Multi-unit Bayesian Auction with Demand or Budget Constraints

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Abstract. In this work, we study multiple item auctions in a setting where items are distinguished by their relative values; any pair of items have the same ratio of values to all buyers. Buyers have additive valuations for multiple items. The application we have in mind is that items are *positions* in an ad auction, and an item's value corresponds to its click-through rate. Buyers have various per-click valuations, which is their private information. We consider a Bayesian model for the values of buyers on the positions.

We develop the optimal (maximum revenue) auction for a *relaxed demand* model (where each buyer i wants at most d_i items) and a *sharp demand* model (where buyer i wants exactly d_i items). We also find a 1/2 approximation for the case when the buyers are budget constrained.

1 Introduction

Internet markets have opened up many opportunities for applications of different marketing models. Search engine advertising, as an example, makes the matching market model practical: Advertisement slots of Google and Yahoo! are created as products for advertisers, the buyers, who want to display their ads to users searching keywords related to their business. Each such ad slot may be of different importance which is measured by the Click Through Rates (CTR), the average number of clicks on the ad placed at the slot for a unit time. Slots with higher CTR are more likely to be clicked by customers. On the other hand, user interest expressed by a click on an ad may have different values to different advertisers. Combining the two major factors, we have a standard model of sponsored search market [6, 12]. Note that the private information of the value for each advertiser creates an asymmetry among the participants and the market maker. Truthful market design relies on the general revelation principle [10] to simplify the search for mechanisms with desirable properties, such as one that brings in the maximum revenue. Therefore, our focus will be considering market mechanisms that bring in the (approximate) optimum revenue, and yet ensuring the participants' incentives to speak truth about their private values.

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Our study considers a more general setting where one advertiser may request more than one slot: in one version an advertiser requests a fixed number of slots; in the other version the request is for a number of slots up to some maximum. This is intended to model setting such as TV ads in TV networks [5].

Similar to the position auction, we consider the problem of assigning m slots or items to n buyers where buyer i's valuation for slot i is given by $v_i c_j$, where v_i is i's per-click value and c_j is position j's CTR. Differing from the standard sponsored search auction, we consider two extra constraints: the demand constraint and the budget constraint. These are natural considerations for advertisers. We study the problem for two types of demand constraint, the *relaxed* demand constraint (where each buyer i requests a number of positions not exceeding a given number d_i) and the *sharp* demand constraint (where each buyer buys his desired number of positions or buys nothing). The budget constraint requires that no buyer can pay more than his budget.

Instead of the standard deterministic model, we consider a Bayesian model for the private values v_i of the buyers, $i = 1, 2, \cdots, n$. The private value v_i of advertiser i follows a publicly known distribution. Therefore, an advertiser knows its exact per-click value v_i but other advertisers as well as the seller of the slots only know that v_i is generated by the given probability distribution. Therefore, we adapt Myerson's classic setting [11], where each buyer's private value is independently drawn from a publicly known distribution. We focus on truthful mechanisms, i.e., bidding his true value is a Bayesian dominant strategy for every buyer. We are interested in obtaining mechanisms to optimize or approximate the expected optimal revenue, taking into consideration the demand constraints and the budget constraints for all the buyers. Our main results are summarized as follows:

Theorem 1. *For the relaxed demand or the sharp demand case without budget constraints, an optimal mechanism can be constructed efficiently; for the case with the budget constraint but without demand constraint, a* 2*-approximate mechanism can be constructed efficiently.*

Related Work

The theoretical study of sponsored search under the generalized second price auction was initiated in [6, 12]. There has been a series of studies of position auctions in deterministic settings [9]. Our consideration of Bayesian settings of position auction fits in the general one dimensional auction design framework [7] with demand constraints and budget constraints. The work in [7] compares the VCG revenue with reserved prices versus optimal revenues in many one dimensional settings. For a single item with budget constraints, the problem is reduced to the case of single item without budget [4] where a 2-approximate mechanism was introduced. Their method does not directly apply to our problem.

Our study considers continuous distributions on buyers' values. For discrete distribution, Cai et al. [3] presents an optimal mechanism for budget constraint buyers without demand constraints in multi-parameter settings; for buyers with both budget constraints and demand constraints, 2-approximate mechanisms [1] and 4-approximate mechanisms [2] exist in the literature.

