Non-isotopic Seifert surfaces in the 4-ball and new computational tools for Khovanov homology



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Abstract

One of the central unsolved questions in low-dimensional topology is whether there exists an exotic smooth structure on S^4. A related problem, called the unknotting conjecture for surfaces, asks if there exists an embedded, closed, oriented surface in S^4 that is exotically knotted. This thesis originates from investigating a special case of the conjecture involving Seifert surfaces. In Chapter 1, we review the background and discuss our results on this special case. The subsequent work branches into three distinct directions, unified by a recurring theme of using algorithms in algebra to prove topological theorems.

In Chapter 2, we extend the known examples of Seifert surfaces that are non-isotopic in D^4. We distinguish these genus one surfaces topologically by examining their symmetrized Seifert forms. This leads to Conway's theory of topographs and our implementation of Conway's algorithm to distinguish integral binary quadratic forms up to isomorphism over $\mathbb{Z}$.

In Chapter 3, we describe a computer program that we developed to calculate cobordism maps induced on Khovanov homology, filling a notable gap in existing tools. We present a few applications, including the calculation of cobordism maps for Seifert surfaces for all prime knots up to 10 crossings and the smooth distinction of ribbon disks obtained through symmetries.

Finally, Chapter 4 explores 3-dimensional handlebodies with a subsurface on their boundary and examines when two such handlebodies can be glued along a homeomorphism of the subsurface to form another handlebody. This study leads to an algebraic analogue about amalgamating free groups and to the algorithm of Diao–Feighn.

Introduction

Surfaces in 4-manifolds

One of the motivating questions of this thesis is the unknotting conjecture for surfaces, which asks whether a closed, oriented surface $\Sigma \subseteq S^4$ with $\pi_1(S^4 \setminus \Sigma) = \mathbb{Z}$ must be smoothly unknotted (that is, smoothly isotopic to the standard embedding).

In the first chapter, we investigate a special case of this conjecture arising from Seifert surfaces. Let $S_0, S_1 \subseteq S^3$ be Seifert surfaces for an oriented link L, and push the interior of S_0 slightly into D^4 to obtain a surface S_0^+. Then $\Sigma = S_0^+ \cup S_1$ is a closed, embedded surface with $\pi_1(S^4 \setminus \Sigma) = \mathbb{Z}$ (Proposition 1.4). This naturally raises the question: is Σ smoothly unknotted?

We show that Σ is indeed smoothly unknotted if S_0 and S_1 have disjoint interiors, or more generally, if they intersect transversely 'in segments' (Proposition 1.7). The result also holds if S_0 can be smoothly isotoped in D^4, relative to its boundary, to a surface $S_0' \subseteq S^3$ intersecting S_1 in segments. Using this, we verify the conjecture for several classes of links, including fibred links, 2-bridge knots, certain pretzel links, and all knots with at most 10 crossings. These cases are treated by reducing to incompressible Seifert surfaces (Lemma 1.6) and applying classification results specific to each link family. In each instance, we find that all incompressible Seifert surfaces are isotopic in D^4.

This leads to a broader question of whether non-isotopic Seifert surfaces can exist. This is the focus of the next chapter.

Non-isotopic Seifert surfaces in the 4-ball

In 1982, Livingston [Liv82] asked whether there exists a knot with two Seifert surfaces of the same genus that are not isotopic in D^4. This was answered positively in 2022 by Hayden–Kim–Miller–Park–Sundberg [HKM+22]. Their main example (based on Lyon's construction [Lyo74b]) comes from attaching a band to two annuli bounded by the torus

link $T_{4,-6}$. They showed that the symmetrized Seifert form obstructs these surfaces from being topologically isotopic, and that the induced cobordism maps in Khovanov homology obstructs smooth isotopy.

Here we study when the symmetrized Seifert form can be used to distinguish Seifert surfaces that are modifications of this example, which can be denoted by $\Sigma_i(2,-3,0,-1)$ in our 4-parameter family.

In Section 2.1, we define a family of pairs of Seifert surfaces $\Sigma_i(p,q,k,n)$, $i=0,1$, and prove that they are topologically non-isotopic in D^4 under certain conditions on (p,q,k,n) (Theorems 2.2 and 2.4). We also see many instances where the surfaces in this family are topologically isotopic (Theorem 2.3). A related, more elegant family of Seifert surfaces that are bounded by 2-component links is given in Theorems 2.1 and 2.5. These results extend the currently known examples of non-isotopic Seifert surfaces.

In Section 2.2, we summarize Conway's theory of topographs, a diagrammatic way to view quadratic forms in two variables, and give an algorithm that decides whether two integral binary quadratic forms are isomorphic over $\mathbb{Z}$. We use this algorithm to find further examples of non-isotopic surface-pairs $\Sigma_i(p,q,k,n)$ that are not covered by our previous theorems. Furthermore, the structure of the topographs also explains the various conditions in our theorems.

Khovanov homology

Khovanov homology assigns an algebraic object $\mathrm{Kh}(L)$ to every oriented link $L \subseteq S^3$. Computer programs [KJob, Sage] are available to calculate this invariant, which can be used to distinguish links.

To any smoothly embedded surface $\Sigma \subseteq S^3 \times [0,1]$ between two links L_0 and L_1, Khovanov homology assigns a map $\mathrm{Kh}(\Sigma)\colon \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1)$. These maps provide a method to distinguish surfaces up to smooth isotopy. However, despite the combinatorial nature of their definition, no computational tool was available for calculating these maps. We developed a program to compute them, as detailed in Section 3.2. This Python module is available on GitHub [KhCC].

As applications of our program, we compute these maps for all incompressible Seifert surfaces for prime knots up to 10 crossings (Table 3.5), and distinguish many ribbon disks arising from symmetries of the boundary knots (Table 3.6). For instance, we present five distinct ribbon disks for the knot 10_{123}, obtained via a rotational symmetry, which are distinguished by Khovanov homology (Figure 3.1).

Gluing handlebodies and amalgamating free groups

The final chapter investigates a question that arose from the unknotting conjecture in Section 1.2.4, but which is also of independent interest. Let M_0, M_1 be handlebodies, and $S_0 \subseteq \partial M_0$, $S_1 \subseteq \partial M_1$ be homeomorphic compact surfaces with one boundary component. The question is: does there exist a homeomorphism $\phi\colon S_0 \to S_1$ such that the gluing $M_0 \cup_\phi M_1$ is a handlebody?

In Section 4.1, we use topological methods to establish a sufficient condition (Theorem 4.2) and a necessary condition (Theorem 4.3) for a positive answer. We also provide a concrete counterexample in Proposition 4.6 using a clever application of Whitehead's cut-vertex lemma.

In Section 4.2, we approach the problem from an algebraic perspective by examining an algebraic analogue about amalgamating free groups into a free group (Question 6). We derive necessary conditions for a positive answer to Question 6 (Theorems 4.9 and 4.16), and hence, to Question 5. This approach involves the theory of Gersten representatives, graphs of groups, and the algorithm of Diao–Feighn. Remarkably, this algebraic method culminates in a result (Proposition 4.17) that is essentially equivalent to the topological condition in Theorem 4.3, suggesting a deeper connection between the two perspectives.

Chapter 1

Surfaces in 4-manifolds

1.1 Exotic 4-manifolds

Two smooth manifolds are called *exotic* if they are homeomorphic but not diffeomorphic. Many examples of exotic smooth structures on 4-manifolds are known (e.g. on $\mathbb{R}^4$ or $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$ for $2 \leq k \leq 9$); moreover, infinite families of pairwise exotic structures have also been found in all these cases. While topological manifolds of dimension $n \leq 3$ always have a unique smooth structure [Moi52, HM74], and closed manifolds of dimension $n \geq 5$ have only finitely many smooth structures [KS77], it is conjectured that a 4-manifold either admits zero or infinitely many smooth structures [Sco05]. It is not known whether there are exotic smooth structures on some of the simplest 4-manifolds such as S^4, $S^2 \times S^2$, $\mathbb{C}P^2$, or $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. The case of S^4 is also known as the smooth 4-dimensional Poincaré conjecture.

1.1.1 Exotic embeddings and exotic knottings

Two surfaces smoothly embedded in a 4-manifold are called *exotically embedded* if they are ambiently homeomorphic but not ambiently diffeomorphic. Two surfaces smoothly embedded in a 4-manifold are called *exotically knotted* if they are topologically ambiently isotopic but not smoothly isotopic.

Understanding exotically knotted surfaces in 4-manifolds is of great interest since they are closely related to exotic structures on 4-manifolds (e.g. Gluck twist along an exotically embedded S^2 in S^4 might yield an exotic S^4). There are many examples of exotically embedded or exotically knotted surfaces in closed oriented 4-manifolds (see a list e.g. in [BS16]). There also exist exotically knotted 2-spheres in a 4-manifold which are ambiently diffeomorphic [Sch19], giving an example of an exotic knotting which is not an exotic

embedding. There are exotic embeddings of $\#k\mathbb{R}P^2$ in S^4 for all $k \geq 5$ [Fin09, MÖR+24] and $k = 1$ [Miy23], but there are no known oriented surfaces in S^4 that are exotically embedded (or knotted). An often used tactic to show non-diffeomorphism of surfaces is to take their double branched covers, and then use a smooth 4-manifold invariant (Donaldson's invariants, Seiberg–Witten invariants) to distinguish them.

We can also consider surfaces with boundary, properly embedded in a 4-manifold with boundary, and define exotic embeddings and knottings in the same way (or, for a slightly different notion, by requiring that the automorphisms and isotopies fix boundaries). There are examples of exotically knotted slice surfaces of any genus in D^4 or in other 4-manifolds with boundary [Hay21].

1.1.2 Stabilizations

A *stabilization* of a 4-manifold X is $X\#(S^2 \times S^2)$. A *stabilization* of a surface embedded in a 4-manifold means removing $S^0 \times D^2$ and gluing $D^1 \times S^1$ to the surface, according to a 1-handle $D^1 \times D^2$ embedded in the 4-manifold. A *trivial stabilization* (also known as internal stabilization) of a surface means taking connected sum with a small unknotted torus in a small ball neighbourhood of a point on the surface.

Gompf [Gom84] proved that any exotic pair of compact, oriented 4-manifolds becomes diffeomorphic after finitely many stabilizations (see also Wall [Wal64] in the simply connected case). Perron [Per86] and Quinn [Qui86] showed that exotically knotted surfaces become smoothly isotopic after finitely many stabilizations of the 4-manifold. Baykur–Sunukjian [BS16] showed that exotically knotted surfaces $\Sigma, \Sigma' \subseteq X$ become smoothly isotopic after finitely many stabilizations of the surfaces (moreover, trivial stabilizations under certain conditions, e.g. if $\pi_1(X \setminus \Sigma)$ is cyclic).

It has been a long-standing open question whether one stabilization is enough in these results. Baykur–Sunukjian [BS13] showed that all currently known infinite families of exotic 4-manifolds become diffeomorphic after taking one connected sum with $S^2 \times S^2$ or $S^2 \mathbin{\widetilde{\times}} S^2 = \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. They proved in [BS16] that most known constructions of exotically knotted surfaces in a 4-manifold become smoothly isotopic after one trivial stabilization of the surfaces. Auckly–Kim–Melvin–Ruberman–Schwartz [AKM+19] proved that exotically knotted surfaces $\Sigma, \Sigma' \subseteq X$ with $\pi_1(X \setminus \Sigma) = 1$ become smoothly isotopic in $X\#(S^2 \times S^2)$ or in $X\#(S^2 \mathbin{\widetilde{\times}} S^2)$.

However, Lin–Mukherjee [LM21] recently gave a negative answer in the non-closed case: they showed that there exist exotically knotted slice disks in a punctured K3 which do not

become smoothly isotopic after one stabilization of the 4-manifold. Juhász–Zemke [JZ24] also obtained lower bounds on the number of surface stabilizations needed for certain (non-exotically knotted) slice disks in D^4 to become smoothly isotopic.

1.2 Unknotting conjecture

The knot $K \subseteq S^3$ is called the *unknot* if it bounds an embedded disk in S^3. Note that K is the unknot if and only if K can be isotoped to lie entirely in S^2. This idea provides an analogy that we use to define unknotted surfaces in one dimension higher.

Let $\Sigma \subseteq S^4$ be a closed, oriented, connected, genus g surface that is locally flatly embedded. Σ is called *topologically unknotted* if it bounds a locally flatly embedded handlebody in S^4. By a genus g *handlebody*, we mean the oriented 3-manifold U_g that is obtained from the 3-ball by attaching g 1-handles. Let $\Sigma \subseteq S^4$ be a closed, oriented, connected, genus g surface that is smoothly embedded. Σ is called *smoothly unknotted* if it bounds a smoothly embedded handlebody in S^4.

An isotopy $\varphi\colon N \times I \to M$ is called a *topological isotopy* if $(\varphi, id_I)\colon N \times I \to M \times I$ is a locally flat embedding.

If $S \subseteq S^3$ is a surface (possibly with boundary), then a *compressing disk* for S is a disk $D \subseteq S^3$ such that $D \cap S = \partial D$ and ∂D does not bound a disk in S. S is called *compressible* if it has a compressing disk; otherwise, it is called *incompressible*. If D is a compressing disk for S, then we may *compress* S along D to get another surface. This is obtained by removing a small neighbourhood of ∂D in S and then attaching two parallel copies of D to the resulting boundary.

The following two lemmas are classical results that establish the equivalent characterizations for a surface being unknotted, analogous to the case of knots.

Lemma 1.1. *Let $\Sigma \subseteq S^3$ be a closed, oriented (not necessarily connected) surface.*

(a) *If Σ is locally flatly embedded, then Σ bounds a locally flatly embedded disjoint union of handlebodies in D^4.*

(b) *If Σ is smoothly embedded, then Σ bounds a smoothly embedded disjoint union of handlebodies in D^4.*

Proof. (a) Note that in dimension 3, any locally flatly embedded surface is topologically isotopic to a smoothly embedded one. Hence, (a) follows trivially from (b).

(b) By the Loop Theorem, if $\Sigma \subseteq S^3$ has a non-sphere component, then Σ is compressible [Hem76, Lemma 6.1]. The trace of a compression gives a 3-dimensional cobordism

$M_0 \subseteq S^3 \times [0,1]$ between Σ and some surface Σ_1 (the compressed Σ) with $\chi(\Sigma_1) > \chi(\Sigma)$. If $\Sigma_1 \subseteq S^3 \times \{1\}$ has a non-sphere component, it is again compressible, giving a cobordism $M_1 \subseteq S^3 \times [1,2]$ between Σ_1 and some Σ_2. We continue until we get a surface Σ_k that is a union of spheres, where we take $M_k \subseteq S^3 \times [k, k+1]$ to be a union of balls that is bounded by Σ_k. Then $M_0 \cup \cdots \cup M_k$ (smoothed out) is an embedded handlebody in $S^3 \times [0, k+1] \subseteq D^4$ bounded by Σ. □

Lemma 1.2. (a) *A locally flatly embedded, closed, oriented, connected surface $\Sigma \subseteq S^4$ is topologically unknotted if and only if it can be topologically isotoped to lie entirely in S^3.*

(b) *A smoothly embedded, closed, oriented, connected surface $\Sigma \subseteq S^4$ is smoothly unknotted if and only if it can be smoothly isotoped to lie entirely in S^3.*

Proof of (b). If Σ can be isotoped into S^3, then Σ is smoothly unknotted by Lemma 1.1.

For the reverse direction, we will show that any embedding of a handlebody is isotopic to the standard embedding. Assume that Σ bounds a smoothly embedded genus g handlebody $h\colon U_g \to S^4$. Then U_g deformation retracts to its core R, a wedge of g circles, by some homotopy φ_t that is an isotopy for $0 \leq t < 1$. As $h(R) \subseteq S^4$ is 1-dimensional, there is an ambient isotopy ψ of S^4 that sends $h(R)$ to a standard embedding $Q \subseteq S^3$. By considering $h(\varphi_t(\partial U_g))$ for $t \in [0, 1-\varepsilon]$ and then ψ, we get an isotopy from Σ to the boundary of a regular 3-dimensional neighbourhood N of Q. Thus, it remains to show that the framing of N can be isotoped to the standard framing.

N can be viewed as g 1-handles C_i attached to a small ball B. The centre of B has a 3-dimensional tangent subspace $V \subseteq \mathbb{R}^4$, which is isotopic to the standard $\mathbb{R}^3 \subseteq \mathbb{R}^4$, hence B can be assumed to lie in S^3. Let c_i be the core of C_i, then the normal bundle of c_i is trivial, so we can write $\mathbb{R}^3$ for its fibres. The tangent spaces of C_i along c_i can be viewed as a family of 2-dimensional oriented subspaces $V_t \subseteq \mathbb{R}^3$ ($t \in [0,1]$). Consider the normal vector n_t of V_t, giving a path n in S^2. Let $N' \subseteq S^3$ be the standard neighbourhood of Q, and n' be obtained similarly from C_i' along c_i. Since $\pi_1(S^2) = 1$, n is isotopic to n' relative to the boundary. As C_i and C_i' were both regular neighbourhoods of c_i, this means that N and N' are isotopic to each other. Hence, Σ is smoothly isotopic to $\partial N' \subseteq S^3$. □

See also [HK79] for a proof of this lemma in the piecewise-linear category.

Theorem 1.3 (Freedman–Quinn [FQ90], Conway–Powell [CP23])**.** *Let $g \neq 1, 2$. A locally flatly embedded, connected, closed, genus g surface $\Sigma \subseteq S^4$ is topologically unknotted if and only if $\pi_1(S^4 \setminus \Sigma) \cong \mathbb{Z}$.*

Conjecture 1 (Unknotting conjecture for surfaces)**.** *A smoothly embedded, connected, closed, genus g surface $\Sigma \subseteq S^4$ is smoothly unknotted if and only if $\pi_1(S^4 \setminus \Sigma) \cong \mathbb{Z}$.*

By the proof of Lemma 1.2, there is only one embedding of a handlebody in S^4 up to isotopy. Therefore, an unknotted surface is isotopic to the standard embedding, whether in the locally flat or smooth category. Hence, the forward direction of Conjecture 1 is clear. Furthermore, by Theorem 1.3, Conjecture 1 is equivalent (for $g \neq 1, 2$) to the question: is there a genus g surface $\Sigma \subseteq S^4$ that is exotically knotted with the standard embedding?

1.2.1 Unknotting conjecture for two Seifert surfaces

A *spanning surface* for an oriented link $L \subseteq S^3$ is a compact, oriented, smoothly embedded surface $S \subseteq S^3$ with no closed components and oriented boundary $\partial S = L$. (Links are always assumed to be smooth.) A *Seifert surface* for L is a connected spanning surface for L.

We write $S^4 = D^4_+ \cup D^4_-$ for the two hemispheres that the 3-sphere divides the 4-sphere into. By slightly pushing the interior of S into $D^4_\pm$, we get a surface $S^\pm$. The main question that motivated this thesis is the following special case of the unknotting conjecture:

Conjecture 2 (Mark Powell)**.** *Let $L \subseteq S^3$ be an oriented link, and let $S_0, S_1 \subseteq S^3$ be two Seifert surfaces for L. Let $\Sigma \subseteq S^4$ be the union of S_0^+ and S_1^- (smoothed out along L). Then Σ is smoothly unknotted.*

We denote this surface by $\Sigma(S_0, S_1)$, which can also be viewed as

$$\Sigma(S_0, S_1) = (S_0 \times \{0\}) \cup (L \times [0, 1]) \cup (S_1 \times \{1\}).$$

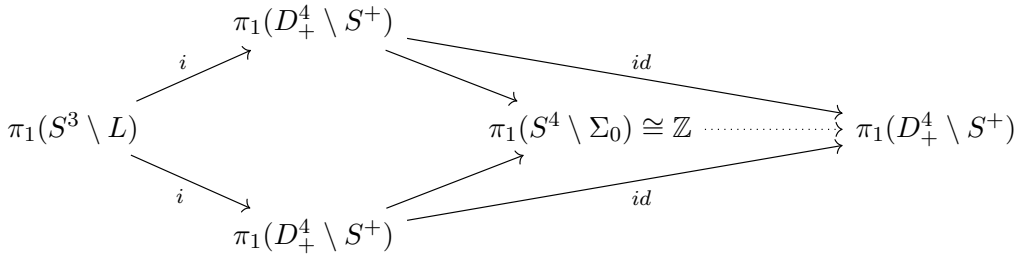
We now calculate the fundamental group of the complement of such a surface, following the approach in [Che21, Lemma 3.1].

Proposition 1.4. (a) *For any Seifert surface $S \subseteq S^3$, $\pi_1(D^4_+ \setminus S^+) \cong \mathbb{Z}$.*

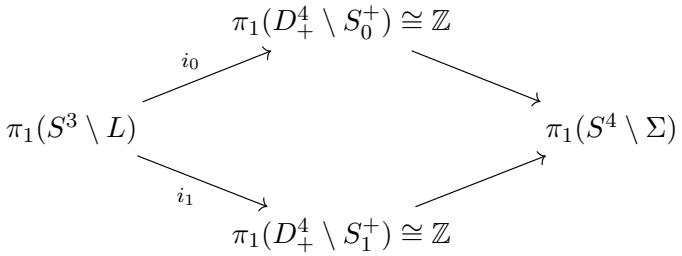
(b) *Let $S_0, S_1 \subseteq S^3$ be two Seifert surfaces for the same link, and $\Sigma = \Sigma(S_0, S_1)$. Then $\pi_1(S^4 \setminus \Sigma) \cong \mathbb{Z}$.*

Proof. (a) Consider $\Sigma_0 = \Sigma(S, S)$, and let $i\colon \pi_1(S^3 \setminus L) \to \pi_1(D^4_+ \setminus S^+)$ be the map induced by inclusion. By the Seifert–van Kampen theorem, $\pi_1(S^4 \setminus \Sigma_0)$ is obtained from the pushout diagram below. By the universal property of pushouts, there exists a surjective map $\pi_1(S^4 \setminus \Sigma_0) \to \pi_1(D^4_+ \setminus S^+)$. Since Σ_0 bounds a handlebody (namely $S \times I$), it is isotopic to the standard embedding. Hence, $\pi_1(S^4 \setminus \Sigma_0) \cong \mathbb{Z}$, implying that $\pi_1(D^4_+ \setminus S^+)$ is cyclic.

By Alexander duality, $H_1(D^4_+ \setminus S^+) \cong H^2(S^+ \cup D^4_-) \cong \mathbb{Z}$, and therefore, $\pi_1(D^4_+ \setminus S^+) \cong \mathbb{Z}$. Note also that i must be surjective, as otherwise the pushout could not be $\mathbb{Z}$.



(b) If $\Sigma = \Sigma(S_0, S_1)$, then by the Seifert–van Kampen theorem and using part (a), we obtain the following pushout diagram:



Since i_0 is surjective, this means that $\pi_1(S^4 \setminus \Sigma)$ is cyclic. By Alexander duality, $H_1(S^4 \setminus \Sigma) \cong H^2(\Sigma) \cong \mathbb{Z}$, and therefore, $\pi_1(S^4 \setminus \Sigma) \cong \mathbb{Z}$. □

By part (b) of this proposition, Conjecture 2 is indeed a special case of Conjecture 1. Note, however, that if Conjecture 1 is true, then any closed surface Σ with $\pi_1(S^4 \setminus \Sigma) \cong \mathbb{Z}$ is smoothly unknotted, and can therefore be smoothly isotoped into a surface $\Sigma' \subseteq S^3$ by Lemma 1.2. By taking a separating curve on Σ' that divides Σ' into the Seifert surfaces S_0 and S_1, we see that Σ is isotopic to $\Sigma(S_0, S_1)$. This provides good reason to believe that Conjecture 2 is not, in fact, a restrictive special case.

1.2.2 Isotopy in D^4

Let $S, S' \subseteq S^3$ be two surfaces. We write $S \approx_3 S'$ if they are smoothly isotopic in S^3 relative to the boundary, and $S \approx_4 S'$ if they are smoothly isotopic in D^4 relative to the boundary.

Lemma 1.5. *If $S, S', T, T' \subseteq S^3$ are Seifert surfaces for the same link and $S \approx_4 S'$, $T \approx_4 T'$, then $\Sigma(S, T)$ is smoothly unknotted if and only if $\Sigma(S', T')$ is smoothly unknotted.*

Proof. If $S \approx_4 S'$, then there is a smooth ambient isotopy of D^4_+ between S^+ and $(S')^+$ that fixes S^3 throughout the isotopy. Similarly, T^- and $(T')^-$ are ambiently isotopic in D^4_-

with S^3 fixed. Gluing these two isotopies, we get a smooth ambient isotopy of S^4 between $\Sigma(S,T)$ and $\Sigma(S',T')$. Thus, $\Sigma(S,T)$ bounds a smoothly embedded handlebody if and only if $\Sigma(S',T')$ does. □

Hence, for investigating Conjecture 2, we wish to understand the set of Seifert surfaces of a link up to the equivalence relation $\approx_4$. We now show that we can restrict this study to incompressible surfaces.

Lemma 1.6. *Suppose every incompressible spanning surface for the link L is connected, and Conjecture 2 holds for any incompressible Seifert surfaces for L. Then Conjecture 2 holds for any Seifert surfaces for L.*

Proof. We first claim that any Seifert surface S for L is a finite trivial stabilization of an incompressible Seifert surface, up to $\approx_4$. We prove this by induction on $g(S)$. Assume that the claim holds for all surfaces of smaller genus.

The claim obviously holds if S is incompressible. Otherwise, S has a compressing disk D, and compressing along D produces a surface S'. Then S is obtained from S' by a stabilization via a 1-handle $D \times I$. By shrinking D, we can replace this stabilization with a thin tube along an arc α, without changing S up to $\approx_3$.

If S' is connected, then the endpoints of α can be connected on S' by an arc σ. Sliding one endpoint of α along σ shows that $S \approx_3 S' \# F$, where $F \subseteq S^3 \setminus S'$ is a thin torus following $\alpha \cup \sigma$. As $F \subseteq S^3$ is smoothly unknotted (Lemma 1.1), it can be isotoped in D^4 (relative to a disk in F) into a small unknotted torus $T^2 \subseteq S^3$. Hence, $S \approx_4 S' \# T^2$, which is a trivial stabilization of a smaller genus surface S'. By the induction hypothesis, S' is isotopic to a finite trivial stabilization of an incompressible Seifert surface, and so S satisfies the claim.

If S' is disconnected, then one of the two components of S' must be closed. Otherwise, by compressing S' until we get an incompressible surface S'', the non-closed components of S'' would form an incompressible spanning surface for L that is not connected, contradicting our assumptions. Thus, $S \approx_3 E \# F$, where F is a closed, non-sphere surface and E is a Seifert surface for L. Since $F \subseteq S^3$ is smoothly unknotted (Lemma 1.1), it can be isotoped in D^4 (relative to a disk in F) into a small unknotted genus f surface $\#fT^2$. Therefore, $S \approx_4 E \# fT^2$, which is a finite trivial stabilization of a smaller genus surface E, and the induction hypothesis finishes the proof of the claim.

