

RANDOM MULTI-HOPPER MODEL. SUPER-FAST RANDOM WALKS ON GRAPHS

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Abstract. We develop a model for a random walker with long-range hops on general graphs. This random multi-hopper jumps from a node to any other node in the graph with a probability that decays as a function of the shortest-path distance between the two nodes. We consider here two decaying functions in the form of the Laplace and Mellin transforms of the shortest-path distances. Remarkably, when the parameters of these transforms approach zero asymptotically, the multi-hopper’s hitting times between any two nodes in the graph converge to their minimum possible value, given by the hitting times of a normal random walker on a complete graph. Stated differently, for small parameter values the multi-hopper explores a general graph as fast as possible when compared to a random walker on a full graph. Using computational experiments we show that compared to the normal random walker, the multi-hopper indeed explores graphs with clusters or skewed degree distributions more efficiently for a large parameter range. We provide further computational evidence of the speed-up attained by the random multi-hopper model with respect to the normal random walker by studying deterministic, random and real-world networks.

1. Introduction. Few mathematical models have found so many applications in the physical, chemical, biological, social and economical sciences as the random walk model [20, 27]. The term “random walk” was first proposed by K. Pearson [38] in an informal question posted in Nature in 1905. Pearsons description reads as “*A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times.*” Among the first respondents to Pearson, Lord Rayleigh [40] already pointed out a connection existing between random walks and other physical processes, namely that a random walk “*is the same as that of the composition of n iso-periodic vibrations of unit amplitude and of phases distributed at random*”. Connections like these between random walks and many physico-chemical, biological and socio-economic processes are what guarantees the great vitality of this research topic. This includes relations between random walks, Brownian motion and diffusive processes in general [9], the connection between random walks and the classical theory of electricity [10, 34], and the formulation of the efficient market hypothesis [26], among others. In order to distinguish the originally-proposed random walk model from its many varieties discussed in the literature we will use the term normal random walk (NRW).

Many real world complex systems are more faithfully represented as a network than as continuous system. This includes ecological and biomolecular, social and eco-

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nomical as well as infrastructural and technological networks [13]. The use of random walk models in these systems also provides a large variety of possibilities ranging from the analysis of the diffusion of information and navigability on these networks to the exploration of their structures to detect their fine-grained organization [35]. From a mathematical point of view, these networks are nothing but (weighted) graphs. We will use both terms interchangeably here with preference to the term network for the case they are representing some real-world system.

The first work exploring the use of random walks on graphs is credited to Pólya in 1921 when he was walking in a park and he crossed the same couple very often [39]. Thus, he asked the important question of the recurrence of a random walker in an infinite regular lattice. Known today as *Pólya's recurrence theorem*, it states that a simple random walk on a d -dimensional lattice is recurrent for $d = 1, 2$ and transient for $d > 2$. Random walks have been studied also on (regular) lattices, where many papers in the physics literature have used this method for the study of diffusion of atoms on a lattice. The classical papers of E. W. Montroll in the 1960's and 1970's paved the way for the study of these phenomena [33, 31, 32].

In the mid 1990's the group of G. Ehrlich [43] observed experimentally the self-diffusion of weakly bounded Pd atoms and made an interesting observation: there were significant contributions to the thermodynamical properties of the system from jumps spanning second and third nearest-neighbors in the metallic surface, which can be considered as a regular lattice. In 1997 the group of F. Besenbacher [23] observed experimentally that for the self-diffusion of Pt atoms on a Pt(110) surface, the jumps from non-nearest neighbors also contribute to the diffusion. Even more surprising are the results of the same group, when they studied the diffusion of two large organic molecules on a Cu(110) surface. In this case, using scanning tunneling microscopy, they observed that long jumps play a dominating role in the diffusion of the two organic molecules, with root-mean-square jump lengths as large as 3.9 and 6.8 lattice spacings [42]. Since then the role of long-jumps in adatom and molecules diffusing on metallic surface has been both theoretically and experimentally confirmed in many different systems [48, 2]. Due to these experimental evidence there has been some attempts to consider long-range jumps in the diffusion of a particle on a regular lattice. The first of them was the paper entitled "*Lattice walks by long jumps*" by Wrigley, Twigg and Ehrlich [47]. Other works have considered the space in which the diffusion takes place to be continuous and applied the random-walk model with Lévy flights to model these long-range effects (see below). However, the development of a general multi-hopper model, in which a random walker hops to any node of a general graph with probabilities depending on the distance separating the corresponding nodes is still missing in the literature. Apart from the physical scenarios related to the diffusion of adatoms and admolecules on metallic surface, long-range jumps on graphs are of a general interest. For instance, in social networks one can take advantage of the full or partial knowledge of the network beyond first acquaintances to diffuse information in a swifter way than can be done by the traditional nearest-neighbor only strategy. In exploring technological and infrastructural networks we can exploit our knowledge of the topology of the network to jump from a position to non-nearest-neighbors in such a way that we reach vaster regions of the system explored in shorter times.

In 2012 two groups published independently models that are designed to account for all potential long jumps that a random walker can take on a graph. Mateos and Riascos [41] proposed a random walk model where jumps occur to non-nearest neighbors with a probability that decays as a power-law of the shortest-path distance separat-

ing the two nodes (all formal definitions are done in the Preliminaries section of this paper). Estrada [12] generalized the concept of graph Laplacian by introducing the k -path Laplacians, and used it to study generalized diffusion equations to assess the influence of long-range jumps on graphs. While the first paper by Mateos and Riascos provides a probabilistic approach to the problem, the latter one by Estrada provides the algebraic tools needed for its generalization and mathematical formalization.

The combination of short- and long-range jumps in random walks is usually considered in the continuous space. This process is known as Lévy flight (LF) [44]. LFs are widely used to model efficient search processes, for instance, of animals searching for sparse food. Due to the scale-free nature of the underlying distribution of jump lengths, the combination of more local search and occasional long, decorrelating excursions leads to less oversampling and thus reduces typical search times, prompting the Lévy foraging hypothesis [46]. This efficiency may, however, be compromised in the presence of an external bias [37]. The efficiency of LFs in the random search context is, of course, also a function of the dimensionality of the embedding space. In dimensions three and above, when regular random walks become transient, LFs gradually lose their advantage as an efficient strategy. LFs are Markovian random walks. At every jump the step length is drawn from a long-tailed probability density function with the power-law decay [28, 29, 18]. The Markovian requirement means that at every jump all memory to previous jumps is erased, and thus the jump lengths x are independent and identically distributed random variables [6]. What happens when the Markovian character of the random walk is broken? A concrete example is an effective LF along a long polymer chain when shortcuts are allowed at locations where the polymer loops back to itself. The analogy with graphs runs in the following way. A long polymer can be considered as an infinite path graph—a graph in which all the nodes have degree two but two nodes are of degree one. Then, when the polymer folds, some regions approach closely to others forming a loop. In the graph this corresponds to the creation of cycles in the graph. In the polymer it has been observed that as the length stored in such loops has a power-law distribution, in the coordinate system of the arc length along the polymer the shortcuts effectively lead to a jump process with long tail [24, 45]. When the chain configurations relax quickly and, after taking a shortcut, the random walker cannot use the same shortcut again, individual jumps are indeed independent and thus the process is a true LF. However, when the chain configurations are much slower compared to the rate of shortcut events, correlations between present and previous jumps become important: the random walker may use a shortcut repeatedly, or it may become stuck in one of the loops (cycles). In the latter case, a scaling argument shows that long jump lengths are combined with long, power-law trapping times in cycles, and effectively the overall motion along the polymer chain is characterized by a *linear time dependence* of the mean squared displacement [45]. Thus, this last process cannot be considered as a true LF. A similar situation is observed in complex networks and in many general graphs where the random walker can get stuck in small regions of the graph due to heterogeneities in degree distribution or due to local structural heterogeneities, such as the existence of network communities, which will destroy the LF nature. To what extent this happens and what the remaining advantage of long-ranged hops is, must be analysed in the actual setting.

In this work we will consider the generalized formulation of a random walk model with long-range jumps that decay as a function—not necessarily a power-law—of the shortest path distance between the nodes. We propose to call this model the *ran-*

dom multi-hopper (RMH). We proceed by using the random walk–electrical networks connection as discussed in detail by Doyle [10] in order to formulate mathematically the RMH model and some of its main parameters, namely the hitting and commute times. We prove that for certain asymptotic values of the parameters of the model, the average hitting time of the random walker is the smallest possible. We further study some deterministic graphs for which extremal properties of random walks are known, such as lollipop, barbell and path graphs. Then, we move to the analysis of random networks, in particular we explore the Erdős-Rényi and the Barabási-Albert models. We finally study a few real-world networks representing a variety of complex systems. In all cases we compare the RMH with the NRW model and conclude that the multi-hopper overcomes several of the difficulties that a normal random walker has to explore a graph. In particular, the multi-hopper explores graphs with clusters of highly interconnected nodes and graphs with very skewed degree distributions more efficiently than a normal random walker. As these characteristics are omnipresent in real-world networks this makes the random multi-hopper an excellent choice for transport and search on complex systems.

