilon)|^\alpha \\
&\le C\delta_\varepsilon \mathbb{E}\left[(2\tau)^{\frac{\mathrm{d}}{\alpha}}(R^\varepsilon_\tau)^{-\frac{\mathrm{d}+1}{2}}\mathbb{1}_{\tau\ge\delta_\varepsilon^\alpha}\right] + 2I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log(\varepsilon)|^\alpha \qquad \text{(A.8)}
\end{aligned}
$$

where $V(B_x)$ denotes the volume of B_x which is proportional to $(\tau^{\frac{1}{\alpha}} + \delta_\varepsilon)^{\mathrm{d}}$, and, conditional on $\tau \ge \delta_\varepsilon^\alpha$, is bounded above by $(2\tau)^{\frac{\mathrm{d}}{\alpha}}$. In Lemma A.4.3, we will bound the $-p$-th moments of $(R^\varepsilon_s)_{s\ge 0}$. Using this, and letting D_1, D_2 change from line to line, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\tau^{\frac{\mathrm{d}}{\alpha}}(R^\varepsilon_\tau)^{-\frac{\mathrm{d}+1}{2}}\mathbb{1}_{\tau\ge\delta_\varepsilon^\alpha}\right] &\le \mathbb{E}\left[\tau^{\frac{\mathrm{d}}{\alpha}}(R^\varepsilon_\tau)^{-\frac{\mathrm{d}+1}{2}}\right] \\
&\le D_1\mathbb{E}\left[\tau^{\frac{\mathrm{d}}{\alpha}}e^\tau\right] + D_2\mathbb{E}\left[\tau^{-\frac{1}{\alpha}}e^\tau\right] \\
&\le D_1\varepsilon^{\frac{2\mathrm{d}}{\alpha}} + D_2\varepsilon^{-\frac{2}{\alpha}}.
\end{aligned}
$$

Substituting this back into (A.8), and again letting the constants change from line to line, we obtain

$$
\begin{aligned}
&\sup_{|w|\le\delta_\varepsilon} \mathbb{E}\left[\int_{B_x} |f_\tau(x,z-w) - f_\tau(x,z)| dz \mathbb{1}_{\tau\le t\wedge t^*}\right] \\
&\le C\delta_\varepsilon\left(\varepsilon^{\frac{2\mathrm{d}}{\alpha}} + \varepsilon^{-\frac{2}{\alpha}}\right) + 2I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log(\varepsilon)|^\alpha \\
&= C_1\varepsilon^{\frac{2\mathrm{d}}{\alpha}} I(\varepsilon)^2|\log(\varepsilon)| + C_2 I(\varepsilon)^2\varepsilon^{-\frac{2}{\alpha}}|\log\varepsilon| + 2I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log(\varepsilon)|^\alpha \\
&\le C_2 I(\varepsilon)^2\varepsilon^{-\frac{2}{\alpha}}|\log\varepsilon|, \qquad \text{(A.9)}
\end{aligned}
$$

where the final inequality follows by Assumptions 3.1.2 (B)-(C) and we allow C_2 to change from line to line. By Assumption 3.1.2 (C), (A.9) goes to 0 with ε.

To bound the second term in (A.7) we use [CK03, Theorem 1.1], which provides upper bounds on the transition density of subordinated Brownian motion, with truncated stable subordinator that has truncation level independent of ε. To apply this result, we rewrite $W(R^\varepsilon_s)$ in terms of a 1-truncated subordinated Brownian motion as follows. Let $(U^a_s)_{s\ge 0}$ denote an $\frac{\alpha}{2}$-stable subordinator with truncation level a (and no speed change). Let $\stackrel{D}{=}$ denote equality in distribution. Then

$$
R^\varepsilon_s \stackrel{D}{=} U^{I(\varepsilon)^2}_{sI(\varepsilon)^{\alpha-2}} \stackrel{D}{=} I(\varepsilon)^2 U^1_{sI(\varepsilon)^{-2}}
$$

where the first equality follows by definition and the second equality follows by showing that, if Ψ and Ψ' are the characteristic exponents of $(U^{I(\varepsilon)^2}_s)_{s\ge 0}$ and $(U^1_s)_{s\ge 0}$ respectively, then $I(\varepsilon)^\alpha\Psi(\theta) = \Psi'(\theta I(\varepsilon)^2)$, which follows easily from the Lévy-Khintchine formula (Theorem 2.1.5). Then, by the scaling property of the Brownian motion,

$$
W(R^\varepsilon_s) \stackrel{D}{=} I(\varepsilon) W(U^1_{sI(\varepsilon)^{-2}}). \qquad \text{(A.10)}
$$

Denote the transition density of $(W(U_t^1))_{t\geq 0}$ by $\hat{f}_t(x,y)$. By (A.10), $\hat{f}_t(x,\cdot)$ is related to the transition density of $W(R_s^\varepsilon)$ by

$$f_t(x,y) = \hat{f}_{I(\varepsilon)^{-2}t}(I(\varepsilon)^{-1}x, I(\varepsilon)^{-1}y).$$

Therefore the second term in (A.7) can be rewritten as

$$\sup_{|w|\leq\delta_\varepsilon} \mathbb{E}\left[\int_{B_x^{\mathrm{c}}} \left|\hat{f}_{I(\varepsilon)^{-2}\tau}(I(\varepsilon)^{-1}x, I(\varepsilon)^{-1}(z-w)) - \hat{f}_{I(\varepsilon)^{-2}\tau}(I(\varepsilon)^{-1}x, I(\varepsilon)^{-1}z)\right| dz \mathbb{1}_{\tau\leq t\wedge t^*}\right]$$

which, by [CK03, Theorem 1.1], is bounded above by

$$\begin{aligned}
&\sup_{|w|\leq\delta_\varepsilon} \mathbb{E}\left[\int_{B_x^{\mathrm{c}}} \frac{D\tau I(\varepsilon)^{-2}}{((z-w)I(\varepsilon)^{-1})^{\mathrm{d}+\alpha}}dz\right] + \mathbb{E}\left[\int_{B_x^{\mathrm{c}}} \frac{D\tau I(\varepsilon)^{-2}}{(zI(\varepsilon)^{-1})^{\mathrm{d}+\alpha}}dz\right] \\
&= \sup_{|w|\leq\delta_\varepsilon} \mathbb{E}\left[\int_{B_x^{\mathrm{c}}} \frac{D\tau I(\varepsilon)^{\alpha-1}}{(z-w)^{\mathrm{d}+\alpha}}dz\right] + \mathbb{E}\left[\int_{B_x^{\mathrm{c}}} \frac{D\tau I(\varepsilon)^{\alpha-1}}{z^{\mathrm{d}+\alpha}}dz\right] \\
&\leq DI(\varepsilon)^{\alpha-1}\mathbb{E}\left[\tau\int_{\tau^{\frac{1}{\alpha}}}^{\infty} \frac{1}{z^{\alpha+1}}dz\right] \\
&\leq DI(\varepsilon)^{\alpha-1}\mathbb{E}[\tau(\tau^{-1})] \\
&= DI(\varepsilon)^{\alpha-1}, \qquad\qquad (A.11)
\end{aligned}$$

for some $D>0$ that we allow to change from line to line. Here we have applied the change of variables $z-w\mapsto z$ in the third line. Note that, since $\alpha>1$ the last quantity goes to 0 with ε. Recall from (3.5) that

$$F(\varepsilon) = I(\varepsilon)^2\varepsilon^{-\frac{2}{\alpha}}|\log\varepsilon| + I(\varepsilon)^{\alpha-1}.$$

Then, choosing $m>0$ sufficiently large and ε sufficiently small so that

$$mF(\varepsilon) \geq \varepsilon^k + DI(\varepsilon)^{\alpha-1} + C_2 I(\varepsilon)^2\varepsilon^{-\frac{2}{\alpha}}|\log\varepsilon|,$$

by (A.11) and (A.9) we can bound (A.7) to obtain

$$\begin{aligned}
&\mathbb{E}_x\left[g_\times(u(t-\tau, Z_\tau^+))\mathbb{1}_{\tau\leq t}\right] \\
&\leq \mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}} g_\times(u(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\leq t\wedge t^*}\right] + mF(\varepsilon). \qquad (A.12)
\end{aligned}$$