Now, let S_0, S_1 be two Seifert surfaces for L. By the claim, $S_i \approx_4 S_i' \# n_i T^2$, where S_i' are incompressible Seifert surfaces. By assumption, $\Sigma(S_0', S_1')$ is smoothly unknotted, so

it bounds a smooth handlebody H. Attaching n_0 trivial 1-handles to H in D^4_+ and n_1 trivial 1-handles in D^4_-, we get a smooth handlebody bounded by $\Sigma(S'_0\#n_0T^2, S'_1\#n_1T^2)$. Therefore, $\Sigma(S_0, S_1)$ is smoothly unknotted by Lemma 1.5. □

Proposition 1.7. *Let S, T be Seifert surfaces for the link L.*

(a) *If $S \approx_4 T$, then $\Sigma(S,T)$ is smoothly unknotted.*

(b) *If S and T have disjoint interiors, then $\Sigma(S,T)$ is smoothly unknotted.*

(c) *Assume that S and T intersect transversely in their interiors, and apart from L, their intersection is a finite collection of segments with endpoints on L (i.e. there are no closed-loop intersections). Then $\Sigma(S,T)$ is smoothly unknotted.*

Proof. Although (c) implies (b) and (b) implies (a), we also provide simpler, direct proofs for (a) and (b).

(a) This follows from Lemma 1.5 and the fact that $\Sigma(S,S)$ bounds the handlebody $S \times I$.

(b) If S and T have disjoint interiors, then $\Sigma(S,T)$ is isotopic to $S \cup T \subseteq S^3$, hence $\Sigma(S,T)$ is smoothly unknotted by Lemma 1.2.

(c) Let β be the closure of $(S \cap T) \setminus L$. By our assumption, β is a set of closed arcs, each connecting two points of L. Figure 1.1 shows a local picture around one component of β. Let B_S, B_T be two small neighbourhoods of β in S and in T with $B_S \cap L = B_T \cap L = \ell$. Then $(S \setminus B_S) \cup (T \setminus B_T)$ is an embedded, oriented surface with boundary $(\partial B_S \cup \partial B_T) \setminus \ell$, which is a collection of circles. Let D be a collection of disks bounded by these circles, with disjoint interior from $S \cup T$. Consider $\Sigma = (S \setminus B_S) \cup (T \setminus B_T) \cup D$, a closed, oriented (not necessarily connected) surface, smoothly embedded in S^3. By Lemma 1.1, Σ bounds a smoothly embedded disjoint union of handlebodies $H \subseteq D^4_+$. $B_S \cup (\mathrm{Cl}\, B_T)^- \cup D$ is a collection of spheres that bounds a collection of balls $E \subseteq D^4_-$. Attaching E to H has the effect of gluing 1-handles to H, along disks of D. Hence, $H \cup E$ is a handlebody in S^4, with boundary $S \cup T^-$. Therefore, $\Sigma(S,T) \approx_4 S \cup T^-$ is smoothly unknotted. □

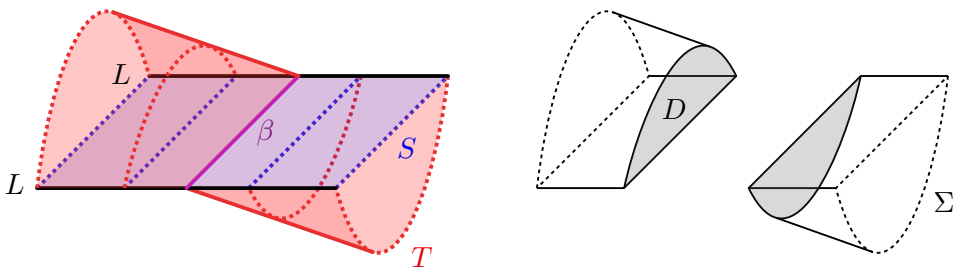


Figure 1.1: Local picture around a segment of intersection.

Note that by Lemma 1.5, the conditions on S and T in the previous proposition can be considered up to $\approx_4$, the conclusion that $\Sigma(S,T)$ is smoothly unknotted will still hold. This leads us to the following questions.

Question 3. *Can any two Seifert surfaces S, T for the same link be smoothly isotoped in D^4 relative to the boundary*

(a) *into each other, if they have the same genus?*

(b) *into Seifert surfaces S', T' with disjoint interiors?*

(c) *into Seifert surfaces S', T' that intersect transversely in segments?*

Note that question (a), originally posed by Livingston in [Liv82], was unsolved for 40 years. It was recently answered negatively with the first counterexample in [HKM+22], where they constructed knots with pairs of Seifert surfaces that are not even topologically isotopic in D^4. We also present large families of knots and links in Chapter 2 with such pairs of Seifert surfaces.

Questions (b) and (c) remain open. A positive answer to either would prove Conjecture 2.

1.2.3 Special cases of links

We have seen partial results of Conjecture 2 by imposing certain conditions on the Seifert surfaces S_0, S_1 (Proposition 1.7). We now shift our focus to restrictions on the link L itself to obtain additional partial results. Specifically, we investigate special families of knots and links for which all incompressible spanning surfaces have been classified.

Fibred links

We call a link L *fibred*, if there is a smooth one-parameter family S_t ($t \in S^1$) of spanning surfaces for L, with pairwise disjoint interiors (called *fibre surfaces*). For example, all torus links $T_{p,q}$ are fibred (for $|p|, |q| \geq 1$). More generally, any (p,q)-cable of a fibred link is fibred (for $|p|, |q| \geq 1$). See [Bak09] for nice visualizations of these facts.

Proposition 1.8. *Conjecture 2 holds for any fibred link L.*

Proof. Fibred links have only one incompressible spanning surface up to $\approx_3$ (see [Kob89], or [Lyo74a] for knots). Hence, Conjecture 2 holds for incompressible Seifert surfaces for L by Proposition 1.7 (a). Therefore, it holds in the general (compressible) case by Lemma 1.6, as fibre surfaces are connected and incompressible [Kaw96, Theorem 4.1.10]. □

2-bridge knots

We say that a surface $S \subseteq S^3$ is a *plumbing* of compact oriented surfaces $S_1, S_2 \subseteq S^3$ if there are 3-balls $V_1, V_2 \subseteq S^3$ satisfying $V_1 \cup V_2 = S^3$, $V_1 \cap V_2 = \partial V_i$, $S_i \subseteq V_i$ $(i = 1, 2)$, $S = S_1 \cup S_2$, and $D = S_1 \cap S_2$ is a square. If S is a plumbing of S_1, S_2, written as $S = S_1 \cup_D S_2$, then we can define a *dual* of S along $P = \partial V_1$ as $S^* = (S \setminus \operatorname{Int} D) \cup (P \setminus D)$. Then S^* is another embedded surface with the same boundary as S.

Hatcher–Thurston [HT85] classified the incompressible Seifert surfaces for the 2-bridge knots $K_{p/q}$. First, write $p/q = r+1/(b_1+1/(b_2+\cdots 1/b_k))$ with $r \in \mathbb{Z}$, $b_i \in 2\mathbb{Z}\setminus\{0\}$. (There is a unique way to do that, for q odd. For q even, $K_{p/q}$ is a link.) Then take an iterated plumbing of k annuli in a row, where the i-th annulus has b_i half-twists $(1 \le i \le k)$, and the plumbing spheres are $P_i = S^2 \times \{i\}$ $(1 \le i \le k-1)$, viewing $S^2 \times \mathbb{R} \subseteq S^3$ (Figure 1.2). This gives an oriented surface S bounded by $K_{p/q}$. Denote by $S(n_1, \dots, n_{k-1})$ $(n_i = 0, 1)$ the surface obtained by dualizing S along P_i for each $n_i = 0$. Then these 2^{k-1} surfaces give all the incompressible Seifert surfaces up to $\approx_3$. (Moreover, the isotopy relations among this set of surfaces is generated by $S(\dots, n_{i-2}, 0, 0, n_{i+1}, \dots)$ being isotopic to $S(\dots, n_{i-2}, 1, 1, n_{i+1}, \dots)$ for each $b_i = \pm 2$.)

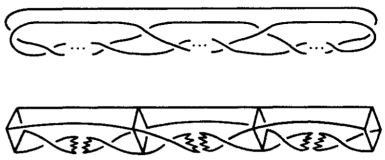


Figure 1.2: Seifert surfaces for 2-bridge knots as iterated plumbing (taken from [HT85]).

Proposition 1.9. *Conjecture 2 holds for any 2-bridge knot $L = K_{p/q}$.*

Proof. Using the classification above, any two incompressible Seifert surfaces for $K_{p/q}$ differ by a sequence of taking duals along some spheres. If S and S^* are dual along P, then $S \approx_4 S^*$, because using the fourth dimension, $P \cap S$ can be isotoped into $P \cap S^*$. Hence, $K_{p/q}$ has only one incompressible Seifert surface up to $\approx_4$. Thus, Conjecture 2 holds for incompressible Seifert surfaces for $K_{p/q}$ by Proposition 1.7 (a), and we are done by Lemma 1.6, as spanning surfaces for a knot are inherently connected. □

If we do not assume that $K_{p/q}$ is a knot, then q must be even, and $K_{p/q}$ is a 2-component link. In this case, we might need to consider multiple different continued fraction representations $p/q = r + 1/(b_1 + 1/(b_2 + \cdots 1/b_k))$ $(r \in \mathbb{Z}, b_i \in 2\mathbb{Z} \setminus \{0\})$ to obtain all incompressible Seifert surfaces. Hence, it is unclear whether they are all the same up to isotopy in D^4.

Prime knots up to 10 crossings

Proposition 1.10. *Let L be a prime knot with at most 10 crossings. Then Conjecture 2 holds for L.*

Proof. Kakimizu [Kak05] classified the incompressible Seifert surfaces for prime knots up to 10 crossings: there are at most 4 different (up to $\approx_3$) surfaces for each such knot. Apart from the 2-bridge knots already covered, only the knots 10_{53}, 10_{67}, 10_{68}, 10_{74} have more than one incompressible Seifert surface. For the first three of these knots, there are 2 incompressible Seifert surfaces, which again come from a dualization of a plumbing. For 10_{74}, there is a Seifert surface which can be written as a plumbing in two ways: $S = S_1 \cup_D S_2 = T_1 \cup_E T_2 = T$, and dualizing along either spheres $P \supseteq D$ or $Q \supseteq E$ gives the 3 incompressible Seifert surfaces: S, S^*, T^*. (Note that P and Q are not disjoint.) As $S^* \approx_4 S \approx_4 T^*$, we again see that there is only one incompressible Seifert surface up to isotopy in D^4 for any such link L. Hence, we are done by Proposition 1.7 (a) and Lemma 1.6. □

Note, however, that there exist prime knots with more than 10 crossings that have infinitely many incompressible Seifert surfaces (up to $\approx_3$).

The Kakimizu complex

Let L be an oriented non-split link. The *Kakimizu complex* $IS(L)$ is defined as the simplicial complex where the vertices are the different incompressible spanning surfaces for L (up to $\approx_3$), and $S_0, \dots, S_k$ span a k-simplex if and only if there are surfaces $S_i' \approx_3 S_i$ such that the interiors of $S_0', \dots, S_k'$ are pairwise disjoint. $MS(L)$ is defined similarly but with the minimal genus spanning surfaces as vertices.

Kakimizu [Kak05] described these simplicial complexes for any prime knot K with at most 10 crossings. In all cases, $IS(K) = MS(K)$ is a subdivided line segment: a graph of at most 4 vertices S_i, with edges between S_i and S_{i+1}. Furthermore, they proved that $IS(L)$ and $MS(L)$ are connected for any oriented link L [Kak92]. Schultens [Sch10] showed that $MS(K)$ is simply connected for any knot K. Przytycki–Schultens [PS12] proved that $MS(K)$ is contractible for any knot K.

Proposition 1.11. *If $IS(L)$ has at most two vertices, which represent connected surfaces, then Conjecture 2 holds for the link L.*

Proof. Let S, T be two incompressible Seifert surfaces for L. If they correspond to the same vertex in $IS(L)$, then $S \approx_3 T$, thus $\Sigma(S, T)$ is smoothly unknotted by Proposition 1.7 (a).

If they correspond to different vertices, then since $IS(L)$ is connected [Kak92] and has only two vertices by assumption, they are connected by an edge. This means that there are surfaces $S' \approx_3 S$, $T' \approx_3 T$ with disjoint interiors. By Proposition 1.7 (b), $\Sigma(S', T')$ is smoothly unknotted, and hence, $\Sigma(S, T)$ is smoothly unknotted by Lemma 1.5. Therefore, we are done by Lemma 1.6. □

Proposition 1.12. *Let L be a pretzel link $P(p, q, r)$ with $pqr \neq 0$ and any orientation. Then Conjecture 2 holds for L.*

Proof. Goda–Ishiwata [GI06] showed that for such pretzel links, L has only one or two different incompressible spanning surfaces (up to $\approx_3$), which are connected. Hence, we are done by Proposition 1.11. □

Note that pretzel links of more than 3 rows of braids may have infinitely many incompressible Seifert surfaces up to $\approx_3$. For example, $P(5, 5, 5, 5, 5)$ has this property, as it has incompressible Seifert surfaces of arbitrarily high genus [Par78, Chapter 4, Section 2] (see also [Wil08]).

We note a few more known results about the structure of the Kakimizu complex. A knot K is called *atoroidal* if $S^3 \setminus N(K)$ does not contain an essential (i.e. incompressible and not boundary parallel) torus; this is also known as a simple knot, or not satellite knot. If K is atoroidal, then $MS(K)$ is finite [SS64, Wil08]. If K is atoroidal and has a genus one Seifert surface, then $MS(K)$ has diameter at most 2 [SS09].

Connected sums

If $K = K_1 \# K_2$, then K may have infinitely many incompressible Seifert surfaces, even if it has relatively few crossings. In some cases (e.g. $K = 5_2 \# 5_2$), the structure of the infinite family of surfaces is still well-understood. To explain that, we now describe a way of modifying a Seifert surface, which is called a *spin around a torus*.

Let S be a Seifert surface for a knot K, and let T be a torus in $S^3 \setminus K$, with a non-trivial closed curve m on T. Let $i\colon S^1 \times S^1 \times I \to S^3 \setminus K$ be an embedding onto a closed regular neighbourhood of T such that $i(S^1 \times \{*\} \times \{*\}) = m$. Then $S^1 \times S^1 \times I$ has a diffeomorphism ψ defined by $\psi(e^{2\pi i x}, e^{2\pi i y}, z) = (e^{2\pi i (x+z)}, e^{2\pi i y}, z)$. This extends to a diffeomorphism f of S^3 by defining it as $\psi \circ i^{-1}$ on the image of i, and as the identity elsewhere. Then $S_n = f^n(S)$ $(n \in \mathbb{Z})$ are also Seifert surfaces for K.

If $K = K_1 \# K_2$, then let T be a swallow-follow torus, and let m be a meridian of K. Eisner [Eis77] showed that if K_i are non-fibred, then the surfaces S_n $(n \in \mathbb{Z})$ obtained by

spinning around T are pairwise different (up to $\approx_3$). Moreover, if K_i are non-fibred and have a unique incompressible Seifert surface, then S_n ($n \in \mathbb{Z}$) are all the incompressible Seifert surfaces for K [Kak92]. In this case, $IS(K) = MS(K)$ is an infinite line: a graph with vertices S_n and an edge between S_n and S_{n+1} for all $n \in \mathbb{Z}$.

Kakimizu [Kak05] proved that if K_1 is fibred, then there is a canonical isomorphism of simplicial complexes: $IS(K_1\#K_2) = IS(K_2)$ and $MS(K_1\#K_2) = MS(K_2)$. Let $|\cdot|$ denote the topological realization of a simplicial complex. Banks [Ban13] showed that if K_1, K_2 are non-fibred, then $|IS(K_1\#K_2)|$ is homeomorphic to $|IS(K_1)| \times |IS(K_2)| \times \mathbb{R}$. The same holds for $|MS(K_1\#K_2)|$.

Proposition 1.13. *Let K_1, K_2 be knots with unique incompressible Seifert surfaces (up to $\approx_3$). Then Conjecture 2 holds for $L = K_1\#K_2$.*

Proof. If K_1 is fibred, then $IS(K_1\#K_2) = IS(K_2)$ by [Kak05], which has only one vertex. Hence, we are done by Proposition 1.11. The same argument holds if K_2 is fibred.

If K_1 and K_2 are both non-fibred, then since they have a unique incompressible Seifert surface, we can use the classification of Eisner above [Eis77]. By this, any incompressible Seifert surface for L is isotopic to one of the surfaces S_n for some $n \in \mathbb{Z}$, obtained by spinning a Seifert surface S around a torus n times.

First, consider the surfaces S_0 and S_1 of this family. Then $A_0 = \mathrm{Cl}(S_0 \setminus S_1)$ and $A_1 = \mathrm{Cl}(S_1 \setminus S_0)$ are two annuli, and $T = A_0 \cup A_1$ is a knotted torus. Moreover, T bounds an embedded solid torus U in S^3, and the core of A_i is a longitude of U. Hence, $A_0 \approx_3 A_1$, and doing this isotopy in the fourth dimension ('above' $S_0 \cap S_1$) yields $S_0 \approx_4 S_1$.

Applying the torus spin diffeomorphism f of S^3, we get $S_n = f^n(S_0) \approx_4 f^n(S_1) = S_{n+1}$ for any $n \in \mathbb{Z}$. Hence, any two surfaces S_i, S_j in this family are the same up to $\approx_4$, and therefore, we are done by Proposition 1.7 (a) and Lemma 1.6. □

Note that a key step in the proof was that $A_0 \approx_3 A_1$ if the core of the annulus A_i is a longitude of the solid torus U. Without this assumption, A_0 and A_1 might not be isotopic, not even in D^4. See Theorems 2.1 and 2.5 in the next chapter.

Proposition 1.14. *Let L be a knot with at most 10 crossings (not necessarily prime). Then Conjecture 2 holds for L.*

Proof. Note that if $L = K_1\# \cdots \#K_n$ with K_i prime and L has at most 10 crossings, then the crossing numbers satisfy $c(L) = c(K_1) + \cdots + c(K_n)$. This follows from the classification of all knot diagrams with up to 10 crossings; see e.g. [Thi25].

If L is prime, the claim was shown in Proposition 1.10.

If $L = K_1 \# K_2$ with K_i prime, then $c(K_1) + c(K_2) \leq 10$. If $c(K_i) \leq 7$ and $K_i \neq \pm 7_4$, then K_i has a unique incompressible Seifert surface [Kak05]. As $c(K_i) \geq 3$, this means that we can use Proposition 1.13, except in the case $L = \pm 3_1 \# \pm 7_4$. For that, since $\pm 3_1$ is fibred, we have $IS(\pm 3_1 \# \pm 7_4) = IS(\pm 7_4)$, which has two vertices, and hence, we are done by Proposition 1.11.

If $L = K_1 \# \cdots \# K_n$ with K_n prime and $n \geq 3$, then $c(K_1) + \cdots + c(K_n) \leq 10$ implies $L = \pm 3_1 \# \pm 3_1 \# K$ with $K = \pm 3_1$ or $K = 4_1$. In each case, since $\pm 3_1$ is fibred, $IS(\pm 3_1 \# \pm 3_1 \# K) = IS(\pm 3_1 \# K) = IS(K)$, which has one vertex, and again, Proposition 1.11 completes the argument. □

The smallest non-fibred knot is 5_2, so the previous proof does not apply, for instance, to $5_2 \# 7_4$ or $5_2 \# 5_2 \# 5_2$.

1.2.4 A sequence of Seifert surfaces

Let $S, T \subseteq S^3$ be two Seifert surfaces for the knot K. Then there is a sequence of Seifert surfaces $S_0, S_1, \ldots, S_n \subseteq S^3$ between $S_0 = S$ and $S_n = T$ such that S_i and S_{i+1} have disjoint interiors for all i by [ST88]. (Alternatively, for S, T incompressible, we can use that $IS(K)$ is connected.) Moreover, there is a sequence such that for all i, S_i is obtained from S_{i-1} by either a compression, a stabilization, or an ambient isotopy of S^3 [Lic97, Theorem 8.2]. Furthermore, the order of these operations can be rearranged in a way that stabilizations come first ($1 \leq i \leq p$), then compressions, then an ambient isotopy.

Let $M_0 \subseteq S^3 \times [0,1]$ be the cobordism between S_0 and S_p induced by this sequence of stabilization moves, and let $M_1 \subseteq S^3 \times [1,2]$ be the cobordism between S_p and S_n induced by the sequence of compressions and ambient isotopy. Then M_0 is a handlebody bounded by $\Sigma(S_0, S_p)$ and M_1 is a handlebody bounded by $\Sigma(S_p, S_n)$. If it were possible to glue the handlebodies M_0, M_1 together along S_p, potentially after isotoping S_p to itself, such that the gluing yields a handlebody, then we would obtain a handlebody bounded by $\Sigma(S, T)$.

We investigate this question in Chapter 4. We give sufficient (Theorem 4.2) and necessary (Theorem 4.3) conditions for such a gluing to exist. However, it turns out that there are many examples of pairs (M_0, S_p), (M_1, S_p) such that there is no gluing homeomorphism $\phi \colon S_p \to S_p$ for which $M_0 \cup_\phi M_1$ is a handlebody (Proposition 4.17).

Chapter 2

Non-isotopic Seifert surfaces in the 4-ball

In this chapter, we give families of knots and links with pairs of Seifert surfaces that are topologically non-isotopic in D^4. This generalizes the main example of Hayden–Kim–Miller–Park–Sundberg [HKM+22] and the proof is similarly based on the double branched cover and the Seifert form. Furthermore, using Conway's theory of topographs, we implement an algorithm that decides whether two integral quadratic forms in two variables are isomorphic over $\mathbb{Z}$.

2.1 Families of surface-pairs

2.1.1 Main results

We write $S \sim_4 S'$ if the surfaces $S, S' \subseteq S^3$ are topologically isotopic in D^4 relative to the boundary.

Theorem 2.1. *Let p, q be coprime integers. The torus link $T_{2p,2q}$ cuts the torus into two annuli, giving two Seifert surfaces $A_0(p,q)$ and $A_1(p,q)$ for $T_{2p,2q}$ (where the two components of $T_{2p,2q}$ are oriented oppositely). Then $A_0(p,q) \not\sim_4 A_1(p,q)$ if and only if $|p|, |q| > 1$.*

Note that if p, q are not coprime, then we can still cut the torus into two surfaces bounded by $T_{2p,2q}$, but these surfaces are not connected, and hence are trivially non-isotopic as they connect different component-pairs of $T_{2p,2q}$.

This theorem will be proven as a consequence of Theorem 2.2. There we investigate whether the surfaces $A_0(p,q)$ and $A_1(p,q)$ remain non-isotopic if we attach a common band $B \subseteq S^3$ to their boundary, giving two Seifert surfaces for a knot. Note that if $F_0 \sim_4 F_1$,

then $F_0 \cup B \sim_4 F_1 \cup B$, because the isotopy between the F_i can be done 'above' the band B in the fourth dimension. However, the opposite is not true: $F_0 \not\sim_4 F_1$ does not imply $F_0 \cup B \not\sim_4 F_1 \cup B$. We will show that for a family of bands $B(k,n)$, the surfaces $A_i(p,q) \cup B(k,n)$ for $i = 0, 1$ are non-isotopic. Then it follows that $A_i(p,q)$ are non-isotopic as well.

Let (p,q,k,n) be integers with p,q coprime. Consider a band $B(k,n)$ as shown in Figure 2.1, that (viewed as a framed, oriented, closed curve) has linking number k with the core of the solid torus and has self-linking number n. Let

$$\Sigma_i(p,q,k,n) = A_i(p,q) \cup B(k,n).$$

Note that $B(k,n)$ can refer to any knotted band with linking numbers k and n, so $\Sigma_i(p,q,k,n)$ $(i = 0,1)$ actually refers to any member of an infinite family of pairs of surfaces.

Theorem 2.2. *Let $p,q > 1$ be coprime, k an integer such that $2kp \not\equiv 1 \pmod{q}$, and n an integer with*

$$n \geq \frac{k(kp-1)}{q} + \left(\frac{pq}{12} - \frac{1}{6} + \frac{1}{2pq}\right).$$

Then after pushing the interior of $\Sigma_i(p,q,k,n) = \Sigma_i$ into D^4, the pairs (D^4, Σ_0) and (D^4, Σ_1) are not homeomorphic. In particular, $\Sigma_0 \not\sim_4 \Sigma_1$.

We will see that in some cases the lower bound on n can be improved, e.g. if $k = 0$, or if $q \leq 3$, then $n > \frac{k(kp-1)}{q}$ is sufficient. However, the next theorem shows that we cannot go lower than that.

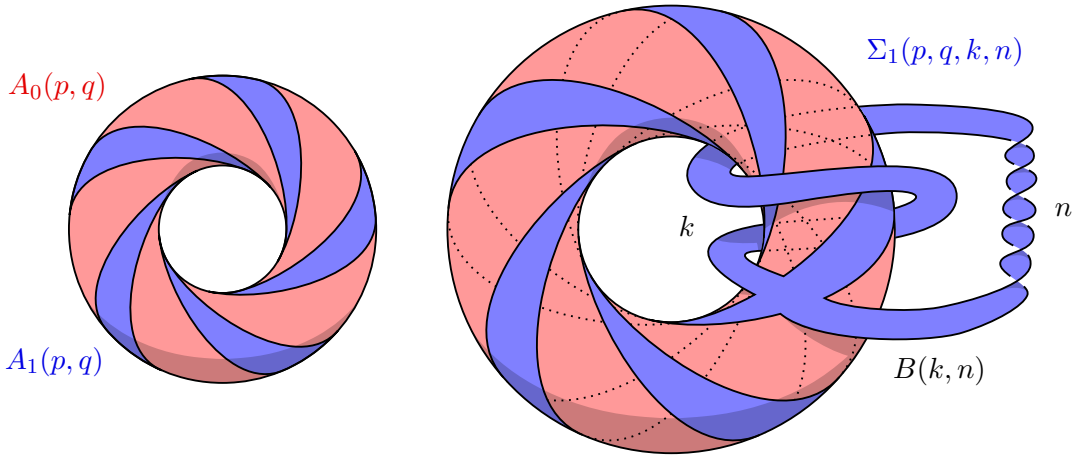


Figure 2.1: The surfaces in the case $(p,q) = (3,5)$, $k = 2$, and $n = 3$.

Theorem 2.3. *Let $p, q > 1$ be coprime, and suppose that integers k, n satisfy*

$$n = \frac{k(kp-1)}{q}.$$

Then $\Sigma_0(p,q,k,n) \sim_4 \Sigma_1(p,q,k,n)$.

Theorem 2.4. *Let $p, q > 1$ be coprime and $n = 0$. Then there exists an integer k_0 such that if $|k| > k_0$ is an integer satisfying $2kp \not\equiv 1 \pmod{q}$, then after pushing the interior of $\Sigma_i(p,q,k,n) = \Sigma_i$ into D^4, the pairs (D^4, Σ_0) and (D^4, Σ_1) are not homeomorphic. In particular, $\Sigma_0 \not\sim_4 \Sigma_1$.*

See Section 2.2.5 for more discussion about the conditions in Theorems 2.2 to 2.4. In particular, the diagrams in Figure 2.9 display these conditions very clearly.

