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# Developing foundational number sense: Number line examples from Poland and Russia

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*For a variety of reasons, children start school with differing number-related skills, leading to differences in later mathematics achievement. Such differences prompt the question, what number-related experiences are necessary if the first year of school is to prepare children appropriately for their learning of mathematics? In this paper, we discuss the development of an eight dimensional framework, foundational number sense (FoNS), that characterises those learning experiences. We then demonstrate the framework's analytical efficacy by evaluating episodes from two sequences of lessons, one Polish and one Russian, focused on the use of the number line. The results show that the FoNS framework is cross-culturally sensitive, simply operationalised and analytically powerful.*

**Keywords:** Foundational number sense, grade one mathematics, Poland, Russia, number line.

## INTRODUCTION

Evidence internationally shows that the depth of a child's number sense predicts later mathematical success (Aubrey, Dahl, & Godfrey, 2006; Aunola, Leskinen, Lerkkanen, & Nurmi, 2004). For example, basic counting competence has predicted later successes in, for example, Canada, England, Finland, Flanders, Taiwan and the USA respectively (LeFevre et al., 2006; Aubrey & Godfrey, 2003; Aunola et al., 2004; Desoete, Stock, Schepens, Baeyens, & Roeyers, 2009; Yang & Li, 2008; Jordan, Kaplan, Locuniak, & Ramineni, 2007). Also, under-developed number sense leads to later mathematical failure (Jordan, Kaplan, Ramineni, & Locuniak, 2009; Gersten, Jordan, & Flojo, 2005; Malofeeva, Day, Saco, Young, & Ciancio, 2004). Thus, understanding number sense and how it can be promoted seems a sensible goal.

However, despite its importance, number sense has been poorly defined (Griffin, 2004, not least because mathematics educators and psychologists work with different definitions (Berch, 2005). Indeed, "no two researchers have defined number sense in precisely the same fashion" (Gersten et al., 2005, p. 296). Over the last two years we have been working on overcoming this definitional impasse. At the same time, we have tried to develop a classroom-focused analytical framework that is simple to operationalise and sensitive to different cultural practices. In this paper, we summarise our progress before evaluating the framework, foundational number sense (FoNS), against case study lessons from Poland and Russia.

## EARLIER WORK ON FOUNDATIONAL NUMBER SENSE

As indicated above, this paper draws on earlier work, including a paper presented at CERME8 (Back, Sayers, & Andrews, 2014). Since then our understanding of number sense in general and foundational number sense in particular has developed further. For example, our initial reading identified two broad conceptions of number sense. Today we argue for three, including the earlier two. The first, preverbal number sense, refers to those number insights innate to all humans and comprises an understanding of small quantities in ways that allow for comparison (Ivrendi, 2011; Lipton & Spelke, 2005). For example, young babies can discern 1:2 but not 2:3 ratios (Feigenson, Dehaene, & Spelke, 2004). The second, applied number sense, concerns those number-related competences that make mathematics sensible for all learners and prepares them for an adult world (McIntosh, Reys, & Reys, 1992). It underpins many curricular specifications and much of the material written on number sense (See, for example, Anghileri, 2000). Finally, the primary focus of this paper, is foundational number

sense (FoNS). This comprises those understandings that require instruction and typically arise during the first year of school (Ivrendi, 2011; Jordan & Levine, 2009). Unlike preverbal number sense, it is something children acquire rather than possess. Unlike applied number sense, its focus is not a world beyond school but later arithmetical and mathematical competence.

When developing the FoNS framework, our intention was not to construct an extensive list of characteristic learning outcomes but a small set of simple to operationalise components amenable to cross-cultural application. Our view was that extensive lists of number sense components and typically comprising around thirty components (Berch, 2005), would be unwieldy. Consequently, we exploited the constant comparison approach of the grounded theorists. Articles and book chapters typically addressing grade one students' acquisition of number-related competence were identified. These were read and broad FoNS-related categories identified. With each new category, previous articles were re-examined for evidence of the new. This approach, placed, for example, *rote counting to five* and *rote counting to ten*, two narrow categories discussed by Howell and Kemp (2005), within a broad category of systematic counting. Among the works examined in this process were (Aubrey & Godfrey, 2003; Aunola et al., 2004; Berch, 2005; Booth & Siegler, 2006; Clarke & Shinn, 2004; Dehaene, 2001; Desoete et al., 2009; Gersten et al., 2005; Griffin, 2004; Howell & Kemp, 2005; Hunting, 2003; Ivrendi, 2011; Jordan et al., 2007; Jordan & Levine, 2009; Lembke & Foegen, 2009; LeFevre et al., 2006; Lipton & Spelke, 2005; Malofeeva et al., 2004; Noël, 2005; Yang & Li, 2008).

