On Fusion 2-Categories

Thibault D. Décoppet

St Hugh's College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Hilary 2023

À la mémoire d'Odette Michoud.

Acknowledgements

First and foremost, I would like to thank my advisors Christopher Douglas and André Henriques for their support and guidance. I am especially thankful towards them for introducing me to the Mathematics I have become passionate about.

I am also particularly grateful towards David Reutter, Christoph Weis, and Matthew Yu, not only for the many stimulating mathematical conversations we have partaken in throughout the years, but also for being great friends.

My time at the Mathematical Institute was livened up by Sofía Marlasca Aparicio, Luciana Basualdo Bonatto, Christoph Dorn, Adele Jackson, Adrià Marín Salvador, Nivedita, Jan Steinebrunner, Filippos Sytilidis, and Jane Tan. Thank you for the countless tea breaks and cake seminars. Outside of Oxford, I would also like to thank my good friends Thomas Blech and Haiping Yang for their support.

I am thankful to the Fondation Zdenek et Michaela Bakala for partially funding my studies.

Finally, I am forever indebted to my parents Josiane and Didier for their unwaning support during these past years. None of this would have been possible without them. I would also like to thank my grandparents Cécile and Bernard, and my brother Tristan for their encouragements.

Abstract

The theory of fusion 1-categories has received much attention, and is therefore well established. Fusion 2-categories were recently introduced as a categorification of the notion of a fusion 1-category. The present thesis is devoted to the study of the general properties of fusion 2-categories.

At the heart of the definition of a fusion 2-category is the concept of finite semisimple 2-category over an algebraically closed field of characteristic zero. We generalize this concept by presenting the notion of a compact semisimple 2-category, which is sensible over an arbitrary field. Further, over an algebraically closed field or a real closed field, we prove that compact semisimple 2-categories are always finite.

Algebras in fusion 1-categories play a key role in the theory of fusion 1-categories. Accordingly, it is natural to study algebras in fusion 2-categories. We focus our attention on rigid algebras in fusion 2-categories, a notion which generalizes the definition of a fusion 1-category. In particular, we study the 2-categories of bimodules over rigid algebras. We also consider separable algebras, which are particularly well-behaved rigid algebras, and prove that a rigid algebra in a fusion 2-category is separable if and only if the associated 2-category of bimodules is finite semisimple.

Categorifying the notion of Morita equivalence between fusion 1-categories, we set up the Morita theory of fusion 2-categories. In particular, we give three equivalent characterizations of Morita equivalence between fusion 2-categories based on the 3-category of separable module 2-categories, the dual fusion 2-category, and the fusion 2-category of bimodules.

A crucial property of fusion 1-categories over an algebraically closed field of characteristic zero is that their Drinfeld centers are finite semisimple 1-categories. Categorifying the aforementioned result, we show that the Drinfeld center of any fusion 2-category over an algebraically closed field of characteristic zero is a finite semisimple 2-category. Our proof consists in a careful analysis of the Morita equivalence classes of fusion 2-categories.

Contents

Introduction

Fusion 1-categories are prominent objects of study in contemporary Mathematics. One reason is their truly ubiquitous nature. Namely, fusion 1-categories naturally arise in representation theory [BK01], as well as in functional analysis via the study of subfactors [JMS14]. Moreover, fusion 1-categories also feature outstandingly in low dimensional topology [Tur94], where they serve as the input of various 3-dimensional topological quantum field theories. Relatedly, fusion 1-categories have also found many applications in quantum Physics [MS89], more specifically in conformal field theory [FRS02], and in condensed matter Physics [Kit06].

A systematic study of the properties of fusion 1-categories was undertaken in [ENO05], where the authors coined the term fusion 1-category for a finite semisimple monoidal 1-category with duals and simple monoidal unit. Moreover, working over an algebraically closed field of characteristic zero, Etingof, Nikshych, and Ostrik prove that the global dimension of a fusion 1-category is always non-vanishing. This property is absolutely crucial in applications. For instance, to any monoidal 1-category, one can associate a braided monoidal 1-category called its Drinfeld center [Dri87, JS91]. It was established in [Müg03b] that the Drinfeld center of a fusion 1-category of non-zero global dimension is a braided fusion 1-category. Thanks to the non-vanishing of the global dimension, this implies that the Drinfeld center of any fusion 1-category over an algebraically closed field of characteristic zero is a braided fusion 1-category. Further, the aforementioned results also ensures that there is a well-behaved notion of Morita equivalence between such fusion 1-categories [ENO05].

The non-vanishing of the global dimension of fusion 1-categories is also a crucial property for the constructions of invariants of 3-manifolds. Namely, this hypothesis is needed in order to extend the Turaev-Viro construction to spherical fusion 1-categories [BW96], and to extend the Reshetikhin-Turaev construction to modular 1-categories [Tur92]. Fusion 1-categories are also related to topological quantum field theories through the cobordism hypothesis of [BD95] and [Lur10], which asserts that framed fully extended TQFTs can be constructed by specifying a fully dualizable object in a symmetric monoidal higher category. Conceptually, full dualizability should be thought of as a very strong finiteness condition. In [DSPS21], the authors considered the symmetric monoidal 3-category **TC** of finite tensor 1-categories, and exhibited fully dualizable objects, i.e. 3-dualizable objects, therein. More precisely, they showed that a fusion 1-category $\mathcal{C}$, or more generally a finite semisimple monoidal 1-category with duals, is a fully dualizable object of **TC** if and only if it is separable, meaning that $\mathcal{C}$ as a $\mathcal{C}$-$\mathcal{C}$-bimodule 1-category is equivalent to the 1-category of modules over a separable algebra in $\mathcal{C}\boxtimes\mathcal{C}^{mop}$. In addition, it was shown in [DSPS21] that a fusion 1-category is separable if and only if its Drinfeld center is finite semisimple. But, over an algebraically closed field of characteristic zero, this last condition is always satisfied as we have recalled above, so that every fusion 1-category over such a field is separable, and therefore yields a framed fully extended 3-dimensional TQFT. Moreover, two fusion 1-categories are equivalent as objects of **TC** if and only if they are Morita equivalent. In particular, Morita equivalent fusion 1-categories yield equivalent 3-dimensional TQFTs.

Fusion 2-categories were introduced in [DR18] as a categorification of the notion of a fusion 1-category over an algebraically closed field of characteristic zero. More precisely, over such a field, fusion 2-categories are by definition finite semisimple monoidal 2-categories with duals and simple monoidal unit. Since their first appearance, these objects have found many applications ranging from condensed matter physics [JF22] to quantum algebra [JFR24] by way of high energy physics [BBFP22, BBSNT22]. Categorifying the result of [DSPS21], Douglas and Reutter have conjectured that fusion 2-categories are fully dualizable objects, that is 4-dualizable objects in an appropriate symmetric monoidal 4-category, and therefore yield 4-dimensional TQFTs through the cobordism hypothesis.

Using the theory of higher condensations introduced in [GJF19], and under some assumptions on the behaviour of colimits in higher categories, the construction of the appropriate symmetric monoidal 4-category was given in [JF22]. Moreover, Johnson-Freyd sketched a characterization of the fully dualizable objects of this 4-category. In a different context, it was proven in [BJS21] that every braided fusion 1-category is a fully dualizable object of the symmetric monoidal 4-category **BrFus** of braided fusion 1-categories. This is related to the theory of fusion 2-categories. Namely, to any braided fusion 1-category $\mathcal{B}$, one can associate the fusion 2-category $\mathbf{Mod}(\mathcal{B})$ of finite semisimple $\mathcal{B}$-module 1-categories. Such fusion 2-categories are called connected, and it was shown in [JFR24] that connected fusion 2-categories have finite semisimple Drinfeld centers. One of the main contributions of the present thesis is to generalize this last result to all fusion 2-categories over an algebraically closed field of characteristic zero. More precisely, we prove that every fusion 2-category over an algebraically closed field of characteristic zero is separable, and show that the Drinfeld center of any such fusion 2-category is a braided fusion 2-category. We expect that these results will play a key role in the proof of the conjecture raised in [DR18]. More broadly, we develop the algebraic theory of fusion 2-categories.

Compact Semisimple 2-Categories

Working over an algebraically closed field of characteristic zero, Douglas and Reutter introduced in [DR18] the notion of a finite semisimple 2-category. Their definition is not only a categorification of the notion of a finite semisimple 1-category, but also gives an abstract characterization of the 2-categories $\mathbf{Mod}(\mathcal{C})$ of finite semisimple module 1-categories over a multifusion 1-category $\mathcal{C}$. Now, in the decategorified setting, the definition of a finite semisimple 1-category makes sense over any field. We explain how to generalize the notion of finite semisimple 2-category over an algebraically closed field of characteristic zero to a corresponding notion over an arbitrary field $\Bbbk$.

One might expect that the definition of finite semisimple 2-category from [DR18] translates directly to arbitrary fields. However, this is not quite the case because the finiteness requirements are too strong over a general field. Namely, given a finite semisimple tensor 1-category $\mathcal{C}$ over $\Bbbk$, we certainly want the 2-category $\mathbf{Mod}(\mathcal{C})$ of separable module 1-categories over $\mathcal{C}$ to be an example of the desired categorified notion. In fact, the appropriate definition should give an abstract characterization of these 2-categories. However, even though $\mathbf{Mod}(\mathcal{C})$ is always a semisimple 2-category in the sense of [DR18], it might not be finite, i.e. it can have infinitely many equivalence classes of simple objects. This explains why we introduce the notion of a compact semisimple 2-category, which relaxes slightly the finiteness assumptions to accommodate our desired examples. More precisely, the existence of non-zero 1-morphisms between non-equivalent simple objects organizes the set of simple objects of a semisimple 2-categories into connected components. We say that a semisimple 2-category is compact if it is locally finite and has only finitely many connected components. Using this definition, we prove the following result.

Theorem 2.1.4.6. *Let $\mathfrak{C}$ be a compact semisimple 2-category. Then, there exists a finite semisimple tensor 1-category $\mathcal{C}$ such that $\mathfrak{C} \simeq \mathbf{Mod}(\mathcal{C})$.*

We also introduce the notion of a locally separable compact semisimple 2-category, provided $\mathbb{k}$ is perfect, which abstracts the features of the semisimple 2-category $\mathbf{Mod}(\mathcal{C})$ when $\mathcal{C}$ is a separable tensor 1-category. In particular, every compact semisimple 2-category over a field of characteristic zero is locally separable.

We then investigate the conditions that are necessary to ensure that a compact semisimple 2-category is finite. We prove that, if $\mathbb{k}$ is a field whose separable closure is an infinite extension, there exists no finite semisimple 2-category over $\mathbb{k}$. Now, if $\mathbb{k}$ is a perfect field whose separable closure is a finite extension, then $\mathbb{k}$ is either real closed or algebraically closed. In this case, we obtain the following theorem.

Theorem 2.2.2.1. *Let $\mathcal{C}$ be a multifusion 1-category over an algebraically closed field or a real closed field. The 2-category $\mathbf{Mod}(\mathcal{C})$ of separable $\mathcal{C}$-module 1-categories is a finite semisimple 2-category.*

Corollary 2.2.2.2. *Over an algebraically closed field or a real closed field, a semisimple 2-category is finite if and only if it is compact.*

We go on to establish a categorification of the well-known equivalence between the 2-category of finite semisimple 1-categories and the Morita 2-category of finite semisimple algebras. More precisely, we show that, over a perfect field, the 3-category of locally separable compact semisimple 2-categories is equivalent to the Morita 3-category of separable finite tensor 1-categories. We use this equivalence to show that the Yoneda embedding of any locally separable compact semisimple 2-category $\mathfrak{C}$ is an equivalence, a result which was established in [DR18] over an algebraically closed field of characteristic zero. Then, generalizing the notion of fusion 2-categories to an arbitrary field, we introduce compact semisimple tensor 2-categories, and explore their fundamental properties. In addition, recall that, in the decategorified setting, the Deligne tensor product provides a canonical operation to combine finite semisimple tensor 1-categories together [Del90]. Categorifying this operation, we review the construction and the properties of the 2-Deligne tensor product of compact semisimple tensor 2-categories.

Rigid and Separable Algebras in Fusion 2-Categories

A key ingredient in the study of fusion 1-categories is the notion of an algebra in a fusion 1-category [EGNO15]. This provides motivation for the study of algebras in a fusion 2-category, or, more generally, a compact semisimple tensor 2-category. However, this notion turns out to be too general. For instance, let **2Vect** denote the fusion 2-category of finite semisimple 1-categories over an algebraically closed field. Then, algebras in **2Vect** are finite semisimple monoidal 1-categories. Instead of studying all algebras, we focus on the notion of a rigid algebra as introduced in [Gai12]. A finite semisimple monoidal 1-category $\mathcal{A}$ is rigid in the sense of [Gai12] if the monoidal product of $\mathcal{A}$ has a right adjoint as an $\mathcal{A}$-$\mathcal{A}$-bimodule functor. This notion of rigidity agrees with the classical one, which would require that $\mathcal{A}$ has duals. Namely, it was established in [BJS21] that an algebra in **2Vect** is rigid in the sense of [Gai12] if and only if it is a multifusion 1-category, i.e. has duals. We are interested in replacing **2Vect** by an arbitrary fusion 2-category, and, more generally, by a suitable monoidal 2-category $\mathfrak{C}$ with product $\Box$. In this more general context, we say that an algebra A is rigid if the multiplication 1-morphism $m : A\Box A \to A$ of A has a right adjoint m^* as a 1-morphism of A-A-bimodules. We obtain a characterization of rigidity for algebras in $\mathfrak{C}$.

Theorem 3.1.2.8. *Let $\mathfrak{C}$ be a monoidal 2-category that has left and right adjoints for 1-morphisms, and let A be an arbitrary algebra in $\mathfrak{C}$. Then, A is rigid if and only if* $\mathbf{Bimod}_{\mathfrak{C}}(A)$*, the 2-category of A-A-bimodules in $\mathfrak{C}$, has right adjoints. In this case,* $\mathbf{Bimod}_{\mathfrak{C}}(A)$ *has left adjoints.*

Now, one of the main take aways of [DSPS21] is that, over an algebraically closed field, separable multifusion 1-categories are much better behaved than arbitrary multifusion 1-categories. Abstracting the definition of a separable multifusion 1-category, we are led to consider the notion of separable algebra in a compact semisimple monoidal 2-category $\mathfrak{C}$ over an arbitrary field, that is a rigid algebra A such that the counit of the adjunction between m and m^* has a section as an A-A-bimodule 2-morphism. The following theorem gives a characterization of separability for rigid algebras in a compact semisimple monoidal 2-category.

Theorem 3.2.1.7. *Let A be a rigid algebra in a compact semisimple monoidal 2-category $\mathfrak{C}$, and write $Z(A)$ for the 1-category of A-A-bimodule 1-endomorphisms of A in $\mathfrak{C}$. Then, A is separable if and only if $Z(A)$ is finite semisimple. If either of these conditions is satisfied,* $\mathbf{Bimod}_{\mathfrak{C}}(A)$ *is a compact semisimple 2-category.*

We note that the above result is crucial for the definition of Morita equivalence between compact semisimple tensor 2-categories. Namely, all the compact semisimple 2-categories that are Morita equivalent to a compact semisimple tensor 2-category $\mathfrak{C}$ are of the form $\mathbf{Bimod}_{\mathfrak{C}}(A)$ for some algebra A in $\mathfrak{C}$. The above theorem allows us to focus our attention on separable algebras.

Over an algebraically closed field of characteristic zero, it follows from [DSPS21] that every rigid algebra in **2Vect** is separable. Based on this evidence, the question of whether every rigid algebra in a fusion 2-category over an algebraically closed field of characteristic zero is separable was raised in [JFR24]. This motivates the introduction of an invariant generalizing the global dimension of fusion 1-categories. To this end, we call an algebra in a compact semisimple monoidal 2-category $\mathfrak{C}$ connected provided that its unit 1-morphism is simple. Over an algebraically closed field and with $\mathfrak{C} = \mathbf{2Vect}$, connected rigid algebras are precisely fusion 1-categories. Then, working over an algebraically closed field, to any connected rigid algebra A in a finite semisimple monoidal 2-category $\mathfrak{C}$, we associate a scalar $\mathrm{Dim}_{\mathfrak{C}}(A)$, called its dimension, and we establish the following theorem.

Theorem 3.2.2.4. *Over an algebraically closed field, a connected rigid algebra A in a finite semisimple monoidal 2-category $\mathfrak{C}$ is separable if and only if its dimension $\mathrm{Dim}_{\mathfrak{C}}(A)$ is non-zero.*

In the case $\mathfrak{C} = \mathbf{2Vect}$, we show that our notion of dimension coincides with the global dimension of a fusion 1-category. Further, we give a formula to compute the dimension of a connected rigid algebra in a connected fusion 2-category. Over an arbitrary field, there is no well-defined notion of dimension for a general connected rigid algebra in a compact semisimple monoidal 2-categories. Nonetheless, we do give a related criterion that can be used to check that such an algebra is separable.

The Morita Theory of Fusion 2-Categories

In the theory of fusion 1-categories, the notion of Morita equivalence plays an essential role. For instance, it is used in the study of the Drinfeld center [Müg03b, ENO05], group-graded extensions [ENO10], and group-theoretical fusion 1-categories [ENO11]. The first definition of Morita equivalence between fusion 1-categories, which uses Frobenius algebras, was introduced in [FRS02], and [Müg03a], where it was used to study conformal field theories, and subfactors, respectively. Categorifying the original definition of Morita equivalence between algebras introduced in [Mor58], it is also natural to say that two fusion 1-categories are Morita equivalent if the associated 2-categories of module 1-categories are equivalent. More precisely, over an algebraically closed field of characteristic zero, we say that two multifusion 1-categories $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if the finite semisimple 2-categories $\mathbf{Mod}(\mathcal{C})$ and $\mathbf{Mod}(\mathcal{D})$ are equivalent. Alternatively, given $\mathcal{M}$ a finite semisimple left $\mathcal{C}$-module 1-category, we can consider $End_{\mathcal{C}}(\mathcal{M})$, the multifusion 1-category of left $\mathcal{C}$-module endofunctors of $\mathcal{M}$. Following [EO04], we use $\mathcal{C}^*_{\mathcal{M}}$ to denote $End_{\mathcal{C}}(\mathcal{M})$, and call it the dual tensor 1-category to $\mathcal{C}$ with respect to $\mathcal{M}$. We may then say that $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if there exists a faithful finite semisimple left $\mathcal{C}$-module 1-category $\mathcal{C}$ together with a monoidal equivalence between $\mathcal{C}^*_{\mathcal{M}}$ and $\mathcal{D}^{mop}$, that is $\mathcal{D}$ equipped with the opposite monoidal structure. It follows from [ENO10] that this notion coincides with the notion of Morita equivalence recalled above. Moreover, it follows from [Ost03b] that there exists an algebra A in $\mathcal{C}$ such that $\mathcal{M}$ is equivalent to $Mod_{\mathcal{C}}(A)$, the 1-category of right A-modules in $\mathcal{C}$. This implies that there is a monoidal equivalence between $End_{\mathcal{C}}(\mathcal{M})$ and $Bimod_{\mathcal{C}}(A)^{mop}$, the monoidal 1-category of A-A-bimodules in $\mathcal{C}$. Let us also note that, by [ENO05], the algebra A is necessarily separable, i.e. A is a special Frobenius algebra. It then follows that $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if and only if there exists a faithful separable algebra A in $\mathcal{C}$ together with an equivalence $\mathcal{D} \simeq Bimod_{\mathcal{C}}(A)$ of monoidal 1-categories. This recovers the original notion of Morita equivalence introduced in [FRS02] and [Müg03a]. Let us also remark that, over an arbitrary field, the above discussion remains sensible provided that all the module 1-categories under consideration are assumed to be separable in the sense of [DSPS21], that is are equivalent to $Mod_{\mathcal{C}}(A)$ for some separable algebra A in $\mathcal{C}$. In fact, a finite semisimple left $\mathcal{C}$-module 1-category $\mathcal{M}$ is separable if and only if $\mathcal{C}^*_{\mathcal{M}}$ is a finite semisimple 1-category.

We now explain how to categorify the equivalent characterizations of Morita equivalence between multifusion 1-categories recalled above. Given a compact semisimple tensor 2-category $\mathfrak{C}$, we begin by showing that the 2-category $\mathbf{Bimod}_{\mathfrak{C}}(A)$ of bimodules over a separable algebra A in $\mathfrak{C}$ has a canonical monoidal structure. Namely, we show that the relative tensor product over A exists in $\mathfrak{C}$. More generally, we establish the existence of the relative tensor product of modules over a separable algebra in any monoidal 2-category that is Karoubi complete in the sense of [GJF19]. In fact, we go further and construct the Morita 3-category $\mathbf{Mor}^{sep}(\mathfrak{C})$ of separable algebras, bimodules, and their morphisms in $\mathfrak{C}$.

We then switch perspective and study left $\mathfrak{C}$-module 2-categories as defined in [Déc21], for an arbitrary monoidal 2-category $\mathfrak{C}$. Namely, building on the construction of the 3-category of 2-categories given in [Gur13], we show that left $\mathfrak{C}$-module 2-categories form a 3-category, which we denote by $\mathbf{LMod}(\mathfrak{C})$. Further, provided that $\mathfrak{C}$ has duals, we prove that a left $\mathfrak{C}$-module 2-functor, which has a 2-adjoint as a plain 2-functor, has a 2-adjoint as a left $\mathfrak{C}$-module 2-functor.

Now, it was shown in [Déc21] that the 2-category $\mathbf{Mod}_{\mathfrak{C}}(A)$ of right A-modules in $\mathfrak{C}$ admits a canonical left $\mathfrak{C}$-module structure. By analogy with the decategorified setting, we wish to compare the monoidal 2-categories $\mathbf{Bimod}_{\mathfrak{C}}(A)$ and $\mathbf{End}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A))$. If $\mathfrak{C}$ is a compact semisimple tensor 2-category, we say that a left $\mathfrak{C}$-module 2-category $\mathfrak{M}$ is separable if it is equivalent as a left $\mathfrak{C}$-module 2-category to $\mathbf{Mod}_{\mathfrak{C}}(A)$ for some separable algebra A in $\mathfrak{C}$. Further, we write $\mathbf{LMod}^{sep}(\mathfrak{C})$ for the full sub-3-category of $\mathbf{LMod}(\mathfrak{C})$ on the separable module 2-categories. We note that, if our base field is perfect and $\mathfrak{C}$ is locally separable, it was established in [Déc21] that any compact semisimple left $\mathfrak{C}$-module 2-category $\mathfrak{M}$ is equivalent as a left $\mathfrak{C}$-module 2-category to $\mathbf{Mod}_{\mathfrak{C}}(A)$ for some rigid algebra A in $\mathfrak{C}$. This categorifies a theorem of [Ost03b], and shows that separability is a reasonable condition to impose on a compact semisimple left $\mathfrak{C}$-module 2-category in this case. Using the above concepts, we prove the following twice categorified version of the classical Eilenberg-Watts theorem.

Theorem 4.3.1.2. *Let $\mathfrak{C}$ be a compact semisimple tensor 2-category. There is a linear 3-functor, contravariant on 1-morphisms,*

$$\mathbf{Mod}_{\mathfrak{C}} : \mathbf{Mor}^{sep}(\mathfrak{C}) \to \mathbf{LMod}^{sep}(\mathfrak{C})$$

that sends a separable algebra in $\mathfrak{C}$ to the associated separable left $\mathfrak{C}$-module 2-category of right modules. Moreover, this 3-functor is an equivalence.

Let us now assume that the compact semisimple tensor 2-category $\mathfrak{C}$ is locally separable. Bringing together various of our results, we obtain the following theorem.

Theorem 4.3.3.2. *Let $\Bbbk$ be a perfect field, and A a separable algebra in a locally separable compact semisimple tensor 2-category $\mathfrak{C}$. Then, the monoidal 2-category*

$$\mathbf{End}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A)) \simeq \mathbf{Bimod}_{\mathfrak{C}}(A)^{mop}$$

is a compact semisimple tensor 2-category.

In particular, given a separable $\mathfrak{C}$-module 2-category $\mathfrak{M}$, we call $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$ the dual tensor 2-category to $\mathfrak{C}$ with respect to $\mathfrak{M}$, which we denote by $\mathfrak{C}^*_{\mathfrak{M}}$. Using the main result of [Déc21] recalled above, we obtain three equivalent characterizations of Morita equivalence between locally separable compact semisimple tensor 2-categories.

Theorem 4.3.4.3. *For any two locally separable compact semisimple tensor 2-categories $\mathfrak{C}$ and $\mathfrak{D}$ over a perfect field $\Bbbk$, the following are equivalent:*

1. *The 3-categories $\mathbf{LMod}^{sep}(\mathfrak{C})$ and $\mathbf{LMod}^{sep}(\mathfrak{D})$ are equivalent.*

2. *There exists a faithful separable left $\mathfrak{C}$-module 2-category $\mathfrak{M}$, and an equivalence of monoidal 2-categories $\mathfrak{D}^{mop} \simeq \mathfrak{C}^*_{\mathfrak{M}}$.*

3. *There exists a faithful separable algebra A in $\mathfrak{C}$, and an equivalence of monoidal 2-categories $\mathfrak{D} \simeq \mathbf{Bimod}_{\mathfrak{C}}(A)$.*

If any of the above equivalent conditions is satisfied, we say that $\mathfrak{C}$ and $\mathfrak{D}$ are Morita equivalent.

Drinfeld Centers and Morita Equivalence Classes of Fusion 2-Categories

Given an arbitrary monoidal 2-category $\mathfrak{C}$, its Drinfeld center, denoted by $\mathscr{Z}(\mathfrak{C})$, was defined in [BN95]. Its objects are given by objects of $\mathfrak{C}$ equipped with a half braiding and coherence data. In particular, $\mathscr{Z}(\mathfrak{C})$ is canonically a braided monoidal 2-category. We now work over an algebraically closed field of characteristic zero. Let $\mathfrak{C}$ be a multifusion 2-category, that is a finite semisimple monoidal 2-category with duals. We prove that the Drinfeld center of $\mathfrak{C}$ is invariant under Morita equivalence. Unlike in the case of fusion 1-categories [Ost03a], Morita equivalence classes of fusion 2-categories are not completely characterized by their Drinfeld centers. This failure can be attributed to the existence of fusion 2-categories that are not Morita trivial but are Morita invertible. Namely, given any non-degenerate braided fusion 1-category $\mathcal{B}$, the fusion 2-category $\mathbf{Mod}(\mathcal{B})$ is invertible [DN21, JFR24]. But, $\mathbf{Mod}(\mathcal{B})$ is Morita trivial if and only if the class of $\mathcal{B}$ is trivial in the Witt group of non-degenerate braided fusion 1-categories. As this Witt group is non-zero [DMNO13], there does indeed exist Morita invertible fusion 2-categories that are not Morita trivial.

We then investigate the notion of Morita equivalence between connected fusion 2-categories over an algebraically closed field of characteristic zero, that is fusion 2-categories of the form $\mathbf{Mod}(\mathcal{B})$ with $\mathcal{B}$ a braided fusion 1-category. Specifically, for any symmetric fusion 1-category $\mathcal{E}$, the concept of Witt equivalence between braided fusion 1-categories with symmetric center $\mathcal{E}$ was introduced in [DNO13]. Given two braided fusion 1-categories $\mathcal{B}_1$ and $\mathcal{B}_2$, we show that the fusion 2-categories $\mathbf{Mod}(\mathcal{B}_1)$ and $\mathbf{Mod}(\mathcal{B}_2)$ are Morita equivalent if and only if $\mathcal{B}_1$ and $\mathcal{B}_2$ both have symmetric center $\mathcal{E}$, and we have that $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent over $\mathcal{E}$.

We go on to study Morita equivalence classes of fusion 2-categories. Recall from [JFY21] that a strongly fusion 2-category is a fusion 2-category whose braided fusion

1-category of endomorphisms of the monoidal unit is the symmetric monoidal 1-category of (super) vector spaces. We show that every fusion 2-category over an algebraically closed field of characteristic zero is Morita equivalent to the 2-Deligne tensor product of a strongly fusion 2-category and an invertible fusion 2-category. Our proof of this result is inspired by the classification of topological orders [JF22], and relies on the minimal non-degenerate extension conjecture established in [JFR24]. Building on this result, we go on to prove the following theorem.

Theorem 5.3.2.1. *Over an algebraically closed field of characteristic zero, every fusion 2-category is Morita equivalent to a connected fusion 2-category.*

In particular, we obtain a complete classification of the Morita equivalence classes of fusion 2-categories in term of quotients of the Witt-groups $\mathcal{W}(\mathcal{E})$ of braided fusion 1-categories with symmetric center $\mathcal{E}$ as introduced in [DNO13].

We continue by explaining various consequences of our last theorem. For instance, we affirmatively answer the question raised in [JFR24] and recalled earlier.

Theorem 5.4.1.1. *Every rigid algebra in a fusion 2-category over an algebraically closed field of characteristic zero is separable.*

The above theorem holds more generally for any multifusion 2-category over an algebraically closed field of characteristic zero. Let us call a multifusion 2-category $\mathfrak{C}$ separable if it is separable when viewed as a left $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-category. Thanks to our last theorem above, we obtain the following categorification of corollary 2.6.8 of [DSPS21].

Corollary 5.4.2.3. *Over an algebraically closed field of characteristic zero, every multifusion 2-category is separable.*

Further, by relying on a sketch of proof given in [JF22], we outline an argument showing that separability is equivalent to 4-dualizability for multifusion 2-categories.

Using the notion of dimension of a connected rigid algebra we have developed, we then define the dimension $\mathrm{Dim}(\mathfrak{C})$ of a fusion 2-category, and we derive the following result, which is a categorification of theorem 2.3 of [ENO05].

Corollary 5.4.2.6. *Over an algebraically closed field of characteristic zero, the dimension of any fusion 2-category is non-zero.*

Finally, we explain how to derive the following corollary from our previous theorems.

Corollary 5.4.3.1. *The Drinfeld center of any multifusion 2-category over an algebraically closed field of characteristic zero is a finite semisimple 2-category.*

Structure of the Thesis

In chapter 1, we review the graphical language from [Déc21] and [Déc23d] that we use to work with monoidal 2-categories. We go on to recall the definitions of algebras and modules in a monoidal 2-category, and establish a useful coherence result.

Then, chapter 2 follows [Déc23a] with a few additions from [Déc22b] and [Déc24]. We define compact semisimple 2-categories over an arbitrary field, and explore their elementary properties. Using criteria on the base field, we precisely explain when compact semisimple 2-categories are finite. We study the 3-category of compact semisimple 2-categories, and give properties of compact semisimple tensor 2-categories.

Chapter 3 consists of [Déc23d]. We study rigid and separable algebras in monoidal 2-categories. In particular, we give an equivalent characterizations of both rigidity and separability for an algebra in a compact semisimple monoidal 2-category in terms of the associated 2-category of bimodules. Then, we define the dimension of a connected rigid algebra in a compact semisimple monoidal 2-category, and explain its relation with separability.

Next, chapter 4 is taken from [Déc23b]. We give three equivalent characterizations of Morita equivalence between locally separable compact semisimple tensor 2-categories. In order to do so, working over a fixed compact semisimple tensor 2-category, we study and compare the Morita 3-category of separable algebras with the 3-category of separable module 2-categories.

Finally, chapter 5 follows [Déc22a]. Working over an algebraically closed field of characteristic zero, we study the Drinfeld centers and Morita equivalence classes of fusion 2-categories. In particular, we give two equivalent sets of representatives for the Morita equivalence classes of fusion 2-categories. Using one of them, we show that the dimension of any fusion 2-category over an algebraically closed field of characteristic zero is non-zero, and that its Drinfeld center is a finite semisimple 2-category.

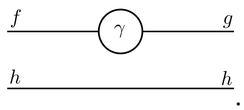
Chapter 1

Preliminaries

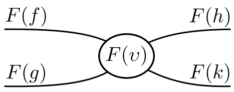
1.1 Graphical Conventions

1.1.1 2-Categories, 2-Functors, and 2-Natural Transformations

Throughout this thesis, we will be working within certain (weak) 2-categories. In this context, there is a well-established graphical calculus of string diagram developped in [GS16]. More precisely, regions correspond to objects, strings to 1-morphisms, and coupons to 2-morphisms. We take the convention that our string diagrams are read from top to bottom (composition of 1-morphisms) and from left to right (composition of 2-morphisms). We use the symbols Id_A and 1 to denote the identity 1-morphism on a given object A, but we will occasionally omit such 1-morphisms. As an example, the composite of the 2-morphisms $\gamma : f \Rightarrow g$ and the 1-morphism h in some 2-category $\mathfrak{C}$, provided it is defined, is depicted using the following diagram:



In section 4.2, we will also consider 2-functors and 2-natural transformations. We now recall from [Déc21] how to extend the above graphical calculus to these objects. In detail, given $F : \mathfrak{A} \to \mathfrak{B}$ a (weak) 2-functor, we write $\phi^F_A : Id_{F(A)} \cong F(Id_A)$ for the 2-isomorphism witnessing that F preserves the identity 1-morphism on the object A in $\mathfrak{A}$, and $\phi^F_{g,f} : F(g) \circ F(f) \cong F(g \circ f)$ for the 2-isomorphism witnessing that F preserves the composition of the two composable 1-morphisms f and g in $\mathfrak{A}$. These 2-isomorphisms satisfy well-known compatibility conditions. Now, given any 2-morphism $\upsilon : g \circ f \Rightarrow k \circ h$ in $\mathfrak{A}$, we set:



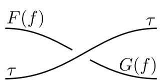
$$:= (\phi^F_{k,h})^{-1} \cdot F(\upsilon) \cdot \phi^F_{g,f}.$$

We extend this convention in the obvious way to the image under a 2-functor of a general 2-morphism, and note that it is well-defined thanks to the coherence axioms for a 2-functor.

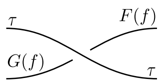
Now, let $F, G : \mathfrak{A} \to \mathfrak{B}$ be two 2-functors, and let $\tau : F \Rightarrow G$ be 2-natural transformation. This means that, for every object A in $\mathfrak{A}$, we have a 1-morphism $\tau_A : F(A) \to G(A)$, and for every 1-morphism $f : A \to B$ in $\mathfrak{A}$, we have a 2-isomorphism

$$\begin{tikzcd}
F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\overset{\tau_f}{\Nearrow}"] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\tau_B"'] & G(B),
\end{tikzcd}$$

The collection of these 2-isomorphisms has to satisfy the obvious coherence relations. In our graphical language, we will depict the 2-isomorphism τ_f using the following diagram on the left, and its inverse using the diagram on the right:



,



.

1.1.2 Monoidal 2-Categories

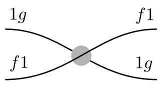
In fact, we will generally work with a monoidal 2-category $\mathfrak{C}$, that is a 2-category $\mathfrak{C}$ equipped with a 2-functor $\Box : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$, and unit object I in $\mathfrak{C}$, as well as various adjoint 2-natural equivalences and modifications ensuring the unitality of I and that $\Box$ is coherently associative (see [SP11], [GS16], or [Déc21] for details). Thanks to the coherence theorem for monoidal 2-categories proven by Gurski in [Gur13], every monoidal 2-category is equivalent to one that is strict cubical. Recall that a strict cubical monoidal 2-category $\mathfrak{C}$ is a strict 2-category, equipped with a monoidal structure for which the unit I is strict, and $\Box$ is strictly associative. In particular, we may omit the symbol I as well as the parentheses. Further, the 2-functor $\Box$ becomes strict when it is restricted to either variable. However, it is not a strict 2-functor in general. Namely, given a pair of composable 1-morphisms f_1, f_2, and g_1, g_2 in $\mathfrak{C}$, the interchanger

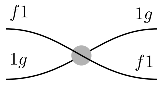
$$\phi^{\Box}_{(f_2,g_2),(f_1,g_1)} : (f_2 \Box g_2) \circ (f_1 \Box g_1) \cong (f_2 \circ f_1) \Box (g_2 \circ g_1),$$

that is the 2-isomorphism ensuring that $\Box$ respects composition, is non-trivial in general. The strict cubical hypothesis on $\mathfrak{C}$ guarantees that $\phi^{\Box}_{(f_2,g_2),(f_1,g_1)}$ is an identity 2-morphism if either $f_2 = 1$ or $g_1 = 1$. Given 1-morphisms f and g in $\mathfrak{C}$, the composite

$$(\phi^{\Box}_{(1,g),(f,1)})^{-1} \cdot \phi^{\Box}_{(f,1),(1,g)} : (f\Box 1) \circ (1\Box g) \cong (1\Box g) \circ (f\Box 1)$$

will be depicted in our graphical language by the string diagram below on the left, and its inverse by that on the right:

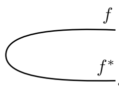


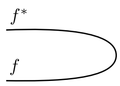
,


.

Note that we have omitted the symbol $\Box$ to make the diagrams more easily readable.

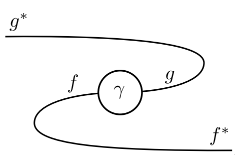
We will be interested in studying right (and left) adjoints for 1-morphisms in the 2-category $\mathfrak{C}$. If f is a 1-morphism in $\mathfrak{C}$ that has a right adjoint f^*, we will use the string diagram depicted below on the left for the unit η^f of this adjunction, and the one on the right for the counit ϵ^f:



,


.

Note that the triangle identities for the adjunction correspond graphically to the so-called snake-equations. We use a similar convention for left adjoints. Finally, given f as above, the 1-morphism $1f$ obtained by taking the monoidal product of f with some identity 1-morphism has a canonical right adjoint given by $1f^*$ with unit $1\eta^f$ and counit $1\epsilon^f$. We shall always consider adjuntion data of this form on such a 1-morphism, and we adopt the analogous convention for the 1-morphism $f1$. Finally, given two 1-morphisms f and g in $\mathfrak{C}$ with right adjoints f^* and g^* together with a 2-morphism $\gamma : f \Rightarrow g$, we set

$\gamma^* :=$ 

.

1.2 Algebras & Modules in Monoidal 2-Categories

1.2.1 Algebras

Throughout, we assume that $\mathfrak{C}$ is strict cubical. Let us begin by recalling the definition of an algebra (also called a pseudo-monoid in [DS97]). For the definition presented using the same graphical calculus in a general monoidal 2-category, we invite the reader to consult [Déc21].

Definition 1.2.1.1. An algebra in $\mathfrak{C}$ consists of:

1. An object A of $\mathfrak{C}$;

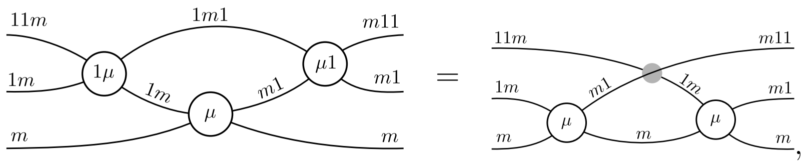
2. Two 1-morphisms $m : A\Box A \to A$ and $i : I \to A$;

3. Three 2-isomorphisms

$$\begin{tikzcd} A \arrow[rr, equal] \arrow[dr, "i1"'] & \arrow[d, phantom, "\lambda\Downarrow"] & A \\ & AA, \arrow[ur, "m"'] & \end{tikzcd} \quad \begin{tikzcd} AAA \arrow[r, "m1"] \arrow[d, "1m"'] \arrow[dr, phantom, "\overset{\mu}{\Nearrow}"] & AA \arrow[d, "m"] \\ AA \arrow[r, "m"'] & A, \end{tikzcd} \quad \begin{tikzcd} & AA \arrow[dr, "m"] \arrow[d, phantom, "\Downarrow\rho"] & \\ A \arrow[ur, "1i"] \arrow[rr, equal] & & M, \end{tikzcd}$$

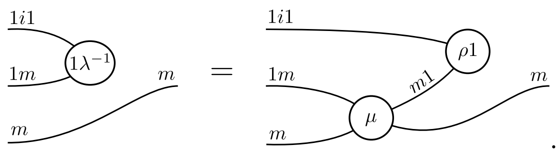
satisfying:

a. We have:



(1.1)

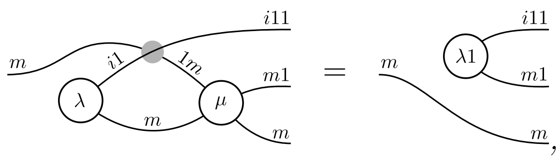
b. We have:



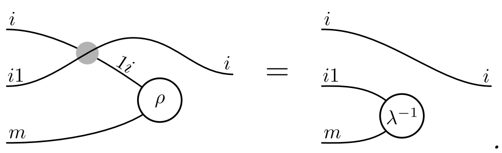
(1.2)

We will use the following coherence properties in what follows.

Lemma 1.2.1.2. *Given any algebra A, the following two equalities hold:*



(1.3)



(1.4)

Proof. See section 6.3 of [Hou07]. $\square$

Example 1.2.1.3. Let $\Bbbk$ be an arbitrary field. Recall that a $\Bbbk$-linear 1-category is called finite if it is equivalent to the 1-category $Mod(R)$ of finite dimensional left R-modules with R a finite dimensional $\Bbbk$-algebra. We write **FinCat** for the 2-category of finite $\Bbbk$-linear 1-categories, right exact functors, and natural transformations. It is well-known that the Deligne tensor product $\boxtimes$ endows **FinCat** with a symmetric monoidal structure (for instance, see [EGNO15]). It is easy to check that algebras in **FinCat** are precisely finite monoidal 1-categories, whose product is right exact in both arguments separately.

1.2.2 Right Modules

Let us now recall the definition of a right A-module in $\mathfrak{C}$, which we assume is strict cubical. Once again, the definition in a general monoidal 2-category may be found in [Déc21].

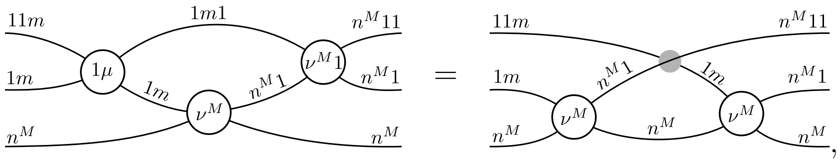
Definition 1.2.2.1. A right A-module in $\mathfrak{C}$ consists of:

1. An object M of $\mathfrak{C}$;
2. A 1-morphism $n^M : M\Box A \to M$;
3. Two 2-isomorphisms

$$\begin{tikzcd} MAA \arrow[r, "n^M1"] \arrow[d, "1m"'] \arrow[dr, phantom, "\nu^M \Nearrow"] & MA \arrow[d, "n^M"] \\ MA \arrow[r, "n^M"'] & M, \end{tikzcd} \qquad \begin{tikzcd} & MA \arrow[dr, "n^M"] \arrow[d, phantom, "\Downarrow\rho^M"] & \\ M \arrow[ur, "1i"] \arrow[rr, equal] & {} & M, \end{tikzcd}$$

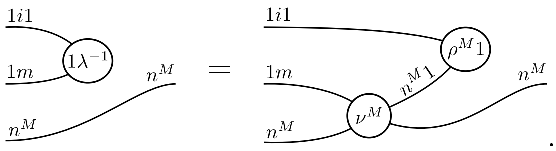
satisfying:

a. We have:



(1.5)

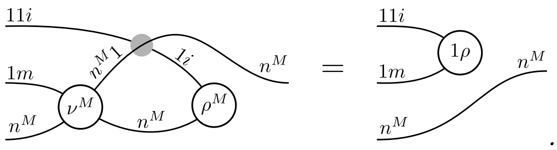
b. We have:



(1.6)

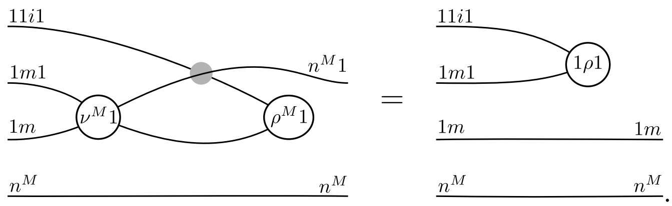
We will need the following coherence result for right A-modules. As we have found no proof in the literature, we include one for completeness.

Lemma 1.2.2.2. *Given any right A-module M, we have the following equality:*



(1.7)

Proof. We begin by proving the following equality:



In order to establish this equality, we use the diagrams depicted in section 1.3.1. Let us begin with figure 1.1. Applying equation (1.6) to the blue coupon, we arrive at figure 1.2. Then, moving the coupon labelled $1\lambda^{-1}$ up, we get to figure 1.3. Using equation (1.5) on the blue coupons yields figure 1.4. Finally, equation (1.2) applied to the blue coupons brings us to figure 1.5. The desired equality follows from that between figures 1.1 and 1.5 by cancelling the bottom coupons labelled ν^M.

Let us now prove that the coherence equation (1.7) holds. Figure 1.6 depicts the left hand-side of equation (1.7). Creating a pair of cancelling coupons labelled $\rho^{M^{-1}}$ and ρ^M in the blue region, we obtain figure 1.7. Moving the coupon labelled $\rho^{M^{-1}}$ to the left and the top string attached to it up, we arrive at figure 1.8. Now, we can use the equality derived above on the blue coupons to get to figure 1.9. Bringing the indicated string down gets us to figure 1.10. Finally, we can cancel the pair of blue coupons, and this brings us to the right hand-side of equation (1.7). □

Remark 1.2.2.3. As A is canonically a right A-module, we get a coherence relation for A.

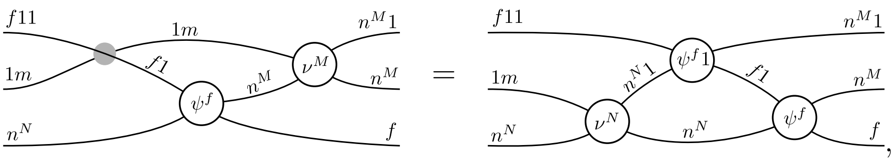
We also recall the definitions 1-morphism and 2-morphism of right A-modules.

Definition 1.2.2.4. Let M and N be two right A-modules. A right A-module 1-morphism consists of a 1-morphism $f : M \to N$ in $\mathfrak{C}$ together with an invertible 2-morphism

$$\begin{tikzcd} MA \arrow[r, "n^M"] \arrow[d, "f1"'] \arrow[dr, phantom, "\overset{\psi^f}{\Nearrow}"] & M \arrow[d, "f"] \\ NA \arrow[r, "n^N"'] & N, \end{tikzcd}$$

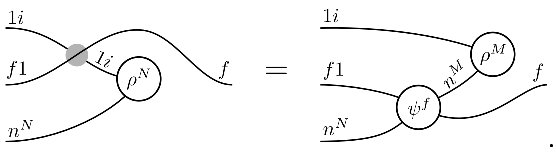
subject to the coherence relations:

a. We have:



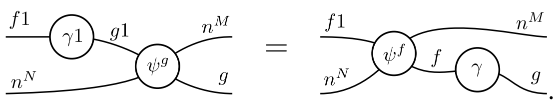
(1.8)

b. We have:



(1.9)

Definition 1.2.2.5. Let M and N be two right A-modules, and $f, g : M \to M$ two right A-module 1-morphisms. A right A-module 2-morphism $f \Rightarrow g$ is a 2-morphism $\gamma : f \Rightarrow g$ in $\mathfrak{C}$ that satisfies the following equality:



Finally, let us recall lemma 3.2.10 of [Déc21].

Lemma 1.2.2.6. *Right A-modules, right A-module 1-morphisms, and right A-module 2-morphisms form a 2-category, which we denote by* $\mathbf{Mod}_{\mathfrak{C}}(A)$.

Example 1.2.2.7. Let us fix a finite monoidal 1-category $\mathcal{A}$, whose monoidal product is right exact in both variable separately. This is precisely the data of an algebra in **FinCat**. Then, right $\mathcal{A}$-modules in **FinCat** are exactly finite right $\mathcal{A}$-module 1-categories, for which the action is right exact separately in both variables.

1.2.3 Left Modules

Let A be an algebra in a strict cubical monoidal 2-category $\mathfrak{C}$.

Definition 1.2.3.1. A left A-module in $\mathfrak{C}$ consists of:

1. An object M of $\mathfrak{C}$;

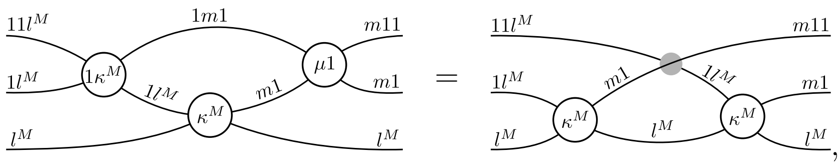
2. A 1-morphism $l^M : A\Box M \to M$;

3. Two 2-isomorphisms

$$\begin{tikzcd} M \arrow[rr, equal] \arrow[dr, "i1"'] & \arrow[d, phantom, "\lambda^M \Downarrow"] & M \\ & AM, \arrow[ur, "l^M"'] & \end{tikzcd} \qquad \begin{tikzcd} AAM \arrow[r, "m1"] \arrow[d, "1l^M"'] \arrow[dr, phantom, "\overset{\kappa^M}{\Nearrow}"] & AM \arrow[d, "l^M"] \\ AM \arrow[r, "l^M"'] & M, \end{tikzcd}$$

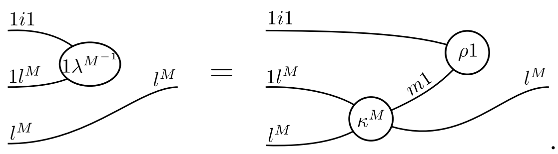
satisfying:

a. We have:



(1.10)

b. We have:



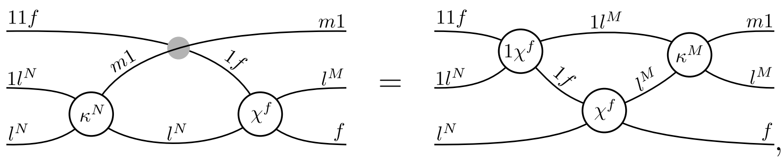
(1.11)

Definition 1.2.3.2. Let M and N be two left A-modules. A left A-module 1-morphism consists of a 1-morphism $f : M \to N$ in $\mathfrak{C}$ together with an invertible 2-morphism

$$\begin{tikzcd} AM \arrow[r, "l^M"] \arrow[d, "1f"'] \arrow[dr, phantom, "\overset{\chi^f}{\Nearrow}"] & M \arrow[d, "f"] \\ AN \arrow[r, "l^N"'] & N, \end{tikzcd}$$

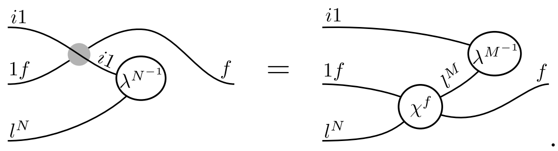
subject to the coherence relations:

a. We have:



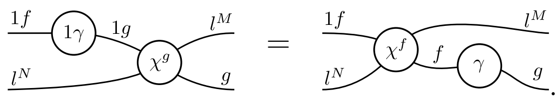
(1.12)

b. We have:



(1.13)

Definition 1.2.3.3. Let M and N be two left A-modules, and $f, g : M \to M$ two left A-module 1-morphisms. A left A-module 2-morphism $f \Rightarrow g$ is a 2-morphism $\gamma : f \Rightarrow g$ in $\mathfrak{C}$ that satisfies the following equality:



1.2.4 Bimodules

Let now A and B be two algebras in the strict cubical monoidal 2-category $\mathfrak{C}$. In the definition below, we write m^A for the multiplication 1-morphism of A and m^B for that of B. We now recall the definition of an A-B-bimodule in $\mathfrak{C}$. Below, we review the definitions of 1-morphisms and 2-morphisms of A-B-bimodules.

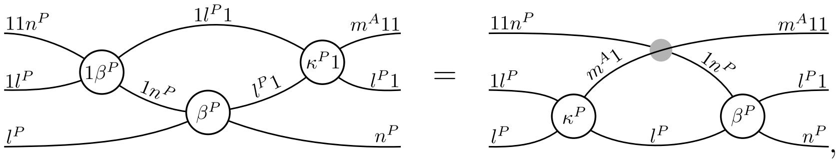
Definition 1.2.4.1. An A-B-bimodule in $\mathfrak{C}$ consists of:

1. An object P of $\mathfrak{C}$;
2. The data $(P, l^P, \lambda^P, \kappa^P)$ of a left A-module structure on P;
3. The data (P, n^P, ν^P, ρ^P) of a right B-module structure on P;
4. A 2-isomorphism

$$\begin{tikzcd} APB \arrow[r, "l^P 1"] \arrow[d, "1n^P"'] \arrow[dr, phantom, "\overset{\beta^P}{\Nearrow}"] & PB \arrow[d, "n^P"] \\ AP \arrow[r, "l^P"'] & M, \end{tikzcd}$$

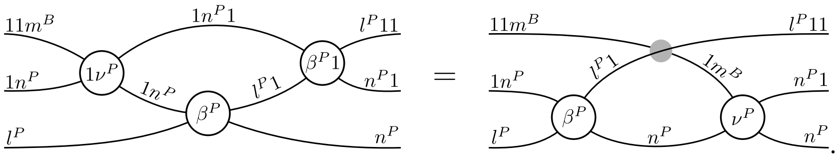
satisfying:

a. We have



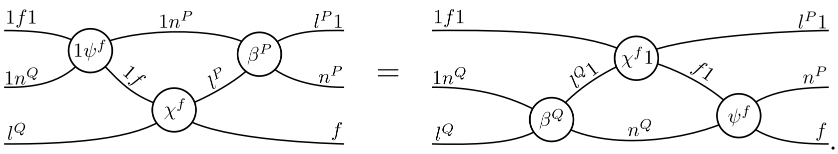
(1.14)

b. We have:



(1.15)

Definition 1.2.4.2. Let P and Q be two A-B-bimodules. An A-B-bimodule 1-morphism consists of a 1-morphism $f : P \to Q$ in $\mathfrak{C}$ together with the data (f, ξ^f) of a left A-module structure and (f, ψ^f) of a right B-module structure satisfying:



(1.16)

Definition 1.2.4.3. Let P and Q be two A-B-bimodules, and $f, g : P \to Q$ two A-B-bimodule 1-morphisms. An A-B-bimodule 2-morphism $f \Rightarrow g$ is a 2-morphism $\gamma : f \Rightarrow g$ in $\mathfrak{C}$, which is both a left A-module 2-morphism and a right B-module 2-morphism.

Lemma 1.2.4.4. *Given two algebras A and B in $\mathfrak{C}$, A-B-bimodules, A-B-bimodule 1-morphisms, and A-B-bimodule 2-morphisms form a 2-category, which we denote by* $\mathbf{Bimod}_{\mathfrak{C}}(A, B)$.

Proof. The proof is easy and left to the reader. We only point out that, as we have assumed that $\mathfrak{C}$ is strict cubical, $\mathbf{Bimod}_{\mathfrak{C}}(A, B)$ is in fact a strict 2-category. $\square$

1.3 Diagrams

1.3.1 Proof of Lemma 1.2.2.2

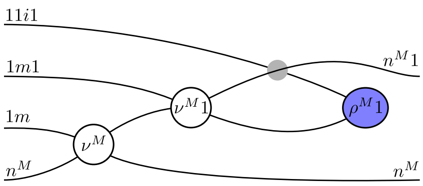


Figure 1.1: Equality (Part 1)

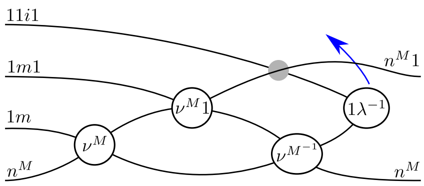


Figure 1.2: Equality (Part 2)

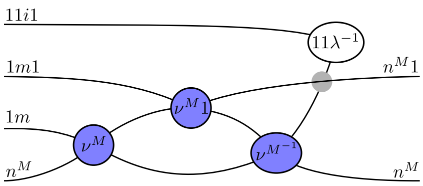


Figure 1.3: Equality (Part 3)

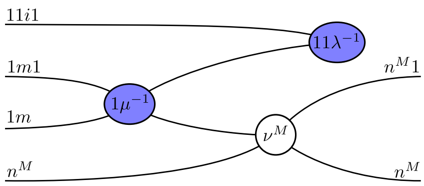


Figure 1.4: Equality (Part 4)

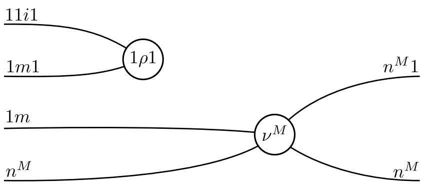


Figure 1.5: Equality (Part 5)

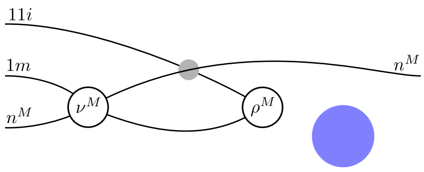


Figure 1.6: Coherence (Part 1)

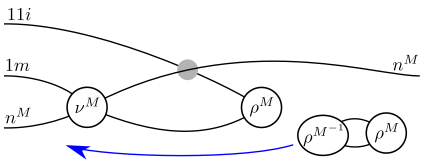


Figure 1.7: Coherence (Part 2)

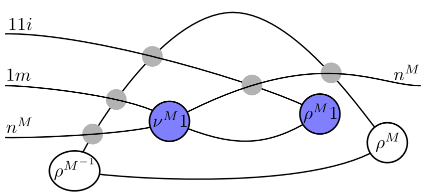


Figure 1.8: Coherence (Part 3)

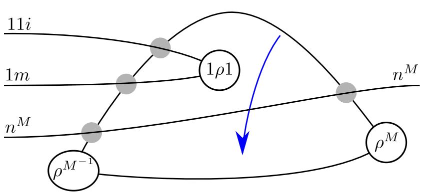


Figure 1.9: Coherence (Part 4)

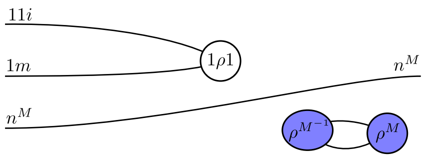


Figure 1.10: Coherence (Part 5)

Chapter 2

Compact Semisimple 2-Categories

2.1 Definition

2.1.1 Cauchy Complete 2-Categories

The notion of an n-condensation monad in an n-category was introduced in [GJF19]. These objects categorify the notion of an idempotent in a 1-category to an n-category. In particular, a 1-condensation monad is an idempotent in a 1-category. Below, we recall the definition in the case $n = 2$.

Definition 2.1.1.1. Let $\mathfrak{C}$ be a (not necessarily linear) 2-category. A 2-condensation monad in $\mathfrak{C}$ consists of:

1. An object A;
2. A 1-morphism $e : A \to A$;
3. Two 2-morphisms $\mu : e \circ e \Rightarrow e$ and $\delta : e \Rightarrow e \circ e$,

such that

a. The pair (e, μ) is a (non-unital) associative algebra;

b. The pair (e, δ) is a (non-counital) coassociative coalgebra;

c. The Frobenius relations hold, i.e. δ is an (e, e)-bimodule map;

d. We have: $\mu \cdot \delta = Id_e$.

Remark 2.1.1.2. More concisely, a 2-condensation monad on A in $\mathfrak{C}$ is a non-unital non-counital special Frobenius algebra in the monoidal category $End_{\mathfrak{C}}(A)$. If the special Frobenius algebra is unital, this recovers the notion of separable monad considered in [DR18].

Generalizing the notion of a split surjection in a 1-category, [GJF19] also introduced the notion of an n-condensation in an n-category. The case $n = 2$ is reviewed in the definition below.

Definition 2.1.1.3. Let $\mathfrak{C}$ be a 2-category. A 2-condensation in $\mathfrak{C}$ consists of:

1. Two objects A and B;
2. Two 1-morphism $f : A \to B$ and $g : B \to A$;
3. Two 2-morphisms $\phi : f \circ g \Rightarrow Id_B$ and $\gamma : Id_B \Rightarrow f \circ g$

such that

a. We have: $\phi \cdot \gamma = Id_{Id_B}$.

Given a 2-condensation as above, observe that the induced structure on the object A alone is precisely that of a 2-condensation monad.

Definition 2.1.1.4. Given a 2-condensation monad (A, e, μ, δ) in $\mathfrak{C}$, a splitting, or, more precisely, an extension of this 2-condensation monad to a 2-condensation, is a 2-condensation $(A, B, f, g, \phi, \gamma)$ together with a 2-isomorphism $\theta : g \circ f \cong e$ such that

$$\mu = \theta \cdot (g \circ \phi \circ f) \cdot (\theta^{-1} \circ \theta^{-1}),$$

$$\delta = (\theta \circ \theta) \cdot (g \circ \gamma \circ f) \cdot \theta^{-1}.$$

Remark 2.1.1.5. Extensions of a 2-condensation monad to a 2-condensation are preserved by 2-functors (upon insertion of the relevant coherence 2-isomorphisms).

Categorifying the notion of being idempotent complete [GJF19] makes the following definition.

Definition 2.1.1.6. A 2-category $\mathfrak{C}$ is Karoubi complete if it is locally idempotent complete, and every 2-condensation monad can be extended to a 2-condensation.

If $\Bbbk$ is a field, and we are considering $\Bbbk$-linear 2-categories, the following definition gives the appropriate notion of completion.

Definition 2.1.1.7. A $\Bbbk$-linear 2-category $\mathfrak{C}$ is Cauchy complete if it is locally Cauchy complete, it has direct sums for objects, and every 2-condensation monad can be extended to a 2-condensation.

Remark 2.1.1.8. Theorem 2.3.2 of [GJF19] states that if $\mathfrak{C}$ is a locally idempotent complete 2-category, then the 2-category of extensions of any 2-condensation monad to a 2-condensation is either empty or contractible. (This is the categorified version of the statement that if an idempotent in a 1-category splits, it does so uniquely up to unique isomorphism.)

Remark 2.1.1.9. In [DR18], the authors introduced a different notion of Cauchy completeness for locally Cauchy complete linear 2-categories, which is based on the splittings of separable monads. It follows from theorem 3.1.7 of [GJF19] that these two notions of Cauchy completeness coincide for locally Cauchy complete 2-categories that have right adjoints for 1-morphisms.

Remark 2.1.1.10. Given a locally Cauchy complete linear 2-category $\mathfrak{C}$, [GJF19] constructs a Cauchy completion for $\mathfrak{C}$, i.e. a 2-category $Cau(\mathfrak{C})$ that is Cauchy complete and is equipped with a canonical fully faithful inclusion $\iota_{\mathfrak{C}} : \mathfrak{C} \hookrightarrow Cau(\mathfrak{C})$. More precisely, $Cau(\mathfrak{C})$ is given by the 2-category whose objects are the 2-condensation monads in $\mathfrak{C}$, 1-morphism are bimodules, and 2-morphisms are bimodule maps. It was established in [Déc22b] that the linear 2-functor $\iota_{\mathfrak{C}} : \mathfrak{C} \hookrightarrow Cau(\mathfrak{C})$ enjoys a 3-universal property. Namely, $\iota_{\mathfrak{C}}$ is an initial object in the 3-category of linear 2-functors from $\mathfrak{C}$ to a Cauchy complete linear 2-category.

2.1.2 Semisimple 2-Categories

Let $\Bbbk$ be an arbitrary field. All of the categories and functors we will consider from now until the end of this chapter are $\Bbbk$-linear.

Definition 2.1.2.1. Recall the following standard definitions:

- An object of an abelian 1-category is called simple if it is not zero, and it has no nontrivial subobject. The endomorphism space of a simple object is a division algebra over $\Bbbk$.

- An abelian 1-category is called semisimple provided that every object splits as a finite direct sum of simple objects.

- An abelian 1-category is called finite it is is equivalent to the category of finite-dimensional modules over a finite-dimensional $\Bbbk$-algebra.

Remark 2.1.2.2. There is an intrinsic characterization of finite 1-categories (for instance, see [DSPS19]). In particular, such a 1-category has finitely many equivalence classes of simple objects, and finite-dimensional Hom-spaces.

Definition 2.1.2.3. A semisimple 2-category is a locally semisimple $\Bbbk$-linear 2-category that has right and left adjoints for 1-morphisms and that is Cauchy complete.

Remark 2.1.2.4. Observe that by theorem 3.1.7 of [GJF19], a locally semisimple 2-category with adjoints is Cauchy complete if and only if it has direct sums and every separable monad splits. Thus, our definition of a semisimple 2-category is equivalent to definition 1.4.1 of [DR18] over an arbitrary field.

Definition 2.1.2.5. An object X of a semisimple 2-category $\mathfrak{C}$ is called simple if its identity 1-morphism Id_X is a simple object of $End_{\mathfrak{C}}(X)$.

The categorical Schur lemma (proposition 1.2.19 of [DR18]) continues to hold in our more general context. In fact, we record an even more general version.

Lemma 2.1.2.6. *Let $f : X \to Y$, and $g : Y \to Z$ be two nonzero 1-morphisms of a semisimple 2-category such that Y is simple. Then, $g \circ f$ is nonzero.*

Proof. Let $g^* : Z \to Y$ be a right adjoint to g, and write $\epsilon : g^* \circ g \Rightarrow Id_Y$ for the counit. As g is non-zero and Id_Y is simple, there exists a section γ of ϵ. In particular, the composite 2-morphism $(\gamma \circ f) \cdot (\epsilon \circ f)$ is the identity on f, and factors through $g^* \circ g \circ f$. As f is non-zero, this shows that $g \circ f$ is non-zero. □

In fact, it turns out that most of the theory of semisimple 2-categories developed in [DR18] holds over an arbitrary field. Let us make explicit the following results.

Lemma 2.1.2.7. *Let C be an object of a semisimple 2-category $\mathfrak{C}$. Then, a decomposition $Id_C \cong \oplus_{i=1}^n f_i$ of Id_C into a direct sum of non-zero 1-morphisms induces a decomposition $C \simeq \boxplus_{i=1}^n C_i$ of C into a direct sum of non-zero objects C_i of $\mathfrak{C}$.*

Proof. By remark 2.1.2.4, our notion of a semisimple 2-category is equivalent to definition 1.4.1 of [DR18] over an arbitrary field. In particular, the proof of proposition 1.3.16 of [DR18] also applies in our context. □

Proposition 2.1.2.8. *In a semisimple 2-category $\mathfrak{C}$, every object splits as a direct sum of simple objects. Further, this splitting is unique up to permutation and equivalence.*

Proof. Let C be an object of $\mathfrak{C}$. Then, as $End_{\mathfrak{C}}(C)$ is a semisimple 1-category, we can pick $Id_C \cong \oplus_{i=1}^n f_i$ a decomposition of Id_C into a direct sum of simple objects of $End_{\mathfrak{C}}(C)$. Lemma 2.1.2.7 then yields the first part of the result. To get the second part, observe that, in fact, splittings $C \simeq \boxplus_{i=1}^n C_i$ of C into a direct sum of simple objects C_i up to equivalence, correspond to splittings $Id_C \cong \oplus_{i=1}^n f_i$ of Id_C into a direct sum of simple objects of $End_{\mathfrak{C}}(C)$. Thence, the second part of the result follows from the uniqueness of the splitting of an object as a direct of simple objects in a semisimple 1-category. $\square$

Finally, over algebraically closed fields of characteristic zero, the following objects play a central role in [DR18]. We shall say more about their existence over arbitrary fields in section 2.2.

Definition 2.1.2.9. A semisimple 2-category is finite if it is locally finite, i.e. its Hom-categories are finite 1-categories, and has finitely many equivalence classes of simple objects.

2.1.3 2-Categories of Separable Module 1-Categories

Recall the following definitions from [DSPS21]:

- A tensor 1-category is a linear 1-category equipped with a rigid monoidal structure.
- Over an algebraically closed field or a real closed field, a multifusion 1-category is a finite semisimple tensor 1-category. A fusion 1-category is a multifusion 1-category whose monoidal unit is a simple object.
- Let $\mathcal{C}$ be a monoidal 1-category. A right $\mathcal{C}$-module 1-category is a 1-category $\mathcal{M}$ equipped with a bilinear right action $\mathcal{M} \times \mathcal{C} \to \mathcal{M}$ that is suitably coherent (see [DSPS21] definition 2.2.6).

Now, let us fix $\mathcal{C}$ a finite semisimple tensor 1-category. Given a finite semisimple right $\mathcal{C}$-module 1-category $\mathcal{M}$, Ostrik's theorem (in the generality of theorem 2.18 of [DSPS19]) asserts that there is an algebra object A in $\mathcal{C}$, and an equivalence of right $\mathcal{C}$-module 1-categories between $\mathcal{M}$ and $\mathrm{Mod}_{\mathcal{C}}(A)$ the 1-category of left A-modules in $\mathcal{C}$.

Definition 2.1.3.1. A finite semisimple right $\mathcal{C}$-module 1-category $\mathcal{M}$ is called separable if the algebra A is separable, i.e. the multiplication map $m_A : A \otimes A \to A$ admits a section as a map of A-A-bimodules. We then say that $\mathcal{M}$ is a separable right $\mathcal{C}$-module 1-category.

Remark 2.1.3.2. There is also an intrinsic characterization of separable right $\mathcal{C}$-module 1-categories. Namely, it is shown in theorem 1.5.4 of [DSPS21] that a finite semisimple right $\mathcal{C}$-module 1-category $\mathcal{M}$ is separable if and only if the 1-category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ of linear right $\mathcal{C}$-module functors $\mathcal{M}\to\mathcal{M}$ is finite semisimple.

Lemma 2.1.3.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two separable right $\mathcal{C}$-module 1-categories. Then,* $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ *is finite semisimple.*

Proof. Let A, B be separable algebras in $\mathcal{C}$ such that $\mathrm{Mod}_{\mathcal{C}}(A)\simeq\mathcal{M}$, and $\mathrm{Mod}_{\mathcal{C}}(B)\simeq\mathcal{N}$ as right $\mathcal{C}$-module 1-categories. Theorem 2.2.17, corollary 2.4.11, and corollary 2.4.14 of [DSPS21] together prove that

$$\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{N})\simeq\mathrm{Bimod}_{\mathcal{C}}(A,B).$$

Furthermore, the right hand-side is a finite 1-category, so it only remains to prove that every object is projective. Let M be an A-B-bimodule in $\mathcal{C}$. We write m_A for the multiplication of A, and Δ_A for its splitting as a map of A-A-bimodules. Similarly, we write m_B for the multiplication of B, and Δ_B for its splitting as a map of B-B-bimodules. The bimodule map

$$A\otimes_A M\otimes_B B\xrightarrow{\Delta_A\otimes_A Id_M\otimes_B\Delta_B}(A\otimes A)\otimes_A M\otimes_B(B\otimes B)$$

is a splitting for the canonical bimodule map

$$(A\otimes A)\otimes_A M\otimes_B(B\otimes B)\to A\otimes_A M\otimes_B B.$$

Thus, we find that M is a direct summand of a free bimodule, which proves the claim. $\square$

We are now ready to given our first examples of semisimple 2-categories.

Proposition 2.1.3.4. *The 2-category $\mathbf{Mod}(\mathcal{C})$ of separable right $\mathcal{C}$-module 1-categories is a semisimple 2-category that is locally finite.*

Proof. The proof follows using the argument given in the proof of theorem 1.4.8 of [DR18]. We give some key points for the reader's convenience. Firstly, observe that given two separable right $\mathcal{C}$-module 1-categories $\mathcal{M}$, and $\mathcal{N}$, then $Hom_{\mathbf{Mod}(\mathcal{C})}(\mathcal{M},\mathcal{N})$ is finite semisimple by lemma 2.1.3.3. The existence of adjoints for 1-morphisms follows from [DSPS19]. Moreover, it is clear that $\mathbf{Mod}(\mathcal{C})$ has a zero object and direct sums for objects. Thus, it only remains to prove that every 2-condensation monad splits. Now, it follows from sections 2.2 and 2.4 of [DSPS21] that $\mathbf{Mod}(\mathcal{C})$ is equivalent to the 2-category of separable algebras, bimodules, and bimodule morphisms in $\mathcal{C}$. Finally, by theorem 3.1.7 of [GJF19], the 2-category of separable algebras in $\mathcal{C}$ is equivalent to $Cau(\mathrm{B}\mathcal{C})$, the 2-category 2-condensation monads in $\mathrm{B}\mathcal{C}$, which is Cauchy complete by construction. □

It is useful to describe the simple objects of $\mathbf{Mod}(\mathcal{C})$ more explicitly.

Lemma 2.1.3.5. *A separable right $\mathcal{C}$-module 1-category $\mathcal{M}$ is simple as an object of $\mathbf{Mod}(\mathcal{C})$ if and only if it is indecomposable.*

Proof. By definition, an object $\mathcal{M}$ of $\mathbf{Mod}(\mathcal{C})$ is simple if and only if the monoidal unit of $\mathrm{End}_{\mathbf{Mod}(\mathcal{C})}(\mathcal{M})$ is simple. But, this finite semisimple tensor 1-category is exactly $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$, whose monoidal unit is simple if and only if $\mathcal{M}$ is indecomposable as a right $\mathcal{C}$-module 1-category. □

Proposition 2.1.3.4 gives us plenty of examples of semisimple 2-categories. Let us examine one in particular.

