

Russo–Seymour–Welsh estimates for the Kostlan ensemble of random polynomials

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Abstract. Beginning with the predictions of Bogomolny–Schmit for the random plane wave, in recent years the deep connections between the level sets of smooth Gaussian random fields and percolation have become apparent. In classical percolation theory a key input into the analysis of global connectivity are scale-independent bounds on crossing probabilities in the critical regime, known as Russo–Seymour–Welsh (RSW) estimates. Similarly, establishing RSW-type estimates for the nodal sets of Gaussian random fields is a major step towards a rigorous understanding of these relations.

The Kostlan ensemble is an important model of Gaussian homogeneous random polynomials. The nodal set of this ensemble is a natural model for a ‘typical’ real projective hypersurface, whose understanding can be considered as a statistical version of Hilbert’s 16th problem. In this paper we establish RSW-type estimates for the nodal sets of the Kostlan ensemble in dimension two, providing a rigorous relation between random algebraic curves and percolation. The estimates are uniform with respect to the degree of the polynomials, and are valid on all relevant scales; this, in particular, resolves an open question raised recently by Beffara–Gayet. More generally, our arguments yield RSW estimates for a wide class of Gaussian ensembles of smooth random functions on the sphere or the flat torus.

Résumé. Partant des prédictions de Bogomolny–Schmit pour les ondes planaires aléatoires, dans les années récentes des relations profondes sont apparues entre les lignes de niveau des champs aléatoires Gaussiens réguliers et la percolation. En théorie de la percolation, un ingrédient clé dans l’analyse de la connectivité globale est la famille des bornes indépendantes en échelle sur les probabilités de croisement dans le régime critique, connues sous le nom d’estimées de Russo–Seymour–Welsh (RSW). De la même façon, établir des estimées du type RSW pour les ensembles nodaux des champs aléatoires Gaussiens est une étape majeure dans la compréhension rigoureuse de ces relations.

L’ensemble de Kostlan est un modèle important de polynômes aléatoires homogènes Gaussiens. L’ensemble nodal des polynômes de Kostlan est un modèle naturel pour une hypersurface projective réelle typique, dont la compréhension peut être vu comme une version statistique du 16ème problème de Hilbert. Dans cet article, nous établissons une estimées du type RSW pour les ensembles nodaux des polynômes de Kostlan en dimension 2, montrant ainsi une relation rigoureuse entre les courbes algébriques aléatoires et la percolation. Les estimées sont uniformes en le degré du polynôme, et sont valables dans toutes les échelles pertinentes ; ceci, en particulier, résout la question posée récemment par Beffara–Gayet. Plus généralement, nos arguments conduisent à des estimées RSW pour une large classe d’ensembles Gaussiens de fonctions régulières aléatoires sur la sphère ou sur le tore plat.

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1. Introduction

In this paper we study the global connectivity of random algebraic curves of large degree, and more generally establish a rigorous connection between the nodal sets of sequences of smooth Gaussian random fields on the sphere and percolation.



Fig. 1. A sample of the $m = 2$ -dimensional Kostlan ensemble of degree 300, with black (resp. white) representing the positive (resp. negative) nodal domains.

1.1. The Kostlan ensemble

The Kostlan ensemble of homogeneous degree- n polynomials in $m + 1 \geq 2$ variables is the Gaussian random field $f_n : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ defined as

$$f_n(x) = f_{n;m}(x) = \sum_{|J|=n} \sqrt{\binom{n}{J}} a_J x^J, \quad (1.1)$$

where $J = (j_0, \dots, j_m)$ is the multi-index, $|J| = j_0 + \dots + j_m$, $\binom{n}{J} = \frac{n!}{j_0! \dots j_m!}$, and $\{a_J\}$ are i.i.d. standard Gaussian random variables. Since f_n is homogeneous, it is also natural to view the Kostlan ensemble as the Gaussian random field on the unit m -dimensional sphere \mathbb{S}^m that is the restriction of (1.1) to \mathbb{S}^m . The natural extension of (1.1) to \mathbb{C}^{m+1} is known as the ‘complex Fubini–Study’ ensemble.

In this paper we are interested in the global connectivity of the *nodal set* of the Kostlan ensemble, i.e. the zero set of f_n , particularly when the degree n is large. Figure 1 depicts the nodal domains of a sample of the $m = 2$ -dimensional Kostlan ensemble of degree 300 on \mathbb{S}^2 . Since f_n is either even or odd depending on n , its nodal set can be naturally considered as a degree- n *hypersurface* (i.e. algebraic variety of co-dimension one) on the projective space \mathbb{RP}^m . As we explain below, the Kostlan ensemble is a natural model for a ‘typical’ homogeneous polynomial, and hence one may think of its nodal set as a ‘typical’ real projective hypersurface. As such, the study of the nodal set can be considered as a statistical version of Hilbert’s 16th problem.

The Kostlan ensemble can be equivalently defined as the canonical Gaussian element in the Hilbert space \mathcal{H}_n of homogeneous degree- n polynomials in $m + 1$ variables spanned by the collection

$$\left\{ \sqrt{\binom{n}{J}} x^J \right\}_{|J|=n}$$

as its orthonormal basis. Restricted to \mathcal{H}_n , the associated scalar product is, up to the constant $\sqrt{n!}$, equal to the scalar product in the Bargmann–Fock space [3], i.e. the space of all analytic functions on \mathbb{C}^{m+1} such that

$$\|f\|_{\text{BF}}^2 = \frac{1}{\pi^{m+1}} \int_{\mathbb{C}^{m+1}} |f(z)|^2 e^{-\|z\|^2} dz < \infty$$

with the scalar product

$$\langle f, g \rangle_{\text{BF}} = \frac{1}{\pi^{m+1}} \int_{\mathbb{C}^{m+1}} f(z) \bar{g}(z) e^{-\|z\|^2} dz, \quad (1.2)$$

playing an important role in quantum mechanics. The restriction of the scalar product (1.2) to \mathcal{H}_n satisfies the following important property, relevant in our setting: it is the unique (up to a scale factor) scalar product on the space of degree- n

homogeneous polynomials on \mathbb{C}^{m+1} that is invariant w.r.t. the unitary group. In other words, the Kostlan ensemble (1.1) is the real trace of the *unique* unitary invariant Gaussian ensemble of homogeneous polynomials (although there exist many other ensembles invariant w.r.t. the orthogonal transformations [10,11]). In particular, the induced distribution on the space of hypersurfaces on \mathbb{RP}^m is also invariant w.r.t. the unitary group, which justifies our description of the nodal set of the Kostlan ensemble as a natural model for a ‘typical’ real projective hypersurface.

As mentioned above, it will be convenient to consider f_n as a Gaussian random field on the unit sphere \mathbb{S}^m , and henceforth we take exclusively this view. Computing explicitly from (1.1), one may evaluate its covariance kernel $\kappa_n : \mathbb{S}^m \times \mathbb{S}^m \rightarrow \mathbb{R}$ to be

$$\kappa_n(x, y) = \mathbb{E}[f_n(x) \cdot f_n(y)] = (\langle x, y \rangle)^n = (\cos \theta(x, y))^n, \quad (1.3)$$

where for $x, y \in \mathbb{S}^m$ we denote $\theta(x, y)$ to be the angle between x and y , also equal to the spherical distance between these points; this covariance kernel determines f_n uniquely via Kolmogorov’s Theorem.

The random field f_n on \mathbb{S}^m is of high merit since it is rotationally invariant and also admits a natural scaling around every point understood in the following way. Let us fix $x_0 \in \mathbb{S}^m$, and define the *scaled* covariance kernel on $\mathbb{R}^m \times \mathbb{R}^m$

$$K_{x_0;n}(x, y) = \kappa_n\left(\exp_{x_0}\left(\frac{x}{\sqrt{n}}\right), \exp_{x_0}\left(\frac{y}{\sqrt{n}}\right)\right), \quad (1.4)$$

where $\exp_{x_0} : \mathbb{R}^m \rightarrow \mathbb{S}^m$ is the exponential map on the sphere based at x_0 . Then, as is shown formally in Section 3.3 below, the scaled covariance $K_{x_0;n}(x, y)$ satisfies the convergence

$$K_{x_0;n}(x, y) \rightarrow K_\infty(x, y) = e^{-\|x-y\|^2/2} \quad (1.5)$$

along with all its derivatives, locally uniformly in $x, y \in \mathbb{R}^m$; the r.h.s. of (1.5) is the defining covariance kernel of the Bargmann–Fock field on \mathbb{R}^m , discussed further below.

1.2. RSW estimates for random subsets of Euclidean space

In percolation theory the RSW estimates [15,16] are uniform lower bounds for *crossing probabilities* of various percolation processes, most fundamentally for Bernoulli percolation. These are a crucial input into establishing the more refined properties of percolation processes, such as the sharpness of the phase transition and scaling limits for the interfaces of percolation clusters.

Let \mathcal{T} be a periodic lattice (i.e. a periodic set of nodes and edges/bonds between each pair of adjacent nodes), and $p \in [0, 1]$ a number. In Bernoulli bond percolation each edge of \mathcal{T} is independently either open with probability p or closed with probability $1 - p$. This defines a (random) percolation subgraph \mathcal{G} of \mathcal{T} containing all vertices and only open edges. Alternatively one can think of colouring edges independently black (with probability p) or white (with probability $1 - p$). In this case \mathcal{G} is the black sub-graph.

A rather simple argument shows that there exists a *critical probability*: a number $p_c \in (0, 1)$ such that for all $p > p_c$ the graph \mathcal{G} a.s. contains an infinite *percolation cluster* (connected component of \mathcal{G}), and for all $p < p_c$ a.s. no such component exists. The more subtle behaviour of the percolation process for $p = p_c$, *critical percolation*, is of high intrinsic interest. Apart from being one of the most studied lattice models, it is also believed [6] to represent the nodal structure of Laplace eigenfunctions on ‘generic’ chaotic manifolds, in the high energy limit. For \mathcal{T} possessing sufficient symmetries, the corresponding critical probability should be equal $p_c = 1/2$; for the square lattice this was established rigorously by Kesten [9].

Let us assume that the lattice \mathcal{T} is regularly embedded in \mathbb{R}^2 , e.g. the canonical embedding of the square lattice as \mathbb{Z}^2 in \mathbb{R}^2 . For $\rho > 1$, $s > 0$ and $x_0 \in \mathbb{R}^2$ a *box-crossing event* is the event that a rectangle

$$R = x_0 + [-\rho s/2, \rho s/2] \times [-s/2, s/2]$$

centred at x_0 of size $s \times \rho s$ is traversed horizontally by a black cluster, i.e. there exists a connected component \mathcal{C} of \mathcal{G} such that \mathcal{C} , restricted to R , intersects both $\{x_0 - \rho s/2\} \times [-s/2, s/2]$ and $\{x_0 + \rho s/2\} \times [-s/2, s/2]$. The basic RSW estimates for critical percolation are the assertion that, for every $\rho > 1$ the corresponding crossing probability is bounded away from 0 uniformly in the scale $s > 0$, i.e. there exists a number $c(\rho) > 0$ such that the probability of a box-crossing event is $\geq c(\rho)$ for all $s > 0$, $x_0 \in \mathbb{R}^2$. The analogous estimates hold for *quads*, i.e. triples $Q = (D; \gamma, \gamma')$, where D is a piecewise-smooth domain, and $\gamma, \gamma' \subseteq \partial U$ are two disjoint boundary curves; in this case the RSW estimates assert that there exists a constant $c(D; \gamma, \gamma') > 0$ such that the probability $p(D; \gamma, \gamma'; s)$ that $sD = \{sx : x \in D\}$ contains a black cluster intersecting both $s\gamma$ and $s\gamma'$ is at least $c(D; \gamma, \gamma')$ for every $s > 0$.

In the more general setting of random subsets of Euclidean space, Tassion [17] recently showed the validity of RSW estimates for the *Voronoi percolation*. Let $\mathcal{P} \subseteq \mathbb{R}^2$ be a Poisson point process on \mathbb{R}^2 with unit intensity, and for each $x \in \mathcal{P}$ construct the associated (random) Voronoi cell

$$\mathcal{C}_x = \{z \in \mathbb{R}^2 : \forall y \in \mathcal{P} \setminus \{x\} \rightarrow d(z, y) \geq d(z, x)\};$$

the various Voronoi cells tile the plane disjointly save for boundary overlaps. Each of the cells is coloured black or white independently with probabilities p and $1 - p$ respectively; here again, by a duality argument, the critical probability is $p_c = 1/2$ [7]. In this setting Tassion [17] proved that RSW estimates hold on all scales; a somewhat weaker version due to Bolobas–Riordan [7] established that the RSW estimates hold for an unbounded subsequence of scales.

1.3. RSW estimates for the Bargmann–Fock field

Our starting point is the recent work of Beffara–Gayet [4], who considered whether bounds analogous to the RSW estimates hold for the Kostlan ensemble. The main result of [4] was that RSW estimates hold for the *local scaling limit* of the Kostlan ensemble on \mathbb{R}^2 , which, in particular, implies the analogous estimates for the Kostlan ensemble on *mesoscopic* scales, but leaves open the question of whether these estimates hold *globally* on the sphere. To the best of our knowledge, along with the very recent announcement of Nazarov–Sodin on the variance of the number of nodal domains (to be published), Beffara–Gayet’s result is the only heretofore known rigorous evidence or manifestation for the conjectured connections [6] between percolation theory and nodal patterns.

Let $g_\infty : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the Bargmann–Fock field, that is, the planar centred Gaussian field corresponding to the covariance kernel K_∞ on the r.h.s. of (1.5), i.e., the *local scaling limit* of the Kostlan ensemble. The Bargmann–Fock field is isotropic, a.s. smooth, and may be constructed explicitly as the series

$$g_\infty(x) = \sum_{i,j=0}^{\infty} a_{ij} \frac{1}{\sqrt{i!j!}} x_1^i x_2^j \quad (1.6)$$

with $\{a_{ij}\}$ i.i.d. standard Gaussian random variables, where the convergence is understood locally uniformly; hence the sample paths of g_∞ are a.s. *real analytic*. Equivalently, recall the Bargmann–Fock space in Section 1.1 above, and define the space \mathcal{F} of analytic functions on \mathbb{R}^2 that admit an analytic extension to \mathbb{C}^2 which lies in the Bargmann–Fock space; equip this space with the scalar product $\langle \cdot, \cdot \rangle_{\text{BF}}$ induced from (1.2). We may then think of g_∞ in (1.6) as the canonical Gaussian element of \mathcal{F} (cf. [4, Appendix A.1]), normalised to have unit variance.

Define the nodal components $\{\mathcal{C}_i\}_i$ of g_∞ to be the connected components of the nodal set $g_\infty^{-1}(0)$, and the nodal domains $\{\mathcal{D}_i\}_i$ of g_∞ to be the connected components of the complement $\mathbb{R}^2 \setminus g_\infty^{-1}(0)$ of the nodal set; a.s. all the nodal components $\{\mathcal{C}_i\}$ are simple smooth curves. Nazarov and Sodin [13] proved that the number of nodal components \mathcal{C}_i entirely contained in the disk of radius R is asymptotic to $c_{\text{NS}} \cdot R^2$ with $c_{\text{NS}} > 0$ the ‘Nazarov–Sodin constant of g_∞ ’. The main result of Beffara–Gayet [4, Theorem 1.1] was that the RSW estimates hold for the complement of the nodal set on all scales, and for the nodal set itself on all sufficiently large scales. The restriction to sufficiently large scales is natural, since the probability that the nodal set intersects a domain tends to zero with the size of the domain.

More generally, the methods of [4] yield RSW estimates for the nodal sets of a family of stationary smooth Gaussian random fields on \mathbb{R}^2 , with *positive* and *rapidly decaying* correlations satisfying *sufficient symmetry*. In particular, these are fields for which Tassion’s aforementioned techniques and ideas are applicable.

1.4. Statement of the principal result: RSW estimates for the Kostlan ensemble

Our aim is to prove the analogous RSW-type estimates for the $m = 2$ -dimensional Kostlan ensemble (1.1) *without passing to the limit*, a natural question stemming from the work of Beffara–Gayet (see, e.g., [4, Question 3] and Corollary 1.6 below). In light of the discussion in Section 1.1 above, these estimates can be interpreted as uniform lower bounds on crossing probabilities for a ‘typical’ algebraic curve on \mathbb{RP}^2 .

We begin by formally defining the RSW estimates as they apply to general sequences of random sets on the sphere; later this will be extended in an analogous way to the flat torus, see Section 2.1 below. Let us start by introducing ‘quads’ and their associated crossing events (cf. the discussion in Section 1.2 above).

Definition 1.1 (Quads and crossing events). A quad $Q = (D; \gamma, \gamma')$ is a piecewise-smooth simply-connected (spherical) domain $D \subset \mathbb{S}^2$ and the choice of two disjoint boundary arcs $\gamma, \gamma' \subset \partial D$. When we consider a quad Q as a set, we will identify it with the closure of D . For each $X \subseteq \mathbb{S}^2$ we denote by Quad_X the collection of quads $Q \subseteq X$.

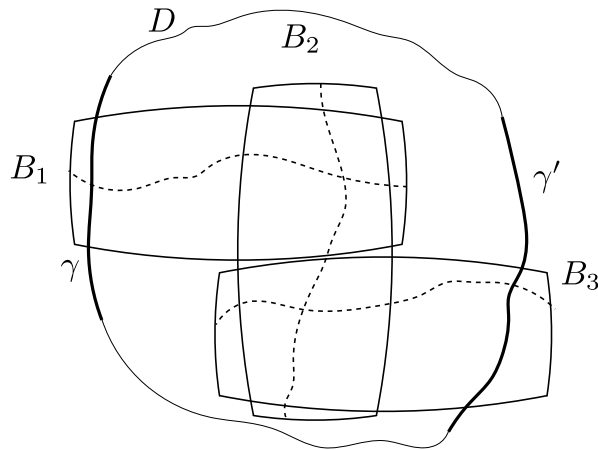


Fig. 2. A box-chain (B_i) of length three that crosses a quad $Q = (D; \gamma, \gamma')$; if each of the boxes B_i are crossed by a set S then so is the quad Q .