We can generalize Theorem 2.1 for 2-component cables in the following way.

Theorem 2.5. *Let $p, q > 1$ be coprime and K be any knot. Let L be the $(2p, 2q)$-cable of K, with the two components oriented oppositely. Then L has non-isotopic Seifert surfaces. Namely, L cuts the boundary of a tubular neighbourhood of K into two annuli, giving two Seifert surfaces A_0 and A_1 for L. Then $A_0 \not\sim_4 A_1$.*

In fact, just like we allowed $B(k,n)$ to be any knotted band in the definition of the surface-pairs $\Sigma_i(p,q,k,n)$, we can also allow $A_0(p,q) \cup A_1(p,q)$ to be any knotted torus (bounding a solid torus disjoint from the inside of $B(k,n)$), and Theorems 2.2 to 2.4 remain true. We will see this in the proof of Theorem 2.5.

2.1.2 Proofs

Two quadratic forms $Q_0, Q_1 \colon \mathbb{Z}^n \to \mathbb{Z}$ are called *isomorphic* over $\mathbb{Z}$ if there exists an invertible integral matrix P such that $Q_0(\mathbf{v}) = Q_1(P\mathbf{v})$ for all $\mathbf{v} \in \mathbb{Z}^n$.

To prove Theorem 2.2, we will need the following lemma.

Lemma 2.6. *Let u, v_0, v_1 be integers such that $v_0 \not\equiv \pm v_1 \pmod{u}$. For an integer t, consider the quadratic forms*

$$Q_i(x,y) = (ux + v_i y)^2 + ty^2.$$

Then for all

$$t > t_0 = \frac{(u/2-1)^2}{3},$$

the quadratic forms Q_0, Q_1 are not isomorphic over $\mathbb{Z}$.

Remark 2.7. The lower bound on t might be weakened, but at the cost of having a more cumbersome, number-theoretic formula. For an integer v, let us denote by $[v]_u$ the integer that satisfies $0 \leq [v]_u \leq u/2$ and $[v]_u \equiv \pm v \pmod{u}$. If $[v_0]_u < [v_1]_u$ (swap v_0 and v_1 if not), then this formula is

$$t > t_1 = \max_{c \geq 2 \text{ integer}} \frac{[v_0]_u^2 - [cv_1]_u^2}{c^2 - 1}.$$

Remark 2.8. The condition on t cannot be omitted: e.g. the quadratic forms

$$(30x + 7y)^2 + ty^2 \quad \text{and} \quad (30x + 13y)^2 + ty^2$$

are not isomorphic for $t \geq 12$, but are isomorphic for $t = 11$. The formula in the previous Remark gives $t_1 = 11$, while $t_0 = 65 + \frac{1}{3}$.

Proof. We can assume $|v_i| \leq u/2$, as taking $Q_i(x \pm y, y)$ (which is isomorphic to Q_i) increases/decreases the value of v_i by u. We can also assume $0 \leq v_i$, as taking $Q_i(x, -y)$ replaces v_i by $-v_i$. These manipulations also do not change the condition $v_0 \not\equiv \pm v_1 \pmod{u}$. Hence, without loss of generality, we can assume $0 \leq v_0 < v_1 \leq u/2$.

Let H_i denote the set of integers (x, y) for which

$$Q_i(x, y) = v_0^2 + t.$$

If $t > 0$, then Q_i is positive definite, hence H_i is finite. We will show $|H_0| > |H_1|$, which proves that Q_0 and Q_1 are not isomorphic (as an isomorphism between Q_0 and Q_1 would induce a bijection between H_0 and H_1).

For $y = 0$, the values (x, y) contribute the same to H_0 and H_1, as $Q_0(x, 0) = Q_1(x, 0)$. The case $|y| = 1$ gives the elements $(0, \pm 1) \in H_0$. However, it gives no elements in H_1, as $v_0 < v_1 \leq u/2$ implies $Q_1(x, \pm 1) > v_0^2 + t$ for all x.

We show that for t large enough, there are no other elements in H_1. As above, $[v]_u$ denotes the integer that satisfies $0 \leq [v]_u \leq u/2$ and $[v]_u \equiv \pm v \pmod{u}$. If $|y| = c \geq 2$, then the smallest value that $Q_1(x, y)$ can take is $[cv_1]_u^2 + tc^2$. Hence, it suffices to have

$$\begin{aligned} [cv_1]_u^2 + tc^2 &> v_0^2 + t, \\ t &> \frac{v_0^2 - [cv_1]_u^2}{c^2 - 1} \end{aligned}$$

for all integers $c \geq 2$. Since $v_0 \leq v_1 - 1 \leq u/2 - 1$ and $[cv_1]_u^2 \geq 0$,

$$\frac{v_0^2 - [cv_1]_u^2}{c^2 - 1} \leq \frac{(u/2-1)^2}{2^2-1} = t_0$$

is suitable. If $t > t_0$, then $|H_0| > |H_1|$, and so Q_0 and Q_1 are not isomorphic over $\mathbb{Z}$. □

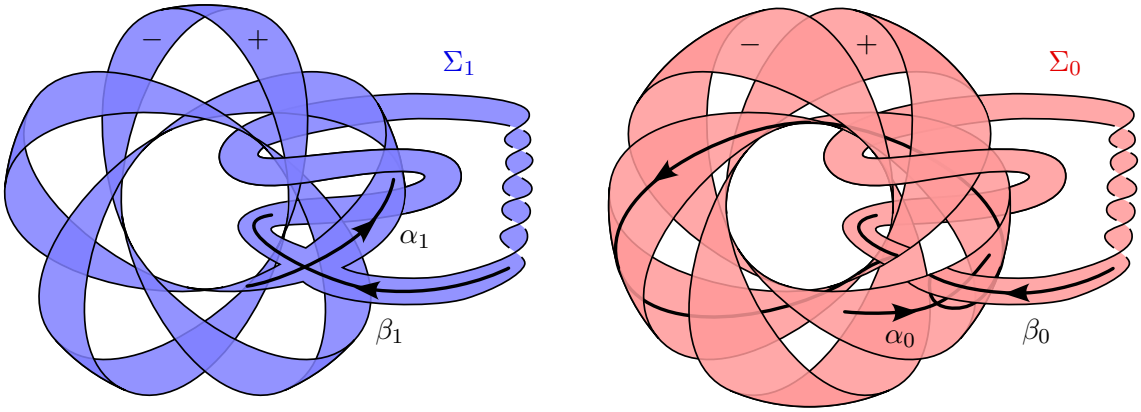


Figure 2.2: Orientations for the basis of $H_1(\Sigma_i)$, $(r, s) = (1, 2)$ pictured.

Proof of Theorem 2.2. Let X_i be the double branched cover of D^4 along Σ_i. Then the intersection form of $H_2(X_i)$ is given by the symmetrized Seifert form of $\Sigma_i \subseteq S^3$ [GL78].

A basis for $H_1(\Sigma_1)$ is given by α_1, β_1 (see Figure 2.2), where α_1 is the core of $A_1(p, q)$ and β_1 is the core of the band $B(k, n)$ with opposite orientation and its endpoints joined by a small arc in $A_1(p, q)$. Then α_1 is a torus knot $T_{p,q}$ and has self-linking number pq. Thus, the Seifert matrix of Σ_1 is

$$V_1 = \begin{bmatrix} pq & -kp \\ 1-kp & n \end{bmatrix}.$$

A basis for $H_1(\Sigma_0)$ is α_0, β_0, where α_0 is a torus knot $T_{p,q}$ in $A_0(p, q)$, but for β_0 we join together the endpoints of the band $B(k, n)$ by some longer arc in $A_0(p, q)$. Then $\beta_0 - \beta_1$ (meant as a union of oriented curves, cancelling the oppositely oriented overlapping parts) is a closed curve on the torus, hence it is a torus knot $T_{r,s}$. Since $T_{r,s}$ intersects $T_{p,q}$ once positively, the integers r, s satisfy

$$ps - qr = 1.$$

Using $\beta_0 = T_{r,s} + \beta_1$, we get

$$\begin{aligned} \operatorname{lk}(\beta_0, \alpha_0^+) &= \operatorname{lk}(T_{r,s}, \alpha_0^+) + \operatorname{lk}(\beta_1, \alpha_0) = ps - kp, \\ \operatorname{lk}(\beta_0, \beta_0^+) &= \operatorname{lk}(T_{r,s}, T_{r,s}^+) + 2\operatorname{lk}(T_{r,s}, \beta_1^+) + \operatorname{lk}(\beta_1, \beta_1^+) = rs - 2kr + n. \end{aligned}$$

Therefore, the Seifert matrix of Σ_0 is

$$V_0 = \begin{bmatrix} pq & qr - kp \\ ps - kp & rs - 2kr + n \end{bmatrix}.$$

The quadratic forms associated to the symmetrized Seifert matrices are

$$\begin{aligned} Q_1(x,y) &= \begin{bmatrix} x & y \end{bmatrix} (V_1 + V_1^T) \begin{bmatrix} x \\ y \end{bmatrix} \\ &= 2pq \cdot x^2 + 2(1 - 2kp) \cdot xy + 2n \cdot y^2 \\ &= \frac{1}{2pq} \left((2pq \cdot x + (1 - 2kp) \cdot y)^2 + \left(4pqn - (2kp - 1)^2\right) y^2 \right), \\ Q_0(x,y) &= \begin{bmatrix} x & y \end{bmatrix} (V_0 + V_0^T) \begin{bmatrix} x \\ y \end{bmatrix} \\ &= 2pq \cdot x^2 + 2(1 - 2kp + 2qr) \cdot xy + 2(rs - 2kr + n) \cdot y^2 \\ &= \frac{1}{2pq} \left((2pq \cdot x + (1 - 2kp + 2qr) \cdot y)^2 + \left(4pqn - (2kp - 1)^2\right) y^2 \right), \end{aligned}$$

where we used that

$$(1 - 2kp + 2qr)^2 - (2kp - 1)^2 = 2qr(2 - 4kp + 2qr) = 2qr(2ps - 4kp) = 4pq(rs - 2kr).$$

By Lemma 2.6, the forms $2pqQ_0$ and $2pqQ_1$ are not isomorphic if

$$\begin{gathered} 1 - 2kp \not\equiv 1 - 2kp + 2qr \pmod{2pq}, \\ 1 - 2kp \not\equiv -(1 - 2kp + 2qr) \pmod{2pq}, \\ 4pqn - (2kp - 1)^2 > \frac{(pq - 1)^2}{3}. \end{gathered}$$

The first condition is equivalent to $0 \not\equiv r \pmod{p}$, which holds because $p > 1$ and $ps - qr = 1$. The second condition is $0 \not\equiv -2 + 4kp - 2qr \pmod{2pq}$, which holds by our assumption $2kp \not\equiv 1 \pmod{q}$ (to which it is in fact equivalent). The third condition is equivalent to

$$\begin{gathered} 4pqn - (2kp - 1)^2 \geq \frac{(pq - 1)^2 + 2}{3}, \\ n \geq \frac{(2kp - 1)^2}{4pq} + \frac{(pq - 1)^2 + 2}{12pq} = \frac{k(kp - 1)}{q} + \left(\frac{pq}{12} - \frac{1}{6} + \frac{1}{2pq} \right), \end{gathered}$$

which was also an assumption. Therefore, the Q_i are not isomorphic over $\mathbb{Z}$.

Moreover, Q_0 and $-Q_1$ are not isomorphic either, as the Q_i are positive definite. Hence,

the X_i are not homeomorphic, implying that the pairs (D^4, Σ_i) are not homeomorphic, and therefore, $\Sigma_0 \not\sim_4 \Sigma_1$. □

Proof of Theorem 2.1. If $|p| = 1$ or $|q| = 1$, then the two annuli are easily seen to be isotopic in S^3 (even smoothly isotopic).

If $|p|, |q| \neq 1$, then we can assume $p, q > 1$. Suppose $A_0(p,q) \sim_4 A_1(p,q)$. Let $k = 0$, then $q > 1$ implies $2kp \not\equiv 1 \pmod{q}$, and let n be large enough (in fact, $n = 1$ works). Then attaching the band $B(k,n)$ to $A_i(p,q)$, we get the surfaces $\Sigma_i(p,q,k,n) = \Sigma_i$. Viewing D^4 as $(S^3 \times [0,1]) \cup D^4_{1/2}$, we can isotope Σ_i into

$$(B(k,n) \times \{0\}) \cup (\partial A_i(p,q) \times [0,1]) \cup (A_i(p,q) \times \{1\}),$$

and then do the isotopy $A_0(p,q) \sim_4 A_1(p,q)$ in $D^4_{1/2}$. We get $\Sigma_0 \sim_4 \Sigma_1$, contradicting Theorem 2.2. Hence, $A_0(p,q) \not\sim_4 A_1(p,q)$. □

Proof of Theorem 2.3. Conway–Powell proved in [CP23, Theorem 1.9] that for an Alexander polynomial one knot, any two Seifert surfaces of the same genus are topologically isotopic in the 4-ball. In our case, using the Seifert matrix of Σ_1 from above, its boundary K has Alexander polynomial

$$\Delta_K(t) = \det(tV_1 - V_1^T) = \begin{vmatrix} tpq - pq & -tkp - (1-kp) \\ t(1-kp) + kp & tn - n \end{vmatrix} =$$
$$= p(nq - k(kp-1))(t-1)^2 + t.$$

Hence, for $n = \frac{k(kp-1)}{q}$, we have $\Delta_K(t) = t \doteq 1$. Therefore, $\Sigma_0 \sim_4 \Sigma_1$. □

Proof of Theorem 2.4. When $n = 0$, the quadratic forms can be factorized:

$$Q_0(x,y) = 2 \cdot (px + ry)\,(qx + (s-2k)\,y)\,,$$
$$Q_1(x,y) = 2 \cdot x\,(pqx + (1-2kp)\,y)\,.$$

If Q_0 and Q_1 are isomorphic over $\mathbb{Z}$, then $Q_0(ax+by, cx+dy) = Q_1(x,y)$ for some invertible integral matrix $P = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Two factorized polynomials agree if and only if their linear factors agree up to multiplication by a constant, hence

$$p(ax+by) + r(cx+dy) = \lambda x \quad \text{or} \quad \lambda\,(pqx + (1-2kp)\,y)$$

for some real number $\lambda \neq 0$. Writing this in matrix form,

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{or} \quad \lambda \begin{bmatrix} pq \\ 1-2kp \end{bmatrix}.$$

Combining this with the similar equation for the other linear factor, we get

$$P^T M_0 = M_1 L_j \quad \text{for } j = 0 \text{ or } 1,$$

where

$$P^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad M_0 = \begin{bmatrix} p & q \\ r & s-2k \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & pq \\ 0 & 1-2kp \end{bmatrix},$$
$$L_0 = \begin{bmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{bmatrix}, \quad \text{and} \quad L_1 = \begin{bmatrix} 0 & 1/\lambda \\ \lambda & 0 \end{bmatrix}.$$

If Q_0 and $-Q_1$ are isomorphic, then similarly, we get that $P^T M_0 = M_1 L_j$ for $j = 2$ or 3, where

$$L_2 = \begin{bmatrix} \lambda & 0 \\ 0 & -1/\lambda \end{bmatrix}, \quad \text{and} \quad L_3 = \begin{bmatrix} 0 & -1/\lambda \\ \lambda & 0 \end{bmatrix}.$$

Hence, we want to show that for large enough k with $2kp \not\equiv 1 \pmod{q}$, there do not exist a j and a real number $\lambda \neq 0$ such that $P^T = M_1 L_j M_0^{-1}$ is integral and invertible over $\mathbb{Z}$. Assume that there exist such a j and $\lambda > 0$ (without loss of generality).

If $j = 0$, then

$$P^T M_0 = M_1 L_0 = \begin{bmatrix} \lambda & pq/\lambda \\ 0 & (1-2kp)/\lambda \end{bmatrix}$$

is integral, hence λ is an integer. Since

$$(P^T)^{-1} M_1 = M_0 L_0^{-1} = \begin{bmatrix} p/\lambda & q\lambda \\ r/\lambda & (s-2k)\lambda \end{bmatrix}$$

is integral, we get $\lambda \mid p$ and $\lambda \mid r$, but $(p, r) = 1$ then implies $\lambda = 1$. Thus

$$P^T = M_1 L_0 M_0^{-1} = \begin{bmatrix} \dfrac{2k - s + pqr}{2kp - 1} & \dfrac{q(1-p^2)}{2kp-1} \\ -r & p \end{bmatrix}$$

can only be integral if $2kp - 1 \mid q(1 - p^2)$. Since $p > 1$, this does not hold for $|k|$ large

enough.

If $j = 1$, then

$$P^T M_0 = M_1 L_1 = \begin{bmatrix} pq\lambda & 1/\lambda \\ (1-2kp)\lambda & 0 \end{bmatrix}$$

is integral, hence $\mu = 1/\lambda$ is an integer. Then

$$P^T = M_1 L_1 M_0^{-1} = \begin{bmatrix} \frac{2kpq - pqs + r\mu^2}{(2kp-1)\mu} & \frac{pq^2 - p\mu^2}{(2kp-1)\mu} \\ \frac{s-2k}{\mu} & \frac{-q}{\mu} \end{bmatrix}$$

is integral. The bottom right entry implies $\mu \mid q$. By the top right entry, $2kp-1 \mid pq^2/\mu - p\mu$. Since $(2kp-1, p) = 1$, we have

$$2kp - 1 \mid q^2/\mu - \mu.$$

If $|k|$ is large enough, then $|2kp-1| \geq q^2$. Since $|q^2/\mu - \mu| \leq q^2 - 1$, the only possibility is $q^2/\mu - \mu = 0$, hence $\mu = q$. But the bottom left entry in $P^T M_0$ implies $\mu \mid 1 - 2kp$, contradicting $2kp \not\equiv 1 \pmod{q}$.

If $j = 2$, then the same argument as for $j = 0$ shows that we must have $2kp-1 \mid q(1+p^2)$, which does not hold for $|k|$ large enough.

If $j = 3$, then the same argument as for $j = 1$ shows that $\mu = 1/\lambda$ is an integer, $\mu \mid q$ and $2kp - 1 \mid q^2/\mu + \mu$, which does not hold if $|2kp-1| \geq q^2 + 2$.

Hence, Q_0 is not isomorphic to $\pm Q_1$ over $\mathbb{Z}$ if $2kp \not\equiv 1 \pmod{q}$ and

$$|k| > k_0 = \max\left(\frac{q(1+p^2)+1}{2p}, \frac{q^2+2}{2p}\right).$$

Then the pairs (D^4, Σ_0) and (D^4, Σ_1) are not homeomorphic, and $\Sigma_0 \not\sim_4 \Sigma_1$. □

Proof of Theorem 2.5. The proof is very similar to Theorem 2.1, so we only highlight the differences.

$A_0 \cup A_1$ is a knotted torus T in S^3 that bounds a solid torus. We attach a band $B(k, n)$ to A_0 and A_1 that is disjoint from the inside of the solid torus and has linking number k with the core of the solid torus and has self-linking number n. We call the resulting surfaces Σ_0 and Σ_1, and will show that their Seifert matrices are the same as in the above proof.

The $(2p, 2q)$-cable of the knot K is defined using the longitude λ of T that satisfies

$\mathrm{lk}(\lambda, \lambda^+) = 0$ and a meridian μ. Then the core α_1 of A_1 is $p\lambda + q\mu$ in $H_1(T)$, hence

$$\mathrm{lk}(\alpha_1, \alpha_1^+) = \mathrm{lk}(p\lambda, p\lambda^+ + q\mu^+) = pq,$$

where λ^+, μ^+ denote the push-off away from the solid torus. A basis for $H_1(\Sigma_i)$ is α_i, β_i similarly as in Figure 2.2. Then the Seifert matrix of Σ_1 is the same V_1 as above. For the Seifert matrix of Σ_0, we now have that $\beta_0 - \beta_1 = t_{r,s}$ is an (r, s)-cable of K with $ps - qr = 1$. Hence, we have

$$\begin{aligned}\mathrm{lk}(t_{r,s}, t_{r,s}^+) &= rs,\\ \mathrm{lk}(t_{r,s}, \alpha_0^+) &= \mathrm{lk}(r\lambda^+ + s\mu^+, p\lambda) = ps,\end{aligned}$$

and the rest of the calculation for V_0 is the same.

Therefore, for $p, q > 1$ and suitably chosen k, n, the symmetrized Seifert forms are not isomorphic over $\mathbb{Z}$, hence $\Sigma_0 \not\sim_4 \Sigma_1$, and $A_0 \not\sim_4 A_1$. □

We conclude this section with the following question.

Question 4. *Let $p > 1$ and let K be a knot. The $(2p, 2)$-cable of K cuts the boundary of a tubular neighbourhood of K into two annuli A_0 and A_1. Is it true that $A_0 \sim_4 A_1$?*

Note that Theorem 2.1 was stated as an equivalence, while Theorem 2.5 was phrased as an implication. When $|p| = 1$, it is clear that $A_0 \sim_4 A_1$ for any $(2p, 2q)$-cable. Thus, a positive answer to Question 4, which addresses the $|q| = 1$ case, would allow Theorem 2.5 to be restated as an equivalence as well.

2.2 Conway's algorithm

There is a diagrammatic way to think about quadratic forms in two variables due to Conway [Con97]. This can be used to solve generalized Pell's equations simply by looking at the corresponding diagram and it also gives an algorithm that decides whether two integral binary quadratic forms are isomorphic over $\mathbb{Z}$. We describe this algorithm and we will use it to study the symmetrized Seifert forms of the surfaces $\Sigma_i(p, q, k, n)$ with values of p, q, k, n that are not covered by Theorems 2.2 to 2.4.

2.2.1 Primitive vectors

We begin the theory by discussing the structure of primitive vectors of $\mathbb{Z}^2$ and how it is related to an infinite 3-regular tree.

We call the vector $\mathbf{v} = (x, y)$ *primitive* if x, y are coprime. We say $(\mathbf{v}_1, \mathbf{v}_2)$ is a *base* if their integral linear combinations give all of $\mathbb{Z}^2$ (then $\mathbf{v}_i$ must be primitive). We call $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ a *superbase* if $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0$ and $(\mathbf{v}_1, \mathbf{v}_2)$ is a base. Then $(\mathbf{v}_1, \mathbf{v}_3)$ and $(\mathbf{v}_2, \mathbf{v}_3)$ are bases as well.

For a vector $\mathbf{v}$, we call the set $\pm\mathbf{v} = \{\mathbf{v}, -\mathbf{v}\}$ a *lax vector*. If $(\mathbf{v}_1, \mathbf{v}_2)$ is a base, then we say $\{\pm\mathbf{v}_1, \pm\mathbf{v}_2\}$ is a *lax base*, and if $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a superbase, then we say $\{\pm\mathbf{v}_1, \pm\mathbf{v}_2, \pm\mathbf{v}_3\}$ is a *lax superbase*.

Note that each lax superbase $\{\pm\mathbf{v}_1, \pm\mathbf{v}_2, \pm\mathbf{v}_3\}$ contains 3 lax bases, and each lax base $\{\pm\mathbf{v}_1, \pm\mathbf{v}_2\}$ is contained in 2 lax superbases, given by $\pm\mathbf{v}_3 = \pm(\mathbf{v}_1 + \mathbf{v}_2)$ and $\pm\mathbf{v}_3 = \pm(\mathbf{v}_1 - \mathbf{v}_2)$. Hence, the lax bases and lax superbases of $\mathbb{Z}^2$ can be visualized as a 3-regular graph, where the vertices are the lax superbases, and the edges are the lax bases, with an edge connecting to a vertex if the lax base is contained in the lax superbase. Conway then proved [Con97] that this graph is connected and contains no cycles, hence it is an infinite 3-regular tree.

If $\pm\mathbf{v}_1$ is a primitive lax vector, then it is in a lax base $\{\pm\mathbf{v}_1, \pm\mathbf{v}_2\}$, which is contained in a lax superbase $\{\pm\mathbf{v}_1, \pm\mathbf{v}_2, \pm\mathbf{v}_3\}$. From any such vertex that contains $\pm\mathbf{v}_1$, there are exactly 2 edges that contain $\pm\mathbf{v}_1$, namely $\{\pm\mathbf{v}_1, \pm\mathbf{v}_2\}$ and $\{\pm\mathbf{v}_1, \pm\mathbf{v}_3\}$. Hence, we get an infinite path of edges that contain $\pm\mathbf{v}_1$. By drawing the tree in a suitable way on the plane, we can regard primitive lax vectors as 'regions' of the tree that are bounded by these infinite paths. See Figure 2.3, where the $\pm$ signs have been omitted.

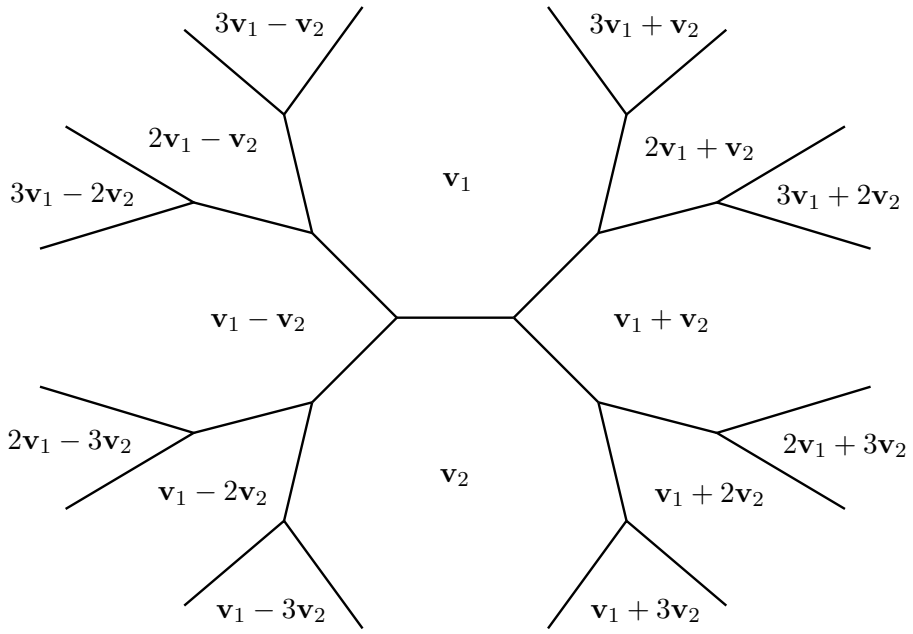


Figure 2.3: The primitive vectors as the regions of a 3-regular tree, for $\mathbf{v}_1, \mathbf{v}_2$ a base of $\mathbb{Z}^2$.

2.2.2 The values of a quadratic form

Consider an integral binary quadratic form: $Q(x, y) = Ax^2 + Hxy + By^2$ with $A, B, H \in \mathbb{Z}$. We wish to understand the structure of values of Q on $\mathbb{Z}^2$.

