Multiple zeta values and iterated Eisenstein integrals

Alex Saad
St Hilda's College
University of Oxford

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Abstract

The affine ring of the motivic path torsor ${}_0\Pi_1^{\mathrm{mot}} := \pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_0, -\vec{1}_1)$ is an ind-object in the Tannakian category $\mathsf{MT}(\mathbb{Z})$ of mixed Tate motives over the integers [16]. Its periods are $\mathbb{Q}[(2\pi i)^{\pm}]$-linear combinations of multiple zeta values (MZVs). Brown showed that $\mathcal{O}({}_0\Pi_1^{\mathrm{mot}})$ generates $\mathsf{MT}(\mathbb{Z})$ by exhibiting a specific basis for the $\mathbb{Q}$-vector space of motivic MZVs [5].

Brown also introduced a class of periods of fundamental groups called multiple modular values [7]. They are periods of the relative completion of the fundamental group of the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves [22]. Among such quantities are iterated integrals of Eisenstein series along elements of the topological fundamental group of $\mathcal{M}_{1,1}$ based at the tangential basepoint $\partial/\partial q$ at the cusp, which is isomorphic to $SL_2(\mathbb{Z})$.

In this thesis we prove that all motivic MZVs may be expressed as certain $\mathbb{Q}[2\pi i]$-linear combinations of motivic iterated Eisenstein integrals (Theorem 12.0.1). This uses a construction relating the (relative) de Rham fundamental groups of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and $\mathcal{M}_{1,1}$ via the de Rham fundamental group of the fiber $\mathcal{E}^{\times}_{\partial/\partial q}$ of the punctured Tate curve over $\partial/\partial q$. We explain how the coefficients in this linear combination may be partially determined using the Galois coaction on motivic periods.

As a consequence we also obtain a new Tannakian generator for $\mathsf{MT}(\mathbb{Z})$ constructed from the universal monodromy representation of the relative fundamental group of $\mathcal{M}_{1,1}$ on the fundamental group of $\mathcal{E}^{\times}_{\partial/\partial q}$ (Theorem 13.1.1).

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Multiple zeta values

Multiple zeta values (MZVs) are real numbers generalising the special values of the Riemann zeta function $\zeta(s)$ at positive integers. They are defined by nested series of the form

$$\zeta(k_1, \ldots, k_r) = \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},$$

where each of $k_1, \ldots, k_r$ is a positive integer with $k_r \geq 2$ to ensure convergence. The integer $k_1 + \cdots + k_r$ is called the *weight*, and the *depth* is r. The $\mathbb{Q}$-vector space spanned by all MZVs[1] is a $\mathbb{Q}$-subalgebra $\mathcal{Z} \subseteq \mathbb{R}$; for example, for integers $k, l \geq 2$ we have

$$\zeta(k)\zeta(l) = \zeta(k,l) + \zeta(l,k) + \zeta(k+l). \tag{1}$$

This formula can be checked directly from the definition above, and one may write down a general product formula (called the *stuffle* or *harmonic* product) by decomposing the product of two nested sums as a $\mathbb{Z}$-linear combination of similar nested sums. The weight and depth define filtrations on $\mathcal{Z}$ that are compatible with this product, and conjecturally the weight defines a grading on $\mathcal{Z}$.

The $\mathbb{Q}$-algebra $\mathcal{Z}$ is of significant arithmetic interest, even at the level of single zeta values (i.e. MZVs of depth 1). For example, Euler's classical formula for the even zeta values is

$$\zeta(2k) = -\frac{B_{2k}(2\pi i)^{2k}}{2(2k)!},$$

where $B_{2k} \in \mathbb{Q}$ is the $2k$th Bernoulli number. This implies that $\zeta(2k)$ is transcendental over $\mathbb{Q}$ and that all even zeta values are algebraically dependent, with $\zeta(2k) = b_k\zeta(2)^k$ for an explicit rational number b_k.

The situation is very different for the odd zeta values $\zeta(2k+1)$. In this case there are no analogous formulas, no algebraicity results and only very few rationality results. It is expected that the numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent

[1] Together with the "empty" zeta value $\zeta(\varnothing) := 1$ of depth $r = 0$.

over $\mathbb{Q}$, meaning that every odd zeta value should be a "new" transcendental number. This difference in behaviour between odd and even zeta values is a consequence of the interpretation of MZVs as periods of mixed Tate motives [5, 15, 16]. The even zeta values are periods of trivial extensions of Tate motives, while the odd zeta values are periods of nontrivial extensions.[2]

This leads us to consider a more geometric interpretation of MZVs as periods of the category $\mathsf{MT}(\mathbb{Z})$ of mixed Tate motives over the integers, which, by [5], is generated by the motivic fundamental path torsor of $\mathbb{P}^1 \setminus \{0, 1, \infty\} / \mathbb{Q}$. Every MZV may be expressed as an iterated integral on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ of length equal to its weight. For example, the single zeta values may be written

$$\zeta(k) = \int_{0<x_1<\cdots<x_k<1} \frac{dx_1}{1-x_1} \frac{dx_2}{x_2} \cdots \frac{dx_k}{x_k}. \tag{2}$$

This formula can be shown by expanding $(1 - x_1)^{-1}$ as a geometric series (which converges inside the domain of integration) and then integrating term-by-term to obtain the series $\zeta(k) = \sum_{n \geq 1} n^{-k}$. In the general case we may write

$$\zeta(k_1, \ldots, k_r) = \int_0^1 \omega_1 \omega_0^{k_1 - 1} \cdots \omega_1 \omega_0^{k_r - 1}, \tag{3}$$

where $\omega_0 = dz/z$ and $\omega_1 = dz/(1-z) \in \Omega^1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$.

The product of two such iterated integrals may be written as a $\mathbb{Z}$-linear combination of iterated integrals of greater length. This produces a different formula for the product of MZVs called the *shuffle* product. This, for example, gives the formula

$$\zeta(2)\zeta(2) = 2\zeta(2,2) + 4\zeta(1,3). \tag{4}$$

Comparing the formulae (1) and (4) produces the relation

$$4\zeta(1,3) = \zeta(4).$$

In general the comparison between the shuffle and stuffle product formulae produces a collection of $\mathbb{Z}$-linear relations in $\mathcal{Z}$ called the *double shuffle relations*. Conjecturally, an extended[3] version of the double shuffle relations is sufficient to describe all relations in $\mathcal{Z}$ [28].

[2] This behaviour also reflects the interpretation of $\zeta(s)$ as an L-function.
[3] This corresponds to regularising divergent sums and integrals.

Multiple modular values

Multiple modular values [7] are another interesting class of numbers. They are periods of the relative completion of the fundamental group of the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves [23]. The fundamental group of its space of complex points is isomorphic to the full modular group $SL_2(\mathbb{Z})$. As in the case of MZVs this means that multiple modular values are iterated integrals, this time on the space

$$\mathcal{M}_{1,1}(\mathbb{C}) = \text{(orbifold) quotient of the upper half plane } \mathfrak{H} \text{ by the action of } SL_2(\mathbb{Z}).$$

However, instead of taking integrals of the differential forms $\omega_0, \omega_1 \in \Omega^1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ we take iterated integrals of certain vector-valued differential forms constructed from modular forms for $SL_2(\mathbb{Z})$ (see (6.1)). The iterated integral expression equips the $\mathbb{Q}$-span M of multiple modular values with the structure of a $\mathbb{Q}$-algebra. This algebra is very rich, containing L-values of modular forms, amongst other quantities (e.g. [11, §7]).

Like $\mathcal{Z}$, the algebra M has many relations. In this case they arise most accessibly from the relations in $SL_2(\mathbb{Z})$, or equivalently from the transformation properties of modular forms. For example, the functional equation for (the special values of) the completed L-function of a cusp form f of weight $2k$ for $SL_2(\mathbb{Z})$ can be interpreted as a relation in M. Finding a minimal generating set of relations in M, as an analogue to the extended double shuffle equations in $\mathcal{Z}$, is an interesting problem. In §15.3 we consider a set of relations called the *cocycle equations* and demonstrate that these already determine many relations in M.

Main result

Our main result in this thesis implies that all multiple zeta values may be expressed as $\mathbb{Q}[2\pi i]$-linear combinations of multiple modular values of level 1.[4] More specifically, we show that every MZV of weight n and depth r can be expressed as $(2\pi i)^n$ times a $\mathbb{Q}$-linear combination of iterated integrals (along a specific element of $\pi_1(\mathcal{M}_{1,1}(\mathbb{C}), \partial/\partial q)$, where $\partial/\partial q$ is a tangential basepoint at the cusp) of level-1 Eisenstein series

$$\mathbb{G}_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n, \quad q := e^{2\pi i \tau}.$$

[4] In fact we prove a stronger statement, where periods are replaced by motivic periods [8]. The advantage of working with motivic periods is that they are equipped with an action of a "motivic" Galois group, with the caveat that this group is only known to be truly *motivic* (rather than Hodge-theoretic) when working with periods of mixed Tate motives. See §4.2.

Under the isomorphism $\pi_1(\mathcal{M}_{1,1}(\mathbb{C}), \partial/\partial q) \cong SL_2(\mathbb{Z})$, the element we integrate along corresponds to

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By definition, integrals along S are regularised integrals along the imaginary axis on the upper half plane $\mathfrak{H}$ (we describe this regularisation procedure in §6.1). Our main result, Theorem 12.0.1, then implies the following theorem.[5]

Theorem. *Every MZV of weight n and depth r can be expressed as a $\mathbb{Q}$-linear combination of regularised iterated integrals on $\mathfrak{H}$ of the form*

$$(2\pi i)^n \int_S \mathbb{G}_{2n_1+2}(\tau_1)\tau_1^{b_1} d\tau_1 \cdots \mathbb{G}_{2n_s+2}(\tau_s)\tau_s^{b_s} d\tau_s, \quad 0 \le b_i \le 2n_i,$$

where $s \le r$ and the total modular weight $N := (2n_1+2) + \cdots + (2n_s+2)$ is bounded above by $n+s$.

One hopes that this result may shed some light on difficult problems in the theory of multiple zeta values by placing them within a modular framework.

Examples

The simplest nontrivial example is the case of $\zeta(3)$. It is a multiple zeta value of weight 3 and depth 1, and can therefore be expressed as an iterated integral on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ of length 3. By (2) we have

$$\zeta(3) = \int_{0<x<y<z<1} \frac{dxdydz}{(1-x)yz}.$$

By our result it may be written as a *single* integral on $\mathfrak{H}$ of an Eisenstein series of weight at most $3+1=4$; the only option is $\mathbb{G}_4$. In Chapter 15 we show that $\int_S \mathbb{G}_4(\tau)\tau^2 d\tau = -\int_S \mathbb{G}_4(\tau)d\tau$ and that $(2\pi i)^3 \int_S^{\mathfrak{m}} \mathbb{G}_4(\tau)\tau d\tau$ is a rational multiple of $(2\pi i)^3$, and hence is imaginary. It follows that $\zeta(3)$ may be expressed as a rational multiple of $(2\pi i)^3 \int_S \mathbb{G}_4(\tau)d\tau$, and we only need to pin down the rational coefficient to obtain

$$\zeta(3) = -(2\pi i)^3 \int_S \mathbb{G}_4(\tau)d\tau.$$

This expresses $\zeta(3)$ as a rapidly converging Lambert series after expanding $\mathbb{G}_4$ in $q = \exp(2\pi i\tau)$. Similar formulae exist for all odd zeta values [3] – for example

$$\zeta(5) = -\frac{1}{12}(2\pi i)^5 \int_S \mathbb{G}_6(\tau)d\tau.$$

[5]This follows from Theorem 12.0.1 by applying the period map. Note that Theorem 12.0.1 is stated in the de Rham normalisation, which differs by a power of $2\pi i$.

Of course these formulae are well-known; they follow from the fact that the critical L-values of $\mathbb{G}_{2n+2}$ are $\zeta(2n+1)$ and powers of $2\pi i$. However, it is interesting to note that there *are* exact formulae for the odd zeta values if one allows for expressions involving modular forms.

A more involved combination occurs in depth 2. Brown gave the first example of an expression of this type for the value $\zeta(3,5)$ [11, Example 7.2]. His formula is[6]

$$\zeta(3,5) = -\frac{5}{12}(2\pi i)^8 \int_S \mathbb{G}_6(\tau_1)d\tau_1 \mathbb{G}_4(\tau_2)d\tau_2 + \frac{503}{2^{13}3^5 5^2 7}(2\pi i)^8. \tag{5}$$

This agrees with our result, which states that $\zeta(3,5)$ may be expressed as a linear combination of at most double Eisenstein integrals of total modular weight at most 10. Compare this to the expression (3) for $\zeta(3,5)$ as an iterated integral on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ of length 8.[7]

However, not all linear combinations of iterated Eisenstein integrals are equal to MZVs. The formula

$$\begin{aligned} &600\pi \int_S \mathbb{G}_4(\tau_1)\tau_1 d\tau_1 \mathbb{G}_{10}(\tau_2)\tau_2^4 d\tau_2 + 480\pi \int_S \mathbb{G}_4(\tau_1)\tau_1^2 d\tau_1 \mathbb{G}_{10}(\tau_2)\tau_2^3 d\tau_2 \\ &= \int_0^{i\infty} \Delta(\tau)\tau^{11} d\tau = \Lambda(\Delta, 12), \end{aligned}$$

given in [11, Example 7.3], exhibits a linear combination of iterated Eisenstein integrals equal to a *noncritical* L-value of the Ramanujan cusp form $\Delta \in S_{12}(SL_2(\mathbb{Z}))$. This multiple modular value is not expected to be a multiple zeta value. This formula is an example of a general principle, guided by Beilinson's conjectures [1], suggesting that noncriticial L-values should be periods of simple extensions of motives (i.e. mixed motives). This necessitates a deeper study of the underlying algebraic and geometric narratives behind these period algebras.

Algebraic structure

As indicated by the preceding examples, the subspace of M spanned by iterated Eisenstein integrals has a rich algebraic structure. The subspace of linear combinations equal to MZVs is spanned by periods[8] of a certain pro-nilpotent Lie subalgebra

[6] There is a difference in normalisation between Brown's formulae and ours; namely, we have $\Lambda(\mathbb{G}_{k_1},\ldots,\mathbb{G}_{k_s}; b_1,\ldots,b_s) = i^{b_1+\cdots+b_s}\int_S \mathbb{G}_{k_1}(\tau_1)\tau_1^{b_1-1}d\tau_1 \cdots \mathbb{G}_{k_s}(\tau_s)\tau_s^{b_s-1}d\tau_s$.

[7] In general the depth of an MZV is always smaller than its weight. Our result shows that there is a substantial simplification to the length filtration on iterated integrals in passing from $\mathbb{P}^1\setminus\{0,1,\infty\}$ to $\mathcal{M}_{1,1}$.

[8] Here we have slightly abused the concept of a period; to be more precise we are referring to periods of the affine ring of the associated pro-unipotent group scheme, which is an ind-object in the category of mixed Tate motives over $\mathbb{Z}$.

$\mathfrak{u}^{\mathrm{geom}} \subseteq \mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$. It is generated by derivations $\varepsilon^{\vee}_{2n+2}$, for each $n \geq 1$, which were originally studied by Tsunogai [52] in relation to the pro-ℓ fundamental group of a punctured elliptic curve. They are defined by the formulae

$$\varepsilon^{\vee}_{2n+2}(\mathsf{a}) = \mathrm{ad}(\mathsf{a})^{2n+2}(\mathsf{b})$$
$$\varepsilon^{\vee}_{2n+2}(\mathsf{b}) = \frac{1}{2}\sum_{i+j=2n+1}(-1)^i\left[\mathrm{ad}(\mathsf{a})^i(\mathsf{b}), \mathrm{ad}(\mathsf{a})^j(\mathsf{b})\right].$$

There are many arithmetic relations in $\mathfrak{u}^{\mathrm{geom}}$, some of which were studied by Pollack in his honours' thesis [44]. In each depth in the lower central series filtration on $\mathfrak{u}^{\mathrm{geom}}$ there is a family of relations whose coefficients are connected to period polynomials of cusp forms for $SL_2(\mathbb{Z})$. Some examples in depth 2 are

$$[\varepsilon^{\vee}_{10}, \varepsilon^{\vee}_{4}] - 3[\varepsilon^{\vee}_{8}, \varepsilon^{\vee}_{6}] = 0$$
$$2[\varepsilon^{\vee}_{14}, \varepsilon^{\vee}_{4}] - 7[\varepsilon^{\vee}_{12}, \varepsilon^{\vee}_{6}] + 11[\varepsilon^{\vee}_{10}, \varepsilon^{\vee}_{8}] = 0.$$

The Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ has a geometric interpretation. It is the Lie algebra of the image of the prounipotent radical of the relative de Rham fundamental group $\mathcal{G}^{\mathrm{dR}}_{1,1} := \pi_1^{\mathrm{rel,dR}}(\mathcal{M}_{1,1})$ under its monodromy representation on the de Rham fundamental group of the infinitesimal punctured Tate curve (see §3.4 and §8.2). The group $\mathcal{G}^{\mathrm{dR}}_{1,1}$ is generated by symbols corresponding to a basis for the space of modular forms for $SL_2(\mathbb{Z})$, tensored with all irreducible representations of the reductive group $SL_2/\mathbb{Q}$. The generators $\varepsilon^{\vee}_{2n+2}$ are the images of Eisenstein symbols, while the cuspidal symbols act trivially.

The Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ is a pro-object in the category $\mathsf{MT}(\mathbb{Z})$ of mixed Tate motives over the integers. The de Rham fiber functor $\omega^{\mathrm{dR}}\colon \mathsf{MT}(\mathbb{Z}) \to \mathsf{Vect}_{\mathbb{Q}}$, sending a mixed Tate motive to its de Rham realisation (algebraic de Rham cohomology), equips $\mathsf{MT}(\mathbb{Z})$ with the structure of a neutral Tannakian category over $\mathbb{Q}$ [15, 5]. Consequently, the *motivic Galois group* $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} := \mathrm{Aut}^{\otimes}_{\mathsf{MT}(\mathbb{Z})}(\omega^{\mathrm{dR}})$, and its unipotent radical $U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$, act on $\mathfrak{u}^{\mathrm{geom}}$.

As a corollary of our result, which crucially relies upon its validity at the level of motivic periods, we obtain the following theorem (Theorem 13.1.1):

Theorem. *The action of $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ on $\mathfrak{u}^{\mathrm{geom}}$ is faithful.*

This is a "modular" analogue of Brown's result [5], which implies that $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ acts faithfully on the motivic fundamental path torsor $\pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_0, -\vec{1}_1)$, or of Belyi's Theorem [2], which implies that $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the geometric fundamental group $\hat{\pi}_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}, b)$ for any rational basepoint b.

By the Tannakian formalism, Theorem 13.1.1 is equivalent to the statement that the full Tannakian subcategory of $\mathsf{MT}(\mathbb{Z})$ generated by $\mathfrak{u}^{\mathrm{geom}}$ is equivalent to $\mathsf{MT}(\mathbb{Z})$. This confirms a conjecture of Brown [7, Remark 14.6] suggesting that every mixed Tate motive over $\mathbb{Z}$ may be constructed from modular forms.

The faithfulness of the Galois action on $\mathfrak{u}^{\mathrm{geom}}$ is connected to the Pollack relations as follows. Let $\mathcal{E}^{\times}_{\partial/\partial q}$ be the fiber of the punctured Tate curve over the tangent vector $\partial/\partial q$ based at the cusp of $\mathcal{M}_{1,1}$ (see §3.4 and §3.5). It is equipped with a canonical tangential basepoint $\partial/\partial w$ at the punctured origin. The Lie algebra of its de Rham fundamental group $\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ is canonically isomorphic to the completed free Lie algebra $\mathrm{Lie}(\mathsf{a},\mathsf{b})^\wedge$.

Although $\mathcal{E}^{\times}_{\partial/\partial q}$ is not algebraic, its de Rham fundamental group (or, equivalently, the Lie algebra $\mathrm{Lie}(\mathsf{a},\mathsf{b})^\wedge$ of the latter) has a motivic structure, and $\mathrm{Lie}(\mathsf{a},\mathsf{b})^\wedge$ is a pro-object of $\mathsf{MT}(\mathbb{Z})$ [9, 27]. Consequently, it has an action by the Lie algebra $\mathfrak{k} = \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})})$, which is non-canonically isomorphic to the completed free Lie algebra on generators σ_{2n+1} for all $n \geq 1$ [5]. This action is described by a Lie algebra homomorphism

$$\rho\colon \mathfrak{k} \to \mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b}).$$

In [9], Brown proved that ρ is injective and that there is a choice of generators $\sigma_{2n+1} \in \mathfrak{k}$ that act to lowest order through $\mathfrak{u}^{\mathrm{geom}} \subseteq \mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$ via

$$\rho(\sigma_{2n+1}) \equiv \varepsilon^{\vee}_{2n+2} \pmod{W_{-2n-3}}, \tag{6}$$

where $W_\bullet\,\mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$ is the negative of the lower central series filtration. The image of ρ is contained in the normaliser of $\mathfrak{u}^{\mathrm{geom}}$ within $\mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$, and the $\mathfrak{k}$-action on $\mathfrak{u}^{\mathrm{geom}}$ is described by a homomorphism $\tilde\rho\colon \mathfrak{k} \to \mathrm{Der}(\mathfrak{u}^{\mathrm{geom}})$ satisfying $\tilde\rho(\sigma) = \mathrm{ad}(\rho(\sigma))$.

If $\mathfrak{u}^{\mathrm{geom}}$ were free, equation (6) would trivially imply that $\tilde\rho$ is injective (i.e. that the $\mathfrak{k}$-action on $\mathfrak{u}^{\mathrm{geom}}$ is faithful). However, the Pollack relations in $\mathfrak{u}^{\mathrm{geom}}$ prevent this, and can be viewed as a potential obstruction to a faithful Galois action arising from cusp forms. The situation is further complicated because the Pollack relations are connected to certain relations in the associated graded Lie algebra $\mathrm{gr}\,\mathfrak{k}$ with respect to the lower central series filtration [27].

Our result implies that $\tilde\rho$ is still injective despite the Pollack relation obstruction. This result can be viewed as "orthogonal" to a consequence of Oda's conjecture[9] [43], which implies a different injectivity result for $\mathfrak{k}$. This is discussed in §13.2.

[9]Now a theorem by [51, 5].

Context

There are close links between MZVs and other modular and elliptic periods. For example, every MZV may be written as a $\mathbb{Q}[\log(2)^{\pm}]$-linear combination of iterated integrals of certain weight 2 modular forms for $\Gamma_0(4)$ [10, Theorem 8.1].

In a different direction, Lochak-Matthes-Schneps [37] have shown that the algebra of MZVs is contained within the algebra of elliptic multiple zeta values modulo $2\pi i$. Elliptic multiple zeta values are functions on $\mathfrak{H}$ given by *indefinite* iterated Eisenstein integrals. They describe the variation of iterated integrals on elliptic curves as the elliptic curve is deformed. They have an expansion in $q = \exp(2\pi i\tau)$ whose constant term is a $\mathbb{Q}[(2\pi i)^{\pm}]$-linear combination of multiple zeta values. This fact is intimately connected to the fact that the fundamental group of the infinitesimal punctured Tate curve is mixed Tate.

Proof idea

In this section we explain the heuristics of the proof, leaving technicalities and precise definitions for the relevant sections within the main text.

Firstly, we must use a slightly modified moduli space $\mathcal{M}_{1,\vec{1}}$ classifying elliptic curves E together with a choice of nonzero tangent vector $\vec{v}$ at the origin. In contrast to $\mathcal{M}_{1,1}$, which is a Deligne-Mumford stack, the space $\mathcal{M}_{1,\vec{1}}$ is an affine scheme. Its associated analytic space is a complex manifold whose topological fundamental group is isomorphic to the braid group B_3 on three strands. Forgetting the tangential basepoint induces a morphism $\mathcal{M}_{1,\vec{1}} \to \mathcal{M}_{1,1}$ equipping $\mathcal{M}_{1,\vec{1}}$ with the structure of a principal $\mathbb{G}_m$-bundle over $\mathcal{M}_{1,1}$. This morphism induces a natural homomorphism of fundamental groups $B_3 \to SL_2(\mathbb{Z})$.

Aside from avoiding stacky technicalities, the benefits of working with $\mathcal{M}_{1,\vec{1}}$ are twofold. Firstly, the space of iterated integrals on $\mathcal{M}_{1,\vec{1}}$ is essentially the same as that on $\mathcal{M}_{1,1}$; the only extra quantities introduced are powers of $2\pi i$ (see (5.16)). Secondly, the addition of basepoint data to the moduli problem equips $\pi_1(E^{\times}, \vec{v})$ with a natural "monodromy" action by $\pi_1(\mathcal{M}_{1,\vec{1}}, b)$ for a particular choice of basepoint b (see §8.1). In particular, we can choose $E = \mathcal{E}_{\partial/\partial q}$ to be the fiber of the Tate curve over $\partial/\partial q$ and $\vec{v}$ to be the tangential basepoint $\partial/\partial w$ at the origin, defined in §3.5.2. Associated to these data is a canonical tangential basepoint $\vec{\mathsf{v}}$ on $\mathcal{M}_{1,\vec{1}}$ defined in §3.5.3. The monodromy action is then an action

$$\pi_1(\mathcal{M}_{1,\vec{1}}, \vec{\mathsf{v}}) \times \pi_1(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w) \to \pi_1(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$$

that may be described with generators and relations. The group $\pi_1(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ is free on two generators α and β. The group $\pi_1(\mathcal{M}_{1,\vec{1}}, \vec{\mathsf{v}})$ is generated by certain elements of the braid group, denoted $\tilde{S}$ and $\tilde{T}$. Under the map $B_3 \to SL_2(\mathbb{Z})$ these are sent to the well-known generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

respectively. The elements $\tilde{S}$ and $\tilde{T}$ act on α and β via combinations of Dehn twists, and this action may be computed explicitly. In particular we have

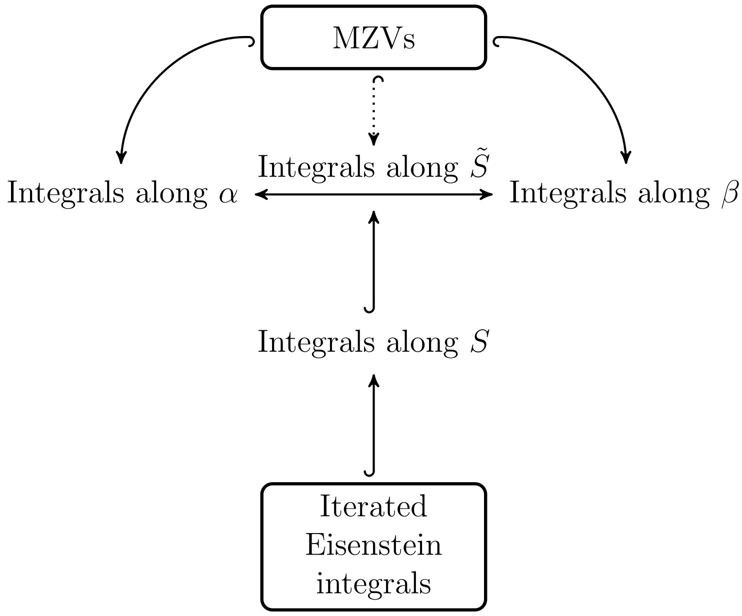
$$\tilde{S}(\beta) = \alpha^{-1}. \tag{7}$$

The importance of this equation is that it relates the iterated integrals on the infinitesimal punctured Tate curve along the path α to those along β via the iterated integrals on $\mathcal{M}_{1,\vec{1}}(\mathbb{C})$ along the path $\tilde{S}$.

We show in Chapter 11 that all MZVs occur as iterated integrals along both α and β, but that their distributions within these two spaces of iterated integrals differ. Equation (7) induces an equation on spaces of iterated integrals, and by carefully filtering this equation we show that the "filtered difference" between the spaces of iterated integrals on β and on α^{-1} is contained in the space of iterated integrals on $\tilde{S}$. In this way we are able to show that all MZVs occur as iterated integrals of modular forms along the element $\tilde{S}$.

Finally we must show that the MZVs within the space of iterated integrals along $\tilde{S}$ are contained within the subspace of iterated integrals of Eisenstein series. This uses the structure of the relative completion of $SL_2(\mathbb{Z})$ and its monodromy action; in particular, it uses the fact that the cuspidal generators act trivially (see Proposition 9.2.1).

The heuristics of the proof are summarised in the following diagram:



In order to formalise this argument and to work rigorously with the "space of iterated integrals along a path", we make judicious use of the notion of the (relative) de Rham fundamental group of a scheme[10] $X/\mathbb{Q}$. This is an affine group scheme over $\mathbb{Q}$. The points of its unipotent radical with values in the algebra $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ of motivic $\mathcal{H}$-periods receive a map[11] from the fundamental group of $X(\mathbb{C})$. Suitably interpreted, this map sends $\gamma \in \pi_1(X(\mathbb{C}))$ to a noncommutative formal generating series for motivic iterated integrals of algebraic forms on X along γ (i.e. the motivic parallel transport along γ of a suitable universal connection on X). Below we define the main spaces, topological paths and generating series of periods used in the argument.

- Let $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$. The straight line path dch between the tangential basepoints at 0 and 1 is mapped to the *Drinfeld associator* $\Phi^{\mathfrak{m}}_{01} = \sum_w \zeta^{\mathfrak{m}}(w)w$, where w ranges over all words in the letters $\mathsf{x}_0, \mathsf{x}_1$.

- Let $X = \mathcal{E}^{\times}_{\partial/\partial q}$. The two generators α, β for $\pi_1(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ are mapped to power series $\alpha^{\mathfrak{m}}, \beta^{\mathfrak{m}}$ in the letters a, b. They are products of exponentials and Drinfeld associators.

- Let $X = \mathcal{M}_{1,1}$. The element S of $\pi_1(\mathcal{M}_{1,1}, \partial/\partial q)$ is mapped to a power series $\mathcal{C}^{\mathfrak{m}}_S$ in symbols corresponding to a basis for modular forms for $SL_2(\mathbb{Z})$ tensored

[10]On occasion, we also consider the relative de Rham fundamental group of a Deligne-Mumford stack.

[11]When the relative de Rham fundamental group is itself unipotent, this map is a homomorphism. In the general case it is a cocycle for $\pi_1(X(\mathbb{C}))$.

with SL_2-representations. It is the value at S of a "canonical cocycle" for $\pi_1(\mathcal{M}_{1,1}, \partial/\partial q)$ taking values in the unipotent radical of the relative completion of $\pi_1(\mathcal{M}_{1,1}, \partial/\partial q)$, and by definition its coefficients are motivic multiple modular values. See [7, Definition 15.4].

- Let $X = \mathcal{M}_{1,\vec{1}}$. Under the map from $\pi_1(\mathcal{M}_{1,\vec{1}}, \vec{\mathsf{v}})$ into the unipotent radical of its relative de Rham completion, the element $\tilde{S}$ is mapped to a series Ψ called the *modular inverter*. It is equal to $\exp(\eta \mathbf{e}_2)\mathcal{C}_S^{\mathfrak{m}}$. The symbol $\mathbf{e}_2$ is dual to the Eisenstein series $\mathbb{G}_2$ of weight 2 (a quasimodular form for $SL_2(\mathbb{Z})$ of weight 2); η is a motivic period whose value we compute as $\eta = \mathbb{L}/8$ in Corollary 14.4.2, where $\mathbb{L} = (2\pi i)^{\mathfrak{m}}$ is the motivic period analogue of $2\pi i$.

The relationship $\tilde{S}(\beta) = \alpha^{-1}$ induces an equation between these power series:[12]

$$\mu(\Psi)(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})) = (\alpha^{\mathfrak{m}})^{-1}. \tag{8}$$

Here μ is a morphism of group schemes called the monodromy morphism (see §8.2). Its image is a subgroup of the automorphism group of the de Rham fundamental group of $\mathcal{E}^{\times}_{\partial/\partial q}$ consisting of noncommutative power series in elements of $\mathfrak{u}^{\mathrm{geom}}$ together with the additional central derivation $\varepsilon_2 = -\operatorname{ad}([\mathsf{a}, \mathsf{b}])$. Since μ kills cuspidal symbols (Lemma 9.2.1), the coefficients of the series $\mu(\Psi)$ are *a priori* motivic iterated Eisenstein integrals. Moreover, these coefficients are contained within the mixed Tate subalgebra $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} \subseteq \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ because the image of μ is a pro-object in $\mathsf{MT}(\mathbb{Z})$ (Proposition 9.5.2). In other words, $\mu(\Psi)$ is a generating series for all linear combinations of motivic iterated Eisenstein integrals equal to $\mathbb{Q}[\mathbb{L}^{\pm}]$-linear combinations of motivic MZVs.

Our goal is to use (8) to understand the coefficients of $\mu(\Psi)$ in terms of those of $S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})$ and $(\alpha^{\mathfrak{m}})^{-1}$. We show that the coefficients of these series are motivic MZVs, and that *all* motivic MZVs occur. To do this formally we introduce the notion of a coefficient space (Definition 7.3.1). We exhibit explicit bounds for the growth of these coefficient spaces in terms of natural filtrations defined in terms of the degrees in a and b (Lemmas 11.4.1 and 11.4.2).

We then compare the difference in the filtered coefficient spaces of $S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})$ and $(\alpha^{\mathfrak{m}})^{-1}$ with respect to the filtration by b-degree. By equation (8), the difference is contained within the filtered coefficient space of $\mu(\Psi)$ with respect to the length filtration. In this way we are able to show that all motivic MZVs occur within the coefficient space of $\mu(\Psi)$. To complete the proof we must relate the MZV weights

[12] The automorphism $S_0^{\mathfrak{m}} \colon (\mathsf{a}, \mathsf{b}) \mapsto (-\mathbb{L}^{-1}\mathsf{b}, \mathbb{L}\mathsf{a})$ in (8) is the degree-0 part of the image of $\tilde{S}$ in the relative fundamental group.

with the modular weights, which is done in Theorem 12.0.1 by comparing the Hodge filtrations on fundamental groups.

Coefficients

Our proof is nonconstructive; we are able to show that every motivic MZV may be written as a linear combination of iterated Eisenstein integrals, and moreover can provide certain constraints on the allowed lengths and modular weights of these integrals, but in general we cannot determine the exact expression using Theorem 12.0.1 alone. Nevertheless, in Chapter 15 we show that some coefficients in this linear combination may be determined using additional information coming from f-alphabet decompositions of motivic periods [8, §5.4] [7, §22].

As an example, consider the formula (5) expressing $\zeta^{\mathfrak{m}}(3,5)$ as a double Eisenstein integral. The coefficient $-5/12$ of the longest word in this expression may be computed as follows: Theorem 12.0.1 implies that $\zeta^{\mathfrak{m}}(3,5)$ may be written as a linear combination of motivic iterated Eisenstein integrals of length at most 2 and total modular weight at most 10. There are many possible iterated Eisenstein integrals satisfying these criteria, but the space they span in fact has many relations. By a computation given in Example 15.4.3, we are able to show that

$$\zeta^{\mathfrak{m}}(3,5) = A\mathbb{L}^6 I^{0,0}_{6,4} + B\mathbb{L}^8, \tag{9}$$

where $A, B \in \mathbb{Q}$ are the coefficients to be determined and where $I^{b_1,\dots,b_s}_{2n_1+2,\dots,2n_s+2}$ is the motivic analogue of the iterated integral

$$\begin{aligned}&\int_S \mathbb{G}_{2n_1+2}(q_1)\log(q_1)^{b_1}\frac{dq_1}{q_1}\cdots\mathbb{G}_{2n_s+2}(q_s)\log(q_s)^{b_s}\frac{dq_s}{q_s}\\ =&(2\pi i)^{b_1+\cdots+b_s+s}\int_S \mathbb{G}_{2n_1+2}(\tau_1)\tau_1^{b_1}d\tau_1\cdots\mathbb{G}_{2n_s+2}(\tau_s)\tau_s^{b_s}d\tau_s.\end{aligned}$$

To leading order in the coradical filtration [8, §2.5, §3.8] the f-alphabet decomposition of $\zeta^{\mathfrak{m}}(3,5)$ is $-5f_5f_3$ [6, §3]. By Lemma 15.2.2 the f-alphabet decomposition of $I^{0,0}_{6,4}$ is $12\mathbb{L}^{-6}f_5f_3$. Comparing coefficients in (9) determines $A = -5/12$. The coefficient B cannot be determined in this manner because of an inherent ambiguity associated with splitting the coradical filtration.

This computation is covered in Chapter 15, together with the outline of a general procedure for determining coefficients. We also pose some open problems as avenues of potential future work in §15.4.1. One broad question in this area asks for a sort of "converse" to Theorem 12.0.1: when is a linear combination of iterated Eisenstein integrals — or even a single iterated Eisenstein integral of the form $I^{b_1,\ldots,b_s}_{2n_1+2,\ldots,2n_s+2}$ — a period of a mixed Tate motive? We propose some conjecturally sufficient conditions, although the problem is in fact rather subtle. We also consider how Theorem 12.0.1 may be used to find (potentially) new relations in the algebra M of multiple modular values.

Chapter 1

Notation and conventions

The purpose of this chapter is to establish conventions and notation in any cases where these may be nonstandard.

1.1 Semidirect products

Let G and Π be groups, and suppose that G acts on Π on the *left* via $\pi \mapsto g(\pi)$. The semidirect product $\Pi \rtimes G$ with respect to this action is the group with underlying set $\Pi \times G$ and product

$$(\pi_1, g_1)(\pi_2, g_2) = (\pi_1 g_1(\pi_2), g_1 g_2).$$

Let us suppose instead that G acts on Π on the *right* via $\pi \mapsto \pi|_g$. The semidirect product $G \ltimes \Pi$ with respect to this action is the group with underlying set $G \times \Pi$ and product

$$(g_1, \pi_1)(g_2, \pi_2) = (g_1 g_2, \pi_1|_{g_2} \pi_2).$$

We also use this notation when G and Π are affine group schemes over a field K.

1.2 Fundamental groups

We use the topologists' convention regarding path multiplication in fundamental groups. If α, β are two elements in a fundamental group, this means that the product $\alpha\beta$ is homotopic to the result of first traversing α, and then traversing β.

1.3 Hopf algebras

If H is a complete Hopf algebra over a field of characteristic zero, let

$$\mathcal{G}(H) = \{x \in H : x \text{ invertible and } \Delta(x) = x \otimes x\}$$

be its group of grouplike elements.

1.3.1 Specific examples

We record two general cases of (complete) Hopf algebras that are widely used in this thesis.

1.3.1.1 The shuffle algebra

Let Z be a set. We define the *shuffle algebra*[1] or *tensor coalgebra* $T^c(Z)$ on Z to be the $\mathbb{Q}$-span of all words $w = z_1 \cdots z_n$ where each $z_k \in Z$, including the empty word $\varnothing$. It is a commutative $\mathbb{Q}$-algebra equipped with the shuffle product $\sqcup\!\sqcup$, where the element $1 \in \mathbb{Q}$ corresponds to the empty word in the alphabet Z. It is naturally graded by the length of words, and we denote the subspace of length n words by $T^c(Z)_n$. It has an increasing length filtration $L_\bullet T^c(Z)$ defined by $L_r T^c(Z) := \bigoplus_{n=0}^{r} T^c(Z)_n$.

It can be given the structure of a Hopf algebra over $\mathbb{Q}$ when equipped with the deconcatenation coproduct

$$\Delta(z_1 \cdots z_n) = \sum_{i=0}^{n} z_1 \cdots z_i \otimes z_{i+1} \cdots z_n,$$

the antipode given by signed reversal $S(z_1 \cdots z_n) = (-1)^n z_n \cdots z_1$, and the counit ε sending all nonempty words to 0.

We occasionally denote the element $z_1 \cdots z_n$ by $[z_1|\cdots|z_n]$ when we wish to emphasise the connection to elements of the reduced bar construction, discussed in [21, §7]. We discuss this further in §5.2.6.

1.3.1.2 The Hopf algebra of noncommutative formal power series

Let Z be a set and R a $\mathbb{Q}$-algebra. The ring $R\langle\langle Z\rangle\rangle$ of formal power series with noncommuting indeterminates in Z consists all formal power series with coefficients in R whose indeterminates are words in the elements of Z, equipped with the (noncommutative) concatenation product. It is the I-adic completion of the free associative algebra $R\langle Z\rangle$ on Z, where $I = (Z)$ is the maximal ideal of $R\langle Z\rangle$ generated by the elements of Z.

The ring $R\langle\langle Z\rangle\rangle$ has the structure of a complete Hopf algebra over $\mathbb{Q}$ when equipped with the coproduct for which elements of Z are primitive. The antipode and counit are defined as in §1.3.1.1. See [45, Appendix A2] for a detailed treatment.

[1] In the literature this is sometimes denoted $\mathbb{Q}\langle Z\rangle$, though this can be confused with the free associative algebra on Z.

1.3.1.3 Duality

Let us assume that $Z = \bigcup_{k\geq 1} Z_k$ is a graded set with each Z_n finite. This induces a canonical grading on the free associative algebra $R\langle Z\rangle$, denoted by $R\langle Z\rangle = \bigoplus_{n\geq 0} R\langle Z\rangle_n$. The R-module $R\langle Z\rangle_n$ consists of all words of total weight n, where elements of Z_k are assigned weight k. By the assumption on Z, each $R\langle Z\rangle_n$ is a finite-rank R-module and has a canonical basis. For example, we have $R\langle Z\rangle_0 = R$; $R\langle Z\rangle_1$ has the basis Z_1; $R\langle Z\rangle_2$ has the basis $\{w_1w_2 : w_1, w_2 \in W_1\} \cup W_2$, etc.

'The dual $R\langle Z\rangle_n^\vee = \mathrm{Hom}(R\langle Z\rangle_n, R)$ has the same (finite) rank as $R\langle Z\rangle_n$. Since $R\langle Z\rangle_n$ has a canonical choice of basis, so does $R\langle Z\rangle_n^\vee$. It consists of linear functionals selecting the coefficient of the appropriate word. Note that $R\langle Z\rangle_0^\vee = R$.

This allows us to speak of the *dual basis*, which is the basis of the graded dual

$$R\langle Z\rangle^* := \bigoplus_{n\geq 0} R\langle Z\rangle_n^\vee$$

of $R\langle Z\rangle$ dual to the natural basis just described. If w denotes a word in the alphabet Z, let $w^\vee$ denote the associated dual basis element.

For each $n \geq 0$ there is a decreasing length filtration $L^\bullet R\langle Z\rangle_n$, where $L^r R\langle Z\rangle_n$ is spanned by words in Z of length at least r and total weight equal to n. Dually, define an increasing length filtration $L_\bullet R\langle Z\rangle_n^\vee$ by

$$L_r R\langle Z\rangle_n := \left\{f\colon R\langle Z\rangle_n \to R : f(L^{r+1}R\langle Z\rangle_n) = 0\right\}.$$

This defines a natural increasing length filtration $L_\bullet R\langle Z\rangle^*$ on the graded dual by

$$L_r R\langle Z\rangle^* := \bigoplus_{n\geq 0} L_r R\langle Z\rangle_n^\vee.$$

Let $\mathcal{B}_n$ be the canonical basis for $L_1 R\langle Z\rangle_n^\vee$. It consists of the linear functionals selecting coefficients of elements of Z_n. The *dual alphabet* is defined as

$$Z^\vee := \bigcup_{n\geq 1} \mathcal{B}_n.$$

The graded dual $R\langle Z\rangle^*$ is isomorphic to the shuffle algebra $T^c(Z^\vee)$ by the isomorphism identifying the dual basis element $w^\vee \in T^c(Z^\vee)$ with the "coefficient of w" linear functional. The shuffle product on $T^c(Z^\vee)$ is dual to the coproduct on $R\langle Z\rangle$ for which elements of Z are primitive. The filtration $L_\bullet R\langle Z\rangle^*$ coincides with the length filtration $L_\bullet T^c(Z^\vee)$ defined in §1.3.1.1, with Z replaced by $Z^\vee$.

The functor sending a $\mathbb{Q}$-algebra R to the group $\mathcal{G}(R\langle\langle Z\rangle\rangle)$ defines an affine group scheme over $\mathbb{Q}$. Its coordinate ring is isomorphic to $T^c(Z^\vee)$. The isomorphism

$\mathrm{Hom}(T^c(Z^\vee), R) \xrightarrow{\sim} \mathcal{G}(R\langle\langle Z\rangle\rangle)$ is given by sending a homomorphism $s\colon T^c(Z^\vee) \to R$ to the series

$$\sum_{w\in M(Z)} s(w^\vee)w,$$

where $M(Z)$ denotes the free monoid on Z. One verifies that this series is grouplike using the fact that s is a homomorphism for the shuffle product.

1.4 Iterated integrals

Let M be a differentiable manifold and let $\omega_1, \ldots, \omega_n$ be smooth 1-forms on M. Let $\gamma\colon [0,1] \to M$ be a smooth path. Since $\Omega^1([0,1]) = \mathcal{O}([0,1])dt$, it follows that $\gamma^*(\omega_k) = f_k(t)dt$ for some $f_k \in \mathcal{O}([0,1])$.

Definition 1.4.1. The iterated integral of the forms $\omega_1, \ldots, \omega_n$ along γ is defined as

$$\int_\gamma \omega_1 \ldots \omega_n := \int_{0\le t_1\le\cdots\le t_n\le 1} f_1(t_1)dt_1 \cdots f_n(t_n)dt_n.$$

When M is a curve, such integrals may be regularised with respect to tangential basepoints. In §6.1 we provide a brief description of how to do this explicitly when $M = \mathcal{M}_{1,1}(\mathbb{C})$.

Chapter 2

Modular background

In this chapter we review basic background material on the modular group, modular forms and representations of the reductive group $SL_2/\mathbb{Q}$. This material is essential to work with the relative completion of $SL_2(\mathbb{Z})$, defined in Chapter 5.

2.1 The modular group

For us the modular group $SL_2(\mathbb{Z}) = \{\gamma \in GL_2(\mathbb{Z}) : \det(\gamma) = 1\}$ plays a fundamental role because it may be identified with the orbifold fundamental group of the moduli space of elliptic curves $\mathcal{M}_{1,1}$, whose space of complex points is the orbifold quotient $[SL_2(\mathbb{Z})\backslash\mathfrak{H}]$ (we recall these notions in Chapter 3). In this chapter we record some basic group-theoretic facts about $SL_2(\mathbb{Z})$, referring to §5.1.1.3 for details on its interpretation as a fundamental group.

2.1.1 Group theory

It is well known that $SL_2(\mathbb{Z})$ is generated by the two matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The matrix S has order 4, while T generates an infinite cyclic subgroup of $SL_2(\mathbb{Z})$. Setting $U := ST$ gives the presentation [48]

$$SL_2(\mathbb{Z}) = \langle S, U | S^2 = U^3, S^4 = \mathrm{id}\rangle. \tag{2.1}$$

2.1.2 Action on $\mathfrak{H}$

A crucial feature of $SL_2(\mathbb{Z})$ is its action on the upper half plane $\mathfrak{H} := \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$ by fractional linear transformations. For a matrix $\gamma \in SL_2(\mathbb{Z})$ as above, and a point

$\tau \in \mathfrak{H}$, this action is defined by

$$\gamma(\tau) := \frac{a\tau + b}{c\tau + d}. \tag{2.2}$$

The matrix $-I \in SL_2(\mathbb{Z})$ acts trivially, and the quotient $PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z})/\{\pm I\}$ acts faithfully on $\mathfrak{H}$. The generators $S, T \in SL_2(\mathbb{Z})$ act via

$$S(\tau) = -\frac{1}{\tau}, \quad T(\tau) = \tau + 1.$$

This action naturally extends to $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$. One observes that S stabilises the point $\tau = i$, U stabilises the point $\tau = \rho = \exp(2\pi i/3)$, and T stabilises the *cusp* $\tau = i\infty$. The stabilisers of these points are

$$SL_2(\mathbb{Z})_i = \langle S \rangle \cong \mathbb{Z}/4\mathbb{Z}, \quad SL_2(\mathbb{Z})_\rho = \langle U \rangle \cong \mathbb{Z}/6\mathbb{Z}, \quad SL_2(\mathbb{Z})_\infty = \langle \pm T \rangle \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

2.2 Representations of SL_2

A crucial object in this paper is the relative completion of $SL_2(\mathbb{Z})$ relative to the inclusion $SL_2(\mathbb{Z}) \hookrightarrow SL_2(\mathbb{Q})$, which is discussed in detail in §5.2. The relative completion is in part constructed from the irreducible representations of the reductive group $SL_2/\mathbb{Q}$. For this reason we give a basic account of the representation theory of SL_2 here.

For $n \geq 0$, let V_n be the $\mathbb{Q}$-vector space of homogeneous polynomials in X and Y of degree n. It is equipped with a *right* action of the group scheme SL_2 as follows. Let R be a $\mathbb{Q}$-algebra, let $p(\mathsf{X}, \mathsf{Y}) \in V_n \otimes_{\mathbb{Q}} R$, and let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R).$$

Then γ acts on $p(\mathsf{X}, \mathsf{Y})$ via

$$p(\mathsf{X}, \mathsf{Y})|_\gamma = p(a\mathsf{X} + b\mathsf{Y}, c\mathsf{X} + d\mathsf{Y}).$$

The collection $\{V_n : n \geq 0\}$ comprises all irreducible representations of SL_2. The representation V_n can be identified with the nth symmetric power $\operatorname{Sym}^n H$ of the standard representation H of SL_2.

2.2.1 Tensor products

Let $m, n \geq 0$. There is a natural isomorphism of rational SL_2-representations

$$V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{|m-n|}.$$

The projection $V_m \otimes V_n \to V_{m+n}$ is given by multiplication of homogeneous polynomials.

2.3 Modular forms

Our main result concerns iterated integrals of *Eisenstein series*, which are examples of modular forms for $SL_2(\mathbb{Z})$. We therefore briefly recall the definition and basic properties of modular forms in this section. Note that we always consider modular forms of *level* 1 – i.e. modular forms for the full modular group.

Definition 2.3.1 (Modular form)**.** Let $k \geq 0$ be an integer. A modular form for $SL_2(\mathbb{Z})$ of weight k is a holomorphic function $f\colon \mathfrak{H} \to \mathbb{C}$ such that for all $\gamma \in SL_2(\mathbb{Z})$ we have

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau),$$

and such that f is holomorphic at $\tau = i\infty$.

Taking $\gamma = -I$ and applying the modular transformation property shows that $f(\tau) = f(-I(\tau)) = (-1)^k f(\tau)$. This implies that all modular forms of odd weight are 0. Hence the weight is always taken to be even, and we write it as $2k$.

The equation $f(\tau + 1) = f(\tau)$, which follows from the modular transformation property applied to the matrix $\gamma = T$, implies that f has a Fourier expansion in the variable $q = \exp(2\pi i\tau)$. Since f is holomorphic everywhere, including at $\tau = i\infty$ (where $q = 0$), this Fourier expansion may be written as a power series

$$f(\tau) = \sum_{n\geq 0} a_n q^n \tag{2.3}$$

involving only nonnegative powers of q. When $a_0 = 0$, f is called a *cusp form* (because it vanishes at the cusp $\tau = i\infty$ corresponding to $q = 0$).

The $\mathbb{Q}$-vector space of modular forms for $SL_2(\mathbb{Z})$ of weight $2k$ and rational Fourier coefficients is denoted by $M_{2k}(SL_2(\mathbb{Z}))$, and the subspace of cusp forms of weight $2k$ is denoted by $S_{2k}(SL_2(\mathbb{Z}))$. A basic fact in the theory of modular forms is that $M_{2k}(SL_2(\mathbb{Z}))$ is finite-dimensional. We also define

$$M(SL_2(\mathbb{Z})) := \bigoplus_{k\geq 0} M_{2k}(SL_2(\mathbb{Z})).$$

Pointwise multiplication equips $M(SL_2(\mathbb{Z}))$ with the structure of a weight-graded $\mathbb{Q}$-algebra.

2.3.1 Eisenstein series

The simplest examples of modular forms to write down are *Eisenstein series.* For each even integer $2k \geq 4$ there is a unique (up to normalisation) Eisenstein series $\mathbb{G}_{2k} \in M_{2k}(SL_2(\mathbb{Z}))$, and $M_{2k}(SL_2(\mathbb{Z})) \cong S_{2k}(SL_2(\mathbb{Z})) \oplus \mathbb{Q}\mathbb{G}_{2k}$. Eisenstein series have an explicit q-expansion, given as follows:

Definition 2.3.2 (Eisenstein series)**.** The Hecke-normalised Eisenstein series of weight $2k \geq 4$ is the modular form $\mathbb{G}_{2k} \in M_{2k}$ with q-expansion

$$\mathbb{G}_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n\geq 1} \sigma_{2k-1}(n) q^n. \tag{2.4}$$

Here $B_k \in \mathbb{Q}$ is the kth Bernoulli number and $\sigma_r(n) := \sum_{d|n} d^r \in \mathbb{Z}$.

The Eisenstein series $\mathbb{G}_4$ and $\mathbb{G}_6$ are of particular importance because they freely generate the graded ring $M(SL_2(\mathbb{Z}))$. In other words, there is an isomorphism of graded rings $M(SL_2(\mathbb{Z})) \cong \mathbb{Q}[\mathbb{G}_4, \mathbb{G}_6]$, where $\mathbb{G}_4$ is assigned weight 4 and $\mathbb{G}_6$ weight 6.

2.3.1.1 The Eisenstein series of weight 2

The q-series (2.4) is equally valid for $2k = 2$, but for convergence reasons the resulting function $\mathbb{G}_2$ is not modular. It is a *quasimodular* form [54, §5.3], and satisfies the transformation property

$$\mathbb{G}_2(\gamma(\tau)) = (c\tau + d)^2 \mathbb{G}_2(\tau) + \frac{ic(c\tau + d)}{4\pi}.$$

In other words, the failure of modularity is measured by a polynomial in τ of degree 1. This is relevant for calculations in §14.5. More detailed information on $\mathbb{G}_2$ can be found in [54, §2.3].

Chapter 3

Geometric background

In this chapter we recall the main geometric objects used in the thesis: $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, the moduli spaces $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,\vec{1}}$ and the infinitesimal punctured Tate curve $\mathcal{E}^{\times}_{\partial/\partial q}$.

3.1 The scheme $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

The thrice-punctured projective line is the affine scheme

$$\mathbb{P}^1 \setminus \{0, 1, \infty\} = \operatorname{Spec} \mathbb{Z}[t, t^{-1}, (1-t)^{-1}].$$

It is defined over $\mathbb{Z}$. It represents the moduli problem assigning to a scheme X the set of global sections $u \in \mathcal{O}_X(X)$ such that both u and $1-u$ are units. Its space of complex points is a genus 0 surface punctured at 0, 1 and ∞.

The scheme $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is also the moduli space $\mathcal{M}_{0,4}$ of genus 0 curves with 4 marked points. This is because for any such 4-marked curve $(C; c_1, \dots, c_4)$ there is a unique choice of embedding into $\mathbb{P}^1$ sending $\{c_1, c_2, c_3\}$ to $\{0, 1, \infty\}$; this embedding sends the fourth marked point c_4 to a point on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ corresponding to the moduli point of $(C; c_1, \dots, c_4)$ on $\mathcal{M}_{0,4}$.

Let $\operatorname{Aut}(\mathbb{P}^1, \{0, 1, \infty\})$ denote the group of automorphisms of $\mathbb{P}^1$ that preserve the set $\{0, 1, \infty\}$. Restricting such an isomorphism to the subset $\{0, 1, \infty\}$ determines an isomorphism

$$\operatorname{Aut}(\mathbb{P}^1, \{0, 1, \infty\}) \xrightarrow{\sim} \operatorname{Aut}(\{0, 1, \infty\}) \cong S_3.$$

The group S_3 is generated by two transpositions: namely, the transposition swapping 0 and ∞ and the transposition swapping 0 and 1. On $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, these correspond to the mappings $t \mapsto t^{-1}$ and $t \mapsto 1-t$ respectively.

3.2 The moduli spaces $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,\vec{1}}$

The moduli stack of elliptic curves is the stack $\mathcal{M}_{1,1}$ over $\mathrm{Spec}(\mathbb{Z})$ whose points $\mathcal{M}_{1,1}(X)$ parametrise isomorphism classes $[E,O]$ of elliptic curves (E,O) over a scheme X [30]. Here $E \to X$ is an elliptic curve over X and $O\colon X \to E$ is a distinguished section of the structure morphism.

Consider the functor $F\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ that assigns to a scheme X the set of isomorphism classes $[E,O,\omega]$ of triples (E,O,ω), where:

- (E,O) is an elliptic curve over X, as above;
- $\omega \in \Omega^1_{E/X}(E)$ is a nonzero differential.

Over $S = \mathrm{Spec}\,\mathbb{Z}[1/6]$, this functor is representable by the affine scheme

$$\mathcal{M}_{1,\vec{1}} := \mathrm{Spec}\,\mathbb{Z}[1/6][u,v,\Delta^{-1}], \quad \text{where } \Delta = u^3 - 27v^2,$$

because every isomorphism class $[E,O,\omega] \in F(X)$ may be represented locally over X by a Weierstrass equation

$$E : y^2 = 4x^3 - g_2 x - g_3 \tag{3.1}$$

where g_2, g_3 are local sections of $\mathcal{O}_X$ with $g_2^3 - 27v^2$ invertible. This Weierstrass equation determines a morphism $X \to \mathcal{M}_{1,\vec{1}}$ via $u \mapsto g_2, v \mapsto g_3$ as well as determining $\omega = dx/y$ and O uniquely [22, §8.3].

Morally, the functor F is representable because the triple (E,O,ω) has no automorphisms; the only allowed changes of variables to the Weierstrass equation (3.1) are $(x,y) \mapsto (\lambda^2 x, \lambda^3 y)$ for $\lambda \in \overline{\mathbb{Q}}^{\times}$ [50, §III.1]. Such an isomorphism sends ω to $\lambda^{-1}\omega$. Thus ω detects automorphisms of (E,O), which means that the only automorphism of the triple (E,O,ω) is the identity (i.e. $\lambda = 1$). Fixing a choice of ω has *rigidified* the moduli problem defining $\mathcal{M}_{1,1}$.