To each quad $Q = (D; \gamma, \gamma')$ and random subset S of \mathbb{S}^2 we associate the ‘crossing event’ $\mathcal{C}_Q(S)$ that a connected component of S , restricted to D , intersects both γ and γ' . We shall sometimes use a phrase such as ‘ Q is crossed by S ’ to describe the event $\mathcal{C}_Q(S)$.

Rather than stating the RSW estimates for *rescaled* boxes or quads (as was done in [4] and [17] for instance), in non-Euclidean settings it is natural to state these estimates for a more general class of quads that can be ‘uniformly crossed by chains of boxes’; we introduce this concept now as it applies to the sphere. The following definition is rather technical but it is well illustrated by Figure 2.

Definition 1.2 (Spherical boxes and box-chains).

- (1) For each $a, b > 0$, an $a \times b$ (spherical) rectangle $D \subset \mathbb{S}^2$ is a simply-connected domain that is bounded by four geodesic line-segments, with all four internal angles equal, and such that the non-adjacent pairs of boundary components have length a and b respectively. We refer to the four boundary components of a rectangle as its ‘sides’, and shall call a rectangle with equal side-lengths a ‘square’.
- (2) An $a \times b$ box B is a quad $Q = (D; \gamma, \gamma')$ in \mathbb{S}^2 such that D is an $a \times b$ rectangle and such that γ and γ' are the opposite sides of length a . We refer to the sides of B other than γ and γ' as the ‘lateral’ sides. For each $X \subseteq \mathbb{S}^2$, $c \geq 1$ and $s > 0$, we denote by $\text{Box}_{X;c}(s)$ the collection of all $a \times b$ boxes $B \subseteq X$ such that $s \leq a, b \leq cs$.
- (3) A curve $\eta \subset D$ is said to ‘transversally cross’ a box $B = (D; \gamma, \gamma')$ if a connected component of η , restricted to D , intersects both of the lateral sides of D ; in particular γ and γ' always transversally cross B .
- (4) A box $B = (D; \gamma, \gamma')$ is said to ‘transversally cross’ another box \hat{B} if both of the lateral sides of D transversally cross \hat{B} ; this definition is symmetric in the sense that it also implies that \hat{B} transversally crosses B .
- (5) A ‘box-chain’ of length n is a finite set $\{B_i\}_{1 \leq i \leq n}$ of boxes such that, for each $i = 2, \dots, n$, B_i transversally crosses B_{i-1} . A quad $Q = (D; \gamma, \gamma')$ is said to be ‘crossed’ by a box-chain $\{B_i\}_{1 \leq i \leq n}$ if γ transversally crosses B_1 , γ' transversally crosses B_n , and $\bigcup_{2 \leq i \leq n-1} B_i \subseteq D$.

The relevance of box-chains to RSW estimates can be seen from the following. Let Q be a quad that is crossed by a box-chain $\{B_i\}$, and let S be a random subset of \mathbb{S}^2 . Then if the event $\mathcal{C}_{B_i}(S)$ holds for each i , so does the event $\mathcal{C}_Q(S)$ (see Figure 2). In other words, one may bound the probabilities of crossings of quads by controlling the crossings of box-chains instead. This motivates the following definition.

Definition 1.3 (Quads that are uniformly crossed by box-chains). For each $X \subseteq \mathbb{S}^2$, $c \geq 1$ and $s > 0$, we denote by $\text{Unif}_{X;c}(s)$ the collection of all quads $Q \in \text{Quad}_X$ that are crossed by a box-chain $\{B_i\}_{1 \leq i \leq n}$ of length $n \leq c$ such that $B_i \in \text{Box}_{X;c}(s)$ for each i .

The property of quads being uniformly crossed by box-chains generalises the notion of scale invariance on the sphere, with the parameter c in the definition of $\text{Unif}_{X;c}(s)$ playing the role of the ‘aspect ratio’. One can check, for instance, that for each quad $Q = (D; \gamma, \gamma')$ there is a $c > 1$ such that $\text{Unif}_{\mathbb{S}^2;c}(s/c)$ contains the rescaled quad $sQ = (sD; s\gamma, s\gamma')$ for each $s \in (0, 1]$, where sA denotes linear rescaling of the set A along the unique geodesic to the origin (deleting the

antipodal point if necessary). This can be seen by observing that, although rescaling does not preserve geodesics on the sphere, the resulting distortion is uniformly controlled on all small enough scales.

The property of being uniformly crossed by box-chains is also closely related to conformal invariants. One can check, for instance, that if a quad $Q = (D; \gamma, \gamma')$ is crossed by a box-chain of length n consisting of boxes from $\text{Box}_{X;c}(s)$, then the extremal distance from γ to γ' in D (which is the only conformal invariant of Q) is bounded above by cn , independently of s . In particular, for $Q \in \text{Unif}_{X;c}(s)$ the extremal distance is uniformly bounded above by c^2 .

We next introduce the RSW estimates as they apply to the sphere; these give a uniform lower bound on crossing probability for quads that are uniformly crossed by box-chains. We state the RSW estimates for arbitrary sequences of random subsets.

Definition 1.4 (RSW estimates). Let $(\mathcal{S}_n)_{n \in \mathbb{N}}$ be a sequence of random subsets of \mathbb{S}^2 , let $X \subseteq \mathbb{S}^2$, and let $s_n \geq 0$ be a sequence satisfying $s_n \rightarrow 0$ as $n \rightarrow \infty$. We say that the sequence $(\mathcal{S}_n)_{n \in \mathbb{N}}$ ‘satisfies the RSW estimates on X down to the scale s_n ’ if for every $c > 1$ there exists a $C > 0$ such that

$$\liminf_{n \rightarrow \infty} \inf_{s > Cs_n} \inf_{Q \in \text{Unif}_{X;c}(s)} \mathbb{P}(\mathcal{C}_Q(\mathcal{S}_n)) > 0. \quad (1.7)$$

We say that the sequence $(\mathcal{S}_n)_{n \in \mathbb{N}}$ ‘satisfies the RSW estimates on X on all scales’ if (1.7) holds for $s_n \equiv 0$.

Strictly speaking, we should restrict the definition of the RSW estimates in (1.7) to only hold for quads Q such that $\mathcal{C}_Q(\mathcal{S}_n)$ is measurable. However, since we work only with \mathcal{S}_n being level sets or excursion sets of a.s. C^2 Gaussian random fields, the events $\mathcal{C}_Q(\mathcal{S}_n)$ are always measurable and so we will ignore this technicality.

We are now ready to state our main result. Recall that the *nodal sets* of the Kostlan ensemble are the (random) subsets $\mathcal{N}_n = f^{-1}(0)$ of the sphere; we also consider their complements, $\mathbb{S}^2 \setminus \mathcal{N}_n$. Our principal result asserts that the RSW estimates in Definition 1.4 hold down to the scale $n^{-1/2}$ for the nodal sets of the Kostlan ensemble, and on all scales for their complements (the latter estimates give a lower bound for the probability of a domain being crossed by a single nodal domain).

Theorem 1.5 (RSW estimates for the nodal sets of the Kostlan ensemble). Let $X \subset \mathbb{S}^2$ be a subset whose closure does not contain pairs of antipodal points, and let $s_n = n^{-1/2}$. Then the following hold:

- (1) The nodal sets of the Kostlan ensemble (1.1) on \mathbb{S}^2 satisfy the RSW estimates on X down to the scale s_n .
- (2) The complements of the nodal sets of the Kostlan ensemble (1.1) on \mathbb{S}^2 satisfy the RSW estimates on X on all scales.

We constrain the RSW estimates to apply only to a set X whose closure does not contain pairs of antipodal points since the Kostlan ensemble is naturally defined on the projective space; indeed, the RSW estimates do not hold on the whole of the sphere, as certain crossing events on the sphere are impossible due to the identification of points on the projective space.

The scales on which we prove the RSW estimates in Theorem 1.5 are optimal in the sense that these estimates fail for the nodal set on smaller scales than $s_n = n^{-1/2}$. To see this, recall that s_n is the scale on which the local uniform convergence of the ensemble in (1.5) takes place (in what follows, we often refer to this as the ‘microscopic scale’), and since the probability that a nodal set crosses a quad in the limit field tends to zero as the size of the quad tends to zero, the same is true for the Kostlan ensemble on scales smaller than s_n .

1.5. An application to random planar polynomials

Our next observation is that Theorem 1.5 implies analogous RSW-type estimates for the planar projection of the Kostlan ensemble, which was the original setting of [4]. Recall that the homogeneous polynomials f_n defining the Kostlan ensemble (1.1) are naturally indexed by the projective coordinates $(x_1 : x_2 : x_3)$. If one restricts f_n to the hyperplane $x_3 = 1$ rather than the sphere, i.e. expresses the projective coordinates $(x_1 : x_2 : x_3)$ as $(x_1, x_2, 1)$, then f_n can also be represented as a planar random field

$$g_n(x_1, x_2) = \sum_{i+j=n} a_{i,j} \frac{\sqrt{n!}}{\sqrt{i!j!(n-i-j)!}} x_1^i x_2^j, \quad (1.8)$$

where $a_{i,j} = a_{i,j,n-i-j}$ are i.i.d. standard Gaussian random variables; the Gaussian random field $g_\infty(x)$ defined in (1.6) is the scaling limit of g_n .

Analogously to Definition 1.1, define a *planar quad* $Q = (D; \gamma, \gamma')$ to be a bounded, piecewise-smooth, simply-connected domain $D \subset \mathbb{R}^2$ and the choice of two disjoint boundary arcs $\gamma, \gamma' \subset \partial D$. Beffara–Gayet [4, Question 3, p. 139]) raised the question of whether RSW estimates hold for the sequence g_n , that is, for every planar quad Q does there exist a positive constant $c = c_Q$ such that, for any degree n large enough, the probability that Q is crossed by the nodal set of g_n is at least c ? A straightforward corollary of Theorem 1.5 resolves Beffara–Gayet’s question.

Corollary 1.6. *For every planar quad Q there exists a number $c_Q > 0$ such that Q is crossed by the nodal set of g_n with probability at least c_Q for all sufficiently large n .*

Proof. Define π to be the radial projection from the upper hemisphere onto the plane

$$P = \{(x_1, x_2, x_3) : x_3 = 1\} \subset \mathbb{R}^3,$$

and observe that, for each homogeneous polynomial f of degree n on \mathbb{R}^3 , the projection of the nodal set of f under π is equal to the nodal set of the projection of f under π . In particular, this implies that the quad Q is crossed by the nodal set of g_n if and only if the spherical quad $Q' = \pi^{-1}(Q)$ is crossed by the nodal set of the Kostlan ensemble f_n .

Observe that the quad Q' lies inside the upper open hemisphere, and so in particular is contained in a subset $X \subset \mathbb{S}^2$ whose closure does not contain pairs of antipodal points. Moreover, it is easy to see that there exists a finite box chain that crosses Q' ; in other words, $Q' \in \text{Unif}_{D,c}(s)$ for some $c > 0$ and all sufficiently small s . Theorem 1.5 implies that there exists a number $c_{Q'} > 0$ such that the probability of Q' being crossed by the nodal set of f_n is bounded from below by $c_{Q'}$ for all sufficiently large n . The same $c_Q = c_{Q'}$ is then also a lower bound on the probability that Q is crossed by the nodal set of g_n . \square

Remark that Corollary 1.6 establishes RSW estimates for the sequence of planar random polynomials g_n via a more general result (Theorem 1.5) for their spherical representations f_n . The primary difficulty with working directly with the sequence g_n is that, unlike the scaling limit g_∞ , these fields are neither stationary nor symmetric, which is fatal to the arguments in [4]. On the other hand, the spherical realisation f_n of the Kostlan ensemble is completely symmetric (in the spherical sense), which ultimately allows us to prove RSW estimates in the spherical setting.

2. Outline of the paper

Theorem 1.5 is a particular case of a more general result, Theorem 2.5 below, which asserts the RSW estimates for the nodal sets, and their complements, of general sequences of centred Gaussian random fields defined on smooth compact Riemannian manifolds \mathbb{X} satisfying sufficient symmetries. In turn, the proof of Theorem 2.5 has, as its main ingredient, the even more general Theorem 2.8, which asserts RSW estimates for abstract sequences of random sets obeying a natural scaling (as well as certain other conditions). We believe that both the general Theorem 2.5 and the abstract Theorem 2.8 are of independent interest.

2.1. General RSW estimates for nodal sets of sequences of Gaussian random fields

Let \mathbb{X} be either the flat torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ or the unit sphere \mathbb{S}^2 , and equip \mathbb{X} with a marked origin $0 \in \mathbb{X}$ and its natural metric $d(\cdot, \cdot)$. We consider a sequence $(f_n)_{n \in \mathbb{N}}$ of Gaussian random fields defined on \mathbb{X} .

Our first task is to define the relevant RSW estimates, which will be the natural generalisation of Definition 1.4 to \mathbb{X} . To begin we can define ‘quads’ and ‘crossing events’ analogously to Definition 1.1 (i.e. replacing \mathbb{S}^2 everywhere with \mathbb{X}). Before discussing ‘boxes’ as in Definition 1.2, we need to alter slightly the definition of ‘rectangles’ in the case $\mathbb{X} = \mathbb{T}^2$, namely restricting their sides to be parallel to the axes; this is so that we may work with fields on \mathbb{T}^2 that are not assumed to be rotationally symmetric. For clarity, we restate this definition (with the difference to Definition 1.2 emphasised).

Definition 2.1 (Toral rectangles). For each $a, b > 0$, an $a \times b$ (toral) rectangle $D \subset \mathbb{T}^2$ is a simply-connected domain that is bounded by four geodesic line-segments that are parallel to the axes, with all four internal angles equal, and such that the non-adjacent pairs of boundary components have length a and b respectively.

With this definition of toral rectangles, ‘boxes’ are defined as in Definition 1.2; the notion of ‘box-crossings’ of quads, as well as the set $\text{Unif}_{X,c}(s)$, are then analogous to in Definitions 1.2 and 1.3. Finally, RSW estimates are defined analogously to Definition 1.4.

We next state various conditions that we impose on the Gaussian random fields $(f_n)_{n \in \mathbb{N}}$ that we consider; these are most conveniently framed in terms of their covariance kernels. We first describe a set of relevant symmetries that these covariance kernels must satisfy. These symmetries naturally limit the choice of the underlying space to \mathbb{T}^2 and \mathbb{S}^2 .

Definition 2.2 (Symmetry). We say that a covariance kernel on \mathbb{X} is ‘symmetric’ if:

- (1) In the case $\mathbb{X} = \mathbb{S}^2$, it is rotationally invariant and symmetric w.r.t. reflection in any great circle;
- (2) In the case $\mathbb{X} = \mathbb{T}^2$, it is stationary and possesses the D_4 symmetry, i.e., it is invariant w.r.t. horizontal reflection and rotation by $\pi/2$.

Next we impose certain smoothness and non-degeneracy conditions on symmetric covariance kernels (in the sense of Definition 2.2). When we work with symmetric covariance kernels, we often naturally consider them as functions of one variable, i.e. setting $\kappa(x) = \kappa(0, x)$ (with a slight abuse of notation).

Assumption 2.3 (Smoothness and non-degeneracy). A symmetric covariance kernel κ on \mathbb{X} satisfies the following:

- (1) The function $\kappa(x)$ is C^6 ;
- (2) The Hessian $H_\kappa(0)$ of κ at the origin is positive-definite.

By the standard theory [1], Assumption 2.3 guarantees that the associated random field is a.s. C^2 , and its nodal set a.s. consists of C^2 curves diffeomorphic to circles.

Finally, we define the concept of ‘local uniform convergence’ for a sequence of covariance kernels on \mathbb{X} , generalising our discussion of the local limit (1.5) of the Kostlan ensemble above. Let $\Phi : \mathbb{R}^2 \rightarrow \mathbb{X}$ denote a smooth map that is locally a linear isometry (i.e. such that $\Phi(0) = 0$ and the differential $d_0\Phi$ is a linear isometry); in the case $\mathbb{X} = \mathbb{T}^2$ one may take the covering map for instance, whereas in the case $\mathbb{X} = \mathbb{S}^2$ one may take the exponential map based at the origin, as in (1.4).

Definition 2.4 (Local uniform convergence of the covariance kernels near the origin; cf. [13], Definition 2). For a sequence $s_n > 0$ satisfying $s_n \rightarrow 0$ as $n \rightarrow \infty$ we say that covariance kernels $(\kappa_n)_{n \in \mathbb{N}}$ on \mathbb{X} ‘converge locally uniformly near the origin on the scale s_n ’ if there exists a symmetric covariance kernel K_∞ on \mathbb{R}^2 , satisfying Assumption 2.3, and an open set $U \subseteq \mathbb{R}^2$ containing the origin such that, as $n \rightarrow \infty$, for $x, y \in U$ uniformly,

$$K_n(x, y) = \kappa_n(\Phi(s_n x), \Phi(s_n y)) \rightarrow K_\infty(x - y). \quad (2.1)$$

We say that the covariance kernels $(\kappa_n)_{n \in \mathbb{N}}$ on \mathbb{X} ‘converge locally uniformly near the origin on the scale s_n along with their first four derivatives’ if the above holds also for all partial derivatives K_n of order up to 4.

We are now ready to state our general result Theorem 2.5; the proof that Theorem 1.5 is a special case of Theorem 2.5 is given in Section 3.3 below.