Since $Q(\lambda \mathbf{v}) = \lambda^2 Q(\mathbf{v})$ for any integer λ, and $Q(-\mathbf{v}) = Q(\mathbf{v})$, it is enough to understand the values of Q on primitive lax vectors. We will write the value $Q(\mathbf{v})$ on the region corresponding to $\pm\mathbf{v}$. We can start by writing A, B and $A + H + B$ on the regions corresponding to $(1, 0)$, $(0, 1)$ and $(1, 1)$; these regions have a common vertex. Then we can continue filling in the diagram by using the following observation: direct calculation shows that for any two vectors $\mathbf{v}_1, \mathbf{v}_2$,

$$Q(\mathbf{v}_1 + \mathbf{v}_2) + Q(\mathbf{v}_1 - \mathbf{v}_2) = 2(Q(\mathbf{v}_1) + Q(\mathbf{v}_2)).$$

This means that if the 4 values around an edge are $a = Q(\mathbf{v}_1)$, $b = Q(\mathbf{v}_2)$, $c = Q(\mathbf{v}_1 + \mathbf{v}_2)$ and $d = Q(\mathbf{v}_1 - \mathbf{v}_2)$, then d, $a + b$, c form an arithmetic progression.

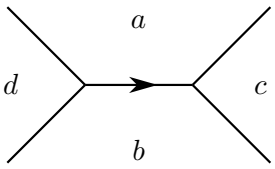


Figure 2.4: $d < a + b < c$ is an arithmetic progression.

We also give an orientation to edges denoting the direction of increase of the arithmetic progression (if it is non-constant). Using the arithmetic progression rule, we can complete the labelling of all the regions and orientations of the edges, resulting in the diagram $\mathcal{T}(Q)$, called the *topograph* of Q.

Two integral binary quadratic forms Q_0 and Q_1 are isomorphic over $\mathbb{Z}$ if and only if they represent the same function on $\mathbb{Z}^2$ but with respect to different bases. Equivalently, there exists a vertex in $\mathcal{T}(Q_0)$ whose surrounding 3 regions have the same numbers as a vertex in $\mathcal{T}(Q_1)$. This shows that the topographs can be used to decide whether two forms are isomorphic over $\mathbb{Z}$, but in order to turn this into an algorithm, we will need to examine the structures of the topographs more closely.

2.2.3 Classifying forms by signs

We get different types of structures depending on the signs of the values that a quadratic form takes. See Figures 2.5 to 2.8 for examples. Conway proved the following:

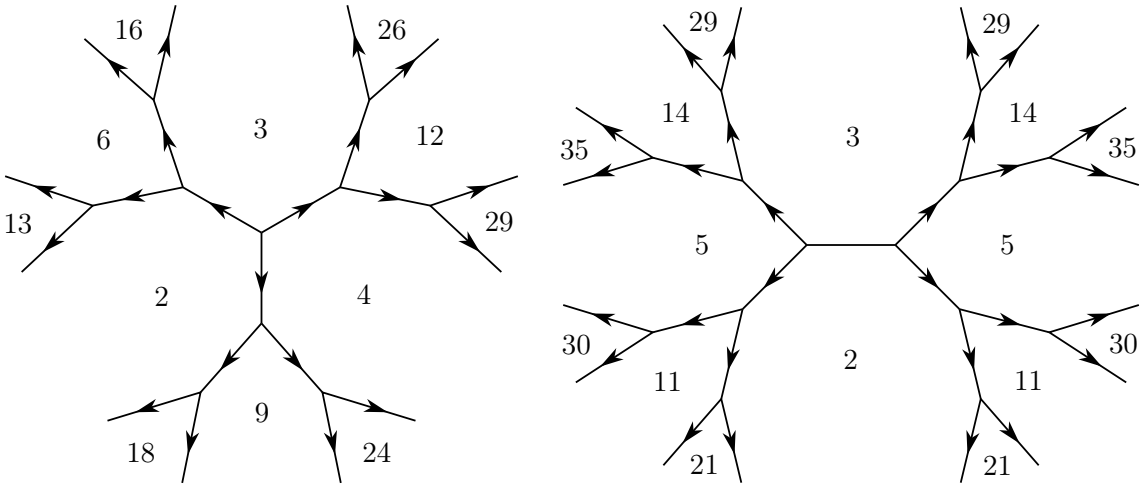


Figure 2.5: Topograph of $2x^2 + xy + 3y^2$ (simple well) and $2x^2 + 3y^2$ (double well).

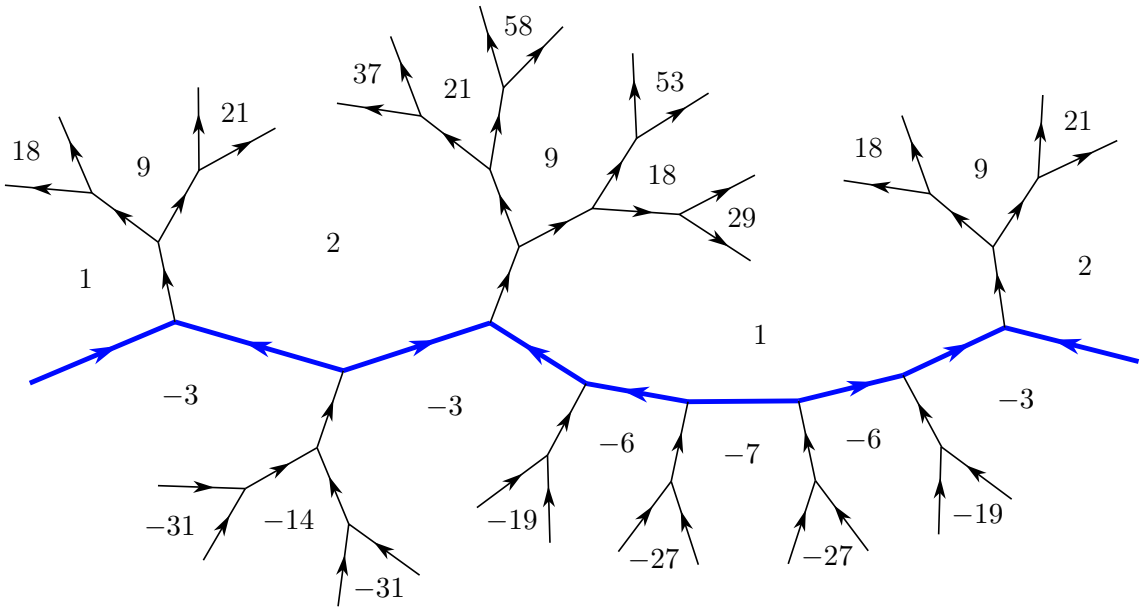


Figure 2.6: Topograph of $x^2 - 7y^2$ (river).

- $+$ forms: If Q is positive definite, then $\mathcal{T}(Q)$ has a vertex such that all edges in the tree are directed away from that vertex, or $\mathcal{T}(Q)$ has two adjacent vertices with a non-oriented edge between them such that all other edges are directed away from them. These 1 or 2 vertices are surrounded by the 3 or 4 smallest values in the topograph, hence we call this a (simple or double) *well*.

- $-$ forms: The negative definite forms are similar.

- $+-$ forms: For indefinite forms, we consider the edges that separate a positive and a negative value. If 0 is not taken as a value, then every vertex has 0 or 2 such edges, hence we get an infinite path, which we call a *river*. There is only one river, and the other edges are directed away from the river on its positive side and towards the

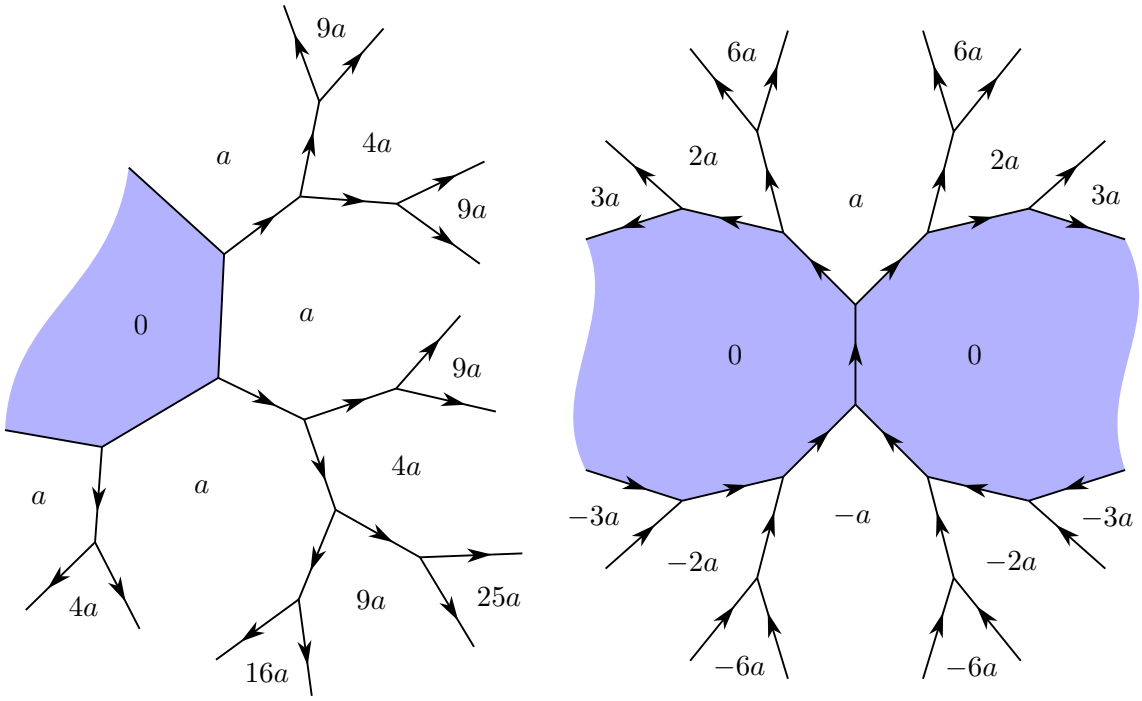


Figure 2.7: Topograph of ax^2 (lake) and axy (weir) for $a > 0$.

river on its negative side. Moreover, for integral forms, the river is periodic, i.e. there exists an m such that moving along the river m steps, we get the same values in the regions next to the river. This is the key fact that allows us to turn this into an algorithm.

- 0 form: For the 0 form, all the values in the topograph are 0.
- 0+ forms: For positive semi-definite forms (that are not the 0 form), there is exactly one region labelled 0, called the *lake*, and all the regions adjacent to the lake have the same value. Moreover, all the edges not in the boundary of the lake are directed away from the lake.
- 0− forms: The negative semi-definite forms are similar.
- 0+− forms: For indefinite forms that take 0 as a value, there are two lakes, with a finite river connecting the two lakes. The positive and negative values in the topograph are separated by the river. All the edges not on the river are directed away from the river on its positive side and towards the river on its negative side. When the river has length 0, the lakes are adjacent, and we also call this a *weir*.

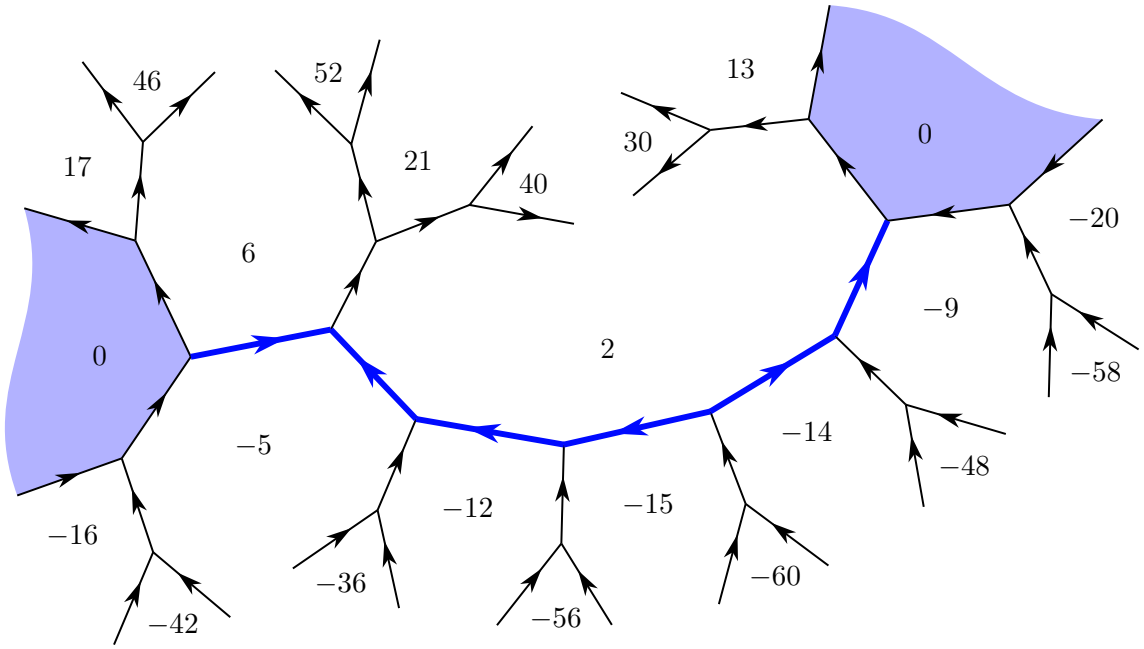


Figure 2.8: Topograph of $6x^2 + 11xy$ (lake-pair).

2.2.4 The algorithm

Using the observations in the previous section, we can give an algorithm that decides whether two integral binary quadratic forms are isomorphic over $\mathbb{Z}$. In fact, we calculate a complete invariant of integral binary quadratic forms, i.e. an invariant that is the same for two forms if and only if the forms are isomorphic over $\mathbb{Z}$.

We start with a triplet of numbers, the values of the form on a superbase, and write these numbers on the regions around a vertex in a 3-regular tree. If the signs of these three numbers are not all the same, then we have found a river or a lake. Otherwise, e.g. if they are all positive, then we calculate the orientations of the three edges using the arithmetic progression rule. (If the numbers are all negative, then we multiply everything by (-1) and refer to the positive case.) If all edges are directed away from the vertex, then we have found a well. Otherwise, there is an edge directed towards the vertex, and we move along that edge and repeat these steps with the triplet around the new vertex. Continuing this procedure of descending, eventually we will find a well, a river, or a lake. If we have a river, then walking along the river, we either get to a lake or find that the river is periodic. If we have a lake, then descending along the boundary of the lake, we either get to a river or find that all the values are the same next to the lake. In each case, using the values around a well, river, or lake, we can clearly define a complete invariant.

We give an implementation of this algorithm in Wolfram Mathematica [Math]. In the following code, `invariant[tri_]` calculates a complete invariant of integral binary

quadratic forms. It takes a triplet of numbers `tri = {u, v, w}` as input (the values of the form on a superbase), and it gives an output that has the format `{"WELL", {a, b, c}}`, `{"RIVER", {a, b, c}}`, `{"LAKE", {a}}`, `{"LAKE-PAIR", {{a, b}, {c, d}}}`, or `{"WEIR", {a}}`, with some integers a, b, c, d that are canonical in some sense (e.g. smallest triplet along the river).

```
lastpos[list_] := Block[{i = -2}, While[list[[i]] < 0, i = i - 1]; list[[i]]]
lastneg[list_] := Block[{i = -2}, While[list[[i]] > 0, i = i - 1]; list[[i]]]
triplet[list_] := {Last@list, lastpos@list, lastneg@list}
rivernext[list_] := (
  {last, lastpositive, lastnegative} = triplet[list];
  Append[list, If[Positive[last],
    2 (lastnegative + last) - lastpositive,
    2 (lastpositive + last) - lastnegative]])
river[tri_] :=
  NestWhile[rivernext, tri, Last@# != 0 &&
    (Length@# <= 4 || {last, lastpositive, lastnegative} != triplet[tri]) &]
riverinvariant[river_] :=
  Block[{t = Table[Sort@triplet[river[[;; i]]], {i, 3, Length@river - 2}]},
    t[[First@Ordering[t, 1]]]]
descendnext[tri_] := Block[{min, middle, max},
  {min, middle, max} = Sort[tri];
  If[max > min + middle,
    {min, middle, 2 (min + middle) - max},
    {"WELL", {min, middle, max}}]]
descend[tri_] := NestWhile[descendnext, tri, Length@# == 3 && Min@# > 0 &]
invariant[tri_] := Block[{d = tri, r, min, middle, max, a, b, u, v},
  If[Min[tri] > 0, d = descend[tri]; If[First@d == "WELL", Return[d]]];
  If[Max[tri] < 0, d = -descend[-tri];
    If[First@d == -"WELL", Return[{"WELL", Last@d}]]];
  If[! MemberQ[d, 0],
    {min, middle, max} = Sort[d];
    r = river[{min, max, middle}];
    If[Last@r == 0, d = triplet[r], Return[{"RIVER", riverinvariant[r]}]]];
  {a, b} = Sort@Rest@SortBy[d, Abs];
  If[a == b, Return[{"LAKE", {a}}]];
  {a, b} = {Mod[a, b - a, -b + a + 1], Mod[a, b - a, 1]};
  If[a == 0, Return[{"WEIR", {b}}]];
  r = river[{a, b, 2 (a + b)}];
  {u, v} = Sort@DeleteCases[triplet[r], 0];
  Return[{"LAKE-PAIR", Sort@{{a, b}, {u, v}}}]]
```

2.2.5 Application to the Seifert forms

We can use the algorithm in the previous section on our quadratic forms Q_0, Q_1 that arose as the symmetrized Seifert forms of the surfaces $\Sigma_i(p, q, k, n) = \Sigma_i$. For instance, Lyon's family of examples [Lyo74b] is $\Sigma_i(3, -4, 1, -n)$ in our notation, which is the reflection of $\Sigma_i(3, 4, 1, n)$. For $n = 1$, this is not covered by Theorem 2.2, but Conway's algorithm can be used to show that the symmetrized Seifert forms are non-isomorphic over $\mathbb{Z}$, and thus Lyon's surface-pairs are non-isotopic in D^4 for $n > 0$.

We can assume $p, q > 0$. Recall that the matrix for Q_i is $V_i + V_i^T$, where

$$V_0 = \begin{bmatrix} pq & qr - kp \\ ps - kp & rs - 2kr + n \end{bmatrix}, \quad V_1 = \begin{bmatrix} pq & -kp \\ 1 - kp & n \end{bmatrix}, \quad ps - qr = 1.$$

The following code calculates whether the invariants of Q_0, Q_1 agree for concrete values of p, q, k, n (in the example below, $(p, q) = (2, 3)$ is fixed and $|k|, |n| \leq 30$ varies), and plots a diagram of the result:

```
{p, q, r, s, k, n} =.;
tri0 = {p q, p q + (p s + q r - 2 k p) + (r s - 2 k r + n), r s - 2 k r + n};
tri1 = {p q, p q + (1 - 2 k p) + n, n};
transform[k_, n_] :=
  Sort[{{Mod[k, q], n - (k (k p - 1))/q + (Mod[k, q] (p Mod[k, q] - 1))/q},
    {Mod[s - k, q], n - (k (k p - 1))/q + (Mod[s - k, q] (p Mod[s - k, q] - 1))/q}}]
parallelgetsol[kmin_, kmax_, nmin_, nmax_] := AbsoluteTiming[
  unknown = Flatten[Table[If[! MemberQ[Keys@sol, transform[k, n]], {k, n}, Nothing],
    {n, nmax, nmin, -1}, {k, kmin, kmax}], 1];
  paralleltable = ParallelTable[{k, n} = value;
    {k, n, invariant[tri0] != invariant[tri1]},
    {value, unknown}, ProgressReporting -> False];
  (sol[transform[#[[1]], #[[2]]]] = #[[3]]) & /@ paralleltable;]
normalize[sol_] := Block[{normalsol = <||>},
  Do[If[MemberQ[Keys@sol, transform[k, n]],
    normalsol[{k, n}] = sol[transform[k, n]]], {k, -size, size}, {n, -size, size}];
  normalsol]
plot[sol_] := Block[{normalsol = normalize[sol]},
  Column[{{p, q},
    MatrixPlot[Normal@SparseArray[
      Table[{-x[[2]] + size + 1, x[[1]] + size + 1} ->
        If[normalsol[x], 1, -1], {x, Keys@normalsol}],
      {2 size + 1, 2 size + 1}], FrameTicks -> None, ImageSize -> 250],
    Iconize[normalsol]}, Center]]

{p, q} = {2, 3};
s = ModularInverse[p, q];
r = (p s - 1)/q;
sol = <||>;
size = 30;
Table[parallelgetsol[k, k, -size, size], {k, -size, size}]; // AbsoluteTiming // First
plot[sol]
```

Figure 2.9 shows the resulting diagrams for a few different values of (p, q). An orange square in these diagrams means that Q_0 and Q_1 are not isomorphic over $\mathbb{Z}$, hence the pairs (D^4, Σ_0) and (D^4, Σ_1) are not homeomorphic in an orientation-preserving way, and $\Sigma_0 \not\sim_4 \Sigma_1$. A blue square means that the Seifert form does not give an answer to whether the surfaces are isotopic. Apart from when the Alexander polynomial is 1 (the points on the parabola, see Theorem 2.3), we do not know if they are isotopic or not.

Topograph type

We can also split the diagrams by the type of the topographs to gain more insight, see Figure 2.10.

We show that the type of topograph (river, well, lake, or lake-pair) is the same for the two quadratic forms. Q_0 and Q_1 are always isomorphic over $\mathbb{Q}$, because

$$P^T V_0 P = V_1 \quad \text{for} \quad P = \begin{bmatrix} 1 & -r/p \\ 0 & 1 \end{bmatrix}.$$

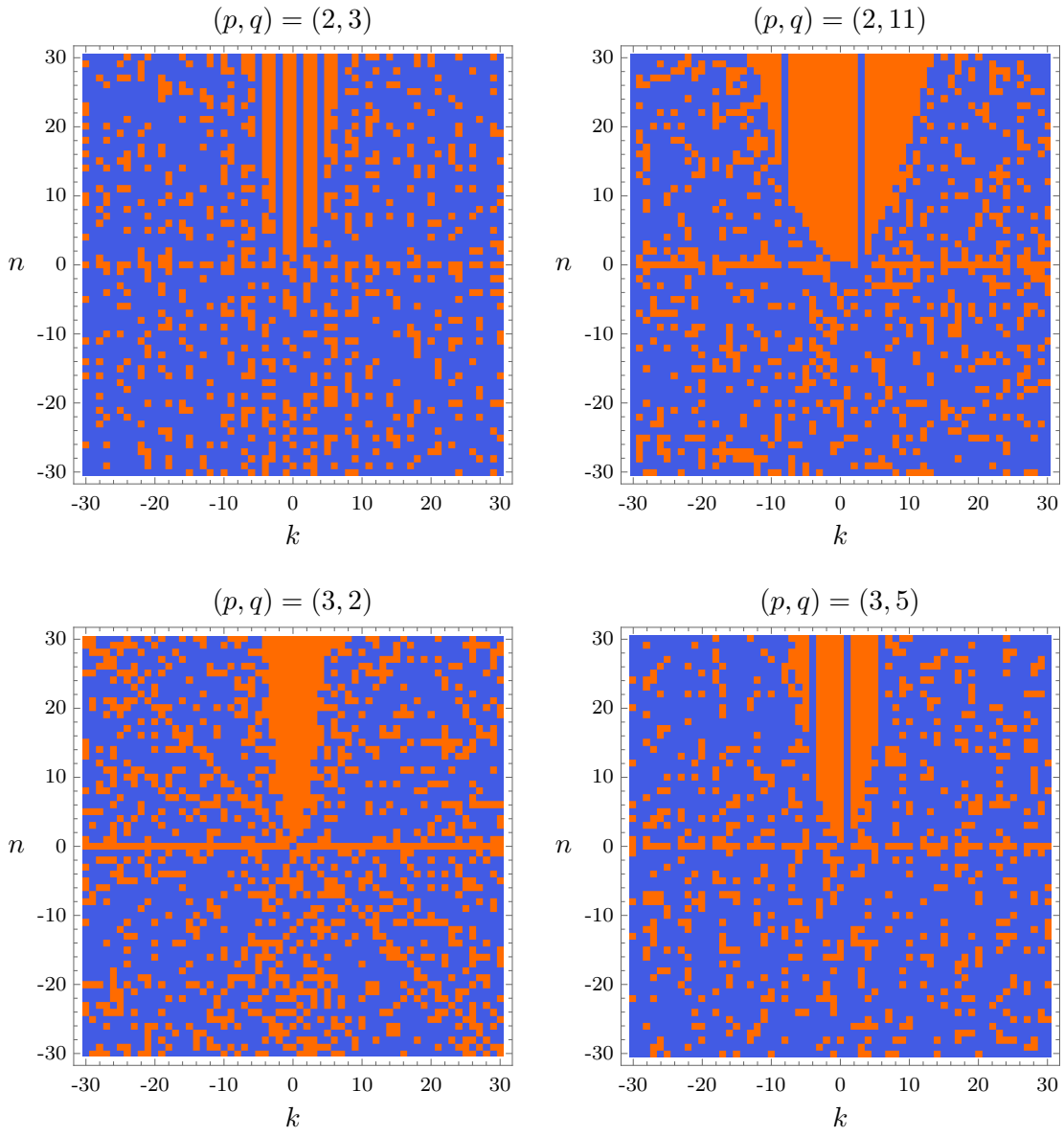


Figure 2.9: Diagrams showing when the symmetrized Seifert forms are isomorphic (blue) and non-isomorphic (orange) over $\mathbb{Z}$.

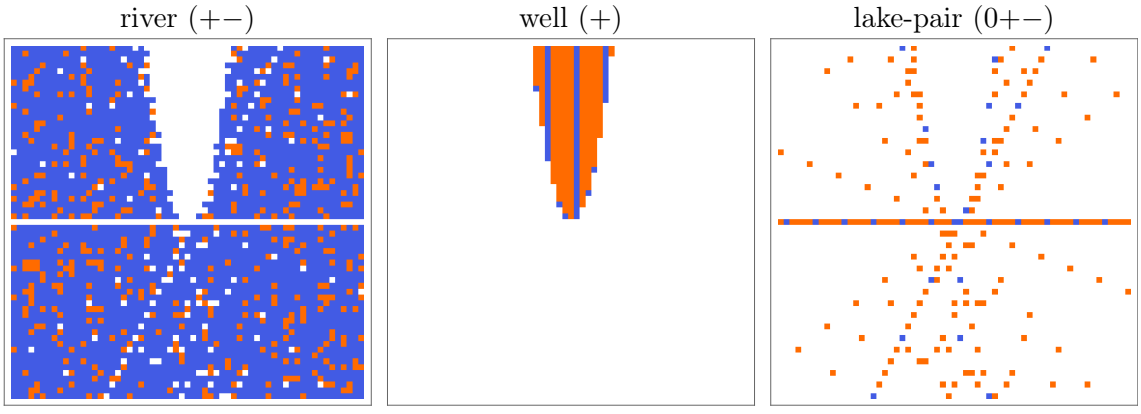


Figure 2.10: Splitting the diagram by topograph type in the case $(p,q) = (3,5)$.