Finally, we note that

$$\begin{aligned}
\mathbb{E}_x\left[g_\times(u(t-\tau, W(R_\tau^\varepsilon))\mathbb{1}_{\tau\leq t}\right] &\geq \mathbb{E}_x\left[g_\times(u(t-\tau, W(R_\tau^\varepsilon))\mathbb{1}_{\tau\leq t\wedge t^*}\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}} g_\times(u(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\leq t\wedge t^*}\right],
\end{aligned}$$

which, together with (A.12) and (A.5) gives

$$\begin{aligned}
&u(t,x)-v(t,x)\\
&\leq \mathbb{E}\left[\int g_\times(u(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\leq t\wedge t^*}\right]-\mathbb{E}\left[\int g_\times(v(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\leq t\wedge t^*}\right]\\
&\quad+e^{-\frac{t}{\varepsilon^2}}+mF(\varepsilon).
\end{aligned}$$

Using the same approach we can obtain the lower bound on $u(t,x)-v(t,x)$. Namely, by analogy with (A.6),

$$\begin{aligned}
&\mathbb{E}_x\left[g_\times(u(t-\tau,Z_t^+))\mathbb{1}_{\tau\leq t}\right]\\
&\geq \mathbb{E}\left[\int_{\mathbb{R}^d} g(u(t-\tau,z))[f_\tau(x,z-w)-f_\tau(x,z)]\,\mathbb{1}_{\tau\leq t\wedge t^*}dz\right]\\
&\quad+\mathbb{E}\left[g(u(t-\tau,W(R_\tau^\varepsilon)))\mathbb{1}_{\tau\leq t\wedge t^*}\right]\\
&\geq -\sup_{|w|\leq\delta_\varepsilon}\left|\mathbb{E}\left[\int_{\mathbb{R}^d} g(u(t-\tau,z))[f_\tau(x,z-w)-f_\tau(x,z)]\,\mathbb{1}_{\tau\leq t\wedge t^*}dz\right]\right|\\
&\quad+\mathbb{E}\left[g(u(t-\tau,W(R_\tau^\varepsilon)))\mathbb{1}_{\tau\leq t\wedge t^*}\right]
\end{aligned}$$

and the final term can be bounded identically as before to give us that

$$\begin{aligned}
&u(t,x)-v(t,x)\\
&\geq \mathbb{E}\left[\int g_\times(u(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\leq t\wedge t^*}\right]-\mathbb{E}\left[\int g_\times(v(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\leq t\wedge t^*}\right]\\
&\quad-\left(e^{-\frac{t}{\varepsilon^2}}+mF(\varepsilon)\right).
\end{aligned}$$

Therefore, using that $g_\times$ is Lipschitz with constant $\frac{3}{2}$, we obtain

$$\begin{aligned}
&|u(t,x)-v(t,x)|\\
&\leq \mathbb{E}\left[\left|\int(g(u(t-\tau,z))-g(v(t-\tau,z)))f_\tau(x,z)dz\right|\mathbb{1}_{\tau\leq t}\right]+e^{-\frac{t}{\varepsilon^2}}+mF(\varepsilon)\\
&\leq \tfrac{3}{2}\mathbb{E}\left[|u(t-\tau,W(R_\tau^\varepsilon))-v(t-\tau,W(R_\tau^\varepsilon))|\mathbb{1}_{\tau\leq t}\right]+e^{-\frac{t}{\varepsilon^2}}+mF(\varepsilon)\\
&\leq \tfrac{3}{2}\int_0^t ||u(\rho,\cdot)-v(\rho,\cdot)||_\infty e^{-(t-\rho)\varepsilon^{-2}}\varepsilon^{-2}d\rho+e^{-\frac{t}{\varepsilon^2}}+mF(\varepsilon).
\end{aligned}$$

Finally, using the Gronwall's inequality [MF72] (see also [Dra03, Theorem 15]) we deduce that

$$\begin{aligned}
||u(t,\cdot)-v(t,\cdot)||_\infty &\leq \left(e^{-\frac{t}{\varepsilon^2}}+mF(\varepsilon)\right)\exp\left(\tfrac{3}{2}\int_0^t\exp(-u\varepsilon^{-2})\varepsilon^{-2}du\right),\\
&\leq \left(e^{-\frac{t}{\varepsilon^2}}+mF(\varepsilon)\right)\exp\left(\tfrac{3}{2}\right),
\end{aligned}$$

which gives the desired bound by choosing an appropriate $m_1,m_2>0$. ◻

A.3 Proof of Proposition 5.2.7

To begin we restate Proposition 5.2.7.

Proposition 5.2.7. *Let $0 < t < \mathscr{T}$ and G be as in (3.12). Then, for any $x \in \mathbb{R}^{\mathrm{d}}$, there exists $m_1, m_2 > 0$ such that, for n sufficiently large,*

$$\left|\mathbb{P}_x[\mathbb{V}_p(\boldsymbol{\Xi}^n(t)) = 1] - \mathbb{P}_x[\mathbb{V}_p^{\times}(\boldsymbol{W}_{R^{\varepsilon_n}}(t)) = 1]\right| \leq m_1 e^{-\frac{t}{\varepsilon^2}} + m_2 G(n). \tag{A.13}$$

Proof. First note that, by Lemma 5.2.2, we may choose $k \in \mathbb{N}$ and $n \geq n_*$ such that, with probability $1 - \varepsilon_n^k$, $\boldsymbol{\Xi}^n(t) \overset{D}{=} \boldsymbol{\Psi}^n(t)$, for $\boldsymbol{\Psi}^n(t)$ defined in Definition 5.2.1. That is, we may work with the branching jump process $(\boldsymbol{\Psi}^n(t))_{t\geq 0}$ throughout. Define

$$u(t,x) := \mathbb{P}_x[\mathbb{V}_p(\boldsymbol{\Psi}^n(t)) = 1] \text{ and } v(t,x) := \mathbb{P}_x\left[\mathbb{V}_p^{\times}(\boldsymbol{W}_{R^{\varepsilon}}(t)) = 1\right].$$

Abuse notation and let τ denote the time of the first branching event in both $\boldsymbol{\Psi}(t)$ and $\boldsymbol{W}_{R^{\varepsilon}}(t)$. Consider the three offspring of the ancestor particle in $\boldsymbol{\Psi}^n$, $\xi_1^n(\tau), \xi_2^n(\tau)$ and $\xi_3^n(\tau)$. Define

$$g(u(\xi_i^n(\tau))\star) := g(u(t-\tau, \xi_1^n(\tau)), u(t-\tau, \xi_2^n(\tau)), u(t-\tau, \xi_3^n(\tau))).$$

By the Markov property at time $g(u(\xi_i^n(\tau))\star)$, u and v can be written as

$$u(t,x) = \mathbb{E}_x[g(u(\xi_i^n(\tau))\star)\mathbb{1}_{\tau\leq t}] + \mathbb{E}_x[p(\xi_1^n(t))\mathbb{1}_{\tau>t}] \tag{A.14}$$

$$v(t,x) = \mathbb{E}_x[g_{\times}(v(t-\tau, W(R_\tau^{\varepsilon_n})))\mathbb{1}_{\tau\leq t}] + \mathbb{E}_x[p(W(R_t^{\varepsilon_n}))\mathbb{1}_{\tau>t}]. \tag{A.15}$$

To bound the difference of u and v, we control the difference of the first terms and second terms in (A.14) and (A.15) separately. First, since $\tau \sim exp(\eta\varepsilon_n^{-2})$,

$$\mathbb{E}_x[p(\xi_1^n(t))\mathbb{1}_{\tau>t}] - \mathbb{E}_x[p(W(R_t^{\varepsilon_n}))\mathbb{1}_{\tau>t}] \leq \mathbb{P}[t < \tau] \leq e^{-\frac{t}{{\varepsilon_n}^2}}. \tag{A.16}$$