Theorem 2.5 (RSW estimates for general sequences of Gaussian random fields). Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of centred Gaussian random fields on \mathbb{X} with respective covariance kernels κ_n . Suppose that there exists a constant $\eta > 0$, a set $X \subseteq \mathbb{X}$, and a sequence $s_n > 0$ satisfying $s_n \rightarrow 0$ as $n \rightarrow \infty$, such that the following hold:

- (1) *Symmetry:* The covariance kernels κ_n are symmetric in the sense of Definition 2.2.
- (2) *Smoothness and non-degeneracy:* The covariance kernels κ_n satisfy Assumption 2.3.
- (3) *Local uniform convergence near the origin:* The covariance kernels κ_n converge locally uniformly near the origin on the scale s_n along with their first four derivatives.
- (4) *Asymptotically non-negative correlations:*

$$\lim_{n \rightarrow \infty} s_n^{-12-\eta} \sup_{x, y \in X} (\kappa_n(x, y) \wedge 0) = 0. \quad (2.2)$$

- (5) *Uniform rapid decay of correlations:*

$$\lim_{C \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{\substack{x, y \in X, \\ d(x, y) > Cs_n}} (d(x, y)s_n^{-1})^{18+\eta} |\kappa_n(x, y)| = 0. \quad (2.3)$$

Then the nodal sets of f_n satisfy the RSW estimates on X down to the scale s_n , and the complements of the nodal sets of f_n satisfy the RSW estimates on X on all scales.

In Theorem 2.5 the covariance kernels are, in principle, allowed to be negative, unlike for the Gaussian random field considered in [4]; this is crucial for our application to the Kostlan ensemble since, for n odd, the Kostlan ensemble is only positively correlated within a subset of the sphere. Nevertheless, since the negative correlations in the Kostlan ensemble decay exponentially rapidly as a function of n for any subset X whose closure does not contain antipodal points, condition (2.3) is satisfied.

In regards to the nature of the exponents 12 and 18 in (2.2) and (2.3) respectively, these are certainly not optimal for the claimed results, and are chosen mainly for simplicity. In fact, using the somewhat more sophisticated methods in [5], if we additionally assume local uniform convergence of the first six derivatives of the covariance kernel we could reduce these exponents to 8 and 12 respectively. For simplicity, we do not implement these improvements here; on the other hand the question of the optimal exponents in (2.2) and (2.3) is of considerable importance.

To complete Section 2.1, we give an example of an application of Theorem 2.5 to a sequence of Gaussian random fields defined on the flat torus; that this example falls under the scope of Theorem 2.5 is established in Section 3.3.

Example 2.6. Let $(f_n)_{n \in \mathbb{N}}$ be the sequence of centred stationary Gaussian random fields on the torus \mathbb{T}^2 with respective covariance kernels

$$\kappa_n(x, y) = (\cos(2\pi(x_1 - y_1)) \cdot \cos(2\pi(x_2 - y_2)))^n, \quad (x, y) = ((x_1, x_2), (y_1, y_2)) \in \mathbb{T}^2 \times \mathbb{T}^2.$$

Let $X \subseteq \mathbb{T}^2$ be subset whose closure contains no distinct points (x_1, y_1) and (x_2, y_2) such that $2(x_1 - y_1)$ and $2(x_2 - y_2)$ are integers. Then the nodal sets of f_n satisfy the RSW estimates on X down to the scale $s_n = n^{-1/2}$, and the complements of the nodal sets of f_n satisfy the RSW estimates on X on all scales.

The restriction on X is imposed, once again, since the nodal sets are naturally defined on a quotient space of \mathbb{T}^2 , and indeed the RSW estimates fail on the whole space.

2.2. Overview of the proof of Theorem 2.5

Similar to [4], the overall structure of the proof of Theorem 2.5 consists of three main steps:

- (1) First (see Section 4) we adapt an argument borrowed from [17] to establish general RSW estimates for abstract sequences of random sets on \mathbb{X} satisfying certain key assumptions. These general estimates are stated as Theorem 2.8 below.
- (2) Next, we develop a sufficiently robust perturbation analysis that allows us to apply the abstract RSW estimates to the complement of the nodal sets (in fact, separately to the positive and negative excursion sets $f_n^{-1}(0, \infty)$ and $f_n^{-1}(-\infty, 0)$ respectively) of the Gaussian random fields f_n in the setting of Theorem 2.5 (see Section 3.1). This perturbation analysis is used in two key places in the proof, namely in establishing (i) that (2.3) guarantees the ‘asymptotic independence’ of crossing events in well-separated domains, and (ii) that negative correlations satisfying (2.2) have a negligible effect on crossing probabilities.
- (3) Finally, we again apply the ‘asymptotic independence’ of crossing events to infer the RSW estimates for the nodal sets from the RSW estimates for the complements of the nodal sets; this follows from similar arguments to those presented in [4] (see the second part of the proof of Theorem 2.5 in Section 3.2).

Despite the structural similarities between our approach and [4], we record three significant modifications that we make here. First, it is necessary to adapt the argument in [17] to handle the differences in our setting, namely: (i) the presence of a sequence of random sets rather than just a single random set; (ii) the fact that we work on (bounded) manifolds rather than the Euclidean plane; and (iii) in the spherical case, the positive curvature of the sphere. We believe these modifications to be of independent interest, since, to the best of our knowledge, no theory of RSW estimates exists outside the scope of Euclidean space, and our approach is the first step in this direction.

Second, compared to [4] we make a crucial simplification in how [17] is applied in the setting of Gaussian random fields. In [4], the argument in [17] was applied to *discretised* versions of the model defined by restricting the model to symmetric lattices. While this approach works well in the Euclidean setting, it fails in the spherical case since spherical lattices lack the necessary symmetries. Instead, our approach is to apply [17] *directly in the continuum*, which both allows us to exploit the symmetries of the sphere and also yields a significant simplification of the argument compared to [4]. On the other hand, working in the continuum requires a modified treatment of the asymptotic independence of crossing events (see also the comments at the end of Section 2.3).

Finally, our argument is able to handle negative correlations, as long as these are asymptotically negligible; negative correlations were absent from the model considered in [4].

2.3. RSW estimates for abstract sequences of random sets

We give here the statement of the abstract RSW estimates for general sequences of random sets on \mathbb{X} ; establishing this abstract result is the first step towards the proof of Theorem 2.5. To this end, we first need to define the analogues of Euclidean annuli and their related crossing events. Recall the definition of a ‘square’ (definitions 1.2 and 2.1), and observe that a square has a natural ‘centre’, being the unique interior point equidistant from each side.

Definition 2.7 (Annuli and circuit crossing events).

- (1) For $b > a > 0$, an $a \times b$ ‘annulus’ is a domain bounded between concentric squares with side-lengths a and b that are ‘parallel’, i.e. such that there is a single geodesic that intersects both boundary squares at the mid-points of opposite sides.
- (2) For each $X \subseteq \mathbb{X}$, $c \geq 1$, $r > 1$ and $s > 0$, we denote by $\text{Ann}_{X;c;r}(s)$ the collection of all $a \times b$ annuli $A \subset X$ such that $s \leq a < b \leq cs$ and $b/a = r$.
- (3) To each annulus A and random subset \mathcal{S} of \mathbb{X} we associate the ‘crossing event’ $\mathcal{C}_A(\mathcal{S})$ that a connected component of \mathcal{S} , restricted to A , contains a ‘circuit’ around A .

For each $r > 0$ and $v \in \mathbb{S}^1$, let $B(r) \subseteq \mathbb{X}$ denote the centred open ball of radius r , and let $\mathcal{L}_v(r)$ denote the geodesic line-segment of length r , based at the origin, in direction v . Our abstract RSW estimates are the following.

Theorem 2.8 (RSW estimates for abstract sequences of random sets). *Let $(\mathcal{S}_n)_{n \in \mathbb{N}}$ be a collection of random sets on \mathbb{X} . Suppose that there exists a set $X \subseteq \mathbb{X}$ and a sequence $s_n > 0$ satisfying $s_n \rightarrow 0$ such that the following hold:*

- (1) *Non-degeneracy: For every $n \in \mathbb{N}$, $\mathbb{P}(0 \in \partial \mathcal{S}_n) = 0$. Moreover, for every $v \in \mathbb{S}^1$,*

$$\lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}(\mathcal{L}_v(rs_n) \cap \partial \mathcal{S}_n = \emptyset) = 1. \quad (2.4)$$

- (2) *Symmetry: For every $n \in \mathbb{N}$, the law of \mathcal{S}_n satisfies the following symmetries: In the case $\mathbb{X} = \mathbb{S}^2$, invariance w.r.t. rotations and reflections w.r.t. great circles; in the case $\mathbb{X} = \mathbb{T}^2$, invariance w.r.t. translations, horizontal reflections and rotation by $\pi/2$.*
- (3) *Positive associations: For every $n \in \mathbb{N}$, all events measurable on X and increasing w.r.t. the indicator function of \mathcal{S}_n are positively correlated.*
- (4) *Crossing of square boxes on arbitrary scales:*

$$\liminf_{n \rightarrow \infty} \inf_{s > 0} \inf_{B \in \text{Box}_{X;1}(s)} \mathbb{P}(\mathcal{C}_B(\mathcal{S}_n)) > 0.$$

- (5) *Arbitrary crossings on the microscopic scale: There exists a number $\delta > 0$ such that*

$$\liminf_{n \rightarrow \infty} \inf_{Q \in \text{Quad}_{B(\delta s_n)}} \mathbb{P}(\mathcal{C}_Q(\mathcal{S}_n)) > 0.$$

- (6) *Annular crossings of a ‘thick’ annulus with high probability: For each $c > 0$, $r > 1$ and $\varepsilon \in (0, 1)$, there exist $C_1, C_2 > 1$ such that, for all sufficiently large $n \in \mathbb{N}$ and all $s > C_1 s_n$, if*

$$\inf_{A \in \text{Ann}_{X;C_2;r}(s)} \mathbb{P}(\mathcal{C}_A(\mathcal{S}_n)) > c,$$

then, for any $s \times C_2 s$ annulus $A \subseteq X$,

$$\mathbb{P}(\mathcal{C}_A(\mathcal{S}_n)) > 1 - \varepsilon.$$

Then the collection of sets $(\mathcal{S}_n)_{n \in \mathbb{N}}$ satisfies the RSW estimates on X on all scales.

Compared to the original setting in [17], and also its application in [4], we have made two important modifications in the formulation of Theorem 2.8. First, since we are dealing with a *sequence* of random sets rather than a single random set, the conditions are all stated in a way that guarantees *uniform* control over all necessary quantities.

Second, we have formulated a general condition guaranteeing annular crossings with high probability (see condition (6)), rather than working under the more constraining assumption that the random sets in disjoint domains are asymptotically independent (as was done in [17] and [4] for instance). This reformulation is necessary because, as described in Section 2.2 above, we want to work *directly in the continuum* rather than with *discretised* versions of the model.

2.4. Summary of the remaining part of the paper

The remainder of the paper is structured as follows. In Section 3 we develop the perturbation analysis that is the crucial ingredient in applying the abstract Theorem 4 to the setting of Gaussian random fields. We then combine this analysis with Theorem 2.8 to complete the proof of Theorem 2.5. We conclude the section by showing that Theorem 1.5 and Example 2.6 fall within the scope of Theorem 2.5.

In Section 4 we give the proof of the abstract Theorem 2.8. This is similar to the argument in [17], but with suitable modifications to adapt to our setting. Finally, in Section 5 we complete the proof of the auxiliary results used in the perturbation analysis developed in Section 3.

3. Proof of Theorem 1.5: RSW estimates for Kostlan ensemble

In this section we complete the proof of Theorem 2.5, which implies Theorem 1.5 as a special case. The main ingredient will be a perturbation analysis that allows us to apply the abstract RSW estimates in Theorem 2.8 to the positive (resp. negative) excursion sets of the Gaussian random fields in our setting; these have similarities to the methods in [4] and [13].

The set-up for the perturbation analysis is the following. Let $(f_n)_{n \in \mathbb{N}}$ be a collection of centred Gaussian random fields on \mathbb{X} whose respective covariance kernels κ_n are symmetric in the sense of Definition 2.2 and satisfy Assumption 2.3. Let

$$\mathcal{S}_n^+ = \{x \in \mathbb{X} : f_n(x) > 0\} \quad \text{and} \quad \mathcal{S}_n^- = \{x \in \mathbb{X} : f_n(x) < 0\} \quad (3.1)$$

denote the positive and negative excursion sets of f_n respectively. Without loss of generality we may assume that f_n has unit variance, since a normalisation does not affect \mathcal{S}_n^+ or \mathcal{S}_n^- . We also assume that there exists a sequence $s_n > 0$ satisfying $s_n \rightarrow 0$ as $n \rightarrow \infty$ such that the covariance kernels κ_n converge locally uniformly (in the sense of Definition 2.4) near the origin on the scale s_n along with their first four derivatives; let K_∞ be the limiting covariance kernel. Let $\delta_0 > 0$ be sufficiently small that this uniform convergence holds on the ball $B(\delta_0)$.

3.1. Perturbation analysis

Our perturbation analysis proceeds in two steps. First we argue that, outside an event of a small probability, crossing events for the positive excursion set are determined by the signs of a Gaussian random field on a (deterministic) set of points of finite cardinality. Second, we control the effect of perturbations of the field on the probability of crossing events by controlling their impact on the finite-dimensional law associated to the signs of the random field on the finitely many points described above (which, up to an event of a small probability, determine the crossing probabilities).

To state the main propositions of the perturbation analysis, we shall need to define an analogue of Euclidean ‘polygons’ for the manifold \mathbb{X} .

Definition 3.1. A polygon is a quad whose boundary consists of a finite number of geodesic line-segments. Similarly to boxes, we refer to the boundary components as ‘sides’, and their length as ‘side-lengths’. For each $X \subseteq \mathbb{X}$, $c > 0$ and $s > 0$, we denote by $\text{Poly}_{X;c}(s)$ the collection of polygons in X with at most c sides and with side-lengths at most cs .

The main propositions of the perturbation analysis are the following.

Proposition 3.2 (Crossing events are determined by the signs of finitely many points). *For sufficiently large $n \in \mathbb{N}$ the following holds. Fix $c, r > 1$. Then there exists a constant*

$$c_1 = c_1(c; r; K_\infty; \delta_0) > 0,$$

such that, for all error thresholds $\varepsilon \in (0, 1)$, scales $s > 0$, and

$$Q \in \text{Poly}_{\mathbb{X};c}(s) \cup \text{Ann}_{\mathbb{X};c;r}(s),$$

there exists a finite set $\mathcal{P} = \mathcal{P}(Q; K_\infty; \delta_0) \subset Q$ of cardinality at most

$$|\mathcal{P}| < c_1(\varepsilon^{-2}(s/s_n)^6 \vee 1),$$

such that, outside an event of probability less than ε , the crossing event $\mathcal{C}_Q(\mathcal{S}_n^+)$ is determined by the signs of f_n restricted to \mathcal{P} .

Lemma 3.3 (Effect of perturbation on the signs of Gaussian vectors). *Fix $\eta > 0$. Let X and Y be centred Gaussian vectors of dimension n with respective covariance matrices Σ_X and Σ_Y , and let \mathbb{P}_X and \mathbb{P}_Y denote their respective laws. Suppose that X is normalised to have unit variance, and define*

$$\delta = \max_{i,j \leq n} |(\Sigma_X)_{i,j} - (\Sigma_Y)_{i,j}|.$$

Then there exists a constant $c > 0$, depending only on η , such that, for all events A that are measurable in \mathbb{P}_X and \mathbb{P}_Y w.r.t. the signs of X and Y respectively, the following hold:

(1) *If the diagonal entries of $\Sigma_Y - \Sigma_X$ are non-negative, then*

$$|\mathbb{P}_X(A) - \mathbb{P}_Y(A)| < c(n^{3+\eta}\delta)^{1/4}.$$

(2) *If in addition $\Sigma_Y - \Sigma_X$ is positive-definite, then*

$$|\mathbb{P}_X(A) - \mathbb{P}_Y(A)| < c(n^{2+\eta}\delta)^{1/4}.$$

The first statement of Lemma 3.3 is an improved version of [4, Theorem 4.3] and [5, Proposition C.1], implementing an idea from [12].

We stress that in Proposition 3.2, once K_∞ and δ_0 are prescribed, neither the constant c_1 nor the set \mathcal{P} , whose existence is established in Proposition 3.2, depend on any other properties of κ_n . Hence we may choose a set \mathcal{P} that works simultaneously for two different sequences of fields whose covariance kernels converge locally uniformly to K_∞ on $B(\delta_0)$; this fact will be crucial in Section 3.1.2 below.

The proof of Proposition 3.2 and Lemma 3.3 are given in Section 5. We mention here that the proof of Proposition 3.2 proceeds by controlling the event that the nodal set intersects any of the edges of a certain graph more than once (see Lemma 5.1). We then argue that, outside this event, all crossing events are determined by the signs of the field restricted to the vertices of the graph. An analogous result for Gaussian random fields on \mathbb{R}^2 was established in [4,5]. We now outline the two key consequences of the perturbation analysis in our setting.

3.1.1. Asymptotic independence of crossing events

The first consequence is that crossing events in disjoint polygons or annuli are asymptotically independent in the limit $n \rightarrow \infty$, as long as their respective polygons or annuli are sufficiently well-separated; this follows in particular from the condition (2.3) of Theorem 2.5.