2 Preliminaries

Throughout the paper, we use the notation v_i to represent the ith buyer's value, $i = 1, 2, \cdots, n$ and c_j for the jth position's CTR, $j = 1, 2, \cdots, m$. Thus, the ith buyer's value for position j is $v_i c_j$. Since we consider truthful mechanisms in this work, the private value v_i is also the bid of buyer i. Therefore, we may use bid and value interchangeably when there is no ambiguity. The vector of all the buyers' bids is denoted by v (called the bid vector) or sometimes $(v_i; v_{-i})$ where v_{-i} is the joint bids of all bidders other than i. We also assume that all buyers' bids are distributed independently according to publicly known bounded distributions, i.e. $v_i \in [\underline{v}_i, \overline{v}_i]$ and $V = \prod_i [\underline{v}_i, \overline{v}_i]$. For each buyer i, let F_i be the Cumulative Distribution Function (CDF) of buyer i's value distribution and let f_i be the Probability Density Function (PDF) of this distribution. In addition, we assume that the concave closure or convex closure or integration of those functions can be computed efficiently.

A mechanism M consists of allocation and payment functions (X, p), that is, buyer i receives position j iff $X_{ij}(v) = 1$ and pays $p_i(v)$. Since the mechanism can be randomized, we use x_{ij} to denote $\Pr[X_{ij} = 1]$. Thus, the expected revenue of the mechanism is $Rev(M) = \mathrm{E}_v \left[\sum_i p_i(v)\right]$ where E_v denotes the expectation with respect to components of v sampled from their respective distributions.

In our model, a buyer may buy more than one position from the mechanism. The buyers' valuation functions are additive, i.e. $v_i(S) = \sum_{j \in S} v_{ij}$. Each buyer i is associated with an integer d_i related to its demand of the number of positions. We consider the following two kinds of constraints on buyers' demands.

Definition 1 (Relaxed Demand Constraint). *Buyer i's demand is relaxedly constrained by d_i if i may buy any number of positions up to a maximum d_i in this auction, i.e. $\sum_j X_{ij} \leq d_i$.*

Definition 2 (Sharp Demand Constraint). *Buyer i's demand is sharply constrained by d_i if i must buy exactly d_i positions in this auction or alternatively buys nothing, i.e. $\sum_j X_{ij} = 0$ or $\sum_j X_{ij} = d_i$.*

In addition to the demand constraints, we also consider the budget constraints. That is, buyers can be restricted by their budget in what they purchase.

Definition 3 (Budget Constraint). *Buyer i's budget is constrained by a publicly known number B_i if i cannot pay more than B_i, i.e. $p_i(v) \leq B_i$.*

Let $U_i(v) = \sum_j x_{ij}(v) v_{ij} - p_i(v)$ be the expected utility of buyer i when all buyers' bids are v. We proceed to define the Bayesian Incentive Compatibility of a mechanism and the ex-interim Individual Rationality:

Definition 4. *A mechanism M is called* Bayesian Incentive Compatible *(BIC) iff the following inequalities hold for all i, v_i, v_i'.*

$$\mathrm{E}_{v_{-i}}[U_i(v)] \geq \mathrm{E}_{v_{-i}}\left[\sum_j x_{ij}(v_i'; v_{-i})v_{ij} - p_i(v_i'; v_{-i})\right] \tag{1}$$

If $U_i(v) \geq \sum_j x_{ij}(v_i'; v_{-i})v_{ij} - p_i(v_i'; v_{-i})$, for all v, i, v_i', we say M is Incentive Compatible*.*

It is not hard to see that all Incentive Compatible mechanisms are Bayesian Incentive Compatible.

Definition 5. *A mechanism M is called* ex-interim Individual Rational *(IR) iff the following inequalities hold for all i, v_i.*

$$\mathrm{E}_{v_{-i}}[U_i(v)] \geq 0 \tag{2}$$

If $U_i(v) \geq 0$ for all v, i, we say M is ex-post Individual Rational*.*

Obviously, an ex-post Individual Rational mechanism must be ex-interim Individual Rational.
Finally, we present the formal definition of approximate mechanism.