There is a natural embedding $\mathcal{M}_{1,\vec{1}} \hookrightarrow \mathbb{A}^2$. The multiplicative group $\mathbb{G}_m$ acts on $\mathbb{A}^2$ via $\lambda \cdot (u,v) = (\lambda^4 u, \lambda^6 v)$. The formula $\lambda \cdot \Delta = \lambda^{12}\Delta$ implies that the $\mathbb{G}_m$-action restricts to an action on $\mathbb{A}^2 \backslash \Delta^{-1}(0) = \mathcal{M}_{1,\vec{1}}$.

We may then write $\mathcal{M}_{1,1}$ as the stack quotient $\mathcal{M}_{1,1} \cong [\mathbb{G}_m \backslash \mathcal{M}_{1,\vec{1}}]$ [22, Remark 8.5]. The quotient morphism $\mathcal{M}_{1,\vec{1}} \to \mathcal{M}_{1,1}$ can be identified with the map $[E,O,\omega] \mapsto [E,O]$ that forgets the choice of differential. It is clear that this map is a principal $\mathbb{G}_m$-bundle over $\mathcal{M}_{1,1}$ because $\Omega^1_{E/X}(E)$ is one dimensional and $\omega \neq 0$.

Remark 3.2.1 (Alternative moduli problem). The notation $\mathcal{M}_{1,\vec{1}}$ is suggestive of the fact that this is also the moduli space parametrising isomorphism classes of triples $(E, O, \vec{v})$, where (E, O) is as before and $\vec{v}$ is a nonzero tangential basepoint at O. The two notations are related by choosing $\vec{v} \in T_O(E)$ to be the element of the tangent space at O satisfying $\vec{v}(\omega) = 1$.

3.2.1 Analytic formulation

To study periods it is useful to have a description of the associated analytic spaces $\mathcal{M}_{1,\vec{1}}^{an}$ and $\mathcal{M}_{1,1}^{an}$. They are examples of complex-analytic orbifolds [22]. As we generally work with moduli spaces as orbifolds rather than as algebraic stacks, we abuse notation and from this point forward use the same symbols to denote the stacks and their associated orbifolds.

Let $\mathfrak{H}$ be the complex upper half plane. The modular group $SL_2(\mathbb{Z})$ acts on $\mathfrak{H}$ by (2.2). The moduli space of elliptic curves is the orbifold quotient $\mathcal{M}_{1,1} := [SL_2(\mathbb{Z})\backslash\mathfrak{H}]$ under this action. Define a left-action of $SL_2(\mathbb{Z})$ on $\mathbb{C}^\times \times \mathfrak{H}$ by

$$\gamma \cdot (\xi, \tau) = \left((c\tau + d)^{-1}\xi, \gamma \cdot \tau\right), \tag{3.2}$$

where $\gamma_2(\mathbb{Z})$ acts on $\tau \in \mathfrak{H}$ as in (2.2). Then $\mathcal{M}_{1,\vec{1}}$ is the orbifold quotient

$$\mathcal{M}_{1,\vec{1}} := [SL_2(\mathbb{Z})\backslash(\mathbb{C}^\times \times \mathfrak{H})].$$

The action (3.2) has no fixed points, so $\mathcal{M}_{1,\vec{1}}$ is an analytic variety [23, §14].

Projection onto the second factor induces a morphism $\mathcal{M}_{1,\vec{1}} \to \mathcal{M}_{1,1}$. This is a $\mathbb{C}^\times$-bundle over $\mathcal{M}_{1,1}$. Let $\mathcal{L}$ denote the associated analytic line bundle over $\mathcal{M}_{1,1}$. The global sections of $\mathcal{L}^{\otimes k}$ are modular forms of weight k.

The moduli space $\mathcal{M}_{1,1}$ has a Deligne-Mumford compactification $\overline{\mathcal{M}}_{1,1}$. This corresponds to compactifying the orbifold $\mathcal{M}_{1,1}$ by patching basic orbifolds together around the *cusp* of $\mathcal{M}_{1,1}$ [22, §4].

3.3 The universal elliptic curve

By the moduli property of $\mathcal{M}_{1,1}$, the identity element id $\in \mathrm{Hom}(\mathcal{M}_{1,1}, \mathcal{M}_{1,1})$ corresponds to an elliptic curve

$$\pi \colon \mathcal{E} \to \mathcal{M}_{1,1}$$

with distinguished section $O\colon \mathcal{M}_{1,1} \to \mathcal{E}$. This space is called the *universal elliptic curve* or *universal family* over $\mathcal{M}_{1,1}$. The universal punctured elliptic curve is $\mathcal{E}^\times := \mathcal{E} \setminus \{O\}$.

Let (E, O_E) be an elliptic curve over a scheme X. Because $\mathcal{M}_{1,1}$ is a fine moduli space for elliptic curves, this data is equivalent to a morphism of stacks $\pi_{(E,O_E)}\colon X \to \mathcal{M}_{1,1}$. The fact that $\pi\colon \mathcal{E} \to \mathcal{M}_{1,1}$ is the universal family means that $E \to X$ (resp. $O_E\colon X \to E$) is the pullback of $\pi\colon \mathcal{E} \to \mathcal{M}_{1,1}$ (resp. $O\colon \mathcal{M}_{1,1} \to \mathcal{E}$) along $\pi_{(E,O_E)}$.

3.3.1 Analytic formulation

The following description of $\mathcal{E}$ as a complex-analytic orbifold is given in [26, §1]. The group $SL_2(\mathbb{Z})$ acts naturally on $\mathbb{Z}^2$ on the *right*. Let $\widehat{\Gamma} := SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ denote the semidirect product associated to this action (see §1.1 for conventions regarding semidirect products). Then $\widehat{\Gamma}$ acts on $\mathbb{C} \times \mathfrak{H}$ as follows: for $v = (m, n) \in \mathbb{Z}^2$ and $\gamma \in SL_2(\mathbb{Z})$, set

$$
\begin{aligned}
v \cdot (\xi, \tau) &= (\xi + m\tau + n, \tau) \\
\gamma \cdot (\xi, \tau) &= \left((c\tau + d)^{-1}\xi, \gamma \cdot \tau\right).
\end{aligned}
$$

This definition is compatible with the action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2$, and therefore defines a left-action of $\widehat{\Gamma}$ on $\mathbb{C} \times \mathfrak{H}$. The universal elliptic curve is the orbifold quotient

$$
\mathcal{E} := [\widehat{\Gamma} \backslash (\mathbb{C} \times \mathfrak{H})].
$$

The section $O\colon \mathcal{M}_{1,1} \to \mathcal{E}$ is induced by the section $\mathfrak{H} \to \mathbb{C} \times \mathfrak{H}$ sending $\tau \mapsto (0, \tau)$.

For any $\tau \in \mathcal{M}_{1,1}$, the fiber $\mathcal{E}_\tau$ is isomorphic to the elliptic curve E_τ whose complex points are isomorphic to the quotient $\mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$. The fiber $\mathcal{E}_\tau^\times$ is isomorphic to the affine curve $E_\tau^\times$.

It is possible to extend $\mathcal{E} \to \mathcal{M}_{1,1}$ over $\overline{\mathcal{M}}_{1,1}$ to obtain a generalised universal elliptic curve $\overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$. The fiber $\overline{\mathcal{E}}_0$ over the cusp of $\overline{\mathcal{M}}_{1,1}$ is the nodal cubic.

The moduli space $\mathcal{M}_{1,\vec{1}}$ may be identified with the normal bundle of the image of the zero section $O\colon \mathcal{M}_{1,1} \to \mathcal{E}$ [26, Proposition 1.1-Corollary 1.3].

3.4 The Tate curve

The Tate curve [49, Chapter V, §3] is an elliptic curve defined over the ring $\mathbb{Z}[[q]]$ of formal power series in an indeterminate q. It arises naturally when considering the behavior of the fiber of $\overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$ close to the cusp. Its affine part is given by the cubic equation

$$
y^2 + xy = x^3 + a_4(q)x + a_6(q),
$$

where the formal power series $a_4(q), a_6(q) \in \mathbb{Q}[[q]]$ are defined by

$$a_4(q) := -5\sum_{n\geq 1} n^3 \cdot \frac{q^n}{1-q^n},$$
$$a_6(q) := -\sum_{n\geq 1} \frac{7n^5+5n^3}{12} \cdot \frac{q^n}{1-q^n}.$$

These power series are contained in $\mathbb{Z}[[q]]$ because $12|(7n^5+5n^3)$ for every positive integer n, which is why the Tate curve is defined over $\mathbb{Z}[[q]]$. The special fiber $\overline{\mathcal{E}}_0$ (where $q=0$) is given by the equation $y^2+xy=x^3$, and is a nodal cubic. Hence, the Tate curve has split multiplicative reduction [41].

The change of variables $x \mapsto x-1/12$, $y \mapsto (2y-2x+1)/4$ produces the following short Weierstrass equation for the Tate curve:

$$y^2 = 4x^3 - g_2(q)x - g_3(q),$$

where $g_2(q) := 20\mathbb{G}_4(q)$, $g_3(q) := \frac{7}{3}\mathbb{G}_6(q)$ and $\mathbb{G}_{2k}(q) \in \mathbb{Q}[[q]]$ is the q-expansion of the Hecke-normalised Eisenstein series of weight $2k$ defined in (2.4), considered as a formal power series in q with rational coefficients.

3.4.1 Analytic formulation

If we instead wish to work with the analytic descriptions of $\mathcal{M}_{1,1}$ and $\mathcal{E}$ given above, we may view the parameter q as taking numerical values in the disk $D = \{q \in \mathbb{C} : |q| < 1\}$ by setting $q = \exp(2\pi i\tau)$. This realises the Tate curve as a family of elliptic curves close to the degenerate fiber over the cusp $q=0$, which is the nodal cubic [26, §16].

3.5 Tangential basepoints

We regularly use fundamental groups equipped with *tangential basepoints* [15, §15]. The following definition of tangential basepoint is attributed to Nakamura [41, Definition 1.1].

Definition 3.5.1 (Tangential basepoint)**.** Let X be a connected scheme and let K be a field of characteristic 0. A K-rational tangential basepoint is a morphism $\vec{v}\colon \operatorname{Spec} K((q)) \to X$. For reasons of notation we define the set of K-rational basepoints on X to be

$$X(K)_{\mathrm{bp}} := X(K) \cup \{K\text{-rational tangential basepoints}\}.$$

A K-rational tangential basepoint on X consists of a scheme-theoretic point $p \in X$ (not necessarily contained in $X(K)$) together with an inclusion $\kappa(p) \hookrightarrow K((q))$, where $\kappa(p)$ is the residue field of p.

In this section we define tangential basepoints on each of the previous spaces. These will be used as basepoints for fundamental groups in Chapter 5. The material in this section is taken largely from [27, §4.2].

3.5.1 Tangential basepoint on $\mathcal{M}_{1,1}$

The Tate curve corresponds to a morphism $\operatorname{Spec} \mathbb{Z}[[q]] \to \overline{\mathcal{M}}_{1,1}$. The parameter q defines a local parameter at the cusp e_0 of $\overline{\mathcal{M}}_{1,1}$, [22, §4]. The fiber $\mathcal{E}_{\partial/\partial q}$ of the Tate curve over $\partial/\partial q$ has equation

$$y^2 + xy = x^3 - 5\bar{q}x - \bar{q},$$

where $\bar{q}$ denotes the image of q in $\mathbb{Z}[q]/(q^2)$. The discriminant of the Tate curve is

$$\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24} \equiv q \pmod{q^2},$$

which is nonzero modulo all primes p. Thus the tangent vector $\partial/\partial q$ based at the cusp of $\overline{\mathcal{M}}_{1,1}$ is defined over $\mathbb{Z}$. It defines a choice of tangential basepoint $\partial/\partial q$ on $\mathcal{M}_{1,1}$, based at the cusp e_0.

The tangential basepoint $\partial/\partial q$ can be realised analytically. It corresponds to the image of the imaginary axis under

$$\{iy : y > 0\} \hookrightarrow \mathfrak{H} \to \mathcal{M}_{1,1}. \tag{3.3}$$

3.5.2 Tangential basepoint on the fiber of the punctured Tate curve over $\partial/\partial q$

By identifying $\overline{\mathcal{M}}_{1,1}$ with the image of the identity section in $\overline{\mathcal{E}}$, the cusp e_0 is identified with the identity on the nodal cubic $\overline{\mathcal{E}}_0$. The nodal cubic is normalised by $\mathbb{P}^1_{\mathbb{Z}} \to \overline{\mathcal{E}}_0$.

Let w be a parameter on $\overline{\mathcal{E}}_0$ such that its pullback to $\mathbb{P}^1_{\mathbb{Z}}$ takes the values 0 and ∞ on the preimage of the double point and the value 1 on (the preimage of) e_0. It is unique up to $w \mapsto w^{-1}$. It defines a tangent vector $\partial/\partial w$ on $\overline{\mathcal{E}}_0$ based at e_0. It therefore defines a tangential basepoint on $\mathcal{E}_0 := \overline{\mathcal{E}}_0 \setminus \{\text{double point}\}$ based at the identity e_0.

The inclusion $\mathcal{E}_0 \hookrightarrow \mathcal{E}_{\partial/\partial q}$ means one can regard $\partial/\partial w$ as a tangent vector on $\mathcal{E}_{\partial/\partial q}$ based at the identity. Hence we have a tangential basepoint on the punctured infinitesimal Tate curve $\mathcal{E}^{\times}_{\partial/\partial q}$ based at the punctured identity section O. It is defined over $\mathbb{Z}$.

3.5.3 Tangential basepoint on $\mathcal{M}_{1,\vec{1}}$

The tangential basepoints $\partial/\partial q$ on $\mathcal{M}_{1,1}$ and $\partial/\partial w$ on $\mathcal{E}_0$ determine a tangential basepoint $\vec{\mathsf{v}} := \partial/\partial q + \partial/\partial w$ on $\mathcal{M}_{1,\vec{1}}$. It can be viewed as the data of the tangential basepoint $\partial/\partial w$ based at the identity of $\mathcal{E}_{\partial/\partial q}$. It is nonzero modulo all primes.

The morphism $\mathcal{M}_{1,\vec{1}} \to \mathcal{M}_{1,1}$ extends over the cusp. It therefore induces a morphism of tangent spaces at the cusp under which $\vec{\mathsf{v}}$ maps to $\partial/\partial q$.

3.5.4 Tangential basepoints on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

There is an isomorphism $\mathbb{G}_m \xrightarrow{\sim} \mathcal{E}_0$ under which the identity 1 on $\mathbb{G}_m$ maps to e_0 (the parameter w on $\mathcal{E}_0$ pulls back to a parameter on $\mathbb{G}_m$ that we also denote by w). The tangent vector $\partial/\partial w$ corresponds to the tangent vector $\partial/\partial w$ on $\mathbb{G}_m$ based at the identity.

Recall from §3.1 that $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is equipped with a global section t. The inclusion $\mathbb{P}^1 \setminus \{0, 1, \infty\} \hookrightarrow \mathbb{G}_m$ is given by the restriction mapping $w \mapsto t$. The tangential basepoint $\partial/\partial w$ on $\mathbb{G}_m$ then pulls back to a tangential basepoint $\partial/\partial t$ on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, based at the puncture at 1. We denote this tangential basepoint by $\vec{1}_1$. By construction, is it defined over $\mathbb{Z}$ with good reduction at every prime.

There are in fact six natural tangential basepoints on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. They correpond to the orbits of $\vec{1}_1$ under the S_3-action on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ described in §3.1. To define them concretely, set $-\vec{1}_1 = -\partial/\partial t$. It is another tangent vector on $\mathbb{P}^1$ based at 1. The action of S_3 on $\mathbb{P}^1$ restricts to an action on $\{0, 1, \infty\}$ and the images of $\pm\vec{1}_1$ define two further tangential basepoints $\pm\vec{1}_0$ based at 0 and two further tangential basepoints $\pm\vec{1}_\infty$ based at ∞.

Chapter 4

Motives and periods

4.1 Motives

The concept of a *motive* is originally due to Grothendieck. Although the subject is still in development, and there is currently no formal definition of a completely general category $\mathsf{MM}_{\mathbb{Q}}$ of (mixed) motives over $\mathbb{Q}$, the basic "yoga" is as follows.

The category $\mathsf{Sch}_{\mathbb{Q}}$ of schemes over $\mathbb{Q}$ is equipped with a range of Weil cohomology theories [32, §1.2]. These are contravariant functors into various linear categories having certain properties. For example, a variety $X/\mathbb{Q}$ has Betti cohomology $H^n_{\mathrm{B}}(X, \mathbb{Q}) = H^n_{sing}(X(\mathbb{C}), \mathbb{Q})$, given by the singular cohomology of the space of complex points $X(\mathbb{C})$; it also has algebraic de Rham cohomology $H^n_{\mathrm{dR}}(X, \mathbb{Q}) = \mathbb{H}^n(X, \Omega^\bullet_X)$, given by the hypercohomology of the complex of differentials.[1]

These cohomology theories are all defined very differently, and are vector spaces over different (characteristic zero) fields of coefficients. Despite this, the resulting vector spaces share many structural similarities. For example, they have the same dimension, and the traces of geometric endomorphisms are rational numbers. There are also certain *comparison isomorphisms* between cohomology theories. For example, Grothendieck's comparison isomorphism [19] implies the existence of a natural isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}} \colon H^n_{\mathrm{dR}}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^n_{\mathrm{B}}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.1}$$

This may be viewed as an abstract version of integration of algebraic differential forms (representing de Rham cohomology classes) along cycles in singular homology.

These similarities and comparisons suggest that a "universal cohomology theory" exists that keeps track of the underlying structure behind the various Weil cohomology

[1] There are other examples of Weil cohomology theories – namely, ℓ-adic and crystalline cohomology – but we do not require these for the topics in this thesis.

theories. Objects of the hypothetical category $\mathsf{MM}_{\mathbb{Q}}$ should be considered as "universal cohomology groups" $H^n(X, \mathbb{Q}(m))$ for algebraic varieties $X/\mathbb{Q}$. There should be a universal functor $\mathsf{Sch}_{\mathbb{Q}}^{\mathrm{op}} \to \mathsf{MM}_{\mathbb{Q}}$ and a collection of "realisation functors" on $\mathsf{MM}_{\mathbb{Q}}$ with the property that every Weil cohomology theory factors as the composition of $\mathsf{Sch}_{\mathbb{Q}} \to \mathsf{MM}_{\mathbb{Q}}$ with one of these realisations. Additionally, $\mathsf{MM}_{\mathbb{Q}}$ should a $\mathbb{Q}$-linear abelian tensor category. Equipping it with any realisation functor $\mathsf{MM}_{\mathbb{Q}} \to \mathsf{Vect}_K$ should give it the structure of a neutral Tannakian category over K [17].

Although $\mathsf{MM}_{\mathbb{Q}}$ is currently not defined, there is a triangulated category $\mathsf{DMM}_{\mathbb{Q}}$ that is expected to be its bounded derived category [53]. If a t-structure is shown to exist on $\mathsf{DMM}_{\mathbb{Q}}$, the heart of this t-structure could be extracted to obtain an abelian category having the properties required of $\mathsf{MM}_{\mathbb{Q}}$. Unfortunately, however, no such t-structure is currently known.

One way to circumvent this issue is to work with motives as a collection of *realisations*, together with their comparisons. This point of view was introduced by Deligne [15], and it is sufficient for the purpose of studying periods. Assuming it is possible to construct $\mathsf{MM}_{\mathbb{Q}}$ formally, one expects a category of "motives" defined by systems of realisations to be equivalent to the formal definition. In the following section we introduce a useful category of "generalised Hodge realisations".

4.1.1 The category $\mathcal{H}$

Based on Deligne's work [15, §1], Brown [8, §3.1] defined a category of generalised Hodge realisations. We briefly recall the definition, referring the reader to [8] for more detail.

Definition 4.1.1. Let $\mathcal{H}$ be the category whose objects are triples $V = (V^{\mathrm{B}}, V^{\mathrm{dR}}, c_V)$ consisting of the following data:

1. A finite dimensional $\mathbb{Q}$-vector space V^{B} equipped with a finite, increasing *weight* filtration $M_\bullet V^{\mathrm{B}}$;

2. A finite dimensional $\mathbb{Q}$-vector space V^{dR} equipped with a finite, increasing *weight* filtration $M_\bullet V^{\mathrm{dR}}$ and a finite, decreasing *Hodge* filtration $F^\bullet V^{\mathrm{dR}}$;

3. An isomorphism of filtered vector spaces $c_V \colon M_\bullet V^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} M_\bullet V^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$;

4. An involution $F_\infty \colon V^{\mathrm{B}} \xrightarrow{\sim} V^{\mathrm{B}}$ called the real Frobenius.

These data satisfy the following conditions:

1. Let ς_{dR} (resp. ς_{B}) denote the $\mathbb{C}$-antilinear involution on $V^{\mathrm{dR}} \otimes \mathbb{C}$ (resp. on $V^{\mathrm{B}} \otimes \mathbb{C}$) given by complex conjugation on coefficients: $v \otimes \lambda \mapsto v \otimes \bar{\lambda}$. Then $(F_\infty \otimes \varsigma_{\mathrm{B}}) \circ c_v = c_v \circ \varsigma_{\mathrm{dR}}$.

2. The weight filtration $M_\bullet V^{\mathrm{B}}$ and the Hodge filtration $c_V F^\bullet(V^{\mathrm{dR}} \otimes \mathbb{C})$ equip V^{B} with the structure of a $\mathbb{Q}$-mixed Hodge structure that we further assume to be graded-polarisable.

The morphisms $f\colon V_1 \to V_2$ in $\mathcal{H}$ are pairs $(f^{\mathrm{B}}\colon V_1^{\mathrm{B}} \to V_2^{\mathrm{B}}, f^{\mathrm{dR}}\colon V_1^{\mathrm{dR}} \to V_2^{\mathrm{dR}})$ that are compatible with the structures given in Definition 4.1.1.

There are fiber functors $\omega_{\mathcal{H}}^{\mathrm{B}}, \omega_{\mathcal{H}}^{\mathrm{dR}}\colon \mathcal{H} \to \mathsf{Vect}_{\mathbb{Q}}$ projecting an object V onto the obvious component. The category $\mathcal{H}$ is neutral Tannakian over $\mathbb{Q}$ with respect to both fiber functors [15].

We now consider a general example of an object in $\mathcal{H}$. Let X be a variety over $\mathbb{Q}$ and define a triple

$$H^n(X) := \left(H^n_{\mathrm{B}}(X,\mathbb{Q}), H^n_{\mathrm{dR}}(X,\mathbb{Q}), \mathrm{comp}_{\mathrm{B,dR}}\right),$$

where $\mathrm{comp}_{\mathrm{B,dR}}$ is the comparison isomorphism defined in (4.1). Then $H^n(X)$ defines an object of $\mathcal{H}$ using the natural weight and Hodge filtrations on cohomology. The real Frobenius $F_\infty\colon H^n_{\mathrm{B}}(X,\mathbb{Q}) \xrightarrow{\sim} H^n_{\mathrm{B}}(X,\mathbb{Q})$ is the map on cohomology induced by the continuous map $X(\mathbb{C}) \xrightarrow{\sim} X(\mathbb{C})$ induced by complex conjugation.

4.1.2 Mixed Tate motives

Despite the difficulty in constructing a completely general category of all mixed motives, there are cases in which well-defined subcategories exist; namely, *mixed Tate motives*, which are constructed from iterated extensions of the cohomology of projective spaces. The existence of mixed Tate motives follows from the existence of a t-structure on a suitable mixed Tate triangulated subcategory of $\mathsf{DMM}_{\mathbb{Q}}$. The relevant abelian category in our case is the category of mixed Tate motives over $\mathbb{Z}$, which we now describe.

There exists a $\mathbb{Q}$-linear rigid monoidal abelian category $\mathsf{MT}(\mathbb{Z})$ called the category of *mixed Tate motives* over $\mathbb{Z}$. It was constructed by Deligne and Goncharov [16] building on fundamental work on triangulated categories of mixed motives [53, 35, 36]. It is determined by simple objects $\mathbb{Q}(n)$ for all $n \in \mathbb{Z}$ together with isomorphisms

$\mathbb{Q}(m) \otimes \mathbb{Q}(n) \xrightarrow{\sim} \mathbb{Q}(m+n)$, and extension groups

$$\mathrm{Ext}^i_{\mathsf{MT}(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \begin{cases} \mathbb{Q} & i = n = 0 \\ K_{2n-1}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} & i = 1, n > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Here $K_m(\mathbb{Z})$ refers to the mth algebraic K-theory of the integers.

The category $\mathsf{MT}(\mathbb{Z})$ is neutral Tannakian over $\mathbb{Q}$ with respect to two different fiber functors: the Betti realisation ω^{B} and the de Rham realisation ω^{dR}. These send a mixed Tate motive $V = H^n(X)$ to the Betti cohomology $V^{\mathrm{B}} := H^n_{\mathrm{B}}(X)$ (resp. de Rham cohomology $V^{\mathrm{dR}} := H^n_{\mathrm{dR}}(X)$) of the associated algebraic variety X. Of course the geometry of X must be fairly specific for $H^n(X)$ to be mixed Tate over $\mathbb{Z}$ – meaning that $H^n(X)$ is some iterated extension of the cohomology of projective space. For example, we could take $X = \mathbb{P}^k$, or X could be a product of schemes of the following types: $\mathbb{A}^1$, $\mathbb{G}_m$, $\mathcal{M}_{0,n}$ for $n \geq 3$, $\overline{\mathcal{M}}_{0,n}$ for $n \geq 3$, Grassmanians, ...

The Betti realisation has an increasing *weight* filtration $M_\bullet V^{\mathrm{B}}$. The de Rham realisation has an increasing *weight* filtration $M_\bullet V^{\mathrm{dR}}$ and the comparison isomorphism between Betti and de Rham cohomology of X respects these filtrations. The de Rham realisation also has a decreasing *Hodge filtration* $F^\bullet V^{\mathrm{dR}}$.

All these structures may be written down explicitly on the simple objects of $\mathsf{MT}(\mathbb{Z})$. The object $\mathbb{Q}(-1) \in \mathsf{MT}(\mathbb{Z})$ can be identified with the cohomology $H^1(\mathbb{G}_m)$; consequently, $\mathbb{Q}(-1)^{\mathrm{B}} \cong \mathbb{Q}$ and $\mathbb{Q}(-1)^{\mathrm{dR}} \cong \mathbb{Q}$. The comparison isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}} \colon \mathbb{Q}(-1)^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathbb{Q}(-1)^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$$

is multiplication by $2\pi i$. The isomorphism $\mathbb{Q}(-n) \cong \mathbb{Q}(-1)^{\otimes n}$ then determines that the comparison map for $\mathbb{Q}(n)$ is multiplication by $(2\pi i)^n$. Finally, the weight and Hodge filtrations are trivial:

$$\begin{aligned} 0 = M_{-2n-1}\mathbb{Q}(n)^{\mathrm{B}} &\subseteq M_{-2n}\mathbb{Q}(n)^{\mathrm{B}} = \mathbb{Q}(n)^{\mathrm{B}} \\ 0 = F^{1-n}\mathbb{Q}(n)^{\mathrm{dR}} &\subseteq F^{-n}\mathbb{Q}(n)^{\mathrm{dR}} = \mathbb{Q}(n)^{\mathrm{dR}}. \end{aligned}$$

It is a deep fact [16] that $\mathsf{MT}(\mathbb{Z})$ embeds as a full subcategory of $\mathcal{H}$ via

$$\omega^{\mathcal{H}} \colon \mathsf{MT}(\mathbb{Z}) \to \mathcal{H}, \quad V \mapsto (V^{\mathrm{B}}, V^{\mathrm{dR}}, \mathrm{comp}_{\mathrm{B,dR}}). \tag{4.3}$$

This means that its structure is entirely determined by the Betti and de Rham realisations, together with their comparisons. The more "arithmetic" ℓ-adic and crystalline realisations are determined entirely from these data. Understanding the Betti and de Rham realisations is equivalent to understanding the action of the motivic Galois group on the periods of these motives (see §4.6). Taken together, this gives some justification of the importance of understanding periods. This is the subject of the next section.

4.2 Motivic periods

The isomorphism (4.1) can be described by an invertible matrix of complex numbers whose entries are called *periods*. The study of these numbers is our main focus.

Certain problems in transcendence theory – the most notable of which is the *period conjecture* of Grothendieck [19, Note (10)] [4] – mean that it is currently easier to work with formal analogues of periods called *motivic periods*. These have a rich algebraic structure and are equipped with an action of an affine group scheme called a *motivic Galois group*. They surject onto the usual algebra of "numerical" periods, and the period conjecture predicts that this is an isomorphism. This would transfer all the useful abstract structures on motivic periods to their numerical analogues.

4.2.1 Tannakian theory of motivic periods

The categories $\mathsf{MT}(\mathbb{Z})$ and $\mathcal{H}$ are each equipped with Betti and de Rham fiber functors

$$\omega^{\mathrm{B}}, \omega^{\mathrm{dR}} \colon \mathsf{MT}(\mathbb{Z}) \to \mathsf{Vect}_{\mathbb{Q}}, \quad \omega^{\mathrm{B}}_{\mathcal{H}}, \omega^{\mathrm{dR}}_{\mathcal{H}} \colon \mathcal{H} \to \mathsf{Vect}_{\mathbb{Q}},$$

with respect to which they are neutral Tannakian over $\mathbb{Q}$. As remarked in (4.3) there is also a fully-faithful $\mathcal{H}$-realisation functor $\omega^{\mathcal{H}} \colon \mathsf{MT}(\mathbb{Z}) \to \mathcal{H}$ satisfying $\omega^{\bullet} = \omega^{\bullet}_{\mathcal{H}} \circ \omega^{\mathcal{H}}$ for $\bullet \in \{\mathrm{B}, \mathrm{dR}\}$.[2]

Definition 4.2.1 (Motivic Galois group)**.** The (de Rham) motivic Galois groups of $\mathsf{MT}(\mathbb{Z})$ and $\mathcal{H}$ are the affine group schemes over $\mathbb{Q}$ defined by

$$G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} := \mathrm{Aut}^{\otimes}_{\mathsf{MT}(\mathbb{Z})}(\omega^{\mathrm{dR}}), \quad G^{\mathrm{dR}}_{\mathcal{H}} := \mathrm{Aut}^{\otimes}_{\mathcal{H}}(\omega^{\mathrm{dR}}_{\mathcal{H}}).$$

The $\mathbb{Q}$-algebras of *de Rham periods* are the affine rings

$$\mathcal{P}^{\mathfrak{dr}}_{\mathsf{MT}(\mathbb{Z})} := \mathcal{O}(G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}), \quad \mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}} := \mathcal{O}(G^{\mathrm{dR}}_{\mathcal{H}}).$$

[2]Throughout, we work almost entirely with the de Rham fiber functors $\omega^{\mathrm{dR}}, \omega^{\mathrm{dR}}_{\mathcal{H}}$, but the story is a mirror image in the Betti case.

To each category one may associate a $\mathbb{Q}$-algebra of motivic periods in the following way. Let $\mathcal{T}_{\mathsf{MT}(\mathbb{Z})} := \mathrm{Isom}^{\otimes}_{\mathsf{MT}(\mathbb{Z})}(\omega^{\mathrm{dR}}, \omega^{\mathrm{B}})$. It is a right $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$-torsor. Similarly, $\mathcal{T}_{\mathcal{H}} := \mathrm{Isom}^{\otimes}_{\mathcal{H}}(\omega^{\mathrm{dR}}_{\mathcal{H}}, \omega^{\mathrm{B}}_{\mathcal{H}})$ is a right $G^{\mathrm{dR}}_{\mathcal{H}}$-torsor.

Definition 4.2.2 (Motivic period)**.** The $\mathbb{Q}$-algebras of motivic periods of $\mathsf{MT}(\mathbb{Z})$ and $\mathcal{H}$ are the affine rings

$$\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} := \mathcal{O}(\mathcal{T}_{\mathsf{MT}(\mathbb{Z})}), \quad \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} := \mathcal{O}(\mathcal{T}_{\mathcal{H}}).$$

Since $\omega^{\mathcal{H}}$ is fully faithful, there is an inclusion

$$\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} \hookrightarrow \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}. \tag{4.4}$$

Elements of the ring $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ should really be referred to as $\mathcal{H}$-periods because general objects of $\mathcal{H}$ are constructed entirely from linear algebra, with no reference to motives. The "true" ring of motivic periods would apply the construction above to the hypothetical Tannakian category $\mathsf{MM}_{\mathbb{Q}}$ to obtain a ring $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MM}_{\mathbb{Q}}}$ containing the motivic periods of all possible motives (in particular, it would contain the ring $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$).

As discussed in [8, Introduction], however, any reasonable definition of $\mathsf{MM}_{\mathbb{Q}}$ will be equipped with a functor $\mathsf{MM}_{\mathbb{Q}} \to \mathcal{H}$ that is expected to be fully faithful. As above, this would correspond to an inclusion $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MM}_{\mathbb{Q}}} \hookrightarrow \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$. The ring $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ can therefore be regarded as a general ambient ring containing all motivic periods that actually arise from the cohomology of algebraic varieties.

4.2.1.1 The motivic coaction

Since $\mathcal{T}_{\mathsf{MT}(\mathbb{Z})}$ (resp. $\mathcal{T}_{\mathcal{H}}$) is a right torsor under $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ (resp. $G^{\mathrm{dR}}_{\mathcal{H}}$), the algebra $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ (resp. $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$) has a right coaction by $\mathcal{P}^{\mathfrak{dr}}_{\mathsf{MT}(\mathbb{Z})}$ (resp. $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}}$):

$$\Delta_{\mathsf{MT}(\mathbb{Z})} \colon \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} \to \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{dr}}_{\mathsf{MT}(\mathbb{Z})}, \quad \Delta_{\mathcal{H}} \colon \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}}. \tag{4.5}$$

They may be written down explicitly in terms of matrix coefficients (see §4.2.2). Understanding these coactions combinatorially (or otherwise) is a major aspect of current research because they may be used to find relations between periods; see e.g. [5, Lemma 2.7]. We use this idea in Chapter 15 to determine explicit coefficients for motivic iterated Eisenstein integrals.

4.2.1.2 The Galois action on periods

Dual to the coactions are (left) group actions

$$G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \times \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} \to \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}, \quad G^{\mathrm{dR}}_{\mathcal{H}} \times \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}, \tag{4.6}$$

defined as follows. Let C be either $\mathsf{MT}(\mathbb{Z})$ or $\mathcal{H}$. Then $g \in G^{\mathrm{dR}}_{\mathsf{C}}(\mathbb{Q}) = \mathrm{Hom}(\mathcal{P}^{\mathfrak{dr}}_{\mathsf{C}}, \mathbb{Q})$ acts via the automorphism

$$\mathcal{P}^{\mathfrak{m}}_{\mathsf{C}} \xrightarrow{\Delta_{\mathsf{C}}} \mathcal{P}^{\mathfrak{m}}_{\mathsf{C}} \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{dr}}_{\mathsf{C}} \xrightarrow{\mathrm{id} \otimes g} \mathcal{P}^{\mathfrak{m}}_{\mathsf{C}} \otimes_{\mathbb{Q}} \mathbb{Q} \xrightarrow{\sim} \mathcal{P}^{\mathfrak{m}}_{\mathsf{C}}. \tag{4.7}$$

In this way, the rings of motivic periods acquire a "Galois action". In §4.2.2 we will write down an explicit formula for the Galois action on motivic periods using matrix coefficients.

4.2.1.3 The period map

Grothendieck's comparison isomorphism [19] may be interpreted as a natural isomorphism $\omega^{\mathrm{dR}}_{\mathcal{H}} \otimes \mathbb{C} \xrightarrow{\sim} \omega^{\mathrm{B}}_{\mathcal{H}} \otimes \mathbb{C}$. Precomposing with the embedding $\omega^{\mathcal{H}} \colon \mathsf{MT}(\mathbb{Z}) \to \mathcal{H}$ induces a natural isomorphism $\omega^{\mathrm{dR}} \otimes \mathbb{C} \xrightarrow{\sim} \omega^{\mathrm{B}} \otimes \mathbb{C}$. Both of these natural isomorphisms are compatible with the tensor structures on the categories of motives, and hence they define canonical complex points[3]

$$c \in \mathcal{T}_{\mathsf{MT}(\mathbb{Z})}(\mathbb{C}) = \mathrm{Hom}(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}, \mathbb{C}), \quad c \in \mathcal{T}_{\mathcal{H}}(\mathbb{C}) = \mathrm{Hom}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}, \mathbb{C}).$$

In terms of the category $\mathcal{H}$, the component of the natural isomorphism c at an object $V \in \mathcal{H}$ is precisely the comparison isomorphism $c_V \colon V^{\mathrm{dR}} \otimes \mathbb{C} \xrightarrow{\sim} V^{\mathrm{B}} \otimes \mathbb{C}$ built into the data of V. That this defines a natural isomorphism as above is a consequence of the definition of morphisms in $\mathcal{H}$.

By evaluating c on elements of $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} = \mathcal{O}(\mathcal{T}_{\mathsf{MT}(\mathbb{Z})})$ (resp. $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} = \mathcal{O}(\mathcal{T}_{\mathcal{H}})$), we obtain canonical $\mathbb{Q}$-algebra homomorphisms

$$\mathrm{per} \colon \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} \to \mathbb{C}, \quad \mathrm{per} \colon \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \to \mathbb{C}, \tag{4.8}$$

called *period maps*. Grothendieck's period conjecture is equivalent to the statement that the hypothetical period map $\mathrm{per} \colon \mathcal{P}^{\mathfrak{m}}_{\mathsf{MM}_{\mathbb{Q}}} \to \mathbb{C}$ is injective. Another interpretation of the period conjecture states that all relations between periods should be of *geometric origin*.

Below, we give an interpretation of the period maps in terms of matrix coefficients. It turns out that they may be interpreted as computing integrals.

[3] By abuse of notation, we refer to both of these points as c by making use of the embedding $\omega^{\mathcal{H}}$.

4.2.2 Matrix coefficients

The embedding (4.4) means that $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ may be studied within the algebra of motivic $\mathcal{H}$-periods, which has an elementary description in terms of *matrix coefficients* that we recall in this section. For this reason, all motivic periods will be viewed as contained in $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$, and when considering periods of mixed Tate motives we use (4.4) implicitly.

Matrix coefficients are formal analogues of integrals. They are symbols $[V, w, \sigma]^{\mathfrak{m}}$, where $V \in \mathcal{H}$, $w \in \omega^{\mathrm{dR}}(V)$ and $\sigma \in \omega^{\mathrm{B}}(V)^{\vee}$. These symbols are required to satisfy various relations, such as bilinearity and compatibility with morphisms in $\mathcal{H}$. Details of these properties can be found in [8, §2.2 and §3.2].

There is also a description of elements of the de Rham period ring $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}}$ in terms of matrix coefficients. These are symbols $[V, w, f]^{\mathfrak{dr}}$ where $V \in \mathcal{H}$, $w \in \omega^{\mathrm{dR}}(V)$ and $f \in \omega^{\mathrm{dR}}(V)^{\vee}$, and satisfy similar relations.

The period maps (4.8) have a simple description in terms of matrix coefficients:

$$\mathrm{per}([V, w, \sigma]^{\mathfrak{m}}) = \sigma(c_V(w)) \in \mathbb{C}.$$

This formula can be further understood as follows. Let X be a scheme over $\mathbb{Q}$ and let $V = H^n(X) \in \mathcal{H}$. Let $\omega \in H^0(X, \Omega^n)$ represent a class in $H^n_{\mathrm{dR}}(X)$ and let σ be a cycle in $X(\mathbb{C})$ representing a class in $H_n^{sing}(X(\mathbb{C}), \mathbb{Q}) \cong H^n_{\mathrm{B}}(X)^{\vee}$. Then the triple $[V, [\omega], [\sigma]]^{\mathfrak{m}}$ defines a matrix coefficient, and (4.8) may be interpreted as the map

$$[M, [\omega], [\sigma]]^{\mathfrak{m}} \mapsto \int_{\sigma} \omega.$$

In this way, we see that the period map sends motivic periods to periods in the sense of [34]. For this reason we denote the matrix coefficient $[V, [\omega], [\sigma]]^{\mathfrak{m}}$ by $\int_{\sigma}^{\mathfrak{m}} \omega$.

A large part of this thesis deals with motivic *iterated* integrals. These may also be interpreted as motivic periods, and can therefore be represented by matrix coefficients as above. In this case, however, the relevant motive is not the cohomology $H^1(X)$ but rather the affine ring of the *relative fundamental group* of X (with a particular choice of basepoint) that will be defined in Definition 5.2.3. We use the notation $\int_{\sigma}^{\mathfrak{m}} \omega_1 \cdots \omega_n$ for a motivic iterated integral, in analogy to the notation for iterated integrals given in Definition 1.4.1. This notation will be justifed in §5.2.6.1 in the context of the affine ring of the relative de Rham fundamental group.

4.2.2.1 The coaction on matrix coefficients

The coaction (4.5) on matrix coefficients is very simple to write down explicitly. Let $[V, w, \sigma]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ and let $e_1, \ldots, e_n$ be a choice of basis for $V^{\mathrm{dR}} = \omega^{\mathrm{dR}}(V)$. Then

$$\Delta([V, w, \sigma]^{\mathfrak{m}}) = \sum_{i=1}^{n} [V, e_i, \sigma]^{\mathfrak{m}} \otimes [V, w, e_i^{\vee}]^{\mathfrak{dr}} \tag{4.9}$$

One may check that this definition is independent of the choice of basis and compatible with all structures on $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$.

4.2.2.2 The Galois action on matrix coefficients

Dually, one may write down the Galois action on matrix coefficients as follows. Let R be a $\mathbb{Q}$-algebra. An element $g \in G^{\mathrm{dR}}_{\mathcal{H}}(R)$ is, by definition, a natural isomorphism $\omega^{\mathrm{dR}}_{\mathcal{H}} \otimes R \xrightarrow{\sim} \omega^{\mathrm{dR}}_{\mathcal{H}} \otimes R$ compatible with the tensor structure on $\mathcal{H}$. For every object $V \in \mathcal{H}$ we therefore obtain an isomorphism $g_V : V^{\mathrm{dR}} \otimes R \xrightarrow{\sim} V^{\mathrm{dR}} \otimes R$ of R-modules. Recall that $G^{\mathrm{dR}}_{\mathcal{H}}(R) = \mathrm{Hom}(\mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}}, R)$. Therefore, for any $g \in G^{\mathrm{dR}}_{\mathcal{H}}(R)$ and any de Rham period $[V, w, f]^{\mathfrak{dr}}$ we obtain an element of R defined by

$$g([V, w, f]^{\mathfrak{dr}}) = f(g_V(w)) \in R.$$

Combining (4.7) with (4.9) then gives a formula for the Galois action on $\mathcal{H}$-periods:

$$g([V, w, \sigma]^{\mathfrak{m}}) = \sum_{i=1}^{n} e_i^{\vee}(g_V(w))[V, e_i, \sigma]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \otimes_{\mathbb{Q}} R. \tag{4.10}$$

In this way the "Galois representation" on the "motive" V is made explicit, and it is possible to write down the representation in terms of a period matrix.

4.2.3 The universal comparison isomorphism

The identity map on $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ defines a canonical point $c^{\mathfrak{m}} \in \mathcal{T}_{\mathcal{H}}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}})$ i.e. a natural isomorphism

$$c^{\mathfrak{m}} : \omega^{\mathrm{dR}}_{\mathcal{H}} \otimes \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \xrightarrow{\sim} \omega^{\mathrm{B}}_{\mathcal{H}} \otimes \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}.$$

Taking the component at $V = (V^{\mathrm{B}}, V^{\mathrm{dR}}, c_V) \in \mathcal{H}$ gives a *universal* comparison isomorphism[4]

$$c^{\mathfrak{m}}_V : V^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \xrightarrow{\sim} V^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}.$$

Its image under $\mathrm{per} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \to \mathbb{C}$ is the comparison isomorphism c constructed in §4.2.1.3. In other words it satisfies $(\mathrm{id} \otimes \mathrm{per}) \circ c^{\mathfrak{m}}_V = c_V \circ (\mathrm{id} \otimes \mathrm{per})$.

[4]If $V \in \mathsf{MT}(\mathbb{Z})$, this restricts to an isomorphism with $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ replacing $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$.

4.3 Mixed Tate periods

In this section we review some of the specifics of periods of mixed Tate motives.

4.3.1 The unipotent radical of $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$

Let $\mathsf{MT}(\mathbb{Z})^{\mathrm{ss}} \hookrightarrow \mathsf{MT}(\mathbb{Z})$ be the full subcategory of semi-simple objects. It is generated as a rigid abelian tensor category by $\mathbb{Q}(-1) = H^1(\mathbb{G}_m)$, and is neutral Tannakian with respect to the restricted fiber functor $\omega^{\mathrm{dR}}|_{\mathsf{MT}(\mathbb{Z})^{\mathrm{ss}}}$. Any tensor-compatible automorphism of this functor is determined by its action on $\omega^{\mathrm{dR}}(\mathbb{Q}(-1)) = \mathbb{Q}$. Therefore, for any $\mathbb{Q}$-algebra R we have

$$\mathrm{Aut}^{\otimes}_{\mathsf{MT}(\mathbb{Z})^{\mathrm{ss}}}(\omega^{\mathrm{dR}}|_{\mathsf{MT}(\mathbb{Z})^{\mathrm{ss}}})(R) = \mathrm{Aut}_R(R) = R^{\times}. \tag{4.11}$$

This implies that the motivic Galois group of the semisimple subcategory $\mathsf{MT}(\mathbb{Z})^{\mathrm{ss}}$ is $\mathbb{G}_m$. The inclusion $\mathsf{MT}(\mathbb{Z})^{\mathrm{ss}} \hookrightarrow \mathsf{MT}(\mathbb{Z})$ is equivalent to a character $\chi\colon G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \to \mathbb{G}_m$.

Consider the weight-graded pieces functor $\mathrm{gr}^M\colon\ \mathsf{MT}(\mathbb{Z}) \to \mathsf{MT}(\mathbb{Z})^{\mathrm{ss}}$. After applying the de Rham fiber functor ω^{dR}, the Hodge filtration canonically splits the weight filtration in the mixed Tate case. This means that for every object $V \in \mathsf{MT}(\mathbb{Z})$ there is a natural isomorphism

$$\mathrm{gr}^M_n V^{\mathrm{dR}} \cong M_{2n}V^{\mathrm{dR}} \cap F^n V^{\mathrm{dR}}. \tag{4.12}$$

It follows that the character χ has a section $\mathbb{G}_m \to G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$. Define

$$U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} := \ker(\chi).$$

By definition, $U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ consists of automorphisms that act trivially on the simple objects $\mathbb{Q}(n)$, which are the weight graded pieces of all objects of $\mathsf{MT}(\mathbb{Z})$. Therefore $U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ is pro-unipotent, since each such automorphism respects the weight filtration and acts trivially on gr^M. We obtain an exact sequence of affine group schemes

$$1 \to U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \to G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \xrightarrow{\chi} \mathbb{G}_m \to 1,$$

The section $\mathbb{G}_m \to G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ splits this exact sequence and defines an isomorphism

$$G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \cong U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \rtimes \mathbb{G}_m.$$

Currently, it is known that the Lie algebra $\mathfrak{k} := \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})})$ is non-canonically isomorphic to the completed free Lie algebra $\mathrm{Lie}(\sigma_3, \sigma_5, \dots)^{\wedge}$ on generators σ_{2n+1} in degree $-(2n+1)$. These σ elements are mysterious and obtaining canonical elements is a subject of current research; see [9], where they are connected to the action of $\mathfrak{k}$ on the fundamental Lie algebra of the infinitesimal punctured Tate curve.

4.3.2 The Lefschetz period

The most basic example of a motivic period is the Lefschetz period[5] $\mathbb{L} = \mathbb{L}^{\mathfrak{m}}$. It satisfies $\mathrm{per}(\mathbb{L}) = 2\pi i$. We recall the definition here as it is widely used in this paper.

Definition 4.3.1 (Lefschetz period)**.** The Lefschetz period is

$$\mathbb{L} = \mathbb{L}^{\mathfrak{m}} := [H^1(\mathbb{G}_m), [dz/z], \gamma_0]^{\mathfrak{m}},$$

where $H^1(\mathbb{G}_m) \cong \mathbb{Q}(-1) \in \mathsf{MT}(\mathbb{Z})$ and $\gamma_0 \in H_1(\mathbb{C}^\times, \mathbb{Q}) \cong H^1_{\mathrm{B}}(\mathbb{G}_m)^\vee$ is the class of a small counterclockwise loop around the origin in $\mathbb{C}^\times$.

Using (4.10) we may compute the action of $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ on $\mathbb{L}$ explicitly. Recall that $H^1_{\mathrm{dR}}(\mathbb{G}_m)$ is one dimensional and spanned by the cohomology class of $\omega_0 = dz/z$. Any $g \in G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}(\mathbb{Q})$ acts on $\mathbb{Q}(-1)$ through the character $\chi \colon G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \to \mathbb{G}_m$. Hence we have

$$g(\mathbb{L}) := [\omega_0]^\vee \left(g_{\mathbb{Q}(-1)}\left([\omega_0]\right)\right) \mathbb{L} = \chi(g)\mathbb{L}.$$

4.3.3 Motivic multiple zeta values

Let

$${}_0\Pi_1^{\mathrm{mot}} := \pi_1^{\mathrm{mot}}(\mathbb{P}^1 \backslash \{0, 1, \infty\}, \vec{1}_0, -\vec{1}_1)$$

denote the motivic path torsor between the tangential basepoints $\vec{1}_0$ and $\vec{1}_1$ on $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ defined in §3.5.4. Its affine ring $\mathcal{O}({}_0\Pi_1^{\mathrm{mot}})$ is an ind-object of $\mathsf{MT}(\mathbb{Z})$ [15]. Via the fully faithful functor $\omega^{\mathcal{H}}$ it can be considered as an ind-object of $\mathcal{H}$:

$$\mathcal{O}({}_0\Pi_1^{\mathrm{mot}}) = (\mathcal{O}({}_0\Pi_1^{\mathrm{B}}), \mathcal{O}({}_0\Pi_1^{\mathrm{dR}}), c).$$

Let $\mathrm{dch} \in {}_0\Pi_1^{\mathrm{B}}(\mathbb{R})$ denote the straight line path from $\vec{1}_0$ to $-\vec{1}_1$.

Definition 4.3.2 (Motivic multiple zeta value)**.** The $\mathbb{Q}$-algebra of motivic multiple zeta values is the ring $\mathcal{Z}^{\mathfrak{m}}$ generated by matrix coefficients $[\mathcal{O}({}_0\Pi_1^{\mathrm{mot}}), w, \mathrm{dch}]^{\mathfrak{m}}$, where w ranges over elements of $\mathcal{O}({}_0\Pi_1^{\mathrm{dR}})$.[6]

There is a canonical isomorphism $\mathcal{O}({}_0\Pi_1^{\mathrm{dR}}) \cong T^c(e_0, e_1)$ with the shuffle algebra on two symbols [16] (we define the shuffle algebra in §1.3.1.1). Here e_0 and e_1 are formal symbols corresponding to the algebraic differential forms $\omega_0 = dz/z$ and $\omega_1 = dz/(1-z)$ generating $H^1_{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0, 1, \infty\})$. They are described in more detail in §5.3.1.

[5] Typically, motivic periods are distinguished by a superscript $\mathfrak{m}$. However, we use the Lefschetz period so frequently that it eases notation to drop the superscript in this case.

[6] The ind-object $\mathcal{O}({}_0\Pi_1^{\mathrm{mot}})$ has a weight filtration by finite-dimensional subobjects $M_r\mathcal{O}({}_0\Pi_1^{\mathrm{mot}}) \in \mathsf{MT}(\mathbb{Z})$. A motivic MZV $\zeta^{\mathfrak{m}}(w)$ of *weight* n can be written as the matrix coefficient $[M_n\mathcal{O}({}_0\Pi_1^{\mathrm{mot}}), w, \mathrm{dch}]^{\mathfrak{m}}$.

Remark 4.3.3 (Admissible words). Let $w \in T^c(e_0, e_1)$ be a single word. The motivic multiple zeta value $[\mathcal{O}({}_0\Pi_1^{\mathrm{mot}}), w, \mathrm{dch}]^{\mathfrak{m}} \in \mathcal{Z}^{\mathfrak{m}}$ is denoted by $\zeta^{\mathfrak{m}}(w)$. If w is *admissible*, meaning it is of the form $w = e_1 v e_0$ for some other word v, then it may be written in the form

$$w = e_1 e_0^{k_1 - 1} \cdots e_1 e_0^{k_r - 1} \quad \text{with } k_1 \dots k_{r-1} \geq 1 \text{ and } k_r \geq 2. \tag{4.13}$$

We use the notation $\zeta^{\mathfrak{m}}(k_1, \dots, k_r) := \zeta^{\mathfrak{m}}(w)$ for an admissible motivic MZV.

Remark 4.3.4 (Shuffle-regularisation). There is a unique shuffle-algebra homomorphism $\zeta^{\mathfrak{m}} \colon T^c(e_0, e_1) \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$, such that whenever w is admissible of the form (4.13) we have $\zeta^{\mathfrak{m}}(w) = \zeta^{\mathfrak{m}}(k_1, \dots, k_r)$, and such that $\zeta^{\mathfrak{m}}(e_0) = \zeta^{\mathfrak{m}}(e_1) = 0$ [13, Proposition 1.173]. We have $\mathrm{per}\, \zeta^{\mathfrak{m}}(w) = \zeta(w)$, where $\zeta(w) \in \mathbb{R}$ is a shuffle-regularised real MZV.

It is possible to write down the motivic coaction on motivic multiple zeta values in terms of a combinatorial formula [5, Theorem 2.4]. This coaction may be used to find relations between motivic MZVs. In a related vein, we will show in Chapter 15 that the coaction may be used to find explicit expressions for motivic MZVs in terms of motivic iterated Eisenstein integrals.

4.3.4 Brown's theorem

In the influential paper [5], Brown proved that the action of $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ on $\mathcal{O}({}_0\Pi_1^{\mathrm{mot}})$ is faithful by exhibiting a specific basis for $\mathcal{Z}^{\mathfrak{m}}$ (the *Hoffman basis*). Brown's theorem implies that there is an isomorphism $\mathcal{Z}^{\mathfrak{m}}[\mathbb{L}^{-1}] \xrightarrow{\sim} \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$.

In this thesis we use a similar idea – namely, we demonstrate that all elements of $\mathcal{Z}^{\mathfrak{m}}$ may be obtained from a certain set of periods (in this case, iterated Eisenstein integrals). As a corollary, we show that $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ acts faithfully on the affine ring of a certain group scheme related to iterated Eisenstein integrals. See Theorem 13.1.1.

Chapter 5

Fundamental groups

For us, fundamental groups are a convenient tool for packaging and manipulating generating series of periods. The relevant fundamental groups for this purpose are the Betti and de Rham fundamental groups, as well as the usual topological fundamental group. For technical reasons, we will also have to consider *relative* versions of these groups. These notions are defined in this chapter.

5.1 The topological fundamental group

Let X be a scheme[1] and let $x \in X(\mathbb{C})_{\mathrm{bp}}$. (The set $X(K)_{\mathrm{bp}}$ of K-rational basepoints on X is defined in Definition 3.5.1. It consists of $X(K)$ together with the K-rational tangential basepoints on X.) The topological fundamental group of X, based at x, is

$$\pi_1^{\mathrm{top}}(X, x) := \pi_1(X(\mathbb{C}), x).$$

5.1.1 Quick computations of fundamental groups

In this section we provide brief and informal computations of the main fundamental groups we use. This is mainly done for interest and to explain the ways in which their elements can be interpreted.

5.1.1.1 $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

The space of complex points of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is $\mathbb{C} \setminus \{0, 1\}$. For any choice of basepoint the fundamental group is a free group on two generators. We provide a description of these generators for a particular tangential basepoint.

[1]When X is an algebraic stack, $X(\mathbb{C})$ is a complex-analytic orbifold and the fundamental group here refers to the orbifold fundamental group [22].

Recall the definitions of the tangential basepoints $\vec{1}_0$ and $\vec{1}_1$ given in §3.5.4. Let σ_p denote a small counterclockwise loop based at $\vec{1}_p$. Let $\gamma_p^{\pm,\mp}$ denote the homotopy class of a *counterclockwise* semicircle from $\pm\vec{1}_p$ to $\mp\vec{1}_p$, so that $\gamma_p^{+,-} \cdot \gamma_p^{-,+} = \sigma_p$. We also let $(\gamma_p^{\pm,\mp})^{-1}$ be the homotopy class of a *clockwise* semicircle from $\mp\vec{1}_p$ to $\pm\vec{1}_p$, so that $\gamma_p^{\pm,\mp} \cdot (\gamma_p^{\pm,\mp})^{-1} = 1$. Finally, let dch denote the canonical straight line path from $\vec{1}_0$ to $-\vec{1}_1$. Then define

$$\gamma_0 := \gamma_1^{+,-} \cdot \mathrm{dch}^{-1} \cdot \sigma_0 \cdot \mathrm{dch} \cdot (\gamma_1^{+,-})^{-1}, \quad \gamma_1 := \sigma_1.$$

They generate the free group $\pi_1^{\mathrm{top}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_1)$. The element γ_p is the homotopy class of a counterclockwise loop based at $\vec{1}_1$ encircling the puncture at p only. We also define $\gamma_\infty := (\gamma_p\gamma_1)^{-1}$. It is the homotopy class of a counterclockwise loop based at $\vec{1}_1$ around the puncture at ∞. These elements satisfy the relation $\gamma_0\gamma_1\gamma_\infty = 1$.

5.1.1.2 $E^\times$

Here $E^\times$ denotes the fiber of $\mathcal{E}^\times \to \mathcal{M}_{1,1}$ over any basepoint in $\mathcal{M}_{1,1}(\mathbb{C})_{\mathrm{bp}}$. In particular, $E^\times$ can be a punctured elliptic curve (taking a basepoint $[\tau]$ for $\tau \in \mathfrak{H}$) or the punctured Tate curve (taking the basepoint $\partial/\partial q$ defined in §3.5.1). The space of complex points $E^\times(\mathbb{C})$ has the homotopy type of a punctured torus, so its fundamental group is free on two generators, α and β, for any choice of basepoint.