Proposition 3.4. *Suppose that there exists $X \subseteq \mathbb{X}$ and $\eta > 0$ such that (2.3) holds. Then for each $c, r, k > 1$ and $\varepsilon > 0$ there is a $C > 0$ such that the following hold for all sufficiently large $n \in \mathbb{N}$:*

$$\sup_{s > Cs_n} \sup_{X_1, X_2 \subset X, \substack{P_1 \in \text{Poly}_{X_1; c}(s), \\ d(X_1, X_2) > s}} \sup_{P_2 \in \text{Poly}_{X_2; c}(s)} |\mathbb{P}(\mathcal{C}_{P_1}(\mathcal{S}_n^+) \cap \mathcal{C}_{P_2}(\mathcal{S}_n^-)) - \mathbb{P}(\mathcal{C}_{P_1}(\mathcal{S}_n^+)) \cdot \mathbb{P}(\mathcal{C}_{P_2}(\mathcal{S}_n^-))| < \varepsilon, \quad (3.2)$$

and, for each $1 \leq j \leq k$,

$$\sup_{s > Cs_n} \sup_{X_1, X_2 \subset X, \substack{\{A_i\}_{0 \leq i \leq j-1} \subset \text{Ann}_{X_1; c; r}(s), \\ d(X_1, X_2) > s}} \sup_{A_j \in \text{Ann}_{X_2; c; r}(s)} \left| \mathbb{P}\left(\bigcap_{0 \leq i \leq j} \mathcal{C}_{A_i}^c(\mathcal{S}_n^+)\right) - \mathbb{P}\left(\bigcap_{0 \leq i \leq j-1} \mathcal{C}_{A_i}^c(\mathcal{S}_n^+)\right) \cdot \mathbb{P}(\mathcal{C}_{A_j}^c(\mathcal{S}_n^+)) \right| < \varepsilon. \quad (3.3)$$

Observe that while (3.3) is stated for the positive excursion sets (and for the complements of the events \mathcal{C}_{A_i}), (3.2) is formulated to control the asymptotic independence *between* crossing events \mathcal{C}_{P_i} for the positive and negative excursion sets. This difference is solely due to how we intend to apply these results, and does not reflect limitations in their generality. Before giving a proof for Proposition 3.4, let us state a crucial corollary of (3.3), namely that condition (2.3) of Theorem 2.5 implies the ‘thick’ annular crossing condition (6) of Theorem 2.8.

Corollary 3.5. *Suppose that there exist $X \subseteq \mathbb{X}$ and $\eta > 0$ such that (2.3) holds. Then for each $c > 0$, $r > 1$ and $\varepsilon \in (0, 1)$, there exists $C_1, C_2 > 1$ such that, for all sufficiently large $n \in \mathbb{N}$ and all $s > C_1 s_n$, if*

$$\inf_{A \in \text{Ann}_{X; C_2; r}(s)} \mathbb{P}(\mathcal{C}_A(\mathcal{S}_n^+)) > c$$

then, for every $s \times C_2 s$ annulus $A \subseteq X$,

$$\mathbb{P}(\mathcal{C}_A(\mathcal{S}_n^+)) > 1 - \varepsilon.$$

Proof of Corollary 3.5 assuming Proposition 3.4. The idea of the proof is straightforward (and well-known [4]). If we take a large number of concentric well-separated annuli, then crossing events in these annuli are almost independent and have the same lower bound. This implies a crossing in one of them with high probability, and hence a crossing in a ‘thick’ annulus with high probability.

Fix $c > 0$, $r > 1$ and $\varepsilon \in (0, 1)$. Since establishing the corollary for a $r > 1$ implies the corollary holds for every smaller $\bar{r} \in (1, r)$, we can and will assume that $r \geq 2$. We work with the collection $(A_{a,b})_{a < b}$ of $a \times b$ annuli centred at the origin that are ‘parallel’, i.e. such that there is a single geodesic that intersects all boundary squares at the mid-points of opposite sides. In particular, for each $s > 0$ we introduce the sequence of disjoint annuli $\{A_i^s\}_{i \geq 0}$ defined by $A_i^s = A_{r^{2i}s, r^{2i+1}s}$. Since $r \geq 2$ it holds that $d(A_i^s, A_j^s) > s$ for all $i \neq j$.

Let k be an integer to be determined later, and set C_2 larger than r^{2k+1} . Fix $s > 0$ and consider an $cs \times C_2 s$ annulus $A \subseteq X$. By symmetry we may assume $A = A_{s, C_2 s}$, and hence $A_i^s \in \text{Ann}_{X; C_2; r}(s)$ for each $0 \leq i \leq k$, which by assumption implies that

$$\mathbb{P}(\mathcal{C}_{A_i^s}(\mathcal{S}_n^+)) > c. \quad (3.4)$$

Now, since $d(A_i^s, A_j^s) > s$, an application of (3.3) in Proposition 3.4 yields a $C_1 > 0$ such that, for sufficiently large n and all $s > C_1 s_n$ and $j = 0, \dots, k$,

$$\left| \mathbb{P}\left(\bigcap_{i=1, \dots, j} \mathcal{C}_{A_i^s}^c(\mathcal{S}_n^+)\right) - \mathbb{P}\left(\bigcap_{i=0, \dots, j-1} \mathcal{C}_{A_i^s}^c(\mathcal{S}_n^+)\right) \cdot \mathbb{P}(\mathcal{C}_{A_j^s}^c(\mathcal{S}_n^+)) \right| < \varepsilon/(2c).$$

Combined with (3.4) this implies that

$$\mathbb{P}(\mathcal{C}_{A_i^s}(\mathcal{S}_n^+) \text{ does not occur for } i = 0, \dots, k) < f_{c; \varepsilon}^k(1 - c),$$

where $f_{c; \varepsilon}^k(x)$ denotes the k -fold iteration of the map $x \mapsto (1 - c)x + \varepsilon/(2c)$. One may check that $f_{c; \varepsilon}^k(1 - c) \rightarrow \varepsilon/2$ as $k \rightarrow \infty$, and hence we may choose a k sufficiently large such that

$$\mathbb{P}(\mathcal{C}_{A_i^s}(\mathcal{S}_n^+) \text{ does not occur for } i = 0, \dots, k) < \varepsilon.$$

Since the occurrence of any one of $\mathcal{C}_{A_i^s}(\mathcal{S}_n^+)$, $i = 0, \dots, k$, implies the occurrence of $\mathcal{C}_A(\mathcal{S}_n^+)$, we have the corollary. \square

Proof of Proposition 3.4. In what follows we prove (3.2); the proof of (3.3) is essentially identical. Fix $c > 1$ and $\varepsilon > 0$ and take C and n sufficiently large that the conclusion of Proposition 3.2 holds, and

$$\sup_{\substack{x, y \in X, \\ d(x, y) > Cs}} (d(x, y)s_n^{-1})^{18+\eta} |\kappa_n(x, y)| < \varepsilon^{10+\eta/3}; \quad (3.5)$$

this latter is possible by (2.3).

Now let $s > Cs_n$, subsets $X_1, X_2 \subset X$ such that $d(X_1, X_2) > s$, and polygons $P_1 \in \text{Poly}_{X_1; c}(s)$ and $P_2 \in \text{Poly}_{X_2; c}(s)$ be given. By Proposition 3.2, there exists a number $c_1 > 0$, independent of ε, s, P_1 and P_2 , such that the events $\mathcal{C}_{P_1}(\mathcal{S}_n^+)$ and $\mathcal{C}_{P_2}(\mathcal{S}_n^-)$ are determined, outside an event of probability less than ε , by the signs of sets $\mathcal{P}_1 \subset X_1$ and $\mathcal{P}_2 \subset X_2$ respectively, each of cardinality at most

$$|\mathcal{P}_1|, |\mathcal{P}_2| < c_1 \varepsilon^{-2} (s/s_n)^6.$$

Applying the first statement of Lemma 3.3 to compare between the joint law on one hand and the product laws on the other hand for the field restricted on $\mathcal{P}_1 \cup \mathcal{P}_2$, we have, for some constant $c_2 > 0$ independent of ε , s , P_1 and P_2 ,

$$\begin{aligned} & |\mathbb{P}(\mathcal{C}_{P_1}(\mathcal{S}_n^+) \cap \mathcal{C}_{P_2}(\mathcal{S}_n^-)) - \mathbb{P}(\mathcal{C}_{P_1}(\mathcal{S}_n^+)) \cdot \mathbb{P}(\mathcal{C}_{P_2}(\mathcal{S}_n^-))| \\ & < \varepsilon + c_2 \left(\varepsilon^{-6-\eta/3} (s/s_n)^{18+\eta} \sup_{\substack{x, y \in X, \\ d(x, y) > s}} |\kappa_n(x, y)| \right)^{1/4} \\ & < \varepsilon + c_2 \left(\varepsilon^{-6-\eta/3} \sup_{\substack{x, y \in X, \\ d(x, y) > C s_n}} (d(x, y) s_n^{-1})^{18+\eta} |\kappa_n(x, y)| \right)^{1/4} \\ & < \varepsilon + c_2 \varepsilon, \end{aligned}$$

where in the last line we used (3.5). Since $\varepsilon > 0$ was arbitrary, we conclude the proof. \square

3.1.2. Perturbation on macroscopic scales

The second consequence of the perturbation analysis is controlling the perturbations on macroscopic scales, the key step in handling asymptotically negligible negative correlations.

Proposition 3.6. *Let $\eta > 0$ and fix a sequence $p_n > 0$ of positive numbers satisfying*

$$\lim_{n \rightarrow \infty} p_n s_n^{-12-\eta} = 0. \quad (3.6)$$

Define the sequence of centred Gaussian random fields $(\tilde{f}_n)_{n \in \mathbb{N}}$ on \mathbb{X} with respective covariance kernels

$$\tilde{\kappa}_n = \kappa_n + p_n;$$

this is a valid covariance kernel since the constant function is positive-definite. Let $\tilde{\mathcal{S}}_n^+$ denote the positive excursion set of \tilde{f}_n . Then for every $c > 0$,

$$\lim_{n \rightarrow \infty} \sup_{s > 0} \sup_{P \in \text{Poly}_{\mathbb{X}; c}(s)} |\mathbb{P}(\mathcal{C}_P(\mathcal{S}_n^+)) - \mathbb{P}(\mathcal{C}_P(\tilde{\mathcal{S}}_n^+))| = 0.$$

Proof. Fix $c > 0$ and $\varepsilon > 0$, and take n sufficiently large that the conclusion of Proposition 3.2 holds, and

$$s_n^{-12-\eta} p_n < \varepsilon^{8+\eta/3}, \quad (3.7)$$

possible by (3.6). Now let $s > 0$ and $P \in \text{Poly}_{\mathbb{X}; c}(s)$ be given. Observe that the sequence of covariance kernels $\tilde{\kappa}_n$ also converge locally uniformly on $B(\delta_0)$, along with their first four derivatives, to the same limit K_∞ . By Proposition 3.2, there exists a number $c_1 > 0$, independent of ε , s and P , such that the events $\mathcal{C}_P(\mathcal{S}_n^+)$ and $\mathcal{C}_P(\tilde{\mathcal{S}}_n^+)$ are determined, outside an event of probability less than ε , by the signs of a set $\mathcal{P} \subseteq P$ of cardinality at most

$$c_1 \varepsilon^{-2} s_n^{-6};$$

for this recall that \mathcal{P} can be chosen to be the same set for all κ_n that converge locally uniformly on $B(\delta_0)$ to the same limit K_∞ (see the comments after the statement of Proposition 3.2). Applying the second statement of Lemma 3.3 to the law on \mathcal{P} of the fields f_n and \tilde{f}_n respectively, for some constant $c_2 > 0$ independent of ε , s and P

$$|\mathbb{P}(\mathcal{C}_P(\mathcal{S}_n^+)) - \mathbb{P}(\mathcal{C}_P(\tilde{\mathcal{S}}_n^+))| < \varepsilon + c_2 (\varepsilon^{-4-\eta/3} s_n^{-12-\eta} p_n)^{1/4} < \varepsilon + c_2 \varepsilon,$$

where to obtain the last inequality we used (3.7). Since $\varepsilon > 0$ was arbitrary, we conclude the proof. \square

3.2. Concluding the proof of Theorem 2.5

We are now almost ready to conclude the proof of Theorem 2.5. Before we begin, we state some simple geometric lemmas and show how to verify the ‘microscopic’ conditions (1) and (5) of Theorem 2.8 in the setting of Theorem 2.5.

We work in the same set-up as for the perturbation analysis given at the beginning of Section 3. Recall that \mathcal{S}_n^+ and \mathcal{S}_n^- denote, respectively, the positive and negative excursion sets of f_n ; we denote by \mathcal{N}_n the nodal set of f_n .

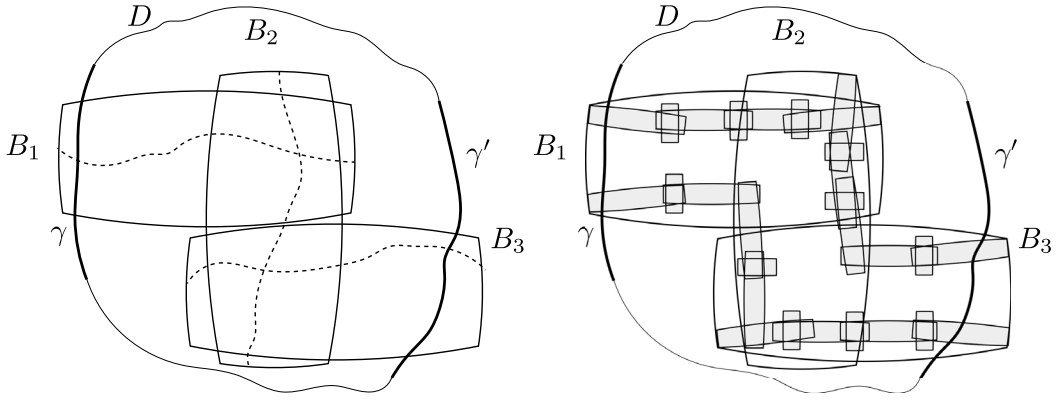


Fig. 3. Left: The union of a box-chain that crosses a quad $Q = (D, \gamma, \gamma')$ forms a polygon such that a crossing of the polygon implies a crossing of the quad. Right: Within this polygon one can choose two well separated box chains of smaller rectangles such that a crossing of these by \mathcal{S}_n^+ and \mathcal{S}_n^- respectively implies a crossing by \mathcal{N}_n of the polygon.

3.2.1. Geometric lemmas

In the proof of Theorem 2.5 we shall need the following. Recall the definition of polygons in Definition 3.1.

Lemma 3.7. Fix $X \subseteq \mathbb{X}$ and $c > 0$. Then there exists a number $c_1 > 0$ such that for each $s > 0$ and quad $Q \in \text{Unif}_{X;c}(s)$ the following hold:

- (1) There exists a polygon $P \in \text{Poly}_{X;c_1}(s) \cap \text{Unif}_{X;c}(s)$ such that the event $\mathcal{C}_P(\mathcal{S}_n^+)$ implies the event $\mathcal{C}_Q(\mathcal{S}_n^+)$.
- (2) There exist disjoint domains $X_1, X_2 \subset X$ satisfying $d(X_1, X_2) > s/c_1$ and polygons $P_1 \in \text{Poly}_{X_1;c_1}(s/c_1) \cap \text{Unif}_{X;c_1}(s/c_1)$ and $P_2 \in \text{Poly}_{X_2;c_1}(s/c_1) \cap \text{Unif}_{X;c_1}(s/c_1)$ such that if the events $\mathcal{C}_{P_1}(\mathcal{S}_n^+)$ and $\mathcal{C}_{P_2}(\mathcal{S}_n^-)$ both hold, then so does $\mathcal{C}_Q(\mathcal{N}_n)$.

Proof. For the first statement of Lemma 3.7, one can simply take the polygon that is the union of the boxes comprising one of the box-chains that cross Q guaranteed by the Definition 1.3 (see Figure 3, left). For the second statement of Lemma 3.7, we observe that the statement is true for any box $B \in \text{Unif}_{X;c}(s)$, since B can be ‘divided’ along two well-spaced geodesics into three parts, and the top and bottom parts can be crossed by box-chains using smaller boxes. Then for any quad $Q \in \text{Unif}_{X;c}(s)$ we can take the box-chain that crosses Q and decompose each constituent box using these geodesics (see Figure 3, right). \square

3.2.2. Verifying the microscopic conditions

Here we argue that the two ‘microscopic’ conditions (1) and (4) of Theorem 2.8 are satisfied; we begin by verifying the non-degeneracy condition (1). For use in this subsection we introduce F_n (resp. F_∞) as the centred, unit variance Gaussian random field on \mathbb{R}^2 with the rescaled covariance kernel K_n (resp. K_∞), as in Definition 2.4.

Lemma 3.8.

- (1) For every $n \in \mathbb{N}$, $\mathbb{P}(f_n(0) = 0) = 0$.
- (2) Recall that for $v \in \mathbb{S}^1$ and $r > 0$ the $\mathcal{L}_v(r)$ is the length- r geodesic segment based at the origin in direction v . For every $v \in \mathbb{S}^1$ we have

$$\lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\exists x \in \mathcal{L}_v(rs_n). f_n(x) = 0) = 0.$$

Proof. The first statement is clear upon recalling that f_n is symmetric and non-degenerate. For the second statement, observe that by the Kac–Rice formula [2, Theorem 6.3], the symmetries of f_n , and since Assumption 2.3 guarantees that $\nabla f_n(0)$ is independent of $f_n(0)$, for each $r > 0$,

$$\mathbb{E}[|\{x \in \mathcal{L}_v(rs_n) : f_n(x) = 0\}|] = \frac{r}{\sqrt{2\pi}} s_n \mathbb{E}\left[\left|\frac{\partial f_n(0)}{\partial v}\right|\right].$$

Since κ_n is C^2 , it holds that

$$\mathbb{E}\left[\left|\frac{\partial f_n(0)}{\partial v}\right|\right] = \sqrt{\frac{2 - \partial^2 \kappa_n(0)}{\pi \partial^2 v}}.$$

Hence by the local uniform convergence of the second derivatives of κ_n , and since F_∞ satisfies Assumption 2.3,

$$\lim_{n \rightarrow \infty} s_n \mathbb{E}\left[\left|\frac{\partial f_n(0)}{\partial v}\right|\right] = \lim_{n \rightarrow \infty} \sqrt{\frac{2 - s_n^2 \partial^2 \kappa_n(0)}{\pi \partial^2 v}} = \sqrt{\frac{2 - \partial^2 K_\infty(0)}{\pi \partial^2 v}} < \infty.$$

Taking $r \rightarrow 0$ yields the result. \square

Next we verify condition (4) of Theorem 2.8 guaranteeing arbitrary crossings on microscopic scales; to this end we formulate the following lemma.

Lemma 3.9. *There exists a number $\delta > 0$ such that*

$$\liminf_{n \rightarrow \infty} \mathbb{P}(F_n(x) > 0 \text{ for all } x \in B(\delta)) > 0.$$

Before we state the proof of Lemma 3.9, we show that it implies condition (4) of Theorem 2.8.