Example 2.1.3.6. Let $\mathbb{k} = \mathbb{F}_p$ for a prime p. By the above proposition, $\mathbf{2Vect}_{\mathbb{k}} = \mathbf{Mod}(\mathbf{Vect}_{\mathbb{k}})$ is a semisimple 2-category. Let us prove that it has infinitely many equivalence classes of simple objects. Given any positive integer q, write $\mathcal{V}_q$ for the finite semisimple 1-category of finite-dimensional $\mathbb{F}_{p^q}$-vector spaces. The 1-category $\mathcal{V}_q$ admits an obvious coherent right action by $\mathbf{Vect}_{\mathbb{k}}$. Further, these 1-categories are pairwise non-equivalent. Therefore, it suffices to show that the right $\mathbf{Vect}_{\mathbb{k}}$-module 1-categories $\mathcal{V}_q$ are separable. This follows from the fact that $\mathbb{F}_{p^q}$ is a separable algebra in $\mathbf{Vect}_{\mathbb{k}}$ (see corollary 4.5.8 of [For17]). Namely, we have that $\mathrm{Mod}_{\mathbf{Vect}_{\mathbb{k}}}(\mathbb{F}_{p^q}) \simeq \mathcal{V}_q$ as right $\mathbf{Vect}_{\mathbb{k}}$-module 1-categories, which proves the claim. In fact, this describes all the finite semisimple $\mathbf{Vect}_{\mathbb{k}}$-module 1-categories.

More generally, given a finite group G, we can consider $\mathcal{C}$ the finite semisimple tensor 1-category of G-graded finite-dimensional $\mathbb{F}_p$-vector spaces. We have that $\mathbf{Mod}(\mathcal{C})$ is a semisimple 2-category. Given a subgroup $H \subseteq G$, a positive integer q, an action σ of H on $\mathbb{F}_{p^q}$ by field automorphisms, and a 2-cocycle γ for H with coefficients in $\mathbb{F}_{p^q}^{\times}$ twisted by σ, then we can consider the twisted group algebra $\mathbb{F}_{p^q}^{\gamma}[H]$. Equipped with the tautological grading, this defines an algebra in $\mathcal{C}$. We write $\mathcal{M}(H,\gamma) := \mathrm{Mod}_{\mathcal{C}}(\mathbb{F}_{p^q}^{\gamma}[H])$, and note that this is a finite semisimple right $\mathcal{C}$-module 1-category. In fact, this yields all indecomposable finite semisimple right $\mathcal{C}$-module 1-categories, but there is some redundancy in this description (see [Nat17] for a related result over an algebraically closed field of characteristic zero). Further, one can check that $\mathbb{F}_{p^q}^{\gamma}[H]$ is separable if and only if $p \nmid |H|$. This follows from a slight elaboration on the argument used in the proof of lemma 1.8 of [Déc23c].

2.1.4 First Properties of Compact Semisimple 2-Categories

Example 2.1.3.6 shows that we can not expect $\mathbf{Mod}(\mathcal{C})$ to be a finite semisimple 2-category in general. Nevertheless, such 2-categories still have reasonable finiteness properties. In order to capture them, we need the following definition.

Definition 2.1.4.1. Let $\mathfrak{C}$ be a semisimple 2-category. Two simple objects X and Y of $\mathfrak{C}$ are said to be connected if there is a non-zero 1-morphism between them.

As is explained in [DR18] over an algebraically closed field of characteristic zero, the categorical Schur lemma shows that being connected defines an equivalence relation on the set of simple objects of a given semisimple 2-category. But the categorical Schur lemma holds over an base field (see lemma 2.1.2.6), so we can make the following definitions.

Definition 2.1.4.2. The set $\pi_0(\mathfrak{C})$ is the quotient of the set of simple objects of $\mathfrak{C}$ under the equivalence relation of being connected. A semisimple 2-category $\mathfrak{C}$ is connected if $\pi_0(\mathfrak{C})$ is a singleton.

Definition 2.1.4.3. A semisimple 2-category $\mathfrak{C}$ is compact if it is locally finite and $\pi_0(\mathfrak{C})$ is finite.

We now show that $\mathbf{Mod}(\mathcal{C})$ is a compact semisimple 2-category for every finite semisimple tensor 1-category $\mathcal{C}$.

Theorem 2.1.4.4. *Let $\mathcal{C}$ be a finite semisimple tensor 1-category. The 2-category of separable right $\mathcal{C}$-module 1-categories is a compact semisimple 2-category.*

Proof. Let us begin by observing that for any nonzero separable right $\mathcal{C}$-module $\mathcal{M}$, there exists a right $\mathcal{C}$-module functor $F : \mathcal{C} \to \mathcal{M}$. Namely, pick an object M in $\mathcal{M}$, and set $F(C) = M \otimes C$ with the canonical right $\mathcal{C}$-module structure. By proposition 2.1.2.8, we know that $\mathcal{C}$ splits as a direct sum of finitely many simple objects. Thus, we have shown that any simple object of $\mathbf{Mod}(\mathcal{C})$ admits a nonzero 1-morphism from a direct summand of $\mathcal{C}$. This finishes the proof. □

In fact, the converse also holds.

Definition 2.1.4.5. We say that a semisimple 2-category $\mathfrak{C}$ has a generator if there exists X in $\mathfrak{C}$ such that the inclusion $\mathrm{B}End_{\mathfrak{C}}(X) \hookrightarrow \mathfrak{C}$ is a Cauchy completion.

Theorem 2.1.4.6. *Let $\mathfrak{C}$ be a compact semisimple 2-category. Then, $\mathfrak{C}$ has a generator. In particular, there exists a finite semisimple tensor 1-category $\mathcal{C}$ such that $\mathfrak{C} \simeq \mathbf{Mod}(\mathcal{C})$.*

Proof. Let us pick a simple object X_i in every equivalence class of $\pi_0(\mathfrak{C})$. We define $X := \boxplus_i X_i$, which is an object of $\mathfrak{C}$, and write $\mathcal{C} := End_{\mathfrak{C}}(X)$. Now, the inclusion

$$\begin{array}{ccc} \mathrm{B}\mathcal{C} & \hookrightarrow & \mathfrak{C} \\ * & \mapsto & X \end{array}$$

is full on 1-morphisms and fully faithful on 2-morphisms. As the right hand-side is Cauchy complete, we get a linear 2-functor

$$F : \mathbf{Mod}(\mathcal{C}) \simeq Cau(\mathrm{B}\mathcal{C}) \hookrightarrow \mathfrak{C},$$

which is still full on 1-morphisms and fully faithful on 2-morphisms. Thence, it only remains to prove essential surjectivity. Given that $\mathbf{Mod}(\mathcal{C})$ has direct sums, it is enough to show that every simple object in $\mathfrak{C}$ is in the essential image of F. In order to do so, we prove that for any simple object Y of $\mathfrak{C}$, there exists a 2-condensation monad on X, whose splitting is Y. Let $f : X \to Y$ be a simple 1-morphism, which exists by construction of X. We write f^* for its right adjoint, $\epsilon^f : f \circ f^* \Rightarrow Id_Y$ for the corresponding counit. As f is non-zero, and Id_Y is simple, ϵ^f admits a section, which we denote by γ. It follows from the definitions that the data $(X, Y, f, f^*, \epsilon, \gamma)$ defines a 2-condensation. In particular, there exists a 2-condensation monad on X whose splitting in $\mathfrak{C}$ is Y as desired. □

2.1.5 Locally Separable Compact semisimple 2-Categories

We now assume that $\Bbbk$ is a perfect field. Let us recall the following definition from [DSPS21].

Definition 2.1.5.1. A finite semisimple tensor 1-category is called separable if it is separable as a right $\mathcal{C}^{mop} \boxtimes \mathcal{C}$-module 1-category.

Remark 2.1.5.2. It is shown in [DSPS21] that a finite semisimple tensor 1-category is separable if and only if its Drinfeld center is finite semisimple. Furthermore, they show that over a field of characteristic zero, every finite semisimple tensor 1-category is separable.

Let $\mathcal{C}$ be a finite semisimple tensor 1-category. Following [EGNO15], a decomposition of the unit into simple summands induces an equivalence of finite semisimple tensor 1-categories

$$\mathcal{C} \simeq \begin{pmatrix} {}_1\mathcal{C}_1 & \cdots & {}_1\mathcal{C}_n \\ \vdots & \ddots & \vdots \\ {}_n\mathcal{C}_1 & \cdots & {}_n\mathcal{C}_n \end{pmatrix}$$

for some integer n, where the monoidal unit of ${}_i\mathcal{C}_i$ is simple for all i.

Lemma 2.1.5.3. *A finite semisimple tensor 1-category $\mathcal{C}$ is separable if and only if the finite semisimple tensor sub-1-categories ${}_i\mathcal{C}_i$ are separable for all i.*

Proof. Let us begin by assuming that $\mathcal{C}$ is connected, i.e. ${}_i\mathcal{C}_j$ is non-zero for every i, j. In this case, $\mathcal{C}$ is Morita equivalent to ${}_1\mathcal{C}_1$, so the statement follows from the fact that the Drinfeld center is a Morita invariant. More generally, $\mathcal{C}$ is Morita equivalent to a finite semisimple tensor 1-category of the form $\oplus_{j\in J\,j}\mathcal{C}_j$ for some subset $J \subseteq \{1, ..., n\}$. It follows from the definitions that $\oplus_{j\in J\,j}\mathcal{C}_j$ is separable if and only if each of the finite semisimple tensor 1-category ${}_j\mathcal{C}_j$ is. □

Definition 2.1.5.4. A compact semisimple 2-category $\mathfrak{C}$ is locally separable if $End_{\mathfrak{C}}(X)$ is a separable finite semisimple tensor 1-category for every simple object X of $\mathfrak{C}$.

Remark 2.1.5.5. Thanks to lemma 2.1.5.3, in order to check that a compact semisimple 2-category $\mathfrak{C}$ is locally separable, it is enough to prove that $End_{\mathfrak{C}}(X_i)$ is a separable finite semisimple tensor 1-category, where the X_i are simple objects representing the equivalence classes in $\pi_0(\mathfrak{C})$.

Theorem 2.1.5.6. *Let $\mathcal{C}$ be a separable finite semisimple tensor 1-category. The 2-category of separable right $\mathcal{C}$-module 1-categories is a locally separable compact semisimple 2-category.*

Proof. Let $\mathcal{M}$ be an indecomposable separable right $\mathcal{C}$-module 1-category. Then, $End_{\mathcal{C}}(\mathcal{M})$ is Morita equivalent (via $\mathcal{M}$) to a finite semisimple tensor sub-1-category of $\mathcal{C}$. As all such finite semisimple tensor sub-1-categories are separable the statement follows. □

Theorem 2.1.5.7. *Let $\mathfrak{C}$ be a locally separable compact semisimple 2-category. Then, there exists a separable finite semisimple tensor 1-category $\mathcal{C}$ such that $\mathfrak{C} \simeq$ $\mathbf{Mod}(\mathcal{C})$.*

Proof. The argument used to prove theorem 2.1.4.6 applies, but, thanks to our hypothesis, the finite semisimple tensor 1-category we construct is separable. □

Remark 2.1.5.8. As $\Bbbk$ is a perfect field, and $\mathcal{C}$ is separable, $\mathbf{Mod}(\mathcal{C})$ may also be described as the 2-category of finite semisimple right $\mathcal{C}$-module 1-categories as can be seen using proposition 2.5.10 of [DSPS21].

Example 2.1.5.9. Let G be a finite group, and $\Bbbk$ be an algebraically closed field. Define $\mathcal{C}$ to be the finite semisimple tensor 1-category of finite dimensional G-graded $\Bbbk$-vector spaces. We now describe the structure of the compact semisimple 2-category $\mathbf{Mod}(\mathcal{C})$. Proposition 4.1 of [EO04] (see also [Nat17]) shows that the equivalence classes of indecomposable finite semisimple right $\mathcal{C}$-module 1-categories are given by pairs consisting of a subgroup $H \subseteq G$ and a cohomology class $\phi \in H^2(H; \Bbbk^\times)$, under a suitable equivalence relation induced by conjugation. Further, it follows from lemma 1.8 of [Déc23b] that the indecomposable finite semisimple right $\mathcal{C}$-module 1-category associated to (H, ϕ) is separable if and only if $char(\Bbbk) \nmid |H|$. Thus, there are only finitely many equivalences classes of simple objects in $\mathbf{Mod}(\mathcal{C})$, i.e. it is a finite semisimple 2-category. Finally, $\mathcal{C}$ is separable if and only if $char(\Bbbk) \nmid |G|$, then $\mathcal{C}$ is separable, so that $\mathbf{Mod}(\mathcal{C})$ is locally separable if and only if $char(\Bbbk) \nmid |G|$.

2.2 Finite Semisimple 2-Categories

The objective of this section is to give examples of finite semisimple 2-categories. Our first task is to identify when the base field allows for such 2-categories to exist. Having done so, we prove that over an appropriate base field, every compact semisimple 2-category is finite.

2.2.1 Non-examples

Given $\mathcal{C}$ a finite semisimple tensor 1-category. Example 2.1.3.6 suggests that there is, in general, no hope for the 2-category of separable right $\mathcal{C}$-module 1-categories to be finite semisimple if our base field has infinitely many non-equivalent finite separable extensions. We begin by making this precise.

Proposition 2.2.1.1. *Let $\Bbbk$ be a field whose separable closure $\Bbbk^{sep}$ is an infinite extension. Then, given any non-zero finite semisimple tensor 1-category $\mathcal{C}$ over $\Bbbk$, the compact semisimple 2-category $\mathbf{Mod}(\mathcal{C})$ has infinitely many equivalence classes of simple objects.*

Proof. Let $\mathcal{M}_0$ be any separable right $\mathcal{C}$-module 1-category. For instance, take $\mathcal{M}_0 = \mathcal{C}$. We let

$$N(\mathcal{M}_0) := \min\{\dim_{\Bbbk}(End_{\mathcal{M}}(X))|X \in \mathcal{M},\ X \text{ simple}\} \in \mathbb{N}.$$

We claim that there exists indecomposable separable right $\mathcal{C}$-module 1-categories $\mathcal{M}_i$ for every non-negative integer i, such that $N(\mathcal{M}_i)$ tends to infinity as i tends to infinity. This readily implies the statement of the proposition.

To this end, let $\Bbbk_i$ be an increasing sequence of separable finite field extensions of $\Bbbk$, which must exist by assumption. In particular, $\Bbbk_i$ is a separable $\Bbbk$-algebra for all i by corollary 4.5.8 of [For17]. We claim that the Deligne tensor product

$\widehat{\mathcal{M}_i} := \mathbf{Vect}_{\Bbbk_i} \boxtimes \mathcal{M}$ over $\Bbbk$ is a separable right $\mathcal{C}$-module 1-category. Namely, by hypothesis, there exists a separable algebra A in $\mathcal{C}$ such that $\mathcal{M} \simeq \mathrm{Mod}_{\mathcal{C}}(A)$ as right $\mathcal{C}$-module 1-categories. We then find that

$$\mathbf{Vect}_{\Bbbk_i} \boxtimes \mathcal{M} \simeq \mathrm{Mod}_{\mathcal{C}}(\Bbbk_i \otimes A),$$

as right $\mathcal{C}$-module 1-categories, and as $\Bbbk_i \otimes A$ is a separable algebra in $\mathcal{C}$, this proves the claim.

Note that $\widehat{\mathcal{M}_i}$ might not be indecomposable, so we pick an indecomposable summand $\mathcal{M}_i$. But, by construction, $\widehat{\mathcal{M}_i}$, and therefore also $\mathcal{M}_i$, is a $\Bbbk_i$-linear 1-category. Thus, we find that

$$\dim_{\Bbbk}(\Bbbk_i) \leq N(\mathcal{M}_i).$$

As the sequence $\dim_{\Bbbk}(\Bbbk_i)$ tends to infinity as i tends to infinity, this finishes the proof of the result. $\square$

It turns out that we can describe those fields whose separable closure is a finite extension quite explicitly using a slight elaboration on the Artin-Schreier theorem. This is carried out in theorem 4.1 of [Con], which we now recall.

Theorem 2.2.1.2. *Let $\Bbbk$ be a field whose separable closure is a finite extension. Then, either $\Bbbk$ is separably closed, or $\Bbbk$ is a real closed field, i.e. char$(\Bbbk) = 0$, $\Bbbk$ is not algebraically closed, and $\Bbbk[\sqrt{-1}]$ is algebraically closed.*

Remark 2.2.1.3. Examples of real closed fields include $\mathbb{R}$, but also $\mathbb{R}^{alg} = \overline{\mathbb{Q}} \cap \mathbb{R}$ the subfield of algebraic numbers. There are more exotic examples such as the fields of hyperreal numbers.

Finally, it is well-known that there is no nontrivial finite dimensional division algebra over an algebraically closed field. We need a weaker version of this statement for finite dimensional division algebras over real closed fields.

Lemma 2.2.1.4. *Let $\Bbbk$ be a real closed field. There are finitely many equivalences classes of finite dimensional division algebras over $\Bbbk$.*

Proof. The proof of the Frobenius theorem on division algebras given in [Pal68] applies verbatim to any real closed field, showing that there are three equivalence classes of finite division rings occurring in this case: $\Bbbk$, $\Bbbk[\sqrt{-1}]$, and the ring of quaternions. This completes the proof. $\square$

2.2.2 Examples

Let us fix a field $\Bbbk$, which satisfies the condition of theorem 2.2.1.2. Further, let us assume that $\Bbbk$ is perfect, as we are mainly interested in developing our theory over such fields. This means that $\Bbbk$ is either algebraically closed or real closed, as a separably closed field is algebraically closed if and only if it is perfect.

Theorem 2.2.2.1. *Let $\mathcal{C}$ be a multifusion 1-category over an algebraically closed field or a real closed field. The 2-category of separable right $\mathcal{C}$-module 1-categories is a finite semisimple 2-category.*

Proof. By theorem 2.1.4.4, it is enough to prove that there are finitely many equivalence classes of indecomposable separable right $\mathcal{C}$-module 1-categories. We mimick the argument of corollary 9.1.6 of [EGNO15].

By proposition 3.4.6 of [EGNO15], there are only finitely many indecomposable $\mathbb{Z}_+$-modules over $Gr(\mathcal{C})$, the Grothendieck ring of $\mathcal{C}$. Moreover, as there are only finitely many equivalence classes of finite division algebras over $\Bbbk$, there are only finitely many non-equivalent ways to categorify every such $\mathbb{Z}_+$-module. Therefore, it is enough to prove that for any finite semisimple 1-category $\mathcal{M}$, there are, up to equivalence, only finitely many separable right $\mathcal{C}$-module structures on $\mathcal{M}$. In order to do so, we will reduce this problem to a question about tensor functors, and appeal to theorem 2.2.3.3 below.

Supplying a right $\mathcal{C}$-module structure on $\mathcal{M}$ is equivalent to giving the data of a tensor functor $F : \mathcal{C}^{mop} \to Fun(\mathcal{M}, \mathcal{M})$. Observe that the right hand-side is a multifusion 1-category as $\Bbbk$ is perfect. If the chosen right $\mathcal{C}$-module structure on $\mathcal{M}$ is separable, we claim that the tensor functor F above exhibits $Fun(\mathcal{M}, \mathcal{M})$ as a separable $\mathcal{C}^{mop} \boxtimes \mathcal{C}$-module 1-category. Namely, proposition 2.4.10 of [DSPS21] gives an equivalence

$$Fun(\mathcal{M}, \mathcal{M}) \simeq Fun(\mathcal{M}, \mathbf{Vect}_{\Bbbk}) \boxtimes \mathcal{M},$$

as $\mathcal{C}$-$\mathcal{C}$-bimodule 1-categories, which we can view as an equivalence of right $\mathcal{C}^{mop} \boxtimes \mathcal{C}$-module 1-categories. Now, because the right $\mathcal{C}$-module 1-category $\mathcal{M}$ is separable, there exists a separable algebra A in $\mathcal{C}$ such that $\mathcal{M} \simeq \mathrm{Mod}_{\mathcal{C}}(A)$ as right $\mathcal{C}$-module 1-categories. Combining proposition 2.4.9 and corollary 2.4.13 of [DSPS21], there is an equivalence of left $\mathcal{C}$-module 1-categories between $Fun(\mathcal{M}, \mathbf{Vect}_{\Bbbk})$ and the 1-category of right A-modules in $\mathcal{C}$. Equivalently, there is an equivalence of right $\mathcal{C}^{mop}$-module 1-categories $Fun(\mathcal{M}, \mathbf{Vect}_{\Bbbk}) \simeq \mathrm{Mod}_{\mathcal{C}^{mop}}(A^{op})$, where A^{op} is A viewed as a separable algebra in $\mathcal{C}^{mop}$. Finally, proposition 3.8 of [DSPS19] shows that

$$Fun(\mathcal{M}, \mathbf{Vect}_{\Bbbk}) \boxtimes \mathcal{M} \simeq Mod_{\mathcal{C}^{mop}\boxtimes\mathcal{C}}(A^{op} \boxtimes A),$$

and it is straightforward to check that $A^{op} \boxtimes A$ is a separable algebra in $\mathcal{C}^{mop} \boxtimes \mathcal{C}$.

Consequently, it is enough to prove that there are only finitely many equivalence classes of tensor functors $F : \mathcal{C}^{mop} \to Fun(\mathcal{M}, \mathcal{M})$, which exhibits the right hand-side as a separable right $\mathcal{C}^{mop} \boxtimes \mathcal{C}$-module 1-category. This is precisely the content of theorem 2.2.3.3. This proves that there are finitely many equivalences classes of indecomposable separable right $\mathcal{C}$-module 1-categories. □

Corollary 2.2.2.2. *Over an algebraically closed field or a real closed field, a semisimple 2-category is finite if and only if it is compact.*

Over algebraically closed fields of positive characteristic, example 2.1.5.9 demonstrates that local separability is an independent property. When $char(\Bbbk) = 0$, more can be said.

Corollary 2.2.2.3. *Over an algebraically closed field of characteristic zero or a real closed field, every compact semisimple 2-category is finite and locally separable.*

Proof. Over a field of characteristic zero, every multifusion 1-category is separable by corollary 2.6.8 of [DSPS21]. □

Example 2.2.2.4. We work over $\mathbb{R}$. Lemma 2.2.1.4 implies that the simple object of $\mathbf{Mod}(\mathbf{Vect}_{\mathbb{R}})$ are given by $\mathbf{Vect}_{\mathbb{R}}$, $\mathbf{Vect}_{\mathbb{C}}$, and $\mathbf{Vect}_{\mathbb{H}}$. But, note that $\mathbf{Mod}(\mathbf{Vect}_{\mathbb{C}})$ has only one simple object, $\mathbf{Vect}_{\mathbb{C}}$ itself.

Let us conclude by examining an example of a compact semisimple 2-category over a separably closed but not algebraically closed field.

Example 2.2.2.5. Let $\Bbbk = \mathbb{F}_p(x)^{sep}$, the separable closure of the function field of $\mathbb{F}_p$. We claim that $\mathbf{Mod}(\mathbf{Vect}_{\Bbbk})$ has only one equivalence class of simple object, $\mathbf{Vect}_{\Bbbk}$ itself. Namely, as $\Bbbk$ is separably closed, so is every finite extension. Further, the Brauer group of a separably closed field is trivial. Thus, the equivalence classes of indecomposable finite semisimple right $\mathbf{Vect}_{\Bbbk}$-module 1-categories are given by the 1-categories $\mathbf{Vect}_{\Bbbk'}$ for all finite extensions $\Bbbk'$ of $\Bbbk$. But, if $\Bbbk'$ is a non-trivial extension, the right $\mathbf{Vect}_{\Bbbk}$-module 1-category $\mathbf{Vect}_{\Bbbk'}$ is clearly not separable.

This leads us to ask the following question.

Question 2.2.2.6. Over a separably closed field, is every compact semisimple 2-category finite?

2.2.3 Absence of Deformations for Tensor Functors

Throughout, we work over $\Bbbk$ a perfect field. Let $\mathcal{C}$, $\mathcal{D}$ be finite semisimple tensor 1-categories, and F a tensor functor $\mathcal{C} \to \mathcal{D}$. The tensor functor F endows $\mathcal{D}$ with the structure of a $\mathcal{C}$-$\mathcal{C}$-bimodule 1-category, or equivalently of a right $\mathcal{C}^{mop} \boxtimes \mathcal{C}$-module 1-category.

Following [Dav97], [Yet98], and [Yet03], we define $C^\bullet_{DY}(F)$, the Davydov-Yetter cochain complex of F by setting for every integer $n \geq 0$:

$$C^n_{DY}(F) := \mathrm{Nat}(\otimes^{n-1} \circ F^{\times n}, \otimes^{n-1} \circ F^{\times n});$$

$$\begin{aligned} \delta(f)_{X_0,\ldots,X_n} :=& Id_{F(X_0)} \otimes f_{X_1,\ldots,X_n} \\ &+ \sum_{i=1}^{n} (-1)^i f_{X_0,\ldots,X_{i-1}\otimes X_i,\ldots,X_n} \\ &+ (-1)^{n+1} f_{X_0,\ldots,X_{n-1}} \otimes Id_{F(X_n)}. \end{aligned}$$

The cohomology groups of the cochain complex $C^\bullet_{DY}(F)$ are denoted by $H^\bullet_{DY}(F)$, and called the Davydov-Yetter cohomology groups of F. These cohomology groups describe the deformations of the tensor functor F.

Proposition 2.2.3.1. *Provided that $\mathcal{D}$ is separable when viewed as a right $\mathcal{C}^{mop} \boxtimes \mathcal{C}$-module 1-category, the group $H^n_{DY}(F)$ is trivial for every integer $n \geq 1$.*

Proof. We claim that the argument of corollary 3.18 of [GHS23] applies verbatim in our more general context. Namely, the only thing we have to check is that Z_F-Mod is finite semisimple (see subsection 3.3 of [GHS23] for the definition). In order to do so, observe that $Z_F\text{-Mod} \simeq \mathcal{Z}_{\mathcal{C}}(\mathcal{D})$ the relative Drinfel'd center by proposition 3.10 of [GHS23]. Now, we have assumed that $\mathcal{D}$ is separable as a $\mathcal{C}^{mop} \boxtimes \mathcal{C}$-module 1-category, thus $\mathcal{Z}_{\mathcal{C}}(\mathcal{D}) \simeq Fun_{\mathcal{C}^{mop}\boxtimes\mathcal{C}}(\mathcal{C}, \mathcal{D})$ is a finite semisimple 1-category, which concludes the proof. □

Remark 2.2.3.2. The condition of proposition 2.2.3.1 is always satisfied if $\mathcal{C}$ is separable. Namely, under this hypothesis, $\mathcal{C}^{mop} \boxtimes \mathcal{C}$ is separable by corollary 2.5.11 of [DSPS21]. Thus, as $\mathcal{D}$ is finite semisimple, it it separable as a right $\mathcal{C}^{mop} \boxtimes \mathcal{C}$-module 1-category.

Our goal is now to generalize theorem 2.31 of [ENO05] (see also theorem 9.1.5 of [EGNO15]), which holds over an algebraically closed field of characteristic zero. We wish to point out that it is asserted in section 9 of [ENO05] that their theorem 2.31

holds over an arbitrary algebraically closed field, so long as $\mathcal{C}$ is separable. When the base field is algebraically closed, our proof is inspired by that given in section 9 of [EGNO15] in which they work over an algebraically closed field of characteristic zero, though we give a detailed explanation as to why certain orbits of an action of an algebraic group are open. When the base field is real closed, the setup of classical algebraic geometry is no longer adequate, and we need to use ideas coming from real algebraic geometry (see [BCR98]).

Theorem 2.2.3.3. *Let $\Bbbk$ be an algebraically closed, or a real closed field. A tensor functor $F : \mathcal{C} \to \mathcal{D}$ satisfying the hypothesis of proposition 2.2.3.1 does not admit any non-trivial deformation. In particular, the number of such functors up to equivalence (for fixed source and target) is finite.*

Proof. **Case 1: $\Bbbk$ is algebraically closed**

Defining the algebraic group action: We may assume that $\mathcal{C}$ and $\mathcal{D}$ are skeletal. We write C_i, $i = 0, ..., n$ for the set of simple objects of $\mathcal{C}$. We think of the assignment $C_i \mapsto D_i := F(C_i)$ as fixed. As $\Bbbk$ is algebraically closed, this is exactly the data of a linear functor $F' : \mathcal{C} \to \mathcal{D}$. A choice of tensor structure for the functor $F' : \mathcal{C} \to \mathcal{D}$ corresponds to the choice of isomorphism $\phi_{C_i,C_j} : F'(C_i) \otimes F'(C_j) \to F'(C_i \otimes C_j)$ in finite dimensional $\Bbbk$-vector space for every i, j satisfying finitely many polynomial conditions. Thus, there is a (potentially non-irreducible) variety X over $\Bbbk$ whose closed points are exactly the possible choices of tensor structure on F'.

Similarly, we define G to be the algebraic group over $\Bbbk$ of natural isomorphisms $\lambda : F' \cong F'$. As the underlying scheme of G is again a (potentially non-irreducible) variety, it is reduced, which implies that G is smooth (see proposition 1.18 of [Mil17]). The algebraic group G acts on the variety X by sending $(\lambda, \{\phi_{C_i,C_j}\}_{i,j})$ in $G \times X$ to $\{\lambda_{C_i \otimes C_j} \phi_{C_i,C_j}(\lambda_{C_i}^{-1} \otimes \lambda_{C_j}^{-1})\}_{i,j}$ in X.

Openness of orbits: Fix x a closed point of X, and write F_x for the corresponding tensor functor. We wish to consider the action map $l_x : G \to X$ given by sending g to $g \cdot x$. The differential of l_x at $e \in G$ is a map $(dl_x)_e : T_eG \to T_xX$. It follows from the definition of the Davydov-Yetter chain complex that $T_xX = Z^2_{DY}(F_x)$, the set of 2-cochains, and that $(dl_x)_e(T_eG) = B^2_{DY}(F_x)$, the set of 2-coboundaries. Now, if F_x exhibits $\mathcal{D}$ as a separable right $\mathcal{C}^{mop} \boxtimes \mathcal{C}$-module 1-category, $H^2_{DY}(F_x) = Z^2_{DY}(F_x)/B^2_{DY}(F_x)$ is trivial by proposition 2.2.3.1, so $(dl_x)_e$ is a surjection.

We now wish to prove that every x in X such that $(dl_x)_e$ is a surjection is a smooth point. Let x in X be such a point. We claim that there exists a subvariety Y in X containing x, which is stable under the action of G, and for which x is a smooth point. Namely, if x is not a smooth point of X, it belongs to the singular locus Y_1 of X, which is stable under the action of G. If x is a smooth point of Y_1, we are done. If not, x belongs to the singular locus of Y_1, etc. This process eventually stops because the singular locus of an irreducible variety is a proper closed subset of the original variety by theorem I.5.3 of [Har06]. Now, let Y be the desired variety. The inclusion $Y \hookrightarrow X$ induces a commutative diagram

$$\begin{tikzcd} T_eG \arrow[rr, twoheadrightarrow, "(dl_y)_e"] \arrow[dr] & & T_xX. \\ & T_xY \arrow[ur, hook] & \end{tikzcd}$$

In particular, the map $T_xY \to T_xX$ is an isomorphism. This implies that the inclusion induces an isomorphism $\mathcal{O}_{Y,x} \cong \mathcal{O}_{X,x}$ of local rings (the first ring is regular), and so X and Y agree locally around x, which proves that x is a smooth point of X.

We can now put our discussion together to prove the first part of the result. Namely, fix x a closed point of X such that $(dl_x)_e$ is surjective. As G is an algebraic group, this implies that for every closed point g of G, the map $(dl_x)_g : T_gG \to T_{g\cdot x}X$ is surjective. Thanks to the discussion above, this implies that $l_x : G \to X$ lands in a smooth subvariety of X. But, by proposition III.10.4 of [Har06], this implies that l_x is smooth, and by corollary 14.34 of [GW10], we get that l_x is open. Thus, we find that the set-theoretic image $l_x(G)$, i.e the orbit $G \cdot x$, is open in X. In particular, this applies to those closed points x for which the tensor functor $F_x : \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{D}$ as a separable right $\mathcal{C}^{mop} \boxtimes \mathcal{C}$-module 1-category, in which case this shows that F_x admits no non-trivial deformations. In particular, this applies to the tensor functor F itself.

Conclusion: Let us write X' for the smooth subscheme of X given by the union of the orbit under G of the closed points x of X for which $(dl_x)_e$ is surjective. By definition, we can write

$$X' = \bigcup_{x \in X'} G \cdot x.$$

But, as X' is an open subset of a variety, it is quasicompact, so X' is the union of finitely many orbit. The second part of the theorem follows from this last fact by combining it with lemma 2.4.1.7. Namely, this lemma asserts that there are finitely many homomorphisms of $\mathbb{Z}_+$-rings from $Gr(\mathcal{C})$ to $Gr(\mathcal{D})$, i.e. there are only finitely many assignments $C_i \mapsto D_i$, as we have fixed at the start of the proof, that can be made into tensor functors.

Case 2: $\Bbbk$ is a real closed field

Deformations of linear functors: We may assume that $\mathcal{C}$ and $\mathcal{D}$ are skeletal. We write C_i, $i = 0, ..., n$ for the set of simple objects of $\mathcal{C}$. We fix an assignment $C_i \mapsto D_i$ an objects of $\mathcal{D}$. Now, $\Bbbk$ is not algebraically closed so this does not determine a linear functor $\mathcal{C} \to \mathcal{D}$ uniquely. But, we claim that there are only finitely many equivalence classes of such functors. Namely, upgrading the above assignment to a functor amounts to choosing a homomorphism of $\Bbbk$-algebras $f_i : End(C_i) \to End(F(C_i))$ for all i. Note that it is enough to prove that up to conjugation by an invertible element of $End(F(C_i))$, there are only finitely many such homomorphisms. Namely, if f_i and f_i' are two choices of homomorphisms such that for every i, we have $f_i = a_i f_i' a_i^{-1}$ for some invertible a_i in $End(F(C_i))$, then the a_i define a natural isomorphism between the functor defined using the homomorphisms f_i and that defined using the homomorphisms f_i'.

So it is enough to prove that for every finite dimensional division algebra A over $\Bbbk$, and finite semisimple $\Bbbk$-algebra B, there are finitely many $\Bbbk$-algebra homomorphisms $A \to B$ up to conjugation by invertible elements of B. Without loss of generality, we may assume that B is simple, i.e. B is the algebra of $m \times m$-matrices on $\Bbbk$, $\Bbbk[\sqrt{-1}]$, or the quaternions. Now, if B is the algebra of $m \times m$-matrices on $\Bbbk$, or the quaternions, the claim follows from the Skolem-Noether theorem (see theorem 2.7.2 of [GS06]). Thus, it only remains to check the claim when B is the algebra of $m \times m$-matrices on $\Bbbk[\sqrt{-1}]$. If $A = \Bbbk$, the statement is trivial. If $A = \Bbbk[\sqrt{-1}]$, then a $\Bbbk$-algebra homomorphism $f : A \to B$ is uniquely determined by the image of $\sqrt{-1}$. But $f(\sqrt{-1})^2 = Id$, so $f(\sqrt{-1})$ is conjugate to a diagonal matrix with entries $\pm\sqrt{-1}$. There are clearly finitely many of those up to conjugation. Finally, if A is the quaternions, let $f' : A \otimes \Bbbk[\sqrt{-1}] \to B$ be the extension of f to a homomorphism of $\Bbbk[\sqrt{-1}]$-algebras. It is enough to prove that there are finitely many $\Bbbk[\sqrt{-1}]$-algebra homomorphisms $A \otimes \Bbbk[\sqrt{-1}] \to B$ up to conjugation by invertible elements of B. But $A \otimes \Bbbk[\sqrt{-1}] \cong Mat_2(\Bbbk[\sqrt{-1}])$ is simple, so the claim follows again from the Skolem-Noether theorem.

Defining the algebraic group action: Let us now fix a $\Bbbk$-linear functor $F' : \mathcal{C} \to \mathcal{D}$. A choice of tensor structure on F' amounts to the choice of an isomorphism ϕ_{C_i,C_j} : $F'(X_i) \otimes F'(X_j) \to F'(X_i \otimes X_j)$ in a finite dimensional $\Bbbk$-vector space for every i, j. Furthermore, these isomorphisms have to satisfy certain polynomial equations guaranteeing that ϕ is a natural transformation and suitably coherent. Thus, there is a (potentially non-irreducible) Zariski-open subset X of an algebraic set in $\Bbbk^{\times N}$ for some positive integer N, whose points are exactly the possible tensor structures on F. Similarly, we define G to be the Zariski-open subset $\Bbbk^{\times M}$ with M a positive integer, whose points are the natural isomorphisms $\lambda : F' \cong F'$. Observe that the Zariski-open subset G carries a natural continuous action given by composition. Moreover, G acts on X by sending $(\lambda, \{\phi_{C_i,C_j}\}_{i,j})$ in $G \times X$ to $\{\lambda_{C_i \otimes C_j}\phi_{C_i,C_j}(\lambda_{C_i}^{-1} \otimes \lambda_{C_j}^{-1})\}_{i,j}$ in X. This action is continuous in the Zariski topology.

Openness of orbits: We claim that X is smooth. To see this, fix a closed point $x \in X$, and write F_x for the corresponding tensor functor. We consider the action map $l_x : G \to X$ given by sending g to $g \cdot x$. Its differential at $e \in G$ is a map $(dl_x)_e : T_eG \to T_xX$ between the Zariski tangent spaces. Now, it follows from the definition of the Davydov-Yetter chain complex that $T_xX = Z^2_{DY}(F_x)$ the set of 2-cochains, and that $(dl_x)_e(T_eG) = B^2_{DY}(F_x)$ the set of 2-coboundaries. By corollary 2.6.9 of [DSPS21], the separability assumption of proposition 2.2.3.1 is satisfied, so that $H^2_{DY}(F_x) = Z^2_{DY}(F_x)/B^2_{DY}(F_x)$ is trivial, and $(dl_x)_e$ is surjective.

Using an inductive argument similar to the one we have used in the previous case together with proposition 3.3.14 of [BCR98], we can construct an algebraic subset Y of X containing x as a smooth point, and stable under the action of G. This implies that l_x factors through Y, so that $(dl_x)_e$ factors through T_xY. But $(dl_x)_e$ is surjective, which implies that the canonical inclusion $T_xY \hookrightarrow T_xX$ is an isomorphism. This shows that Y is a Zariski-open subset of X containing x, and proves that X is smooth at x.

In order to finish our argument, we need to consider a finer topology than the Zariski topology. Namely, as $\Bbbk$ is a real closed field, it is in particular well-ordered by theorem 1.2.2 of [BCR98]. This allows us to consider the Euclidean topology on the spaces $\Bbbk^{\times N}$ and $\Bbbk^{\times M}$, which is strictly finer than the Zarisky topology. For the remainder of the proof, we will exclusively work with the Euclidean topology. In particular, we think of the algebraic sets X, and G as being endowed with the Euclidean topology, i.e. we think of X, and G as semi-algebraic sets in the terminology of [BCR98].

Let us fix a point x in X, we will now prove that the orbit of x in X under the action of G is open in the Euclidean topology. As x is a smooth point of an algebraic set, proposition 3.3.11 of [BCR98] shows that there exists an open subset U of x in X, an open subset V of $\Bbbk^{\times P}$, and a Nash diffeomorphism $\phi : U \cong V$. (A Nash function between semi-algebraic sets is a smooth semi-algebraic map.) Let us now consider the Nash function $\phi \circ l_x : l_x^{-1}(U) \to V$, which maps g in $l_x^{-1}(U) \subseteq G$ to $\phi(g \cdot x)$. But $(dl_x)_e$ is surjective, thence it follows from proposition 2.9.7 of [BCR98] that the image of $\phi \circ l_x$ contains an open neighborhood of $\phi(x)$. Given that ϕ is a Nash diffeomorphism, this implies that the image of $l_x : G \to X$ contains an open neighborhood of x. As

x was arbitrary we find that $G \cdot x$ is an open subset of X, i.e. the associated tensor functor F_x admits no non-trivial deformations.

Conclusion: Proposition 2.2.7 of [BCR98] shows that for any x in X, $G \cdot x$ is semi-agebraic. As the orbits in X under the action of G are disjoint, we therefore find that $G \cdot x$ is an open and closed semi-algebraic subset of X, i.e. a union of semi-algebraically connected components of X. But,

$$X = \bigcup_{x \in X} G \cdot x,$$

and X is a finite union of semi-algebraically connected components by theorem 2.4.4 of [BCR98]. Thus, X is the union of finitely many orbits under the action of G. The second part of the statement now follows from lemma 2.4.1.7. □

Example 2.2.3.4. Let $\Bbbk = \mathbb{Q}$, and write $\mathcal{C}$ for the separable finite semisimple tensor 1-category of finite dimensional $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces. Then, there are infinitely many non-equivalent tensor functors $\mathcal{C} \to \mathbf{Vect}_{\Bbbk}$. Namely, such functors up to equivalence correspond exactly to classes in $H^2(\mathbb{Z}/2; \mathbb{Q}^\times) \cong \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$, and this group is infinite. In the notation of the proof of theorem 2.2.3.3, $X = \mathbb{Q}^\times$, $G = \mathbb{Q}^\times$, and G acts on X by $(g, x) \mapsto g^2 x$. If we view X and G as sets, the orbits of this action are manifestly not open. However, if we view them as schemes, it is possible to show that the orbits are indeed open. The reason for this discrepancy is that schemes over $\mathbb{Q}$ have many closed points which are not $\mathbb{Q}$-points.

2.2.4 Absence of Deformations for Separable Multifusion 1-Categories

Let us assume that $\Bbbk$ is an algebraically closed field or a real closed field. We begin by briefly explaining how the methods of the previous section can be used to generalize theorem 2.30 of [ENO05], which holds over an algebraically closed field of characteristic zero. Further, over an algebraically closed field of positive characteristic, this statement appears in section 9 of [ENO05], and a proof can be obtained using the techniques developed in their article. We sketch a proof in the algebraically closed case both for completeness, and in order to introduce some notations that will be used in what follows. We emphasize that the real closed case has not been previously considered in the literature.

Theorem 2.2.4.1. *Let $\mathcal{C}$ be a separable multifusion 1-category over an algebraically closed or real closed field. Then, $\mathcal{C}$ has no non-trivial deformations. Consequently,* *the number of equivalence classes of separable multifusion 1-categories with a fixed Grothendieck ring is finite.*

Proof. The proof is entirely analogous to that of theorem 2.2.3.3, so that we only indicate the key steps.

Case 1: $\Bbbk$ is algebraically closed

We may assume that $\mathcal{C}$ is skeletal. We write C_i, $i = 0, ..., n$ for its simple objects, and assume that $C_0 = I$ is the monoidal unit. The tensor structure of $\mathcal{C}$ establishes for us an assignment $(C_i, C_j) \mapsto \oplus_k n_{ij}^k C_k$ for some non-negative integers n_{ij}^k, which we think as fixed. As $\mathcal{C}$ is rigid and skeletal, for any i, the left dual C_i^* is equal to C_{i^*}, for some integer i^* in $\{0, ..., n\}$. We fix left and right coevalutation morphisms $coev_i : I \to C_i \otimes C_{i^*}$ and $coev_i' : I \to C_{i^*} \otimes C_i$ for every i in $\mathcal{C}$. Let us now define Z to be the (potentially non-irreducible) variety over $\Bbbk$ whose closed points are precisely the data necessary to turn the above assignment into a tensor 1-category, i.e. the coherent choices of associators, and evaluation morphisms compatible with the associators and the fixed coevaluation morphisms. Similarly, we define a smooth algebraic group H whose closed points are the automorphisms of the identity functor on $\mathcal{C}$ (viewed as a linear 1-category). The group structure is given by composition, and H acts on Z.

Given a point z in Z, write $\mathcal{C}_z$ for the corresponding multifusion 1-category. We consider the action map $l_z : H \to Z$ given by $h \mapsto h \cdot z$. The differential of l_z at e is a map $(dl_z)_e : T_eG \to T_zX$. If $\mathcal{C}_z$ is separable, the Davydov-Yetter cohomology group $H^3_{DY}(\mathcal{C}_z)$ vanishes. This can be seen by following the original argument given in [ENO05] carefully. More straightforwardly, it is also also possible use the fact that the proof of theorem 9.1.3 of [EGNO15] applies to any separable fusion 1-category over an algebraically closed field. (One may alternatively appeal to proposition 2.2.3.1 above.) As a consequence, we find that $(dl_z)_e$ is surjective. As in the proof of theorem 2.2.3.3, any point z for which $(dl_z)_e$ is surjective is smooth. Moreover, the orbit $H \cdot z$ of any such point z is open, so that $\mathcal{C}_z$ admits no nontrivial deformations. This applies in particular to $\mathcal{C}$.

Finally, write Z' for the smooth subvariety of Z given by the union of the orbits under H of the closed points z in Z such that $(dl_z)_e$ is surjective. Then, we have

$$Z' = \bigcup_{z \in Z'} H \cdot z.$$

But, Z' being an open subset of a variety, it is quasi-compact, which shows that Z' is the union of finitely many orbits. This proves the second part of the statement.

Case 2: $\Bbbk$ is a real closed field

Without loss of generality, we may assume that $\mathcal{C}$ is skeletal. Analogously to the algebraically closed case, we define a (potentially non-irreducible) Zariski open subset Z of an algebraic set in $\Bbbk^{\times N}$ for some positive integer N, whose points parameterise the tensor structure on $\mathcal{C}$. We also define H to be the Zariksi-open subset of $\Bbbk^{\times M}$ with M a positive integer, whose points are the automorphisms of the identity functor on the linear 1-category $\mathcal{C}$. Observe that H acts on Z in a canonical way.

Given a point z in Z, write $\mathcal{C}_z$ for the corresponding multifusion 1-category. We consider the action map $l_z : H \to Z$ given by $h \mapsto h \cdot z$. The differential of l_z at e, the identity of H, is a map $(dl_z)_e : T_eG \to T_zX$. Thanks to proposition 2.2.3.1 and the fact that every multifusion 1-category over $\Bbbk$ is separable, the Davydov-Yetter cohomology group $H^3_{DY}(\mathcal{C}_z)$ vanishes. As a consequence, we find that $(dl_z)_e$ is surjective for every z, and the argument used in the proof of theorem 2.2.3.3 then shows that Z is smooth. Further, by passing to the Euclidean topology, which is possible as $\Bbbk$ is a real closed field, one shows that the orbit $H \cdot z$ of any point z in Z is an open semi-algebraic subset. This implies that $\mathcal{C}_z$ admits no nontrivial deformations.

Finally, for any z in Z, the subset $H \cdot z$ is open and closed, meaning that it is a union of semi-algebraically connected components of Z. Further, we have

$$Z = \bigcup_{z\in Z} H \cdot z.$$

But, by theorem 2.4.4 of [BCR98], Z is a finite union of semi-algebraically connected components. This implies that Z is the union of finitely many orbits under the action of H, thereby establishing the second part of the theorem. □

Remark 2.2.4.2. Over a general field, the statement of theorem 2.2.3.3 does not hold. Namely, working over $\mathbb{F}_p$, $\mathbf{Vect}_{\mathbb{F}_{pq}}$ is a finite semisimple tensor 1-category for every positive integer q. These finite semisimple tensor 1-categories are manifestly separable and non-equivalent. Furthermore, they all share the same Grothendieck ring, $\mathbb{Z}$.

Over an algebraically closed field, corollary 2.2.4.3 first appeared in [Ste97], and corollary 2.2.4.4 follows from remark 2.33 of [ENO05]. Over a real closed field, both corollaries are new.

Corollary 2.2.4.3. *There are only finitely many equivalence classes of semisimple and cosemisimple Hopf algebras of a fixed dimension d.*

Proof. Recall that, for any Hopf algebra H of dimension d, the Frobenius-Perron dimension of the fusion 1-category $\mathbf{Rep}(H)$ is d. Further, from the reconstruction theorem, it follows that there is a bijection between equivalence classes of semisimple Hopf algebras of dimension d and fusion 1-categories equipped with a tensor functor to $\mathbf{Vect}_{\Bbbk}$. But, for a semisimple Hopf algebra H, it is argued in section 9.1 of [EO04], that $\mathbf{Rep}(H)$ is separable if and only H is cosemisimple. The proof is finished by appealing to theorems 2.2.3.3 and 2.2.4.1. $\square$

Corollary 2.2.4.4. *There are only finitely many equivalence classes of braidings on a given separable fusion 1-category.*

Proof. A braiding on a fusion 1-category $\mathcal{C}$ can equivalently be described as an tensor functor $\mathcal{C} \to \mathcal{Z}(\mathcal{C})$ whose composite with the forgetful tensor functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is the identity. Now, if $\mathcal{C}$ is separable, $\mathcal{Z}(\mathcal{C})$ is a finite semisimple 1-category. Furthermore, $\mathcal{Z}(\mathcal{C})$ is a separable fusion 1-category. Over an algebraically closed field, this follows from theorem 2.15 of [ENO05], and, over a real closed field, every fusion 1-category is separable. In particular, theorem 2.2.3.3 applies, which proves the result. $\square$

Finally, let us recall corollary 9.1.8 of [EGNO15] as corollary 2.2.4.5 below, and give the appropriate generalizations in corollaries 2.2.4.7 and 2.2.4.8. Let us also mention that corollary 2.2.4.7 is certainly known to experts, as it is mentioned in subsection 4.5 of [Eti18].

Corollary 2.2.4.5. *Let $\Bbbk$ be an algebraically closed field of characteristic zero, then any multifusion 1-category, tensor functor between them, finite semisimple module 1-category over them, and semisimple Hopf algebra is defined over $\overline{\mathbb{Q}}$.*

Proof. Clearly, $\overline{\mathbb{Q}}$ is a subfield of $\Bbbk$. Now, observe that the variety Z over $\Bbbk$ used in the proof of theorem 2.2.4.1 is defined over $\mathbb{Z}$. Let us write $P(Z)$ for the corresponding algebraic set in $\Bbbk^{\times m}$, which is also defined over $\mathbb{Z}$. Then, using lemma 2.2.4.6 below, we see that the set of $\overline{\mathbb{Q}}$-points of $P(Z)$ is dense in the Zariski topology. Let $\mathcal{C}$ be a multifusion 1-category corresponding to a point $z \in P(Z)$ (i.e. a closed point of Z). As the orbit $H \cdot z$ is open in $P(Z)$, there exists a point z' in $H \cdot z$ whose coordinates are in $\overline{\mathbb{Q}}$. This proves the claim for multifusion 1-categories.

In order to establish the claim for tensor functors, one can use the result we have just proven, and therefore assume that our multifusion 1-categories are defined over $\overline{\mathbb{Q}}$. This implies that the variety X over $\Bbbk$ used in the proof of theorem 2.2.3.3 is defined over $\overline{\mathbb{Q}}$. Then, we can apply the same reasoning as above to deduce the desired result.

Let $\mathcal{C}$ be a multifusion 1-category, which we can assume to be defined over $\overline{\mathbb{Q}}$. Note that for any finite semisimple 1-category $\mathcal{M}$, $End(\mathcal{M})$ is canonically defined over $\overline{\mathbb{Q}}$. Now, observe that a finite semisimple right $\mathcal{C}$-module 1-category $\mathcal{M}$ is defined over $\overline{\mathbb{Q}}$ if and only if the corresponding tensor functor $\mathcal{C}^{mop} \to End(\mathcal{M})$ is defined over $\overline{\mathbb{Q}}$. Thus, the result for finite semisimple module 1-categories follows from that for tensor functors.

Finally, let H be a semisimple Hopf algebra. By what we have prove above, there exists a fusion 1-category $\mathcal{C}$ defined over $\overline{\mathbb{Q}}$, which is equivalent to $\mathbf{Rep}(H)$. Furthermore, there is a tensor functor $F : \mathcal{C} \to \mathbf{Vect}$ defined over $\overline{\mathbb{Q}}$, which is equivalent to the canonical tensor functor $\mathcal{C} \simeq \mathbf{Rep}(H) \to \mathbf{Vect}$. Applying the reconstruction theorem to F yields a Hopf algebra H', which is defined over $\overline{\mathbb{Q}}$, and equivalent to H. $\square$

The following lemma is certainly well-known. We include a proof for completeness.

Lemma 2.2.4.6. *Let $\Bbbk \subseteq \mathbb{K}$ be algebraically closed fields. Let $f_1, ..., f_m$ be polynomials in $\Bbbk[x_1, ..., x_n]$, and write $V_{\Bbbk}$, $V_{\mathbb{K}}$ respectively, for the vanishing sets of the f_i's in $\Bbbk^{\times n}$, $\mathbb{K}^{\times n}$ respectively. Then, $V_{\Bbbk}$ is dense in $V_{\mathbb{K}}$ with respect to the Zariski topology.*

Proof. We have to show that if a polynomial in $\mathbb{K}[x_1, ..., x_n]$ vanishes on $V_{\Bbbk}$, then it vanishes on $V_{\mathbb{K}}$. We prove the converse, i.e. that any polynomial g in $\mathbb{K}[x_1, ..., x_n]$ which does not vanish on $V_{\mathbb{K}}$, does not vanish on $V_{\Bbbk}$. Let $(p_1, ..., p_n)$ be a point in $V_{\mathbb{K}}$ for which $g(p_1, ..., p_n) \neq 0$. Let us consider A the $\Bbbk$-subalgebra of $\mathbb{K}$ generated by the coefficients of g, the p_i's, and $g(p_1, ..., p_n)^{-1}$. Let $\mathfrak{m}$ be a maximal ideal of A. As A is finitely generated and $\Bbbk$ is algebraically closed, we have $A/\mathfrak{m} \cong \Bbbk$. Let $q_1, ..., q_n$ be the images of $p_1, .., p_n$ along the quotient $A \twoheadrightarrow A/\mathfrak{m} \cong \Bbbk$. We have $g(q_1, ..., q_n) = g(p_1, ..., p_n)$ in $A/\mathfrak{m}$, which is a unit by construction. $\square$

Corollary 2.2.4.7. *Let $\Bbbk$ be an algebraically closed field of characteristic p, any separable multifusion 1-category, separable module 1-category over a separable multifusion 1-category, tensor functor between separable multifusion 1-categories, semisimple and cosemisimple Hopf algebra is defined over $\overline{\mathbb{F}}_p$.*

Proof. With the obvious modifications, one can use the argument used to prove corollary 2.2.4.5. $\square$

Corollary 2.2.4.8. *Let $\Bbbk$ be a real closed field, then any multifusion 1-category, tensor functor between them, finite semisimple module 1-category over them, and finite semisimple Hopf algebra is defined over $\mathbb{R}^{alg} = \overline{\mathbb{Q}} \cap \mathbb{R}$.*

Proof. Modifying the argument used to prove corollary 2.2.4.5 by replacing $\overline{\mathbb{Q}}$ by $\mathbb{R}^{alg}$ and by appealing to proposition 5.3.5 of [BCR98] instead of lemma 2.2.4.6 yields the desired result. □

2.3 Additional Properties of Compact Semisimple 2-Categories

We explain how a number of important results in the theory of finite semisimple 2-categories generalize to compact semisimple 2-categories over an arbitrary field $\Bbbk$.

2.3.1 An Equivalence of 3-categories

In the spirit of [Déc22b], we establish an equivalence between two 3-categories. In order to introduce the first one, recall from [DSPS21] following [JFS17] that, over any base field, there is a 3-category $\mathbf{TC}$ of finite tensor 1-categories, finite bimodule 1-categories, bimodule functors, and bimodule natural transformations.

Theorem 2.3.1.1. *Finite semisimple tensor 1-categories, right separable bimodule 1-categories, bimodule functors, and bimodule natural transformations define a sub-3-category* $\mathbf{TC}^{rsep}$ *of* $\mathbf{TC}$.

Proof. It is enough to prove that right separable bimodule 1-categories compose. This is precisely what the proof of theorem 2.5.5 of [DSPS21] shows. □

Remark 2.3.1.2. Let $\Bbbk$ be a perfect field, the full sub-3-category of $\mathbf{TC}^{rsep}$ on the separable finite semisimple tensor 1-categories is denoted by $\mathbf{TC}^{sep}$ in [DSPS21]. Namely, proposition 2.5.10 of [DSPS21] shows that $\mathbf{TC}^{sep}$ can also be described as the 3-category of separable finite semisimple tensor 1-categories, right separable bimodule 1-categories, bimodule functors, and bimodule natural transformations.

Definition 2.3.1.3. We write **CSS2C** for the 3-category of compact semisimple 2-categories, linear 2-functors, 2-natural transformations, and modifications.

Theorem 2.3.1.4. *Over an arbitrary field* $\Bbbk$*, there is an equivalence of linear 3-categories*

$$\mathbf{Mod} : \mathbf{TC}^{rsep} \xrightarrow{\simeq} \mathbf{CSS2C}.$$

Proof. The proof follows that of theorem 2.2.2 of [Déc22b]. As $\mathbf{Vect}_{\Bbbk}$ is a finite semisimple tensor 1-category, we have a linear 3-functor

$$Hom_{\mathbf{TC}^{rsep}}(\mathbf{Vect}_{\Bbbk},-):\mathbf{TC}^{rsep}\rightarrow\mathbf{2Cat}_{\Bbbk},$$

to the 3-category of $\Bbbk$-linear 2-categories. Given a finite semisimple tensor 1-category $\mathcal{C}$, $Hom_{\mathbf{TC}^{rsep}}(\mathbf{Vect}_{\Bbbk},\mathcal{C})$ is by definition the 2-category of separable right $\mathcal{C}$-module 1-categories, whence the above 3-functor factors through the full sub-2-category $\mathbf{CSS2C}$. We denote this linear 3-functor by

$$\mathbf{Mod}:\mathbf{TC}^{rsep}\rightarrow\mathbf{CSS2C}.$$

We will prove that $\mathbf{Mod}$ is an equivalence by showing that it is essentially surjective and induces equivalences on Hom-2-categories. Essential surjectivity follows from theorem 2.1.4.6. Now, let $\mathcal{C}$, $\mathcal{D}$ be two finite semisimple tensor 1-categories. It remains to prove that the linear 2-functor

$$T:Hom_{\mathbf{TC}^{rsep}}(\mathcal{C},\mathcal{D})\rightarrow\mathbf{Fun}(\mathbf{Mod}(\mathcal{C}),\mathbf{Mod}(\mathcal{D}))$$

induced by $\mathbf{Mod}(-)$ is an equivalence. By construction, this 2-functor sends the $\mathcal{C}$-$\mathcal{D}$-bimodule 1-category $\mathcal{M}$, which is separable as a right module 1-category, to the 1-functor

$$\begin{array}{ccc}\mathbf{Mod}(\mathcal{C}) & \rightarrow & \mathbf{Mod}(\mathcal{D})\\ \mathcal{N} & \mapsto & \mathcal{N}\boxtimes_{\mathcal{C}}\mathcal{M}.\end{array}$$

A pseudo-inverse to T is given by the linear 2-functor

$$R:\mathbf{Fun}(\mathbf{Mod}(\mathcal{C}),\mathbf{Mod}(\mathcal{D}))\rightarrow Hom_{\mathbf{TC}^{rsep}}(\mathcal{C},\mathcal{D})$$

defined by sending the 2-functor $F:\mathbf{Mod}(\mathcal{C})\rightarrow\mathbf{Mod}(\mathcal{D})$ to $F(\mathcal{C})$. Note that $F(\mathcal{C})$ inherits a left action by $\mathcal{C}$ from the canonical left action of $\mathcal{C}$ on itself. The functoriality of F ensures that it is compatible with the given right action by $\mathcal{D}$, so that $F(\mathcal{C})$ is a $\mathcal{C}$-$\mathcal{D}$-bimodule 1-category, which is separable as a right $\mathcal{D}$-module 1-category. Then, it is not hard to check directly that $R\circ T$ is equivalent to the identity 2-functor on $Hom_{\mathbf{TC}^{rsep}}(\mathcal{C},\mathcal{D})$. Finally, similarly to what is done in the proof of theorem 2.2.2 of [Déc22b], one can use the 3-universal property of the Cauchy completion (see definition 1.2.1 of [Déc22b]) to show that $T\circ R$ is equivalent to the identity 2-functor on $\mathbf{Fun}(\mathbf{Mod}(\mathcal{C}),\mathbf{Mod}(\mathcal{D}))$. $\square$

Corollary 2.3.1.5. *Over a perfect field $\Bbbk$, the equivalence of theorem 2.3.1.4 restricts to an equivalence between the full sub-3-categories on the separable tensor 1-categories and on the locally separable compact semisimple 2-categories.*

Proof. This is a consequence of theorems 2.1.5.6 and 2.1.5.7 □

Corollary 2.3.1.6. *If $\Bbbk$ is an algebraically closed field or a real closed field, the equivalence of theorem 2.3.1.4 becomes an equivalence between the 3-category of multifusion 1-categories and that of finite semisimple 2-categories.*

Proof. This follows from theorem 2.2.2.1. □

2.3.2 The Yoneda Embedding

In [DR18], the authors prove that the absolute Yoneda embedding of a finite semisimple 2-category over an algebraic closed field of characteristic zero is an equivalence (proposition 1.4.11). Assuming that $\Bbbk$ is a perfect field, their proof generalizes to locally separable compact semisimple 2-categories. We present here a slightly different argument using theorem 2.3.1.4.

Proposition 2.3.2.1. *Le $\Bbbk$ be a perfect field, and $\mathfrak{C}$ be a locally separable compact semisimple 2-category. The Yoneda embedding*

$$よ : \mathfrak{C} \to \mathbf{Fun}(\mathfrak{C}^{1op}, \mathbf{2Vect}_{\Bbbk})$$

is an equivalence.

Proof. Thanks to theorem 2.1.4.6, we may assume that $\mathfrak{C} = \mathbf{Mod}(\mathcal{C})$ for some separable tensor 1-category $\mathcal{C}$. Moreover, we have

$$\mathbf{Mod}(\mathcal{C}^{mop}) \simeq Cau(\mathrm{B}(\mathcal{C}^{mop})) \simeq Cau(\mathrm{B}(\mathcal{C})^{1op}) \simeq Cau(\mathrm{B}(\mathcal{C}))^{1op} \simeq \mathbf{Mod}(\mathcal{C})^{1op}.$$

Under this equivalence, the finite semisimple right $\mathcal{C}$-module 1-category $\mathrm{Mod}_{\mathcal{C}^{mop}}(A^{op})$ is sent to $\mathrm{Mod}_{\mathcal{C}}(A)$.

Now, by theorem 2.3.1.4 and the above equivalence, $\mathbf{Fun}(\mathfrak{C}^{1op}, \mathbf{2Vect}_{\Bbbk})$ is equivalent to the 2-category $\mathbf{LMod}(\mathcal{C}^{mop})$ of finite semisimple left $\mathcal{C}^{mop}$-module 1-categories via evaluation at $\mathcal{C}$. More precisely, given a 2-functor $F : \mathfrak{C}^{1op} \to \mathbf{2Vect}_{\Bbbk}$, we view $F(\mathcal{C})$ as a left $\mathcal{C}^{mop}$-module 1-category by precomposition using the left action of $\mathcal{C}$ on itself. Under these identifications, the Yoneda embedding is given by

$$\mathcal{M} \mapsto Fun_{\mathcal{C}}(\mathcal{C}, \mathcal{M}).$$

Observe that the right hand-side is equivalent to $\mathcal{M}$ viewed as a left $\mathcal{C}^{mop}$-module 1-category. This 2-functor is therefore manifestly an equivalence, so the proof of the result is complete. □

Remark 2.3.2.2. We have to assume that $\Bbbk$ is perfect in the statement of proposition 2.3.2.1. Namely, if $\Bbbk$ is arbitrary, the 2-functor よ may not make sense as $Hom_{\mathfrak{C}}(C, D)$ is a priori only a finite semisimple $\Bbbk$-linear 1-category, and not a separable $\Bbbk$-linear 1-category. Explicitly, let us set $\Bbbk = \mathbb{F}_p(x)$, and define $\mathbb{K} = \mathbb{F}_p(\sqrt[p]{x})$. If we take $\mathfrak{C} = \mathbf{2Vect}_{\Bbbk}$, and $C = \mathbf{Vect}_{\Bbbk}$, $D = \mathbf{Vect}_{\mathbb{K}}$, then $Hom_{\mathfrak{C}}(C, D) = \mathbf{Vect}_{\mathbb{K}}$, which is not a separable $\Bbbk$-linear 1-category.

Similarly, assuming that $\Bbbk$ is perfect alone is not enough. Namely, theorem 2.3.1.4 identifies $\mathbf{Fun}(\mathbf{Mod}(\mathcal{C})^{1op}, \mathbf{2Vect}_{\Bbbk})$ with the 2-category of left $\mathcal{C}^{mop}$-module 1-categories whose underlying 1-category is separable. Note that, in this case, the 2-functor よ does indeed exist. Taking $\Bbbk = \mathbb{F}_p$, and $\mathcal{C}$ to be the finite semisimple tensor 1-category of $\mathbb{Z}/p$-graded finite dimensional $\Bbbk$-vector spaces, we see that $\mathbf{Vect}_{\Bbbk}$ is a left $\mathcal{C}^{mop}$-module 1-category whose underlying 1-category is separable. However, it cannot be in the image of よ as it is not separable as a left $\mathcal{C}^{mop}$-module 1-category.

Corollary 2.3.2.3. *Let $\Bbbk$ be a perfect field, $\mathfrak{C}$, $\mathfrak{D}$ be two compact semisimple 2-categories, and $P : \mathfrak{D}^{1op} \times \mathfrak{C} \to \mathbf{2Vect}_{\Bbbk}$ be a bilinear 2-functor. If $\mathfrak{D}$ is locally separable, there exists a linear 2-functor $F : \mathfrak{C} \to \mathfrak{D}$ such that P and $Hom_{\mathfrak{D}}(-, F(-))$ are 2-naturally equivalent.*

Proof. By hypothesis, P defines a linear 2-functor $Q : \mathfrak{C} \to \mathbf{Fun}(\mathfrak{D}^{1op}, \mathbf{2Vect}_{\Bbbk})$. Composing Q with a pseudo-inverse of よ, we obtain a linear 2-functor $F : \mathfrak{C} \to \mathfrak{D}$. For every C in $\mathfrak{C}$, and D in $\mathfrak{D}$, there are natural equivalences

$$\begin{aligned} Hom_{\mathfrak{D}}(D, F(C)) &\simeq Hom_{Fun(\mathfrak{D}^{1op}, \mathbf{2Vect}_{\Bbbk})}(\text{よ}(D), \text{よ}(F(C))) \\ &\simeq Hom_{Fun(\mathfrak{D}^{1op}, \mathbf{2Vect}_{\Bbbk})}(\text{よ}(D), Q(C)) \\ &\simeq (Q(C))(D) = P(D, C). \end{aligned}$$

This proves the result. □

2.3.3 Compact Semisimple Tensor 2-Categories

The following definitions are straightfoward categorifications of the definitions of tensor 1-category and of (multi)fusion 1-category recalled in 2.1.3. In this section, we work over an arbitrary field $\Bbbk$.

Definition 2.3.3.1. A tensor 2-category is a rigid monoidal linear 2-category.

Definition 2.3.3.2. Over an algebraically closed field or a real closed field, a multifusion 2-category is a rigid monoidal finite semisimple 2-category. A fusion 2-category is a multifusion 2-category whose monoidal unit is a simple object.

Example 2.3.3.3. Let $\Bbbk$ be any field. Let us call a $\Bbbk$-linear 1-category perfect if it is finite semisimple and the endomorphism algebra of any object is separable. We write $\mathbf{2Vect}_{\Bbbk}$ for the 2-category of perfect 1-categories, linear functors, and natural transformations. The Deligne tensor product $\boxtimes$ preserves perfect categories, so that $\mathbf{2Vect}$ is a symmetric monoidal full sub-2-category of $\mathbf{FinCat}$, the 2-category of finite 1-categories. Further, we note that $\mathbf{2Vect}_{\Bbbk}$ is equivalent to $\mathbf{Mod}(\mathbf{Vect}_{\Bbbk})$, so that $\mathbf{2Vect}_{\Bbbk}$ is a symmetric compact semisimple tensor 2-category. If the base field $\Bbbk$ is clear from the context, we will often omit it from the notation.

Remark 2.3.3.4. If we assume that the field $\Bbbk$ is perfect, then every finite semisimple 1-category is perfect, so that $\mathbf{2Vect}$ coincides with the symmetric monoidal 2-category of finite semisimple 1-categories. In fact, it follows from the fact that a $\Bbbk$-algebra R is separable if and only $R \otimes R$ is semisimple that a finite semisimple 1-category $\mathcal{A}$ is perfect if and only if $\mathcal{A} \boxtimes \mathcal{A}$ is finite semisimple. However, for an arbitrary field, $\mathbf{FinCat_{ss}}$, the full sub-2-category on the finite semisimple 1-categories, is not closed under the Deligne tensor product.

The results of section 2.4 of [Déc22c] also apply to compact semisimple tensor 2-categories. We highlight the generalized versions of some of them below. We omit the proofs as they are virtually identical to those given in [Déc22c].

Proposition 2.3.3.5. *Let $\mathcal{B}$ be a braided finite semisimple tensor 1-category. Then,* $\mathbf{Mod}(\mathcal{B})$ *is a compact semisimple tensor 2-category with monoidal structure given by the balanced Deligne tensor product $\boxtimes_{\mathcal{B}}$.*

Definition 2.3.3.6. Let $\mathfrak{C}$ be a compact semisimple tensor 2-category with monoidal unit I. We write $\Omega\mathfrak{C}$ for the braided finite semisimple tensor 1-category $End_{\mathfrak{C}}(I)$ of endomorphisms of the monoidal unit.

Proposition 2.3.3.7. *Let $\mathfrak{C}$ be a compact semisimple tensor 2-category. Then, $\mathfrak{C}^0$, the connected component of the identity of $\mathfrak{C}$, is a compact semisimple tensor 2-category. Moreover, there is an equivalence of monoidal 2-categories $\mathfrak{C}^0 \simeq \mathbf{Mod}(\Omega\mathfrak{C})$.*

Example 2.3.3.8. Let us work over $\mathbb{R}$, and examine the fusion rule of the fusion 2-categories $\mathbf{Mod}(\mathbf{Vect}_{\mathbb{R}})$ and $\mathbf{Mod}(\mathbf{Vect}_{\mathbb{C}})$. From example 2.2.2.4, we know that the latter has only one simple object, which is the monoidal unit, so the fusion rule is completely determined. We also know that the former has three equivalence classes of simple objects given by $\mathbf{Vect}_{\mathbb{R}}$, $\mathbf{Vect}_{\mathbb{C}}$, and $\mathbf{Vect}_{\mathbb{H}}$. Clearly, $\mathbf{Vect}_{\mathbb{R}}$ is the monoidal unit. Now, the remaining products are given by:

$$\mathbf{Vect}_{\mathbb{C}} \boxtimes_{\mathbf{Vect}_{\mathbb{R}}} \mathbf{Vect}_{\mathbb{C}} \simeq \mathbf{Vect}_{\mathbb{C}} \boxplus \mathbf{Vect}_{\mathbb{C}},$$

$$\mathbf{Vect}_{\mathbb{C}} \boxtimes_{\mathbf{Vect}_{\mathbb{R}}} \mathbf{Vect}_{\mathbb{H}} \simeq \mathbf{Vect}_{\mathbb{H}} \boxtimes_{\mathbf{Vect}_{\mathbb{R}}} \mathbf{Vect}_{\mathbb{C}} \simeq \mathbf{Vect}_{\mathbb{C}},$$

$$\mathbf{Vect}_{\mathbb{H}} \boxtimes_{\mathbf{Vect}_{\mathbb{R}}} \mathbf{Vect}_{\mathbb{H}} \simeq \mathbf{Vect}_{\mathbb{R}}.$$

Example 2.3.3.9. Let us examine the structure of the compact semisimple tensor 2-category $\mathbf{Mod}(\mathbf{Vect}_{\mathbb{F}_p})$ for some fixed prime p. Equivalence classes of simple objects are classified by finite separable division algebras over $\mathbb{F}_p$. But, by the Wedderburn theorem, every such algebra is isomorphic to $\mathbb{F}_{p^q}$ for some positive integer q. This means that a set of representatives for the equivalence classes of simple objects of $\mathbf{Mod}(\mathbf{Vect}_{\mathbb{F}_p})$ is given by $\mathbf{Vect}_{\mathbb{F}_{p^q}}$ for all positive integer q. For any two positive integers $q \geq r$, we have

$$\mathbf{Vect}_{\mathbb{F}_{p^q}} \boxtimes_{\mathbf{Vect}_{\mathbb{F}_p}} \mathbf{Vect}_{\mathbb{F}_{p^r}} \simeq \boxplus_{i=1}^{r} \mathbf{Vect}_{\mathbb{F}_{p^q}}.$$

Namely, $\mathbb{F}_{p^q} \otimes \mathbb{F}_{p^r} = \oplus_{i=1}^{r} \mathbb{F}_{p^q}$. A similar identity holds if $r \leq q$.