Definition 6 (α-approximate Mechanism). *Given any feasible mechanism set $\mathbb{M}$, we say a mechanism M is an α-approximate mechanism in $\mathbb{M}$ iff $M \in \mathbb{M}$ and for each mechanism $M' \in \mathbb{M}$, $Rev(M') \leq \alpha \cdot Rev(M)$. We say a mechanism is optimal in $\mathbb{M}$ if it is a 1-approximate mechanism in $\mathbb{M}$.*

3 Optimal Mechanism for Demand Constraints

In this section, we will describe an optimal (in revenue) mechanism among all feasible mechanisms that are Bayesian Incentive Compatible, ex-interim Individual Rational and satisfying buyers' demand constraints (relaxed or sharp). We will show the optimal auctions for the relaxed demand constraints and sharp demand constraints in Section 3.1 and 3.2 respectively. Our mechanism is inspired by Myerson's prominent optimal auction in [11] which has proven fruitful for the one-dimensional setting.

3.1 Relaxed Demand Constraints

Recall that a mechanism $M = (X, p)$ satisfies the relaxed demand constraint d_i for buyer i iff $\sum_j (X_{ij}) \leq d_i$. By the Birkhoff-Von Neumann theorem [8], it suffices to satisfy $\sum_j (x_{ij}) \leq d_i$ where $x_{ij} = \Pr[X_{ij} = 1]$. Thus, a mechanism M can be represented by (x, p). Let $U_i(x, p, v)$ denote the utility of buyer i if the mechanism is (x, p) and all buyers' profile is v. Hence, we formalize our problem as the following optimization problem denoted by RELAXED.

$$
\begin{aligned}
\text{Maximize:} \quad & \sum_{i\in[n]} \mathrm{E}_v[p_i(v)] \\
\text{s.t.} \quad & \sum_j x_{ij}(v) \le d_i, \sum_i x_{ij}(v) \le 1, x_{ij}(v) \ge 0 && \forall i, j, v \quad (*) \\
& \mathrm{E}_{v_{-i}}[U_i(x, p, v)] \ge \mathrm{E}_{v_{-i}}[\sum_j v_{ij} x_{ij}(v_i'; v_{-i}) - p_i(v_i'; v_{-i})] && \forall\ i, v, v_i' \\
& \mathrm{E}_{v_{-i}}[U_i(x, p, v)] \ge 0 && \forall i, v
\end{aligned}
\tag{RELAXED}
$$

Now let g_i be the virtual valuation function for buyer i, i.e., $g_i(t) = t - \frac{1-F_i(t)}{f_i(t)}$. We assume $g_i(t)$ is monotone increasing. This assumption is without loss of generality because if $g_i(t)$ is not monotone, Myerson's ironing technique can be utilized to make $g_i(t)$ monotone — it is here that we invoke our assumption that we can efficiently compute the convex closure of a continuous function.

For convenience of presentation, let $q_i(v) = \sum_j x_{ij}(v)c_j$ and $Q_i(v_i) = \mathrm{E}_{v_{-i}}[q_i(v_i; v_{-i})]$. Similar to Myerson's work, we have the following lemma.

Lemma 1. *Suppose x is the function that maximizes*

$$\mathrm{E}_v \left[\sum_{i\in[n]} \sum_{j\in[m]} c_j g_i(v_i) x_{ij}(v) \right]$$

subject to the constraints that $Q_i(v_i)$ is non-decreasing monotone and inequalities $(*)$. Suppose also that

$$p_i(v) = v_i q_i(v) - \int_{\underline{v}_i}^{v_i} q_i(v_{-i}, s_i) ds_i \tag{3}$$

Then (x, p) represents an optimal mechanism for RELAXED*, the relaxed demand case.*

By Lemma 1, given buyers' bids v, the optimal mechanism always maximize $\sum_i \sum_j c_j g_i(v_i) x_{ij}(v)$. This problem can be solved by the following greedy algorithm.