In this paper, we summarise these eight FoNS components before showing how they play out in two post-Soviet educational contexts. This is the third case study pilot of the emergent FoNS framework, undertaken to ensure its viability for a large scale international study. The first case study examined two teachers, one in each of England and Hungary, working with grade one children on number sequences (Back et al., 2014). The analyses, based on an earlier seven component FoNS framework, indicated not only that the original framework's categories were sensitive to culturally different classroom traditions but also that the ways in which the categories combined resonated with earlier studies' showing high levels of didactical sophistication in Hungary and, in relative terms, low levels in England. The second case study,

involving two teachers, one in each of Hungary and Sweden, focused on the ways in which children were encouraged to acquire the skills of conceptual substituting (Sayers, Andrews, & Björklund Boistrup, 2014). In this case, the findings, based on the revised eight component framework, again showed a sensitivity to cultural context and highlighted well how different approaches to the same topic yield different FoNS-related outcomes, pertaining again to different levels of didactical sophistication. This paper reflects a third, and final, pilot evaluation of the framework. Before presenting the analyses, however, we present a summary of the eight components, which derived from the literature review described above. To avoid repetition and save space, each component is summarised independently of the literature on which it is based. The components of foundational number sense are:

**Number recognition:** Children recognise number symbols and know their vocabulary and meaning. They can identify a particular number symbol from a collection of number symbols and name a number when shown that symbol;

**Systematic counting:** Children are able to count systematically and understand ordinality. They count to twenty and back, or count upwards and backwards from arbitrary starting points, knowing that each number occupies a fixed position in the sequence of all numbers.

**Awareness of the relationship between number and quantity:** Children understand the correspondence between a number's name and the quantity it represents, and that the last number in a count represents the total number of objects, its cardinality.

**Quantity discrimination:** Children understand magnitude and can compare different magnitudes. They deploy language like *bigger than* or *smaller than* and understand that eight represents a quantity that is bigger than six but smaller than ten.

**An understanding of different representations of number:** Children understand that numbers can be represented differently, including the number line, different partitions, various manipulatives and fingers.

**Estimation:** Children can estimate, whether it be the size of a set or an object. Estimation involves moving

between representations of number; for example, placing a number on an empty number line.

**Simple arithmetic competence:** Children perform simple arithmetical operations, which Jordan and Levine (2009) describe as the transformation of small sets through addition and subtraction.

**Awareness of number patterns:** Children extend and are able to identify a missing number in a simple.

Importantly, the eight FoNS components, while distinct, are not unrelated. This is because number sense “relies on many links among mathematical relationships, mathematical principles..., and mathematical procedures” (Gersten et al., 2005, p. 297), links that help avoid situations where children can count but not know that five is bigger than three.

## METHODS AND RESULTS

The data examined in this study derived serendipitously from video-based teacher professional development programmes. However, both sets of lessons exploited the number line with grade one children and, therefore, proved amenable to a topic-based FoNS-related analysis. Both teachers, construed locally as effective, were video-recorded in ways that would optimise capturing their actions and utterances. Lessons, with transcripts, were scrutinised by at least two of the three authors with the intention of identifying episodes suitable for demonstrating a range of FoNS-related opportunities. In the following we present three episodes from each teacher’s lessons as examples of the ways in which they worked with the number line.

### The Russian episodes

The Russian teacher, Olga, began by sketching a horizontal line across the board, telling her class that this was a number line before asking what was missing. Over the next two or three minutes, four volunteers, with appropriate commentary from Olga, completed the number line as follows. The first drew a small arrow at the right hand end of the line to signify that numbers go from left to right also extend indefinitely. The second drew a small flag near to the line’s left hand end to represent the start or zero point. The third, using what looked like a postcard, added regular intervals, as shown in Figure 1. The fourth added the integers correctly.



**Figure 1:** Marking the number line with an arbitrary unit

**Commentary:** In this first episode can be discerned several FoNS components. The use of the number line reflected an expectation that students would engage with different representations of number. The manner in which the line was constructed, using a repeated measure, implicitly addresses the relationship between a number and the quantity it represents, while the process of numbering the line, including the emphasis on the placement of a zero, highlight both number recognition and systematic counting.

The lesson now progressed to the class using this new number line. A girl came to the board and was told to show three. This she did by pointing to the flag (zero) with the index finger of her left hand and three with her right. Next, she was asked to show five, which she did in the same way. This was followed by Olga asking the girl to show how she would get from three to five, which she did by counting on two units.

**Commentary:** Within this episode, which was no more than two minutes in length, can be seen at evidence of at least five FoNS components. The manner in which three, and other numbers, was demonstrated highlighted not only an identification of the symbol but also how three’s position reflected a measure of units, essentially arbitrary, along an axis. In other words, it reflected the relationship between number and quantity. The task included an expectation that learners would count systematically, work with a particular representation of number, and engage with simple arithmetical operations.