To bound the difference of the first terms in (A.14) and (A.15), set $t^* := k\varepsilon_n^2|\log(\varepsilon_n)|$ and $\delta_n := 2n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)}$. Denote the transition density of $W(R_t^{\varepsilon_n})$ started at x by $f_t(x,\cdot)$. Then, using that g is bounded above by one and is increasing in each variable, by Corollary 5.2.6, there exists a subordinated Brownian motion $(W(R_t^{\varepsilon_n}))_{t\geq 0}$ such that

$$\begin{aligned}
&\mathbb{E}_x\left[g(u(\xi_i^n(\tau))\star)\mathbb{1}_{\tau\leq t}\right] \\
&\leq \mathbb{E}_x\left[g(u(\xi_i^n(\tau))\star)\mathbb{1}_{\tau\leq t\wedge t^*}\right] + \mathbb{P}[\tau > t^*] \\
&\leq \sup_{|w|\leq\delta_n} \mathbb{E}_x\left[g(u(t-\tau, W(R_\tau^{\varepsilon_n}) + w))\mathbb{1}_{\tau\leq t\wedge t^*}\right] + n^{-\beta\left(1-\frac{\alpha}{2}\right)} I(\varepsilon_n)^{\alpha-2} + b_{\varepsilon_n} + \varepsilon_n^k \\
&= \sup_{|w|\leq\delta_n} \mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}} g(u(t-\tau, z+w)) f_\tau(x,z)dz\, \mathbb{1}_{\tau\leq t\wedge t^*}\right] + n^{-\beta\left(1-\frac{\alpha}{2}\right)} I(\varepsilon_n)^{\alpha-2} + b_{\varepsilon_n} + \varepsilon_n^k
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{|w|\le\delta_n} \mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}} g(u(t-\tau,z))f_\tau(x,z-w)dz\,\mathbb{1}_{\tau\le t\wedge t^*}\right] + n^{-\beta\left(1-\frac{\alpha}{2}\right)}I(\varepsilon_n)^{\alpha-2} + b_{\varepsilon_n} + \varepsilon_n^k \\
&\le \sup_{|w|\le\delta_n} \mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}} |f_\tau(x,z-w)-f_\tau(x,z)|\,\mathbb{1}_{\tau\le t\wedge t^*}dz\right] + n^{-\beta\left(1-\frac{\alpha}{2}\right)}I(\varepsilon_n)^{\alpha-2} + b_{\varepsilon_n} + \varepsilon_n^k \\
&\quad + \mathbb{E}_x\left[g(u(t-\tau,W(R_\tau^{\varepsilon_n})))\mathbb{1}_{\tau\le t\wedge t^*}\right]. \qquad \text{(A.17)}
\end{aligned}
$$

To bound the expected value of the integral of $|f_\tau(x,z-w)-f_\tau(x,z)|$, we partition the integral over $\mathbb{R}^{\mathrm{d}} \equiv B_x + B_x^{\mathsf{c}}$ (where B_x^{c} denotes the complement of B_x in $\mathbb{R}^{\mathrm{d}}$) for

$$
B_x := B_x\left(\tau^{\frac{1}{\alpha}}+\delta_n\right) = \left\{z\in\mathbb{R}^{\mathrm{d}} : |z-x| \le \tau^{\frac{1}{\alpha}}+\delta_n\right\}.
$$

Note that

$$
\mathbb{P}[\tau<\delta_n^\alpha] \le 1-\exp\left(-\delta_n^\alpha\varepsilon_n^{-2}\right) \le n^{-\frac{\beta\alpha}{2}\left(1-\frac{\alpha}{2}\right)}\varepsilon_n^{-2},
$$

so by Proposition A.2.1, proceeding exactly as in (A.9), there exists $C>0$ such that

$$
\sup_{|w|\le\delta_n} \mathbb{E}\left[\int_{B_x} |f_\tau(x,z-w)-f_\tau(x,z)|dz\,\mathbb{1}_{\tau\le t\wedge t^*}\right] \le C\left(n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)}\varepsilon_n^{-2}\right)^{\frac{1}{\alpha}} \qquad \text{(A.18)}
$$

and $D>0$ such that

$$
\sup_{|w|\le\delta_n} \mathbb{E}\left[\int_{B_x^{\mathsf{c}}} |f_\tau(x,z-w)-f_\tau(x,z)|dz\,\mathbb{1}_{\tau\le t\wedge t^*}\right] \le DI(\varepsilon_n)^{\alpha-1}. \qquad \text{(A.19)}
$$

Recall from (3.12) that

$$
G(n) = n^{-\beta\left(1-\frac{\alpha}{2}\right)}I(\varepsilon_n)^{\alpha-2} + b_{\varepsilon_n} + \left(n^{-\frac{\beta}{2}\left(1-\frac{\alpha}{2}\right)}\varepsilon_n^{-2}\right)^{\frac{1}{\alpha}} + I(\varepsilon_n)^{\alpha-1}.
$$

Then, choosing $m>0$ and ε_n sufficiently small, by (A.18) and (A.19) we can bound (A.17) to obtain

$$
\mathbb{E}_x\left[g(u(\xi_i^n(\tau))\star)\mathbb{1}_{\tau\le t}\right] \le \mathbb{E}\left[g(u(t-\tau,W(R_\tau^{\varepsilon_n})))\mathbb{1}_{\tau\le t\wedge t^*}\right] + mG(n). \qquad \text{(A.20)}
$$

It follows from (A.20) and (A.16) that

$$
\begin{aligned}
u(t,x)-v(t,x) &\le \mathbb{E}\left[g(u(t-\tau,W(R_\tau^{\varepsilon_n})))\mathbb{1}_{\tau\le t\wedge t^*}\right] - \mathbb{E}\left[g_\times(v(t-\tau,W(R_\tau^{\varepsilon_n})))\mathbb{1}_{\tau\le t\wedge t^*}\right] \\
&\quad + e^{-\frac{t}{\varepsilon_n^2}} + mG(n). \qquad \text{(A.21)}
\end{aligned}
$$

Using the same approach we can obtain the lower bound on $u(t,x)-v(t,x)$. Namely, by analogy with (A.17),

$$
\begin{aligned}
&\mathbb{E}_x\left[g(u(\psi_i^n(\tau))\star)\mathbb{1}_{\tau\le t}\right] \\
&\ge \mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}} g(u(t-\tau,z))[f_\tau(x,z-w)-f_\tau(x,z)]\,\mathbb{1}_{\tau\le t\wedge t^*}dz\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}\left[g(u(t-\tau, W(R_\tau^{\varepsilon_n})))\mathbb{1}_{\tau\le t\wedge t^*}\right] \\
&\geq -\sup_{|w|\le\delta_n}\left|\mathbb{E}\left[\int_{\mathbb{R}^d} g(u(t-\tau,z))[f_\tau(x,z-w)-f_\tau(x,z)]\,\mathbb{1}_{\tau\le t\wedge t^*}dz\right]\right| \\
&+\mathbb{E}\left[g(u(t-\tau, W(R_\tau^{\varepsilon_n})))\mathbb{1}_{\tau\le t\wedge t^*}\right]
\end{aligned}
$$

and the final term can be bounded exactly as before to give us that

$$
\begin{aligned}
u(t,x)-v(t,x) \geq \mathbb{E}\left[g(u(t-\tau, W(R_\tau^{\varepsilon_n})))\mathbb{1}_{\tau\le t\wedge t^*}\right] - \mathbb{E}\left[g_\times(v(t-\tau, W(R_\tau^{\varepsilon_n})))\mathbb{1}_{\tau\le t\wedge t^*}\right] \\
- e^{-\frac{t}{\varepsilon_n{}^2}} - mG(n). \qquad \text{(A.22)}
\end{aligned}
$$