2. Preliminaries. We introduce in this section some definitions and properties associated with random walks on graphs and set the notation used throughout the work. A *graph* $G = (V, E)$ is defined by a set of n nodes (vertices) V and a set of m edges $E = \{(u, v) | u, v \in V\}$ between the nodes. All the graphs considered in this work are finite, undirected, simple, without self-loops, and connected. A *path* of length k in G is a sequence of different nodes $u_1, u_2, \dots, u_k, u_{k+1}$ such that for all $1 \leq l \leq k$, $(u_l, u_{l+1}) \in E$. The length of a shortest path between two nodes i and j constitutes a distance function, here designated by $d(i, j)$, which is known as the *shortest-path distance* between the nodes i and j . Let $A = (a_{ij})_{n \times n}$ be the *adjacency matrix* of the graph where $a_{ij} = 1$ if and only if $(i, j) \in E$ and it is zero otherwise. The *degree* of a node i is the number of nodes adjacent to it and it is designated here by $k(i) = \sum_j a_{ij}$.

A *random walk* on a graph is a random sequence of vertices generated as follows. Given a starting vertex i we select a neighbor j at random, and move to this neighbor. Then we select a neighbor k of j uniformly at random, move to it, and so on [25, 3]. This sequence of random nodes $v_t : t = 0, 1, \dots$ is a Markov chain, with probability distribution encoded in the following vector:

$$(2.1) \quad \mathbf{p}_t(q) = \Pr(v_t = q).$$

The *matrix of transition probabilities* is $P = (p_{ij})_{i,j \in V}$ is defined via:

$$(2.2) \quad p_{ij} = \frac{a_{ij}}{k(i)}.$$

Let K be the diagonal matrix whose entries $K_{ii} = k(i)$. With this definition we can write the above matrix compactly as $P = K^{-1}A$. The vector containing the probability of finding the walker at a given node of the graph at time t is

$$(2.3) \quad \mathbf{p}_t = (P^T)^t \mathbf{p}_0,$$

where T represents the transpose of the matrix and \mathbf{p}_0 is the initial probability distribution of the random walker.

The following are important characteristics of a random walk on a graph which are of direct utility in the current work. For a random walk starting at the node i ,

the expected number of steps before it reaches the node j is known as the *hitting time* and it is denoted by $H(i, j)$. The expected number of steps in a random walk starting at node i , before the random walker visits the node j , and then visits node i again is known as the *commute time*, and it is denoted by $\kappa(i, j)$. Both quantities are related by

$$(2.4) \quad \kappa(i, j) = H(i, j) + H(j, i).$$

3. Random Multi-Hopper Model.

3.1. Intuition of the model. For a normal random walk on a graph, the random walker makes steps of length one in terms of the number of edges traveled. After each step she throws again the dice to decide where to move. As analogy we may think of a drunkard who does not remember where her home is. Thus she stops at every junction that she finds and takes a decision of where to go next in a random way.

Let us now suppose the existence of a random walker who does not necessarily stop at an adjacent node of her current position. That is, suppose that the random walker placed at the node i of the graph selects any node q of the graph to which she wants to move. Let us consider that the shortest path distance between i and q is $d(i, q)$. If $d(i, q) > 1$ the random walker will not stop at any of the intermediate nodes between i and q , but she will go directly to that node. In our analogy this would correspond to a drunkard who thinks she remembers where her home is, then she walks a few blocks without stopping at any junction until she arrives at a given point where she realizes she is lost. After this she repeats the process again. Therefore the movements of the drunkard can be represented by a graph, the edges of correspond to shortest paths in the original graph G , and the probability for the random walker to jump straight from node i to node q is proportional to a certain weight $\omega(i, q)$, which is a function of the distance $d(i, q)$ in G . As examples of decaying functions of the shortest path distance, we mention $\omega(i, q) = \exp(-l \cdot d(i, q))$, $\omega(i, q) = (d(i, q))^{-s}$ and $\omega(i, q) = z^{-d(i, q)}$, for $l > 0$, $s > 0$, and $z > 1$, respectively. Hereafter we will consider the first two for the analysis, called respectively the Laplace transform case and the Mellin transform case, by analogy with these transforms. The specific details in which we implement these transforms are given in the next section, where we used a slight variation of the Laplace transform.

As an example let us consider a one-dimensional linear chain of 5 nodes labeled as 1—2—3—4—5, and a weight function $\omega(i, q) = \exp(-0.5 \cdot d(i, q))$. If the drunkard is placed at the node 1 she has the following probabilities of having a walk of length 1, 2, 3 or 4: 0.46, 0.28, 0.18 and 0.10, respectively. That is, the probability that she stops at the nearest node from her current position is still higher than that for the rest of nodes, but the last ones are not zero like in the classical random walk. If $l \rightarrow \infty$, the probabilities approach those of the classical random walker. For instance if $\omega(i, q) = \exp(-5 \cdot d(i, q))$ the probabilities of having a walk of length 1, 2, 3 or 4 are: 0.9933, 0.0067, 0.000 and 0.000, respectively. For $l > 10$ we effectively recover the classical random walk model, and have numerical probabilities of 1, 0, 0, and 0 for the walks of length 1, 2, 3, and 4, respectively. This shows how the current model is a generalization of the classical random walk model.

3.2. Mathematical formulation. Consider a connected graph $G = (V, E)$ with n nodes. Let d_{\max} be the graph diameter, i.e., the maximum shortest path distance in the graph. Let us now define the d -path adjacency matrix ($d \leq d_{\max}$), denoted by

A_d , as the symmetric $n \times n$ matrix, with entries:

$$(3.1) \quad A_d(i, j) = \begin{cases} 1 & \text{if } d_{ij} = d, \\ 0 & \text{otherwise,} \end{cases}$$

where d_{ij} is the shortest path distance, i.e., the number of edges in the shortest path connecting the nodes i and j . The d -path degree of the node i is given by [12, 41]

$$(3.2) \quad k_d(i) = (A_d \mathbf{1})_i$$

where $\mathbf{1}$ is an all-ones column vector.

Let us now consider the following transformed k -path adjacency matrices [12]:

$$(3.3) \quad \hat{A}^\tau = \begin{cases} \sum_{d=1}^{d_{\max}} d^{-s} A_d & \text{for } \tau = \text{Mel}, \\ A_1 + \sum_{d=2}^{d_{\max}} \exp(-l \cdot d) A_d & \text{for } \tau = \text{Lapl}, \end{cases}$$

with $s > 0$ and $l > 0$, which are the Mellin and Laplace transforms, respectively.

Let us define the generalized degree of a given node as

$$(3.4) \quad \hat{k}^\tau(i) = \left(\hat{A}^\tau \mathbf{1} \right)_i.$$

Now we define the probability that a particle staying at node i hops to the node j as

$$(3.5) \quad P^\tau(i, j) = \frac{\hat{A}^\tau(i, j)}{\hat{k}^\tau(i)}.$$

Notice that if do not consider any long-range interaction, then $\hat{A}^\tau = A$ and $P^\tau(i, j) = P$. That is, we recover the classical random walk probability.

Let us denote by \hat{K}^τ the diagonal matrix with $\hat{K}^\tau(i, i) = \hat{k}^\tau(i)$ and let us define the matrix $\hat{P}^\tau = (\hat{K}^\tau)^{-1} \hat{A}^\tau$. Then, the evolution equation ruling the states of the walker at a given time step is given by

$$(3.6) \quad \mathbf{p}_{t+1} = \left(\hat{P}^\tau \right)^T \mathbf{p}_t.$$

3.3. k -path Laplacians, Hitting and Commute Times. In a similar way as the Laplacian matrix is introduced for graph we define the Laplacian matrix corresponding to Eq. (3.3). That is,

$$(3.7) \quad \hat{L}^\tau = \hat{K}^\tau - \hat{A}^\tau,$$

where K^τ is the diagonal matrix of generalized degree $\hat{k}^\tau(i)$ defined in Eq. (3.4) and A^τ is the generalized adjacency matrix defined in Eq. (3.3). This is the Laplacian of the graph G^τ , the (weighted) adjacency matrix of which is A^τ . As a result, this generalized Laplacian \hat{L}^τ is positive semi-definite [12].