Corollary 3.10. *There exists a number $\delta > 0$ such that*

$$\liminf_{n \rightarrow \infty} \inf_{Q \in \text{Quad}_{B(\delta s_n)}} \mathbb{P}(\mathcal{C}_Q(\mathcal{S}_n^+)) > 0,$$

where \mathcal{S}_n^+ are the positive excursion sets (3.1) of f_n .

Proof. Recall that $\Phi : \mathbb{R}^2 \rightarrow \mathbb{X}$ is the locally linear isometry, as used in Definition 2.4. By Lemma 3.9, there is a number $\delta > 0$ such that

$$\liminf_{n \rightarrow \infty} \mathbb{P}(\Phi(B(\delta s_n)) \subseteq \mathcal{S}_n^+) > 0.$$

Since Φ is locally an isometry, for any $\delta_1 < \delta$, $\Phi(B(\delta s_n))$ eventually contains the disk $B(\delta_1 s_n)$. Finally, since the occurrence of the event $\{B(\delta_1 s_n) \subseteq \mathcal{S}_n^+\}$ implies the crossing event $\mathcal{C}_Q(\mathcal{S}_n)$ for any $Q \subset B(\delta_1 s_n)$, we have the result. \square

Proof of Lemma 3.9. Recall that K_∞ denotes the limit of K_n , well-defined as a stationary C^2 covariance kernel on $B(\delta_0)$. Define a stationary covariance kernel \tilde{K}_∞ on $B(\delta_0/2)$ by

$$\tilde{K}_\infty(x, y) = K_\infty(x/2, y/2),$$

and let \tilde{F}_∞ denote the centred, unit variance Gaussian random field on $B(0, \delta_0/2)$ with covariance kernel \tilde{K}_∞ . By the local uniform convergence of K_n and its first three derivatives to K_∞ and its respective derivatives, and the strictly negative second derivatives of K_∞ (since K_∞ satisfies Assumption 2.3), there exists a $\delta_1 \in (0, \delta_0/2)$ such that, for sufficiently large n ,

$$K_n(x, y) > \tilde{K}_\infty(x, y) \quad \text{for all } x, y \in B(\delta_1).$$

Hence by Slepian's lemma [1, Theorem 2.2.1], for sufficiently large n , every $\delta \in (0, \delta_1)$ satisfies

$$\mathbb{P}(F_n(x) > 0 \text{ for all } x \in B(\delta)) \geq \mathbb{P}(\tilde{F}_\infty(x) \text{ for all } x \in B(\delta)). \quad (3.8)$$

It then remains to prove the existence of a $\delta \in (0, \delta_1)$ such that the latter probability is positive.

By the Borell-TIS Theorem [1, Theorem 2.1.1] and Markov's inequality, there exists a number $c_1 > 0$ such that for every $\lambda > 0$,

$$\mathbb{P}\left(\sup_{v \in \mathbb{S}^1} \max_{x \in B(\delta_0/2)} \left|\frac{\partial \tilde{F}_\infty}{\partial v}(x)\right| > \lambda\right) < c_1/\lambda.$$

Hence, by taking $\lambda_1, \lambda_2 > 0$ sufficiently small, the event

$$E = \{\tilde{F}_\infty(0) > \lambda_1\} \cap \left\{ \sup_{v \in \mathbb{S}^1} \max_{x \in B(\delta_0/2)} \left| \frac{\partial \tilde{F}_\infty}{\partial v}(x) \right| < \lambda_2 \right\}$$

has positive probability. By Taylor's theorem we can choose $\delta > 0$ sufficiently small that

$$E \subseteq \{\tilde{F}_\infty(x) \text{ for all } x \in B(\delta)\};$$

which, since $\mathbb{P}(E) > 0$ and in light of (3.8), yields Lemma 3.9. \square

3.2.3. Proof of Theorem 2.5 assuming Theorem 2.8, Proposition 3.2 and Lemma 3.3

Let $(f_n)_{n \in \mathbb{N}}$ be given as in Theorem 2.5; with no loss of generality we assume that f_n has unit variance. Let $\eta > 0$, $X \subset \mathbb{X}$ and s_n satisfy the conditions of Theorem 2.5.

We begin by slightly perturbing the covariance kernels κ_n of f_n to eliminate possible negative correlations. Define a collection of centred Gaussian random fields $(\tilde{f}_n)_{n \in \mathbb{N}}$ on \mathbb{X} with respective covariance kernels

$$\tilde{\kappa}_n(x, y) = \kappa_n(x, y) + s_n^{12+\eta/2}. \quad (3.9)$$

Observe that, by condition (2.2), $\tilde{\kappa}_n$ is everywhere positive on X for n sufficiently large. Moreover, the choice (3.9) of perturbation means that the conclusion of Proposition 3.6 is valid.

We now argue that the positive excursion sets $\tilde{\mathcal{S}}_n^+$ of \tilde{f}_n satisfy all the conditions of Theorem 2.8 for the set X and sequence s_n ; by symmetry, the same conclusion holds also for the negative excursion sets. The justification for the validity of conditions (2), (3) and (5) of Theorem 2.8 is via standard arguments: the symmetry of the excursion sets follows from the symmetry of the kernel, positive associations on X follow from the positivity of the covariance kernels on X by the well-known result of Pitt [14] (although [14] deals only with Gaussian vectors, the result readily extends to arbitrary increasing events on X by approximation arguments), and the probability of crossing square-boxes is exactly 1/2 by the symmetry of the kernel and the symmetry of a Gaussian random field w.r.t. sign changes. Moreover, conditions (1), (4) and (6) in Theorem 2.8 follow from the analysis we developed above, namely Lemma 3.8, and Corollaries 3.10 and 3.5 respectively. Hence all the conditions of Theorem 2.8 are satisfied, and an application of Theorem 2.8 yields the desired conclusions for \tilde{f}_n , i.e. that the positive (resp. negative) excursion sets of \tilde{f}_n satisfy the RSW estimates on X on all scales. In particular, for all $c > 0$,

$$\liminf_{n \rightarrow \infty} \inf_{s > 0} \inf_{Q \in \text{Unif}_{X;c}(s)} \mathbb{P}(\mathcal{C}_Q(\tilde{\mathcal{S}}_n^+)) > 0. \quad (3.10)$$

Next we use Proposition 3.6 to infer that the positive excursion sets of f_n also satisfy the RSW estimates on X on all scales (the same statement for the negative excursion sets \mathcal{S}_n^- then holds by an identical argument). Fix $c > 0$, and let $c_1 > 0$ be the constant prescribed by Lemma 3.7. Also let $\varepsilon > 0$ be such that, for all sufficiently large $n \in \mathbb{N}$, both

$$\inf_{s > 0} \inf_{Q \in \text{Unif}_{X;c}(s)} \mathbb{P}(\mathcal{C}_Q(\tilde{\mathcal{S}}_n^+)) > 2\varepsilon, \quad (3.11)$$

and

$$\sup_{s > 0} \sup_{P \in \text{Poly}_{X;c_1}(s)} |\mathbb{P}(\mathcal{C}_P(\mathcal{S}_n^+)) - \mathbb{P}(\mathcal{C}_P(\tilde{\mathcal{S}}_n^+))| < \varepsilon, \quad (3.12)$$

hold; possible by (3.10) and Proposition 3.6 respectively. Now let $s > 0$ and $Q \in \text{Unif}_{X;c}(s)$ be given. By Lemma 3.7, there exists a polygon $P \in \text{Poly}_{X;c_1}(s) \cap \text{Unif}_{X;c}(s)$ such that the event $\mathcal{C}_P(\mathcal{S}_n^+)$ is contained in the event $\mathcal{C}_Q(\mathcal{S}_n^+)$. In particular, since $P \in \text{Unif}_{X;c}(s)$, by (3.11)

$$\mathbb{P}(\mathcal{C}_P(\tilde{\mathcal{S}}_n^+)) > 2\varepsilon,$$

and in light of (3.12), applicable since $P \in \text{Poly}_{X;c_1}(s)$, we obtain

$$\mathbb{P}(\mathcal{C}_P(\mathcal{S}_n^+)) > \varepsilon.$$

Finally, since $\mathcal{C}_P(\mathcal{S}_n^+) \subseteq \mathcal{C}_Q(\mathcal{S}_n^+)$, we conclude that

$$\mathbb{P}(\mathcal{C}_Q(\mathcal{S}_n^+)) \geq \mathbb{P}(\mathcal{C}_P(\mathcal{S}_n^+)) > \varepsilon,$$

the RSW estimates for \mathcal{S}_n^+ on all scales.

The final step of the proof of Theorem 2.5 is using the first statement (3.2) of Proposition 3.4 to infer the RSW estimates for the *nodal sets* \mathcal{N}_n of f_n from the already established RSW estimates for the *excursion sets* of f_n (as was done in [4]). Again fix $c > 0$, and let $c_1 > 0$ be the corresponding constant appearing from Lemma 3.7. Let $\varepsilon > 0$ and $C > 0$ be such that, for all sufficiently large $n \in \mathbb{N}$,

$$\inf_{s>0} \inf_{\substack{Q_1 \in \text{Unif}_{X;c_1}(s/c_1), \\ Q_2 \in \text{Unif}_{X;c_1}(s/c_1)}} \mathbb{P}(\mathcal{C}_{Q_1}(\mathcal{S}_n^+)) \cdot \mathbb{P}(\mathcal{C}_{Q_2}(\mathcal{S}_n^-)) > 2\varepsilon, \quad (3.13)$$

and

$$\sup_{s>Cs_n} \sup_{\substack{X_1, X_2 \subset X, \\ d(X_1, X_2) > s/c_1}} \sup_{\substack{P_1 \in \text{Poly}_{X_1;c_1}(s/c_1), \\ P_2 \in \text{Poly}_{X_2;c_1}(s/c_1)}} |\mathbb{P}(\mathcal{C}_{P_1}(\mathcal{S}_n^+) \cap \mathcal{C}_{P_2}(\mathcal{S}_n^-)) - \mathbb{P}(\mathcal{C}_{P_1}(\mathcal{S}_n^+)) \cdot \mathbb{P}(\mathcal{C}_{P_2}(\mathcal{S}_n^-))| < \varepsilon. \quad (3.14)$$

both hold; possible since the RSW estimates hold for the excursion sets of f_n on all scales and by (3.2) respectively.

Now let $s > Cs_n$ and $Q \in \text{Unif}_{X;c}(s)$ be given. By Lemma 3.7 there exist disjoint domains $X_1, X_2 \subset X$ satisfying $d(X_1, X_2) > s/c_1$ and polygons $P_1 \in \text{Poly}_{X_1;c_1}(s/c_1) \cap \text{Unif}_{X;c_1}(s/c_1)$ and $P_2 \in \text{Poly}_{X_2;c_1}(s/c_1) \cap \text{Unif}_{X;c_1}(s/c_1)$ such that if the events $\mathcal{C}_{P_1}(\mathcal{S}_n^+)$ and $\mathcal{C}_{P_2}(\mathcal{S}_n^-)$ both occur, then so does $\mathcal{C}_Q(\mathcal{N}_n)$, i.e.,

$$\mathcal{C}_{P_1}(\mathcal{S}_n^+) \cap \mathcal{C}_{P_2}(\mathcal{S}_n^-) \subseteq \mathcal{C}_Q(\mathcal{N}_n).$$

In particular, since $P_1, P_2 \in \text{Unif}_{X;c_1}(s/c_1)$, by (3.13)

$$\mathbb{P}(\mathcal{C}_{P_1}(\mathcal{S}_n^+)) \cdot \mathbb{P}(\mathcal{C}_{P_2}(\mathcal{S}_n^-)) > 2\varepsilon.$$

Since also $P_1 \in \text{Poly}_{X_1;c_1}(s/c_1)$, $P_2 \in \text{Poly}_{X_2;c_1}(s/c_1)$ and $d(X_1, X_2) > s/c_1$, in light of (3.14) we deduce that

$$\mathbb{P}(\mathcal{C}_{P_1}(\mathcal{S}_n^+) \cap \mathcal{C}_{P_2}(\mathcal{S}_n^-)) > \varepsilon.$$

Finally, since $\mathcal{C}_{P_1}(\mathcal{S}_n^+) \cap \mathcal{C}_{P_2}(\mathcal{S}_n^-) \subseteq \mathcal{C}_Q(\mathcal{N}_n)$, we conclude that

$$\mathbb{P}(\mathcal{C}_Q(\mathcal{N}_n)) \geq \mathbb{P}(\mathcal{C}_{P_1}(\mathcal{S}_n^+) \cap \mathcal{C}_{P_2}(\mathcal{S}_n^-)) > \varepsilon,$$

which validates the RSW estimates for \mathcal{N}_n down to the scale s_n .

3.3. Proof of Theorem 1.5 and the validity of Example 2.6

In this section we show that Theorem 1.5 and Example 2.6 are within the scope of the more general Theorem 2.5.

Proof of Theorem 1.5. Observe that the covariance kernels κ_n are symmetric in sense of Definition 2.2 and satisfy Assumption 2.3. Next we check the local uniform convergence of κ_n together with all its derivatives on the scale s_n (previously stated at (1.5)). For this, define the smooth functions $G_n : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $F_n : \mathbb{R} \rightarrow \mathbb{R}$ by

$$G_n(x, y) = \sqrt{n} \cdot \|\Phi(x/\sqrt{n}) - \Phi(y/\sqrt{n})\| \quad \text{and} \quad F_n(t) = (\cos(t/n))^n.$$

An explicit computation shows that G_n and F_n converge locally uniformly together with all of their derivatives to the respective limits

$$G_\infty(x, y) = \|x - y\| \quad \text{and} \quad F_\infty(t) = e^{-t^2/2},$$

and hence so does their composition $F_n \circ G_n$. Since

$$K_n(x, y) = F_n \circ G_n(x, y) = \kappa_n(\Phi(s_n x), \Phi(s_n y)) \quad \text{and} \quad K_\infty(x, y) = F_\infty \circ G_\infty(x, y),$$

we have the stated convergence.

It remains to show that conditions (4) and (5) of Theorem 2.5 hold for any constant $\eta > 0$, scale $s_n = n^{-1/2}$ and $X \subset \mathbb{S}^2$ whose closure does not contain antipodal points. For condition (4), we observe that since the closure of X does

not contain antipodal points, there exists a number $c_1 < \pi$ such that $\theta(x, y) < c_1$ for each $x, y \in X$. Therefore there exists a $c_2 > 0$ such that, for all $n \in \mathbb{N}$ and $x, y \in X$,

$$\kappa_n(x, y) = \cos(\theta(x, y))^n > -e^{-c_2 n},$$

and so, for any $\eta > 0$, as $n \rightarrow \infty$,

$$s_n^{-12-\eta} \inf_{x, y \in X} (\kappa_n(x, y) \wedge 0) = -n^{6+\eta/2} e^{-c_2 n} \rightarrow 0.$$

For condition (5), let $c_1 < \pi$ be as above, and choose a $c_2 \in (0, 1/c_1^2)$ such that $|\cos(t)| \leq 1 - c_2 t^2$ for each $|t| < c_1$. Together with the inequality $\log(1 - x) \leq -x$, valid on $x \in (0, 1)$, we have for all $x, y \in X$,

$$|\kappa_n(x, y)| = |\cos(\theta(x, y))|^n \leq e^{n \log(1 - c_2 \theta(x, y)^2)} \leq e^{-nc_2 \theta(x, y)^2} = e^{-c_2 (d(x, y) s_n^{-1})^2}.$$

Hence for any $\eta > 0$ and $C > 1$,

$$\limsup_{n \rightarrow \infty} \sup_{\substack{x, y \in X, \\ \theta(x, y) > C s_n}} (\theta(x, y) s_n^{-1})^{18+\eta} |\kappa_n(x, y)| \leq \limsup_{n \rightarrow \infty} \sup_{t > C} t^{18+\eta} e^{-c_2 t^2} = \sup_{t > C} t^{18+\eta} e^{-c_2 t^2},$$

which tends to zero as $C \rightarrow \infty$. □

Validity of Example 2.6. Remark first that κ_n is a valid covariance kernel since $\cos^n(x) \cos^n(y)$ can be written as a Fourier series $\sum_{i,j} a_{i,j} \cos(ix) \cos(jy)$ for non-negative coefficients $a_{i,j}$, which implies that κ_n is positive-definite.

Similarly to in the proof of Theorem 1.5 above, it is sufficient that the conditions of Theorem 2.5 hold for any constant $\eta > 0$, scale $s_n = n^{-1/2}$ and subset $X \subseteq \mathbb{T}^2$ such that the closure of X does not contain distinct $x, y \in X$ having $2(x_1 - y_1)$ and $2(x_2 - y_2)$ as integers. The proof of this is similar to the proof of Theorem 1.5, so we omit the details. □

4. Proof of Theorem 2.8: RSW estimates for sequences of random sets

In this section we prove the abstract RSW estimates in Theorem 2.8, following the argument in [17] that established the analogous estimates for planar Voronoi percolation. For the benefit of a reader familiar with [17], we explain the four main differences in our setting, and well as briefly describing the necessary modifications to the argument.

- (1) Recall that Theorem 2.8 is stated for either the unit sphere \mathbb{S}^2 or the flat torus \mathbb{T}^2 . The first difference is due to the non-Euclidean geometry of \mathbb{S}^2 ; indeed, since the interior angles of spherical squares (see Definition 1.2) depend on their scale, many of the simple geometric arguments in [17] fail in the spherical case and need to be derived from scratch or modified significantly. Interestingly, these modifications apparently fail in the case of negatively curved manifolds and it is an interesting question whether a RSW theory can also be developed in that case. On the other hand, on the flat torus these arguments work as in [17].
- (2) Second, we work with a sequence of random sets rather than a single set. Hence we rely on extra ‘uniform’ conditions on the covariance kernels in the statement of Theorem 2.8, which ensure that all the inputs into the argument are uniformly controlled.
- (3) Third, the random set considered in [17], arising from planar Voronoi percolation, is asymptotically independent in a very strong sense: the Voronoi percolation restricted to disjoint domains is independent as long as there are no Voronoi cells intersecting both of them, see the discussion in Section 1.2. Since we wish to apply Theorem 2.8 to Gaussian random fields, we do not have this type of strong mixing of the model, and instead we work with a much weaker notion of asymptotic independence as was done in [4] (see condition (6) in the statement of Theorem 2.8).
- (4) Finally, unlike [17], the property of ‘positive associations’ only applies inside a subset $X \subseteq \mathbb{X}$; this is essential in order to include the Kostlan ensemble (1.1). As a result, we need to take extra care in the argument to ensure our geometric constructions take place exclusively in this set.