Combining (A.21) and (A.22), using that $g(\cdot)$ is Lipschitz with constant $\frac{3}{2}$ and $g_\times(x) := g((1-b_{\varepsilon_n})x + \frac{1}{2}b_{\varepsilon_n})$ we obtain

$$
\begin{aligned}
&|u(t,x)-v(t,x)| \\
&\leq \mathbb{E}\left[\int_{\mathbb{R}^d} |g(u(t-\tau,z)) - g_\times(v(t-\tau,z))|\, f_\tau(x,z)\mathbb{1}_{\tau\le t}dz\right] + e^{-\frac{t}{\varepsilon_n^2}} + mG(n) \\
&\leq \tfrac{3}{2}\mathbb{E}\left[|u(t-\tau, W(R_\tau^{\varepsilon_n})) - v(t-\tau, W(R_\tau^{\varepsilon}))|\mathbb{1}_{\tau\le t}\right] + e^{-\frac{t}{\varepsilon_n^2}} + mG(n) + \tfrac{3}{2}b_{\varepsilon_n} \\
&\leq \tfrac{3}{2}\int_0^t ||u(\rho,\cdot)-v(\rho,\cdot)||_\infty e^{-(t-\rho)\varepsilon_n^{-2}}\varepsilon_n^{-2}d\rho + e^{-\frac{t}{\varepsilon_n^2}} + mG(n) + \tfrac{3}{2}b_{\varepsilon_n}.
\end{aligned}
$$

Finally, using the Gronwall's inequality [MF72] (see also [Dra03, Theorem 15]) we deduce that

$$
\begin{aligned}
||u(t,\cdot)-v(t,\cdot)||_\infty &\leq \left(e^{-\frac{t}{\varepsilon_n^2}} + mG(n) + \tfrac{3}{2}b_{\varepsilon_n}\right)\exp\left(\tfrac{3}{2}\int_0^t \exp(-u\varepsilon_n^{-2})\varepsilon_n^{-2}du\right), \\
&\leq \left(e^{-\frac{t}{\varepsilon_n^2}} + mG(n) + \tfrac{3}{2}b_{\varepsilon_n}\right)\exp\left(\tfrac{3}{2}\right),
\end{aligned}
$$

which gives the desired bound by choosing an appropriate $m_1, m_2 > 0$. □

A.4 Truncated subordinator calculations

Throughout this section, let $(R_s^\varepsilon)_{s\geq0}$ be the $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator with Lévy measure given by

$$
\tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I(\varepsilon)^{\alpha-2}y^{-1-\frac{\alpha}{2}}\mathbb{1}_{\left\{0\le y\le\frac{2-\alpha}{\alpha}I(\varepsilon)^2\right\}}dy. \qquad \text{(A.23)}
$$

Denote by $\mathbb{P}$ the probability measure under which $(R_s^\varepsilon)_{s\geq0}$, started at $R_0^\varepsilon = 0$ has this distribution, and let $\mathbb{E}$ denote the corresponding expectation. For all $s, \lambda \geq 0$ the Laplace transform $\phi(\lambda) := \mathbb{E}\left[\exp\left(-\lambda R_s^\varepsilon\right)\right]$ of R_s^ε is

$$
\phi(\lambda) = \exp\left(\tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I(\varepsilon)^{\alpha-2}s\int_0^{\frac{2-\alpha}{\alpha}I(\varepsilon)^2}\frac{e^{-\lambda y}-1}{y^{\frac{\alpha}{2}+1}}dy\right). \qquad \text{(A.24)}
$$

The following lemma will enable us to show, in Proposition A.4.2, that R_s^ε is close to s for small times s, which is a crucial component of our proof of Theorem 3.3.18.

Lemma A.4.1. *Let $(R_s^\varepsilon)_{s\geq 0}$ be as above. For all $s \geq 0$, $\mathbb{E}[R_s^\varepsilon] = s$.*

Proof. The expected value of R_s^ε can be calculated explicitly by considering the derivative of its Laplace transform, namely $\mathbb{E}[R_s^\varepsilon] = -\frac{d}{d\lambda}\phi(\lambda)|_{\lambda=0}$. Denote $I := I(\varepsilon)$ throughout. Fix $\lambda, s \geq 0$. Using integration by parts,

$$\int_0^{\frac{2-\alpha}{\alpha}I^2} \frac{e^{-\lambda y}-1}{y^{\frac{\alpha}{2}+1}}dy = -\tfrac{2}{\alpha}\left(\tfrac{2-\alpha}{\alpha}\right)^{-\frac{\alpha}{2}} I^{-\alpha}(e^{-\frac{2-\alpha}{\alpha}\lambda I^2}-1) - \tfrac{2}{\alpha}\lambda^{\frac{\alpha}{2}}\gamma\left(1-\tfrac{\alpha}{2}, \tfrac{2-\alpha}{\alpha}\lambda I^2\right)$$

where $\gamma(s,x) := \int_0^x t^{s-1}e^{-t}dt$ is the lower incomplete gamma function. So, using (A.24), $\phi(\lambda)$ can be written as

$$\phi(\lambda) = \exp\left(-I^{-2}s\left(e^{-\frac{2-\alpha}{\alpha}\lambda I^2}-1\right) - \left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2}\lambda^{\frac{\alpha}{2}}s\gamma\left(1-\tfrac{\alpha}{2}, \tfrac{2-\alpha}{\alpha}\lambda I^2\right)\right).$$

By differentiating this quantity with respect to λ, we obtain

$$\begin{aligned}&\mathbb{E}[R_s^\varepsilon]\\ &= -\tfrac{2-\alpha}{\alpha}s + \left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2}s\left(\tfrac{\alpha}{2}\lambda^{\frac{\alpha}{2}-1}\gamma\left(1-\tfrac{\alpha}{2}, \tfrac{2-\alpha}{\alpha}\lambda I^2\right) + \lambda^{\frac{\alpha}{2}}\frac{d\gamma}{d\lambda}\left(1-\tfrac{\alpha}{2}, \tfrac{2-\alpha}{\alpha}\lambda I^2\right)\right)\Bigg|_{\lambda=0}.\end{aligned}$$

This can be calculated by considering the following identities for γ. First, by the definition of γ and a change of variables

$$\gamma\left(1-\tfrac{\alpha}{2}, \tfrac{2-\alpha}{\alpha}\lambda I^2\right) = I^{2-\alpha}\int_0^{\frac{2-\alpha}{\alpha}\lambda} z^{-\frac{\alpha}{2}}e^{-zI^2}dz.$$

Therefore

$$\begin{aligned}\lambda^{\frac{\alpha}{2}}\frac{d\gamma}{d\lambda}\left(1-\tfrac{\alpha}{2}, \tfrac{2-\alpha}{\alpha}\lambda I^2\right)\Big|_{\lambda=0} &= I^{2-\alpha}\left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}} e^{-\frac{2-\alpha}{\alpha}\lambda I^2}\Big|_{\lambda=0}\\ &= I^{2-\alpha}\left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}}.\end{aligned}$$

Under a different change of variables, γ also satisfies

$$\begin{aligned}\lambda^{\frac{\alpha}{2}-1}\gamma\left(1-\tfrac{\alpha}{2}, \tfrac{2-\alpha}{\alpha}\lambda I^2\right)\Big|_{\lambda=0} &= \left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}}\int_0^{I^2} z^{-\frac{\alpha}{2}}e^{-\frac{2-\alpha}{\alpha}\lambda z}dz\Big|_{\lambda=0}\\ &= \left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}}\left(\tfrac{2}{2-\alpha}\right)I^{2-\alpha}.\end{aligned}$$

Putting this all together we obtain

$$\begin{aligned}\mathbb{E}[R_s^\varepsilon] &= -\tfrac{2-\alpha}{\alpha}s + \left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2}s\left(\tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}}\left(\tfrac{2}{2-\alpha}\right)I^{2-\alpha} + \left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}}I^{2-\alpha}\right)\\ &= s.\end{aligned}$$

□

With this, we can now prove Proposition A.4.2.

Proposition A.4.2. *Let $k \in \mathbb{N}$ and $(R^\varepsilon_s)_{s\geq 0}$ be as above. There exists $\varepsilon_k > 0$ such that, for all $\varepsilon \in (0, \varepsilon_k)$ and $s \leq \varepsilon^2 |\log \varepsilon|$,*

$$\mathbb{P}\left[|R^\varepsilon_s - s| \geq (k+1)I(\varepsilon)^2|\log\varepsilon|\right] \leq \varepsilon^k.$$

Proof. Let $\varepsilon > 0$. As before we set $I := I(\varepsilon)$. Fix $k \in \mathbb{N}$ and $s \leq \varepsilon^2|\log\varepsilon|$. Note that

$$\begin{aligned}&\mathbb{P}\left[|R^\varepsilon_s - s| \geq (k+1)I^2|\log\varepsilon|\right]\\ &= \mathbb{P}\left[R^\varepsilon_s \geq (k+1)I^2|\log\varepsilon| + s\right] + \mathbb{P}\left[R^\varepsilon_s \leq s - (k+1)I^2|\log\varepsilon|\right]. \qquad \text{(A.25)}\end{aligned}$$

By Assumption 3.1.2 (b), $\varepsilon I^{-1} \to 0$ as $\varepsilon \to 0$, and since $s \leq \varepsilon^2|\log\varepsilon|$, we may decrease ε as necessary to ensure that $s - (k+1)I^2|\log\varepsilon| \leq 0$, and the second term in (A.25) equals zero.