The graph G^τ can be seen as a network of resistances, with the entry of A^τ representing the conductance (inverse resistance) of the connection between two nodes. By assuming that an electric current is flowing through the network G^τ by entering at node i and leaving at node j we can calculate the effective resistance between these two nodes as follows:

$$(3.8) \quad \hat{\Omega}^\tau(i, j) = \hat{L}^\tau(i, i) + \hat{L}^\tau(j, j) - 2\hat{L}^\tau(i, j),$$

where $\hat{\mathcal{L}}^\tau$ is the Moore-Penrose pseudo-inverse of the generalized Laplacian matrix. It is well known that the analogous of this effective resistance for the simple graph is a distance between the corresponding pair of nodes [21]. It is straightforward to show that this is also the case here and we will call $\hat{\Omega}^\tau(i, j)$ the *generalized effective resistance* between the nodes i and j in a graph.

The sum of all resistance distances in a graph is known as the Kirchhoff index of the graph [21]. In the context of the multi-hopper model it can be defined in a similar way as

$$(3.9) \quad \hat{\Omega}_{\text{tot}}^\tau = \sum_{i < j} \hat{\Omega}^\tau(i, j) = \frac{1}{2} \mathbf{1}^T \hat{\Omega}^\tau \mathbf{1}.$$

Then, an extension of a result obtained by Nash-Williams [34] and by Chandra *et al.* [8] allows us to calculate the commute and hitting times based on the generalized resistance distance. That is, the commute time between the corresponding nodes is given by

$$(3.10) \quad \hat{\kappa}^\tau(i, j) = \text{vol}(G^\tau) \hat{\Omega}^\tau(i, j),$$

where $\text{vol}(G^\tau)$ is the sum of all the weights of the edges of G^τ (see for instance, Ref. [14]). Using the *scaled generalized Fundamental Matrix* (SGFM) (details about the definition and notation for this matrix can be found in Refs. [15, 5]) we can express the hitting and commute times in matrix form as

$$(3.11) \quad \hat{H}^\tau = \mathbf{1} \left[\text{diag}(\tilde{Z}^{\tau^{-1}}) \right]^T - \tilde{Z}^\tau,$$

$$(3.12) \quad \hat{\kappa}^\tau = \mathbf{1} \left[\text{diag}(\tilde{Z}^{\tau^{-1}}) \right]^T + \left[\text{diag}(\tilde{Z}^{\tau^{-1}}) \right] \mathbf{1}^T - \tilde{Z}^\tau - \tilde{Z}^{\tau^T},$$

where \tilde{Z}^τ is just the SGFM for the graph G^τ . The expected commute time averaged over all pairs of nodes can be easily obtained from the multi-hopper Kirchhoff index as

$$(3.13) \quad \langle \hat{\kappa}^\tau \rangle = \frac{4(\mathbf{1}^T \mathbf{w}^\tau)}{n(n-1)} \hat{\Omega}_{\text{tot}}^\tau,$$

where \mathbf{w}^τ is the vector containing the weight of each edge in the graph G^τ . In a similar way we can obtain the expected hitting time averaged over all pairs of nodes

$$(3.14) \quad \langle \hat{H}^\tau \rangle = \frac{2(\mathbf{1}^T \mathbf{w}^\tau)}{n(n-1)} \hat{\Omega}_{\text{tot}}^\tau.$$

In order to understand the mechanism behind the efficiency of the multi-hopper random walker to explore networks, it is important to relate the structure of the network with dynamical quantities. In the following part we study the stationary probability distribution and the mean-first return time and its relation with the distances in the network.

The stationary probability distribution vector $\boldsymbol{\pi}^\tau$ is obtained as follows:

$$(3.15) \quad \boldsymbol{\pi}^\tau = \frac{\hat{P}^\tau \mathbf{1}}{\mathbf{1}^T \hat{P}^\tau \mathbf{1}}.$$

Now, having into account the definition of the matrix \hat{P}^τ , we obtain for the elements $\pi^\tau(i)$ with $i = 1, 2, \dots, n$ of the stationary probability distribution:

$$(3.16) \quad \pi^\tau(i) = \frac{k^\tau(i)}{\sum_{j=1}^n k^\tau(j)},$$

In this way, we obtain for the Mellin transformation with parameter s :

$$(3.17) \quad k^{\tau=\text{Mel}}(i) = k(i) + k_2(i) \frac{1}{2^s} + k_3(i) \frac{1}{3^s} + \dots + k_{d_{\max}}(i) \frac{1}{d_{\max}^s},$$

and the Laplace transformation with parameter l :

$$(3.18) \quad k^{\tau=\text{Lapl}}(i) = k(i) + k_2(i) \frac{1}{e^{2l}} + k_3(i) \frac{1}{e^{3l}} + \dots + k_{d_{\max}}(i) \frac{1}{e^{ld_{\max}}}.$$

The stationary probability distribution in Eq. (3.16) determines the probability to find the random walker at the node i in the limit t large. The expressions (3.17) and (3.18) allow to identify how the structure of the networks and the long-range strategy controlled by the parameters s or l combine in order to change the stationary probability distribution. In the limit of $s, l \rightarrow \infty$ the long-range contribution is null and the result $\pi(i) = k_i / \sum_{l=1}^n k_l$ for the normal random walker is recovered. On the other hand, when $s, l \rightarrow 0$, the dynamics includes, in the same proportion, contributions of first-, second-, third-, ..., and d_{\max} -nearest neighbors. In this limit case the stationary probability distribution is the same for all the nodes and $\pi(i) = 1/n$.

4. On the hitting time in the random multi-hopper model. The most important result of this work is related to the average hitting time of the random multi-hopper walk when the parameters of the corresponding transforms tends to zero.

LEMMA 4.1. *Let us consider the transformed k -path adjacency matrices:*

$$(4.1) \quad \hat{A}^\tau = \sum_{d=1}^{d_{\max}} c_d^\tau A_d,$$

with $c_d^{\tau=\text{Mel}} = d^{-s}$ for $s > 0$ and $c_d^{\tau=\text{Lapl}} = \exp(-l \cdot d)$ for $l > 0$. Then, when $s \rightarrow 0$ or $l \rightarrow 0$ the average hitting time $\langle \hat{H}^\tau \rangle \rightarrow (n-1)$, independently of the topology of the graph, which is the minimum for any graph of n nodes.

Proof. First, let G_n be any connected, simple graphs with n nodes. Then, $\langle \hat{H}(G_n) \rangle \leq \langle \hat{H}(K_n) \rangle = n-1$, with equality if and only if $G_n = K_n$, where K_n is the complete graph with n nodes (see Ref. [36]). Then, when $s \rightarrow 0$ or $l \rightarrow 0$ we have that $\hat{A}^{\text{Mell}}(i, j) \rightarrow 1$ and $\hat{A}^{\text{Lapl}}(i, j) \rightarrow 1$, respectively. This means that $\hat{A}^\tau(i, j) = 1 \forall i \neq j$ and $\hat{A}^\tau(i, j) = 0 \forall i = j$. In other words, $\hat{A}^\tau = \mathbf{1}\mathbf{1}^T - I$, which is the adjacency matrix of the complete graph K_n . In closing, when $s \rightarrow 0$ or $l \rightarrow 0$, $\hat{A}^\tau \rightarrow A(K_n)$. As it has been previously proved $\langle H(K_n) \rangle = n-1$ [36], which proves the result. \square

4.1. Case study: hitting times of the multi-hopper on all 8-node graphs.

We study the average hitting time of all 11,117 connected graphs with 8 nodes. The average hitting time has mean 10.036 ± 1.932 for all the graphs with $n = 8$, with a maximum value of 21.071. These values converge quickly to $n-1$ as soon as $s, l \rightarrow 0$. For instance for $s = 0.5$ the mean of the average hitting time is already 7.062 ± 0.024 , and this value drops up to 7.00253 ± 0.00096 for $s = 0.1$. The situation is very similar



FIG. 1. Illustration of the lollipop graph $L(8,4)$ (left) and the graph displaying the maximum value of $\langle \hat{H}^{\text{Mell}} \rangle$ (right).

for $l \rightarrow 0$, and the mean of the average hitting time is 7.0037 ± 0.0045 for $l = 0.1$ and $7.000072 \pm 4.67 \cdot 10^{-5}$ for $l = 0.01$. In all cases the minimum value of the average hitting time is obtained for the complete graph K_8 .

However, this is where the similarities between the classical RW and the multi-hopper with the Mellin and the Laplace transforms end. For instance, the graph with the maximum value of $\langle H \rangle$ for the NRW is the lollipop graph $L(8,4)$ illustrated in the left panel of Fig. 1. However, for $\langle \hat{H}^{\text{Mell}} \rangle$ with the values of s studied here the maximum is reached for the graph formed by a diamond graph with two paths connected to opposite nodes as illustrated in the right panel of Fig. 1. For $\langle \hat{H}^{\text{Lapl}} \rangle$ with the values of l studied here, the graph displaying the maximum is the path graph P_8 . We unfortunately do not have an analytic or otherwise explanation for this phenomenon.