Hence, the signs $(0{+}{-})$ of the values of Q_0 on $\mathbb{Q}^2$ are the same as of Q_1, and thus, their signs on $\mathbb{Z}^2$ agree as well. It follows that $\mathcal{T}(Q_0)$ and $\mathcal{T}(Q_1)$ have the same type.

Symmetries

We note two important symmetries of these diagrams. Consider the affine transformations

$$\tau(k,n) = (k+q,\ n+2kp+pq-1),$$
$$\rho(k,n) = (s-k,\ n-2kr+rs).$$

Note that the value of $n - \frac{k(kp-1)}{q}$ is invariant under τ and ρ. Intuitively, this means that the vertical distance from the parabola $n = \frac{k(kp-1)}{q}$ is unchanged, hence τ is translation to the right by q 'along the parabola', and ρ is reflection in the vertical line $k = s/2$ along the parabola.

Direct calculation shows that the isomorphism classes of the quadratic forms Q_0 and Q_1 are invariant under τ, while ρ swaps them. Hence, the diagrams in Figure 2.9 are invariant under τ and ρ. In fact, the surface-pair $\Sigma_i(p,q,k,n)$ is isotopic in S^3 to a surface-pair $\Sigma_i(p,q,\tau(k,n))$ and to a $\Sigma_{1-i}(p,q,\rho(k,n))$. This can be seen by sliding the attaching region of the band $B(k,n)$ along the full circle boundary of $A_i(p,q)$.

Vertical lines

The diagrams in Figure 2.9 show vertical blue lines q distance apart, but only when q is odd. Indeed, we prove that the lines $2kp \equiv 1 \pmod q$ are blue. The quadratic forms $(x,y) \mapsto (ux+v_iy)^2 + ty^2$ for $i = 0, 1$ are clearly isomorphic over $\mathbb{Z}$ if $v_0 \equiv -v_1 \pmod u$ (c.f. Lemma 2.6). We have seen that the forms $2pqQ_i$ are of this form with

$$u = 2pq, \quad v_0 = 1-2kp+2qr, \quad v_1 = 1-2kp, \quad t = 4pqn-(2kp-1)^2.$$

Hence, we need to show $v_0 + v_1 = 2(1-2kp+qr) \equiv 0 \pmod{2pq}$. As p, q are coprime, this holds if and only if $1-2kp+qr$ is divisible by p and q. It is always divisible by p because $1+qr = ps$, and it is divisible by q when $2kp \equiv 1 \pmod q$ holds. Then $2pqQ_i$ are indeed isomorphic.

Another way to see this is by observing that every point (k,n) on these vertical lines is fixed under a transformation $\tau^{2m} \circ \rho$ for some $m \in \mathbb{Z}$. Since τ fixes the isomorphism class

of Q_i and ρ swaps them, we get

$$Q_0^{(k,n)} = Q_0^{\tau^{2m}(\rho(k,n))} \cong Q_0^{\rho(k,n)} \cong Q_1^{(k,n)}.$$

The parabola

The most prominent feature of the diagrams in Figure 2.9 is the parabolic region: we have proved in Theorem 2.2 that $2kp \not\equiv 1 \pmod{q}$ and

$$n \geq \frac{k(kp-1)}{q} + \left(\frac{pq}{12} - \frac{1}{6} + \frac{1}{2pq}\right)$$

gives a set of orange squares. On the other hand, a consequence of Theorem 2.3 is that the points along the parabola $n = \frac{k(kp-1)}{q}$ are blue. Note that since $pq > 0$, the quadratic forms Q_i are positive definite if and only if $0 < \det(V_1 + V_1^T) = 4pqn - (2kp-1)^2$, or equivalently, $n > \frac{k(kp-1)}{q}$.

It is not true that $n > \frac{k(kp-1)}{q}$ and $2kp \not\equiv 1 \pmod{q}$ gives a fully orange region, as the following example shows: let $(p, q, k, n) = (3, 5, -1, 1)$. Then

$$V_0 = \begin{bmatrix} 15 & 8 \\ 9 & 5 \end{bmatrix} \quad \text{and} \quad V_1 = \begin{bmatrix} 15 & 3 \\ 4 & 1 \end{bmatrix}$$

yield isomorphic quadratic forms. The isomorphism $P^T V_0 P = V_1$ is given by $P = \left[\begin{smallmatrix} 2 & 1 \\ -5 & -2 \end{smallmatrix}\right]$. In fact, the quadratic forms $2pqQ_1(x, y)$ and $2pqQ_0(x + y, -y)$ are the ones in Remark 2.8 with $t = 11$, which was the borderline case. See also the second diagram in Figure 2.10, which displays the positive definite forms, showing a few blue squares at the bottom of the region.

However, we prove that if $k = 0$ or $q \leq 3$, then the assumption on n in Theorem 2.2 can be weakened to $n > \frac{k(kp-1)}{q}$. For that, we use the improved bound on t described in Remark 2.7.

If $k = 0$, then the forms Q_0, Q_1 have $u = 2pq$, $v_0 = 1 + 2qr$, and $v_1 = 1$. Since $[v_0]_u \geq 1 = [v_1]_u$ and $[cv_0]_u^2 \geq 0$ for all c, we have

$$t_1 = \max_{c \geq 2 \text{ integer}} \frac{[v_1]_u^2 - [cv_0]_u^2}{c^2 - 1} \leq \max_{c \geq 2 \text{ integer}} \frac{1}{c^2 - 1} = \frac{1}{3}.$$

We have seen that $[v_0]_u \neq [v_1]_u$ if $p > 1$ and $2kp \not\equiv 1 \pmod{q}$. Hence, Q_0 and Q_1 are non-isomorphic for every $0 < t = 4pqn - (2kp-1)^2$, which is what we wanted to prove.

If $q \leq 3$, then we show that using the symmetries τ and ρ, it can be assumed that

$k = 0$. Indeed, if $q = 2$, then $ps - qr = 1$ implies that s is odd, hence (using ρ) one of k or $s - k$ is even, which can be translated (using τ) to 0 by some multiple of q. If $q = 3$, then the condition $2kp \not\equiv 1 \pmod{q}$ and $(q, s) = 1$ implies $2k \equiv 2kps \not\equiv s \pmod{q}$. Since ρ is a reflection in the line $2k = s$, it means that it swaps the other two remainder classes modulo 3. Hence, k can be sent to 0 by τ and ρ, and we are done.

Horizontal and diagonal lines

Lastly, the third diagram in Figure 2.10 explains why the horizontal line $n = 0$ was special in Theorem 2.4. The diagram shows when the quadratic forms take 0 as a value, which means that each of the forms can be written as a product of two linear terms. This is what we used in the proof.

There often are other lines visible on this diagram too, but in the case of $(p, q) = (3, 5)$, all the other lines are actually consequences of the line $n = 0$ and the symmetries τ and ρ. For instance, the line $n = 2k - 2$ is the ρ-image of the line $n = 0$ (with $(r, s) = (1, 2)$). The line $n = -2k/5$ appears because the point $(5m, -2m)$ is the τ^{2m}-image of the point $(-5m, 0)$ on the $n = 0$ line.

In other cases though, e.g. for $(p, q) = (3, 10)$, there are other factorizable quadratic forms that do not arise in this way.

Gauss composition

Aka–Feller–Miller–Wieser [AFMW24] investigated which pairs of integral binary quadratic forms arise as the symmetrized Seifert forms of a pair of genus one Seifert surfaces for the same knot that have disjoint interiors. They obtained a complete number-theoretic characterization in terms of the Gauss composition $*$. Through the oriented class group $(\mathcal{G}_D^+, *)$, they established a sufficient condition on D for which there exists a knot K with determinant D that has non-isotopic Seifert surfaces $S \not\sim_4 S'$. Knots have a positive determinant if and only if their symmetrized Seifert forms are positive definite, corresponding to $n > \frac{k(kp-1)}{q}$ in our family.

Chapter 3

Khovanov homology

3.1 Background

We give a brief overview of the definition of Khovanov homology, following [HS24]. Our focus is on the combinatorial details that are crucial for implementing the computer program. For a broader discussion of the context and philosophy underlying Khovanov homology, see [BN05, Kho00]. Throughout, we work with $\mathbb{Z}$ coefficients.

3.1.1 Khovanov homology of links

Let $L \subseteq S^3$ be an oriented link. Choose a diagram D of L, and an enumeration $X = (x_1, \dots, x_n)$ of its crossings. We define a chain complex $(\mathcal{C}\mathrm{Kh}(D, X), d)$ associated to the data (D, X) as follows.

For any binary sequence $\sigma = (\sigma_1, \dots, \sigma_n) \in \{0, 1\}^n$, we consider the *smoothing* D_σ, which is obtained by replacing each crossing x_i  with a 0-smoothing  or 1-smoothing , depending on σ_i. We call the connected components of D_σ *loops*. A *labelled smoothing* α_σ is a labelling of the loops of D_σ with a 1 or x. Let $\mathcal{C}\mathrm{Kh}(D, X)$ be the free $\mathbb{Z}$-module generated by all the labelled smoothings α_σ of D.

For simplicity, we describe the differential map d using the Morse induced chain maps that are defined in Table 3.1. Let α_σ be a labelled smoothing with $\sigma = (\sigma_1, \dots, \sigma_n)$. For an i with $\sigma_i = 0$, let m_i denote the chain map induced by the Morse saddle move  that turns the 0-smoothing of x_i into the 1-smoothing. (Thus, $m_i(\alpha_\sigma)$ is a sum of 0, 1, or 2 labelled smoothings.) We define the differential map $d\colon \mathcal{C}\mathrm{Kh}(D, X) \to \mathcal{C}\mathrm{Kh}(D, X)$ on the generators by

$$d(\alpha_\sigma) = \sum_{\substack{i \\ \sigma_i = 0}} (-1)^{\xi_i} m_i(\alpha_\sigma), \qquad \xi_i = \sum_{\substack{j \\ j < i}} \sigma_j,$$

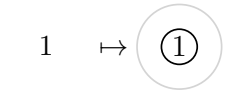
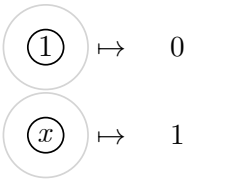
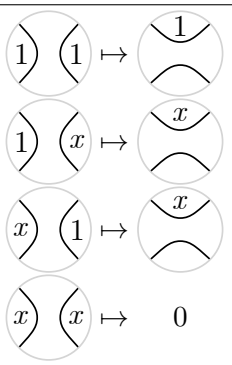
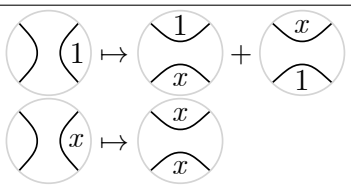
Morse move	Ornament	Definition of chain map
birth		
death		
saddle		
		

Table 3.1: Chain maps induced by a Morse move.

and extend it linearly to $\mathcal{C}\mathrm{Kh}(D,X)$. Then it can be verified that $d^2 = 0$, showing that $(\mathcal{C}\mathrm{Kh}(D,X), d)$ is indeed a chain complex. Its homology is denoted by $\mathrm{Kh}(D,X)$. It turns out that up to isomorphism, $\mathrm{Kh}(D,X)$ is unaffected by our particular choices of D and X [Kho00], and thus, it can be denoted by $\mathrm{Kh}(L)$, the *Khovanov homology* of the link L.

The chain complex $\mathcal{C}\mathrm{Kh}(D,X)$ is bigraded by homological grading h and quantum grading q. Let n_+, n_- be the number of positive and negative crossings in D. For a labelled smoothing α_σ, let $|\sigma|$ be the number of 1-smoothings in σ, and let $v_+(\alpha_\sigma), v_-(\alpha_\sigma)$ be the number of 1-labels and x-labels in α_σ. The grading (h,q) of α_σ is defined by

$$\begin{aligned} h(\alpha_\sigma) &= |\sigma| - n_-, \\ q(\alpha_\sigma) &= v_+(\alpha_\sigma) - v_-(\alpha_\sigma) + h(\alpha_\sigma) + n_+ - n_-, \end{aligned}$$

and $\mathcal{C}\mathrm{Kh}^{h,q}(D,X)$ is defined as the submodule generated by such α_σ. Then the differential map is also bigraded, $d\colon \mathcal{C}\mathrm{Kh}^{h,q}(D,X) \to \mathcal{C}\mathrm{Kh}^{h+1,q}(D,X)$. Hence, the bigrading descends to homology, and we have that $\mathrm{Kh}(L) = \bigoplus_{h,q} \mathrm{Kh}^{h,q}(L)$ is a bigraded $\mathbb{Z}$-module.

3.1.2 Crossing order change

We explicitly construct the isomorphism of $\mathcal{C}\mathrm{Kh}(D, X)$ for a different choice of X, as this will be essential for computations. Consider two enumerations $X = (x_1, \ldots, x_n)$ and $X' = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ for some permutation π. Then any labelled smoothing $\alpha_\sigma \in \mathcal{C}\mathrm{Kh}(D, X)$ with $\sigma = (\sigma_1, \ldots, \sigma_n)$ naturally corresponds to a labelled smoothing $\alpha'_{\sigma'} \in \mathcal{C}\mathrm{Kh}(D, X')$ that has $\sigma' = (\sigma_{\pi(1)}, \ldots, \sigma_{\pi(n)})$ and the same labels. By definition, the differential d' on $\mathcal{C}\mathrm{Kh}(D, X')$ is given by

$$d'(\alpha'_{\sigma'}) = \sum_{\substack{i \\ \sigma_i = 0}} (-1)^{\xi'_i} m_i(\alpha'_{\sigma'}), \qquad \xi'_i = \sum_{\substack{j \\ \pi^{-1}(j) < \pi^{-1}(i)}} \sigma_j.$$

We define the isomorphism $\Phi \colon \mathcal{C}\mathrm{Kh}(D, X) \to \mathcal{C}\mathrm{Kh}(D, X')$ on the generators by

$$\Phi(\alpha_\sigma) = (-1)^{\pi_\sigma} \alpha'_{\sigma'}, \qquad \pi_\sigma = \sum_{\substack{j,k \\ j<k \\ \pi^{-1}(j) > \pi^{-1}(k)}} \sigma_j \sigma_k,$$

and extend it linearly. Then it can be checked that $\Phi(d(\alpha_\sigma)) = d'(\Phi(\alpha_\sigma))$. Therefore, $(\mathcal{C}\mathrm{Kh}(D, X), d)$ and $(\mathcal{C}\mathrm{Kh}(D, X'), d')$ are isomorphic chain complexes, and $\mathrm{Kh}(D, X) \cong \mathrm{Kh}(D, X')$.

We also provide a more intuitive perspective on this isomorphism, by describing the differential map in a way that is independent of the crossing order, as follows. For a labelled smoothing α_σ, let $i_1 < \cdots < i_k$ represent all indices i where $\sigma_i = 1$, and replace α_σ with the formal expression $\alpha_\sigma \cdot \omega_\sigma$, where $\omega_\sigma = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. By substituting each labelled smoothing (α_σ and every $m_i(\alpha_\sigma)$) in this way within the definition of d, the differential map can be written formally as

$$d = m_1 \cdot dx_1 + \cdots + m_n \cdot dx_n,$$

where $m_i \cdot dx_i$ acts on a labelled smoothing as

$$(m_i \cdot dx_i)(\alpha_\sigma \cdot \omega_\sigma) = m_i(\alpha_\sigma) \cdot (dx_i \wedge \omega_\sigma).$$

Indeed, the summation restricts to the indices i with $\sigma_i = 0$ because $dx_i \wedge dx_i = 0$, and the signs $(-1)^{\xi_i}$ in our original definition of d come from $dx_i \wedge dx_j = -dx_j \wedge dx_i$, when we change $dx_i \wedge \omega_\sigma$ to have increasing indices (as $m_i(\alpha_\sigma)$ has a new 1-smoothed crossing x_i).

With this order-independent view of d, it is clear what happens when we change the

order of crossings to X'. The new $\omega'_{\sigma'}$ that uses the ordering X' has $\omega'_{\sigma'} = \pm\omega_\sigma$, where the $\pm$ sign depends on the parity of the permutation of $dx_{i_1}, \dots, dx_{i_k}$ in $\omega'_{\sigma'}$. In the above definition of Φ, α_σ gets this sign, whose explicit definition is $(-1)^{\pi_\sigma}$. For example, if $X = (x_1, x_2, x_3)$ and $X' = (x_1, x_3, x_2)$, then

$$\Phi(\alpha_{110} + \alpha_{101} + \alpha_{011}) = \alpha'_{101} + \alpha'_{110} - \alpha'_{011}.$$

3.1.3 Khovanov homology of cobordisms

Let L_0, L_1 be links. A *cobordism* from L_0 to L_1 is a smoothly embedded compact surface $\Sigma \subseteq S^3 \times [0,1]$ whose boundary is $L_0 \times \{0\} \cup L_1 \times \{1\}$. Any cobordism can be represented as a *movie*: a sequence of diagrams $D_0, \dots, D_N$, where D_{i+1} is obtained from D_i by an *elementary move* (Reidemeister or Morse move), and D_0, D_N represent the links L_0, L_1, respectively. We often write the cobordism as a function $\Sigma\colon L_0 \to L_1$.

Given a cobordism $\Sigma\colon L_0 \to L_1$, we will associate to Σ a group homomorphism $\mathrm{Kh}(\Sigma)\colon \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1)$. Moreover, if Σ is orientable, with oriented boundary $\partial\Sigma = (L_0 \times \{0\}) \cup (L_1^r \times \{1\})$ (where L_1^r denotes L_1 with reversed orientation), then $\mathrm{Kh}(\Sigma)$ will be a bigraded map:

$$\mathrm{Kh}(\Sigma)\colon \mathrm{Kh}^{h,q}(L_0) \to \mathrm{Kh}^{h,q+\chi(\Sigma)}(L_1).$$

We now give the definition of this map, using the movie $D_0, \dots, D_N$. We do this by defining a chain map $\mathcal{C}\mathrm{Kh}(D_i, X_i) \to \mathcal{C}\mathrm{Kh}(D_{i+1}, X_{i+1})$ associated to every Reidemeister and Morse move, and $\mathcal{C}\mathrm{Kh}(\Sigma)$ will be the composition of these maps.

Morse moves

There are three types of Morse moves: *births*, *deaths*, and *saddles*. A birth ⌑ adds a small new loop to the diagram, a death ⊗ removes a small loop, and a saddle ⋋ merges or splits loops, by changing)(to ⋌ locally.

Let X_i be an enumeration of the crossings of D_i. If D_{i+1} is obtained from D_i by a Morse move, then the list of crossings does not change. We let $X_{i+1} = X_i$, and we define the chain map $\mathcal{C}\mathrm{Kh}(D_i, X_i) \to \mathcal{C}\mathrm{Kh}(D_{i+1}, X_{i+1})$ on labelled smoothings according to Table 3.1 (taken from [HS24]).

Reidemeister moves

There are three types of Reidemeister moves: R1 removes or adds 1 crossing, R2 removes or adds 2 crossings, and R3 manipulates the diagram around 3 crossings. We define the

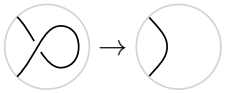
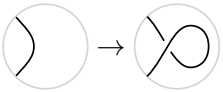
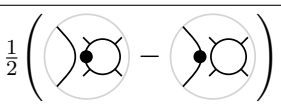
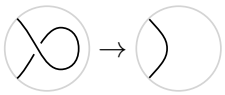
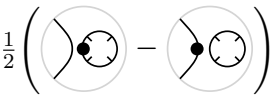
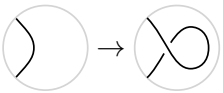
Reidemeister move	Smoothing	Induced map
		
		0
		
		
		0
		

Table 3.2: Chain maps induced by a Reidemeister 1 move.

Reidemeister induced chain maps using the Morse induced chain maps defined previously. In addition, we will use  to denote two saddle moves on an arc (one splitting, then one re-merging). Then $\frac{1}{2}$ sends a 1-labelled loop to an x-labelled loop, and an x-labelled loop to 0.

Let X_i be an enumeration of the crossings of D_i. If D_{i+1} is obtained from D_i by an R1 move that adds a crossing x, then add x to the enumeration in the last position: $X_{i+1} = (X_i, x)$. We define the chain map $\mathcal{C}\mathrm{Kh}(D_i, X_i) \to \mathcal{C}\mathrm{Kh}(D_{i+1}, X_{i+1})$ on labelled smoothings according to Table 3.2. If the R1 move removes a crossing x, then consider first an enumeration X_i' that has x as its last element, and apply the isomorphism $\mathcal{C}\mathrm{Kh}(D_i, X_i) \to \mathcal{C}\mathrm{Kh}(D_i, X_i')$ described in Section 3.1.2. Then we define $\mathcal{C}\mathrm{Kh}(D_i, X_i') \to \mathcal{C}\mathrm{Kh}(D_{i+1}, X_{i+1})$ according to the same Table, where $X_i' = (X_{i+1}, x)$.

If D_{i+1} is obtained from D_i by an R2 move that adds the crossings x_1, x_2, then let $X_{i+1} = (X_i, x_1, x_2)$, and define the chain map $\mathcal{C}\mathrm{Kh}(D_i, X_i) \to \mathcal{C}\mathrm{Kh}(D_{i+1}, X_{i+1})$ on labelled smoothings according to Table 3.3. If the R2 move removes two crossings x_1, x_2, then first apply an isomorphism $\mathcal{C}\mathrm{Kh}(D_i, X_i) \to \mathcal{C}\mathrm{Kh}(D_i, X_i')$ that moves x_1, x_2 to the last two positions in X_i', then define $\mathcal{C}\mathrm{Kh}(D_i, X_i') \to \mathcal{C}\mathrm{Kh}(D_{i+1}, X_{i+1})$ according to the same Table.

Finally, assume that D_{i+1} is obtained from D_i by an R3 move. The chain maps are presented in table form (Table 3.4), interpreted as follows. (These tables are adapted from [HS24], but with corrected signs.) The top-left cell of each table depicts the Reidemeister

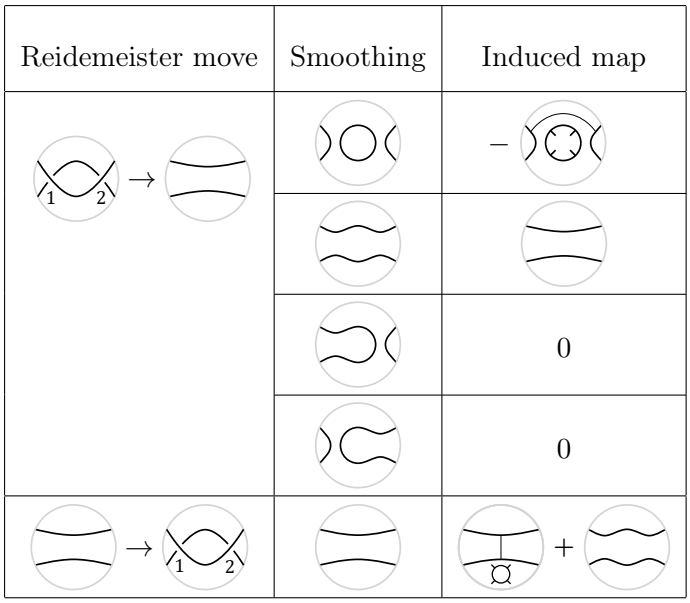


Table 3.3: Chain maps induced by a Reidemeister 2 move.

move, showing the starting diagram with crossings x_1, x_2, x_3 in the left corner, and the ending diagram with crossings y_1, y_2, y_3 in the right corner. As usual, we first apply an isomorphism $\mathcal{C}\mathrm{Kh}(D_i, X_i) \to \mathcal{C}\mathrm{Kh}(D_i, X_i')$, which reorders the crossings x_1, x_2, x_3 to the last three positions. Write $X_i' = (X, x_1, x_2, x_3)$, and let $X_{i+1} = (X, y_1, y_2, y_3)$. The map $\mathcal{C}\mathrm{Kh}(D_i, X_i') \to \mathcal{C}\mathrm{Kh}(D_{i+1}, X_{i+1})$ is then defined as follows. The left column of the table lists the smoothings of the starting diagram, while the top row lists the smoothings of the ending diagram. The image of a labelled smoothing is obtained by adding together the non-empty cells in its corresponding row, where I denotes an isotopy of the labelled smoothing.

This completes the definition of the maps induced by elementary moves. Now, let $\Sigma\colon L_0 \to L_1$ be a cobordism with movie $D_0, \dots, D_N$, and let X_0 and X_N' be enumerations of the crossings of D_0 and D_N, respectively. We obtain enumerations $X_1, \dots, X_N$ and maps $\phi_i\colon \mathcal{C}\mathrm{Kh}(D_i, X_i) \to \mathcal{C}\mathrm{Kh}(D_{i+1}, X_{i+1})$ from the description above. We define

$$\mathcal{C}\mathrm{Kh}(\Sigma, X_0, X_N') = \Phi \circ \phi_{N-1} \circ \cdots \circ \phi_0\colon \mathcal{C}\mathrm{Kh}(D_0, X_0) \to \mathcal{C}\mathrm{Kh}(D_N, X_N'),$$

where $\Phi\colon \mathcal{C}\mathrm{Kh}(D_N, X_N) \to \mathcal{C}\mathrm{Kh}(D_N, X_N')$ is the isomorphism induced by reordering the crossings. It can be checked that any ϕ_i is a chain map. Therefore, $\mathcal{C}\mathrm{Kh}(\Sigma, X_0, X_N')$ is also a chain map, which induces a map on homology: $\mathrm{Kh}(\Sigma)\colon \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1)$.

The power of using Khovanov homology for cobordisms comes from the fact that the map $\mathrm{Kh}(\Sigma)$ is invariant (up to a global sign) under smooth isotopy of Σ relative to the boundary. In fact, the same holds if we only look at a diffeomorphism of the ambient space:

	000	100	010	001	110	101	011	111
000	I							
100		I		I				
010			I					
001								
110					I			
101							$-I$	
011								
111								

	000	100	010	001	110	101	011	111
000								
100			I					
010								
001				I				
110							I	
101						I		
011							I	
111								I

Table 3.4: Chain maps induced by a Reidemeister 3 move.

Theorem 3.1 ([LS22, Proposition 3.7]). *Let $\Sigma \subseteq S^3 \times [0,1]$ be a cobordism from L_0 to L_1. Then the map $\mathrm{Kh}(\Sigma)\colon \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1)$ is invariant up to multiplication by ± 1 under any diffeomorphism of $S^3 \times [0,1]$ that restricts to the identity on the boundary.*

As often is the case, if $\Sigma \subseteq D^4$ is a slice surface bounded by the link $L \subseteq S^3$, then Σ can be viewed as a cobordism $\Sigma\colon L \to \emptyset$, or as $\Sigma^\star\colon \emptyset \to L$. Thus, to distinguish the surfaces $\Sigma_0, \Sigma_1 \subseteq D^4$ smoothly, it suffices to distinguish the maps $\mathrm{Kh}(\Sigma_i)$, or the maps $\mathrm{Kh}(\Sigma_i^\star)$. Recent examples in the literature have demonstrated the use of this method to distinguish surfaces smoothly or exotically [SS22, HS24, HKM+22, Hay23].