To bound the first term in (A.25), we note that, by [Sat99, Theorem 25.17], and since the Lévy measure of R^ε_s has compact support, the Laplace transform $\phi(\lambda)$ from (A.24) can be extended to all of $\mathbb{R}$, and the exponential moments of R^ε_s exist and satisfy $\mathbb{E}\left[\exp\left(\lambda R^\varepsilon_s\right)\right] = \phi(\lambda)$ for all $\lambda \in \mathbb{R}$. It is straightforward to verify using Lemma A.4.1 that $\mathbb{E}\left[R^\varepsilon_s\right]$ satisfies

$$\mathbb{E}\left[R^\varepsilon_s\right] = \tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2} s \int_0^{\frac{2-\alpha}{\alpha}I^2} y^{-\frac{\alpha}{2}}dy,$$

therefore for any $s, \lambda \geq 0$, by Taylor's theorem

$$\begin{aligned}\mathbb{E}\left[\exp\left(\lambda R^\varepsilon_s - \lambda\mathbb{E}[R^\varepsilon_s]\right)\right] &= \exp\left(\tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2} s \int_0^{\frac{2-\alpha}{\alpha}I^2} \frac{e^{\lambda v} - 1 - \lambda v}{v^{1+\frac{\alpha}{2}}}dv\right)\\ &\leq \exp\left(\tfrac{\alpha}{4}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2} e^{\frac{2-\alpha}{\alpha}I^2\lambda}\lambda^2 s \int_0^{\frac{2-\alpha}{\alpha}I^2} v^{1-\frac{\alpha}{2}}dv\right)\\ &= \exp\left(\tfrac{1}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^2 \tfrac{\alpha}{4-\alpha} e^{\frac{2-\alpha}{\alpha}I^2\lambda} I^2\lambda^2 s\right). \qquad \text{(A.26)}\end{aligned}$$

Choose $\lambda = I^{-2}$. Then (A.26) becomes

$$\begin{aligned}\mathbb{E}\left[\exp\left(I^{-2}R^\varepsilon_s - I^{-2}\mathbb{E}[R^\varepsilon_s]\right)\right] &\leq \exp\left(\tfrac{1}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^2 \tfrac{\alpha}{4-\alpha} e^{\frac{2-\alpha}{\alpha}} I^{-2} s\right)\\ &\leq \exp\left(\tfrac{1}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^2 \tfrac{\alpha}{4-\alpha} e^{\frac{2-\alpha}{\alpha}} \frac{\varepsilon^2|\log\varepsilon|}{I^2}\right).\end{aligned}$$

This, together with Lemma A.4.1 and Markov's inequality gives us

$$\begin{aligned}\mathbb{P}\left[R^\varepsilon_s \geq (k+1)I^2|\log\varepsilon| + s\right] &= \mathbb{P}\left[R^\varepsilon_s - \mathbb{E}[R^\varepsilon_s] \geq (k+1)I^2|\log\varepsilon|\right]\\ &= \mathbb{P}\left[\exp\left(I^{-2}R^\varepsilon_s - I^{-2}\mathbb{E}[R^\varepsilon_s]\right) \geq \varepsilon^{-k-1}\right]\\ &\leq \varepsilon^{k+1}\exp\left(\tfrac{1}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^2 \tfrac{\alpha}{4-\alpha} e^{\frac{2-\alpha}{\alpha}} \frac{\varepsilon^2|\log\varepsilon|}{I^2}\right).\end{aligned}$$

By Assumption 3.1.2 (B), $\frac{\varepsilon^2|\log\varepsilon|}{I^2} \to 0$ as $\varepsilon \to 0$, so the previous inequality implies that, for ε sufficiently small,

$$\mathbb{P}\left[R_s^\varepsilon > (k+1)I^2|\log\varepsilon| + s\right] \leq \varepsilon^k,$$

which, by (A.25) gives us the result. $\square$

Lemma A.4.3. *Let $(R_s^\varepsilon)_{s\geq 0}$ be as above. Then, for any $q, s > 0$, there exists $D_1 = D_1(q,\alpha) > 0$ and $D_2 = D_2(q,\alpha) > 0$ such that*

$$\mathbb{E}\left[(R_s^\varepsilon)^{-q}\right] \leq e^s(D_1 + D_2 s^{-\frac{2q}{\alpha}}).$$

Proof. Fix $s \geq 0$, $\varepsilon > 0$ and $I := I(\varepsilon)$. Let γ be the lower incomplete gamma function. The Laplace transform (A.24) of R_s^ε can be written as

$$\begin{aligned}
\phi(\lambda) &= \exp\left(-I^{-2}s\left(e^{-\frac{2-\alpha}{\alpha}\lambda I^2} - 1\right) - \left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2}\lambda^{\frac{\alpha}{2}} s\gamma\left(1-\tfrac{\alpha}{2}, \tfrac{2-\alpha}{\alpha}\lambda I^2\right)\right) \\
&= \exp\left(I^{-2}s\int_0^{\frac{2-\alpha}{\alpha}\lambda I^2} e^{-z}dz - \left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2}\lambda^{\frac{\alpha}{2}} s\int_0^{\frac{2-\alpha}{\alpha}\lambda I^2} z^{-\frac{\alpha}{2}}e^{-z}dz\right) \\
&= \exp\left(\tfrac{2-\alpha}{\alpha}s\int_0^{\lambda} e^{-\frac{2-\alpha}{\alpha}I^2 z}\left(1-\lambda^{\frac{\alpha}{2}}z^{-\frac{\alpha}{2}}\right)dz\right).
\end{aligned}$$

By [Sch02, Theorem 1.1], if X is a non-negative random variable with Laplace transform ρ, then for any $q > 0$, $\mathbb{E}\left[X^{-q}\right] = \frac{1}{q\Gamma(q)}\int_0^\infty \rho\left(t^{\frac{1}{q}}\right)dt$. Therefore, using the above expression for the Laplace transform of R_s^ε, for any $q > 0$

$$\mathbb{E}\left[(R_s^\varepsilon)^{-q}\right] = \frac{1}{q\Gamma(q)}\int_0^\infty \exp\left(\tfrac{2-\alpha}{\alpha}s\int_0^{\lambda^{\frac{1}{q}}} e^{-\frac{2-\alpha}{\alpha}I^2 z}\left(1-\lambda^{\frac{\alpha}{2q}}z^{-\frac{\alpha}{2}}\right)dz\right)d\lambda.$$

When $\lambda \in [0,1]$,

$$\int_0^{\lambda^{\frac{1}{q}}} e^{-\frac{2-\alpha}{\alpha}I^2 z}\left(1-\lambda^{\frac{\alpha}{2q}}z^{-\frac{\alpha}{2}}\right)dz \leq 1$$

and if $\lambda > 1$, since the integrand is negative

$$\int_0^{\lambda^{\frac{1}{q}}} e^{-\frac{2-\alpha}{\alpha}I^2 z}\left(1-\lambda^{\frac{\alpha}{2q}}z^{-\frac{\alpha}{2}}\right)dz \leq \int_0^1 1-\lambda^{\frac{\alpha}{2q}}z^{-\frac{\alpha}{2}}dz = 1 - \tfrac{2}{2-\alpha}\lambda^{\frac{\alpha}{2q}}.$$

Therefore

$$\mathbb{E}\left[(R_s^\varepsilon)^{-q}\right] \leq \frac{1}{q\Gamma(q)}e^{\frac{2-\alpha}{\alpha}s}\left(1+\int_1^\infty \exp\left(-\tfrac{2}{\alpha}s\lambda^{\frac{\alpha}{2q}}\right)d\lambda\right).$$