Theorem 2. *The mechanism that applies the allocation rule according to Algorithm 1 and payment rule according to Equation (3) is an optimal mechanism for the position auction design problem with relaxed demand constrained buyers.*

Proof. By Lemma 1, it suffices to prove that $Q_i(v_i)$ is non-decreasing monotone. We prove strong monotonicity by proving $q_i(v_{-i}, v_i)$ is non-decreasing as v_i increases. In Algorithm 1, given v_{-i}, let $X_i(v_i)$ denote the set of positions allocated to buyer i when i declares v_i, then the monotonicity of $q_i(v_{-i}, v_i)$ is equivalent to $\sum_{j\in X_i(v_i)} c_j \le \sum_{j\in X_i(v_i')} c_j$ given $v_i' > v_i$. If $v_i' > v_i$, w.l.o.g.,

Input: Demands d_i, CDFs F_i, PDFs f_i, CTRs c_j and bids v
Output: Allocation x_{ij}
$g_i \leftarrow v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$;
Sort buyers with decreasing order on g_i;
Sort positions with decreasing order on c_j;
$x_{ij} \leftarrow 0$;
for *each buyer* i **do**
  for *each position* j **do**
    if $g_i > 0$ *and* $\sum_i x_{ij} < 1$ *and* $\sum_j x_{ij} < d_i$ **then**
      $x_{ij} \leftarrow 1$;
    end
  end
end
return x;

Algorithm 1: RELAXED

suppose $g_i(v_i) < g_i(v_i')$ (otherwise if $g_i(v_i) = g_i(v_i')$ since g_i is regular, then $X_i(v_i) = X_i(v_i')$ since the algorithm is deterministic, all done). Let M and M' denote the total quantities obtained by all the other buyers except buyer i in Algorithm 1 when buyer i bids v_i and v_i' respectively. Then, we have

$$\sum_{j \in X_i(v_i')} g_i(v_i')c_j + M' \geq^{(a)} \sum_{j \in X_i(v_i)} g_i(v_i')c_j + M$$
$$\geq^{(b)} \sum_{j \in X_i(v_i)} g_i(v_i)c_j + M \geq^{(c)} \sum_{j \in X_i(v_i')} g_i(v_i)c_j + M'.$$

(a) and (c) is due to the optimality of allocations found by the greedy Algorithm 1 when i bids v_i and v_i' respectively and (b) is due to $g_i(v_i) < g_i(v_i')$. From (a) and (c), we obtain

$$g_i(v_i')(\sum_{j \in X_i(v_i)} c_j - \sum_{j \in X_i(v_i')} c_j) \leq M' - M \leq g_i(v_i)(\sum_{j \in X_i(v_i)} c_j - \sum_{j \in X_i(v_i')} c_j).$$

Since $g_i(v_i) < g_i(v_i')$, $\sum_{j \in X_i(v_i)} c_j - \sum_{j \in X_i(v_i')} c_j \leq 0$. □

3.2 Sharp Demand Constraints

Recall that if a buyer is sharply constrained by d_i, he only wants to buy exactly d_i positions or nothing. Thus the only difference between this problem with RELAXED is that the inequalities $(*)$ should be replaced by the following inequalities.

$$\sum_j x_{ij}(v) = d_i y_i(v), \quad \sum_i x_{ij}(v) \leq 1, \quad x_{ij}(v) \geq 0, \quad y_i(v) \in \{0,1\} \quad \forall i,j,v \tag{**}$$

Similar to the relaxed demand case, we have the following lemma (recall that w.l.o.g. $g_i(t)$ is monotone non-decreasing).

Lemma 2. *Suppose x be the function maximizes*

$$\mathrm{E}_v \left[\sum_{i\in[n]} \sum_{j\in[m]} c_j g_i(v_i) x_{ij}(v) \right]$$

subject to the constraints that $Q_i(v_i)$ is non-decreasing monotone and inequalities $(**)$. Suppose also that

$$p_i(v) = v_i q_i(v) - \int_{\underline{v}_i}^{v_i} q_i(v_{-i}, s_i) ds_i.$$

Then (x, p) represents an optimal mechanism for the sharp demand case.

By Lemma 2, we observe that the optimal mechanism always maximizes $\sum_i \sum_j c_j g_i(v_i) x_{ij}(v)$ subject to sharp demand constraints. W.l.o.g., suppose $g_1(v_1) \geq g_2(v_2) \geq \cdots \geq g_n(v_n) > 0$ (otherwise allocating nothing to buyers i with $g_i(v_i) \leq 0$) and $c_1 \geq c_2 \geq \cdots \geq c_m$. If we view $c_j g_i(v_i)$ as weights between buyer i and position j, then the problem is equivalent to finding the maximum matchings, which can be solved by dynamic programming precisely.