Consider the Jacobi uniformisation $E(\mathbb{C}) \cong \mathbb{C}^\times/q^{\mathbb{Z}}$, where $q = \exp(2\pi i\tau)$. Let us assume that τ is imaginary, so that q is real and $0 < q < 1$. As a basepoint on the elliptic curve we use the tangent vector $\partial/\partial w$ on $\mathbb{C}^\times$ based at 1, where w denotes the coordinate on $\mathbb{C}^\times$. On $\mathbb{C}^\times$ the element α may be represented by a path consisting of a positively-oriented semicircle around 1 beginning at $\partial/\partial w$, followed by a positively-oriented circle around 0, followed by a positively-oriented semicircle around 1 returning to $\partial/\partial w$. The element β may be represented by a path consisting of a negatively-oriented semicircle around 1 starting at $\partial/\partial w$, followed by a straight line to $\partial/\partial w$ based at q. The point q is equivalent to 1 in $\mathbb{C}^\times/q^{\mathbb{Z}}$, so this defines a loop.

Suppose now that $E = \mathcal{E}_{\partial/\partial q}$ is the fiber of the Tate curve over $\partial/\partial q$. Its complex points can be uniformised in a similar way. The basepoint $\partial/\partial w$ on $\mathbb{C}^\times$ based at 1 corresponds to the tangential basepoint $\partial/\partial w$ on $\mathcal{E}^\times_{\partial/\partial q}$ defined in §3.5.2.

Hain describes how to work with this space topologically [26, §16]: take a real-oriented blowup of $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$ at the three punctured points, so that tangential basepoints on the set $\{0,1,\infty\}$ become regular basepoints along the boundary, and then identify the boundary circles around 0 and ∞. The resulting space is homotopy equivalent to $\mathcal{E}^{\times}_{\partial/\partial q}$. Figure 5.1 depicts Hain's model for the fundamental groupoid of $\mathcal{E}^{\times}_{\partial/\partial q}$ together with the generators α and β.

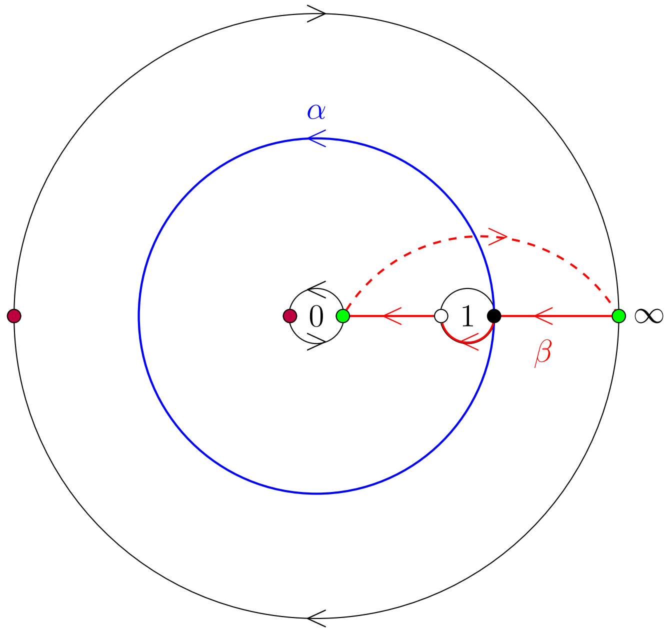


Figure 5.1: The fundamental groupoid of $\mathcal{E}^{\times}_{\partial/\partial q}$. The boundaries around 0 and ∞ are identified, taking into account their orientations. The dots represent the tangential basepoints $\pm\vec{1}_p$ for each $p \in \{0, 1, \infty\}$; those of the same colour are identified in the fundamental groupoid of $\mathcal{E}^{\times}_{\partial/\partial q}$. The black dot is the tangential basepoint $\vec{1}_1$ corresponding to $\partial/\partial w$. The generators α and β for $\pi_1^{\mathrm{top}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ are indicated.

5.1.1.3 $\mathcal{M}_{1,1}$

The basic orbifold $\mathcal{M}_{1,1}(\mathbb{C})$ is the orbifold quotient $[SL_2(\mathbb{Z})\backslash\mathfrak{H}]$, [22, §3.5]. The "universal covering" map $b_0 \colon \mathfrak{H} \to \mathcal{M}_{1,1}$ defines a basepoint[2] b_0 on $\mathcal{M}_{1,1}$, and a canonical isomorphism $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, b_0) \cong SL_2(\mathbb{Z})$. This isomorphism identifies $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, b_0)$ with the group of deck transformations of the universal cover $b_0 \colon \mathfrak{H} \to \mathcal{M}_{1,1}$, which equals $SL_2(\mathbb{Z})$ by definition.

The inclusion $\{iy : y > 0\} \hookrightarrow \mathfrak{H}$, which is a homotopy equivalence corresponding to the basepoint $\partial/\partial q$, induces an isomorphism $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, b_0)$. Finally, the inclusion of any point $\{\tau\} \hookrightarrow \mathfrak{H}$, which is also a homotopy equivalence, induces an isomorphism $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, [\tau]) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, b_0)$. To conclude, for x of any of these basepoints there is a natural isomorphism

$$\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x) \cong SL_2(\mathbb{Z}).$$

[2]See [22, §3.3] for an explanation of the homotopy theory of orbifolds.

We now give a description of the fundamental group in the case $x = \partial/\partial q$.

The group $SL_2(\mathbb{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

as noted in §2.1.1. Setting $U = ST$ gives the presentation (2.1).

The points i and ρ on $\mathfrak{H}$ are stabilised by the subgroups $\langle S\rangle \cong \mathbb{Z}/4\mathbb{Z}$ and $\langle U\rangle \cong \mathbb{Z}/6\mathbb{Z}$ respectively. Under the isomorphism $\pi_1(\mathcal{M}_{1,1}, \partial/\partial q) \cong SL_2(\mathbb{Z})$ the element S corresponds to a loop around $[i]$ based at $\partial/\partial q$, U corresponds to a loop around $[\rho]$ based at $\partial/\partial q$, and T corresponds to a small loop around the cusp based at $\partial/\partial q$. The orbifold points $[i]$ and $[\rho]$ may be thought of as "between a point and a puncture"; for example, the equation $S^4 = \mathrm{id}$ means that the loop around $[i]$ corresponding to S must be traversed 4 times before it is possible to "pull" it through the point $[i]$.

In a more concrete manner, we may regard the element of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q)$ corresponding to S as the imaginary axis on $\mathfrak{H}$. It is a straight line path on the extended upper half plane $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$ from a unit tangent vector at $\tau = 0$ to the unit tangent vector at $\tau = i\infty$, which may be identified with $\partial/\partial q$. A procedure for regularising integrals along this path is given in §6.1.

Now let x be any of the basepoints above. We consider a different interpretation for the fundamental group $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x)$ that is described further in §8.1. By the moduli property of $\mathcal{M}_{1,1}$, the basepoint x corresponds to an isomorphism class $[E]$ of an elliptic curve E. The fibration $E \to \mathcal{E} \to \mathcal{M}_{1,1}$ and the canonical section $O\colon \mathcal{M}_{1,1} \to \mathcal{E}$ imply the existence of a natural monodromy action of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x)$ on $\pi_1^{\mathrm{top}}(E, O) \cong H_1(E(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}^2$. This corresponds to the inclusion $SL_2(\mathbb{Z}) \hookrightarrow GL_2(\mathbb{Z})$ up to conjugation.

The fact that the automorphisms of $H_1(E(\mathbb{C}), \mathbb{Z})$ coming from this action lie in the subgroup $SL_2(\mathbb{Z}) \subseteq GL_2(\mathbb{Z})$ corresponds to the fact that the monodromy action preserves the orientation of the elliptic curve.

5.1.1.4 $\mathcal{M}_{1,\vec{1}}$

Let $x \in \mathcal{M}_{1,\vec{1}}(\mathbb{C})_{\mathrm{bp}}$. By the algebraic description of $\mathcal{M}_{1,\vec{1}}$ given in §3.2, the basepoint x corresponds to an isomorphism class $[E, O, \omega]$ where $E/\mathbb{C}$ is an elliptic curve with identity section O and ω is a trivialisation of $\Omega^1_{E/\mathbb{C}}$.

As E is defined over $\mathbb{C}$ it has a Weierstrass equation of the form

$$y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3),$$

where $\alpha_i \in \mathbb{C}$ are pairwise distinct. This equation uniquely determines O and ω, and therefore determines an isomorphism class as above. This isomorphism class is equivalent to the triple $(\alpha_1, \alpha_2, \alpha_3)$ in the configuration space of 3 unordered points in $\mathbb{C}$, whose fundamental group is the braid group B_3 on 3 strands. This may be seen as follows: the homotopy class of a loop in the configuration space above, based at $(\alpha_1, \alpha_2, \alpha_3)$, is equivalent to a *braid* on 3 strings in $\mathbb{C} \times [0,1]$ with start-points $\{(\alpha_1, 0), (\alpha_2, 0), (\alpha_3, 0)\}$ and endpoints $\{(\alpha_1, 1), (\alpha_2, 1), (\alpha_3, 1)\}$. This is because throughout the deformation along $[0,1]$ the three points in $\mathbb{C}$ must remain distinct in order to define an element of the configuration space. Figure 5.2 shows an example of a braid representing an element of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, x)$:

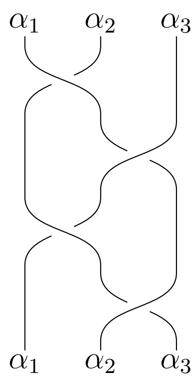


Figure 5.2: A braid.

The braid group has the presentation

$$B_3 = \langle t_A, t_B | t_A t_B t_A = t_B t_A t_B \rangle. \tag{5.1}$$

In Figure 5.2 the element t_A represents crossing the first strand above the second, and t_B represents crossing the second strand above the third (so that the element depicted is $t_A t_B^{-1} t_A t_B^{-1}$).

These generators may also be interpreted in terms of the monodromy action of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, x)$ on the fundamental group of the punctured elliptic curve $E^\times$ (see §8.1). The elements t_A and t_B may be identified with Dehn twists on simple closed curves $A, B \subseteq E(\mathbb{C})$ intersecting transversally at one point. One such choice is (the images of) α and β. This point of view realises $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, x)$ as the mapping class group of a genus 1 surface with one puncture. See §8.1 for a detailed description of the monodromy action.

5.2 The relative fundamental group

The unipotent completion of a discrete group Γ over a characteristic zero field K is an affine group scheme Γ^{un}/K that is is universal[3] with respect to homomorphisms $\Gamma \to U(K)$ into the K-points of unipotent groups U/K. When $K \subseteq \mathbb{C}$ and Γ is the topological fundamental group of a scheme X/K with respect to some basepoint, the unipotent fundamental group can be viewed as a nonabelian generalisation of the (co)homology of X and is, in a sense, constructed from the cohomology $H^1(X, K)$ with coefficients in K. It does not contain quite as much information as the profinite fundamental group $\hat{\pi}_1(X)$ or the pro-algebraic fundamental group $\pi_1^{\mathrm{alg}}(X)$ when $K = \mathbb{Q}$. This makes it a more tractable object to work with explicitly [31].

The relative completion [21] of Γ generalises the notion of unipotent completion. It occupies an intermediate zone between the unipotent completion and the profinite/pro-algebraic completions, and may be used as a substitute in cases when the unipotent completion is trivial. When R/K is a reductive group and $\rho : \Gamma \to R(K)$ is a Zariski-dense homomorphism, the completion of Γ relative to ρ is constructed from the cohomology $H^1(X, V)$ where V ranges over *all* irreducible representations of R.

Our main use of the general notion of relative completion is for $\Gamma = SL_2(\mathbb{Z})$, with $\rho\colon SL_2(\mathbb{Z}) \hookrightarrow SL_2(\mathbb{Q})$ the natural inclusion. It is used because the unipotent completion of $SL_2(\mathbb{Z})$ over $\mathbb{Q}$ is trivial, since there are no modular forms of weight 2 for $SL_2(\mathbb{Z})$ and consequently $H^1_{\mathrm{B}}(\mathcal{M}_{1,1}, \mathbb{Q}) = H^1(SL_2(\mathbb{Z}), \mathbb{Q}) = 0$, [11, §4]. The profinite completion of $SL_2(\mathbb{Z})$, on the other hand, is extremely large (by Belyi's theorem [2]). The remedy is to use the relative fundamental group, which is built not just from $H^1(SL_2(\mathbb{Z}), \mathbb{Q})$ but also from $H^1(SL_2(\mathbb{Z}), V_{2n})$ for *all* irreducible rational representations V_{2n} of $SL_2/\mathbb{Q}$ defined in §2.2. This naturally incorporates level 1 modular forms of *all* weights.

In the following subsections we define the relative fundamental group and review some of its structure. Crucially, in §5.2.1 we define two different *realisations* of the relative fundamental group, called Betti and de Rham realisations, generalising the analogous notions in cohomology; in §5.2.3 we note an important decomposition theorem for the relative fundamental group; in §5.2.4 we introduce a comparison isomorphism for relative fundamental groups; and in §5.2.5 we show how relative completion generalises unipotent completion.

[3]Here universal means that $\Gamma \to \Gamma^{\mathrm{un}}$ is left adjoint to the functor $U \mapsto U(K)$ from the category of pro-unipotent groups over K to the category of abstract groups.

5.2.1 Abstract relative completion

Here we use a geometric variant of relative completion, due to Brown [7, §12]. The advantage of this method is that it is possible to define all realisations of the relative completion simultaneously from the following categorical construction.

Definition 5.2.1 (Abstract relative completion)**.** Let (C, ω) be a neutral Tannakian category over a field K of characteristic zero, and let $\mathsf{S} \hookrightarrow \mathsf{C}$ be a full semisimple[4] Tannakian subcategory with fiber functor $\omega|_{\mathsf{S}}$. Define a subcategory $\mathsf{F}(\mathsf{C}, \mathsf{S}) \hookrightarrow \mathsf{C}$ whose objects are objects $V \in \mathsf{C}$ with a (finite, exhaustive, separated) increasing filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

by C-subobjects of V, such that each graded piece V_{i+1}/V_i is isomorphic to an object in S. Then $\mathsf{F}(\mathsf{C}, \mathsf{S})$ is a Tannakian category equipped with the fiber functor $\omega|_{\mathsf{F}(\mathsf{C},\mathsf{S})}$. We define the fundamental group of (C, ω) relative to S to be the Tannakian fundamental group

$$\mathcal{G} = \pi_1(\mathsf{C}, \mathsf{S}, \omega) := \mathrm{Aut}^{\otimes}_{\mathsf{F}(\mathsf{C},\mathsf{S})}\left(\omega|_{\mathsf{F}(\mathsf{C},\mathsf{S})}\right).$$

Let $S = \mathrm{Aut}^{\otimes}_{\mathsf{S}}(\omega|_{\mathsf{S}})$. It is pro-reductive because S is semisimple [17, Proposition 2.23]. There is an inclusion $\mathsf{S} \hookrightarrow \mathsf{F}(\mathsf{C}, \mathsf{S})$ given by equipping an object $V \in \mathsf{S}$ with the trivial filtration $0 = V_0 \subseteq V_1 = V$. This inclusion induces a faithfully flat morphism of affine group schemes $\mathcal{G} \to S$ whose kernel is a pro-unipotent group $\mathcal{U}$, exhibiting $\mathcal{G}$ as an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to S \to 1. \tag{5.2}$$

Remark 5.2.2. The inclusion $\mathsf{F}(\mathsf{C}, \mathsf{S}) \hookrightarrow \mathsf{C}$ is equivalent to a faithfully flat morphism of affine group schemes $\pi_1(\mathsf{C}, \omega) \to \mathcal{G}$. We do not make use of this morphism in the sequel.

5.2.2 Relative fundamental groups

Let X be a smooth, connected scheme[5] over a field $K \subseteq \mathbb{C}$, and let $x \in X(K)_{\mathrm{bp}}$. To this data we associate two neutral Tannakian categories over K:

[4] "Semisimple" means that every object of S is a direct sum of simple objects of C.

[5] In one case we must consider the relative fundamental group of the *algebraic stack* $\mathcal{M}_{1,1}$. As described in §3.2, this is the stack quotient $[\mathbb{G}_m \backslash \mathcal{M}_{1,\vec{1}}]$. We define its relative fundamental groups by looking at objects of $\mathsf{LS}(\mathcal{M}_{1,\vec{1}})$ and $\mathsf{Con}(\mathcal{M}_{1,\vec{1}})$ with a $\mathbb{G}_m$-action that are $\mathbb{G}_m$-invariant and trivial on $\mathbb{G}_m$-orbits. We then applying the same Tannakian machinery as for schemes.

- The category $\mathsf{LS}(X) = \mathsf{LS}_K(X)$ of local systems V of finite-dimensional K-vector spaces on X, equipped with the *Betti* fiber functor ω_x^{B} sending each local system V to the stalk V_x.
- The category $\mathsf{Con}(X) = \mathsf{Con}_K(X)$ of algebraic K-vector bundles with a flat connection $(\mathcal{V}, \nabla)$ on X and regular singularities at infinity, equipped with the *de Rham* fiber functor ω_x^{dR} sending $(\mathcal{V}, \nabla)$ to the fiber $\mathcal{V}(x) = \mathcal{V}_x/\mathfrak{m}_x\mathcal{V}_x$.

Let $\mathsf{S}^{\mathrm{B}} \hookrightarrow \mathsf{LS}(X)$ (resp. $\mathsf{S}^{\mathrm{dR}} \hookrightarrow \mathsf{Con}(X)$) be a full semisimple Tannakian subcategory of $\mathsf{LS}(X)$ (resp. $\mathsf{Con}(X)$) with fiber functor given by the restriction of ω_x^{B} (resp. ω_x^{dR}). Denote their Tannaka groups by S^{B} and S^{dR} respectively.

Definition 5.2.3 (Relative fundamental group)**.** Applying Definition 5.2.1 to the above setup produces two affine group schemes over K,

$$\pi_1^{\mathrm{rel,B}}(X, x) := \pi_1(\mathsf{LS}(X), \mathsf{S}^{\mathrm{B}}, \omega_x^{\mathrm{B}}), \quad \pi_1^{\mathrm{rel,dR}}(X, x) := \pi_1(\mathsf{Con}(X), \mathsf{S}^{\mathrm{dR}}, \omega_x^{\mathrm{dR}}), \tag{5.3}$$

respectively called the *Betti* and *de Rham* relative fundamental groups of X with basepoint x. They depend on the choice of the semisimple subcategories.

5.2.3 Splitting

For relative completions of fundamental groups, the exact sequence (5.2) always splits [23, Proposition 3.1]. This means that for $\bullet \in \{\mathrm{B}, \mathrm{dR}\}$ the relative fundamental group is isomorphic to a semidirect product

$$\pi_1^{\mathrm{rel},\bullet}(X, x) \cong \mathcal{U}^\bullet \rtimes S^\bullet, \tag{5.4}$$

where $S^\bullet$ is pro-reductive and $\mathcal{U}^\bullet$ is pro-unipotent. This decomposition is non-canonical because it depends upon the choice of splitting.

5.2.4 Comparison isomorphism

Let $X_{\mathbb{C}} = X \times_K \mathbb{C}$ and let $\hat{x} \in X_{\mathbb{C}}(\mathbb{C})$ be a lift of x. The Riemann-Hilbert correspondence induces a tensor-equivalence $\mathsf{Con}_{\mathbb{C}}(X_{\mathbb{C}}) \xrightarrow{\sim} \mathsf{LS}_{\mathbb{C}}(X_{\mathbb{C}})$. Let us assume that this further induces an equivalence $\mathsf{S}_{\mathbb{C}}^{\mathrm{dR}} \xrightarrow{\sim} \mathsf{S}_{\mathbb{C}}^{\mathrm{B}}$. Tannakian duality then produces a canonical isomorphism $\pi_1^{\mathrm{rel},B}(X_{\mathbb{C}}, \hat{x}) \xrightarrow{\sim} \pi_1^{\mathrm{rel,dR}}(X_{\mathbb{C}}, \hat{x})$. Because relative completion commutes with base change [23, §3.3] we obtain a canonical isomorphism

$$\pi_1^{\mathrm{rel},B}(X, x) \times_K \mathbb{C} \xrightarrow{\sim} \pi_1^{\mathrm{rel,dR}}(X, x) \times_K \mathbb{C}. \tag{5.5}$$

Definition 5.2.4 (Comparison isomorphism)**.** The isomorphism (5.5) is called the comparison isomorphism between relative Betti and de Rham fundamental groups.

The isomorphism (5.5) is induced by viewing the affine rings as ind-objects in $\mathcal{H} \otimes_{\mathbb{Q}} K$, whose ring of motivic periods is $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \otimes K$. By §4.2.3 it lifts to a *universal comparison isomorphism*

$$\pi_1^{\mathrm{rel},B}(X,x) \times_K (\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \otimes_{\mathbb{Q}} K) \xrightarrow{\sim} \pi_1^{\mathrm{rel,dR}}(X,x) \times_K (\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \otimes_{\mathbb{Q}} K). \tag{5.6}$$

Remark 5.2.5 (Generating series)*.* The Betti fiber functor $\omega_x^{\mathrm{B}} \colon \mathsf{LS}(X) \to \mathsf{Vect}_K$ factors through the category $\mathsf{Rep}_K(\pi_1^{\mathrm{top}}(X,x))$ of K-representations of the topological fundamental group[6]. This induces a canonical Zariski-dense homomorphism

$$\pi_1^{\mathrm{top}}(X,x) \to \pi_1^{\mathrm{rel,B}}(X,x)(K). \tag{5.7}$$

Consider the composition

$$\pi_1^{\mathrm{top}}(X,x) \to \pi_1^{\mathrm{rel,B}}(X,x)(K) \to \pi_1^{\mathrm{rel,dR}}(X,x)(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \otimes_{\mathbb{Q}} K) \tag{5.8}$$

where the first map is (5.7) and the second is induced by the universal comparison isomorphism. It may be viewed as the map sending an element $\gamma \in \pi_1^{\mathrm{top}}(X,x)$ to the "motivic parallel transport" along γ of a horizontal section of a suitable universal connection on X. By finding explicit generators for $\pi_1^{\mathrm{rel,dR}}(X,x)$ this map may also be viewed as sending γ to the generating series of all homotopy-invariant motivic iterated integrals on X along γ.

Definition 5.2.6 (Unipotent and reductive parts)**.** We denote the homomorphism (5.8) by the formula

$$\gamma \mapsto \gamma^{\mathfrak{m}}.$$

Using the choice of splitting $\pi_1^{\mathrm{rel,dR}}(X,x) \cong \mathcal{U}^{dR} \rtimes S^{\mathrm{dR}}$ chosen in (5.4), write

$$\gamma^{\mathfrak{m}} = (\gamma_u^{\mathfrak{m}}, \gamma_0^{\mathfrak{m}}).$$

The element $\gamma_u^{\mathfrak{m}} \in \mathcal{U}^{\mathrm{dR}}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \otimes_{\mathbb{Q}} K)$ refers to the "unipotent part" of $\gamma^{\mathfrak{m}}$. It depends upon the choice of splitting. The element $\gamma_0^{\mathfrak{m}} \in S^{\mathrm{dR}}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \otimes_{\mathbb{Q}} K)$ refers to the "reductive" or "degree 0" part of $\gamma^{\mathfrak{m}}$. It is canonically defined.

The map $\gamma \mapsto \gamma_u^{\mathfrak{m}}$ defines a nonabelian cocycle $\pi_1^{\mathrm{top}}(X,x) \to \mathcal{U}^{\mathrm{dR}}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \otimes_{\mathbb{Q}} K)$. The map $\gamma \mapsto \gamma_0^{\mathfrak{m}}$ is a homomorphism. When S is the trivial group we have $\gamma^{\mathfrak{m}} = \gamma_u^{\mathfrak{m}}$. In this case the cocycle $\gamma \mapsto \gamma_u^{\mathfrak{m}}$ reduces to a group homomorphism.

Remark 5.2.7*.* In general, $\gamma_u^{\mathfrak{m}}$ is not canonically defined, as it depends upon the choice of splitting (5.4). When such a splitting is canonical, however, $\gamma_u^{\mathfrak{m}}$ is canonically defined. This is of relevance for the sequel; see (8.6).

[6]The category $\mathsf{Rep}_K(\pi_1^{\mathrm{top}}(X,x))$ is equipped with the forgetful fiber functor.

5.2.5 Relation to unipotent completion

Let $\bullet \in \{\mathrm{B}, \mathrm{dR}\}$. When $\mathsf{S}^\bullet$ is the semisimple subcategory generated by the unit object $\mathbf{1}$, the relative fundamental group $\pi_1^{\mathrm{rel},\bullet}(X,x)$ is pro-unipotent. In this case we denote $\pi_1^{\mathrm{rel},\bullet}(X,x)$ by $\pi_1^\bullet(X,x)$.

In the Betti case the unit object is the constant sheaf $\mathbf{1} = \underline{K}$ and $\pi_1^{\mathrm{B}}(X,x)$ is the unipotent completion of $\pi_1^{\mathrm{top}}(X,x)$. The canonical homomorphism (5.7) is none other than the natural map from $\pi_1^{\mathrm{top}}(X,x)$ to the rational points of its unipotent completion. In the de Rham case the unit object is the vector bundle with trivial connection $\mathbf{1} = (\mathcal{O}_X, d)$.

Throughout this paper, we only make use of a nontrivial semisimple subcategory $\mathsf{S}^\bullet$ in the cases $(X,x) = (\mathcal{M}_{1,1}, \partial/\partial q)$ or $(\mathcal{M}_{1,\vec{1}}, \vec{\mathsf{v}})$, where $\vec{\mathsf{v}} := \partial/\partial q + \partial/\partial w$ was defined in §3.5.3. In these cases $\mathsf{S}^\bullet$ is generated by an explicit object depending on the universal family over $\mathcal{M}_{1,1}$ (resp. $\mathcal{M}_{1,\vec{1}}$). This is because the unipotent completion of $SL_2(\mathbb{Z})$ is trivial. In all other situations we only require relative completion with respect to $\mathsf{S}^\bullet = \langle \mathbf{1} \rangle_\otimes$.

5.2.6 The coordinate ring of $\pi_1^{\mathrm{rel},\mathrm{dR}}(X,x)$ and iterated integrals

The coordinate ring of the de Rham relative fundamental group $\pi_1^{\mathrm{rel},\mathrm{dR}}(X,x)$ is defined in [21, §8]. As described in §7 of *loc. cit.*, it may be defined independently of the Tannakian definition using the reduced bar construction on a commutative differential graded algebra (CDGA) constructed from "locally constant iterated integrals" on X.

In particular, the coordinate ring of its unipotent radical $\mathcal{U}^{\mathrm{dR}}$ is defined as the 0th cohomology of the reduced bar construction on the CDGA

$$\bigoplus_{V \in \mathcal{I}} \Omega^\bullet(X, C(V)^\vee) \otimes_K V,$$

where $\mathcal{I}$ denotes a complete set of representatives for isomorphism classes of irreducible representations of S^{dR} over K and $C(V) \in \mathsf{Con}(X)$ is the associated vector bundle with connection on X. If x is a finite[7] basepoint $x\colon \mathrm{Spec}(K) \to X$, the CDGA inherits an augmentation via the composition

$$\begin{aligned} \Omega^\bullet(X, C(V)^\vee) \otimes_K V &\to \Omega^\bullet(\mathrm{Spec}(K), x^*C(V)^\vee) \otimes_K V \\ &\cong H^0(\mathrm{Spec}(K), x^*C(V)^\vee) \otimes_K V \\ &\cong C(V)_x^\vee \otimes_K V \end{aligned}$$

[7] A similar computation works for tangential basepoints.

$$\cong V^\vee \otimes_K V \to K,$$

where the first map is induced by pullback along x. The reduced bar construction acquires a CDGA structure via the connection on $C(V)$ and the wedge product of differential forms.

The coordinate ring $\mathcal{O}(\mathcal{U}^{\mathrm{dR}})$ is isomorphic to H^0 of the reduced bar construction on the above CDGA, which is naturally a Hopf algebra; details are described in [21]. It is the shuffle algebra (see §1.3.1.1) on elements w of positive degree in $\bigoplus_{V\in\mathcal{I}} \Omega^\bullet(X, C(V)^\vee) \otimes_K V$. Its elements may be written as words $[w_1|\cdots|w_s]$. It is equipped with the shuffle product, the deconcatenation coproduct and the counit making use of the augmentation described above.

As it is of use in the sequel, we now define an important increasing filtration on $\mathcal{O}(\mathcal{U}^{\mathrm{dR}})$.

Definition 5.2.8. The length filtration $L_\bullet\mathcal{O}(\mathcal{U}^{\mathrm{dR}})$ is the increasing filtration on the index s. The word $[w_1|\cdots|w_s]$ lies in $L_s\mathcal{O}(\mathcal{U}^{\mathrm{dR}})$.

5.2.6.1 Motivic iterated integrals

We may view the element $[w_1|\cdots|w_s]$ as an abstract "integrand" of an indefinite iterated integral of the forms $w_1,\ldots,w_s$, where iterated integrals were defined in Definition 1.4.1 and generalised to higher dimensional forms in [21, §6]. The unipotent part $\gamma_u^{\mathfrak{m}} \in \mathcal{U}^{\mathrm{dR}}(\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}})$ of an element $\gamma \in \pi_1^{\mathrm{top}}(X,x)$, defined in Definition 5.2.6, can be viewed as the element of $\mathrm{Hom}(\mathcal{O}(\mathcal{U}^{\mathrm{dR}}), \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}})$ that evaluates indefinite iterated integrals along γ. The motivic iterated integral $\int_\gamma^{\mathfrak{m}} w_1\cdots w_s$ is, by definition, the $\mathcal{H}$-period of $\mathcal{O}(\mathcal{U}^{\mathrm{dR}})$ defined as

$$\int_\gamma^{\mathfrak{m}} w_1\cdots w_s := \gamma_u^{\mathfrak{m}}\left([w_1|\cdots|w_s]\right).$$

Its image under the period map is the iterated integral $\int_\gamma w_1\cdots w_s \in \mathbb{C}$ as defined in Definition 1.4.1.

5.2.7 Functoriality

The relative fundamental groups can be viewed as functors as follows. Let K be a field of characteristic zero and define categories

- LS_K^*: the category of triples (X,x,V) where X is a scheme over K, $x\in X(K)_{\mathrm{bp}}$ is a K-rational basepoint and $V\in\mathsf{LS}_K(X)$. The morphims are pairs consisting of a smooth morphism of schemes $f\colon X_1\to X_2$ satisfying $f(x_1)=x_2$, together with a monomorphism $\hat{f}^{\mathrm{B}}\colon f^*V_2 \hookrightarrow V_1$ in $\mathsf{LS}_K(X_1)$.

- Con_K^*: the category of triples $(X, x, \mathcal{V})$ where X is a scheme over K, $x \in X(K)_{\mathrm{bp}}$ is a K-rational basepoint and $\mathcal{V} \in \mathsf{Con}_K(X)$. The morphisms are pairs consisting of a smooth morphism of schemes $f\colon X_1 \to X_2$ satisfying $f(x_1) = x_2$, together with a monomorphism $\hat{f}^{\mathrm{dR}}\colon f^*\mathcal{V}_2 \hookrightarrow \mathcal{V}_1$ in $\mathsf{Con}_K(X_1)$.

Proposition 5.2.9. *The assignments*

$$\begin{aligned} \mathsf{LS}_K^* &\to \mathsf{GrpSch}_K \\ (X, x, V) &\mapsto \pi_1(\mathsf{LS}_K(X), \langle V \rangle_\otimes, \omega_x^{\mathrm{B}}) \end{aligned}$$

and

$$\begin{aligned} \mathsf{Con}_K^* &\to \mathsf{GrpSch}_K \\ (X, x, \mathcal{V}) &\mapsto \pi_1(\mathsf{Con}_K(X), \langle \mathcal{V} \rangle_\otimes, \omega_x^{\mathrm{dR}}) \end{aligned}$$

define functors to the category of group schemes over K.

Proof. We prove the statement only for the de Rham relative fundamental group, since the proofs are direct parallels. By the Tannaka theorem, the result follows by checking that the map $(X, x, \mathcal{V}) \mapsto (\mathsf{F}(\mathsf{Con}(X), \langle \mathcal{V} \rangle_\otimes), \omega_x^{\mathrm{dR}})$ defines a functor from $(\mathsf{Con}_K^*)^{\mathrm{op}}$ to the category of neutral Tannakian categories over K, whose morphisms are additive tensor functors compatible with the fiber functors on each Tannakian category.

Let $(f, \hat{f}^{\mathrm{dR}})\colon (X_1, x_1, \mathcal{V}_1) \to (X_2, x_2, \mathcal{V}_2)$ be a morphism in Con_K^*. Pullback by f defines a functor $f^*\colon \mathsf{Con}_K(X_2) \to \mathsf{Con}_K(X_1)$. The monomorphism $\hat{f}^{\mathrm{dR}}\colon f^*\mathcal{V}_2 \hookrightarrow \mathcal{V}_1$ implies that this induces a functor $f^*\colon \langle \mathcal{V}_2 \rangle_\otimes \to \langle \mathcal{V}_1 \rangle_\otimes$ between the associated subcategories. These two properties define a functor

$$f^*\colon \mathsf{F}(\mathsf{Con}_K(X_2), \langle \mathcal{V}_2 \rangle_\otimes) \to \mathsf{F}(\mathsf{Con}_K(X_1), \langle \mathcal{V}_1 \rangle_\otimes).$$

This is a tensor functor because pullback commutes with tensor products. The property $f(x_1) = x_2$ implies that $\omega_{x_1}^{\mathrm{dR}} \circ f^* = \omega_{x_2}^{\mathrm{dR}}$. Therefore f^* is a functor between Tannakian categories. By duality we obtain a morphism of group schemes

$$\pi_1(\mathsf{Con}_K(X_1), \langle \mathcal{V}_1 \rangle_\otimes, \omega_{x_1}^{\mathrm{dR}}) \to \pi_1(\mathsf{Con}_K(X_2), \langle \mathcal{V}_2 \rangle_\otimes, \omega_{x_2}^{\mathrm{dR}}).$$

□

5.2.8 Lie algebraic structure

Let us now consider the Betti relative completion $\mathcal{G}^{\mathrm{B}}/\mathbb{Q}$ of scheme $X/\mathbb{Q}$ with respect to the semisimple category $\mathsf{S}^{\mathrm{B}} = \langle L \rangle_{\otimes}$ generated by a local system L of finite-dimensional vector spaces on X. Let S^{B} be the associated reductive group. By the equivalence between local systems and representations of the topological fundamental group we obtain a homomorphism $\Gamma := \pi_1^{\mathrm{top}}(X, x) \to S^{\mathrm{B}}(\mathbb{Q})$ for any choice of basepoint $x \in X(\mathbb{Q})_{\mathrm{bp}}$. This requires an identification between S^{B} and the automorphism group (scheme) of the stalk L_x.

Let us assume that for every representation V of Γ over $\mathbb{Q}$ the vector space $H^j(\Gamma, V)$ is finite-dimensional. This condition is satisfied when Γ is finitely presented, and hence when $X \times_{\mathbb{Q}} \mathbb{C}$ is a complex algebraic variety because in this case $X(\mathbb{C})$ is homotopy-equivalent to a finite complex.

Let $\mathcal{U}^{\mathrm{B}}$ be the unipotent radical of $\mathcal{G}^{\mathrm{B}}$ and let $\mathfrak{u}^{\mathrm{B}} = \mathrm{Lie}(\mathcal{U}^{\mathrm{B}})$. Let $\mathcal{I}$ be a complete set of representatives for isomorphism classes of finite-dimensional irreducible representations of S^{B} over $\mathbb{Q}$. Under certain basic assumptions (see [23, §3.2, 3.4]) there is an isomorphism

$$H_1(\mathfrak{u}^{\mathrm{B}}) \cong \prod_{V \in \mathcal{I}} H^1(\Gamma, V)^{\vee} \otimes_{\mathbb{Q}} V \tag{5.9}$$

and a continuous S^{B}-invariant surjection

$$\prod_{V \in \mathcal{I}} H^2(\Gamma, V)^{\vee} \otimes_{\mathbb{Q}} V \to H_2(\mathfrak{u}^{\mathrm{B}}),$$

where in each case the product is equipped with the product topology.

Such information can be used to determine the structure of $\mathfrak{u}^{\mathrm{B}}$. For example, when Γ has cohomological dimension 1, as when Γ is a finitely generated free group or when $\Gamma = SL_2(\mathbb{Z})$, we have $H^j(\Gamma, V) = 0$ for every $j \geq 2$. Consequently $H_2(\mathfrak{u}^{\mathrm{B}}) = 0$, which implies that $\mathfrak{u}^{\mathrm{B}}$ is non-canonically isomorphic to the completed free Lie algebra on $H_1(\mathfrak{u}^{\mathrm{B}}) \cong (\mathfrak{u}^{\mathrm{B}})^{\mathrm{ab}}$. By identifying $H^1(\Gamma, V) \cong H^1_{\mathrm{B}}(X, L(V))$ where $L(V) \in \mathsf{LS}(X)$ is the associated local system we obtain the description

$$\mathfrak{u}^{\mathrm{B}} \cong \mathrm{Lie}\left(\prod_{V \in \mathcal{I}} H^1_{\mathrm{B}}(X, L(V))^{\vee} \otimes_{\mathbb{Q}} V \right)^{\wedge}. \tag{5.10}$$

Note that this description is non-canonical because the generators for $(\mathfrak{u}^{\mathrm{B}})^{\mathrm{ab}}$ may be modified by any element of the commutator $[\mathfrak{u}^{\mathrm{B}}, \mathfrak{u}^{\mathrm{B}}]$.

An identical calculation on the de Rham side also holds relative to a choice of semisimple subcategory $\mathsf{S}^{\mathrm{dR}} = \langle \mathcal{L} \rangle_{\otimes}$ defined by an object $\mathcal{L} \in \mathsf{Con}(X)$. We obtain

$$\mathfrak{u}^{\mathrm{dR}} \cong \mathrm{Lie}\left(\prod_{V \in \mathcal{J}} H^1_{\mathrm{dR}}(X, C(V))^{\vee} \otimes_{\mathbb{Q}} V \right)^{\wedge}, \tag{5.11}$$

where $\mathcal{J}$ is now a complete set of representatives for isomorphism classes of finite-dimensional irreducible representations of S^{dR} over $\mathbb{Q}$ and $C(V) \in \mathsf{Con}(X)$ is the associated vector bundle with connection. Once again this description depends upon a choice of section of $\mathfrak{u}^{\mathrm{dR}} \to H_1(u^{\mathrm{dR}})$.

Let us assume that the choices of S^{B} and S^{dR} are compatible under the Riemann-Hilbert correspondence. We obtain a canonical isomorphism $\mathcal{U}^{\mathrm{B}}_{\mathbb{C}} \xrightarrow{\sim} \mathcal{U}^{\mathrm{dR}}_{\mathbb{C}}$ and hence a canonical isomorphism $\mathfrak{u}^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathfrak{u}^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$. It describes the mixed Hodge structure on the relative fundamental group [21]. We discuss this further in §5.2.9.

5.2.9 Mixed Hodge structure

Let us continue with the same assumptions and notation as in §5.2.8. In particular, these imply the existence of a bijection $\mathcal{I} \cong \mathcal{J}$. For every $V \in \mathcal{I}$ we obtain a local system $L(V) \in \mathsf{LS}(X)$, a vector bundle with connection $C(V) \in \mathsf{Con}(X)$, and a comparison isomorphism $c_V \colon H^1_{\mathrm{dR}}(X, C(V)) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{B}}(X, L(V)) \otimes_{\mathbb{Q}} \mathbb{C}$. Let

$$H^1(X, V) := \left(H^1_{\mathrm{B}}(X, L(V)), H^1_{\mathrm{dR}}(X, C(V), c_V \right),$$

considered as an object of $\mathcal{H}$.

As discussed in §5.2.8, there is a canonical comparison isomorphism[8]

$$\mathrm{comp} \colon \mathfrak{u}^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathfrak{u}^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}. \tag{5.12}$$

This comparison isomorphism induces a canonical (limit) mixed Hodge structure (MHS) on $\mathfrak{u} := (\mathfrak{u}^{\mathrm{B}}, \mathfrak{u}^{\mathrm{dR}}, \mathrm{comp})$, considered as a pro-object of $\mathcal{H}$. This, in turn, is induced from the canonical (limit) MHS on the relative fundamental group [21, 23]. In this section we briefly describe a general procedure to define this MHS.

There is a canonical MHS on the Lie algebra

$$H_1(\mathfrak{u}^{\mathrm{dR}}) \cong \prod_{V \in \mathcal{J}} H^1_{\mathrm{dR}}(X, C(V))^{\vee} \otimes_{\mathbb{Q}} V$$

[8]Note that a version of (5.12) also holds with $\mathbb{C}$ replaced by the ring $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ because $\mathcal{O}(\pi_1^{\mathrm{rel}}(X, x))$ defines an ind-object of $\mathcal{H}$.

induced by the MHS on $H^1_{\mathrm{dR}}(X, V)$. Choose a section of the abelianisation map $\mathfrak{u}^{\mathrm{dR}} \to H_1(\mathfrak{u}^{\mathrm{dR}})$ that preserves the Hodge filtration $F^\bullet$, the weight filtration $W_\bullet$ and, if one exists, the relative weight filtration $M_\bullet$. It is possible to choose such a section because abelianisation is a morphism of MHS and is therefore strict with respect to these filtrations. This section induces a Lie algebra surjection

$$\mathrm{Lie}(H_1(\mathfrak{u}^{\mathrm{dR}}))^\wedge \to \mathfrak{u}^{\mathrm{dR}}.$$

The Hodge filtration $F^\bullet \mathfrak{u}^{\mathrm{dR}}$ and weight filtrations $M_\bullet \mathfrak{u}^{\mathrm{dR}}$, $W_\bullet \mathfrak{u}^{\mathrm{dR}}$ are the images of those on $\mathrm{Lie}(H_1(\mathfrak{u}^{\mathrm{dR}}))^\wedge$ under this surjection. The Lie algebra $\mathfrak{u}^{\mathrm{dR}}$ therefore inherits a (bi)grading with respect to the weight filtration(s).

An analogous situation occurs in the Betti case, and when base changed to $\mathbb{C}$ the two cases are compatible via the comparison isomorphism.

Remark 5.2.10 (Uniqueness of bigrading). Any section of the abelianisation map preserving all these filtrations is unique for the Eisenstein quotient of the relative completion. This includes all cases we consider except that of $\mathfrak{u}_{1,1}$ (discussed in §5.5.1). See [27, §23] and [27, Theorem 23.1] for a precise statement. Hence, there is a canonical bigrading on $\mathfrak{u}$ in all cases considered except $\mathfrak{u}_{1,1}$.

5.3 De Rham fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

In this section we use the preceeding discussion to describe the de Rham fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ explicitly.

5.3.1 Structure

As described in §3.5.4, there are six tangential basepoints on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with good reduction at every prime: each puncture $p \in \{0, 1, \infty\}$ is equipped with two tangent vectors $\pm\vec{1}_p$ based at p. The fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with respect to these six tangential basepoints is built from two essentially different types of elements: loops based at a *single* tangent vector at a puncture p, circling that puncture only (or a semicircle between the positive and negative tangent vectors at p); and paths between tangent vectors at two *different* punctures. The de Rham fundamental groups of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with respect to these basepoints are all canonically isomorphic, because of the canonical de Rham splitting for mixed Tate motives; in the sequel, we use the basepoint $\vec{1}_1$.

The group $\pi_1^{\mathrm{top}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_1)$ is free on two generators (see §5.1.1.1). Therefore, $\pi_1^{\mathrm{top}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_1)$ has cohomological dimension 1. The de Rham analogue of (5.9) produces an isomorphism

$$\mathrm{Lie}(\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_p))^{\mathrm{ab}} \cong H_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, \mathbb{Q})^\vee = \mathbb{Q}\mathsf{x}_0 \oplus \mathbb{Q}\mathsf{x}_1$$

for any $p \in \{0,1,\infty\}$. The elements x_0 and x_1 each span a copy of $\mathbb{Q}(1)$; as explained in §5.2.9, this follows from the MHS on $H_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, \mathbb{Q})$.

There is therefore an isomorphism

$$\mathrm{Lie}\, \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_p) \cong \mathrm{Lie}(\mathsf{x}_0, \mathsf{x}_1)^\wedge.$$

This isomorphism is canonical by the results of [27, §23]; this follows from the triviality of the de Rham path torsor of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ between any two basepoints. Hence, under the comparison isomorphism

$$\mathrm{Lie}\, \pi_1^{\mathrm{B}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_p) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathrm{Lie}\, \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_p) \otimes_{\mathbb{Q}} \mathbb{C}$$

of (5.12), we have $\log(\sigma_p) \mapsto 2\pi i \mathsf{x}_p$. The elements x_0 and x_1 define the unique bigrading on $\mathrm{Lie}\, \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_p)$ that is compatible with the grading on $\mathfrak{k} = \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})})$ (see Remark 5.2.10).

The completed universal enveloping algebra of $\mathrm{Lie}\, \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_p)$ is the Hopf algebra $\mathbb{Q}\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$ described in §1.3.1.2. The group of R-points of the de Rham fundamental group $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_p)$ is isomorphic to its group of grouplike elements over R:

$$\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_p)(R) \cong \mathcal{G}(R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle), \tag{5.13}$$

By [16], the coordinate ring $\mathcal{O}(\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_p))$ is canonically isomorphic to the shuffle algebra $T^c(e_0, e_1)$ on the dual alphabet $\{e_0, e_1\}$ (these notions were defined in §1.3.1.1 and §1.3.1.3), and is the de Rham realisation of an ind-object in $\mathsf{MT}(\mathbb{Z})$. Its motivic periods are therefore contained in the subalgebra $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ of mixed Tate periods of $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$, as described in (4.4). The computations in §5.3.2 demonstrate this explicitly.

5.3.2 Specific $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$-points

In §5.3.1 we explained that the fundamental groupoid of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ with respect to its six natural tangential basepoints is generated by two distinct types of elements. In terms of the isomorphism (5.13), these two different types of elements lead to two different types of generating series of motivic periods for $\mathbb{P}^1 \setminus \{0,1,\infty\}$: exponential series and Drinfeld associators.

5.3.2.1 Exponentials

The simplest nontrivial element of $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_p)(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$ is the exponential series. It is the image $\sigma_p^{\mathfrak{m}}$ of the element σ_p of $\pi_1^{\mathrm{top}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_p)$, defined in §5.1.1.1, under the natural map $\pi_1^{\mathrm{top}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_p) \to \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_p)(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$ described in (5.8). Using the notation for matrix coefficients given in §4.2.2, and for motivic iterated integrals in §5.2.6.1, the map is given by the following formula, valid for $p = 0, 1$:

$$\begin{aligned}
\sigma_p \mapsto & \sum_{w \text{ basis element of } \mathcal{O}(\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_p))} [\mathcal{O}(\pi_1^{\mathrm{mot}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_p)), w, \gamma_p]^{\mathfrak{m}} w^{\vee} \\
&= \sum_{w \in M(\mathsf{x}_0, \mathsf{x}_1)} \left(\int_{\sigma_p}^{\mathfrak{m}} w^{\vee} \right) w \\
&= \sum_{k \geq 0} \left(\int_{\sigma_p}^{\mathfrak{m}} \omega_p^k \right) \mathsf{x}_p^k \\
&= \sum_{k \geq 0} \frac{\mathbb{L}^k}{k!} \mathsf{x}_p^k = \exp(\mathbb{L}\mathsf{x}_p),
\end{aligned}$$

where $M(\mathsf{x}_0, \mathsf{x}_1)$ denotes the free monoid on $\{\mathsf{x}_0, \mathsf{x}_1\}$ and $\omega_0 := dz/z$, $\omega_1 := dz/(1-z)$ are 1-forms on $\mathbb{P}^1 \backslash \{0,1,\infty\}$ generating its first de Rham cohomology. In going from the second to the third line we use that $\int_{\sigma_p}^{\mathfrak{m}} \omega_{j_1} \cdots \omega_{j_k} = 0$ unless all $j_1, \ldots, j_k$ are equal to p. This follows from Cauchy's theorem.

Finally, one verifies that $\exp(\mathbb{L}\mathsf{x}_p)$ is grouplike using that x_p is a primitive element of the Hopf algebra $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$.

5.3.2.2 Drinfeld associators

Let $j, k \in \{0,1,\infty\}$ be distinct and adjacent in the cyclic ordering $0 \to 1 \to \infty \to 0$. There is a natural "straight line path"

$$\mathrm{dch}_{jk} \in {}_j\Pi_k := \pi_1^{\mathrm{top}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_j, -\vec{1}_k)$$

contained within $\mathbb{P}^1(\mathbb{R})$. For the same choice of j, k, there is also a natural straight line path

$$\mathrm{dch}_{kj} \in \pi_1^{\mathrm{top}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, -\vec{1}_k, \vec{1}_j)$$

These two definitions then define a canonical straight line path between any two distinct points $j, k \in \{0,1,\infty\}$, depending on the cyclic ordering of the pair (j, k).

Under the map from a topological path torsor of $\mathbb{P}^1 \backslash \{0,1,\infty\}$ into the $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$-points of de Rham completion, described in (5.8), the element dch_{jk} is sent to an element $\mathrm{dch}^{\mathfrak{m}}_{jk} \in {}_j\Pi^{\mathrm{dR}}_k(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$. The canonical de Rham splitting of mixed Tate motives (4.12) produces a canonical isomorphism of $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_1)$-torsors

$$ {}_j\Pi^{\mathrm{dR}}_k \xrightarrow{\sim} \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_1). $$

The image of $\mathrm{dch}^{\mathfrak{m}}_{jk}$ under this isomorphism, followed by (5.13), is a grouplike power series $\Phi^{\mathfrak{m}}_{jk} \in \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$. As in the case for the real Drinfeld associator $\Phi_{01} \in \mathbb{R}\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$, $\Phi^{\mathfrak{m}}_{01}$ may be viewed as the generating series for shuffle-regularised motivic multiple zeta values[9] $\zeta^{\mathfrak{m}}(w)$ (see §4.3.3) as w ranges over all elements of the free monoid $M(\mathsf{x}_0, \mathsf{x}_1)$:

$$ \Phi^{\mathfrak{m}}_{01} = \Phi^{\mathfrak{m}}(\mathsf{x}_0, \mathsf{x}_1) = \sum_{w \in M(\mathsf{x}_0, \mathsf{x}_1)} \zeta^{\mathfrak{m}}(w) w. \tag{5.14} $$

It satisfies $\mathrm{per}(\Phi^{\mathfrak{m}}_{01}) = \Phi_{01}$. The other series $\Phi^{\mathfrak{m}}_{jk}$ are obtained making by the change of variables $\Phi^{\mathfrak{m}}_{jk} := \Phi^{\mathfrak{m}}(\mathsf{x}_j, \mathsf{x}_k)$, where $\mathsf{x}_\infty := -(\mathsf{x}_0 + \mathsf{x}_1)$.

It follows from the properties defining an associator that $\Phi^{\mathfrak{m}}_{jk}$ is grouplike [18].

5.4 De Rham fundamental group of $\mathcal{E}^{\times}_{\partial/\partial q}$

In this section we describe the de Rham fundamental group of the infinitesimal punctured Tate curve $\mathcal{E}^{\times}_{\partial/\partial q}$ (see §3.4) based at the tangential basepoint $\partial/\partial w$ (see §3.5.2) in more detail.

5.4.1 Structure

The group $\pi_1^{\mathrm{top}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ is free on the two generators α and β that were defined in §5.1.1.2. By the de Rham analogue of (5.9) there is an isomorphism

$$ \mathrm{Lie}(\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w))^{\mathrm{ab}} \cong H^1(\mathcal{E}^{\times}_{\partial/\partial q}, \mathbb{Q})^{\vee} \cong \mathbb{Q}\mathsf{a} \oplus \mathbb{Q}\mathsf{b}. $$

The elements a, b span $\mathbb{Q}(1)$ and $\mathbb{Q}(0)$ respectively, which follows from the fact that $H^1(\mathcal{E}^{\times}_{\partial/\partial q}, \mathbb{Q}) \cong \mathbb{Q}(-1) \oplus \mathbb{Q}(0)$. See [26, Lemma 15.3] and [7, §13.6]. Consequently, there is an isomorphism $\mathrm{Lie}\,\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w) \cong \mathrm{Lie}(\mathsf{a}, \mathsf{b})^{\wedge}$. By [27, §23], $\mathrm{Lie}\,\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ is canonically bigraded and this is the unique isomorphism compatible with the bigrading and with the grading on the Lie algebra $\mathfrak{k}$.

Under the comparison isomorphism

$$ \mathrm{Lie}\,\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathrm{Lie}\,\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w) \otimes_{\mathbb{Q}} \mathbb{C} $$

[9]Here and onward we also make an obvious abuse of notation and write $\zeta^{\mathfrak{m}}(w)$ for $w \in M(\mathsf{x}_0, \mathsf{x}_1)$, even though $\zeta^{\mathfrak{m}}(w)$ is strictly defined for words in the dual alphabet $\{e_0, e_1\}$.

of (5.12), we have

$$\log(\alpha) \mapsto 2\pi i \mathsf{a} \quad \text{(mod commutators)}$$
$$\log(\beta) \mapsto -\mathsf{b} \quad \text{(mod commutators).}$$

As explained in the analogous case of §5.3.1, $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)(R)$ is isomorphic to the group of grouplike elements in the completed universal enveloping algebra of its Lie algebra:

$$\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)(R) \cong \mathcal{G}(R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle). \tag{5.15}$$

By [27, §23], this isomorphism is canonical – it preserves the bigrading on each side and is compatible with the action of $\mathfrak{k}$. See Remark 5.2.10.

5.4.2 Specific $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$-points

Under the natural map $\pi_1^{\mathrm{top}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w) \mapsto \pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}})$ defined in (5.8), the generators α and β are sent to grouplike power series

$$\alpha^{\mathfrak{m}}, \beta^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle.$$

We compute explicit expressions for these power series in Lemma 11.2.1. They may be written as products of the exponential series and Drinfeld associators that were described in §5.3.2.1 and §5.3.2.2; however, in this case the variables x_p are replaced by modified variables $\phi(\mathsf{x}_p) \in \mathrm{Lie}(\mathsf{a}, \mathsf{b})^\wedge$. Here

$$\phi\colon \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \vec{1}_1) \to \pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$$

is a morphism of de Rham fundamental groups called the *Hain morphism*. It is studied in detail in §11.1. One immediate consequence of its definition is that the abelianisations of $\alpha^{\mathfrak{m}}$ and $\beta^{\mathfrak{m}}$ are precisely $\exp(\mathbb{L}\mathsf{a})$ and $\exp(-\mathsf{b})$ respectively, as they should be by the formulas above.

Because they are essentially constructed from the image of the Hain morphism, the coefficients of $\alpha^{\mathfrak{m}}$ and $\beta^{\mathfrak{m}}$ are products of motivic multiple zeta values and powers of $\mathbb{L}$, and are thus contained within the subalgebra[10] $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ of $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ consisting of periods of mixed Tate motives over $\mathbb{Z}$. This follows because $\mathcal{O}(\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w))$ is an ind-object of $\mathsf{MT}(\mathbb{Z})$ [9, 27]. In Lemma 11.2.1 we give explicit formulae for these series.

[10] In fact, they are even contained within the smaller subalgebra of *effective* periods of $\mathsf{MT}(\mathbb{Z})$.

5.5 Relative fundamental groups of $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,\vec{1}}$

Let $\pi\colon \mathcal{E} \to \mathcal{M}_{1,1}$ be the universal family of §3.3. Let $V := R^1\pi_*\underline{\mathbb{Q}}$ be the local system over $\mathcal{M}_{1,1}$ whose fiber over τ is the Betti cohomology $H^1_{\mathrm{B}}(E_\tau, \mathbb{Q})$. Associated to V is a vector bundle $\mathcal{V}$ over $\mathcal{M}_{1,1}$ equipped with the Gauss-Manin connection ∇.

Let S^{B} be the full semisimple subcategory of $\mathsf{LS}_{\mathbb{Q}}(\mathcal{M}_{1,1})$ generated by V, and let S^{dR} be the full subcategory of $\mathsf{Con}_{\mathbb{Q}}(\mathcal{M}_{1,1})$ generated by $(\mathcal{V}, \nabla)$. Their Tannakian fundamental groups are SL_2^{B} and SL_2^{dR} respectively. Definition 5.2.3 yields the Betti and de Rham relative fundamental groups:

$$\mathcal{G}^{\mathrm{B}}_{1,1} := \pi_1^{\mathrm{rel,B}}(\mathcal{M}_{1,1}, \partial/\partial q), \quad \mathcal{G}^{\mathrm{dR}}_{1,1} := \pi_1^{\mathrm{rel,dR}}(\mathcal{M}_{1,1}, \partial/\partial q).$$

Let V' (resp. $\mathcal{V}'$) denote the pullback of V (resp. $\mathcal{V}$) by $\mathcal{M}_{1,\vec{1}} \to \mathcal{M}_{1,1}$. There is an associated map on compactifications and the induced morphism of tangent spaces at the cusp sends $\vec{\mathsf{v}} = \partial/\partial q + \partial/\partial w$ to $\partial/\partial q$. The same procedure thus defines

$$\mathcal{G}^{\mathrm{B}}_{1,\vec{1}} := \pi_1^{\mathrm{rel,B}}(\mathcal{M}_{1,\vec{1}}, \vec{\mathsf{v}}), \quad \mathcal{G}^{\mathrm{dR}}_{1,\vec{1}} := \pi_1^{\mathrm{rel,dR}}(\mathcal{M}_{1,\vec{1}}, \vec{\mathsf{v}}).$$

The decomposition (5.4) gives non-canonical isomorphisms

$$\mathcal{G}^{\bullet}_{1,1} \cong \mathcal{U}^{\bullet}_{1,1} \rtimes SL_2^{\bullet}, \quad \mathcal{G}^{\bullet}_{1,\vec{1}} \cong \mathcal{U}^{\bullet}_{1,\vec{1}} \rtimes SL_2^{\bullet},$$

where $\mathcal{U}^{\bullet}_{1,1}$ and $\mathcal{U}^{\bullet}_{1,\vec{1}}$ are pro-unipotent. By [23, Proposition 14.2] there is a canonical decomposition[11]

$$\mathcal{G}^{\mathrm{dR}}_{1,\vec{1}} \cong \mathcal{G}^{\mathrm{dR}}_{1,1} \times \mathbb{G}_a(1). \tag{5.16}$$

The factor $\mathbb{G}_a$ can be interpreted as the de Rham fundamental group of the fiber of the morphism $\mathcal{M}_{1,\vec{1}} \to \mathcal{M}_{1,1}$ over τ. This fiber is isomorphic to $\mathbb{G}_m$, and hence its de Rham fundamental group is $\mathbb{G}_a$. The notation $\mathbb{G}_a(1)$ means that the MHS on this fundamental group (in the sense of [20]) is $\mathbb{Q}(1)$.