Example 2.3.3.10. Let p be a prime number, and $\Bbbk$ be an algebraically closed field such that $char(\Bbbk) \neq p$. We work with $\mathbf{Vect}_{\mathbb{Z}/p}$, the fusion 1-category of finite dimensional $\mathbb{Z}/p$-graded $\Bbbk$-vector spaces. A braiding b on $\mathbf{Vect}_{\mathbb{Z}/p}$ is completely determined by

$$b_{\Bbbk_1,\Bbbk_1} : \Bbbk_1 \otimes \Bbbk_1 \to \Bbbk_1 \otimes \Bbbk_1,$$

where $\Bbbk_1$ is the one dimensional $\Bbbk$-vector space concentrated in degree $1 \in \mathbb{Z}/p$. Furthermore, the braiding axioms imply that $b_{\Bbbk_1,\Bbbk_1}$ has to be a p-th root of unity ζ. Conversely, any choice of a p-th root of unity ζ describes a braiding on $\mathbf{Vect}_{\mathbb{Z}/p}$ via the assignment $b_{\Bbbk_1,\Bbbk_1} = \zeta$.

On one hand, if $p = char(\Bbbk)$, then it follows from example 2.1.5.9 that the finite semisimple 2-category $\mathbf{Mod}(\mathbf{Vect}_{\mathbb{Z}/p})$ has a unique equivalence class of simple objects given by $\mathbf{Vect}_{\mathbb{Z}/p}$. Further, in this case, 1 is the unique p-th root of unity, so there is a unique braiding $triv$ on $\mathbf{Vect}_{\mathbb{Z}/p}$. We write $\mathcal{C}$ for the corresponding braided fusion 1-category. The fusion rule of $\mathbf{Mod}(\mathcal{C})$ is then completely described by $\mathbf{Vect}_{\mathbb{Z}/p} \boxtimes_{\mathcal{C}} \mathbf{Vect}_{\mathbb{Z}/p} \simeq \mathbf{Vect}_{\mathbb{Z}/p}$, as $\mathbf{Vect}_{\mathbb{Z}/p}$ is the monoidal unit.

On the other hand, if $p \neq char(\Bbbk)$, then we have seen how to describe the equivalence classes of simple of $\mathbf{Mod}(\mathbf{Vect}_{\mathbb{Z}/p})$ in example 2.1.5.9. In this setting, there are two equivalence classes of simple objects, $\mathbf{Vect}_{\mathbb{Z}/p}$ with the canonical right action (corresponding to the zero subgroup with the trivial cohomology class), and $\mathbf{Vect}$ with the trivial right action (corresponding to the group $\mathbb{Z}/p$ with the trivial cohomology class).

Let us denote by $\mathcal{C}$ the braided fusion 1-category obtained by giving $\mathbf{Vect}_{\mathbb{Z}/p}$ the braiding specified by 1 in $\Bbbk$, and by $\mathcal{D}$ the braided fusion 1-category obtained by endowing $\mathbf{Vect}_{\mathbb{Z}/p}$ with the braiding β specified by a fixed primitive p-th root of unity ζ in $\Bbbk$. We now describe the fusion rules of the associated compact semisimple tensor 2-categories. In either case, the simple object $\mathbf{Vect}_{\mathbb{Z}/p}$ is the monoidal unit, so we only have to describe the product of $\mathbf{Vect}$ by $\mathbf{Vect}$. In $\mathbf{Mod}(\mathcal{C})$, we have

$$\mathbf{Vect} \boxtimes_{\mathcal{C}} \mathbf{Vect} \simeq \boxplus_{i=1}^{p} \mathbf{Vect},$$

and in $\mathbf{Mod}(\mathcal{D})$, we have

$$\mathbf{Vect} \boxtimes_{\mathcal{D}} \mathbf{Vect} \simeq \mathbf{Vect}_{\mathbb{Z}/p}.$$

This can be seen by generalizing the argument of section 2.5 of [Déc22c] to the characteristic p case. Namely, the argument only relies on the field being algebraically closed. More generally, the fusion rule of $\mathbf{Mod}(\mathbf{Vect}_A^q)$ with A a finite abelian group, and $q : A \to \Bbbk^\times$ a quadratic form may be found in [Déc23c].

Example 2.3.3.11. Let G be a finite group whose order is coprime to $char(\Bbbk)$. We write $\mathrm{B}G$ for the 2-category with one object $*$, and $End_{\mathrm{B}G}(*) = G$. We may consider the compact semisimple 2-category $\mathbf{Fun}(\mathrm{B}G, \mathbf{2Vect})$ of (finite perfect) 2-representations of G, denoted by $\mathbf{2Rep}(G)$. Said differently, the objects of $\mathbf{2Rep}(G)$ are perfect 2-vector spaces equipped with a G-action. The symmetric monoidal structure of $\mathbf{2Vect}$ endows $\mathbf{2Rep}(G)$ with the structure of a symmetric compact semisimple 2-category. More precisely, given V and W two 2-vector spaces with a G-action, their monoidal product is given by the Deligne tensor product $V \boxtimes W$ endowed with the diagonal G-action. The compact semisimple 2-category $\mathbf{2Rep}(G)$ is in fact rigid as can be seen either directly or from lemma 2.3.3.12 below. We also note that this construction associated more generally a fusion 2-category to any 2-group $\mathcal{G}$ provided that the order of $\pi_1(\mathcal{G})$, the group of equivalence classes of objects of $\mathcal{G}$, is coprime to $char(\Bbbk)$.

The next lemma gives an alternative description of the symmetric monoidal 2-category $\mathbf{2Rep}(G)$ of perfect 2-representations of a finite group G. To this end, let us write $\mathbf{Rep}(G)$ for the symmetric fusion 1-category of finite dimensional representations of G.

Lemma 2.3.3.12. *Let G be a finite group whose order is coprime to $char(\Bbbk)$. The symmetric monoidal compact semisimple 2-categories* $\mathbf{Mod}(\mathbf{Rep}(G))$ *and* $\mathbf{2Rep}(G)$ *are equivalent. In particular,* $\mathbf{2Rep}(G)$ *is rigid.*

Proof. This follows from a slight elaboration on theorem 8.5 of [Gre10]. For completeness, we give a proof using the theory of compact semisimple tensor 2-categories. By definition, the monoidal unit I of $\mathbf{2Rep}(G)$ is $\mathbf{Vect}$, the 1-category of finite $\Bbbk$-vector spaces, equipped with the trivial G-action, and inspection shows that $\Omega\mathbf{2Rep}(G) \simeq \mathbf{Rep}(G)$ as symmetric finite semisimple tensor 1-categories. Finally, note that the compact semisimple 2-category $\mathbf{2Rep}(G)$ is connected, so that the desired equivalence of symmetric monoidal compact semisimple 2-categories follows from proposition 2.3.3.7 above. □

Example 2.3.3.13. Given a finite 2-group $\mathcal{G}$, we can consider the fusion 2-category $\mathbf{2Vect}_{\mathcal{G}}$ of $\mathcal{G}$-graded 2-vector spaces as in construction 2.1.13 of [DR18]. Succinctly, $\mathbf{2Vect}_{\mathcal{G}}$ is constructed as follows. We define a monoidal 2-category by $\mathfrak{X} := \mathcal{G} \times \mathrm{B}^2\Bbbk$, the free $\Bbbk$-linear monoidal 2-category on $\mathcal{G}$. We then write $\widetilde{\mathfrak{X}}$ for its local Cauchy completion, and finally set $\mathbf{2Vect}_{\mathcal{G}} := Cau(\widetilde{\mathfrak{X}})$. More generally, given a 4-cocycle π for $\mathcal{G}$ with coefficients in $\Bbbk^\times$, we can consider the fusion 2-category $\mathbf{2Vect}^\pi_{\mathcal{G}}$ of π-twisted $\mathcal{G}$-graded 2-vector spaces (see construction 2.1.16 of [DR18]). Namely, the 4-cocycle π provides us with the Postnikov data necessary to form the 3-group $\mathcal{G} \times^\pi \mathrm{B}^2\Bbbk^\times$, which we think of as a non-linear monoidal 2-category. After linearization, we obtain a linear monoidal 2-category $\mathfrak{Y} := \mathcal{G} \times^\pi \mathrm{B}^2\Bbbk$, which is a twisted variant of $\mathfrak{X}$ above. We then set $\mathbf{2Vect}^\pi_{\mathcal{G}} := Cau(\widetilde{\mathfrak{Y}})$, with $\widetilde{\mathfrak{Y}}$ the local Cauchy completion of $\mathfrak{Y}$.

2.3.4 Algebras in Compact Semisimple Tensor 2-Categories

Let $\Bbbk$ be a field. We give various examples of algebras in monoidal 2-categories, and in compact semisimple monoidal 2-categories over $\Bbbk$.

Example 2.3.4.1. Algebras in $\mathbf{2Vect}$ correspond precisely to perfect monoidal 1-categories. Fixing such an algebra $\mathcal{A}$, right $\mathcal{A}$-modules in $\mathbf{2Vect}$ are exactly perfect right $\mathcal{A}$-module 1-categories.

The above example can be generalized in various different directions.

Example 2.3.4.2. Let us fix G a finite group. We write $\mathbf{2Vect}_G$ for the compact semisimple tensor 2-category of G-graded perfect 1-categories. It is easy to check that algebras in $\mathbf{2Vect}_G$ are exactly given by G-graded perfect monoidal categories such that the monoidal structure preserves the grading. Right modules admit a similar explicit description. More generally, if we are in addition given a 4-cocycle ω for G with coefficient in $\Bbbk^\times$, we can use ω to twist the coherence of the monoidal 2-category $\mathbf{2Vect}_G$ (see construction 2.1.16 of [DSPS21] and [Del22]). Doing so, we get another compact semisimple tensor 2-category, which we denote by $\mathbf{2Vect}_G^\omega$. If $H \subseteq G$ is a subgroup such that there exists a 3-cochain $\gamma : H \times H \times H \to \mathbb{k}^\times$ with $d\gamma = \omega|_H$, then $\mathbf{Vect}_H^\gamma$, the monoidal category of finite dimensional H-graded $\mathbb{k}$-vector spaces with associator twisted by γ, is an algebra in $\mathbf{2Vect}_G^\omega$. The 2-category of right modules over $\mathbf{Vect}_H^\gamma$ is equivalent to $\mathbf{2Vect}_{G/H}$.

Example 2.3.4.3. Let $\mathcal{B}$ be a braided finite semisimple tensor 1-category. We wish to describe algebras in the compact semisimple tensor 2-category $\mathbf{Mod}(\mathcal{B})$ of separable right $\mathcal{B}$-module 1-categories. Following [BJS21], a $\mathcal{B}$-central monoidal 1-category is a monoidal linear 1-category $\mathcal{C}$ equipped with a braided monoidal functor $F : \mathcal{B} \to \mathcal{Z}(\mathcal{C})$ to the Drinfeld center of $\mathcal{C}$. This notion has also appeared under different names in [DGNO10], [HPT23] and [MPP18]. Note that this provides both a right and a left $\mathcal{B}$-module structure on $\mathcal{C}$. We will call a finite semisimple $\mathcal{B}$-central monoidal 1-category $\mathcal{C}$ separable if $\mathcal{C}$ is separable as a (left, or equivalently right) $\mathcal{B}$-module 1-category. It follows from proposition 3.2 of [BJS21] that algebras in $\mathbf{Mod}(\mathcal{B})$ correspond precisely to separable $\mathcal{B}$-central monoidal 1-categories. If $\mathcal{C}$ is such an algebra, it is easy to check that right $\mathcal{C}$-modules in $\mathbf{Mod}(\mathcal{B})$ are exactly finite semisimple right $\mathcal{C}$-module 1-categories, which are separable as a right $\mathcal{B}$-module 1-category.

Example 2.3.4.4. Let G be a finite group of order coprime to $char(\mathbb{k})$. Algebras in $\mathbf{2Rep}(G)$ are given exactly by perfect monoidal 1-categories with a G-action. Further, algebra 1-homomorphisms are monoidal functors preserving the G-actions, and monoidal natural transformations preserving the G-action.

2.3.5 The 2-Deligne Tensor Product

The Deligne tensor product is well-known operation between finite (semisimple) 1-categories. In this section, we survey the construction and the properties of the 2-Deligne tensor product of compact semisimple 2-categories introduced in [Déc24], to which we refer the reader for further details. Given two Cauchy complete linear 2-category $\mathfrak{C}$ and $\mathfrak{D}$, one can form their Cauchy completed tensor product $\mathfrak{C}\widehat{\otimes}\mathfrak{D} := Cau(\mathfrak{C} \otimes \mathfrak{D})$. Further, there is a canonical bilinear 2-functor $\widehat{\otimes} : \mathfrak{C} \times \mathfrak{D} \to \mathfrak{C}\widehat{\otimes}\mathfrak{D}$ that is 3-universal with respect to bilinear 2-functor from $\mathfrak{C} \times \mathfrak{D}$ to a Cauchy complete linear 2-category (see proposition 3.4 of [Déc24]).

Moreover, if $\mathfrak{C}$ and $\mathfrak{D}$ are compact semisimple 2-categories over the perfect field $\mathbb{k}$, then it was shown in theorem 3.7 of [Déc24] that $\mathfrak{C}\widehat{\otimes}\mathfrak{D}$ is again a compact semisimple 2-category, which we denote by $\mathfrak{C} \boxtimes \mathfrak{D}$, and call the 2-Deligne tensor product of $\mathfrak{C}$

and $\mathfrak{D}$. More precisely, if $\mathfrak{C} \simeq \mathbf{Mod}(\mathcal{C})$ and $\mathfrak{D} \simeq \mathbf{Mod}(\mathcal{D})$ for some finite semisimple tensor 1-categories $\mathcal{C}$ and $\mathcal{D}$, then we have

$$\mathfrak{C} \boxtimes \mathfrak{D} \simeq \mathbf{Mod}(\mathcal{C} \boxtimes \mathcal{D}).$$

In particular, provided $\Bbbk$ is perfect, it follows from theorem 2.1.5.7, that locally separable compact semisimple 2-categories are closed under $\boxtimes$. For later use, we also record the following proposition from [Déc24].

Proposition 2.3.5.1. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two compact semisimple 2-categories. For any objects C_1, C_2 in $\mathfrak{C}$ and D_1, D_2 in $\mathfrak{D}$, there is an equivalence of categories:*

$$Hom_{\mathfrak{C}\boxtimes\mathfrak{D}}(C_1 \boxtimes D_1, C_2 \boxtimes D_2) \simeq Hom_{\mathfrak{C}}(C_1, C_2) \boxtimes Hom_{\mathfrak{C}}(D_1, D_2).$$

Moreover, this equivalence commutes with composition up to natural isomorphism.

We note however that not every object of $\mathfrak{C}\boxtimes\mathfrak{D}$ is of the form $C\boxtimes D$ in general! For instance, over an algebraically closed field of characteristic zero, it follows from [Nat17] that $\mathbf{Mod}(\mathbf{Vect}_{\mathbb{Z}/2}) \boxtimes \mathbf{Mod}(\mathbf{Vect}_{\mathbb{Z}/2})$ has exactly six equivalence classes of simple objects, whereas $\mathbf{Mod}(\mathbf{Vect}_{\mathbb{Z}/2})$ has two. On the other hand, if $\Bbbk$ is an algebraically closed field, then $\pi_0(\mathfrak{C} \boxtimes \mathfrak{D}) \cong \pi_0(\mathfrak{C}) \times \pi_0(\mathfrak{D})$.

Using the 3-universal property of the 2-Deligne tensor product, one may show that the 2-Deligne tensor product of two compact semisimple monoidal 2-categories inherits a canonical monoidal structure. Building on this result, the following theorem was established in [Déc24].

Theorem 2.3.5.2. *Given $\mathfrak{C}$ and $\mathfrak{D}$ two compact semisimple tensor 2-categories over a perfect field. Their 2-Deligne tensor product $\mathfrak{C} \boxtimes \mathfrak{D}$ is a compact semisimple tensor 2-category. Further, the 2-functor $\boxtimes : \mathfrak{C} \times \mathfrak{D} \to \mathfrak{C} \boxtimes \mathfrak{D}$ is monoidal.*

In addition, we note that, for any compact semisimple tensor 2-categories $\mathfrak{C}$ and $\mathfrak{D}$, we have $\Omega(\mathfrak{C} \boxtimes \mathfrak{D}) \simeq \Omega\mathfrak{C} \boxtimes \Omega\mathfrak{D}$ as finite semisimple braided tensor 1-categories.

2.4 Weak Based Rings

The goal of the last section of this chapter is to state a version of proposition 3.4.6 of [EGNO15] that applies to weak based rings. As indicated in exercise 3.8.1 therein, such a statement does hold. We provide a proof for completeness. Further, we also prove lemma 2.4.1.7, as it features prominently in the proof of our main theorem, and we have not been able to locate a proof in the literature. We begin by recalling the necessary definitions from [EGNO15] section 3.1 and section 3.8.

Definition 2.4.1.1. A $\mathbb{Z}_+$-ring is a ring A, which is free as a $\mathbb{Z}$-module, together with a basis $\{b_i\}_{i\in I}$ such that $1 = \sum_i a^i b_i$ for non-negative integers a^i and, for every i, j we have $b_i b_j = \sum_k c_{ij}^k b_k$ for non-negative integers c_{ij}^k.

Given a $\mathbb{Z}_+$-ring A, we write I_0 for the finite subset of I corresponding to the non-zero a^i's. We define $\tau : A \to \mathbb{Z}$ to be the group homomorphism given by

$$\tau(b_i) = \begin{cases} 1 \text{ if } i \in I_0 \\ 0 \text{ otherwise.} \end{cases}$$

Definition 2.4.1.2. A $\mathbb{Z}_+$-ring A with basis $\{b_i\}_{i\in I}$ is called a weak based ring if there exists an involution $i \mapsto i^*$ of I such that the induced map on A is an anti-involution, and

$$\tau(b_i b_j) = \begin{cases} t_i > 0 \text{ if } i = j^* \\ 0 \text{ if } i \neq j^*. \end{cases}$$

If $t_i = 1$ for all i, then we say that A is a based ring.

The following example also appears in [EG12].

Example 2.4.1.3. The Grothendieck ring $Gr(\mathcal{C})$ of a finite semisimple tensor 1-category $\mathcal{C}$ with unit I is a weak based ring with basis given by the equivalence classes of simple objects and involution induced by the dual functor. Namely, the usual proof shows that right and left duals agree up to a non-canonical isomorphism. Namely, for any simple object C of $\mathcal{C}$, we have

$$Hom_{\mathcal{C}}(C^*, {}^*C) \cong Hom_{\mathcal{C}}(C \otimes C^*, I) \cong Hom_{\mathcal{C}}(I, C \otimes C^*) \cong Hom_{\mathcal{C}}(C, C).$$

For any simple objects C, D in $\mathcal{C}$, there is an isomorphism

$$Hom_{\mathcal{C}}(C^* \otimes D, I) \cong Hom_{\mathcal{C}}(D, C),$$

and, as the right hand-side is non-zero if and only if C and D are isomorphic, this shows that $Gr(\mathcal{C})$ is a weak based ring. Note that if $\Bbbk$ is algebraically closed and I is simple, then the Grothendieck ring of $\mathcal{C}$ is a based ring.

Definition 2.4.1.4. Let A, C be two $\mathbb{Z}_+$-rings with bases $\{b_i\}_{i\in I}$, and $\{d_j\}_{j\in J}$ respectively. A homomorphism of $\mathbb{Z}_+$-rings $f : A \to C$ is a homomorphism of rings such that for every i, $f(b_i) = \sum_{j\in J} g_i^j d_j$ for some non-negative integers g_i^j.

Definition 2.4.1.5. Let A, C be two weak based $\mathbb{Z}_+$-rings. A homomorphism of weak based $\mathbb{Z}_+$-rings $f : A \to C$ is a homomorphism of $\mathbb{Z}_+$-rings which preserves the involution.

Example 2.4.1.6. A tensor functor $F : \mathcal{C} \to \mathcal{D}$ between two finite semisimple tensor 1-categories induces a homomorphism of weak based $\mathbb{Z}_+$-rings $Gr(F) : Gr(\mathcal{C}) \to Gr(\mathcal{D})$ by sending the equivalence class of the simple object C to $F(C)$.

Lemma 2.4.1.7. *The set of homomorphisms of weak based $\mathbb{Z}_+$-rings between two weak based $\mathbb{Z}_+$-rings of finite rank is finite.*

Proof. Let A, C be two $\mathbb{Z}_+$-rings with bases $\{b_i\}_{i\in I}$, and $\{d_j\}_{j\in J}$ respectively. For any j, k in J, we have $d_j d_k = \sum_{l\in J} c^l_{jk} d_l$ for some non-negative integers c^l_{jk}. We write $f : A \to C$ for some homomorphism of $\mathbb{Z}_+$-rings. Let us define $b := \sum_{i\in I} b_i$. We have $b^2 = \sum_{j\in I} n^i b_i$, for some non-negative integers n_i, and we write $N := \max\{n_i\}$. Further, we also have $f(b_i) = \sum_{j\in J} g^j_i d_j$ for some non-negative integers g^j_i. Summing over i, we write $f(b) = \sum_{j\in J} g^j d_j$ for some non-negative integers g^j. We now compute $f(b^2) = \sum_{j\in J} e^j d_j$ in two different ways. Firstly, we have

$$f(b^2) = f(\sum_{i\in I} n^i b_i) = \sum_{i\in I, j\in J} g^j_i n^i d_j,$$

from which we find $\sum_{j\in J} e^j \leq N \sum_j g^j$. Secondly, we have

$$f(b^2) = (\sum_{j\in J} g^j d_j)(\sum_{k\in J} g^k d_k) = \sum_{j,k,l} c^l_{jk} g^j g^k d_l.$$

As f is a homomorphism of weak based $\mathbb{Z}_+$-rings, and $b = b^*$, we have $f(b) = f(b)^*$. This together with the above equation implies that $\sum_{j\in J} e^j \geq \sum_j (g^j)^2$. Putting both inequalities together, we have $\sum_j (g^j)^2 \leq N \sum_j g^j$, which forces $g^j \leq |J|N$ for all j, whence the proof is complete. $\square$

We now recall two definitions from section 3.4 of [EGNO15].

Definition 2.4.1.8. Let A be a $\mathbb{Z}_+$-ring with basis $\{b_i\}_{i\in I}$. A $\mathbb{Z}_+$-module over A is a right A-module M, which is free as a $\mathbb{Z}$-module, together with a basis $\{m_l\}_{l\in L}$ such that for all i, l, we have $m_l b_i = \sum_k a^k_{il} m_k$ for some non-negative integers a^k_{il}.

Definition 2.4.1.9. A $\mathbb{Z}_+$-module M over A is called indecomposable if it cannot be written as the direct sum of two non-zero $\mathbb{Z}_+$-modules over A. A $\mathbb{Z}_+$-module M over A is called irreducible if no proper non-empty subset of the basis generates a proper submodule over A.

Example 2.4.1.10. Let $\mathcal{C}$ be a finite semisimple tensor 1-category, and $\mathcal{M}$ a finite semisimple right $\mathcal{C}$-module 1-category. Then, the action of $Gr(\mathcal{C})$ on $Gr(\mathcal{M})$ the Grothendieck group of $\mathcal{M}$, endows $Gr(\mathcal{M})$ with the structure of $\mathbb{Z}_+$-module over $Gr(\mathcal{C})$. Further, there is a bijective correspondence between $\mathbb{Z}_+$-submodules of $Gr(\mathcal{M})$ that are closed under the action of $Gr(\mathcal{C})$, and finite semisimple right $\mathcal{C}$-module sub-1-categories of $\mathcal{M}$. This shows that $\mathcal{M}$ is indecomposable as a right $\mathcal{C}$-module 1-category if and only if $Gr(\mathcal{M})$ is indecomposable as a $\mathbb{Z}_+$-module over $Gr(\mathcal{C})$.

The following lemma generalizes proposition 7.7.2 of [EGNO15].

Lemma 2.4.1.11. *Let $\mathcal{M}$ be a finite semisimple right $\mathcal{C}$-module 1-category. Then, the $\mathbb{Z}_+$-module $Gr(\mathcal{M})$ over $Gr(\mathcal{C})$ is irreducible.*

Proof. Let N be a proper $\mathbb{Z}_+$-submodule of $Gr(\mathcal{M})$ stable under $Gr(\mathcal{C})$. These conditions ensures that there exists a proper finite semisimple right $\mathcal{C}$-submodule 1-category $\mathcal{N}$ of $\mathcal{M}$ such that $Gr(\mathcal{N}) = N$. Similarly, if we write $N^{\perp}$ for the proper $\mathbb{Z}_+$-submodule of $Gr(\mathcal{M})$ spanned by the basis vector of $Gr(\mathcal{M})$ not in N, there exists a proper finite semisimple subcategory $\mathcal{N}^{\perp}$ of $\mathcal{M}$ such that $Gr(\mathcal{N}^{\perp}) = N^{\perp}$. We claim that $\mathcal{N}^{\perp}$ is stable under the right action of $\mathcal{C}$. Provided this is true, we have exhibited $\mathcal{M}$ as a direct sum of the two non-trivial finite semisimple right $\mathcal{C}$-module 1-categories $\mathcal{N}$ and $\mathcal{N}^{\perp}$, a contradiction.

Let us now prove the claim. Let C in $\mathcal{C}$, M in $\mathcal{N}^{\perp}$, and N in $\mathcal{N}$ be arbitrary objects, there is an isomorphism

$$Hom_{\mathcal{M}}(M \otimes C, N) \cong Hom_{\mathcal{M}}(M, N \otimes C^*).$$

Thus, we see that $\mathcal{N}$ is stable under the action of $\mathcal{C}$ if and only if $\mathcal{N}^{\perp}$ is, which proves the claim. □

The next proposition is proposition 3.4.6 of [EGNO15] stated in the setting of weak based rings. We follow their proof very closely.

Proposition 2.4.1.12. *Let A be a weak based ring of finite rank. Then there are only finitely many equivalences classes of irreducible $\mathbb{Z}_+$-modules over A.*

Proof. As A has finite rank, so will any irreducible $\mathbb{Z}_+$-modules over A. Namely, let M be an irreducible $\mathbb{Z}_+$-modules over A with basis $\{m_l\}_{l\in L}$. Given any l in L and i in I, we write $m_l b_i = \sum_{k\in L} f_{li}^k m_k$ for some non-negative integers f_{li}^k. If we let $b := \sum_{i\in I} b_i$, we have $m_l b = \sum_{k\in L} f_l^k m_k$ with $f_l^k = \sum_{i\in I} f_{li}^k$, which is positive as A has a unit. We let $\langle l\rangle := \{k \in L | f_l^k \neq 0\}$, which is finite, and claim that the $\mathbb{Z}_+$-submodule of M generated by m_k with $k \in \langle l\rangle$ is stable under the action of A. Namely, for any k

in $\langle l \rangle$, m_k appears in $m_l b$. On the other hand, $(m_l b)b = m_l(b^2) = \sum_{i\in I, k\in\langle l\rangle} n_i f_{il}^k m_k$. This proves the claim. Further, as M is irreducible, we have $L = \langle l \rangle$.

In A, we have $b^2 = \sum_{i\in I} n_i b_i$ for some non-negative integers n_i, and we define $N := \max\{n_i\}$, which is a positive integer independent of M. We also set $f_l := \sum_{k\in L} f_l^k$. Now, pick l_0 in L such that f_{l_0} is minimal amongst the f_l's. We now compute $m_{l_0} b^2 = \sum_{k\in L} c_k m_k$ in two different ways. Firstly, we have

$$(m_{l_0} b)b = \sum_{j,k\in L} f_{l_0}^j f_j^k m_k,$$

from which we deduce $\sum_{k\in L} c_k \geq f_{l_0}^2$. Secondly, we have

$$m_{l_0}(b^2) = \sum_{i\in I, k\in L} n_i f_{l_0 i}^k m_k,$$

from which we get $N f_{l_0} \geq \sum_{k\in L} c_k$. Putting everything together, we find $N f_{l_0} \geq f_{l_0}^2$, and so $N \geq l_0$. As $L = \langle l_0 \rangle$ because M is irreducible, this bounds the rank of M. Moreover, we also get $N^2 \geq \sum_{k\in L} c_k = \sum_{j,k\in L} f_{l_0}^j f_j^k$. As $f_{l_0}^j$ is positive for every j, there are only finitely many non-negative integers f_j^k satisfying this inequality. This completes the proof of the proposition. $\square$

Chapter 3

Rigid and Separable Algebras in Fusion 2-Categories

3.1 Properties of Rigid Algebras

3.1.1 Definitions in Detail

We fix $\mathfrak{C}$ a monoidal 2-category, which we assume to be strict cubical without loss of generality. Following [Gai12] (see also [JFR24]), we say that an algebra A in $\mathfrak{C}$ is rigid if its multiplication map $m : A\Box A \to A$ has a right adjoint as a map of A-A-bimodules. For later use, we spell out what this means in detail.

Definition 3.1.1.1. A rigid algebra in $\mathfrak{C}$ consists of:

1. An algebra A in $\mathfrak{C}$ as in definition 1.2.1.1;

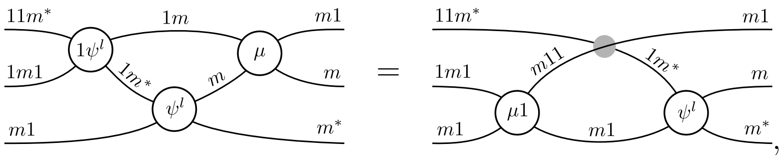
2. A right adjoint $m^* : A \to A\Box A$ in $\mathfrak{C}$ to the multiplication map m with unit η^m and counit ϵ^m (depicted below as a cup and a cap);

3. Two 2-isomorphisms

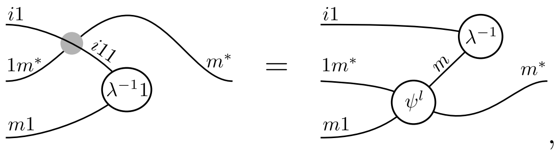
$$\begin{tikzcd} AA \arrow[r, "m"] \arrow[d, "m^*1"'] \arrow[dr, phantom, "\overset{\psi^r}{\Nearrow}"] & A \arrow[d, "m^*"] \\ AAA \arrow[r, "1m"'] & AA, \end{tikzcd} \qquad \begin{tikzcd} AA \arrow[r, "m"] \arrow[d, "1m^*"'] \arrow[dr, phantom, "\overset{\psi^l}{\Nearrow}"] & A \arrow[d, "m^*"] \\ AAA \arrow[r, "m1"'] & AA; \end{tikzcd}$$

   satisfying:

a. The 2-morphism ψ^l endow m^* with the structure of a left A-module 1-morphism:

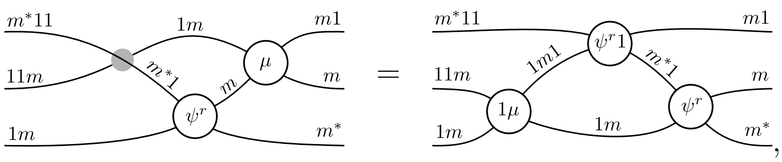


$$(3.1)$$

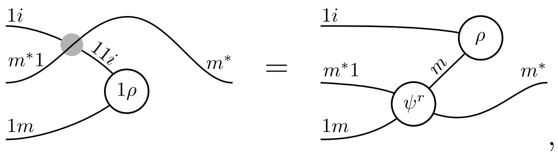


$$(3.2)$$

b. The 2-morphism ψ^r endow m^* with the structure of a right A-module 1-morphism:

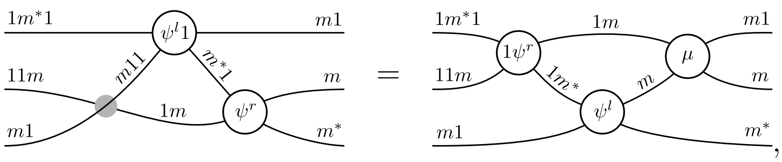


$$(3.3)$$



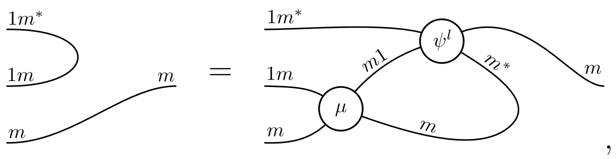
$$(3.4)$$

c. The structures of left and right A-module 1-morphisms on m^* constructed above are compatible, i.e. they turn m^* into an A-A-bimodule 1-morphism:

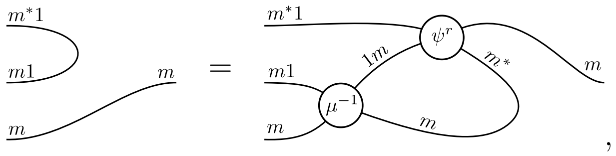


$$(3.5)$$

d. The 2-morphism ϵ^m is an A-A-bimodule 2-morphism:

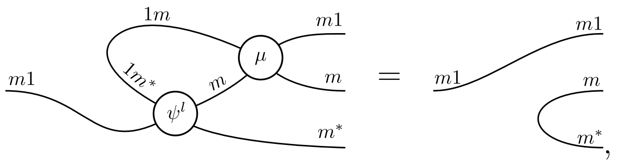


$$(3.6)$$

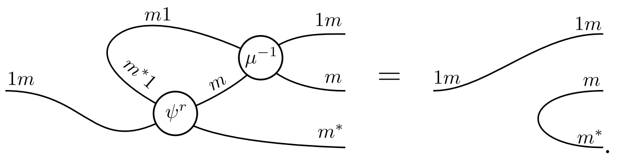


$$(3.7)$$

e. The 2-morphism η^m is an A-A-bimodule 2-morphism:



(3.8)



(3.9)

As we now explain, many familiar objects are rigid algebras in a certain monoidal 2-category.

Example 3.1.1.2. By proposition 1.3 of [BJS21], a finite monoidal 1-category $\mathcal{A}$ whose monoidal product is right exact in both variables separately is tensor, i.e. has right and left duals for objects, if and only if $\mathcal{A}$ is a rigid algebra in the sense of the above definition. In particular, every finite semisimple tensor 1-category is a rigid algebra in **FinCat**.

Example 3.1.1.3. Fixing G a finite group, a slight elaboration on the argument of the above example shows that rigid algebras in $\mathbf{2Vect}_G$ correspond exactly to perfect G-graded tensor 1-categories.

Let $\mathcal{B}$ be a finite semisimple braided tensor 1-category. We have seen in example 2.3.4.3 above that algebras in $\mathbf{Mod}(\mathcal{B})$ are exactly given by separable $\mathcal{B}$-central monoidal 1-categories. We now characterize those algebras that are rigid.

Lemma 3.1.1.4. *A separable $\mathcal{B}$-central monoidal 1-category $\mathcal{C}$ is a rigid algebra in* $\mathbf{Mod}(\mathcal{B})$ *if $\mathcal{C}$ is tensor. If the underlying 1-category of $\mathcal{C}$ is perfect, this holds if and only if $\mathcal{C}$ is tensor.*

Proof. As $\mathcal{C}$ is separable as a $\mathcal{B}$-module 1-category, $\mathcal{C}\boxtimes_{\mathcal{B}}\mathcal{C}$ is finite semisimple. In particular, the canonical functor $m:\mathcal{C}\boxtimes_{\mathcal{B}}\mathcal{C}\rightarrow\mathcal{C}$ has a right adjoint. Thus, if $\mathcal{C}$ is rigid, it follows from [DSPS19] that m has a right adjoint as a $\mathcal{C}$-$\mathcal{C}$-bimodule functor. This proves the first part of the statement.

Let us now assume that $\mathcal{C}$ is perfect, but not a priori tensor. Observe that it is enough to prove that the canonical $\mathcal{C}$-$\mathcal{C}$-bimodule functor $G:\mathcal{C}\boxtimes\mathcal{C}\rightarrow\mathcal{C}\boxtimes_{\mathcal{B}}\mathcal{C}$ has a $\mathcal{C}$-$\mathcal{C}$-bimodule right adjoint. As both the target and the domain of G are finite semisimple, we find that G is exact, so that it has a right adjoint G^* as a linear functor. Further, by separability of $\mathcal{C}$ as a $\mathcal{B}$-module 1-category, $End_{\mathcal{B}}(\mathcal{C})$ is a finite semisimple tensor 1-category, thence it follows from [DSPS19] that G^* has an $End_{\mathcal{B}}(\mathcal{C})$-$End_{\mathcal{B}}(\mathcal{C})$-bimodule structure. Restricting this bimodule structure on G^* along the monoidal functor $\mathcal{C} \to End_{\mathcal{B}}(\mathcal{C})$ proves the claim. □

Remark 3.1.1.5. We now record one particularly noteworth consequence of lemma 2.3.3.12. Let G be a finite group whose order is coprime to $char(\Bbbk)$. As $\mathbf{Mod}(\mathbf{Rep}(G))$ and $\mathbf{2Rep}(G)$ are equivalent as symmetric monoidal 2-categories, the associated (symmetric monoidal) 2-categories of algebra, algebra 1-homomorphisms and algebra 2-homomorphisms are equivalent. In particular, this induces an equivalence between the full sub-2-categories on the rigid algebras. Thanks to the last lemma above, we therefore get an equivalence between the 2-category of perfect tensor 1-categories with a G-action, and the 2-category of perfect $\mathbf{Rep}(G)$-central tensor 1-categories. Over an algebraically closed field of characteristic zero, this is a well-known result (see theorem 4.18 of [DGNO10]). In addition, we get an equivalence between the (symmetric monoidal) 2-categories of braided rigid algebras. That is there is an equivalence between the 2-categories of perfect braided tensor 1-categories with a braided G-action and perfect braided tensor 1-categories equipped with a braided functor from $\mathbf{Rep}(G)$ into its Müger center. Over an algebraically closed field of characteristic zero, this is also a classical result (see proposition 4.22 of [DGNO10]).

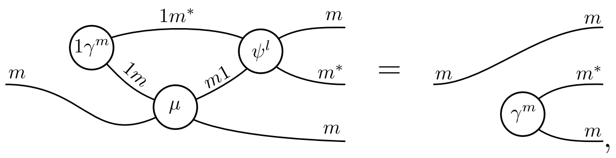
Remark 3.1.1.6. It is well-known that algebras in **Cat** the 2-category of 1-categories equipped with the Cartesian product as monoidal structure are precisely monoidal 1-categories. One can check directly that rigid algebras in **Cat** are all equivalent to the monoidal 1-category with one object and one morphism. This also follows from the general observation that the object underlying a rigid algebra is always self-dual. Thus, in this context, the classical notion of rigidity is more appropriate.

We also give in detail the definition of a separable algebra

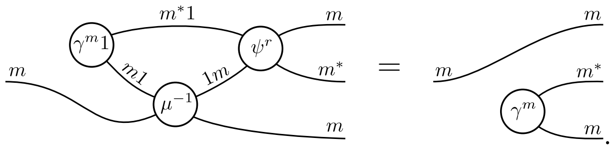
Definition 3.1.1.7. A separable algebra in $\mathfrak{C}$ is a rigid algebra A in $\mathfrak{C}$ equipped with a 2-morphism $\gamma^m : Id_A \Rightarrow m \circ m^*$ such that:

a. The 2-morphism γ^m is a section of ϵ^m, i.e. $\epsilon^m \cdot \gamma^m = Id_{Id_A}$,

b. The 2-morphism γ^m is an A-A-bimodule 2-morphism:



(3.10)



(3.11)

We will explore this notion further in the next section, which will allow us to give many examples in section 3.2.3.

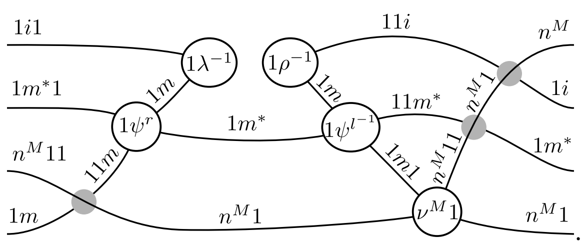
Remark 3.1.1.8. Any separable algebra is a 3-condensation monad in the sense of [GJF19]. In fact, if $\mathfrak{C}$ is a compact semisimple tensor 2-category, then it follows from the proof of theorem 3.17 of [GJF19] that every 3-condensation monad is equivalent to a separable algebra. We decide to work with separable algebras because they are more familiar algebraic objects.

3.1.2 Adjoints in 2-Categories of (Bi)Modules

Let us fix A a rigid algebra in the strict cubical monoidal 2-category $\mathfrak{C}$, and let M be a right A-module. Note that the 1-morphism n^M has a canonical structure of a right A-module 1-morphism supplied by ν^M. Our first goal is to show that n^M has a right adjoint as a right A-module 1-morphism. We begin by defining

$$p^M := (n\Box A)\circ(M\Box m^*)\circ(M\Box i) : M \to M\Box A, \tag{3.12}$$

a 1-morphism in $\mathfrak{C}$, together with a 2-morphism in $\mathfrak{C}$ given by

$\psi^p :=$ 

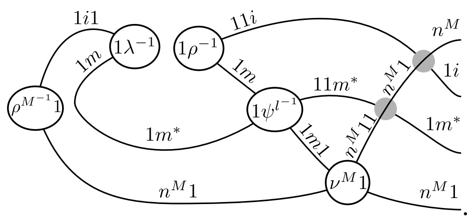
Lemma 3.1.2.1. *The 2-morphism ψ^p endows the 1-morphism p^M with the structure of a right A-module 1-morphism.*

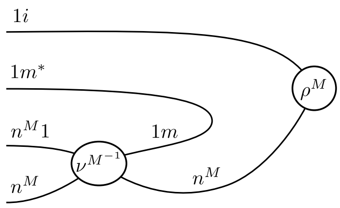
Proof. We need to check that ψ^p satisfies the two coherence axioms of definition 1.2.2.4. In order to do this, we will use the diagrams depicted in appendix 3.3.1. We begin by checking axiom a: Figure 3.1 depicts the right hand-side of axiom a for ψ^p. Using naturality to move the coupons labelled 1μ and $\nu^M 11$, we find that this 2-isomorphism is equal to that depicted in figure 3.2. Now, we use equation (3.3) on the blue coupons, and equation (1.5) on the green ones to get that this is equal to the 2-isomorphism given in figure 3.3. In order to get to figure 3.4, we apply equation (3.1) to the blue coupons, and use naturality to move the coupon labelled $\nu^M 11$. Using equation (3.5) on the blue coupons, we find that this 2-isomorphism is equal to that depicted in figure 3.5. Equation (1.2) shows that this string diagram represents the same 2-isomorphism as that given in figure 3.6. Using equation (1.3) on the blue coupons, and equation (1.7) for A on the green coupons, we get that this is equal to figure 3.7. But, figure 3.7 manifestly depicts the left hand-side of axiom a of definition 1.2.2.4.

We proceed to check that axiom b also holds. Firstly, note that figure 3.8 depicts the right hand-side of axiom b of definition 1.2.2.4. Naturality implies that the 2-isomorphism described by this string diagram is the same as that described by figure 3.9. Using equation (1.6) on the blue coupons, we find that this also corresponds to the figure 3.10. Equation (3.2) above applied to the blue coupons shows further that it is equal to the 2-isomorphism depicted by figure 3.11. Using equation 1.4 twice, once on the blue coupon, and once on the green coupons, we find that figure 3.12 also describes the same 2-isomorphism. To derive the equality with figure 3.13, we use equation (3.4) on the blue coupons. Finally, naturality allows us to assert that the 2-isomorphism described by figure 3.14 agrees with the previous ones. As this last string diagram also corresponds to the left hand-side of axiom b of definition 1.2.2.4, we are done. $\square$

We now define two 2-morphisms in $\mathfrak{C}$, which will be the unit and counit of the desired adjunction:

$\eta^M :=$



$\epsilon^M :=$ 

,

Lemma 3.1.2.2. *The 2-morphisms η^M and ϵ^M in $\mathfrak{C}$ depicted above are right A-module 2-morphisms.*

Proof. The proof that ϵ^M is right A-module 2-morphisms follows by successively applying equations (1.5), (3.6), (1.7), (3.7), and finally (1.6). We now prove that η^M is right A-module 2-morphisms. As this is more subtle, we explain the proof in details using the figures given in appendix 3.3.2. Figure 3.15 depicts the left hand-side of the equation of definition 1.2.2.5 for ϵ^M. Using naturality, we move the indicated coupons along the blue arrows, which gets us to figure 3.16. Then, applying equation (1.5) to the blue coupons, we arrive at figure 3.17. Using equation (3.1) on the blue coupons, leads us to contemplate figure 3.18. In order to get to figure 3.19, we use equation (3.5) on the blue coupons, as well as equation (1.7) for the right A-module A on the green ones. Applying equation (1.4) to the blue coupons produces figure 3.20. Now, we can use equation (1.3) in the blue region, to obtain figure 3.21. Then, the two blue coupons are manifestly each other's inverse, so we can fuse them together to obtain figure 3.22. Using equation (3.9) on the blue coupons gets us to figure 3.23. Finally, we can use naturality to move the left most coupon to the right, which produces the right hand-side of the equation of definition 1.2.2.5 for ϵ^M. The proof is therefore complete. □

Lemma 3.1.2.3. *The right A-module 2-morphisms η^M and ϵ^M witness that p^M is a right adjoint of n^M in the 2-category $\mathbf{Mod}_{\mathfrak{C}}(A)$ of right A-modules.*

Proof. We have to check that ϵ^M and η^M satisfy the triangle identities. In order to do so, we use the diagrams depicted in appendix 3.3.3. One of the triangle identities corresponds to showing that the 2-isomorphism depicted in figure 3.24 is the identity on n^M. Using equation (1.5) on the blue coupons, we find that the 2-isomorphism depicted by this string diagram is equal to that given in figure 3.25. The latter is equal to the one depicted in figure 3.26, as can be seen by first using naturality to move the cap to the left, and then equation (3.6). Using the triangle identity to straighten the blue line, we get to the string diagram depicted in figure 3.27. Finally, the blue coupons cancel each other out by equation (1.9), and similarly the green coupons cancel each other out by equation (1.7). Thus we are left with a single string labelled n^M, which establishes one of the triangle identity.

We now show that the second triangle identity holds, that is we show that the 2-isomorphism depicted in figure 3.28 is the identity on p^M. Using naturality to move some coupons, we find that this first string diagram represents the same 2-morphism as that given in figure 3.29. Now, equation (1.7) applied to the blue coupon, and equation (1.5) applied to the green ones show that this is also equal to the 2-morphism described by figure 3.30. Cancelling the two blue coupons out, we arrive at figure 3.31. We make use equation (3.8) to arrive at figure 3.32. Using equation (1.2) on the blue coupons, and equation (1.6) on the green ones gets us to figure 3.33. Equation (3.1) proves that the 2-morphism described by this last figure is equal to that represented by figure 3.34. Now, applying equation (1.7) to the blue coupons, and equation (3.2) to the green ones imply that this is equal to figure 3.35. Finally, we can naturality followed by the triangle equation for m to straighten the blue line, and equation (1.4) on the green coupons, resulting in a string diagrams representing the identity 2-morphism on p. This proves the second triangle identity holds. $\square$

Putting the above lemmas together, we get the following result, which generalizes proposition 3.11 of [BZBJ18] (see also proposition D.2.2 of [Gai12] for a similar result in a different context).

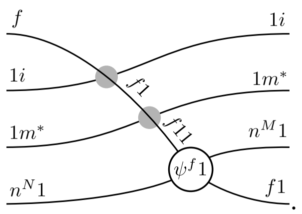
Proposition 3.1.2.4. *Let A be a rigid algebra in $\mathfrak{C}$, and M a right A-module with right action $n^M : M\Box A \to M$. Then, the right A-module 1-morphism $p^M : M \to M\Box A$ defined in equation (3.12) is a right adjoint to n^M in $\mathbf{Mod}_{\mathfrak{C}}(A)$.*

We will use the above proposition in our proof of the following result.

Proposition 3.1.2.5. *Let A be a rigid algebra in $\mathfrak{C}$, and $f : M \to N$ a 1-morphism of right A-modules. If f has a left adjoint in $\mathfrak{C}$, then f has a left adjoint in $\mathbf{Mod}_{\mathfrak{C}}(A)$.*

Proof. Let $f : M \to N$ be a right A-module 1-morphism, which has a left ${}^*f$ adjoint as a 1-morphism in $\mathbf{Mod}_{\mathfrak{C}}(A)$. We have to endow the 1-morphism ${}^*f$ in $\mathfrak{C}$ with a right A-module structure such that the unit $\eta^f : Id_N \Rightarrow f \circ {}^*f$ and counit $\epsilon^f : {}^*f \circ f \Rightarrow Id_M$ of this adjunction in $\mathfrak{C}$ are right A-module 2-morphisms. We begin by making the following definition:

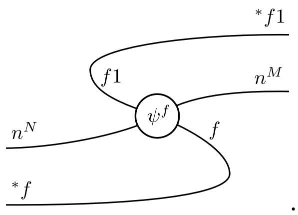
$\zeta^f :=$



Then, by proposition 3.1.2.4, we have explicit left adjoints for p^M and p^N. Using these, we let

$$\psi^{*f} := {}^*(\zeta^f).$$

As ζ^f is clearly invertible, so is ψ^{*f}. It is possible to check the coherence axioms of definition 1.2.2.4 for ψ^{*f} directly, but this is tedious. We give an alternative approach. Let us set

$\xi^f :=$ 

We claim that ξ^f is the inverse of ψ^{*f}. Assuming this is true, it is easy to see that ξ^f satisfy the inverse of the equations giving the axioms of definition 1.2.2.4, and so proves that ${}^*f$ is a right A-module map. Moreover, it is not hard to check that η^f and ϵ^f are right A-module 2-morphisms.

We prove the above claim using the diagrams depicted in appendix 3.3.4. The composite $\xi^f \cdot \psi^{*f}$ is depicted in figure 3.36. By moving the coupon labelled ψ^f down and to the left, whilst using the triangle identity for f, we arrive at figure 3.37. Equation (1.8) applied to the blue coupons yields figure 3.38. Moving the coupon labelled $1\psi^{l^{-1}}$ to the right along the blue arrow, and using equation (1.5) on the green coupons brings us to figure 3.39. Now, we can move the cap to the left along the blue arrow, and then appeal to equation (3.6) on the green coupons. Then, taking the coupon labelled $1\rho^{-1}$ to the right along the red arrows leads us to contemplate figure 3.40. At this point, we can move the coupon labelled $1\lambda^{-1}$ to the right along the blue arrow, straighten the green curve, and use equation (1.7) on the red coupons to arrive at figure 3.41. Appealing to equation (1.9) on the blue coupons produces figure 3.42. Finally, straightening the blue curve, and using equation (1.6) on the green coupons shows that the last figure depicts the identity 2-morphism of $n^M \circ {}^*f1$, which proves that $\xi^f \cdot \psi^{*f} = Id_{n^M \circ {}^*f1}$.

We now consider the composite $\psi^{*f} \cdot \xi^f$ depicted in figure 3.43. Dragging the coupon labelled ψ^f to the right as well as straightening the green curve brings us to figure 3.44. Appealing to equation (1.8) on the blue coupons produces figure 3.45 for us. Then, we move the indicated cap along the blue arrow, which gives us figure 3.46 after rearranging the strings a little. Using equation (1.5) for N on the blue coupons gives figure 3.47. We can now apply equation (3.6) to the blue coupons, and arrive at figure 3.48. Straightening the curve in blue, and using equation (1.9) on the green coupons gives figure 3.48. Finally, applying equation (1.6) to the blue coupons, equation (1.7) applied to the green ones, and straightening the red curve shows that this last figure depicts the identity 2-morphism on ${}^*f \circ n^N$. Thus, we have prove that $\psi^{*f} \cdot \xi^f = Id_{{}^*f \circ n^N}$, which finishes the proof of the claim, whence of the proposition. □

Remark 3.1.2.6. If we take $\mathfrak{C} = \mathbf{FinCat}$, $\mathcal{A}$ a finite tensor category, and $F : \mathcal{M} \to \mathcal{N}$ a right $\mathcal{A}$-module functor between finite categories, then proposition 3.1.2.5 recovers the statement that if F has a left adjoint as a linear functor (i.e. is right exact), it has a left adjoints as an $\mathcal{A}$-module functor (see section 3.3 of [EO04]). This also generalizes theorem 5.21 of [BJS21].

Adapting the proof of proposition 3.1.2.5, one can prove the following proposition.

Proposition 3.1.2.7. *Let A, B be rigid algebras in $\mathfrak{C}$, and $f : P \to Q$ an A-B-bimodule 1-morphism. If f has a left adjoint in $\mathfrak{C}$, then it has a left adjoint as an A-B-bimodule 1-morphism.*

Proof. We leave the details to the zealous reader. □

The following theorem is the main application of the above results.

Theorem 3.1.2.8. *Assume that the monoidal 2-category $\mathfrak{C}$ has left and right adjoints, and let A be an arbitrary algebra in $\mathfrak{C}$. Then, A is rigid if and only if $\mathbf{Bimod}_{\mathfrak{C}}(A)$ has right adjoints. In this case, $\mathbf{Bimod}_{\mathfrak{C}}(A)$ has left adjoints, and $\mathbf{Mod}_{\mathfrak{C}}(A)$ has left and right adjoints.*

Proof. Let us begin by the following observation: We can consider the strict opcubical monoidal 2-category $\mathfrak{C}^{2op}$ obtained from $\mathfrak{C}$ by reversing the direction of the 2-morphisms. If A is an algebra in $\mathfrak{C}$, then $(A, m, i, \alpha^{-1}, \rho^{-1}, \lambda^{-1})$ is an algebra in $\mathfrak{C}^{2op}$, which we denote by A^{2op}. Furthermore, inspection shows that

$$\big(\mathbf{Mod}_{\mathfrak{C}}(A)\big)^{2op} = \mathbf{Mod}_{\mathfrak{C}^{2op}}(A^{2op}) \text{ and } \big(\mathbf{Bimod}_{\mathfrak{C}}(A)\big)^{2op} = \mathbf{Bimod}_{\mathfrak{C}^{2op}}(A^{2op}).$$

Let us now assume that A is a rigid algebra, it follows from proposition 3.1.2.7 that $\mathbf{Bimod}_{\mathfrak{C}}(A)$ has left adjoints. Via the second of the above equivalences, this implies that the 2-category $\mathbf{Bimod}_{\mathfrak{C}^{2op}}(A^{2op})$ has right adjoints, so that A^{2op} is a rigid algebra. Propositions 3.1.2.5 and 3.1.2.7 then show that $\mathbf{Mod}_{\mathfrak{C}^{2op}}(A^{2op})$ and $\mathbf{Bimod}_{\mathfrak{C}^{2op}}(A^{2op})$ have left adjoints. The first of the two equivalences above proves that $\mathbf{Mod}_{\mathfrak{C}}(A)$ has right adjoints, while the second demonstrates that $\mathbf{Bimod}_{\mathfrak{C}}(A)$ has right adjoints.

Assume that $\mathbf{Bimod}_{\mathfrak{C}}(A)$ has right adjoints. Then as $m : A\Box A \to A$ is an A-A-bimodule map, it has a right adjoint. This shows that A is rigid, and completes the present proof. □

3.2 Separable Algebras in Compact Semisimple Monoidal 2-Categories

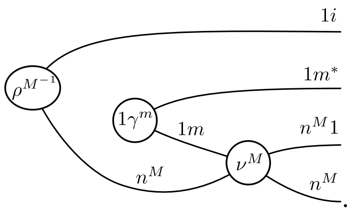
Throughout, we fix a compact semisimple monoidal 2-category $\mathfrak{C}$ over an arbitrary field $\Bbbk$. Without loss of generality, we assume that $\mathfrak{C}$ is strict cubical.

3.2.1 Characterizing Separable Algebras

We begin by the following lemma, which elaborates on the results of the previous section.

Lemma 3.2.1.1. *Let A be a separable algebra, there exists a right A-module 2-morphism γ^M such that $(M\Box A, M, n^M, p^M, \epsilon^M, \gamma^M)$ is a 2-condensation in $\mathbf{Mod}_{\mathfrak{C}}(A)$.*

Proof. We let γ^M be the 2-morphism in $\mathfrak{C}$ given by the following string diagram:

$\gamma^M :=$ 

It follows immediately from the definitions that $\epsilon^M \cdot \gamma^M = Id_{n^M \circ p^M}$. Thus, in order to show that $(M\Box A, M, n^M, p^M, \epsilon^M, \gamma^M)$ is a 2-condensation in $\mathbf{Mod}_{\mathfrak{C}}(A)$, it only remains to prove that γ^M is a right A-module 2-morphism. Starting from the left hand-side of the equation of definition 1.2.2.5 for γ^M, we can successively apply equations (1.5), (3.11), (1.5), (1.6), (3.10), and (1.7) to get to the right hand-side. This finishes the proof of the lemma. □

Proposition 3.2.1.2. *Let A be a separable algebra in $\mathfrak{C}$. Then, $\mathbf{Mod}_{\mathfrak{C}}(A)$ is a compact semisimple 2-category.*

Proof. It follows from theorem 3.1.2.8 that $\mathbf{Mod}_{\mathfrak{C}}(A)$ has right and left adjoints. Further, $\mathbf{Mod}_{\mathfrak{C}}(A)$ is Cauchy complete as was proven in proposition 3.3.8 of [Déc21].

We now show that $\mathbf{Mod}_{\mathfrak{C}}(A)$ is locally finite semisimple. Given any right A-module N, and object C of $\mathfrak{C}$, lemma 3.2.13 of [Déc21] provides us with the following equivalence of linear 1-categories

$$Hom_A(C\Box A, N) \simeq Hom_{\mathfrak{C}}(C, N).$$

By definition, the right hand-side is a finite semisimple 1-category. Now, we use the fact proven in lemma 3.2.1.1 that every right A-module M is the splitting of a 2-condensation monad on the free right A-module $M\Box A$. Together with the above equivalence and the fact that 2-condensations are preserved by every 2-functor, this implies that $Hom_A(M, N)$ is the splitting of a 2-condensation monad on $Hom_{\mathfrak{C}}(M, N)$ in $\mathbf{FinCat_{ss}}$. But we show in lemma 3.2.1.3 below that $\mathbf{FinCat_{ss}}$ is Cauchy complete, so that $Hom_A(M, N)$ is indeed a finite semisimple 1-category.

We have just proven that $\mathbf{Mod}_{\mathfrak{C}}(A)$ is a semisimple 2-category, so it only remains to prove that $\pi_0(\mathbf{Mod}_{\mathfrak{C}}(A))$ is finite. Let us write C_i for a finite set of simple objects of $\mathfrak{C}$ representing the equivalence classes in $\pi_0(\mathfrak{C})$. We claim that, for every simple object M in $\mathbf{Mod}_{\mathfrak{C}}(A)$, there exists a non-zero 1-morphism $C_i\Box A \to M$ for some i. Assuming that this holds, it follows from lemma 2.1.2.7 that every equivalence classes in $\pi_0(\mathbf{Mod}_{\mathfrak{C}}(A))$ is represented by a simple right A-module summand of $C_i\Box A$ for some i. But, as $\mathbf{Mod}_{\mathfrak{C}}(A)$ is locally finite semisimple and $\pi_0(\mathfrak{C})$ is finite, there are only finitely many such summands.

Let us prove the claim. For any simple summand D of M in $\mathfrak{C}$, the composite $D\Box A \hookrightarrow M\Box A \xrightarrow{n^M} M$ is a non-zero right A-module 1-morphism, as precomposition with $D\Box i$ is 2-isomorphic to the inclusion $D \hookrightarrow M$. By construction, there exists a non-zero 1-morphism $C_i \to D$ for some i, and the composite $C_i \to M$ is non-zero. Thus, the composite $C_i\Box A \to M\Box A \xrightarrow{n^M} M$ is a non-zero right A-module 1-morphism. This concludes the proof of the claim. □

Lemma 3.2.1.3. *The 2-category* $\mathbf{FinCat_{ss}}$ *of finite semisimple 1-categories is Cauchy complete.*

Proof. Let us begin by noting that $\mathbf{FinCat_{ss}}$ has right and left adjoints for 1-morphisms, as every functor is exact. Further, recall that $\mathbf{FinCat_{ss}}$ is equivalent to the 2-category of finite semisimple $\Bbbk$-algebras, finite dimensional $\Bbbk$-bimodules, and homomorphisms of bimodules. Under this perspective, it is clear that $\mathbf{FinCat_{ss}}$ is locally Cauchy complete and has direct sum for objects. It therefore only remains to prove that every 2-condensation monad splits. By theorem 3.1.4 of [GJF19], it is in fact enough to check that unital 2-condensation monads splits.

Let $(A, e, \mu, \delta, \iota)$ be a unital 2-condensation monad in $\mathbf{FinCat_{ss}}$, that is a 2-condensation monad such that ι is a unit for μ. Unfolding what this means, we find that the A-A-bimodule e has the structure of a finite dimensional $\Bbbk$-algebra, which we denote by E. Further, the algebra E is equipped with a morphism $A \to E$ of $\Bbbk$-algebras such that the canonical map $E \otimes_A E \to E$ is split by δ as a map of E-E-bimodules. We claim that E is a semisimple algebra. Namely, let M be a left E-module. Then, the canonical map $E \otimes_A M \to M$ splits. But, as A is semisimple, M viewed as a left A-module is a direct summand of a A^n for some $n \geq 0$. This implies that M is a direct summand of E^n, so that E is semisimple.

In order to conclude the proof, let $B := E$ as an object of $\mathbf{FinCat_{ss}}$, f be the 1-morphism from A to B given by the A-E-bimodule E, g the 1-morphism from B to A given by the E-A-bimodule E, ϕ the 2-morphism corresponding to the map of E-E-bimodules ${}_EE \otimes_A E_E \to {}_EE_E$ induced by the multiplication of E, and γ the 2-morphism corresponding to the map of E-E-bimodules ${}_EE_E \to {}_EE \otimes_A E_E$ given by δ. This defines a 2-condensation, which extends $(A, e, \mu, \delta, \iota)$ as desired. ☐

Proposition 3.2.1.4. *Let A and B be separable algebras in the compact semisimple monoidal 2-category $\mathfrak{C}$. Then, $\mathbf{Bimod}_{\mathfrak{C}}(A, B)$ is a compact semisimple 2-category.*

Proof. The proof of proposition 3.2.1.2 can be adapted to the bimodule case. We leave the details to the reader. ☐

Definition 3.2.1.5. Let B be an arbitrary algebra in an arbitrary monoidal 2-category $\mathfrak{D}$. The partial center of B, denoted by $Z(B)$, is the monoidal 1-category of B-B-bimodule endomorphisms of B.

Remark 3.2.1.6. We call $Z(B)$ the partial center of B to distinguish it from the full center, which is a braided algebra in the Drinfeld center of $\mathfrak{D}$. We will explore the properties of the full centre in an upcoming article, in the meantime, we refer the reader to [Dav10] for the decategorified version of this story. Now, if B is a rigid algebra in a compact semisimple 2-category, then, by theorem 3.1.2.8, $Z(B)$ is a rigid monoidal 1-category. If, in addition, $\mathfrak{D}$ is compact semisimple, we expect that $Z(B)$ is a finite 1-category.

Theorem 3.2.1.7. *Let A be a rigid algebra in a compact semisimple monoidal 2-category $\mathfrak{C}$. Then, A is separable if and only if $Z(A)$ is finite semisimple. If either of these conditions is satisfied, then $\mathbf{Bimod}_{\mathfrak{C}}(A)$ is a compact semisimple 2-category.*

Proof. If A is separable, it follows from proposition 3.2.1.4 that $\mathbf{Bimod}_{\mathfrak{C}}(A)$ is compact semisimple, so that, a fortiori, $Z(A)$ is finite semisimple. Now, for the backward direction, let us assume that $Z(A)$ is a finite semisimple 1-category. We first establish the result under the additional assumption that A is indecomposable, i.e. that $Id_A : A \to A$ is a simple object of $Z(A)$. In that case, because $\mathbf{Bimod}_{\mathfrak{C}}(A)$ has right adjoint by theorem 3.1.2.8, the A-A-bimodule 1-morphism $m : A \Box A \to A$ has a right adjoint (as an A-A-bimodule 1-morphism), which we denote by m^*. Let us write ϵ^m for the counit of this adjunction. We claim that ϵ^m has a section in $\mathbf{Bimod}_{\mathfrak{C}}(A)$. Namely, we have assumed that Id_A is a simple object of $Z(A)$, so that either $\epsilon^m : m \circ m^* \Rightarrow Id_A$ has a section or it is zero. But, as m is not the zero map, the latter is not possible. This proves that A is a separable algebra in this case.

Moving on to the general case, let us write

$$Id_A = f_1 \oplus ... \oplus f_n$$

for a decomposition of Id_A into a direct sum of simple objects of $Z(A)$. As $\mathbf{Bimod}_{\mathfrak{C}}(A)$ is Cauchy complete (this follows from proposition 3.3.8 of [Déc21]), we can use proposition 1.3.16 of [DR18] to obtain a direct sum decomposition

$$A = A_1 \boxplus ... \boxplus A_n$$

of A in $\mathbf{Bimod}_{\mathfrak{C}}(A)$. In particular, each object A_i defines an algebra in $\mathfrak{C}$ using the restriction of the algebra structure on A. Further, it follows from the above direct sum decomposition that the restriction of $m : A \Box A \to A$ to $A_i \Box A_j$ can only be non-zero if $i = j$. Thence, A is the direct sum of the algebras A_i in $\mathfrak{C}$. This implies that $Z(A) \cong Z(A_1) \oplus ... \oplus Z(A_n)$, so that $Id_{A_i} = f_i$ is simple as an A-A-bimodule if and only if it is simple as an A_i-A_i-bimodule. In particular, each A_i is indecomposable. Finally, as A was assumed to be rigid, so is each A_i. The special case proven above allows us to conclude the proof. □

The following result is well-known when $\mathbb{k}$ is a perfect field (see corollary 2.5.9 of [DSPS21]).

Corollary 3.2.1.8. *Over any field $\mathbb{k}$, a perfect tensor 1-category $\mathcal{C}$ is separable as a rigid algebra in* **2Vect** *if and only if its Drinfeld center $\mathcal{Z}(\mathcal{C})$ is finite semisimple.*

Proof. Recall that $\mathcal{Z}(\mathcal{C})$ is identified with the 1-category of $\mathcal{C}$-$\mathcal{C}$-bimodule endofunctors of $\mathcal{C}$. Thus, the result follows from theorem 3.2.1.7. □

3.2.2 The Dimension of a Rigid Algebra

Definition 3.2.2.1. An algebra A in $\mathfrak{C}$ is called connected if the unit $i : I \to A$ is a simple 1-morphism.

To help with our intuition, let us examine an example.

Example 3.2.2.2. A perfect monoidal 1-category $\mathcal{A}$ with unit I viewed as an algebra in **2Vect** is connected if and only if $End_{\mathcal{A}}(I)$ is a simple $\Bbbk$-algebra. Note that this implies that $End_{\mathcal{A}}(I)$ is a field, as it is necessarily a commutative algebra.

We now fix a connected rigid algebra A in $\mathfrak{C}$. In particular, $End_{\mathfrak{C}}(i)$, the finite dimensional $\Bbbk$-algebra of 2-endomorphisms of i in $\mathfrak{C}$, is a division $\Bbbk$-algebra. In fact, as A is an algebra, $End_{\mathfrak{C}}(i)$ has two compatible composition operations, so that the product is necessarily commutative by the Eckman-Hilton argument. Thence, $End_{\mathfrak{C}}(i)$ is a finite field extension of $\Bbbk$. Further, under the equivalence

$$\begin{array}{ccc} Hom_{A-A}(A\Box A, A) & \xrightarrow{\simeq} & Hom_{\mathfrak{C}}(I, A) \\ f & \mapsto & f \circ (i \Box i) \end{array}$$

the unit $i : I \to A$ corresponds to $m : A\Box A \to A$ viewed as an A-A-bimodule 1-morphism. This implies that $\mathbb{K} := End_{A-A}(m)$, the endomorphism algebra of m in $\mathbf{Bimod}_{\mathfrak{C}}(A)$, is a finite field extension of $\Bbbk$.

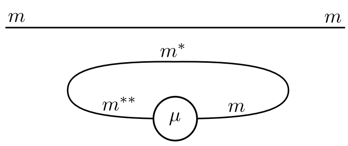
As $\mathfrak{C}$ is a compact semisimple 2-category, it has right adjoints. In particular, $m^* : A \to A\Box A$ has a right adjoint in $\mathfrak{C}$, which we denote by $m^{**} : A\Box A \to A$. Thanks to theorem 3.1.2.8, m^{**} is right adjoint to m^* as an A-A-bimodule 1-morphism. But, in any finite semisimple tensor 1-category, the right and left dual of any object are non-canonically isomorphic. This means that we have $m^{**} \cong m$ in $\mathfrak{C}$. This can be substantially refined. Namely, the equivalence above factors as

$$Hom_{A-A}(A\Box A, A) \to Hom_{\mathfrak{C}}(A\Box A, A) \to Hom_{\mathfrak{C}}(I, A),$$

and m, m^{**} are both objects of the left hand-side, which become isomorphic in the middle step. As the composite is an equivalence of 1-categories, there must exist a 2-isomorphism $\mu : m^{**} \cong m$ of A-A-bimodule 1-morphisms. Note that μ is not unique, it is determined up to a non-zero scalar in $\mathbb{K}$.

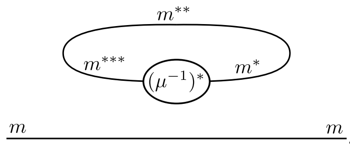
Let us now fix an A-A-bimodule 2-isomorphism $\mu : m^{**} \cong m$. We define a 2-morphism $\mathrm{Tr}(\mu)$, the trace of μ, by

$\mathrm{Tr}(\mu) :=$



.

As all of the 2-morphisms used in the definition of $\mathrm{Tr}(\mu)$ are A-A-bimodule 2-morphisms, so is $\mathrm{Tr}(\mu)$. Thus, we think of $\mathrm{Tr}(\mu)$ as a scalar in $\mathbb{K}$. Analogously, we define the A-A-bimodule 2-morphism $\mathrm{Tr}((\mu^{-1})^*)$ by

$\mathrm{Tr}((\mu^{-1})^*) :=$ 

.

The scalars $\mathrm{Tr}(\mu)$ and $\mathrm{Tr}((\mu^{-1})^*)$ depend not only on the 2-isomorphism $\mu : m \cong m^{**}$, but also on the adjunction data for m, m^*, and m^{**}. However, if $\Bbbk$ is algebraically closed, then by multiplying them together we obtain a scalar which is independent of any choice. This is precisely the same trick that is used to define the squared norm of a simple object in a multifusion 1-category (see definition 7.12.2 of [EGNO15]).

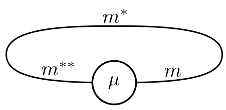
Definition 3.2.2.3. Over an algebraically closed field $\Bbbk$, the dimension of a connected rigid algebra A in $\mathfrak{C}$ is the scalar in $\mathbb{K} \cong \Bbbk$ defined by

$$\mathrm{Dim}_{\mathfrak{C}}(A) := \mathrm{Tr}(\mu) \cdot \mathrm{Tr}((\mu^{-1})^*).$$

The importance of the dimension is witnessed by the following result, which is a generalization of theorem 2.6.7 of [DSPS21].

Theorem 3.2.2.4. *A connected rigid algebra A in a compact semisimple monoidal 2-category $\mathfrak{C}$ is separable if and only if there exists an A-A-bimodule 2-isomorphism $\mu : m \cong m^{**}$ such that $\mathrm{Tr}(\mu)$ is non-zero. Over an algebraically closed field, this holds if and only if $\mathrm{Dim}_{\mathfrak{C}}(A)$ is non-zero.*

Proof. Let us define a 2-morphism of A-A-bimodules τ by

$\tau :=$ 

.

As $\mathrm{Tr}(\mu)$ is a scalar in $\mathbb{K}$, if it is non-zero, then it is in fact invertible. Now, $\mathrm{Tr}(\mu)$ is obtained by precomposing τ with m. But, at the level of 2-morphisms in $\mathfrak{C}$, precomposition along m induces a split inclusion $End_{\mathfrak{C}}(Id_A) \hookrightarrow End_{\mathfrak{C}}(m)$ of $\Bbbk$-algebras. This implies that τ is invertible as a 2-morphism in $\mathfrak{C}$. A direct check then shows that τ is invertible as a 2-morphism of A-A-bimodules. Thus, we can set

$$\gamma^m := (\mu \circ m^*) \cdot \eta^{m^*} \cdot \tau^{-1},$$

which is an A-A-bimodule 2-morphism, and satisfies $\epsilon^m \cdot \gamma^m = Id_A$. Thus, γ^m upgrades the rigid algebra A to a separable algebra. Moreover, if $\Bbbk$ is algebraically closed, and $\mathrm{Dim}_{\mathfrak{C}}(A)$ is non-zero, then $\mathrm{Tr}(\mu)$ is necessarily non-zero.