Dynamic Programming
Let $w[i, j]$ denote the weight of the maximum weighted matchings with buyers $1, 2, \cdots, i$ and exactly all the positions indexed by $\{1, 2, \cdots, j\}$ being sold. Then we have the transition function,

$$w[i, j] = \max \left\{ w[i-1, j], w[i-1, j-d_i] + \sum_{k=j-d_i+1}^{j-d_i} g_i(v_i) c_k \right\}$$

Finding the maximum $w[i, j]$ over $i \in [n]$ and $j \in [m]$ gives the maximum weighted matchings and optimal solutions. We decribe the mechanism in Algorithm 2.

Theorem 3. *The mechanism which applies the allocation rule w.r.t. the above Dynamic Programming and payment rule w.r.t equation (3) is an optimal mechanism for position auction design problem with sharp demand constrained buyers.*

The proof of Theorem 3 is similar to the relaxed demand case.

4 Approximate Mechanism for Budget Constraints

In this section, we will present a 2-approximate mechanism for the position auction with budget constrained buyers. It should be noted that there is no demand constraints for all the buyers considered in this section. Recall that a mechanism

```
Input: Demands d_i, CDFs F_i, PDFs f_i, CTRs c_j and bids v
Output: Allocation x_ij
g_i ← v_i − (1−F_i(v_i))/f_i(v_i);
Sort buyers with decreasing order on g_i;
Sort positions with decreasing order on c_j;
w[i, j] ← −∞; w[0, 0] ← 0;
t[i, j] ← 0; x_ij ← 0;
for each buyer i with positive g_i do
    for each position j do
        tmp ← w[i − 1, j − d_i] + Σ_{k=j−d_i+1}^{j−d_i} g_i c_k;
        w[i, j] ← w[i − 1, j];
        if w[i, j] < tmp then
            t[i, j] ← 1;
            w[i, j] ← tmp;
        end
    end
end
w[i*, j*] = max_{i,j}{w[i, j]};
while i* > 0 do
    if t[i*, j*] = 1 then
        for each item k from j* − d_{i*} + 1 to j* do
            x_{i*,k} ← 1;
        end
        j* ← j* − d_{i*};
    end
    i* ← i* − 1;
end
return x;
```

Algorithm 2: SHARP

$M = (x, p)$ satisfies the buyer i's budget constraint iff $p_i(v) \leq B_i$ for all buyer profiles v. If $m = 1$, i.e. the auctioneer only has one slot, a 2- approximate mechanism has been suggested in [1] and [2]. Thus, our approach is to reduce the Position Auction to Single-item Auction, i.e. the case for $m = 1$. Recall that B_i denotes bidder i's budget and $x_{ij}(v)$ denote the probability of allocating the position j to the buyer when the buyers' bids revealed type is v. Then the position auction problem can be formalized as the following optimization problem.

$$
\begin{aligned}
\text{Max:}\quad & \mathrm{E}_v\left[\sum_i p_i(v)\right] & & \\
\text{s.t.}\quad & \sum_j v_i c_j x_{ij}(v) - p_i(v) \geq \sum_j v_i c_j x_{ij}(v_i', v_{-i}) - p_i(v'), & \forall v, i, v_i' \quad & \text{(BIC)} \\
& \sum_j v_i c_j x_{ij}(v) - p_i(v) \geq 0, & \forall v, i \quad & \text{(IR)} \\
& p_i(v) \leq B_i, & \forall v, i \quad & \text{(Budget)} \\
& x_{ij}(v) \geq 0 & \forall v, i, j \quad & \text{(Positive)} \\
& \sum_i x_{ij}(v) \leq 1 & \forall v, j \quad & \text{(Supply)}
\end{aligned}
$$

(Position)

Let $y_i(v) = \sum_j x_{ij}(v) c_j$. Observe that the condition (BIC) and (IR) can be rewritten as

$$v_i y_i(v) - p_i(v) \geq v_i y_i(v_i', v_{-i}) - p_i(v_i'; v_{-i})$$

$$v_i y_i(v) - p_i(v) \geq 0$$

Then we need to refine the constraints (Positive) and (Supply) to be represented by $y_i(v)$.