Let $\Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}dt$ be the upper incomplete gamma function. Then it is straightforward to show that the above is equivalent to

$$\begin{aligned}\mathbb{E}\left[(R^\varepsilon_s)^{-q}\right] &\le \frac{1}{q\Gamma(q)}e^{\frac{2-\alpha}{\alpha}s}\left(1+\frac{2q}{\alpha}\left(\frac{\alpha}{2s}\right)^{\frac{2q}{\alpha}}\Gamma\left(\tfrac{2q}{\alpha},\tfrac{2s}{\alpha}\right)\right)\\ &\le \frac{1}{q\Gamma(q)}e^{\frac{2-\alpha}{\alpha}s}\left(1+\frac{2q}{\alpha}\left(\frac{\alpha}{2s}\right)^{\frac{2q}{\alpha}}\Gamma\left(\tfrac{2q}{\alpha}\right)\right).\end{aligned}$$

Since $\frac{2-\alpha}{\alpha}\le 1$, the result follows. □

Lemma A.4.4. *Let $(R^\varepsilon_s)_{s\ge 0}$ be as above and $(B_s)_{s\ge 0}$ be a standard one-dimensional Brownian motion. Let $T^*\in(0,\infty)$, $\varepsilon>0$ and define z_ε implicitly by the relation*

$$\mathbb{P}_{z_\varepsilon}[B(R^\varepsilon_{T^*})\ge 0] = \tfrac{1}{2} + (u_+ - u_-)^{-1}\varepsilon \tag{A.27}$$

where u_+ and u_- are the fixed points of $g_\times$ from Proposition A.1.1. Then, for ε sufficiently small,

$$z_\varepsilon \le 8\sqrt{2\pi(T^*+2)}\,\varepsilon.$$

Proof. By symmetry of $(B_s)_{s\ge0}$, (A.27) is equivalent to

$$\mathbb{P}_0[B(R^\varepsilon_{T^*})\in(0,z_\varepsilon)] = (u_+-u_-)^{-1}\varepsilon.$$

It is a standard fact for Brownian motion that, if $t>0$ and $x>0$,

$$\mathbb{P}_0\left[B_t\in(0,x)\right] \ge \frac{x}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}.$$

Let $f_{R^\varepsilon_{T^*}}(\cdot)$ denote the transition density of $R^\varepsilon_{T^*}$. Then

$$\begin{aligned}\mathbb{P}_0[B(R^\varepsilon_{T^*})\in(0,z_\varepsilon)] &= \int_0^\infty \mathbb{P}_0[B(r)\in(0,z_\varepsilon)]f_{R^\varepsilon_{T^*}}(r)dr\\ &\ge \int_0^\infty \frac{z_\varepsilon}{\sqrt{2\pi r}}e^{-\frac{z_\varepsilon^2}{2r}}f_{R^\varepsilon_{T^*}}(r)dr\\ &\ge \int_{\frac{1}{2}T^*}^{2I(\varepsilon)^2|\log\varepsilon|+T^*} \frac{z_\varepsilon}{\sqrt{2\pi r}}e^{-\frac{z_\varepsilon^2}{2r}}f_{R^\varepsilon_{T^*}}(r)dr\\ &\ge \frac{z_\varepsilon e^{-\frac{z_\varepsilon^2}{T^*}}}{\sqrt{2\pi(2I(\varepsilon)^2|\log\varepsilon|+T^*)}}\mathbb{P}\left[R^\varepsilon_{T^*}\in\left(\tfrac{1}{2}T^*, 2I(\varepsilon)^2|\log\varepsilon|+T^*\right)\right].\end{aligned}$$

Note that

$$\begin{aligned}&\mathbb{P}\left[R^\varepsilon_{T^*}\in\left(\tfrac{1}{2}T^*, 2I(\varepsilon)^2|\log\varepsilon|+T^*\right)\right]\\ &= \mathbb{P}\left[R^\varepsilon_{T^*}< 2I(\varepsilon)^2|\log\varepsilon|+T^*\right] - \mathbb{P}\left[R^\varepsilon_{T^*}\le \tfrac{1}{2}T^*\right].\end{aligned} \tag{A.28}$$

We bound each of these terms separately. First, by the proof of Proposition A.4.2, for ε sufficiently small,

$$\mathbb{P}\left[R^{\varepsilon}_{T^*} < 2I(\varepsilon)^2|\log\varepsilon| + T^*\right] \geq 1-\varepsilon.$$

Let $\phi(\lambda) := \mathbb{E}\left[\exp(-\lambda R^{\varepsilon}_{T^*})\right]$ be the Laplace transform of $R^{\varepsilon}_{T^*}$. Now, by Markov's inequality,

$$\begin{aligned}\mathbb{P}\left[R^{\varepsilon}_{T^*} \leq \tfrac{1}{2}T^*\right] &= \mathbb{P}\left[\exp\left(-\tfrac{\alpha}{2-\alpha}I(\varepsilon)^{-2}R^{\varepsilon}_{T^*}\right) \geq \exp\left(-\tfrac{\alpha}{2-\alpha}\tfrac{1}{2}I(\varepsilon)^{-2}T^*\right)\right] \\ &\leq \exp\left(\tfrac{\alpha}{2-\alpha}\tfrac{1}{2}I(\varepsilon)^{-2}T^*\right)\phi\left(\tfrac{\alpha}{2-\alpha}I(\varepsilon)^{-2}\right).\end{aligned}$$

Now, using the expression for $\phi(\lambda)$ from the proof of Lemma A.4.1,

$$\phi\left(\tfrac{\alpha}{2-\alpha}I(\varepsilon)^{-2}\right) = \exp\left(\left(1-e^{-1}\right)T^*I(\varepsilon)^{-2} - \gamma\left(1-\tfrac{\alpha}{2},1\right)T^*I(\varepsilon)^{-2}\right),$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function. Using that $1-e^{-x}\leq x$,

$$\begin{aligned}\gamma\left(1-\tfrac{\alpha}{2},1\right) &= \int_0^1 t^{-\frac{\alpha}{2}}e^{-t}dt \\ &\geq \int_0^1 t^{-\frac{\alpha}{2}}(1-t)dt \\ &= \frac{1}{(1-\frac{\alpha}{2})(2-\frac{\alpha}{2})},\end{aligned}$$

therefore

$$\phi\left(\tfrac{\alpha}{2-\alpha}I(\varepsilon)^{-2}\right) \leq \exp\left(\left(1-e^{-1}-\frac{1}{(1-\frac{\alpha}{2})(2-\frac{\alpha}{2})}\right)T^*I(\varepsilon)^{-2}\right)$$

and

$$\mathbb{P}\left[R^{\varepsilon}_{T^*} \leq \tfrac{1}{2}T^*\right] \leq \exp\left(\left(1-e^{-1}-\frac{1}{(1-\frac{\alpha}{2})(2-\frac{\alpha}{2})}+\frac{\alpha}{2(2-\alpha)}\right)T^*I(\varepsilon)^{-2}\right).$$

It is straightforward to verify algebraically that, since $\alpha\in(1,2)$,

$$\frac{\alpha}{2(2-\alpha)} - \frac{1}{(1-\frac{\alpha}{2})(2-\frac{\alpha}{2})} \leq -\frac{2}{3},$$

so

$$\mathbb{P}\left[R^{\varepsilon}_{T^*} \leq \tfrac{1}{2}T^*\right] \leq \exp\left(\left(\tfrac{1}{3}-e^{-1}\right)T^*I(\varepsilon)^{-2}\right),$$

where we note that $\frac{1}{3}-e^{-1}<0$. Returning to (A.28), we obtain

$$\begin{aligned}\mathbb{P}\left[R^{\varepsilon}_{T^*} \in \left(\tfrac{1}{2}T^*, 2I(\varepsilon)^2|\log\varepsilon|+T^*\right)\right] &\geq 1-\varepsilon-\exp\left(\left(\tfrac{1}{3}-e^{-1}\right)T^*I(\varepsilon)^{-2}\right) \\ &\geq \tfrac{1}{2}\end{aligned}$$

where the last inequality holds by choosing ε sufficiently small. Returning to our original inequality, we have

$$\mathbb{P}_0[B(R^{\varepsilon}_{T^*}) \in (0, z_\varepsilon)] \geq \frac{1}{2} \frac{z_\varepsilon e^{-\frac{z_\varepsilon^2}{T^*}}}{\sqrt{2\pi(2I(\varepsilon)^2 |\log \varepsilon| + T^*)}},$$

so by assumption

$$(u_+ - u_-)^{-1}\varepsilon \geq \frac{1}{2} \frac{z_\varepsilon e^{-\frac{z_\varepsilon^2}{T^*}}}{\sqrt{2\pi(2I(\varepsilon)^2 |\log \varepsilon| + T^*)}}.$$