Before embarking on the proof of Theorem 2.8, we first build up a collection of preliminary results that hold for arbitrary $n \in \mathbb{N}$. The first result (Lemma 4.1) can be viewed as a modification of the ‘standard theory’ of RSW: this shows how to transform the bounds on the probability of crossing a small fixed box to infer the bounds on the probability of crossing large domains. The second set of results (Section 4.2) contains our modification of Tassion’s argument in [17].

Throughout the rest of this section we fix a set $X \subseteq \mathbb{X}$ as in the statement of Theorem 2.8. Our preliminary results depend only on the conditions of Theorem 2.8 that hold for each $n \in \mathbb{N}$, namely the first non-degeneracy statement in

condition (1), the symmetry in condition (2), and the guarantee of positive associations in X in condition (3). We stress that all the preliminary estimates that we state give lower bounds on various crossing probabilities depending on $n \in \mathbb{N}$ in terms of a positive power of the quantity

$$c_0(n) = \inf_{s>0} \inf_{B \in \text{Box}_{X;1}(s)} \mathbb{P}(C_B(\mathcal{S}_n)); \quad (4.1)$$

importantly these are monotone increasing in c_0 . By condition (4) of Theorem 2.8, $c_0(n)$ is uniformly bounded away from zero for sufficiently large $n \in \mathbb{N}$, which, in light of the above, yields a uniform control over the crossing probabilities for varying n . For the next two sections we work with arbitrary fixed $n \in \mathbb{N}$, and for notational convenience we drop all dependencies on n and on the random set \mathcal{S}_n .

4.1. The ‘standard theory’ of RSW: From a fixed box to larger domains

One of the most fundamental tools in percolation theory is the FKG inequality, which implies positive associations for the percolation subgraph, and in particular implies that crossing events are positively correlated. In the classical theory (i.e. on the plane), the FKG property is used to infer bounds on the probability of crossing larger domains from assumed bounds on the probability of crossing a fixed small box; we call this the ‘standard theory’ of RSW. For instance, in [17, Corollary 1.3] ‘horizontal’ crossings of two overlapping rectangles are connected via a ‘vertical’ crossing of a square to deduce a ‘horizontal’ crossing of a longer rectangle.

In our setting the property of positive associations is true in the set X by assumption, and by analogy we shall refer to this fact as the ‘FKG property’. We next state a version of the ‘standard theory’ of RSW that is valid in the spherical setting. On the sphere, the construction used in [17, Corollary 1.3] fails, since two spherical rectangles cannot be overlapped in a way that the overlapping region is a square. Instead, we connect ‘horizontal’ crossings using a third ‘vertical’ rectangle.

Let us introduce a fixed box, $\bar{B}(s)$, which denotes, for each $s > 0$, an $s \times 2s$ box chosen arbitrarily. Recall also that, for each $c, r \geq 1$ and $s > 0$, the collection of boxes and annuli $\text{Box}_{X;c}(s)$ and $\text{Ann}_{X;c;r}(s)$ were introduced in Definitions 1.2, 2.1 and 2.7, and note in particular that $\text{Ann}_{X;6;6}(s)$ consists exclusively of $s \times 6s$ annuli.

Lemma 4.1 (From a fixed box to arbitrary boxes and annuli; cf. [17, Corollary 1.3]). *There exists a sufficiently small $s^* > 0$ such that the following holds for every $c > 1$ and $s < s^*$: there exists a monotone increasing function f_c , depending only on c , and an absolute monotone increasing function g such that*

$$\inf_{B \in \text{Box}_{X;c}(3s)} \mathbb{P}(C_B) > f_c(\mathbb{P}(C_{\bar{B}(s)})) \quad \text{and} \quad \inf_{A \in \text{Ann}_{X;6;6}(s)} \mathbb{P}(C_A) > g(\mathbb{P}(C_{\bar{B}(s)})).$$

The value of s^* depends only on the geometry of \mathbb{X}^2 ; in the case $\mathbb{X} = \mathbb{T}^2$ it could be arbitrary, whereas in the case $\mathbb{X} = \mathbb{S}^2$ it must be sufficiently small so that the distortions due to the spherical geometry are controlled on a ball $B(s^*)$. This constant could be computed explicitly, but its precise value is irrelevant. In the Euclidean case, the numbers 3 and 6 in the statement of Lemma 4.1 could be replaced by 2 and 5 respectively. Using 3 and 6 provides a bit more ‘space’ in the spherical case to account for distortions.

Proof. The proof is based on the observation that, for sufficiently small $s^* > 0$ and $s < s^*$, it is possible to form a box-chain out of alternating ‘horizontal’ and ‘vertical’ copies of $\bar{B}(s)$ that are aligned along a single geodesic (see Figure 4, left).

For the first statement, fix $3s \leq a, b \leq 3cs$ and consider an $a \times b$ box $B \subseteq X$. Let $\{B_i\}$ be a box-chain consisting of horizontal and vertical copies of $\bar{B}(s)$ aligned along the geodesic joining the mid-points of the opposite sides of B . Since the shortest sides of B are longer than the longest sides of $\bar{B}(s)$, for sufficiently small $s^* > 0$ and $s < s^*$ we can find such a $\{B_i\}$ that both crosses B and lies inside B (cf. the Euclidean case, where we could replace the number 3 with 2); moreover, the number of boxes required depends only on c . Since the FKG property holds in B , this establishing the bound.

The second statement is proved similarly, working instead with four inter-connecting box-chains aligned along the four ‘median’ geodesics that bisect orthogonally the geodesic line segments joining the mid-points of the boundary squares of any $s \times 6s$ annuli A (see Figure 4, right). Such box-chains can be formed inside A since $\bar{B}(s)$ fits inside A when aligned with its shortest sides perpendicular to a geodesic bisecting A (cf. the Euclidean case, where we could replace the number 6 with 5). \square

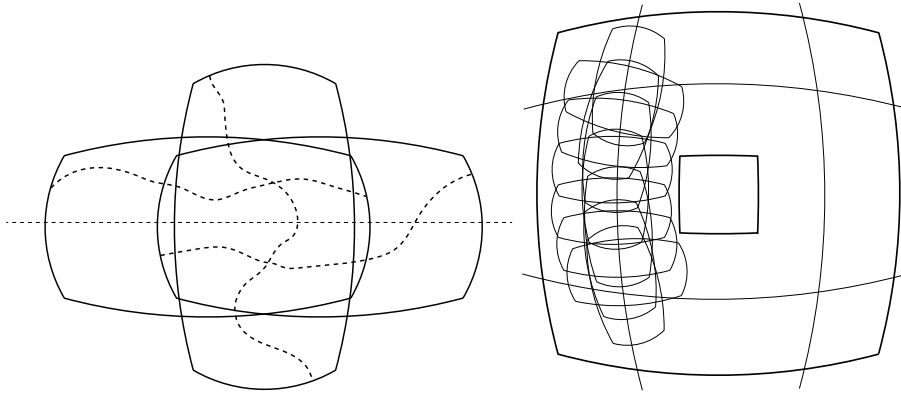


Fig. 4. Left: A crossing in a ‘vertical’ rectangle connects crossings in two ‘horizontal’ rectangles that are copies of each other shifted along a (dashed) geodesic. Right: By repeating this construction many times we can obtain a long crossing. Combining four of them we obtain a circuit in the annulus.

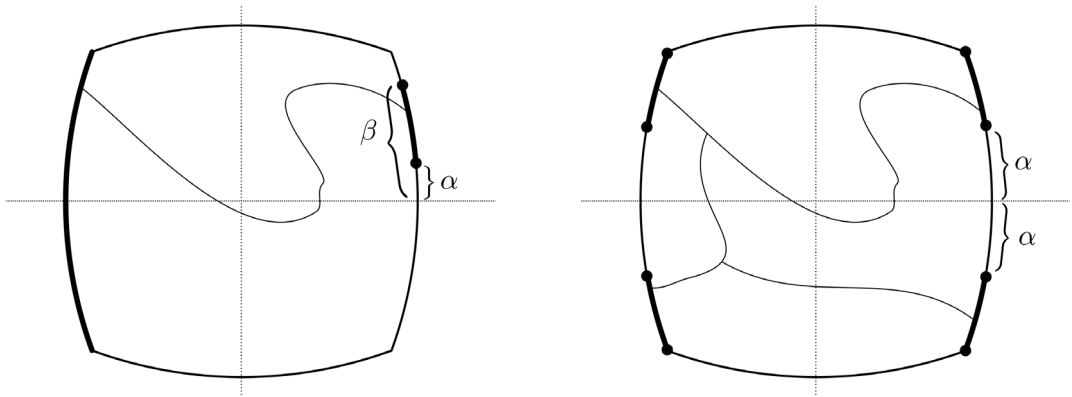


Fig. 5. An illustration of an H -crossing (left) and an X -crossing (right) of a square box in the spherical case.

4.2. Tassion’s argument

In this section we develop Tassion’s argument from [17], with suitable modifications to account for the difference in our setting. We begin by introducing, following Tassion, the concept of H -crossings and X -crossings of square boxes (see Figure 5 for an illustration in the spherical case).

Throughout this section, when the parameter s^* in the statement of a lemma may be set sufficiently small, we always implicitly set it so that the conclusion of Lemma 4.1, as well as the conclusion of any proceeding lemmas in this section, is valid. Since if X has an empty interior Theorem 2.8 has no content, we may assume that X has non-empty interior, and, by symmetry, that X contains an open ball $B(\delta_0)$ centred at the origin. Therefore we may assume that s^* is sufficiently small so that all of the (finite) collections of domains that we manipulate in the proofs of the following lemmas are contained inside $B(\delta_0)$; we may thereby always assume the FKG property holds.

Definition 4.2.

- (1) For each $s > 0$ and $\alpha, \beta \in [0, s/2]$, an H -crossing of an $s \times s$ square box $B = (D; \gamma, \gamma')$, denoted by $\mathcal{H}_s(\alpha, \beta) = \mathcal{H}_{s;B}(\alpha, \beta)$, is the event that a connected component of \mathcal{S}_n , restricted to B , intersects both γ and the segment of γ' of length $\beta - \alpha$ at distance α from the mid-point of γ' (see Figure 5, left).
- (2) For every $s > 0$ and $\alpha \in [0, s/2]$, an X -crossing of an $s \times s$ square box $B = (D; \gamma, \gamma')$, denoted by $\mathcal{X}_s(\alpha) = \mathcal{X}_{s;B}(\alpha)$, is the event that a connected component of \mathcal{S}_n , restricted to B , intersects the four segments of $\gamma \cup \gamma'$ obtained by removing from each of γ and γ' the centred intervals of length 2α (see Figure 5, right).

Observe that, by the symmetry condition (2) of Theorem 2.8 and the definitions of $\mathcal{H}_s(\alpha, \beta)$ and $\mathcal{X}_s(\alpha)$, both $\mathbb{P}[\mathcal{H}_s(\alpha, \beta)]$ and $\mathbb{P}[\mathcal{X}_s(\alpha)]$ are independent of the choice of the square box B . Hence the function

$$\phi_s(\alpha) = \mathbb{P}[\mathcal{H}_s(0, \alpha)] - \mathbb{P}[\mathcal{H}_s(\alpha, s/2)] \quad (4.2)$$

is well-defined, and is a continuous function of α by the first statement of the non-degeneracy condition (1) of Theorem 2.8. Recalling the definition (4.1) of c_0 , for every scale $s > 0$ we may fix the constant

$$\alpha_s = \min(\phi_s^{-1}(c_0/4), s/4) \leq s_4,$$

where $\phi_s^{-1}(x) = \inf\{y \geq 0 : \phi_s(y) = x\}$ is the generalised inverse. The following lemma contains the essential consequences of the definition of α_s , cf. [17, Lemma 2.1].

Lemma 4.3. *There exists a sufficiently small $s^* > 0$, and absolute numbers $a_1 > 0$ and $k_1 \in \mathbb{N}$, such that if $s < s^*$, the following two properties hold:*

(P1) *For all $0 \leq \alpha \leq \alpha_s$, $\mathbb{P}[\mathcal{X}_s(\alpha)] > a_1 \cdot c_0^{k_1}$.*

(P2) *If $\alpha_s < s/4$, then for all $\alpha_s \leq \alpha \leq s/2$, $\mathbb{P}[\mathcal{H}_s(0, \alpha)] \geq c_0/4 + \mathbb{P}[\mathcal{H}_s(\alpha, s/2)]$.*

Proof. The proof of Lemma 4.3 is independent of the geometry of the ambient space and the argument from [17] works in our setting unimpaired. \square

The next three lemmas are the heart of Tassion's argument. Recall the fixed $s \times 2s$ box $\bar{B}(s)$ in the statement of Lemma 4.1. We think of $s > 0$ as being a 'good' scale if it satisfies $\alpha_s \leq 2\alpha_{2s/3}$, and proceed to formulate a few consequences of a good scale. As a corollary, we deduce, for a fixed $n \in \mathbb{N}$, the existence of uniform bounds on crossing probabilities on all large scales, provided that certain inputs into the argument are also controlled.

As in the proof of Corollary 3.5, in this section we work with the collection $(A_{a,b})_{a < b}$ of $a \times b$ annuli centred at the origin that are 'parallel', i.e. such that there is a single geodesic that passes through both mid-points of both pairs of opposite sides. When working with square boxes, we shall sometimes abuse notation by referring to these simply as 'squares'.

Lemma 4.4 (Good scales imply crossings of the fixed box; cf. [17, Lemma 2.2]). *There exists a sufficiently small $s^* > 0$ and absolute numbers $a_2 > 0$ and $k_2 \in \mathbb{N}$, such that if $s < s^*$ and $\alpha_s \leq 2\alpha_{2s/3}$ then $\mathbb{P}(C_{\bar{B}(2s)}) > a_2 \cdot c_0^{k_2}$.*

The proof of Lemma 4.4 is similar to the proof of [17, Lemma 2.2], with certain modifications needed to handle the spherical geometry in the case $\mathbb{X} = \mathbb{S}^2$; here we only give a sketch of the argument while explaining in detail the necessary modifications.

Proof. We consider separately two cases, $\alpha_s = s/4$ and $\alpha_s = \phi_s^{-1}(c_0/4) < s/4$, beginning with the first case. In light of (P1) from Lemma 4.3, we have a lower bound on $\mathbb{P}[\mathcal{X}_s(s/4)]$ of the form $a_2 \cdot c_0^{k_2}$. Hence, by the FKG property, it suffices to construct a finite collection of $s \times s$ squares S_i such that if $\mathcal{X}_s(s/4)$ holds for each S_i then so does $C_{\bar{B}(2s)}$.

In what follows we refer to the labelling in Figure 6, which illustrates the argument in the spherical case. Consider the $s \times s$ square $ABCD$ and its translation $A'B'C'D'$ by $s/2$ along the geodesic AB . Observe that if the event $\mathcal{X}_s(s/4)$ holds for both squares $ABCD$ and $A'B'C'D'$ then there is a crossing of S inside the union of the squares that intersects the two sub-intervals of the geodesic AB' formed by removing a centred interval of length s . Repeating this construction along the top edge of $\bar{B}(2s)$ we obtain a crossing of $\bar{B}(2s)$ using only X -crossings of $s \times s$ squares.

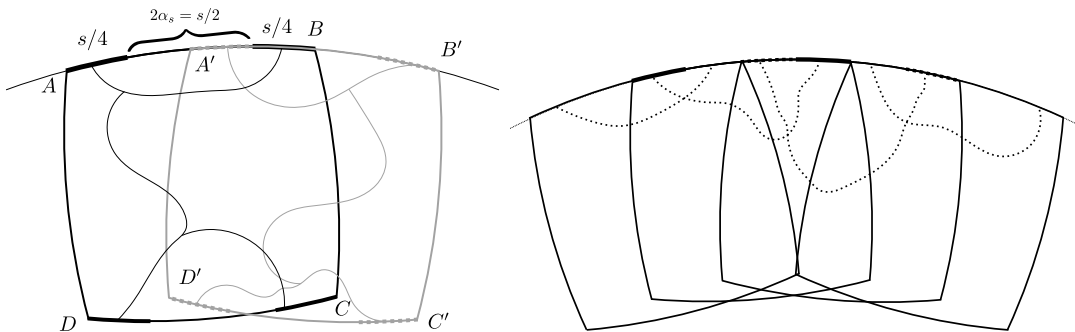


Fig. 6. Left: The grey $s \times s$ square $A'B'C'D'$ is a translation of the black square $ABCD$ by $s/2$ along the side AB . If the event $\mathcal{X}_s(s/4)$ holds in both of these squares (indicated by the black/grey connecting sets), then there is a connection between the top-left and top-right ends of AB' . Right: Repeating the construction we obtain an arbitrary long crossing which is s -close to a given geodesics.