In order to describe $\mathcal{G}^{\mathrm{dR}}_{1,1}$ and $\mathcal{G}^{\mathrm{dR}}_{1,\vec{1}}$ it is therefore sufficient to describe the Lie algebra $\mathfrak{u}_{1,1} := \mathrm{Lie}(\mathcal{U}^{\mathrm{dR}}_{1,1})$ together with its SL_2^{dR}-action. The Lie algebra $\mathfrak{u}_{1,\vec{1}} := \mathrm{Lie}(\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}})$ is canonically isomorphic to the direct product $\mathfrak{u}_{1,1} \oplus \mathbb{Q}\mathbf{e}_2$ with a central generator $\mathbf{e}_2$ spanning a copy of $\mathbb{Q}(1)$. This generator is acted on trivially by SL_2.

[11] No decomposition of the form (5.16) exists in the topological setting. There is an exact sequence of fundamental groups $1 \to \mathbb{Z} \to B_3 \to SL_2(\mathbb{Z}) \to 1$ associated to the $\mathbb{G}_m$-torsor $\mathcal{M}_{1,\vec{1}} \to \mathcal{M}_{1,1}$, but this exact sequence does not split.

5.5.1 Explicit description of $\mathfrak{u}_{1,1}$

It remains to describe $\mathfrak{u}_{1,1}$. It is equipped with a limit MHS[12] [23], and therefore has two different weight filtrations: a relative weight filtration $M_\bullet\mathfrak{u}_{1,1}$ and a geometric weight filtration $W_\bullet\mathfrak{u}_{1,1}$, together with a Hodge filtration $F^\bullet\mathfrak{u}_{1,1}$. The geometric weight filtration is constructed from the limit of the weight filtration on the fibers of V as the smooth fiber degenerates to the nodal cubic. The relative weight filtration is a new object determined by the weight filtration on the smooth fibers together with the local monodromy action associated to the degeneration. The data $(\mathfrak{u}_{1,1}, M_\bullet, F^\bullet)$ defines a MHS, and each subobject $W_m\mathfrak{u}_{1,1}$ defines a sub-MHS.

We provide a brief description of the two weight filtrations in this section, referring the reader to [23, §12] and [7] for more detail. Note that the MHS on $\mathfrak{u}^{\mathrm{geom}}$ defined in §9.3 is the image of the MHS on $\mathfrak{u}_{1,1}$ under the monodromy morphism that will be defined in §8.2.

As explained in §5.2.8, the Lie algebra $\mathfrak{u}_{1,1}$ is constructed from $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathcal{V}_\alpha)$ as $\mathcal{V}_\alpha$ ranges over a set of vector bundles over $\mathcal{M}_{1,1}$ corresponding to all isomorphism classes of irreducible representations of $SL_2^{\mathrm{dR}}/\mathbb{Q}$.

The fundamental group $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q)$ has cohomological dimension 1. Therefore, by (5.11), $\mathfrak{u}_{1,1}$ may be written as the free Lie algebra on its abelianisation $H_1(\mathfrak{u}_{1,1}) = \mathfrak{u}_{1,1}^{\mathrm{ab}}$. A complete set of representatives of the irreducible representations of SL_2^{dR} is given by the collection of V_n^{dR} for all $n \geq 0$, where V_{2n}^{dR} is the representation defined in §2.2, distinguished by denoting its generators with the sans-serif symbols[13] X, Y. The representation V_n^{dR} corresponds to $\mathcal{V}_n := \mathrm{Sym}^n\,\mathcal{V}$.

Brown and Hain [12] provide a description of $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathcal{V}_n)$ in terms of modular forms. They construct a natural isomorphism of $\mathbb{Q}$-vector spaces

$$\varpi\colon M^!_{n+2}/\mathcal{D}^{n+1}M^!_{-n} \xrightarrow{\sim} H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathcal{V}_n),$$

where $M^!_n$ is the space of weakly holomorphic modular forms for $SL_2(\mathbb{Z})$ with rational Fourier coefficients, and $\mathcal{D} := qd/dq$. It does not preserve modularity, but its $(n+1)$-fold iterate induces a linear map $\mathcal{D}^{n+1}\colon M^!_{-n} \to S^!_{n+2}$, where $S^!_n \subseteq M^!_n$ is the subspace of weakly holomorphic cusp forms [12, Proposition 2.2].

The space of modular forms of weight $n+2$ is a subspace $M_{n+2} \subseteq M^!_{n+2}/\mathcal{D}^{n+1}M^!_{-n}$. Via ϖ it is mapped isomorphically to $F^{n+1}H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathcal{V}_n)$, where the Hodge filtration

[12] The limit MHS is in fact defined on $\mathfrak{g}_{1,1} = \mathrm{Lie}(\mathcal{G}^{\mathrm{dR}}_{1,1})$. It satisfies $W_{-1}\mathfrak{g}_{1,1} = \mathfrak{u}_{1,1}$ and $\mathrm{gr}^W_0\,\mathfrak{g}_{1,1} = \mathfrak{sl}_2$, where $\mathfrak{sl}_2 = \mathrm{Lie}(SL_2^{\mathrm{dR}})$ is the Lie algebra of the reductive quotient [7, §13.5]. This justifies the notation $\gamma_0^{\mathfrak{m}}$ for the reductive part given in Definition 5.2.6.

[13] The distinction between V_{2n}^{dR} and V_{2n} is explained in §5.5.2.2.

on $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathcal{V}_n)$ is

$$0 = F^{n+2} \subseteq F^{n+1} \subseteq \cdots \subseteq F^1 \subseteq F^0 = H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathcal{V}_n).$$

The isomorphism ϖ implies that

$$\dim H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathcal{V}_n) = \dim(M^!_{n+2}/\mathcal{D}^{n+1}M^!_{-n}) = \begin{cases} 2\dim(S_{n+2}) + 1, & n \geq 1 \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

The cohomology $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathcal{V}_n)$ is semisimple, and the action of Hecke operators induces an isomorphism

$$H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, \mathcal{V}_n)^\vee \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \mathbb{Q}(1) \oplus \bigoplus_f M^{\mathcal{H}}_f(1),$$

where f ranges over normalised Hecke eigenforms of weight $n+2$ and $M^{\mathcal{H}}_f$ denotes the $\mathcal{H}$-realisation of the motive [47] of f, which has rank 2. Let $\{\mathbf{e}'_f, \mathbf{e}''_f\}$ denote a (non-canonical!) choice of basis for $M^{\mathcal{H}}_f(1)$. The object $M^{\mathcal{H}}_f(1)$ lies in $W_{-1} \cap M_{2n-1}$. It is defined over the number field generated by the Fourier coefficients of f, which is why one must extend scalars to $\overline{\mathbb{Q}}$.

The object $\mathbb{Q}(1)$ corresponds to the $\mathcal{H}$-realisation of the motive of $\mathbb{G}_{n+2}$, twisted by $\mathbb{Q}(1)$. It has a basis consisting of an element $\mathbf{e}_{2n+2}$ corresponding to $\mathbb{G}_{n+2}$. This element lies in $W_{-2n-2} \cap M_{-2}$.

The discussion in §5.2.8 then implies that

$$H_1(\mathfrak{u}_{1,1}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \prod_{n \geq 1} \left(\mathbb{Q}(1) \oplus \bigoplus_f M^{\mathcal{H}}_f(1) \right) \otimes V^{\mathrm{dR}}_{2n}.$$

The Lie algebra $\mathfrak{u}_{1,1}$ is then the completed free Lie algebra $\mathfrak{u}_{1,1} = \mathrm{Lie}(H_1(\mathfrak{u}_{1,1}))^\wedge$.

5.5.1.1 Totally holomorphic quotient

The associated bigraded Lie algebra $\mathrm{gr}^M \mathrm{gr}^W \mathfrak{u}_{1,1} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is isomorphic to the free Lie algebra on the symbols

$$\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}, \quad \mathbf{e}'_f\mathsf{X}^k\mathsf{Y}^{2n-k}, \quad \mathbf{e}''_f\mathsf{X}^k\mathsf{Y}^{2n-k} \tag{5.17}$$

where $n \geq 1$, $0 \leq k \leq 2n$ and f ranges over normalised Hecke eigenforms of weight $2n+2$. The symbols X and Y span copies of $\mathbb{Q}(0)$ and $\mathbb{Q}(1)$ respectively [7, §13.3]. This free Lie algebra is the graded Lie algebra of a quotient of $\mathcal{U}^{\mathrm{dR}}_{1,1}$ called the *totally holomorphic quotient* [7, §3.5 and §13.7] and denoted $\mathcal{U}^{\mathrm{dR,hol}}_{1,1}$. This is because its periods are a subset of the periods of the full relative completion coming from regularised iterated integrals of holomorphic modular forms. Its points over a $\mathbb{Q}$-algebra r can be identified with grouplike power series in the symbols (5.17). Note that while the symbols $\mathbf{e}_{2n+2}$ are canonically defined (up to normalisation), the symbols $\mathbf{e}'_f$ and $\mathbf{e}''_f$ are not. Hence we must fix such a choice and note that it is not unique.

The affine ring $\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR,hol}})$ is a subalgebra of $\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}})$, but it is not motivic (it does not have a natural MHS).

5.5.1.2 Free Eisenstein quotient

The projection

$$\mathrm{Lie}\left(\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}, \mathbf{e}'_f\mathsf{X}^k\mathsf{Y}^{2n-k}, \mathbf{e}''_f\mathsf{X}^k\mathsf{Y}^{2n-k}\right) \to \mathrm{Lie}\left(\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}\right)$$

onto the free Lie algebra generated by the Eisenstein symbols $\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}$ (for all $n \geq 1$ and $0 \leq k \leq 2n$) is equivalent to a quotient mapping $\mathcal{U}_{1,1}^{\mathrm{dR,hol}} \to \mathcal{U}_E^{\mathrm{dR}}$, where $\mathcal{U}_E^{\mathrm{dR}}$ is the *free* Eisenstein quotient[14] of $\mathcal{U}_{1,1}^{\mathrm{dR}}$. Its points can be identified with grouplike power series in the symbols $\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}$. These symbols are canonically defined (unlike the symbols $\mathbf{e}'_f$ and $\mathbf{e}''_f$ associated to cusp forms f).

It therefore makes sense to consider the shuffle algebra on the *dual alphabet* of the Eisenstein symbols in the sense of §1.3.1.3. The dual of $\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}$ is denoted by $E_{2n+2}(k)$. As discussed in §1.3.1.3, the shuffle algebra on the generators $E_{2n+2}(k)$ is naturally isomorphic to the coordinate ring $\mathcal{O}(\mathcal{U}_E^{\mathrm{dR}})$. We denote the word $E_{2n_1+2}(k_1)\cdots E_{2n_s+2}(k_s) \in \mathcal{O}(\mathcal{U}_E^{\mathrm{dR}})$ by the bar notation $[E_{2n_1+2}(k_1)|\cdots|E_{2n_s+2}(k_s)]$.

5.5.1.3 Description of $\mathfrak{u}_{1,1}$ and $\mathfrak{u}_{1,\vec{1}}$

The $M_\bullet$ and $W_\bullet$ filtrations on $\mathfrak{u}_{1,1}$ can be simultaneously split [27, Appendix B]. Note, however, that there is not a canonical choice of simultaneous splitting as there is mixed Tate motives by (4.12). Therefore we can identify $\mathfrak{u}_{1,1}$ with the completed free Lie algebra on the symbols (5.17), but must emphasise that this isomorphism is non-canonical and depends upon the choice of splitting.

The isomorphism $\mathfrak{u}_{1,\vec{1}} \cong \mathfrak{u}_{1,1} \oplus \mathbb{Q}\mathbf{e}_2$ implies that $\mathfrak{u}_{1,\vec{1}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ may be written as a completed free Lie algebra generated by the elements in (5.17), where we also allow the value $n = 0$ and recall that $\mathbf{e}_2$ is central. This means that a general element of $\mathcal{U}_{1,\vec{1}}^{\mathrm{dR}}(R)$ may be written as $u \cdot \exp(r\mathbf{e}_2)$, where $u \in \mathcal{U}_{1,1}^{\mathrm{dR}}(R)$ and $r \in R$. Once again, the expression for the element u in the symbols (5.17) is non-canonical and depends upon the choice of simultaneous splitting of the weight filtrations.

[14] We distinguish the free Eisenstein quotient from *the* Eisenstein quotient in the sense of [23, §16]. The former does not have a natural MHS.

5.5.2 Actions of SL_2^{dR} and other groups

As remarked above, in order to understand $\mathcal{G}_{1,1}^{\mathrm{dR}}$ it suffices to understand $\mathfrak{u}_{1,1}$ together with its SL_2^{dR}-action. In this section we define this action, which has a simple formula, as well as some other relevant group actions.

The reductive group SL_2^{dR} acts on $\mathfrak{u}_{1,1}$ naturally on the *right* as follows: fix a splitting of the M and W filtrations as in §5.5.1.3, so that we may write down elements of $\mathfrak{u}_{1,1}$ (non-canonically!) in terms of the elements (5.17). Let R be a $\mathbb{Q}$-algebra and let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2^{\mathrm{dR}}(R).$$

Then γ acts on $\mathfrak{u}_{1,1} \otimes_{\mathbb{Q}} R$ via

$$P(\mathsf{X},\mathsf{Y})|_\gamma := P(a\mathsf{X} + b\mathsf{Y}, c\mathsf{X} + d\mathsf{Y}). \tag{5.18}$$

It also acts on $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$ via right multiplication on the frame (a,b):

$$(\mathsf{a},\mathsf{b})|_\gamma := (\mathsf{a},\mathsf{b})\gamma = (a\mathsf{a} + c\mathsf{b}, b\mathsf{a} + d\mathsf{b}) \tag{5.19}$$

In contrast, the unipotent group $\mathcal{U}_{1,\vec{1}}^{\mathrm{dR}}$ acts on $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$ on the *left* by the *monodromy action* defined in §8.2. This action involves applying certain derivations on $\mathrm{Lie}(\mathsf{a},\mathsf{b})$ to elements of its completed universal enveloping algebra. The derivations are defined in §9.1.

It is convenient to consider only left actions, so we let SL_2^{dR} act on the *left* by taking the inverse action of each right action. For $\gamma \in SL_2^{\mathrm{dR}}(R)$ and x an R-point of either $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$, $\mathcal{U}_{1,\vec{1}}^{\mathrm{dR}}$ or $\mathcal{U}_{1,1}^{\mathrm{dR}}$, this is defined by $\gamma(x) = x|_{\gamma^{-1}}$.

The group $\mathcal{G}_{1,1}^{\mathrm{dR}}$ (resp. $\mathcal{G}_{1,\vec{1}}^{\mathrm{dR}}$) splits non-canonically as the semidirect product

$$\mathcal{G}_{1,1}^{\mathrm{dR}} \cong \mathcal{U}_{1,1}^{\mathrm{dR}} \rtimes SL_2^{\mathrm{dR}} \quad (\text{resp. } \mathcal{G}_{1,\vec{1}}^{\mathrm{dR}} \cong \mathcal{U}_{1,\vec{1}}^{\mathrm{dR}} \rtimes SL_2^{\mathrm{dR}}), \tag{5.20}$$

where SL_2^{dR} acts on the left as above.

5.5.2.1 Induced $\mathfrak{sl}_2$-actions

The SL_2^{dR}-action on $\mathfrak{u}_{1,1}$ described in §5.5.2 induces an action of $\mathrm{Lie}(SL_2^{\mathrm{dR}}) = \mathfrak{sl}_2$ on $\mathfrak{u}_{1,1}$. It acts via the operators $\mathsf{X}\partial/\partial\mathsf{Y}$ and $-\mathsf{Y}\partial/\partial\mathsf{X}$.

The SL_2^{dR}-action on $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$ also equips $\mathrm{Lie}(\mathsf{a},\mathsf{b})$ with an action of SL_2^{dR}, and hence of its Lie algebra $\mathfrak{sl}_2$. This, in turn, furnishes $\mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$ with an inner action of $\mathfrak{sl}_2$ via the operators $\mathsf{b}\partial/\partial\mathsf{a}$ and $-\mathsf{a}\partial/\partial\mathsf{b}$. This is described further in §9.1.1.

5.5.2.2 Two forms of SL_2

The comparison isomorphism $\mathcal{G}^{\mathrm{B}}_{1,1} \times_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \xrightarrow{\sim} \mathcal{G}^{\mathrm{dR}}_{1,1} \times_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ defined in (5.6) induces a canonical comparison isomorphism

$$SL_2^{\mathrm{B}} \times_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \xrightarrow{\sim} SL_2^{\mathrm{dR}} \times_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$$

on reductive quotients. By [7, §13.3], the coordinate ring $\mathcal{O}(SL_2^{\bullet})$, viewed as an object of $\mathcal{H}$, is an infinite direct sum of Tate objects $\mathbb{Q}(n)$. Consequently, the comparison isomorphism is defined over $\mathbb{Q}[\mathbb{L}^{\pm}]$. It is given by conjugation by an element of $GL_2(\mathbb{Q}[\mathbb{L}^{\pm}])$; namely,

$$\gamma \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{L}^{-1} \end{pmatrix} \gamma \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{L}^{-1} \end{pmatrix}^{-1}.$$

We may write down this homomorphism explicitly by precomposing with the Zariski-dense inclusion $SL_2(\mathbb{Z}) \hookrightarrow SL_2^{\mathrm{B}}(\mathbb{Q})$ defined in (5.7). The resulting composition

$$SL_2(\mathbb{Z}) \hookrightarrow SL_2^{\mathrm{B}}(\mathbb{Q}) \to SL_2^{\mathrm{dR}}(\mathbb{Q}[\mathbb{L}^{\pm}])]$$

is given by $\gamma \mapsto \gamma_0^{\mathfrak{m}}$, where the reductive part $\gamma_0^{\mathfrak{m}}$ was defined in Definition 5.2.6. It may be written down on the generators S and T for $SL_2(\mathbb{Z})$ defined in §2.1.1 by the formula

$$\begin{aligned} S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &\mapsto S_0^{\mathfrak{m}} = \begin{pmatrix} 0 & -\mathbb{L} \\ \mathbb{L}^{-1} & 0 \end{pmatrix} \\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &\mapsto T_0^{\mathfrak{m}} = \begin{pmatrix} 1 & \mathbb{L} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Recall the definition of the representation V_n of SL_2 defined in §2.2. We consider two different forms of this representation for the two forms $SL_2^{\mathrm{B}}, SL_2^{\mathrm{dR}}$ of $SL_2/\mathbb{Q}$, denoted V_n^{B} and V_n^{dR} respectively. The generators for V_n^{B} are denoted by $\mathsf{X}_{\mathrm{B}}, \mathsf{Y}_{\mathrm{B}}$, while those for V_n^{dR} are denoted by X, Y. They are related by an isomorphism $V_n^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{L}^{\pm}] \xrightarrow{\sim} V_n^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{L}^{\pm}]$ given by

$$(\mathsf{X}_{\mathrm{B}}, \mathsf{Y}_{\mathrm{B}}) \mapsto (\mathsf{X}, \mathbb{L}\mathsf{Y}).$$

Elements of $SL_2^{\mathrm{B}}(\mathbb{Q})$ act on V_n^{dR} via their image under the map $SL_2^{\mathrm{B}}(\mathbb{Q}) \to SL_2^{\mathrm{dR}}(\mathbb{Q}[\mathbb{L}])$ described above.

5.5.3 Specific $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$-points

Recall that we are identifying $SL_2(\mathbb{Z})$ with $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q)$. As described in §5.1.1.3, the matrix S corresponds to the imaginary axis on $\mathcal{H}$. To work with motivic iterated Eisenstein integrals along this path formally, it is necessary to understand the image of S under the map

$$SL_2(\mathbb{Z}) \to \mathcal{G}^{\mathrm{dR}}_{1,1}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}) \cong \mathcal{U}^{\mathrm{dR}}_{1,1}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}) \rtimes SL_2^{\mathrm{dR}}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}). \tag{5.21}$$

Using the splitting (5.20), the following diagram commutes

$$\begin{tikzcd}
SL_2(\mathbb{Z}) \arrow[r] \arrow[d, hook] & \mathcal{U}^{\mathrm{dR}}_{1,1}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}) \rtimes SL_2^{\mathrm{dR}}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}) \arrow[d] \\
SL_2^{\mathrm{B}}(\mathbb{Q}) \arrow[r] & SL_2^{\mathrm{dR}}(\mathbb{Q}[\mathbb{L}^{\pm}])
\end{tikzcd}$$

where the right vertical map is the projection onto the second factor and the bottom map $\gamma \mapsto \gamma^{\mathfrak{m}}$ is induced from the isomorphism $SL_2^{\mathrm{B}} \times_{\mathbb{Q}} \mathbb{Q}[\mathbb{L}^{\pm}] \xrightarrow{\sim} SL_2^{\mathrm{dR}} \times_{\mathbb{Q}} \mathbb{Q}[\mathbb{L}^{\pm}]$ described in §5.5.2.2. The map (5.21) is given by

$$\gamma \mapsto \gamma^{\mathfrak{m}} = (\mathcal{C}^{\mathfrak{m}}_{\gamma}, \gamma^{\mathfrak{m}}_0),$$

where the unipotent part $\mathcal{C}^{\mathfrak{m}}_{\gamma} := \gamma^{\mathfrak{m}}_u$ and reductive part $\gamma^{\mathfrak{m}}_0$ of $\gamma^{\mathfrak{m}}$ are defined in Definition 5.2.6. The association $\gamma \mapsto \mathcal{C}^{\mathfrak{m}}_{\gamma}$ defines a nonabelian cocycle

$$\mathcal{C}^{\mathfrak{m}} \in Z^1(SL_2(\mathbb{Z}), \mathcal{U}^{\mathrm{dR}}_{1,1}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}))$$

called the *canonical cocycle* [7, Definition 15.4]. The element $\mathcal{C}^{\mathfrak{m}}_{\gamma}$ is a generating series for motivic iterated integrals along $\gamma \in \pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q)$.

Definition 5.5.1 (Motivic multiple modular values)**.** The $\mathbb{Q}$-algebra $M^{\mathfrak{m}}$ of motivic multiple modular values [7] is the $\mathbb{Q}$-subalgebra of $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ generated by the coefficients of $\mathcal{C}^{\mathfrak{m}}_{\gamma}$ for all $\gamma \in SL_2(\mathbb{Z})$. Let $M = \mathrm{per}(M^{\mathfrak{m}}) \subseteq \mathbb{C}$. It is the $\mathbb{Q}$-algebra of multiple modular values.

Proposition 5.5.2. *The cocycle $\mathcal{C}^{\mathfrak{m}}$ satisfies the equations*

$$\mathcal{C}^{\mathfrak{m}}_S|_{S^{\mathfrak{m}}_0} \cdot \mathcal{C}^{\mathfrak{m}}_S = 1$$
$$\mathcal{C}^{\mathfrak{m}}_U|_{(U^{\mathfrak{m}}_0)^2} \cdot \mathcal{C}^{\mathfrak{m}}_U|_{U^{\mathfrak{m}}_0} \cdot \mathcal{C}^{\mathfrak{m}}_U = 1,$$

where $U := ST \in SL_2(\mathbb{Z})$ and $\mathcal{C}^{\mathfrak{m}}_U := \mathcal{C}^{\mathfrak{m}}_S|_{T^{\mathfrak{m}}_0} \cdot \mathcal{C}^{\mathfrak{m}}_T$.

Proof. The canonical cocycle $\mathcal{C}^{\mathfrak{m}}$ is a cocycle for $SL_2(\mathbb{Z})$, and $SL_2(\mathbb{Z})$ acts on $\mathcal{U}_{1,1}^{\mathrm{dR}}(\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}})$ on the right via the composition $SL_2(\mathbb{Z}) \to SL_2^{\mathrm{B}}(\mathbb{Q}) \to SL_2^{\mathrm{dR}}(\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}})$ sending γ to $\gamma_0^{\mathfrak{m}}$. This implies that $\mathcal{C}_{\gamma\delta}^{\mathfrak{m}} = \mathcal{C}_{\gamma}^{\mathfrak{m}}|_{\delta_0^{\mathfrak{m}}} \cdot \mathcal{C}_{\delta}^{\mathfrak{m}}$ for all $\gamma, \delta \in SL_2(\mathbb{Z})$. The result follows from the presentation $SL_2(\mathbb{Z}) = \langle S, U | S^2 = U^3 = -I \rangle$ together with the fact that $\mathcal{C}_{-I}^{\mathfrak{m}} = 1$; the latter follows because there are no modular forms of odd weight for $SL_2(\mathbb{Z})$. $\square$

Remark 5.5.3. Proposition 5.5.2, together with the fact that $SL_2(\mathbb{Z})$ is generated by the matrices S and T, implies that $M^{\mathfrak{m}}$ is generated by the $\mathbb{Q}$-subalgebras of coefficients of $\mathcal{C}_S^{\mathfrak{m}}$ and $\mathcal{C}_T^{\mathfrak{m}}$. It is known [7, Lemma 15.6] that the coefficients of $\mathcal{C}_T^{\mathfrak{m}}$ are contained in $\mathbb{Q}[\mathbb{L}]$, which follows because the element T corresponds to a small loop around the cusp of $\mathcal{M}_{1,1}$ (see §5.1.1.3). Hence all the interesting information is contained in the coefficients of the series $\mathcal{C}_S^{\mathfrak{m}}$.

In Chapter 6 we study the algebra of motivic multiple modular values in more detail. In particular, we will be interested in specific elements called motivic iterated Eisenstein integrals. In Chapter 10 we discuss a "lift" of the cocycle $\mathcal{C}^{\mathfrak{m}}$ via $B_3 \to SL_2(\mathbb{Z})$ having particular importance for the monodromy action described in Chapter 8.

Chapter 6

Multiple modular values

The algebras M and $M^{\mathfrak{m}}$ of (motivic) multiple modular values, defined in Definition 5.5.1 as the coefficients of the canonical cocycle $\mathcal{C}^{\mathfrak{m}}$, are very rich. In this chapter we explore some of the elements in these algebras in more detail. Of particular importance to this thesis are motivic *iterated Eisenstein integrals*, defined in Definition 6.3.1.

Recall that for each $\gamma \in SL_2(\mathbb{Z})$, the value $\mathcal{C}^{\mathfrak{m}}_{\gamma}$ of the canonical cocycle on γ can be viewed as a generating series for motivic iterated integrals along γ, where we identify $SL_2(\mathbb{Z})$ with the topological fundamental group $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q)$. The astute reader may wonder what is meant by an integral whose endpoints are tangential basepoints. In the following section, therefore, we recall the notion of "regularisation" for iterated integrals on $\mathcal{M}_{1,1}$ with respect to the tangential basepoint $\partial/\partial q$. Details can be found in [7, §4].

6.1 Regularised iterated integrals of modular forms

Let f be a modular form for $SL_2(\mathbb{Z})$ of weight $2k+2 \geq 4$. We identify $SL_2(\mathbb{Z})$ with $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q)$ as in §5.1.1.3. Define a global section $\underline{f}(\tau)$ of $\Omega^1(\mathfrak{H}; V^{\mathrm{B}}_{2k} \otimes_{\mathbb{Q}} \mathbb{C})$ by

$$\underline{f}(\tau) := (2\pi i)^{2k+1} f(\tau)(\mathsf{X}_{\mathrm{B}} - \tau \mathsf{Y}_{\mathrm{B}})^{2k} d\tau. \tag{6.1}$$

It is invariant under the action of $SL_2(\mathbb{Z})$ via $\underline{f}(\tau) \mapsto \underline{f}(\gamma(\tau))|_{\gamma}$, where $SL_2(\mathbb{Z})$ acts on V^{B}_{2n} via the inclusion $SL_2(\mathbb{Z}) \hookrightarrow SL^{\mathrm{B}}_2(\mathbb{Q})$ given in (5.7). By the definition of the orbifold quotient $\mathcal{M}_{1,1} = [SL_2(\mathbb{Z})\backslash\mathfrak{H}]$, this means that it defines an element of $\Omega^1(\mathcal{M}_{1,1}, \mathbb{V}_{2k})$, where $\mathbb{V}_{2k}$ is a (nontrivial) vector bundle on $\mathcal{M}_{1,1}$.

We can also write the 1-form $\underline{f}(\tau)$ in terms of the de Rham representation V^{dR}_{2k} using the image under the period map of the isomorphism $(\mathsf{X}_{\mathrm{B}}, \mathsf{Y}_{\mathrm{B}}) \mapsto (\mathsf{X}, \mathbb{L}\mathsf{Y})$ described in §5.5.2.2. The coordinate $q = \exp(2\pi i\tau)$ defines a local coordinate at the cusp of $\mathcal{M}_{1,1}$. It satisfies $d\log q = 2\pi i d\tau$. We obtain a de Rham normalised form

$$\underline{f}(q) = (2\pi i)^{2k} f(q)(\mathsf{X} - \log(q)\mathsf{Y})^{2k} \frac{dq}{q}. \tag{6.2}$$

6.1.1 Explicit regularisation

Recall from §5.1.1.3 that we are identifying $SL_2(\mathbb{Z})$ with $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q)$. Under this identification the matrix $S \in SL_2(\mathbb{Z})$ corresponds to the imaginary axis on $\mathfrak{H}$. To make sense of the coefficients of $\mathcal{C}_S^{\mathfrak{m}}$, therefore, we require a definition of iterated integrals of the 1-forms $\underline{f}(\tau)$ along the imaginary axis. This notion makes sense if f is a cusp form, in which case $\underline{f}(\tau)$ is holomorphic at $\tau = i\infty$ (i.e. $\underline{f}(q)$ is holomorphic at the cusp $q = 0$). However, in this thesis we are principally interested in iterated integrals of the Eisenstein forms $\underline{\mathbb{G}_{2k}}(\tau)$, and the q-expansion given in (2.4) demonstrates that these are not holomorphic at $\tau = i\infty$. Consequently, the naive integrals of Eisenstein series from any point on $\mathfrak{H}$ to $i\infty$ will diverge.

In order to deal with these divergent integrals we must *regularise* with respect to the tangential basepoint $\partial/\partial q$ at the cusp defined in §3.5.1. The cusp on $\overline{\mathcal{M}}_{1,1}$ corresponds to the point $i\infty$ on the extended upper half plane $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$. In practice this regularisation procedure refers to subtracting off an appropriate linear combination of products of iterated integrals of the constant term in the Fourier expansion of the modular forms (thought of as the "value at $i\infty$") from the remaining part of the Fourier expansion (which vanishes at $i\infty$). The resulting integrals always converge. Below we explain how this may be done systematically. The procedure is described in detail in [7, §4].

Recall that $q = \exp(2\pi i \tau)$ defines a local parameter around the cusp of $\overline{\mathcal{M}}_{1,1}$. We can therefore regard the tangent vector $\partial/\partial q$ at the cusp (defined in §3.5.1) as a "tangent vector at $i\infty$" on $\mathfrak{H}^*$.

Let W to be the $\mathbb{C}$-vector space spanned by the forms $\underline{f}(\tau)$ where f ranges over all modular forms for $SL_2(\mathbb{Z})$ with $\mathbb{C}$-coefficients. Let $f(\tau) = \sum_{n\geq 0} a_n q^n$ be the q-expansion of f and define

$$\underline{f}^{\infty}(\tau) := (2\pi i)^{2k+1} a_0 (\mathsf{X}_{\mathrm{B}} - \tau \mathsf{Y}_{\mathrm{B}})^{2k} d\tau, \quad \underline{f}^{0}(\tau) := \underline{f}(\tau) - \underline{f}^{\infty}(\tau).$$

The form $\underline{f}^{\infty}(\tau)$ can be viewed as a differential form on the tangent space of $\mathfrak{H}^*$ at $i\infty$. It vanishes when f is a cusp form.

Consider the shuffle algebra $T^c(W)$ on W that was defined in §1.3.1.1. It is a Hopf algebra over $\mathbb{C}$ equipped with the shuffle product $⧢$, the deconcatenation coproduct

Δ and the antipode S given by signed reversal of words. Let π^∞ be the projection sending $\underline{f}(\tau)$ to $\underline{f}^\infty(\tau)$. It extends naturally to $T^c(W)$. Define

$$R := ⧢ \circ (\mathrm{id} \otimes \pi^\infty S) \circ \Delta \colon \mathbb{Q}\langle W\rangle \to \mathbb{Q}\langle W\rangle.$$

Definition 6.1.1. Let $z \in \mathfrak{H}$ and let $f_1, \dots, f_n \in M(SL_2(\mathbb{Z})) \otimes_{\mathbb{Q}} \mathbb{C}$. Then the regularised integral from z to $\partial/\partial q$ is

$$\int_z^{\partial/\partial q} \underline{f_1}(\tau)\cdots\underline{f_n}(\tau) = \sum_{i=0}^{n}(-1)^{n-i}\int_z^{i\infty} R\left(\underline{f_1}(\tau)\cdots\underline{f_i}(\tau)\right)\int_0^z \underline{f_n}^\infty(\tau)\cdots\underline{f_{i+1}}^\infty(\tau), \tag{6.3}$$

where we use the notation for iterated integrals as in Definition 1.4.1.

One immediately verifies that when all the f_i are cusp forms the regularised integral (6.3) reduces to the usual integral $\int_z^{i\infty}\underline{f_1}(\tau)\cdots\underline{f_n}(\tau)$.

We then define the operator

$$\int_S := \int_i^{\partial/\partial q}(\mathrm{id} - S^*), \tag{6.4}$$

where S^* is the automorphism of $\mathbb{Q}\langle W\rangle \otimes_{\mathbb{Q}} \mathbb{C}$ induced by the image of $S \in SL_2(\mathbb{Z})$ in $\mathrm{Aut}(\mathfrak{H})$. This is precisely the formula for regularised integrals along the element S, viewed as an element of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q)$ as in §5.1.1.3.

The definition (6.4) relies on the following facts:

- The element $S \in SL_2(\mathbb{Z})$, which acts on $\mathfrak{H}$ via $S(\tau) = -1/\tau$, fixes $i \in \mathfrak{H}$.
- The imaginary axis may be decomposed as the segment from 0 to i, followed by the segment from i to $i\infty$. The action of S interchanges these two segments, along with their orientation.

6.2 Totally holomorphic multiple modular values

Let $\gamma \in SL_2(\mathbb{Z})$. Some of the coefficients of $\mathcal{C}_\gamma^{\mathfrak{m}}$ are the coefficients of motivic iterated integrals of the form

$$\int_\gamma^{\mathfrak{m}} \underline{f_1}(\tau_1)\cdots\underline{f_s}(\tau_s),$$

where the 1-form $\underline{f}(\tau)$ is defined in §6.1 for $f \in M_{2n+2}(SL_2(\mathbb{Z}))$, and motivic iterated integrals are defined in §5.2.6.1. These coefficients can be expressed as motivic iterated integrals of 1-forms of the shape

$$f(q)\log(q)^b\frac{dq}{q} = \mathbb{L}^{b+1}f(\tau)\tau^b d\tau, \quad 0 \le b \le 2n. \tag{6.5}$$

Such integrals are called *totally holomorphic* motivic multiple modular values [11, Definition 5.1]. They are periods of the totally holomorphic quotient $\mathcal{U}_{1,1}^{\mathrm{dR,hol}}$ of $\mathcal{U}_{1,1}^{\mathrm{dR}}$ defined in §5.5.1.

Recall from Definition 5.2.8 that the coordinate ring $\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}})$ is equipped with an increasing length filtration $L_\bullet\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}})$. It restricts to a length filtration on the Hopf subalgebra $\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR,hol}})$ where it coincides with the increasing filtration on words $[w_1|\cdots|w_s]$ induced by the index s. The value of such a word under the homomorphism $\mathcal{C}_\gamma^{\mathfrak{m}}$ is a motivic iterated integral of length at most s. In the following sections we examine the numbers that occur in length 1 and in higher lengths.

6.2.1 Length 1

It is easy to see that for any cusp form $f \in S_{2n+2}(SL_2(\mathbb{Z}))$, the completed L-value $\Lambda(f,b)$ is a totally holomorphic multiple modular value for $1 \leq b \leq 2n+1$. This follows from the integral expression

$$\Lambda(f,b) = i^{-b}\int_0^{i\infty} f(\tau)\tau^{b-1}d\tau$$

for L-values within the critical strip given in [7, Proof of Lemma 7.1]. In fact, this formula holds more generally for *any* $f \in M_{2n+2}(SL_2(\mathbb{Z}))$ as long as one regularises the integral as in §6.1, giving

$$\Lambda(f,b) = i^{-b}\int_S f(\tau)\tau^{b-1}d\tau.$$

To illustrate the richness of the numbers that occur as multiple modular values even in length 1, we consider the case of Eisenstein series $f = \mathbb{G}_{2k}$. As in *loc. cit.* we have

$$\int_S \mathbb{G}_{2k}(\tau)d\tau = -\frac{(2k-2)!}{2(2\pi i)^{2k-1}}\zeta(2k-1).$$

This shows that $M[2\pi i]$ already contains all single zeta values within its length 1 filtered piece (i.e. those iterated integrals of length 1). The essential goal of this thesis is to show that $M[2\pi i]$ contains the entire algebra $\mathcal{Z}$ of multiple zeta values, and to understand the distribution of these within the length filtration on M.

Other interesting examples occur in length 1 – for example one obtains the periods and quasi-periods of cusp forms in this manner, as can be seen from Brown's computation of the length-1 filtered piece of $\mathcal{C}_S^{\mathfrak{m}}$ [7, §15.4]. We refer the reader to Chapter 14 for a detailed account of the length 1 part of $\mathcal{C}_S^{\mathfrak{m}}$ and its image under the monodromy morphism of §8.2.

6.2.2 Higher lengths

We can also say something about higher lengths. For example, in [11, Example 7.3], Brown gives the following formula:

$$\begin{aligned} &600\pi \int_S \mathbb{G}_4(\tau_1)\tau_1 d\tau_1 \mathbb{G}_{10}(\tau_2)\tau_2^4 d\tau_2 + 480\pi \int_S \mathbb{G}_4(\tau_1)\tau_1^2 d\tau_1 \mathbb{G}_{10}(\tau_2)\tau_2^3 d\tau_2 \\ &= \int_0^{i\infty} \Delta(\tau)\tau^{11} d\tau = \Lambda(\Delta, 12). \end{aligned} \tag{6.6}$$

Here $\Delta \in S_{12}(SL_2(\mathbb{Z}))$ is the Ramanujan cusp form. The L-value $\Lambda(\Delta, 12)$ is *non-critical* and so should be a period of a simple extension of pure motives of modular forms [47] by Beilinson's conjectures [1]. This corresponds to its expression in (6.6) as a length 2 iterated integral of Eisenstein series. Note that the integrand $\Delta(\tau)\tau^{11}d\tau$ is not of the shape (6.5) because the power of τ is greater than $12 - 2 = 10$, so this is a genuinely new type of period.

In the greatest generality, the elements of $M^{\mathfrak{m}}$ should be the motivic periods associated to "mixed modular motives for $SL_2(\mathbb{Z})$" [7, §1.2.2]. Although such a category is yet to be defined formally, it should correspond to *mixed* versions (iterated extensions) of the pure modular motives M_f constructed in [47].

6.3 Iterated Eisenstein integrals

The central purpose of this thesis is to relate multiple zeta values with iterated Eisenstein integrals. In this section we define the latter explicitly and record some of their key features.

Here, and in the sequel, we consider *motivic* iterated Eisenstein integrals. Our notation for general motivic iterated integrals uses the notation of the reduced bar construction that was introduced in §5.2.6.1. Motivic iterated Eisenstein integrals are motivic periods of the affine ring of $\mathcal{U}_{1,1}^{\mathrm{dR}}$. They may be written in terms of the generators $E_{2n+2}(k)$ for the affine ring $\mathcal{O}(\mathcal{U}_E^{\mathrm{dR}})$ of the free Eisenstein quotient[1], which is a Hopf subalgebra of $\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}})$. The generator $E_{2n+2}(k)$ is dual to $\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}$.

Definition 6.3.1 (Motivic iterated Eisenstein integrals)**.** Motivic iterated Eisenstein integrals are totally holomorphic motivic multiple modular values of the form

$$\int_S^{\mathfrak{m}} E_{2n_1+2}(b_1) \dots E_{2n_s+2}(b_s). \tag{6.7}$$

[1]The affine ring of the free Eisenstein quotient does not have a natural $\mathcal{H}$-structure, and therefore does not have motivic periods. However, it does allow us to single out specific "Eisenstein elements" $E_{2n+2}(k)$ in the larger algebra $\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}})$, which does have a $\mathcal{H}$-structure.

In other words, they are the image of the element $[E_{2n_1+2}(k_1)|\cdots|E_{2n_s+2}(k_s)] \in \mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1})$ under the homomorphism $\mathcal{C}^{\mathfrak{m}}_S \colon \mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1}) \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$. The length of this integral is s and the total modular weight is $N := \sum_{i=1}^{s}(2n_i+2)$.

As its name suggests, the period map takes the motivic iterated Eisenstein integral (6.7) to the complex number

$$\begin{aligned}&\int_S \mathbb{G}_{2n_1+2}(q_1)\log(q_1)^{b_1}\frac{dq_1}{q_1}\cdots\mathbb{G}_{2n_s+2}(q_s)\log(q_s)^{b_s}\frac{dq_s}{q_s}\\ =&(2\pi i)^{b_1+\cdots+b_s+s}\int_S \mathbb{G}_{2n_1+2}(\tau_1)\tau_1^{b_1}d\tau_1\cdots\mathbb{G}_{2n_s+2}(\tau_s)\tau_s^{b_s}d\tau_s.\end{aligned}$$

Remark 6.3.2. Our definition only considers the (motivic) *values* of iterated Eisenstein integrals along the specific element $S \in SL_2(\mathbb{Z})$, viewed as an element of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1},\partial/\partial q)$. The homomorphism $\mathcal{C}^{\mathfrak{m}}_S$ is interpreted as the motivic "integration along S" operator on $\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1})$ and is a motivic analogue of the operator $\int_S$ defined in (6.4).

More generally, one defines iterated Eisenstein integrals along each $\gamma \in SL_2(\mathbb{Z})$ as the image under the homomorphism $\mathcal{C}^{\mathfrak{m}}_\gamma \in \mathrm{Hom}(\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1}),\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}})$ defined in §5.5.3. By applying this homomorphism to $[E_{2n_1+2}(k_1)|\cdots|E_{2n_s+2}(k_s)]$ we obtain a motivic period $\int_\gamma^{\mathfrak{m}} E_{2n_1+2}(k_1)\cdots E_{2n_s+2}(k_s) \in \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$.

Recall, however, that $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1},\partial/\partial q) \cong SL_2(\mathbb{Z})$ is generated by the matrices S and T. By [7, Lemma 15.6], $\int_T^{\mathfrak{m}} E_{2n_1+2}(k_1)\cdots E_{2n_s+2}(k_s)$ is a rational polynomial in $\mathbb{L}$ for any choice of integrand. It follows that the only really interesting motivic iterated Eisenstein integrals are along the element S. This is the motivation for restricting Definition 6.3.1 to integrals along S.

The length filtration $L_\bullet\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1})$ induces a natural filtration on the space of motivic iterated Eisenstein integrals, which we also refer to as the length filtration. In §15.1.1 we relate this to the *coradical filtration* $C_\bullet\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ on the algebra of $\mathcal{H}$-periods.

As shown in §6.2.1, the space of motivic iterated Eisenstein integrals of length 1 already contains all motivic single zeta values. For example, we have

$$\int_S^{\mathfrak{m}} E_{2n+2}(0) = -\frac{(2n)!}{2}\frac{\zeta^{\mathfrak{m}}(2n+1)}{\mathbb{L}^{2n}}, \qquad \int_S^{\mathfrak{m}} E_{2n+2}(2n) = \frac{(2n)!}{2}\zeta^{\mathfrak{m}}(2n+1),$$

and furthermore we have that $\int_S^{\mathfrak{m}} E_{2n+2}(b) \in \mathbb{Q}\mathbb{L}^{b+1}$ when $1 \le b \le 2n-1$. This provides a complete description of the iterated Eisenstein integrals of length 1. In particular, they are all elements of $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$.

This is not true in higher lengths, as demonstrated by (6.6). Here a peculiar phenomenon known as "transference of periods" begins to emerge, whereby periods of different modular forms are related [7, §8], [11, §8.3].

However, there are some linear combinations of motivic iterated Eisenstein integrals of length 2 within $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$. For example, in Chapter 15 we show that

$$\zeta^{\mathfrak{m}}(3,5) = -\frac{5}{12}\mathbb{L}^6 \int_S^{\mathfrak{m}} E_6(0)E_4(0) + \lambda\mathbb{L}^8, \quad \text{for some } \lambda \in \mathbb{Q}.$$

In [11, Example 7.2] Brown determines that $\lambda = 503/2^{13}3^5 5^2 7$ and provides a range of other examples. In Chapter 15 we explain how the coefficient of the longest motivic iterated Eisenstein integral in a linear combination equal to a given motivic multiple zeta value may be determined using the motivic coaction of §4.2.1.1.

As we have seen, the $\mathbb{Q}$-subalgebra $\mathcal{E}^{\mathfrak{m}}$ of $M^{\mathfrak{m}}$ constisting of motivic iterated Eisenstein integrals is both rich and rigid. Theorem 12.0.1, our main result, implies that $\mathcal{Z}^{\mathfrak{m}} \subseteq \mathcal{E}^{\mathfrak{m}}[\mathbb{L}]$. However, formulae such as (6.6), and the general phenomenon of transference, indicate that $\mathcal{E}^{\mathfrak{m}}$ contains more than just periods of mixed Tate motives.

Chapter 7

Filtrations on fundamental groups and coefficient spaces

The fundamental groups of Chapter 5 are equipped with various *decreasing* filtrations. They arise by identifying the R-points of these groups with subgroups of certain non-commutative power series rings. Truncating a power series in one of these filtrations defines an *increasing* filtration on the space of coefficients of the series. The technical core of this thesis uses this idea to relate the depth filtration on motivic multiple zeta values to the length filtration on motivic iterated Eisenstein integrals.

In this chapter, which is largely technical, we define the decreasing filtrations on fundamental groups that are at our disposal. We also introduce some new notation and concepts, such as coefficient spaces, and show how these inherit structure from the associated power series.

This chapter is something of a transition point in this thesis. The previous chapters have mostly focused on theoretical background. The later chapters use the material developed in this chapter to focus on new results.

7.1 Formalities on filtrations

Let Z be a set and $M(Z) = M(z : z \in Z)$ the free monoid on Z. Throughout this section we assume that $Z = \bigcup_{k \geq 1} Z_k$ is a graded set with each Z_k finite. Elements of Z_k are assigned weight k, and this induces a grading on $M(Z)$.

Let R be a $\mathbb{Q}$-algebra. In §1.3.1.2 we defined the free associative R-algebra $R\langle Z \rangle$ and its I-adic completion $R\langle\langle Z \rangle\rangle$, where $I = (Z)$ is the maximal ideal generated by elements of Z. Elements of $R\langle\langle Z \rangle\rangle$ are formal power series in words in the alphabet

Z. They may be written as

$$s = \sum_{w \in M(Z)} s_w w, \quad s_w \in R.$$

Remark 7.1.1. Let $T^c(Z^\vee)$ denote the free shuffle algebra on the dual alphabet $Z^\vee$ that was defined in §1.3.1.1. Recall from §1.3.1.3 that $T^c(Z^\vee)$ is isomorphic to the graded dual[1] of $R\langle\langle Z\rangle\rangle$ with respect to the total weight grading $R\langle\langle Z\rangle\rangle = \bigoplus_{n\geq 0} R\langle\langle Z\rangle\rangle_n$ induced by the grading on Z.

Let $F^\bullet R\langle\langle Z\rangle\rangle$ be a decreasing filtration. We define an associated *increasing* filtration $F_\bullet T^c(Z^\vee)$ by letting $F_r T^c(Z^\vee)$ be the elements f of the graded dual such that $f(F^{r+1}R\langle\langle Z\rangle\rangle) = 0$. This means that $f(F^{r+1}R\langle\langle Z\rangle\rangle_n) = 0$ for each $n \geq 0$.

7.1.1 Filtered pieces of series

Let $F^\bullet R\langle\langle Z\rangle\rangle$ be a separated and exhaustive decreasing filtration that we furthermore assume is obtained from a grading. These assumptions hold in all cases considered in §7.2 by the results of [27, Appendix B], which implies that the Hodge filtration and both weight filtrations on the fundamental group can be simultaneously split.

Let $s \in R\langle\langle Z\rangle\rangle$. Then we may write

$$s = \sum_{k\geq 0} s_k, \quad \text{where } s_k \in (F^k \backslash F^{k+1}) \cup \{0\}\,.$$

Definition 7.1.2 (Filtered piece of series)**.** The rth F-filtered piece of s is

$$\mathrm{Fil}^r_F(s) := \sum_{k=0}^{r} s_k.$$

The definition implies that $s \equiv \mathrm{Fil}^r_F(s) \pmod{F^{r+1}R\langle\langle Z\rangle\rangle}$.

7.2 Decreasing filtrations on fundamental groups

In this section we define certain decreasing filtrations on the fundamental groups $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)$, $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$, $\mathcal{U}^{\mathrm{dR}}_{1,1}$ and $\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}}$.

[1] In §1.3.1.3, we stated this for the free associative algebra $R\langle Z\rangle$ rather than for its completion $R\langle\langle Z\rangle\rangle$. However, the assumption that Z is graded by finite sets Z_k implies that $R\langle\langle Z\rangle\rangle_n = R\langle Z\rangle$ is also a finite-rank R-module, generated by words of total weight equal to n.

7.2.1 Filtrations on $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)$

In §5.3.1 we saw that the group of R-points of $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)$ is the group of grouplike power series

$$\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)(R) \cong \mathcal{G}\left(R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle\right).$$

The filtrations on $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)(R)$ are induced by the following filtrations on the full power series ring $R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$.

Definition 7.2.1. The ring $R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$ is equipped with two natural decreasing filtrations: the *weight* filtration $\mathcal{W}^\bullet R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$ is the decreasing filtration on the total degree in both x_0 and x_1, and the *depth* filtration $\mathcal{D}^\bullet R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$ is the decreasing filtration on the x_1-degree.

These restrict to well-defined filtrations on $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)$, denoted by $\mathcal{W}^\bullet \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)$ and $\mathcal{D}^\bullet \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)$ respectively, because there is a unique bigrading on (the Lie algebra of) $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)$ that is compatible with the action of $\mathfrak{k} = \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})})$ by [27, §23 and §28]. See Remark 5.2.10.

Remark 7.2.2. The definition of the *decreasing* weight filtration $\mathcal{W}^\bullet R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$ is non-standard. This filtration satisfies $\mathcal{W}^r R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle = W_{-r} R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$, where $W_\bullet R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$ is the standard *increasing* weight filtration on the completed group algebra of the fundamental group afforded by its mixed Hodge structure [20].

We use the decreasing weight filtration $\mathcal{W}^\bullet R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$ in this thesis in order to relate it more easily to the decreasing depth filtration $\mathcal{D}^\bullet R\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$ and the other decreasing filtrations outlined in the following subsections. However, in all other cases weight filtrations will be considered to be increasing.

Remark 7.2.3. The depth filtration is induced by the inclusion $\mathbb{P}^1 \backslash \{0,1,\infty\} \hookrightarrow \mathbb{G}_m$ in the sense that $\mathcal{D}^1 \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)$ is the kernel of the induced morphism $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1) \to \pi_1^{\mathrm{dR}}(\mathbb{G}_m, \vec{1}_1)$. When $r \geq 1$, $\mathcal{D}^r \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)$ is the rth term of the lower central series filtration on $\mathcal{D}^1 \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)$.

7.2.2 Filtrations on $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$

In §5.4.1 we saw that the group of R-points of $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$ is canonically isomorphic to the group of grouplike power series

$$\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)(R) \cong \mathcal{G}\left(R\langle\langle \mathsf{a}, \mathsf{b} \rangle\rangle\right).$$

The filtrations on $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)(R)$ are induced by the following filtrations on the power series ring.

Definition 7.2.4. The ring $R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle$ is equipped with three natural decreasing filtrations: the A-filtration $A^\bullet R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle$ is the decreasing filtration on the a-degree; the B-filtration $B^\bullet R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle$ is the decreasing filtration on the b-degree; and the *elliptic depth* filtration $\mathcal{D}^\bullet R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle$ is the decreasing filtration on the $[\mathsf{a},\mathsf{b}]$-degree, defined by setting $\mathcal{D}^0 R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle = R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle$, $\mathcal{D}^1 R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle = \ker\left(R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle \to R[[\mathsf{a},\mathsf{b}]]\right)$, and, for $r \geq 1$, $\mathcal{D}^r R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle$ is the rth power of the two-sided ideal $\mathcal{D}^1 R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle$ in $R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle$. It is clear that $\mathcal{D}^\bullet \subseteq A^\bullet \cap B^\bullet$.

These restrict to well-defined filtrations on the fundamental group $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$, denoted by $A^\bullet\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$, $B^\bullet\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$ and $\mathcal{D}^\bullet\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$ respectively, because there is a unique bigrading on (the Lie algebra of) $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$ that is compatible with the isomorphism $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)(R) \cong \mathcal{G}\left(R\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle\right)$ and the action of $\mathfrak{k}$ by [27, Proposition 23.2].

Remark 7.2.5. The elliptic depth filtration arises geometrically from the inclusion $\mathcal{E}^\times_{\partial/\partial q} \hookrightarrow \mathcal{E}_{\partial/\partial q}$, in analogy to the depth filtration on $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \vec{1}_1)$. The piece $\mathcal{D}^1\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$ is the kernel of the induced morphism $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w) \to \pi_1^{\mathrm{dR}}(\mathcal{E}_{\partial/\partial q}, \partial/\partial w)$ and $\mathcal{D}^r\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$ is the rth term in the lower central series on $\mathcal{D}^1\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$.

7.2.3 Filtrations on $\mathcal{U}_{1,1}$ and $\mathcal{U}_{1,\vec{1}}$

In §5.5 we discussed the groups $\mathcal{U}_{1,1}$ and $\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}}$, and have a description of their Lie algebras. Their elements may be written in terms of the symbols $\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}$, $\mathbf{e}'_f\mathsf{X}^k\mathsf{Y}^{2n-k}$ and $\mathbf{e}''_f\mathsf{X}^k\mathsf{Y}^{2n-k}$ that were defined in (5.17). Recall from §5.5.1.1 that the totally holomorphic subgroup $\mathcal{U}^{\mathrm{dR,hol}}_{1,1}(R)$ of $\mathcal{U}^{\mathrm{dR}}_{1,1}(R)$ is non-canonically isomorphic to the group of grouplike power series in the above symbols.

Recall from (5.16) that there is a canonical MHS-preserving isomorphism $\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}} \cong \mathcal{U}^{\mathrm{dR}}_{1,1} \times \mathbb{G}_a(1)$. The Lie algebra of the factor $\mathbb{G}_a$ is spanned by the canonical element $\mathbf{e}_2$. It is central in $\mathfrak{u}_{1,\vec{1}} = \mathrm{Lie}(\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}}$.

Definition 7.2.6. The *length* filtration $L^\bullet\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}}$ is the decreasing filtration given by the lower central series. By restriction one defines $L^\bullet\mathcal{U}^{\mathrm{dR}}_{1,1}$.

Remark 7.2.7. The length filtration is so named for the following reason. It induces a filtration $L^\bullet\mathcal{U}^{\mathrm{dR,hol}}_{1,1}$ on the totally holomorphic quotient (resp. on the free Eisenstein quotient $\mathcal{U}^{\mathrm{dR}}_E$). The dual increasing filtration $L_\bullet\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1})$ (resp. $L_\bullet\mathcal{O}(\mathcal{U}^{\mathrm{dR,hol}}_{1,1})$, $L_\bullet\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_E)$) on the affine ring is the length filtration on the reduced bar construction defined in Definition 5.2.8.

7.3 Coefficient spaces

De Rham fundamental groups are a convenient tool for packaging iterated integrals into generating series. The notion of coefficient space extracts the coefficients from these series.

Definition 7.3.1 (Coefficient space)**.** Let $s \in R\langle\langle Z\rangle\rangle$ be a formal power series. The coefficient space $\mathrm{Co}(s)$ is vector subspace of R spanned by the coefficients of s.

Extra structure on a series s induces structure on $\mathrm{Co}(s)$. As described in §1.3.1.2 the ring $R\langle\langle Z\rangle\rangle$ may be equipped with the structure of a Hopf algebra by taking the coproduct Δ for which each $z \in Z$ is primitive. As described in §1.3.1.3, the shuffle algebra $T^c(Z^\vee)$ on the dual alphabet is isomorphic to the coordinate ring of the affine group scheme over $\mathbb{Q}$ representing the functor $R \mapsto \mathcal{G}(R\langle\langle Z\rangle\rangle)$. The isomorphism

$$\mathrm{Hom}_{\mathsf{Alg}_{\mathbb{Q}}}(T^c(Z^\vee), R) \xrightarrow{\sim} \mathcal{G}\left(R\langle\langle Z\rangle\rangle\right) \tag{7.1}$$

is given by sending a shuffle algebra homomorphism s to $\sum s(w^\vee)w$, where w ranges over all words in th alphabet Z. It follows that if $s \in R\langle\langle Z\rangle\rangle$ is grouplike then it defines an algebra homomorphism $T^c(Z) \to R$. The image of this algebra homomorphism is equal to $\mathrm{Co}(s)$, and therefore $\mathrm{Co}(s)$ is a $\mathbb{Q}$-subalgebra of R.

Remark 7.3.2. Let X be a connected scheme over $\mathbb{Q}$ and let $x \in X(\mathbb{Q})_{\mathrm{bp}}$. Let $\mathcal{U}^{\mathrm{dR}}$ be the unipotent radical of $\pi_1^{\mathrm{rel,dR}}(X, x)$ with respect to some choice of semisimple subcategory of $\mathsf{Con}(X)$. By the discussion in §5.2.6, the affine ring $\mathcal{O}(\mathcal{U}^{\mathrm{dR}})$ is equipped with the shuffle product on iterated integrals. A minimal set of generators for $\mathcal{O}(\mathcal{U}^{\mathrm{dR}})$ is given by a choice of basis for $H^1(\mathcal{U}^{\mathrm{dR}})$. Under some basic assumptions (e.g. those considered in §5.2.8) $H^1(\mathcal{U}^{\mathrm{dR}})$ is naturally graded and there is a non-canonical isomorphism $\mathcal{O}(\mathcal{U}^{\mathrm{dR}}) \cong T^c(H^1(\mathcal{U}^{\mathrm{dR}})^*)$, where $H^1(\mathcal{U}^{\mathrm{dR}})^*$ denotes the graded dual.