If $\Sigma\colon \emptyset \to L$ is a cobordism, then the map $\mathrm{Kh}(\Sigma)\colon \mathrm{Kh}(\emptyset) \to \mathrm{Kh}(L)$ is entirely determined by the image of $1 \in \mathbb{Z} = \mathrm{Kh}(\emptyset)$. We call this homology class $\mathrm{Kh}(\Sigma)(1)$ the *Khovanov–Jacobsson class* of Σ.

To determine whether two Khovanov–Jacobsson classes are the same up to sign, part (b) of the following well-known statement often provides a simple diagrammatic proof that a chain element is non-zero in homology.

Proposition 3.2. *Let $\alpha \in \mathcal{C}\mathrm{Kh}(D, X)$ be a chain element, given as a linear combination $\alpha = n_1\alpha_1 + \cdots + n_s\alpha_s$ of distinct labelled smoothings α_i, $0 \neq n_i \in \mathbb{Z}$.*

(a) *If, for every α_i, each 0-smoothed crossing in α_i connects two distinct x-labelled loops, then $d\alpha = 0$, i.e. α is a cycle.*

(b) *If there exists an α_i where each 1-smoothed crossing in α_i connects two distinct 1-labelled loops, then $\alpha \neq d\beta$ for any β, i.e. α is not a boundary.*

Proof. This is straightforward from the definition of d and the Morse saddle induced chain maps (Table 3.1). □

3.1.4 Mirroring

Let $-L$ denote the mirror of the link L, and $-D$ the mirror of the diagram D, obtained by swapping overstrands and understrands at every crossing. Then there is an isomorphism of chain complexes $\mathcal{C}\mathrm{Kh}(-D, X) \cong \mathcal{C}\mathrm{Kh}(D, X)^\star$, where $C^\star = \mathrm{Hom}(C, \mathbb{Z})$ is the dual of the $\mathbb{Z}$-module C [Kho00, Proposition 32]. This isomorphism Ψ_D is given by $\alpha_\sigma \mapsto (\beta_{-\sigma})^\star$, where $(-\sigma)_i = 1 - \sigma_i$ is the same smoothing on the mirrored diagram, $\beta_{-\sigma}$ is obtained from α_σ by swapping 1-labels and x-labels, and for a $\mathbb{Z}$-module C freely generated by $c, c_1, c_2, \ldots \in C$, the homomorphism $c \mapsto 1, c_i \mapsto 0$ is denoted by $c^\star \in C^\star$.

If $f\colon C \to B$ is a map, then the dual $f^\star\colon B^\star \to C^\star$ is defined by $\phi \mapsto \phi \circ f$ for $\phi \in B^\star$. Then it is straightforward to check that $\Psi_D \circ d_{-D} = d_D^\star \circ \Psi_D$, showing the isomorphism of

chain complexes $(\mathcal{C}\mathrm{Kh}(-D,X), d_{-D})$ and $(\mathcal{C}\mathrm{Kh}(D,X)^\star, d_D^\star)$. Taking homology, this shows that $\mathrm{Kh}(-L)$ is related to $\mathrm{Kh}(L)$ in the usual way that homology is related to cohomology. If Tor denotes the torsion subgroup, this means that, since Ψ multiplies both gradings by -1,

$$\mathrm{Kh}^{h,q}(-L) \otimes \mathbb{Q} \cong \mathrm{Kh}^{-h,-q}(L) \otimes \mathbb{Q},$$
$$\mathrm{Tor}(\mathrm{Kh}^{h,q}(-L)) \cong \mathrm{Tor}(\mathrm{Kh}^{1-h,-q}(L)).$$

Let $\Sigma\colon L_0 \to L_1$ be a cobordism. The reverse of the cobordism is $\Sigma^\star\colon L_1 \to L_0$, and the mirror of the cobordism is $-\Sigma\colon -L_0 \to -L_1$. Then the cobordism maps induced on Khovanov homology are compatible with the previous isomorphism [Sun22]. Specifically, the following diagram commutes up to chain homotopy:

$$\begin{array}{ccc} \mathcal{C}\mathrm{Kh}(-D_0,X_0) & \xrightarrow[\Psi_{D_0}]{\cong} & \mathcal{C}\mathrm{Kh}(D_0,X_0)^\star \\ {\scriptstyle \mathcal{C}\mathrm{Kh}(-\Sigma)}\downarrow & & \downarrow{\scriptstyle \mathcal{C}\mathrm{Kh}(\Sigma^\star)^\star} \\ \mathcal{C}\mathrm{Kh}(-D_1,X_1) & \xrightarrow[\Psi_{D_1}]{\cong} & \mathcal{C}\mathrm{Kh}(D_1,X_1)^\star \end{array}$$

3.2 The computer program

We have developed a Python module that can calculate the map $\mathrm{Kh}(\Sigma)$ for any cobordism Σ (given as a movie). This addresses an important gap in the existing computational tools for Khovanov homology. The source code is provided in the file `khovanov.py` attached to this thesis, and the most up-to-date version of the program is available on GitHub [KhCC].

As a basic example, we show that $A_0(2,3) \not\approx_4 A_1(2,3)$ from Theorem 2.1 using the following code:

```
from khovanov import *

L = Link(braid_closure = [1, 2, 3] * 6)
c, d = L.crossings[0], L.crossings[3]
L.orient((c, 0), (c, 1))

A0 = Cobordism(L)
A0.morse_saddle((c, 0), (c, 3))
A0.finish()

A1 = Cobordism(L)
A1.morse_saddle((d, 0), (d, 3))
A1.finish()

compare(A0.KJ_class(), A1.KJ_class())
```

Lines 3–5 define the link $L = T_{4,6}$ using a braid closure, denote two of its crossings c and d, and define its orientation (needed only when the link has more than one component). Lines 7–9 define the annulus A_0 combinatorially, viewed as a cobordism from L to $\emptyset$. Its movie starts with a Morse saddle move between two strands of the diagram that are identified by the 0th and 3rd strands around the crossing c. The next moves of the movie would then be added as subsequent function calls (e.g. `A0.morse_birth()`, `A0.reidemeister_2((d, 3), (d, 0))`, `A0.reidemeister_1_up((c, 2))`). In this example, however, the resulting link after the first move is the unknot, so we use the convenient `finish()` function to automatically finish the rest of the cobordism. Lines 11–13 define the cobordism A_1 similarly, starting with a different Morse saddle move. In line 15, we calculate chain representatives of the Khovanov–Jacobsson classes of $A_0^\star, A_1^\star$, and compare them in homology. In this case, the program outputs that both their sum and difference are trivially outside the image of the differential map d (Proposition 3.2 (b)), giving a proof that $A_0 \not\approx_4 A_1$ by Theorem 3.1.

3.2.1 Classes and methods

We provide a brief overview of the program's classes, highlighting key properties and methods that might be useful to users. The program builds on SnapPy's `Link` class [Snap], with modifications. This class represents a link diagram combinatorially as a graph in which each vertex has degree 4 or 2.

Crossing

A `Crossing` represents a crossing in the link diagram, modelled as a degree-4 vertex. The strands are labelled 0, 1, 2, and 3 in positive orientation around the vertex, with 0 and 2 corresponding to the understrand. Its properties:

- `adjacent` specifies the connection for each strand.
- `label` is used to identify a crossing within a `Link` (usually an integer).

For example, `a = Crossing(7)` creates a `Crossing` object with `label` 7, and connecting the strand `(a, 3)` to `(b, 0)` can be achieved by writing `a[3] = b[0]`.

Strand

A `Strand` is similar to a `Crossing`, but represents a degree-2 vertex. While SnapPy automatically fuses these and removes unknotted unlinked components, we retain them as they are essential for representing cobordism movies.

Link

A `Link` represents a link diagram with an enumeration of its crossings. Its property:

- `crossings` provides a list of `Crossing` and `Strand` objects that form the vertices of the graph.

Links can be created using PD codes, braid closures, or a list of `Crossing` and `Strand` objects, e.g. `L = Link([a, b, c])`. Its methods:

- `L.orient(cs0, cs1, ...)`: if the link L has multiple components, it may be necessary to define its orientation to set its `n_plus` and `n_minus` properties (representing the number of positive and negative crossings).
- `L.differential_matrix(h, q)` returns a matrix of the differential map d in grading $(h, q) \to (h + 1, q)$.
- `L.homology_with_generators(h, q)` calculates the Khovanov homology $\mathrm{Kh}^{h,q}(L)$, expressing it as a direct sum of cyclic groups and returning chain representatives for the generators of each group (this method needs Sage [Sage]).

For computing Khovanov homology without generators, significantly faster programs exist, such as KnotJob [KJob], which implements Bar-Natan's efficient algorithm [BN07].

CrossingStrand

A `CrossingStrand` represents one of the four strands around a `Crossing` (or one of the two strands around a `Strand`). Its properties:

- `crossing` references the associated `Crossing` or `Strand` object.
- `strand_index` indicates the strand's position, e.g. 0, 1, 2, or 3.

For example, `cs = CrossingStrand(a, 2)` sets `cs.crossing` to the object `a`, and `cs.strand_index` to 2.

`CrossingStrand` objects are crucial for defining cobordisms, as they allow precise specification of strands involved in an elementary move. Several methods are available to navigate within a `Link`:

- `cs + 1`, `cs.next()`, and `cs - 1` give the adjacent strands around the same crossing.
- `cs.opposite()` returns the strand located at the opposite end of the edge connected to `cs`.

It is important to understand that while `cs` and `cs.opposite()` lie on the same edge of the underlying graph of the link diagram, they represent distinct portions of the edge. Care must be taken to use the appropriate one when specifying strands for defining an

elementary move.

SmoothLink

A `SmoothLink` represents a smoothing of a link diagram. Its properties:

- `link` refers to the underlying `Link`.
- `smoothing` is a dictionary mapping each `Crossing` to 0 or 1.
- `loops` is a list of loops, where each loop is represented as a tuple of `CrossingStrand` objects (two at each crossing) traversed along the loop.

LabelledSmoothing

A `LabelledSmoothing` represents a labelled smoothing with a coefficient. Its properties:

- `smooth_link` refers to the underlying `SmoothLink`.
- `labels` is a dictionary mapping each loop to `"1"` or `"x"`.
- `coefficient` is an integer that will represent the coefficient in a linear combination.

There are several methods for examining the relationship between a `LabelledSmoothing` `LS` and the image of the differential map (without computing the entire differential matrix):

- `LS.differential()` returns its differential as a list of `LabelledSmoothing` objects, interpreted as a linear combination.
- `LS.is_outside()` checks if every 1-smoothed crossing connects two distinct 1-labelled loops. If so, then `LS` lies outside the image of the differential by Proposition 3.2 (b).
- `LS.row_size()` and `LS.reverse_differential()` give how many and which elements contain `LS` in their differential.

CKhElement

A `CKhElement` represents a chain element of $\mathcal{C}\mathrm{Kh}(D, X)$. It is a list of `LabelledSmoothing` objects, interpreted as a linear combination using their `coefficient` properties.

Each of the elementary moves can be applied to a `CKhElement` `CKH` by calling the corresponding method and specifying the location with appropriate `CrossingStrand` objects (see the source code for more details and pictures):

- `CKH.morse_birth()` adds a new loop.
- `CKH.morse_death(cs)` removes a loop containing `cs`.
- `CKH.morse_saddle(cs0, cs1)` adds a saddle move to connect `cs0` with `cs1`.

- `CKH.reidemeister_1(cs)` removes a twist containing `cs`.
- `CKH.reidemeister_1_up(cs, True)` adds a positive twist to the left of `cs`.
- `CKH.reidemeister_2(cs0, cs1)` removes the common crossing of `cs0` and `cs1`, and the common crossing of `cs0.opposite()` and `cs1.opposite()`.
- `CKH.reidemeister_2_up(cs0, cs1)` moves `cs0` to the right and over `cs1`.
- `CKH.reidemeister_3(cs0, cs1, cs2)` moves `cs0` over `cs1`, `cs2`, and the common crossing of `cs1.opposite()` and `cs2.opposite()`.

These change the underlying `LabelledSmoothing`, `SmoothLink`, and `Link` objects as well by the elementary move. Further methods:

- `CKH.fuse(strand)` fuses a `Strand`, combining the two edges from it.
- `CKH.differential()` returns the differential.
- `CKH.reorder_crossings(new_order)` reorders the crossings (see Section 3.1.2).
- `CKH.replace_link(link, flipping)` might be useful if `CKH` is on a different copy of the same link.
- `CKH.simplify()` simplifies the linear combination by combining equal terms and removing 0-coefficient elements.
- `CKH0 + CKH1` and `CKH0 - CKH1` calculate the sum and difference of two chain elements and simplify the result.
- `compare(CKH0, CKH1)` returns whether two chain elements are the same up to sign in homology.

Cobordism

A `Cobordism` represents a movie of a cobordism between two links. Its properties:

- `movie` contains a list of elementary moves and their locations.
- `links` gives the list of links corresponding to each stage of the movie.

A new cobordism $C\colon L \to L$ with empty movie can be created by `C = Cobordism(L)`, and moves can be added by subsequent calls of methods. Crossing labels can also be used in arguments instead of `CrossingStrand` objects, e.g. `C.morse_death((7, 1))`. In addition to the above methods of `CKhElement`, there are two convenient methods for adding multiple elementary moves at once: `band_move(...)` and `finish()`.

- `C.band_move(n, cs0, (cs1, True), (cs2, False), ..., csk)` adds a band with n half-twists, starting at `cs0`, passing over `cs1`, under `cs2`, and so on, before ending at `csk`.
- `C.finish()` tries to simplify the last link using Reidemeister 1–3 moves (similar to

SnapPy's `simplify("level")` algorithm), then does a Morse death on each crossingless component.

- `C.reverse()` returns the reversal of the cobordism.
- `C.mirror()` returns the mirror of the cobordism.
- `C.chi()` returns the Euler characteristic of the cobordism.
- `C.map(CKH)` applies the map induced by the cobordism to `CKH` and changes it in place.
- `C.KJ_class()` returns the Khovanov–Jacobsson class of `C` or `C.reverse()`, if one of the ends of `C` is the empty link.
- `C.matrix(h, q)` calculates the cobordism map induced on homology in grading $(h, q) \to (h, q + \chi(C))$ in matrix form (this method needs Sage).

All classes support the standard `print(...)` method to display information about an object. For instance, `print(L)` outputs the adjacency structure of a `Link`, `print(C)` displays the sequence of moves in a `Cobordism` movie, and `print(CKH)` represents a `CKhElement` by showing the labels of each loop in every element of the linear combination. Additionally, `CKH.print_short()` provides a concise representation that omits printing the loops.

3.3 Applications

As two applications of our program, we compute cobordism maps for Seifert surfaces and ribbon disks bounded by small knots.

3.3.1 Seifert surfaces up to 10 crossings

For a prime knot L with at most 10 crossings, there is a unique incompressible Seifert surface Σ up to $\approx_4$ (as shown in the proof of Proposition 1.10). Viewing Σ as a cobordism $L \to \emptyset$, the induced map on Khovanov homology is $\mathrm{Kh}(\Sigma)\colon \mathrm{Kh}(L) \to \mathrm{Kh}(\emptyset)$. As $\mathrm{Kh}(\emptyset) \cong \mathbb{Z}$ is supported in grading $(0, 0)$, and $\mathrm{Kh}(\Sigma)$ shifts grading by $(0, \chi)$, where $\chi = \chi(\Sigma)$, the only non-zero part of this cobordism map is $\mathrm{Kh}^{0,-\chi}(L) \to \mathrm{Kh}^{0,0}(\emptyset)$. For all knots L considered, $\mathrm{Kh}^{0,-\chi}(L) \cong \mathbb{Z}^k$ is torsion-free. When $k \geq 1$, this map is represented by a $1 \times k$ matrix. As the isomorphism with $\mathbb{Z}^k$ is not canonical, the only information contained in this matrix is the greatest common divisor of its entries. We now calculate this non-negative integer for all such knots L and their mirror.

If D is an alternating diagram of L, then Seifert's algorithm produces a minimal genus Seifert surface for L [Gab86]. This algorithm constructs the surface from the orientation-

preserving smoothing σ by attaching a half-twisted band at each crossing to D_σ. This method provides a straightforward way to algorithmically generate incompressible Seifert surfaces for most prime knots with at most 10 crossings.

For non-alternating knots, Seifert's algorithm often still yields minimal genus Seifert surfaces. Only 18 knots out of 249 required additional adjustments. For these cases, we repeatedly perturbed the diagram using Reidemeister moves (occasionally increasing the crossing number by 1 or 2) until we successfully obtained a minimal genus Seifert surface using Seifert's algorithm in every instance. The PD codes and minimal genus data were sourced from KnotInfo [KInfo].

Using this data of Seifert surfaces, we computed the cobordism maps in matrix form for both Σ and $-\Sigma$, the mirror of the surface (bounded by the mirror of the knot). Table 3.5 shows the resulting matrices, where an empty cell means $k = 0$, and an entry such as 2 0 0 indicates $k = 3$.

Viewing the cobordism Σ in the opposite direction, denoted $\Sigma^\star$, we obtain a map $\mathrm{Kh}^{0,0}(\emptyset) \to \mathrm{Kh}^{0,\chi}(L)$, which is determined by the Khovanov–Jacobsson class of $\Sigma^\star$. However, these maps $\mathrm{Kh}(\Sigma^\star)$ provide no new information, since $\mathcal{C}\mathrm{Kh}(-\Sigma)$ and $\mathcal{C}\mathrm{Kh}(\Sigma^\star)^\star$ are canonically related as described in Section 3.1.4.

There have been studies on the Khovanov–Jacobsson class of surfaces obtained via Seifert's algorithm [SS22, Ell10, Ell09], but only on the chain level rather than homology. These studies help interpret some of the results in Table 3.5, using the state graph Γ_σ of the Seifert smoothing σ of a diagram. This graph's vertices represent the loops of D_σ, while its edges correspond to the crossings of the link diagram connecting two loops. The edges are coloured red or blue based on whether the crossing is positive or negative (i.e. 0-smoothed or 1-smoothed in σ). The Khovanov–Jacobsson chain element's structure only depends on the Betti numbers b_0, b_1 of the red subgraph of Γ_σ [SS22].

3.3.2 Ribbon disks from symmetries

Let L be a ribbon knot with ribbon disk Σ and suppose that L has a non-trivial symmetry: a diffeomorphism $\varphi\colon S^3 \to S^3$ such that $\varphi(L) = L$ (disregarding orientation of L) and φ is not isotopic to the identity through diffeomorphisms that fix L setwise. Viewing Σ as an immersed surface in S^3 with ribbon singularities, consider $\varphi(\Sigma)$. This is also a ribbon disk for L, which often differs from Σ (up to $\approx_4$). For instance, the two exotically knotted ribbon disks in [HS24, Figure 1] and the annuli in Theorem 2.1 are both related by symmetry and distinguished smoothly by Khovanov homology.

L	Σ	$-\Sigma$													
3_1	1		9_{16}	1		10_{17}		10_{67}	2 0	10_{117}		
4_1	2	2	9_{17}			10_{18}	2 0 0 0	10_{68}		10_{118}		
5_1	1		9_{18}	1		10_{19}		10_{69}		10_{119}		
5_2	1		9_{19}			10_{20}	2	10_{70}		10_{120}	1	
6_1	2	2 0	9_{20}	2 0		10_{21}	2 0	10_{71}		10_{121}		
6_2	2		9_{21}		2 0 0	10_{22}		10_{72}	2 0	10_{122}		
6_3			9_{22}			10_{23}		10_{73}		10_{123}		
7_1	1		9_{23}	1		10_{24}	2 0	10_{74}	2 0 0	10_{124}	1	
7_2	1		9_{24}			10_{25}	2 0	10_{75}		10_{125}		
7_3	1		9_{25}	2 0		10_{26}		10_{76}	2	10_{126}		
7_4	1		9_{26}			10_{27}		10_{77}		10_{127}	2	
7_5	1		9_{27}			10_{28}		10_{78}	2 0 0	10_{128}	1	
7_6	2 0		9_{28}			10_{29}		10_{79}		10_{129}		
7_7			9_{29}			10_{30}	2 0 0	10_{80}	1	10_{130}		
8_1	2	2 0	9_{30}			10_{31}		10_{81}		10_{131}	2	
8_2	2		9_{31}			10_{32}		10_{82}		10_{132}		2
8_3	2 0	2 0	9_{32}			10_{33}		10_{83}		10_{133}	2	
8_4		2 0	9_{33}			10_{34}		10_{84}		10_{134}	1	
8_5	2		9_{34}			10_{35}		10_{85}		10_{135}		
8_6	2		9_{35}	1		10_{36}	2 0	10_{86}		10_{136}		0 0
8_7			9_{36}		2 0	10_{37}		10_{87}		10_{137}		
8_8			9_{37}			10_{38}	2 0	10_{88}		10_{138}		
8_9			9_{38}	1		10_{39}	2 0	10_{89}		10_{139}	1	
8_{10}			9_{39}		2 0 0	10_{40}		10_{90}		10_{140}		
8_{11}	2 0		9_{40}			10_{41}		10_{91}		10_{141}		
8_{12}			9_{41}			10_{42}		10_{92}	2 0 0	10_{142}	1	
8_{13}			9_{42}		0	10_{43}		10_{93}		10_{143}		
8_{14}	2 0		9_{43}	2		10_{44}		10_{94}		10_{144}	2 0	
8_{15}	1		9_{44}			10_{45}		10_{95}		10_{145}		1
8_{16}			9_{45}		2	10_{46}	2	10_{96}		10_{146}		
8_{17}			9_{46}	2 0	2	10_{47}		10_{97}	2 0 0	10_{147}	2 0 0	
8_{18}			9_{47}			10_{48}		10_{98}	2 0 0	10_{148}		
8_{19}	1		9_{48}		2 0 0	10_{49}	1	10_{99}		10_{149}	2	
8_{20}			9_{49}	1		10_{50}	2 0	10_{100}		10_{150}	2 0	
8_{21}	2		10_1	2	2 0	10_{51}		10_{101}	1	10_{151}		
9_1	1		10_2	2		10_{52}		10_{102}		10_{152}	1	
9_2	1		10_3	2 0	2 0 0	10_{53}	1	10_{103}		10_{153}		
9_3	1		10_4		2 0 0	10_{54}		10_{104}		10_{154}	1	
9_4	1		10_5			10_{55}	1	10_{105}		10_{155}		
9_5	1		10_6	2		10_{56}	2 0	10_{106}		10_{156}		
9_6	1		10_7	2 0		10_{57}		10_{107}		10_{157}		2
9_7	1		10_8	2 0		10_{58}		10_{108}		10_{158}		
9_8	2 0 0		10_9			10_{59}		10_{109}		10_{159}		
9_9	1		10_{10}			10_{60}		10_{110}		10_{160}	2 0	
9_{10}	1		10_{11}	2 0 0		10_{61}	2 0	10_{111}	2 0 0	10_{161}	1	
9_{11}		2 0	10_{12}			10_{62}		10_{112}		10_{162}		2 0
9_{12}	2 0		10_{13}			10_{63}	1	10_{113}		10_{163}		
9_{13}	1		10_{14}	2 0		10_{64}		10_{114}		10_{164}		
9_{14}			10_{15}			10_{65}		10_{115}		10_{165}		2
9_{15}		2 0	10_{16}	2 0 0		10_{66}	1	10_{116}				

Table 3.5: Khovanov cobordism maps induced by the unique incompressible Seifert surfaces $\Sigma\colon L \to \emptyset$ and their mirror $-\Sigma\colon -L \to \emptyset$.

In this section, we study ribbon disks obtained via symmetries for all prime ribbon knots with at most 9 crossings and for certain 10-crossing ribbon knots, using Khovanov homology.

The group of diffeomorphisms of the pair (S^3, L) up to isotopy is called the *symmetry group* of L. If L is hyperbolic, this group is finite, and can be computed from the hyperbolic triangulation of the knot complement. KnotInfo [KInfo] provides data on these symmetry groups, summarized in Table 3.6. Here, D_n denotes the dihedral group of $2n$ elements, corresponding to the symmetry group of a regular n-gon. The table also lists the number of different ribbon disks we identified, distinguished by Khovanov homology.

Knot	6_1	8_8	8_9	8_{20}	9_{27}	9_{41}	9_{46}	10_3	10_{123}
Symmetry group	D_2	D_2	D_4	D_1	D_2	D_3	D_2	D_2	D_{10}
Ribbon disks found	2	2	4	1	2	2	2	2	5

Table 3.6: Ribbon disks distinguished by Khovanov homology.

Knowing the symmetry group of a knot does not always make it straightforward to identify its symmetries. A knot L is called *reversible*, if there exists a symmetry φ that reverses the orientation of L. For any reversible prime knot with at most 10 crossings, there exists a *symmetric diagram*, where an orientation-reversing symmetry φ is realized as a 180° rotation, as listed in [Sak86, Lam22]. Every ribbon knot up to 10 crossings that admits a non-trivial symmetry is reversible [KInfo].

Eisermann–Lamm [EL11] provide band diagrams for one ribbon disk for each ribbon knot up to 10 crossings (given as symmetric union diagrams). Using KLO [KLO], we manually transformed these into symmetric diagrams to find additional ribbon disks derived from the symmetries. In certain cases (8_9 and 9_{27}), further manual manipulations were performed to search for the remaining symmetries. These ribbon disks were then compared using our Khovanov homology program. The source code for these computations is available in the attached `ribbon_disks.py` file and on GitHub [KhCC].

Figure 3.1 shows the band diagrams for ribbon disks obtained via symmetries, layered over the same knot diagram. With the exception of 10_{123}, each band diagram consists of a single band. Solid lines represent ribbon disks that are distinguished by Khovanov homology. Dotted lines indicate disks that are isotopic to others, while dashed lines represent disks that cannot be distinguished by Khovanov homology (both these disks and their mirrors share the same Khovanov–Jacobsson class with another disk in the figure). Disks sharing the same Khovanov homology are shown in the same colour.