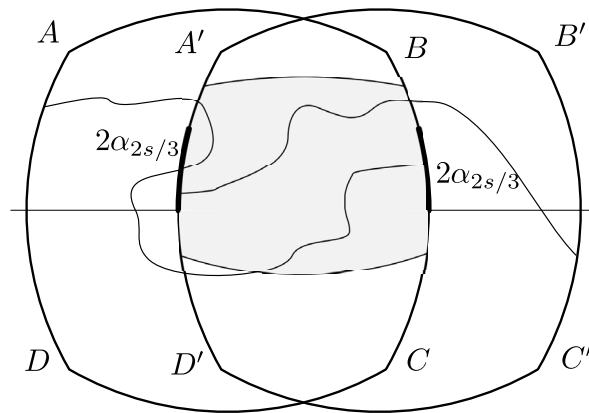


Fig. 7. An $s \times s$ square $ABCD$ and its translation $A'B'C'D'$ along the line joining the mid-points of AD and BC by a distance $d > 0$. In the planar case, if $d = 4s/3$ then the shaded area is a $(2s/3) \times (2s/3)$ square; this construction was used in [17] to connect horizontal crossings of the $s \times s$ squares via an X -crossing of the shaded region. In the spherical case, the shaded area is not a square for any choice of d , so we use a different construction to connect horizontal crossings of the large squares.

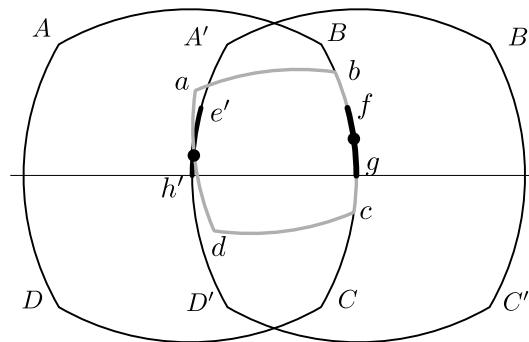


Fig. 8. After the small square $abcd$ is fixed, the translation distance is chosen such that the left side of the large square $A'B'C'D'$ intersects the mid-point of ad .

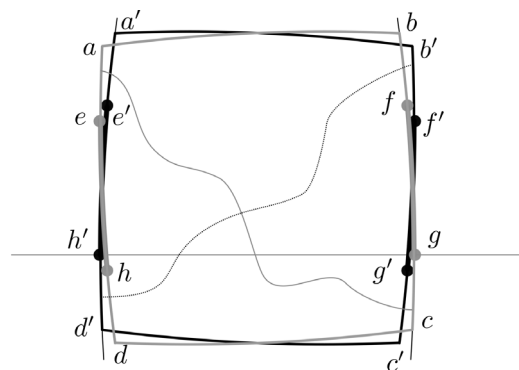


Fig. 9. Two $(2s/3) \times (2s/3)$ squares with marked intervals. The right side of the grey square $abcd$ is on the right side of the left $s \times s$ square $ABCD$, the left side of the black square is on the left side of the right square $A'B'C'D'$ (see Figure 8). Any curve connecting ae with cg (grey solid line) inside the grey square $abcd$ must intersect a curve connecting $h'd'$ with $b'f'$ (solid black line) inside the black square $a'b'c'd'$.

We turn to the second case. Since $\alpha_s \leq 2\alpha_{2s/3}$ and in light of Lemma 4.3, in this case we have lower bounds on both $\mathbb{P}[\mathcal{X}_{2s/3}(\alpha_{2s/3})]$ and $\mathbb{P}[\mathcal{H}_s(0, 2\alpha_{2s/3})]$ of the form $a_2 \cdot c_0^{k_2}$. Hence, by the FKG property, it suffices to construct a finite collection of $s \times s$ squares S_i , and $(2s/3) \times (2s/3)$ squares T_i , such that if $\mathcal{H}_s(0, 2\alpha_{2s/3})$ and $\mathcal{X}_{2s/3}(\alpha_{2s/3})$ holds for each S_i and T_i respectively, then so does $\mathcal{C}_{\tilde{B}(s)}$.

In what follows we refer to the labelling in Figures 7–9, which illustrate the argument in the spherical case. Consider the $s \times s$ square $ABCD$ and its translation $A'B'C'D'$ by a distance d (to be determined) along the geodesic joining

the mid-points of the sides AD and BC . Our aim is to deduce a horizontal crossing of the union of these squares (i.e. between AD and $B'C'$) by assuming $\mathcal{H}_s(0, 2\alpha_{2s/3})$ holds for both the squares, and assuming also $\mathcal{X}_{2s/3}(\alpha_{2s/3})$ holds for two suitably chosen $(2s/3) \times (2s/3)$ squares. In the planar case [17] we may let $d = 4s/3$, since then the shaded region in Figure 7 forms a $(2s/3) \times (2s/3)$ square which is sufficient for this purpose. In the spherical case this shaded region is not a square for any choice of translation distance, and so we shall need a slightly different construction.

We consider the $(2s/3) \times (2s/3)$ square $abcd$ such that its ‘right’ side bc lies on BC with its mid-point coinciding with the mid-point of the marked thick interval fg of length $2\alpha_{2s/3}$ (see Figure 8). Note that bc is a subset of BC since $\alpha_s \leq s/4$ for each s , and hence $s/3 + \alpha_{2s/3}$ is at most $s/2$. Once this square is fixed, we consider the unique geodesic which passes through the middle of the side ad of the small square $abcd$ and orthogonal to the geodesic connecting mid-points of AD and BC . We define the second $s \times s$ square $A'B'C'D'$ to be the square such that its left side is on this geodesic. The second $(2s/3) \times (2s/3)$ square $a'b'c'd'$ (not shown in Figure 8, but magnified in Figure 9) is constructed as the symmetric image of $abcd$ and its left side is on $A'D'$. Observe that the mid-point of ad (marked by a dot) will lie on the marked interval $e'h'$. We also notice, that since the distance between the mid-points of the opposite sides of a $(2s/3) \times (2s/3)$ square is $2s/3 + O(s^3)$ where $O(s^3)$ term depends on s only, the square $A'B'C'D'$ is a copy of $ABCD$ shifted by $d = s/3 + O(s^3)$. In particular, there is s^* such that for all $s < s^*$ it holds that $d \in (s/4, s/2)$.

Next let us consider the two small squares $abcd$ and $a'b'c'd'$, shown in more detail in Figure 9. We mark the middle parts of length $2\alpha_{2s/3}$ on ‘vertical’ sides of both small squares. Two of these marked intervals fg and $e'h'$ are the marked intervals on Figures 7 and 8. As mentioned above, the intervals eh and $e'h'$ intersect. By symmetry, the intervals fg and $f'g'$ intersect as well. This implies that any curve in $abcd$ connecting ae to cg must intersect $a'h'$ and $c'f'$ and thus disconnect $h'd'$ from $b'f'$ inside $a'b'c'd'$. This shows that if $\mathcal{X}_{2s/3}(\alpha_{2s/3})$ holds for both squares, then the connecting curves must intersect.

We also notice that a curve connecting $a'e'$ with $h'd'$ separates $e'h'$ from the right side of the $s \times s$ square $A'B'C'D'$. Similarly, a curve connecting bf and cg separates fg from AD . This implies that in the event that there are crossings from AD to fg , from $e'h'$ to $B'C'$, and two X-crossings in $abcd$ and $a'b'c'd'$ there is a crossing connecting AD to $B'C'$.

All in all, we infer a horizontal crossing of the union of the $s \times s$ squares (i.e. between AD and $B'C'$) that are translated a distance $d \in (s/4, s/2)$ apart. We finish the proof of Lemma 4.4 by using a similar construction to the one in the proof of Lemma 4.1, using multiple copies of such a crossing (i.e. alternating ‘horizontally’ and ‘vertically’) and as long as s^* is sufficiently small, to infer a crossing of $\bar{B}(2s)$. \square

Lemma 4.5 (Good scales imply annular crossings on larger scales; cf. [17, Lemma 3.1]). *There exists a sufficiently small $s^* > 0$ and absolute numbers $a_3 > 0$ and $k_3 \in \mathbb{N}$, such that if, for some s and t such that $12s \leq t < s^*$, $\alpha_s \leq 2\alpha_{2s/3}$ and $\alpha_t \leq s$ both hold, then $\mathbb{P}(C_{A_{t,6t}}) > a_3 \cdot c_0^{k_3}$.*

Proof. Since $\alpha_s \leq 2\alpha_{2s/3}$, Lemma 4.4 yields a lower bound on $\mathbb{P}[C_{\bar{B}(2s)}]$ of the form $a_3 \cdot c_0^{k_3}$. By Lemma 4.1 we then conclude the same for $C_{A_{2s,12s}}$. Since also $\mathbb{P}[\mathcal{H}_t(0, s)] \geq c_0/4$ (implied by $\alpha_t \leq s$), by the FKG property it suffices to find two $t \times t$ squares S_1 and S_2 such that if $\mathcal{H}_t(0, s)$ holds for S_1 and S_2 , and $C_{A_{2s,12s}}$ also holds, then we may deduce $C_{\bar{B}(t)}$.

The proof is identical to [17, Lemma 3.1], and we only briefly sketch it. Consider the two $t \times t$ squares $S_1 = (D_1; \gamma_1, \gamma')$ and $S_2 = (D_2; \gamma_2, \gamma')$ whose common side γ' has the origin as its mid-point. Observe that if $C_{A_{2s,12s}}$ holds simultaneously with the event $\mathcal{H}_t(0, s)$ for S_1 and S_2 , then there exists a crossing of $S_1 \cup S_2$; for this observe that the distance between mid-points of a $s \times s$ square is at least s (in the spherical case it is precisely $\arccos(1 - 2\tan^2(s/2)) = s + \frac{1}{8}s^3 + O(s^5) > s$), and so the line-segment of length s in the definition of $\mathcal{H}_t(0, s)$ lies inside the inner square bounding $A_{2s,12s}$. Since such a crossing of $S_1 \cup S_2$ also implies a crossing of two squares that are translated by any smaller amount along the geodesic joining the mid-points of the opposite sides of S_1 and S_2 , we infer a crossing of $\bar{B}(t)$. Finally, from Lemma 4.1 we deduce the statement. \square

To state the final lemma in Tassion’s argument, we need a certain assumption that is related to condition (6) of Theorem 2.8, stated for a fixed $n \in \mathbb{N}$.

Assumption 4.6. For a quadruple (c, ε, C, s) with $c > 0$, $\varepsilon \in (0, 1)$, $C \geq 1$ and $s > 0$, we assume that the following holds: If

$$\inf_{A \in \text{Ann}_{X; C, 6}(s)} \mathbb{P}(C_A(\mathcal{S}_n)) > c,$$

then, for each $s \times Cs$ annulus $A \subseteq X$,

$$\mathbb{P}(C_A) > 1 - \varepsilon.$$

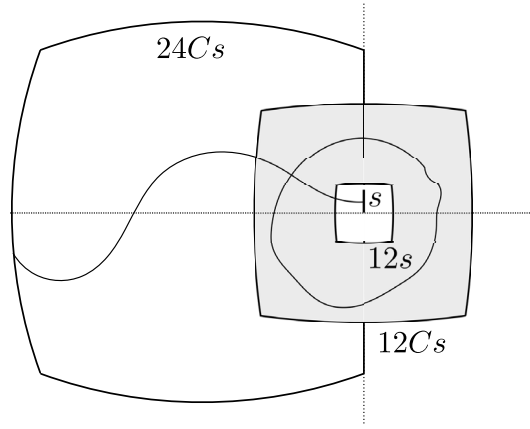


Fig. 10. A crossing from the left side of the white square to a small interval of length s on the right side (event \mathcal{A}) intersects with a circuit in the grey annulus (event \mathcal{B}), which implies a crossing from the left side of the white square to the part of the right side lying above the interval (event \mathcal{C}). Hence, if the event $\mathcal{A} \setminus \mathcal{C}$ occurs, then the event \mathcal{B} does not. The labels refer to the sizes of the squares in the proof of Lemma 4.7.

It is clear that if Assumption 4.6 is valid for a quadruple (c, ε, C, s) , then it is also valid for the quadruple (c', ε', C, s) for any $c' > c$ and $\varepsilon' > \varepsilon$. For the final two lemmas, we let a_3 and k_3 be the constants proscribed by Lemma 4.5 and fix

$$c_3 = a_3 \cdot c_0^{k_3}. \quad (4.3)$$

Lemma 4.7 (Good scales imply larger good scales; cf. [17, Lemma 3.2]). Fix $C > 1$. Then there exist a number $C_1 > 12$, depending only on C , and a sufficiently small $s^* > 0$, such that if $s < s^*$, $\alpha_s \leq 2\alpha_{2s/3}$ and Assumption 4.6 holds for the quadruple $(c_3, c_0/8, C, 12s)$, then there exists a number $t \in [12s, C_1s]$ such that $\alpha_t \leq 2\alpha_{2t/3}$.

Proof. Suppose $s < s^*$ and $\alpha_s \leq 2\alpha_{2s/3}$. The first step is to show that $\alpha_{t_i} > s$ for at least one of $t_1 = 12s$ or

$$t_2 = 2Ct_1 = 24Cs. \quad (4.4)$$

Let A be a $t_1 \times 12Ct_1$ annulus. Arguing by contradiction, if $\alpha_{t_1} \leq s$, then from Lemma 4.5 we deduce that

$$\inf_{A \in \text{Ann}_{X; C; 6}(t_1)} \mathbb{P}(\mathcal{C}_A) > c_3.$$

Hence, since we make Assumption 4.6 for the quadruple $(c_3, c_0/8, C, t_1)$, it holds that

$$\mathbb{P}(\mathcal{C}_A) > 1 - c_0/8.$$

On the other hand, let S be a $t_2 \times t_2$ square whose centre coincides with the centre of A and such that one side of S lies on a geodesic bisecting A . Define the event $E = \mathcal{H}_{t_2}(0, s) \setminus \mathcal{H}_{t_2}(s, t_2/2)$ for square S , and remark that the occurrence of the event E implies that \mathcal{C}_A does not occur (see Figure 10). Recalling the definition (4.2) of $\phi_s(\alpha)$ it is clear that $\mathbb{P}(E) \geq \phi_{t_2}(s)$. If also $\alpha_{t_2} \leq s$, this implies $\alpha_{t_2} < t_2/4$ by (4.4), and by (P2) of Lemma 4.3 we have $\phi_{t_2}(s) \geq c_0/4$. Since $E \subseteq \mathcal{C}_A^c$, this shows that $\mathbb{P}(\mathcal{C}_A)$ is at most $1 - c_0/4$, which is a contradiction.

To conclude the proof of Lemma 4.7, recall that α_s is sub-linear in the sense that $\alpha_s < s$ for all $s > 0$. Hence if $\alpha_{t_i} > s$ for at least one of $t_1 = 12s$ or $t_2 = 24Cs$, then there exist sufficiently large C_1 , depending only on C , such that $\alpha_t \geq 2\alpha_{2t/3}$ for at least one $t \in [12s, C_1s]$. \square

To conclude this section we combine the preceding lemmas into a form most convenient for completing the proof of Theorem 2.8.

Corollary 4.8. Fix the constants $C > 0$ and $\bar{c}_1, \bar{c}_2 > 0$. Then there exists a sufficiently small $s^* > 0$ and numbers $a_4 > 0$, $k_4 \in \mathbb{N}$ and $C_1 > 0$, depending only on C , \bar{c}_1 and \bar{c}_2 , such that, if $s < s^*$, $\alpha_s > \bar{c}_1 s$, and Assumption 4.6 holds for the quadruple $(c_3, c_0/8, C, t)$ for all $t > s$, then

$$\inf_{s' > C_1 s} \inf_{B \in \text{Box}_{X; \bar{c}_2}(s')} \mathbb{P}(\mathcal{C}_B) > a_4 \cdot c_0^{k_4}.$$

Proof. In light of lemmas 4.1 and 4.4, it suffices to exhibit constants $C_1, C_2 > 0$, depending only on C and \bar{c}_1 , and a sequence of ‘good’ scales $\{s^{(i)}\}_{1 \leq i \leq k}$ such that

$$s^{(1)} < C_1 s / 6, \quad 12 \leq s^{(i+1)} / s^{(i)} \leq C_2 \quad \text{and} \quad 12s^{(k)} \leq s^*,$$

and such that $\alpha_{s^{(i)}} \leq 2\alpha_{2s^{(i)}/3}$ holds for each $1 \leq i \leq k$. We argue by induction. For the base case, we argue as in the proof of Lemma 4.7: since $\alpha_s > \bar{c}_1 s$ and α_s is sub-linear (in the sense that $\alpha_s \leq s/4$ for all s), there exists a sufficiently large C_1 , depending only on C and \bar{c}_1 , such that $\alpha_t \leq 2\alpha_{2t/3}$ for at least one $t \in [12s, C_1 s / 6]$. Next suppose we have a scale $s^{(i)}$ such that $s^{(i)} < s^*$ and $\alpha_{s^{(i)}} \leq 2\alpha_{2s^{(i)}/3}$. We may suppose that Assumption 4.6 holds for the quadruple $(c_3, c_0/8, C, 12s^{(i)})$. Hence by Lemma 4.7 there exists a number $t \in [12s^{(i)}, 12Cs^{(i)}]$ such that $\alpha_t \leq 2\alpha_{2t/3}$, which concludes the induction step, and thus also Corollary 4.8. \square

4.3. Concluding the proof of Theorem 2.8

Fix $s^* > 0$ to be sufficiently small such that the conclusions of Lemma 4.1 and Corollary 4.8 are valid. Before continuing, we discuss the roles of conditions (1) and (6) of Theorem 2.8 in ensuring that the conclusion of Corollary 4.8 holds on all necessary scales and is uniform for sufficiently large n .

We first claim that (2.4) in condition (1) implies that, for each $C > 0$, α_{Cs_n}/s_n is uniformly bounded from below. In fact, we prove the stronger statement that $\alpha_{Cs_n} > rs_n$ for any $r \in (0, C/4)$ such that,

$$\mathbb{P}(\mathcal{L}_v(rs_n) \cap \partial S = \emptyset) > 1 - c_0/4 \quad (4.5)$$

for all directions v in the spherical case (resp. x and y directions in the toral case); the existence of a single such $r > 0$ for n sufficiently large is then guaranteed by (2.4). Similarly to the proof of Lemma 4.7, consider the event

$$E = \mathcal{H}_{Cs_n}(0, rs_n) \setminus \mathcal{H}_{Cs_n}(rs_n, Cs_n)$$

corresponding to a $Cs_n \times Cs_n$ square S , and let L denote the line-segment of length rs_n on the boundary of S used to define the event $\mathcal{H}_{Cs_n}(0, rs_n)$. It is then clear that $\mathbb{P}(E) \geq \phi_{Cs_n}(rs_n)$. If we now assume, for contradiction, that (4.5) holds and $\alpha_{Cs_n} \leq rs_n$, then since $r < C/4$, by (P2) of Lemma 4.3 it must be true that $\phi_{Cs_n}(rs_n) \geq c_0/4$. Since E implies that ∂S intersects L , we have that

$$\mathbb{P}(\mathcal{L}_v(rs_n) \cap \partial S = \emptyset) = \mathbb{P}(|L \cap \partial S| = \emptyset) \leq 1 - c_0/4,$$

which is a contradiction.