Conversely, if A is separable, then it follows from theorem 3.2.1.7 that the 2-category $\mathbf{Bimod}_{\mathfrak{C}}(A)$ is compact semisimple. Moreover, as A is connected, we have seen above that $End_{A-A}(m) \cong End_{\mathfrak{C}}(i)$ is a field, which establishes that m is a simple 1-morphism in $\mathbf{Bimod}_{\mathfrak{C}}(A)$. Now, left and right adjoints are non-canonically isomorphic in a compact semisimple 2-category. Given that m is a simple A-A-bimodule, and γ^m is non-zero, we find that $\gamma^m : Id_A \Rightarrow m \circ m^*$ is a unit witnessing that m^* is left adjoint to m. Let us write $\delta^m : m^* \circ m \Rightarrow Id_{AA}$ for a compatible counit. Taking $\mu = Id_m$, we find that $\mathrm{Tr}(\mu) = Id_m$. This concludes the first part of the proof.

Let us now assume that $\Bbbk$ is algebraically closed. It is enough to show that $\mathrm{Tr}((\mu^{-1})^*) = m \circ (\delta^m \cdot \eta^m)$ is non-zero. In order to do so, we will argue that the $\Bbbk$-linear pairing

$$\begin{array}{rccc} \mathrm{P}: & Hom_{A-A}(m^* \circ m, Id_{AA}) \times Hom_{A-A}(Id_{AA}, m^* \circ m) & \rightarrow & End_{A-A}(m) \\ & (\alpha, \beta) & \mapsto & m \circ (\alpha \cdot \beta) \end{array}$$

is non-degenerate. Namely, recall that $\mathbb{K} = End_{A-A}(m)$, but as $\Bbbk$ is algebraically closed, we have $\mathbb{K} = \Bbbk$. This implies that $Hom_{A-A}(m^* \circ m, Id_{AA}) \cong \Bbbk$ and that $Hom_{A-A}(Id_{AA}, m^* \circ m) \cong \Bbbk$. Further, there is a right action

$$\begin{array}{ccc} Hom_{A-A}(m^* \circ m, Id_{AA}) \times End_{A-A}(m) & \rightarrow & Hom_{A-A}(m^* \circ m, Id_{AA}). \\ (\alpha, \gamma) & \mapsto & \alpha \cdot (m^* \circ \gamma) \end{array}$$

Likewise, there is a left action of $End_{A-A}(m)$ on $Hom_{A-A}(Id_{AA}, m^* \circ m)$, and the pairing P is balanced under these actions of $End_{A-A}(m)$. It is therefore enough to show that P is non-zero. We claim that the map

$$\begin{array}{ccc} Hom_{A-A}(m^* \circ m, Id_{AA}) & \rightarrow & Hom_{A-A}(m \circ m^* \circ m, m). \\ \alpha & \mapsto & m \circ \alpha \end{array}$$

is non-zero. Namely, δ^m is a counit witnessing an adjunction between m and m^*, so that $m \circ \delta^m$ is non-zero. In particular, there exists a simple A-A-bimodule 1-morphism $f : A \Box A \rightarrow A \Box A$, which is a summand of both Id_{AA} and $m^* \circ m$, and such that $m \circ f$ is non-zero. This demonstrates that P is non-zero. $\Box$

Remark 3.2.2.5. Let us assume that $\Bbbk$ is algebraically closed. For later use, let us record a few additional facts about the pairing P considered in the second half of the proof of the theorem above. Given that m is a simple A-A-bimodule 1-morphism, we find that there are unique, up to scalar in $\Bbbk^\times$, non-zero A-A-bimodule 2-morphisms $\alpha : m^* \circ m \Rightarrow Id_{AA}$ and $\beta : Id_{AA} \Rightarrow m^* \circ m$. In particular, we have that the composite $\alpha{\cdot}\beta : Id_{AA} \Rightarrow Id_{AA}$ is a scalar multiple $\lambda \in \Bbbk$ of the projection onto a simple summand f of Id_{AA}. This holds because m is a simple object of the finite semisimple tensor 1-category $End_{A-A}((A\Box A) \boxplus A)$, where the result is standard. Further, the pairing P precisely measures the scalar λ.

Example 3.2.2.6. Over a general field, the dimension of a connected rigid algebra might not be well-defined as we now explain. Let us take $\Bbbk = \mathbb{R}$, and $\mathfrak{C} = \mathbf{2Vect}$. We consider the fusion 1-category $\mathcal{C}$ of $\mathbb{C}$-$\mathbb{C}$-bimodules in **Vect**. We have that $End_{\mathcal{C}-\mathcal{C}}(\mathcal{C}) \simeq \mathbf{Vect}$. In particular, $\mathcal{C}$ is a separable algebra in **2Vect** by theorem 3.2.1.7. On the other hand, for any choice of $\mathcal{C}$-$\mathcal{C}$-bimodule natural isomorphism μ, the scalar τ defined in the proof of theorem 3.2.2.4 lies in $\mathbb{R}$. But μ is defined up to a non-zero element of $\mathbb{C}$, so that there must exist a non-zero μ such that $\mathrm{Tr}(\mu) = 0$. Moreover, a slight generalization of the second part of the proof of theorem 3.2.2.4, shows that the $\mathbb{R}$-bilinear pairing $\mathrm{P} : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ is $\mathbb{C}$-balanced. Thus, as $\mathcal{C}$ is separable, it corresponds equivalently to a non-zero $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{R}$. This shows that there exists choices of μ for which $\mathrm{Tr}(\mu) \cdot \mathrm{Tr}((\mu^{-1})^*)$ is non-zero.

Remark 3.2.2.7. It has been conjectured in [JFR24] that every rigid algebra in a fusion 2-category over an algebraically closed field of characteristic zero is separable. Thanks to our result above, in order to establish their conjecture, it would suffice to prove that the dimension of every rigid algebra is non-zero. We discuss some examples below.

Remark 3.2.2.8. The following question was brought to our attention by Theo Johnson-Freyd: Is the dimension of a connected rigid algebra a Morita invariant. More precisely, let A and B be two connected rigid algebras in a finite semisimple monoidal 2-category $\mathfrak{C}$ over an algebraically closed field. The associated 2-categories of right modules $\mathbf{Mod}_{\mathfrak{C}}(A)$, and $\mathbf{Mod}_{\mathfrak{C}}(B)$ are left $\mathfrak{C}$-module 2-categories by proposition 3.3.1 of [Déc21]. Then, the precise question is: If $\mathbf{Mod}_{\mathfrak{C}}(A) \simeq \mathbf{Mod}_{\mathfrak{C}}(B)$ as left $\mathfrak{C}$-module 2-categories, do we have $\mathrm{Dim}_{\mathfrak{C}}(A) = \mathrm{Dim}_{\mathfrak{C}}(B)$? In the case $\mathfrak{C} = \mathbf{2Vect}$, this is a standard result in the theory of fusion 1-categories (see [ENO05]). As evidence for the viability of this question, we will see in remark 4.3.1.8 below that, under additional assumptions on $\mathfrak{C}$, the non-vanishing of the dimension of a connected rigid algebra is Morita invariant.

Remark 3.2.2.9. For the sake of completeness, let us also comment on the case of non-connected rigid algebras. Let A be an algebra with unit i in a compact semisimple monoidal 2-category $\mathfrak{C}$. In particular, we may decompose $i = i_1 \oplus ... \oplus i_n$ into a direct sum of simple 1-morphisms in $\mathfrak{C}$, and this induces a decomposition of the algebra A as

$$\begin{pmatrix} {}_1A_1 & \cdots & {}_1A_n \\ \vdots & \ddots & \vdots \\ {}_nA_1 & \cdots & {}_nA_n \end{pmatrix},$$

where ${}_jA_j$ is a connected algebra with unit i_j, and ${}_jA_k$ is an ${}_jA_j$-${}_kA_k$-bimodule for every j, k. Now, if we assume that A is rigid, then it is easy to check that ${}_jA_j$ is a rigid algebra for every j. In particular, if $\mathbb{k}$ is algebraically closed and $\mathfrak{C} = \mathbf{2Vect}$, then this is precisely the well-known decomposition of a mutlifusion 1-category into a matrix of finite semisimple bimodule categories with fusion categories along the diagonal. Over an arbitrary compact semisimple 2-category, we expect that A is separable if and only if ${}_jA_j$ is separable for every j.

3.2.3 Examples

When working with fusion 1-categories over an algebraically closed field, there is a well-known notion of global (or categorical) dimension. We begin relating the notion of dimension we have just introduced with this classical notion.

Proposition 3.2.3.1. *Assume that $\mathbb{k}$ is algebraically closed, and let $\mathcal{C}$ be a fusion 1-category with simple unit I, i.e. a connected rigid algebra in* $\mathbf{2Vect}$*. Then, the dimension of $\mathcal{C}$ in the sense of definition 3.2.2.3 coincides with* $\dim(\mathcal{C})$*, its global dimension in the sense of definition 2.6.3 of [DSPS21].*

Proof. Instead of working with $\mathcal{C}$-$\mathcal{C}$-bimodules, we will equivalently work with left $\mathcal{C} \boxtimes \mathcal{C}^{mop}$-modules. As $\mathcal{C}$ is a fusion 1-category, so is $\mathcal{C} \boxtimes \mathcal{C}^{mop}$. We write I for the simple unit of $\mathcal{C} \boxtimes \mathcal{C}^{mop}$ and $\otimes$ for its monoidal product. By subsection 2.6 of [DSPS21], there exists a Frobenius algebra R in $\mathcal{C} \boxtimes \mathcal{C}^{mop}$ with multiplication $m : R \otimes R \to R$, unit $u : I \to R$, comultiplication $\Delta : R \to R \otimes R$, and counit $\lambda : R \to I$ such that $\mathcal{C}$ is equivalent to $Mod_{\mathcal{C}\boxtimes\mathcal{C}^{mop}}(R)$, the 1-category of right R-modules in $\mathcal{C} \boxtimes \mathcal{C}^{mop}$, as a left $\mathcal{C} \boxtimes \mathcal{C}^{mop}$-module category. By the construction given in [DSPS21], it follows that $End_R(R)$, the algebra of endomorphisms of R as a left R-module, is given by $\mathbb{k}$. Furthermore, we may assume that $\lambda \circ u = 1$, so that the R-R-bimodule map $m \circ \Delta : R \to R$ is given by multiplication by $\dim(\mathcal{C})$.

Now, let us also identify $\mathcal{C} \boxtimes \mathcal{C}$ with $Mod_{\mathcal{C}\boxtimes\mathcal{C}^{mop}}(I)$ as a left $\mathcal{C} \boxtimes \mathcal{C}^{mop}$-module 1-category. Under these identifications, the left $\mathcal{C} \boxtimes \mathcal{C}^{mop}$-module functor $\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ is given by $(-) \otimes R_R$. In particular, its right adjoint (as a $\mathcal{C} \boxtimes \mathcal{C}^{mop}$-module functor) is given by $(-) \otimes_R R_I : Mod_{\mathcal{C}\boxtimes\mathcal{C}^{mop}}(R) \to Mod_{\mathcal{C}\boxtimes\mathcal{C}^{mop}}(I)$. The unit for this adjunction is induced by the unit $u : I \to R$ of R, and the counit of this adjunction is induced by the multiplication $m : R \otimes R \to R$ viewed as a map of R-R-bimodules. The triangle identities follow from the fact that R is an algebra.

But, $(-) \otimes_R R_I$ is also the left adjoint of $(-) \otimes R_R$. Namely, Δ supplies a map of R-R-bimodules $R \to R \otimes R$, which gives us the desired unit, and λ provides us with the desired counit. These maps clearly satisfy the triangle identities. This means that the double right adjoint of $(-) \otimes R_R$ is $(-) \otimes R_R$ itself. Thus, we may take μ to be the natural isomorphism of left $\mathcal{C} \boxtimes \mathcal{C}^{mop}$-module induced by the identity map on R (viewed as a map of left R-modules). Using the above data, we find that $\mathrm{Tr}(\mu) = \dim(\mathcal{C})$ and $\mathrm{Tr}((\mu^{-1})^*) = 1$, where we have used $End_R(R) = \Bbbk$. Putting everything together, we find $\mathrm{Dim}(\mathcal{C}) = \dim(\mathcal{C})$ as desired. □

Remark 3.2.3.2. If $\Bbbk$ is algebraically closed, it follows from theorem 2.6.7 of [DSPS21] that the fusion 1-category $\mathcal{C}$ is separable as an algebra in **2Vect** if and only if it is separable in the sense of definition 2.5.8 of [DSPS21]. In particular, if we assume in addition that $\Bbbk$ has characteristic zero, then every fusion category has non-zero global dimension by [ENO05], and so is separable.

Proposition 3.2.3.3. *Let $\mathcal{B}$ be a perfect braided tensor 1-category with simple unit, and $\mathcal{C}$ be a separable $\mathcal{B}$-central tensor 1-category with simple unit. If $\mathcal{B}$ is a separable tensor 1-category, then $\mathcal{C}$ is separable as an algebra in* $\mathbf{Mod}(\mathcal{B})$ *if and only if $\mathcal{C}$ is a separable tensor 1-category.*

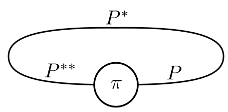
Proof. We begin by proving that $\mathcal{C}$ is perfect. In order to do so, it is enough to show that $\mathcal{C} \boxtimes \mathcal{C}$ is a finite semisimple 1-category. Now, $\mathcal{C}$ is a separable right $\mathcal{B}$-module category, thence it follows from the definition that there exists a separable algebra R in $\mathcal{B}$ such that $\mathcal{C} \simeq Mod_{\mathcal{B}}(R)$ as right $\mathcal{B}$-module 1-categories. By proposition 3.8 of [DSPS19], $\mathcal{C} \boxtimes \mathcal{C} \simeq Mod_{\mathcal{B}\boxtimes\mathcal{B}}(R \boxtimes R)$ is a separable right $\mathcal{B} \boxtimes \mathcal{B}$-module 1-category. But, $\mathcal{B} \boxtimes \mathcal{B}$ is finite semisimple, so we find that $\mathcal{C} \boxtimes \mathcal{C}$ is finite semisimple.

Let us denote by $P : \mathcal{B} \boxtimes \mathcal{B} \to \mathcal{B}$ the monoidal structure of $\mathcal{B}$, and $T : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ the monoidal structure of $\mathcal{C}$. As $\mathcal{C}$ is a $\mathcal{B}$-central tensor 1-category, we can factor the $\mathcal{C}$-$\mathcal{C}$-bimodule functor T as

$$\mathcal{C} \boxtimes \mathcal{C} \xrightarrow{Q} \mathcal{C} \boxtimes_{\mathcal{B}} \mathcal{C} \xrightarrow{\widetilde{T}} \mathcal{C}.$$

In fact, the canonical $\mathcal{C}$-$\mathcal{C}$-bimodule functor $Q : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C} \boxtimes_{\mathcal{B}} \mathcal{C}$ coincides with the functor obtained by applying $\mathcal{C} \boxtimes_{\mathcal{B}} (-) \boxtimes_{\mathcal{B}} \mathcal{C}$ to P.

Let us pick $\pi : P^{**} \cong P$ a natural isomorphism of $\mathcal{B}$-$\mathcal{B}$-bimodule functors. As $\mathcal{B}$ is separable, we may assume that

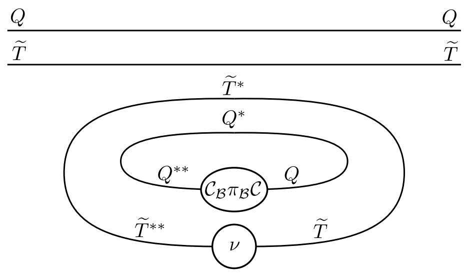


$$= 1$$

.

We also pick $\nu : \widetilde{T}^{**} \cong \widetilde{T}$ a natural isomorphism of $\mathcal{C}$-$\mathcal{C}$-bimodule functors. Then, $\mu := \nu \cdot (\mathcal{C} \boxtimes_{\mathcal{B}} \pi \boxtimes_{\mathcal{B}} \mathcal{C}) : T^{**} \cong T$ is a natural isomorphism of $\mathcal{C}$-$\mathcal{C}$-bimodule functors. Thus, $\mathrm{Tr}(\mu)$ is given by

$$\mathrm{Tr}(\mu) =$$



.

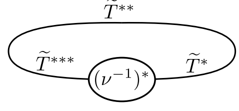
By construction, the inner loop of the last diagram above evaluates to 1, so we may remove it, and we find that $\mathrm{Tr}(\mu) = \mathrm{Tr}(\nu) \circ Q$. As precomposition by Q is faithful, we find that $\mathrm{Tr}(\mu)$ is non-zero if and only if $\mathrm{Tr}(\nu)$ is non-zero, which concludes the proof of the result. $\square$

Over an algebraically closed field, more can be said. More precisely, we can compute the dimension of most connected rigid algebras in a connected fusion 2-category.

Proposition 3.2.3.4. *Assume that $\Bbbk$ is an algebraically closed field. Let $\mathcal{B}$ be a braided fusion 1-category, and $\mathcal{C}$ be a separable $\mathcal{B}$-central fusion 1-category, then we have*

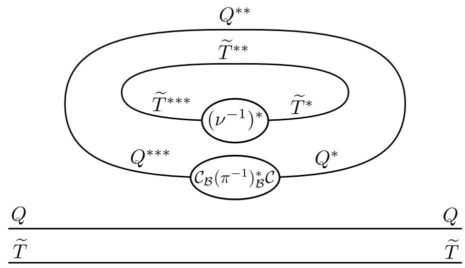
$$\dim(\mathcal{C}) = \mathrm{Dim}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \cdot \dim(\mathcal{B}).$$

Proof. We continue to use the notations introduced in the proof of proposition 3.2.3.3. In particular, we have picked $\pi : P^{**} \cong P$ a natural isomorphism of $\mathcal{B}$-$\mathcal{B}$-bimodule functors. If $\dim(\mathcal{B}) = 0$, then $\mathrm{Tr}(\pi) = 0$, so that $\mathrm{Tr}(\mu) = 0$, and therefore $\dim(\mathcal{C}) = 0$ as we are working over an algebraically closed field. Thus, we may assume that $\dim(\mathcal{B})$ is non-zero, and choose π such that $\mathrm{Tr}(\mu) = \mathrm{Tr}(\nu) \circ Q$ as in the proof of the previous proposition. Further, it follows from remark 3.2.2.5 that the diagram



viewed in $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$ corresponds to a scalar $\lambda \in \Bbbk$ times the projection onto a simple summands of the identity $\mathcal{C}$-$\mathcal{C}$-bimodule 1-morphism on $\mathcal{C} \boxtimes_{\mathcal{B}} \mathcal{C}$. If λ is zero, then both $\dim(\mathcal{C})$ and $\mathrm{Dim}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$ are zero, so we are done. We will therefore assume that λ is non-zero. In fact, it follows from remark 3.2.2.5 that we can chose ν such that $\lambda = 1$. This implies that $\mathrm{Tr}(\mu) = \mathrm{Dim}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$. Moreover, we also find that

$$\mathrm{Tr}((\mu^{-1})^*) =$$



$$= \dim(\mathcal{B}).$$

Namely, the inner loops evaluate to 1, as $\lambda = 1$. Finally, we find that

$$\dim(\mathcal{C}) = \mathrm{Tr}(\mu) \cdot \mathrm{Tr}((\mu^{-1})^*) = \mathrm{Dim}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \cdot \dim(\mathcal{B}),$$

as desired. $\square$

Remark 3.2.3.5. Let $\mathcal{B}$ be any braided finite semisimple tensor 1-category, then there exists connected separable algebras in $\mathbf{Mod}(\mathcal{B})$. For instance, $\mathcal{B}$ viewed as a $\mathcal{B}$-central tensor 1-category is such a connected separable algebra.

Remark 3.2.3.6. Let us assume that $\Bbbk$ has characteristic zero, and let $\mathcal{B}$ be a finite semisimple braided tensor 1-category. We claim that every finite semisimple $\mathcal{B}$-central tensor 1-category $\mathcal{C}$ is separable. We use $\mathcal{C}^{mop}$ to denote the multifusion 1-category obtained from $\mathcal{C}$ by reversing the order of the monoidal product. Note that there is a canonical $\mathcal{B}$-central structure on $\mathcal{C}^{mop}$. Then, inspection shows that there is an equivalence

$$\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \simeq \mathbf{Mod}(\mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C})$$

of 2-categories. Moreover, it follows from theorem 6.2 of [Gre10] that $\mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C}$ is a rigid monoidal 1-category. Further, it is finite semisimple by corollaries 2.5.11 and 2.6.8 of [DSPS21]. This shows that $\mathcal{C}$ is indeed separable as an algebra in $\mathcal{B}$.

The 2-category $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$ is the 2-category of finite semisimple $\mathcal{B}$-balanced $\mathcal{C}$-$\mathcal{C}$-bimodule 1-categories in the sense of definition 3.24 [Lau20]. Namely, the author assumes that $\mathcal{C}$ is a $\mathcal{B}$-augmented monoidal 1-category as in definition 3.12 therein, but only the underlying $\mathcal{B}$-central structure on $\mathcal{C}$ is relevant for the definition of $\mathcal{B}$-balanced $\mathcal{C}$-$\mathcal{C}$-bimodule 1-categories. Similarly, let us also point out that the equivalence of 2-categories given above is essentially theorem 3.27 of [Lau20].

Now, let $\Bbbk$ be algebraically closed, and let $F : \mathfrak{C} \to \mathfrak{D}$ be a monoidal linear 2-functor between finite semisimple monoidal 2-categories. As F is monoidal and right adjoints are preserved by 2-functors, it follows that $F(A)$ is a rigid algebra in $\mathfrak{D}$. If we assume that $F(A)$ is connected, then it is natural to ask how the dimensions of A and $F(A)$ compare. More precisely, F induces a map of $\Bbbk$-algebras from $End_{A-A}(m) \cong \Bbbk$ to $End_{F(A)-F(A)}(F(m)) \cong \Bbbk$, which is an isomorphism.

Lemma 3.2.3.7. *Assume that $\Bbbk$ is algebraically closed. Let $F : \mathfrak{C} \to \mathfrak{D}$ be a monoidal linear 2-functor between finite semisimple monoidal 2-categories, and A a connected rigid algebra in $\mathfrak{C}$ such that $F(A)$ is connected. Then, we have* $\mathrm{Dim}_{\mathfrak{C}}(A) = \mathrm{Dim}_{\mathfrak{D}}(F(A))$ *in* $\Bbbk$.

Proof. This follows from an explicit computation. □

Corollary 3.2.3.8. *Let G be a finite group, and $\Bbbk$ an algebraically closed field of characteristic zero. Then, every connected rigid algebra in $\mathbf{2Vect}_G$ is separable.*

Corollary 3.2.3.9. *In the notation of example 2.3.4.2, the connected algebra $\mathbf{Vect}_H^{\gamma}$ in $\mathbf{2Vect}_G^{\omega}$ is rigid. Further, if $\Bbbk$ is algebraically closed, it has dimension $|H|$.*

Proof. As γ trivializes $\omega|_H$, we can use it to construct a monoidal linear 2-functor $F : \mathbf{2Vect}_H \hookrightarrow \mathbf{2Vect}_G^{\omega}$, which induces the inclusion $H \subseteq G$ on equivalence classes of simple objects. In particular, we have $F(\mathbf{Vect}_H) = \mathbf{Vect}_H^{\gamma}$. But $\mathbf{Vect}_H$ is clearly rigid as a monoidal 1-category, so that it is a connected rigid algebra in $\mathbf{2Vect}_H$ by example 3.1.1.3. Now, let us assume that $\Bbbk$ is algebraically closed, and write $U : \mathbf{2Vect}_H \to \mathbf{2Vect}$ for the monoidal linear 2-functor forgetting the grading. The rigid algebra $U(\mathbf{Vect}_H^{\kappa})$ in $\mathbf{2Vect}$ is exactly the tensor category of finite H-graded $\Bbbk$-vector spaces, which has dimension $|H|$. This proves the result. □

Corollary 3.2.3.10. *If $\Bbbk$ is algebraically closed, and char$(\Bbbk) \nmid |H|$, then the 2-category of $\mathbf{Vect}_H^{\gamma}$-$\mathbf{Vect}_H^{\gamma}$-bimodules in $\mathbf{2Vect}_G^{\omega}$ is compact semisimple.*

3.3 Diagrams

3.3.1 Proof of Lemma 3.1.2.1

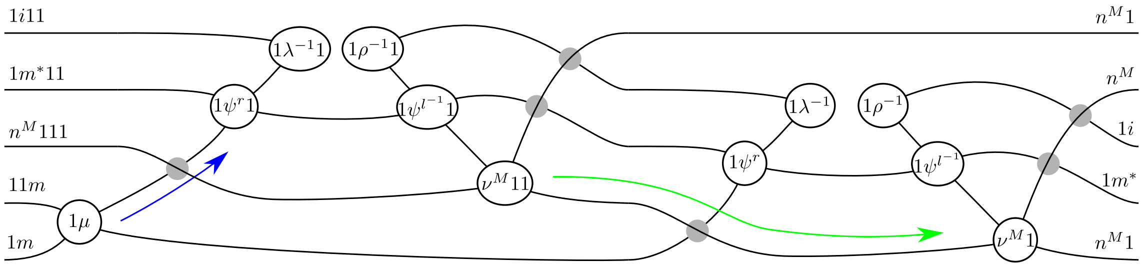


Figure 3.1: Axiom a (Part 1)

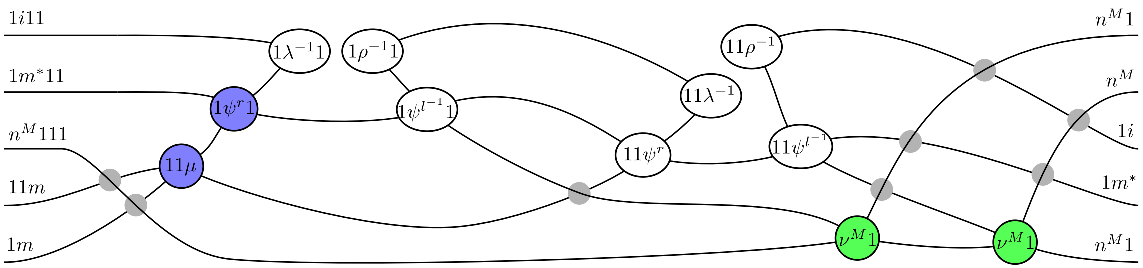


Figure 3.2: Axiom a (Part 2)

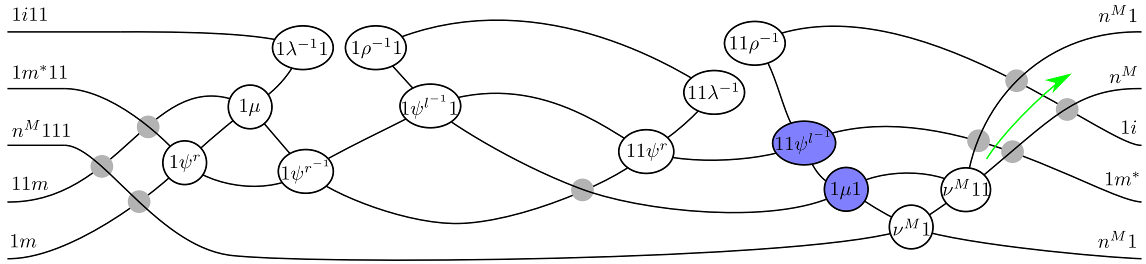


Figure 3.3: Axiom a (Part 3)

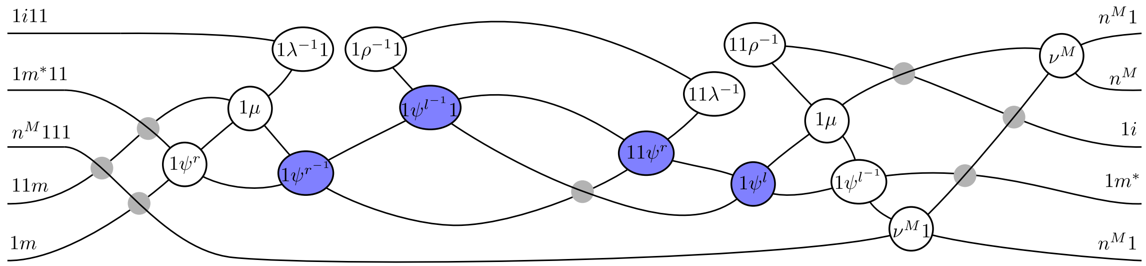


Figure 3.4: Axiom a (Part 4)

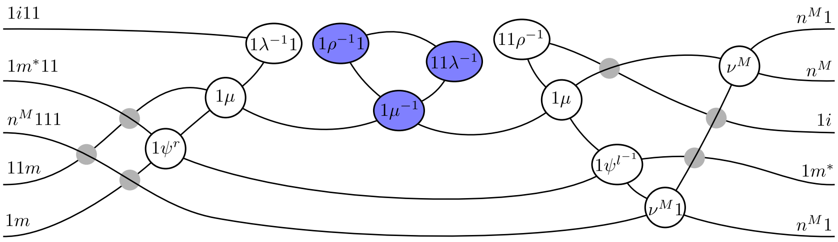


Figure 3.5: Axiom a (Part 5)

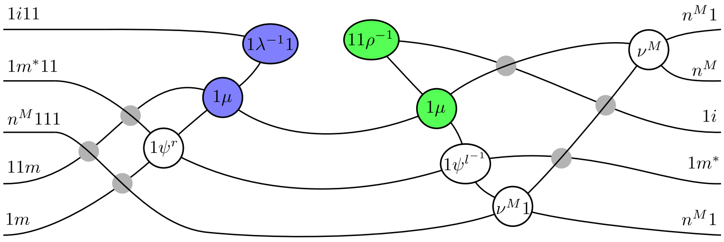


Figure 3.6: Axiom a (Part 6)

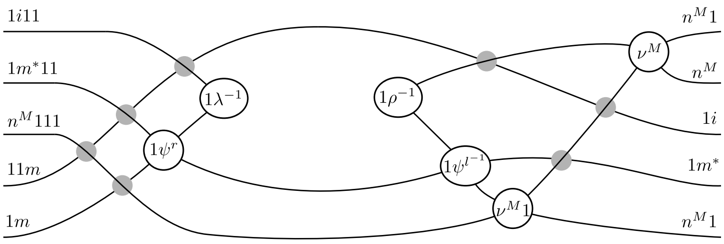


Figure 3.7: Axiom a (Part 7)

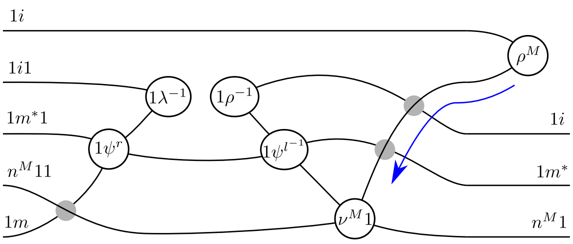


Figure 3.8: Axiom b (Part 1)

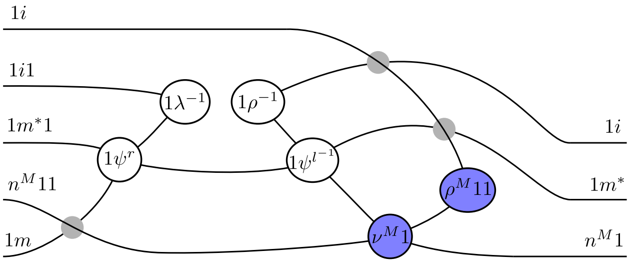


Figure 3.9: Axiom b (Part 2)

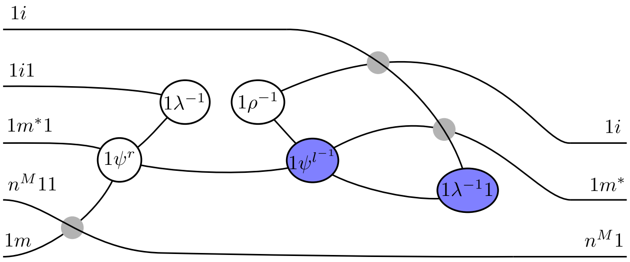


Figure 3.10: Axiom b (Part 3)

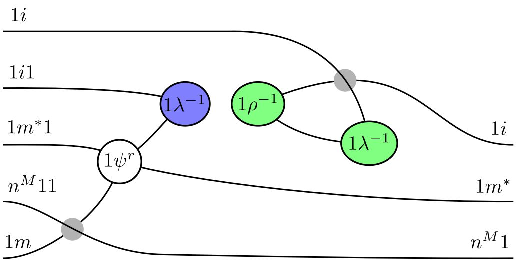


Figure 3.11: Axiom b (Part 4)

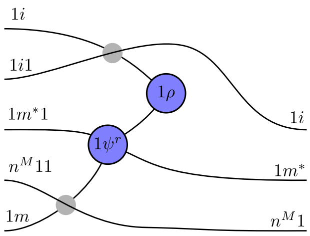


Figure 3.12: Axiom b (Part 5)

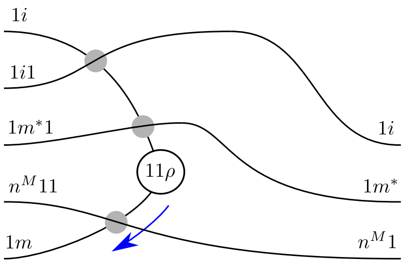


Figure 3.13: Axiom b (Part 6)

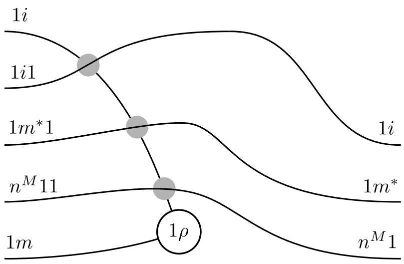


Figure 3.14: Axiom b (Part 7)

3.3.2 Proof of Lemma 3.1.2.2

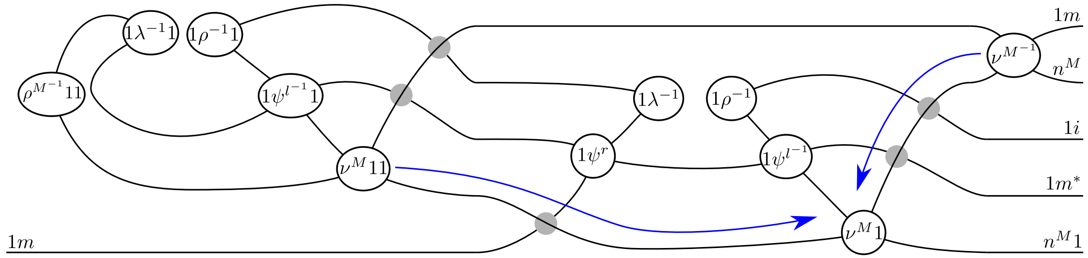


Figure 3.15: Module 2-Morphism (Part 1)

$\substack{\infty \\ \infty}$

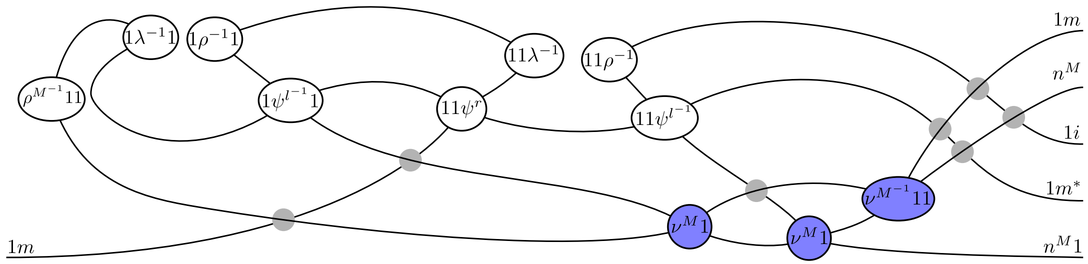


Figure 3.16: Module 2-Morphism (Part 2)

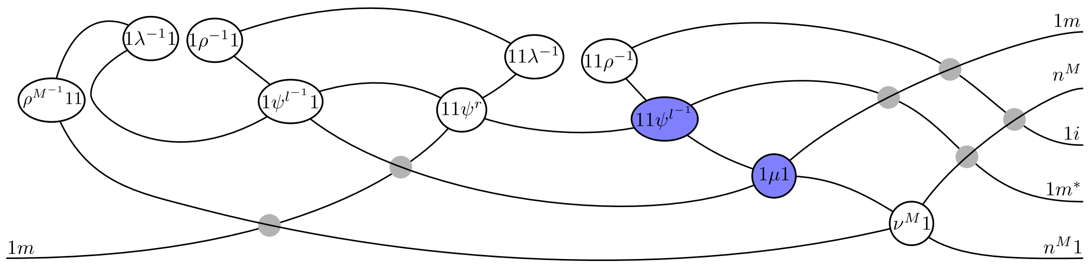


Figure 3.17: Module 2-Morphism (Part 3)

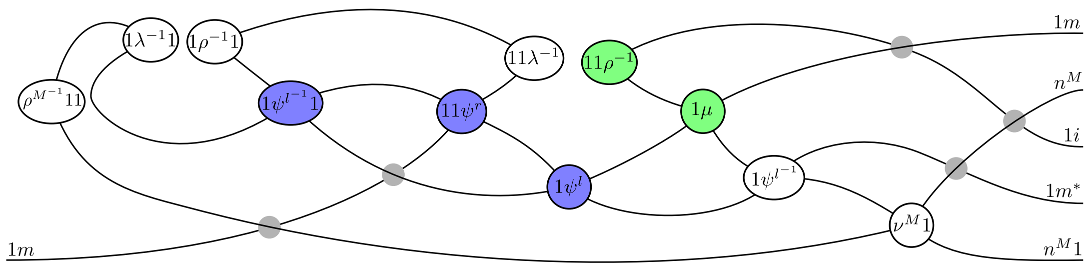


Figure 3.18: Module 2-Morphism (Part 4)

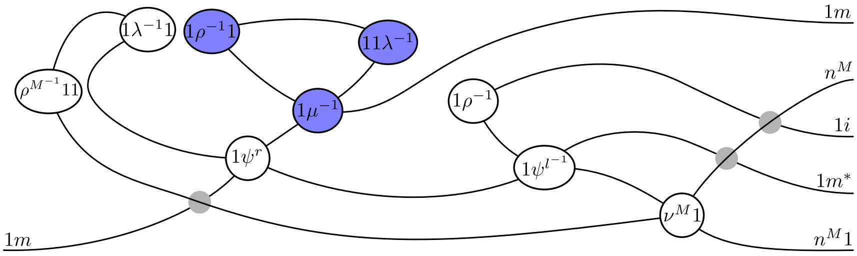


Figure 3.19: Module 2-Morphism (Part 5)

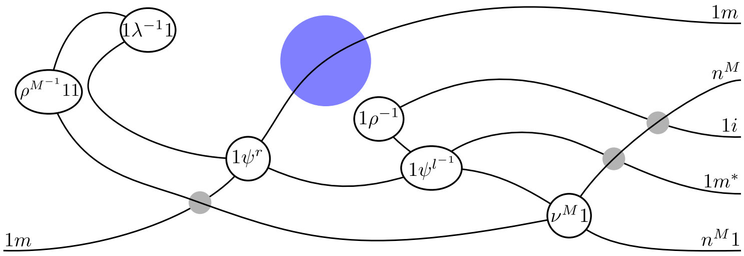


Figure 3.20: Module 2-Morphism (Part 6)

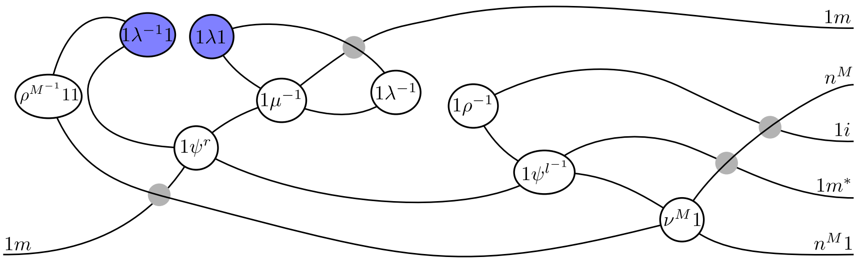


Figure 3.21: Module 2-Morphism (Part 7)

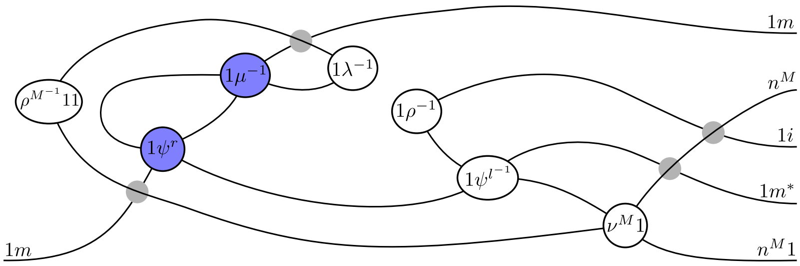


Figure 3.22: Module 2-Morphism (Part 8)

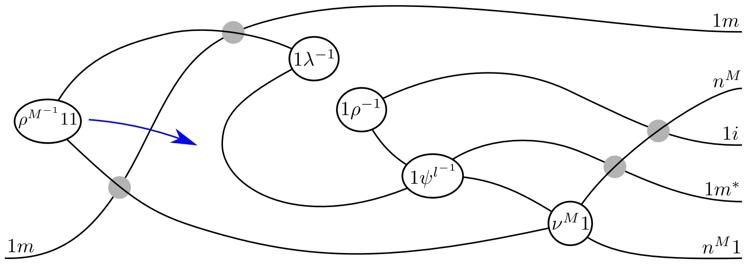


Figure 3.23: Module 2-Morphism (Part 9)

3.3.3 Proof of Lemma 3.1.2.3

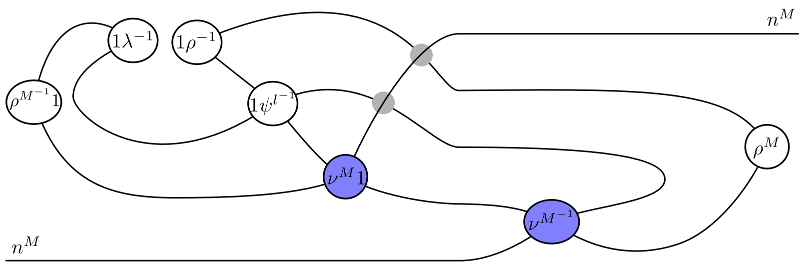


Figure 3.24: Triangle 1 (Part 1)

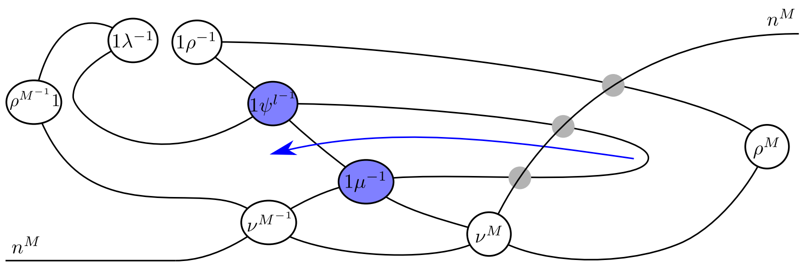


Figure 3.25: Triangle 1 (Part 2)

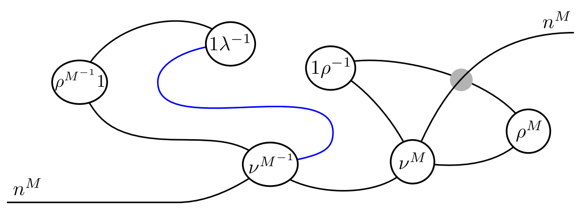


Figure 3.26: Triangle 1 (Part 3)

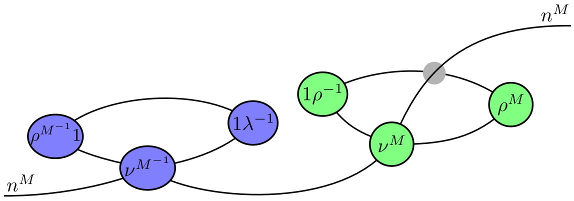


Figure 3.27: Triangle 1 (Part 4)

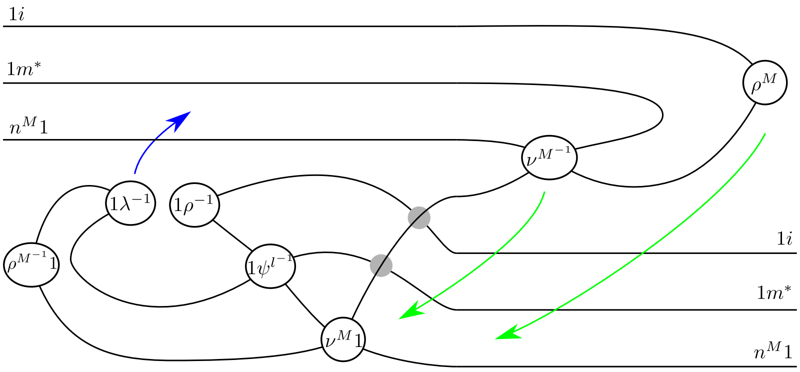


Figure 3.28: Triangle 2 (Part 1)

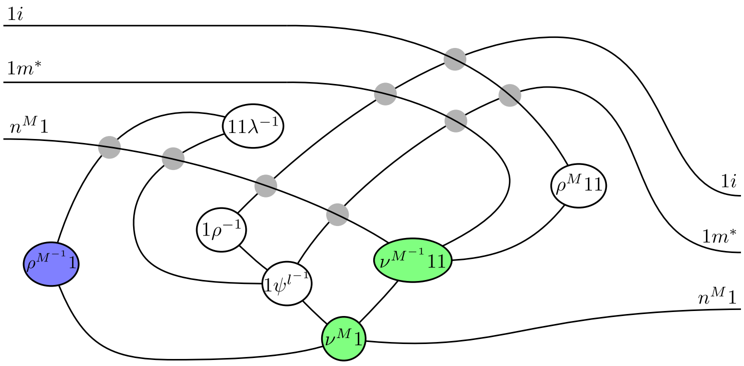


Figure 3.29: Triangle 2 (Part 2)

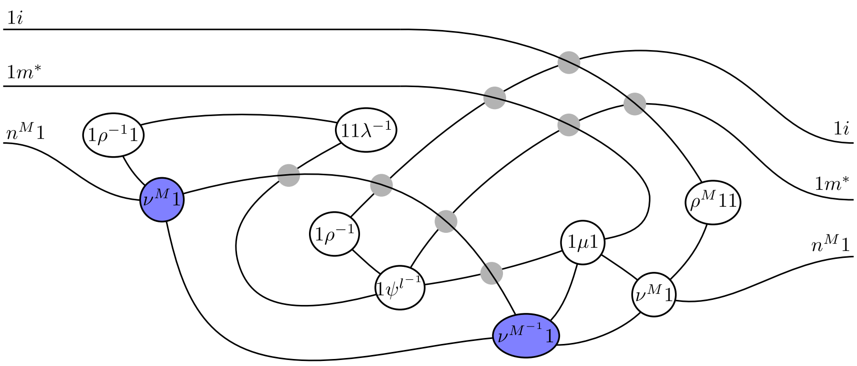


Figure 3.30: Triangle 2 (Part 3)

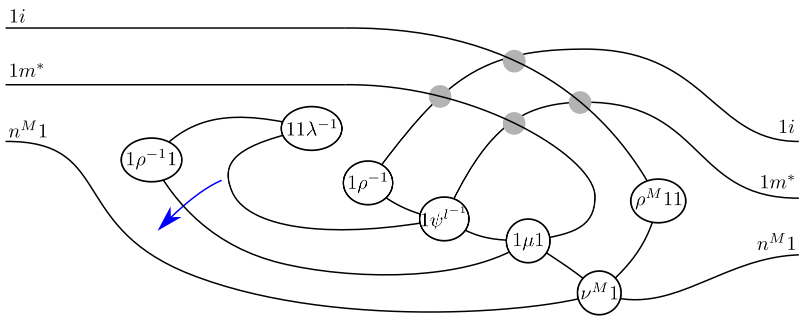


Figure 3.31: Triangle 2 (Part 4)

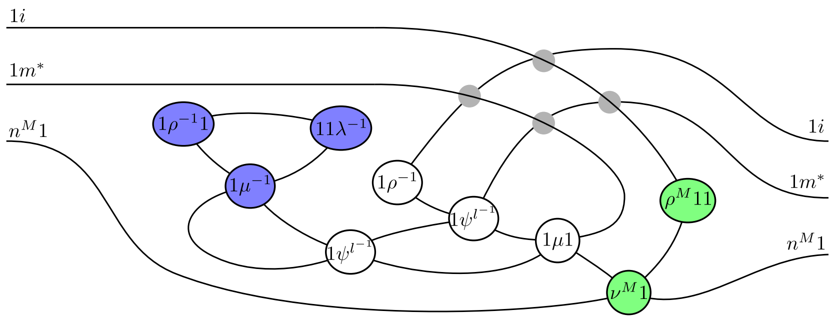


Figure 3.32: Triangle 2 (Part 5)

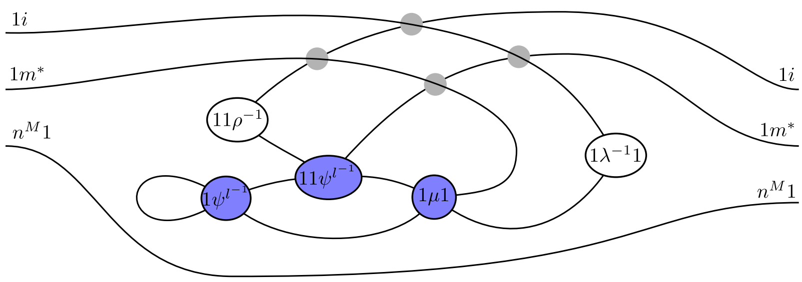


Figure 3.33: Triangle 2 (Part 6)

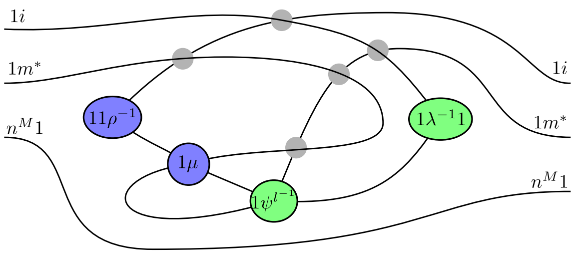


Figure 3.34: Triangle 2 (Part 7)

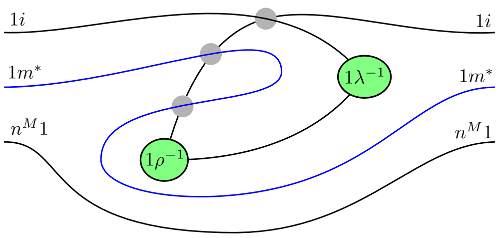


Figure 3.35: Triangle 2 (Part 8)

3.3.4 Proof of Proposition 3.1.2.5

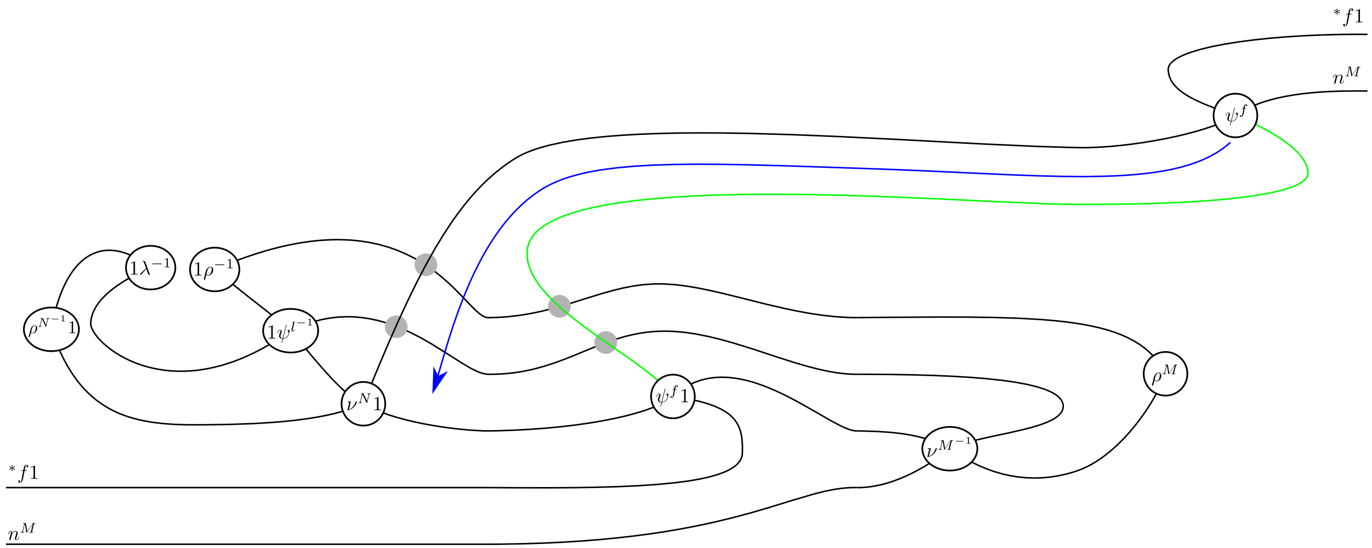


Figure 3.36: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part I.1)

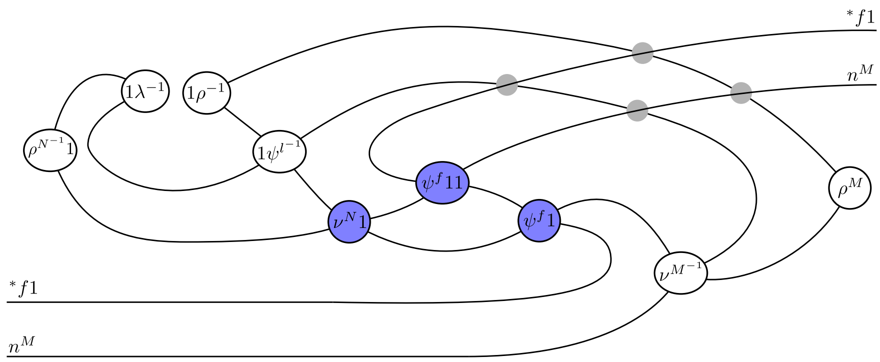


Figure 3.37: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part I.2)

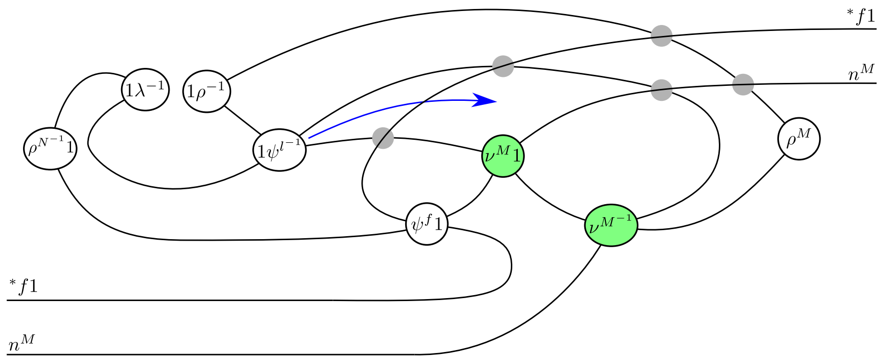


Figure 3.38: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part I.3)

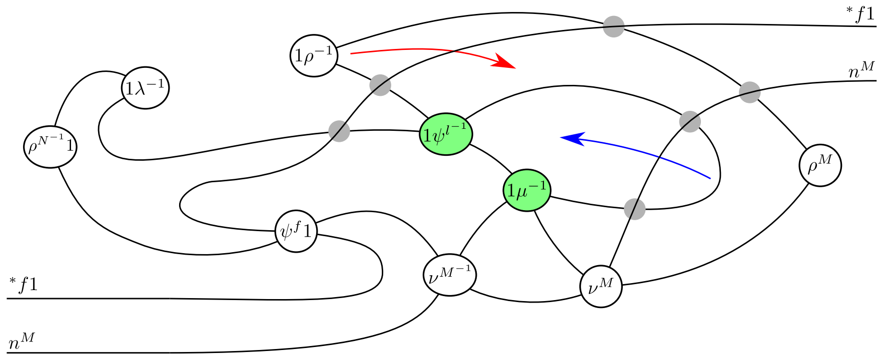


Figure 3.39: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part I.4)

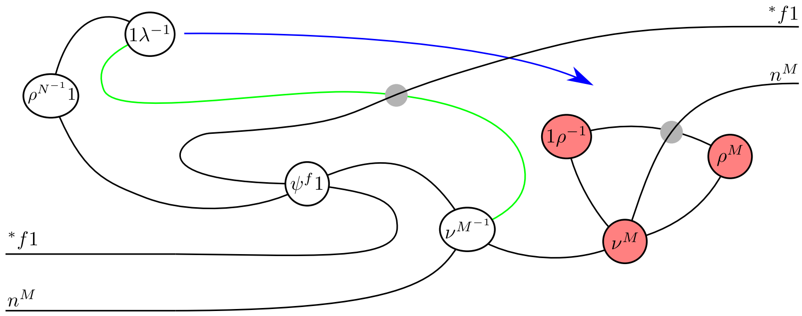


Figure 3.40: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part I.5)

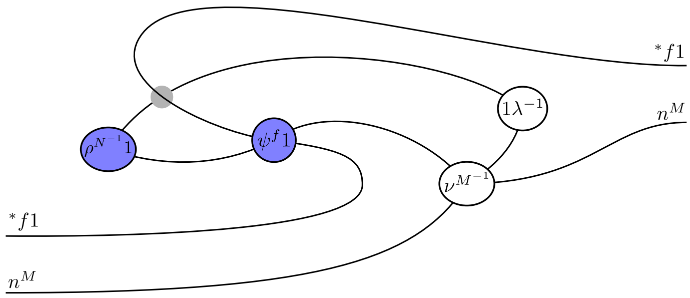


Figure 3.41: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part I.6)

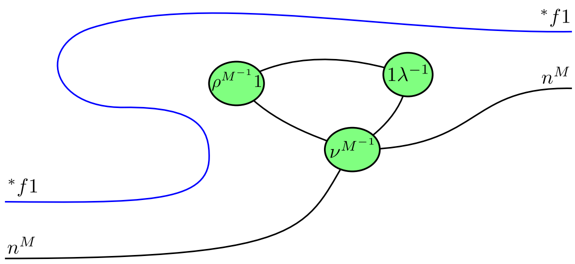


Figure 3.42: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part I.7)

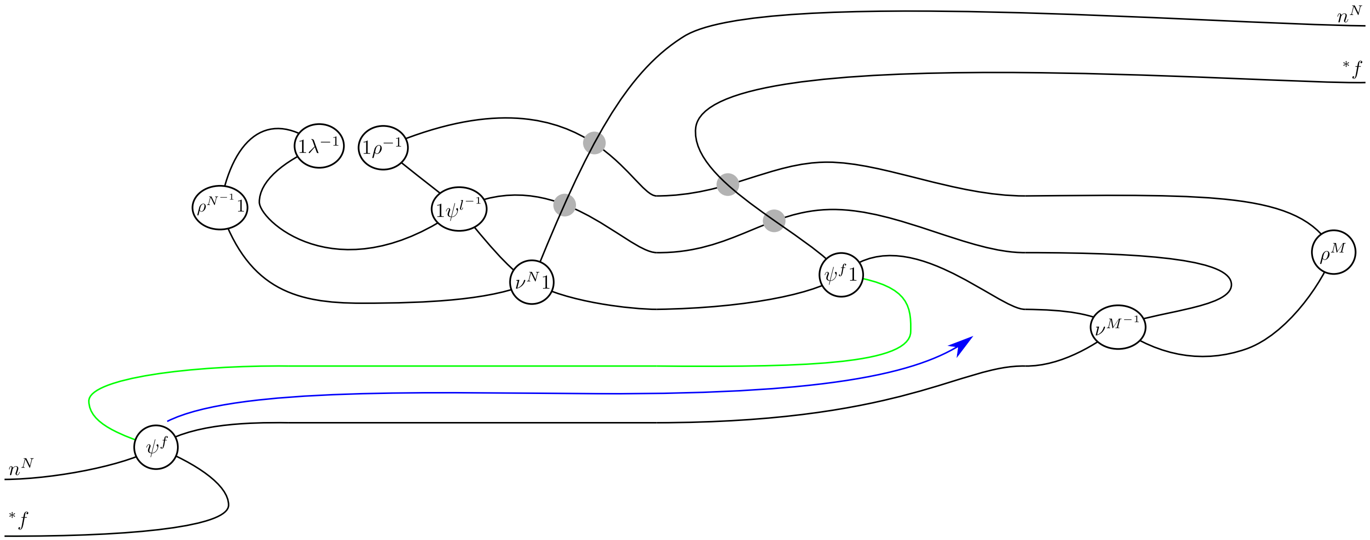


Figure 3.43: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part II.1)

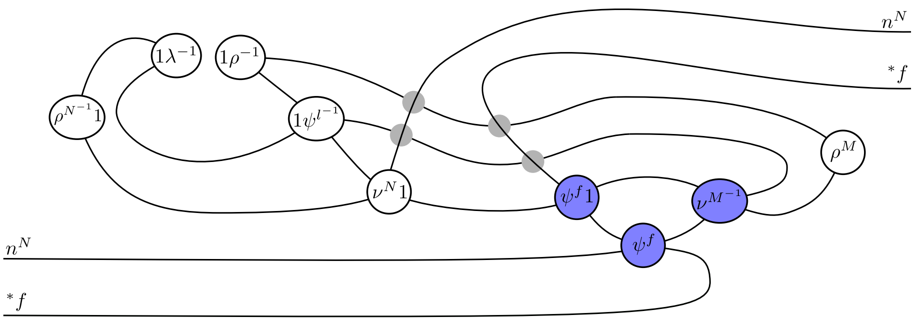


Figure 3.44: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part II.2)

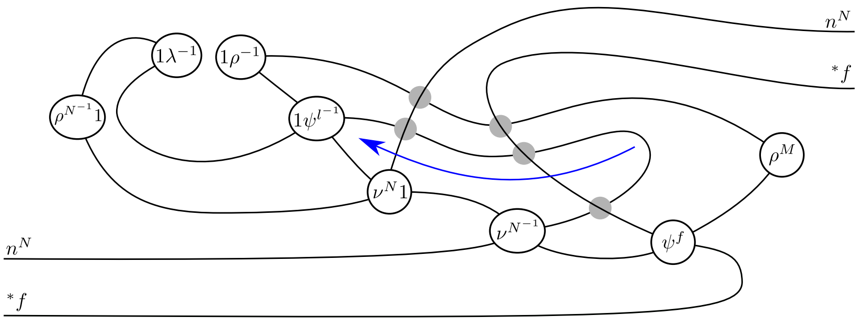


Figure 3.45: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part II.3)

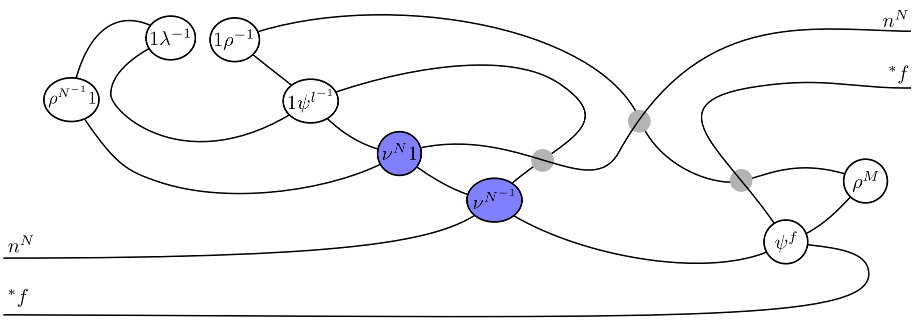


Figure 3.46: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part II.4)

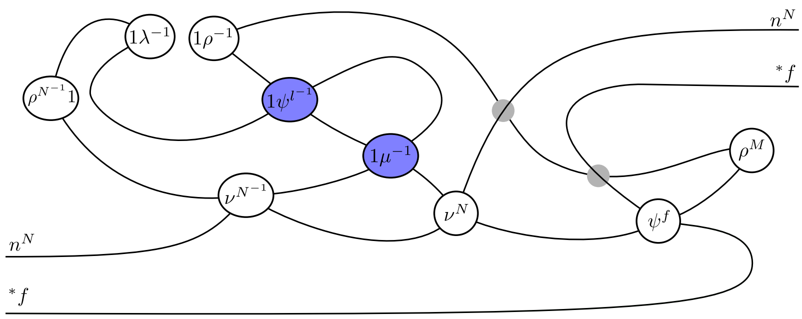


Figure 3.47: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part II.5)

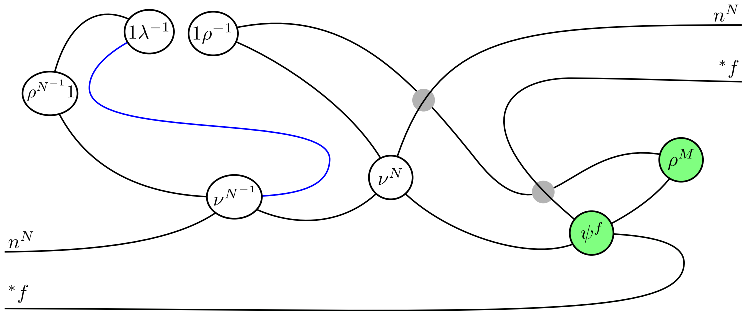


Figure 3.48: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part II.6)

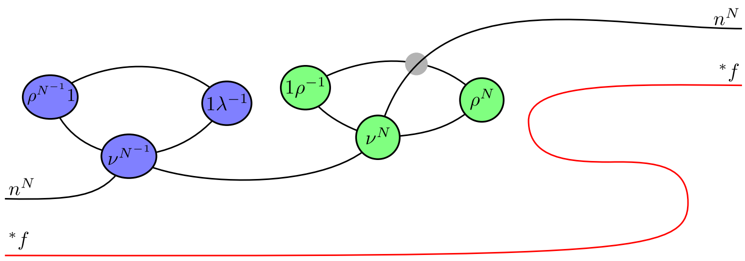


Figure 3.49: The 2-morphisms ψ^{*f} and ξ^f are inverses. (Part II.7)

Chapter 4

The Morita Theory of Fusion 2-Categories

4.1 The Relative Tensor Product over Separable Algebras

Throughout this section, we work with a fixed monoidal 2-category $\mathfrak{C}$, which we assume to be strict cubical without loss of generality. Our first goal is to explain the 2-universal property of the relative tensor product of a right and a left module over an arbitrary algebra A. We then prove that if $\mathfrak{C}$ is Karoubi complete and A is separable, then the relative tensor product over A always exists. Using this fact, we construct the Morita 3-category of separable algebras, bimodules, and their morphisms in $\mathfrak{C}$.

4.1.1 Definition and Existence

Let A be an algebra in $\mathfrak{C}$. We fix M a right A-module in $\mathfrak{C}$, N a left A-module in $\mathfrak{C}$. We begin by defining A-balanced morphisms out of the pair (M, N).

Definition 4.1.1.1. Let C be an object of $\mathfrak{C}$. An A-balanced 1-morphism $(M, N) \to C$ consists of:

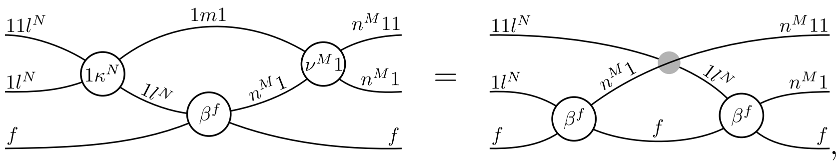
1. A 1-morphism $f : M\Box N \to C$ in $\mathfrak{C}$;

2. A 2-isomorphisms

$$\begin{tikzcd} MAN \arrow[r, "n^M 1"] \arrow[d, "1l^N"'] \arrow[dr, phantom, "\overset{\beta^f}{\Nearrow}"] & MN \arrow[d, "f"] \\ MN \arrow[r, "f"'] & C, \end{tikzcd}$$

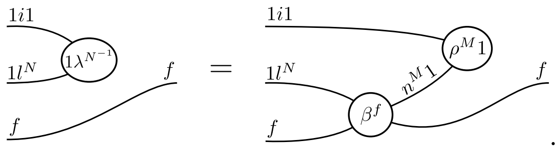
satisfying:

a. We have:



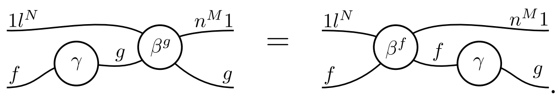
(4.1)

b. We have:



(4.2)

Definition 4.1.1.2. Let C be an object of $\mathfrak{C}$, and $f, g : (M, N) \to C$ be two A-balanced 1-morphisms. An A-balanced 2-morphism $f \Rightarrow g$ is a 2-morphism $\gamma : f \Rightarrow g$ in $\mathfrak{C}$ such that



Using the above definitions of A-balanced morphisms, we can give the definition of the relative tensor product over A.

Definition 4.1.1.3. The relative tensor product of M and N over A, if it exists, is an object $M\Box_A N$ of $\mathfrak{C}$ together with an A-balanced 1-morphism $t_A : (M, N) \to M\Box_A N$ satisfying the following 2-universal property:

1. For every A-balanced 1-morphism $f : (M, N) \to C$, there exists a 1-morphism $\widetilde{f} : M\Box_A N \to C$ in $\mathfrak{C}$ and an A-balanced 2-isomorphism $\xi : \widetilde{f} \circ t_A \cong f$.

2. For any 1-morphisms $g, h : M\Box_A N \to C$ in $\mathfrak{C}$, and any A-balanced 2-morphism $\gamma : g \circ t_A \Rightarrow h \circ t_A$, there exists a unique 2-morphism $\zeta : g \Rightarrow h$ such that $\zeta \circ t_A = \gamma$.

Remark 4.1.1.4. Observe that, for any object C in $\mathfrak{C}$, A-balanced 1-morphisms and 2-morphisms out of (M, N) form a 1-category, which we denote by $Bal_A(M, N; C)$. Furthermore, this assignment is functorial in M, N, and C. Definition 4.1.1.3 may be rephrased as asserting that precomposition with t_A induces an equivalence of 1-categories

$$Hom_{\mathfrak{C}}(M\Box_A N, C) \simeq Bal_A(M, N; C),$$

which is natural in the object C in $\mathfrak{C}$. Let us also note that it follows readily from the definition that the 2-category of relative tensor products $M\Box_A N$ is either empty or a contractible 2-groupoid.

Remark 4.1.1.5. Over an algebraically closed field, with $\mathfrak{C} = \mathbf{2Vect}$, and $\mathcal{C}$ a multifusion 1-category, then definition 4.1.1.3 recovers the relative tensor product over $\mathcal{C}$ given in definition 3.3 of [ENO10]. As $\mathcal{C}$ is automatically separable in this case, theorem 4.1.1.6 below recovers the well-known statement that the relative tensor product of two finite semisimple $\mathcal{C}$-module 1-categories exists and is a finite semisimple 1-category. Other particular cases of definition 4.1.1.3 have already appeared in definition 3.2 [DSPS19] and definition 3.3 of [BZBJ18].