In related fashion, the next task involved starting with seven and subtracting three. While the girl concerned initially struggled to stretch her arms sufficiently to reach seven, as shown in Figure 2, she seemed confi-

dent with the mathematics. On Olga's invitation, she counted out to seven, while keeping her left hand forefinger at the flag (zero). Next, she was invited to count back three spaces, which she did. Olga asked the class what the girl's action signified and was told that she had subtracted three from seven to get four.



**Figure 2:** A demonstration of seven on the number line

Commentary: Within this third episode, which lasted less than a minute, can also be discerned at least five FoNS components. The manner in which seven was demonstrated highlighted not only an identification of the number symbol but also the relationship between number and quantity. The task included an expectation that learners would count systematically, work with a particular representation of number, and engage with simple arithmetical operations.

### The Polish episodes

Maria began her lesson by inviting each child randomly to the front to receive a sticker placed on his or her chest bearing a number, with Maria beginning by giving herself zero. She then asked the class to arrange itself in numerical order in a line down the middle of the room. Once this was done, she asked the class to return to their seats before asking them to repeat the task as quickly as possible. On this occasion, with

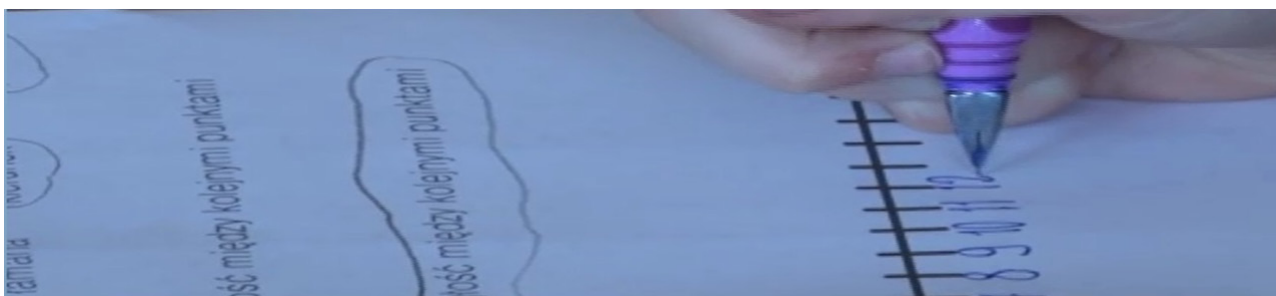
great excitement, the class arranged itself within a few seconds.

Commentary: With respect to the FoNS components, Maria's actions were commensurate with her encouraging her students to recognise numbers and, essentially, count systematically. It also highlighted the extent to which the units of the number line are arbitrary and the use of the number line as a representation of number.

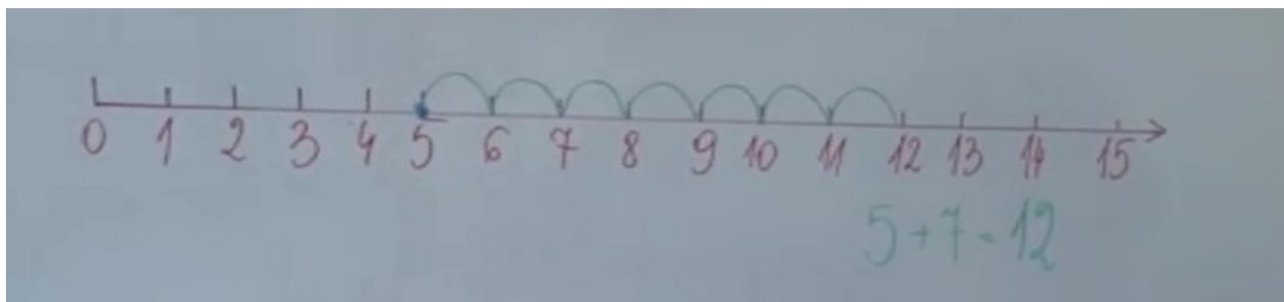
Later in the lesson Maria presented a number line with units but no numbers. She asked what should be placed at the end and was told zero, before being told that this should be followed by one, two and so on. At this point she asked her students to complete the number line on their sheet, as shown in Figure 3. Next she asked what would happen if the first marked point had been two and not one. This initiated a discussion on the importance of each unit being a representation of the same value with the consequence that the line would now show even numbers, 0, 2, 4, 6 and so on. Finally, in response to her asking what would happen if the first number had been three, it was agreed that the sequence would go 0, 3, 6, 9, 12, ... with each successive number being found by counting on three.

Commentary: Within this episode could be discerned five FoNS categories. At the most obvious level, Maria was attending to number recognition, different representations of number and systematic counting. Also, the introduction of the multiples focused attention on number patterns and, in the counting on of threes, simple arithmetic.