The lollipop graphs appear in many extremal properties related to random walks on graphs. In 1990 Brightwell and Winkler [7] proved that the hitting time between a pair of nodes i and j in a graph is maximum for the lollipop graph $L(n, \lfloor \frac{2n}{3} \rfloor)$ consisting of a clique of $\lfloor \frac{2n}{3} \rfloor$ nodes including i to which a path on the remaining nodes, ending in j , is attached. The same graph was found by Jonasson as the one containing the pair of nodes maximizing the commute time among all graphs [19].

Then, we investigate the average hitting time of the lollipops $L(n, \lfloor \frac{n}{2} \rfloor)$ —the one having the largest value of $\langle H \rangle$ for the NRW among the graphs with $n = 8$ —and $L(n, \lfloor \frac{2n}{3} \rfloor)$, as well as the graphs with the structure displayed in the right panel of Fig. 1 and the paths P_n for $8 \leq n \leq 1000$. Our investigation produces the following results: (i) among the four types of graphs studied the lollipop $L(n, \lfloor \frac{n}{2} \rfloor)$ has always the largest value of $\langle H \rangle$; (ii) for $n \geq 10$, the lollipop $L(n, \lfloor \frac{2n}{3} \rfloor)$ has always the largest value of $\langle \hat{H}^{\text{Mell}} \rangle$ for $s = 0.5$ —the graph of the type illustrated in the right panel of Fig. 1 is maximum for $n = 8, 9$; (iii) for $8 \leq n \leq 29$ the path graph P_n has the largest value of $\langle \hat{H}^{\text{Lapl}} \rangle$ for $l = 0.1$. For $n \geq 30$ the lollipop $L(n, \lfloor \frac{n}{2} \rfloor)$ has the largest value of $\langle \hat{H}^{\text{Lapl}} \rangle$ for $l = 0.1$ among the four types of graphs investigated.

On the basis of the previous results one may conjecture that the graphs with the maximum value of the average hitting time correspond to certain type of lollipop graph. This is intuitively justified by the fact that in a lollipop the random walker spend a lot of time hopping among the nodes of the clique and it has a little chance of going through the path. However, on the same intuitively basis the barbell graphs are also good candidates for having the maximum average hitting time among graphs with n nodes (see for instance Ref. [3]). A barbell graph $B(n, k_1, k_2)$ is a graph with n nodes and two cliques of sizes k_1 and k_2 connected by a path on the remaining nodes. The problem that arises is to determine which of the many lollipop or barbell graphs has the largest average hitting time. This problem is out of the scope of the current work, but constitutes an important open problem. We remark that even for the case of the NRW with its many years of investigation, it is still unknown which is the graph with the maximum average hitting time. In case of the multi-hopper the

problem is even more complicated as the maximum value of the average hitting time may depend on the value of the parameters s and l of the transform used. We thus formulate the following open problem.

PROBLEM 1. *Determine the graph(s) with the maximum value of $\langle \hat{H}^\tau \rangle$ for different transforms of the multi-hopper random walk.*

5. Deterministic graphs. In this section we study some of the properties of the multi-hopper model for some classes of graphs which have deterministic structure.

5.1. Lollipop and barbell graphs. The first classes of graphs that we study here are the so-called lollipop and barbell graphs. These graphs appear in many extremal properties related to random walks on graphs as we have discussed in the previous section. Here we consider lollipop graphs $L(n, \lfloor \frac{n}{2} \rfloor)$ and $L(n, \lfloor 2\frac{n}{3} \rfloor)$, which have appeared already in the previous section and the symmetric barbell graphs $B(n, \lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor)$. We want to remark that these graphs are not necessarily the extremal ones for the hitting time as discussed in the previous section but they can be considered as representative of their classes.

We observe that the three graphs display $\langle H \rangle \approx an^3$ for the normal random walk. The coefficients a obtained by using nonlinear fitting of the hitting times with n are: $a \approx 0.01387$ for $L(n, \lfloor \frac{n}{3} \rfloor)$, $a \approx 0.0179$ for $B(n, \lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor)$ and $a \approx 0.01928$ for $L(n, \lfloor \frac{2n}{3} \rfloor)$.

We then study the variation of the parameters s and l in the Mellin and Laplace transforms of the multi-hopper model for the three graphs previously studied. In Fig. 2 we illustrate the results of these calculations. The important things to remark in this point are the obvious differences between the use of the Mellin and Laplace transforms in the multi-hopper model. First, it is easily observed that the Mellin transform produces a faster decay of the average hitting time than the Laplace one for the three graphs. For instance the 50% reduction in the average hitting time of the three graphs occurs for values of $3.5 < s < 4.0$ for the Mellin transform, but it happens for $1.5 < l < 2.0$ for the Laplace. This implies that the Laplace transform converges to the average hitting time of $n - 1$ at much smaller values than the Mellin transform. The second important difference is observed in the insets of Fig. 2. For the Laplace-transformed multi-hopper, the lollipop graph $L(n, \lfloor \frac{n}{2} \rfloor)$ always has the largest value of the average hitting time at any value of l among the three graphs studied. However, for the Mellin-transformed case, the lollipop graph $L(n, \lfloor \frac{n}{2} \rfloor)$ has the largest hitting time for large values of s , but for $s \lesssim 2.1$ the lollipop graph $L(n, \lfloor 2\frac{n}{3} \rfloor)$ is the one with the largest value of the average hitting time (see the crossing in the inset of Fig. 2). This confirms the complexity of the analysis of the extremal graphs for the multi-hopper model as we have hinted in the previous section.

We now concentrate on the variation of the average hitting time with the number of nodes in the lollipop $L(n, \lfloor \frac{2n}{3} \rfloor)$ for different values of the parameters s and l (see Fig. 3). For the Mellin transformed multi-hopper model with a fixed value of the exponent s , the average hitting time can always be fit well by a power-law of the number of nodes: $\langle \hat{H}^{\text{Mel}}(s) \rangle \approx an^\gamma$, where $\gamma \rightarrow 3$ when $s \rightarrow \infty$ and $\gamma \rightarrow 1$ when $s \rightarrow 0$. For instance, $\gamma = 2.831$ for $s = 3$; $\gamma = 2.129$ for $s = 2$; $\gamma = 1.772$ for $s = 1$; $\gamma = 1.291$ for $s = 0.5$; $\gamma = 1.011$ for $s = 0.1$. The situation is quite similar for the Laplace transformed multi-hopper model. This observation indicates that for small values of the parameters s and l the average hitting time changes linearly with the number of nodes. This important observation is repeated for every family of graphs as we will see in further sections of this work.

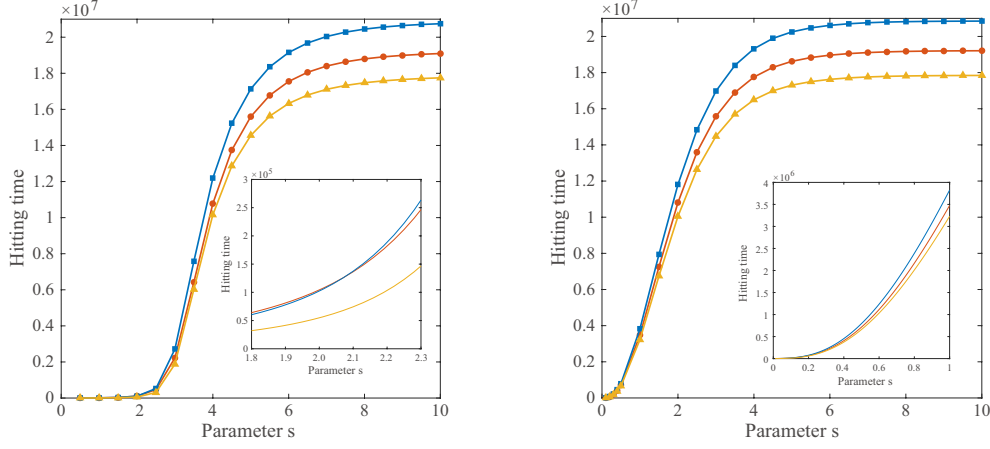


FIG. 2. Hitting time as a function of the parameter s for the Mellin (left) and the Laplace (right) transforms in lollipops $L(n, \lfloor \frac{n}{2} \rfloor)$ (blue squares), $L(n, \lfloor \frac{2n}{3} \rfloor)$ (red circles) and barbell $B(n, \lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor)$ (yellow triangles) graphs for $n = 999$. In the inset panels we zoom the plot for the region $1.8 \leq s \leq 2.3$ and $0.01 \leq l \leq 1$, respectively.