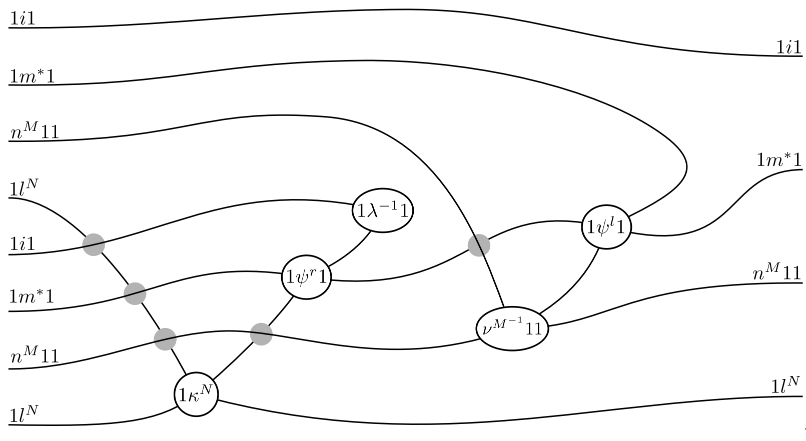
Theorem 4.1.1.6. *Let A be a separable algebra in a Karoubi complete monoidal 2-category $\mathfrak{C}$. Then, the relative tensor product of any right A-module M and any left A-module N in $\mathfrak{C}$ exists, and is given by the splitting of a 2-condensation monad on $M\Box N$.*

Proof. Let us consider the 2-condensation monad $(M\Box N, e, \mu, \delta)$ in $\mathfrak{C}$ given by

$$e := \big(M\Box l^N\big) \circ \big(n^M \Box A \Box N\big) \circ \big(M\Box(m^* \circ i)\Box N\big),$$

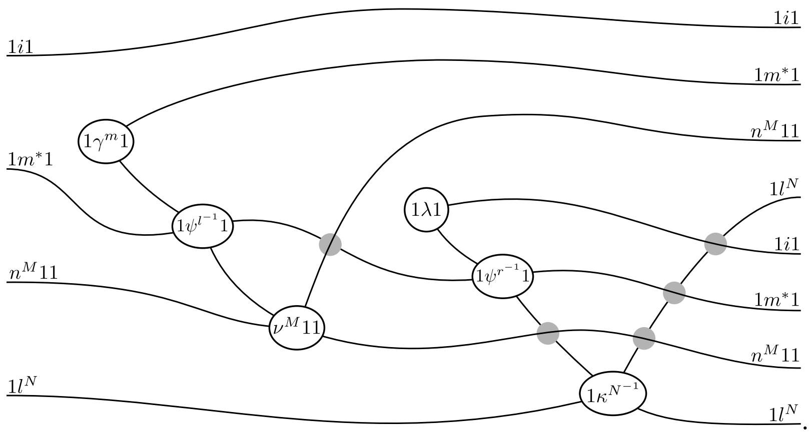
and

$\mu :=$



,

$\delta :=$

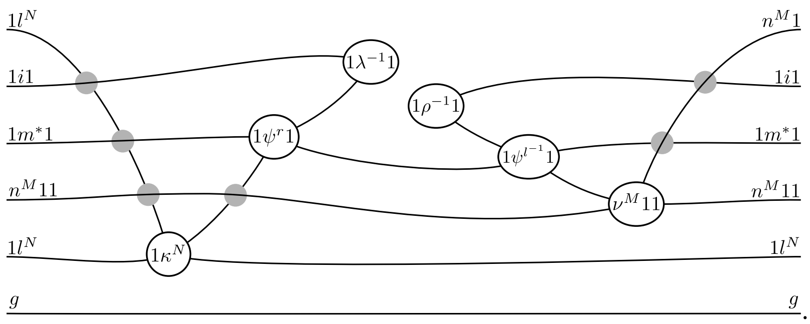


Clearly, $\mu \cdot \delta = Id_e$. We now prove that μ is associative using the diagrams depicted in section 4.4.1. Figure 4.1 depicts the composite $\mu \cdot (e \circ \mu)$. We begin by moving the two indicated coupons labeled $1\kappa^N$ and $\nu^{M^{-1}}11$ to the left along the corresponding arrows, which brings us to figure 4.2. We then using equation (1.10) on the blue coupons and equation (1.5) on the green coupons to arrive at figure 4.3. We go on by moving the coupon labeled $11\kappa^N$ up, as well as the coupons labeled $\nu^{M^{-1}}1111$, $\nu^{M^{-1}}111$, and $1\mu^{-1}111$ to the left along the green arrow. Having arrived at figure 4.4, we move the coupon labeled $\nu^{M^{-1}}11111$ up, and that labelled $\nu^{M^{-1}}1111$ down. Further, we also move the left most cap along the red arrow, and in doing so, use equations (3.7) and (3.6), which brings us to figure 4.5. Now, we use equations (3.5) on the blue coupons to get to figure 4.6. We then move the coupon labeled $1\mu^{-1}1111$ to the right, as well as the coupon labeled $11\mu1$ up in order to apply equation (3.3) to the green coupons, and use equation (3.1) on the red coupons, bringing us to figure 4.7. We can then make use of equation (1.3) on the blue coupons, and cancel the green coupons to arrive at figure 4.8. Finally, reorganising the diagram along the depicted arrows leads us to figure 4.9, which represents $\mu \cdot (\mu \circ e)$. Thence, we have established the associativity of μ as desired. The coassociativity of δ can be proven similarly.

Let us now move on to proving that $(\mu \circ e) \cdot (e \circ \delta) = \delta \cdot \mu$ using diagrams depicted in section 4.4.1. Figure 4.10 depicts the left hand-side of this equality. By moving the coupons labeled $1\kappa^{N^{-1}}$ and $\nu^M 11$ to the right, we arrive at figure 4.11. Then, applying equation (1.10) to the blue coupons, and equation (1.5) to the green ones, we get to contemplate figure 4.12. We proceed to move some coupons along the depicted arrows, and use equation (3.3) on the blue coupons, and equation (3.1) on the green coupons, which brings us to figure 4.13. Using equation (1.3) on the blue coupons, and moving the coupons labeled $1\psi^{r^{-1}}1$ and $1\kappa^{N^{-1}}$ to the right yields the diagram given in figure 4.14. Then, we first apply equation (3.5) to the blue coupons, and then equation (1.1) on the green coupon together with the coupon labeled $1\mu 1$, which was just created. This brings us to figure 4.15. Finally, using in succession equation (3.7) on the blue coupons, equation (3.6) on the green coupons, and equation (1.3) on the red coupons, leads us to figure 4.16, which depicts $\delta \cdot \mu$. This proves the desired equality. The equality $(e \circ \mu) \cdot (\delta \circ e) = \delta \cdot \mu$ can be proven using a similar argument.

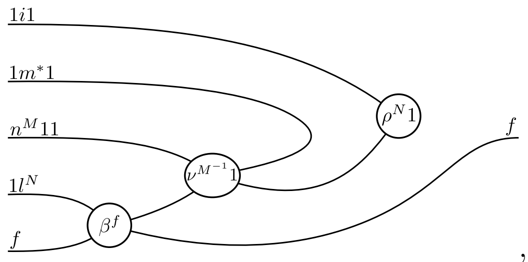
In order to prove that the relative tensor product of M and N over A exists, we will use the reformulation given in remark 4.1.1.4. To this end, recall that 2-condensation monads are preserved by all 2-functors, so that applying $Hom_{\mathfrak{C}}(-, C)$ to $(M\Box N, e, \mu, \delta)$ yields a 2-condensation monad on the 1-category $Hom_{\mathfrak{C}}(M\Box N, C)$. In fact, this yields a 2-condensation monad on the 2-functor $Hom_{\mathfrak{C}}(M\Box N, -)$. We claim that $Bal_A(M, N; C)$ is a splitting this 2-condensation monad. Namely, to see this, let $U : Bal_A(M, N; C) \to Hom_{\mathfrak{C}}(M\Box N, C)$ be the forgetful functor, and let $E : Hom_{\mathfrak{C}}(M\Box N, C) \to Bal_A(M, N; C)$ be the functor given by $g \mapsto g \circ e$, with A-balanced structure on the composite $g \circ e$ supplied by the 2-isomorphism $\beta^{g\circ e}$ given by

$\beta^{g\circ e} :=$



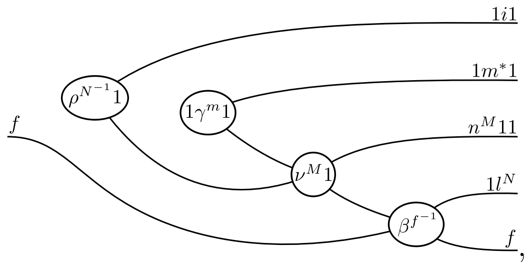
The fact that this defines an A-balanced structure can be seen as follows. Let us start with the right hand-side of equation (4.1) for $\beta^{g\circ e}$. We begin by applying equation (3.5) after having moved some coupons, then we use equation (1.2). We continue by appealing to equations (1.5) and (1.10), followed by (3.3) and (3.1). At last, we can use equations (1.3) and (1.7) for A as well as reorganise the string diagram to get to the left hand-side of (4.1). Equation (4.2) for $\beta^{g\circ e}$ follows similarly. Now, observe that both U and E are 2-natural in C. Further, let us define natural transformations $p : E \circ U \Rightarrow Id$ and $s : Id \Rightarrow E \circ U$ by

$p_f :=$



,

$s_f :=$



,

for every A-balanced 1-morphism $f : (M, N) \to C$. Again, note that s and p are 2-natural in C. Further, we have $p \cdot s = Id$, so that

$$(Hom_{\mathfrak{C}}(M\Box N, -), Bal_A(M, N; -), E, U, p, s)$$

is a 2-condensation. It remains to check that this splits the 2-condensation monad on $Hom_{\mathfrak{C}}(M\Box N, -)$ induced by $(M\Box N, e, \mu, \delta)$. To see this, it is enough to prove that for every 1-morphism $g : M\Box N \to C$ in $\mathfrak{C}$, we have $p_{g\circ e} = g \cdot \mu$ and $s_{g\circ e} = g \cdot \delta$. The first equality follows by applying equations (1.5), (3.1), followed by equation (1.7) for A, and then by using successively equations (1.6), (3.2), and (1.4). The second equality is obtained in a similar fashion.

Finally, as $\mathfrak{C}$ is Karoubi complete, the 2-condensation monad $(M\Box N, e, \mu, \delta)$ admits a splitting in $\mathfrak{C}$, which we denote by $M\Box_A N$. Now, the splitting of a 2-condensation monad is preserved by any 2-functor, so that $Hom_{\mathfrak{C}}(M\Box_A N, -)$ is also a splitting of the 2-condensation monad on $Hom_{\mathfrak{C}}(M\Box N, -)$ induced by $(M\Box N, e, \mu, \delta)$. But, the 2-category of splittings of a 2-condensation monad is a contractible 2-groupoid, so that we get the desired equivalence. $\Box$

Remark 4.1.1.7. In the language of [CMV02], the 1-category $Bal_A(M, N; C)$ is the pseudo-coequalizer for the descent object

$$\begin{tikzcd} Hom_{\mathfrak{C}}(MN, C) \arrow[r, shift left=2] \arrow[r, shift right=2] & Hom_{\mathfrak{C}}(MAN, C) \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Hom_{\mathfrak{C}}(MAAN, C) \end{tikzcd}$$

obtained by applying $Hom_{\mathfrak{C}}(-, C)$ to the canonical codescent object

$$\begin{tikzcd} M\Box A\Box A\Box N \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & M\Box A\Box N \arrow[r, shift left=2] \arrow[r, shift right=2] & M\Box N. \arrow[l] \end{tikzcd}$$

Theorem 4.1.1.6 shows that the 2-functor $Bal_A(M, N; -)$ is corepresented by $M\Box_A N$, so that $M\Box_A N$ is the pseudo-coequalizer of the above codescent object.

Thanks to the definition of the relative tensor product using a 2-universal property, the following result is an immediate consequence of the above theorem.

Corollary 4.1.1.8. *If $\mathfrak{C}$ is a Karoubi complete 2-category, and A is a separable algebra, the relative tensor product over A defines a 2-functor*

$$\Box_A : \mathbf{Mod}_{\mathfrak{C}}(A) \times \mathbf{LMod}_{\mathfrak{C}}(A) \to \mathfrak{C}.$$

Remark 4.1.1.9. For completeness, let us note that if $\mathfrak{C}$ is a linear monoidal 2-category, then it follows from the 2-universal property of $\Box_A$ and the fact that $\Box$ is a bilinear 2-functor that $\Box_A$ is a bilinear 2-functor.

4.1.2 The Morita 3-Category

Our goal is now to explain how to construct the Morita 3-category of separable algebras in a Karoubi complete monoidal 2-category. In order to do so, we need to generalize the setup of the previous section to bimodules.

Definition 4.1.2.1. Let A, B, C be algebras in $\mathfrak{C}$, and let M be an A-B-bimodule, N be a B-C-bimodule, and P be an A-C-bimodule. A B-balanced A-C-bimodule 1-morphism $(M, N) \to P$ is an A-C-bimodule 1-morphism $f : M\Box N \to P$ together with an A-C-bimodule 2-isomorphism $\beta^f : f \circ (M\Box l^N) \cong f \circ (n^M\Box N)$ providing f with an A-balanced structure. A B-balanced A-C-bimodule 2-morphism is an A-C-bimodule 2-morphism that is also B-balanced.

Proposition 4.1.2.2. *Let A, B, C be algebras in $\mathfrak{C}$, with B separable. Let M be an A-B-bimodule, and N be a B-C-bimodule, the relative tensor product $t_B : M\Box N \to M\Box_B N$ can be endowed with an A-C-bimodule structure such that it is 2-universal with respect to B-balanced A-C-bimodule morphisms.*

Proof. Note that if M and N are bimodules in the proof of theorem 4.1.1.6, then the 2-condensation monad $(M\Box N, e, \mu, \delta)$ in $\mathfrak{C}$ can be upgraded to a 2-condensation monad in $\mathbf{Bimod}_{\mathfrak{C}}(A, C)$. The remainder of the proof can be straightforwardly adapted to accommodate for the bimodule case. The only noteworthy change is that one needs to use the fact that $\mathbf{Bimod}_{\mathfrak{C}}(A, C)$ is Karoubi complete, which follows from the proof of proposition 3.3.4 of [Déc21] as $\mathfrak{C}$ is Karoubi complete. In particular, this constructs a 2-universal B-balanced A-C-bimodule 1-morphism $\widetilde{t}_B : M\Box N \to M\Box_B N$. But, as splittings of 2-condensation monads are preserved by all 2-functors, the underlying B-balanced 1-morphism $\widetilde{t}_B : M\Box N \to M\Box_B N$ in $\mathfrak{C}$ satisfies the 2-universal property of definition 4.1.1.3. This finishes the proof of the proposition. $\Box$

Remark 4.1.2.3. Let us sketch an alternative proof of proposition 4.1.2.2. It follows from the construction of theorem 4.1.1.6 and the fact that 2-condensation are preserved by all 2-functors that $A\Box t_B : M\Box N \to A\Box(M\Box_B N)$ is 2-universal with respect to B-balanced 1-morphisms. The 2-universal property of the relative tensor product over B can then be used repeatedly to endow $t_B : M\Box N \to M\Box_B N$ with a left A-module structure. Similarly, we can construct a right C-module structure on t_B, which is compatible with the left A-module structure. Finally, one can directly check that the B-balanced A-C-bimodules 1-morphism t_B is 2-universal with respect to B-balanced A-C-bimodule morphisms.

Corollary 4.1.2.4. *Let A, B, C be arbitrary algebras in $\mathfrak{C}$ with B separable. The relative tensor product over B induces a 2-functor*

$$\Box_B : \mathbf{Bimod}_{\mathfrak{C}}(A,B)\times \mathbf{Bimod}_{\mathfrak{C}}(B,C)\to \mathbf{Bimod}_{\mathfrak{C}}(A,C).$$

We now prove a unitality property of the relative tensor product that will play a crucial role later on.

Lemma 4.1.2.5. *Let A and B be arbitrary algebras in $\mathfrak{C}$. There is a 2-natural adjoint equivalence*

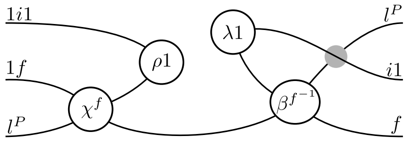
$$l^{\mathbf{M}}_P : A\Box_A P \simeq P$$

for any A-B-bimodule P in $\mathfrak{C}$.

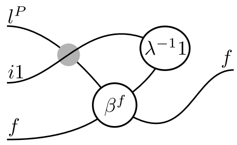
Proof. Let P, Q be two A-B-bimodule in $\mathfrak{C}$. Observe that $l^P : A\Box P \to P$ is an A-balanced A-B-bimodule 1-morphism via $\beta^{l^P} := \kappa^P$. We claim that this 1-morphism satisfies the 2-universal property defining the relative tensor product. Namely, given $f : A\Box P \to Q$ an A-balanced A-B-bimodule 1-morphism, we define g as the composite right B-module 1-morphism

$$g : P \xrightarrow{i\Box P} A\Box P \xrightarrow{f} Q.$$

In addition, the 2-isomorphism

$\chi^g :=$ 

endows g with a compatible left A-module structure. Further, it follows from the definitions that the 2-isomorphism $\xi : g \circ l^P \cong f$ given by

$\xi :=$ 

is an A-balanced A-B-bimodule 2-morphism as desired. Now, let $g, h : P \to Q$ be two A-B-bimodule 1-morphisms, and $\gamma : g \circ l^P \Rightarrow h \circ l^P$ be an A-balanced A-B-bimodule 2-morphisms, then it is not hard to check that $\zeta := \gamma \circ (i \Box P)$ is an A-B-bimodule 2-morphism satisfying $\zeta \circ l^p = \gamma$. This finishes the proof of the claim. Finally, using the 2-universal property of the relative tensor product, one can readily construct the desired adjoint 2-natural equivalence $l^{\mathbf{M}}$. $\Box$

For our purposes, it is also necessary to examine the relative tensor product of multiple bimodules.

Definition 4.1.2.6. Let A, B, C, D be algebras in $\mathfrak{C}$, and let M be an A-B-bimodule, N be a B-C-bimodule, P be a C-D-bimodule, and Q an A-D-bimodule in $\mathfrak{C}$. A (B, C)-balanced A-D-bimodule 1-morphism $(M, N, P) \to Q$ consists of:

1. An A-D-bimodule 1-morphism $f : M \Box N \Box P \to Q$,
2. Two A-D-bimodule 2-isomorphisms β^f_B and β^f_C given by

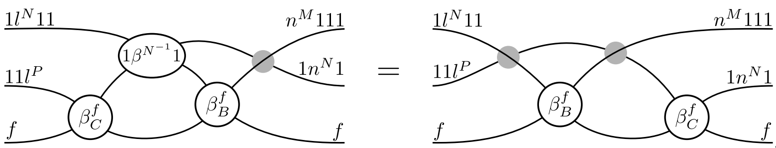
$$\begin{tikzcd} MBNP \arrow[r, "n^M 11"] \arrow[d, "1l^N1"'] \arrow[dr, phantom, "\beta^f_B \Nearrow"] & MNP \arrow[d, "f"] \\ MNP \arrow[r, "f"'] & Q, \end{tikzcd} \qquad \begin{tikzcd} MBNP \arrow[r, "1n^N1"] \arrow[d, "11l^P"'] \arrow[dr, phantom, "\beta^f_C \Nearrow"] & MNP \arrow[d, "f"] \\ MNP \arrow[r, "f"'] & Q, \end{tikzcd}$$

satisfying:

a. The 2-isomorphism β^f_B endows $f : (M, N \Box P) \to Q$ with a B-balanced structure,

b. The 2-isomorphism β^f_C endows $f : (M \Box N, P) \to Q$ with a C-balanced structure,

c. The 2-isomorphisms β^f_B and β^f_C commute in the sense that



(4.3)

A (B, C)-balanced A-D-bimodule 2-morphism is an A-D-bimodule 2-morphism that is both B-balanced and C-balanced.

Lemma 4.1.2.7. *Let A, B, C, D be algebras in $\mathfrak{C}$, and let M be an A-B-bimodule, N be a B-C-bimodule, and P be a C-D-bimodule. If $\mathfrak{C}$ is Karoubi complete, and B, C are separable algebras, then both $M\Box_B(N\Box_C P)$ and $(M\Box_B N)_C P$ are 2-universal with respect to (B,C)-balanced A-D-bimodule morphisms. In particular, there exists an adjoint 2-natural equivalence*

$$\alpha^{\mathbf{M}}_{M,N,P} : (M\Box_B N)\Box_C P \simeq M\Box_B(N\Box_C P).$$

Proof. Let us show that the (B,C)-balanced A-D-bimodule 1-morphism

$$M\Box N\Box P \xrightarrow{M\Box t_C} M\Box(N\Box_C P) \xrightarrow{t_B} M\Box_B(N\Box_C P)$$

is 2-universal with respect to (B,C)-balanced A-D-bimodule morphisms. In order to prove this, note that the C-balanced A-D-bimodule 1-morphism $M\Box t_C : M\Box N\Box P \to M\Box(N\Box_C P)$ is 2-universal with respect to C-balanced A-D-bimodule morphism.

Now, let Q be an A-D-bimodule, and let $f : (M,N,P) \to Q$ be a (B,C)-balanced A-D-bimodule 1-morphism. This gives us the solid arrow part of the diagram below

$$\begin{tikzcd} & M\Box(N\Box_C P) \arrow[dd, dotted, "f'"] \arrow[dr, "t_B"] & \\ M\Box N\Box P \arrow[ur, "M\Box t_C"] \arrow[dr, "f"'] & & M\Box_B(N\Box_C P) \arrow[dl, dotted, "\widetilde{f}"] \\ & Q. & \end{tikzcd}$$

As f is in particular a C-balanced A-D-bimodule 1-morphisms, there exists an A-D-bimodule 1-morphism $f' : M\Box(N\Box_C P) \to Q$, and a C-balanced A-D-bimodule 2-isomorphism $\xi_C : f'\circ(M\Box t_C) \cong f$. But, thanks to equation (4.3) and the 2-universal property of $M\Box t_C$, the B-balanced structure of f induces a B-balanced structure on f'. Thus, there exists an A-D-bimodule 1-morphisms $\widetilde{f} : M\Box_B(N\Box_C P) \to Q$, and a B-balanced A-D-bimodule 2-isomorphism $\xi_B : \widetilde{f}\circ t_B \cong f'$. It follows from the definitions that the composite A-D-bimodule 2-isomorphism $\xi_C\cdot(\xi_B\circ(M\Box t_C))$ is (B,C)-balanced, so that $\widetilde{f}$ is the sought after factorization of f.

Finally, let $g,h : M\Box_B(N\Box_C P) \to Q$ be two A-D-bimodule 1-morphisms, and $\gamma : g\circ t_B\circ(M\Box t_C) \Rightarrow h\circ t_B\circ(M\Box t_C)$ a (B,C)-balanced A-D-bimodule 2-morphism. It follows immediately from the 2-universal property of $M\Box t_C$ that there exists an A-D-bimodule 2-morphism $\zeta' : g\circ t_B \Rightarrow h\circ t_B$ such that $\zeta\circ(M\Box t_C) = \gamma$. But, using the 2-universal property of $M\Box t_C$ again together with the fact that γ is B-balanced, we find that ζ' is necessarily B-balanced. Thence, by the 2-universal property of t_B, there exists an A-D-bimodule 2-morphism $\zeta : g \Rightarrow h$ such that $\zeta \circ t_B = \zeta'$. Putting everything together, we find that $\zeta \circ t_B \circ (M\Box t_C) = \gamma$ as desired. This proves that $M\Box_B(N\Box_C P)$ is 2-universal with respect to (B,C)-balanced A-D-bimodule morphisms.

One proceeds analogously to show that the (B,C)-balanced A-D-bimodule 1-morphism

$$M\Box N\Box P \xrightarrow{t_B\Box P} (M\Box_B N)\Box P \xrightarrow{t_C} (M\Box_B N)\Box_C P$$

is 2-universal with respect to (B,C)-balanced A-D-bimodule morphisms. The second part of the statement then follows readily by appealing to the 2-universal property. $\Box$

We are now ready to explain the main construction of this section.

Theorem 4.1.2.8. *Let $\mathfrak{C}$ be a Karoubi complete monoidal 2-category. Separable algebras in $\mathfrak{C}$, bimodules, bimodule 1-morphisms, and bimodule 2-morphisms form a 3-category, which we denote by $\mathbf{Mor}^{sep}(\mathfrak{C})$.*

Proof. Let A, B, C, be separable algebras in $\mathfrak{C}$. We set

$$Hom_{\mathbf{Mor}^{sep}(\mathfrak{C})}(B,A) := \mathbf{Bimod}(A,B).$$

Then, the bilinear 2-functor

$$\Box_B : \mathbf{Bimod}_{\mathfrak{C}}(A,B) \times \mathbf{Bimod}_{\mathfrak{C}}(B,C) \rightarrow \mathbf{Bimod}_{\mathfrak{C}}(A,C)$$

of corollary 4.1.2.4 provides us with the necessary composition 2-functor. Further, the identity 1-morphism on the algebra A is given by the canonical A-A-bimodule A. It remains to prove that these operations can be made suitably coherent in the sense of definition 4.1 of [Gur13]. Firstly, note that lemma 4.1.2.5 provides us with an adjoint 2-natural equivalence $l^{\mathbf{M}}$. Using a similar argument, one can construct a 2-natural equivalence $r^{\mathbf{M}}$ given on the A-B-bimodule P by $r^{\mathbf{M}}_P : P\Box_B B \simeq P$. Moreover, lemma 4.1.2.7 provides us with an adjoints 2-natural equivalence $\alpha^{\mathbf{M}}$ witnessing associativity of the composition of 1-morphisms.

Secondly, we have to supply invertible modifications $\lambda^{\mathbf{M}}$, $\mu^{\mathbf{M}}$, $\rho^{\mathbf{M}}$, and $\pi^{\mathbf{M}}$ between specific composites of $l^{\mathbf{M}}$, $r^{\mathbf{M}}$, and $\alpha^{\mathbf{M}}$. Let us explain how to construct $\lambda^{\mathbf{M}}$. Let M

be an A-B-bimodule and N a B-C-bimodule in $\mathfrak{C}$, and consider the diagram

$$\begin{tikzcd}
 & & A\Box M\Box N \arrow[dll] \arrow[dd] \arrow[drr] & & \\
(A\Box_A M)\Box_B N \arrow[rrrr, "l^{\mathbf{M}}_M\Box_B N"] \arrow[drr, "\alpha^{\mathbf{M}}_{A,M,N}"'] & & & & M\Box_B N \\
 & & A\Box_A(M\Box_B N), \arrow[urr, "l^{\mathbf{M}}_{M\Box_B N}"'] & &
\end{tikzcd}$$

where the three unlabeled arrows are the canonical (A,B)-balanced A-B-bimodule 1-morphisms, and the three top triangles are filled by canonical (A,B)-balanced A-B-bimodule 2-isomorphisms. Thanks to the 2-universal property of $A\Box M\Box N \to (A\Box_A M)\Box_B N$, there exists an A-B-bimodule 2-isomorphism

$$\lambda^{\mathbf{M}}_{M,N} : l^{\mathbf{M}}_M\Box_B N \cong l^{\mathbf{M}}_{M\Box_B N}\circ \alpha^{\mathbf{M}}_{A,M,N}.$$

Using the 2-universal property again, it is easy to check that these 2-isomorphisms define an invertible modification. The invertible modifications $\mu^{\mathbf{M}}$ and $\rho^{\mathbf{M}}$ are constructed similarly.

It remains to construct the invertible modification $\pi^{\mathbf{M}}$. Given separable algebras A, B, C, D, E, one defines (B,C,D)-balanced A-E-bimodule morphisms by adapting definition 4.1.2.6 in the obvious way. Following the proof of lemma 4.1.2.7, one then shows that for any A-B-bimodule M, B-C-bimodule N, C-D-bimodule P, and D-E-bimodule Q, the canonical (B,C,D)-balanced A-E-bimodule 1-morphisms to the different ways of parenthesising $M\Box_B N\Box_C P\Box_D Q$ are all 2-universal with respect (B,C,D)-balanced A-E-bimodule morphisms. Analogously to the above arguments, $\pi^{\mathbf{M}}$ is constructed using this 2-universal property.

Finally, one has to check that the equation between these invertible modifications given in definition 4.1 of [Gur13] are satisfied. All of them follow readily from the 2-universal property of the relative tensor product over either three or four algebras. $\Box$

Remark 4.1.2.9. Over a perfect field, the 3-category $\mathbf{Mor}^{sep}(\mathbf{2Vect})$ constructed above is the underlying 3-category of the symmetric monoidal 3-category $\mathbf{TC}^{sep}$ of separable multifusion 1-categories considered in [DSPS21]. Over an algebraically closed field of characteristic zero, and given $\mathcal{B}$ a braided fusion 1-category, the 3-category $\mathbf{Mor}^{sep}(\mathbf{Mod}(\mathcal{B}))$ corresponds to the Hom-3-category from $\mathcal{B}$ to $\mathbf{Vect}$ in the symmetric monoidal 4-category $\mathbf{BrFus}$ of braided fusion 1-categories considered in [BJS21].

Remark 4.1.2.10. Let $\mathfrak{C}$ be a Karoubi complete monoidal 2-category. In [GJF19], the authors outlined the construction of a 3-category $Kar(\mathrm{B}\mathfrak{C})$ of 3-condensation monads, condensation bimodules, and their morphisms. Using variants of the results proven in section 3 of [GJF19], we expect that one can prove that the 3-category $\mathbf{Mor}^{sep}(\mathfrak{C})$ considered above is equivalent to $Kar(\mathrm{B}\mathfrak{C})$. In particular, this would show that $\mathbf{Mor}^{sep}(\mathfrak{C})$ satisfies a 4-universal property. (We refer the reader to [Déc22b] for a precise discussion of the 2-categorical case.)

Remark 4.1.2.11. Our proof of theorem 4.1.2.8 also applies to other setups. Namely, given any monoidal 2-category $\mathfrak{C}$ and any set $\mathscr{A}$ of algebras in $\mathfrak{C}$ such that for any algebras A, B, and C in $\mathscr{A}$ the relative tensor product over B of any A-B-bimodule and B-C-bimodule exists. The above proof constructs a 3-category $\mathbf{Mor}^{\mathscr{A}}(\mathfrak{C})$ of algebras in $\mathscr{A}$, bimodules between them, and their bimodule morphisms. In particular, if every codescent diagram admits a pseudo-coequalizer in $\mathfrak{C}$, and that $\Box$ commutes with them, then it follows from remark 4.1.1.7 that the relative tensor product over any algebra in $\mathfrak{C}$ exists. In this case, we can therefore consider the 3-category $\mathbf{Mor}(\mathfrak{C})$ of all algebras in $\mathfrak{C}$, bimodules and their bimodule morphims. We note that this last example has already been thoroughly examined in [Hau17] in an ∞-categorical context.

For later use, let us also record the following corollary.

Corollary 4.1.2.12. *Let A be an algebra in $\mathfrak{C}$. Giving an algebra B in the monoidal 2-category $\mathbf{Bimod}_{\mathfrak{C}}(A)$ is equivalent to giving an algebra B in $\mathfrak{C}$ together with an algebra 1-homomorphism $A \to B$. Furthermore, if $\mathfrak{C}$ is compact semisimple, then the algebra B in $\mathbf{Bimod}_{\mathfrak{C}}(A)$ is rigid, respectively separable, if and only if the underlying algebra B in $\mathfrak{C}$ is rigid, respectively separable.*

Proof. Inspection of the proof of theorem 4.1.2.8 shows that the forgetful 2-functor $\mathbf{Bimod}_{\mathfrak{C}}(A) \to \mathfrak{C}$ is lax monoidal. This yields the forward direction of the first part. The backward direction follows by the 2-universal property of the balanced tensor product over A. For the second, note that, if we write $\mathfrak{D} := \mathbf{Bimod}_{\mathfrak{C}}(A)$, then it follows 2-universal property of the relative tensor product over A that the forgetful 2-functor $\mathbf{Bimod}_{\mathfrak{D}}(B) \to \mathbf{Bimod}_{\mathfrak{C}}(B)$ is an equivalence. The result then follows from theorems 3.1.2.8 and 3.2.1.7 above. $\Box$

4.2 Module 2-Categories

We recall the definitions of a module 2-category, module 2-functor, module 2-natural transformation, and module modification and show that, over a fixed monoidal 2-category, these objects assemble into a 3-category. We then review the definition of a 2-adjunction between two 2-functors, and explain how this concepts interacts with that of a module 2-functor over a rigid monoidal 2-category. Theses results are quite technical in nature, but will play a determining role in the last part of the present thesis.

4.2.1 The 3-Category of Module 2-Categories

Let $\mathfrak{C}$ be a cubical monoidal 2-category. Our goal is to construct a 3-category whose objects are left $\mathfrak{C}$-module 2-categories in the sense of definition 2.1.3 of [Déc21]. Now, it follows from proposition 2.2.8 of [Déc21] that every pair $(\mathfrak{C}, \mathfrak{M})$ consisting of a monoidal 2-category $\mathfrak{C}$ and a left $\mathfrak{C}$-module 2-category $\mathfrak{M}$ is equivalent to a pair in which both $\mathfrak{C}$ and $\mathfrak{M}$ are strict cubical (see definition 4.2.1.1 below). Thus, there is no loss of generality in assuming that $\mathfrak{C}$ and $\mathfrak{M}$ are strict cubical. In fact, by remark 2.2.9 of [Déc21], this strictification procedure holds for any set of module 2-categories.

Definition 4.2.1.1. Let $\mathfrak{M}$ be a strict 2-category. A strict cubical left $\mathfrak{C}$-module 2-category structure on $\mathfrak{M}$ is a strict cubical 2-functor $\Box : \mathfrak{C} \times \mathfrak{M} \to \mathfrak{M}$ such that:

1. The induced 2-functor $I\Box(-) : \mathfrak{M} \to \mathfrak{M}$ is exactly the identity 2-functor,
2. The two 2-functors

   $$\big((-)\Box(-)\big)\Box(-) : \mathfrak{C} \times \mathfrak{C} \times \mathfrak{M} \to \mathfrak{M}, \text{ and } (-)\Box\big((-)\Box(-)\big) : \mathfrak{C} \times \mathfrak{C} \times \mathfrak{M} \to \mathfrak{M}$$

   are equal on the nose.

Notation 4.2.1.2. It is straightforward to extend the graphical conventions introduced in 1.1 for strict cubical monoidal 2-categories to strict cubical left $\mathfrak{C}$-module 2-categories. Throughout this section, we use this extended graphical language.

Remark 4.2.1.3. If $\Bbbk$ is a field, and $\mathfrak{C}$ is a monoidal $\Bbbk$-linear 2-category, then, by definition, $\Box : \mathfrak{C}\Box\mathfrak{C} \to \mathfrak{C}$ is a bilinear 2-functor. Likewise, if $\mathfrak{M}$ is a $\Bbbk$-linear 2-category left $\mathfrak{C}$-module 2-category, we require that $\Box : \mathfrak{C} \times \mathfrak{M} \to \mathfrak{M}$ is a bilinear 2-functor.

Definition 4.2.1.4. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two strict cubical left $\mathfrak{C}$-module 2-categories. A left $\mathfrak{C}$-module 2-functor is a (not necessarily strict) 2-functor $F : \mathfrak{M} \to \mathfrak{N}$ together with:

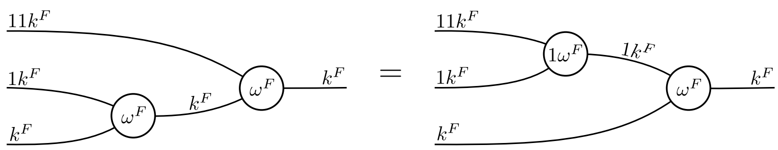
1. An adjoint 2-natural equivalence k^F given on A in $\mathfrak{C}$, and M in $\mathfrak{M}$ by
$$k^F_{A,M} : A\Box F(M) \to F(A\Box M);$$
2. Two invertible modifications ω^F, and γ^F given on A, B in $\mathfrak{C}$ and M in $\mathfrak{M}$ by

$$\begin{tikzcd} & A\Box F(B\Box M) \arrow[dr, "k^F_{A,B\Box M}"] \arrow[d, phantom, "\Downarrow \omega^F_{A,B,M}"] & \\ A\Box B\Box F(M) \arrow[ur, "Id_A\Box k^F_{B,M}"] \arrow[rr, "k^F_{A\Box B,M}"'] & & F(A\Box B\Box M), \end{tikzcd}$$

$$\gamma^F_M : k^F_{I,M} \Rightarrow Id_{F(M)};$$

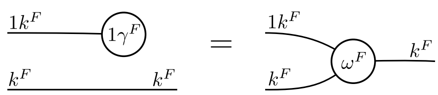
Subject to the following relations:

a. For every A, B, C in $\mathfrak{C}$, and M in $\mathfrak{M}$, the equality



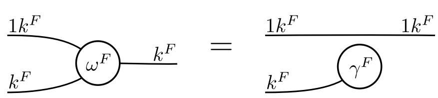
holds in $Hom_{\mathfrak{N}}(A\Box B\Box C\Box F(M), F(A\Box B\Box C\Box M))$,

b. For every A in $\mathfrak{C}$, and M in $\mathfrak{M}$, the equality



holds in $Hom_{\mathfrak{N}}(A\Box I\Box F(M), F(A\Box M))$;

c. For every B in $\mathfrak{C}$, and M in $\mathfrak{M}$, the equality



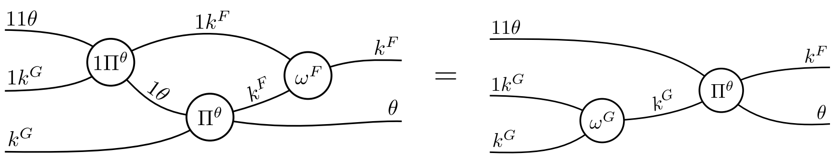
holds in $Hom_{\mathfrak{N}}(I\Box B\Box F(M), F(B\Box M))$.

Definition 4.2.1.5. Let $F, G : \mathfrak{M} \to \mathfrak{N}$ be two left $\mathfrak{C}$-module 2-functors as in definition 4.2.1.4. A left $\mathfrak{C}$-module 2-natural transformation is a 2-natural transformation $\theta : F \Rightarrow G$ equipped with an invertible modification Π^θ given on A in $\mathfrak{C}$, and M in $\mathfrak{M}$ by

$$\begin{tikzcd}
AG(M) \arrow[d, "k^G"'] & AF(M) \arrow[l, "1\theta"'] \arrow[d, "k^F"] \arrow[dl, phantom, "\Pi^\theta"] \\
G(AM) & F(AM); \arrow[l, "\theta"]
\end{tikzcd}$$

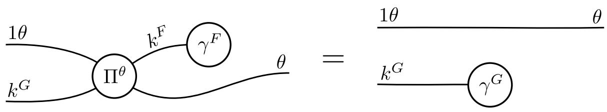
Subject to the following relations:

a. For every A, B in $\mathfrak{C}$, and M in $\mathfrak{M}$, the equality



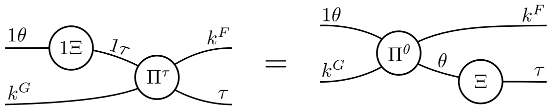
holds in $Hom_{\mathfrak{N}}(A\Box B\Box F(M), G(A\Box B\Box M))$;

b. For every M in $\mathfrak{M}$, the equality



holds in $Hom_{\mathfrak{N}}(I\Box F(M), G(M))$.

Definition 4.2.1.6. Let $\theta, \tau : F \Rightarrow G$ be two left $\mathfrak{C}$-module 2-natural transformations. A left $\mathfrak{C}$-module modification is a modification $\Xi : \theta \Rrightarrow \tau$ such that for every A in $\mathfrak{C}$, and M in $\mathfrak{M}$ the equality



holds in $Hom_{\mathfrak{N}}(A\Box F(M), G(A\Box M))$.

Fixing two strict cubical left $\mathfrak{C}$-module 2-categories $\mathfrak{M}$ and $\mathfrak{N}$, it was shown in proposition 2.2.1 of [Déc21] that left $\mathfrak{C}$-module 2-functors, left $\mathfrak{C}$-module 2-natural transformation, and left $\mathfrak{C}$-module modifications form a 2-category, which we denote by $\mathbf{Fun}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{N})$. In particular, given $\theta,\overline{\theta},\overline{\overline{\theta}}:F\Rightarrow G$ left $\mathfrak{C}$-module 2-natural transformations, and two left $\mathfrak{C}$-module modifications $\Xi:\theta\Rrightarrow\overline{\theta}$, $Z:\overline{\theta}\Rrightarrow\overline{\overline{\theta}}$, the vertical composite $Z\bullet\Xi$ is a left $\mathfrak{C}$-module modification. Further, given two left $\mathfrak{C}$-module 2-natural transformations $\theta:F\Rightarrow G$ and $\tau:G\Rightarrow H$, their composite is the 2-natural transformation $\tau\cdot\theta$ equipped with the invertible modification

$$\Pi^{\tau\cdot\theta}:=(\Pi^{\tau}\cdot\theta)\bullet(\tau\cdot\Pi^{\theta}).$$

Thanks to our strictness hypotheses, the above composition of left $\mathfrak{C}$-module 2-natural transformations is in fact strictly associative and unital. Thence, $\mathbf{Fun}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{N})$ is in fact a strict 2-category. For later use, we now assemble all of these 2-categories together.

Theorem 4.2.1.7. *Let $\mathfrak{C}$ be a monoidal 2-category. Left $\mathfrak{C}$-module 2-categories, left $\mathfrak{C}$-module 2-functors, left $\mathfrak{C}$-module 2-natural transformations, and left $\mathfrak{C}$-module modifications form a 3-category, which we denote by* $\mathbf{LMod}(\mathfrak{C})$.

Proof. In section 5.1 of [Gur13], the author constructs a 3-category of 2-categories. Our proof is precisely a left $\mathfrak{C}$-module version of this argument. In order to do this, it is enough to upgrade the structures defined in section 5.1 of [Gur13] with suitable left $\mathfrak{C}$-module actions. Furthermore, thanks to proposition 2.2.8 and remark 2.2.9 of [Déc21], we may assume without loss of generality that $\mathfrak{C}$ and every left $\mathfrak{C}$-module 2-category is strict cubical.

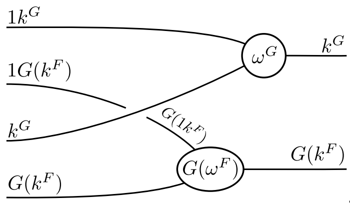
Let $\mathfrak{M}$, $\mathfrak{N}$, and $\mathfrak{P}$ be strict cubical left $\mathfrak{C}$-module 2-categories. We begin by constructing the 2-functor

$$\circ:\mathbf{Fun}_{\mathfrak{C}}(\mathfrak{N},\mathfrak{P})\times\mathbf{Fun}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{N})\to\mathbf{Fun}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{P})$$

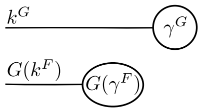
providing us with the composition of left $\mathfrak{C}$-module 2-functors. Given two left $\mathfrak{C}$-module 2-functors $F:\mathfrak{M}\to\mathfrak{N}$ and $G:\mathfrak{N}\to\mathfrak{P}$, we endow their composite $G\circ F$ with a left $\mathfrak{C}$-module structure using the following assignments. We define the adjoint 2-natural equivalence $k^{G\circ F}$ by

$$k^{G\circ F}_{A,M}:=G(k^{F}_{A,M})\circ k^{G}_{A,F(M)},$$

for every A in $\mathfrak{C}$ and M in $\mathfrak{M}$, and the two invertible modifications $\omega^{G\circ F}$, and $\gamma^{G\circ F}$ by

$\omega^{G\circ F} :=$ 

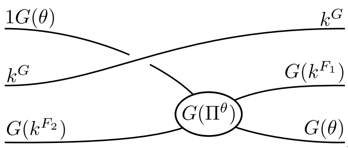
,

$\gamma^{G\circ F} :=$ 

.

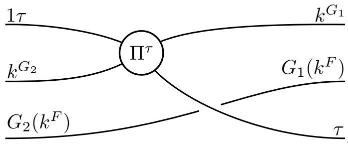
It is not hard to show that the above data satisfies the axioms of definition 4.2.1.4.

Then, given a left $\mathfrak{C}$-module 2-natural transformation $\theta : F_1 \Rightarrow F_2$ between two left $\mathfrak{C}$-module 2-functors $F_1, F_2 : \mathfrak{M} \to \mathfrak{N}$, we endow the 2-natural transformation $G \circ \theta$ with a left $\mathfrak{C}$-module structure by setting

$\Pi^{G\circ\theta} :=$ 

.

Likewise, given a left $\mathfrak{C}$-module 2-natural transformation $\tau : G_1 \Rightarrow G_2$ between two left $\mathfrak{C}$-module 2-functors $G_1, G_2 : \mathfrak{N} \to \mathfrak{P}$, we may similarly define a left $\mathfrak{C}$-module structure on the 2-natural transformation $\tau \circ F$ by

$\Pi^{\tau\circ F} :=$ 

.

Now, recall from the proof of proposition 5.1 of [Gur13] that $\tau \circ \theta = (G_2 \circ \theta) \cdot (\tau \circ F_1)$, so that the 2-natural transformation $\tau \circ \theta$ inherits a $\mathfrak{C}$-module structure. These assignments can be straightforwardly extended to left $\mathfrak{C}$-module modifications, so that we obtain a functor

$$Nat_{\mathfrak{C}}(G_1, G_2) \times Nat_{\mathfrak{C}}(F_1, F_2) \to Nat_{\mathfrak{C}}(G_1 \circ F_1, G_2 \circ F_2)$$

between the 1-categories of left $\mathfrak{C}$-module 2-natural transformations. The additional structure constraints needed to define a 2-functor are the invertible modifications given in proposition 5.1 of [Gur13], and one checks easily that they respect the relevant left $\mathfrak{C}$-module structures defined above. Thus, we obtain the desired 2-functor $\circ : \mathbf{Fun}_{\mathfrak{C}}(\mathfrak{N}, \mathfrak{P}) \times \mathbf{Fun}_{\mathfrak{C}}(\mathfrak{M}, \mathfrak{N}) \to \mathbf{Fun}_{\mathfrak{C}}(\mathfrak{M}, \mathfrak{P})$. Moreover, the unit on $\mathfrak{M}$ for the composition of left $\mathfrak{C}$-module 2-functors is given by the identity 2-functor $Id : \mathfrak{M} \to \mathfrak{M}$ with its canonical left $\mathfrak{C}$-module structure.

Proposition 5.3 of [Gur13] defines an adjoint 2-natural equivalence α witnessing the associativity of the composition of (plain) 2-functors. Now, let $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$, and $\mathfrak{Q}$ be strict cubical left $\mathfrak{C}$-module 2-categories, and let $F : \mathfrak{M} \to \mathfrak{N}$, $G : \mathfrak{N} \to \mathfrak{P}$, and $H : \mathfrak{P} \to \mathfrak{Q}$ be left $\mathfrak{C}$-module 2-functors. It follows from proposition 5.3 of [Gur13] that $\alpha_{H,G,F} : (H \circ G) \circ F \simeq H \circ (G \circ F)$ is the identity 2-natural transformation. Further, the left $\mathfrak{C}$-module structures of $(H \circ G) \circ F$ and $H \circ (G \circ F)$ are equal, so that we can upgrade $\alpha_{H,G,F}$ to a left $\mathfrak{C}$-module adjoint 2-natural equivalence using the identity modification. The collection of these assignments promote α to an adjoint 2-natural equivalence witnessing the associativity of the composition of left $\mathfrak{C}$-module 2-functors.

Analogously, the adjoint 2-natural equivalences l, and r constructed in proposition 5.5 of [Gur13], witnessing that composition of 2-functors is unital, can be promoted to left $\mathfrak{C}$-module adjoint 2-natural equivalences. Namely, as we are working with strict 2-categories, these adjoint 2-natural equivalences are in fact both given by the identity 2-natural adjoint equivalence. Thus, given a $\mathfrak{C}$-module 2-functor $F : \mathfrak{M} \to \mathfrak{N}$ between strict cubical left $\mathfrak{C}$-module 2-categories, the 2-natural transformations l_F and r_F can canonically be upgraded to left $\mathfrak{C}$-module adjoint 2-natural equivalences. With these additional pieces of data, l and r define adjoint 2-natural equivalence witnessing the unitality of the composition of left $\mathfrak{C}$-module 2-functors. The proof is then completed by checking that the invertible modification π, μ, λ, and ρ given in proposition 5.6 of [Gur13] are compatible with the left $\mathfrak{C}$-module structures we have defined. This is immediate as it follows from our strictness assumptions that π, μ, λ, and ρ are all identity modifications. $\square$

Remark 4.2.1.8. It follows immediately from our proof of theorem 4.2.1.7 that there is a forgetful 3-functor $\mathbf{LMod}(\mathfrak{C}) \to \mathbf{2Cat}$ to the 3-category of 2-categories.

Corollary 4.2.1.9. *Let $\mathfrak{M}$ be a left $\mathfrak{C}$-module 2-category. Then, $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$ is a monoidal 2-category. Further, given $\mathfrak{M}$ be any left $\mathfrak{C}$-module 2-category, the 2-category $\mathbf{Fun}_{\mathfrak{C}}(\mathfrak{M}, \mathfrak{N})$ is an $\mathbf{End}_{\mathfrak{C}}(\mathfrak{N})$-$\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$-bimodule 2-category.*

We end this first section on module 2-categories with the following proposition, which will constitute a key ingredient in our study of the Morita theory of fusion 2-categories.

Proposition 4.2.1.10. *Let $\mathfrak{M}$ be a left $\mathfrak{C}$-module 2-category. The action of $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$ on $\mathfrak{M}$ given by evaluation defines a left $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$-module structure on $\mathfrak{M}$. Further, this structure is compatible with the left $\mathfrak{C}$-module one, so that $\mathfrak{M}$ is a left $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M}) \times \mathfrak{C}$-module 2-category.*

Proof. One may directly check that evaluation of 2-functors $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M}) \times \mathfrak{M} \to \mathfrak{M}$ provides $\mathfrak{M}$ with a left $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$-module structure. By definition, this left $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$-module structure on $\mathfrak{M}$ is compatible with the left $\mathfrak{C}$-module structure, so that $\mathfrak{M}$ has a left $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M}) \times \mathfrak{C}$-module structure. □

4.2.2 Module 2-Functors and 2-Adjunctions

The goal of this section is study the interaction between the notion of a module 2-functor recalled above, and that of a 2-adjunction between 2-functors, which we now recall.

Definition 4.2.2.1. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two 2-categories, and $F : \mathfrak{M} \to \mathfrak{N}$ and $G : \mathfrak{N} \to \mathfrak{M}$ be two 2-functors. A 2-adjunction between F and G consists of two 2-natural transformations u^F, called the unit, and c^F, called the counit, given on M in $\mathfrak{M}$ and N in $\mathfrak{N}$ by

$$u^F_M : M \to G(F(M)), \text{ and } c^F_N : F(G(N)) \to N,$$

together with two invertible modifications Φ^F and Ψ^F, called triangulators, given on M in $\mathfrak{M}$ and N in $\mathfrak{N}$ by

$$\Phi^F_M : c^F_{F(M)} \circ F(u^F_M) \cong Id_{F(M)},$$

$$\Psi^F_N : G(c^F_N) \circ u^F_{G(N)} \cong Id_{G(N)}.$$

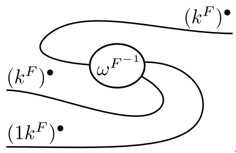
We also say that F is a left 2-adjoint to G, or that G is a right 2-adjoint to F.

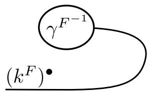
Definition 4.2.2.2. Let $\mathfrak{C}$ be a monoidal 2-category, $\mathfrak{M}$ and $\mathfrak{N}$ be two left $\mathfrak{C}$-module 2-categories, and $F : \mathfrak{M} \to \mathfrak{N}$ and $G : \mathfrak{N} \to \mathfrak{M}$ be two left $\mathfrak{C}$-module 2-functors. A left $\mathfrak{C}$-module 2-adjunction between F and G is a 2-adjunction between F and G as in definition 4.2.2.1 such that u^F and c^F are left $\mathfrak{C}$-module 2-natural transformations, and Φ^F and Ψ^F are left $\mathfrak{C}$-module modifications.

Let $\mathfrak{C}$ be a rigid monoidal 2-category, and assume that both $\mathfrak{M}$ and $\mathfrak{N}$ are left $\mathfrak{C}$-module 2-categories. The categorified version of corollary 2.13 of [DSPS19] holds, as we show in the next two propositions. In fact, our proof also establishes the categorifications of their lemmas 2.10 and 2.11.

Proposition 4.2.2.3. *Let $\mathfrak{C}$ be a rigid monoidal 2-category, and let $F : \mathfrak{M} \to \mathfrak{N}$ be a left $\mathfrak{C}$-module 2-functor between left $\mathfrak{C}$-module 2-categories. If F has a right 2-adjoint G, then G can canonically be upgraded to a left $\mathfrak{C}$-module right 2-adjoint to F.*

Proof. Thank to proposition 2.2.8 and remark 2.2.9 of [Déc21], we may assume that $\mathfrak{C}$ is strict cubical, and that both $\mathfrak{M}$ and $\mathfrak{N}$ are strict cubical left $\mathfrak{C}$-module 2-categories. Further, let us denote by $(\omega^{F^{-1}})^{\bullet}$ and $(\gamma^{F^{-1}})^{\bullet}$ the 2-isomorphisms given by

$(\omega^{F^{-1}})^{\bullet} :=$ 

, $(\gamma^{F^{-1}})^{\bullet} :=$ 

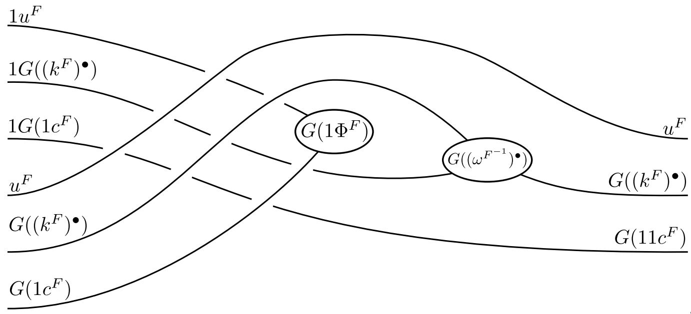
,

where the cups and the caps denote the unit and counit 2-isomorphisms witnessing that k^F and $(k^F)^{\bullet}$ form an adjoint 2-natural equivalence.

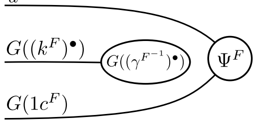
We begin by proving that G can be endowed with a lax left $\mathfrak{C}$-module structure. Given A in $\mathfrak{C}$, and M in $\mathfrak{M}$, we let the 2-natural transformation k^G be given by

$$k^G_{A,M} : A\Box G(M) \xrightarrow{u^F} G(F(A\Box G(M))) \xrightarrow{G((k^F)^{\bullet})} G(A\Box F(G(M))) \xrightarrow{G(1c^F)} G(A\Box M),$$

where $(k^F)^{\bullet}$ is the pseudo-inverse of k^F provided in the data of a left $\mathfrak{C}$-module 2-functor. The invertible modifications ω^G and γ^G are given by

$\omega^G :=$ 

,

$\gamma^G :=$ 

.

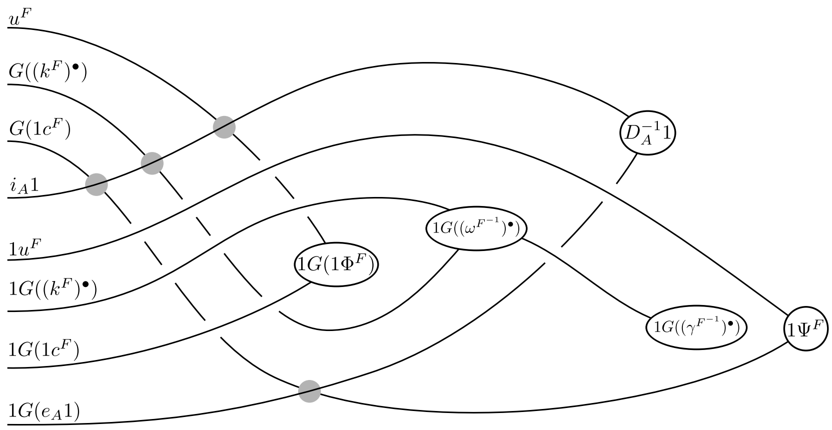
Using the axioms of definition 4.2.1.4 for F, it is easy to check that ω^G and γ^G satisfy the axioms of 4.2.1.4.

We now show that k^G can be upgraded to an adjoint 2-natural equivalence. As every 2-natural equivalence can be upgraded to an adjoint 2-natural equivalence (see section 1 of [Gur12]), it is enough to exhibit for every A in $\mathfrak{C}$ and M in $\mathfrak{M}$, a pseudo-inverse $(k^G)^{\bullet}_{A,M}$ for the 1-morphism $k^G_{A,M}$. Let $^{\sharp}A$ be a left dual for A in $\mathfrak{C}$ with unit

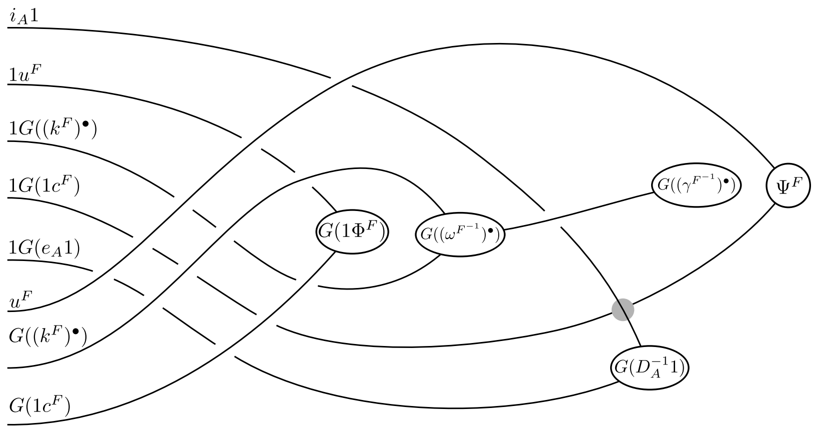
1-morphism $i_A : I \to A\Box{}^\sharp A$, counit 1-morphism $e_A : {}^\sharp A\Box A \to I$ and 2-isomorphisms $C_A : (e_A\Box{}^\sharp A) \circ (({}^\sharp A)\Box i_A) \Rightarrow Id_{{}^\sharp A}$, and $D_A : Id_A \Rightarrow (A\Box e_A) \circ (i_A\Box A)$. We define

$$\begin{aligned}(k^G)^\bullet_{A,M} : G(A\Box M) &\xrightarrow{i_A 1} A\Box({}^\sharp A)\Box G(A\Box M) \xrightarrow{1u^F} A\Box GF({}^\sharp A\Box G(A\Box M))\\ &\xrightarrow{1G((k^F)^\bullet)} A\Box G({}^\sharp A\Box FG(A\Box M))\\ &\xrightarrow{1G(1c^F)} A\Box G({}^\sharp A\Box A\Box M) \xrightarrow{1G(e_A 1)} A\Box G(M),\end{aligned}$$

where $(k^F)^\bullet$ denotes the canonical pseudo-inverse of k^F supplied by the definition of a module 2-functor. The two 2-isomorphisms



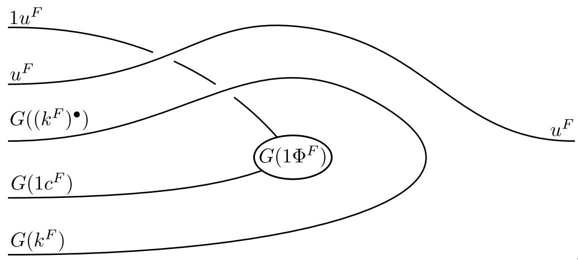
,



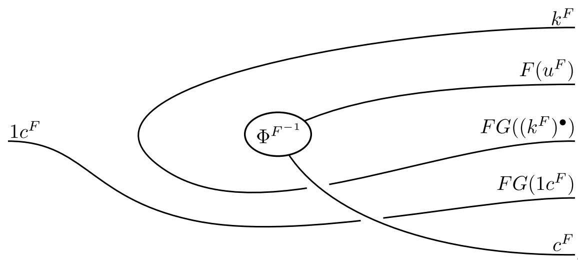
,

witness that $(k^G)^\bullet_{A,M}$ is a pseudo-inverse for $k^G_{A,M}$ as desired.

It remains to upgrade u^F and c^F to left $\mathfrak{C}$-module 2-natural transformations, and show that Φ^F and Ψ^F define invertible left $\mathfrak{C}$-module modifications. For the first part, we endow u^F and c^F with left $\mathfrak{C}$-module structures using the modifications Π^{u^F} and Π^{c^F} specified by

$\Pi_A^{u^F} :=$ 

,

$\Pi_A^{c^F} :=$ 

.

It is easy to check that Π^{u^F} and Π^{c^F} satisfy the axioms of definition 4.2.1.5. Finally, using naturality together with axioms a and b of definition 4.2.1.4 for G, one can readily check that Φ^F and Ψ^F are compatible with the left $\mathfrak{C}$-module structure on u^F and c^F defined above, which finishes the proof of the result. $\square$

The analogue of proposition 4.2.2.3 for left 2-adjoints also holds.

Proposition 4.2.2.4. *Let $G : \mathfrak{N} \to \mathfrak{M}$ be a left $\mathfrak{C}$-module 2-functor. If G has a left 2-adjoint F, then F can canonically be upgraded to left $\mathfrak{C}$-module left 2-adjoint to G.*

Proof. This follows by applying proposition 4.2.2.3 to $\mathfrak{C}^{1op}$, the monoidal 2-category obtained from $\mathfrak{C}$ by reversing the direction of the 1-morphisms. $\square$

4.3 Dual Tensor 2-Categories and Morita Equivalence

In general, it is difficult to work with arbitrary compact semisimple module 2-categories over a fixed compact semisimple tensor 2-category $\mathfrak{C}$. Motivated by the decategorified situation studied in [DSPS21], we therefore restrict our attention to a particularly nice class of compact semisimple module 2-categories called separable module 2-categories. We prove that the 3-category of separable algebras in $\mathfrak{C}$ is equivalent to the 3-category of separable left $\mathfrak{C}$-module 2-categories. Under the assumption that $\mathfrak{C}$ be locally separable, we then show that the monoidal 2-category of bimodules over a separable algebra in $\mathfrak{C}$ is a compact semisimple tensor 2-category. This allows us to define the dual tensor 2-category of $\mathfrak{C}$ with respect to a separable module 2-category. We end by giving three equivalent characterizations of Morita equivalence between locally separable compact semisimple tensor 2-categories. Throughout, we work over a fixed field $\Bbbk$, meaning that all (monoidal) categories and functors under consideration are $\Bbbk$-linear.

4.3.1 Separable Module 2-Categories

Let us fix $\mathfrak{C}$ a compact semisimple tensor 2-category.

Definition 4.3.1.1. A compact semisimple left $\mathfrak{C}$-module 2-category $\mathfrak{M}$ is called separable if there exists a separable algebra A in $\mathfrak{C}$ such that

$$\mathfrak{M} \simeq \mathbf{Mod}_{\mathfrak{C}}(A)$$

as left $\mathfrak{C}$-module 2-categories.

In theorem 4.2.1.7, we have proven that left $\mathfrak{C}$-module 2-categories form a 3-category, which we denote by $\mathbf{LMod}(\mathfrak{C})$. We will write $\mathbf{LMod}^{sep}(\mathfrak{C})$ for the full sub-3-category whose objects are the separable module 2-categories. We are now ready to state our next theorem, of which a closely related variant was conjectured in remark 5.4.9 of [Déc21]. Let us mention that if $\Bbbk$ is algebraically closed of characteristic zero and $\mathfrak{C} = \mathbf{2Vect}$, we recover the main result of [Déc22b] as every multifusion 1-category is separable. More generally, if $\Bbbk$ is perfect and $\mathfrak{C} = \mathbf{2Vect}$, the theorem below also recovers corollary 2.3.1.5 above thanks to proposition 2.5.10 of [DSPS21].

Theorem 4.3.1.2. *Let $\mathfrak{C}$ be a compact semisimple tensor 2-category. There is a linear 3-functor, contravariant on 1-morphisms,*

$$\mathbf{Mod}_{\mathfrak{C}} : \mathbf{Mor}^{sep}(\mathfrak{C}) \to \mathbf{LMod}^{sep}(\mathfrak{C})$$

that sends a separable algebra in $\mathfrak{C}$ to the associated separable left $\mathfrak{C}$-module 2-category of right modules. Moreover, this 3-functor is an equivalence.

Proof. Without loss of generality, we may assume that $\mathfrak{C}$ is strict cubical. The monoidal unit I of $\mathfrak{C}$ is canonically a separable algebra in $\mathfrak{C}$. Thanks to theorem 4.1.2.8, this yields a contravariant linear 3-functor

$$\mathbf{Hom}_{\mathbf{Mor}^{sep}(\mathfrak{C})}(-,I) : \mathbf{Mor}^{sep}(\mathfrak{C}) \to \mathbf{2Cat}_{\Bbbk}$$

to the 3-category of $\Bbbk$-linear 2-categories. But, $\mathbf{End}_{\mathbf{Mor}^{sep}(\mathfrak{C})}(I,I) = \mathbf{Bimod}_{\mathfrak{C}}(I) = \mathfrak{C}$ as monoidal 2-categories thanks to our strictness hypothesis. In particular, for every separable algebra A in $\mathfrak{C}$, the 2-category $\mathbf{Hom}_{\mathbf{Mor}^{sep}(\mathfrak{C})}(A,I)$ has a canonical left $\mathfrak{C}$-module structure, which is compatible with bimodule morphisms in the variable A. Further, $\mathbf{Hom}_{\mathbf{Mor}^{sep}(\mathfrak{C})}(A,I) = \mathbf{Bimod}_{\mathfrak{C}}(I,A) = \mathbf{Mod}_{\mathfrak{C}}(A)$ is a separable left $\mathfrak{C}$-module 2-category. Thus, the 3-functor $\mathbf{Hom}_{\mathbf{Mor}^{sep}(\mathfrak{C})}(-,I)$ can canonically be lifted to a 3-functor

$$\mathbf{Mod}_{\mathfrak{C}} : \mathbf{Mor}^{sep}(\mathfrak{C}) \to \mathbf{LMod}^{sep}(\mathfrak{C}).$$

It remains to prove that $\mathbf{Mod}_{\mathfrak{C}}$ is an equivalence of 3-categories, i.e. that it is essentially surjective and induces equivalences on Hom-2-categories. Essential surjectivty follows immediately from the definition of a separable left $\mathfrak{C}$-module 2-category. Therefore, it is only left to prove that for every separable algebras A, B in $\mathfrak{C}$, the 2-functor

$$\begin{array}{rccc} \mathbf{F}: & \mathbf{Bimod}_{\mathfrak{C}}(A,B) & \to & \mathbf{Fun}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A), \mathbf{Mod}_{\mathfrak{C}}(B)) \\ & P & \mapsto & (-)\Box_A P \end{array}$$

induced by $\mathbf{Mod}_{\mathfrak{C}}$ is an equivalence of 2-categories. In order to exhibit a pseudo-inverse to $\mathbf{F}$, consider the following 2-functor

$$\begin{array}{rccc} \mathbf{B}: & \mathbf{Fun}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A), \mathbf{Mod}_{\mathfrak{C}}(B)) & \to & \mathbf{Bimod}_{\mathfrak{C}}(A,B), \\ & F & \mapsto & F(A) \end{array}$$

where the left A-module structure on $F(A)$ arises from the canonical A-A-bimodule structure on A. This assignment can straightforwardly be extended to left $\mathfrak{C}$-module 2-natural transformations and left $\mathfrak{C}$-module modifications. Further, for any A-B-bimodule P in $\mathfrak{C}$, we have that $\mathbf{B}\circ\mathbf{F}(P) = A\Box_A P$ as an A-B-bimodule in $\mathfrak{C}$. Thence, by lemma 4.1.2.5, we find that $\mathbf{B}\circ\mathbf{F} \simeq Id$ as desired.

Now, let $F : \mathbf{Mod}_{\mathfrak{C}}(A) \to \mathbf{Mod}_{\mathfrak{C}}(B)$ be a left $\mathfrak{C}$-module 2-functor. By definition, for any right A-module M, we have that $\big(\mathbf{F}\circ\mathbf{B}(F)\big)(M) = M\Box_A F(A)$. We claim that

$M\Box_A F(A) \simeq F(M)$ as A-B-bimodules. Namely, as splittings of 2-condensation monads are preserved by all 2-functors, it follows from the last part of theorem 4.1.1.6 that $F(A\Box M) \to F(A\Box_A M)$ is 2-universal with respect to A-balanced A-B-bimodule morphisms. Then, by comparing the 2-universal properties, we find that the 1-morphism $k^F_{M,A} : M\Box F(A) \simeq F(M\Box A)$ induces an equivalence $M\Box_A F(A) \simeq F(M\Box_A A)$ in $\mathbf{Bimod}_{\mathfrak{C}}(A,B)$. Thanks to lemma 4.1.2.5, we have $F(M\Box_A A) \simeq F(M)$ as A-B-bimodules, which established the claim. Finally, it follows from its construction that the equivalence $M\Box_A F(A) \simeq F(M)$ is 2-natural both in M and in F, so that we get $\mathbf{F}\circ\mathbf{B} \simeq Id$. This finishes proving that $\mathbf{F}$ and $\mathbf{B}$ are pseudo-inverses. □

The above theorem yields two equivalent characterizations of Morita equivalence for separable algebras in $\mathfrak{C}$. In specific examples, the second one can be unfolded further as we explain below.

Corollary 4.3.1.3. *Let A and B be two separable algebras in $\mathfrak{C}$, then the following are equivalent:*

1. *The left $\mathfrak{C}$-module 2-categories $\mathbf{Mod}_{\mathfrak{C}}(A)$ and $\mathbf{Mod}_{\mathfrak{C}}(B)$ are equivalent.*
2. *The separable algebras A and B are equivalent as objects of $\mathbf{Mor}^{sep}(\mathfrak{C})$.*

If either of these conditions is satisfied, we say that A and B are Morita equivalent.

Example 4.3.1.4. Let $\mathfrak{C} = \mathbf{2Vect}_G$ for some finite group G, and, for simplicity, let us also assume that $\Bbbk$ is algebraically closed of characteristic zero. Then, two G-graded multifusion 1-categories $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent in the sense of 2 of corollary 4.3.1.3 if and only if there exists an invertible G-graded finite semisimple $\mathcal{C}$-$\mathcal{D}$-bimodule 1-category $\mathcal{M}$. It follows from proposition 4.2 of [ENO10] that $\mathcal{M}$ is invertible if and only if $\mathcal{D}^{mop} \simeq End_{\mathcal{C}}(\mathcal{M})$ as G-graded multifusion 1-categories.

In particular, if $\mathcal{C}$ and $\mathcal{D}$ are faithfully G-graded, then they are Morita equivalent in the sense of 2 of corollary 4.3.1.3 if and only if they are graded Morita equivalent in the sense of definition 4.10 of [GJS22]. Then, the finite semisimple case of theorem 4.16 of [GJS22] asserts that $\mathcal{C}$ and $\mathcal{D}$ are graded Morita equivalent if and only if $\mathbf{Mod}_{\mathbf{2Vect}_G}(\mathcal{C})$ and $\mathbf{Mod}_{\mathbf{2Vect}_G}(\mathcal{D})$ are equivalent as 2-categories with a G-action. This is exactly the content of corollary 4.3.1.3 with $\mathfrak{C} = \mathbf{2Vect}_G$.

Example 4.3.1.5. If $\mathfrak{C} = \mathbf{2Rep}(G)$ for some finite group G, and $\Bbbk$ is algebraically closed of characteristic zero, then point 2 of corollary 4.3.1.3 can be unpacked further. Namely, let $\mathcal{C}$ be a multifusion 1-category $\mathcal{C}$ with G-action, and let $\mathcal{M}$ be finite semisimple 1-category equipped with a G-action and a compatible left $\mathcal{C}$-module structure. Then, the multifusion 1-category $End_{\mathcal{C}}(\mathcal{M})$ of all left $\mathcal{C}$-module endofunctors on $\mathcal{M}$ carries a canonical G-action. It follows from proposition 4.2 of [ENO10] that two multifusion 1-categories $\mathcal{C}$ and $\mathcal{D}$ with G-actions are Morita equivalent if and only if there exists a left $\mathcal{C}$-module 1-category $\mathcal{M}$ as above such that $\mathcal{D}^{mop} \simeq End_{\mathcal{C}}(\mathcal{M})$ as multifusion 1-categories with a G-action.

We now prove an alternative characterization of separability for a compact semisimple left $\mathfrak{C}$-module 2-category $\mathfrak{M}$ under mild assumptions on the underlying 2-categories of $\mathfrak{C}$ and $\mathfrak{M}$. More precisely, following [Déc23a], if $\Bbbk$ is perfect, we say that a compact semisimple 2-category $\mathfrak{A}$ is locally separable if for every simple object A of $\mathfrak{A}$, the finite semisimple tensor 1-category $End_{\mathfrak{A}}(A)$ is separable in the sense of [DSPS21].

Proposition 4.3.1.6. *Let $\Bbbk$ be a perfect field, and $\mathfrak{C}$ be a locally separable compact semisimple tensor 2-category. A locally separable compact semisimple left $\mathfrak{C}$-module 2-category $\mathfrak{M}$ is separable if and only if $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$ is a compact semisimple 2-category.*

Proof. The forward direction follows by combining theorem 4.3.1.2 above with proposition 3.2.1.4. Conversely, let us assume that $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$ is compact semisimple. Thanks to theorem 5.1.5 and remark 5.1.11 of [Déc21], there exists an algebra A in $\mathfrak{C}$ such that $\mathbf{Mod}_{\mathfrak{C}}(A) \simeq \mathfrak{M}$ as left $\mathfrak{C}$-module 2-categories. We will use the 2-functor

$$\mathbf{B} : \mathbf{End}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A)) \to \mathbf{Bimod}_{\mathfrak{C}}(A),$$

sending a left $\mathfrak{C}$-module 2-functor to its value on the canonical A-A-bimodule A. Firstly, observe that the image under $\mathbf{B}$ of the identity 2-functor Id on $\mathbf{Mod}_{\mathfrak{C}}(A)$ is given by A. Further, if we write $F : \mathbf{Mod}_{\mathfrak{C}}(A) \to \mathbf{Mod}_{\mathfrak{C}}(A)$ for the canonical left $\mathfrak{C}$-module 2-functor given by $M \mapsto M\Box A$, then we have $\mathbf{B}(F) = F(A) = A\Box A$. Secondly, observe that for any right A-module M, $n^M : M\Box A \to M$ defines a left $\mathfrak{C}$-module 2-natural transformation $n : F \Rightarrow Id$ such that $\mathbf{B}(n) = m : A\Box A \to A$ with its canonical A-A-bimodule structure. But $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$ has right adjoints for 1-morphisms by hypothesis, so that n has a right adjoint n^* with counit ϵ^n. As right adjoint are preserved by 2-functors, $\mathbf{B}(n^*)$ is a right adjoint for m as an A-A-bimodule 1-morphism with counit $\mathbf{B}(\epsilon^n)$. This implies that A is rigid. Thirdly, note that the 2-morphism $\epsilon^n : n \cdot n^* \Rightarrow Id$ is surjective. Namely, for every simple object M of $\mathfrak{M} \simeq \mathbf{Mod}_{\mathfrak{C}}(A)$, the 2-morphism ϵ^n_M is surjective as $n^M : M\Box A \to M$ is a non-zero 1-morphism. But $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$ is compact semisimple, so that ϵ^n has a section γ^n as a left $\mathfrak{C}$-module modification. Then, $\mathbf{B}(\gamma^n)$ is a section of $\mathbf{B}(\epsilon^n)$ as an A-A-bimodule 2-morphism so that A is separable, and the proof is complete. $\Box$

The proof of the above proposition also establishes the following result (over any field $\mathbb{k}$ and compact semisimple tensor 2-category $\mathfrak{C}$).