Let $s \in \mathcal{U}^{\mathrm{dR}}(R)$, regarded as an algebra homomorphism $s\colon T^c(H^1(\mathcal{U}^{\mathrm{dR}})^*) \to R$. By (7.1), it follows that $\mathrm{Co}(s)$ is a $\mathbb{Q}$-subalgebra of R. Note that $\mathrm{Co}(s)$ does not depend upon the choice of identification $\mathcal{O}(\mathcal{U}^{\mathrm{dR}}) \cong T^c(H^1(\mathcal{U}^{\mathrm{dR}})^*)$.

7.3.1 Induced filtrations on coefficient spaces

As before, let Z be a set satisfying the assumptions in §7.1. Suppose that $R\langle\langle Z\rangle\rangle$ is equipped with an exhaustive, separated, decreasing filtration $F^\bullet R\langle\langle Z\rangle\rangle$ induced from a grading, and let $s \in R\langle\langle Z\rangle\rangle$. To the pair $(s, F^\bullet)$ one can associate an *increasing* filtration on $\mathrm{Co}(s)$ as follows:

Definition 7.3.3. Define an increasing filtration of subspaces $\mathrm{Co}^F_\bullet(s)$ of $\mathrm{Co}(s)$ by

$$\mathrm{Co}^F_r(s) := \mathrm{Co}\left(\mathrm{Fil}^r_F(s)\right).$$

Proposition 7.3.4. *If s is grouplike, the filtration $\mathrm{Co}^F_\bullet(s)$ is compatible with the algebra structure on $\mathrm{Co}(s)$ i.e. $\mathrm{Co}^F_m(s)\,\mathrm{Co}^F_n(s) \subseteq \mathrm{Co}^F_{m+n}(s)$.*

Proof. The isomorphism (7.1) implies that $s\colon T^c(Z^\vee) \to R$ is a shuffle-algebra homomorphism. Let $c_1 \in \mathrm{Co}^F_m(s)$ and $c_2 \in \mathrm{Co}^F_n(s)$. By linearity, we may assume $c_i = s(u_i)$ are the coefficients of words u_1, u_2 in the dual alphabet $Z^\vee$. Then

$$c_1c_2 = s(u_1)s(u_2) = s(u_1 \sqcup\!\sqcup\, u_2),$$

where $\sqcup\!\sqcup$ denotes the shuffle product. Let $F_\bullet T^c(Z^\vee)$ be the associated increasing filtration on the shuffle algebra defined in Remark 7.1.1. The assumption on c_1 and c_2 implies that $u_1 \in F_kT^c(Z^\vee)$ for $k \leq m$ and $u_2 \in F_lT^c(Z^\vee)$ for $l \leq n$. The shuffle product $u_1 \sqcup\!\sqcup\, u_2 = \sum v$ is a sum of words $v \in F_{k+l}T^c(Z^\vee)$, which implies that $c_1c_2 = \sum s(v) \in \mathrm{Co}^F_{k+l}(s) \subseteq \mathrm{Co}^F_{m+n}(s)$. $\square$

7.3.2 Induced filtrations

Let $\mathcal{A}$ be an R-algebra equipped with a decreasing filtration $F^\bullet\mathcal{A}$ satisfying the same assumptions as in §7.1.1. The filtration $F^\bullet\mathcal{A}$ induces a canonical filtration $F^\bullet\,\mathrm{End}(\mathcal{A})$ on the endomorphism algebra of $\mathcal{A}$ via

$$F^r\,\mathrm{End}(\mathcal{A}) := \left\{e \in \mathrm{End}(\mathcal{A}) : e\left(F^k\mathcal{A}\right) \subseteq F^{k+r}\mathcal{A}\right\}.$$

Let E be an R-algebra acting on $\mathcal{A}$. This action is equivalent to an R-algebra homomorphism $E \to \mathrm{End}(\mathcal{A})$. The filtration $F^\bullet\,\mathrm{End}(\mathcal{A})$ induces a filtration $F^\bullet E$ by taking the preimages of the filtered pieces under this homomorphism.

The following statement explains how the coefficient space filtration may be decomposed in the case where one power series ring $R\langle\langle Z'\rangle\rangle$ acts on another $R\langle\langle Z\rangle\rangle$.

Proposition 7.3.5. *Let Z, Z' be sets satisfying the assumptions in §7.1. Suppose we have a filtration $F^\bullet R\langle\langle Z\rangle\rangle$ and an action of $E := R\langle\langle Z'\rangle\rangle$ on the filtered algebra $F^\bullet R\langle\langle Z\rangle\rangle$. Let $F^\bullet E$ be the induced filtration on E and suppose that $L^\bullet E$ is another filtration on E such that for each r, $L^rE \subseteq F^rE$. Then for $e \in E$ and $s \in R\langle\langle Z\rangle\rangle$ we have*

$$\mathrm{Co}^F_r(e(s)) = \sum_{i+j=r} \mathrm{Co}^L_i(e)\,\mathrm{Co}^F_j(s).$$

Proof. Write $e = \sum_{k\geq 0} e_k$ and $s = \sum_{l\geq 0} s_l$, where $e_k \in L^k \backslash L^{k+1}$ and $s_l \in F^l\backslash F^{l+1}$. It is clear that

$$\mathrm{Fil}^r_F(e(s)) = \sum_{\substack{k,l\geq 0\\ k+l\leq r}} e_k(s_l),$$

because $e_k(s_l) \in F^{k+l}R\langle\langle Z\rangle\rangle$. The result follows by taking coefficient spaces. □

7.3.3 Depth filtration and weight grading on motivic MZVs

The filtrations on coefficient spaces defined in §7.3.1 can be used to equip the algebra $\mathcal{Z}^{\mathfrak{m}}$ of motivic multiple zeta values with natural filtrations. The notions in this section were introduced in §4.3.3.

Let $\Phi^{\mathfrak{m}}_{01} \in \mathcal{G}(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle)$ be the motivic Drinfeld associator defined in §5.3.2.2. It is a generating series for motivic MZVs, and therefore $\mathrm{Co}(\Phi^{\mathfrak{m}}_{01}) = \mathcal{Z}^{\mathfrak{m}}$. The weight and depth filtrations on $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle$ defined in Definition 7.2.1 can be used to equip $\mathcal{Z}^{\mathfrak{m}}$ with weight and depth filtrations, as follows:

Definition 7.3.6. The algebra $\mathcal{Z}^{\mathfrak{m}}$ has two increasing filtrations: the *weight* filtration $\mathfrak{W}_\bullet \mathcal{Z}^{\mathfrak{m}}$, and the *depth* filtration $\mathfrak{D}_\bullet \mathcal{Z}^{\mathfrak{m}}$. They are defined by

$$\mathfrak{W}_\bullet \mathcal{Z}^{\mathfrak{m}} := \mathrm{Co}^{\mathcal{W}}_\bullet(\Phi^{\mathfrak{m}}_{01});$$
$$\mathfrak{D}_\bullet \mathcal{Z}^{\mathfrak{m}} := \mathrm{Co}^{\mathcal{D}}_\bullet(\Phi^{\mathfrak{m}}_{01}).$$

Since $\Phi^{\mathfrak{m}}_{01}$ is grouplike, Proposition 7.3.4 implies that these filtrations are compatible with the shuffle product on $\mathcal{Z}^{\mathfrak{m}}$.

Remark 7.3.7. The weight (resp. depth) filtration is so named because its image under the period map is precisely the weight (resp. depth) filtration on numerical multiple zeta values. We recall that the weight of an admissible multiple zeta value $\zeta(k_1, \ldots, k_r)$ is $k_1 + \cdots + k_r$ and the depth is r.

Proposition 7.3.8. *The weight filtration satisfies* $\mathfrak{W}_\bullet \mathcal{Z}^{\mathfrak{m}} = \mathrm{Co}^{\mathcal{W}}_\bullet(\Phi^{\mathfrak{m}}_{ij})$ *whenever* $i, j \in \{0, 1, \infty\}$ *are distinct.*

Proof. The series $\Phi^{\mathfrak{m}}_{ij}$ is obtained from $\Phi^{\mathfrak{m}}_{01}$ by making the change of variables $(\mathsf{x}_0, \mathsf{x}_1) \mapsto (\mathsf{x}_i, \mathsf{x}_j)$. This change of variables induces an automorphism $\mathcal{W}^\bullet \xrightarrow{\sim} \mathcal{W}^\bullet$, and therefore the coefficient space remains unchanged. □

The weight filtration $\mathfrak{W}_\bullet\,\mathcal{Z}^{\mathfrak{m}}$ is induced from a *grading*. This is particular to motivic multiple zeta values; numerical MZVs are only conjecturally graded.

The weight grading is defined as follows: let

$$\mathcal{Z}^{\mathfrak{m}}_k := \langle \zeta^{\mathfrak{m}}(w) : \deg(w) = k\rangle_{\mathbb{Q}}$$

be the subspace of motivic MZVs of weight k. Then $\mathcal{Z}^{\mathfrak{m}}_k \cong \mathfrak{W}_k\,\mathcal{Z}^{\mathfrak{m}}\,/\mathfrak{W}_{k-1}\,\mathcal{Z}^{\mathfrak{m}}$ and

$$\mathfrak{W}_r\,\mathcal{Z}^{\mathfrak{m}} = \bigoplus_{k=0}^{r} \mathcal{Z}^{\mathfrak{m}}_k.$$

The ideal of motivic MZVs of positive weight is

$$\mathcal{Z}^{\mathfrak{m}}_{>0} := \bigoplus_{r>0} \mathcal{Z}^{\mathfrak{m}}_r.$$

Let $\mathcal{P} := \mathcal{Z}^{\mathfrak{m}}[\mathbb{L}]$. Brown's result [5] implies that $\mathcal{P}$ is the ring $\mathcal{P}^{\mathfrak{m},+}_{\mathsf{MT}(\mathbb{Z})}$ of effective motivic periods for mixed Tate motives over $\mathbb{Z}$, and that $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} \cong \mathcal{P}[\mathbb{L}^{-1}]$.

7.3.4 Length filtration on motivic multiple modular values and iterated Eisenstein integrals

The length filtration $L_\bullet\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1})$ may be used to define an increasing length filtration on the $\mathbb{Q}$-algebra $M^{\mathfrak{m}}$ of multiple modular values.

Definition 7.3.9. The *length* filtration $\mathfrak{L}_\bullet M^{\mathfrak{m}}$ is the increasing filtration defined by letting $\mathfrak{L}_r M^{\mathfrak{m}}$ be the image of $L_r\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1})$ under $\mathcal{C}^{\mathfrak{m}}_S$, followed by tensoring with $\mathbb{Q}[\mathbb{L}^{\pm}]$.

The "totally holomorphic" subspace of $\mathfrak{L}_r M^{\mathfrak{m}}$ is spanned by totally holomorphic motivic iterated integrals of modular forms of length $s \le r$ as considered in §6.2.

7.3.5 Action of $S \in SL_2^{\mathrm{dR}}(\mathbb{Q})$

In this subsection we record some simple but important facts about the action of SL_2^{dR} on the filtrations $A^\bullet$ and $B^\bullet$ on $R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$, as described in §5.5.2. The general formula for this action, given in equation (5.19), implies that the matrix

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2^{\mathrm{dR}}(\mathbb{Q})$$

acts via

$$S(\mathsf{a}, \mathsf{b}) = (-\mathsf{b}, \mathsf{a}). \tag{7.2}$$

Proposition 7.3.10. *Let R be a $\mathbb{Q}$-algebra. The action of $S \in SL_2^{\mathrm{dR}}(\mathbb{Q})$ on $R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$ induces isomorphisms of filtered R-algebras $A^\bullet R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle \rightleftarrows B^\bullet R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$.*

Proof. This is immediate from equation (7.2). □

The following useful lemma is an immediate consequence of Proposition 7.3.10.

Lemma 7.3.11. *On $R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$ there is an equality of operators*

$$S \circ \mathrm{Fil}^r_A = \mathrm{Fil}^r_B \circ S.$$

The same formula also holds with A and B interchanged.

Proof. An element $w \in R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$ is contained in A^r iff $S(w) \in B^r$. □

7.3.5.1 The action of $S_0^{\mathfrak{m}}$

Let $S_0^{\mathfrak{m}} \in SL_2^{\mathrm{dR}}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}})$ be the reductive part of $S^{\mathfrak{m}} = (\mathcal{C}^{\mathfrak{m}}_S, S_0^{\mathfrak{m}})$ defined in Definition 5.2.6. It is contained within $SL_2^{\mathrm{dR}}(\mathbb{Q}[\mathbb{L}^{\pm}])$ and is conjugate to $S \in SL_2^{\mathrm{dR}}(\mathbb{Q})$ by an element of $GL_2(\mathbb{Q}[\mathbb{L}^{\pm}])$ by the formula

$$S_0^{\mathfrak{m}} = \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{L}^{-1} \end{pmatrix} S \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{L}^{-1} \end{pmatrix}^{-1}.$$

By (7.2) we obtain the formula

$$S_0^{\mathfrak{m}}(\mathsf{a}, \mathsf{b}) = (-\mathbb{L}^{-1}\mathsf{b}, \mathbb{L}\mathsf{a}). \tag{7.3}$$

Proposition 7.3.10 and Lemma 7.3.11 are also true when S is replaced by $S_0^{\mathfrak{m}}$ as long as R is a $\mathbb{Q}[\mathbb{L}^{\pm}]$-algebra.

Chapter 8

The monodromy action

In this chapter we explore the relationship between the fundamental groups of the moduli spaces $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,\vec{1}}$ and the fundamental group of the infinitesimal punctured Tate curve.

The essential connection comes in the form of a *monodromy action*. This action is present at the topological level, and is described in §8.1. An important property of the monodromy action is that it relates the two generators α, β of $\pi_1^{\mathrm{top}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$. This provides an important extra structure on this free group.

The topological monodromy action extends to an action of the relative fundamental group $\mathcal{G}^{\mathrm{dR}}_{1,\vec{1}}$ on $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$. We describe this action in §8.2. Proposition 9.2.1 gives a formula for this action in terms of the Lie algebra $\mathfrak{u}^{\mathrm{geom}}$. Crucially, the monodromy action factors through the "Eisenstein quotient" [23, §16] of $\mathcal{U}^{\mathrm{dR}}_{1,1}$, which is the maximal quotient of $\mathcal{U}^{\mathrm{dR}}_{1,1}$ on which $\mathfrak{k} = \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})})$ acts.

8.1 The topological monodromy action

Let E be an elliptic curve over a field $K \subseteq \mathbb{C}$. This determines a point $x = [E] \in \mathcal{M}_{1,1}(K)$. By equipping $E^\times$ with a nonzero K-rational tangential basepoint $\vec{v}$ at the punctured origin O we also determine a point $\hat{x} \in \mathcal{M}_{1,\vec{1}}(K)$ mapping to x under the morphism $\mathcal{M}_{1,\vec{1}} \to \mathcal{M}_{1,1}$. We may also choose x to be a tangential basepoint on $\mathcal{M}_{1,1}$; e.g. $x = \partial/\partial q$. In this case the corresponding elliptic curve is the Tate curve $\mathcal{E}_{\partial/\partial q}$, which can be equipped with the tangent vector $\vec{v} = \partial/\partial w$ at the origin.

Throughout this section we work with an arbitrary (possibly tangential) basepoint x. From §8.2 onward, however, we specialise to $x = \partial/\partial q$. This is because our argument crucially relies on $\mathcal{O}(\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w))$ being mixed Tate.

8.1.1 The outer monodromy action

The fiber of the universal punctured family $\mathcal{E}^\times \to \mathcal{M}_{1,1}$ over x is isomorphic to $E^\times$. The homotopy exact sequence of this fibration produces a short exact sequence

$$1 \to \pi_1^{\mathrm{top}}(E^\times, \vec{v}) \to \pi_1^{\mathrm{top}}(\mathcal{E}^\times, [E^\times, \vec{v}]) \to \pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x) \to 1$$

exhibiting $\pi_1^{\mathrm{top}}(\mathcal{E}^\times, [E^\times, \vec{v}])$ as an extension of $SL_2(\mathbb{Z})$ by a free group on two generators [26, Proposition 1.4]. Conjugation determines an outer action

$$\bar{\mu}_0 \colon \pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x) \to \operatorname{Out} \pi_1^{\mathrm{top}}(E^\times, \vec{v}). \tag{8.1}$$

Remark 8.1.1. This outer action does not lift to a genuine action on $\pi_1^{\mathrm{top}}(E^\times, \vec{v})$ for a geometric reason. The basepoint x on $\mathcal{M}_{1,1}$ corresponds to an isomorphism class of elliptic curves $[E]$. It could therefore be represented by a different *model* of E, say $x = [E']$, corresponding to an isomorphism $E \xrightarrow{\sim} E'$ defined over K. But there is no natural way to choose a tangential basepoint $\vec{v}$ on all models of E simultaneously such that the $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x)$-action respects this choice.

For example, let $\lambda \in K^\times$. The change of variables $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$ defines an isomorphism between two models of the elliptic curve

$$E : y^2 = 4x^3 - g_2 x - g_3$$

that is defined over K. As described in §3.2 this automorphism scales $\omega = dx/y$ by $\omega \mapsto \lambda^{-1}\omega$. Dual to ω is a nonzero tangential basepoint $\vec{v}$ at the origin that is scaled by $\vec{v} \mapsto \lambda \vec{v}$. Any two models of E over $\bar{K}$ are related by a change of variables of the above type for some $\lambda \in \bar{K}^\times$, [50, §III.1]. Moreover, choosing $\lambda = -1$ defines an automorphism of every elliptic curve E which sends $\vec{v} \mapsto -\vec{v}$. It is therefore not possible to fix a particular *nonzero* tangential basepoint $\vec{v}$ on the isomorphism class $[E]$, and thus there is no natural monodromy action of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x)$ on $\pi_1^{\mathrm{top}}(E^\times, \vec{v})$. Note, however, that there is a natural monodromy action of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x)$ on $\pi_1^{\mathrm{top}}(E, O)$, where O is the origin on the elliptic curve. Up to conjugation this coincides with the action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2$.

The moduli space $\mathcal{M}_{1,\vec{1}}$ overcomes this issue because a basepoint $\hat{x} \in \mathcal{M}_{1,\vec{1}}$ determines an elliptic curve E *together* with a choice of tangential basepoint $\vec{v}$. This is equivalent to determining a Weierstrass equation for E. This allows us to define an action of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x})$ on $\pi_1^{\mathrm{top}}(E^\times, \vec{v})$.

8.1.2 The topological monodromy action

We may identify $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x}) \cong B_3$ with the mapping class group of $E^\times(\mathbb{C})$, which is a genus 1 surface with one puncture. The generators t_A and t_B correspond to Dehn twists on simple closed curves $A, B \subseteq E(\mathbb{C})$ intersecting transversally at one point. For example, A and B could be (the images of) the two generators α and β for the fundamental group of $E^\times(\mathbb{C})$ that were defined in §5.1.1.2.

The *left* action of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x})$ on $\pi_1^{\mathrm{top}}(E^\times, \vec{v})$ via the Dehn twists t_A and t_B defines a homomorphism

$$\mu_0 \colon \pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x}) \to \operatorname{Aut} \pi_1^{\mathrm{top}}(E^\times, \vec{v}). \tag{8.2}$$

Lemma 8.1.2. *Let A and B denote the images of generators α and β for $\pi_1^{top}(E^\times, \vec{v})$. Then μ_0 is given by the following explicit action:*

$$t_A(\alpha) = \alpha, \quad t_A(\beta) = \beta\alpha, \quad t_B(\alpha) = \alpha\beta^{-1}, \quad t_B(\beta) = \beta. \tag{8.3}$$

Proof. This follows from the description of t_A and t_B as Dehn twists along the images of α and β by considering the shape of α, β in the Jacobi uniformisation $E(\mathbb{C}) \cong \mathbb{C}^\times/q^{\mathbb{Z}}$ [24] (see also Figure 5.1, which is representative for any choice of E.). It may also be derived group-theoretically by viewing the free group $F_2 = \langle \alpha, \beta \rangle$ as a subgroup of the braid group B_4 on four strands, with $B_3 \subseteq B_4$ acting on F_2 via conjugation [42, §9]. $\square$

Definition 8.1.3. Let $\Theta := \alpha\beta\alpha^{-1}\beta^{-1}$. It is the homotopy class of the boundary circle on the punctured torus.

One verifies that the action described in (8.3) fixes Θ. This is because elements of the mapping class group $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x})$ act by automorphisms fixing the boundary circle.

8.1.3 Main topological equation

Let $\tilde{S} := (t_A t_B t_A)^{-1} \in B_3$. Under (8.2), this element acts by

$$\tilde{S}(\alpha) = \alpha\beta\alpha^{-1}, \quad \tilde{S}(\beta) = \alpha^{-1}.$$

Up to conjugation and inverses, the element $\tilde{S}$ interchanges α and β. The essential idea of this paper is to use this element to show how the periods of α and β are related to the periods of $\tilde{S}$. The notation $\tilde{S}$ is justified in the following subsection.

8.1.4 Induced morphism to $SL_2(\mathbb{Z})$

There is an induced action of $\pi_1^{\text{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x})$ on the abelianisation

$$\pi_1^{\text{top}}(E^\times, \vec{v})^{\text{ab}} \cong H_1(E(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}^2.$$

This may be described by a homomorphism $\pi_1^{\text{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x}) \to GL_2(\mathbb{Z})$. This morphism factors through $SL_2(\mathbb{Z})$ because $\pi_1^{\text{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x})$ fixes Θ, and is thus orientation-preserving. We now describe this morphism explicitly.

Denote the images of α and β in $H_1(E^\times(\mathbb{C}), \mathbb{Z})$ by $[\alpha]$ and $[\beta]$ respectively. The fundamental group $\pi_1^{\text{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x})$ acts on $H_1(E^\times(\mathbb{C}), \mathbb{Z})$ by the abelianisation of the action (8.3). This can be written as a *right* action on frames, given by

$$(t_A([\alpha]), t_A([\beta])) = ([\alpha], [\beta]) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (t_B([\alpha]), t_B([\beta])) = ([\alpha], [\beta]) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Proposition 8.1.4. *The abelianisation of the action* (8.3) *defines a homomorphism* $f\colon B_3 \to SL_2(\mathbb{Z})$ *by the formula*

$$t_A \mapsto T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad t_B \mapsto (TST)^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

It is surjective and its kernel is an infinite cyclic group generated by $\tilde{S}^4 = (t_A^{-1} t_B^{-1})^6$.

Proof. It is clear from the definitions that the abelianisation of (8.3) factors through f. One can also verify directly that $f\colon B_3 \to SL_2(\mathbb{Z})$ is a homomorphism.

Let $\tilde{S} = (t_A t_B t_A)^{-1}$ as above, and let $\tilde{T} := t_A$. We have $f(\tilde{S}) = S$ and $f(\tilde{T}) = T$. Since $SL_2(\mathbb{Z})$ is generated by S and T, f is surjective.

It is easy to verify that $f(\tilde{S}^4) = I$. Set $\tilde{U} = \tilde{S}\tilde{T}$. From the presentation for the braid group in terms of the generators t_A and t_B given in (5.1), we obtain the following alternate presentation:

$$B_3 \cong \langle \tilde{S}, \tilde{U} | \tilde{S}^2 = \tilde{U}^3 \rangle.$$

There is a well-known presentation for $SL_2(\mathbb{Z})$ in terms of S and $U := ST$:

$$SL_2(\mathbb{Z}) = \langle S, U | S^2 = U^3, S^4 = I \rangle.$$

From this it is clear the kernel is precisely generated by $\tilde{S}^4$. □

Remark 8.1.5 (Geometric interpretation). The kernel is isomorphic to $\mathbb{Z}$ because f may be identified with the map on fundamental groups induced by the $\mathbb{G}_m$-torsor $\mathcal{M}_{1,\vec{1}} \to \mathcal{M}_{1,1}$.

Recall that $\ker(f)$ is generated by $\tilde{S}^4$ and that Θ denotes the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ as defined in Definition 8.1.3. One verifies that $\mu_0(\tilde{S}^4) = \mathrm{Ad}_\Theta$. Therefore, $\ker(f)$ consists of elements acting on $\pi_1^{\mathrm{top}}(E^\times, \vec{v})$ as a power of Ad_Θ.

The action of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x})$ on $\pi_1^{\mathrm{top}}(E^\times, \vec{v})^{\mathrm{ab}}$ factors through the natural homomorphism $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x}) \to \pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x)$. In turn, the action of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x)$ on $\pi_1^{\mathrm{top}}(E^\times, \vec{v})^{\mathrm{ab}}$ factors through the outer action (8.1). This gives a geometric interpretation for why elements in $\ker(f)$ act as a power of the inner automorphism Ad_Θ.

The situation is summarised in the following commutative diagram:

$$\begin{tikzcd}
\ker(f) = \langle \tilde{S}^4 \rangle \arrow[r, "\mu_0|_{\ker(f)}"] \arrow[d, hook] & \langle \mathrm{Ad}_\Theta \rangle \arrow[d, hook] \\
\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \hat{x}) \cong B_3 \arrow[r, "\mu_0"] \arrow[d, "f"'] & \mathrm{Aut}\, \pi_1^{\mathrm{top}}(E^\times, \vec{v}) \arrow[d] \\
\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x) \arrow[r, "\bar{\mu}_0"] \arrow[d, "\sim"'] & \mathrm{Out}\, \pi_1^{\mathrm{top}}(E^\times, \vec{v}) \arrow[d] \\
SL_2(\mathbb{Z}) \arrow[r, hook] & \mathrm{Aut}\left(\pi_1^{\mathrm{top}}(E^\times, \vec{v})^{\mathrm{ab}}\right) \cong GL_2(\mathbb{Z})
\end{tikzcd}$$

The left column is exact at B_3 and the final vertical map in the right column is the natural map $\mathrm{Out}(G) \to \mathrm{Aut}(G^{\mathrm{ab}})$.

Remark 8.1.6. Although there is only an outer action of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, x)$ on $\pi_1^{\mathrm{top}}(E^\times, \vec{v})$, there is an action of the de Rham relative completion $\pi_1^{\mathrm{rel,dR}}(\mathcal{M}_{1,1}, x)$ on the de Rham fundamental group $\pi_1^{\mathrm{dR}}(E^\times, \vec{v})$. See Remark 8.2.4.

8.2 The relative monodromy action

Because relative completion is functorial (see §5.2.7), the topological action described in §8.1 extends to an action of the de Rham relative completion $\mathcal{G}_{1,\vec{1}}^{\mathrm{dR}}$ on the de Rham fundamental group $\pi_1^{\mathrm{dR}}(\mathcal{E}_{\partial/\partial q}^\times, \partial/\partial w)$. This is described by a morphism of groip schemes

$$\mu \colon \mathcal{G}_{1,\vec{1}}^{\mathrm{dR}} \to \mathrm{Aut}(\pi_1^{\mathrm{dR}}(\mathcal{E}_{\partial/\partial q}^\times, \partial/\partial w)) \tag{8.4}$$

called the *monodromy morphism*.

In terms of the semidirect product decomposition $\mathcal{G}_{1,\vec{1}}^{\mathrm{dR}} \cong \mathcal{U}_{1,\vec{1}}^{\mathrm{dR}} \rtimes SL_2^{\mathrm{dR}}$ given in (5.20), the monodromy action may be written as follows. Fix a $\mathbb{Q}$-algebra R. Let

$u \in \mathcal{U}^{\mathrm{dR}}_{1,\vec{1}}(R)$, $\gamma \in SL_2^{\mathrm{dR}}(R)$ and $\pi \in \pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)(R)$. Then (u, γ) defines an element of $\mathcal{G}^{\mathrm{dR}}_{1,\vec{1}}(R)$ that acts on π via the formula

$$(u, \gamma)(\pi) = \mu(u)(\gamma(\pi)). \tag{8.5}$$

Here γ acts on π as in §5.5.2. Note that the expression (u, γ) depends upon the choice of splitting $\mathcal{G}^{\mathrm{dR}}_{1,\vec{1}} \cong \mathcal{U}^{\mathrm{dR}}_{1,\vec{1}} \rtimes SL_2^{\mathrm{dR}}$; the result of acting on π, however, does not. We explain this below.

8.2.1 Induced action of $\mathcal{G}^{\mathrm{dR}}_{1,1}$

Recall that $\mathcal{G}^{\mathrm{dR}}_{1,\vec{1}} \cong \mathcal{G}^{\mathrm{dR}}_{1,1} \times \mathbb{G}_a$. Precomposing with the corresponding inclusion produces a morphism

$$\mathcal{G}^{\mathrm{dR}}_{1,1} \hookrightarrow \mathcal{G}^{\mathrm{dR}}_{1,\vec{1}} \xrightarrow{\mu} \mathrm{Aut}(\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)),$$

which we also denote by μ by abuse of notation.

Definition 8.2.1. Let $\mathcal{G}^{\mathrm{geom}}$ be the image of $\mathcal{G}^{\mathrm{dR}}_{1,1}$ under μ. It is equipped with a morphism $\mathcal{G}^{\mathrm{geom}} \to SL_2^{\mathrm{dR}}$, where SL_2^{dR} is viewed as a subgroup of $\mathrm{Aut}(\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w))$ via the formula (5.19).

Let $\mathcal{U}^{\mathrm{geom}} := \ker(\mathcal{G}^{geom} \to SL_2^{\mathrm{dR}})$. It is a pro-unipotent affine group scheme over $\mathbb{Q}$.

The sequence $1 \to \mathcal{U}^{\mathrm{geom}} \to \mathcal{G}^{\mathrm{geom}} \to SL_2^{\mathrm{dR}} \to 1$ is canonically split [27, §23]. Therefore, there is a canonical isomorphism

$$\mathcal{G}^{\mathrm{geom}} \cong \mathcal{U}^{\mathrm{geom}} \rtimes SL_2^{\mathrm{dR}}. \tag{8.6}$$

Hence, although there is no canonical expression for an element of $\mathcal{G}^{\mathrm{dR}}_{1,\vec{1}}(R)$ as a pair $(u, \gamma) \in \mathcal{U}^{\mathrm{dR}}_{1,\vec{1}}(R) \rtimes SL_2^{\mathrm{dR}}(R)$, the result $(u, \gamma)(\pi) = \mu(u)(\gamma(\pi))$ does not depend upon the choice.

Remark 8.2.2. This allows us to define elements of $\mathcal{U}^{\mathrm{geom}}$ directly. One important case to note is that while $\mathcal{C}^{\mathfrak{m}}_S$ (defined in §5.5.3) is not canonically defined, its image $\mu(\mathcal{C}^{\mathfrak{m}}_S)$ is. This allows us to consider its space of coefficients as a well-defined object.

Definition 8.2.3. Let $\mathfrak{g}^{\mathrm{geom}} = \mathrm{Lie}(\mathcal{G}^{\mathrm{geom}})$ and $\mathfrak{u}^{\mathrm{geom}} = \mathrm{Lie}(\mathcal{U}^{\mathrm{geom}})$. They are Lie subalgebras of the derivation Lie algebra $\mathrm{Der\,Lie\,Aut}(\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w))$. By [27] there is a canonical decomposition

$$\mathfrak{g}^{\mathrm{geom}} = \mathfrak{u}^{\mathrm{geom}} \rtimes \mathfrak{sl}_2. \tag{8.7}$$

Because of the canonical decompositions (8.6) and (8.7), we will be mostly work from this point onward with the unipotent group $\mathcal{U}^{\mathrm{geom}}$ and its Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ in the sequel. These will be studied in detail in Chapter 9, where we will also provide an explicit formula for the monodromy morphism μ.

Remark 8.2.4. It is possible to define an action of $\mathcal{G}_{1,1}^{\mathrm{dR}}$ on $\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ because, for any choice of basepoint $x \in \mathcal{M}_{1,1}(\mathbb{C})$ and any choice of lift $\hat{x} \in \mathcal{M}_{1,\vec{1}}(\mathbb{C})$, the exact sequence

$$1 \to \mathbb{G}_a(1) \to \pi_1^{\mathrm{rel,dR}}(\mathcal{M}_{1,\vec{1}}, \hat{x}) \to \pi_1^{\mathrm{rel,dR}}(\mathcal{M}_{1,1}, x) \to 1$$

associated to the $\mathbb{G}_m$-torsor $\mathcal{M}_{1,\vec{1}} \to \mathcal{M}_{1,1}$ is canonically split and respects mixed Hodge structures [23, Proposition 14.2]. Compare this to the topological situation, where the exact sequence of topological fundamental groups

$$1 \to \mathbb{Z} \to B_3 \to SL_2(\mathbb{Z}) \to 1$$

does not split. As discussed in §8.1 this means that there is only an outer action of $SL_2(\mathbb{Z})$ on $\pi_1^{\mathrm{top}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$. This is an example of how the relative completion simplifies certain structures that are inaccessible at the topological level.

Chapter 9

Geometric derivations

In this chapter we study the group scheme $\mathcal{U}^{\mathrm{geom}}$, defined in Definition 8.2.3, and its Lie algebra $\mathfrak{u}^{\mathrm{geom}}$, defined in Definition 8.2.3, in detail. As described in §8.2, $\mathcal{U}^{\mathrm{geom}}$ is a subgroup of $\mathrm{Aut}(\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w))$ coming from the monodromy action of $\mathcal{U}_{1,1}^{\mathrm{dR}}$ on $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$.

Recall from §5.4.1 that $\mathrm{Lie}\,\mathrm{Aut}(\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w))$ is canonically isomorphic to the lower central series completion of $\mathrm{Lie}(\mathsf{a},\mathsf{b})$. It follows that $\mathfrak{u}^{\mathrm{geom}}$ may be described as a certain Lie subalgebra of the lower central series completion of the derivation algebra $\mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$.

The derivations in $\mathfrak{u}^{\mathrm{geom}}$ were first written down by Tsunogai [52, §3] in the context of Galois actions on the pro-ℓ fundamental group of a punctured elliptic curve $E^\times$ over a number field K. Tsunogai studied how the Galois action of $\mathrm{Gal}(\bar{K}/K)$ on $\pi_1^{(\ell)}(E^\times \times_K \bar{K})$ interacts with these derivations. The approach we take uses the de Rham fundamental group instead of the pro-ℓ fundamental group, and the Galois action is replaced by the motivic Galois group $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ in the case $E = \mathcal{E}_{\partial/\partial q}$. Nevertheless the Galois and monodromy actions are intertwined in either case; we discuss the motivic story further in §13.2.

In a related direction, the Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ is closely connected to modular forms for $SL_2(\mathbb{Z})$. Pollack [44] showed that there is a family of relations in $\mathfrak{u}^{\mathrm{geom}}$ corresponding to period polynomials of cusp forms. We expand on this in §9.6.

9.1 The derivations $\varepsilon^\vee_{2n+2}$ and ε_{2n+2}

Let $\mathrm{Lie}(\mathsf{a},\mathsf{b})$ denote the free Lie algebra on two letters a and b over $\mathbb{Q}$ [46]. Its lower central series completion $\mathrm{Lie}(\mathsf{a},\mathsf{b})^\wedge$ is canonically isomorphic to $\mathrm{Lie}\,\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$.

Let $\theta = [\mathsf{a},\mathsf{b}]$ and consider the Lie subalgebra $\mathrm{Der}^\theta\,\mathrm{Lie}(\mathsf{a},\mathsf{b}) \subseteq \mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$ consisting of derivations δ for which $\delta(\theta) = 0$. Within this subalgebra there is a distinguished family of derivations $\varepsilon^\vee_{2n+2}$, defined for every $n \geq -1$, that were originally studied by Tsunogai in [52]. They are defined as follows:

Definition 9.1.1 (Geometric derivations)**.** For $n \geq -1$ define $\varepsilon^\vee_{2n+2} \in \operatorname{Der}^\theta \operatorname{Lie}(\mathsf{a}, \mathsf{b})$ by

$$\begin{aligned}
\varepsilon^\vee_{2n+2}(\mathsf{a}) &= \operatorname{ad}(\mathsf{a})^{2n+2}(\mathsf{b}) \\
\varepsilon^\vee_{2n+2}(\mathsf{b}) &= \frac{1}{2} \sum_{i+j=2n+1} (-1)^i \left[\operatorname{ad}(\mathsf{a})^i(\mathsf{b}), \operatorname{ad}(\mathsf{a})^j(\mathsf{b})\right].
\end{aligned}$$

The derivation $\varepsilon^\vee_0$ may also be written as $\varepsilon^\vee_0 = \mathsf{b}\partial/\partial\mathsf{a}$.

One verifies that $\varepsilon^\vee_{2n+2}(\theta) = 0$, and hence that $\varepsilon^\vee_{2n+2} \in \operatorname{Der}^\theta \operatorname{Lie}(\mathsf{a}, \mathsf{b})$.

9.1.1 Action of SL_2 and $\mathfrak{sl}_2$

The group SL_2^{dR} acts on the *right* of $\operatorname{Lie}(\mathsf{a}, \mathsf{b})$ by right-multiplying the row vector (a, b) as described in §5.5.2. The derivations $\varepsilon^\vee_{2n+2}$ can be twisted by this action to obtain new derivations

$$\varepsilon_{2n+2} := (-)|_S \circ \varepsilon^\vee_{2n+2} \circ (-)|_{S^{-1}}. \tag{9.1}$$

We may also write $\varepsilon_0 = -\mathsf{a}\partial/\partial\mathsf{b}$. There is an inner action of $\mathfrak{sl}_2$ on $\operatorname{Der}^\theta \operatorname{Lie}(\mathsf{a}, \mathsf{b})$ by $\varepsilon^\vee_0$, ε_0 and $h = [\varepsilon_0, \varepsilon^\vee_0] =$ multiplication by $(\deg_{\mathsf{b}} - \deg_{\mathsf{a}})$.

Proposition 9.1.2. *The derivation $\varepsilon^\vee_{2n+2}$ generates an irreducible representation of $\mathfrak{sl}_2$ of dimension $2n+1$, with basis $\left\{\operatorname{ad}(\varepsilon^\vee_0)^k(\varepsilon^\vee_{2n+2}) : 0 \leq k \leq 2n\right\}$.*

Proof. See [52, Lemma 4.5] or [7, Lemma 5.2]. □

These representations could equivalently be defined using the elements ε_{2n+2}. The derivations $\varepsilon^\vee_{2n+2}$ are lowest-weight vectors and the derivations ε_{2n+2} are highest-weight vectors with respect to the $\mathfrak{sl}_2$-action. They are related by the following lemma.

Lemma 9.1.3. *We have*

$$\operatorname{ad}(\varepsilon^\vee_0)^k(\varepsilon^\vee_{2n+2}) = \frac{k!}{(2n-k)!} \operatorname{ad}(\varepsilon_0)^{2n-k}(\varepsilon_{2n+2}). \tag{9.2}$$

Proof. By comparing $\mathfrak{sl}_2$-weights, one knows that the quantities $\operatorname{ad}(\varepsilon^\vee_0)^k(\varepsilon^\vee_{2n+2})$ and $\operatorname{ad}(\varepsilon_0)^{2n-k}(\varepsilon_{2n+2})$ must be proportional. The scale factor can be found by applying both sides of (9.2) to either a or b. □

9.2 Explicit formula for μ

By [27, §22] the monodromy morphism $\mu\colon \mathcal{U}_{1,\vec{1}}^{\mathrm{dR}} \to \mathrm{Aut}(\pi_1^{\mathrm{dR}}(\mathcal{E}_{\partial/\partial q}^{\times}, \partial/\partial w))$ factors through the totally holomorphic quotient $\mathcal{U}_{1,\vec{1}}^{\mathrm{dR,hol}}$ defined in §5.5.1. In other words, it is determined by the associated bigraded Lie algebra homomorphism

$$\mathrm{gr}^M \mathrm{gr}^W \mathfrak{u}_{1,\vec{1}} \to \mathrm{gr}^M \mathrm{gr}^W \mathrm{Der}\,\mathrm{Lie}\,\pi_1^{\mathrm{dR}}(\mathcal{E}_{\partial/\partial q}^{\times}, \partial/\partial w),$$

which, by the descriptions given in §5.4.1 and §5.5.1, is given by a Lie algebra morphism

$$\mathrm{Lie}(\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}, \mathbf{e}'_f\mathsf{X}^k\mathsf{Y}^{2n-k}, \mathbf{e}''_f\mathsf{X}^k\mathsf{Y}^{2n-k}) \oplus \mathbb{Q}\mathbf{e}_2 \to \mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b}), \tag{9.3}$$

where the symbols $\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}$, $\mathbf{e}'_f\mathsf{X}^k\mathsf{Y}^{2n-k}$ and $\mathbf{e}''_f\mathsf{X}^k\mathsf{Y}^{2n-k}$ are defined in (5.17). In this section we give an explicit formula for this morphism.

Proposition 9.2.1. *The monodromy morphism induces a morphism of graded Lie algebras*

$$\mathrm{Lie}(\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}, \mathbf{e}'_f\mathsf{X}^k\mathsf{Y}^{2n-k}, \mathbf{e}''_f\mathsf{X}^k\mathsf{Y}^{2n-k}) \oplus \mathbb{Q}\mathbf{e}_2 \to \mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b}),$$

It kills the symbols $\mathbf{e}'_f, \mathbf{e}''_f$ corresponding to cusp forms. On the Eisenstein symbols $\mathbf{e}_{2n+2}$ it acts via

$$\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k} \mapsto \frac{2(2n-k)!}{[(2n)!]^2}\,\mathrm{ad}\,(\varepsilon_0^{\vee})^k\left(\varepsilon_{2n+2}^{\vee}\right)$$

Proof. The cuspidal symbols are contained in $\ker(\mu)$ for Hodge-theoretic reasons [23, Theorem 15.4]. A proof of the formula for an Eisenstein symbol with $k=0$ is given in [23, Theorem 15.7]. It remains to compute the action on a general Eisenstein symbol $\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k}$ for any $0 \le k \le 2n$.

The SL_2-equivariance of $\mu\colon \mathcal{U}_{1,\vec{1}}^{\mathrm{dR}} \to \mathrm{Aut}(\pi_1^{\mathrm{dR}}(\mathcal{E}_{\partial/\partial q}^{\times}, \partial/\partial w))$ implies that the induced map on Lie algebras is $\mathfrak{sl}_2$-equivariant. The two representations of $\mathfrak{sl}_2$ are identified via

$$\mathsf{X}\frac{\partial}{\partial\mathsf{Y}} \mapsto \mathrm{ad}(\varepsilon_0^{\vee}), \quad \mathsf{Y}\frac{\partial}{\partial\mathsf{X}} \mapsto -\mathrm{ad}(\varepsilon_0).$$

The $\mathfrak{sl}_2$-action may be used to write a general Eisenstein symbol in the form

$$\mathbf{e}_{2n+2}\mathsf{X}^k\mathsf{Y}^{2n-k} = \frac{(2n-k)!}{(2n)!}\left(\mathsf{X}\frac{\partial}{\partial\mathsf{Y}}\right)^k\left(\mathbf{e}_{2n+2}\mathsf{Y}^{2n}\right).$$

Applying μ and using the $\mathfrak{sl}_2$-equivariance gives the result. □

Remark 9.2.2. Proposition 9.2.1 implies that $\mu(\mathbf{e}_2) = 2\varepsilon_2$, where $\varepsilon_2 = \varepsilon_2^{\vee} = -\,\mathrm{ad}([\mathsf{a},\mathsf{b}])$.

9.3 $\mathfrak{u}^{\text{geom}}$ revisited

The Lie algebra $\mathfrak{u}^{\text{geom}}$ is the image of $\mathfrak{u}_{1.1}$ under μ. It follows from Proposition 9.2.1 that $\mathfrak{u}^{\text{geom}}$ may be identified with the completion of the Lie subalgebra of $\operatorname{Der}^\theta \operatorname{Lie}(\mathsf{a}, \mathsf{b})$ generated by $\operatorname{ad}(\varepsilon_0^\vee)^k(\varepsilon_{2n+2}^\vee)$ for all $n \geq 1$ and $0 \leq k \leq 2n$. This bigrading on $\operatorname{Der}^\theta \operatorname{Lie}(\mathsf{a}, \mathsf{b})$ is canonical [27, Theorem 23.1], and defines a canonical bigrading on $\mathfrak{u}^{\text{geom}}$; see Remark 5.2.10.

Proposition 9.1.2 implies that $\mathfrak{u}^{\text{geom}}$ carries a canonical $\mathfrak{sl}_2$-action via the adjoint action of $\varepsilon_0^\vee$ and ε_0, and defines its decomposition into irreducible representations. This $\mathfrak{sl}_2$-action defines the Lie algebra $\mathfrak{g}^{\text{geom}} \cong \mathfrak{u}^{\text{geom}} \rtimes \mathfrak{sl}_2$ that was introduced in Definition 8.2.3. It also defines a canonical bigrading on $\mathfrak{g}^{\text{geom}}$ extending the bigrading on $\mathfrak{u}^{\text{geom}}$.

9.3.1 Geometric interpretation

The fact that elements of $\mathfrak{u}^{\text{geom}}$ annihilate $\theta = [\mathsf{a}, \mathsf{b}]$ corresponds to the fact that elements of $\mathcal{U}_{1,1}^{\text{dR}}$ fix a small loop around the puncture on $\mathcal{E}_{\partial/\partial q}^\times$. This loop is homotopic to the commutator $\Theta = \alpha\beta\alpha^{-1}\beta^{-1}$, Under the comparison isomorphism

$$\operatorname{Lie} \pi_1^{\mathrm{B}}(\mathcal{E}_{\partial/\partial q}^\times, \partial/\partial w) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \operatorname{Lie} \pi_1^{\mathrm{dR}}(\mathcal{E}_{\partial/\partial q}^\times, \partial/\partial w) \otimes_{\mathbb{Q}} \mathbb{C}$$

we have $\log(\Theta) \mapsto 2\pi i\theta$.

Note that $\mathfrak{u}^{\text{geom}}$ does not contain the derivation $\varepsilon_2^\vee = \varepsilon_2 = -\operatorname{ad}([a, b])$. This element is central in $\operatorname{Der}^\theta \operatorname{Lie}(\mathsf{a}, \mathsf{b})$. It corresponds to the logarithm of an element of $\mathcal{U}_{1,\vec{1}}^{\text{dR}}$ that acts on $(\mathcal{E}_{\partial/\partial q}^\times, \partial/\partial w)$ by rotating the tangent vector $\partial/\partial w$. The element ε_2 is fixed by the action of $\mathfrak{sl}_2$, and the Lie algebra $\mathfrak{u}^{\text{geom}} \oplus \mathbb{Q}\varepsilon_2$ is the graded Lie algebra of the image of the unipotent radical of $\pi_1^{\text{rel,dR}}(\mathcal{M}_{1,\vec{1}}, \vec{\mathsf{v}})$ in $\operatorname{Aut}(\pi_1^{\text{dR}}(\mathcal{E}_{\partial/\partial q}^\times, \partial/\partial w))$, where the tangential basepoint $\vec{\mathsf{v}} := \partial/\partial q + \partial/\partial w$ was defined in §3.5.3.

9.4 Mixed Hodge structure on $\mathfrak{u}^{\text{geom}}$

The fundamental group $\pi_1^{\text{dR}}(\mathcal{E}_{\partial/\partial q}^\times, \partial/\partial w)$, and its automorphism group, are equipped with limiting mixed Hodge-Tate structures [7, §13.6]. Therefore $\mathfrak{u}^{\text{geom}}$ is equipped with a limiting MHS, and has two increasing weight filtrations – $M_\bullet\, \mathfrak{u}^{\text{geom}}$ and $W_\bullet\, \mathfrak{u}^{\text{geom}}$ – and a decreasing Hodge filtration $F^\bullet\, \mathfrak{u}^{\text{geom}}$, all of which can be simultaneously split [27, §23]. As described in [26, §15.2], these filtrations are induced from filtrations on $\mathbb{Q}\langle\langle \mathsf{a}, \mathsf{b} \rangle\rangle$ as follows.

Let $W_{-n}\mathbb{Q}\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle := I^n$, where $I := (\mathsf{a},\mathsf{b})$ is the maximal ideal of $\mathbb{Q}\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle$. Let $M_{-2m}\mathbb{Q}\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle := \{w \in \mathbb{Q}\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle : \deg_{\mathsf{a}}(w) \geq m\}$. Finally, let $F^{-p}\mathbb{Q}\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle := \{w \in \mathbb{Q}\langle\langle \mathsf{a},\mathsf{b}\rangle\rangle : \deg_{\mathsf{a}}(w) \leq p\}$. These define natural filtrations $W_\bullet$, $M_\bullet$, $F^\bullet$ on $\mathrm{Lie}(\mathsf{a},\mathsf{b})$ by restriction. They therefore induce natural filtrations on $\mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$, and consequently on $\mathfrak{u}^{\mathrm{geom}}$.

The M and W filtrations on $\mathfrak{u}^{\mathrm{geom}}$ may be simultaneously and canonically split [27, Theorem 23.1]. This splitting can be written down as follows: recall that the Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ is generated by $\mathrm{ad}(\varepsilon_0^\vee)^k(\varepsilon_{2n+2}^\vee)$ for all $n \geq 1$ and $0 \leq k \leq 2n$. The element $\mathrm{ad}(\varepsilon_0^\vee)^k(\varepsilon_{2n+2}^\vee)$ lies in $M_{2k-2-4n}\,\mathfrak{u}^{\mathrm{geom}} \cap W_{-2n-2}\,\mathfrak{u}^{\mathrm{geom}}$; in other words, $\mathrm{ad}(\varepsilon_0^\vee)^k(\varepsilon_{2n+2}^\vee)$ has M-weight $2k-2-4n$ and W-weight $-2n-2$.

One should note that the MHS on $\mathfrak{u}^{\mathrm{geom}}$ is the image of the MHS on $\mathfrak{u}_{1,1}$ under the monodromy morphism.

9.5 Motivic structure

The MHS on $\mathfrak{u}^{\mathrm{geom}}$ is induced by a deeper motivic structure. In fact $\mathfrak{u}^{\mathrm{geom}}$ is the $\mathcal{H}$-realisation of a pro-object in the category $\mathsf{MT}(\mathbb{Z})$ under the fully faithful functor $\omega^{\mathcal{H}} : \mathsf{MT}(\mathbb{Z}) \hookrightarrow \mathcal{H}$. This follows from the following general statement about Tannakian categories.

Lemma 9.5.1. *Let $(\mathsf{C}, \omega_{\mathsf{C}})$ and $(\mathsf{D}, \omega_{\mathsf{D}})$ be neutral Tannakian categories over $\mathbb{Q}$, and let $F\colon \mathsf{C} \to \mathsf{D}$ be a fully faithful additive tensor functor satisfying $\omega_{\mathsf{D}} \circ F = \omega_{\mathsf{C}}$. Let $M \in \mathsf{C}$. Then $F(\langle M\rangle_\otimes) \simeq \langle F(M)\rangle_\otimes$.*

Proof. Recall that the fiber functors ω_C and ω_D are exact and faithful. Since $\omega_C = \omega_D \circ F$, one may show that F is exact.

Let $V \in F(\langle M\rangle_\otimes)$ so that $V \cong F(W)$, where W is a subquotient of $M^{\otimes n}$ for some $n \in \mathbb{Z}$. Since F is an exact tensor functor, it follows that $F(W)$ is a subquotient of $F(M)^{\otimes n}$. Thus $V \in \langle F(M)\rangle_\otimes$.

Conversely, suppose that $V \in \langle F(M)\rangle_\otimes$, so that V is a subquotient of $F(W)$ for some $W \in \langle M\rangle_\otimes$. We first assume that V is a D-subobject of $F(W)$, and will show that $V \cong F(U)$ for some C-subobject $U \subseteq W$. The general case, where V is a quotient of a subobject of $F(W)$, then follows from this.

Therefore, assume that $V \subseteq F(W)$. By Tannakian duality there are affine group schemes G and H over $\mathbb{Q}$ such that $\mathsf{C} \simeq \mathsf{Rep}(G)$ and $\mathsf{D} \simeq \mathsf{Rep}(H)$. The objects V and $F(W)$ are equipped with H-actions that are compatible with the inclusion $V \hookrightarrow F(W)$.

The functor F is equivalent to a faithfully flat (surjective) morphism of group schemes $\varphi_F\colon H \to G$. Since F is fully faithful, the action of H on $F(W)$ factors through φ_F. Let $K = \ker(\varphi_F)$. Then K acts trivially on $F(W)$, and therefore acts trivially on any subobject of $F(W)$. In particular, it acts trivially on V. Therefore the H-action on V factors through φ_F, so V is a representation of G. But then V is the image under F of a C-subobject $U \subseteq W$.

Now assume that V is a general subquotient of $F(W)$. If V is a quotient of a subobject $V' \subseteq F(W)$ then the previous part of the proof shows that $V' \cong F(U')$ for some subobject $U' \subseteq W$. Hence we may assume that $V \cong F(W)/V_0$ for some subobject $V_0 \subseteq F(W)$. Once again, it follows that $V_0 \cong F(W_0)$ for a subobject $W_0 \subseteq W$. Then $V \cong F(W)/F(W_0)$. Since F is exact, the natural map $F(W)/F(W_0) \to F(W/W_0)$ is an isomorphism. Therefore $V \in F(\langle M\rangle_\otimes)$. $\square$

The $\mathcal{H}$-realisation functor $\omega^{\mathcal{H}}\colon \mathsf{MT}(\mathbb{Z}) \to \mathcal{H}$ is fully faithful and compatible with the respective de Rham fiber functors. The Lie algebra $\operatorname{Lie}\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ is a pro-object within its essential image [9, 27]. Recall from §5.4.1 that $\operatorname{Lie}\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ is canonically isomorphic to the completed free Lie algebra on its abelianisation

$$\operatorname{Lie}(\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w))^{\mathrm{ab}} \cong H^1(\mathcal{E}^{\times}_{\partial/\partial q})^{\vee} \cong \mathbb{Q}(1) \oplus \mathbb{Q}(0),$$

which is mixed Tate. Consequently,

$$\begin{aligned}\operatorname{Der}\operatorname{Lie}\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w) &\cong \operatorname{Hom}(H^1(\mathcal{E}^{\times}_{\partial/\partial q}), \operatorname{Lie}\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w))\\ &\cong \operatorname{Hom}(\mathbb{Q}(1) \oplus \mathbb{Q}(0), \operatorname{Lie}\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w))\end{aligned}$$

is also a pro-object in the essential image[1] of $\omega^{\mathcal{H}}$.

By definition, $\mathfrak{u}^{\mathrm{geom}}$ is isomorphic to a $\mathcal{H}$-subobject of $\operatorname{Der}\operatorname{Lie}\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ by a canonical isomorphism compatible with the $\mathcal{H}$-structure [27]. Applying Lemma 9.5.1 with $\mathsf{C} = \mathsf{MT}(\mathbb{Z})$, $\mathsf{D} = \mathcal{H}$, $F = \omega^{\mathcal{H}}$ and $F(M) = \operatorname{Der}\operatorname{Lie}\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ proves the following important fact:

Proposition 9.5.2. *The Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ is the $\mathcal{H}$-realisation of a pro-object in $\mathsf{MT}(\mathbb{Z})$.*

Proposition 9.5.2 is equivalent to the statement that the motivic Galois group $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ acts on $\mathfrak{u}^{\mathrm{geom}}$. In Theorem 13.1.1 we prove that this action is faithful.

[1] This follows because $\mathsf{MT}(\mathbb{Z})$ is a Tannakian category, and is therefore rigid [17]. In particular, for any two objects X, Y of a Tannakian category C, rigidity implies that the internal hom $\operatorname{Hom}(X, Y)$ is an object of C. The internal hom represents the presheaf $\operatorname{Hom}_{\mathsf{C}}((-) \otimes X, Y)$ on C.

9.6 The Pollack relations

The Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ is not free; for example, we have the following quadratic relation:

$$[\varepsilon_{10}^\vee, \varepsilon_4^\vee] - 3[\varepsilon_8^\vee, \varepsilon_6^\vee] = 0. \tag{9.4}$$

This relation is the simplest in an infinite family of relations in $\mathfrak{u}^{\mathrm{geom}}$ studied by Pollack in his honours' thesis [44], who showed they are related to cusp forms for $SL_2(\mathbb{Z})$. Equation (9.4) is associated to the first such cusp form, the Ramanujan cusp form $\Delta \in S_{12}(SL_2(\mathbb{Z}))$. More generally, each cusp form $f \in S_{2n}(SL_2(\mathbb{Z}))$ determines a relation in $\mathfrak{u}^{\mathrm{geom}}$ of lower central series depth 2 and W-weight $-2n-2$.

9.6.1 Relationship to motives

Let $\mathfrak{k} = \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})})$ be the Lie algebra of the unipotent radical of the Galois group of $\mathsf{MT}(\mathbb{Z})$ as described in §4.3.1. It is non-canonically isomorphic to the completed free Lie algebra $\mathrm{Lie}(\sigma_3, \sigma_5, \ldots,)^\wedge$. Recall from §5.4 that $\mathrm{Lie}\,\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$ is a pro-object of $\mathsf{MT}(\mathbb{Z})$ and is canonically isomorphic to $\mathrm{Lie}(\mathsf{a}, \mathsf{b})^\wedge$ by an isomorphism compatible with the motivic structure. Therefore the associated bigraded $\mathrm{Lie}(\mathsf{a}, \mathsf{b})$ is also a pro-object of $\mathsf{MT}(\mathbb{Z})$. This implies the existence of a Lie algebra homomorphism

$$\rho\colon \mathfrak{k} \to \mathrm{Der}\,\mathrm{Lie}(\mathsf{a}, \mathsf{b}).$$

Brown [9] gives a formula for ρ modulo lower-order terms in $W_\bullet\, \mathfrak{u}^{\mathrm{geom}}$; namely,

$$\rho(\sigma_{2n+1}) \equiv \varepsilon_{2n+2}^\vee \pmod{W_{-2n-3}\,\mathrm{Der}\,\mathrm{Lie}(\mathsf{a}, \mathsf{b})}.$$

Amazingly, Pollack's relation (9.4) is related to the Ihara-Takao congruence [29]

$$[\sigma_9, \sigma_3] - 3[\sigma_7, \sigma_5] \equiv 0 \quad (\text{mod depths} \geq 3).$$

Applying ρ to the Ihara-Takao relation produces Pollack's relation (9.4) modulo higher-depth terms. See [27, §29 and Theorem 29.6] for more information on the connection between the Ihara-Takao congruences, Pollack relations and universal mixed elliptic motives.