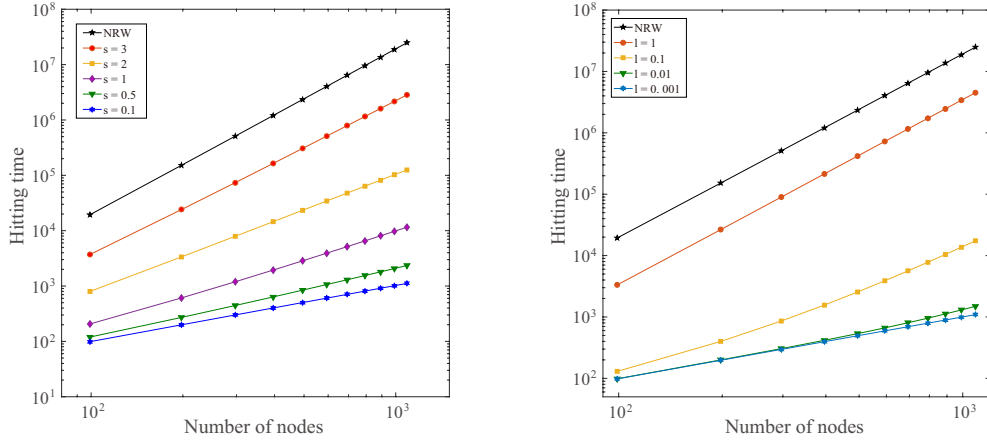


FIG. 3. Average hitting times for the random multi-hopper model with Mellin (left) and Laplace (right) transforms in the lollipop graph $L(n, \lfloor \frac{2n}{3} \rfloor)$ as a function of the number of nodes n in the graph.

We then study the variation of the average hitting time with the number of nodes for the lollipop $L(n, \lfloor \frac{2n}{3} \rfloor)$ for $0.001 \leq s \leq 0.05$ and obtain a linear dependence of the type: $\langle \hat{H}^{\text{Mell}}(G), s \rangle \approx \alpha n + \beta$.

Using these linear fits we can estimate the critical number of nodes n_c below which $(n-1) \leq \langle \hat{H}^{\text{Mell}}(L(n, \lfloor \frac{2n}{3} \rfloor)), s \rangle \leq n$ for a given value of the parameter s . Clearly, $n_c \leq \beta/(1-\alpha)$. However, we can simplify this expression by observing that $\beta < -1$ and that $\alpha \approx 1 + 2.751s^2$. Then,

$$(5.1) \quad n_c \leq \frac{1}{2.751s^2}, \quad s \leq 0.05.$$

The values of the critical number of nodes range from 145 for $s = 0.05$ to 363,504 for $s = 0.01$. This means, for instance, that any lollipop $L(n, \lfloor \frac{2n}{3} \rfloor)$ having less than

58,160 nodes will display average hitting time bounded between $n - 1$ and n in the random multi-hopper model with a Mellin transform and parameter $s \leq 0.0025$. The previous inequality can also be used in the other way around, namely in order to estimate what is the value of s that should be used such that a lollipop $L(n, \lfloor \frac{2n}{3} \rfloor)$ has average hitting time bounded as $(n - 1) \leq \langle \hat{H}^{\text{Mell}}(L(n, \lfloor \frac{2n}{3} \rfloor)), s \rangle \leq n$. For instance, if we would like to know the value of s for which any graph with less than 100,000 nodes has hitting time below $n - 1$, we use

$$(5.2) \quad s \geq \frac{1}{\sqrt{2.751n_c}}, \quad s \leq 0.05,$$

and obtain $s \approx 0.0019$. We venture out here and make some extrapolations to roughly estimate the value of s for which any lollipop $L(n, \lfloor \frac{2n}{3} \rfloor)$ with less than 1 million nodes has hitting time bounded as before. This estimation gives a value of $s \lesssim 0.0006$.

The importance of the previous investigation is the following. Currently we do not know what are the graphs with the largest value of the average hitting time among all the graphs with n nodes. However, we have evidence that it should be either a lollipop or a barbell graph. For these graphs the average hitting time is of the order n^3 for the NRW. Then, we can use the previous values obtained for the lollipop $L(n, \lfloor \frac{2n}{3} \rfloor)$ as rough indications of the worse case scenarios that can be expected for any graph. In other words, if we consider a graph of any structure having 1 million nodes we should expect that its average hitting time is bounded below n for $s \lesssim 0.0006$. We will see that for the case of real-world networks, these values of s are orders of magnitude over-estimated in relation to this upper bound. A first flavor of these differences is obtained by the analysis of random graphs in the next section of this work.

5.1.1. Time evolution. In this section we are not interested in a detailed description of the time evolution of the random walker or the multi-hopper in the lollipop or barbell graphs. We rather will make a comparison between the evolution of them at different times in such a way that we remark the main difference between the two models. Consequently we consider a lollipop $L(n, \lfloor \frac{2n}{3} \rfloor)$ and a barbell $B(n, \lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor)$ graph, both with $n = 999$ nodes. In both cases we place the random walker at a node of one of the two cliques. This node is selected not to be the one attached to the path. Let any node of a clique in the lollipop (respectively, a clique in the barbell) which is not the one connected to the path be designated as the node i . Let the endpoint of the path be named j . We remark that the node j does not belong to the clique. Let the node connecting the clique to which i belongs and the path be designated as k . Then, we have placed the random walker and the multi-hopper at the node i of the lollipop and the barbell and explore the probability at each node after different times using

$$(5.3) \quad \mathbf{p}_t = \left(\tilde{P}^T \right)^t \mathbf{p}_0,$$

where \tilde{P}^T is the transpose of \hat{P}^{Mell} .

As can be seen in Fig. 4 the classical random walker spends most of its time in the clique of the graphs, taking on average $\lfloor \frac{2n}{3} \rfloor - 1$ steps to visit the node k in the lollipop and $\lfloor \frac{n}{3} \rfloor - 1$ to visit it in the barbell. Once the walker visits the node k she can walk to the node j only with probability $1 / \lfloor \frac{2n}{3} \rfloor$ in the lollipop and $1 / \lfloor \frac{n}{3} \rfloor$ in the barbell graph. Then it can be seen in Fig. 4 that for times as large as $t = 10^6$ the random walker is still stacked in the clique of the lollipop graph. In the case of the

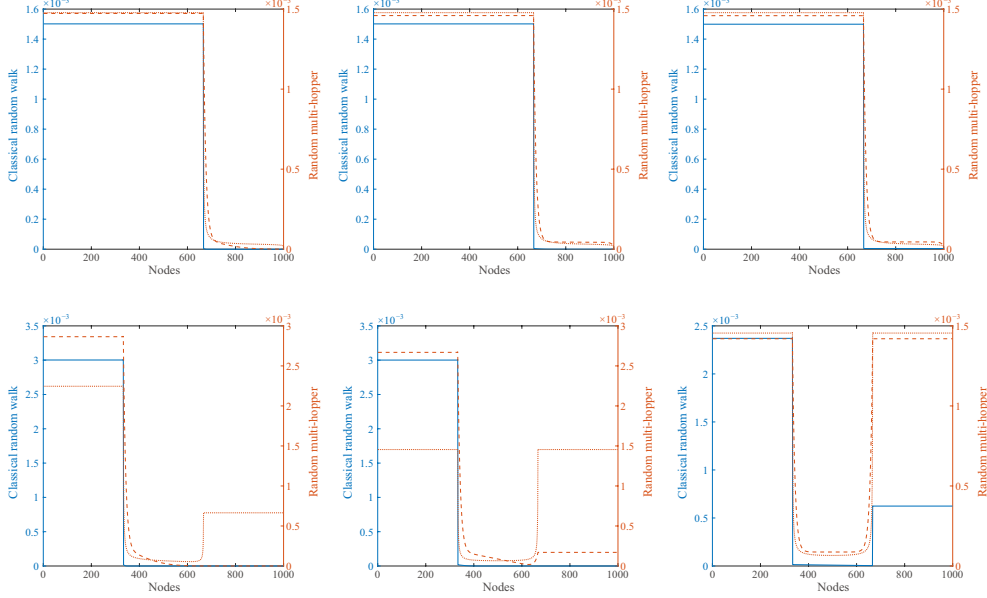


FIG. 4. Probability distribution at the different nodes of a lollipop $L(n, \lfloor \frac{2n}{3} \rfloor)$ (top) and barbell $B(n, \lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor)$ (bottom) graph with $n = 999$ nodes. Classical random walk (blue solid line) and the multi-hopper using the Mellin transform with $s = 1$ (red broken line) and using the Laplace transform with $l = 0.1$ (red dotted line). The evolution of the probabilities is shown at three different times for a random walk starting at node i (see text) at $t = 50$ (left), $t = 1000$ (center) and $t = 10^6$ (right).

barbell when $t = 10^3$ the walker has visited only the nodes of the clique in which she started and when $t = 10^6$ she starts to visit the nodes of the other clique.