Later, Maria sketched a number line from zero to fifteen on the board. She marked a point at five, and asked her students to do the same. She then wrote  $5 + 7 =$  before showing how the sum can be counted out, as in Figure 4. Following this she asked her students to do the same on their sheets. The students were



**Figure 3:** Completing the number line



**Figure 4:** Using the number line to undertake simple arithmetic

then invited to repeat the process for  $6 + 6 =$ ,  $14 - 9 =$ ,  $5 + 6 - 4 =$ . After each one, Maria repeated the process on the board, with instructions provided by different students. She counted very slowly and deliberately as she marked off each number on the way.

**Commentary:** In this final episode, in addition to the already familiar number recognition, systematic counting and number representations, Maria was attending to simple arithmetical operations and how they can be modelled on the number line. Interestingly, she did not just focus on addition but included subtraction and, with the third problem, both operations.

**DISCUSSION**

The analyses above, summarised in Table 1, indicate both similarities and differences in the ways in which FoNS was addressed. In respect of similarities, both teachers addressed, over the course of the analysed episodes, five categories. Both encouraged, throughout their respective excerpts, students’ recognition of number alongside, systematic counting and awareness of different representations of number. Both also, but not to the same degree, focused attention on simple arithmetic operations. Overall, and bearing in mind the number line focus, such findings were of little surprise and, from the perspective of validating the framework, helpful – an analytical framework

that failed to identify the expected would be of little use.

While it is always important to acknowledge similarities, differences are frequently more enlightening. On the one hand, Olga emphasised, through her insisting that students make a bodily link between a number and zero, the connection between number and quantity. On the other hand, through her discussion of multiples of two and three, Maria was seen to use the number line to support children’s engagement with simple sequences. However, it is our view that such findings reflect not insignificant qualitative differences in the teachers’ emphases. Olga’s exploitation of the bodily link avoided too early a shift to working with numbers as abstract entities. This seems a more significant didactical decision when compared to Maria’s emphasis on sequences, albeit a key category of FoNS in its own right. Such qualitative differences show that the FoNS framework has the propensity for highlighting, in much the way that generic learning outcomes exploited in other studies have shown, both simple analyses based on the frequencies of particular events and more sophisticated analyses based on the interactions of those events (see Andrews & Sayers 2013).

In this paper, we have shown how opportunities for students to acquire FoNS played out in two post-Soviet classrooms. This is of particular interest in the light

	Olga’s episodes			Maria’s episodes		
Number recognition	X	X	X	X	X	X
Systematic counting	X	X	X	X	X	X
Relationships between numbers and quantities	X	X	X			
Quantity discrimination						
Different representations of number	X	X	X	X	X	X
Estimation						
Simple arithmetical operations		X	X			X
Number patterns and missing numbers				X		

**Table 1:** the distribution of the categories across the episodes

of recent evidence that the rise of the free market had had markedly different impacts on student achievement on international tests (Bodovski, Kotok, & Henck, 2014). In this respect, Poland's PISA mathematics scores, reflecting students' real-world application of mathematical knowledge at age 15, have risen from significantly below to significantly above the international mean at a time when Russia's have remained largely static, constantly significantly below the international mean. These scores are interesting when set against Russia's grade 8 TIMSS scores, assessments of students' technical competence, has been consistently above the international mean. With respect to TIMSS grade 4, Russia has been consistently one of the higher achieving nations, while Poland has remained significantly below the international mean. In other words, and putting the case crudely, if such tests tell us anything it is that Polish students are increasingly competent on real-world mathematical tasks located in text and requiring a degree of interpretation, which Russian students are not, while Russian students are strong on mathematics tasks located in world of technical mathematics, which Polish students are not. In light of this, what do our limited analyses have to say?

Firstly, acknowledging the limited sample presented here, Olga's teaching seemed more focused on the structural properties of number and mathematics than Maria's. Not once did Olga make any number line-related reference to a world outside the classroom. Her efforts were focused constantly on a world contained solely within mathematics. Maria, on the other hand, although not reported above for lack of space, made frequent use of different representations of the number line drawn from the real world. For example, at different times she made reference to several thermometers, each of which presented a different scale and starting value, different tailors' measuring tapes, a carpenter's retractable tape, a measuring jug, various skeletons of fish and snakes showing, in particular, the regular spread of ribs. Each one elicited a brief discussion as to its purpose and relationship to the number line. On a separate occasion she based a counting activity on the use of a representation of a hotel lift travelling between the many floors of a very tall hotel. Thus, these differing emphases, essentially unrelated to the mathematics being taught, may have profound implications for students' successes on international tests of achievement. Olga's teaching seems unrelated to PISA expectations and Maria's to

TIMSS. Such matters clearly require further examination.

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