9.7 $\mathcal{U}^{\mathrm{geom}}$ revisited

In this section we revisit the group $\mathcal{U}^{\mathrm{geom}}$ defined in Definition 8.2.1 with the more concrete knowledge of its Lie algebra developed in the previous sections.

9.7.1 The affine ring of $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$

The affine ring $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ is a Hopf subalgebra of $\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}})$. Since μ is a morphism of MHS by [23, Proposition 15.1], $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ is furthermore a sub-ind-$\mathcal{H}$ object of $\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}})$. In fact, Proposition 9.5.2 implies that it is actually the $\mathcal{H}$-realisation of an object in $\mathsf{MT}(\mathbb{Z})$.

Recall that μ factors through the totally holomorphic quotient $\mathcal{U}_{1,1}^{\mathrm{dR}} \to \mathcal{U}_{1,1}^{\mathrm{dR,hol}}$ by [27, §22]. By Proposition 9.2.1, it also factors through the free Eisenstein quotient $\mathcal{U}_{1,1}^{\mathrm{dR,hol}} \to \mathcal{U}_{E}^{\mathrm{dR}}$. As discussed in §5.5.1.2, the affine ring $\mathcal{O}(\mathcal{U}_{E}^{\mathrm{dR}})$ is isomorphic to the shuffle algebra on the alphabet $Z_E := \{E_{2n+2}(k) : n \geq 1, 0 \leq k \leq 2n\}$, where the symbols $E_{2n+2}(k)$ are defined in §5.5.1.2. It is a Hopf subalgebra of $\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}})$, but does not have a natural $\mathcal{H}$-structure.

It follows that $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ is a Hopf subalgebra of $\mathcal{O}(\mathcal{U}_{E}^{\mathrm{dR}})$. We may represent elements of $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ by elements of the shuffle algebra $T^c(Z_E)$, and identify these with their images in $\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}})$. Note, however, that $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ is strictly contained within $\mathcal{O}(\mathcal{U}_{E}^{\mathrm{dR}})$; see §15.4.1 for a conjectural description of this subalgebra.

9.7.2 Decreasing filtrations on $\mathcal{U}^{\mathrm{geom}}$

The group $\mathcal{U}^{\mathrm{geom}}$ is equipped with a decreasing *length* filtration $L^{\bullet}\mathcal{U}^{\mathrm{geom}}$ defined by the lower central series. It can also be defined as the image of the length filtration $L^{\bullet}\mathcal{U}_{1,1}$ under μ. Equivalently, by making use of the canonical bigrading afforded by [27, §23], it can be defined as the decreasing filtration induced by the total degree, where $\mathrm{ad}(\varepsilon_0^{\vee})^k(\varepsilon_{2n+2}^{\vee})$ is assigned degree 1 for all $n \geq 1$ and all $0 \leq k \leq 2n$.

Remark 9.7.1. The lower central series filtration on $\mathcal{U}^{\mathrm{geom}}$ is a source of possible confusion. Recall that its Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ is generated by the elements $\mathrm{ad}(\varepsilon_0^{\vee})^k(\varepsilon_{2n+2}^{\vee})$ for all $n \geq 1$ and all $0 \leq k \leq 2n$. The derivation $\varepsilon_0^{\vee} = \mathsf{b}\partial/\partial\mathsf{a}$, which is contained within $\mathrm{Der}^{\theta}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$, is *not* itself contained within $\mathfrak{u}^{\mathrm{geom}}$. Therefore, the element $\mathrm{ad}(\varepsilon_0^{\vee})^k(\varepsilon_{2n+2}^{\vee})$ lies in $L^1\,\mathfrak{u}^{\mathrm{geom}}$, but is contained in $\mathrm{LCS}^{k+1}\,\mathrm{Der}^{\theta}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$.

We refer to the lower central series filtration as the length filtration to emphasise the following fact. It defines a dual increasing filtration $L_{\bullet}\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ on the affine ring by letting $L_r\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ be the functions on $\mathcal{U}^{\mathrm{geom}}$ vanishing on $L^r\,\mathcal{U}^{\mathrm{geom}}$. By the isomorphism $\mathcal{O}(\mathcal{U}_{E}^{\mathrm{dR}}) \cong T^c(Z_E)$ discussed in §9.7.1, $L_r\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ is spanned by certain words $[E_{2n_1+2}(k_1)|\cdots|E_{2n_s+2}(k_s)]$ with $s \leq r$ i.e. certain iterated Eisenstein integrals of length at most r. By the discussion above, however, not all such linear combinations of iterated Eisenstein integrals will be contained in $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$. One necessary condition is that they evaluate to periods of $\mathsf{MT}(\mathbb{Z})$ under the homomorphism $\mathcal{C}_S^{\mathfrak{m}}\colon \mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}}) \to \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$. See §15.4.1.

There is another filtration on $\mathcal{U}^{\mathrm{geom}}$ (and on its Lie algebra $\mathfrak{u}^{\mathrm{geom}}$) that is of significant interest. Recall that we defined a filtration $B^{\bullet}R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$ in Definition 7.2.4. It is the decreasing filtration on the b-degree.

There is an inclusion $\mathcal{U}^{\mathrm{geom}} \hookrightarrow \mathrm{End}(R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle)$ induced by the inclusion $\mathfrak{u}^{\mathrm{geom}} \hookrightarrow \mathrm{Der}\,\mathrm{Lie}(\mathsf{a}, \mathsf{b})$. By the discussion in §7.3.2 there is a natural induced filtration $B^{\bullet}\,\mathcal{U}^{\mathrm{geom}}$ and, similarly, one obtains a filtration $B^{\bullet}\,\mathfrak{u}^{\mathrm{geom}}$ on its Lie algebra. Elements of $B^{r}\,\mathcal{U}^{\mathrm{geom}}$ act on $R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$ by increasing the b-degree by at least r.

Both $L^{\bullet}\,\mathfrak{u}^{\mathrm{geom}}$ and $B^{\bullet}\,\mathfrak{u}^{\mathrm{geom}}$ extend naturally to filtrations on $\mu(\mathfrak{u}_{1,\vec{1}}) = \mathfrak{u}^{\mathrm{geom}} \oplus \mathbb{Q}\varepsilon_2$. The central derivation $\varepsilon = -\,\mathrm{ad}([\mathsf{a}, \mathsf{b}])$ is contained in $L^1 \cap B^1$.

The following proposition describes how these two filtrations interact.

Proposition 9.7.2. *For each $r \geq 1$ we have $L^{r}\left(\mathfrak{u}^{\mathrm{geom}} \oplus \mathbb{Q}\varepsilon_2\right) \subseteq B^{r}\left(\mathfrak{u}^{\mathrm{geom}} \oplus \mathbb{Q}\varepsilon_2\right)$.*

Proof. For each $n \geq 1$ and $0 \leq k \leq 2n$ we have $\mathrm{ad}(\varepsilon_0^{\vee})^k(\varepsilon_{2n+2}^{\vee}) \in L^1\,\mathfrak{u}^{\mathrm{geom}}$. By the definition of $\varepsilon_{2n+2}^{\vee}$ given in Definition 9.1.1 one verifies that $\mathrm{ad}(\varepsilon_0^{\vee})^k(\varepsilon_{2n+2}^{\vee})$ raises the b-degree by $k+1$ and is therefore contained within $B^{k+1}\,\mathfrak{u}^{\mathrm{geom}}$, and hence within $B^1\,\mathfrak{u}^{\mathrm{geom}}$. It follows that $L^1\left(\mathfrak{u}^{\mathrm{geom}} \oplus \mathbb{Q}\varepsilon_2\right) \subseteq B^1\left(\mathfrak{u}^{\mathrm{geom}} \oplus \mathbb{Q}\varepsilon_2\right)$ because $\mathfrak{u}^{\mathrm{geom}}$ is generated by $\mathrm{ad}(\varepsilon_0^{\vee})^k(\varepsilon_{2n+2}^{\vee})$ as $n \geq 1$ and $0 \leq k \leq 2n$, and ε_2 is contained within $L^1 \cap B^1$. The result follows by induction on r, noting that the Lie bracket of derivations is given by $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$. □

Chapter 10

The series Ψ and its coefficients

As explained in Chapter 6, we are mainly interested in the iterated integrals along the element $S \in SL_2(\mathbb{Z})$, considered as an element of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q)$. A particular subset of these are iterated Eisenstein integrals. However, the monodromy action of §8.1 is an action of $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \vec{\mathrm{v}}) \cong B_3$ rather than of its quotient $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q) \cong SL_2(\mathbb{Z})$ by the homomorphism $f \colon B_3 \to SL_2(\mathbb{Z})$ of Proposition 8.1.4. Therefore we must also understand the iterated integrals on $\mathcal{M}_{1,1}(\mathbb{C})$ along the element of its fundamental group corresponding to $\tilde{S} = (t_A t_B t_A)^{-1} \in B_3$, which satisfies $f(\tilde{S}) = S$.

To work with these iterated integrals formally, we must study the images of the elements S and $\tilde{S}$ under the natural maps (5.8) sending an element γ to its "generating series of motivic iterated integrals". This leads us to the commutative diagram

$$\begin{array}{ccc} B_3 & \longrightarrow & (\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}} \rtimes SL_2^{\mathrm{dR}})(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ SL_2(\mathbb{Z}) & \longrightarrow & \left(\mathcal{U}^{\mathrm{dR}}_{1,1} \rtimes SL_2^{\mathrm{dR}}\right)(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}) \end{array} \tag{10.1}$$

in which the horizontal maps are the compositions of the universal comparisons (5.8) followed by the decompositions $\mathcal{G}^{\mathrm{dR}}_{1,\bullet} \xrightarrow{\sim} \mathcal{U}^{\mathrm{dR}}_{1,\bullet} \rtimes SL_2^{\mathrm{dR}}$ given in (5.20), where $\bullet \in \left\{1, \vec{1}\right\}$. The right hand vertical map is induced from the projection $\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}} \cong \mathcal{U}^{\mathrm{dR}}_{1,1} \times \mathbb{G}_a \to \mathcal{U}^{\mathrm{dR}}_{1,1}$ given in (5.16). Concretely it is just the map $\mathbf{e}_2 \mapsto 0$.

We studied the image of $S \in SL_2(\mathbb{Z})$ under the bottom map in Diagram (10.1) in §5.5.3; this led to the definition of the unipotent part $\mathcal{C}^{\mathfrak{m}}_S \in \mathcal{U}^{\mathrm{dR}}_{1,1}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}})$. In this chapter we study the image of $\tilde{S}$ under the top horizontal map in Diagram (10.1).

10.1 The series Ψ

Recall from (5.16) that there is a canonical decomposition $\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}} \cong \mathcal{U}^{\mathrm{dR}}_{1,1} \times \mathbb{G}_a(1)$, where $\mathrm{Lie}(\mathbb{G}_a)$ is spanned by an element $\mathbf{e}_2$. Dually, there is a decomposition

$$\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}}) \cong \mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1}) \otimes_{\mathbb{Q}} \mathcal{O}(\mathbb{G}_a)(-1).$$

The coordinate ring $\mathcal{O}(\mathbb{G}_a)$ is generated by an element $E_2(0)$ dual to $\mathbf{e}_2$. Define a homomorphism $r^{\mathfrak{m}} \colon B_3 \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ by the composition

$$B_3 \cong \pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \vec{\mathsf{v}}) \to \mathcal{G}^{\mathrm{dR}}_{1,\vec{1}}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}) \to \mathbb{G}_a(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}) = \mathrm{Hom}(\mathcal{O}(\mathbb{G}_a), \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}) \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$$

where the first arrow is (5.8), the second is the canonical projection afforded by (5.16), and the third is evaluation at the element $E_2(0) \in \mathcal{O}(\mathbb{G}_a)$. For every $\sigma \in B_3$ the motivic period $r^{\mathfrak{m}}(\sigma)$ is a rational multiple of $\mathbb{L}$ because $E_2(0)$ spans $\mathbb{Q}(-1)$. It is a motivic lift of a homomorphism r constructed by Matthes [38, 39] that we shall revisit in §14.5.

The top horizontal homomorphism in Diagram 10.1 is then given by the formula

$$\sigma \mapsto \left(\exp(r^{\mathfrak{m}}(\sigma)\mathbf{e}_2)\mathcal{C}^{\mathfrak{m}}_{f(\sigma)}, f(\sigma)^{\mathfrak{m}}_0\right).$$

Here $f \colon B_3 \to SL_2(\mathbb{Z})$ is the homomorphism defined in Proposition 8.1.4, and the unipotent and reductive parts of the image of an element $\gamma \in SL_2(\mathbb{Z})$ under the natural map $SL_2(\mathbb{Z}) \to \mathcal{G}^{\mathrm{dR}}_{1,1}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}})$, as defined in Definition 5.2.6, are denoted $\mathcal{C}^{\mathfrak{m}}_{\gamma}$ and $\gamma^{\mathfrak{m}}_0$ respectively. The assignment $\sigma \mapsto \exp(r^{\mathfrak{m}}(\sigma)\mathbf{e}_2)\mathcal{C}^{\mathfrak{m}}_{f(\sigma)}$ defines an element of $Z^1(B_3, \mathcal{U}^{\mathrm{dR}}_{1,\vec{1}}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}))$ because $\mathbf{e}_2 \in \mathfrak{u}_{1,\vec{1}}$ is central.

Definition 10.1.1 (Series Ψ)**.** Let $\tilde{S} = (t_A t_B t_A)^{-1} \in B_3$, so that $f(\tilde{S}) = S$. Composition along the top row of Diagram (10.1) sends $\tilde{S}$ to

$$\tilde{S}^{\mathfrak{m}} = (\Psi, S^{\mathfrak{m}}_0),$$

where $\Psi := \exp(r^{\mathfrak{m}}(\tilde{S})\mathbf{e}_2)\mathcal{C}^{\mathfrak{m}}_{S}$. The element Ψ is the motivic *modular inverter.*

Definition 10.1.2. Let $\eta := \int^{\mathfrak{m}}_{\tilde{S}} E_2(0)$. Then η is precisely

$$r^{\mathfrak{m}}(\tilde{S}) = \Psi[\mathbf{e}_2] := \text{coefficient of the word } \mathbf{e}_2 \text{ in the series } \Psi.$$

We compute $\eta = \mathbb{L}/8$ in Corollary 14.4.2.

Remark 10.1.3 (Integrals along S and $\tilde{S}$). The algebra $M^{\mathfrak{m}}$ is generated by $\mathrm{Co}(\mathcal{C}_S^{\mathfrak{m}})$ and $\mathrm{Co}(\mathcal{C}_T^{\mathfrak{m}}) = \mathbb{Q}[\mathbb{L}]$. Since $\Psi = \exp(\eta \mathbf{e}_2)\mathcal{C}_S^{\mathfrak{m}}$, where $\eta \in \mathbb{Q}\mathbb{L}$ and $\mathcal{C}_S^{\mathfrak{m}}$ is invertible, it follows that $M^{\mathfrak{m}} = \mathrm{Co}(\Psi)$. Elements of $\mathrm{Co}(\Psi)$ may be decomposed into elements of $\mathrm{Co}(\mathcal{C}_S^{\mathfrak{m}})$ multiplied by powers of η as follows.

The projection $\mathcal{U}_{1,\vec{1}}^{\mathrm{dR}} \cong \mathcal{U}_{1,1}^{\mathrm{dR}} \times \mathbb{G}_a \to \mathcal{U}_{1,1}^{\mathrm{dR}}$ sending $\mathbf{e}_2 \mapsto 0$ is dual to an inclusion

$$j : \mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}}) \hookrightarrow \mathcal{O}(\mathcal{U}_{1,\vec{1}}^{\mathrm{dR}}) \cong \mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}}) \otimes \mathbb{Q}[E_2(0)]$$

satisfying $\Psi \circ j = \mathcal{C}_S^{\mathfrak{m}} \in \mathrm{Hom}(\mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}}), \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}})$. This implies that $\int_{\tilde{S}}^{\mathfrak{m}} E_{2n+2}(b) = \int_S^{\mathfrak{m}} E_{2n+2}(b)$ whenever $n > 0$. The integral of any element of $\mathcal{O}(\mathcal{U}_{1,\vec{1}}^{\mathrm{dR}})$ can therefore be decomposed as an integral along S multiplied by a power of $\eta = \int_{\tilde{S}}^{\mathfrak{m}} E_2(0)$.

10.2 The coefficients of $\mu(\Psi)$

The discussion in Remark 8.2.2 implies that that $\mu(\Psi)$ is a canonical element of $\mathcal{U}^{\mathrm{geom}}(\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}})$. The coefficient space $\mathrm{Co}(\mu(\Psi))$ is of central interest.

Recall that the monodromy morphism $\mu\colon \mathcal{U}_{1,\vec{1}}^{\mathrm{dR}} \to \mathrm{Aut}(\pi_1^{\mathrm{dR}}(\mathcal{E}_{\partial/\partial q}^{\times}, \partial/\partial w))$ factors through the totally holomorphic quotient $\mathcal{U}_{1,1}^{\mathrm{dR,hol}}$, whose points are grouplike power series in the symbols 5.17. Proposition 9.2.1 implies that μ vanishes on words involving a cuspidal symbol $\mathbf{e}_f'$ or $\mathbf{e}_f''$. This means that the coefficients of the series $\mu(\Psi)$ (and therefore $\mu(\mathcal{C}_S^{\mathfrak{m}})$) are *a priori* totally holomorphic motivic iterated Eisenstein integrals.

However, the image of the monodromy morphism $\mu\colon \mathcal{U}_{1,\vec{1}}^{\mathrm{dR}} \to \mathrm{Aut}(\pi_1^{\mathrm{dR}}(\mathcal{E}_{\partial/\partial q}^{\times}, \partial/\partial w))$ is a subgroup of $\mathrm{Aut}(\pi_1^{\mathrm{dR}}(\mathcal{E}_{\partial/\partial q}^{\times}, \partial/\partial w))$, which is a pro-object of $\mathsf{MT}(\mathbb{Z})$ [9, 27]. Therefore $\mathrm{Co}(\mu(\Psi))$ is also a $\mathbb{Q}$-subalgebra of $\mathcal{P}_{\mathsf{MT}(\mathbb{Z})}^{\mathfrak{m}} \cong \mathcal{Z}^{\mathfrak{m}}[\mathbb{L}^{\pm}]$. In other words, the coefficient space of $\mu(\Psi)$ consists of linear combinations of motivic iterated Eisenstein integrals that are elements of $\mathcal{P}_{\mathsf{MT}(\mathbb{Z})}^{\mathfrak{m}} \cong \mathcal{Z}^{\mathfrak{m}}[\mathbb{L}^{\pm}]$. It is somewhat remarkable that the monodromy morphism μ cuts out precisely the space of motivic iterated integrals that we are interested in!

Our main goal is to show that $\mathrm{Co}(\mu(\Psi))$ is as large as possible within $\mathcal{P}_{\mathsf{MT}(\mathbb{Z})}^{\mathfrak{m}}$ i.e. that its image contains all motivic MZVs. We describe the strategy in more detail at the start of §12.

Chapter 11

The Hain morphism and the series $\alpha^{\mathfrak{m}}$ and $\beta^{\mathfrak{m}}$

In this chapter we study how the fundamental groups of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ and $\mathcal{E}^{\times}_{\partial/\partial q}$ are related. The main link comes in the form of a morphism $\phi\colon \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0, 1, \infty\}, \vec{1}_1) \to \pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ called the *Hain morphism*. This establishes a link between multiple zeta values and the periods of the Tate curve.

11.1 The Hain morphism

The Hain morphism [26, §12.2, §16-18], [9, §3.3] is a morphism of de Rham fundamental groups $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0, 1, \infty\}, \vec{1}_1) \to \pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$. It is obtained by pulling back the universal elliptic Knizhnik-Zamolodchikov-Bernard (KZB) connection [26, 14] to $\mathbb{P}^1(\mathbb{C}) \backslash \{0, 1, \infty\}$. One obtains a version of the Knizhnik-Zamolodchikov (KZ) connection [33] with modified residues at the poles $\{0, 1, \infty\}$. The values of these residues define the Hain morphism, as explained below.

Definition 11.1.1 (Hain morphism)**.** The Hain morphism is a morphism of affine group schemes $\phi\colon \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0, 1, \infty\}, \vec{1}_1) \to \pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ over $\mathbb{Q}$. It is equivalent to the continuous Lie algebra homomorphism $\phi\colon \operatorname{Lie}(\mathsf{x}_0, \mathsf{x}_1)^\wedge \to \operatorname{Lie}(\mathsf{a}, \mathsf{b})^\wedge$ given by

$$\begin{aligned} \mathsf{x}_0 &\mapsto \frac{\operatorname{ad}(\mathsf{b})}{e^{\operatorname{ad}(\mathsf{b})} - 1}(\mathsf{a}) = \sum_{k \geq 0} \frac{B_k}{k!} \operatorname{ad}(\mathsf{b})^k(\mathsf{a}) \\ \mathsf{x}_1 &\mapsto -[\mathsf{a}, \mathsf{b}]. \end{aligned}$$

Remark 11.1.2. A remarkable consequence of the motivic theory is that although there is no algebraic morphism $\mathbb{P}^1 \backslash \{0, 1, \infty\} \to \mathcal{E}^{\times}_{\partial/\partial q}$ defined over $\mathbb{Q}$, there is an inclusion of topological spaces $\mathbb{P}^1(\mathbb{C}) \backslash \{0, 1, \infty\} \hookrightarrow \mathcal{E}^{\times}_{\partial/\partial q}(\mathbb{C})$, and this still induces a morphism of affine group schemes over $\mathbb{Q}$ at the level of Betti/de Rham fundamental groups. The Hain morphism may be defined as follows: let ∇ be the connection on the bundle over $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ obtained as the pullback of the universal KZB connection to $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ and let $p \in \{0,1,\infty\}$. Then $\phi(\mathsf{x}_p) = \mathrm{res}_p(\nabla)$.

The following basic facts about ϕ are stated in [27, §27].

Proposition 11.1.3. *The Hain morphism is injective and* $\mathfrak{k}$*-equivariant, where* $\mathfrak{k} = \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}) \cong \mathrm{Lie}(\sigma_3, \sigma_5, \dots)^\wedge$.

11.1.1 Dictionary between filtrations

The Hain morphism relates the weight and depth filtrations on $R\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle$ to the filtrations $A^\bullet$ and $B^\bullet$ on $R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$. This is made precise in Lemma 11.1.4, which generalises [9, Lemma 3.3]. Lemma 11.1.4 is valuable because it allows us to detect motivic MZVs of a certain weight or depth in the coefficients of elements of $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$ by making use of the filtrations on $\pi_1^{\mathrm{dR}}(\mathcal{E}^\times_{\partial/\partial q}, \partial/\partial w)$ defined in §7.2.2. This is covered in Lemmas 11.4.1 and 11.4.2.

Recall that a morphism of filtered algebras $f\colon F^\bullet\mathcal{A} \to G^\bullet\mathcal{B}$ is said to be *strict* if $f(F^r\mathcal{A}) = f(\mathcal{A}) \cap G^r\mathcal{B}$.

Lemma 11.1.4. *The Hain morphism induces* strict *morphisms of filtered* R*-algebras*

$$\phi\colon \mathcal{W}^\bullet R\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle \to A^\bullet R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$$
$$\phi\colon \mathcal{D}^\bullet R\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle \to B^\bullet R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle.$$

Proof. Let $w \in R\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle$. By linearity we may assume that w is a word in the letters $\mathsf{x}_0, \mathsf{x}_1$, which may be written in the general form

$$w = \mathsf{x}_0^{k_1}\mathsf{x}_1^{l_1}\dots\mathsf{x}_0^{k_n}\mathsf{x}_1^{l_n}, \quad \text{where } k_1, l_n \geq 0 \text{ and } k_2, \dots, k_n, l_1, \dots, l_{n-1} \geq 1. \tag{11.1}$$

Definition 11.1.1 implies that $\phi(\mathsf{x}_0) = \mathsf{a} + d$, where $d \in \mathcal{D}^1\mathbb{L}(\mathsf{a}, \mathsf{b})^\wedge$, and $\phi(\mathsf{x}_1) = -[\mathsf{a}, \mathsf{b}]$. Applying ϕ to (11.1) gives

$$\phi(w) = (-1)^l(\mathsf{a}+d)^{k_1}[\mathsf{a},\mathsf{b}]^{l_1}\cdots(\mathsf{a}+d)^{k_n}[\mathsf{a},\mathsf{b}]^{l_n} \tag{11.2}$$

where $l := l_1 + \cdots + l_n$. We have $\mathsf{a} + d \in A^1 \cap B^0$ and $[\mathsf{a}, \mathsf{b}] \in \mathcal{D}^1 \subseteq A^1 \cap B^1$.

The Lie subalgebra of $\mathrm{Lie}(\mathsf{a}, \mathsf{b})$ generated by a and $[\mathsf{a}, \mathsf{b}]$ is free. Hence the map $R\langle\langle \mathsf{a}, [\mathsf{a}, \mathsf{b}]\rangle\rangle \to R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$ is injective. We can therefore compare (11.1) and (11.2) to obtain

$$w \in \mathcal{W}^r R\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle \iff k_1 + \cdots + k_n + l_1 + \cdots + l_n \geq r \iff \phi(w) \in A^r R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle.$$

Similarly, we have

$$w \in \mathcal{D}^r R\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle \iff l_1 + \cdots + l_n \geq r \iff \phi(w) \in B^r R\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle.$$

□

Remark 11.1.5. For a more conceptual proof that ϕ is strict with respect to the depth see [27, Proposition 28.2].

11.2 The series $\alpha^{\mathfrak{m}}$ and $\beta^{\mathfrak{m}}$

Recall that $\pi_1^{\mathrm{top}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ is a free group on the generators α and β defined in §5.1.1.2. Under the natural map $\pi_1^{\mathrm{top}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w) \to \pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}})$ these elements are sent to two power series

$$\alpha^{\mathfrak{m}}, \beta^{\mathfrak{m}} \in \mathcal{G}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle)$$

which were mentioned in §5.4.2. Using the Hain morphism, it is possible to explicitly write down these series in terms of the exponential series and associators of §5.3.2.1 and §5.3.2.2. Conceptually, this follows because $\mathcal{E}^{\times}_{\partial/\partial q}(\mathbb{C})$ can be constructed topologically from $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ very simply [26, §16]. This has the effect of equipping $(\mathcal{O}(\pi_1^{\mathrm{B}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)), \mathcal{O}(\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)), \mathrm{comp})$ with the structure of an ind-object in $\mathsf{MT}(\mathbb{Z})$ [9, 27]. The following lemma gives equations for $\alpha^{\mathfrak{m}}$ and $\beta^{\mathfrak{m}}$; the reader may wish to refer to Figures 11.1 and 11.2 for diagrams to aid the proof.

Lemma 11.2.1. *The series $\alpha^{\mathfrak{m}}$ and $\beta^{\mathfrak{m}}$ are given explicitly by*

$$\begin{aligned}\alpha^{\mathfrak{m}} &= \phi(\Phi^{\mathfrak{m}}_{1\infty})e^{-\mathbb{L}\phi(\mathsf{x}_\infty)}\phi(\Phi^{\mathfrak{m}}_{\infty 1}),\\ \beta^{\mathfrak{m}} &= e^{-\frac{\mathbb{L}}{2}\phi(\mathsf{x}_1)}\phi(\Phi^{\mathfrak{m}}_{10})e^{-\mathsf{b}}\phi(\Phi^{\mathfrak{m}}_{\infty 1}).\end{aligned}$$

Proof. These formulae are obtained from the topological model for $\mathcal{E}^{\times}_{\partial/\partial q}(\mathbb{C})$ in terms of $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ given in [26, §16], described briefly in §5.1.1.2. (Note that this is a topological construction, and not algebraic.) This induces a homomorphism $\phi_0\colon \pi_1^{\mathrm{top}}(\mathbb{P}^1\setminus\{0,1,\infty\}, \vec{1}_1) \to \pi_1^{\mathrm{top}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ of topological fundamental groups, and a commutative diagram

$$\begin{tikzcd}
\pi_1^{\mathrm{top}}(\mathbb{P}^1\setminus\{0,1,\infty\}, \vec{1}_1) \arrow[r, "\phi_0"] \arrow[d] & \pi_1^{\mathrm{top}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w) \arrow[d] \\
\pi_1^{\mathrm{dR}}(\mathbb{P}^1\setminus\{0,1,\infty\}, \vec{1}_1)(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}) \arrow[r, "\phi"'] & \pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})
\end{tikzcd}$$

where the vertical arrows are the natural morphisms (5.8); they land in the $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$-points of the respective fundamental groups because the affine rings of each are ind-objects in $\mathsf{MT}(\mathbb{Z})$.

Let

$$\gamma := \mathrm{dch}_{1\infty} \cdot \sigma_{\infty}^{-1} \cdot \mathrm{dch}_{\infty 1} \in \pi_1^{\mathrm{top}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1), \tag{11.3}$$

where $\sigma_p \in \pi_1^{\mathrm{top}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_p)$ denotes the homotopy class of a small counterclockwise loop around the punctured point $p \in \{0,1,\infty\}$. We have $\phi_0(\gamma) = \alpha$, [26, §16]. Under the map $\pi_1^{\mathrm{top}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1) \to \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$, we have

$$\gamma \mapsto \gamma^{\mathfrak{m}} := \Phi^{\mathfrak{m}}_{1\infty} e^{-\mathbb{L}\mathsf{x}_\infty} \Phi^{\mathfrak{m}}_{\infty 1}.$$

It follows that $\alpha^{\mathfrak{m}} = \phi(\gamma^{\mathfrak{m}})$.

The path β is slightly different, as it is not in the image of ϕ_0. Instead, it is constructed in two stages. First we take the paths $\delta_1 := (\gamma_1^{-,+})^{-1} \cdot \mathrm{dch}_{10} \in \pi_1^{\mathrm{top}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1, -\vec{1}_0)$ and $\delta_2 := \mathrm{dch}_{\infty 1} \in \pi_1^{\mathrm{top}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_\infty, -\vec{1}_1)$, where $\gamma_p^{\pm,\mp}$ represents the homotopy class of a counterclockwise semicircle from $\pm\vec{1}_p$ to $\mp\vec{1}_p$ on $\mathbb{P}^1(\mathbb{C}) \backslash \{0,1,\infty\}$. Their images in the $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$-points of the respective de Rham path torsors are $\delta_1^{\mathfrak{m}} = \exp(-\mathbb{L}\mathsf{x}_1/2)\Phi^{\mathfrak{m}}_{10}$ and $\delta_2^{\mathfrak{m}} = \Phi^{\mathfrak{m}}_{\infty 1}$.

We then identify the boundary circles around the punctures at 0 and ∞ to obtain a space homotopic to $\mathcal{E}^{\times}_{\partial/\partial q}(\mathbb{C})$ (this identification is done without twisting; see [26, §18.1]). The element β is the result of first traversing δ_1, then identifying the boundary circles, and then traversing δ_2. The identification contributes a factor of $e^{-\mathsf{b}}$ [26, §18.1]. This gives $\beta^{\mathfrak{m}} = \delta_1^{\mathfrak{m}} e^{-\mathsf{b}} \delta_2^{\mathfrak{m}}$. $\square$

The formulae in Lemma 11.2.1 can be seen intuitively by comparing Figure 11.1 to Figure 11.2. Identifying the inner and outer boundary circles as described in Figure 5.1 has the effect of applying the Hain morphism ϕ to elements in the fundamental group of $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_1)(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$. Composition in the fundamental groupoid corresponds to concatenation of power series.

Remark 11.2.2 (Hexagon equation). The element γ is homotopic to the path

$$\gamma_1^{+,-} \cdot \mathrm{dch}_{10} \cdot \sigma_0 \cdot \mathrm{dch}_{01} \cdot \gamma_1^{-,+}.$$

For this reason, or equivalently by directly using the hexagon equation for $\Phi^{\mathfrak{m}}_{ij}$ [18], the element $\alpha^{\mathfrak{m}}$ may also be written

$$\alpha^{\mathfrak{m}} = e^{\frac{\mathbb{L}}{2}\phi(\mathsf{x}_1)} \phi(\Phi^{\mathfrak{m}}_{10}) e^{\mathbb{L}\phi(\mathsf{x}_0)} \phi(\Phi^{\mathfrak{m}}_{01}) e^{\frac{\mathbb{L}}{2}\phi(\mathsf{x}_1)}.$$

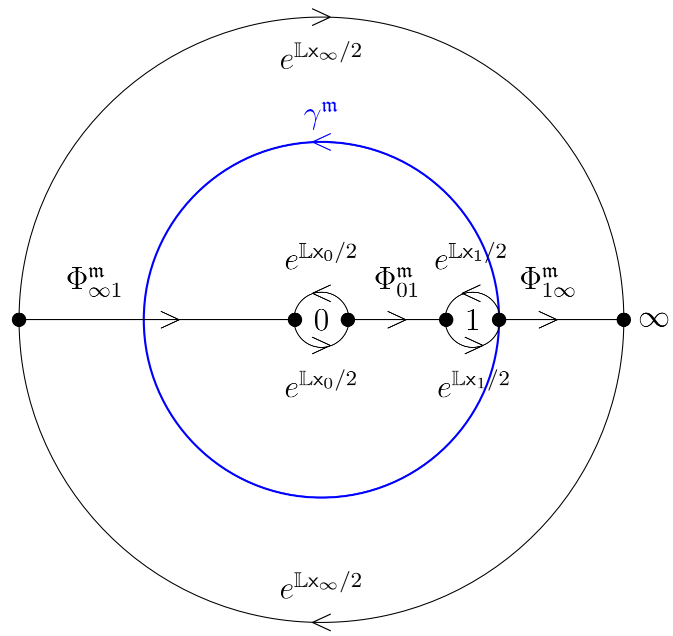


Figure 11.1: The de Rham fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ based at the tangent vectors $\pm\vec{1}_p$ for $p \in \{0, 1, \infty\}$. Homotopies between topological paths induce relations between exponential series and Drinfeld associators. The element $\gamma^{\mathfrak{m}}$ is indicated in blue.

Lemma 11.2.1 implies that the coefficient spaces of $\alpha^{\mathfrak{m}}$ and $\beta^{\mathfrak{m}}$ are contained within the algebra $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$.[1] This follows because the MHS on $\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ is mixed Tate [7, §13.6].

Formulae similar to those in Lemma 11.2.1 appear in various forms in the literature. They are the initial values of solutions to the universal elliptic KZB connection at the cusp [14, Proposition 4.9, Theorem 4.11], [40, Theorem 4.3], [37, §3.1]. In this case their images under the period map are sometimes denoted A_∞ and B_∞. The reader is warned that conventions differ regarding path multiplication and choice of tangential basepoints on the infinitesimal punctured Tate curve.

Similar formulae also appear in the profinite context [41, Theorem 3.4] when describing the Galois representation associated to the fundamental group of $\mathcal{E}^{\times}_{\partial/\partial q}$.

11.3 Strategy

In §8.1 we showed that the topological fundamental group $\pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \vec{\mathsf{v}}) \cong B_3$ acts on $\pi_1^{\mathrm{top}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w) = \langle \alpha, \beta \rangle$. In §8.1.3 we provided a simple formula for the action of the element $\tilde{S} \in B_3$ lifting $S \in SL_2(\mathbb{Z})$; namely, $\tilde{S}(\beta) = \alpha^{-1}$.

[1]In fact they are even contained within the subalgebra $\mathcal{P}^{\mathfrak{m},+}_{\mathsf{MT}(\mathbb{Z})} \subseteq \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ of effective periods.

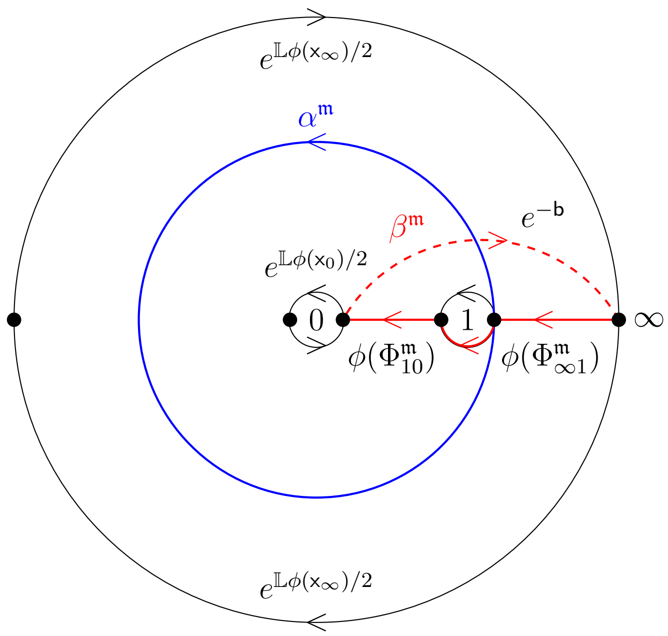


Figure 11.2: The de Rham fundamental groupoid of $\mathcal{E}^{\times}_{\partial/\partial q}$ obtained by applying the Hain morphism to Figure 11.1. The series $\alpha^{\mathfrak{m}}$ and $\beta^{\mathfrak{m}}$ are indicated, and may be written in terms of $\exp(\mathbb{L}\phi(\mathsf{x}_p))$ and $\phi(\Phi^{\mathfrak{m}}_{ij})$. Only some elements are labelled.

We also showed that the monodromy action lifts to an action of the relative fundamental group $\mathcal{G}^{\mathrm{dR}}_{1,\vec{1}}$ on $\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ in §8.2. Because relative completion is functorial and respects composition of paths, there is a corresponding equation relating the images of these elements in the $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$-points of the respective relative de Rham fundamental groups, giving the formula $\tilde{S}^{\mathfrak{m}}(\beta^{\mathfrak{m}}) = (\alpha^{\mathfrak{m}})^{-1}$.

Now write $\tilde{S}^{\mathfrak{m}} = (\Psi, S_0^{\mathfrak{m}})$ using the splitting $\mathcal{G}^{\mathrm{dR}}_{1,\vec{1}} \cong \mathcal{U}_{1,\vec{1}} \rtimes SL_2^{\mathrm{dR}}$, where Ψ was defined in Definition 10.1.1. The semidirect product acts through the monodromy morphism μ as described in §8.2, giving the formula

$$\mu(\Psi)(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})) = (\alpha^{\mathfrak{m}})^{-1}. \tag{11.4}$$

This is an equation holding between noncommutative formal power series. By Proposition 9.2.1, $\mu(\Psi)$ is a power series in elements of $\mathfrak{u}^{\mathrm{geom}}$ whose coefficients are motivic iterated Eisenstein integrals. The elements $S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})$ and $(\alpha^{\mathfrak{m}})^{-1}$ are power series in elements of $\mathrm{Lie}(\mathsf{a}, \mathsf{b})$, on which $\mathfrak{u}^{\mathrm{geom}}$ acts.

Recall the notion of the coefficient space $\mathrm{Co}(s)$ of a series s given in Defintion 7.3.1. Our main goal in this thesis is to show that $\mathrm{Co}(\mu(\Psi)) \subseteq \mathrm{Co}(\Psi)$ contains $\mathcal{Z}^{\mathfrak{m}}$. We do this by understanding the distribution of motivic MZVs within $\mathrm{Co}(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}}))$ and $\mathrm{Co}((\alpha^{\mathfrak{m}})^{-1})$ which, by the previous section, are both contained within $\mathcal{Z}^{\mathfrak{m}}[\mathbb{L}]$. We then use (11.4) to compare the "difference" of the distributions of multiple zeta values within $\mathrm{Co}(S_0^{\mathfrak{m}}(\beta))$ and $\mathrm{Co}((\alpha^{\mathfrak{m}})^{-1})$. By (11.4), this "difference" is the contribution from $\mathrm{Co}(\mu(\Psi))$.

More precisely, in the next section we use Lemma 11.1.4 to compute bounds on $\mathrm{Co}_r^B(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}}))$ and $\mathrm{Co}_r^B((\alpha^{\mathfrak{m}})^{-1})$ in terms of multiple zeta values of a certain weight or depth. In Chapter 12, we will use Proposition 9.7.2 to transfer this to information about $\mathrm{Co}_r^L(\mu(\Psi))$.

11.4 Bounds on coefficients of $S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})$ and $(\alpha^{\mathfrak{m}})^{-1}$

In this section we gain a sharper understanding of the distribution of these spaces within $\mathcal{Z}[\mathbb{L}]$ by using Lemma 11.1.4 to compute bounds on the filtered coefficient spaces $\mathrm{Co}_r^B(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}}))$ and $\mathrm{Co}_r^B((\alpha^{\mathfrak{m}})^{-1})$ in terms of the weight and depth filtrations on $\mathcal{Z}^{\mathfrak{m}}$.

11.4.1 The A^r-filtered pieces of $\beta^{\mathfrak{m}}$

In this section we study $\mathrm{Co}_r^A(\beta^{\mathfrak{m}})$. By Lemma 7.3.11 we have

$$\mathrm{Co}_r^B(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})) = \mathrm{Co}(S_0^{\mathfrak{m}}(\mathrm{Fil}_A^r(\beta^{\mathfrak{m}}))) \subseteq \mathrm{Co}_r^A(\beta^{\mathfrak{m}})[\mathbb{L}^{\pm}]$$

The following lemma gives an *upper bound* on $\mathrm{Co}_r^A(\beta^{\mathfrak{m}})$ in terms of the weight filtration on $\mathcal{Z}^{\mathfrak{m}}$. It therefore gives an upper bound for $\mathrm{Co}_r^B(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}}))$.

Lemma 11.4.1. *For all $r \geq 0$ we have*

$$\mathrm{Co}_r^A(\beta^{\mathfrak{m}}) \subseteq \sum_{0 \leq k \leq r} \left(\bigoplus_{0 \leq j \leq k} \mathbb{L}^j \mathbb{Q} \right) \mathfrak{W}_{r-k}\, \mathcal{Z}^{\mathfrak{m}}\,.$$

Proof. Recall that

$$\beta^{\mathfrak{m}} = e^{-\frac{\mathbb{L}}{2}\phi(\mathsf{x}_1)} \phi(\Phi_{10}^{\mathfrak{m}}) e^{-\mathsf{b}} \phi(\Phi_{\infty 1}^{\mathfrak{m}}) \in \mathcal{G}\left(\mathcal{P}\langle\langle a,b\rangle\rangle^{\times}\right).$$

We have

$$\begin{aligned}
\mathrm{Fil}_A^r(\beta^{\mathfrak{m}}) &= \sum_{r_1+\cdots+r_4=r} \mathrm{Fil}_A^{r_1}\left(e^{-\frac{\mathbb{L}}{2}\phi(\mathsf{x}_1)}\right) \mathrm{Fil}_A^{r_2}\left(\phi(\Phi_{10}^{\mathfrak{m}})\right) \mathrm{Fil}_A^{r_3}\left(e^{-\mathsf{b}}\right) \mathrm{Fil}_A^{r_4}\left(\phi(\Phi_{\infty 1}^{\mathfrak{m}})\right) \\
&= \sum_{r_1+r_2+r_3=r} \phi\left(\mathrm{Fil}_{\mathcal{W}}^{r_1}\left(e^{-\frac{\mathbb{L}}{2}\mathsf{x}_1}\right)\right) \phi\left(\mathrm{Fil}_{\mathcal{W}}^{r_2}\left(\Phi_{10}^{\mathfrak{m}}\right)\right) e^{-\mathsf{b}} \phi\left(\mathrm{Fil}_{\mathcal{W}}^{r_3}\left(\Phi_{\infty 1}^{\mathfrak{m}}\right)\right). \qquad (11.5)
\end{aligned}$$

The second line follows by applying Lemma 11.1.4 and noting that $e^{-\mathsf{b}} \in A^0$.

The Hain morphism is injective and defined over $\mathbb{Q}$. This implies that $\mathrm{Co}(\phi(s)) = \mathrm{Co}(s)$ for any $s \in \mathcal{P}\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle$. It therefore suffices to compute the coefficient spaces of the $\mathcal{W}$-filtered pieces of the factors in (11.5).

It is easy to see that $\mathrm{Co}_r^{\mathcal{W}}(e^{-\mathbb{L}\mathsf{x}_1/2}) = \bigoplus_{j=0}^{r} \mathbb{L}^j \mathbb{Q}$ and $\mathrm{Co}\left(e^{-b}\right) = \mathbb{Q}$. Proposition 7.3.8 states that for any $i \neq j$ we have $\mathrm{Co}_r^{\mathcal{W}}(\Phi_{ij}^{\mathfrak{m}}) = \mathfrak{W}_r \mathcal{Z}^{\mathfrak{m}}$. Therefore

$$\mathrm{Co}_r^{A}(\beta^{\mathfrak{m}}) = \sum_{r_1+r_2+r_3=r} \left(\bigoplus_{j=0}^{r_1} \mathbb{L}^j \mathbb{Q} \right) \cdot \mathfrak{W}_{r_2} \mathcal{Z}^{\mathfrak{m}} \cdot \mathfrak{W}_{r_3} \mathcal{Z}^{\mathfrak{m}}. \tag{11.6}$$

The weight filtration satisfies $\mathfrak{W}_r \mathcal{Z}^{\mathfrak{m}} \cdot \mathfrak{W}_s \mathcal{Z}^{\mathfrak{m}} \subseteq \mathfrak{W}_{r+s} \mathcal{Z}^{\mathfrak{m}}$, which gives the result. □

11.4.2 The B^r-filtered pieces of $(\alpha^{\mathfrak{m}})^{-1}$

In this section we compute a lower bound on $\mathrm{Co}_r^{B}\left((\alpha^{\mathfrak{m}})^{-1}\right)$ in terms of the depth filtration on motivic MZVs.

Lemma 11.4.2. *For all $r \geq 0$ we have $\mathbb{L}\mathfrak{D}_r \mathcal{Z}^{\mathfrak{m}} \subseteq \mathrm{Co}_r^{B}(\alpha^{\mathfrak{m}})^{-1})$.*

Remark 11.4.3 (Proof strategy). While the proof may appear complicated, the strategy is simple: we compute an expression (equation (11.9)) for $\mathrm{Fil}_{\mathcal{D}}^{r}((\gamma^{\mathfrak{m}})^{-1})$ modulo products of MZVs of positive weight, where $\gamma^{\mathfrak{m}} = \Phi_{1\infty}^{\mathfrak{m}} e^{-\mathbb{L}\mathsf{x}_\infty} \Phi_{\infty 1}^{\mathfrak{m}}$ as before. The formula has a leading term consisting of a generating series for motivic MZVs $\zeta^{\mathfrak{m}}(w)$ where w has depth precisely r, followed by a correction term whose coefficients are motivic MZVs of depth strictly less than r. This explicitly demonstrates that $\mathrm{Co}_r^{\mathcal{D}}((\gamma^{\mathfrak{m}})^{-1})$ contains all MZVs of depth at most r, multiplied by $\mathbb{L}$. We then use the fact that $\phi(\gamma^{\mathfrak{m}}) = \alpha^{\mathfrak{m}}$ and that $\phi \colon \mathcal{D}^{\bullet}\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0, 1, \infty\}, \vec{1}_1) \to B^{\bullet}\pi_1^{\mathrm{dR}}(\mathcal{E}_{\partial/\partial q}^{\times}, \partial/\partial w)$ is a strict morphism of filtered objects to conclude.

Throughout this section we fix $\mathcal{P} := \mathcal{P}_{\mathsf{MT}(\mathbb{Z})}^{\mathfrak{m},+}$ to be the subalgebra of effective periods of mixed Tate motives over $\mathbb{Z}$.

Proof of Lemma 11.4.2. Let γ be as in (11.3). Its image in $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0, 1, \infty\}, \vec{1}_1)(\mathcal{P})$ is $\gamma^{\mathfrak{m}} = \Phi_{1\infty}^{\mathfrak{m}} e^{-\mathbb{L}\mathsf{x}_\infty} \Phi_{\infty 1}^{\mathfrak{m}}$. Recall that $\phi(\gamma^{\mathfrak{m}}) = \alpha^{\mathfrak{m}}$. By Lemma 11.1.4, it suffices to prove that

$$\mathbb{L}\mathfrak{D}_r \mathcal{Z}^{\mathfrak{m}} \subseteq \mathrm{Co}_r^{\mathcal{D}}((\gamma^{\mathfrak{m}})^{-1}). \tag{11.7}$$

This is trivially true for $r = 0$, so from now on we assume that $r \geq 1$. Define $I := \mathcal{Z}_{>0}^{\mathfrak{m}} \mathcal{Z}_{>0}^{\mathfrak{m}} + \mathbb{L}\mathcal{Z}_{>0}^{\mathfrak{m}} \mathcal{Z}_{>0}^{\mathfrak{m}}$. This is an ideal of $\mathcal{P}$, and hence induces a homomorphism $\pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0, 1, \infty\}, \vec{1}_1)(\mathcal{P}) \to \pi_1^{\mathrm{dR}}(\mathbb{P}^1 \backslash \{0, 1, \infty\}, \vec{1}_1)(\mathcal{P}/I)$. It sends $(\gamma^{\mathfrak{m}})^{-1}$ to an element $\bar{\gamma}$.

We may regard $\mathrm{Co}_r^{\mathcal{D}}(\bar{\gamma})$ as the subspace of $\mathrm{Co}_r^{\mathcal{D}}((\gamma^{\mathfrak{m}})^{-1}))$ spanned by coefficients that are not products of two positive-weight MZVs. Therefore it suffices to prove (11.7) with $(\gamma^{\mathfrak{m}})^{-1}$ replaced by $\bar{\gamma}$, using the $\mathcal{D}$-filtration defined on $(\mathcal{P}/I)\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle$. We show this by direct calculation.

To begin, write $\Phi^{\mathfrak{m}}_{1\infty} = 1 + u$ and $\Phi^{\mathfrak{m}}_{\infty 1} = 1 + v$, where $u, v \in \mathcal{Z}^{\mathfrak{m}}_{>0}\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle$. The property $\Phi^{\mathfrak{m}}_{1\infty}\Phi^{\mathfrak{m}}_{\infty 1} = 1$ implies that $u \equiv -v \pmod{\mathcal{Z}^{\mathfrak{m}}_{>0}\mathcal{Z}^{\mathfrak{m}}_{>0}}$. Expanding in $(\mathcal{P}/I)\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle$ gives

$$\bar{\gamma} \equiv (1-v)e^{\mathbb{L}\mathsf{x}_\infty}(1+v) \equiv e^{\mathbb{L}\mathsf{x}_\infty} + \mathbb{L}\,\mathrm{ad}(\mathsf{x}_\infty)(v) \pmod{I}. \tag{11.8}$$

We now compute v. Recall that $\Phi^{\mathfrak{m}}_{\infty 1} = \tau_{0\infty}(\Phi^{\mathfrak{m}}_{01})$, where

$$\tau_{0\infty}(\mathsf{x}_0) = \mathsf{x}_\infty = -(\mathsf{x}_0 + \mathsf{x}_1), \quad \tau_{0\infty}(\mathsf{x}_1) = \mathsf{x}_1.$$

It is clear from the formula that $\tau_{0\infty}$ preserves the depth filtration and that the associated depth-graded morphism is multiplication by $(-1)^{(\mathsf{x}_0\text{-degree})} = (-1)^{\mathrm{weight}\,-\,\mathrm{depth}}$. Filtering $\Phi^{\mathfrak{m}}_{01}$ by the depth gives

$$\Phi^{\mathfrak{m}}_{01} = 1 + \sum_{k\geq 1}\sum_{\mathrm{depth}(w)=k} \zeta^{\mathfrak{m}}(w)w.$$

We obtain

$$\begin{aligned} v &= \tau_{0\infty}(\Phi^{\mathfrak{m}}_{01}) - 1 \\ &\equiv \sum_{\mathrm{depth}(w)=r} (-1)^{\mathrm{weight}(w)-\mathrm{depth}(w)}\zeta^{\mathfrak{m}}(w)w + t \pmod{\mathcal{D}^{r+1}\,\mathcal{Z}^{\mathfrak{m}}\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle}, \end{aligned}$$

where $t \in \mathfrak{D}_{r-1}\,\mathcal{Z}^{\mathfrak{m}}\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle$ is the contribution from applying $\tau_{0\infty}$ to $\mathcal{D}^{r-1}\backslash\mathcal{D}^r$.

Using that $\mathsf{x}_\infty \equiv -\mathsf{x}_0 \pmod{\mathcal{D}^1}$, the final term in (11.8) may be approximated:

$$\mathrm{ad}(\mathsf{x}_\infty)(v) \equiv -\,\mathrm{ad}(\mathsf{x}_0)\left(\sum_{\mathrm{depth}(w)=r} (-1)^{\mathrm{weight}(w)-r}\zeta^{\mathfrak{m}}(w)w\right) + t' \pmod{\mathcal{D}^{r+1}} \tag{11.9}$$

where $t' \in \mathfrak{D}_{r-1}\,\mathcal{Z}^{\mathfrak{m}}\langle\langle \mathsf{x}_0, \mathsf{x}_1\rangle\rangle$ is the contribution from applying $\tau_{0\infty}$ to $\mathcal{D}^{r-1}\backslash\mathcal{D}^r$.

Consider the words in this sum beginning with the letter x_1. As w ranges over all words of depth r, the coefficients of these words range over the $\mathbb{Q}$-vector space of all admissible motivic MZVs of depth r (see Remark 4.3.3). This vector space is equal to the space of all motivic MZVs of depth r by shuffle-regularisation.

Induction on all $s < r$ implies that $\mathrm{Co}(t')$ contains $\mathbb{L}\mathfrak{D}_{r-1}\,\mathcal{Z}^{\mathfrak{m}}$. Equations (11.8) and (11.9) imply that $\mathrm{Co}_r^{\mathcal{D}}(\bar{\gamma})$ contains $\mathbb{L}\mathfrak{D}_r\,\mathcal{Z}^{\mathfrak{m}}$, which completes the proof. □

Chapter 12

Main calculation

This chapter is devoted to the proof of our main result:

Theorem 12.0.1. *Every motivic MZV of weight n and depth r is a $\mathbb{Q}$-linear combination of motivic iterated Eisenstein integrals of the form*

$$\mathbb{L}^m \int_S^{\mathfrak{m}} E_{2n_1+2}(b_1)\dots E_{2n_s+2}(b_s)$$

of length $s \leq r$ and total modular weight $N \leq n+s$, where $m := n - s - \sum b_i \geq 0$.

12.1 Proof idea

The main idea behind the proof of Theorem 12.0.1 is to detect coefficients of $\mu(\mathcal{C}_S^{\mathfrak{m}})$ by comparing those of $\alpha^{\mathfrak{m}}$ and $\beta^{\mathfrak{m}}$. This comparison is achieved as follows. As explained in §11.3, the topological equation $\tilde{S}(\beta) = \alpha^{-1}$ given in (8.3) induces the following equation at the level of $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$-points of $\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$:

$$\mu(\Psi)\,(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})) = (\alpha^{\mathfrak{m}})^{-1}.$$

This equation relates $(\alpha^{\mathfrak{m}})^{-1}$ and $S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})$, which are generating series for motivic MZVs, via $\mu(\Psi)$, which is a generating series for linear combinations of mixed Tate motivic iterated Eisenstein integrals.

This equation may be filtered with respect to $B^{\bullet}\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$. Combining with the bounds given in Lemmas 11.4.1 and 11.4.2, we will show that for all $r \geq 0$

$$\mathfrak{D}_r\,\mathcal{Z}^{\mathfrak{m}} \subseteq \langle \text{iterated Eisenstein integrals of length } s \leq r\rangle_{\mathbb{Q}}[\mathbb{L}^{\pm}].$$

The proof concludes by relating the modular weights to the MZV weights using the MHS on $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$. The canonical bigrading discussed in §9.4 splits both weight filtrations and the Hodge filtration on $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$. This extra information also fixes the powers of $\mathbb{L}$ that can occur in the linear combination of motivic iterated Eisenstein integrals equal to a particular motivic MZV. In particular, we show that a precise *nonnegative* power of $\mathbb{L}$ is required for each term in this linear combination.

Remark 12.1.1. The weight filtration $\mathfrak{W}_\bullet\, \mathcal{Z}^{\mathfrak{m}}$ is given by the length filtration of iterated integrals on $\mathbb{P}^1 \backslash \{0,1,\infty\}$. Theorem 12.0.1 therefore implies that there is a large reduction in the length of the iterated integral expressing a given multiple zeta value in passing from $\mathbb{P}^1 \backslash \{0,1,\infty\}$ to $\mathcal{M}_{1,1}$.