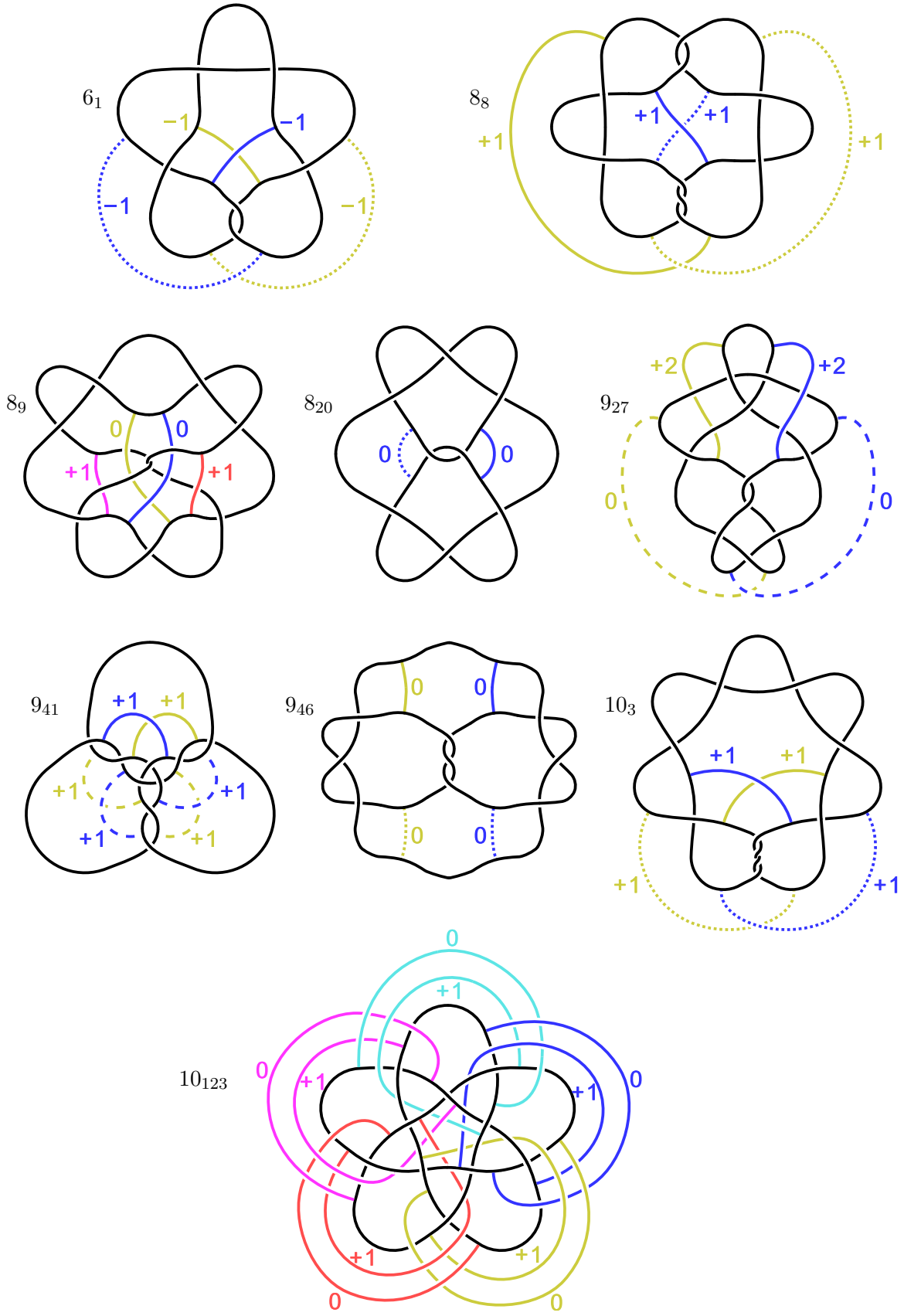


Figure 3.1: Band diagrams for ribbon disks obtained by symmetry. Solid lines of different colours represent non-isotopic disks distinguished by Khovanov homology.

- 6_1, 8_8, 9_{46}, 10_3: Likely all four symmetries were identified, yielding two distinct ribbon disks. The other two disks are equivalent to the first two via band isotopy.
- 8_9: Four of eight symmetries were found, resulting in four distinct ribbon disks.
- 8_{20}: Both symmetries were identified, but they produce the same ribbon disk.
- 9_{27}: Likely all four symmetries were found, yielding at least two distinct ribbon disks. The other two are not distinguished by Khovanov homology but are not evidently the same band diagram.
- 9_{41}: Likely all six symmetries were identified, resulting in at least two distinct ribbon disks. The other four are not distinguished by Khovanov homology.
- 10_{123}: Chosen for its high number of symmetries, likely all 20 were identified, yielding at least five distinct ribbon disks. The remaining disks are not distinguished by Khovanov homology.

Some of these ribbon disks have already been observed to be non-isotopic. For instance, see [Fox61, Examples 10–11] for 6_1 and [MP19, SS22] for 9_{46}.

Chapter 4

Gluing handlebodies and amalgamating free groups

4.1 Gluing handlebodies into a handlebody

Question 5. *Let M_0, M_1 be handlebodies, and $S_0 \subseteq \partial M_0$, $S_1 \subseteq \partial M_1$ be homeomorphic compact surfaces with one boundary component. Is there a homeomorphism $\phi\colon S_0 \to S_1$ such that the gluing $M_0 \cup_\phi M_1$ is a handlebody?*

We are going to show that there exist manifolds (M_i, S_i) for which Question 5 has a negative answer.

3-manifolds with subsurfaces on their boundary are often called *sutured manifolds*, and are usually given as (M, γ), where M is a compact oriented 3-manifold with boundary, and $\gamma \subseteq \partial M$ is an oriented 1-manifold dividing ∂M into two components: R_+ and R_- with the orientations given by $\partial R_+ = \partial R_- = \gamma$. We will use the notation (M, S) where $S = R_+$, and gluing together two sutured manifolds (M_i, S_i) will mean gluing them with a homeomorphism between the surfaces S_i. We ignore orientations.

4.1.1 Positive examples

To illustrate the challenge of finding counterexamples, we begin by presenting a few positive examples, where a suitable gluing results in a handlebody.

Let (M_i, S_i) be as in Question 5, and denote the genera by

$$g_0 = g(M_0), \quad g_1 = g(M_1), \quad s = g(S_0) = g(S_1).$$

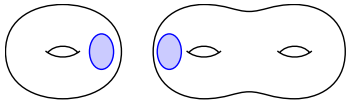
Then $g(\partial M_i \setminus S_i) = g_i - s$, hence $s \leq g_0, g_1$. Also note that

$$\begin{aligned}\chi(M_0 \cup_\phi M_1) &= \chi(M_0) + \chi(M_1) - \chi(S_0) = (1-g_0) + (1-g_1) - (1-2s) \\ &= 1 - (g_0 + g_1 - 2s).\end{aligned}$$

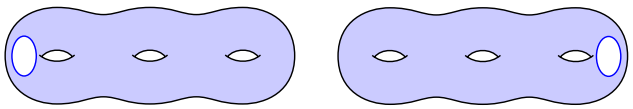
Therefore, if $M_0 \cup_\phi M_1$ is a handlebody, then it must have genus

$$g(M_0 \cup_\phi M_1) = g_0 + g_1 - 2s.$$

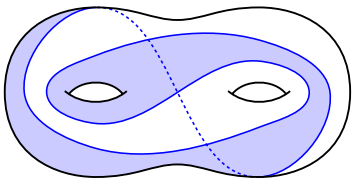
Example 4.1. (a) $s = 0$. Then S_0, S_1 are disks, and g_0, g_1 can be arbitrary. There is only one gluing, which gives a handlebody of genus $g_0 + g_1$.



(b) $s = g_0 = g_1$. In this case, $\partial M_0 \setminus S_0$, $\partial M_1 \setminus S_1$ are disks. While there are many different gluings, we can achieve a handlebody of genus $g_0 + g_1 - 2s = 0$ (i.e. a ball) by the following: take a standard (unknotted) embedding $(M_0, \partial M_0 \setminus S_0) \to (D^3, \partial D^3)$, then the complement $(D^3 \setminus M_0, \partial D^3 \setminus M_0)$ is homeomorphic to (M_1, S_1).



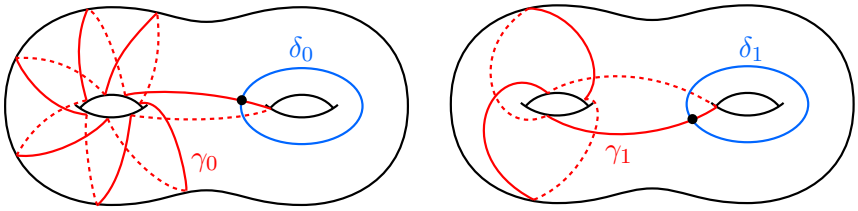
(c) *Product sutured handlebody.* Let $S_0 = S_0 \times \{0\}$ be a surface of genus s, and let $M_0 = S_0 \times I$. Then M_0 is a handlebody of genus $2s$. For any (M_1, S_1), there are many different gluings, each of which gives a handlebody (as there is an obvious homeomorphism between M_1 and $M_0 \cup_\phi M_1$). The figure shows S_0 when $s = 1$ and M_0 is depicted as a standard handlebody.



and any (M_1, S_1)

(d) *p/q-strip with a band.* Let $g_0 = g_1 = 2$, and let γ_i, δ_i be curves on ∂M_i as in the figure below. Define S_i as a small neighbourhood of $\gamma_i \cup \delta_i$. Since γ_i and δ_i intersect once, S_i is a genus 1 surface with one boundary component. In the figure, the curves γ_0, γ_1 are shown with slopes $-2/7$ and $2/3$, but these can take any values $p_i/q_i \in \mathbb{Q} \cup \{\infty\}$. If the gluing map sends γ_0 to δ_1 and δ_0 to γ_1, then $M_0 \cup_\phi M_1$ will be a handlebody of genus 2.

Coincidentally, these sutured handlebodies (M_i, S_i) also arise from the surface-pairs $\Sigma_j = \Sigma_j(p_i, q_i, k, n)$ of Theorems 2.2 to 2.4. For a fixed i, Σ_0 and Σ_1 have disjoint interiors after a small isotopy, and $\Sigma_0 \cup \Sigma_1$ bounds a genus 2 handlebody $H \subseteq S^3$. As we consider H abstractly here, without its specific embedding in S^3, changing k and n does not affect the sutured handlebody (H, Σ_0), which is homeomorphic to (M_i, S_i).



$S_0 = N(\gamma_0 \cup \delta_0)$ $\quad$ $S_1 = N(\gamma_1 \cup \delta_1)$

In particular, (a) and (b) show that for $g_0, g_1 \leq 1$, there always is a gluing. Hence, $g_0 = 1$, $g_1 = 2$, $s = 1$ is the smallest case that could give a negative example.

We now give a broader family of positive examples. Except for (c), the above examples all fall in this family.

A disk D is called a *compressing disk* of a 3-manifold M, if $D \subseteq M$, $\partial D \subseteq \partial M$, and ∂D does not bound a disk in ∂M. We also call ∂D a *compressing curve* of ∂M. A set of embedded closed curves $a_1, b_1, \ldots, a_g, b_g$ in a genus g surface S is called a *symplectic basis* of S if they form a basis of $H_1(S)$, each pair a_j, b_j intersects once for all j, and no other pairs of curves intersect. We use the word *disjoint* to mean pairwise disjoint.

The following theorem is based on an idea of Maggie Miller.

Theorem 4.2. *Let (M, S) and (M', S') be two sutured handlebodies, where S, S' are homeomorphic genus g surfaces with one boundary component. Let $a_1, b_1, \ldots, a_g, b_g$ be a symplectic basis of S, and $a'_1, b'_1, \ldots, a'_g, b'_g$ of S'. Let $d_1, \ldots, d_g$ be a set of disjoint compressing curves of ∂M, and $d'_1, \ldots, d'_g$ of $\partial M'$. Suppose that these curves intersect in the following way, for all j:*

- *d_j intersects a_j once and transversally,*
- *d_j does not intersect a_k for $k > j$,*
- *d_j does not intersect b_k for all k,*
- *d'_j intersects a'_j once and transversally, and*
- *d'_j does not intersect a'_k for $k > j$.*

Then (M, S) and (M', S') can be glued together to a handlebody.

Proof. We consider all our curves to be oriented. Let our gluing homeomorphism $\phi\colon S \to S'$

be the one that satisfies $\phi(a_j) = b'_j$, $\phi(b_j) = a'_j$ for all j (using that a_j, b_j and a'_j, b'_j are symplectic bases).

Let $U \subseteq S$ be a disk containing a neighbourhood of all the intersection points $a_j \cap b_j$ and not intersecting a_j, b_j outside this neighbourhood. Let $U' = \phi(U) \subseteq S'$. Then gluing S to S' by ϕ is the same as connecting U to U' with a 1-handle h, and adding 2-handles along the disjoint curves

$$\alpha_j = a_j h \bar{b}'_j \bar{h}, \quad \beta_j = b_j h \bar{a}'_j \bar{h}$$

for all j, where we slightly abused notation to mean a concatenation of arcs (instead of closed curves), and $\bar{c}$ denotes the inverse of the oriented arc c.

a_j and d_j intersect once, so α_j and d_j also intersect once. Hence, adding a 2-handle along α_j is the same as compressing along a disk D_j with $\partial D_j = d_j$. Similarly, since β_j and d'_j intersect once, adding a 2-handle along β_j is the same as compressing along D'_j with $\partial D'_j = d'_j$. Considering the 2-handles in the order $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$, compression along D_j does not affect the curves $\alpha_{j+1}, \dots, \alpha_g, \beta_1, \dots, \beta_g$, and D'_j does not affect $\beta_{j+1}, \dots, \beta_g$ (using the assumptions about d_j and d'_j not intersecting certain curves).

Therefore, $M \cup_\phi M'$ is obtained by a sequence of compressions from the handlebody $M \cup_h M'$, and thus, it is a handlebody. □

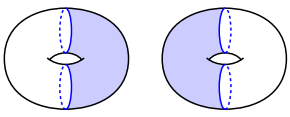
The simplest special case is when d_j does not intersect any of the curves a_k, b_k other than a_j. If both (M_0, S_0), (M_1, S_1) have this property, then they form a positive example. The sutured handlebodies in Example 4.1 (a), (b), (d) all satisfy this.

As an example for a slightly more complicated use of Theorem 4.2, we see that the p/q-strip with a band (Example 4.1 (d)) forms a positive example with (M, S) of Figure 4.1, even though we will later show that (M, S) with itself constitutes a negative example.

4.1.2 Negative examples

If we allowed ∂S_i to have multiple components, straightforward negative examples would arise, such as:

- $M_i = D^2 \times S^1$, $S_i = \partial D^2 \times D^1_+$. There is only one gluing (up to isotopy), resulting in a space homotopy equivalent to $S^2 \vee S^1 \vee S^1$, which is therefore not a handlebody.



The following observation, noted by Marc Lackenby, provides a large family of negative examples.

Theorem 4.3. *Let (M_i, S_i) be two sutured handlebodies, where S_0, S_1 are homeomorphic surfaces with one boundary component, and $g(M_0), g(M_1) \geq 1$. If (M_0, S_0) and (M_1, S_1) can be glued together to form a handlebody, then for some i, there exists a compressing curve of ∂M_i that intersects ∂S_i in at most two points.*

Proof. Suppose $M = M_0 \cup_\phi M_1$ is a handlebody. The statement is clear if S_i or $\partial M_i \setminus S_i$ is a disk for some i, and consequently, if M is a 3-ball (since ∂S_i is connected). Otherwise, take a compressing disk D of M; we can assume that D is transverse to S_0 (which is identified with S_1 in M). Let $X = D \cap S_0$, then X is a 1-manifold in D with ∂X in ∂D.

If X contains a circle, then let E be a disk that an innermost circle from X bounds in D. Then $(E, \partial E)$ is in (M_i, S_i) for some i. If E is a compressing disk of M_i, then we are done; otherwise ∂E bounds a disk F in ∂M_i. Since $\partial M_i \setminus S_i$ is not a disk, F must be in S_i. Then $(D \setminus E) \cup F$ in M can be pushed off along F from M_i so that the resulting compressing disk D' has $X' = D' \cap S_0 = X \setminus \partial E$. Continuing this process with innermost circles, we can assume that X does not contain circles.

If X does not contain circles, then it is a finite union of disjoint arcs. If X is non-empty, then let E be a 2-gon that an innermost arc from X and part of ∂D bounds (where innermost means that the interior of E does not intersect X). Then $(E, \partial E)$ is in $(M_i, \partial M_i)$ for some i, with ∂E intersecting ∂S_i twice. If E is a compressing disk of M_i, then we are done; otherwise ∂E bounds a disk F in ∂M_i. Then F is divided by ∂S_i into two 2-gons (since S_i has one boundary component); let $G = F \cap S_i$. Then $(D \setminus E) \cup G$ in M can be pushed off along G from M_i so that the resulting disk D' has $X' = X \setminus \partial E$. D' is a compressing disk of M, since ∂D and $\partial D'$ are isotopic (using $F \setminus G$). Continuing this process with innermost arcs, we can assume that X is empty.

If X is empty, then D is a compressing disk of some M_i (since S_i is not a disk) such that ∂D intersects ∂S_i zero times, and we are done. □

We say (M, S) contains a *product disk* D if D is a compressing disk of M such that ∂D intersects ∂S twice and transversally. Hence, if neither of (M_0, S_0), (M_1, S_1) contains a product disk, then they form a negative example (as compressing curves with zero intersections with ∂S can be isotoped to have two intersections). Deciding whether (M, S) contains a product disk can be done algorithmically (e.g. by computing the dimension of the sutured Floer homology polytope).

4.1.3 Whitehead's cut-vertex lemma

As a transition into the next algebraic section, we now present the first negative example that we discovered. The proof relies on showing that, for any gluing, the fundamental group of $M_0 \cup_\phi M_1$ is not a free group. While the fact that this construction (Proposition 4.6) is a negative example now follows from Theorem 4.3, the proof provided here establishes additional algebraic results (Proposition 4.7).

We start with a few results from group theory. An element of the free group F_n is called *primitive* if it is a member of some basis of F_n.

Proposition 4.4 ([LS77, Ch. II, Prop. 5.10])**.** *The one-relator group $\langle x_1, \dots, x_n \mid r \rangle$ is free if and only if $r = 1$ or r is a primitive element of F_n.*

Therefore, we need a method to prove that certain elements are not primitive. While this can be decided using Whitehead's algorithm [Whi36b], we will instead use Whitehead's cut-vertex lemma [Whi36a]. To state this lemma, we first review the basic terminology and concepts required to define the Whitehead graph.

A word w representing an element in $F_n = \langle x_1, \dots, x_n \rangle$ is called *reduced* if it does not contain consecutive letters of the form $x\bar{x}$ (where $\bar{x}$ denotes the inverse of x, $\bar{\bar{x}} = x$, and x stands for any of the x_i or $\bar{x}_i$). When w is treated as a cyclic word (i.e. the last and first letters are considered cyclically consecutive), and w does not contain cyclically consecutive letters of the form $x\bar{x}$, then w is called *cyclically reduced*. Any word can be simplified into a reduced word, and any cyclic word can be made cyclically reduced, by removing occurrences of $x\bar{x}$. This reduced or cyclically reduced representation is unique (up to cyclic permutations for cyclic words).

The *Whitehead graph* of a word in the letters $x_1, \dots, x_n$ is defined as follows. First, take the cyclically reduced representation w of the cyclic word. The vertices of the graph are $x_1, \dots, x_n, \bar{x}_1, \dots \bar{x}_n$. For every pair of cyclically consecutive letters uv in w, an (unoriented) edge $u\bar{v}$ is added to the graph, with multiplicities determined by the occurrences of uv.

A vertex v of a graph G is called a *cut-vertex*, if $G \setminus \{v\}$ is disconnected. An edge e of a graph G is called a *cut-edge*, if $G \setminus \{e\}$ is disconnected.

Theorem 4.5 (Whitehead)**.** *If w is primitive, then the Whitehead graph of w has a cut-vertex or a cut-edge.*

Proposition 4.6. *Let M be a genus 2 handlebody, and let γ, δ be the curves on ∂M depicted in Figure 4.1. Define S as a small neighbourhood of $\gamma \cup \delta$ in ∂M. Consider two copies of (M, S), denoted (M_0, S_0) and (M_1, S_1). Then there is no gluing of (M_0, S_0) and (M_1, S_1) that results in a handlebody.*

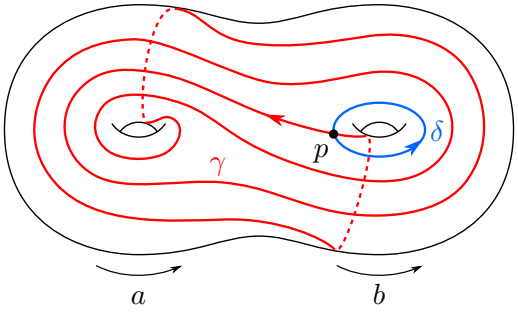


Figure 4.1: (M, S), where $S = N(\gamma \cup \delta)$.

Proof. Since S_i deformation retracts to a wedge of two circles, let

$$\pi_1(S_0) = \langle x, y \rangle, \quad \pi_1(S_1) = \langle u, v \rangle,$$

where x, y and u, v correspond to the curves γ, δ, with basepoint $p = \gamma \cap \delta$. Let

$$\pi_1(M_0) = \langle a, b \rangle, \quad \pi_1(M_1) = \langle c, d \rangle,$$

where a, b (and c, d) are the standard generators based at p, as shown in Figure 4.1. Let $\phi\colon S_1 \to S_0$ be any homeomorphism, which can be assumed to fix the basepoint p (as isotoping ϕ does not affect the gluing), and let ϕ act on the fundamental group as

$$\phi_*(u) = s, \quad \phi_*(v) = t,$$

where s, t are reduced words in x, y. Let $i\colon S_0 \to M_0$, $j\colon S_1 \to M_1$ be the inclusion maps, then

$$\begin{aligned} i_*(x) &= ab\bar{a}ba, & j_*(u) &= cd\bar{c}dc, \\ i_*(y) &= b, & j_*(v) &= d, \\ i_*(s) &= f, \quad i_*(t) = g, \end{aligned}$$

where f, g are reduced words in a, b. Hence, by the Seifert–van Kampen theorem,

$$\begin{aligned}\pi_1(M_0 \cup_\phi M_1) &= \langle a, b, c, d \mid j_*(u) = i_*\phi_*(u),\ j_*(v) = i_*\phi_*(v)\rangle \\ &= \langle a, b, c, d \mid cd\bar{c}dc = f,\ d = g\rangle \\ &= \langle a, b, c \mid cg\bar{c}gc\bar{f}\,\rangle.\end{aligned}$$

Suppose that $M_0 \cup_\phi M_1$ is a handlebody. (It must have genus 2, but we do not need that.) Then $\langle a, b, c \mid cg\bar{c}gc\bar{f}\,\rangle$ is a free group. By Proposition 4.4, we get that $cg\bar{c}gc\bar{f}$ is a primitive element of $\langle a, b, c\rangle$, since it cannot be 1.

To use Whitehead's cut-vertex lemma on the word $cg\bar{c}gc\bar{f}$, we will first transform $g = i_*(t)$ into a cyclically reduced word (in a, b). For this purpose, write

$$t = qr\bar{q},$$

where r is a cyclically reduced word (in x, y), and let

$$i_*(q) = k, \quad i_*(r) = h.$$

Then

$$g = i_*(t) = i_*(q)i_*(r)\overline{i_*(q)} = kh\bar{k}.$$

Note that the i_*-image of a word is obtained by replacing every occurrence of x, $\bar{x}$, y, $\bar{y}$ by $ab\bar{a}ba$, $\bar{a}\bar{b}a\bar{b}\bar{a}$, b, $\bar{b}$, respectively. Therefore, if we do this replacement in a cyclically reduced word in x, y, then we get a cyclically reduced word in a, b, since $ab\bar{a}ba$ starts and ends with a, and b starts and ends with b. Thus, $i_*(r) = h$ is cyclically reduced.

Now, k is a word in a, b, and by applying the automorphism of $\langle a, b, c\rangle$ defined by $c \mapsto kc\bar{k}$ and fixing a, b, we get

$$\langle a, b, c \mid cg\bar{c}gc\bar{f}\,\rangle = \langle a, b, c \mid kc\bar{k}gk\bar{c}\bar{k}gkc\bar{k}\bar{f}\,\rangle = \langle a, b, c \mid ch\bar{c}hc\bar{k}\bar{f}k\rangle.$$

Let w be the reduced form of $ch\bar{c}hc\bar{k}\bar{f}k$. By Theorem 4.5, since w is primitive, the Whitehead graph G of w has a cut-vertex or cut-edge. Since h is a cyclically reduced word in a, b, its first letter h_0 and its last letter h_1 are not inverses of each other. By considering the subword $ch_0 \dots h_1\bar{c}h_0 \dots h_1c$ in w, we obtain the edges $c\bar{h}_0, h_1c, \bar{c}\bar{h}_0, h_1\bar{c}$ in G. This means that there are edges from c to two different vertices in $\{a, b, \bar{a}, \bar{b}\}$, and similarly, from $\bar{c}$.

We claim that there is at least one letter x or $\bar{x}$ in the reduced form of $\bar{q}sq$ or $\bar{q}tq$. Otherwise, $\bar{q}sq = y^n$ and $\bar{q}tq = y^m$ for some integers n, m, and hence $s^m = t^n$. Since u, v form a basis of $\langle u, v\rangle$, and ϕ_* is an isomorphism (as ϕ is a homeomorphism), we get that

$\phi_*(u) = s$ and $\phi_*(v) = t$ form a basis of $\langle x, y\rangle$. Then $s^m = t^n$ implies $m = n = 0$, $\bar{q}sq = 1$, and $s = 1$, which is a contradiction.

Using again the description of i_* as above, since there is an x or $\bar{x}$ in the reduced form of $\bar{q}\bar{s}q$ or $\bar{q}tq = r$, we get that there is a subword $i_*(x) = ab\bar{a}ba$ or its inverse in the reduced form of $i_*(\bar{q}\bar{s}q) = \bar{k}\bar{f}k$ or $i_*(r) = h$. Hence, there is a subword $ab\bar{a}ba$ or its inverse in the cyclically reduced word w. Taking inverse does not affect the Whitehead graph; therefore, the edges $a\bar{b}, ba, \bar{a}\bar{b}, b\bar{a}$ are contained in G (Figure 4.2).

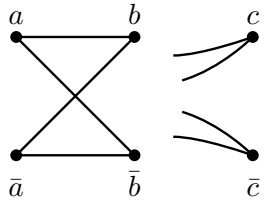


Figure 4.2: The Whitehead graph of $ch\bar{c}hc\bar{k}\bar{f}k$.

Now, by combining the two edges from c and the two from $\bar{c}$ with the circle $ab\bar{a}\bar{b}$, we see that removing any vertex or edge from G leaves the graph connected. Hence, it has no cut-vertex or cut-edge, which is a contradiction. Therefore, $M_0 \cup_\phi M_1$ is not a handlebody. □

4.2 Amalgamated free products

In this section, we investigate the following algebraic analogue to Question 5.

Question 6. *Let $G \leq F_m$ and $H \leq F_n$ be subgroups of free groups with $G \cong H \cong F_k$. Is there an isomorphism $\varphi\colon G \to H$ such that $F_m *_\varphi F_n$ is free?*

Here, we denoted by $F_m *_\varphi F_n$ the quotient of the free product $F_m * F_n$ by the relations $\varphi(g) = g$ for all $g \in G$, called an *amalgamated free product*. This question is both broader and narrower than Question 5: broader because some subgroups $G \leq F_m$ do not correspond to the π_1-image of any subsurface, and narrower because some subsurfaces $S \subseteq \partial M$ do not induce an injective map $\pi_1(S) \to \pi_1(M)$ via the inclusion $S \to M$.