Corollary 4.3.1.7. *Assume that $\mathfrak{M}$ is a separable left $\mathfrak{C}$-module 2-category, and let B be any algebra such that $\mathfrak{M} \simeq \mathbf{Mod}_{\mathfrak{C}}(B)$ as left $\mathfrak{C}$-module 2-categories, then B is separable.*

Remark 4.3.1.8. Let us call two arbitrary algebras A, and B in $\mathfrak{C}$ Morita equivalent if $\mathbf{Mod}_{\mathfrak{C}}(A) \simeq \mathbf{Mod}_{\mathfrak{C}}(B)$ as left $\mathfrak{C}$-module 2-categories. Corollary 4.3.1.7 above may then be succinctly reformulated as the statement that separability is a Morita invariant property. Further, rigidity is also a Morita invariant property. On one hand, if A is a rigid algebra, then $\mathbf{Mod}_{\mathfrak{C}}(A)$ has right adjoints by theorem 3.1.2.8, so that $\mathbf{End}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A))$ has right adjoints. On the other hand, the proof of proposition 4.3.1.6 above, proves that if $\mathbf{End}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A))$ has right adjoints, then A is rigid.

We end this section by examining an example in detail.

Example 4.3.1.9. Let G be a finite group whose order is coprime to $char(\mathbb{k})$. Then, the monoidal forgetful 2-functor $\mathbf{2Vect}_G \to \mathbf{2Vect}$ provides $\mathbf{2Vect}$ with a canonical left $\mathbf{2Vect}_G$-module structure. We claim that $\mathbf{End}_{\mathbf{2Vect}_G}(\mathbf{2Vect}) \simeq \mathbf{2Rep}(G)$ as 2-categories. Namely, let $F : \mathbf{2Vect} \to \mathbf{2Vect}$ be a left $\mathbf{2Vect}_G$-module 2-functor. As $\mathbf{2Vect}$ is generated by $\mathbf{Vect}$ under direct sums and splittings of 2-condensation monads, the underlying linear 2-functor F is determined by $V := F(\mathbf{Vect})$, a perfect 1-category. Further, unfolding the definition, we find that the left $\mathbf{2Vect}_G$-module structure on F yields a G-action on V. But $\mathbf{2Vect}_G$ is the Cauchy completion of the monoidal 2-category $G \times \mathbf{2Vect}$, so that this G-action on V characterizes F completely up to equivalence. A similar argument deals with $\mathbf{2Vect}_G$-module 2-natural transformations and $\mathbf{2Vect}_G$-module modifications, establishing the desired equivalence $\mathbf{End}_{\mathbf{2Vect}_G}(\mathbf{2Vect}) \simeq \mathbf{2Rep}(G)$ of 2-categories. An immediate consequence of the above equivalence is that $\mathbf{2Vect}$ is a separable $\mathbf{2Vect}_G$-module 2-category. Over an algebraically closed field of characteristic zero, this equivalence was first observed in section 3.2 of [Del22].

For later use, we now wish to upgrade this to an equivalence of monoidal 2-categories. Observe that the identity $\mathbf{2Vect}_G$-module 2-endofunctor of $\mathbf{2Vect}$ corresponds to the object $I = \mathbf{Vect}$ of $\mathbf{2Rep}(G)$ under the above equivalence. It follows from the proof of lemma 2.3.3.12 that $\mathbf{2Rep}(G)$ is a connected compact semisimple 2-category. By proposition 2.3.3.7, the monoidal structure on $\mathbf{2Rep}(G)$ is therefore completely determined by the braiding β on the finite semisimple tensor 1-category $End_{\mathbf{2Rep}(G)}(I) \simeq \mathbf{Rep}(G)$. But, by proposition 4.2.1.10, $\mathbf{2Vect}$ is a left $\mathbf{Mod}(\mathbf{Rep}^{\beta}(G))$-module 2-category. Thus, by definition, there exists a braided monoidal functor $\mathbf{Rep}^{\beta}(G) \to \mathcal{Z}(\mathbf{Vect}) = \mathbf{Vect}$. As this functor is necessarily faithful, this forces the braiding β to be the trivial one, so that $\mathbf{End}_{\mathbf{2Vect}_G}(\mathbf{2Vect}) \simeq \mathbf{2Rep}(G)$ as monoidal 2-categories.

4.3.2 Indecomposable Module 2-Categories

Let $\mathfrak{C}$ be a compact semisimple tensor 2-category, and $\mathfrak{M}$ a compact semisimple left $\mathfrak{C}$-module 2-category. It is useful to know when the compact semisimple monoidal 2-category $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$ has simple monoidal unit. We now explain when this is the case.

Definition 4.3.2.1. A compact semisimple left $\mathfrak{C}$-module 2-category $\mathfrak{M}$ is indecomposable if there exists a simple object M of $\mathfrak{M}$ such that for any simple object N of $\mathfrak{M}$, there exists an object C in $\mathfrak{C}$ and a non-zero 1-morphisms $C \Box M \to N$.

Example 4.3.2.2. A left $\mathbf{2Vect}$-module 2-category is indecomposable if and only if the underlying compact semisimple 2-category is connected. More generally, if $\mathfrak{C}$ is a connected compact semisimple tensor 2-category, then a left $\mathfrak{C}$-module 2-category is indecomposable if and only if the underlying compact semisimple 2-category is connected.

Lemma 4.3.2.3. *Let $\mathfrak{M}$ be a compact semisimple left $\mathfrak{C}$-module 2-category. There exists a decomposition*

$$\mathfrak{M} \simeq \boxplus_{i=1}^{n} \mathfrak{M}_i$$

into a direct sum of indecomposable compact semisimple left $\mathfrak{C}$-module 2-categories.

Proof. Let $\mathscr{O}(\mathfrak{M})$ denote the finite set of equivalence classes of simple objects of $\mathfrak{M}$. Let M, N be two (equivalence classes of) simple objects in $\mathfrak{M}$, we write $M \sim N$ if there exists an object C of $\mathfrak{C}$ and a non-zero 1-morphism $f : C \Box M \to N$. This relation is evidently reflexive, symmetry follows from lemma 2.2.10 of [Déc21], and transitivity from lemma 2.2.11 of [Déc21]. Let us write $\mathscr{O}(\mathfrak{M})/\sim = \{X_1, ..., X_n\}$, and let $\mathfrak{M}_i$ be the full compact semisimple sub-2-category of $\mathfrak{M}$ generated under direct sums and splittings of 2-condensation monads by the simple objects in X_i. As the relation $\sim$ is coarser than that of being connected, the sub-2-categories $\mathfrak{M}_i$ and $\mathfrak{M}_j$ do not contain any common simple object. Furthermore, it is immediate from the definition of $\sim$ that $\mathfrak{M}_i$ inherits a left $\mathfrak{C}$-module structure, under which it is indecomposable. Thence, we find $\mathfrak{M} \simeq \boxplus_{i=1}^{n} \mathfrak{M}_i$ as left $\mathfrak{C}$-module 2-categories. □

Lemma 4.3.2.4. *Let $\mathfrak{M}$ be a compact semisimple left $\mathfrak{C}$-module 2-category. Then, the identity left $\mathfrak{C}$-module 2-functor on $\mathfrak{M}$ splits as the direct sum of the projectors onto the $\mathfrak{M}_i$. Further, if $\mathfrak{M}$ is separable, every such projector is a simple object of* $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$.

Proof. The first assertion is immediate. Let us assume that $\mathfrak{M}$ is separable, and $\mathfrak{M} \simeq \boxplus_{i=1}^{n} \mathfrak{M}_i$ be a decomposition of $\mathfrak{M}$ as a direct sum of indecomposable compact semisimple left $\mathfrak{C}$-module 2-categories. We wish to prove that the projection $P_i : \mathfrak{M} \twoheadrightarrow \mathfrak{M}_i \hookrightarrow \mathfrak{M}$ is a simple object of $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$. To this end, let $Q, R : \mathfrak{M} \to \mathfrak{M}$ be two $\mathfrak{C}$-module 2-functor such that $P_i = Q \boxplus R$. Let us additionally assume that $Q(M)$ is non-zero for some M (necessarily in $\mathfrak{M}_i$). Then, it follows from the proof of lemma 4.3.2.3 that given any simple object N of $\mathfrak{M}_i$, there exists an object C of $\mathfrak{C}$ and a non-zero 1-morphism $C \Box N \to M$. In particular, M is the splitting of a 2-condensation monad supported on $C \Box N$. But splittings of 2-condensation monads are preserved by all 2-functors, so that M is the splitting of a 2-condensation monad on $Q(C \Box N)$. As M is non-zero, so must be $Q(C \Box N)$. Now, Q is a left $\mathfrak{C}$-module 2-functor, so that $Q(C \Box N) \simeq C \Box Q(N)$, which implies that $Q(N)$ is non-zero. Finally, we have $N = P_i(N) = Q(N) \boxplus R(N)$, and N is simple, so that $R(N) = 0$ by proposition 2.1.2.8 above. As N was arbitrary, we find that $R = 0$, which finishes the proof of the lemma. $\Box$

Corollary 4.3.2.5. *Let $\mathfrak{M}$ be a separable left $\mathfrak{C}$-module 2-category. Then $\mathfrak{M}$ is indecomposable if and only if* $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$ *has simple monoidal unit.*

Given the equivalence of 3-categories established in theorem 4.3.1.2, it is only natural to examine what property of a separable algebra corresponds to indecomposability of the associated module 2-category.

Definition 4.3.2.6. Let A be a separable algebra. We say that A is indecomposable if A is simple as an A-A-bimodule.

Corollary 4.3.2.7. *A separable algebra A is indecomposable if and only if* $\mathbf{Mod}_{\mathfrak{C}}(A)$ *is indecomposable.*

Remark 4.3.2.8. In particular, this shows that being indecomposable is a Morita invariant property of separable algebras in $\mathfrak{C}$. Further, it follows from lemma 5.2.6 that any separable algebra A may be split into a direct sum of indecomposable separable algebras. A direct proof of this fact is given in the proof of theorem 3.2.1.7 above.

Finally, definition 4.3.2.1 admits an obvious analogue for bimodule 2-categories. This yields a notion of indecomposability for compact semisimple tensor 2-categories.

Definition 4.3.2.9. We say that the compact semisimple tensor 2-category $\mathfrak{C}$ is indecomposable if it is indecomposable as a $\mathfrak{C}$-$\mathfrak{C}$-bimodule 2-category.

The proof of lemma 4.3.2.3 can be adapted in the obvious so as to give the following result.

Lemma 4.3.2.10. *Every compact semisimple tensor 2-category $\mathfrak{C}$ splits as a direct sum of finitely many indecomposable compact semisimple tensor 2-categories.*

4.3.3 Dual Tensor 2-Categories

In this section, we assume throughout that $\mathbb{k}$ is perfect. The following technical result is needed to prove our main theorem over fields of positive characteristic. Recall that a compact semisimple 2-category $\mathfrak{C}$ is locally separable if for every simple object C of $\mathfrak{C}$, the finite semisimple tensor 1-category $End_{\mathfrak{C}}(C)$ is separable in the sense of [DSPS21]. We remark that, over fields of characteristic zero, they show that every finite semisimple tensor 1-category is separable, so that every compact semisimple 2-category is locally separable over such fields.

Proposition 4.3.3.1. *Let $\mathfrak{C}$ be a locally separable compact semisimple monoidal 2-category, and A a separable algebra in $\mathfrak{C}$. Then, $\mathbf{Mod}_{\mathfrak{C}}(A)$ is locally separable.*

Proof. By theorem 2.1.5.7, there exists a separable finite semisimple tensor 1-category $\mathcal{C}$ such that $\mathbf{Mod}(\mathcal{C}) \simeq \mathfrak{C}$ as 2-categories. It follows from theorem 4.3.1.2 that

$$\mathbf{End}(\mathfrak{C}) \simeq \mathbf{Bimod}(\mathcal{C})^{mop}.$$

Further, as $\mathfrak{C}$ is locally separable, theorem 3.2.1.7 above and corollary 2.5.9 of [DSPS21] imply that $\mathbf{Bimod}(\mathcal{C})$ is compact semisimple. Then, thanks to corollary 4.1.2.12, we find that separable algebras in $\mathbf{Bimod}(\mathcal{C})$ are precisely given by separable finite semisimple tensor 1-categories $\mathcal{D}$ equipped with a monoidal functor $\mathcal{C} \to \mathcal{D}$. Now, observe that the separable algebra A in $\mathfrak{C}$ yields a separable algebra $\mathcal{A}$ in $\mathbf{End}(\mathfrak{C}) \simeq \mathbf{Bimod}(\mathcal{C})^{mop}$ via $C \mapsto C\Box A$. Further, it follows from the definition that

$$\mathbf{Mod}_{\mathfrak{C}}(A) \simeq \mathbf{Mod}_{\mathfrak{C}}(\mathcal{A}),$$

where, on the right hand-side, we use the canonical right $\mathbf{End}(\mathfrak{C})^{mop}$-module structure on $\mathfrak{C}$.

Now, note that $End(\mathcal{C})$, the finite semisimple tensor 1-category of linear endofunctors of $\mathcal{C}$, is a separable algebra in $\mathbf{Bimod}(\mathcal{C})$ via the left action of $\mathcal{C}$ on itself. Further, there are equivalences of right $\mathbf{Bimod}(\mathcal{C})$-module 2-categories

$$\mathbf{LMod}_{\mathbf{Bimod}(\mathcal{C})}(End(\mathcal{C})) \simeq \mathbf{Bimod}(End(\mathcal{C}), \mathcal{C}) \simeq \mathbf{Mod}(\mathcal{C}) \simeq \mathfrak{C},$$

as $End(\mathcal{C})$ and $\mathbf{Vect}$ are Morita equivalent finite semisimple tensor 1-categories. Putting everything together, we find that there are equivalences of 2-categories

$$\begin{aligned}\mathbf{Mod}_{\mathfrak{C}}(A) &\simeq \mathbf{Bimod}_{\mathbf{Bimod}(\mathcal{C})}(End(\mathcal{C}), \mathcal{A}) \\ &\simeq \mathbf{Bimod}(End(\mathcal{C}), \mathcal{A}) \simeq \mathbf{Mod}(End(\mathcal{C})^{mop} \boxtimes \mathcal{A}).\end{aligned}$$

But it follows from corollary 2.5.11 of [DSPS21] that $End(\mathcal{C})^{mop} \boxtimes \mathcal{A}$ is a separable finite semisimple tensor 1-category, so that $\mathbf{Mod}(End(\mathcal{C})^{mop} \boxtimes \mathcal{A})$ is locally separable by theorem 2.1.5.6. □

We are now in the position to prove the following theorem.

Theorem 4.3.3.2. *Let $\Bbbk$ be a perfect field, and A a separable algebra in a locally separable compact semisimple tensor 2-category $\mathfrak{C}$. Then, the monoidal 2-category*

$$\mathbf{End}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A)) \simeq \mathbf{Bimod}_{\mathfrak{C}}(A)^{mop}$$

is a compact semisimple tensor 2-category.

Proof. The equivalence of monoidal 2-categories is an immediate consequence of theorem 4.3.1.2. Furthermore, it follows from theorem 3.2.1.7 that the underlying 2-category of $\mathbf{Bimod}_{\mathfrak{C}}(A)$ is compact semisimple. Thus, it only remains to establish the existence of duals. We will show that $\mathbf{End}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A))$ satisfies this property. Namely, it follows from proposition 4.3.3.1 that $\mathbf{Mod}_{\mathfrak{C}}(A)$ is locally separable. Then, corollary 2.3.2.3 shows that every linear 2-functor $\mathbf{Mod}_{\mathfrak{C}}(A) \to \mathbf{Mod}_{\mathfrak{C}}(A)$ has a right 2-adjoint 2-functor. In particular, proposition 4.2.2.3 applies to every object of $\mathbf{End}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A))$, which proves that $\mathbf{End}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A))$ has right duals. By corollary 1.3.4 of [Déc22c], $\mathbf{End}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A))$ also has left duals, which concludes the proof of the result. □

Remark 4.3.3.3. The assumption that $\mathfrak{C}$ be locally separable in theorem 4.3.3.2 might not be necessary. Namely, we believe that it is possible to show directly that for any separable algebra A in a compact semisimple tensor 2-category, the monoidal 2-category $\mathbf{Bimod}_{\mathfrak{C}}(A)$ has duals. We leave it to the interested reader to pursue this line of investigation.

Thanks to the above theorem, the following definition is sensible.

Definition 4.3.3.4. Let $\mathfrak{C}$ be a locally separable compact semisimple tensor 2-category, and let $\mathfrak{M}$ be a separable left $\mathfrak{C}$-module 2-category. We write $\mathfrak{C}^*_{\mathfrak{M}}$ for the compact semisimple tensor 2-category $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$, and call it the dual tensor 2-category to $\mathfrak{C}$ with respect to $\mathfrak{M}$.

Combining the theorem 4.3.3.2 above with corollary 2.2.2.3, we obtain the following result.

Corollary 4.3.3.5. *Let* $\Bbbk$ *be an algebraically closed field of characteristic zero,* $\mathfrak{C}$ *a multifusion 2-category, and* $\mathfrak{M}$ *a separable left* $\mathfrak{C}$*-module 2-category. Then,* $\mathfrak{C}^*_{\mathfrak{M}}$*, the dual tensor 2-category to* $\mathfrak{C}$ *with respect to* $\mathfrak{M}$*, is a multifusion 2-category.*

The following corollary follows from corollary 3.2.3.10 above.

Corollary 4.3.3.6. *Let* G *be a finite group, and* π *a 4-cocycle for* G *with coefficients in* $\Bbbk^{\times}$*. Further, let* $H \subseteq G$ *be a subgroup of order coprime to* $char(\Bbbk)$ *and* γ *a 3-cochain for* H *with coefficients in* $\Bbbk^{\times}$*. Then,* $\mathbf{Bimod}_{\mathbf{2Vect}^{\pi}_{G}}(\mathbf{Vect}^{\gamma}_{H})$ *is a compact semisimple tensor 2-category.*

We end this section by examining some examples of dual tensor 2-categories.

Example 4.3.3.7. Let $\Bbbk$ be an algebraically closed field, and let $\mathcal{C}$ be a separable fusion 1-category. The locally separable compact semisimple 2-category $\mathbf{Mod}(\mathcal{C})$ admits a canonical left $\mathbf{2Vect}$-module structure. We claim that $\mathbf{2Vect}^*_{\mathbf{Mod}(\mathcal{C})} \simeq \mathbf{Mod}(\mathcal{Z}(\mathcal{C}))$ as monoidal 2-categories. Note that $\mathcal{Z}(\mathcal{C})$ is a finite semisimple tensor 1-category thanks to corollary 2.5.9 of [DSPS21]. Theorem 4.3.1.2 provides us with an equivalence of monoidal 2-categories $\mathbf{2Vect}^*_{\mathbf{Mod}(\mathcal{C})} \simeq \mathbf{Bimod}(\mathcal{C})^{mop}$. Provided $\Bbbk$ has characteristic zero, theorem 1.3 of [Gre10] gives an equivalence of monoidal 2-categories $\mathbf{Bimod}(\mathcal{C}) \simeq \mathbf{LMod}(\mathcal{Z}(\mathcal{C}))$, to the monoidal 2-category of finite semisimple left $\mathcal{Z}(\mathcal{C})$-module 1-categories. Finally, we have $\mathbf{LMod}(\mathcal{Z}(\mathcal{C})) \simeq \mathbf{Mod}(\mathcal{Z}(\mathcal{C})^{\beta op})$ as monoidal 2-categories, which concludes the proof.

Alternatively, over an arbitrary algebraically closed field $\Bbbk$, as $\mathcal{C}$ is a separable fusion 1-category, $\mathcal{C}^{mop} \boxtimes \mathcal{C}$ is also a separable fusion 1-category, so that the compact semisimple 2-category $\mathbf{Bimod}(\mathcal{C}) \simeq \mathbf{Mod}(\mathcal{C}^{mop} \boxtimes \mathcal{C})$ is connected. Thus, by proposition 2.3.3.7, in order to determine the monoidal structure on $\mathbf{Bimod}(\mathcal{C})$, it is enough to understand the braiding on the fusion 1-category $End_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C})$ of endomorphisms the monoidal unit. Inspection shows that it is equivalent to $\mathcal{Z}(\mathcal{C})^{mop}$ as a braided monoidal 1-category. But, there is an equivalence of braided monoidal 1-categories $\mathcal{Z}(\mathcal{C})^{mop} \simeq \mathcal{Z}(\mathcal{C})^{\beta op}$. Thus, we have $End_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{C})^{\beta op}$ as braided fusion 1-categories.

Example 4.3.3.8. We now discuss a generalization of example 4.3.3.7. For simplicity, we will assume that $\Bbbk$ is an algebraically closed field of characteristic zero. Let $\mathcal{B}$ be a non-degenerate braided fusion 1-category, and $\mathcal{C}$ a $\mathcal{B}$-central fusion 1-category. We claim that there is an equivalence of monoidal 2-categories $\mathbf{Mod}(\mathcal{B})^*_{\mathbf{Mod}(\mathcal{C})} \simeq \mathbf{Mod}(\mathcal{A})$, where $\mathcal{A}$ is the centralizer of $\mathcal{B}$ in $\mathcal{Z}(\mathcal{C})$, which is non-degenerate by theorem 3.13 of [DGNO10]. By corollary 5.9 of [DMNO13], this implies further that $\mathcal{B}$ and $\mathcal{A}$ are Witt equivalent non-degenerate braided fusion 1-categories.

In order to prove the claim, observe that, as $\mathcal{C}$ is a separable algebra in $\mathbf{Mod}(\mathcal{B})$, corollary 4.3.3.5 establishes that $\mathbf{Mod}(\mathcal{B})^*_{\mathbf{Mod}(\mathcal{C})} \simeq \mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})^{mop}$ is a fusion 2-category. Moreover, after having unfolded the definitions, we find $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \simeq \mathbf{Mod}(\mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C})$ as finite semisimple 2-categories. As $\mathcal{B}$ is non-degenerate, it follows from theorems 2.26 and 3.20 of [BJSS21] that there is an equivalence

$$\mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C} \simeq End_{\mathcal{A}}(\mathcal{C})$$

of multifusion 1-categories. In particular, $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \simeq \mathbf{Mod}(\mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C})$ is a connected finite semisimple 2-category. The above equivalence of multifusion 1-categories implies that $\mathcal{A} \simeq End_{\mathcal{C}^{mop}\boxtimes_{\mathcal{B}}\mathcal{C}}(\mathcal{C})$ as fusion 1-categories, so that the endomorphism fusion 1-category of $\mathcal{C}$ in $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$ is given by $\mathcal{A}$. But $\mathcal{C}$ is the monoidal unit of $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$. Thence, appealing to proposition 2.4.7 of [Déc22c], it is enough to understand the braiding on the fusion 1-category $\mathcal{A}$ of endomorphisms of $\mathcal{C}$ in $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$.

To this end, note that the forgetful 2-functor $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \to \mathbf{Bimod}(\mathcal{C})$ induces the canonical inclusion of fusion 1-categories $\mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{C})^{\beta op}$. But the forgetful monoidal 2-functor

$$\mathbf{End}_{\mathbf{Mod}(\mathcal{B})}(\mathbf{Mod}(\mathcal{C}))^{mop} \to \mathbf{End}(\mathbf{Mod}(\mathcal{C}))^{mop}$$

is identified via theorem 4.3.1.2 to $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \to \mathbf{Bimod}(\mathcal{C})$. This shows that the monoidal inclusion $\mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{C})^{\beta op}$ is in fact braided, thereby establishing the desired equivalence $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \simeq \mathbf{Mod}(\mathcal{A}^{\beta op})$ of monoidal 2-categories.

Remark 4.3.3.9. As one can readily observe from example 4.3.1.9, the non-degeneracy condition in example 4.3.3.8 above can not be omitted in general.

We now generalize example 4.3.1.9 to finite 2-groups.

Proposition 4.3.3.10. *Let* $\Bbbk$ *be a field of characteristic zero, and let* $\mathcal{G}$ *be a finite 2-group. The dual of* $\mathbf{2Vect}_{\mathcal{G}}$ *with respect to* $\mathbf{2Vect}$ *is* $\mathbf{2Rep}(\mathcal{G})$*.*

Proof. Observe that there is a monoidal 2-functor $\mathbf{U} : \mathbf{2Vect}_{\mathcal{G}} \to \mathbf{2Vect}$. Namely, by definition, $\mathbf{2Vect}_{\mathcal{G}}$ is the Cauchy completion of the monoidal 2-category $\mathcal{G} \times \mathrm{B}^2\Bbbk$. The monoidal 2-functor $\mathbf{U}$ is induced by taking the Cauchy completion of the canonical monoidal 2-functor $\mathcal{G} \times \mathrm{B}^2\Bbbk \to \mathrm{B}^2\Bbbk$. In particular, $\mathbf{2Vect}$ is canonically a left $\mathbf{2Vect}_{\mathcal{G}}$-module 2-category. As $\mathbf{2Vect}_{\mathcal{G}}$ is the Cauchy completion of the monoidal 2-category $\mathcal{G} \times \mathrm{B}^2\Bbbk$, it follows from the 3-universal property of the Cauchy completion (see [Déc24]) that the data of a left $\mathbf{2Vect}_{\mathcal{G}}$-module structure on a 2-functor, 2-natural transformation, or modification is equivalent to the data of a left $\mathcal{G} \times \mathrm{B}^2\Bbbk$-module structure. Moreover, all the 2-functors that we consider are linear, so that the latter structures are completely determined by their underlying $\mathcal{G}$-module structures. This means that we have an equivalence of monoidal 2-categories

$$\mathbf{End}_{\mathbf{2Vect}_{\mathcal{G}}}(\mathbf{2Vect}) \simeq \mathbf{End}_{\mathcal{G}}(\mathbf{2Vect}).$$

Furthermore, as $\mathbf{2Vect} \simeq \mathbf{Mod}(\mathbf{Vect})$, any left $\mathcal{G}$-module 2-endofunctor on $\mathbf{2Vect}$ is completely determined by its value on $\mathbf{Vect}$, which is a 2-representation of $\mathcal{G}$. Thus, there is an equivalence of 2-categories

$$\begin{array}{ccc} \mathbf{End}_{\mathcal{G}}(\mathbf{2Vect}) & \to & \mathbf{2Rep}(\mathcal{G}). \\ F & \mapsto & F(\mathbf{Vect}) \end{array}$$

Its pseudo-inverse is given by the 2-functor sending the 2-representation R of $\mathcal{G}$ to the 2-endofunctor F_R on $\mathbf{2Vect}$ given by $F_R(V) := R \boxtimes V$ with left $\mathcal{G}$-module structure induced by the action of $\mathcal{G}$ on R. Direct inspection shows that this pseudo-inverse is a monoidal 2-functor, so that there is an equivalence of monoidal 2-categories

$$\mathbf{End}_{\mathcal{G}}(\mathbf{2Vect}) \simeq \mathbf{2Rep}(\mathcal{G}).$$

This concludes the proof as $\mathbf{2Rep}(\mathcal{G})$ is symmetric monoidal. □

Remark 4.3.3.11. The above proposition can be slightly generalized. Namely, monoidal (linear) 2-functors $\mathcal{G} \times \mathrm{B}^2\Bbbk \to \mathrm{B}^2\Bbbk$ are parametrised up to equivalence by $H^3(\mathrm{B}\mathcal{G}, \Bbbk^\times)$. Thus, given a 3-cocycle γ for $\mathcal{G}$ with coefficients in $\Bbbk^\times$, we can define a monoidal 2-functor $\mathbf{U}^\gamma : \mathbf{2Vect}_{\mathcal{G}} \to \mathbf{2Vect}$. This endows $\mathbf{2Vect}$ with a left $\mathbf{2Vect}_{\mathcal{G}}$-module structure, and we write $\mathbf{2Vect}^\gamma$ for this module 2-category. We again find that the dual of $\mathbf{2Vect}_{\mathcal{G}}$ with respect to $\mathbf{2Vect}^\gamma$ is $\mathbf{2Rep}(G)$. Namely, a left $\mathcal{G}$-module 2-endofunctor on $\mathbf{2Vect}^\gamma$ is completely determined by its evaluation on $\mathbf{Vect}$ together with its left action by $\mathcal{G}$, which is exactly the data of a 2-representation of $\mathcal{G}$. Furthermore, it is not necessary to assume that $\Bbbk$ has characteristic zero. The above proof goes through so long as the characteristic of $\Bbbk$ is coprime to both $|\pi_1(\mathcal{G})|$ and $|\pi_2(\mathcal{G})|$.

4.3.4 Morita Equivalence

Let us fix $\Bbbk$ a perfect field. We give three equivalent characterizations of Morita equivalence between locally separable compact semisimple tensor 2-categories. Before doing so, we need a definition.

Let $\mathfrak{C}$ be a compact semisimple tensor 2-category. It follows from proposition 2.1.2.8 that there is a splitting $I \simeq \boxplus I_i$ of the monoidal unit of $\mathfrak{C}$ into a finite direct sum of simple objects.

Definition 4.3.4.1. Let $\mathfrak{M}$ be a left $\mathfrak{C}$-module 2-category. We say that $\mathfrak{M}$ is faithful if the action of I_i on $\mathfrak{M}$ is non-zero for every i. Let A be an algebra in $\mathfrak{C}$, we say that A is faithful if $\mathbf{Mod}_{\mathfrak{C}}(A)$ is a faithful left $\mathfrak{C}$-module 2-category.

Remark 4.3.4.2. If $\mathfrak{C}$ has simple monoidal unit, then every non-zero module 2-category is faithful. This holds more generally if $\mathfrak{C}$ is indecomposable. In fact, if $\mathfrak{C} \simeq \boxplus \mathfrak{C}_n$ is a splitting of $\mathfrak{C}$ into a finite direct sum of indecomposable compact semisimple tensor 2-categories, then an algebra A in $\mathfrak{C}$ is faithful if and only if its underlying object has a summand in $\mathfrak{C}_n$ for every n.

Theorem 4.3.4.3. *For any two locally separable compact semisimple tensor 2-categories $\mathfrak{C}$ and $\mathfrak{D}$ over a perfect field $\Bbbk$, the following are equivalent:*

1. *The 3-categories $\mathbf{LMod}^{sep}(\mathfrak{C})$ and $\mathbf{LMod}^{sep}(\mathfrak{D})$ are equivalent.*
2. *There exists a faithful separable left $\mathfrak{C}$-module 2-category $\mathfrak{M}$, and an equivalence of monoidal 2-categories $\mathfrak{D}^{mop} \simeq \mathfrak{C}^*_{\mathfrak{M}}$.*
3. *There exists a faithful separable algebra A in $\mathfrak{C}$, and an equivalence of monoidal 2-categories $\mathfrak{D} \simeq \mathbf{Bimod}_{\mathfrak{C}}(A)$.*

If any of the above equivalent conditions is satisfied, we say that $\mathfrak{C}$ and $\mathfrak{D}$ are Morita equivalent.

Proof. The equivalence between 2 and 3 follows from theorem 4.3.1.2. Let us now assume that there is an equivalence of 3-categories $\mathbf{F} : \mathbf{LMod}^{sep}(\mathfrak{C}) \simeq \mathbf{LMod}^{sep}(\mathfrak{D})$, with pseudo-inverse $\mathbf{G}$. As there is a canonical equivalence of monoidal 2-categories $\mathbf{End}_{\mathfrak{D}}(\mathfrak{D}) \simeq \mathfrak{D}^{mop}$, and $\mathbf{G}$ induces an equivalence

$$\mathbf{End}_{\mathfrak{D}}(\mathfrak{D}) \simeq \mathbf{End}_{\mathfrak{C}}(\mathbf{G}(\mathfrak{D}))$$

of monoidal 2-categories, we find that $\mathfrak{D}^{mop} \simeq \mathfrak{C}^*_{\mathbf{G}(\mathfrak{D})}$. Now, suppose that there is a simple summand I_i of the monoidal unit I of $\mathfrak{C}$ that acts as zero on $\mathbf{G}(\mathfrak{D})$. Then, we have $\mathbf{Fun}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(I_i), \mathbf{G}(\mathfrak{D})) = 0$. On the other hand, as $\mathbf{F}$ is an equivalence of 3-categories, we have that

$$\mathbf{Fun}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(I_i), \mathbf{G}(\mathfrak{D})) \simeq \mathbf{Fun}_{\mathfrak{D}}(\mathbf{F}(\mathbf{Mod}_{\mathfrak{C}}(I_i)), \mathfrak{D}),$$

and the right hand-side is non-zero by proposition 4.2.2.3, so that $\mathbf{G}(\mathfrak{D})$ is a faithful left $\mathfrak{C}$-module 2-category.

Conversely, let A be a faithful separable algebra A in $\mathfrak{C}$ such that $\mathfrak{D} := \mathbf{Bimod}_{\mathfrak{C}}(A)$ is a locally separable compact semisimple tensor 2-category. Without loss of generality, we may assume that $\mathfrak{C}$ is strict cubical. Firstly, we claim that there is a faithful separable algebra B in $\mathfrak{D}$ such that

$$\mathbf{Bimod}_{\mathfrak{D}}(B) \simeq \mathfrak{C}$$

as monoidal 2-categories.

To see this, recall from corollary 4.1.2.12 that separable algebras in $\mathfrak{D}$ are precisely separable algebras in $\mathfrak{C}$ equipped with an algebra 1-homomorphism from A. Now, we fix C an object of $\mathfrak{C}$ that has a simple summand in every connected component of $\mathfrak{C}$, and write $({}^\sharp C, C, i_C, e_C, E_C, F_C)$ for a coherent left dual for C in the sense of [Pst14] (see also [Déc22c]). We take B to be the algebra in $\mathfrak{C}$ whose underlying object is $A\Box C\Box({}^\sharp C)\Box A$, and with unit and mutliplication 1-morphisms given by

$$i^B : I \xrightarrow{i} A \xrightarrow{m^*} AA \xrightarrow{1i_C1} AC({}^\sharp C)A,$$

$$\begin{aligned} m^B : AC({}^\sharp C)AAC({}^\sharp C)A \xrightarrow{111m111} AC({}^\sharp C)AC({}^\sharp C)A \xrightarrow{111i^*111} AC({}^\sharp C)C({}^\sharp C)A \\ \xrightarrow{11e_C11} AC({}^\sharp C)A, \end{aligned}$$

where i^* is a right adjoint in $\mathfrak{C}$ for the unit i of A. The coherence 2-isomorphisms are defined in the obvious way. Further, it follows from the definition that $(1i_C1) \circ m^* : A \to B$ is an algebra 1-homomorphism in $\mathfrak{C}$. Moreover, we have that $\mathfrak{C} \simeq \mathbf{Mod}_{\mathfrak{C}}(B)$ as left $\mathfrak{C}$-module 2-categories via $D \mapsto D\Box({}^\sharp C)\Box A$ for every D in $\mathfrak{C}$. Namely, as A is faithful, $A\Box C$ is a $\mathfrak{C}$-generator of $\mathfrak{C}$ in the sense of definition 5.1.2 of [Déc21].

In particular, we can apply theorem 5.1.5 of [Déc21] with $M := A\Box C$, or more precisely the generalization given in remark 5.1.11. The assertion then follows by combining example 4.1.3 of [Déc21] with the fact that A is a self-dual object of $\mathfrak{C}$ with coevaluation 1-morphism $m^* \circ i$ and evaluation 1-morphism $i^* \circ m$. For later use, we also record that the 2-functor $\mathfrak{C} \to \mathbf{LMod}_{\mathfrak{C}}(B)$ given by $D \mapsto A\Box C\Box D$ is an equivalence of right $\mathfrak{C}$-module 2-categories. This follows from the above argument applied to $\mathfrak{C}^{mop}$. Finally, theorem 4.3.1.2 implies that there is an equivalence of monoidal 2-categories $\mathfrak{C} \to \mathbf{Bimod}_{\mathfrak{C}}(B)$, which is given by $D \mapsto A\Box C\Box D\Box(^\sharp C)\Box A$. This concludes the proof of the claim.

Secondly, observe that there is a 3-functor

$$\mathbf{Fun}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A), -) : \mathbf{LMod}^{sep}(\mathfrak{C}) \to \mathbf{LMod}(\mathfrak{D}).$$

Up to the equivalence of 3-categories of theorem 4.3.1.2, this 3-functor is equivalent to $\mathbf{Bimod}_{\mathfrak{C}}(A, -) : \mathbf{Mor}^{sep}(\mathfrak{C}) \to \mathbf{LMod}(\mathfrak{D})$. Thanks to the claim above, there is also a 3-functor

$$\mathbf{Fun}_{\mathfrak{D}}(\mathbf{Mod}_{\mathfrak{D}}(B), -) : \mathbf{LMod}^{sep}(\mathfrak{D}) \to \mathbf{LMod}(\mathfrak{C}).$$

This last 3-functor is equivalent to $\mathbf{Bimod}_{\mathfrak{D}}(B, -) : \mathbf{Mor}^{sep}(\mathfrak{D}) \to \mathbf{LMod}(\mathfrak{C})$ via the equivalence of 3-categories of theorem 4.3.1.2. Further, up to the identification $\mathfrak{C} \simeq \mathbf{Bimod}_{\mathfrak{C}}(B)$, it follows from the claim above and its proof that $\mathbf{Bimod}_{\mathfrak{D}}(B, -) \simeq \mathbf{Mod}_{\mathfrak{C}}(-)$ as left $\mathfrak{C}$-module 2-categories. In particular, $\mathbf{Fun}_{\mathfrak{D}}(\mathbf{Mod}_{\mathfrak{D}}(B), -)$ factors through $\mathbf{LMod}^{sep}(\mathfrak{C})$. Putting the above discussion together, we find that there are equivalences of 3-functors

$$\begin{aligned} &\mathbf{Fun}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A), \mathbf{Fun}_{\mathfrak{D}}(\mathbf{Mod}_{\mathfrak{D}}(B), \mathbf{Mod}_{\mathfrak{D}}(-))) \\ &\qquad\qquad \simeq \mathbf{Fun}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A), \mathbf{Bimod}_{\mathfrak{D}}(B, -)) \\ &\qquad\qquad \simeq \mathbf{Fun}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A), \mathbf{Mod}_{\mathfrak{C}}(-)) \\ &\qquad\qquad \simeq \mathbf{Bimod}_{\mathfrak{C}}(A, -) \simeq \mathbf{Mod}_{\mathfrak{D}}(-). \end{aligned}$$

This shows that the composite of the 3-functor $\mathbf{Fun}_{\mathfrak{D}}(\mathbf{Mod}_{\mathfrak{D}}(B), -)$ together with $\mathbf{Fun}_{\mathfrak{C}}(\mathbf{Mod}_{\mathfrak{C}}(A), -)$ is equivalent to the identity on $\mathbf{LMod}^{sep}(\mathfrak{D})$.

Finally, one can run the above argument starting with $\mathfrak{D}$ and B. This shows that the 3-functor $\mathbf{Fun}_{\mathfrak{D}}(\mathbf{Mod}_{\mathfrak{D}}(B), -)$ has both a left and a right pseudo-inverse, so that it induces an equivalences of 3-categories $\mathbf{LMod}^{sep}(\mathfrak{D}) \simeq \mathbf{LMod}^{sep}(\mathfrak{C})$ as desired. □

Let us record the following corollary of the proof of theorem 4.3.4.3.

Corollary 4.3.4.4. *Let $\mathfrak{C}$ be a locally separable compact semisimple tensor 2-category, and let $\mathfrak{M}$ be a separable left $\mathfrak{C}$-module 2-category. Then, $\mathfrak{M}$ is a separable left $\mathfrak{C}^*_{\mathfrak{M}}$-module 2-category. Furthermore, if $\mathfrak{M}$ is faithful, the canonical monoidal 2-functor $\mathfrak{C} \to (\mathfrak{C}^*_{\mathfrak{M}})^*_{\mathfrak{M}}$ is an equivalence.*

Proof. The first part is immediate. For the second part, we use the notations of the proof of theorem 4.3.4.3. In particular, there is an equivalence of monoidal 2-categories

$$(\mathfrak{C}^*_{\mathfrak{M}})^*_{\mathfrak{M}} \simeq \mathbf{Bimod}_{\mathfrak{C}}(B).$$

Under this equivalence, the canonical monoidal 2-functor $\mathfrak{C} \to (\mathfrak{C}^*_{\mathfrak{M}})^*_{\mathfrak{M}}$ is identified with the 2-functor $\mathfrak{C} \to \mathbf{Bimod}_{\mathfrak{C}}(B)$ given by $D \mapsto A\Box C\Box D\Box(^\sharp C)\Box A$. But, this 2-functor was shown to be an equivalence in the course of the proof of theorem 4.3.4.3. □

We now examine two examples.

Example 4.3.4.5. Let G be a finite group whose order is not divisible by $char(\mathbb{k})$. It follows from example 4.3.1.9 above that the locally separable compact semisimple tensor 2-categories $\mathbf{2Vect}_G$ and $\mathbf{2Rep}(G)$ are Morita equivalent. In particular, the 3-categories of separable left $\mathbf{2Vect}_G$-module 2-categories and of separable left $\mathbf{2Rep}(G)$-module 2-categories are equivalent.

Example 4.3.4.6. Let $\mathbb{k}$ be algebraically closed of characteristic zero, and $\mathcal{B}_1$, $\mathcal{B}_2$ be two non-degenerate braided fusion 1-categories. It follows from example 4.3.3.8 that the fusion 2-categories $\mathbf{Mod}(\mathcal{B}_1)$ and $\mathbf{Mod}(\mathcal{B}_2)$ are Morita equivalent if and only if the non-degenerate braided fusion 1-categories $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent in the sense of [DMNO13].

For later use, we also record the following technical lemmas, which show that Morita equivalence is compatible with the 2-Deligne tensor product.

Lemma 4.3.4.7. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two compact semisimple monoidal 2-categories. For any separable algebras A in $\mathfrak{C}$ and B in $\mathfrak{D}$, there is an equivalence*

$$\mathbf{Mod}_{\mathfrak{C}}(A) \boxtimes \mathbf{Mod}_{\mathfrak{D}}(B) \simeq \mathbf{Mod}_{\mathfrak{C}\boxtimes\mathfrak{D}}(A \boxtimes B)$$

of left $\mathfrak{C} \boxtimes \mathfrak{D}$-module 2-categories.

Proof. Thanks to the 3-universal property of the 2-Deligne tensor product obtained in theorem 3.7 of [Déc24], the bilinear 2-functor

$$\begin{array}{ccc} \mathbf{Mod}_{\mathfrak{C}}(A) \times \mathbf{Mod}_{\mathfrak{D}}(B) & \rightarrow & \mathbf{Mod}_{\mathfrak{C}\boxtimes\mathfrak{D}}(A \boxtimes B). \\ (M, N) & \mapsto & M \boxtimes N \end{array}$$

induces a linear 2-functor $\mathbf{K} : \mathbf{Mod}_{\mathfrak{C}}(A) \boxtimes \mathbf{Mod}_{\mathfrak{D}}(B) \rightarrow \mathbf{Mod}_{\mathfrak{C}\boxtimes\mathfrak{D}}(A \boxtimes B)$. Furthermore, this 2-functor is compatible with the left $\mathfrak{C} \boxtimes \mathfrak{D}$-module structures. Now, it follows from the proof of proposition 3.2.1.2 above that $\mathbf{Mod}_{\mathfrak{C}\boxtimes\mathfrak{D}}(A \boxtimes B)$ is generated under Cauchy completion by the full sub-2-category on the objects of the form $(C\Box D) \boxtimes (A\Box B)$ with C in $\mathfrak{C}$ and D in $\mathfrak{D}$. Likewise, it follows from its construction that $\mathbf{Mod}_{\mathfrak{C}}(A) \boxtimes \mathbf{Mod}_{\mathfrak{D}}(B)$ is generated under Cauchy completion by the full sub-2-category on the objects of the form $(C\Box A) \boxtimes (D\Box B)$ for every C in $\mathfrak{C}$ and D in $\mathfrak{D}$. But, we have that $\mathbf{K}\big((C\Box A) \boxtimes (D\Box B)\big) = (C\Box D) \boxtimes (A\Box B)$. Furthermore, it follows by combining proposition 4.1 of [Déc24] and lemma 3.2.13 of [Déc21] that $\mathbf{K}$ induces an equivalence between these two full sub-2-categories. This establishes that $\mathbf{K}$ is an equivalence as desired. □

Lemma 4.3.4.8. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two compact semisimple tensor 2-categories. For any separable algebras A in $\mathfrak{C}$ and B in $\mathfrak{D}$, there is an equivalence*

$$\mathbf{Bimod}_{\mathfrak{C}}(A) \boxtimes \mathbf{Bimod}_{\mathfrak{D}}(B) \simeq \mathbf{Bimod}_{\mathfrak{C}\boxtimes\mathfrak{D}}(A \boxtimes B)$$

of compact semisimple tensor 2-categories.

Proof. The proof is analogous to that of the previous lemma. □

Remark 4.3.4.9. In particular, the 2-Deligne tensor product is compatible with Morita equivalences of locally separable compact semisimple 2-categories.

4.4 Diagrams

4.4.1 Proof of Theorem 4.1.1.6

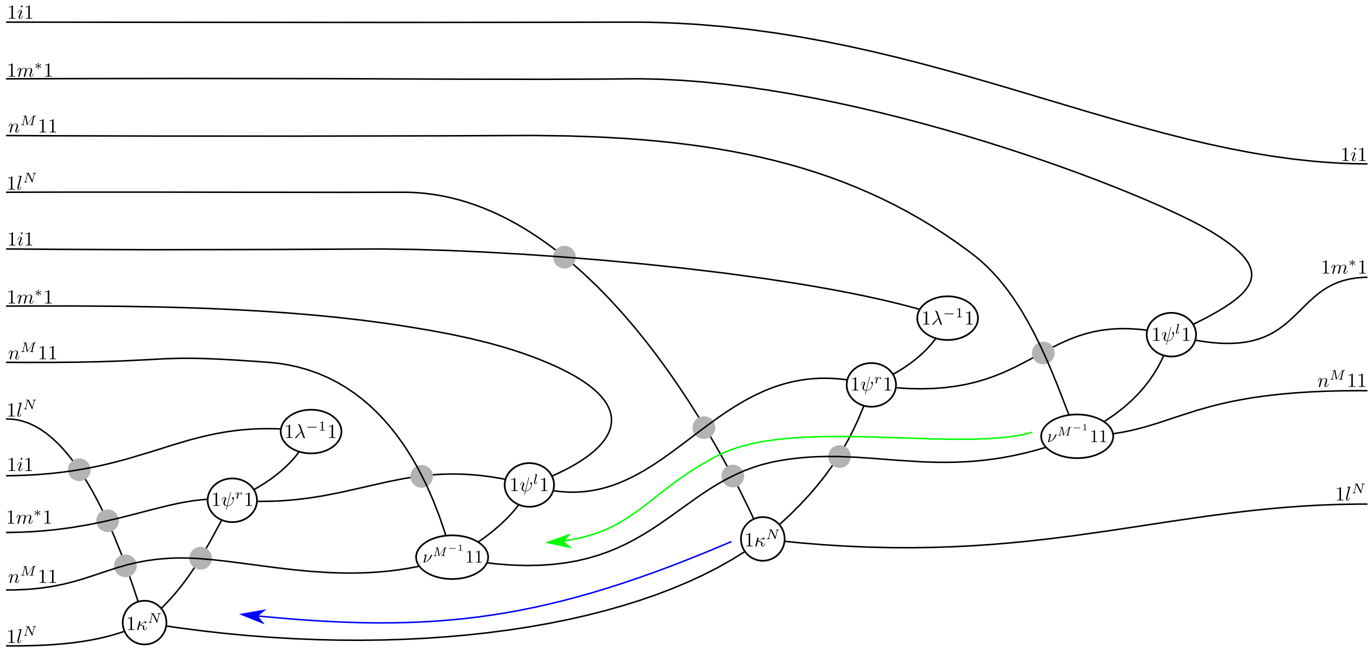


Figure 4.1: Associativity (Part 1)

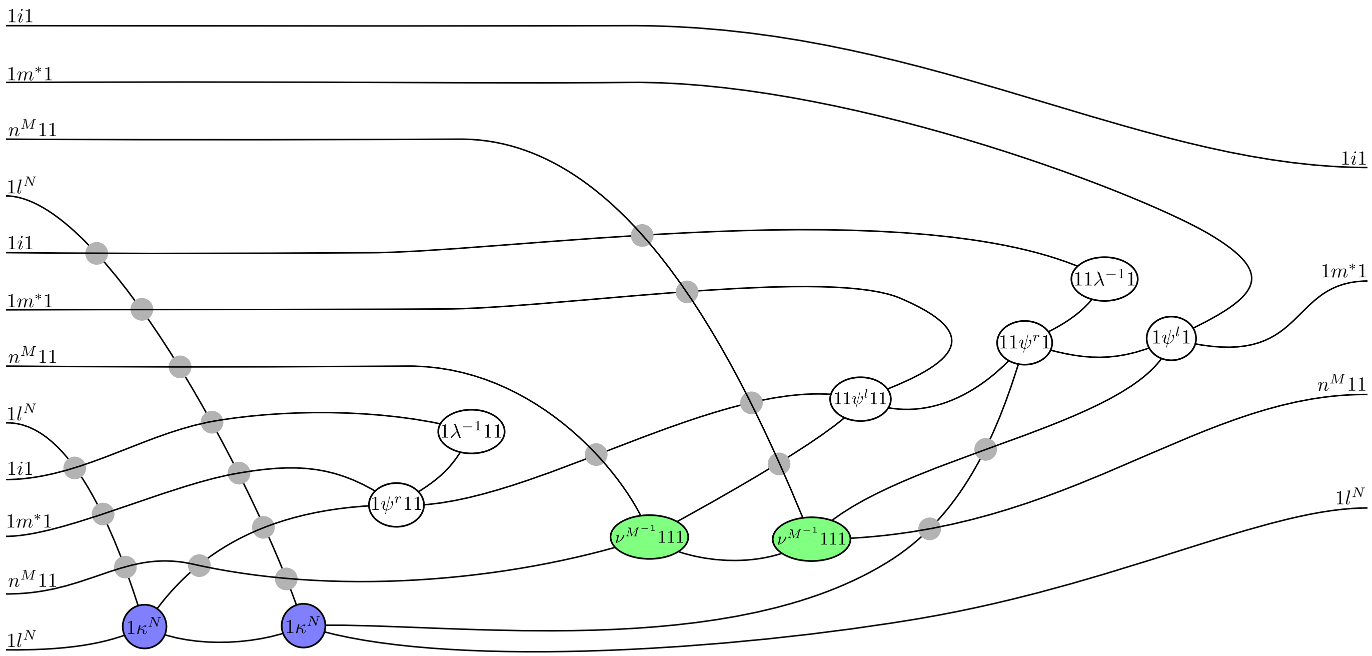


Figure 4.2: Associativity (Part 2)

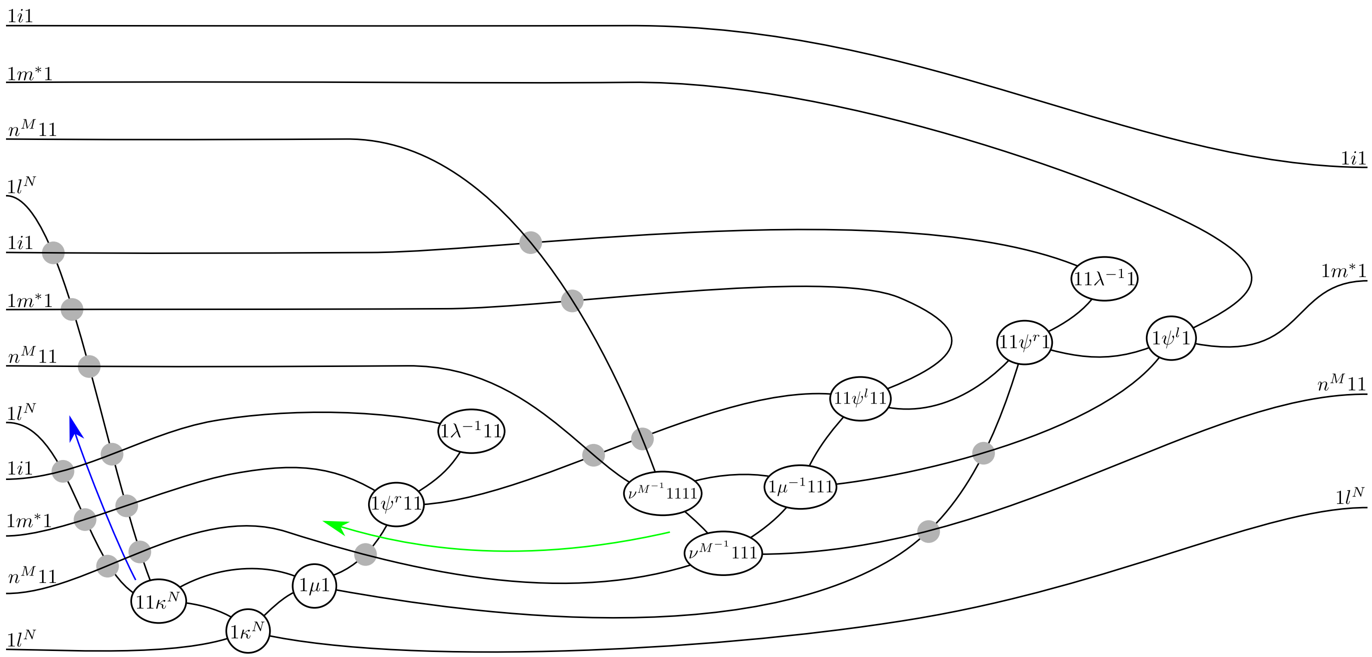


Figure 4.3: Associativity (Part 3)

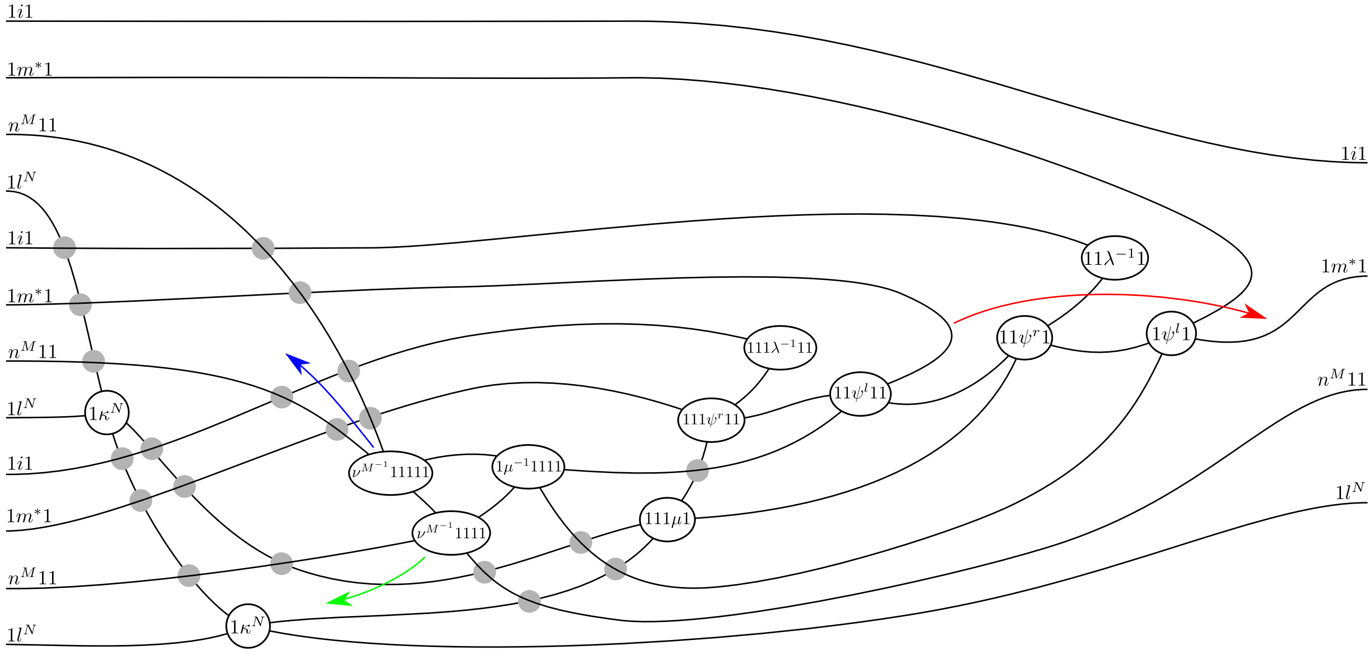


Figure 4.4: Associativity (Part 4)

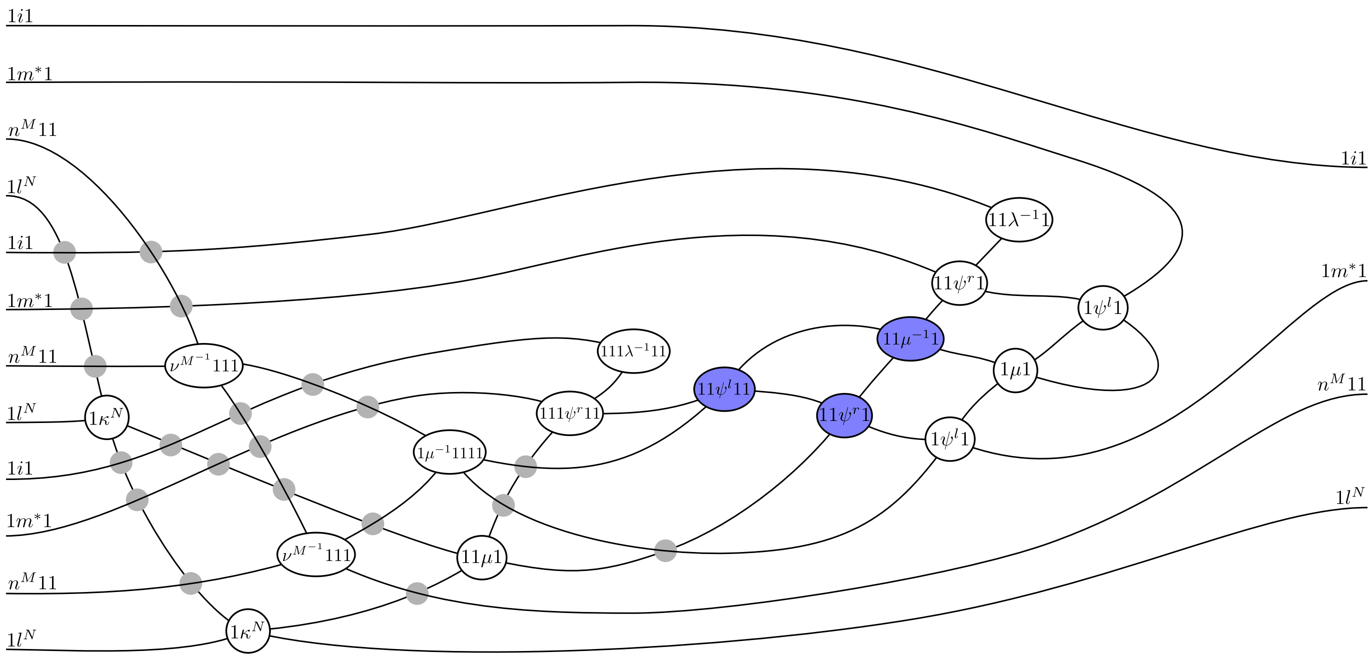


Figure 4.5: Associativity (Part 5)

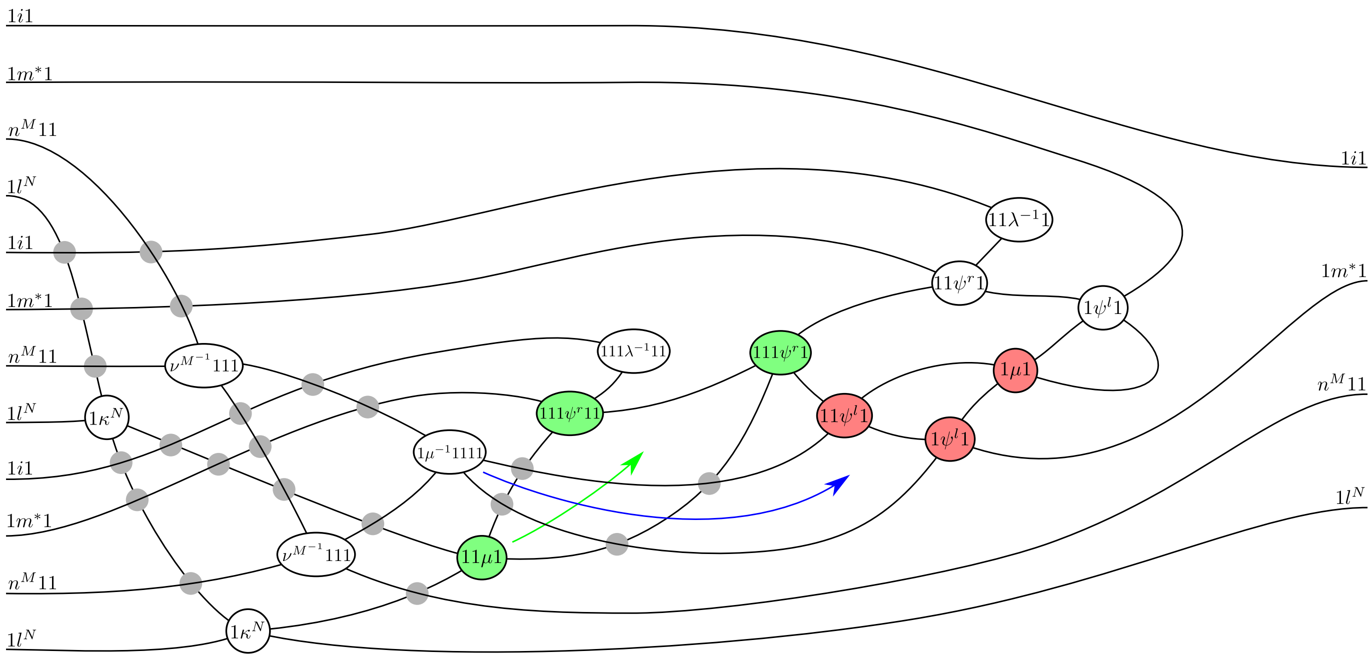


Figure 4.6: Associativity (Part 6)

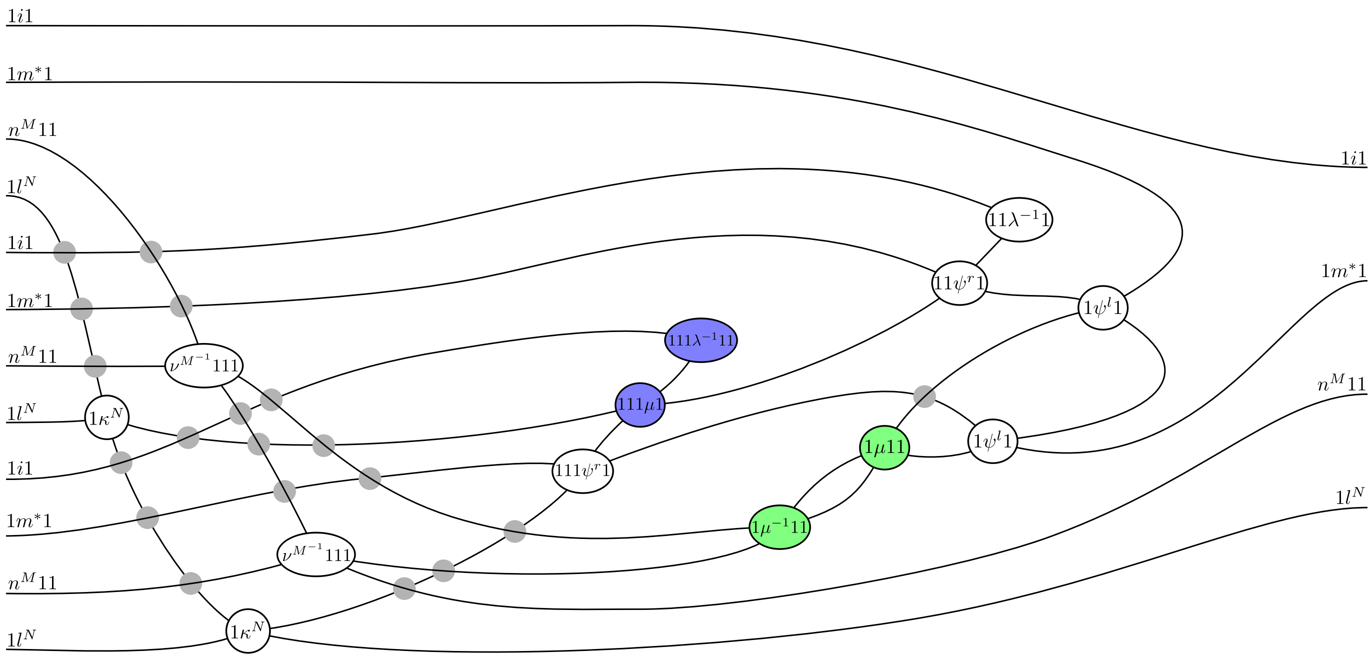


Figure 4.7: Associativity (Part 7)

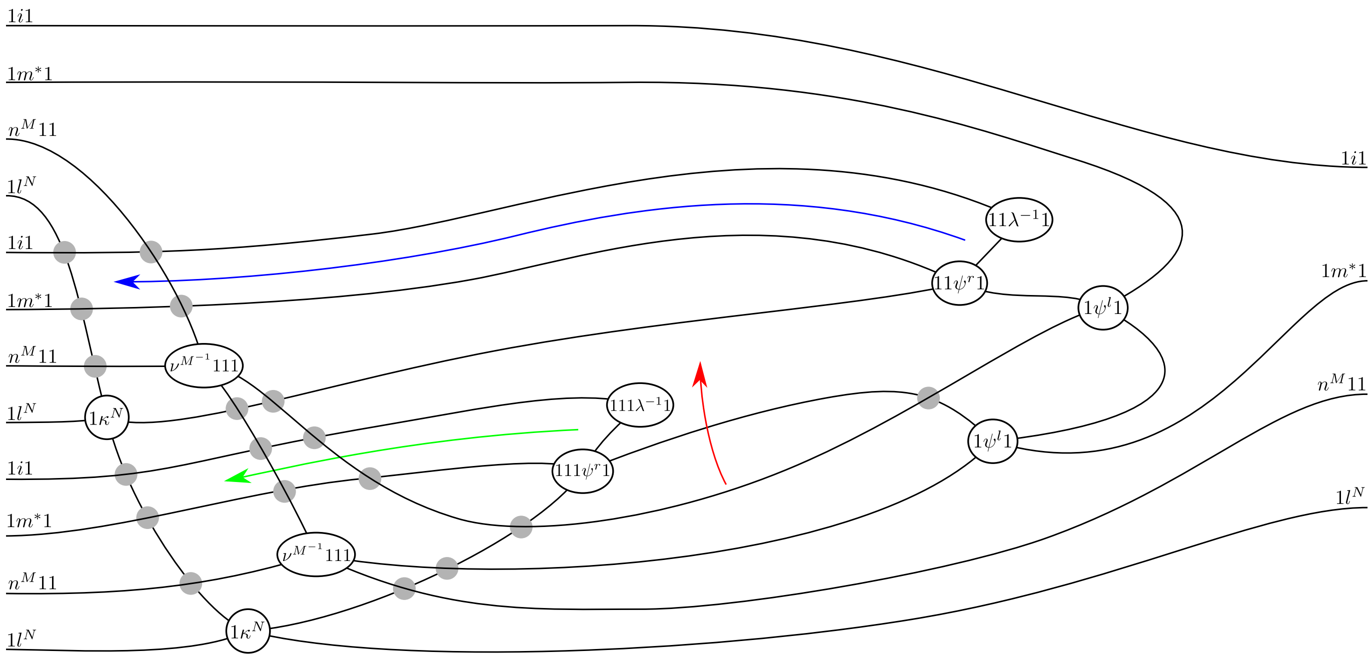


Figure 4.8: Associativity (Part 8)

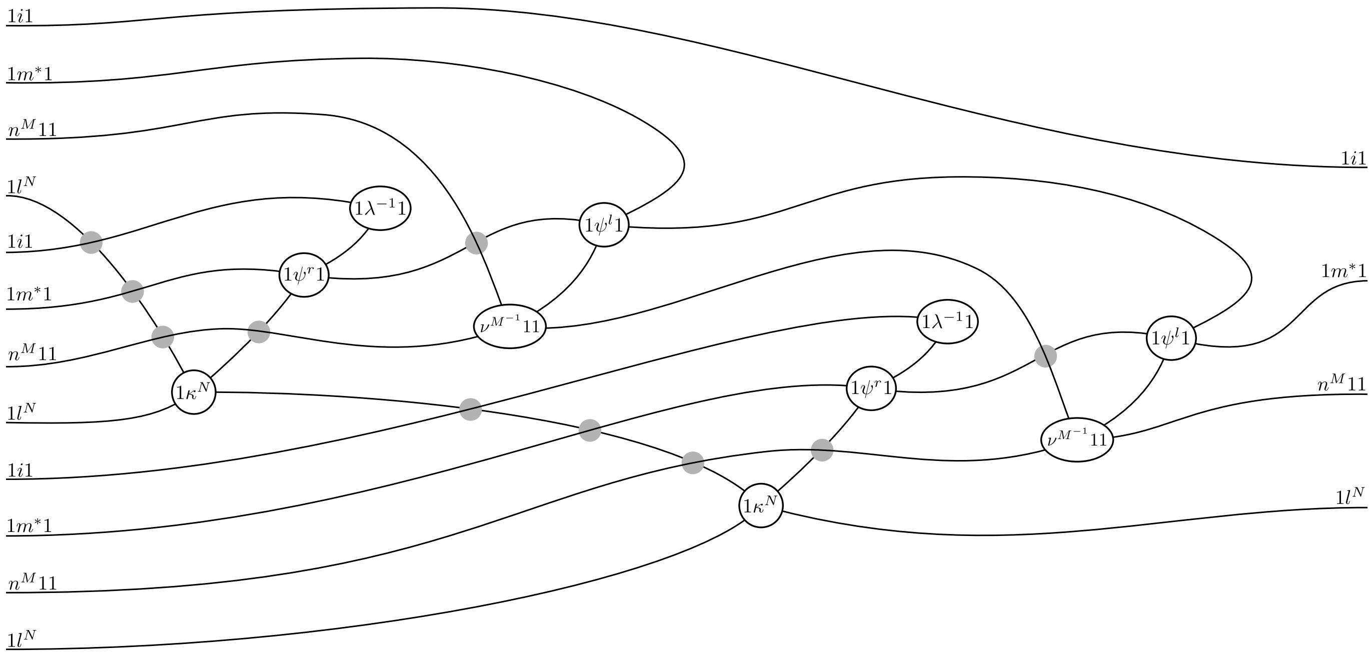


Figure 4.9: Associativity (Part 9)