On the other hand, the random multi-hopper has a non-zero probability of escaping directly from the clique at very short times. As can be seen in the right panels of Fig. 4 even for the small time $t = 50$ the multi-hopper with Mellin transform has already visited all the nodes of the graphs. For this short time, however, the multi-hopper with Laplace transform has visited all the nodes of the cliques plus the initial nodes of the path, but she has not arrived yet at the node j . For time $t = 1000$ the multi-hopper with Mellin transform is already in the stationary state and the one with Laplace transform has already visited all the nodes of the graphs. At $t = 10^6$ the multi-hopper has reached the stationary state for both transforms. This significant difference with the classical RW is due to the fact that the random multi-hopper is not trapped in the cliques due to the fact that she can go directly from i to any node of the graph with a probability that decays as a function of the distance from i . Then, the first few nodes of the path are frequently visited by the multi-hopper as they are at relatively short distances from the node i . Once, on these nodes the multi-hopper can visit the most extreme nodes of the graphs in an easier way overtaking the classical RW even at relatively short times. The way in which a random multi-hopper is propagated through a path is analyzed in the next subsection of this work.

5.2. Path graphs. Another interesting graph to consider is the path P_n . A path P_n is the graph having n nodes all of degree 2 but two which are of degree 1. As proved by Palacios [36] this graph has the maximum Kirchhoff index. For the normal RW, Palacios proved that $\Omega_{\text{tot}} \sim n^3$. As the number of edges in P_n is $n - 1$ one can

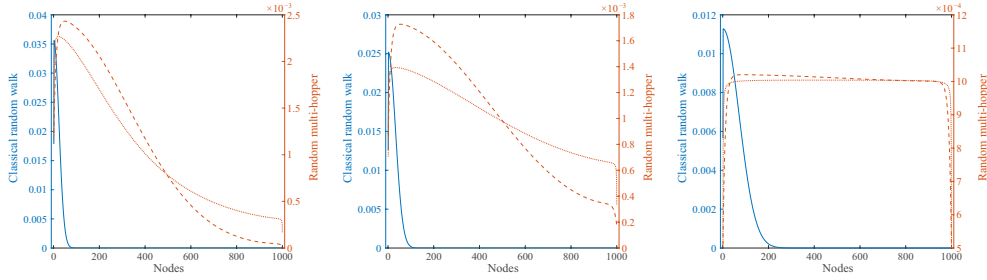


FIG. 5. Probabilities of finding the random walker at a given node of P_{1000} at $t = 500$ (left), $t = 1000$ (center), and $t = 5000$ (right) for the classical (blue solid line) and multi-hopper random walk model with Mellin transform with parameter $s = 2$ and with the Laplace transform with parameter $l = 0.1$.

easily see that $\langle H(P_n) \rangle \sim n^2$. In Fig. 5 we illustrate the evolution of the probabilities of being at a given node of the path of 1000 nodes labelled in consecutive order from 1 to 1000, in which we have placed the random walker at the node 1. As can be seen in the left panel of Fig. 5 for $t = 500$, the classical random walker has visited only the first 100 nodes of the path while the random multi-hoppers for both transforms has already visited all the nodes. As the time increases, the random multi-hopper model gives almost identical probabilities of finding the walker at any node of the path, but the classical random walker still shows close to zero probability of finding the walker at the other side of the path for times as large as $t = 5000$ (see the right panel of Fig. 5).

As in the previous subsection we study here the influence of the graph size over the hitting time in paths for both the Mellin and Laplace transforms. In particular, we compare both transformations in the multi-hopper random walk with the classical one for the path graph with $100 \leq n \leq 1000$. As expected the average hitting time in the classical random walk follows a quadratic dependence with the number of nodes, $\langle H \rangle \approx 0.3333n^2$. However, for the multi-hopper one it follows power-laws with exponent smaller than 2. For instance, for the dependence is of the form $\langle H^\tau \rangle \approx an^b$ with $b < 2$.

The most interesting thing here is that as for the barbell and lollipop graphs the average hitting time of paths also increases linearly with the number of nodes for relatively small values of the parameters s and l .

5.3. Some remarks. It is intuitive to think that the average shortest-path distance plays a fundamental role in explaining the average hitting time of graphs in the normal RW model. Then, because we allow for long-range jumps in the multi-hopper model we would intuitively expect that such influence of the shortest-path distance is diminished in this model. However, one important thing that we have learn from the analysis of the lollipop, barbell and path graphs is the following. Although the average shortest path distance plays some role in the determination of the average hitting times, it is the existence of large, relatively isolated, clusters which plays the major role. That is, although in a path graph we can have the largest possible average shortest path distance of any graph with n nodes—it has average path length equal to $\frac{n+1}{3}$ —they display average hitting time one order of magnitude smaller than the lollipop and barbell graphs, which may have relatively small average shortest path distances—particularly for the ones analyzed in this section. This role of large clusters in graphs, which we discussed in this section, is a great importance for the

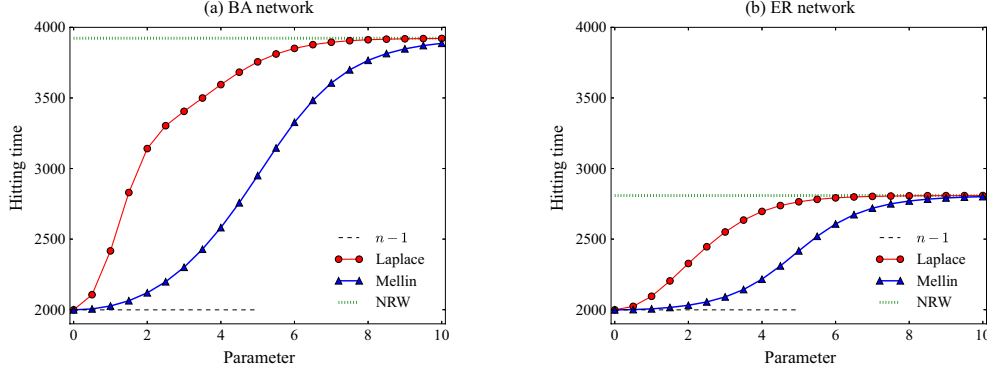


FIG. 6. Influence of the degree distribution on the average hitting time for random networks with $n = 2000$ nodes. (a) Barabási-Albert (BA) network, (b) connected Erdős-Rényi (ER) networks with probability $p = \log(n)/n$. We plot the results for the hitting time as a function of the parameter l for the Laplace transform and s for the Mellin transformation. We depict, with dashed lines, the results for the normal random walk (NRW) and $n - 1$ obtained for a complete graph.

analysis of real-world networks. Although these networks have relatively small average shortest-path distance due to their small-world properties, they contain many communities—clusters of tightly connected nodes, which are poorly connected among them—which resemble the extremal situation of barbell and lollipop graphs.

6. Random graphs. In this section we explore the multi-hopper model for two types of random networks: Barabási-Albert (BA) [4] and Erdős-Rényi (ER) [11] types. We use the exact result obtained for the expected hitting time averaged over all pairs of nodes $\langle \hat{H}^\tau \rangle$ in Eq. (3.14).

The analysis of the Barabási-Albert (BA) and Erdős-Rényi (ER) random networks shows that the hitting time increases linearly with the number of nodes in the graph (data not shown). In all cases the use of the Mellin transform in the multi-hopper drops the slope of the lines $\langle \hat{H}^\tau \rangle \approx an + b$ expressing the dependence of the hitting time with the number of nodes. Thus, we found that in general, using the long-range strategies, the resulting random walker reaches more efficiently any site on the network in comparison with the normal random walk.

In Fig. 6 we fixed the number of nodes to $n = 2000$ in ER and BA networks and we calculate the average hitting time as a function of the parameters l and s for the Laplace and Mellin transform, respectively. The results confirm previous findings [41] that in the limit for $l, s \rightarrow 0$ the hitting times reach the value $n - 1$. However, for parameters in the interval $(0, 10)$, we see how the two types of strategies present a strong variation in the average hitting time $\langle \hat{H}^\tau \rangle$. This is a direct consequence of how the random walk strategies assign weights to small, intermediate and large steps. Finally, for large values of the parameters, long-range transitions appear with low probability and the values of $\langle \hat{H}^\tau \rangle$ are equal to the results for the normal random walk strategy with transitions only to nearest neighbors. These results are of great significance for the further analysis of real-world networks in the next section of this work.