12.2 Comparing depth and length

Proposition 12.2.1 below, which proves the first part of Theorem 12.0.1, relates the depth filtration on motivic MZVs to the length filtration on iterated Eisenstein integrals as follows:

Proposition 12.2.1. *Let $r \geq 0$ and let $H_r := \mathrm{Co}_r^L(\mu(\Psi))$. There is an inclusion*

$$\mathfrak{D}_r\, \mathcal{Z}^{\mathfrak{m}} \subseteq H_r[\mathbb{L}^{\pm}].$$

Proof. The equation $\tilde{S}(\beta) = \alpha^{-1}$ of §8.1.3 implies that

$$\mu(\Psi)\left(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})\right) = (\alpha^{\mathfrak{m}})^{-1}. \tag{12.1}$$

By Proposition 9.7.2 we have $L^r\, \mathcal{U}^{\mathrm{geom}} \subseteq B^r\, \mathcal{U}^{\mathrm{geom}}$ for each $r \geq 0$. Therefore we may apply Co_r^B to both sides of equation (12.1) and decompose the coefficient space using Proposition 7.3.5 to obtain

$$\begin{aligned}\mathrm{Co}_r^B((\alpha^{\mathfrak{m}})^{-1}) &= \sum_{i+j=r} \mathrm{Co}_i^L(\mu(\Psi)) \cdot \mathrm{Co}_j^B(S_0^{\mathfrak{m}}(\beta)) \\ &= \sum_{i+j=r} H_i \cdot \mathrm{Co}(S_0^{\mathfrak{m}}(\mathrm{Fil}_A^j(\beta^{\mathfrak{m}}))).\end{aligned}$$

In going from the first to the second line we use Lemma 7.3.11, which implies that $\mathrm{Fil}_B^r \circ S_0^{\mathfrak{m}} = S_0^{\mathfrak{m}} \circ \mathrm{Fil}_A^r$. We then combine the bounds on the coefficient spaces $\mathrm{Co}_r^A(\beta^{\mathfrak{m}})$ and $\mathrm{Co}_r^B((\alpha^{\mathfrak{m}})^{-1})$ given in Lemmas 11.4.1 and 11.4.2 with the previous formula to obtain

$$\begin{aligned}
\mathbb{L}\mathfrak{D}_r\,\mathcal{Z}^{\mathfrak{m}} &\subseteq \mathrm{Co}_r^B((\alpha^{\mathfrak{m}})^{-1})\\
&= \sum_{i+j=r} H_i \cdot \mathrm{Co}(S_0^{\mathfrak{m}}(\mathrm{Fil}_A^j(\beta^{\mathfrak{m}})))\\
&\subseteq \sum_{i+j=r} H_i \cdot \mathrm{Co}_j^A(\beta^{\mathfrak{m}})[\mathbb{L}^{\pm}]\\
&\subseteq \sum_{i+j=r} H_i \cdot \left(\sum_{0\le k\le j}\left(\bigoplus_{0\le l\le k}\mathbb{L}^l\mathbb{Q}\right)\mathfrak{W}_{j-k}\,\mathcal{Z}^{\mathfrak{m}}\right)[\mathbb{L}^{\pm}] \qquad (12.2)\\
&\subseteq \sum_{i+j=r} H_i \cdot \left(\sum_{0\le k\le j}\mathfrak{W}_{j-k}\,\mathcal{Z}^{\mathfrak{m}}\right)[\mathbb{L}^{\pm}]\\
&\subseteq \sum_{i+j=r} H_i \cdot \mathfrak{W}_j\,\mathcal{Z}^{\mathfrak{m}}[\mathbb{L}^{\pm}].
\end{aligned}$$

The first line uses Lemma 11.4.2 to give a lower bound on $\mathrm{Co}_r^B((\alpha^{\mathfrak{m}})^{-1})$ in terms of the depth filtration on $\mathcal{Z}^{\mathfrak{m}}$. To go from the second to third line we use $\mathrm{Co}(S_0^{\mathfrak{m}}(s)) \subseteq \mathrm{Co}(s)[\mathbb{L}^{\pm}]$ for any series $s \in \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$; this follows from the formula $S_0^{\mathfrak{m}}(\mathsf{a}, \mathsf{b}) = (-\mathbb{L}^{-1}\mathsf{b}, \mathbb{L}\mathsf{a})$ given in (7.3). To go from the third to the fourth line we apply Lemma 11.4.1 which provides an upper bound on $\mathrm{Co}_j^A(\beta^{\mathfrak{m}})$ in terms of the weight filtration on $\mathcal{Z}^{\mathfrak{m}}$. The remaining simplifications follow by allowing multiplication by arbitrary powers of $\mathbb{L}$.

We now use induction to show that $\mathfrak{D}_r\,\mathcal{Z}^{\mathfrak{m}} \subseteq H_r[\mathbb{L}^{\pm}]$ for all r. The case $r = 0$ is trivially true, and the case $r = 1$ is known by Brown's formula for the length 1 part of $\mathcal{C}_S^{\mathfrak{m}}$, [7, §15.4]. So let us assume that for all $k \le r$ we have $\mathfrak{D}_k\,\mathcal{Z}^{\mathfrak{m}} \subseteq H_k[\mathbb{L}^{\pm}]$. Then apply (12.2) and the induction hypothesis to obtain

$$\begin{aligned}
\mathbb{L}\mathfrak{D}_{r+1}\,\mathcal{Z}^{\mathfrak{m}} &\subseteq \sum_{\substack{i+j=r+1\\ j>0}} H_i \cdot \mathfrak{W}_j\,\mathcal{Z}^{\mathfrak{m}}[\mathbb{L}^{\pm}] + H_{r+1}[\mathbb{L}^{\pm}]\\
&\subseteq \sum_{i+j=r} H_i \cdot H_j[\mathbb{L}^{\pm}] + H_{r+1}[\mathbb{L}^{\pm}]\\
&\subseteq H_{r+1}[\mathbb{L}^{\pm}].
\end{aligned}$$

The transition from the first to the second line follows from the trivial inclusion $\mathfrak{W}_j\,\mathcal{Z}^{\mathfrak{m}} \subseteq \mathfrak{D}_{j-1}\,\mathcal{Z}^{\mathfrak{m}}$, coming from the fact that every motivic MZV can be written as a linear combination of admissible MZVs, followed by applying the induction hypothesis. Now invert the leading $\mathbb{L}$ to obtain $\mathfrak{D}_{r+1}\,\mathcal{Z}^{\mathfrak{m}} \subseteq H_{r+1}[\mathbb{L}^{\pm}]$. This proves the inductive step and completes the proof. □

12.3 Proof of Theorem 12.0.1

It is now possible to prove Theorem 12.0.1. The essential input is Proposition 12.2.1, which relates the MZV depth to the iterated Eisenstein integral length, together with information on the MHS of $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$, which relates the MZV weight to the modular weight.

Proof of Theorem 12.0.1. By Proposition 12.2.1, there is an inclusion $\mathfrak{D}_r \mathcal{Z}^{\mathfrak{m}} \subseteq H_r[\mathbb{L}^{\pm}]$ for every $r \geq 0$. Recall that H_r is the image of $L_r(\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \otimes_{\mathbb{Q}} \mathbb{Q}[E_2(0)])$ under the homomorphism

$$\mu(\Psi) \colon \mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \otimes_{\mathbb{Q}} \mathbb{Q}[E_2(0)] \to \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$$

where the length filtration on $\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \otimes_{\mathbb{Q}} \mathbb{Q}[E_2(0)]$ is the restriction of the length filtration $L_\bullet \mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}})$ defined in Definition 5.2.8.

The $\mathcal{H}$-subobject $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ of $\mathcal{O}(\mathcal{U}^{\mathcal{H}}_{1,1})$ is mixed Tate, and the element $E_2(0)$ spans a copy of $\mathbb{Q}(-1)$. Therefore, by Remark 7.2.7, the $\mathbb{Q}$-vector space $H_r \subseteq \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ is spanned by linear combinations of iterated Eisenstein integrals

$$\int^{\mathfrak{m}}_{\tilde{S}} E_{2n_1+2}(b_1) \cdots E_{2n_s+2}(b_s), \quad \text{with } s \leq r \text{ and } n_j \geq 0,$$

that are equal to *mixed Tate* motivic periods. We then use Remark 10.1.3 to split such integrals into products of nonnegative powers of $\mathbb{L}$ and iterated Eisenstein integrals along $S \in \pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q)$. This proves the first part of Theorem 12.0.1.

We now turn to determining the weight. There is a canonical bigrading on $\mathcal{O}({}_0\Pi_1^{\mathrm{dR}})$ and $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})[\mathbb{L}^{\pm}]$ that is compatible with the natural grading on $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \cong U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \rtimes \mathbb{G}_m$ by [27, §23] (see also (4.12)). It is given as follows.

Recall that $\mathcal{O}({}_0\Pi_1^{\mathrm{dR}})$ is canonically isomorphic to the shuffle algebra $T^c(e_0, e_1)$ by [16]. The grading is given by assigning both e_0 and e_1 degree 1. On the other hand, recall from §9.7.1 that $O(\mathcal{U}^{\mathrm{geom}})$ is a subalgebra of the shuffle algebra on the symbols $E_{2n+2}(b)$. The grading on $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})[\mathbb{L}^{\pm}]$ is given by assigning $E_{2n+2}(b)$ degree $b+1$ and $\mathbb{L}$ degree 1 (so that $\mathbb{L}^{-1}$ has degree -1).

Let $\zeta^{\mathfrak{m}}(w) \in \mathcal{Z}^{\mathfrak{m}}_n$ be a motivic multiple zeta value of weight n. It follows that $w \in T^c(e_0, e_1)$ has degree n.

Now let us suppose further that $\zeta^{\mathfrak{m}}(w) \in \mathfrak{D}_r \mathcal{Z}^{\mathfrak{m}}$ (i.e. that it has depth at most r). By the previous part of the proof we may write $\zeta^{\mathfrak{m}}(w)$ as a linear combination of the form

$$\zeta^{\mathfrak{m}}(w) = \sum \lambda \mathbb{L}^m \int^{\mathfrak{m}}_{S} E_{2n_1+2}(b_1) \cdots E_{2n_s+2}(b_s) \tag{12.3}$$

where each[1] $s \leq r$, $m \in \mathbb{Z}$ and for each $1 \leq j \leq s$ we have $0 \leq b_j \leq 2n_j$. We also define $a_j := 2n_j - b_j$ so that $a_j + b_j = 2n_j$.

By construction, $\mathbb{L}^m[E_{2n_1+2}(b_1)|\cdots|E_{2n_s+2}(b_s)]$ is an element of $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})[\mathbb{L}^{\pm}]$. It has degree $m + s + \sum_{j=1}^{s} b_j$.

The gradings described above are compatible with the action of $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ on each side of (12.3). We deduce that

$$n = m + s + \sum_{j=1}^{s} b_j. \tag{12.4}$$

We now show that $m \geq \sum a_j$, from which the result follows.

Let $\gamma \in \pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \partial/\partial q)$. For all $n \geq 0$, the coefficient of $\mathbf{e}_{2n+2}\mathsf{Y}^{2n}$ in $\mathcal{C}^{\mathfrak{m}}_{\gamma}$ (i.e. the value $\int_{\gamma}^{\mathfrak{m}} E_{2n+2}(2n)$) is an effective motivic period.[2] This follows because $E_{2n+2}(2n)$ spans the effective motive $\mathbb{Q}(-2n-1)$.

Recall that the symbols X and Y are a basis for the standard representation V_1^{dR} of SL_2^{dR}. It has the Hodge structure $\mathbb{Q}(0) \oplus \mathbb{Q}(1)$, where X spans $\mathbb{Q}(0)$ and Y spans $\mathbb{Q}(1)$. Its symmetric power $V_{2n}^{\mathrm{dR}} = \mathrm{Sym}^{2n}(V_1^{\mathrm{dR}})$ has a basis consisting of all monomials $\mathsf{X}^a\mathsf{Y}^b$ with $a + b = 2n$. The generator $\mathsf{X}^a\mathsf{Y}^b$ may be obtained from Y^{2n} by applying the $\mathfrak{sl}_2$-generator $\mathbb{L}^{-1}\mathsf{X}\partial/\partial\mathsf{Y}$ a times. This operator has weight zero in $\mathfrak{gl}(V_{2n}^{\mathrm{dR}})$.

It follows that $\int_{\gamma}^{\mathfrak{m}} E_{2n+2}(b)$ is the quotient of an effective motivic period by $\mathbb{L}^a$. This is also true of iterated integrals of such forms. Hence, each integral on the RHS of (12.3) above must be of the form

$$\int_S^{\mathfrak{m}} E_{2n_1+2}(b_1)\cdots E_{2n_s+2}(b_s) = \frac{\kappa}{\mathbb{L}^{a_1+\cdots+a_s}},$$

where $\kappa \in \mathcal{P}^{\mathfrak{m},+}_{\mathcal{H}}$ is effective.

As $\zeta^{\mathfrak{m}}(w)$ is effective, each term on the right hand side of (12.3) is also effective. It follows that each integer m satisfies $m \geq \sum a_i$. In particular this implies that $m \geq 0$, which proves that $\mathfrak{D}_r\, \mathcal{Z}^{\mathfrak{m}} \subseteq H_r[\mathbb{L}]$. Finally, by combining the bound $m \geq \sum a_i$ with (12.4) we obtain

$$n = m + s + \sum_{j=1}^{s} b_j \geq N - s,$$

where the total modular weight is

$$N := \sum_{j=1}^{s}(2n_j + 2) = \sum_{j=1}^{s}(a_j + b_j + 2).$$

This produces the bound $N \leq n + s$ and completes the proof. $\square$

[1] Each term in this linear combination has its own length $s \leq r$ and choice of n_j, a_j, b_j.

[2] See [8, §3.3] for a definition of effective $\mathcal{H}$-periods.

Chapter 13

Galois-theoretic consequences

The benefit of working at the level of motivic periods throughout is that it gives access to structure results for the category of mixed Tate motives over $\mathbb{Z}$. In this chapter we apply this to prove a faithfulness result for the Galois action on $\mathfrak{u}^{\mathrm{geom}}$. We also discuss the relevance of this result to a conjecture of Oda.[1]

13.1 Modular generator for $\mathsf{MT}(\mathbb{Z})$

Recall from Proposition 9.5.2 that $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ is an ind-object of $\mathsf{MT}(\mathbb{Z})$. It is therefore equipped with an action of the motivic Galois group $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$. The following theorem, which follows from Theorem 12.0.1, can be seen as a "modular" analogue of Brown's main result in [5].

Theorem 13.1.1. *The group $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ acts faithfully on $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$.*

By Tannakian duality, Theorem 13.1.1 is equivalent to:

Corollary 13.1.2. *The affine ring $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ generates $\mathsf{MT}(\mathbb{Z})$.*

The central idea of the proof of Theorem 13.1.1 is to exhibit a specific element $\varepsilon \in \mathcal{U}^{\mathrm{geom}}(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$ with sufficiently many coefficients to distinguish elements of $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}(\mathbb{Q})$. By Theorem 12.0.1 the element $\varepsilon = \mu(\mathcal{C}^{\mathfrak{m}}_S)$ has this property.

Proof of Theorem 13.1.1. The action of $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \cong U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \rtimes \mathbb{G}_m$ on $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ is equivalent to the Galois action on the group of points

$$\mathcal{U}^{\mathrm{geom}}(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}) \cong \mathrm{Hom}(\mathcal{O}(\mathcal{U}^{\mathrm{geom}}), \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}).$$

[1] Now a theorem by [51, 5].

For $\varepsilon \in \mathcal{U}^{\mathrm{geom}}(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$ and $g \in G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}(\mathbb{Q})$, this action is given by

$$\Big(\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \xrightarrow{\varepsilon} \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}\Big) \mapsto \Big(\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \xrightarrow{g} \mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \xrightarrow{\varepsilon} \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}\Big).$$

This, in turn, is equivalent to the Galois action on $\mathcal{U}^{\mathrm{geom}}(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$ via the action on *coefficients*; by viewing ε as a series $\varepsilon = \sum_w \varepsilon_w w$ with $\varepsilon_w \in \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ and w ranging over a basis for $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$, it is defined by $g(\varepsilon) = \sum_w g(\varepsilon_w)w$, where g acts on $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ as in (4.10).

Let $\varepsilon = \mu(\mathcal{C}^{\mathfrak{m}}_S) \in \mathcal{U}^{\mathrm{geom}}(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$, and let $g \in U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}(\mathbb{Q})$. Suppose that $g(\varepsilon) = \varepsilon$. Then for each basis element $w \in \mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ we must have

$$g(\varepsilon_w) = \varepsilon_w. \tag{13.1}$$

Since $g \in U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}(\mathbb{Q})$ it also follows that $g(\mathbb{L}) = \mathbb{L}$.

Theorem 12.0.1 implies that there is a $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$-equivariant inclusion $\mathcal{Z}^{\mathfrak{m}} \hookrightarrow \mathrm{Co}(\varepsilon)[\mathbb{L}]$. This means that any $\zeta^{\mathfrak{m}}(w) \in \mathcal{Z}^{\mathfrak{m}}$ may be written as a linear combination

$$\zeta^{\mathfrak{m}}(w) = \sum_w \lambda_w \mathbb{L}^{m_w} \varepsilon_w$$

where w ranges over a set of basis elements for $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$, $\lambda_w \in \mathbb{Q}$ are zero for all but finitely many w, and the integer $m_w \geq 0$ is determined as in Theorem 12.0.1. Since g fixes $\mathbb{L}$ it follows that $g(\mathbb{L}^{m_w}) = \mathbb{L}^{m_w}$ for each w.

Equation (13.1) therefore implies that $g(\zeta^{\mathfrak{m}}(w)) = \zeta^{\mathfrak{m}}(w)$ for each $\zeta^{\mathfrak{m}}(w) \in \mathcal{Z}^{\mathfrak{m}}$. In other words, g acts trivially on $\mathcal{Z}^{\mathfrak{m}}$. But the action of $U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ on $\mathcal{Z}^{\mathfrak{m}}$ is faithful by Brown's theorem [5], so $g = \mathrm{id}$. This proves that the action of $U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ on $\mathcal{U}^{\mathrm{geom}}(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$ is free, and hence faithful. The compatibilities outlined above then imply that the action of $U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$ on $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ is faithful.

The affine ring $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ is an ind-object in $\mathsf{MT}(\mathbb{Z})$. Its weight filtration is canonically split by the Hodge filtration as in (4.12) and the multiplicative group $\mathbb{G}_m$ acts faithfully on the associated weight-graded. It follows that the full motivic Galois group $G^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} = U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} \rtimes \mathbb{G}_m$ also acts faithfully on $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$. □

Remark 13.1.3. Theorem 13.1.1 is equivalent to the statement that the Lie algebra $\mathfrak{u}^{\mathrm{geom}} = \mathrm{Lie}(\mathcal{U}^{\mathrm{geom}})$ generates $\mathsf{MT}(\mathbb{Z})$.

It is *a priori* very surprising that these "modular" generators exist; instead of being constructed directly from the cohomology of projective spaces, as is customary for mixed Tate motives, they emerge from the relative completion of the fundamental group of $\mathcal{M}_{1,1}$ and its action on the fundamental group of the punctured Tate curve.

13.2 Oda's conjecture

Oda's conjecture [43] was proven by Takao [51], building on work of Ihara, Matsumoto and Nakamura, as well as Brown's result [5]. Although it is a statement about relative pro-ℓ completions of mapping class groups and their associated Galois action, Oda's conjecture implies the following statement at the motivic level [25].

Recall that $\mathrm{Lie}\,\pi_1^{\mathrm{dR}}(\mathcal{E}^{\times}_{\partial/\partial q}, \partial/\partial w)$ is canonically isomorphic to $\mathrm{Lie}(\mathsf{a},\mathsf{b})^{\wedge}$. It is a pro-object of $\mathsf{MT}(\mathbb{Z})$ and thus a $\mathfrak{k}$-module, where $\mathfrak{k} = \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})})$. This isomorphism preserves the motivic structure, where we recall that a spans $\mathbb{Q}(1)$ and b spans $\mathbb{Q}(0)$. The associated bigraded is isomorphic to $\mathrm{Lie}(\mathsf{a},\mathsf{b})$, which is also a $\mathfrak{k}$-module. Consequently, there is a homomorphism

$$\rho\colon \mathfrak{k} \to \mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$$

whose image normalises[2] the subalgebra $\mathfrak{u}^{\mathrm{geom}}$. Let $\mathfrak{u}_{1,\vec{1}} = \mathrm{Lie}(\mathcal{U}^{\mathrm{dR}}_{1,\vec{1}})$. The monodromy action of §8.2 induces a homomorphism

$$\mu\colon \mathfrak{u}_{1,\vec{1}} \to \mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b}),$$

whose image is $\mathfrak{u}^{\mathrm{geom}} \oplus \mathbb{Q}\varepsilon_2$.

Proposition 9.5.2 implies that $\mathfrak{u}^{\mathrm{geom}}$ is equipped with a $\mathfrak{k}$-action described by a Lie algebra homomorphism $\tilde{\rho}\colon \mathfrak{k} \to \mathrm{Der}(\mathfrak{u}^{\mathrm{geom}})$. It is compatible with the $\mathfrak{k}$-action on $\mathrm{Lie}(\mathsf{a},\mathsf{b})$ via ρ in the sense that $\tilde{\rho}(\sigma) = \mathrm{ad}(\rho(\sigma))$ as elements of $\mathrm{Der}(\mathfrak{u}^{\mathrm{geom}})$.

Oda's conjecture implies that ρ induces an injection $\mathfrak{k} \hookrightarrow N(\mathfrak{u}^{\mathrm{geom}})/\,\mathfrak{u}^{\mathrm{geom}}$, where N denotes the normaliser [27, Remark 29.3]. Since $\mathfrak{k}$ is a free subalgebra of $N(\mathfrak{u}^{\mathrm{geom}})/\,\mathfrak{u}^{\mathrm{geom}}$, its adjoint action on $N(\mathfrak{u}^{\mathrm{geom}})/\,\mathfrak{u}^{\mathrm{geom}}$ is faithful.

Theorem 13.1.1 can be regarded as an "orthogonal" statement to Oda's conjecture because it says that the action of $\mathfrak{k}$ on $\mathfrak{u}^{\mathrm{geom}}$ is also faithful. This is surprisingly nontrivial, for the following reason. In [9], Brown showed that $\mathfrak{k}$ acts faithfully on $\mathrm{Lie}(\mathsf{a},\mathsf{b})$ and that there is a choice of generators $\sigma_{2n+1} \in \mathfrak{k}$ that act via

$$\rho(\sigma_{2n+1}) \equiv \varepsilon^{\vee}_{2n+2} \pmod{W_{-2n-3}\,\mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})}. \tag{13.2}$$

If $\mathfrak{u}^{\mathrm{geom}}$ were free, (13.2) would trivially imply that $\tilde{\rho}$ is injective, which is equivalent to the statement that $\mathfrak{k}$ acts faithfully on $\mathfrak{u}^{\mathrm{geom}}$. But $\mathfrak{u}^{\mathrm{geom}}$ is *not* free, due to the Pollack relations of §9.6. The Pollack relations therefore pose a potential obstruction to a faithful Galois action on $\mathfrak{u}^{\mathrm{geom}}$ arising from cusp forms for $SL_2(\mathbb{Z})$. Theorem 13.1.1 implies that $\tilde{\rho}$ is still injective despite this obstruction.

[2]This means that $[\mathrm{im}(\rho), \mathfrak{u}^{\mathrm{geom}}] \subseteq \mathfrak{u}^{\mathrm{geom}}$, or equivalently that $\sigma \mapsto \mathrm{ad}(\rho(\sigma))|_{\mathfrak{u}^{\mathrm{geom}}}$ defines a Lie algebra homomorphism $\mathfrak{k} \to \mathrm{Der}(\mathfrak{u}^{\mathrm{geom}})$.

Chapter 14

Example in depth 1

In this chapter we compare our results with Brown's formula for $\mathrm{Fil}^1_L(\mathcal{C}^{\mathfrak{m}}_S)$ [7, §15.4]. Brown's formula explicitly demonstrates that all odd motivic zeta values appear in $\mathrm{Co}(\mu(\mathcal{C}^{\mathfrak{m}}_S))$. This comparison allows us to compute the motivic period η appearing in the formula for Ψ.

Doing so uses several interesting computational tools, such as a depth 1 version of the Baker-Campbell-Hausdorff formula, which has a connection to Euler's formula for the even zeta values $\zeta(2n)$. We also give an interpretation of the value of the motivic period η defined in Definition 10.1.2 in terms of a cocycle for the braid group constructed by Matthes [38, 39]. It therefore seems worthwhile to include an expanded account of the depth 1 case here.

14.1 Brown's computation of $\mathcal{C}^{\mathfrak{m}}_S$ in length 1

In [7, §15.4], Brown gives a formula for $\mathrm{Fil}^1_L(\mathcal{C}^{\mathfrak{m}}_S)$. When written in the de Rham normalisation his formula is

$$\mathrm{Fil}^1_L(\mathcal{C}^{\mathfrak{m}}_S) = 1 + \sum_{n\geq 1} \mathbf{e}_{2n+2}\left[\frac{(2n)!}{2}\zeta^{\mathfrak{m}}(2n+1)\left(\frac{\mathsf{X}^{2n}}{\mathbb{L}^{2n}} - \mathsf{Y}^{2n}\right) + \mathbb{L}e^0_{2n+2,S}\right] + C. \quad (14.1)$$

Here:

- $e^0_{2n+2,S}$ is a homogeneous polynomial of degree $2n$ in X and $\mathbb{L}\mathsf{Y}$. The degree of each individual variable in each term is odd. It is the $\mathbb{Q}(\mathbb{L})$-rational part of the cocycle for $SL_2(\mathbb{Z})$ attached to $\mathbb{G}_{2n+2}$, evaluated at S;
- $C \in \ker(\mu)$ is a contribution from generators associated to cusp forms.

Formula (14.1) already provides a lot of useful information. It is possible to directly read off the fact that all odd motivic zeta values are single motivic integrals of Eisenstein series. The term C is also interesting, and gives information about the periods and quasi-periods of Hecke eigenforms [7].

Brown's formula also gives cohomological information. The term enclosed in square brackets in (14.1) is the value at S of the period polynomial cocycle for $SL_2(\mathbb{Z})$ attached to $\mathbb{G}_{2n+2}$. The motivic zeta values $\zeta^{\mathfrak{m}}(2n+1)$ are coefficients of the coboundary terms $\mathbb{L}^{-2n}\mathsf{X}^{2n} - \mathsf{Y}^{2n} = (S_0^{\mathfrak{m}} - I)\mathsf{Y}^{2n}$. In cohomology these coboundary terms vanish, and the cocycle attached to $\mathbb{G}_{2n+2}$ is rational up to powers of $\mathbb{L}$.

14.2 Explicit calculation

Recall that $\Psi = \exp(\eta\mathbf{e}_2)\mathcal{C}_S^{\mathfrak{m}}$, where $\eta \in \mathbb{Q}\mathbb{L}$. The element Ψ satisfies $\mu(\Psi)(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})) = (\alpha^{\mathfrak{m}})^{-1}$, and Proposition 9.2.1 implies that $\mu(\mathbf{e}_2) = 2\varepsilon_2$. Consequently, we have

$$e^{2\eta\varepsilon_2}\mu(\mathcal{C}_S^{\mathfrak{m}})(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})) = (\alpha^{\mathfrak{m}})^{-1}.$$

We now filter this equation modulo B^2. We may expand the exponential to obtain

$$\begin{aligned}
\mathrm{Fil}_B^1\left((\alpha^{\mathfrak{m}})^{-1}\right) &\equiv e^{2\eta\varepsilon_2}\,\mathrm{Fil}_B^1\left[\mu(\mathcal{C}_S^{\mathfrak{m}})(S_0^{\mathfrak{m}}(\beta))\right] \\
&\equiv (1+2\eta\varepsilon_2)\,\mathrm{Fil}_B^1\left[\mu(\mathcal{C}_S^{\mathfrak{m}})(S_0^{\mathfrak{m}}(\beta))\right] \\
&\equiv \mathrm{Fil}_B^1\left[\mu(\mathcal{C}_S^{\mathfrak{m}})(S_0^{\mathfrak{m}}(\beta))\right] + 2\eta\varepsilon_2\left(\mathrm{Fil}_B^0\left[\mu(\mathcal{C}_S^{\mathfrak{m}})(S_0^{\mathfrak{m}}(\beta))\right]\right) \\
&\equiv \mathrm{Fil}_L^1(\mu(\mathcal{C}_S^{\mathfrak{m}}))\left(\mathrm{Fil}_B^1(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}}))\right) + 2\eta\varepsilon_2(\mathrm{Fil}_L^0(\mathcal{C}_S^{\mathfrak{m}})(\mathrm{Fil}_B^0(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})))) \\
&\equiv \mathrm{Fil}_L^1(\mu(\mathcal{C}_S^{\mathfrak{m}}))\left(S_0^{\mathfrak{m}}(\mathrm{Fil}_A^1(\beta^{\mathfrak{m}}))\right) + 2\eta\varepsilon_2(e^{-\mathbb{L}a}) \pmod{B^2}. \qquad (14.2)
\end{aligned}$$

The second line follows because $\varepsilon_2 \in L^1$. The third line follows by expanding the first factor and noting that $\varepsilon_2(w) \equiv 0 \pmod{B^2}$ for any $w \in B^1\backslash B^0$. The fourth line follows from Proposition 9.7.2. The final line follows by applying Lemma 7.3.11, together with the equations $\mathrm{Fil}_L^0(\mathcal{C}_S^{\mathfrak{m}}) = 1$ and $\mathrm{Fil}_B^0(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}})) = \exp(-\mathbb{L}\mathsf{a})$.

Equation (14.2) determines $\eta = \mathbb{L}/8$; in order to show this, we will compare explicit expressions for the power series $\mathrm{Fil}_L^1(\mu(\mathcal{C}_S^{\mathfrak{m}}))$, $\mathrm{Fil}_A^1(\beta^{\mathfrak{m}})$ and $\mathrm{Fil}_B^1((\alpha^{\mathfrak{m}})^{-1})$.

14.2.1 Formula for $\mathrm{Fil}_L^1(\mu(\mathcal{C}_S^{\mathfrak{m}}))$

Proposition 14.2.1. *Let $Z := 1 - \sum_{n\geq 1}\zeta^{\mathfrak{m}}(2n+1)\varepsilon_{2n+2}^{\vee}$. Then for all $w \in \mathbb{Q}\langle\langle a,b\rangle\rangle$ we have*

$$\mathrm{Fil}_L^1(\mu(\mathcal{C}_S^{\mathfrak{m}}))(w) \equiv Z(w) \pmod{B^2}.$$

Proof. Apply the explicit formula for μ given in Proposition 9.2.1 to Brown's formula (14.1) to obtain

$$\mathrm{Fil}^1_L(\mu(\mathcal{C}^{\mathfrak{m}}_S)) = 1 + \sum_{n\geq 1} \zeta^{\mathfrak{m}}(2n+1)\left[\left(\frac{\varepsilon_{2n+2}}{\mathbb{L}^{2n}} - \varepsilon^\vee_{2n+2}\right) + \mathbb{L}\mu\left(e^0_{2n+2,S}(\mathsf{X}, \mathbb{L}\mathsf{Y})\mathbf{e}_{2n+2}\right)\right].$$

Note that in the above we have applied Proposition 9.2.1 and Lemma 9.1.3 to obtain

$$\mu(\mathbf{e}_{2n+2}\mathsf{X}^{2n}) = \frac{2}{[(2n)!]^2}\,\mathrm{ad}(\varepsilon^\vee_0)^{2n}(\varepsilon^\vee_{2n+2}) = \frac{2}{(2n)!}\varepsilon_{2n+2}.$$

For any $w \in \mathbb{Q}\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$ we have

$$\mu\left(e^0_{2n+2,S}\mathbf{e}_{2n+2}\right)(w) \equiv 0 \pmod{A^2 \cap B^2}.$$

This is for degree reasons: the description of the polynomial $e^0_{2n+2,S}$ above implies that $\mu\left(e^0_{2n+2,S}\mathbf{e}_{2n+2}\right)$ is a linear combination of derivations $\delta^{(k)}_{2n+2} = \mathrm{ad}(\varepsilon^\vee_0)^k\left(\varepsilon^\vee_{2n+2}\right)$ for $1 \leq k \leq 2n-1$. Any such $\delta^{(k)}_{2n+2}$ with k in this range raises both the a- and b-degree by at least 2.

Similarly, the definition of ε_{2n+2} implies that $\varepsilon_{2n+2}(w) \in B^2$ for any w. Therefore the only terms in $\mathrm{Fil}^1_L(\mu(\mathcal{C}^{\mathfrak{m}}_S))$ that act nontrivially modulo B^2 are those involving $\varepsilon^\vee_{2n+2}$. The definition of Z then implies that $\mathrm{Fil}^1_L(\mu(\mathcal{C}^{\mathfrak{m}}_S))(w) \equiv Z(w) \pmod{B^2}$ for all $w \in \mathbb{Q}\langle\langle \mathsf{a}, \mathsf{b}\rangle\rangle$. □

14.2.2 Formulae for $\mathrm{Fil}^1_A(\beta^{\mathfrak{m}})$ and $\mathrm{Fil}^1_B\left((\alpha^{\mathfrak{m}})^{-1}\right)$

In Lemma 11.2.1 we gave explicit formulae for the elements $\alpha^{\mathfrak{m}}$ and $\beta^{\mathfrak{m}}$. These may be used to compute their filtered pieces, which are very simple in low degrees.

Proposition 14.2.2. *We have*

$$\mathrm{Fil}^1_A(\beta^{\mathfrak{m}}) \equiv e^{\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]}e^{-\mathsf{b}} \pmod{A^2}$$
$$\mathrm{Fil}^1_B\left((\alpha^{\mathfrak{m}})^{-1}\right) \equiv e^{-\mathbb{L}\mathsf{a}+\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]} + \mathrm{ad}\left(e^{-\mathbb{L}\mathsf{a}}\right)(t) \pmod{B^2}$$

where

$$t := \sum_{n\geq 2}(-1)^n\zeta^{\mathfrak{m}}(n)\,\mathrm{ad}(\mathsf{a})^n(\mathsf{b}).$$

Proof. The first statement follows from the formula

$$\beta^{\mathfrak{m}} = e^{-\frac{\mathbb{L}}{2}\phi(\mathsf{x}_1)}\phi(\Phi^{\mathfrak{m}}_{10})e^{-\mathsf{b}}\phi(\Phi^{\mathfrak{m}}_{\infty 1}),$$

given in Lemma 11.2.1, by computing Fil^1_A of each factor. Lemma 11.1.4 implies that $\mathrm{Fil}^1_A\left(\phi(\Phi^{\mathfrak{m}}_{ij})\right) \equiv \phi\left(\mathrm{Fil}^1_{\mathcal{W}}(\Phi^{\mathfrak{m}}_{ij})\right) \pmod{A^2}$. The associators are invertible and have no linear term, which implies that $\mathrm{Fil}^1_{\mathcal{W}}(\Phi^{\mathfrak{m}}_{ij}) = \mathrm{Fil}^0_{\mathcal{W}}(\Phi^{\mathfrak{m}}_{ij}) = 1$. Thus the only contributions to $\mathrm{Fil}^1_A(\beta^{\mathfrak{m}})$ come from the exponential factors. The result follows by recalling that $\phi(\mathsf{x}_1) = -[\mathsf{a}, \mathsf{b}]$.

To prove the second statement, recall that $\alpha^{\mathfrak{m}} = \phi(\gamma^{\mathfrak{m}})$, where $\gamma^{\mathfrak{m}} = \Phi^{\mathfrak{m}}_{1\infty} e^{-\mathbb{L}\mathsf{x}_\infty} \Phi^{\mathfrak{m}}_{\infty 1}$. By Lemma 11.1.4 we have

$$\mathrm{Fil}^1_B\left((\alpha^{\mathfrak{m}})^{-1}\right) \equiv \phi\left(\mathrm{Fil}^1_{\mathcal{D}}\left((\gamma^{\mathfrak{m}})^{-1}\right)\right) \pmod{B^2}.$$

We will therefore compute an expression for $(\gamma^{\mathfrak{m}})^{-1}$ modulo $\mathcal{D}^2$ and then apply ϕ.

The series $\Phi^{\mathfrak{m}}_{\infty 1}$ is obtained from $\Phi^{\mathfrak{m}}_{01}$ by exchanging, in each word, all occurrences of x_0 for $\mathsf{x}_\infty = -(\mathsf{x}_0 + \mathsf{x}_1)$. The formula for $\Phi^{\mathfrak{m}}_{01}$ in depth 1 [16, §6.7] gives

$$\begin{aligned} \Phi^{\mathfrak{m}}_{\infty 1} &\equiv 1 + \sum_{n\geq 2} \zeta^{\mathfrak{m}}(n)\,\mathrm{ad}(\mathsf{x}_\infty)^{n-1}(\mathsf{x}_1) \pmod{\mathcal{D}^2} \\ &\equiv 1 - \sum_{n\geq 2} (-1)^n \zeta^{\mathfrak{m}}(n)\,\mathrm{ad}(\mathsf{x}_0)^{n-1}(\mathsf{x}_1) \pmod{\mathcal{D}^2}. \end{aligned}$$

Let $r := \sum_{n\geq 2}(-1)^n \zeta^{\mathfrak{m}}(n)\,\mathrm{ad}(\mathsf{x}_0)^{n-1}(\mathsf{x}_1) \in \mathcal{D}^1$. The property $\Phi^{\mathfrak{m}}_{1\infty}\Phi^{\mathfrak{m}}_{\infty 1} = 1$ implies that $\Phi^{\mathfrak{m}}_{1\infty} \equiv \exp(r) \pmod{\mathcal{D}^2}$. Therefore

$$(\gamma^{\mathfrak{m}})^{-1} \equiv \mathrm{Ad}_{\exp(r)}\left(e^{\mathbb{L}\mathsf{x}_\infty}\right) = e^{\mathrm{ad}(r)}\left(e^{\mathbb{L}\mathsf{x}_\infty}\right) \equiv e^{\mathbb{L}\mathsf{x}_\infty} + \mathrm{ad}\,(r)\left(e^{\mathbb{L}\mathsf{x}_\infty}\right) \pmod{\mathcal{D}^2}.$$

We now apply ϕ to this formula. Note that $\phi(r) \equiv -t \pmod{B^2}$, where $t \in B^1$ is defined in the statement of the proposition. This gives

$$\begin{aligned} (\alpha^{\mathfrak{m}})^{-1} = \phi\left((\gamma^{\mathfrak{m}})^{-1}\right) &\equiv e^{\mathbb{L}\phi(\mathsf{x}_\infty)} - \mathrm{ad}(t)(e^{\mathbb{L}\phi(\mathsf{x}_\infty)}) \pmod{B^2} \\ &\equiv e^{-\mathbb{L}\mathsf{a} + \frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]} + \mathrm{ad}\left(e^{-\mathbb{L}\mathsf{a}}\right)(t) \pmod{B^2}. \end{aligned}$$

The second line follows by noting that $\mathrm{ad}(e^{\mathbb{L}\phi(\mathsf{x}_\infty)})(t) \equiv \mathrm{ad}(e^{-\mathbb{L}\mathsf{a}})(t) \pmod{B^2}$, because $t \in B^1$. □

14.3 Formal Lie algebraic computations

Before proving Theorem 14.4.1 it is helpful to include several elementary lemmas valid for any Lie algebra $\mathfrak{g}$ over a field of characteristic zero. The first of these (Lemma 14.3.1) gives a formula for the adjoint action of elements in the universal enveloping algebra $U(\mathfrak{g})$. The second (Lemma 14.3.2) is an explicit "depth-1" version of the classic Baker-Campbell-Hausdorff formula.

Lemma 14.3.1. *Let u, v be elements in a Lie algebra $\mathfrak{g}$ over a field of characteristic zero, and let $k \geq 1$. Then in the universal enveloping algebra $U(\mathfrak{g})$ we have*

$$\mathrm{ad}(u^k)(v) = \sum_{r=1}^{k} u^{r-1}\,\mathrm{ad}(u)(v)u^{k-r}.$$

Proof. Use the relation $\mathrm{ad}(u)(v) = uv - vu$ in $U(\mathfrak{g})$ and induct on k. □

We also need the following "depth-1" version of the Baker-Campbell-Hausdorff formula. A similar formula was used in [40] to compute the meta-abelian logarithm of the elliptic associator.

Lemma 14.3.2 (Truncated BCH formula)**.** *Let u and v be elements in a Lie algebra $\mathfrak{g}$ over a field of characteristic zero. In the universal enveloping algebra we have*

$$e^u e^v \equiv e^{u+v} + \mathrm{ad}\left(e^u\right)\left(\sum_{n\geq 1} \frac{B_n^+}{n!}\,\mathrm{ad}(u)^{n-1}(v)\right) \pmod{v^2},$$

where $B_n^+ = (-1)^n B_n$ is the sequence of Bernoulli numbers with $B_1^+ = 1/2$.

Proof. The classical BCH formula [46, Corollary 3.24] gives

$$\log(e^u e^v) \equiv u + v + \sum_{n\geq 1} \frac{B_n}{n!}\,\mathrm{ad}(v)^n(u) \pmod{u^2}.$$

Taking exponentials, inverting both sides and replacing (u, v) by $(-v, -u)$ gives

$$e^u e^v \equiv \exp\left(u + v + \tilde{v}\right) \pmod{v^2}, \quad \text{where } \tilde{v} := \sum_{n\geq 1} \frac{B_n^+}{n!}\,\mathrm{ad}(u)^n(v).$$

Expanding the exponential, we obtain

$$\begin{aligned}
e^{u+v+\tilde{v}} &= \sum_{k\geq 0} \frac{1}{k!}\left(u + v + \tilde{v}\right)^k \\
&\equiv \sum_{k\geq 0} \frac{(u+v)^k}{k!} + \sum_{k\geq 1} \frac{1}{k!} \sum_{r=1}^{k} (u+v)^{r-1}\tilde{v}(u+v)^{k-r} \pmod{v^2} \\
&\equiv e^{u+v} + \sum_{k\geq 1} \frac{1}{k!} \sum_{r=1}^{k} u^{r-1}\tilde{v}u^{k-r} \pmod{v^2} \\
&= e^{u+v} + \sum_{n,k\geq 1} \frac{B_n^+}{n!} \frac{1}{k!} \sum_{r=1}^{k} u^{r-1}\,\mathrm{ad}(u)^n(v)u^{k-r} \\
&= e^{u+v} + \sum_{n,k\geq 1} \frac{B_n^+}{n!} \frac{1}{k!}\,\mathrm{ad}(u^k)\left(\mathrm{ad}(u)^{n-1}(v)\right)
\end{aligned}$$

$$
\begin{aligned}
&= e^{u+v} + \mathrm{ad}\left(e^{u} - 1\right)\left(\sum_{n\geq 1} \frac{B_n^+}{n!}\,\mathrm{ad}(u)^{n-1}(v)\right)\\
&= e^{u+v} + \mathrm{ad}\left(e^{u}\right)\left(\sum_{n\geq 1} \frac{B_n^+}{n!}\,\mathrm{ad}(u)^{n-1}(v)\right).
\end{aligned}
$$

In going from the fourth to the fifth line we use Lemma 14.3.1, where v has been replaced by $\mathrm{ad}(u)^{n-1}(v)$. □

14.4 Main calculation

In this section we prove the following result:

Theorem 14.4.1. *We have*

$$
\mathrm{Fil}_B^1\left((\alpha^{\mathfrak{m}})^{-1}\right) \equiv \mathrm{Fil}_B^1\left(\mu(\mathcal{C}_S^{\mathfrak{m}})(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}}))\right) + \frac{\mathbb{L}}{4}\varepsilon_2\left(e^{-\mathbb{L}\mathsf{a}}\right) \pmod{B^2}. \tag{14.3}
$$

Comparing Theorem 14.4.1 to equation (14.2) gives a precise value for the motivic period η that was defined in Definition 10.1.2.

Corollary 14.4.2. *We have $\eta = \mathbb{L}/8$.*

In §14.5 we discuss the significance of the value $\eta = \mathbb{L}/8$ and how it may be computed analytically by integrating a differential form on $\mathcal{M}_{1,\vec{1}}$ constructed from the Eisenstein series of weight 2 along the path $\tilde{S} \in \pi_1^{\mathrm{top}}(\mathcal{M}_{1,\vec{1}}, \vec{\mathsf{v}})$.

We prove Theorem 14.4.1 by directly computing the right hand side of (14.3) using Lemmas 14.3.1 and 14.3.2 and then comparing the result with the formula for $\mathrm{Fil}_B^1((\alpha^{\mathfrak{m}})^{-1})$ given in Proposition 14.2.2. To simplify notation in this section we write $t = t_{\mathrm{even}} - t_{\mathrm{odd}}$, where

$$
t_{\mathrm{even}} := \sum_{n\geq 1} \zeta^{\mathfrak{m}}(2n)\,\mathrm{ad}(\mathsf{a})^{2n}(\mathsf{b}), \quad t_{\mathrm{odd}} := \sum_{n\geq 1} \zeta^{\mathfrak{m}}(2n+1)\,\mathrm{ad}(\mathsf{a})^{2n+1}(\mathsf{b}).
$$

Proposition 14.4.3. *We have*

$$
\mathrm{Fil}_B^1\left(\mu(\mathcal{C}_S^{\mathfrak{m}})(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}}))\right) \equiv e^{\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]}e^{-\mathbb{L}\mathsf{a}} - \mathrm{ad}\left(e^{-\mathbb{L}\mathsf{a}}\right)(t_{\mathrm{odd}}) \pmod{B^2}.
$$

Proof. This is a straightforward calculation, working modulo B^2:

$$
\begin{aligned}
\mathrm{Fil}_B^1\left(\mu(\mathcal{C}_S^{\mathfrak{m}})(S_0^{\mathfrak{m}}(\beta^{\mathfrak{m}}))\right) &\equiv \mathrm{Fil}_L^1(\mu(\mathcal{C}_S^{\mathfrak{m}}))(S_0^{\mathfrak{m}}(\mathrm{Fil}_A^1(\beta^{\mathfrak{m}})))\\
&\equiv Z\left(e^{\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]}e^{-\mathbb{L}\mathsf{a}}\right)\\
&\equiv e^{\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]}Z(e^{-\mathbb{L}\mathsf{a}})
\end{aligned}
$$

$$\equiv e^{\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]}\left(e^{-\mathbb{L}\mathsf{a}} - \sum_{n\geq 1}\zeta^{\mathfrak{m}}(2n+1)\varepsilon^{\vee}_{2n+1}(e^{-\mathbb{L}\mathsf{a}})\right)$$
$$\equiv e^{\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]}e^{-\mathbb{L}\mathsf{a}} - \sum_{n,k\geq 1}\frac{(-\mathbb{L})^k}{k!}\zeta^{\mathfrak{m}}(2n+1)\sum_{r=1}^{k}\mathsf{a}^{r-1}\,\mathrm{ad}(\mathsf{a})^{2n+2}(\mathsf{b})\mathsf{a}^{k-r}$$
$$= e^{\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]}e^{-\mathbb{L}\mathsf{a}} - \mathrm{ad}\left(e^{-\mathbb{L}\mathsf{a}}\right)(t_{\mathrm{odd}})\,.$$

The first line follows from Lemma 9.7.2. The second line follows from Propositions 14.2.1 and 14.2.2. In the third line, the prefactor $e^{\mathbb{L}[\mathsf{a},\mathsf{b}]/2}$ can be moved in front of the operator Z because every term in Z is a geometric derivation $\varepsilon^{\vee}_{2n+2}$, all of which annihilate $[\mathsf{a},\mathsf{b}]$. The fourth line follows from the definition of Z given in Proposition 14.2.1. The fifth line follows by computing $\varepsilon^{\vee}_{2n+2}\left(e^{-\mathbb{L}\mathsf{a}}\right)$ explicitly. We also reduce the product of the prefactor and the sum over n modulo B^2, because $e^{\mathbb{L}[\mathsf{a},\mathsf{b}]/2} - 1 \in B^1$. The sixth line follows by applying Lemma 14.3.1 with $u = \mathsf{a}$ and $v = \mathrm{ad}(\mathsf{a})^n(\mathsf{b})$. □

We may now apply the BCH formula of Lemma 14.3.2 to write the product $e^{\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]}e^{-\mathbb{L}\mathsf{a}}$ in terms of the single exponential term $e^{-\mathbb{L}\mathsf{a}+\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]}$.

Proposition 14.4.4. *We have*

$$e^{\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]}e^{-\mathbb{L}\mathsf{a}} \equiv e^{-\mathbb{L}\mathsf{a}+\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]} + \mathrm{ad}\left(e^{-\mathbb{L}\mathsf{a}}\right)(t_{\mathrm{even}}) - \frac{\mathbb{L}}{4}\varepsilon_2\left(e^{-\mathbb{L}\mathsf{a}}\right) \pmod{B^2}.$$

Proof. To apply Lemma 14.3.2 directly we work with the conjugate $e^{-\mathbb{L}\mathsf{a}}e^{\mathbb{L}[\mathsf{a},\mathsf{b}]/2}$ instead of $e^{\mathbb{L}[\mathsf{a},\mathsf{b}]/2}e^{-\mathbb{L}\mathsf{a}}$. They are related by the formula

$$e^u e^v - e^v e^u \equiv \mathrm{ad}(e^u)(v) \pmod{v^2}$$

with $u = -\mathbb{L}\mathsf{a}$ and $v = \mathbb{L}[\mathsf{a},\mathsf{b}]/2$.

The degree in v corresponds to the B-filtration and $\varepsilon_2 = -\,\mathrm{ad}([\mathsf{a},\mathsf{b}])$. This gives

$$e^{\mathbb{L}[\mathsf{a},\mathsf{b}]/2}e^{-\mathbb{L}\mathsf{a}} \equiv e^{-\mathbb{L}\mathsf{a}}e^{\mathbb{L}[\mathsf{a},\mathsf{b}]/2} - \frac{\mathbb{L}}{2}\varepsilon_2(e^{-\mathbb{L}\mathsf{a}}) \pmod{B^2}, \tag{14.4}$$

Now apply the BCH formula of Lemma 14.3.2 to the first term on the right hand side of (14.4). This gives

$$e^{-\mathbb{L}\mathsf{a}}e^{\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]} \equiv e^{-\mathbb{L}\mathsf{a}+\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]} + \mathrm{ad}\left(e^{-\mathbb{L}\mathsf{a}}\right)(t_{\mathrm{even}}) + \frac{\mathbb{L}}{4}\varepsilon_2\left(e^{-\mathbb{L}\mathsf{a}}\right) \pmod{B^2}. \tag{14.5}$$

The term involving ε_2 appears by separating off the $n = 1$ term from the sum over n in the BCH formula and then rearranging. again using that $\varepsilon_2 = -\,\mathrm{ad}([\mathsf{a},\mathsf{b}])$. The remaining terms in the sum over n are only nonzero when n is even because of the

Bernoulli numbers. This sum can be shown to equal t_{even} by applying Euler's formula for the even zeta values

$$\zeta^{\mathfrak{m}}(2n) = -\frac{B_{2n}\mathbb{L}^{2n}}{2(2n)!}.$$

Combining (14.4) and (14.5) gives the result. $\square$

We are now in a position to prove Theorem 14.4.1.

Proof of Theorem 14.4.1. By combining Proposition 14.4.3 and Proposition 14.4.4, we obtain

$$\mathrm{Fil}^1_B\left(\mu(\mathcal{C}^{\mathfrak{m}}_S)(S^{\mathfrak{m}}_0(\beta^{\mathfrak{m}}))\right) + \frac{\mathbb{L}}{4}\varepsilon_2\left(e^{-\mathbb{L}\mathsf{a}}\right) \equiv e^{-\mathbb{L}\mathsf{a}+\frac{\mathbb{L}}{2}[\mathsf{a},\mathsf{b}]} + \mathrm{ad}\left(e^{-\mathbb{L}\mathsf{a}}\right)(t) \pmod{B^2}.$$

But the expression on the right hand side is precisely $\mathrm{Fil}^1_B((\alpha^{\mathfrak{m}})^{-1})$, as computed in Proposition 14.2.2. $\square$

14.5 Interpretation of η

The motivic period η was defined in Definition 10.1.2 as the motivic integral of $E_2(0)$ over $\tilde{S}$. In this section we explain the significance of the value $\eta = \mathbb{L}/8$ that was computed in Corollary 14.4.2, and how it may be computed by alternative analytic means.

The element $\mathbf{e}_2$ is dual to the cohomology class of a 1-form on $\mathcal{M}_{1,\vec{1}}$, [23, §14]. This element may be written as the following 1-form defined on the partial cover $\mathbb{C}^\times \times \mathfrak{H} \to \mathcal{M}_{1,\vec{1}}$,

$$\underline{\mathbb{G}_2}(\xi,\tau) = 2\pi i\mathbb{G}_2(\tau)d\tau - \frac{1}{2}\frac{d\xi}{\xi},$$

by verifying that it is preserved under the $SL_2(\mathbb{Z})$-action $(\xi,\tau)\mapsto((c\tau+d)^{-1}\xi,\gamma(\tau))$. Here $\mathbb{G}_2(\tau) := -1/24+\sum_{n\geq 1}\sigma(n)q^n$ is the Eisenstein series of weight 2 (see §2.3.1.1).

The universal cover of $\mathcal{M}_{1,\vec{1}}$ factors as $\mathbb{C}\times\mathfrak{H}\to\mathbb{C}^\times\times\mathfrak{H}\to\mathcal{M}_{1,\vec{1}}$, where the first map is $\exp\times\mathrm{id}$ and the second is the partial cover above identifying $SL_2(\mathbb{Z})$-equivalent points. The coordinates on the universal cover are (z,τ). Pulling back $\underline{\mathbb{G}_2}(\xi,\tau)$ to the universal cover produces the B_3-invariant form

$$\underline{\mathbb{G}_2}(z,\tau) = 2\pi i\mathbb{G}_2(\tau)d\tau - \frac{1}{2}dz \in \Omega^1(\mathbb{C}\times\mathfrak{H}).$$

The integral of $\underline{\mathbb{G}_2}(z,\tau)$ over $\tilde{S}$ is the coefficient of $\mathbf{e}_2$ in $\mathrm{per}(\Psi)$, which is

$$\int_{\tilde{S}}\underline{\mathbb{G}_2}(\xi,\tau) = \mathrm{per}(\Psi)[\mathbf{e}_2] = \mathrm{per}(\eta) = \frac{2\pi i}{8}. \tag{14.6}$$

This equals the value, at $\tilde{S}$, of a cocycle $r \in Z^1(B_3, \mathbb{C})$ constructed by Matthes [38]. Up to a sign, the cocycle r is the result of composing the homomorphism $r^{\mathfrak{m}}\colon B_3 \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ constructed in §10.1 with the period map $\mathrm{per}\colon \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \to \mathbb{C}$. Using our conventions, which differ slightly from those of Matthes,[1] it is defined as the map $r\colon B_3 \to \mathbb{C}$ given by

$$\gamma \mapsto \int_{\gamma^{-1}(0,\partial/\partial q)}^{(0,\partial/\partial q)} \underline{\mathbb{G}_2}(z,\tau), \tag{14.7}$$

where $(0, \partial/\partial q)$ is a tangential basepoint on $\mathbb{C} \times \mathfrak{H}$ and elements of B_3 act on the universal cover as deck transformations. It can be viewed as the limiting $n = 0$ case of a family of cocycles $r_{2n+2} \in Z^1(B_3, V_{2n} \otimes \mathbb{C})$. For $n > 0$ these cocycles are pullbacks of the period polynomial cocycles in $Z^1(SL_2(\mathbb{Z}), V_{2n} \otimes \mathbb{C})$ by the homomorphism $f\colon B_3 \to SL_2(\mathbb{Z})$ defined in Proposition 8.1.4. Of course, in the $n = 0$ case the cocycle r is not a lift of any period polynomial cocycle because there are no modular forms for $SL_2(\mathbb{Z})$ of weight 2. Additionally, B_3 acts trivially on $V_0 \otimes \mathbb{C} \cong \mathbb{C}$, which implies that $r \in \mathrm{Hom}(B_3, \mathbb{C})$.

The braid group is a central extension

$$1 \to \mathbb{Z} \to B_3 \xrightarrow{f} SL_2(\mathbb{Z}) \to 1.$$

Here $\mathbb{Z}$ is the automorphism group of the covering $\exp \times \mathrm{id}\colon \mathbb{C} \times \mathfrak{H} \to \mathbb{C}^\times \times \mathfrak{H}$. This group may be identified with the fundamental group of $\mathbb{C}^\times$, which is generated by a counterclockwise loop around 0 denoted by σ.

Proposition 8.1.4 states that $\ker(f)$ is generated by $\tilde{S}^4$, and hence $\tilde{S}^4 = \sigma^{\pm 1}$. It follows that

$$r(\tilde{S}) = \pm\frac{r(\sigma)}{4}. \tag{14.8}$$

It therefore suffices to compute $r(\sigma)$. Matthes' original computation [39] is unpublished, so we record it again here using our own conventions.

Proposition 14.5.1. *We have*

$$r(\sigma) = -\frac{2\pi i}{2}.$$

Proof. We simply compute the action on the universal covering space and then apply (14.7). The generator σ acts on $\mathbb{C} \times \mathfrak{H}$ via $\sigma(z,\tau) = (z + 2\pi i, \tau)$. Since it fixes the second coordinate, we have

$$r(\sigma) = \int_{(-2\pi i,\partial/\partial q)}^{(0,\partial/\partial q)} \left(2\pi i \mathbb{G}_2(\tau) d\tau - \frac{1}{2} dz\right) = -\frac{1}{2}\int_{-2\pi i}^{0} dz = -\frac{2\pi i}{2}.$$

[1]Matthes' convention for the $SL_2(\mathbb{Z})$-action on $\mathbb{C}^\times \times \mathfrak{H}$ differs from the one used here by the reciprocal in the first factor. This only changes the cocycle values by a possible sign. He also defines the cocycle in terms of $E_2 = -24\mathbb{G}_2$.

$\square$

By (14.8) it follows that $r(\tilde{S}) = \pm(2\pi i)/8$. Comparing to (14.6) shows that the two values agree up to a choice of sign. This reason for this is because the coefficient of $\mathbf{e}_{2n+2}$ in $\mathrm{per}(\mathrm{Fil}^1_L(\Psi))$ is $r_{2n+2}(\tilde{S})$, for $n > 0$, and the coefficient of $\mathbf{e}_2$ is the value at $\tilde{S}$ of a cocycle for B_3 associated to $\mathbb{G}_2$ that must be compatibly normalised in order for Ψ to be grouplike. Our choice of normalisation, together with the fact that $H^1(B_3, \mathbb{C})$ is one-dimensional, implies that this cocycle is r.

Chapter 15

Coefficients of iterated Eisenstein integrals

The proof of Theorem 12.0.1 is nonconstructive, and for a given motivic multiple zeta value $\zeta^{\mathfrak{m}}(w)$ the result only implies that there is some linear combination of motivic iterated Eisenstein integrals of certain lengths and weights equal to $\zeta^{\mathfrak{m}}(w)$. Nevertheless it is possible to determine some of the coefficients in this linear combination using additional information. One source of information is the f-alphabet decomposition of mixed Tate motivic periods [6],[7, §22]. This assigns an element of the shuffle algebra $\mathbb{Q}[\mathbb{L}^{\pm}] \otimes_{\mathbb{Q}} T^c(f_{2n+1} : n \geq 1)$ to a given element of $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$. Although this depends on some choices, the highest-length term in this decomposition is canonical.

In this section we use this idea to determine the coefficients of the highest-length iterated Eisenstein integrals appearing in linear combinations equal to some example MZVs. This combines both the combinatorics of the f-alphabet decomposition for motivic MZVs and the modular theory of the canonical cocycle $\mathcal{C}^{\mathfrak{m}}$.