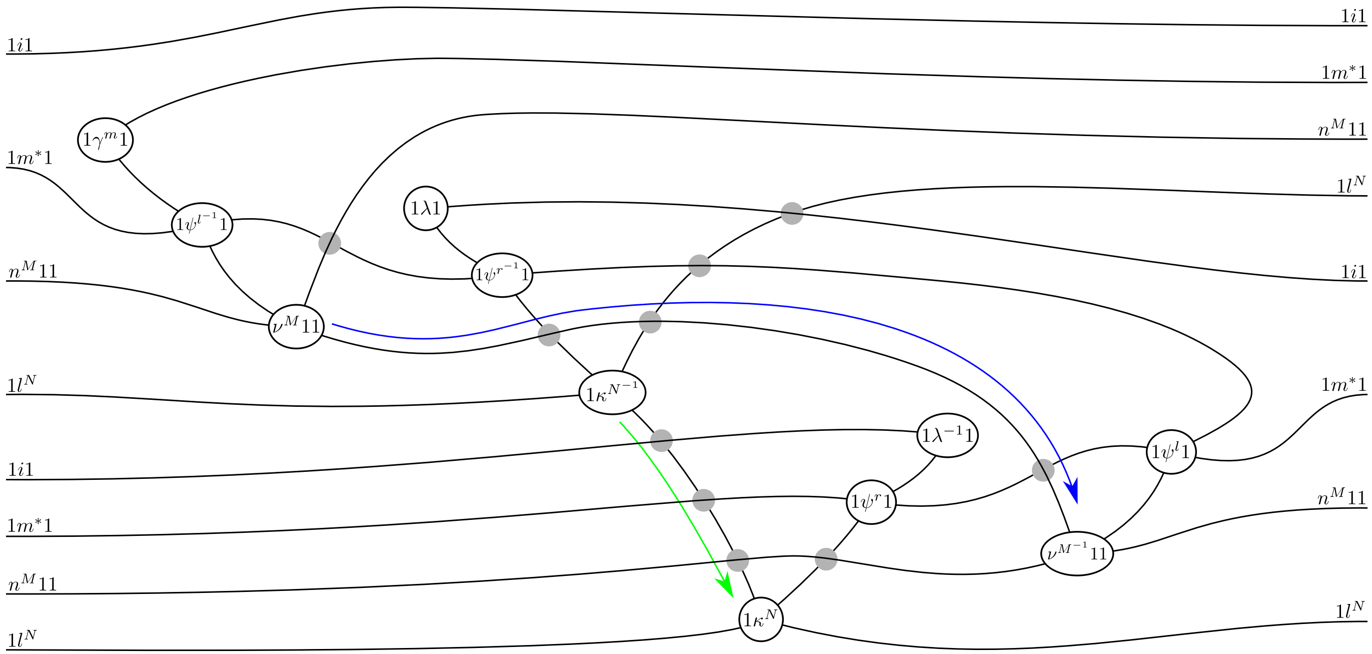


Figure 4.10: Left Frobenius (Part 1)

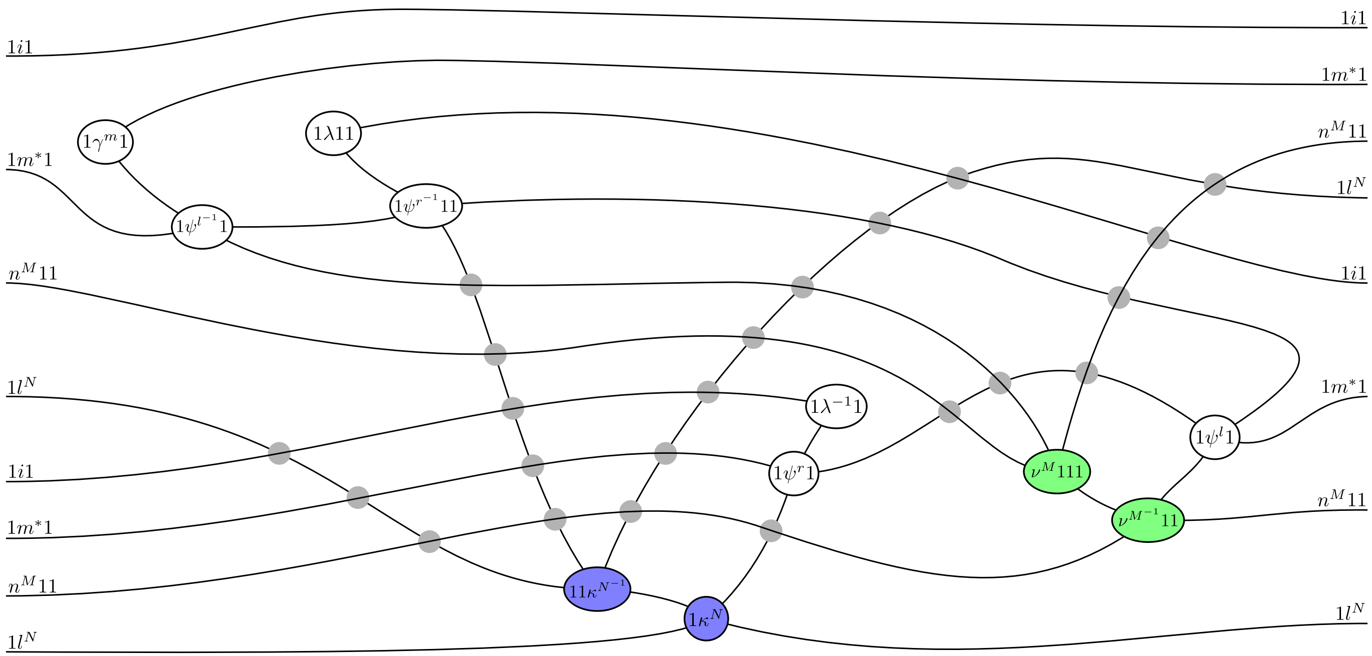


Figure 4.11: Left Frobenius (Part 2)

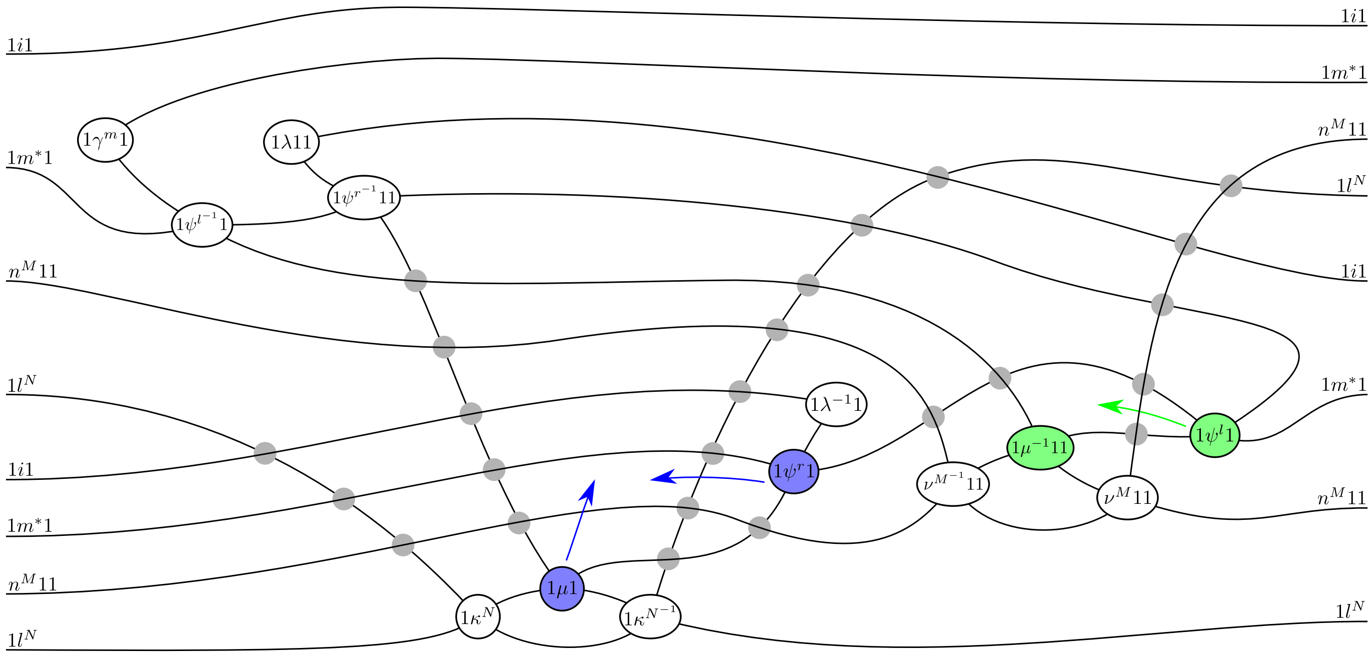


Figure 4.12: Left Frobenius (Part 3)

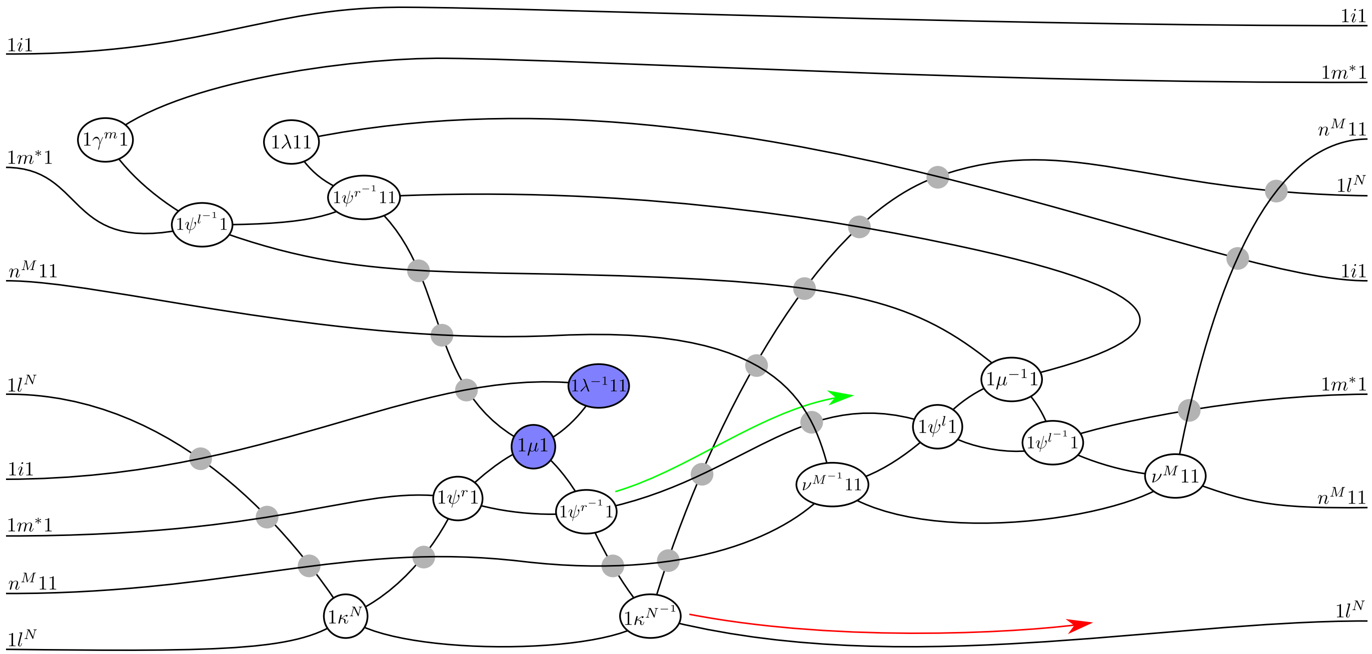


Figure 4.13: Left Frobenius (Part 4)

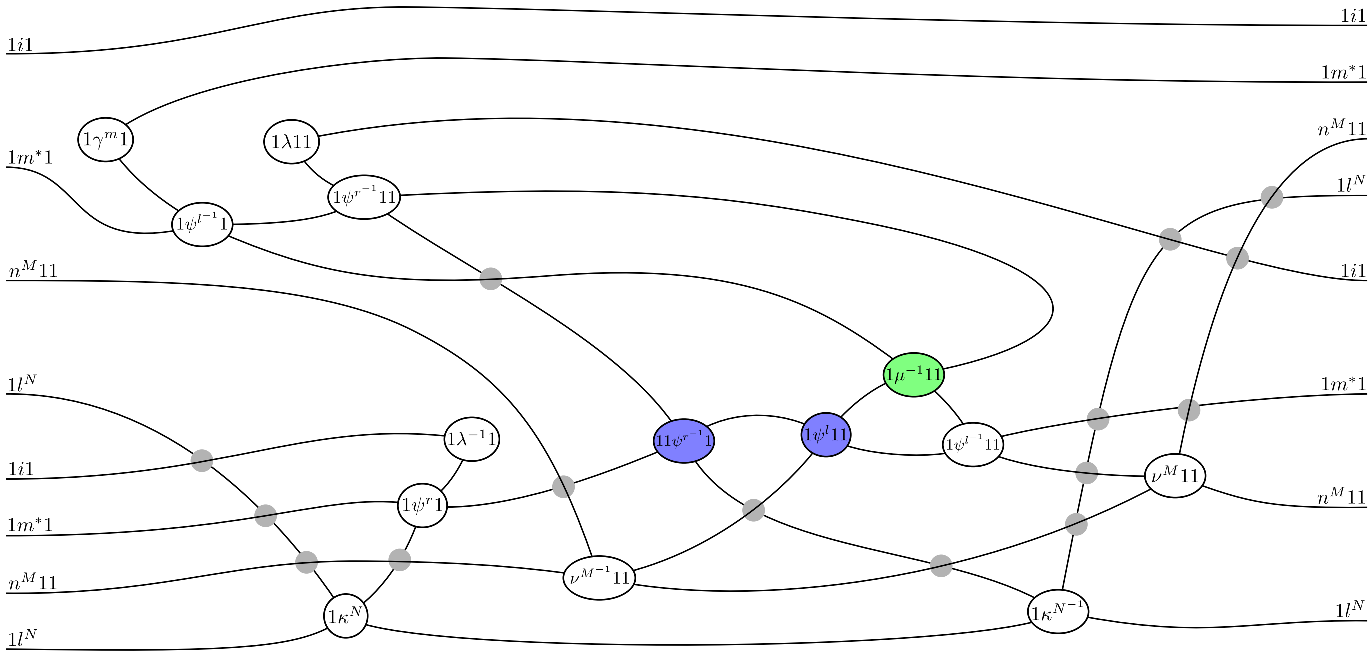


Figure 4.14: Left Frobenius (Part 5)

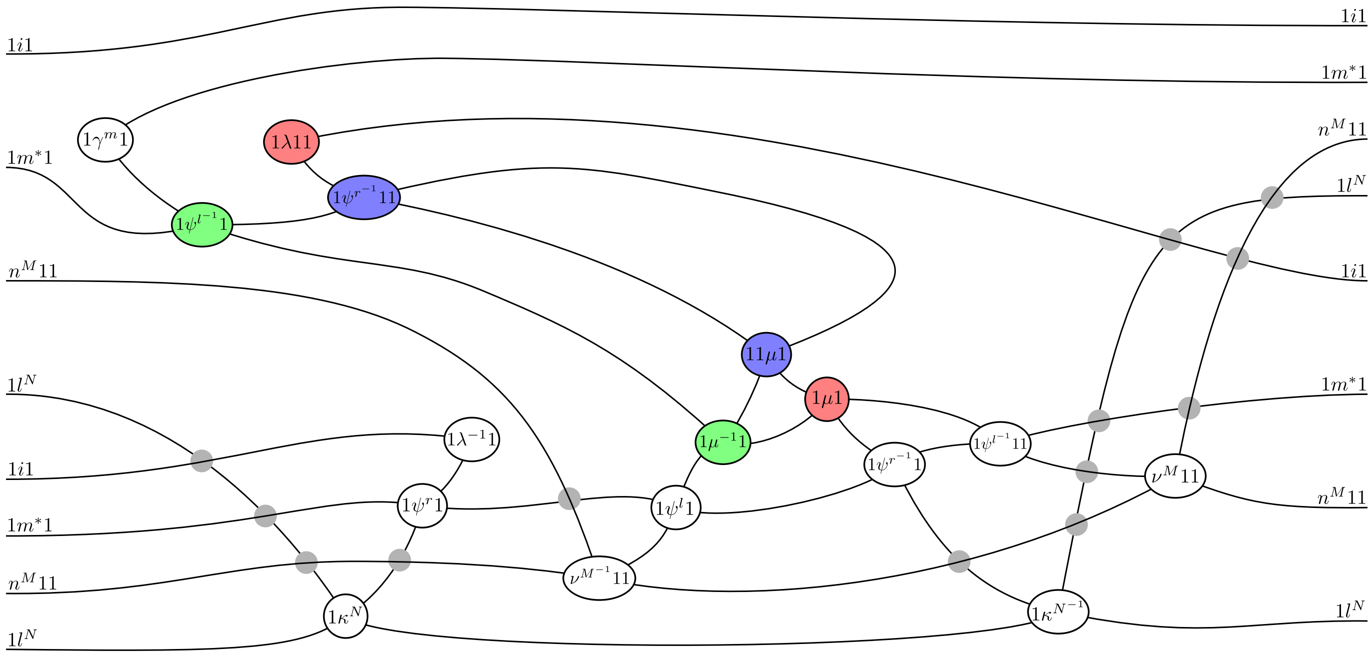


Figure 4.15: Left Frobenius (Part 6)

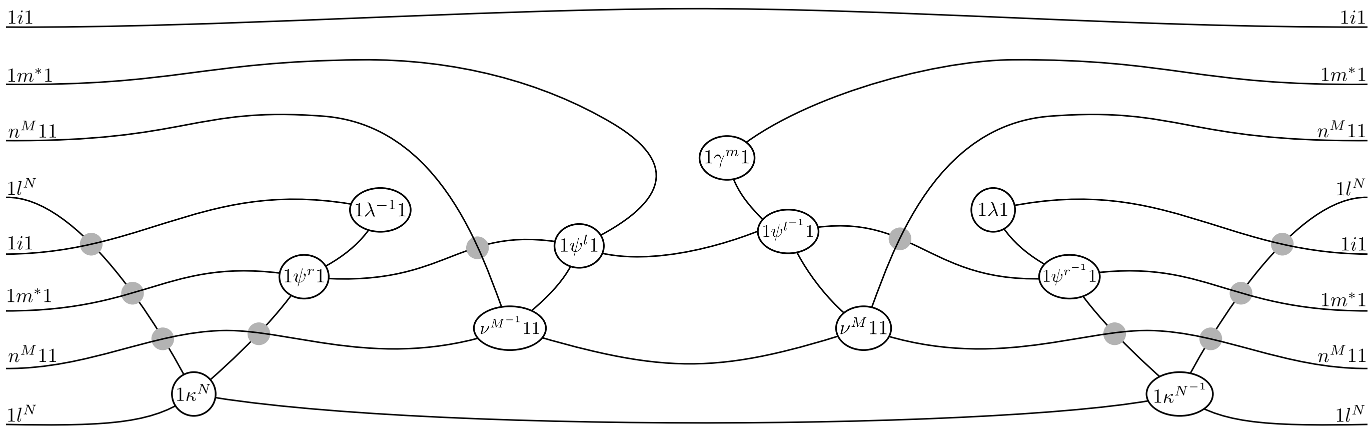


Figure 4.16: Left Frobenius (Part 7)

Chapter 5

Drinfeld Centers and Morita Equivalence Classes of Fusion 2-Categories

5.1 The Drinfeld Center

In this section, we begin be reviewing the definition of the Drinfeld center of a monoidal 2-category. We go on to prove some of its elementary properties. Finally, we show that the Drinfeld center of a multifusion 2-category is invariant under Morita equivalence.

5.1.1 Definition

Let $\mathfrak{C}$ be a strict cubical monoidal 2-category. We recall from [Cra98] the definition of the Drinfeld center $\mathscr{Z}(\mathfrak{C})$ of $\mathfrak{C}$, which is a braided monoidal 2-category (see also [BN95]). For a version the definition of the Drinfeld center of a weak monoidal 2-category, we refer the reader to [KTZ20].

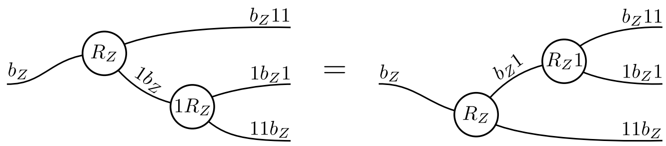
An object of $\mathscr{Z}(\mathfrak{C})$ consists of an object Z of $\mathfrak{C}$ equipped with a half braiding, that is an adjoint 2-natural equivalence b_Z given on C in $\mathfrak{C}$ by

$$(b_Z)_C : Z\Box C \xrightarrow{\simeq} C\Box Z,$$

together with an invertible modification R_Z given on C, D in $\mathfrak{C}$ by

$$\begin{tikzcd} Z\Box C\Box D \arrow[rr, "b_Z"] \arrow[dr, "b_Z 1"'] & \arrow[d, phantom, "\Downarrow R_Z"] & C\Box D\Box Z \\ & C\Box Z\Box D \arrow[ur, "1 b_Z"'] & \end{tikzcd}$$

such that, for every objects C, D, E in $\mathfrak{C}$, we have

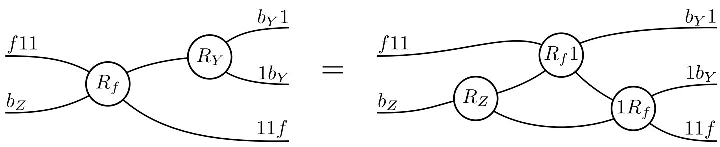
 (5.1)

in $Hom_{\mathfrak{C}}(ZCDE, CDEZ)$.

A 1-morphism in $\mathscr{Z}(\mathfrak{C})$ from (Y, b_Y, R_Y) to (Z, b_Z, R_Z) consists of a 1-morphism $f : Y \to Z$ in $\mathfrak{C}$ together with an invertible modification R_f given on C in $\mathfrak{C}$ by

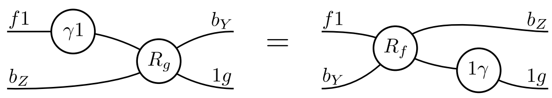
$$\begin{tikzcd} Y\Box C \arrow[r, "b_Y"] \arrow[d, "f1"'] \arrow[dr, phantom, "\overset{R_f}{\Nearrow}"] & C\Box Y \arrow[d, "1f"] \\ Z\Box C \arrow[r, "b_Z"'] & C\Box Z \end{tikzcd}$$

such that, for every objects C, D in $\mathfrak{C}$, we have



in $Hom_{\mathfrak{C}}(YCD, CDZ)$.

A 2-morphism in $\mathscr{Z}(\mathfrak{C})$ from (f, R_f) to (g, R_g), two 1-morphisms from (Y, b_y, R_Y) to (Z, b_Z, R_Z), is a 2-morphism $\gamma : f \Rightarrow g$ in $\mathfrak{C}$ such that, for every object C in $\mathfrak{C}$, we have



in $Hom_{\mathfrak{C}}(YC, CZ)$.

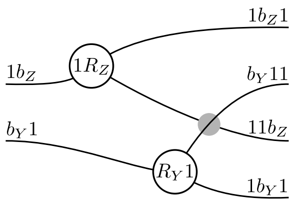
The identity 1-morphism on the object (Z, b_Z, R_Z) of $\mathscr{Z}(\mathfrak{C})$ is given by (Id_Z, Id_{b_Z}). Given two 1-morphisms (f, R_f) from (X, b_X, R_X) to (Y, b_Y, R_Y) and (g, R_g) from (Y, b_Y, R_Y) to (Z, b_Z, R_Z) in $\mathscr{Z}(\mathfrak{C})$ their composite is the 1-morphism $(g \circ f, (R_g \circ f1) \cdot (1g \circ R_f))$. This endows $\mathscr{Z}(\mathfrak{C})$ with the structure of a strict 2-category.

The monoidal structure of $\mathscr{Z}(\mathfrak{C})$ is given as follows. The monoidal unit is I, the monoidal unit of $\mathfrak{C}$, equipped with the identity adjoint 2-natural equivalence, and identity modification. The monoidal product of two objects (Y, b_Y, R_Y) and (Z, b_Z, R_Z) of $\mathscr{Z}(\mathfrak{C})$ is given by

$$(Y, b_Y, R_Y)\Box(Z, b_Z, R_Z) = (Y\Box Z, (b_Y\Box 1) \circ (1\Box b_Z), R_{YZ}),$$

where R_{YZ} is the invertible modification given on C, D in $\mathfrak{C}$ by

$(R_{YZ})_{C,D} :=$



in $Hom_{\mathfrak{C}}(YZCD, CDYZ)$. The monoidal product of two 1-morphisms (f, R_f) and (g, R_g) of $\mathscr{Z}(\mathfrak{C})$ is given by $(f, R_f)\Box(g, R_g) = (f\Box g, (R_f\Box g) \cdot (f\Box R_g))$. Finally, the monoidal product of two 2-morphisms γ and δ in $\mathscr{Z}(\mathfrak{C})$ is given by $\gamma\Box\delta$. Using the interchange $\phi^{\Box}$ of the monoidal product of $\mathfrak{C}$, we can upgrade the above assignment to a 2-functor that defines a strict cubical monoidal structure on $\mathscr{Z}(\mathfrak{C})$ as in [Cra98]. Furthermore, it follows from the definitions that the forgetful 2-functor $\mathscr{Z}(\mathfrak{C}) \to \mathfrak{C}$ is monoidal.

Finally, in the notation of definition 2.3 of [SP11], the braiding on $\mathscr{Z}(\mathfrak{C})$ is given as follows. For any two objects (Y, b_Y, R_Y), and (Z, b_Z, R_Z) of $\mathscr{Z}(\mathfrak{C})$, we define the braiding b by $b_{Y,Z} := (b_Y)_Z$. Further, b is promoted to an adjoint 2-natural equivalence in an obvious using the definitions of the objects and 1-morphisms in $\mathscr{Z}(\mathfrak{C})$. For any three objects (X, b_X, R_X), (Y, b_Y, R_Y), and (Z, b_Z, R_Z) of $\mathscr{Z}(\mathfrak{C})$, we also define invertible modifications $R_{X,Y,Z} := (R_X)_{Y,Z}$ and $S_{X,Y,Z} := Id_{XYZ}$. It follows from [Cra98] that these assignments define a braiding on $\mathscr{Z}(\mathfrak{C})$.

5.1.2 Elementary Properties

As $\mathfrak{C}$ is a monoidal 2-category, it has a canonical structure of $\mathfrak{C}$-$\mathfrak{C}$-bimodule 2-category (see definition 2.1.1 of [Déc21]). We begin by the following familiar observation, which will be generalized in proposition 5.1.3.1 below.

Lemma 5.1.2.1. *There is an equivalence $\mathscr{Z}(\mathfrak{C}) \simeq \mathbf{End}_{\mathfrak{C}-\mathfrak{C}}(\mathfrak{C})$ of monoidal 2-categories.*

Proof. This is a straightforward verification. We leave the details to the keen reader. □

Using $\mathfrak{C}^{mop}$ to denote $\mathfrak{C}$ with the opposite monoidal structure, we may equivalently think of $\mathfrak{C}$ as a $\mathfrak{C}$-$\mathfrak{C}$-bimodule 2-category or as a left $\mathfrak{C} \times \mathfrak{C}^{mop}$-module 2-category. By combining the above lemma with propositions 4.2.2.3 and 4.2.2.4, we therefore obtain the following result.

Corollary 5.1.2.2. *If $\mathfrak{C}$ is a rigid monoidal 2-category, then its Drinfeld center $\mathscr{Z}(\mathfrak{C})$ is a rigid monoidal 2-category.*

We also record the following lemma, where $\mathscr{Z}(\mathfrak{C})^{rev}$ denotes the braided monoidal 2-category obtained from $\mathscr{Z}(\mathfrak{C})$ by using the opposite braiding. More precisely, given (Y, b_Y, R_Y) and (Z, b_Z, R_Z) in $\mathscr{Z}(\mathfrak{C})^{rev}$, the braiding on the objects is given by $b^{\bullet}_{Y,Z} : Y\Box Z \to Z\Box Y$, the prescribed adjoint equivalence of $b_{Y,Z}$.

Lemma 5.1.2.3. *There is an equivalence $\mathscr{Z}(\mathfrak{C}^{mop}) \simeq \mathscr{Z}(\mathfrak{C})^{rev}$ of braided monoidal 2-categories.*

Proof. This follows by direct inspection. □

From now on, we will work over a fixed field $\Bbbk$. In particular, we will assume that $\mathfrak{C}$ is a monoidal $\Bbbk$-linear 2-category.

Lemma 5.1.2.4. *Let $\mathfrak{C}$ be a Cauchy complete monoidal 2-category. Then, $\mathscr{Z}(\mathfrak{C})$ is Cauchy complete.*

Proof. Let (Y, e, μ, δ) be a 2-condensation monad in $\mathscr{Z}(\mathfrak{C})$. By hypothesis, the underlying 2-condensation monad in $\mathfrak{C}$ can be split by a 2-condensation $(Y, Z, f, g, \phi, \gamma)$ and 2-isomorphism $\theta : g \circ f \cong e$. We begin by upgrading Z to an object of $\mathscr{Z}(\mathfrak{C})$. To this end, recall that splittings of 2-condensation monads are preserved by all 2-functors. In particular, for any C in $\mathfrak{C}$, $b_{Y,C} : Y\Box C \simeq C\Box Y$ induces an equivalence between two 2-condensation monads. By the 2-universal property of the splitting of 2-condensation monads (see theorem 2.3.2 of [GJF19]), this induces an equivalence $b_{Z,C} : Z\Box C \simeq C\Box Z$. Further, thanks to the 2-universal property these 1-morphisms assemble to given an adjoint 2-natural equivalence b_Z. Likewise, the invertible modification R_Y induces an invertible modification R_Z on Z, which satisfies equation (5.1). Similarly, one can endow both f and g with the structures of 1-morphisms in $\mathscr{Z}(\mathfrak{C})$, and check that ϕ, γ, and θ are 2-morphisms in $\mathscr{Z}(\mathfrak{C})$. This concludes the proof. □

Let $\mathfrak{C}$ and $\mathfrak{D}$ be two Cauchy complete monoidal linear 2-categories. It follows from theorem 5.2 of [Déc24] that the completed tensor product $\mathfrak{C}\widehat{\otimes}\mathfrak{D}$ inherits a monoidal structure. It is therefore natural to attempt to compare its Drinfeld center with that of $\mathfrak{C}$ and $\mathfrak{D}$. In this direction, we prove the following technical result.

Lemma 5.1.2.5. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two Cauchy complete monoidal 2-categories. Then, there is a canonical braided monoidal 2-functor $\mathscr{Z}(\mathfrak{C})\widehat{\otimes}\mathscr{Z}(\mathfrak{C}) \to \mathscr{Z}(\mathfrak{C}\widehat{\otimes}\mathfrak{D})$ that commutes with the forgetful monoidal 2-functors to $\mathfrak{C}\widehat{\otimes}\mathfrak{D}$ up to equivalence.*

Proof. We claim that there is a bilinear 2-functor

$$\begin{array}{ccc} \mathscr{Z}(\mathfrak{C}) \times \mathscr{Z}(\mathfrak{D}) & \to & \mathscr{Z}(\mathfrak{C}\widehat{\otimes}\mathfrak{D}). \\ (Y, Z) & \mapsto & Y\widehat{\otimes}Z \end{array}$$

Namely, given Y in $\mathscr{Z}(\mathfrak{C})$ and Z in $\mathscr{Z}(\mathfrak{D})$, we can equip the object $Y\widehat{\otimes}Z$ of $\mathfrak{C}\widehat{\otimes}\mathfrak{D}$ with the adjoint 2-natural equivalence $b_Y\widehat{\otimes}b_Z$, which is defined for all objects of $\mathfrak{C}\widehat{\otimes}\mathfrak{D}$ of the form $C\widehat{\otimes}D$. But, by the construction given in the proof of proposition 3.4 of [Déc24], $\mathfrak{C}\widehat{\otimes}\mathfrak{D}$ is the Cauchy completion of its full sub-2-category on the objects of the form $C\widehat{\otimes}D$ with C in $\mathfrak{C}$ and D in $\mathfrak{D}$. Thus, thanks to the 3-universal property of the Cauchy completion (see [Déc22b]), $b_Y\widehat{\otimes}b_Z$ can be canonically extended to an adjoint 2-natural equivalence defined on all the objects of $\mathfrak{C}\widehat{\otimes}\mathfrak{D}$. An analogous argument shows that the invertible modification $R_Y\widehat{\otimes}R_Z$ can be canonically extended to all of $\mathfrak{C}\widehat{\otimes}\mathfrak{D}$. Further, it follows from its construction that this extension satisfies equation (5.1), so that $Y\widehat{\otimes}Z$ is indeed an object of $\mathscr{Z}(\mathfrak{C}\widehat{\otimes}\mathfrak{D})$. This assignment is extended to morphisms using the same techniques.

Thanks to lemma 5.1.2.4 above and the 3-universal property of the completed tensor product obtained in proposition 3.4 of [Déc24], the bilinear 2-functor above induces a linear 2-functor $\mathscr{Z}(\mathfrak{C})\widehat{\otimes}\mathscr{Z}(\mathfrak{D}) \to \mathscr{Z}(\mathfrak{C}\widehat{\otimes}\mathfrak{D})$. By construction, this is a braided monoidal 2-functor, and it is compatible with the forgetful monoidal 2-functors. □

Example 5.1.2.6. If $\Bbbk$ is an algebraically closed field of characteristic zero and G is a finite group. Then, the Drinfeld center $\mathscr{Z}(\mathbf{2Vect}_G)$ of the fusion 2-category $\mathbf{2Vect}_G$ was shown to be a finite semisimple braided monoidal 2-category in [KTZ20]. Thence, it follows from 5.1.2.2 that $\mathscr{Z}(\mathbf{2Vect}_G)$ is in fact a fusion 2-category. Algebras in $\mathscr{Z}(\mathbf{2Vect}_G)$ are in particular G-graded finite semisimple monoidal 1-categories. But, objects of this Drinfeld center are in addition equipped with a coherent action of G, which is given by conjugation on grading. Thus, algebras in $\mathscr{Z}(\mathbf{2Vect}_G)$ are exactly finite semisimple G-crossed monoidal categories (originally introduced in section 2.1 of [Tur00], see also definition 5.1 of [Gal17]). In fact, rigid algebras in $\mathscr{Z}(\mathbf{2Vect}_G)$ are precisely G-crossed multifusion 1-categories. In particular, we see that G-crossed braided fusion 1-categories yield algebras in $\mathscr{Z}(\mathbf{2Vect}_G)$.

5.1.3 Morita Invariance

Let us now fix $\mathfrak{C}$ a multifusion 2-category over an algebraically closed field $\Bbbk$ of characteristic zero, and $\mathfrak{M}$ a separable left $\mathfrak{C}$-module 2-category. Thanks to proposition 2.2.8 of [Déc21], we may assume without loss of generality that $\mathfrak{C}$ and $\mathfrak{M}$ are strict cubical in the sense of definition 2.2.7 of [Déc21]. Now, it is shown in proposition

4.2.1.10 above that $\mathfrak{M}$ can be canonically viewed as a left $\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}$-module 2-category. Further, this left action is multilinear, so that we may view $\mathfrak{M}$ as a left $\mathfrak{C}\boxtimes\mathfrak{C}^*_{\mathfrak{M}}$-module 2-category.

Proposition 5.1.3.1. *Let $\mathfrak{M}$ be a faithful separable left $\mathfrak{C}$-module 2-category. Then, there is an equivalence*

$$(\mathfrak{C}\boxtimes\mathfrak{C}^*_{\mathfrak{M}})^*_{\mathfrak{M}}\simeq\mathscr{Z}(\mathfrak{C})$$

of monoidal 2-categories.

Proof. By definition, we have that $(\mathfrak{C}\boxtimes\mathfrak{C}^*_{\mathfrak{M}})^*_{\mathfrak{M}}=\mathbf{End}_{\mathfrak{C}\boxtimes\mathfrak{C}^*_{\mathfrak{M}}}(\mathfrak{M})$. In fact, as the left action of $\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}$ on $\mathfrak{M}$ is multilinear, it follows from the 3-universal property of the 2-Deligne tensor product obtained in [Déc21] that $(\mathfrak{C}\boxtimes\mathfrak{C}^*_{\mathfrak{M}})^*_{\mathfrak{M}}\simeq\mathbf{End}_{\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}}(\mathfrak{M})$. Further, note that left $\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}$-module 2-endofunctors on $\mathfrak{M}$ are exactly left $\mathfrak{C}^*_{\mathfrak{M}}$-module 2-endofunctors on $\mathfrak{M}$ equipped with a left $\mathfrak{C}$-module structure, whose coherence data consists of left $\mathfrak{C}^*_{\mathfrak{M}}$-module 2-natural transformations and left $\mathfrak{C}^*_{\mathfrak{M}}$-module modifications. More succinctly, $\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}$-module 2-endofunctors are exactly 2-endofunctors equipped with commuting left actions by $\mathfrak{C}$ and $\mathfrak{C}^*_{\mathfrak{M}}$. Similar descriptions hold for left $\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}$-module 2-natural transformations, as well as left $\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}$-module modifications.

Given (Z,b_Z,R_Z) in $\mathscr{Z}(\mathfrak{C})$, we can consider the 2-endofunctor $\mathbf{J}(Z)$ on $\mathfrak{M}$ given by $M\mapsto Z\Box M$. For any F in $\mathfrak{C}^*_{\mathfrak{M}}$, the left $\mathfrak{C}$-module structure on F yields an adjoint 2-natural equivalence $(\chi^F_{Z,M})^\bullet:F(Z\Box M)\simeq Z\Box F(M)$ and invertible modification, which endow $\mathbf{J}(Z)$ with a canonical left $\mathfrak{C}^*_{\mathfrak{M}}$-module structure. Furthermore, $\mathbf{J}(Z)$ carries a compatible left $\mathfrak{C}$-module structure given by $b^\bullet_{Z,C}\Box M: C\Box Z\Box M\simeq Z\Box C\Box M$. This assignment can be extended to morphisms in the obvious way, so that we have a 2-functor $\mathbf{J}:\mathscr{Z}(\mathfrak{C})\to\mathbf{End}_{\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}}(\mathfrak{M})$. Further, it follows by inspection that $\mathbf{J}$ is monoidal.

Corollary 4.3.4.4 implies that the canonical monoidal 2-functor $\mathfrak{C}\to\mathbf{End}_{\mathfrak{C}^*_{\mathfrak{M}}}(\mathfrak{M})$ given by $C\mapsto\{M\mapsto C\Box M\}$ is an equivalence. In particular, objects of $\mathbf{End}_{\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}}(\mathfrak{M})$ are identified with 2-endofunctors $M\mapsto Z\Box M$ for some Z in $\mathfrak{C}$ equipped with a left action by $\mathfrak{C}$. More precisely, for every C in $\mathfrak{C}$, we have a left $\mathfrak{C}^*_{\mathfrak{M}}$-module adjoint 2-natural equivalence $C\Box Z\Box M\simeq Z\Box C\Box M$. By the above equivalence of monoidal 2-categories, this corresponds exactly to an adjoint 2-natural equivalence $b^\bullet_{Z,C}:C\Box Z\simeq Z\Box C$. An analogous argument shows that the invertible modification witnessing the coherence of the left $\mathfrak{C}$-action corresponds precisely to the data of an invertible modification R_Z satisfying (5.1). This shows that $\mathbf{J}$ is essentially surjective on objects. One proceeds similarly to show that $\mathbf{J}$ is essentially surjective on 1-morphisms and fully faithful on 2-morphisms. This concludes the proof. $\Box$

We are now ready to prove the main theorem of this section.

Theorem 5.1.3.2. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be Morita equivalent multifusion 2-categories, then there is an equivalence $\mathscr{Z}(\mathfrak{C}) \simeq \mathscr{Z}(\mathfrak{D})$ of braided monoidal 2-categories.*

Proof. Thanks to theorem 4.3.4.3, there exists a faithful separable left $\mathfrak{C}$-module 2-category $\mathfrak{M}$ and an equivalence $\mathfrak{D} \simeq (\mathfrak{C}^*_{\mathfrak{M}})^{mop}$ of monoidal 2-categories. Thus, by lemma 5.1.2.3 it is enough to prove that $\mathscr{Z}(\mathfrak{C}^*_{\mathfrak{M}})^{rev} \simeq \mathscr{Z}(\mathfrak{C})$. Thanks to the proof of proposition 5.1.3.1, we know that the canonical monoidal 2-functor $\mathbf{J} : \mathscr{Z}(\mathfrak{C}) \to \mathbf{End}_{\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}}(\mathfrak{M})$ that sends Z in $\mathscr{Z}(\mathfrak{C})$ to $M \mapsto Z\Box M$ is an equivalence. But, thanks to corollary corollary 4.3.4.4, we have that $\mathfrak{D}^*_{\mathfrak{M}} \simeq \mathfrak{C}^{mop}$ as monoidal 2-categories. This implies that the canonical monoidal 2-functor $\mathbf{K} : \mathscr{Z}(\mathfrak{C}^*_{\mathfrak{M}}) \to \mathbf{End}_{\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}}(\mathfrak{M})$ that sends F in $\mathscr{Z}(\mathfrak{C}^*_{\mathfrak{M}})$ to $M \mapsto F(M)$ is an equivalence.

Now, given any object Z in $\mathscr{Z}(\mathfrak{C})$, the left $\mathfrak{C}$-module 2-endofunctor $M \mapsto Z\Box M$ can be canonically upgraded to an object of $\mathscr{Z}(\mathfrak{C}^*_{\mathfrak{M}})$. Namely, for every object F of $\mathfrak{C}^*_{\mathfrak{M}}$, the left $\mathfrak{C}$-module structure on F provides us with an adjoint 2-natural equivalence $\chi^F_{Z,M} : Z\Box F(M) \simeq F(Z\Box M)$, and an invertible modification satisfying (5.1). This define a half braiding on $M \mapsto Z\Box M$, and yields a monoidal 2-functor $\mathbf{L} : \mathscr{Z}(\mathfrak{C}) \to \mathscr{Z}(\mathfrak{C}^*_{\mathfrak{M}})$. Further, observe that the following diagram of monoidal 2-functors is weakly commutative:

$$\begin{tikzcd} \mathscr{Z}(\mathfrak{C}) \arrow[rr, "\mathbf{L}"] \arrow[dr, "\mathbf{J}"'] & & \mathscr{Z}(\mathfrak{C}^*_{\mathfrak{M}}) \arrow[dl, "\mathbf{K}"] \\ & \mathbf{End}_{\mathfrak{C}\times\mathfrak{C}^*_{\mathfrak{M}}}(\mathfrak{M}). & \end{tikzcd}$$

It follows from the first part of the proof that both $\mathbf{J}$ and $\mathbf{K}$ are equivalences, so that $\mathbf{L}$ is an equivalence of monoidal 2-categories.

Finally, let (Y, b_Y, R_Y) and (Z, b_Z, R_Z) be objects of $\mathscr{Z}(\mathfrak{C})$. By the definition recalled in section 5.1.1 above, the braiding on $\mathbf{L}(Y)$ and $\mathbf{L}(Z)$ in $\mathscr{Z}(\mathfrak{C}^*_{\mathfrak{M}})$ is given by

$$b_{\mathbf{L}(Y),\mathbf{L}(Z)}(M) = \chi^{\mathbf{L}(Z)}_{Y,M} = b^{\bullet}_{Z,Y}\Box M : Y\Box Z\Box M \to Z\Box Y\Box M.$$

Thus, $\mathbf{L}$ yields an equivalence $\mathscr{Z}(\mathfrak{C}) \to \mathscr{Z}(\mathfrak{C}^*_{\mathfrak{M}})^{rev}$ of braided monoidal 2-categories. This concludes the proof. □

Remark 5.1.3.3. The proof of theorem 5.1.3.2 constructs for every Morita equivalence between $\mathfrak{C}$ and $\mathfrak{D}$ an equivalence $\mathscr{Z}(\mathfrak{C}) \simeq \mathscr{Z}(\mathfrak{D})$ of braided monoidal 2-categories. With $\mathfrak{C} = \mathfrak{D}$, we expect that this assignment defines a homomorphism of groups $\Phi : \mathrm{BrPic}(\mathfrak{C}) \to Aut^{br}(\mathscr{Z}(\mathfrak{C}))$, from the group of Morita autoequivalences of $\mathfrak{C}$ to the group of braided monoidal autoequivalences of $\mathscr{Z}(\mathfrak{C})$. This is a partial categorification of theorem 1.1 of [ENO10]. On the other hand, it is well-known that two fusion 1-categories are Morita equivalent if and only if their Drinfeld centers are braided equivalent (see theorem 3.1 of [ENO11]). This is not the case for fusion 2-categories! Namely, it follows from lemma 2.16 and theorem 2.52 of [JFR24] that, for any non-degenerate braided fusion 1-category $\mathcal{B}$, there is an equivalence $\mathscr{Z}(\mathbf{Mod}(\mathcal{B})) \simeq \mathbf{2Vect}$ of braided fusion 2-category (see also proposition 4.17 of [DN21]). Further, by example 4.3.4.6 or theorem 5.2.1.4 below, $\mathbf{Mod}(\mathcal{B})$ is Morita equivalent to $\mathbf{2Vect}$ if and only if $\mathcal{B}$ is Witt equivalent to $\mathbf{Vect}$. But, the Witt group of non-degenerate braided fusion 1-categories is known to be non-trivial (see [DMNO13]).

Corollary 5.1.3.4. *There is an equivalence of braided fusion 2-categories*

$$\mathscr{Z}(\mathbf{2Vect}_G) \simeq \mathscr{Z}(\mathbf{2Rep}(G)).$$

Remark 5.1.3.5. Corollary 5.1.3.4 has one particularly noteworthy consequence, which we now explain. Namely, as $\mathscr{Z}(\mathbf{2Vect}_G)$ and $\mathscr{Z}(\mathbf{2Rep}(G))$ are equivalent as braided monoidal 2-categories, the associated (monoidal) 2-categories of algebras, and their homomorphisms are equivalent. In particular, this induces an equivalence between the full sub-2-categories on the rigid algebras. On one hand, it follows by inspection that rigid algebras in $\mathscr{Z}(\mathbf{2Vect}_G)$ are precisely G-crossed multifusion 1-categories in the sense of [Gal17] (see [Tur00] for the original definition). On the other hand, rigid algebras in $\mathscr{Z}(\mathbf{2Rep}(G))$ are exactly multifusion 1-categories $\mathcal{C}$ equipped with two braided monoidal functors $\mathbf{Rep}(G) \rightrightarrows \mathcal{Z}(\mathcal{C})$ lifting $\mathbf{Rep}(G) \to \mathcal{C}$. This correspondence appears to be new.

In addition, we also get an equivalence between the 2-categories of braided rigid algebras. More precisely, we get an equivalence between the 2-category of G-crossed braided multifusion 1-categories and that of braided multifusion 1-categories equipped with a braided monoidal functor from $\mathbf{Rep}(G)$ (see table 1 of [DN21] for the second identification). This is a well-known result in the theory of fusion 1-categories, which dates back to [Kir01] and [Müg03a].

5.2 The Morita Theory of Connected Fusion 2-Categories

In this section, we study the notion of Morita equivalence between connected fusion 2-categories. Throughout, we work over an algebraically closed field $\mathbb{k}$ of characteristic zero.

5.2.1 Comparison with Witt equivalence

We begin by introducing a notion of Witt equivalence between arbitrary braided fusion 1-categories.

Given a braided fusion 1-category $\mathcal{B}$ with braiding β. We use $\mathcal{Z}_{(2)}(\mathcal{B})$ to denote the symmetric center of $\mathcal{B}$, that is the full fusion sub-1-category of $\mathcal{B}$ on those objects B for which

$$\beta_{C,B} \circ \beta_{B,C} = Id_{B\otimes C}$$

for every object C of $\mathcal{B}$. It follows from the definition that $\mathcal{Z}_{(2)}(\mathcal{B})$ is a symmetric fusion 1-category.

Let $\mathcal{C}$ be a $\mathcal{B}$-central fusion 1-category $\mathcal{C}$. We write β for the braiding on $\mathcal{Z}(\mathcal{C})$, and $F : \mathcal{B} \to \mathcal{Z}(\mathcal{C})$ for the braided monoidal functor supplying $\mathcal{C}$ with its central structure. Then, we use $\mathcal{Z}(\mathcal{C}, \mathcal{B})$ to denote the centralizer of the image of $\mathcal{B}$ in $\mathcal{Z}(\mathcal{C})$, that is the full fusion sub-1-category of $\mathcal{Z}(\mathcal{C})$ on those objects Z for which

$$\beta_{F(B),Z} \circ \beta_{Z,F(B)} = Id_{Z\otimes F(B)}$$

for every object B of $\mathcal{B}$.

Definition 5.2.1.1. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two braided fusion 1-categories whose symmetric centers are given by the symmetric fusion 1-category $\mathcal{E}$. We say $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent provided that there exists a fusion 1-category $\mathcal{C}$, together with a fully faithful braided embedding $\mathcal{B}_1 \hookrightarrow \mathcal{Z}(\mathcal{C})$ and a braided monoidal equivalence $\mathcal{B}_2 \simeq \mathcal{Z}(\mathcal{C}, \mathcal{B}_1)^{rev}$.

Let $\mathcal{E}$ be a symmetric fusion 1-category. A braided fusion 1-category over $\mathcal{E}$, also known as an $\mathcal{E}$-central braided fusion 1-category, is a pair $(\mathcal{B}, F)$ consisting of a braided fusion 1-category $\mathcal{B}$ and a braided embedding $F : \mathcal{E} \hookrightarrow \mathcal{B}$. We say that $(\mathcal{B}, F)$, a braided fusion 1-category over $\mathcal{E}$, is non-degenerate if F induces an equivalence $\mathcal{E} \simeq \mathcal{Z}_{(2)}(\mathcal{E})$. In [DNO13], the authors introduced a notion of Witt equivalence over $\mathcal{E}$ between non-degenerate braided fusion 1-categories over $\mathcal{E}$. More precisely, $(\mathcal{B}_1, F_1)$ and $(\mathcal{B}_2, F_2)$ are Witt equivalent over $\mathcal{E}$ if there exists a fusion 1-category $\mathcal{C}$, together with a fully faithful braided embedding $\mathcal{B}_1 \hookrightarrow \mathcal{Z}(\mathcal{C})$ and a braided monoidal equivalence $\mathcal{B}_2 \simeq \mathcal{Z}(\mathcal{C}, \mathcal{B}_1)^{rev}$ such that the two braided functors $\mathcal{E} \to \mathcal{Z}(\mathcal{C})$ are equivalent. We now explain how this is related to the notion of Witt equivalence we have introduced above.

Lemma 5.2.1.2. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two braided fusion 1-categories whose symmetric centers are given by the symmetric fusion 1-category $\mathcal{E}$. Then, $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent in the sense of definition 5.2.1.1 if and only if there exists identifications $F_1 : \mathcal{E} \simeq \mathcal{Z}_{(2)}(\mathcal{B}_1)$ and $F_2 : \mathcal{E} \simeq \mathcal{Z}_{(2)}(\mathcal{B}_2)$ such that $(\mathcal{B}_1, F_1)$ and $(\mathcal{B}_2, F_2)$ are Witt equivalent over $\mathcal{E}$ as in definition 5.1 of [DNO13].*

Proof. Let us fix an identification $F_1 : \mathcal{E} \simeq \mathcal{Z}_{(2)}(\mathcal{B}_1)$. If there exists a fusion 1-category $\mathcal{C}$, together with a fully faithful braided embedding $\mathcal{B}_1 \hookrightarrow \mathcal{Z}(\mathcal{C})$ and a braided monoidal equivalence $\mathcal{B}_2 \simeq \mathcal{Z}(\mathcal{C}, \mathcal{B}_1)^{rev}$, then there is a canonical identification $F_2 : \mathcal{E} \simeq \mathcal{Z}_{(2)}(\mathcal{B}_2)$ such that

$$\mathcal{B}_1 \boxtimes_{\mathcal{E}} \mathcal{B}_2^{rev} \simeq \mathcal{B}_1 \boxtimes_{\mathcal{E}} \mathcal{Z}(\mathcal{C}, \mathcal{B}_1) \simeq \mathcal{Z}(\mathcal{C}, \mathcal{E})$$

by proposition 4.3 of [DNO13]. This shows that $(\mathcal{B}_1, F_1)$ and $(\mathcal{B}_2^{rev}, F_2^{rev})$ define opposite elements in the Witt group $\mathcal{W}(\mathcal{E})$ in the sense of [DNO13], so that $(\mathcal{B}_1, F_1)$ and $(\mathcal{B}_2, F_2)$ are Witt equivalent over $\mathcal{E}$ in the sense of definition 5.1 of [DNO13]. The converse follows by running the argument in reverse. $\square$

Remark 5.2.1.3. We emphasize that Witt equivalence over $\mathcal{E}$ does not in general coincide with the notion of Witt equivalence we have introduced above. Namely, given a braided fusion 1-category $\mathcal{B}$ and identifications $F_1, F_2 : \mathcal{E} \simeq \mathcal{Z}_{(2)}(\mathcal{B})$, then $(\mathcal{B}, F_1)$ and $(\mathcal{B}, F_2)$ are Witt equivalent over $\mathcal{E}$ if and only if there exists E, an autoequivalence of $\mathcal{B}$ as a braided fusion 1-category, such that F_1 and $E \circ F_2$ are equivalent as braided functors. But, it was pointed out to us by Dmitri Nikshych that there exists braided fusion 1-categories $\mathcal{B}$ such that the canonical group homomorphism $Aut^{br}(\mathcal{B}) \to Aut^{br}(\mathcal{Z}_{(2)}(\mathcal{B}))$ is not surjective. For instance, given a prime $p > 3$, the braided fusion 1-category $\mathbf{Vect}^q_{\mathbb{Z}/p^2\mathbb{Z}}$ associated to the quadratic form $q(x) := (-1)^{x^2}$ exhibits such behaviour.

We can now state the main theorem of this section, which completely characterizes which connected fusion 2-categories are Morita equivalent. In particular, it generalizes example 4.3.4.6 above. The proof relies on the technical results derived in section 5.2.2 below.

Theorem 5.2.1.4. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two braided fusion 1-categories. Then,* $\mathbf{Mod}(\mathcal{B}_1)$ *and* $\mathbf{Mod}(\mathcal{B}_2)$ *are Morita equivalent fusion 2-categories if and only if $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same symmetric center $\mathcal{E}$ and they are Witt equivalent.*

Proof. Assume that $\mathbf{Mod}(\mathcal{B}_1)$ and $\mathbf{Mod}(\mathcal{B}_2)$ are Morita equivalent fusion 2-categories. Then, it follows from theorem 5.1.3.2 that

$$\mathscr{Z}(\mathbf{Mod}(\mathcal{B}_1)) \simeq \mathscr{Z}(\mathbf{Mod}(\mathcal{B}_2))$$

as braided fusion 2-categories. Thanks to lemma 2.16 of [JFR24] or corollary 5.2.2.3 below, we find that

$$\mathcal{Z}_{(2)}(\mathcal{B}_1) \simeq \Omega\mathscr{Z}(\mathbf{Mod}(\mathcal{B}_1)) \simeq \Omega\mathscr{Z}(\mathbf{Mod}(\mathcal{B}_2)) \simeq \mathcal{Z}_{(2)}(\mathcal{B}_2)$$

as symmetric fusion 1-categories. We denote by $\mathcal{E}$ this symmetric fusion 1-category. Now, as $\mathbf{Mod}(\mathcal{B}_1)$ and $\mathbf{Mod}(\mathcal{B}_2)$ are Morita equivalent fusion 2-categories, there exists a fusion 1-category $\mathcal{C}$ equipped with a braided monoidal functor $F:\mathcal{B}_1\rightarrow\mathcal{Z}(\mathcal{C})$, and a monoidal equivalence $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B}_1)}(\mathcal{C}) \simeq \mathbf{Mod}(\mathcal{B}_2)$ of fusion 2-categories. Let us write $\mathcal{A}$ for the image of $\mathcal{B}_1$ in $\mathcal{Z}(\mathcal{C})$, which is a braided fusion 1-category by definition. Then, it follows from proposition 4.3 of [DNO13] and from the definitions that

$$\mathcal{Z}_{(2)}(\mathcal{A}) \simeq \mathcal{Z}_{(2)}(\mathcal{Z}(\mathcal{C},\mathcal{A})) \simeq \mathcal{Z}_{(2)}(\mathcal{Z}(\mathcal{C},\mathcal{B}_1))$$

as symmetric fusion 1-categories. On the other hand, by lemma 5.2.2.1 below, we find that $\mathcal{Z}(\mathcal{C},\mathcal{B}_1)^{rev}\simeq\mathcal{B}_2$ as braided fusion 1-categories. This implies that $\mathcal{Z}_{(2)}(\mathcal{A})\simeq\mathcal{E}$. Thence, it follows from corollary 3.24 of [DMNO13] that F is fully faithful, so that $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent.

On the other hand, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent, then, by definition, there exists a $\mathcal{B}_1$-central fusion 1-category $\mathcal{C}$ such that $\mathcal{B}_1\rightarrow\mathcal{Z}(\mathcal{C})$ is a fully faithful embedding, and an equivalence $\mathcal{Z}(\mathcal{C},\mathcal{B}_1)^{rev}\simeq\mathcal{B}_2$ of braided fusion 1-categories. By corollary 5.2.2.6 below, the $\mathcal{B}_1$-central fusion 1-category $\mathcal{C}$ witnesses the desired Morita equivalence. □

Corollary 5.2.1.5. *Let $\mathcal{B}$ be a braided fusion 1-category, and $\mathcal{C}$ a fusion 1-category with $\mathcal{B}$-central structure given by $F:\mathcal{B}\rightarrow\mathcal{Z}(\mathcal{C})$. Then,* $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$ *is connected if and only if F is fully faithful.*

Remark 5.2.1.6. More generally, it would be interesting to understand the connected components of the fusion 2-category $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$. For instance, with $\mathcal{B}=\mathbf{Rep}(G)$ for some finite group G, and $\mathcal{C}=\mathbf{Vect}$, it was shown in example 4.3.1.9

that $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \simeq \mathbf{2Vect}_G$, which has $|G|$ connected components. On the other hand, with $\mathcal{B} = \mathbf{Rep}(G \times G^{op})$ and $\mathcal{C} = \mathbf{Rep}(G)$, it follows from lemma 5.1.2.1 and corollary 5.1.3.4 that $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \simeq \mathscr{Z}(\mathbf{2Vect}_G)$. It was shown in [KTZ20] that the connected components of the underlying finite semisimple 2-category of $\mathscr{Z}(\mathbf{2Vect}_G)$ are parameterised by the conjugacy classes of G.

The technical results used in the proof of the above theorem will be established below. But first, we record the following corollary. Let us fix $\mathcal{E}$ a symmetric fusion 1-category. A non-degenerate braided fusion 1-category $\mathcal{B}$ equipped with a fully faithful embedding of $\mathcal{E}$ is called minimal if the centralizer of $\mathcal{E}$ in $\mathcal{B}$ is exactly $\mathcal{E}$. The set of minimal non-degenerate extensions of $\mathcal{E}$ forms a group (see [LKW17]), which is denoted by $Mext(\mathcal{E})$. We say that a minimal non-degenerate extension $\mathcal{B}$ of $\mathcal{E}$ is Witt-trivial if the class of the non-degenerate braided fusion 1-category $\mathcal{B}$ is trivial in the Witt group $\mathcal{W}$. This is equivalent to requiring that $\mathcal{B}$ is braided equivalent to the Drinfeld center of a fusion 1-category. We note that the set of Witt-trivial minimal non-degenerate extensions of $\mathcal{E}$ is a subgroup of $Mext(\mathcal{E})$ thanks to theorem 3.6.20 of [Sun22].

Corollary 5.2.1.7. *Let $\mathcal{E}$ be a symmetric fusion 1-category. Then, there is a bijective correspondence between Morita autoequivalences of $\mathbf{Mod}(\mathcal{E})$ and pairs consisting of a Witt-trivial minimal non-degenerate extension of $\mathcal{E}$ and an equivalence class of symmetric monoidal functor $\mathcal{E} \simeq \mathcal{E}$.*

Proof. It follows readily from corollary 5.2.1.5 that every Morita autoequivalence of $\mathbf{Mod}(\mathcal{E})$ is given by an $\mathcal{E}$-central fusion 1-category $\mathcal{C}$ such that $\mathcal{Z}(\mathcal{C}, \mathcal{E}) = \mathcal{E}$ together with a braided autoequivalence of $\mathcal{E}$. On the other hand, any two $\mathcal{E}$-central fusion 1-categories $\mathcal{C}_1$ and $\mathcal{C}_2$ with $\mathcal{Z}(\mathcal{C}_1, \mathcal{E}) = \mathcal{E} = \mathcal{Z}(\mathcal{C}_2, \mathcal{E})$ can induce the same Morita equivalence only if $\mathbf{Mod}(\mathcal{C}_1)$ and $\mathbf{Mod}(\mathcal{C}_2)$ are equivalent as left $\mathbf{Mod}(\mathcal{E})$-module 2-categories. This implies that $\mathcal{C}_1$ and $\mathcal{C}_2$ are Morita equivalent fusion 1-categories so that we have an equivalence $\mathcal{Z}(\mathcal{C}_1) \simeq \mathcal{Z}(\mathcal{C}_2)$ of braided fusion 1-categories (see theorem 1.1 of [ENO10]). Finally, $\mathbf{Mod}(\mathcal{C}_1)$ and $\mathbf{Mod}(\mathcal{C}_2)$ are equivalent as left $\mathbf{Mod}(\mathcal{E})$-module 2-categories if and only if the diagram of braided monoidal functors

$$\begin{tikzcd} & \mathcal{E} \arrow[dl] \arrow[dr] & \\ \mathcal{Z}(\mathcal{C}_1) \arrow[rr, "\simeq"] & & \mathcal{Z}(\mathcal{C}_2) \end{tikzcd}$$

commutes up to braided natural equivalence. This is the case if and only if $\mathcal{Z}(\mathcal{C}_1)$ and $\mathcal{Z}(\mathcal{C}_2)$ are equivalent minimal non-degenerate extensions of $\mathcal{E}$. This concludes the proof. $\square$

Remark 5.2.1.8. Let G be a finite group. It follows from the above corollary and example 2.1 of [Nik22] that there is a bijection of sets $\mathrm{BrPic}(\mathbf{Mod}(\mathbf{Rep}(G))) \simeq H^3(G;\Bbbk^\times) \times Aut^{br}(\mathbf{Rep}(G)))$. Namely, every minimal non-degenerate extension of $\mathbf{Rep}(G)$ is Witt-trivial. In this case, we expect that the group homomorphism $\Phi : \mathrm{BrPic}(\mathbf{Mod}(\mathbf{Rep}(G))) \to Aut^{br}(\mathscr{Z}(\mathbf{Mod}(\mathbf{Rep}(G))))$ of remark 5.1.3.3 is an isomorphism.

On the other hand, the above corollary shows that $\mathrm{BrPic}(\mathbf{2SVect}) \simeq 0$. Namely, the group of minimal non-degenerate extensions of $\mathbf{SVect}$ is identified with $\mathbb{Z}/16\mathbb{Z}$ by example 2.2 of [Nik22], and only one of the sixteen minimal non-degenerate extensions of $\mathbf{SVect}$ is Witt-trivial (see appendix A.3.2 of [DGNO10]). But, every minimal non-degenerate extension $\mathcal{C}$ of $\mathbf{SVect}$ induces a braided autoequivalence of $\mathscr{Z}(\mathbf{2SVect})$. In detail, the left $\mathbf{2SVect}$-module 2-category $\mathbf{Mod}(\mathcal{C})$ induces a braided monoidal 2-functor $\mathscr{Z}(\mathbf{2SVect}) \simeq \mathscr{Z}(\mathbf{2SVect} \boxtimes \mathbf{Mod}(\mathcal{C}))$, and, thanks to corollary 5.4.3.2 below, the right-hand side is canonically equivalent to $\mathscr{Z}(\mathbf{2SVect}) \boxtimes \mathscr{Z}(\mathbf{Mod}(\mathcal{C})) \simeq \mathscr{Z}(\mathbf{2SVect})$ as a braided monoidal 2-category. In fact, it was sketched in section 2.3 of [JF21] that this construction induces a bijection of sets $Aut^{br}(\mathscr{Z}(\mathbf{2SVect})) \cong \mathbb{Z}/16\mathbb{Z}$. We therefore expect that the map $\Phi : \mathrm{BrPic}(\mathbf{2SVect}) \to Aut^{br}(\mathscr{Z}(\mathbf{2SVect}))$ is not an isomorphism, so that the full version of theorem 1.1 of [ENO10] can not be naïvely categorified.

Let $\mathcal{E}$ be a symmetric fusion 1-category. The group $\mathcal{W}(\mathcal{E})$ of classes of non-degenerate braided fusion 1-categories over $\mathcal{E}$ up to Witt equivalence over $\mathcal{E}$ was introduced in [DNO13]. The group $Aut^{br}(\mathcal{E})$ acts on $\mathcal{W}(\mathcal{E})$. Namely, given $E : \mathcal{E} \simeq \mathcal{E}$ an equivalence of symmetric fusion 1-categories, and $(\mathcal{B}, F)$ a non-degenerate braided fusion 1-category over $\mathcal{E}$, we can consider $(\mathcal{B}, F \circ E)$. Further, it is clear that this assignment is compatible with Witt equivalence over $\mathcal{E}$. We write $\widehat{\mathcal{W}}(\mathcal{E})$ for the quotient of $\mathcal{W}(\mathcal{E})$ under the action of $Aut^{br}(\mathcal{E})$. The next result follows by combing lemma 5.2.1.2 with theorem 5.3.2.1.

Corollary 5.2.1.9. *Morita equivalence classes of connected fusion 2-categories correspond precisely to pairs consisting of a symmetric fusion 1-category $\mathcal{E}$ and a class in $\widehat{\mathcal{W}}(\mathcal{E})$.*

Remark 5.2.1.10. Let us fix $\mathcal{E}$ a symmetric fusion 1-category. A $\mathbf{Mod}(\mathcal{E})$-central fusion 2-category, or a fusion 2-category over $\mathbf{Mod}(\mathcal{E})$, is a fusion 2-category $\mathfrak{C}$ equipped with a braided monoidal 2-functor $\mathbf{F} : \mathbf{Mod}(\mathcal{E}) \to \mathscr{Z}(\mathfrak{C})$. A Morita equivalence between two $\mathbf{Mod}(\mathcal{E})$-central fusion 2-category $(\mathfrak{C}_1, \mathbf{F}_1)$ and $(\mathfrak{C}_2, \mathbf{F}_1)$ is a Morita equivalence between the underlying fusion 2-categories $\mathfrak{C}_1$ and $\mathfrak{C}_2$ such that the induced diagram of braided monoidal 2-functors

$$\begin{tikzcd} & \mathbf{Mod}(\mathcal{E}) \arrow[dl, "\mathbf{F}_1"'] \arrow[dr, "\mathbf{F}_2"] & \\ \mathscr{Z}(\mathfrak{C}_1) \arrow[rr, "\simeq"] & & \mathscr{Z}(\mathfrak{C}_2) \end{tikzcd}$$

commutes up to braided monoidal 2-natural equivalence. Now, the data of a $\mathbf{Mod}(\mathcal{E})$-central structure for a connected fusion 2-category $\mathbf{Mod}(\mathcal{B})$ corresponds exactly to a symmetric monoidal functor $F : \mathcal{E} \to \mathcal{Z}_{(2)}(\mathcal{B})$. Namely, we have $\Omega\mathscr{Z}(\mathbf{Mod}(\mathcal{B})) \simeq \mathbf{Mod}(\mathcal{Z}_{(2)}(\mathcal{B}))$, and any monoidal 2-functor $\mathbf{F} : \mathbf{Mod}(\mathcal{E}) \to \mathscr{Z}(\mathbf{Mod}(\mathcal{B}))$ is completely determined by the braided monoidal functor $\Omega\mathbf{F}$. In particular, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are braided fusion 1-categories, and $F_1 : \mathcal{E} \simeq \mathcal{Z}_{(2)}(\mathcal{B}_1)$ and $F_2 : \mathcal{E} \simeq \mathcal{Z}_{(2)}(\mathcal{B}_2)$ are equivalences of symmetric fusion 1-categories, it follows from theorem 5.2.1.4 that the $\mathbf{Mod}(\mathcal{E})$-central fusion 2-category $(\mathbf{Mod}(\mathcal{B}_1), \mathbf{Mod}(F_1))$ and $(\mathbf{Mod}(\mathcal{B}_2), \mathbf{Mod}(F_1))$ are Morita equivalent if and only if $(\mathcal{B}_1, F_1)$ and $(\mathcal{B}_2, F_2)$ are Witt equivalent over $\mathcal{E}$ in the sense of [DNO13].

5.2.2 Technical Results

We establish the technical lemmas that were used in the proof of theorem 5.2.1.4.

Lemma 5.2.2.1. *Let $\mathcal{B}$ be a braided fusion 1-category, and $\mathcal{C}$ a $\mathcal{B}$-central fusion 1-category. There is an equivalence*

$$\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})^0 \simeq \mathbf{Mod}(\mathcal{Z}(\mathcal{C}, \mathcal{B})^{rev})$$

of monoidal 2-categories.

Proof. We have seen in example 2.3.4.3 above that there is an equivalence of 2-categories

$$\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \simeq \mathbf{Mod}(\mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C}).$$

Further, as observed in remark 3.2.3.6, the 2-category on the left hand-side is the 2-category of finite semisimple $\mathcal{B}$-balanced $\mathcal{C}$-$\mathcal{C}$-bimodule 1-categories as considered in section 3.4 of [Lau20]. Now, $\mathcal{C}$ with its canonical $\mathcal{B}$-balanced $\mathcal{C}$-$\mathcal{C}$-bimodule structure is the monoidal unit of $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$. Let us write $End^{\mathcal{B}}_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C})$ for the braided fusion 1-category of endomorphisms of $\mathcal{C}$ in $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$. Now, observe that the forgetful 2-functor

$$\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \to \mathbf{Bimod}(\mathcal{C})$$

is monoidal. On the monoidal unit, this 2-functor induces a braided monoidal functor $End^{\mathcal{B}}_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C}) \to End_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C})$. But, by example 4.3.3.7, we know that there is an equivalence $End_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{C})^{rev}$ of braided fusion 1-categories. Moreover, it follows from definition 3.24 of [Lau20] that $End^{\mathcal{B}}_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C})$ is the full fusion sub-1-category of $End_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{C})^{rev}$ on those objects centralizing the image of $\mathcal{B}^{rev}$ in $\mathcal{Z}(\mathcal{C})^{rev}$. Thus, we find that $End^{\mathcal{B}}_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{C}, \mathcal{B})^{rev}$ as braided fusion 1-categories. Finally, the proof is concluded by appealing to proposition 2.4.7 of [Déc22c]. □

Remark 5.2.2.2. As already mentioned in remark 3.2.3.6, [Lau20] works with $\mathcal{B}$-augmented tensor 1-categories, which are $\mathcal{B}$-balanced tensor 1-categories equipped with additional structure. Nonetheless, the results of section 3.4 of [Lau20] hold for any $\mathcal{B}$-central tensor 1-category. On the other hand, this is not the case for all the results of section 3 of [Lau20]. More precisely, corollary 3.46 of [Lau20] does not hold for an arbitrary $\mathcal{B}$-central tensor 1-category. Namely, if $\mathcal{B} = \mathbf{Rep}(G)$, and $\mathcal{C} = \mathbf{Vect}$, then $\mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C} \simeq \boxplus_{g \in G} \mathbf{Vect}$ and $\mathcal{C}$ is not a faithful $\mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C}$-module 1-category.

We recover lemma 2.16 of [JFR24].

Corollary 5.2.2.3. *Let $\mathcal{B}$ be a braided fusion 1-category, then we have*

$$\Omega\mathscr{Z}(\mathbf{Mod}(\mathcal{B})) \simeq \mathcal{Z}_{(2)}(\mathcal{B}).$$

Proof. Thanks to lemma 5.1.2.1 we can identify $\mathscr{Z}(\mathbf{Mod}(\mathcal{B}))$ with the dual to the fusion 2-category $\mathbf{Mod}(\mathcal{B}^{rev} \boxtimes \mathcal{B})$ with respect to $\mathbf{Mod}(\mathcal{B})$. It follows from corollary 4.4 of [DNO13] and theorem 3.10 (ii) of [DGNO10] that $\mathcal{Z}(\mathcal{B}, \mathcal{B}^{rev} \boxtimes \mathcal{B}) \simeq \mathcal{Z}_{(2)}(\mathcal{B})$, which establishes the result. □

We now give a sufficient criterion for the fusion 2-category $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$ to be connected.

Lemma 5.2.2.4. *Let $\mathcal{B}$ be a braided fusion 1-category, and $\mathcal{C}$ a $\mathcal{B}$-central fusion 1-category such that the composite $\mathcal{B} \to \mathcal{C}$ is fully faithful. Then, $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$ is a connected fusion 2-category.*

Proof. Let us identify $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}) \simeq \mathbf{Mod}(\mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C})$ as 2-categories. Thanks to proposition 2.3.5 of [Déc22b], in order to prove that this finite semisimple 2-category is connected, it is enough to show that $\mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C}$ is a fusion 1-category. But, the canonical functor $\mathcal{C}^{mop} \boxtimes \mathcal{C} \to \mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C}$ is monoidal. Further, the image of $\mathcal{B}$ in $\mathcal{C}$ may be identified with $\mathcal{B}$ by hypothesis. Thus, the monoidal unit of $\mathcal{C}^{mop} \boxtimes_{\mathcal{B}} \mathcal{C}$ is given by the image of the monoidal unit $I \boxtimes I$ of $\mathcal{B}^{mop} \boxtimes \mathcal{B}$ under the canonical monoidal functor $\mathcal{B}^{mop}\boxtimes\mathcal{B}\rightarrow\mathcal{B}^{mop}\boxtimes_{\mathcal{B}}\mathcal{B}\simeq\mathcal{B}$. As $\mathcal{B}$ is a fusion 1-category by hypothesis, the proof is complete. □

We need the following slightly more general version of the previous lemma.