As we have observed in the previous analysis there are very significant differences between the random networks considered here and the graphs analyzed in the previous section, where the average hitting time increases as a third or second power of the number of nodes. The linear increase observed here for the random graphs

studied cannot be understood only on the basis of the fact that they display relatively small shortest-path distances. For instance, we can construct barbell graphs $B(n, \lfloor \frac{n-k}{2} \rfloor, \lfloor \frac{n-k}{2} \rfloor)$ with small values of k , which have small average shortest-path distance. A graph $B(n, \lfloor \frac{n-k}{2} \rfloor, \lfloor \frac{n-k}{2} \rfloor)$ has only distances $d_{ij} \in [1, k]$. One important difference, however, between the studied random networks and the barbell and lollipop graphs previously considered is the lack of large cliques in these random graphs which may trap the random walker inside them. In the next section we study this problem by using random graphs with different intercommunity density of links. In addition, we study the influence of the degree distribution on the hitting time of these random graphs with the goal of understanding the differences between Erdős-Rényi and Barabási-Albert networks.

6.1. Influence of communities and degree distribution. We start here by considering the influence of the presence of clusters of nodes defined in the following way. Let us consider a network with n nodes. Let us make a partition of the network in k clusters of size $\lfloor \frac{n}{k} \rfloor$. Let C_i and C_j be two of such clusters. Then, the probability that two nodes $p, q \in C_i$ are connected is much larger than the probability that two nodes $r \in C_i$ and $s \in C_j$ are connected. This gives rise to higher internal densities of links in the clusters than the inter-cluster density of links. It is a well-known fact that neither the Erdős-Rényi nor the Barabási-Albert networks contain such kind of clusters. The lack of such clusters—known in network theory as communities—is characterized by the so-called good expansion properties of these graphs. Loosely speaking a graph is an expander if it does not contain any structural bottleneck, i.e., a small group of nodes or edges whose removal separates the network into two connected components of approximately the same size [17]. We remark here that both ER and BA graphs have been proved to be expanders when the number of nodes is very large [17, 30]. Then, we use here an implementation of the algorithm described by Lancichinetti *et al.* [22] to produce undirected random networks with communities with a fixed average degree $\langle k \rangle$. A mixing parameter μ defines the fraction of links that a node share with nodes in other communities. A small value of the mixing parameter produces graphs with tightly connected clusters which are poorly connected among them. That is, it produces very well-defined communities in the graph. As the mixing parameter approaches the value of one, the communities disappear and the graph looks more and more as an expander for sufficiently large number of nodes.

Here, we explore the effect of communities in the capacity of the multi-hopper random walk strategy to reach any site of the network by constructing random graphs with the same number of nodes and edges but changing the mixing parameter. In Fig. 7 we depict the average hitting time $\langle H^\tau \rangle$ for different values of the parameters l and s for networks with communities constructed as described before. As can be seen here the random graphs with well defined communities, i.e., small values of the mixing parameter, makes that the random walker takes significantly longer time to explore the whole network. This is particularly true for relatively large values of the Mellin and Laplace parameters of the multi-hopper model, which indicate that the normal RW is significantly less efficient in networks having communities than in networks not displaying such structural characteristic. Here again, as these parameters approach zero the hitting time decays to the lower bound as expected from the theory. In closing, the small hitting times observed for the random graphs studied in the previous subsection are mainly due to the fact that these graphs are expanders and they lack any community structure, which may trap the random walker for longer times without visiting other clusters. In those cases analyzed here where there are communities, the

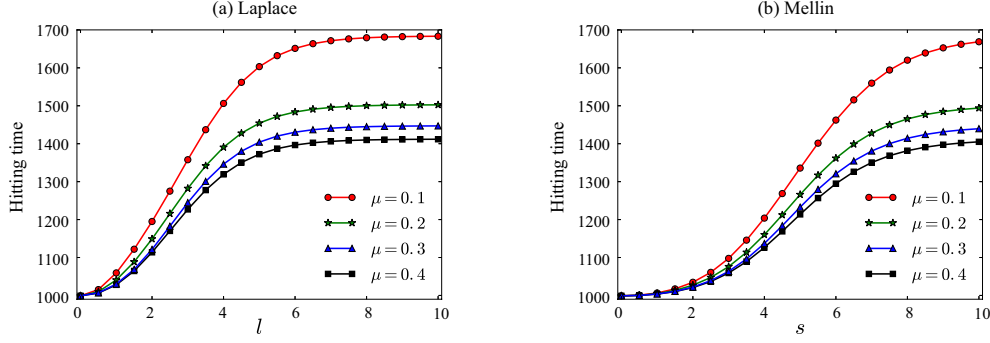


FIG. 7. Average hitting time for the multi hopper random walker in networks with communities. (a) Laplace and (b) Mellin strategies. We explore networks with $n = 1000$ nodes, an average degree $\langle k \rangle = 15$ and different values of the mixing parameter μ that defines the fraction of connection that a node has with nodes in other communities.

multi-hopper solves this trapping problem due to the fact that it is allowed to jump from one community to another, reducing her time inside each of the clusters visited.

Now we move to the consideration of the influence of the degree distribution on the performance of the random multi-hopper. We then study the stationary probability distribution $\pi^\tau(i)$ for the Laplace and Mellin transformation in a Barabási-Albert and an Erdős-Rényi network, with $n = 2000$ nodes. The results are obtained from the calculation of the long-range degrees $k^\tau(i)$ in Eqs. (3.17)–(3.18) and the respective normalization defined in Eq. (3.16).

In Fig. 8 we resume our results. The important aspect of these plot is to consider the slope of the corresponding curves for different values of the Mellin and Laplace transforms in the multi-hopper model. If we compare the slopes for the ER network with that of the BA one, we observe that the first is smaller and closer to the constant line $n - 1$ than the second. The smallest hitting time is obtained when the slope coincides with this line, which represents the fully connected graph. Thus, the ER graphs are already close to this slope and this is the main reason why they display relatively small hitting times. However, in the BA model when s, l are very large the slope of the curves are very steep and far away from the asymptotic result. As soon as these parameters approach zero the slope of the curves become more flat approaching $\pi^\tau(i) = n - 1$ as a consequence of the fact that the graph approaches the fully connected one. That is, the long-range dynamics changes the way in which the random walker reaches the nodes. For small values of the parameter s or l , the stationary probability distribution reaches the value $\pi(i) = 1/n$. On the other hand, the inverse of the stationary probability distribution defines the average time $\langle t^\tau(i) \rangle = \frac{1}{\pi^\tau(i)}$ needed for the random walker to return for the first time to the node i . In this way, the random walker returns to sites with high values of π^τ and, gets trapped in regions with this property. As we can see in Fig. 8, the effect of the long-range strategies is to reduce the probability to revisit sites highly connected and increasing the capacity to reach any site of the network.

In closing, in this section we have seen that a random walker can be trapped in certain regions of a network—i.e., having larger probability of staying at these regions than in other parts of the graph—due to two different factors. The first is the presence of clusters of highly connected nodes in which the random walker is

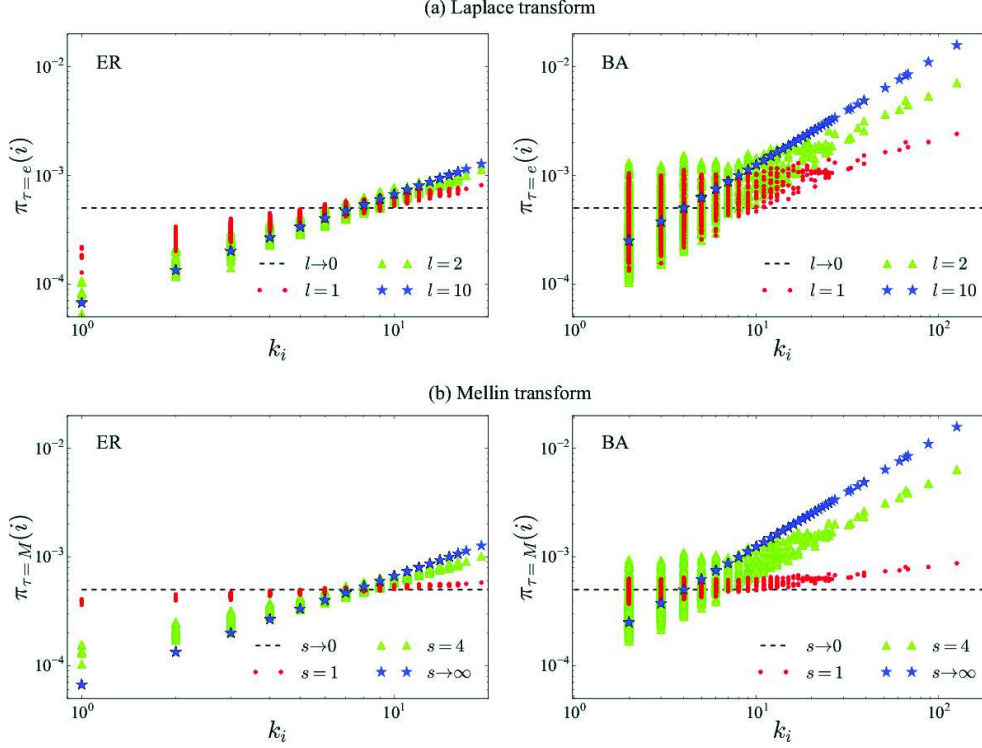


FIG. 8. Stationary probability distribution for multi-hopper random walkers on Barabási-Albert and Erdős-Rényi networks with $n = 2000$ nodes. (a) Laplace transform, (b) Mellin transform.

retained for long times before she visits other clusters of the graph. The second is the existence of hubs—highly connected nodes—which make that the random walker returns frequently to them making her exploration of the network more difficult. These two characteristics, the presence of communities and the existence of fat-tailed degree distributions, are well known to be ubiquitous in real-world networks. Then, the observation that the random multi-hopper overcome both of these traps make this model an important election for the exploration of real-world networks, which is the topic of our next section.