Lemma 3. *There exists x_{ij} such that $\sum_i x_{ij} \leq 1$, and $x_{ij} \geq 0$ iff y satisfies the condition that*

$$\sum_{i=1}^{n} y_i \leq \sum_{j=1}^{m} c_j, y_i \geq 0$$

Proof. For the only if direction, we have

$$\sum_i y_i = \sum_i \sum_j x_{ij} c_j = \sum_j c_j \sum_i x_{ij} \leq \sum_j c_j$$

For the if direction, given y_i, let $x_{ij} = y_i / \sum_k c_k$, then we have,

$$\sum_i x_{ij} = \frac{\sum_i y_i}{\sum_k c_k} \leq 1$$

□

Now consider the following single-item problem. Let $y_i(v)$ be the allocation function for bidder i and $q_i(v)$ be the payment function for bidder i.

$$\begin{aligned}
\text{Maximize:}\quad & \mathrm{E}_v\left[\sum_i q_i(v)\right] \\
\text{s.t.}\quad & v_i y_i(v) - q_i(v) \geq v_i y_i(v_i', v_{-i}) - q_i(v_i'; v_{-i}), && \forall v, i, v_i' \\
& v_i y_i(v) - q_i(v) \geq 0, && \forall v, i \\
& q_i(v) \leq B_i, && \forall v, i \\
& y_i(v) \geq 0 && \forall v, i \\
& \sum_i y_i(v) \leq \sum_j c_j && \forall v
\end{aligned} \qquad \text{(SINGLE)}$$

Our main observation for the above optimization problems is the following proposition.

Proposition 1. *The problems* POSITION *and* SINGLE *are equivalent:*

- *for any feasible mechanism* $\mathbf{M}(v) = (x(v), p(v))$ *of problem* POSITION*, the following mechanism* $\hat{\mathbf{M}}(v) = (y(v), p(v))$ *is a feasible mechanism for problem* SINGLE *where* $y_i(v) = \sum_j c_j x_{ij}(v)$.
- *for any feasible mechanism* $\hat{\mathbf{M}}(v) = (y(v), p(v))$ *of problem* SINGLE*, the following mechanism* $M(v) = (x(v), p(v))$ *is a feasible mechanism for problem* POSITION *where* $x_{ij}(v) = y_i(v)$.

Ultimately, we reduce the position auction design problem to the single-item auction design problem. By the results of [1] and [2], there exists a 2-approximate mechanism for problem SINGLE. Thus, we have a 2-approximate mechanism for problem POSITION.

Remark 1. For the discrete distribution case, Cai et al. [3] presents an optimal mechanism, for multi-buyers with multi-items. Their algorithm can be extended to the case where buyers are budget constrained but not demand constrained. Given buyers' discrete distribution and bid profiles, a revised version of their mechanism is an optimal mechanism and runs in in polynomial time of $\sum_i |T_i|$, where $|T_i|$ is the number of types of buyer i for all the items. Hence, restricting their results to position auction, that optimal mechanism is indeed an optimal mechanism for each buyer having a budget constraint but no demand constraint, with values independently drawn from discrete distribution, running in polynomial time of the input.

5 Conclusion

In this work, we study the optimal mechanism design issues for the generalized position auction problem. We focus on two demand models, the relaxed demand and the sharp demand model. We develop optimal (revenue) mechanisms for the seller of the positions. In addition, for the budget constrained model (without demand constraints), we develop a 2-approximate truthful mechanism. We prove that the solution is polynomial time solvable. Our results have potential to a wide range of application areas, such as sponsored search or TV advertising.

A major open problem is to find a constant approximation scheme when the demand constraints and the budget constraints are used simultaneously. For discrete distribution, Aleai [1] and Bhattacharya et al. [2] suggested a constant approximate mechanism for position auction with budget and relaxed demand constrained buyers. However, their approach which is based on solving an associated linear program cannot be extended to the continuous distribution case. Of course, another direction is to improve the approximation ratio for budget constrained cases.

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