Lemma 5.2.2.5. *Let $\mathcal{B}$ be a braided fusion 1-category, and $\mathcal{C}$ a fusion 1-category with $\mathcal{B}$-central structure $F:\mathcal{B}\rightarrow\mathcal{Z}(\mathcal{C})$ given by a fully faithful braided monoidal functor. Then, $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$ is a connected fusion 2-category.*

Proof. Let us consider the fusion 2-category $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{Z}(\mathcal{C}))$, where $\mathcal{Z}(\mathcal{C})$ is endowed with its canonical $\mathcal{B}$-central structure. Thanks to lemma 5.2.2.4 above, it is connected. On the other hand, by theorem 4.3.3.2 above, there is an equivalence of fusion 2-categories

$$\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{Z}(\mathcal{C}))\simeq\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}^{mop}\boxtimes\mathcal{C}).$$

More precisely, $\mathcal{C}^{mop}\boxtimes\mathcal{C}$ is equipped with the braided monoidal functor

$$\mathcal{B}\xrightarrow{F}\mathcal{Z}(\mathcal{C})\hookrightarrow\mathcal{Z}(\mathcal{C}^{mop})\boxtimes\mathcal{Z}(\mathcal{C})\simeq\mathcal{Z}(\mathcal{C}^{mop}\boxtimes\mathcal{C}).$$

Then, as $\mathcal{Z}(\mathcal{C})$ and $\mathcal{C}^{mop}\boxtimes\mathcal{C}$ are Morita equivalent fusion 1-categories, $\mathbf{Mod}(\mathcal{Z}(\mathcal{C}))$ and $\mathbf{Mod}(\mathcal{C}^{mop}\boxtimes\mathcal{C})$ are equivalent finite semisimple 2-categories. Furthermore, this Morita equivalence of fusion 1-categories is compatible with the $\mathcal{B}$-central structures as the following diagram of braided monoidal functors commute

$$\begin{tikzcd} & \mathcal{B} \arrow[dl] \arrow[dr] & \\ \mathcal{Z}(\mathcal{C}^{mop}\boxtimes\mathcal{C}) \arrow[rr, "\simeq"] & & \mathcal{Z}(\mathcal{Z}(\mathcal{C})). \end{tikzcd}$$

But, as the $\mathcal{B}$-central structure on $\mathcal{C}^{mop}\boxtimes\mathcal{C}$ factors through $\mathcal{C}$, we have that

$$\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C}^{mop}\boxtimes\mathcal{C})\simeq\mathbf{Bimod}(\mathcal{C}^{mop})\boxtimes\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$$

as fusion 2-categories. Finally, $\mathbf{Bimod}(\mathcal{C}^{mop})$ and $\mathbf{Mod}(\mathcal{Z}(\mathcal{C}^{mop}))$ are equivalent fusion 2-categories (see example 4.3.3.7 above). But, thanks to lemma 4.3 of [Déc24], the set of connected components of a 2-Deligne tensor product is the product of the set of connected components, so that $\mathbf{Bimod}_{\mathbf{Mod}(\mathcal{B})}(\mathcal{C})$ is indeed connected. □

Corollary 5.2.2.6. *Let $\mathcal{B}$ be a braided fusion 1-category, and $\mathcal{C}$ be a $\mathcal{B}$-central fusion 1-category such that the braided monoidal functor $\mathcal{B}\rightarrow\mathcal{Z}(\mathcal{C})$ is fully faithful. Then, the dual to $\mathbf{Mod}(\mathcal{B})$ with respect to $\mathbf{Mod}(\mathcal{C})$ is given by $\mathbf{Mod}(\mathcal{Z}(\mathcal{C},\mathcal{B}))$.*

5.3 Fusion 2-Categories up to Morita Equivalence

We analyze the Morita equivalence classes of fusion 2-categories. More precisely, we show that every fusion 2-category is Morita equivalent to the 2-Deligne tensor product of a strongly fusion 2-category and an invertible fusion 2-category. Further, we show that every strongly fusion 2-category is Morita equivalent to a connected fusion 2-category. In particular, this establishes that every fusion 2-category is Morita equivalent to a connected fusion 2-category. Throughout, we work over a fixed algebraically closed field $\Bbbk$ of characteristic zero.

5.3.1 Strongly Fusion and Invertible

We begin by introducing invertible multifusion 2-categories, and we characterize them explicitly.

Definition 5.3.1.1. A multifusion 2-category $\mathfrak{C}$ is invertible if $\mathscr{Z}(\mathfrak{C}) \simeq \mathbf{2Vect}$.

Remark 5.3.1.2. Let $\mathfrak{C}$ be an invertible multifusion 2-category. It follows from proposition 4.3.1.6 that $\mathfrak{C}$ is separable as a left $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$-module 2-category. Then, lemma 5.1.2.1 implies that the Morita equivalence class of $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$ is trivial. This justifies the name. In fact, thanks to theorem 5.1.3.2 the latter gives an equivalent characterization of invertibility.

Lemma 5.3.1.3. *A fusion 2-category is invertible if and only if it is equivalent to* $\mathbf{Mod}(\mathcal{B})$ *with* $\mathcal{B}$ *a non-degenerate braided fusion 1-category.*

Proof. Let $\mathfrak{C}$ be an invertible fusion 2-category. This means that the dual to $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$ with respect to $\mathfrak{C}$ is $\mathbf{2Vect}$. Thanks to corollary 4.3.4.4, this implies that the dual to $\mathbf{2Vect}$ with respect to $\mathfrak{C}$ is $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$. Thus, by combining proposition 4.3.1.6 and corollary 4.3.2.5, we get that $\mathfrak{C}$ is indecomposable as a finite semisimple 2-category, i.e. it is connected. Let $\mathcal{B}$ be a braided fusion 1-category such that $\mathbf{Mod}(\mathcal{B}) \simeq \mathfrak{C}$ as fusion 2-category. As $\mathscr{Z}(\mathbf{Mod}(\mathcal{B})) \simeq \mathbf{2Vect}$, it follows from lemma 2.16 of [JFR24], or corollary 5.2.2.3 above, that $\mathcal{Z}_{(2)}(\mathcal{B}) \simeq \mathbf{Vect}$, so that $\mathcal{B}$ is non-degenerate. □

Let us now recall the following definition from [JFY21].

Definition 5.3.1.4. A bosonic strongly fusion 2-category is fusion 2-category $\mathfrak{C}$ for which $\Omega\mathfrak{C} \simeq \mathbf{Vect}$ as braided fusion 1-categories. A fermionic strongly fusion 2-category is fusion 2-category $\mathfrak{C}$ for which $\Omega\mathfrak{C} \simeq \mathbf{SVect}$, the symmetric fusion 1-category of super vector spaces.

Before stating the first theorem of this section, we will establish a technical lemma. In order to do so, it is convenient to introduce the generalize the above terminology.

Definition 5.3.1.5. We say that a fusion 2-category $\mathfrak{C}$ is bosonic if the symmetric fusion 1-category $\mathcal{Z}_{(2)}(\Omega\mathfrak{C})$ is Tannakian, i.e. there exists a symmetric monoidal functor $\mathcal{Z}_{(2)}(\Omega\mathfrak{C}) \to \mathbf{Vect}$. Otherwise, we say that $\mathfrak{C}$ is fermionic.

Lemma 5.3.1.6. *Every bosonic fusion 2-category $\mathfrak{C}$ is Morita equivalent to a fusion 2-category $\mathfrak{D}$ with $\Omega\mathfrak{D}$ a non-degenerate braided fusion 1-category. Everey fermionic fusion 2-category $\mathfrak{C}$ is Morita equivalent to a fusion 2-category $\mathfrak{D}$ with $\Omega\mathfrak{D}$ a slightly-degenerate braided fusion 1-category, i.e. $\mathcal{Z}_{(2)}(\Omega\mathfrak{D}) = \mathbf{SVect}$.*

Proof. Let $\mathfrak{C}$ be a bosonic fusion 1-category. We write $\mathcal{E} := \mathcal{Z}_{(2)}(\Omega\mathfrak{C})$. By hypothesis, there exists a symmetric monoidal functor $\mathcal{E} \to \mathbf{Vect}$. We define a braided fusion 1-category $\mathcal{D} := \Omega\mathfrak{C}\boxtimes_{\mathcal{E}}\mathbf{Vect}$. By proposition 4.30 of [DGNO10], $\mathcal{D}$ is non-degenerate. Further, we view $\mathcal{D}$ as a connected rigid algebra in $\mathbf{Mod}(\Omega\mathfrak{C}) \simeq \mathfrak{C}^0$ using the canonical braided monoidal functor $\Omega\mathfrak{C} \to \mathcal{D} \to \mathcal{Z}(\mathcal{D})$. It follows from example 2.3.4.3 that $\mathcal{D}$ is a separable algebra. Thus, by theorem 4.3.4.3, the multifusion 2-category $\mathbf{Bimod}_{\mathfrak{C}}(\mathcal{D})$ is Morita equivalent to $\mathfrak{C}$. Further, as $\mathcal{D}$ is connected, it is necessarily indecomposable, so that it follows from corollary 4.3.2.5 that $\mathbf{Bimod}_{\mathfrak{C}}(\mathcal{D})$ is in fact a fusion 2-category, and we set $\mathfrak{D} := \mathbf{Bimod}_{\mathfrak{C}}(\mathcal{D})$. It remains to analyze $\mathfrak{D}^0$. As $\mathcal{D}$ is an algebra in $\mathfrak{C}^0$, corollary 2.3.6 of [Déc22c] shows that the underlying object of any simple $\mathcal{D}$-$\mathcal{D}$-bimodule in $\mathfrak{C}$ is contained in a single connected component of $\mathfrak{C}$. In particular, we have an equivalence

$$\mathfrak{D}^0 \simeq (\mathbf{Bimod}_{\mathfrak{C}^0}(\mathcal{D}))^0 \simeq (\mathbf{Bimod}_{\mathbf{Mod}(\Omega\mathfrak{C})}(\mathcal{D}))^0$$

of fusion 2-categories. Moreover, it was shown in lemma 5.2.2.1 above that

$$(\mathbf{Bimod}_{\mathbf{Mod}(\Omega\mathfrak{C})}(\mathcal{D}))^0 \simeq \mathbf{Mod}(\mathcal{Z}(\mathcal{D}, \Omega\mathfrak{C}))$$

as fusion 2-categories. Finally, as $\mathcal{D}$ is non-degenerate, we have that $\mathcal{Z}(\mathcal{D}) \simeq \mathcal{D}\boxtimes\mathcal{D}^{rev}$ by proposition 3.7 of [DGNO10]. Further, the canonical braided monoidal functor $\Omega\mathfrak{C} \to \mathcal{D}$ is surjective (in the sense of definition 2.1 of [DGNO10]), so that $\mathcal{Z}(\mathcal{D}, \Omega\mathfrak{C}) \simeq \mathcal{D}^{rev}$ as braided fusion 1-categories. This concludes the proof in this case.

If $\mathfrak{C}$ is fermionic, then it follows from [Del02] that there exists a dominant symmetric monoidal functor $\mathcal{E} \to \mathbf{SVect}$. We then define a braided fusion 1-category $\mathcal{D} := \Omega\mathfrak{C} \boxtimes_{\mathcal{E}} \mathbf{SVect}$. By proposition 4.30 of [DGNO10], $\mathcal{D}$ is slightly-degenerate.

Then, the argument used above shows that $\mathfrak{D} := \mathbf{Bimod}_{\mathbf{Mod}(\Omega\mathfrak{C})}(\mathcal{D})$ is a fusion 2-category with $\mathfrak{D}^0 \simeq \mathbf{Mod}(\mathcal{Z}(\mathcal{D}, \Omega\mathfrak{C}))$ as fusion 2-categories. It follows from proposition 4.3 of [DNO13] that $\mathcal{Z}(\mathcal{D}, \Omega\mathfrak{C})$ is slightly-degenerate as desired, thereby finishing the proof. □

Theorem 5.3.1.7. *Every bosonic, respectively fermionic, fusion 2-category is Morita equivalent to the 2-Deligne tensor product of a bosonic, respectively fermionic, strongly fusion 2-category and an invertible fusion 2-category.*

Proof. Let $\mathfrak{C}$ be a fusion 2-category, and write $\mathcal{E}$ for the symmetric fusion 1-category $\mathcal{Z}_{(2)}(\Omega\mathfrak{C})$. We treat the cases of bosonic and fermionic fusion 2-categories separately.
Case 1: We assume that the fusion 2-category $\mathfrak{C}$ is bosonic. In that case, it follows from lemma 5.3.1.6 that there exists a fusion 2-category $\mathfrak{D}$ that is Morita equivalent to $\mathfrak{C}$, and such that $\mathcal{B} := \Omega\mathfrak{D}$ is a non-degenerate braided fusion 1-category. Let us now define $\mathfrak{F} := \mathfrak{D} \boxtimes \mathbf{Mod}(\mathcal{B}^{rev})$, so that $\Omega\mathfrak{F} \simeq \mathcal{B} \boxtimes \mathcal{B}^{rev}$ by proposition 5.1 of [Déc24]. In particular, equipping $\mathcal{B}$ with the canonical braided monoidal functor $\mathcal{B} \boxtimes \mathcal{B}^{rev} \to \mathcal{Z}(\mathcal{B})$, we can view $\mathcal{B}$ as a separable algebra in the fusion 2-category $\mathbf{Mod}(\mathcal{B} \boxtimes \mathcal{B}^{rev}) \simeq \mathfrak{F}^0$. Let us write $\mathfrak{G}$ for the fusion 2-category given by $\mathbf{Bimod}_{\mathfrak{F}}(\mathcal{B})$. By example 4.3.3.7, see also lemma 5.2.2.5, we find that

$$\mathfrak{G}^0 = \mathbf{Bimod}_{\mathfrak{F}^0}(\mathcal{B}) \simeq \mathbf{2Vect}.$$

This shows that $\mathfrak{G}$ is a bosonic strongly fusion 2-category.

Finally, it follows from lemma 4.3.4.8 that Morita equivalence is compatible with the 2-Deligne tensor product. As $\mathbf{2Vect}$ is Morita equivalent to $\mathbf{Mod}(\mathcal{B}^{rev}) \boxtimes \mathbf{Mod}(\mathcal{B})$, we have that $\mathfrak{C}$ is Morita equivalent to the fusion 2-category $\mathfrak{C} \boxtimes \mathbf{Mod}(\mathcal{B}^{rev}) \boxtimes \mathbf{Mod}(\mathcal{B})$. Furthermore, the above discussion shows that $\mathfrak{C} \boxtimes \mathbf{Mod}(\mathcal{B}^{rev}) \boxtimes \mathbf{Mod}(\mathcal{B})$ is Morita equivalent to $\mathfrak{G} \boxtimes \mathbf{Mod}(\mathcal{B})$. But, Morita equivalence is an equivalence relation by theorem 4.3.4.3, so that we get the desired result in this case.
Case 2: We assume that the fusion 2-category $\mathfrak{C}$ is fermionic. In that case, it follows from lemma 5.3.1.6 that there exists a fusion 2-category $\mathfrak{D}$ that is Morita equivalent to $\mathfrak{C}$, and such that $\mathcal{B} := \Omega\mathfrak{D}$ is a slightly degenerate braided fusion 1-category. Now, thank to the main theorem of [JFR24], there exists a non-degenerate braided fusion 1-category $\mathcal{C}$ together with a braided embedding $\mathcal{B} \subseteq \mathcal{C}$, such that the centralizer of $\mathcal{B}$ in $\mathcal{C}$ is exactly $\mathbf{SVect}$. Let us write $\mathfrak{E}$ for the fusion 2-category $\mathbf{Bimod}_{\mathfrak{D}}(\mathcal{C})$. By lemma 5.2.2.5 above, we find that

$$\mathfrak{E}^0 = \mathbf{Bimod}_{\mathfrak{D}^0}(\mathcal{C}) \simeq \mathbf{Mod}(\mathbf{SVect} \boxtimes \mathcal{C}^{rev}).$$

Namely, as $\mathcal{C}$ is a non-degenerate braided fusion 1-category, we have that $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C}\boxtimes\mathcal{C}^{rev}$ by proposition 3.7 of [DGNO10], so that $\mathcal{Z}(\mathcal{C},\mathcal{B}) \simeq \mathbf{SVect}\boxtimes\mathcal{C}^{rev}$ as braided fusion 1-categories. Then, we consider the fusion 2-category $\mathfrak{F} := \mathfrak{E}\boxtimes\mathbf{Mod}(\mathcal{C})$. Using the same argument as the one used in the proof of the first case, we find that $\mathfrak{G} := \mathbf{Bimod}_{\mathfrak{F}}(\mathcal{C})$ is a fermionic strongly fusion 2-category.

Finally, as $\mathbf{2Vect}$ is Morita equivalent to $\mathbf{Mod}(\mathcal{C})\boxtimes\mathbf{Mod}(\mathcal{C}^{rev})$, we have that $\mathfrak{C}$ is Morita equivalent to the fusion 2-category $\mathfrak{C}\boxtimes\mathbf{Mod}(\mathcal{C})\boxtimes\mathbf{Mod}(\mathcal{C}^{rev})$. Furthermore, the above discussion implies that $\mathfrak{C}\boxtimes\mathbf{Mod}(\mathcal{C})\boxtimes\mathbf{Mod}(\mathcal{C}^{rev})$ is Morita equivalent to $\mathfrak{G}\boxtimes\mathbf{Mod}(\mathcal{C}^{rev})$. This concludes the proof. □

Remark 5.3.1.8. We emphasize that the invertible fusion 2-category supplied by theorem 5.3.1.7 is not unique. Namely, if $\mathcal{C}$ is any minimal non-degenerate extension of $\mathbf{SVect}$, then it follows from theorem 5.2.1.4 that $\mathbf{2SVect}\boxtimes\mathbf{Mod}(\mathcal{C})$ is Morita equivalent to $\mathbf{2SVect}$. Further, in the fermionic case, the strongly fusion 2-categories is not quite unique either, as will be explained in [JF]. On the other hand, we expect that in the bosonic case the strongly fusion 2-category supplied by theorem 5.3.1.7 is unique up to monoidal equivalence, and that the invertible fusion 2-category is unique up to Morita equivalence.

5.3.2 Connected

We presently establish the second main theorem of this section.

Theorem 5.3.2.1. *Every fusion 2-category is Morita equivalent to a connected fusion 2-category.*

Proof. By theorem 5.3.1.7, every fusion 2-category is Morita equivalent to the 2-Deligne tensor product of a strongly fusion 2-category and an invertible fusion 2-category. It is therefore enough to establish that every strongly fusion 2-category is Morita equivalent to a connected fusion 2-category.

We fix $\mathfrak{C}$ a strongly fusion 2-category, and write $\mathfrak{C}^\times$ for the monoidal sub-2-category of $\mathfrak{C}$ on the invertible objects and the invertible morphisms. As $\mathfrak{C}$ is a monoidal $\Bbbk$-linear 2-category, there is an equivalence of monoidal 2-categories $\mathfrak{C}^\times \simeq \mathcal{G}\times^{\pi}\mathrm{B}^2\Bbbk^\times$ for some finite 2-group $\mathcal{G}$ and 4-cocycle π for $\mathcal{G}$ with coefficients in $\Bbbk^\times$. We consider the fusion 2-category $\mathbf{2Vect}_{\mathcal{G}}^{\pi}$ from example 2.3.3.13. Then, the monoidal inclusion $\mathfrak{C}^\times \hookrightarrow \mathfrak{C}$ induces a monoidal 2-functor $\mathbf{J}:\mathbf{2Vect}_{\mathcal{G}}^{\pi}\rightarrow\mathfrak{C}$. This follows from the fact $\mathbf{2Vect}_{\mathcal{G}}^{\pi}$ is constructed from $\mathfrak{C}^\times$ through linearization, local Cauchy completion, and Cauchy completion, which are all operations satisfying sufficient universal properties.

Moreover, because $\mathfrak{C}$ is a strongly fusion 2-category, the main results of [JFY21] yield that every simple object of $\mathfrak{C}$ in invertible. As a consequence, we find that $\mathbf{J}$ is essentially surjective on objects.

Now, if follows lemma 5.3.2.2 below that there exists a finite group H and a monoidal 2-functor $H \times \mathrm{B}^2\Bbbk^\times \to \mathcal{G} \times^\pi \mathrm{B}^2\Bbbk^\times \simeq \mathfrak{C}^\times$, which is essentially surjective on objects. After linearization, local Cauchy completion, and Cauchy completion, this yields a monoidal 2-functor $\mathbf{K} : \mathbf{2Vect}_H \to \mathbf{2Vect}_{\mathcal{G}}^\pi$ between fusion 2-categories such that the composite $\mathbf{J} \circ \mathbf{K}$ is essentially surjective on objects. The fusion 1-category $\mathbf{Vect}_H$ equipped with the non-trivial grading by H defines a connected rigid algebra in $\mathbf{2Vect}_H$, which is separable thanks to corollary 3.2.3.8. We write A for the separable algebra $\mathbf{J}(\mathbf{K}(\mathbf{Vect}_H))$ in $\mathfrak{C}$. We note that A is connected. In fact, its unit 1-morphism $i : I \to A$ is given by the inclusion of the monoidal unit of $\mathfrak{C}$. This holds as the unit 1-morphism of the algebra $\mathbf{Vect}_H$ is given by the inclusion of the monoidal unit of $\mathbf{2Vect}_H$. In addition, we also record that, by construction, the underlying object of A contains every simple object of $\mathfrak{C}$ as a direct summand.

Finally, we check that $\mathbf{Bimod}_{\mathfrak{C}}(A)$ is a connected fusion 2-category. As A is connected, it is a fortiori indecomposable, which ensures that $\mathbf{Bimod}_{\mathfrak{C}}(A)$ is a fusion 2-category by corollary 4.3.2.5. Let P be any simple A-A-bimodule. There exists a simple object C of $\mathfrak{C}$ and a non-zero 1-morphism $A\Box C\Box A \to P$ of A-A-bimodules. For instance, pick any summand of P viewed as an object of $\mathfrak{C}$, and consider the induced map of A-A-bimodules. We claim that $A\Box C\Box A$ is a simple A-A-bimodule. Namely, under the adjunction

$$Hom_{A\text{-}A}(A\Box C\Box A, A\Box C\Box A) \simeq Hom_{\mathfrak{C}}(C, A\Box C\Box A),$$

the identity A-A-bimodule 1-morphism on P corresponds to the 1-morphism $i\Box Id_C\Box i$ in $\mathfrak{C}$. But, the 1-morphism $i : I \hookrightarrow A$ is the inclusion of a summand by construction. As every simple object of $\mathfrak{C}$ is invertible, it follows that $i\Box Id_C\Box i$ is a simple 1-morphism in $\mathfrak{C}$, which establishes that $A\Box C\Box A$ is a simple A-A-bimodule. Now, observe that the composite A-A-bimodule 1-morphism $A\Box C\Box A \hookrightarrow A\Box A\Box A \to A\Box A$ is non-zero. Thence, every simple object of $\mathbf{Bimod}_{\mathfrak{C}}(A)$ is in the connected component of the simple object $A\Box A$, concluding the proof of the theorem. □

Lemma 5.3.2.2. *Let $\mathcal{G}$ be any finite 2-group, and π any 4-cocycle for $\mathcal{G}$ with coefficients in $\Bbbk^\times$. Then, there exists a finite group H and an essentially surjective monoidal functor $K : H \to \mathcal{G}$ such that the pullback of π along K is a coboundary.*

Proof. Let us write $G := \pi_1(\mathcal{G})$ for the finite group of equivalence classes of objects of $\mathcal{G}$, and $A := \pi_2(\mathcal{G})$ for the finite abelian group of automorphisms of the monoidal unit of $\mathcal{G}$. Further, G acts on A by conjugation. It was shown in [Sín75] that the finite 2-group $\mathcal{G}$ is uniquely determined by the above data together with ω, a 3-cocycle for G with coefficients in A. In order to prove the lemma, we begin by observing that there exists a finite group $\widetilde{G}$, and a surjective group homomorphism $p : \widetilde{G} \twoheadrightarrow G$ such that $p^*\omega$ is a coboundary. Namely, the case when A is cyclic is proven in [Opo94], and the general case follows from the Lyndon–Hochschild–Serre spectral sequence [HS53] using a similar argument. In particular, the data of p together with a cochain trivializing $p^*\omega$ induces an essentially surjective monoidal functor $p : \widetilde{G} \to \mathcal{G}$. Then, $p^*\pi$, the pullback of π along p, is a 4-cocycle for $\widetilde{G}$ with coefficients in $\Bbbk^\times$. By theorem 1 of [Opo94], there exists a finite group H and a surjective group homomorphism $q : H \twoheadrightarrow \widetilde{G}$ such that $q^*p^*\pi$ is a coboundary. Taking $K := p \circ q$ concludes the proof of the lemma. □

Remark 5.3.2.3. It follows from theorem 5.3.2.1 and corollary 5.2.1.9 that Morita equivalence classes of fusion 2-categories correspond exactly to pairs consisting of a symmetric fusion 1-category $\mathcal{E}$ and a class in $\widehat{\mathcal{W}}(\mathcal{E})$, the quotient of the Witt group $\mathcal{W}(\mathcal{E})$ under the action of $Aut^{br}(\mathcal{E})$. Furthermore, it was shown in theorem 5.5 of [DNO13] that every class in $\mathcal{W}(\mathcal{E})$ is represented by a unique completely $\mathcal{E}$-anisotropic braided fusion 1-category over $\mathcal{E}$. Thus, every class in $\widehat{\mathcal{W}}(\mathcal{E})$ is represented by a unique completely $\mathcal{E}$-anisotropic braided fusion 1-category. Fixing $\mathcal{B}$ a completely $\mathcal{E}$-anisotropic braided fusion 1-category, the fusion 2-categories that are Morita equivalent to $\mathbf{Mod}(\mathcal{B})$ are classified by the indecomposable finite semisimple left $\mathbf{Mod}(\mathcal{B})$-module 2-categories. By theorem 4.3.1.2, the later are exactly given by the Morita equivalence classes of $\mathcal{B}$-central fusion 1-categories.

Our last theorem has the following interesting consequence.

Corollary 5.3.2.4. *The property of being bosonic, respectively fermionic, is preserved under Morita equivalence of fusion 2-categories.*

Proof. Let $\mathcal{B}$ be a braided fusion 1-category, and $\mathcal{C}$ be a $\mathcal{B}$-central fusion 1-category. We will show that the symmetric center of $\mathcal{Z}(\mathcal{C}, \mathcal{B})$ is Tannakian if and only if the symmetric center of $\mathcal{B}$ is Tannakian. By virtue of theorem 5.3.2.1 and lemma 5.2.2.1, this yields the statement of the corollary. Let us write $\mathcal{A}$ for the image of $\mathcal{B}$ in $\mathcal{Z}(\mathcal{C})$, which is a braided fusion 1-category, and note that $\mathcal{Z}(\mathcal{C}, \mathcal{B}) \simeq \mathcal{Z}(\mathcal{C}, \mathcal{A})$ as braided fusion 1-categories. It follows from corollary 3.2.4 of [DMNO13] that the symmetric center of $\mathcal{A}$ is Tannakian if and only symmetric center of $\mathcal{B}$ is Tannakian. Finally, proposition 4.3 of [DNO13] establishes that $\mathcal{Z}_{(2)}(\mathcal{A}) \simeq \mathcal{Z}_{(2)}(\mathcal{Z}(\mathcal{C}, \mathcal{A}))$ as symmetric fusion 1-categories. This concludes the proof of the claim. $\square$

5.4 Separability

We begin by proving that every rigid algebra in a fusion 2-category is separable. As a consequence, we find that every multifusion 2-category is separable. Following [JF22], we comment on the relation between the separability of multifusion 2-categories and the fact that multifusion 2-categories are fully dualizable objects of an appropriate symmetric monoidal 4-category. We also introduce a notion of dimension for fusion 2-categories, and check that it is always non-vanishing. Finally, we show that the Drinfeld center of any fusion 2-category is a finite semisimple 2-category. Throughout, we work over a fixed algebraically closed field $\mathbb{k}$ of characteristic zero.

5.4.1 Rigid Algebras

Thanks to theorem 5.3.2.1, we have an extremely good control over the Morita equivalence classes of fusion 2-categories. We use this to prove the following deep result on the structure of rigid algebras in fusion 2-categories, which internalizes corollary 2.6.8 of [DSPS21], and positively answers a question raised in [JFR24].

Theorem 5.4.1.1. *Every rigid algebra in a fusion 2-category is separable.*

Proof. Let $\mathfrak{C}$ be any fusion 2-category. Thanks to theorems 4.3.4.3 and 5.3.2.1, there exists a connected fusion 2-category $\mathfrak{D}$, a separable algebra A in $\mathfrak{D}$, and a monoidal equivalence $\mathfrak{C} \simeq \mathbf{Bimod}_{\mathfrak{D}}(A)$ of fusion 2-categories. Under the above equivalence, it follows from corollary 4.1.2.12 that any rigid algebra in $\mathfrak{C}$ corresponds to a rigid algebra B in $\mathfrak{D}$ equipped with an algebra 1-homomorphism $A \to B$. But, as $\mathfrak{D}$ is connected, remark 3.2.3.6 implies that B is a separable algebra in $\mathfrak{D}$. Thence, corollary 4.1.2.12 demonstrates that the corresponding rigid algebra in $\mathfrak{C}$ is separable as desired. $\square$

Corollary 5.4.1.2. *Every rigid algebra in a multifusion 2-category is separable.*

Proof. Every multifusion 2-categories can be split into a direct sum of finitely many indecomposable multifusion 2-categories by lemma 4.3.2.10. In particular, this induces a decomposition of any algebra into a finite direct sum of algebras contained in exactly one indecomposable multifusion 2-category. It is therefore enough to show that any rigid algebra in an indecomposable multifusion 2-category is separable. In order to see this, note that every indecomposable multifusion 2-category is Morita equivalent to a fusion 2-category as can be seen from remark 4.3.4.2. The result then follows from the proof of theorem 5.4.1.1 above. $\square$

By combining the last corollary with theorems 5.1.5 and 5.4.7 of [Déc21], we get the following result.

Corollary 5.4.1.3. *Every finite semisimple module 2-category over a multifusion 2-category is separable.*

Remark 5.4.1.4. Provided we are working over an algebraically closed field of characteristic zero, we find a posteriori that all the results of section 4.3 hold even with the separability hypothesis removed. In particular, for any multifusion 2-category $\mathfrak{C}$, there is an equivalence between the Morita 3-category of rigid algebra in $\mathfrak{C}$ and the 3-category of finite semisimple left $\mathfrak{C}$-module 2-categories. This verifies the content of remark 5.4.9 of [Déc21].

Remark 5.4.1.5. Let $\mathfrak{C}$ be a multifusion 2-category. As a consequence of the above results, one can show that the relative 2-Deligne tensor product of any finite semisimple $\mathfrak{C}$-module 2-categories exists. In particular, it then follows from [JFS17] that multifusion 2-categories, finite semisimple bimodule 2-categories, and bimodule morphisms form a symmetric monoidal 4-category, which we denote by **F2C**. Let us also consider **BrFus**, the Morita 4-category of braided multifusion 1-categories studied in [BJS21]. We expect that the assignment

$$\mathbf{Mod}(-) : \mathbf{BrFus} \to \mathbf{F2C}$$

given by sending a braided multifusion 1-category $\mathcal{B}$ to the associated multifusion 2-category $\mathbf{Mod}(\mathcal{B})$ extends to a symmetric monoidal 4-functor (see the second proof of theorem 1 of [JF22] for a related result). Theorem 5.3.2.1 proves that the 4-functor **Mod** is essentially surjective on objects. Moreover, it follows from theorem 4.3.1.2 and lemma 2.2.6 of [Déc21] that **Mod** induces equivalence between Hom-3-categories. Thus, provided it exists, the symmetric monoidal 4-functor **Mod** is an equivalence. As braided multifusion 1-categories are fully dualizable objects of **BrFus** thanks to [BJS21], it would then follow that multifusion 2-categories are fully dualizable objects of **F2C**. We give another approach to establishing the full dualizability of multifusion 2-categories in remark 5.4.2.4 below.

5.4.2 Fusion 2-Categories

We work with a fixed multifusion 2-category $\mathfrak{C}$. The following definition categorifies definition 2.5.8 of [DSPS21].

Definition 5.4.2.1. The multifusion 2-category $\mathfrak{C}$ is called separable if it is separable as a finite semisimple left $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$-module 2-category.

We now associate a canonical rigid algebra to any multifusion 2-category. This construction should be compared with that of section 2.6 of [DSPS21], in which a canonical Frobenius algebra is associated to any fusion 1-category.

Construction 5.4.2.2. As $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$ is a multifusion 2-category acting on the finite semisimple 2-category $\mathfrak{C}$, it follows from theorem 4.2.2 of [Déc21] that $\mathfrak{C}$ can be given the structure of a $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$-enriched 2-category. More precisely, for any C, D in $\mathfrak{C}$, the Hom-object $\underline{Hom}(C, D)$ in $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$ is characterized by the adjoint 2-natural equivalence

$$Hom_{\mathfrak{C}}(H \Box C, D) \simeq Hom_{\mathfrak{C}\boxtimes\mathfrak{C}^{mop}}(H, \underline{Hom}(C, D)),$$

for every H in $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$. It was established in proposition 4.1.1 of [Déc21] that such an object $\underline{Hom}(C, D)$ of $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$ always exists. We write $\mathcal{R}_{\mathfrak{C}}$ for the algebra $\underline{End}(I)$ in $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$, and observe that $\mathcal{R}_{\mathfrak{C}}$ is rigid by theorem 5.4.7 of [Déc21]. Moreover, the monoidal unit I of $\mathfrak{C}$ generates $\mathfrak{C}$ under the action of $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$ in the sense of definition 5.1.2 of [Déc21]. Thus, by theorem 5.1.5 of [Déc21], there is an equivalence

$$\mathfrak{C} \simeq \mathbf{Mod}_{\mathfrak{C}\boxtimes\mathfrak{C}^{mop}}(\mathcal{R}_{\mathfrak{C}})$$

of left $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-categories. For later use, let us also record that, if we assume that $\mathfrak{C}$ is a fusion 2-category, then $\mathcal{R}_{\mathfrak{C}}$ is connected. Namely, its unit 1-morphism corresponds to the identity 1-morphism $I \to I$ under the adjunction above.

Thanks to theorem 5.4.1.1 and corollary 5.4.1.2, the rigid algebra $\mathcal{R}_{\mathfrak{C}}$ is in fact separable. This yields the following result, which categorifies corollary 2.6.8 of [DSPS21].

Corollary 5.4.2.3. *Every multifusion 2-category is separable.*

Remark 5.4.2.4. We now comment on the relation between 4-dualizability and separability for multifusion 2-categories. We begin by recalling the setup considered in section 2 of [JF22]. Let us write $\mathbf{Cau2Cat}_{\Bbbk}$ for the linear 3-category of Cauchy complete $\Bbbk$-linear 2-categories, and note that the completed tensor product $\widehat{\otimes}$ endows this 3-category with a symmetric monoidal structure. We will assume that $\mathbf{Cau2Cat}_{\Bbbk}$ is closed under colimits and that $\widehat{\otimes}$ preserves them. Then, thanks to the results of section 8 of [JFS17], we can consider the symmetric monoidal 4-category $\mathbf{Mor}_1(\mathbf{Cau2Cat}_{\Bbbk})$ of Cauchy complete monoidal 2-categories, Cauchy complete bimodule 2-categories, and bimodule morphisms. Theorem 1 of [JF22] then sketches a proof of the fact that a multifusion 2-category $\mathfrak{C}$ is a fully dualizable object, that is a 4-dualizable object, of $\mathbf{Mor}_1(\mathbf{Cau2Cat}_{\mathbb{k}})$ if and only if it is 2-dualizable. More precisely, let $T : \mathfrak{C}\boxtimes\mathfrak{C}^{mop}\rightarrow\mathfrak{C}$ denote the canonical left $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-functor. The proof of theorem 1 of [JF22] outlines that the multifusion 2-category $\mathfrak{C}$ yields a 4-dualizable object of $\mathbf{Mor}_1(\mathbf{Cau2Cat}_{\mathbb{k}})$ if and only if there exists a left $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-functor $\Delta : \mathfrak{C}\rightarrow\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$ such that the composite left $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-endofunctor $T\circ\Delta$ on $\mathfrak{C}$ can be extended to a 3-condensation in the sense of [GJF19] with splitting $Id_{\mathfrak{C}}$ as a $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-functor.

We now argue that this is possible given that the multifusion 2-category $\mathfrak{C}$ is separable. Through the equivalence of left $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-categories of construction 5.4.2.2, the 2-functor T is identified with the left $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-functor $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}\rightarrow\mathbf{Mod}_{\mathfrak{C}\boxtimes\mathfrak{C}^{mop}}(\mathcal{R}_{\mathfrak{C}})$ given by $H\mapsto H\Box\mathcal{R}_{\mathfrak{C}}$. We let Δ be the forgetful 2-functor $\mathbf{Mod}_{\mathfrak{C}\boxtimes\mathfrak{C}^{mop}}(\mathcal{R}_{\mathfrak{C}})\rightarrow\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$. Then, the composite $T\circ\Delta$: $\mathbf{Mod}_{\mathfrak{C}\boxtimes\mathfrak{C}^{mop}}(\mathcal{R}_{\mathfrak{C}})\rightarrow\mathbf{Mod}_{\mathfrak{C}\boxtimes\mathfrak{C}^{mop}}(\mathcal{R}_{\mathfrak{C}})$ is identified with the 2-functor $H\mapsto H\Box\mathcal{R}_{\mathfrak{C}}$. But, theorem 4.3.1.2 exhibits an equivalence between the Morita 3-category of separable algebras in $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$ and the 3-category of separable left $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-categories. Under this equivalence, the composite left $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-functor $T\circ\Delta$ corresponds to the $\mathcal{R}_{\mathfrak{C}}$-$\mathcal{R}_{\mathfrak{C}}$-bimodule $\mathcal{R}_{\mathfrak{C}}\Box\mathcal{R}_{\mathfrak{C}}$. Conversely, the identity left $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-functor $Id_{\mathfrak{C}}$ corresponds to the $\mathcal{R}_{\mathfrak{C}}$-$\mathcal{R}_{\mathfrak{C}}$-bimodule $\mathcal{R}_{\mathfrak{C}}$. Finally, as $\mathfrak{C}$ is separable, then $\mathcal{R}_{\mathfrak{C}}$ is a separable algebra, so that the $\mathcal{R}_{\mathfrak{C}}$-$\mathcal{R}_{\mathfrak{C}}$-bimodule $\mathcal{R}_{\mathfrak{C}}\Box\mathcal{R}_{\mathfrak{C}}$ can be extended to a 3-condensation with splitting $\mathcal{R}_{\mathfrak{C}}$. We therefore find that $T\circ\Delta$ can be extended to a 3-condensation with splitting $Id_{\mathfrak{C}}$, which proves the desired result.

We explain how to categorify the notion of the global (also called categorical) dimension of a fusion 1-category. If $\mathfrak{C}$ is a fusion 2-category, we have obtained in construction 5.4.2.2 a connected rigid algebra $\mathcal{R}_{\mathfrak{C}}$ in $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$. Then, via the construction given in section 3.2.2, one can associate to $\mathcal{R}_{\mathfrak{C}}$ a well-defined scalar $\mathrm{Dim}_{\mathfrak{C}\boxtimes\mathfrak{C}^{mop}}(\mathcal{R}_{\mathfrak{C}})$, called its dimension.

Definition 5.4.2.5. The dimension of a fusion 2-category $\mathfrak{C}$ is

$$\mathrm{Dim}(\mathfrak{C}) := \mathrm{Dim}_{\mathfrak{C}\boxtimes\mathfrak{C}^{mop}}(\mathcal{R}_{\mathfrak{C}}).$$

Thanks to theorem 5.4.1.1, the rigid algebra $\mathcal{R}_{\mathfrak{C}}$ is actually separable. Combining this observation with theorem 3.2.2.4 yields the following result, which categorifies theorem 2.3 of [ENO05].

Corollary 5.4.2.6. *The dimension* $\mathrm{Dim}(\mathfrak{C})$ *of any fusion 2-category* $\mathfrak{C}$ *is non-zero.*

Remark 5.4.2.7. A notion of dimension for finite semisimple 2-categories was introduced in definition 1.2.28 of [DR18]. We do not know how this notion compares with ours in general, though these two notions agree in the examples considered below. However, we point out that there is a priori no reason to expect that they agree, as the definition given in [DR18] does not take into account the rigid monoidal structure.

Example 5.4.2.8. Let $\mathfrak{C}$ be a bosonic strongly fusion 2-category, so that $\mathfrak{C}$ is equivalent to $\mathbf{2Vect}_G^{\pi}$ for some finite group G and 4-cocycle π for G with coefficients in $\Bbbk^{\times}$. Without loss of generality, we set $\mathfrak{C} := \mathbf{2Vect}_G^{\pi}$. Then, direct inspection shows that we have

$$\mathfrak{C} \boxtimes \mathfrak{C}^{mop} = \mathbf{2Vect}_G^{\pi} \boxtimes (\mathbf{2Vect}_G^{\pi})^{mop} \simeq \mathbf{2Vect}_{G\times G^{op}}^{\pi\times\pi^{op}}$$

as fusion 2-categories. By construction 5.4.2.2, we have that $\mathfrak{C} \simeq \mathbf{Mod}_{\mathfrak{C}\boxtimes\mathfrak{C}^{mop}}(\mathcal{R}_{\mathfrak{C}})$. Now, given two elements $g, h \in G$, we write $\mathbf{Vect}_{(g,h)}$ for the simple object of $\mathbf{2Vect}_{G\times G^{op}}^{\pi\times\pi^{op}}$ given by $\mathbf{Vect}$ with grading (g, h). It follows from the definition above that

$$\mathcal{R}_{\mathfrak{C}} = \boxplus_{g\in G}\mathbf{Vect}_{(g,g^{-1})}$$

as an object of $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$. On the other hand, let $\mathfrak{D}$ be the fusion sub-2-category of $\mathbf{2Vect}_{G\times G^{op}}^{\pi\times\pi^{op}}$ generated by the simple objects of the form $\mathbf{Vect}_{(g,g^{-1})}$ with $g \in G$. It is clear that $\mathfrak{D} \simeq \mathbf{2Vect}_G$, and that $\mathcal{R}_{\mathfrak{C}}$ is a connected rigid algebra in $\mathfrak{D}$. It therefore follows from corollary 3.2.3.9 that $\mathcal{R}_{\mathfrak{C}}$ has dimension $|G|$, so that $\mathrm{Dim}(\mathbf{2Vect}_G^{\pi}) = |G|$. In this case, we note that the dimension of $\mathbf{2Vect}_G^{\pi}$ in the sense of [DR18] is also $|G|$.

Example 5.4.2.9. Let $\mathcal{B}$ be a braided fusion 1-category, and write $\dim(\mathcal{B})$ for its global dimension. We now compute the dimension of the connected fusion 2-category $\mathbf{Mod}(\mathcal{B})$. In order to do so, we consider $\mathbf{Mod}(\mathcal{B})$ as a left module 2-category over $\mathbf{Mod}(\mathcal{B}) \boxtimes \mathbf{Mod}(\mathcal{B})^{mop} \simeq \mathbf{Mod}(\mathcal{B} \boxtimes \mathcal{B}^{rev})$. In this case, we find that the canonical rigid algebra $\mathcal{R}_{\mathbf{Mod}(\mathcal{B})}$ in $\mathbf{Mod}(\mathcal{B} \boxtimes \mathcal{B}^{rev})$ is given by $\mathcal{B}$ equipped with its canonical $\mathcal{B} \boxtimes \mathcal{B}^{rev}$-central structure. In particular, it follows from proposition 3.2.3.4 above that

$$\mathrm{Dim}(\mathbf{Mod}(\mathcal{B})) = 1/\dim(\mathcal{B}).$$

In particular, for connected fusion 2-categories, our notion of dimension coincides with that introduced in [DR18]. Further, the above equality shows that the dimension of a fusion 2-category is not invariant under Morita equivalence. Namely, for any finite group G, we have $\mathrm{Dim}(\mathbf{2Rep}(G)) = 1/|G|$, but we have seen above that $\mathrm{Dim}(\mathbf{2Vect}_G) = |G|$.

5.4.3 Finite Semisimplicity of the Drinfeld Center

In the previous section, we have associated a canonical separable algebra $\mathcal{R}_{\mathfrak{C}}$ to any multifusion 2-category $\mathfrak{C}$ such that $\mathfrak{C} \simeq \mathbf{Mod}_{\mathfrak{C}\boxtimes\mathfrak{C}^{mop}}(\mathcal{R}_{\mathfrak{C}})$ as left $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$-module 2-categories. It then follows from theorem 4.3.1.2 and lemma 5.1.2.1 above that

$$\mathbf{Bimod}_{\mathfrak{C}\boxtimes\mathfrak{C}^{mop}}(\mathcal{R}_{\mathfrak{C}}) \simeq \mathscr{Z}(\mathfrak{C})$$

as monoidal 2-categories. In particular, by appealing to theorem 3.2.1.7 and corollary 2.2.2.2, we get the following corollary.

Corollary 5.4.3.1. *The Drinfeld center of any multifusion 2-category is a finite semisimple 2-category.*

We end by recording some additional properties of the Drinfeld center of a fusion 2-category, which follow from the above theorems.

Corollary 5.4.3.2. *For every two multifusion 2-categories $\mathfrak{C}$ and $\mathfrak{D}$, we have an equivalence $\mathscr{Z}(\mathfrak{C}\boxtimes\mathfrak{D}) \simeq \mathscr{Z}(\mathfrak{C})\boxtimes\mathscr{Z}(\mathfrak{D})$ of braided monoidal 2-categories.*

Proof. Let us write $\mathfrak{E} := \mathfrak{C}\boxtimes\mathfrak{C}^{mop}\boxtimes\mathfrak{D}\boxtimes\mathfrak{D}^{mop} \simeq \mathfrak{C}\boxtimes\mathfrak{D}\boxtimes(\mathfrak{C}\boxtimes\mathfrak{D})^{mop}$. It follows from combining construction 5.4.2.2 and proposition 4.1 of [Déc24] that the separable algebra $\mathcal{R}_{\mathfrak{C}\boxtimes\mathfrak{D}}$ in $\mathfrak{E}$ is equivalent to $\mathcal{R}_{\mathfrak{C}}\boxtimes\mathcal{R}_{\mathfrak{D}}$. In particular, there is an equivalence of monoidal 2-categories $\mathscr{Z}(\mathfrak{C}\boxtimes\mathfrak{D}) \simeq \mathbf{Bimod}_{\mathfrak{E}}(\mathcal{R}_{\mathfrak{C}}\boxtimes\mathcal{R}_{\mathfrak{D}})$. Furthermore, it follows from lemma 4.3.4.8 that there is an equivalence $\mathbf{Bimod}_{\mathfrak{E}}(\mathcal{R}_{\mathfrak{C}}\boxtimes\mathcal{R}_{\mathfrak{D}}) \simeq \mathscr{Z}(\mathfrak{C})\boxtimes\mathscr{Z}(\mathfrak{D})$ of monoidal 2-categories. Finally, one checks that the composite monoidal 2-functor $\mathscr{Z}(\mathfrak{C})\boxtimes\mathscr{Z}(\mathfrak{D}) \simeq \mathbf{Bimod}_{\mathfrak{E}}(\mathcal{R}_{\mathfrak{C}}\boxtimes\mathcal{R}_{\mathfrak{D}}) \simeq \mathscr{Z}(\mathfrak{C}\boxtimes\mathfrak{D})$ agrees with the canonical braided monoidal 2-functor constructed in lemma 5.1.2.5. This finishes the proof of the result. □

Combining the above corollary with theorem 5.3.1.7 yields the following result.

Corollary 5.4.3.3. *The Drinfeld center of any fusion 2-category is equivalent as a braided fusion 2-category to the Drinfeld center of a strongly fusion 2-category.*

Remark 5.4.3.4. We note that there are non-trivial braided monoidal equivalences between the Drinfeld centers of strongly fusion 2-categories. For instance, given a finite group G and a 4-cocycle π for G with coefficients in $\Bbbk^\times$, any outer automorphism α of G induces an equivalence $\mathbf{2Vect}_G^{\pi} \simeq \mathbf{2Vect}_G^{\alpha^*(\pi)}$ of fusion 2-categories. In addition, such an equivalence can be twisted using a 3-cocycle for G with coefficients in $\mathbb{k}^{\times}$. Then, by taking the Drinfeld center, we get an induced braided monoidal equivalence

$$\mathscr{Z}(\mathbf{2Vect}_G^{\pi}) \simeq \mathscr{Z}(\mathbf{2Vect}_G^{\alpha^*(\pi)}).$$

We expect that every braided monoidal equivalence between the Drinfel centers of bosonic strongly fusion 2-categories arises this way. On the other hand, there are more exotic braided monoidal equivalences between the Drinfeld centers of fermionic strongly fusion 2-categories as is explained in [JF]. The case of **2SVect** is discussed above in remark 5.2.1.8.

Remark 5.4.3.5. Let us write $\mathscr{M}$ for the set of Morita equivalence classes of fusion 2-categories. This is a commutative monoidal under the 2-Deligne tensor product. We use $\mathscr{M}^{\times}$ to denote the maximal subgroup of $\mathscr{M}$. It follows from lemma 5.3.1.3 and theorem 5.2.1.4 (see also example 4.3.4.6) that $\mathscr{M}^{\times}$ is isomorphic to $\mathcal{W}$, the Witt group of non-degenerate braided fusion 1-categories. Further, let us use **ZF2C** to denote the set of braided monoidal equivalence classes of Drinfeld centers of fusion 2-categories. Theorem 5.1.3.2 and corollary 5.4.3.2 imply that the map $\mathscr{Z} : \mathscr{M}/\mathcal{W} \to$ **ZF2C** sending a Morita equivalence class of fusion 2-categories to its Drinfeld center is well-defined. As a generalization of proposition 4.1 of [JFR24], we conjecture that the map $\mathscr{Z} : \mathscr{M}/\mathcal{W} \to$ **ZF2C** is a bijection.

Corollary 5.4.3.6. *For every multifusion 2-category $\mathfrak{C}$, the forgetful 2-functor $\mathbf{F}$: $\mathscr{Z}(\mathfrak{C}) \to \mathfrak{C}$ is dominant, i.e. every object of $\mathfrak{C}$ is the splitting of a 2-condensation monad in $\mathfrak{C}$ supported on an object in the image of $\mathbf{F}$.*

Proof. The monoidal 2-functor $\mathbf{F}$ is dominant if and only if $\mathfrak{C}$ is an indecomposable left $\mathscr{Z}(\mathfrak{C})$-module 2-category. But, the dual to $\mathscr{Z}(\mathfrak{C})$ with respect to $\mathfrak{C}$ is $\mathfrak{C} \boxtimes \mathfrak{C}^{mop}$, a fusion 2-category. The result thus follows from corollary 4.3.2.5. □

Corollary 5.4.3.7. *Let $\mathfrak{C}$ be a multifusion 2-category, then $\mathscr{Z}(\mathfrak{C})$ is a factorizable braided multifusion 2-category, that is the canonical braided monoidal 2-functor $\mathscr{Z}(\mathfrak{C}) \boxtimes \mathscr{Z}(\mathfrak{C})^{rev} \to \mathscr{Z}(\mathscr{Z}(\mathfrak{C}))$ is an equivalence.*

Proof. The multifusion 2-categories $\mathscr{Z}(\mathfrak{C})$ and $\mathfrak{C}\boxtimes\mathfrak{C}^{mop}$ are Morita equivalent, so that $\mathscr{Z}(\mathfrak{C}) \boxtimes \mathscr{Z}(\mathfrak{C})^{rev} \simeq \mathscr{Z}(\mathscr{Z}(\mathfrak{C}))$ as braided monoidal 2-category thanks to theorem 5.1.3.2. Inspection shows that the braided monoidal 2-functor constructed in this proof agrees with the canonical one. □

Remark 5.4.3.8. Let $\mathfrak{B}$ be a braided multifusion 2-category. We use $\mathscr{Z}_{(2)}(\mathfrak{B})$ to denote its sylleptic center, as in section 5.1 of [Cra98]. We expect that a braided multifusion 2-category $\mathfrak{B}$ is factorizable if and only if it is non-degenerate in the sense that there is an equivalence $\mathscr{Z}_{(2)}(\mathfrak{B}) \simeq \mathbf{2Vect}$ of sylleptic monoidal 2-categories.

Outlook

In chapter 2, we have defined and studied the notion of compact semisimple 2-categories over an arbitrary field, which is a 2-categorical version of the concept of a finite semisimple 1-category. More generally, finite 1-categories have been defined in [EO04], and it would be interesting to devise a notion of finite 2-categories. The definition of a finite 2-category should in particular include the 2-categories $\mathbf{Mod}^{ex}(\mathcal{C})$ of exact module 1-categories over a finite tensor 1-category $\mathcal{C}$ as examples. Over a field of characteristic zero, we expect that compact semisimple 2-categories are examples of finite 2-categories. Over a general field, we suspect that compact semisimple 2-categories are merely full sub-2-categories of finite 2-categories. Further, it would be invaluable to investigate how to generalize the results that we have obtained in chapters 3 and 4 from compact semisimple tensor 2-categories to finite tensor 2-categories. In particular, we surmise that the 2-category of bimodules over a rigid algebra in a finite tensor 2-category is again a finite tensor 2-category.

We have extensively developed the Morita theory of fusion 2-categories over an algebraically closed field of characteristic zero. It is then natural to study fusion 2-categories up to monoidal equivalence. More precisely, in chapter 5, we have established that Morita equivalence classes of fusion 2-categories correspond exactly to pairs consisting of a symmetric fusion 1-category $\mathcal{E}$ and a class in $\widehat{\mathcal{W}}(\mathcal{E})$, a quotient of the Witt group $\mathcal{W}(\mathcal{E})$. But, for any braided fusion 1-category $\mathcal{B}$, the fusion 2-categories that are in the Morita equivalence class of $\mathbf{Mod}(\mathcal{B})$ are all given by the 2-categories of $\mathcal{C}$-$\mathcal{C}$-bimodules in $\mathbf{Mod}(\mathcal{B})$ for some $\mathcal{B}$-central fusion 1-category $\mathcal{C}$. In particular, every fusion 2-category can be obtained through this construction, albeit not uniquely. Nevertheless, the present discussion does suggest that fusion 2-categories can be classified up to monoidal equivalence using only data associated to braided fusion 1-categories.

It is also natural to examine the properties of fusion 2-categories equipped with additional structure, such as a braiding. In chapter 5, we have encountered some examples of braided fusion 2-categories in the form of the Drinfeld centers of fusion 2-

categories over an algebraically closed field of characteristic zero. Some results about braided fusion 2-categories, such as the existence of fiber 2-functors for symmetric fusion 2-categories, have been obtained in [DY23]. More generally, one ought to attempt to categorify the results derived in [DGNO10] for braided fusion 1-categories. For instance, it would be interesting to study equivariantization and de-equivariantization for braided fusion 2-categories. A noteworthy application would be the classification of braided fusion 2-categories using exclusively group-theoretic data. In fact, as is explained in detail in [JFR24], there is a close relationship between Witt groups of braided fusion 1-categories and Drinfeld centers of fusion 2-categories, and the above classification yields descriptions of these Witt groups.

Fusion 2-categories were originally introduced in [DR18], where spherical fusion 2-categories are used to define a state-sum invariant for oriented 4-dimensional manifolds. We believe that the Drinfeld center of a spherical fusion 2-category over an algebraically closed field of characteristic zero inherits a spherical structure. In particular, it should be an example of a modular fusion 2-category, i.e. a non-degenerate braided fusion 2-category equipped with a spherical structure. As braided fusion 2-categories can be classified, we believe that it is also possible to classify modular fusion 2-categories. In a slightly different direction, unitary fusion 1-categories are of particular interest in applications to quantum Physics. It would be invaluable to come up with a definition of unitary fusion 2-category. We expect that most of the results of this thesis admit generalizations to the unitary setting.

Finally, it is natural to attempt to study higher fusion categories. We expect that most of the results in the present thesis hold for fusion n-categories. In particular, we suspect that it is possible to show that every fusion n-category over an algebraically closed field of characteristic zero is separable by following the outline of our proof in the case $n = 2$. In addition, there should be an interesting relation between separable algebras in a fusion $(n+1)$-category, and flavours of fusion n-categories, generalizing the many examples given in chapter 3. For instance, this may be used to give a short and conceptual proof of the equivalences induced by equivariantization and de-equivariantization for fusion n-categories. We note however that the computations we have carried out using string diagrams in this thesis become unwieldy in monoidal n-categories when n is larger than 3. Thus, in order to effectively study higher fusion categories, one needs to use more refined tools coming from higher category theory and homotopy theory.

Bibliography

[BBFP22] Thomas Bartsch, Mathew Bullimore, Andrea E. V. Ferrari, and Jamie Pearson. Non-invertible symmetries and higher representation theory II, 2022. arXiv:2212.07393.

[BBSNT22] Lakshya Bhardwaj, Lea E. Bottini, Sakura Schafer-Nameki, and Apoorv Tiwari. Non-invertible symmetry webs, 2022. arXiv:2212.06842.

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real Algebraic Geometry*. A Series of Modern Surveys in Mathematics. Springer, 1998.

[BD95] John C. Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *J. Math. Phys.*, 36:6073–6105, 1995.

[BJS21] Adrien Brochier, David Jordan, and Noah Snyder. On dualizability of braided tensor categories. *Compositio Mathematica*, 3:435–483, 2021. arXiv:1804.07538.

[BJSS21] Adrien Brochier, David Jordan, Pavel Safronov, and Noah Snyder. Invertible braided tensor categories. *Algebr. & Geom. Topol.*, 21(4):2107–2140, 2021. arXiv:2003.13812.

[BK01] Bojko Bakalov and Alexander Kirillov, Jr. *Lectures on tensor categories and modular functor*, volume 21 of *University Lecture Series*. AMS, 2001.

[BN95] John C. Baez and Martin Neuchl. Higher-dimensional algebra i: Braided monoidal 2-categories. *Advances in Mathematics*, 121:196–244, 1995. arXiv:q-alg/9511013.

[BW96] John W. Barrett and Bruce W. Westbury. Invariants of piecewise-linear 3-manifolds. *Trans. Amer. Math. Soc.*, 348(10):3997–4022, 1996. arXiv:hep-th/9311155.

[BZBJ18] David Ben-Zvi, Adrien Brochier, and David Jordanr. Integrating quantum groups over surfaces. *Journal of Topology*, 11(4):873–916, 2018. arXiv:1501.04652.

[CMV02] I.J. Le Creurer, F. Marmolejo, and E.M. Vitale. Beck's theorem for pseudo-monads. *Journal of Pure and Applied Algebra*, 173:293–313, 2002.

[Con] Keith Conard. The Artin-Schreier theorem. www.kconrad.math.uconn.edu/blurbs/galoistheory/artinschreier.pdf.

[Cra98] Sjoerd E. Crans. Generalized centers of braided and sylleptic monoidal 2-categories. *Advances in Mathematics*, 136:183–223, 1998.

[Dav97] Alexei A. Davydov. Twisting of monoidal structures, 1997. arXiv:q-alg/9703001.

[Dav10] Alexei Davydov. Centre of an algebra. *Advances in Mathematics*, 225(1):319–348, 2010. arXiv:0908.1250.

[Déc21] Thibault D. Décoppet. Finite semisimple module 2-categories, 2021. arXiv:2107.11037.

[Déc22a] Thibault D. Décoppet. Drinfeld centers and Morita equivalence classes of fusion 2-categories, 2022. arXiv:2211.04917.

[Déc22b] Thibault D. Décoppet. Multifusion categories and finite semisimple 2-categories. *Journal of Pure and Applied Algebra*, 226(8), 2022. arXiv:2012.15774.

[Déc22c] Thibault D. Décoppet. Weak fusion 2-categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, LXIII(1):3–24, 2022. arXiv:2103.15150.

[Déc23a] Thibault D. Décoppet. Compact semisimple 2-categories. *Trans. Amer. Math. Soc.*, 376(12):8309–8336, 2023. arXiv:2111.09080.

[Déc23b] Thibault D. Décoppet. The Morita theory of fusion 2-categories. *Higher Structures*, 7(1):234–292, 2023. arXiv:2208.08722.

[Déc23c] Thibault D. Décoppet. The relative Deligne tensor product over pointed braided fusion categories. *Journal of Algebra*, 620:89–112, 2023. arXiv:2203.10331.

[Déc23d] Thibault D. Décoppet. Rigid and separable algebras in fusion 2-categories. *Advances in Mathematics*, 419:108967, 2023. arXiv:2205.06453.

[Déc24] Thibault D. Décoppet. The 2-Deligne tensor product. *Kyoto J. Math.*, 64(1):1–29, 2024. arXiv:2103.16880.

[Del90] Pierre Deligne. Catégories tannakienne. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Prog. Math.*, pages 111–195. Birkhäuser Boston, 1990.

[Del02] Pierre Deligne. Catégories tensorielles. *Mosc. Math. J.*, 2(2):227–248, 2002.

[Del22] Clement Delcamp. Tensor network approach to electromagnetic duality in (3+1)d topological gauge models. *JHEP*, 149, 2022. arXiv:2112.08324.

[DGNO10] Vladimir Drinfeld, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. On braided fusion categories I. *Selecta Mathematica*, 16:1–119, 2010. arXiv:0906.0620.

[DMNO13] Alexei Davydov, Michael Müger, Dmitri Nikshych, and Victor Ostrik. The Witt group of non-degenerate braided fusion categories. *J. für die Reine und Angew. Math.*, 2013(667), 2013. arXiv:1009.2117.

[DN21] Alexei Davidov and Dimitri Nikshych. Braided Picard groups and graded extensions of braided tensor categories. *Selecta Mathematica*, 27(65), 2021. arXiv:2006.08022.

[DNO13] Alexei Davydov, Dmitri Nikshych, and Victor Ostrik. On the structure of the Witt group of braided fusion categories. *Selecta Mathematica*, 19:237–269, 2013. arXiv:1109.5558.

[DR18] Christopher L. Douglas and David J. Reutter. Fusion 2-categories and a state-sum invariant for 4-manifolds, 2018. arXiv:1812.11933.

[Dri87] Vladimir Drinfeld. Quantum groups. *Proceedings of the International Congress of Mathematicians, Berkeley*, page 798–820, 1987.

[DS97] Brian Day and Ross Street. Monoidal bicategories and Hopf algebroids. *Advances in Mathematics*, 129(AI971649):99–157, 1997.

[DSPS19] Christopher L. Douglas, Christopher Schommer-Pries, and Noah Snyder. The balanced tensor product of module categories. *Kyoto Journal of Mathematics*, 59:167–179, 2019. arXiv:1406.4204.

[DSPS21] Christopher L. Douglas, Christopher Schommer-Pries, and Noah Snyder. *Dualizable tensor categories.* Mem. Amer. Math. Soc. AMS, 2021. arXiv:1312.7188.

[DY23] Thibault D. Décoppet and Matthew Yu. Gauging noninvertible defects: a 2-categorical perspective. *Lett. Math. Phys.*, 113(2):36, 2023. arXiv:2211.08436.

[EG12] Pavel Etingof and Shlomo Gelaki. Descent and forms of tensor categories. *IMRN*, 2021(13):3040–3063, 2012. arXiv:1102.0657.

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor Categories.* Mathematical Surveys and Monographs. AMS, 2015.

[ENO05] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. On fusion categories. *Annals of Mathematics*, 162:581–642, 2005. arXiv:math/0203060.

[ENO10] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. Fusion categories and homotopy theory. *Quantum Topology*, 1(3):209–273, 2010. arXiv:0909.3140.

[ENO11] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. Weakly group-theoretical and solvable fusion categories. *Advances in Mathematics*, 226(1):176–205, 2011. arXiv:0809.3031.

[EO04] Pavel Etingof and Viktor Ostrik. Finite tensor categories. *Mosc. Math. J.*, 4(3):627–654, 2004. arXiv:math/0301027.

[Eti18] Pavel Etingof. On faithfulness of the lifting for Hopf algebras and fusion categories. *Algebra & Number Theory*, 12(3):551–569, 2018. arXiv:1704.07855.

[For17] Timothy J. Ford. *Separable Algebras*, volume 183 of *Graduate Studies in Mathematics.* AMS, 2017.

[FRS02] Jürgen Fuchs, Ingo Runkel, and Christoph Schweiger. Conformal correlation functions, Frobenius algebras and triangulations. *Nuclear Physics B*, 624(3):452–468, 2002. arXiv:hep-th/0110133.

[Gai12] Dennis Gaitsgory. Sheaves of categories and the notion of 1-affineness. In *Stacks and Categories in Geometry, Topology, and Algebra*, volume 643 of *Contemporary Mathematics*, pages 127–226. AMS, 2012. arXiv:1306.4304.

[Gal17] César Galindo. Coherence for monoidal G-categories and braided G-crossed categories. *Journal of Algebra*, 487:118–137, 2017. arXiv:1604.01679.

[GHS23] Azat M. Gainutdinov, Jonas Haferkamp, and Christoph Schweigert. Davydov-Yetter cohomology, comonads and Ocneanu rigidity. *Advances in Mathematics*, 414:108853, 2023. arXiv:1910.06094.

[GJF19] Davide Gaiotto and Theo Johnson-Freyd. Condensations in higher categories, 2019. arXiv:1905.09566v2.

[GJS22] César Galindo, David Jaklitsch, and Christoph Schweigert. Equivariant Morita theory for graded tensor categories. *Bull. Belg. Math. Soc. Simon Stevin*, 29(2):145–171, 2022. arXiv:2106.07440.

[Gre10] Justin Greenough. Monoidal 2-structure of bimodule categories. *Journal of Algebra*, 324(8):1818–1859, 2010. arXiv:0911.4979.

[GS06] Philippe Gille and Tamás Szamuel. *Central Simple Algebras and Galois Cohomology*. Cambridge studies in advanced mathematics. Cambridge University Press, 2006.

[GS16] Richard Garner and Michael Schulman. Enriched categories as a free cocompletion. *Advances in Mathematics*, 289:1–94, 2016. arXiv:1301.3191v2.

[Gur12] Nick Gurski. Biequivalence in tricategories. *Theory and Applications of Categories*, 26, 2012. arXiv:1102.0979.

[Gur13] Nick Gurski. *Coherence in Three-Dimensional Category Theory*. Cambridge Tracts in Mathematics. Cambridge University Press, 2013.

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I*. Advanced Lecture in Mathematics. Vieweg + Teubner, 2010.

[Har06] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 2006.

[Hau17] Rune Haugseng. The higher Morita category of E_n-algebras. *Geom. & Topol.*, 21:1631–1730, 2017. arXiv:1412.8459.

[Hou07] Robin Houston. *Linear Logic without Units*. PhD thesis, University of Manchester, 2007. arXiv:1305.2231.

[HPT23] André Henriques, David Penneys, and James Tener. *Planar algebras in braided tensor categories*. Mem. Amer. Math. Soc. AMS, 2023. arXiv:1607.06041.

[HS53] Gerhard Hochschild and Jean-Pierre Serre. Cohomology of group extensions. *Trans. Amer. Math. Soc.*, 74:110–134, 1953.

[JF] Theo Johnson-Freyd. A 4D TQFT which is not (quite) a higher-form gauge theory. In preparation.

[JF21] Theo Johnson-Freyd. (3+1)d topological orders with only a $\mathbb{Z}/2$-charged particle, 2021. arXiv:2011.11165v1.

[JF22] Theo Johnson-Freyd. On the classification of topological orders. *Commun. Math. Phys.*, 393:989–1033, 2022. arXiv:2003.06663.

[JFR24] Theo Johnson-Freyd and David J. Reutter. Minimal nondegenerate extensions. *Journ. Amer. Math. Soc.*, 37(1):81–150, 2024. arXiv:2105.15167.

[JFS17] Theo Johnson-Freyd and Claudia Scheimbauer. (Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories. *Advances in Mathematics*, 307:147–223, 2017. arXiv:1502.06526.

[JFY21] Theo Johnson-Freyd and Matthew Yu. Fusion 2-categories with no line operators are grouplike. *Bull. Aust. Math. Soc.*, 104(3):434–442, 2021. arXiv:2010.07950.

[JMS14] Vaughan F.R. Jones, Scott Morrison, and Noah Snyder. The classification of subfactors of index at most 5. *Bull. Amer. Math. Soc.*, 21(2):277–327, 2014. arXiv:1304.6141.

[JS91] André Joyal and Ross Street. Tortile yang-baxter operators in tensor categories. *Journal of Pure and Applied Algebra*, 71(1):43–51, 1991.

[Kir01] Alexandre Kirillov Jr. Modular categories and orbifold models II, 2001. arXiv:math/0110221.

[Kit06] Alexei Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321(1):2–111, 2006. arXiv:cond-mat/0506438.

[KTZ20] Liang Kong, Yin Tian, and Shan Zhou. The center of monoidal 2-categories in 3+1d Dijkgraaf-Witten theory. *Advances in Mathematics*, 360(106928), 2020. arXiv:1905.04644.

[Lau20] Robert Laugwitz. The relative monoidal center and tensor products of monoidal categories. *Comm. Contemp. Math.*, 22(8), 2020. arXiv:1803.04403.

[LKW17] Tian Lan, Liang Kong, and Xiao-Gang Wen. Modular extensions of unitary braided fusion categories and 2+1D topological/SPT orders with symmetries. *Commun. Math. Phys.*, 351(2):709–739, 2017.

[Lur10] Jacob Lurie. On the classification of topological field theories. *Curr. Dev. Math. Sci.*, 1:129–280, 2010.

[Mil17] James S. Milnes. *Algebraic Groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2017.

[Mor58] Kiiti Morita. Duality for modules and its applications to the theory of rings with minimum condition. *Science Reports of the Tokyo Kyoiku Daigaku. Section A*, 6(150):83–1427, 1958.

[MPP18] Scott Morrison, David Penneys, and Julia Plavnik. Completion for braided enriched monoidal categories, 2018. arXiv:1809.09782.

[MS89] Gregory Moore and Nathan Seiberg. Lectures on RCFT. In *1989 Banff NATO ASI: Physics, Geometry and Topology*, 1989.

[Müg03a] Michael Müger. From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories. *Journal of Pure and Applied Algebra*, 180:81–157, 2003. arXiv:math/0111204.

[Müg03b] Michael Müger. From subfactors to categories and topology II. the quantum double of tensor categories and subfactors. *Journal of Pure and Applied Algebra*, 180:159–219, 2003. arXiv:math/0111205.

[Nat17] Sonia Natale. On the equivalence of module categories over a group-theoretical fusion category. *SIGMA*, 13, 2017. arxiv:1608.04435.

[Nik22] Dmitri Nikshych. Computing the group of minimal non-degenerate extensions of a super-Tannakian category. *Commun. Math. Phys.*, 2022. arXiv:2201.04076.

[Opo94] Hans Opolka. Group extensions and group cohomology II. *Journal of Algebra*, 169:328–331, 1994.

[Ost03a] Victor Ostrik. Module categories over the Drinfeld double of a finite group. *Int. Math. Res. Not.*, 27:1507–1520, 2003. arXiv:math/0202130.

[Ost03b] Victor Ostrik. Module categories, weak Hopf algebras and modular invariants. *Transformation Groups*, 8:177–206, 2003. arXiv:math/0111139.

[Pal68] Richard S. Palais. The classification of real division algebras. *Amer. Math. Monthly*, 75:366–368, 1968.

[Pst14] Piotr Pstragowski. On dualizable objects in monoidal bicategories, framed surfaces and the cobordism hypothesis, 2014. arXiv:1411.6691.

[Sín75] Hoàng Xuân Sính. *Gr-catégories*. PhD thesis, Université Paris VII, 1975.

[SP11] Christopher J. Schommer-Pries. *The Classification of Two-Dimensional Extended Topological Field Theories*. PhD thesis, UC Berkeley, 2011. arXiv:1112.1000.

[Ste97] Dragos Stefan. The set of types of n-dimensional semisimple and cosemisimple Hopf algebras is finite. *Journal of Algebra*, 193:571–580, 1997.

[Sun22] Long Sun. The symmetry enriched functor is fully faithful. *Commun. Math. Phys.*, 395:1345–1382, 2022. arXiv:2201.00192.

[Tur92] Vladimir G. Turaev. Modular categories and 3-manifolds invariants. *International Journal of Modern Physics B*, 6(11n12):1807–1824, 1992.

[Tur94] Vladimir G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., 1994.

[Tur00] Vladimir G. Turaev. Homotopy field theory in dimension 3 and crossed group-categories, 2000. arXiv:math/0005291.

[Yet98] David N. Yetter. Braided deformations of monoidal categories and Vassiliev invariants. In *Higher category theory*, volume 230, page 117–134. Contemp. Math. AMS, 1998. arXiv:q-alg/9710010.

[Yet03] David N. Yetter. Abelian categories of modules over a (lax) monoidal functor. *Advances in Mathematics*, 174(2):266–309, 2003. arXiv:math/0112307.