7. Real-world networks. One of the areas in which the random multi-hopper can show many potential applications is in the study of large real-world networks. Normal random walks on networks have been previously used as mechanisms of transport and search on networks [1, 16, 35]. These are graphs representing the networked skeleton of complex systems ranging from infrastructural and technological to biological and social systems. As an example of the potential applications of the random multi-hopper for these systems we consider the exploration of the electrical power grid of the western USA. In this case we compare the normal RW with the multi-hopper by placing the walker at the node having the largest closeness centrality in the network. This is the node which is relatively closer to the rest of the nodes of the graph. We compare the results with the selection of the initial node as the ones having the smallest closeness centrality among all the nodes in the graph, i.e., the one relatively farthest from all the other nodes. In Fig. 9 we illustrate the evolution of the probabil-

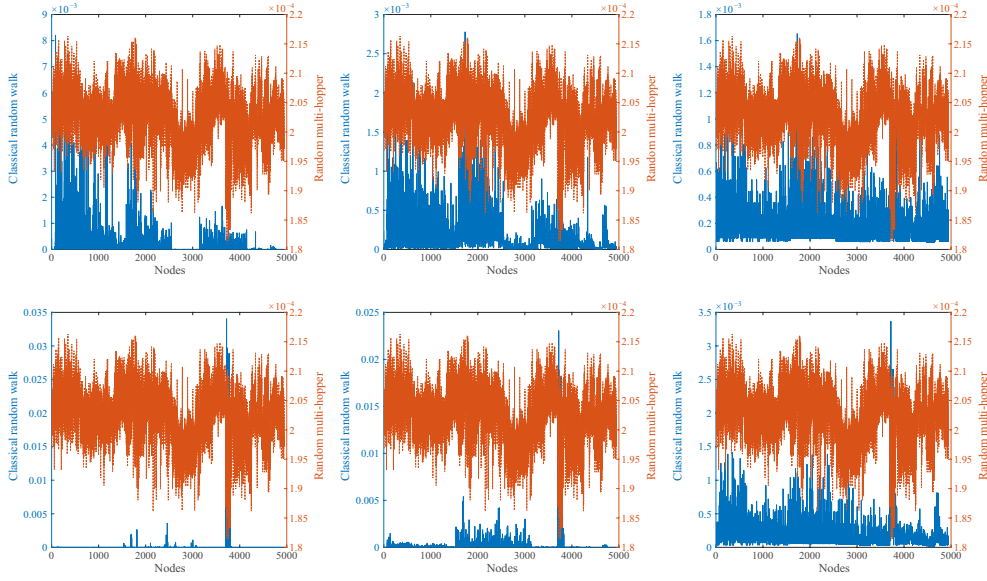


FIG. 9. Probability of finding the random walker at a given node of the western USA power grid for the classical random walker (blue solid line) and for the multi-hopper with $s = 0.1$ (red broken line) and $l = 0.01$ (red dotted line). The random walker started her walk at the node with the largest (top panels) and the smallest closeness centrality (bottom panels). The snapshots are taken at three different times, at $t = 50$ (left), $t = 500$ (center), and $t = 5000$ (right).

ity of finding the random walker at a given node of the power grid at $t = 50$, $t = 500$, and $t = 5000$. When the initial node is the one with the largest closeness centrality, at relatively short times (see the top-left panel of Fig. 9) the normal random walk has left some regions of the power grid unexplored. This situation is more critical when the initial node is the one with the smallest closeness centrality (see the bottom-left panel of Fig. 9), where the graph remains almost totally unexplored. In this last case even when $t = 500$ there are vast regions of the power grid that have not been visited by the walker. On the contrary, the random multi-hopper reaches the stationary state at very early times and at $t = 50$ she already has visited every node of the power grid independently of whether the initial node has the largest or the smallest closeness centrality.

We finally study a few networks representing a variety of real-world complex systems, including biological, communication and infrastructural ones. In Table 1 we report the sizes of these networks as well as the average hitting times using the normal random walk and the multi-hopper with Mellin transform. By using the expression (5.2) we can estimate the lower bound for the value of the Mellin parameter s for which $\langle \hat{H}^{\text{Mell}} \rangle \leq n$. These values are given in Table 1 as s_{lower} for all the networks studied in this section. In addition we calculate the actual value of this parameter for which $\langle \hat{H}^{\text{Mell}} \rangle \leq n$ in these networks and report it as s_c in Table 1. The values of s_c are obtained as follow. We calculate the value of $\langle \hat{H}^{\text{Mell}} \rangle$ for different values of s and obtain a fit of the form: $\langle \hat{H}^{\text{Mell}} \rangle \approx \alpha s^2 + \beta$ for $0.01 \leq s \leq 0.5$. Obviously, $\beta = n - 1$, which is the lowest value obtained by $\langle \hat{H}^{\text{Mell}} \rangle$ for any graph. Using these fitting equations we then calculate the values of s_c reported in Table 1. As can be seen the values of s_c are as average 10 times larger than the lower bound expected from the lollipop graphs of the same size as the studied networks.

network	n	$\langle H \rangle$	s_c	s_B	% impr.
Bio_PPI_yeast	2,224	8,652	0.145	0.0128	389
City_Atlanta	3,234	11,973	0.088	0.0106	370
Colab_Geom	3,621	15,719	0.086	0.0100	434
City_Berlin	4,495	13,752	0.094	0.0090	306
Power_USA	4,941	34,455	0.098	0.0086	697
City_Barcelona	5,575	17,282	0.063	0.0081	310
Colab_AstroPh	17,903	101,072	0.048	0.0045	565
City_Seattle	20,207	108,746	0.041	0.0042	538
Colab_CondMat	21,363	77,298	0.049	0.0041	362
Comm_Enron	33,696	193,714	0.040	0.0033	575

TABLE 1

Real-world networks studied in this work, their number of nodes n and the average hitting time of the normal random walker $\langle H \rangle$. s_c is the value of the Mellin parameter s for which the corresponding network has hitting time smaller than n . The value of s_B is the Mellin parameter s for which the corresponding a lollipop graph $L(n, \lfloor \frac{3n}{2} \rfloor)$ with the same number of nodes as the real-world network has hitting time smaller than n . The last column, % improv., represents the percentage of improvement in the hitting time using the Mellin-transformed multi-hopper respect to the NRW.

8. Conclusions. We develop here a mathematical and computational framework for using random walks with long-range jumps on graphs of any topology. This multi-hopper model allows a random walker positioned at a given node of a simple, connected graph to jump to any other node of the graph with a probability that decays as a function of the shortest-path distance between her original and final positions. The decaying probabilities for long-range jumps are selected as Laplace or Mellin transforms of the shortest-path distances in this work. We prove here that when the parameters of these transforms approach asymptotically zero, the hitting time in the multi-hopper approaches the minimum possible value for a normal random walker. Thus, the multi-hopper represents a super-fast random walker hopping among the nodes of a graph. We show by computational experiments that the multi-hopper overcomes several of the difficulties that a normal random walker has to explore a graph. For instance, the multi-hopper explores more efficiently a graph having clusters of highly interconnected nodes, which are poorly connected to other clusters, i.e., the presence of network communities, than the normal random walker. It also overcomes the normal random walker in those graphs with very skew degree distributions, such as scale-free networks. In these graphs, the normal random walker visits more frequently the hubs than the nodes of low degree, thus getting stacked around the high-degree nodes of the graph. Finally, we illustrate how the multi-hopper can be useful for transport and search problems in real-world networks where these structural heterogeneities, i.e., presence of communities and skew degree distributions, are more a rule than an exception. We hope that the use of the random multi-hopper will open new avenues in the exploration of lattices, graphs and real-world networks.

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