Next we observe that condition (6) of Theorem 2.8 implies Assumption 4.6 on all necessary scales. To see why note that condition (6) guarantees the existence, for any choice of $c > 0$ and $\varepsilon > 0$, of constants $C_1, C_2 > 1$ such that, for all sufficiently large n and all $s > C_1 s_n$, Assumption 4.6 holds for the quadruple (c, ε, C_2, s) ; in particular it also holds for any larger c and ε (see the remark immediately after Assumption 4.6). Recall now that

$$c_0(n) = \inf_{s>0} \inf_{B \in \text{Box}_{X;1}(s)} \mathbb{P}(\mathcal{C}_B(\mathcal{S}_n)),$$

is bounded from below by some constant \hat{c}_0 for sufficiently large n ; hence, by (4.3), the same is true for the number $c_3(n)$ prescribed by Lemma 4.5, monotonically increasing in c_0 . Putting this together, condition (6) guarantees the existence of $C_1, C_2 > 0$ such that, for all sufficiently large n , Assumption 4.6 holds for the quadruple $(c_3(n), c_0(n)/8, C_2, s)$ for all $s > C_1 s_n$. At this point we may fix such $C_1, C_2 > 0$ and n sufficiently large such that the assumption holds for all $s > C s_n$.

We can now finish the proof of Theorem 2.8. Choose $c > 0$ as in the statement of the RSW estimates. Given the definition of $\text{Unif}_{X;c}(s)$, and since the FKG property is valid in X , it is sufficient to show the existence of a constant c_1 such that for sufficiently large n ,

$$\inf_{s>0} \inf_{k \in (0,c)} \inf_{B \text{ a } s \times ks \text{ box}} \mathbb{P}(\mathcal{C}_B(\mathcal{S}_n)) > c_1. \quad (4.6)$$

In turn, it is sufficient to establish (4.6) on both the microscopic scales $s \approx s_n$, and then for all larger scales $s \gg s_n$.

For the microscopic scales $s \approx s_n$, recall that, by condition (4) of Theorem 2.8, there exist numbers $\delta > 0$ and $c_2 > 0$ such that, for all sufficiently large n ,

$$\inf_{s < \delta s_n} \inf_{k \in (0,c)} \inf_{B \text{ a } s \times ks \text{ box}} \mathbb{P}(\mathcal{C}_B(\mathcal{S}_n)) > c_2.$$

By Lemma 4.1, the same conclusion holds for δ replaced by any constant C , i.e. there exists a c_4 , depending on C , such that for sufficiently large n ,

$$\inf_{s < C s_n} \inf_{k \in (0, c)} \inf_{B \text{ a } s \times ks \text{ box}} \mathbb{P}(C_B(\mathcal{S}_n)) > c_4. \quad (4.7)$$

For the larger scales $s \gg s_n$, take the constant C_1 that was fixed above, and recall that $\alpha_{C_1 s_n}/s_n$ is uniformly bound below by some constant \bar{c}_1 . Since also Assumption 4.6 holds for the quadruple $(c_3(n), c_0(n)/8, C_2, t)$ for all $t > C_1 s_n$, by Corollary 4.8 there are numbers $a_4 > 0$, $k_4 \in \mathbb{N}$ and $C_3 > 0$, depending only on c , \bar{c}_1 , C_1 and C_2 , such that

$$\inf_{s' > C_3 s_n} \inf_{B \in \text{Box}_{X; c}(s')} \mathbb{P}(C_B) > a_4 \cdot c_0^{k_4}(n) > a_4 \cdot \hat{c}_0^{k_4},$$

which establishes (4.6) for $s > C_3 s_n$. Combining with (4.7) we conclude the proof.

5. Perturbation analysis

In this section we establish the auxiliary results used in the perturbation analysis in Section 3. In the first part we prove Proposition 3.2, showing that crossing events are determined, outside a small error event, by the signs of the field on a (deterministic) finite set of points. In the second part we prove Lemma 3.3, which controls the effect of a perturbation on the signs of Gaussian vectors.

5.1. Measurability of crossing events on a finite number of points

We use the following preliminary lemma, which bounds the probability that the nodal set crosses any (geodesic) line-segment twice. Recall that for symmetric covariance kernels we often abuse notation by writing $\kappa_n(x) = \kappa_n(0, x)$.

Lemma 5.1 (Two-point estimate of nodal crossings; cf. [5, Proposition 4.4]). *Let f be a Gaussian random field on \mathbb{X} whose covariance kernel κ is C^4 and is symmetric in the sense of Definition 2.2. Suppose that there exists $\delta > 0$ such that, for every $x, y \in \mathbb{X}$ with $0 < d(x, y) < \delta$ the random vector $(f(x), f(y)) \in \mathbb{R}^2$ is non-degenerate. Define*

$$L_2 = \sup_{v \in \mathbb{S}^1} |\kappa_v''(0)| \quad \text{and} \quad L_4 = \sup_{v \in \mathbb{S}^1} \max_{d(0, y) < \delta} |\kappa_v^{(iv)}(y)|,$$

where $\kappa_v^{(ii)}$ and $\kappa_v^{(iv)}$ are the second and the fourth derivatives of κ in direction v respectively. Then there exists a absolute constant $c > 0$ such that, for each geodesic line-segment $\mathcal{L} \subseteq \mathbb{X}$ of length $\varepsilon < \delta$,

$$\mathbb{P}(|\{x \in \mathcal{L} : f(x) = 0\}| \geq 2) < c\varepsilon^3 \sqrt{L_2^3 + L_2^{-1} L_4^2}.$$

Proof. It is convenient to use the arc-length parametrisation of \mathcal{L} , namely let $\tilde{f} : [-\varepsilon/2, \varepsilon/2] \rightarrow \mathbb{R}$ be the restriction $f|_{\mathcal{L}}$ of f to \mathcal{L} , and denote by $\tilde{\kappa} : [-\varepsilon/2, \varepsilon/2] \rightarrow \mathbb{R}$ its covariance kernel. By the symmetry assumption on f , the process \tilde{f} is stationary, and with no loss of generality we may assume that \tilde{f} has unit variance.

Let $N = |\{x \in \mathcal{L} : f(x) = 0\}|$. Applying the Kac–Rice formula [2, Theorem 6.3], valid by the non-degeneracy assumption on $(f(x), f(y))$ in Lemma 5.1 we have

$$\mathbb{E}[N(N-1)] = \int_{x, y \in [-\varepsilon/2, \varepsilon/2]} M_2(x-y) dx dy \quad (5.1)$$

with $M_2(x) \geq 0$ the two-point correlation function of the zeros of \tilde{f} . It is known [8, Corollary 2.5] that M_2 is given by

$$M_2(x) = \frac{1}{\pi^2} \frac{-\tilde{\kappa}''(0) \cdot (1 - \tilde{\kappa}(x)^2) - \tilde{\kappa}'(x)^2}{(1 - \tilde{\kappa}(x)^2)^{3/2}} \cdot (\sqrt{1 - \rho(x)^2} + \rho(x) \cdot \arcsin \rho(x)),$$

with ρ an explicit expression in terms of $\tilde{\kappa}$ and its first two derivatives, irrelevant for our purpose. The upshot is that the function

$$t \mapsto \sqrt{1-t^2} + t \cdot \arcsin t$$

is bounded from above, hence

$$M_2(x) \leq c_1 \cdot \frac{-\tilde{\kappa}''(0) \cdot (1 - \tilde{\kappa}(x)^2) - \tilde{\kappa}'(x)^2}{(1 - \tilde{\kappa}(x)^2)^{3/2}} \quad (5.2)$$

for some absolute constant $c_1 > 0$.

Finally, recall that κ is C^4 , and so Taylor's theorem implies that each $x \in [0, \varepsilon]$ satisfies,

$$\left| \tilde{\kappa}(x) - 1 - \frac{1}{2}\tilde{\kappa}'(0)x^2 \right| \leq \max_{y \in B(\delta)} |\tilde{\kappa}^{(iv)}(y)| x^4 \quad \text{and} \quad |\tilde{\kappa}'(x) - \tilde{\kappa}'(0)x| \leq \max_{y \in B(\delta)} |\tilde{\kappa}^{(iv)}(y)| x^3. \quad (5.3)$$

Expanding (5.2) into the Taylor polynomial of fourth degree around the origin with the help of (5.3), we obtain the bound

$$M_2(x) \leq c_2 \left(\tilde{\kappa}'(0)^{3/2} + \tilde{\kappa}'(0)^{-1/2} \max_{y \in B(\delta)} \tilde{\kappa}^{(iv)}(y) \right) |x|$$

with some absolute constant $c_2 > 0$. Finally, integrating the latter inequality over $x, y \in [-\varepsilon/2, \varepsilon/2]$ as in (5.1) yields that

$$\mathbb{E}[N(N-1)] < c_3 \varepsilon^3 \left(\tilde{\kappa}'(0)^{3/2} + \tilde{\kappa}'(0)^{-1/2} \max_{y \in B(\delta)} \tilde{\kappa}^{(iv)}(y) \right),$$

with $c_3 > 0$ absolute. Since \mathbb{X} has constant curvature, the ratio of the derivatives of $\tilde{\kappa}$ and κ are bounded from above and from below by absolute constants, and so by Markov's inequality we conclude the proof. \square

We now state the main implication of Lemma 5.1 in our setting. Recall the set-up of the perturbation analysis from Section 3, and in particular the constant δ_0 and the limit kernel K_∞ . The following is an easy corollary of Lemma 5.1, the uniform convergence of κ_n on $B(\delta_0)$ to K_∞ along with its first four derivatives, and the fact that K_∞ satisfies Assumption 2.3 (and so in particular has strictly-positive second derivatives at the origin); by the above we can take a single number $\delta > 0$ satisfying the assumptions of Lemma 5.1 applied to $f = f_n$ for n sufficiently large (i.e. the δ corresponding to K_∞).

Corollary 5.2. *There exists a number $0 < \delta < \delta_0$ sufficiently small, and $c_1 > 0$ sufficiently large depending on K_∞ only, such that for $n \in \mathbb{N}$ sufficiently large the following holds. For every geodesic line-segment $\mathcal{L} \subseteq \mathbb{X}$ of length $\ell \in (0, \delta)$,*

$$\mathbb{P}(|\{x \in \mathcal{L} : f_n(x) = 0\}| \geq 2) < c_1 (\ell/s_n)^3.$$

We can now complete the proof of Proposition 3.2. For this we will use the following notion of a ‘triangular decomposition’ of a polygon.

Definition 5.3.

- (1) For a polygon $P = (D; \gamma, \gamma')$ as in Definition 3.1, a *triangular decomposition* \mathbb{T} of P is a (finite) embedded graph on $\mathbb{X} \cap P$ such that each edge is a geodesic line-segment, each face has three boundary edges, and the union of the faces equals P , save for boundaries.
- (2) A triangular decomposition \mathbb{T} of a polygon P is said to be *compatible* with P if both γ and γ' can be expressed as the union of edges of \mathbb{T} .
- (3) A *triangular decomposition* of an annulus A as in Definition 2.7 is defined analogously.

Proof of Proposition 3.2. Fix $n \in \mathbb{N}$ sufficiently large, $\delta > 0$ sufficiently small and c_1 sufficiently large, so that the conclusion of Corollary 5.2 holds, and fix also $c, r > 1$ as in the statement of Proposition 3.2. Let $s > 0$, $\varepsilon \in (0, 1)$ and $Q \in \text{Poly}_{\mathbb{X};c}(s) \cup \text{Ann}_{\mathbb{X};c;r}(s)$ be given. By the definition of the sets $\text{Poly}_{\mathbb{X};c}(s)$ and $\text{Ann}_{\mathbb{X};c}(s)$, there exists a number $c_2 > 0$, depending only on c and r , such that for each $\ell \in (0, s \wedge \delta]$ there exists a triangular decomposition \mathbb{T} of Q with the following properties: (i) if $Q \in \text{Poly}_{\mathbb{X};c}(s)$ then \mathbb{T} is compatible with Q ; (ii) the edges of \mathbb{T} have length at most ℓs ; and (iii) \mathbb{T} has at most $c_2(s/\ell)^2$ vertices.

Fix an edge e in \mathbb{T} and consider the event that e is crossed at least twice by the nodal set. Applying Corollary 5.2, there exists a constant c_2 , depending only on K_∞ , such that this event is of probability at most $c_2(\ell/s_n)^3$. By the union bound, the event E that *all* the edges of \mathbb{T} are crossed at most once by the nodal set has probability bounded from below by

$$1 - c_1 c_2 (s/\ell)^2 (\ell/s_n)^3 = 1 - c_1 c_2 s^2 s_n^{-3} \ell.$$

Setting

$$\ell = \min\{\delta, s, \varepsilon s_n^3 / (c_1 c_2 s^2)\},$$

this is bounded from below by $1 - \varepsilon$. Moreover, with this choice of ℓ , the cardinality of \mathcal{P} is at most

$$|\mathcal{P}| \leq c_2 \max\{\delta^{-2} s^2, 1, (c_1 c_2)^2 \varepsilon^{-2} (s/s_n)^{-6}\}.$$

Since the sets $\text{Poly}_{\mathbb{X};c}(s)$ and $\text{Ann}_{\mathbb{X};c}(s)$ are empty unless s is less than a constant (2π in the spherical case, 1 in the toral case), this in turn is bounded from above by

$$|\mathcal{P}| \leq c_3 (\varepsilon^{-2} (s/s_n)^{-6} \wedge 1),$$

where $c_3 > 0$ is a constant depending only on c, r, δ and K_∞ . Finally, observe that on the event E the crossing event $\mathcal{C}_Q(\mathcal{S}_n^+)$ is determined by the signs of f_n on the vertices of the triangular decomposition \mathbb{T} , which concludes the proof. \square

5.2. Proof of Lemma 3.3

We begin with the first statement. Define the matrices

$$\Sigma_Z = n\delta I_n \quad \text{and} \quad \Sigma_W = n\delta I_n + \Sigma_Y - \Sigma_X,$$

where I_n denotes the $n \times n$ identity matrix. By the Gershgorin circle theorem and the definition of δ , the matrix Σ_W is positive-definite. Hence

$$Y + Z \stackrel{d}{=} X + W,$$

where Z and W are independent Gaussian random vectors with respective covariance matrices Σ_Z and Σ_W .

Fix $\varepsilon > 0$ and define the events

$$\mathcal{E}_1 = \bigcup_{i=1}^n \{|Y_i| < \varepsilon\}, \quad \mathcal{E}_2 = \bigcup_{i=1}^n \{|Z_i| > \varepsilon\}, \quad \mathcal{E}_3 = \bigcup_{i=1}^n \{|X_i| < \varepsilon\} \quad \text{and} \quad \mathcal{E}_4 = \bigcup_{i=1}^n \{|W_i| > \varepsilon\}.$$

Observe that the variance of the components of Y and X are at least one, whereas the variance of the components of Z and W are at most $(n+1)\delta$. Hence by the union bound, standard results on the maximum of Gaussian vectors, and Markov's inequality, there exists an absolute number $c_1 > 0$ such that

$$\mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_3) < c_1 n \varepsilon \quad \text{and} \quad \mathbb{P}(\mathcal{E}_2) + \mathbb{P}(\mathcal{E}_4) < c_1 (\log n \vee 1)^{1/2} \varepsilon^{-1} ((n+1)\delta)^{1/2}.$$

This implies that we may couple the vectors X and Y so that, outside of an event of probability

$$< c_1 (n \varepsilon + (\log n \vee 1)^{1/2} \varepsilon^{-1} ((n+1)\delta)^{1/2}),$$

the signs of all the components of the vectors are equal, and hence all the events measurable w.r.t. the signs of the vectors have the same probability up to the said error. To optimise the result we set

$$\varepsilon = \delta^{1/4} (n+1)^{1/2} n^{-1/2} (\log n \vee 1)^{1/4},$$

which yields the error probability as

$$c_1 n^{1/2} (n+1)^{1/4} (\log n \vee 1)^{1/4} \delta^{1/4} < c_2 (n^{3+\eta} \delta)^{1/4},$$

for a constant c_2 depending only on $\eta > 0$.

For the second statement the argument is similar. Since $\Sigma_Y - \Sigma_X$ is positive-definite, one may write $Y \stackrel{d}{=} X + W$ where W is an independent Gaussian random vector with covariance matrix $\Sigma_Y - \Sigma_X$. Fix $\varepsilon > 0$ and let $\mathcal{E}_1, \mathcal{E}_3$ and \mathcal{E}_4 be defined as before. Since the variance of the components of W are at most δ , as before there exists an absolute $c_1 > 0$ such that

$$\mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_3) < c_1 n \varepsilon \quad \text{and} \quad \mathbb{P}(\mathcal{E}_4) < c_1 (\log n \vee 1)^{1/2} \varepsilon^{-1} \delta^{1/2}.$$

Hence we may couple the vectors X and Y so that, outside of an event of probability

$$< c_1(n\varepsilon + (\log n \vee 1)^{1/2}\varepsilon^{-1}\delta^{1/2}),$$

the signs of all the components of the vectors are equal. Setting

$$\varepsilon = \delta^{1/4}n^{-1/2}(\log n \vee 1)^{1/4},$$

the error is at most $c_2(n^{2+\eta}\delta)^{1/4}$ for a constant c_2 depending only on $\eta > 0$.

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