Since $\lim_{\varepsilon \to 0} z_\varepsilon = 0$, assume ε is sufficiently small so that $e^{-\frac{z_\varepsilon^2}{T^*}} \geq \frac{1}{2}$. Further decrease ε if necessary so that $(u_+ - u_-)^{-1} \leq 2$. Then

$$8\sqrt{2\pi(2I(\varepsilon)^2 |\log \varepsilon| + T^*)}\,\varepsilon \geq z_\varepsilon,$$

so the result follows by decreasing ε if necessary so that $I(\varepsilon)^2 |\log \varepsilon| \leq 1$. $\square$

Appendix B

Chapter 6 supplementary material

We now prove the Feynman–Kac formula, Theorem 6.2.1, and our bounds on the displacement of a stable process, Lemma 6.2.3. Let $(X_s)_{s\geq 0}$ be a one-dimensional α-stable process throughout, where $\alpha \in (0,2)$.

B.1 Proof of Lemma 6.2.3

To prove Lemma 6.2.3, we require the following results.

Lemma B.1.1 ([LS16])**.** *If R_t is a stable subordinator with index $\frac{\alpha}{2}$, there exists $\rho \geq 0$ such that, for all $t, \lambda > 0$*

$$\mathbb{E}\left[e^{-\lambda R_t}\right] = \exp(-\rho t \lambda^{\frac{\alpha}{2}}).$$

Lemma B.1.2 ([Kho11])**.** *Let R_t be a stable subordinator with index $\frac{\alpha}{2}$. There exist positive and finite constants $c_1 = c_1(\alpha)$ and $c_2 = c_2(\alpha)$ such that, for all $z > 0$*

$$1 - \exp\left(-c_1 z^{-\frac{\alpha}{2}}\right) \leq \mathbb{P}[R_1 > z] \leq c_2(1 \wedge z^{-\frac{\alpha}{2}}).$$

Lastly, we will need the following well-known estimate for the displacement of a standard Brownian motion, which can be found in [Rob15, Lemma 5]. We state here the version that can be found in [Pen18] in the proof of Lemma 2.3.

Lemma B.1.3. *Let $(B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion and $T > 0$. There exists a constant $c_T > 0$ such that for any $t \geq T$ and $0 \leq x_0 \leq 1$,*

$$\mathbb{P}_0\left[|B_s| \leq 1 \ \forall\, s \leq t, |B_t| \leq x_0\right] \geq c_T e^{-\pi^2 t/8} \min(x_0, \tfrac{2}{3}).$$

Equipped with these results, we can now restate and prove Lemma 6.2.3.

Lemma 6.2.3. *We have the following two bounds for the displacement of X_s.*

(a) *There exists* $\gamma > 0$ *such that, for any* $c \geq 0, t \geq 0$ *and* $w > 0$,

$$\mathbb{P}_0 \left[\sup_{s \in [0,t]} |X_s| \geq c \right] \leq 4e^{-\frac{c^2}{2w}} + \gamma(1 \wedge tw^{-\frac{\alpha}{2}}).$$

(b) *If* $t \geq 1$ *and* $0 \leq x_0 \leq 1$, *there exist constants* $\eta_1(\alpha), \eta_2(\alpha) > 0$ *such that*

$$\mathbb{P}_0 \left[|X_s| \leq 1 \ \forall \ s \leq t, \ |X_t| \leq x_0 \right] \geq \eta_1 \min(x_0, \tfrac{2}{3}) e^{-\eta_2 t}.$$

Proof. We begin by proving part (a). Let R_t be an $\frac{\alpha}{2}$-stable subordinator, independent of a standard Brownian motion B_t. Denote the transition density of R_t by f_{R_t}. Then, since $B(R_s)$ is equal in distribution to X_s, for $c \geq 0$

$$\begin{aligned}
\mathbb{P}_0 \left[\sup_{s \in [0,t]} |X_s| \geq c \right] &\leq \int_0^\infty \mathbb{P}_0 \left[\sup_{s \in [0,r]} |B_s| \geq c \right] f_{R_t}(r) dr \\
&= 4 \int_0^\infty \mathbb{P}_0 \left[B_r \geq c \right] f_{R_t}(r) dr \\
&\leq 4 \int_0^\infty e^{-c^2/2r} f_{R_t}(r) dr \\
&\leq 4\mathbb{P}[R_t \geq w] + 4e^{-c^2/2w}
\end{aligned}$$

for any $w > 0$, where the second line follows by the symmetry of Brownian motion and the reflection principle, and the third line follows because, for $Z \sim N(0,1)$, $\mathbb{P}[Z > x] \leq e^{-x^2/2}$ if $x > 0$. The result then follows from Lemma B.1.2.

To prove part (b), suppose $t \geq 1$ and $x_0 \in [0,1]$. Define

$$T := \frac{8}{\pi^2} \left(\frac{c_1}{2\rho} \right)^{\frac{2}{\alpha}}.$$

Let $c := c_T$ be the constant from Lemma B.1.3 for this choice of T. Then for $0 \leq x_0 \leq 1$

$$\begin{aligned}
&\mathbb{P}_0 \left[|X_s| \leq 1 \, \forall \, s \leq t, \ |X_t| \leq x_0 \right] \\
&= \mathbb{P}_0 \left[|B(R_s)| \leq 1 \, \forall \, s \leq t, \ |B(R_t)| \leq x_0 \right] \\
&= \int_0^\infty \mathbb{P}_0 \left[|B(R_s)| \leq 1 \, \forall \, s \leq t, \ |B_r| \leq x_0 \ \middle| \ R_t = r \right] f_{R_t}(r) dr \\
&\geq \int_T^\infty \mathbb{P}_0 \left[|B_s| \leq 1 \, \forall \, s \leq r, \ |B_r| \leq x_0 \right] f_{R_t}(r) dr \\
&\geq \int_T^\infty c \min(x_0, \tfrac{2}{3}) e^{-\pi^2 r/8} f_{R_t}(r) dr \\
&\geq c \min(x_0, \tfrac{2}{3}) \left(\mathbb{E}_0 \left[e^{-\pi^2 R_t/8} \right] - \mathbb{P}_0[R_t \leq T] \right) \\
&\geq c \min(x_0, \tfrac{2}{3}) \left(e^{-\rho t (\pi^2/8)^{\alpha/2}} - e^{-2\rho t (\pi^2/8)^{\alpha/2}} \right) \\
&\geq c \min(x_0, \tfrac{2}{3}) \rho (\pi^2/8)^{\alpha/2} e^{-2\rho t (\pi^2/8)^{\alpha/2}}
\end{aligned}$$

where in the second inequality we have used Lemma B.1.3, in the fourth inequality we have used Lemma B.1.1 and Lemma B.1.2, and in the final inequality we use the Mean Value Theorem and that $t \geq 1$. ◻

B.2 Proof of Theorem 6.2.1

Next, we will restate and prove the Feynman–Kac formula from Theorem 6.2.1. We will follow broadly the proof found in [Lig10], where the same result was proved, but for nonlinearities $f \in C(\mathbb{R})$ independent of time. We conclude this section with Proposition B.2.1, which offers an alternative formulation of the Feynman–Kac formula.