We also say that we *amalgamate* the subgroups G and H when we form an amalgamated free product $F_m *_\varphi F_n$. We have already seen our first negative example to Question 6 in the proof of Proposition 4.6:

Proposition 4.7. *The subgroups $\langle ab\bar{a}ba, b\rangle \leq \langle a, b\rangle$ and $\langle cd\bar{c}dc, d\rangle \leq \langle c, d\rangle$ cannot be amalgamated to produce a free group.*

Question 6 is completely understood in the case $k = 1$:

Proposition 4.8. *Let* $u \in \langle a_1, \ldots, a_m \rangle$, $v \in \langle b_1, \ldots, b_n \rangle$, *and*

$$G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid u = v \rangle.$$

Then G *is a free group if and only if* u *or* v *is primitive.*

This follows from the work of Shenitzer [She55], or it is an immediate consequence of the theorem below of Swarup [Swa86] (and it can also be proven from our subsequent Theorem 4.16). A subgroup $H \leq G$ is called a *free factor* of G if there is a subgroup $K \leq G$ with $G = H * K$. H is *non-trivial* if $H \neq 1$.

Theorem 4.9 (Swarup)**.** *Let* $H_1 \leq G_1$, $H_2 \leq G_2$ *be finitely generated, non-trivial subgroups of groups, with an isomorphism* $\varphi \colon H_1 \to H_2$. *If* $G_1 *_\varphi G_2$ *is free, then for some* i, *there exists a non-trivial free factor* K_i *of* H_i *such that* K_i *is a free factor of* G_i.

This theorem gives a family of negative examples to Question 6, and in turn, to Question 5. However, its implications appear to be somewhat limited; for instance, Proposition 4.7 does not follow. Indeed, the subgroup $H = \langle ab\bar{a}ba, b \rangle$ has $K = \langle b \rangle$ as a free factor of H and of $G = \langle a, b \rangle$.

We will show that if G and H are *dark* subgroups, then they give a negative example to Question 6 (Theorem 4.16). As a consequence, we also get a family of negative examples to Question 5 (Proposition 4.17), but this family turns out to be a subset of the negative examples provided by Theorem 4.3. The definition of a dark subgroup will be stated in Definition 4.15, but the intuition is that the subgroup has to be 'sufficiently complicated'. Darkness is algorithmically decidable. The subgroup in Proposition 4.7 is dark.

For a fixed isomorphism $\varphi \colon G \to H$, there is an algorithm to decide if $F_m *_\varphi F_n$ is free, due to Diao and Feighn [DF05]. Moreover, their algorithm gives the Grushko decomposition of a finite graph of finite rank free groups. We now explain these concepts, then sketch the algorithm.

The *Grushko decomposition* of a group G is $G \cong F_m * G_1 * \cdots * G_n$, where F_m is free and G_i is freely indecomposable and non-free. Any finitely generated group admits this decomposition, which is unique up to isomorphism and reordering of the groups G_i.

4.2.1 Graph of groups

Definition 4.10. In the context of this section, a *graph* means a 4-tuple

$$\Gamma = (V, \hat{E}, \mathrm{op}, \partial_0),$$

where $V, \hat{E}$ are sets, $\mathrm{op}\colon \hat{E} \to \hat{E}$ is a function with $\mathrm{op}(\mathrm{op}(e)) = e$, $\mathrm{op}(e) \neq e$ for all $e \in \hat{E}$, and $\partial_0\colon \hat{E} \to V$ is a function.

We also write $\mathrm{op}(e) = \bar{e}$, $E = \{\{e, \bar{e}\} \mid e \in \hat{E}\}$, $\hat{E}(v) = \{e \in \hat{E} \mid \partial_0 e = v\}$, and $\partial_1 = \partial_0 \circ \mathrm{op}$. Then this definition agrees with the usual notion of an unoriented graph (V, E), where $\hat{E}$ is the double of E (with each edge oriented in two different ways), and the oriented edge $e \in \hat{E}$ goes from the vertex $\partial_0 e$ to $\partial_1 e$. A graph can also be viewed as a 1-dimensional CW-complex in the usual way.

Definition 4.11. A *graph of groups* is a 4-tuple

$$\mathcal{G} = (\Gamma, \{G_v\}_{v \in V}, \{G_e\}_{e \in \hat{E}}, \{\varphi_e\}_{e \in \hat{E}}),$$

where $\Gamma = (V, \hat{E}, \mathrm{op}, \partial_0)$ is a connected graph, G_v, G_e are groups with $G_e = G_{\bar{e}}$, and $\varphi_e\colon G_e \to G_{\partial_0 e}$ is an injective homomorphism for all $e \in \hat{E}$.

The *fundamental group* $\pi_1(\mathcal{G})$ of a graph of groups $\mathcal{G}$ can be defined geometrically as follows (though we present this viewpoint only to provide intuition). For each vertex $v \in V$, take a topological space X_v with $\pi_1(X_v) = G_v$, and for each $e \in \hat{E}$, take $X_e = X_{\bar{e}}$ with $\pi_1(X_e) = G_e$. Let $f_e\colon X_e \to X_{\partial_0 e}$ be a continuous map such that the induced map on π_1 is $(f_e)_* = \varphi_e$. For each edge in E, take $X_e \times [0, 1]$, and glue $X_e \times \{0\}$ to $X_{\partial_0 e}$ using f_e, and glue $X_e \times \{1\}$ to $X_{\partial_1 e}$ using $f_{\bar{e}}$. The result is a topological space $X(\mathcal{G})$, and we define $\pi_1(\mathcal{G}) = \pi_1(X(\mathcal{G}))$.

From this description, we see that $\pi_1(\mathcal{G})$ can also be obtained in the following algebraic way, using amalgamated free products and HNN-extensions. Take a spanning tree $T \subseteq E$ of Γ. Form the free product of the vertex groups G_v, and take quotient by the relations $\varphi_e(g) = \varphi_{\bar{e}}(g)$ $(g \in G_e)$ for each edge $\{e, \bar{e}\} \in T$ (hence, this is an iterated amalgamated free product). Then add HNN-extensions along each edge in $E \setminus T$; this means adding new generators t_e to the group presentation together with the relations $t_e \varphi_e(g) t_e^{-1} = \varphi_{\bar{e}}(g)$ for all $g \in G_e$, where e runs over a set of oriented edge representatives of the elements $\{e, \bar{e}\} \in E \setminus T$. The resulting group is $\pi_1(\mathcal{G})$. We see that this algebraic construction is independent of the choice of T by the geometric description above.

In particular, if Γ is a graph of two vertices with one edge between them, and $\mathcal{G}$ is a graph of finite rank free groups, i.e.

$$\Gamma = (\{v_0, v_1\}, \{e, \bar{e}\}, \mathrm{op}, \partial_0), \quad \partial_i(e) = v_i,$$
$$\mathcal{G} = (\Gamma, \{F_m, F_n\}, \{F_k\}, \{\varphi_e, \varphi_{\bar{e}}\}),$$

then

$$\pi_1(\mathcal{G}) = F_m *_\varphi F_n \quad (\varphi = \varphi_{\bar{e}} \circ \varphi_e^{-1} \colon \varphi_e(F_k) \to \varphi_{\bar{e}}(F_k))$$

is the main object of Question 6. We are allowed to change φ_e to $\varphi_e \circ \alpha$ with an automorphism $\alpha \in \mathrm{Aut}(F_k)$, but we cannot change the image of φ_e. The question asks if there is a suitable α that gives a free group $\pi_1(\mathcal{G})$.

4.2.2 Gersten representatives

A crucial notion that we need for the algorithm of Diao–Feighn is that of *Gersten representatives*. It is a way of representing subgroups of F_n by labelled graphs in the simplest way possible.

Let $\Sigma = (V, E)$ be a *labelled graph*, meaning a connected oriented graph equipped with a labelling function $\lambda \colon E \to B$, where B is a finite set. Let R be a wedge of $|B|$ circles, each labelled by a different element of B, and identify $\pi_1(R) = F_B$. Consider the simplicial map $\ell \colon \Sigma \to R$ according to the labelling λ. For some basepoint $p \in V$, we get a subgroup $\ell_*(\pi_1(\Sigma, p)) \leq F_B$. If a subgroup $H \leq F_B$ is conjugate to $\ell_*(\pi_1(\Sigma, p))$ for some (or equivalently, any) choice of p, then we say Σ *represents* the conjugacy class of H.

Definition 4.12. Let $\vec{H} = (H_1, \ldots, H_k)$ be a sequence of subgroups of F_B. A *Gersten representative* $\Sigma_{\mathrm{G}}(\vec{H})$ of $\vec{H}$ is a sequence $\vec{\Sigma} = (\Sigma_1, \ldots, \Sigma_k)$ of labelled graphs such that

- there is an $\alpha \in \mathrm{Aut}(F_B)$ such that for all i, Σ_i represents the conjugacy class of $\alpha(H_i)$, and
- the sum of the number of vertices of the elements of $\vec{\Sigma}$ is minimal among sequences that satisfy the previous property.

See Figures 4.3 and 4.4 for examples of Gersten representatives.

We now describe an algorithm, due to Gersten [Ger84], that computes a Gersten representative of $\vec{H}$ (where each H_i is given as the subgroup generated by a finite set of words in F_B). In particular, consider the case where $\vec{H}$ has only one component H, and $H = \langle w \rangle$ is generated by one element $w \in F_B$. Then $\Sigma_{\mathrm{G}}(H)$ is a graph on one vertex and one edge if and only if there exists an $\alpha \in \mathrm{Aut}(F_B)$ such that $\alpha(w)$ is a single letter. This condition is equivalent to w being a primitive element of F_B. Hence, Gersten's algorithm is a generalization of Whitehead's algorithm mentioned earlier.

We write $\bar{b} = b^{-1}$ and $B^{\pm 1} = B \cup \{\bar{b} \mid b \in B\}$. An *elementary Whitehead automorphism* is an $\alpha \in \mathrm{Aut}(F_B)$ of the following form: there is an $a \in B^{\pm 1}$ and a set $A \subseteq B^{\pm 1} \setminus \{a, \bar{a}\}$

such that for all $b \in B$,

$$\alpha(b) = \begin{cases} b & \text{if } b \notin A,\ \bar{b} \notin A, \\ ab & \text{if } b \in A,\ \bar{b} \notin A, \\ b\bar{a} & \text{if } b \notin A,\ \bar{b} \in A, \\ ab\bar{a} & \text{if } b \in A,\ \bar{b} \in A. \end{cases}$$

Note that the set of elementary Whitehead automorphisms of F_B is finite.

Algorithm 4.13 (Gersten)**.** Given $\vec{H}$, the following produces $\Sigma_{\mathrm{G}}(\vec{H})$:

1. *Start with any* $\vec{\Sigma}$ *representing* $\vec{H}$: For instance, if $H_i = \langle w_1, \dots, w_j \rangle$, then we can set Σ_i to be a wedge of j circles, each divided into a sequence of labelled edges that read w_k along the k-th circle.

2. *Do Stallings folds*: If any Σ_i contains two edges (which can be loops) that have the same endpoint or same starting point, and the same label, then identify these two edges (as in a topological space), and return to Step 2 with the new Σ_i in $\vec{\Sigma}$.

3. *Take core*: This means removing degree-one vertices (and edges connecting to those) repeatedly in each Σ_i.

4. *Elementary Whitehead automorphisms*: For each elementary Whitehead automorphism α, do Steps 1–3 for $\alpha(\vec{H})$. (Or more efficiently, modify our current $\vec{\Sigma}$ by replacing each edge labelled b with a sequence of labelled edges that read $\alpha(b)$ and do Steps 2–3.) If any α produces a $\vec{\Sigma}^\alpha$ that has fewer vertices than $\vec{\Sigma}$, then set $\vec{H} := \alpha(\vec{H})$, $\vec{\Sigma} := \vec{\Sigma}^\alpha$ and return to Step 4.

   Then $\vec{\Sigma}$ is a Gersten representative of (the original) $\vec{H}$.

4.2.3 The algorithm of Diao–Feighn

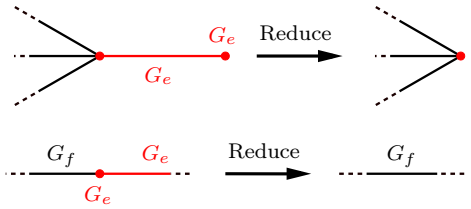
In this section, we present the algorithm that produces the Grushko decomposition of a finite graph of finite rank free groups. Then, in the next section, we observe a simple case when the algorithm ends quickly, which gives a family of negative examples to Question 6, and hence to Question 5.

Algorithm 4.14 (Diao–Feighn)**.** Let $\mathcal{G}$ be a finite graph of finite rank free groups, i.e.

$$\mathcal{G} = ((V, \hat{E}, \mathrm{op}, \partial_0), \{G_v\}_{v \in V}, \{G_e\}_{e \in \hat{E}}, \{\varphi_e\}_{e \in \hat{E}}),$$

where V, $\hat{E}$ are finite, and G_v, G_e are finite rank free groups. Then the following produces the Grushko decomposition of $\pi_1(\mathcal{G})$:

1. *Reduce* $\mathcal{G}$: If a vertex $v \in V$ has degree one and $\varphi_e \colon G_e \to G_v$ is an isomorphism, then delete this vertex and edge from $\mathcal{G}$ and return to Step 1. If a vertex $v \in V$ has degree two, $\hat{E}(v) = \{e, f\}$, and $\varphi_e \colon G_e \to G_v$ is an isomorphism, then remove this vertex and edge from $\mathcal{G}$ and join f to the other endpoint of e by setting $\partial_0 f := \partial_1 e$, $\varphi_f := \varphi_{\bar{e}} \circ \varphi_e^{-1} \circ \varphi_f$.



2. *Try to simplify at each vertex*: For each $v \in V$, do the following. Consider

$$\vec{H} = \{\varphi_e(G_e) \mid e \in \hat{E}(v)\},$$

a sequence of subgroups of $G_v = F_B$. Let $\vec{\Sigma} = \Sigma_{\mathrm{G}}(\vec{H})$ be a Gersten representative (see Algorithm 4.13). Check if $\vec{\Sigma} = (\Sigma_1, \ldots, \Sigma_k)$ satisfies any of the following:

   (a) *Blow up*: If there is a partition $B = B' \sqcup B''$ $(B', B'' \neq \emptyset)$ such that each Σ_i has labels only from B' or only from B'', then go to Step 3 (a).

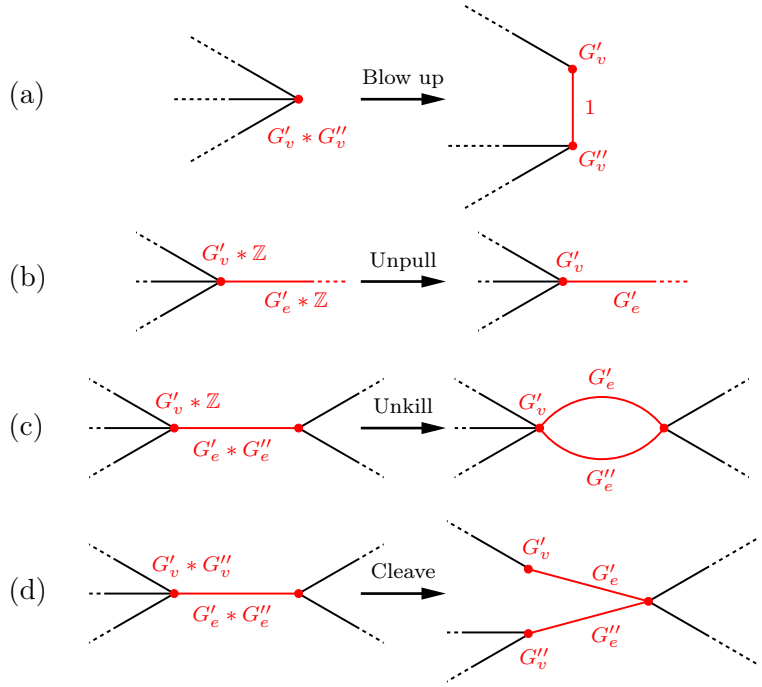
   (b) *Unpull*: If there is a label $b \in B$ that appears only once in $\vec{\Sigma}$, which is a non-separating edge in Σ_i, then go to Step 3 (b).

   (c) *Unkill*: If there is a label $b \in B$ that appears only once in $\vec{\Sigma}$, which is a separating edge in Σ_i, then go to Step 3 (c).

   (d) *Cleave*: If there is a partition $B = B' \sqcup B''$ $(B', B'' \neq \emptyset)$, and there is a Σ_i such that $\Sigma_i = \Sigma' \vee \Sigma''$ (wedge sum of topological spaces) where Σ' is a labelled graph with labels only from B', Σ'' has labels only from B'', and each Σ_j $(j \neq i)$ has labels only from B' or only from B'', then go to Step 3 (d).

   If none of the vertices give a $\vec{\Sigma}$ that has any of these properties, then go to Step 4.

3. *Simplify* $\mathcal{G}$: We do not explain the exact details here (because we will not need them), but provide the figures to give an idea of what might happen. These simplifications do not change $\pi_1(\mathcal{G})$.



Return to Step 1 with the new $\mathcal{G}$.

4. *Read off the decomposition*: Let $E_0 \subseteq E$ be the set of edges with $G_e \neq 1$. Let the restriction of $\mathcal{G}$ to the connected components of (V, E_0) be the graphs of groups $\mathcal{G}_1, \ldots, \mathcal{G}_n$, and let $\pi_1(\mathcal{G}_i) = G_i$. Let the graph obtained from (V, E) by collapsing the edges in E_0 have fundamental group F_m.

   Then $F_m * G_1 * \cdots * G_n$ is the Grushko decomposition of (the original) $\pi_1(\mathcal{G})$.

4.2.4 Dark subgroups

Definition 4.15. We call the subgroup $H \leq F_B$ *dark* if it has a Gersten representative Σ that satisfies the following:

- each label $b \in B$ appears at least twice in Σ, and
- Σ is not of the form $\Sigma = \Sigma' \vee \Sigma''$, where Σ' has labels only from B', and Σ'' has labels only from B'', for some partition $B = B' \sqcup B''$ $(B', B'' \neq \emptyset)$.

If one Gersten representative satisfies these, then all Gersten representatives of H do. We get the following family of negative examples to Question 6.

Theorem 4.16. *Let $G \leq F_m$ and $H \leq F_n$ be subgroups of free groups with $G \cong H \cong F_k$. If G and H are dark, then there is no isomorphism $\varphi\colon G \to H$ such that $F_m *_\varphi F_n$ is free.*

Proof. Let $\varphi\colon G \to H$ be any isomorphism, and consider the graph of groups

$$\Gamma = (\{v_0, v_1\}, \{e, \bar{e}\}, \mathrm{op}, \partial_0), \quad \partial_i(e) = v_i,$$
$$\mathcal{G} = (\Gamma, \{F_m, F_n\}, \{F_k\}, \{\varphi_e, \varphi_{\bar{e}}\}),$$

where $\varphi_e\colon F_k \to G, \varphi_{\bar{e}}\colon F_k \to H$ are some maps with $\varphi = \varphi_{\bar{e}} \circ \varphi_e^{-1}$. We follow Algorithm 4.14 on the input $\mathcal{G}$.

1. Suppose $G = F_m$, then $\Sigma_{\mathrm{G}}(G)$ would be a graph of one vertex and m edges, each with a different label, and hence G would not be dark. Therefore, φ_e and $\varphi_{\bar{e}}$ are not isomorphisms, and we cannot reduce $\mathcal{G}$.
2. The vertex v_0 gives the sequence of subgroups $\vec{H} = (G)$, whose Gersten representative does not satisfy any of the properties (a)–(d), because G is dark. (Note that Definition 4.15 is the direct opposite of these properties. We required that no label appears zero times, as that would imply property (a).) A similar argument applies to v_1, as H is dark, and thus we skip to Step 4.
4. We get $E_0 = \{\{e, \bar{e}\}\}$ (as $k \neq 0$), $\mathcal{G}_1 = \mathcal{G}$, the free part is F_0, and the Grushko decomposition is $\pi_1(\mathcal{G}) = F_0 * \pi_1(\mathcal{G}_1)$.

By definition, this means that $\pi_1(\mathcal{G}_1)$ is not free. Therefore, $\pi_1(\mathcal{G}_1) = \pi_1(\mathcal{G}) = F_m *_\varphi F_n$ is not free. □

As a consequence, we get a family of negative examples to Question 5.

Proposition 4.17. *Let M_0, M_1 be handlebodies, and $S_0 \subseteq \partial M_0$, $S_1 \subseteq \partial M_1$ be homeomorphic compact connected surfaces (with any number of boundary components). Assume that the inclusion $i_j\colon S_j \to M_j$ is π_1-injective, and $(i_j)_*(\pi_1(S_j)) \leq \pi_1(M_j)$ is a dark subgroup for $j = 0, 1$. Then there is no gluing $\phi\colon S_0 \to S_1$ such that $M_0 \cup_\phi M_1$ is a handlebody.*

Proof. This is immediate from the previous theorem. Let

$$G = (i_0)_*(\pi_1(S_0)) \leq \pi_1(M_0) = F_m,$$
$$H = (i_1)_*(\pi_1(S_1)) \leq \pi_1(M_1) = F_n,$$

then $G \cong \pi_1(S_0) = F_k \cong H$. Let $\phi\colon S_0 \to S_1$ be any homeomorphism, then

$$\pi_1(M_0 \cup_\phi M_1) = F_m *_\varphi F_n,$$
$$\varphi\colon G \to H, \quad \varphi = (i_1)_* \circ \phi_* \circ (i_0)_*^{-1}.$$

By Theorem 4.16, since G and H are dark, $F_m *_\varphi F_n$ is not free. Hence, $M_0 \cup_\phi M_1$ is not a handlebody. □

However, this statement (when S_i has one boundary component) also follows from Theorem 4.3, so we do not obtain a larger family of negative examples to Question 5. We now prove this implication.

Second proof (if ∂S_i is connected). We need to show that if $i\colon S \to M$ is π_1-injective, and $H = i_*(\pi_1(S)) \leq \pi_1(M)$ is dark, then (M, S) does not contain a product disk D. Assume that it does.

If D is separating M, then we have a basis $B' \sqcup B''$ ($B', B'' \neq \emptyset$) of $\pi_1(M)$, corresponding to the two components of $M \setminus D$. Thus, the subgroup H can be represented by a graph $\Sigma = \Sigma' \vee \Sigma''$, where Σ' uses B'-labels, and Σ'' uses B''-labels. By [DF05, Lemma 9.23], since H is dark, it follows that $\Sigma_{\mathrm{G}}(H)$ is also of the form $\Sigma'_0 \vee \Sigma''_0$, where Σ'_0 uses B'_0-labels, and Σ''_0 uses B''_0-labels. Hence, H is not dark, a contradiction.

If D is not separating M, then take a closed curve b in M intersecting D once, and extend it to a basis B of $\pi_1(M)$. If $\partial D \cap S$ is separating S, then there is a basis $u_1, \ldots, u_n, bv_1\bar{b}, \ldots, bv_m\bar{b}$ of H, where u_j, v_k are words in $B \setminus \{b\}$. Thus, there is a graph Σ representing H, consisting of a wedge of circles for the words u_j, another wedge of circles for the words v_k, and an edge labelled b between the two wedge points. If $\partial D \cap S$ is not separating S, then there is a basis $u, u_1, \ldots, u_s$ of H, where u uses the letter b once, and u_j are words in $B \setminus \{b\}$. Thus, there is a graph Σ representing H, consisting of a wedge of circles for all the words u, u_j. In either case, Σ has only one edge labelled b. By [DF05, Lemma 9.20], it follows that $\Sigma_{\mathrm{G}}(H)$ also has a label that appears at most once. Hence, H is not dark, a contradiction. □

Therefore, we can view Theorem 4.16 as an algebraic analogue of Theorem 4.3.

4.2.5 Examples

We finish with a few examples of using Theorem 4.16. The first one is another proof of our first negative example to Question 6:

Second proof of Proposition 4.7. By Theorem 4.16, it suffices to show that $H = \langle ab\bar{a}ba, b\rangle$ is a dark subgroup of $\langle a, b\rangle$.

Using Algorithm 4.13, we can construct the Gersten representative of H. Figure 4.3 shows a graph Σ that represents the conjugacy class of H. There are no possible Stallings folds or degree-one vertices. There are 12 elementary Whitehead automorphisms of $\langle a, b\rangle$.

Checking each of them, performing Stallings folds, and taking cores, we find that none of them reduces the number of vertices. Hence, Σ is a Gersten representative.

Each label appears at least twice in Σ, and Σ is not of the form $\Sigma' \vee \Sigma''$, where Σ' uses only a-labels, Σ'' uses only b-labels. Therefore, H is dark, completing the proof. □

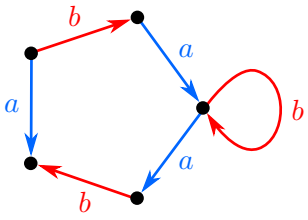


Figure 4.3: $\Sigma_G(\langle ab\bar{a}ba, b\rangle)$.

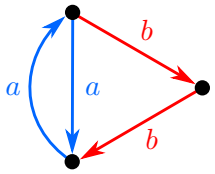


Figure 4.4: $\Sigma_G(\langle aa, bab\rangle)$.

Proposition 4.18. *The subgroups $\langle aa, bab\rangle \leq \langle a, b\rangle$ and $\langle cc, dcd\rangle \leq \langle c, d\rangle$ cannot be amalgamated to produce a free group.*

The first proof (4.6) of Proposition 4.7 does not apply directly in this case, but the second proof does. A Gersten representative of $H = \langle aa, bab\rangle$ is shown in Figure 4.4, hence H is dark.

It also follows, for example, that $\langle ab\bar{a}ba, b\rangle \leq \langle a, b\rangle$ and $\langle cc, dcd\rangle \leq \langle c, d\rangle$ cannot be amalgamated to produce a free group.

This subgroup $\langle aa, bab\rangle \leq \langle a, b\rangle$ also arises from a sutured handlebody (M, S); see Figure 4.5. Specifically, for the inclusion $i\colon S \to M$, i_* is injective, and

$$\langle aa, bab\rangle = i_*(\pi_1(S)) \leq \pi_1(M) = \langle a, b\rangle.$$

By Proposition 4.18, this sutured handlebody cannot be glued to itself to produce a handlebody. Alternatively, this result can also be established using Theorem 4.3.

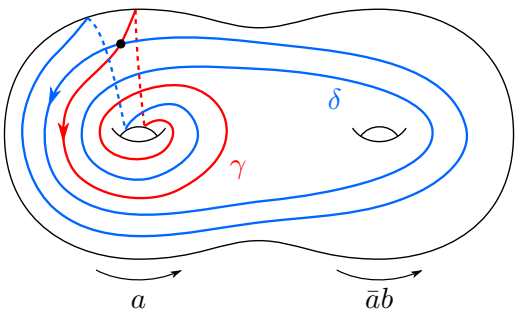


Figure 4.5: (M, S), where $S = N(\gamma \cup \delta)$.

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