15.1 The coradical filtration

The notion of an f-alphabet decomposition relies upon the *coradical filtration* $C_{\bullet}\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$. This is a general concept that can be defined for any right $\mathcal{O}(U)$-comodule (for any unipotent group U) [8, §2.5]. Here we only define it for $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$; the relevant unipotent group in this case is $U = U^{\mathrm{dR}}_{\mathcal{H}}$, the unipotent radical of the motivic Galois group $G^{\mathrm{dR}}_{\mathcal{H}} = \mathrm{Aut}^{\otimes}_{\mathcal{H}}(\omega^{\mathrm{dR}}_{\mathcal{H}})$.

The coaction on motivic periods defined in equation (4.5) induces, via the natural surjection $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}} = \mathcal{O}(G^{\mathrm{dR}}_{\mathcal{H}}) \to \mathcal{O}(U^{\mathrm{dR}}_{\mathcal{H}})$, a "unipotent" coaction

$$\Delta^u \colon \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \otimes_{\mathbb{Q}} \mathcal{O}(U^{\mathrm{dR}}_{\mathcal{H}}).$$

Let $\Delta^{u,\mathrm{red}} := \Delta^u - \mathrm{id}\otimes 1$ be the associated reduced coaction. Following [8, §3.8] we define the coradical filtration on $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ to be the increasing filtration $C_\bullet\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ given by

$$C_r\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} = \left\{\xi \in \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} : (\Delta^{u,\mathrm{red}})^{r+1}(\xi) = 0\right\}.$$

This definition gives $C_0\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} = \{\xi : \Delta^u(\xi) = \xi \otimes 1\}$. We may express this subalgebra using the Tannakian structure on $\mathcal{H}$ as follows. The action of $U^{\mathrm{dR}}_{\mathcal{H}}$ on $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ is given by the formula (4.7); namely, for $g \in U^{\mathrm{dR}}_{\mathcal{H}}(\mathbb{Q}) = \mathrm{Hom}(\mathcal{O}(U^{\mathrm{dR}}_{\mathcal{H}}), \mathbb{Q})$ we have

$$g(\xi) = (\mathrm{id}\otimes g)(\Delta(\xi)).$$

This implies that $C_0\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} = (\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}})^{U^{\mathrm{dR}}_{\mathcal{H}}}$ is the space of motivic periods fixed by the action of $U^{\mathrm{dR}}_{\mathcal{H}}$. Since $U^{\mathrm{dR}}_{\mathcal{H}}$ acts trivially precisely on the full subcategory $\mathcal{H}^{ss} \hookrightarrow \mathcal{H}$ of semisimple objects, $C_0\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} = \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}^{ss}}$ is the subalgebra of semisimple motivic periods.

In general, the coradical filtration is the fastest-growing filtration $C_\bullet\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ that is preserved by $U^{\mathrm{dR}}_{\mathcal{H}}$ such that $U^{\mathrm{dR}}_{\mathcal{H}}$ acts trivially on $\mathrm{gr}^C\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ and $C_{-1}\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} = 0$.

15.1.1 Relationship to the length filtration

The coradical filtration is connected to the length filtration $L_\bullet\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1})$ that was defined in Definition 5.2.8. By [7, Theorem 22.2] the element $\mathcal{C}^{\mathfrak{m}}_S$ of $\mathcal{U}^{\mathrm{dR}}_{1,1}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}})$, which we may interpret as the motivic "integration-along-S" operator $\int^{\mathfrak{m}}_S$, is a morphism of filtered $\mathbb{Q}$-algebras

$$\mathcal{C}^{\mathfrak{m}}_S\colon L_\bullet\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1}) \to C_\bullet\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}. \tag{15.1}$$

Hence, on the subalgebra $M^{\mathfrak{m}}$ of $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ constisting of motivic multiple modular values, the coradical filtration coincides with the length filtration $\mathfrak{L}_\bullet M^{\mathfrak{m}}$ that was defined in Definition 7.3.9.

Recall from §9.7.1 that the Hopf subalgebra $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ of $\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1})$ is an ind-object in $\mathsf{MT}(\mathbb{Z})$. We may write

$$L_r\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) = \langle [E_{2n_1+2}(b_1)|\ldots|E_{2n_s+2}(b_s)] \in \mathcal{O}(\mathcal{U}^{\mathrm{geom}}) : s \le r\rangle_{\mathbb{Q}}.$$

When pulled back via the inclusion $\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \hookrightarrow \mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1})$ the element $\mathcal{C}^{\mathfrak{m}}_S$ equals the homomorphism $\mu(\mathcal{C}^{\mathfrak{m}}_S) \in \mathrm{Hom}(\mathcal{O}(\mathcal{U}^{\mathrm{geom}}), \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$. This may also be interpreted as the operator $\int^{\mathfrak{m}}_S$, this time restricted to mixed Tate iterated Eisenstein integrals, giving a homomorphism of filtered $\mathbb{Q}$-algebras

$$\mu(\mathcal{C}^{\mathfrak{m}}_S)\colon L_\bullet\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \to C_\bullet\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}.$$

15.2 f-alphabet decomposition

In this section we define the notion of an f-alphabet decomposition of motivic periods. We also give a formula for the decomposition of certain motivic iterated Eisenstein integrals to leading order in the coradical filtration.

Definition 15.2.1 (f-alphabet decomposition)**.** An f-alphabet decomposition is a $\mathbb{G}_m$-equivariant isomorphism $\varphi\colon \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} \xrightarrow{\sim} \mathbb{Q}[\mathbb{L}^{\pm}] \otimes T^c(f_3, f_5, \dots)$. It is *normalised* if $\varphi(\zeta^{\mathfrak{m}}(2n+1)) = f_{2n+1}$ for all $n \geq 1$.

There is a more general notion of a decomposition map [8, Definition 3.10]. Recall from Definition 4.2.1 that $G^{\mathrm{dR}}_{\mathcal{H}}$ denotes the motivic Galois group of the category $\mathcal{H}$ based at the de Rham fiber functor. Its unipotent radical is $U^{\mathrm{dR}}_{\mathcal{H}}$ and we denote its reductive quotient by $S^{\mathrm{dR}}_{\mathcal{H}}$. The decomposition map is a canonical isomorphism of $S^{\mathrm{dR}}_{\mathcal{H}}$-modules $\Phi\colon \operatorname{gr}^C \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \xrightarrow{\sim} \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}^{ss}} \otimes T^c(\operatorname{gr}^C_1 \mathcal{O}(U^{\mathrm{dR}}_{\mathcal{H}}))$. It is an isomorphism of $S^{\mathrm{dR}}_{\mathcal{H}}$-modules because $U^{\mathrm{dR}}_{\mathcal{H}}$ acts trivially on both sides.

Recall from equation (4.4) that the fully faithful functor $\omega^{\mathcal{H}}\colon\ \mathsf{MT}(\mathbb{Z}) \hookrightarrow \mathcal{H}$ induces an inclusion $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} \hookrightarrow \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$. As described in [8, §5.4] the decomposition map on $\operatorname{gr}^C \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ restricts to a decomposition map on $\operatorname{gr}^C \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$. Noting that $S^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})} = \mathbb{G}_m$, this restricted decomposition is a canonical isomorphism of $S^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})}$-modules

$$\Phi\colon\ \operatorname{gr}^C \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} \xrightarrow{\sim} \mathbb{Q}[\mathbb{L}^{\pm}] \otimes T^c(f_3, f_5, \dots).$$

A choice of splitting of the coradical filtration $C_{\bullet}\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ (e.g. using the Hoffman elements [5]) determines an isomorphism $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} \cong \operatorname{gr}^C \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ and hence an isomorphism φ as above. Different choices of splittings for the coradical filtration will determine different choices of φ; the map $\Phi = \operatorname{gr}^C \varphi$, however, is canonical.

15.2.1 f-alphabet decomposition for motivic iterated Eisenstein integrals

The morphism of filtered algebras $\mu(\mathcal{C}^{\mathfrak{m}}_S)\colon L_{\bullet}\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \to C_{\bullet}\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ described in §15.1.1 is compatible with the following formula for the f-alphabet decomposition of certain motivic iterated Eisenstein integrals. Recall that $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ is described in §9.7.1; in particular, its elements may be expressed as certain words in the elements $E_{2n+2}(b)$, defined in §5.5.1.2, that integrate to periods of $\mathsf{MT}(\mathbb{Z})$ under $\int^{\mathfrak{m}}_S$.

Lemma 15.2.2. *Let φ be any normalised f-alphabet decomposition and let $\Phi = \operatorname{gr}^C \varphi$. Assume that for all $b_1, \dots, b_s$ with $0 \leq b_j \leq 2n_j$, the word $[E_{2n_1+2}(b_1)| \cdots |E_{2n_s+2}(b_s)]$*

is an element of $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$*. Then* $\Phi\colon \mathrm{Co}(\mu(\mathcal{C}_S^{\mathfrak{m}})) \to \mathbb{Q}[\mathbb{L}^{\pm}] \otimes T^c(f_3, f_5, \dots)$ *is given explicitly on the lowest and highest weight words by*

$$\int_S^{\mathfrak{m}} E_{2n_1+2}(0) \cdots E_{2n_s+2}(0) \mapsto (-1)^s \frac{(2n_1)! \cdots (2n_s)!}{2^s} \frac{f_{2n_1+1} \cdots f_{2n_s+1}}{\mathbb{L}^{2n_1+\cdots+2n_s}}$$

$$\int_S^{\mathfrak{m}} E_{2n_1+2}(2n_1) \cdots E_{2n_s+2}(2n_s) \mapsto \frac{(2n_1)! \cdots (2n_s)!}{2^s} f_{2n_s+1} \cdots f_{2n_1+1}.$$

Proof. We prove the equality

$$\int_S^{\mathfrak{m}} E_{2n_1+2}(0) \cdots E_{2n_s+2}(0) = \frac{(-1)^s}{\mathbb{L}^{2n_1+\cdots+2n_s}} \int_S^{\mathfrak{m}} E_{2n_s+2}(2n_s) \cdots E_{2n_1+2}(2n_1), \quad (15.2)$$

which may be shown by making use of the action of $S_0^{\mathfrak{m}}$ on $\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{L}^{\pm}]$. The result follows by applying [7, Theorem 22.2] to the right hand side of (15.2), although some modifications must be made (namely, the coefficient needs to be inverted and the order of letters reversed).

To proceed with showing (15.2), note that the action of $S_0^{\mathfrak{m}}$ on $\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{L}^{\pm}]$ satisfies

$$S_0^{\mathfrak{m}}\colon [E_{2n_1+2}(2n_1)| \cdots |E_{2n_s+2}(2n_s)] \mapsto \mathbb{L}^{2n_1+\cdots+2n_s} [E_{2n_1+2}(0)| \cdots |E_{2n_s+2}(0)]. \quad (15.3)$$

Recall that $\mu(\mathcal{C}_S^{\mathfrak{m}})$ may be identified with the motivic "integrate along S" operator on $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$. Apply the homomorphism $\mu(\mathcal{C}_S^{\mathfrak{m}}) \in \mathrm{Hom}(\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{L}^{\pm}], \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})})$ to both sides of (15.3) to obtain

$$\mathcal{C}_S^{\mathfrak{m}}(S_0^{\mathfrak{m}}([E_{2n_1+2}(2n_1)| \cdots |E_{2n_s+2}(2n_s)])) = \mathbb{L}^{2n_1+\cdots+2n_s} \mathcal{C}_S^{\mathfrak{m}}([E_{2n_1+2}(0)| \cdots |E_{2n_s+2}(0)]). \quad (15.4)$$

For any $w \in O(\mathcal{U}^{\mathrm{geom}}) \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{L}^{\pm}]$ we have $\mathcal{C}_S^{\mathfrak{m}}(S_0^{\mathfrak{m}}(w)) = \mathcal{C}_S^{\mathfrak{m}}|_{S_0^{\mathfrak{m}}}(w)$. The cocycle equation for $\mathcal{C}^{\mathfrak{m}}$ associated to the equation $S^2 = -I$ implies that $\mathcal{C}_S^{\mathfrak{m}}|_{S_0^{\mathfrak{m}}} = (\mathcal{C}_S^{\mathfrak{m}})^{-1}$. The element $(\mathcal{C}_S^{\mathfrak{m}})^{-1}$ is the generating series for motivic iterated integrals along S^{-1}, where we use the choice of splitting that defines $\mathcal{C}_S^{\mathfrak{m}}$. Equation (15.4) therefore reads

$$\int_{S^{-1}}^{\mathfrak{m}} E_{2n_1+2}(2n_1) \cdots E_{2n_s+2}(2n_s) = \mathbb{L}^{2n_1+\cdots+2n_s} \int_S^{\mathfrak{m}} E_{2n_1+2}(0) \cdots E_{2n_s+2}(0).$$

Applying the reversal of paths formula gives

$$\int_S^{\mathfrak{m}} E_{2n_s+2}(2n_s) \cdots E_{2n_1+2}(2n_1) = (-1)^s \mathbb{L}^{2n_1+\cdots+2n_s} \int_S^{\mathfrak{m}} E_{2n_1+2}(0) \cdots E_{2n_s+2}(0).$$

This implies equation (15.2). Taking f-alphabet decompositions gives the result. $\square$

Remark 15.2.3. Lemma 15.2.2 is actually valid for all $n_1, \ldots, n_s > 0$, even when the words $[E_{2n_1+2}(0)|\ldots|E_{2n_s+2}(0)], [E_{2n_1+2}(2n_1)|\ldots|E_{2n_s+2}(2n_s)] \in \mathcal{O}(\mathcal{U}_E^{\mathrm{dR}})$ are not contained in the mixed Tate subalgebra $\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \subseteq \mathcal{O}(\mathcal{U}_{1,1}^{\mathrm{dR}})$ (the same proof goes through with $\mu(\mathcal{C}_S^{\mathfrak{m}})$ replaced by $\mathcal{C}_S^{\mathfrak{m}}$ in this case). This implies that the leading-order term in the coradical filtration of *any* motivic iterated Eisenstein integral is an element of $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$; see [11, Conjecture 2] for a conjectural description of this term in terms of MZVs. With these integrals of more general words, however, non-mixed Tate motivic periods may appear lower down in the coradical filtration.

15.3 Cocycle equations

To determine coefficients in linear combinations of iterated Eisenstein integrals we must understand relations between such integrals. The *cocycle equations* for $\mathcal{C}^{\mathfrak{m}} \in Z^1(SL_2(\mathbb{Z}), \mathcal{U}_{1,1}^{\mathrm{dR}}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}))$ determine many of these. Set $U = ST \in SL_2(\mathbb{Z})$; then

$$U_0^{\mathfrak{m}} = S_0^{\mathfrak{m}} T_0^{\mathfrak{m}} = \begin{pmatrix} 0 & -\mathbb{L} \\ \mathbb{L}^{-1} & 1 \end{pmatrix} \in SL_2^{\mathrm{dR}}(\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}).$$

We have $S^2 = U^3 = -I$. Furthermore $\mathcal{C}^{\mathfrak{m}}_{-I} = 1$ because all modular forms for $SL_2(\mathbb{Z})$ have even weight. By Proposition 5.5.2, the canonical cocycle therefore satisfies the equations

$$\mathcal{C}_S^{\mathfrak{m}}|_{S_0^{\mathfrak{m}}} \cdot \mathcal{C}_S^{\mathfrak{m}} = 1 \tag{15.5}$$

$$\mathcal{C}_U^{\mathfrak{m}}|_{(U_0^{\mathfrak{m}})^2} \cdot \mathcal{C}_U^{\mathfrak{m}}|_{U_0^{\mathfrak{m}}} \cdot \mathcal{C}_U^{\mathfrak{m}} = 1, \tag{15.6}$$

where $\mathcal{C}_U^{\mathfrak{m}} = \mathcal{C}_{ST}^{\mathfrak{m}} = \mathcal{C}_S^{\mathfrak{m}}|_{T_0^{\mathfrak{m}}} \cdot \mathcal{C}_T^{\mathfrak{m}}$. Taking the coefficient of Eisenstein words in equations (15.5) and (15.6) produces relations between iterated Eisenstein integrals.

When written out in terms of $\mathcal{C}_S^{\mathfrak{m}}$ and $\mathcal{C}_T^{\mathfrak{m}}$, it becomes rapidly more difficult to take coefficients in Equation (15.6) by hand as the length increases. However, since $\mathrm{Co}(\mathcal{C}_T^{\mathfrak{m}}) = \mathbb{Q}[\mathbb{L}] \subseteq C_0\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$, we may focus instead on only the leading-order terms in the coradical filtration. Doing so gives

$$\mathcal{C}_S^{\mathfrak{m}}|_{(TSTST)_0^{\mathfrak{m}}} \cdot \mathcal{C}_S^{\mathfrak{m}}|_{(TST)_0^{\mathfrak{m}}} \cdot \mathcal{C}_S^{\mathfrak{m}}|_{T_0^{\mathfrak{m}}} = 1 \in \mathcal{U}_{1,1}^{\mathrm{dR}}(\mathrm{gr}^C \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}). \tag{15.7}$$

For our purposes this is just as useful and easier to compute; for example, taking the coefficient of a length 2 word in (15.6) produces a relation with 21 terms, whereas taking the coefficient of the same word in (15.7) produces a relation modulo $C_1\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ with only 6 terms.

15.4 Examples

We now consider the two example equations given in the introduction to this paper and show how some of their coefficients may be determined using the above machinery. The essential idea in each case is to first use Theorem 12.0.1 to obtain the (fairly small) finite set of iterated Eisenstein integrals that may appear in a linear combination equal to the given MZV. We then use relations between iterated Eisenstein integrals coming from the cocycle equations to exhibit linear dependencies in this set and reduce its size further. Finally we form a general linear combination with the remaining terms, apply f-alphabet decompositions to both sides and compare coefficients.

Example 15.4.1. Consider the expression for $\zeta(3)$ given in the introduction:

$$\begin{aligned}\zeta(3) &= -(2\pi i)^3 \int_S \mathbb{G}_4(\tau)d\tau \\ &= -(2\pi i)^2 \int_S E_4(0).\end{aligned}$$

We illustrate how to derive this equation from the motivic theory. Theorem 12.0.1 implies that $\zeta^{\mathfrak{m}}(3)$ can be written as a $\mathbb{Q}$-linear combination of the motivic periods $\mathbb{L}^2 \int_S^{\mathfrak{m}} E_4(0)$, $\mathbb{L} \int_S^{\mathfrak{m}} E_4(1)$, $\int_S^{\mathfrak{m}} E_4(2)$ and $\mathbb{L}^3$. The morphism (15.1) implies we may work modulo $C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$, which kills $\mathbb{L}^3 \in C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$. We shortly explain why there can be no $\mathbb{L}^3$ term in the full linear combination, so in fact it makes no difference whether we work modulo $C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ or not in this case.

Taking the coefficient of $\mathbf{e}_4\mathsf{X}^2$ in (15.5) implies that $\int_S^{\mathfrak{m}} E_4(2) = -\mathbb{L}^2 \int_S^{\mathfrak{m}} E_4(0)$. Taking the coefficient of $\mathbf{e}_4\mathsf{XY}$ in (15.7) implies that $\int_S^{\mathfrak{m}} E_4(1) \equiv 0 \pmod{C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}}$.[1] Therefore we may write $\zeta^{\mathfrak{m}}(3) \equiv A\mathbb{L}^2 \int_S^{\mathfrak{m}} E_4(0) \pmod{C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}}$, for some $A \in \mathbb{Q}$. In fact this expression holds "on the nose" even without working modulo $C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$, because the undetermined term in $C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ must have weight 3 and so must be a multiple of $\mathbb{L}^3$, which is anti-invariant under the action of the real Frobenius F_∞ (see Definition 4.1.1). We are left with a simple equation of the form

$$\zeta^{\mathfrak{m}}(3) = A\mathbb{L}^2 \int_S^{\mathfrak{m}} E_4(0), \quad \text{for some } A \in \mathbb{Q}. \tag{15.8}$$

We now apply the normalised decomposition map φ to (15.8). Lemma 15.2.2 gives

$$\varphi\left(\int_S^{\mathfrak{m}} E_4(0)\right) = -\frac{2!}{2}\frac{f_3}{\mathbb{L}^2} + \lambda\mathbb{L}^3, \quad \text{for some } \lambda \in \mathbb{Q}.$$

Comparing coefficients in (15.8) determines $A = -1$ and $\lambda = 0$.

[1] This concisely expresses that the term involving a zeta value is a coboundary and the image of the cocycle of $\mathbb{G}_4$ in cohomology is rational up to a power of $\mathbb{L}$ [7, Equation (7.8)].

Remark 15.4.2. A similar calculation may be used to show that

$$\zeta^{\mathfrak{m}}(2n+1) = -\frac{2}{(2n)!}\mathbb{L}^{2n}\int_S^{\mathfrak{m}} E_{2n+2}(0) = \frac{2}{(2n)!}\int_S^{\mathfrak{m}} E_{2n+2}(2n),$$

which recovers the completed L-values $\Lambda(\mathbb{G}_{2n+2},1)$ and $\Lambda(\mathbb{G}_{2n+2},2n+1)$. Combining this with the relation obtained by taking the coefficient of $\mathbf{e}_{2n+2}\mathsf{X}^{2n-b}\mathsf{Y}^b$ in equation (15.7) also shows that

$$\int_S^{\mathfrak{m}} E_{2n+2}(b) \equiv (\delta_{2n-b}-\delta_b)\frac{(2n)!}{2}\frac{\zeta^{\mathfrak{m}}(2n+1)}{\mathbb{L}^{2n-b}} \pmod{C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}},$$

where $\delta_c = \delta_{c,0}$ is the Kronecker delta. In particular, this implies that the "middle" values $\int_S^{\mathfrak{m}} E_{2n+2}(b)$ for $1 \leq b \leq 2n-1$ are powers of $\mathbb{L}$, which is of course also well-known. From this point onward we will therefore use the fact that the motivic iterated Eisenstein integrals with $b \in \{0, 2n\}$ evaluate to motivic single zeta values.

The second example gives an expression for $\zeta^{\mathfrak{m}}(3,5) \in \mathfrak{D}_2\,\mathcal{Z}^{\mathfrak{m}} \setminus \mathfrak{D}_1\,\mathcal{Z}^{\mathfrak{m}}$ as a double Eisenstein integral.

Example 15.4.3. We have the following expression for $\zeta(3,5)$ [11, Example 7.2][2]:

$$\begin{aligned}\zeta(3,5) &= -\frac{5}{12}(2\pi i)^8\int_S \mathbb{G}_6(\tau_1)d\tau_1\mathbb{G}_4(\tau_2)d\tau_2 + \frac{503}{2^{13}3^5 5^2 7}(2\pi i)^8\\ &= -\frac{5}{12}(2\pi i)^6\int_S E_6(0)E_4(0) + \frac{503}{2^{13}3^5 5^2 7}(2\pi i)^8.\end{aligned}$$

We explain how the coefficient $-5/12$ of the longest iterated Eisenstein integral may be computed abstractly. We first determine the possible such integrals that may appear in a linear combination giving $\zeta^{\mathfrak{m}}(3,5)$. Theorem 12.0.1 implies that these must be iterated integrals of length at most 2 and total modular weight at most 10. By (15.1) we therefore work modulo $C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$, giving

$$\zeta^{\mathfrak{m}}(3,5) \equiv \sum_{\substack{(n_1,n_2)\in\{(1,1),(1,2),(2,1)\}\\ 0\leq b_i\leq 2n_i}} \lambda \int_S^{\mathfrak{m}} E_{2n_1+2}(b_1)E_{2n_2+2}(b_2) \pmod{C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}}, \quad (15.9)$$

where λ is a rational multiple of $\mathbb{L}^{6-b_1-b_2}$. The motivic MZV $\zeta^{\mathfrak{m}}(3,5)$ has weight 8, and the weight 8 subspace[3] of $C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ is spanned by $\zeta^{\mathfrak{m}}(8)$, which is a rational multiple of $\mathbb{L}^8$. Thus the only term that is lost by working modulo $C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ is a

[2] There is a difference in normalisation between Brown's formulae and ours; namely $\Lambda(\mathbb{G}_{k_1},\ldots,\mathbb{G}_{k_s};b_1,\ldots,b_s) = i^{b_1+\cdots+b_s}(2\pi i)^{-(b_1+\cdots+b_s)}\int_S E_{k_1}(b_1-1)\cdots E_{k_s}(b_s-1)$.

[3] Recall that the weight filtration is a grading on motivic MZVs.

rational multiple of $\mathbb{L}^8 \in C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$. In other words working modulo $C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ is equivalent to working modulo $C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ in this case.

Let φ be some choice of normalised f-alphabet decomposition and $\Phi = \operatorname{gr}^C \varphi$. The decomposition algorithm [6, §5] gives

$$\Phi(\zeta^{\mathfrak{m}}(3,5)) \equiv -5 f_5 f_3 \pmod{C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}}. \tag{15.10}$$

The coaction on $T^c(f_3, f_5, \dots) \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{L}^{\pm}]$ is actually a coproduct because scalars have been extended to $\mathbb{Q}[\mathbb{L}^{\pm}]$. It is given by deconcatenation. This gives

$$\Delta(\Phi(\zeta^{\mathfrak{m}}(3,5)) = -5 \otimes f_5 f_3 - 5 f_5 \otimes f_3 - 5 f_5 f_3 \otimes 1. \tag{15.11}$$

In particular we have $\Delta'(\Phi(\zeta^{\mathfrak{m}}(3,5))) \equiv -5 f_5 \otimes f_3 \neq 0 \pmod{C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}}$, where $\Delta' := \Delta - \mathrm{id} \otimes 1 - 1 \otimes \mathrm{id}$ is the reduced coproduct.

We must now compute the f-alphabet decomposition of the iterated Eisenstein integrals on the right hand side of (15.9). To leading order in the coradical filtration $C_\bullet\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$, the coproduct on motivic iterated Eisenstein integrals in $\mathrm{Co}(\mu(\mathcal{C}^{\mathfrak{m}}_S))$ is deconcatenation [7, Theorem 22.2]. This can be written explicitly as follows: let $[E_{2n_1+2}(b_1)|\cdots|E_{2n_s+2}(b_s)] \in \mathcal{O}(\mathcal{U}^{\mathrm{geom}})$. Define an element $I^{b_1,\dots,b_s}_{2n_1+2,\dots,2n_s+2} \in \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ by

$$I^{b_1,\dots,b_s}_{2n_1+2,\dots,2n_s+2} := \int_S^{\mathfrak{m}} E_{2n_1+2}(b_1) \cdots E_{2n_s+2}(b_s).$$

We have

$$\begin{aligned}\Delta\left(I^{b_1,b_2}_{2n_1+2,2n_2+2}\right) \equiv\ & 1 \otimes I^{b_1,b_2}_{2n_1+2,2n_2+2} + I^{b_1,b_2}_{2n_1+2,2n_2+2} \otimes 1 \\ & + I^{b_1}_{2n_1+2} \otimes I^{b_2}_{2n_2+2} \pmod{C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}}.\end{aligned}$$

Note that I^b_{2n+2} is a rational multiple of:

$$\begin{cases} \mathbb{L}^{b-2n}\zeta^{\mathfrak{m}}(2n+1) \in C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}, & b = 0 \text{ or } 2n; \\ \mathbb{L}^{b+1} \in C_0\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}, & \text{otherwise.} \end{cases}$$

Therefore $\Delta'(I^{b_1,b_2}_{2n_1+2,2n_2+2}) \equiv 0 \pmod{C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}}$ unless (b_1,b_2) is one of $(0,0)$, $(0,2n_2)$, $(2n_1,0)$ or $(2n_1,2n_2)$. In any of these cases we have

$$\Delta'(\Phi(I^{b_1,b_2}_{2n_1+2,2n_2+2})) = (-1)^{\delta_{b_1}+\delta_{b_2}} \frac{(2n_1)!(2n_2)!}{4} \mathbb{L}^{b_1+b_2-2n_1-2n_2}(f_{2n_1+1} \otimes f_{2n_2+1}).$$

Thus the only iterated Eisenstein integrals that contribute nontrivially modulo $C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ to the sum on the right hand side of (15.9) are $I^{b_1,b_2}_{2n_1+2,2n_2+2}$ with $\{n_1,n_2\} = \{1,2\}$ and $b_i \in \{0, 2n_i\}$. As described in Lemma 15.2.2, the action of $S_0^{\mathfrak{m}}$ gives an equality $I^{b_1,b_2}_{6,4} = \mathbb{L}^{6-2b_1-2b_2} I^{2-b_2,4-b_1}_{4,6}$ so that we may fix $(n_1, n_2) = (2, 1)$. The sum in (15.9) therefore consists of at most 4 distinct integrals.

We may simplify further using the cocycle equations. Taking the coefficients of $\mathbf{e}_6\mathsf{X}_1^4\mathbf{e}_4\mathsf{X}_2^2$ and $\mathbf{e}_6\mathsf{X}_1^4\mathbf{e}_4\mathsf{Y}_2^2$ in (15.5) respectively produces the two equations

$$\mathbb{L}^6 I^{0,0}_{6,4} + I^{4,2}_{6,4} - 12\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5) = 0; \tag{15.12}$$

$$\mathbb{L}^4 I^{0,2}_{6,4} + \mathbb{L}^2 I^{4,0}_{6,4} + 12\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5) = 0. \tag{15.13}$$

Taking the coefficients of the same words in (15.7) produces the equations[4]

$$2\mathbb{L}^6 I^{0,0}_{6,4} + \mathbb{L}^4 I^{0,2}_{6,4} + \mathbb{L}^2 I^{4,0}_{6,4} + 2I^{4,2}_{6,4} - 12\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5) \equiv 0 \pmod{C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}}; \tag{15.14}$$

$$\mathbb{L}^6 I^{0,0}_{6,4} + 2\mathbb{L}^4 I^{0,2}_{6,4} + \mathbb{L}^2 I^{4,0}_{6,4} + I^{4,2}_{6,4} \equiv 0 \pmod{C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}}. \tag{15.15}$$

Equation (15.14) is implied by combining (15.12) with (15.13), which shows that it holds even without working modulo $C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$. Though this gives no interesting new information for this calculation, it does imply the existence of a further relation in $C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$. What is more useful is (15.15). Combining it with equations (15.12) and (15.13) implies that

$$I^{0,2}_{6,4} \equiv 0 \pmod{C_1\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}}. \tag{15.16}$$

The combination of (15.12), (15.13) and (15.16) together with the previous analysis imply that $\zeta^{\mathfrak{m}}(3,5)$ may be written as a $\mathbb{Q}$-linear combination

$$\zeta^{\mathfrak{m}}(3,5) \equiv A\mathbb{L}^6 I^{0,0}_{6,4} + B\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5) \pmod{\mathbb{Q}\mathbb{L}^8}, \quad A, B \in \mathbb{Q}. \tag{15.17}$$

Applying Lemma 15.2.2 to (15.17) gives

$$\begin{aligned} -5f_5f_3 &\equiv \Phi(\zeta^{\mathfrak{m}}(3,5)) \\ &\equiv \Phi(A\mathbb{L}^6 I^{0,0}_{6,4} + B\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)) \\ &= 12Af_5f_3 + Bf_3 ⧢ f_5 \\ &= (12A+B)f_5f_3 + Bf_3f_5 \pmod{\mathbb{Q}\mathbb{L}^8}. \end{aligned}$$

Comparing coefficients determines $A = -5/12$ and $B = 0$. We obtain

$$\zeta^{\mathfrak{m}}(3,5) = -\frac{5}{12}\mathbb{L}^6 \int_S^{\mathfrak{m}} E_6(0)E_4(0) + \lambda\mathbb{L}^8, \quad \text{for some } \lambda \in \mathbb{Q}.$$

[4]In these equations we have been able to replace the period ring $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ by $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$. This is because Theorem 12.0.1 supplies iterated Eisenstein integrals in $\mathrm{Co}(\mu(\mathcal{C}^{\mathfrak{m}}_S)) \subseteq \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$.

15.4.1 The general case and further questions

Theorem 12.0.1 implies that any $\zeta^{\mathfrak{m}}(w) \in \mathcal{Z}^{\mathfrak{m}}$ may be written as a linear combination of the form

$$\zeta^{\mathfrak{m}}(w) = \sum_i A_i \mathbb{L}^{m_i} \int_S^{\mathfrak{m}} v_i, \tag{15.18}$$

where $v_i \in \mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ and $A_i \in \mathbb{Q}$. Theorem 12.0.1 also places constraints on the elements v_i and values $m_i \in \mathbb{Z}$ in terms of the weight n and depth r of $\zeta^{\mathfrak{m}}(w)$.

The left hand side of (15.18) has an f-alphabet decomposition. In many cases (e.g. for the family of Hoffman MZVs [5]) this decomposition may be computed algorithmically up to a rational multiple of f_n [6], though in general this decomposition is not unique. However the left hand side has a coradical filtration and, in particular, the longest term in the f-alphabet decomposition with respect to the coradical filtration is canonically determined by this algorithm.

The right hand side also has an f-alphabet decomposition. The space of iterated Eisenstein integrals on the right hand side has a length filtration[5] with respect to which the longest part in the f-alphabet decomposition is canonically determined by Lemma 15.2.2. By comparing the f-alphabet decompositions of each side of (15.18) we may then read off the coefficients A_i of the leading order terms in the coradical or length filtrations.

By extending the formula given in Lemma 15.2.2 for the f-alphabet decomposition for motivic iterated Eisenstein integrals past the highest length term it may be possible to determine more of the coefficients A_i. In the most optimistic scenario all coefficients except that of $\zeta^{\mathfrak{m}}(n)$ may be determined by this method. This raises the following question:

Problem 15.4.4. *Is there a recursive procedure to completely decompose motivic iterated Eisenstein integrals in the f-alphabet in a similar way to the procedure given for motivic MZVs defined in [6]?*

A related area of potential future study is suggested by Theorem 13.1.1, which implies that the action of $\mathfrak{k} = \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MT}(\mathbb{Z})})$ on $\mathfrak{u}^{\mathrm{geom}}$ is faithful. The first few terms of this action with respect to the filtration $W_\bullet \, \mathfrak{u}^{\mathrm{geom}}$ have already been written down [9].

Problem 15.4.5. *Describe the action of $\mathfrak{k}$ on $\mathfrak{u}^{\mathrm{geom}}$ completely.*

[5]This coincides with the coradical filtration.

The proof of Theorem 12 implies that the words v_i appearing on the right hand side of (15.18) are actually elements of $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$, viewed as a Hopf subalgebra of $\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1})$. The motivic integral along S of such a word — that is, its value under the homomorphism $\mu(\mathcal{C}^{\mathfrak{m}}_S) \in \mathrm{Hom}(\mathcal{O}(\mathcal{U}^{\mathrm{geom}}), \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}})$ — is an element of $\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ because $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ is an ind-object of $\mathsf{MT}(\mathbb{Z})$. Applying the period map and [5] implies that $\int_S v_i = \mu(\mathcal{C}_S)(v_i) \in \mathcal{Z}[(2\pi i)^{\pm}] \subseteq \mathbb{C}$.

We expect that a form of converse holds as long as one considers integrals of not just a single word v_i but rather the whole (finite) subset of words obtained by varying the values of $b_1, \cdots, b_s$ within v_i in the allowed ranges $0 \le b_j \le 2n_j$. This may be formalised as follows.

Recall the definition of the free Eisenstein quotient[6] $\mathcal{U}^{\mathrm{dR}}_E$ of $\mathcal{U}^{\mathrm{dR}}_{1,1}$ from §5.5.1.2. Its coordiate ring is the shuffle algebra on the Eisenstein forms $E_{2n+2}(b)$, and contains $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ as a Hopf subalgebra. For $n_1, \dots, n_s > 0$ define a linear map

$$[E_{2n_1+2}|\dots|E_{2n_s+2}] \colon V^{\mathrm{dR}}_{2n_1} \otimes \cdots \otimes V^{\mathrm{dR}}_{2n_s} \to \mathcal{O}(\mathcal{U}^{\mathrm{dR}}_E)$$

by the formula

$$\mathsf{X}_1^{2n_1-b_1}\mathsf{Y}_1^{b_1} \otimes \cdots \otimes \mathsf{X}_s^{2n_s-b_s}\mathsf{Y}_s^{b_s} \mapsto [E_{2n_1+2}(b_1)|\dots|E_{2n_s+2}(b_s)].$$

It is SL_2-equivariant. Precomposing with any choice of SL_2-equivariant linear map $g \colon V^{\mathrm{dR}}_{2n} \to V^{\mathrm{dR}}_{2n_1} \otimes \cdots \otimes V^{\mathrm{dR}}_{2n_s}$ defines an SL_2-equivariant linear map

$$V^{\mathrm{dR}}_{2n} \xrightarrow{g} V^{\mathrm{dR}}_{2n_1} \otimes \cdots \otimes V^{\mathrm{dR}}_{2n_s} \xrightarrow{[E_{2n_1+2}|\dots|E_{2n_s+2}]} \mathcal{O}(\mathcal{U}^{\mathrm{dR}}_E). \tag{15.19}$$

and every element of $\mathrm{Hom}_{SL_2}(V^{\mathrm{dR}}_{2n}, \mathcal{O}(\mathcal{U}^{\mathrm{dR}}_E))$ is a $\mathbb{Q}$-linear combination of maps of the form (15.19). With this in mind we pose the following conjecture.

Conjecture 15.4.6. *Let $f \in \mathrm{Hom}_{SL_2}(V^{\mathrm{dR}}_{2n}, \mathcal{O}(\mathcal{U}^{\mathrm{dR}}_E))$. Then $\mathrm{im}(f) \subseteq \mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ if and only if $\mathrm{im}(\mathcal{C}_S \circ f) \subseteq \mathcal{Z}[(2\pi i)^{\pm}]$, where we view $\mathcal{C}_S \in \mathcal{U}^{\mathrm{dR}}_{1,1}(\mathbb{C}) = \mathrm{Hom}(\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_{1,1}), \mathbb{C})$.*

The implications of Conjecture 15.4.6 are somewhat subtle. For example, there are elements of $\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_E)$ that are (conjecturally) *not* contained in $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ whose image under $\mathcal{C}_S$ is contained in $\mathcal{Z}[(2\pi i)^{\pm}]$. Brown gives two numerical examples illustrating this in [11, Examples 7.3 and 7.5]. Their conjectural motivic versions are

$$\frac{\mathbb{L}^{10}}{2^5 \cdot 3^3 \cdot 5} I^{0,0}_{6,8} - \frac{3\mathbb{L}^4}{2^3 \cdot 691} I^{2,4}_{4,10} \overset{?}{=} \zeta^{\mathfrak{m}}_{5,7}, \tag{15.20}$$

$$\frac{\mathbb{L}^{10}}{2^6 \cdot 3^2 \cdot 5 \cdot 7} I^{0,0}_{4,10} + \frac{\mathbb{L}^4}{2^3 \cdot 691} I^{2,4}_{4,10} \overset{?}{=} \zeta^{\mathfrak{m}}_{3,9}. \tag{15.21}$$

[6]Note that $\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_E)$ is not a natural ind-object of $\mathcal{H}$.

Here $\zeta^{\mathfrak{m}}_{5,7}, \zeta^{\mathfrak{m}}_{3,9} \in C_2\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}$ are motivic MZVs whose f-alphabet decompositions are $f_5 f_7 \pmod{\mathbb{Q}\mathbb{L}^{12}}$ and $f_3 f_9 \pmod{\mathbb{Q}\mathbb{L}^{12}}$ respectively. Hence, they are only well-defined up to addition of a rational multiple of $\mathbb{L}^{12}$. Despite lying in coradical degree 2 they cannot be expressed by motivic MZVs of depth less than 4.

A verification of (15.20) and (15.21) would imply that the linear combinations

$$\frac{(2\pi i)^{10}}{2^5 \cdot 3^3 \cdot 5}[E_6(0)|E_8(0)] - \frac{3(2\pi i)^4}{2^3 \cdot 691}[E_4(2)|E_{10}(4)]$$
$$\frac{(2\pi i)^{10}}{2^6 \cdot 3^2 \cdot 5 \cdot 7}[E_4(0)|E_{10}(0)] + \frac{(2\pi i)^4}{2^3 \cdot 691}[E_4(2)|E_{10}(4)],$$

which are elements of $\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_E) \otimes_{\mathbb{Q}} \mathbb{Q}[2\pi i]$, evaluate to elements of $\mathcal{Z}[(2\pi i)^{\pm}]$ under $\mathcal{C}_S$. Conjecturally, however, the individual terms appearing in these linear combinations are not contained in $\mathcal{O}(\mathcal{U}^{\mathrm{geom}}) \otimes_{\mathbb{Q}} \mathbb{Q}[2\pi i]$ since their values under $\mathcal{C}_S$ should also involve the noncritical L-value $\Lambda(\Delta; 12)$ of the cusp form $\Delta \in S_{12}(SL_2(\mathbb{Z}))$ as well as a "new" period $c(\Delta; 12)$ [11, §7.2]; consequently, they are not mixed Tate. By Remark 15.2.3 the motivic versions of these periods associated to Δ must occur in $C_1\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$. They happen to cancel out in the linear combinations on the left hand sides of (15.20) and (15.21) to give the mixed Tate periods $\zeta^{\mathfrak{m}}_{5,7}$ and $\zeta^{\mathfrak{m}}_{3,9}$.

A further implication is that equations (15.20) and (15.21) are inaccessible from Theorem 12.0.1, which only makes use of periods of $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$. This is evidenced by the expressions (15.20) and (15.21) for the depth 4 motivic MZVs $\zeta^{\mathfrak{m}}_{5,7}$ and $\zeta^{\mathfrak{m}}_{3,9}$ as linear combinations of iterated Eisenstein integrals of length $s \leq 2$, while Theorem 12.0.1 only gives the weaker bound $s \leq 4$.

This suggests that it is possible to reduce the length s of the iterated Eisenstein integrals appearing on the right hand side of (15.18) substantially below the bound $s \leq \operatorname{depth} \zeta^{\mathfrak{m}}(w)$ afforded by Theorem 12.0.1 if we allow integrals of elements in the full space $\mathcal{O}(\mathcal{U}^{\mathrm{dR}}_E)$ rather than just the mixed Tate subspace $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$. In this larger space, correction terms in lower coradical degree may account for these "depth defects".

This also raises an interesting question about relations between iterated Eisenstein integrals. Theorem 12.0.1 implies that the depth 4 MZVs appearing on the right hand sides of (15.20) and (15.21) may be written as linear combinations of iterated Eisenstein integrals of length at most 4 that, moreover, come from $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$. Equating with the iterated Eisenstein integrals on the respective left hand sides produces relations between iterated Eisenstein integrals along S of potentially different lengths. It is natural to ask whether these relations may be proven using known relations (e.g. the cocycle relations). We therefore conclude with a final suggestion for future investigation:

Problem 15.4.7. *Consider the relations in $M^{\mathfrak{m}}$ arising as follows: let*

$$\sum_i \int_S^{\mathfrak{m}} w_i = \kappa^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})} = \mathcal{Z}^{\mathfrak{m}}[\mathbb{L}^{\pm}] \tag{15.22}$$

be a mixed Tate linear combination of motivic iterated Eisenstein integrals, where each $w_i \in \mathcal{O}(\mathcal{U}^{\mathrm{dR}}_E)$. Theorem 12.0.1 produces a new expression

$$\kappa^{\mathfrak{m}} = \sum_j \int_S^{\mathfrak{m}} v_j \tag{15.23}$$

where each $v_i \in \mathcal{O}(\mathcal{U}^{\mathrm{geom}})$. Equating (15.22) *and* (15.23) *produces the following relation between mixed Tate linear combinations of iterated Eisenstein integrals:*

$$\sum_i \int_S^{\mathfrak{m}} w_i = \sum_j \int_S^{\mathfrak{m}} v_j. \tag{15.24}$$

Does the relation (15.24) *have a geometric origin?*

Index of notation

$\pi_1^{\mathrm{rel,B}}(X,x)$ Relative Betti fundamental group, page 48

$\pi_1^{\mathrm{rel,dR}}(X,x)$ Relative de Rham fundamental group, page 48

${}_0\Pi_1^{\mathrm{mot}}$ Motivic path torsor $\pi_1^{\mathrm{mot}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{1}_0, -\vec{1}_1)$, page 39

$\mathcal{G}_{1,1}^{\mathrm{dR}}, \mathcal{G}_{1,\vec{1}}^{\mathrm{dR}}$ Relative de Rham fundamental groups of $(\mathcal{M}_{1,1}, \partial/\partial q)$ and $(\mathcal{M}_{1,\vec{1}}, \vec{\mathsf{v}})$, page 60

$\mathcal{U}_{1,1}^{\mathrm{dR}}, \mathcal{U}_{1,\vec{1}}^{\mathrm{dR}}$ Unipotent radicals of $\mathcal{G}_{1,1}^{\mathrm{dR}}, \mathcal{G}_{1,\vec{1}}^{\mathrm{dR}}$, page 60

$\mathcal{U}_{1,1}^{\mathrm{dR,hol}}$ Totally holomorphic quotient of $\mathcal{U}_{1,1}^{\mathrm{dR}}$, page 62

$\mathcal{U}_E^{\mathrm{dR}}$ Free Eisenstein quotient of $\mathcal{U}_{1,1}^{\mathrm{dR}}$, page 63

$\mathcal{U}^{\mathrm{geom}}$ Image of $\mathcal{U}_{1,1}^{\mathrm{dR}}$ under μ, page 89

$\mathsf{MT}(\mathbb{Z})$ Tannakian category of mixed Tate motives over $\mathbb{Z}$, page 31

$\mathcal{H}$ Tannakian category of generalised Hodge realisations, page 30

$G_{\mathsf{MT}(\mathbb{Z})}^{\mathrm{dR}}, G_{\mathcal{H}}^{\mathrm{dR}}$ De Rham motivic Galois groups, page 33

$U_{\mathsf{MT}(\mathbb{Z})}^{\mathrm{dR}}$ Unipotent radical of $G_{\mathsf{MT}(\mathbb{Z})}^{\mathrm{dR}}$, page 38

$\mathfrak{k}$ Lie algebra of $U_{\mathsf{MT}(\mathbb{Z})}^{\mathrm{dR}}$, page 38

$\mathfrak{u}_{1,1}$ Lie algebra of $\mathcal{U}_{1,1}^{\mathrm{dR}}$, page 60

μ The monodromy morphism, page 88

ϕ The Hain morphism, page 103

$\Phi_{jk}^{\mathfrak{m}}$ Motivic Drinfeld associator, page 58

$\alpha^{\mathfrak{m}}, \beta^{\mathfrak{m}}$ Power series in a, b, page 59

$\mathcal{C}^{\mathfrak{m}}$ Canonical motivic cocycle, page 66

Ψ The motivic modular inverter, page 101

$\gamma^{\mathfrak{m}}$ Image of fundamental group element γ in relative completion, page 49

$\gamma_u^{\mathfrak{m}}$ Unipotent part of $\gamma^{\mathfrak{m}}$, non-canonical, page 49

$\gamma_0^{\mathfrak{m}}$ Reductive/degree-0 part of $\gamma^{\mathfrak{m}}$, canonical, page 49

$S^{\mathfrak{m}}, T^{\mathfrak{m}}$ Images of $SL_2(\mathbb{Z})$ generators in relative completion, page 66

$S_0^{\mathfrak{m}}, T_0^{\mathfrak{m}}$ Degree-0 parts of $S^{\mathfrak{m}}, T^{\mathfrak{m}}$, page 65

Θ Commutator $\alpha\beta\alpha^{-1}\beta^{-1}$, page 86

$\mathcal{P}^{\mathfrak{m}}_{\mathsf{MT}(\mathbb{Z})}, \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ Rings of motivic periods, page 34

$\mathcal{P}^{\mathfrak{dr}}_{\mathsf{MT}(\mathbb{Z})}, \mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}}$ Rings of de Rham periods, page 33

$\mathcal{Z}^{\mathfrak{m}}$ Ring of motivic multiple zeta values, page 39

$M^{\mathfrak{m}}$ Ring of motivic multiple modular values, page 66

$\mathfrak{W}_\bullet \mathcal{Z}^{\mathfrak{m}}$ Weight filtration on motivic MZVs, page 81

$\mathfrak{D}_\bullet \mathcal{Z}^{\mathfrak{m}}$ Depth filtration on motivic MZVs, page 81

$C_\bullet \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ Coradical filtration on $\mathcal{H}$-periods, page 131

$\mathbb{L}$ Lefschetz period, equal to $(2\pi i)^{\mathfrak{m}}$, page 39

$\zeta^{\mathfrak{m}}(w)$ Motivic multiple zeta value, page 40

$\int_\gamma^{\mathfrak{m}} E_{2n_1+2}(b_1)\cdots E_{2n_s+2}(b_s)$ Motivic iterated Eisenstein integral, page 73

η Motivic period $\int_{\tilde{S}}^{\mathfrak{m}} E_2(0)$, equal to $\mathbb{L}/8$ by Corollary 14.4.2, page 101

$\mathrm{Fil}^r_F(s)$ rth filtered piece of series s, page 76

$\mathrm{Co}^F_\bullet(s)$ Filtration on $\mathrm{Co}(s)$ induced by F, page 80

$\mathsf{X}_{\mathrm{B}}, \mathsf{Y}_{\mathrm{B}}$ Generators for standard representation of SL_2^{B}, page 65

X, Y Generators for standard representation of SL_2^{dR}, page 65

References

[1] A. Beilinson. Higher regulators and values of L-functions. *Journal of Soviet Mathematics*, 30(2):2036–2070, 1985.

[2] G. V. Belyi. On Galois extensions of a maximal cyclotomic field. *Mathematics of the USSR-Izvestiya*, 14(2):247–256, 1980.

[3] B. C. Berndt and A. Straub. Ramanujan's formula for $\zeta(2n+1)$. In H. Montgomery, A. Nikeghbali, and M. T. Rassias, editors, *Exploring the Riemann Zeta Function*, pages 13–34, Springer, Cham, 2017.

[4] J. B. Bost and F. Charles. Some remarks concerning the Grothendieck period conjecture. *Journal für die reine und angewandte Mathematik*, 2016(714), 2016.

[5] F. Brown. Mixed Tate motives over $\mathbb{Z}$. *Annals of Mathematics*, 175(2):949–976, 2012.

[6] F. Brown. On the decomposition of motivic multiple zeta values. In H. Nakamura, F. Pop, L. Schneps, and A. Tamagawa, editors, *Galois–Teichmüller Theory and Arithmetic Geometry*, pages 31–58, Mathematical Society of Japan, Tokyo, Japan, 2012.

[7] F. Brown. Multiple modular values and the relative completion of the fundamental group of $\mathcal{M}_{1,1}$. *arXiv:1407.5167v4*, 2017.

[8] F. Brown. Notes on motivic periods. *Communications in Number Theory and Physics*, 11(3):557–655, 2017.

[9] F. Brown. Zeta elements in depth 3 and the fundamental Lie algebra of the infinitesimal Tate curve. *Forum of Mathematics, Sigma*, 5:e1, 2017.

[10] F. Brown. A multi-variable version of the completed Riemann zeta function and other L-functions. *arXiv:1904.00190*, 2019.

[11] F. Brown. From the Deligne-Ihara conjecture to multiple modular values. *arXiv:1904.00179*, 2019.

[12] F. Brown and R. Hain. Algebraic de Rham theory for weakly holomorphic modular forms of level one. *Algebra & Number Theory*, 12(3):723–750, 2018.

[13] J. I. Burgos Gil and J. Fresán. *Multiple zeta values: from numbers to motives.* Clay Mathematics Proceedings. (to appear).

[14] D. Calaque, B. Enriquez, and P. Etingof. Universal KZB equations: the elliptic case. In Y. Tschinkel and Y. Zarhin, editors, *Algebra, Arithmetic, and Geometry: Volume I: In Honor of Yu. I. Manin*, pages 165–266, Birkhäuser Boston, 2009.

[15] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In Y. Ihara, K. Ribet, and J-P. Serre, editors, *Galois Groups over* $\mathbb{Q}$, pages 79–297, Springer, New York, 1989.

[16] P. Deligne and A. B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. *Annales scientifiques de l'École Normale Supérieure*, 4e série, 38(1):1–56, 2005.

[17] P. Deligne and J. S. Milne. Tannakian categories. In *Hodge Cycles, Motives, and Shimura Varieties*, pages 101–228, Springer, Berlin, Heidelberg, 1982.

[18] V. G. Drinfel'd. Quasitriangular quasi-Hopf algebras and a group closely connected to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *Algebra i Analiz*, 2(4):149–181, 1990.

[19] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 29(1):95–103, 1966.

[20] R. Hain. The geometry of the mixed Hodge structure on the fundamental group. *Proc. Symp. Pure Math*, 46(2):247–282.

[21] R. Hain. The Hodge de Rham theory of relative Malcev completion. *Annales scientifiques de l'École Normale Supérieure*, Ser. 4, 31(1):47–92, 1998.

[22] R. Hain. Lectures on moduli spaces of elliptic curves. In L. Ji and S-T. Yau, editors, *Transformation Groups and Moduli Spaces of Curves*, volume 16 of *Advanced Lectures in Mathematics*, pages 95–166. Higher Education Press, 2010, 2010.

[23] R. Hain. The Hodge-de Rham theory of modular groups. In M. Kerr and G. Pearlstein, editors, *Recent Advances in Hodge Theory: Period Domains, Algebraic Cycles, and Arithmetic*, London Mathematical Society Lecture Note Series, page 422–514. Cambridge University Press, 2016.

[24] R. Hain. Action of $\Gamma_{1,\vec{1}}$ on $\pi_1(E'_{\partial/\partial q})$. (handwritten notes), 2017.

[25] R. Hain. Personal communication. (email), 2019.

[26] R. Hain. Notes on the universal elliptic KZB connection. *Pure and Applied Mathematics Quarterly*, 16(2):229–312, 2020.

[27] R. Hain and M. Matsumoto. Universal mixed elliptic motives. *Journal of the Institute of Mathematics of Jussieu*, 19(3):663–766, 2020.

[28] K. Ihara, M. Kaneko, and D. Zagier. Derivation and double shuffle relations for multiple zeta values. *Compositio Mathematica*, 142(2):307–338, 2006.

[29] Y. Ihara. Some arithmetic aspects of Galois actions on the pro-p fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. In M. D. Fried and Y. Ihara, editors, *Arithmetic Fundamental Groups and Noncommutative Algebra*, volume 70 of *Proc. Symp. Pure Math.*, pages 247–273, 2002.

[30] N. M. Katz and B. Mazur. *Arithmetic Moduli of Elliptic Curves*. Number 108 in Annals of Mathematics Studies. Princeton University Press, 1985.

[31] M. Kim. Galois theory and Diophantine geometry. In J. Coates, M. Kim, F. Pop, M. Saïdi, and P. Schneider, editors, *Non-abelian Fundamental Groups and Iwasawa Theory*, London Mathematical Society Lecture Note Series, page 162–187. Cambridge University Press, 2011.

[32] S. L. Kleiman. Algebraic cycles and the Weil conjectures. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 359–386. North-Holland, Amsterdam, 1968.

[33] V. G. Knizhnik and A. B. Zamolodchikov. Current algebra and Wess-Zumino model in two-dimensions. *Nucl. Phys.*, B247:83–103, 1984.

[34] M. Kontsevich and D. Zagier. Periods. In B. Engquist and W. Schmid, editors, *Mathematics Unlimited — 2001 and Beyond*, pages 771–808. Springer, Berlin, Heidelberg, 2001.

[35] M. Levine. Tate Motives and the Vanishing Conjectures for Algebraic K-Theory. In P. G. Goerss and J. F. Jardine, editors, *Algebraic K-Theory and Algebraic Topology*, pages 167–188, Springer Netherlands, Dordrecht, 1993.

[36] M. Levine. Mixed motives. In E. M. Friedlander and D. R. Grayson, editors, *Handbook of K-Theory*, pages 429–521, Springer, Berlin, Heidelberg, 2005.

[37] P. Lochak, N. Matthes, and L. Schneps. Elliptic multizetas and the elliptic double shuffle relations. *arXiv:1703.09410*, 2017.

[38] N. Matthes. On the algebraic structure of iterated integrals of quasimodular forms. *Algebra & Number Theory*, 11(9):2113–2130, 2017.

[39] N. Matthes. The canonical cocycle of E_2. (unpublished notes), 2017.

[40] N. Matthes. The meta-abelian elliptic KZB associator and periods of Eisenstein series. *Selecta Mathematica*, 24(4):3217–3239, 2018.

[41] H. Nakamura. Tangential basepoints and Eisenstein power series. In H Völklein, D Harbater, P. Müller, and J. Thompson, editors, *Aspects of Galois Theory*, London Mathematcial Society Lecture Note Series, pages 202–217. Cambridge University Press, 1999.

[42] H. Nakamura. On profinite Eisenstein periods in the monodromy of universal elliptic curves. Available at: `http://www4.math.sci.osaka-u.ac.jp/~nakamura/zoo/fox/EisenRevisited.pdf`, 2016.

[43] T. Oda. The universal monodromy representations on the pro-nilpotent fundamental groups of algebraic curves. In *Mathematische Arbeitstagung (Neue Serie)*. Max-Planck Institute preprint, 1993.

[44] A. Pollack. Relations between derivations arising from modular forms. Honors thesis, Duke University, 2009.

[45] D. Quillen. Rational Homotopy Theory. *Annals of Mathematics*, 90(2):205–295, 1969.

[46] C. Reutenauer. *Free Lie Algebras*. LMS monographs. Clarendon Press, 1993.

[47] A. Scholl. Motives for modular forms. *Inventiones mathematicae*, 100:419–430, 1990.

[48] J-P. Serre. *Trees*. Springer-Verlag, Berlin, 1980.

[49] J. H. Silverman. *Advanced topics in the arithmetic of elliptic curves*. Graduate Texts in Mathematics. Springer-Verlag, New York, 1994.

[50] J. H. Silverman. *The Arithmetic of Elliptic Curves*. Graduate Texts in Mathematics. Springer-Verlag, New York, 2009.

[51] N. Takao. Braid monodromies on proper curves and pro-ℓ Galois representations. *Journal of the Institute of Mathematics of Jussieu*, 11(1):161–188, 2012.

[52] H. Tsunogai. On some derivations of Lie algebras related to Galois representations. *Publications of the Research Institute for Mathematical Sciences*, 31:113–134, 1995.

[53] V. Voevodsky, A. Suslin, and E. M. Friedlander. *Cycles, Transfers, and Motivic Homology Theories. (AM-143)*. Princeton University Press, 2000.

[54] Don Zagier. *Elliptic Modular Forms and Their Applications*, pages 1–103. Springer, Berlin, Heidelberg, 2008.