Theorem 6.2.1. *Take $u_0 \in \mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$ and suppose $f \in C([0,\infty) \times \mathbb{R}, [0,\infty))$ is differentiable in the first variable with measurable derivative. For $x \in \mathbb{R}$ and $t > 0$ consider the equation*

$$\partial_t u(t,x) = -(-\Delta)^{\frac{\alpha}{2}} u(t,x) + f(t,x)u(t,x), \quad u(0,x) = u_0(x). \tag{B.1}$$

Then a unique solution u to (6.7) exists with $u(t,\cdot) \in \mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$ for each $t \geq 0$. Moreover, u satisfies

$$u(t,x) = \mathbb{E}_x \left[\exp\left(\int_0^t f(t-s, X_s) ds \right) u_0(X_t) \right]. \tag{B.2}$$

Proof. The existence and uniqueness of a solution $u(t,\cdot) \in \mathcal{D}((-\Delta)^{\frac{\alpha}{2}})$ follows from the discussion [CR13, p. 697] together with [CR13, Remark 2.6]. In order to show that $u(t,x)$ satisfies (B.1), we adapt [Lig10, Theorem 3.47] and consider $\lim_{\varepsilon \downarrow 0} \frac{u(t+\varepsilon,x)-u(t,x)}{\varepsilon}$. Fix $t \geq 0$ and $x \in \mathbb{R}$. Let $\varepsilon > 0$ and note that $u(t+\varepsilon,x) - u(t,x) = \mathbb{E}_x[\mathcal{G}_1 + \mathcal{G}_2]$ where

$$\mathcal{G}_1 := u_0(X_{t+\varepsilon}) \exp\left(\int_0^{t+\varepsilon} f(t+\varepsilon-v, X_v) dv \right) - u_0(X_t) \exp\left(\int_0^t f(t+\varepsilon-v, X_v) dv \right)$$

$$\mathcal{G}_2 := u_0(X_t) \left(\exp\left(\int_0^t f(t+\varepsilon-v, X_v) dv \right) - \exp\left(\int_0^t f(t-v, X_v) dv \right) \right).$$

For $0 \leq s \leq r \leq t+\varepsilon$ define

$$I^\varepsilon(s,r) := \int_s^r f(t+\varepsilon-v, X_v) dv.$$

When $\varepsilon = 0$ we write $I(s,r) := I^0(s,r)$. We can further decompose $\mathcal{G}_1$ into the sum $\mathcal{G}_1 = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$ where

$$\begin{aligned}
\mathcal{E}_1 &:= (u_0(X_{t+\varepsilon}) - u_0(X_t))\, e^{I^\varepsilon(0,t)} \left(e^{I^\varepsilon(t,t+\varepsilon)} - 1 \right) \\
\mathcal{E}_2 &:= (u_0(X_{t+\varepsilon}) - u_0(X_t))\, e^{I^\varepsilon(0,t)} \\
\mathcal{E}_3 &:= u_0(X_t) e^{I^\varepsilon(0,t)} \left(e^{I^\varepsilon(t,t+\varepsilon)} - 1 \right)
\end{aligned}$$

Following almost identical arguments to [Lig10, Theorem 3.47] we have

$$\mathbb{E}_x\left[|\mathcal{E}_1|\right] \leq e^{t||f||_\infty}(e^{\varepsilon||f||_\infty} - 1)\mathbb{E}_x\left[|u_0(X_{t+\varepsilon}) - u_0(X_t)|\right] = o(\varepsilon)$$

and

$$\lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}_x[\mathcal{E}_2]}{\varepsilon} = \mathbb{E}_x\left[-(-\Delta)^{\frac{\alpha}{2}} u_0(X_t)e^{I(0,t)}\right]$$

which follows by applying the Markov property at time t to $\mathbb{E}_x\left[\mathcal{E}_2\right]$. Lastly, consider $\mathbb{E}_x[\mathcal{E}_3]$. By the Monotone Convergence Theorem, since $f \geq 0$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathbb{E}_x\left[\exp\left(\int_t^{t+\varepsilon} f(t+\varepsilon - v, X_v)dv\right) - 1\right] \\
&= \mathbb{E}_x\left[\frac{d}{dz}\exp\left(\int_t^{t+z} f(t+z-v, X_v)dv\right)\Big|_{z=0}\right] \\
&= \mathbb{E}_x\left[f(0, X_t) + \int_t^{t+z} \partial_t f(t+z-v, X_v)dv\Big|_{z=0}\right] \\
&= \mathbb{E}_x\left[f(0, X_t)\right],
\end{aligned}$$

and it follows that

$$\lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}_x\left[\mathcal{E}_3\right]}{\varepsilon} = \mathbb{E}_x\left[u_0(X_t)f(0, X_t)e^{I(0,t)}\right].$$

We conclude that $\lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}_x[\mathcal{G}_1]}{\varepsilon}$ exists with

$$\lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}_x[\mathcal{G}_1]}{\varepsilon} = \mathbb{E}_x\left[-(-\Delta)^{\frac{\alpha}{2}} u_0(X_t)e^{I(0,t)}\right] + \mathbb{E}_x\left[u_0(X_t)f(0, X_t)e^{I(0,t)}\right].$$

To show that $\lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}_x[\mathcal{G}_2]}{\varepsilon}$ exists, by the Mean Value Theorem and the Measurable Selection Theorem, there exists a measurable function $\zeta : [0, t) \to \mathbb{R}$ with $\zeta(r) \in (t - r, t + \varepsilon - r)$ for which

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\mathbb{E}_x\left[\mathcal{G}_2\right] &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\mathbb{E}_x\left[u_0(X_t)e^{\int_0^t f(t-r,X_r)dr}\left(e^{\varepsilon\int_0^t \partial_t f(\zeta(r),X_r)dr} - 1\right)\right] \\
&= \mathbb{E}_x\left[u_0(X_t)e^{\int_0^t f(t-r,X_r)dr}\int_0^t \partial_t f(t-r, X_r)dr\right].
\end{aligned}$$

We conclude that u is differentiable in t with

$$\begin{aligned}
\partial_t u(t,x) &= \mathbb{E}_x\left[-(-\Delta)^{\frac{\alpha}{2}} u_0(X_t)e^{I(0,t)}\right] + \mathbb{E}_x\left[u_0(X_t)f(0, X_t)e^{I(0,t)}\right] \\
&\quad + \mathbb{E}_x\left[u_0(X_t)e^{\int_0^t f(t-r,X_r)dr}\int_0^t \partial_t u_0(t-r, X_r)dr\right].
\end{aligned}$$

It remains to show that u satisfies (B.1). By the Markov Property at time ε,

$$u(t+\varepsilon, x) = \mathbb{E}_x\left[u(t, X_\varepsilon)e^{\int_0^\varepsilon f(t+\varepsilon-r,X_r)dr}\right]$$

so

$$u(t+\varepsilon,x)-u(t,x)=\mathbb{E}_x\left[u(t,X_\varepsilon)\left(e^{\int_0^\varepsilon f(t+\varepsilon-r,X_r)dr}-1\right)\right]+\mathbb{E}_x\left[u(t,X_\varepsilon)-u(t,x)\right].$$

Dividing the above by ε and taking $\varepsilon\downarrow 0$, the left hand side will exist by the above arguments, and the first term on the right hand side also exists and equal $f(t,x)u(t,x)$. This implies the final term exists and equals $-(-\Delta)^{\frac{\alpha}{2}}u(t,x)$, so u satisfies (B.1). □

Proposition B.2.1. *Under the assumptions of Theorem 6.2.1, (B.2) can equivalently be written as*

$$u(t,x)=\mathbb{E}_x\left[\exp\left(\int_0^r f(t-s,X_s)dr\right)u(t-r,X_r)\right]$$

for any $r\in[0,t]$.

Proof. To see this, let $r\in[0,t]$. Then

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\left(\int_0^r f(t-s,X_s)dr\right)u(t-r,X_r)\right]\\
&=\mathbb{E}_x\left[\exp\left(\int_0^r f(t-s,X_s)dr\right)\mathbb{E}_{X_r}\left[\exp\left(\int_r^t f(t-s,X_{s-r})ds\right)u_0(X_{t-r})\right]\right]\\
&=\mathbb{E}_x\left[\exp\left(\int_0^t f(t-s,X_s)dr\right)u_0(X_t)\right]
\end{aligned}$$

where in the first equality we apply equation (B.2) together with a change of variables in the exponential term, and in the final equality we use the Markov Property. □

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