



The lengths of conjugators in the model filiform groups

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Abstract

The conjugator length function of a finitely generated group Γ gives the optimal upper bound on the length of a shortest conjugator for any pair of conjugate elements in the ball of radius n in the Cayley graph of Γ . We prove that polynomials of arbitrary degree arise as conjugator length functions of finitely presented groups. To establish this, we analyse the geometry of conjugation in the discrete model filiform groups $\Gamma_d = \mathbb{Z}^d \rtimes_{\phi} \mathbb{Z}$ where ϕ is the automorphism of \mathbb{Z}^d that fixes the last element of a basis a_1, \dots, a_d and sends a_i to $a_i a_{i+1}$ for $i < d$. The conjugator length function of Γ_d is polynomial of degree d .

Keywords Nilpotent groups · Model filiform groups · Conjugator length

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1 Introduction

Whereas Dehn (isoperimetric) functions are the most natural measure of the complexity of a direct approach to the word problem in a finitely presented group, *conjugator length functions* are the most obvious measure of the complexity of a direct approach to the conjugacy problem in a finitely generated group. By definition, $CL_G(n)$ is the least integer N such that whenever a pair of words u and v of length at most n in the generators represent conjugate elements of G , there is a word w of length at most N such that $uw = wv$ in G . The study of conjugator length functions is far less developed than the study of Dehn functions. In particular, whereas it is essentially known which functions arise as Dehn functions [2], the class of functions that arise as conjugator length functions is poorly understood. Indeed, until recently, very few sharp estimates on conjugator length functions had been established. In part, this reflects the greater delicacy of the conjugacy problem: for example, there exist pairs of finitely generated

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groups $H < G$ with $|G/H| = 2$ such that H has a solvable conjugacy problem and G does not [9]. Such examples warn us that, in contrast to the study of Dehn functions, we cannot hope to understand conjugacy length functions using the techniques of coarse geometry. On the other hand, the conjugacy problem is fundamentally geometric in nature: understanding conjugacy in a group Γ is closely tied to an understanding of the geometry of annuli with prescribed boundaries in any space with fundamental group Γ . We refer to our article with Andrew Sale [8] for a more detailed introduction to these ideas and a survey of what is known about conjugator length functions.

In this article we will advance the state of the art by showing that polynomials of arbitrary degree arise as conjugator length functions of finitely presented groups. We shall do so by investigating the geometry of the conjugacy problem in the discrete model filiform groups Γ_d . By definition, $\Gamma_d = \mathbb{Z}^d \rtimes_{\phi_d} \mathbb{Z}$, where, with respect to a fixed basis a_1, \dots, a_d for \mathbb{Z}^d , the automorphism ϕ_d fixes a_d and maps $a_i \mapsto a_i a_{i+1}$ for $i = 1, \dots, d - 1$.

$$\Gamma_d = \langle a_1, \dots, a_d, t \mid \forall 1 \leq i < j \leq d, [a_i, a_j] = 1 = [t, a_d], t^{-1} a_i t = a_i a_{i+1} \rangle.$$

The corresponding nilpotent Lie groups $G_d = \mathbb{R}^d \rtimes_{\phi} \mathbb{R}$ have been studied extensively in the context of Carnot geometry, while the lattices $\Gamma_d < G_d$ have served as key examples for nilpotent and polynomial phenomena in geometric group theory. In particular, $(\Gamma_d)_{d \in \mathbb{N}}$ was among the first families of groups used to prove Dehn functions can be polynomial of arbitrary degree; see [1, 3]. Here, we prove that the groups $(\Gamma_d)_{d \in \mathbb{N}}$ illustrate the same diversity of behaviour among conjugator length functions. In a companion article [5], we will show that 2-step nilpotent groups can also have conjugator length functions of arbitrary polynomial degree. A polynomial upper bound on the conjugator length functions of arbitrary finitely generated nilpotent groups was established much earlier [12].

Theorem 1.1 *The conjugator length function of Γ_d is polynomial of degree d .*

Theorem 1.1 provides a counterpoint to our work with Andrew Sale on the conjugator length functions of free-by-cyclic groups: in [7] we proved that $CL(n) \simeq n$ for the group $F_d \rtimes \mathbb{Z}$ obtained by removing the relations $[a_i, a_j] = 1$ from the presentation of Γ_d . It also contrasts with Sale’s result [13, Theorem 5.4] that if θ is diagonalisable over \mathbb{R} with positive eigenvalues, then $\mathbb{Z}^d \rtimes_{\theta} \mathbb{Z}$ has $CL(n) \simeq n$.

An outline of the proof. Whenever one tries to understand a conjugacy problem in a group Γ , one is inevitably led to study the structure of centralisers in Γ , because the set of solutions x to an equation $x^{-1} \gamma_1 x = \gamma_2$ is a coset of the centraliser $C(\gamma_1)$. Thus a feature of our study is that we will need a close understanding of the structure of centralisers in Γ_d ; this is described in Section 4. It will emerge from this analysis of centralisers that we also need tight control on the geometry of roots in Γ_d , by which we mean solutions to equations $x^p = \gamma$; this is the subject of Section 5. With these tools in hand, we will prove Theorem 1.1 by means of an induction on d that exploits the observation that the centre of Γ_d is infinite cyclic, generated by a_d , and the quotient of Γ_d by its centre is isomorphic to Γ_{d-1} , with the images of t and a_i ($i < d$) satisfying the defining relations of Γ_{d-1} . When working with this inductive structure, it is important to keep in mind that if two words u, v in the letters t, a_1, \dots, a_{d-1} define the same element in Γ_{d-1} , they will in general not be equal in Γ_d ; rather, an equality of the form $u = v a_d^r$ will hold in Γ_d . The first lesson to be taken from this observation is that we must be careful to specify in which group equalities between words are taking place. The second point to absorb is that we will have reason to control $|r|$ in expressions of the form $u = v a_d^r$.

All distances in Γ_d will be taken with respect to the generating set of this presentation and the area of any null-homotopic word will be calculated with respect to this set of defining relations. It is worth noting, however, that Γ_d is generated by just two elements, a_1 and t .

We are interested in the *conjugator length function* $CL : \mathbb{N} \rightarrow \mathbb{N}$ of these groups. By definition, $CL(n)$ is the minimal integer N such that whenever a pair of words u and v of length at most n in the generators represent conjugate elements in Γ_d , there is a word w of length at most N such that $uw = vw$ in Γ_d . If we worked with a different finite generating set, the resulting conjugator length function would be equivalent in the sense of the following standard definition.

For functions f and g mapping $\mathbb{N} \rightarrow \mathbb{N}$ write $f \preceq g$ when there exists $C > 0$ such that $f(n) \leq Cg(Cn + C) + C$ for all $n \in \mathbb{N}$, and write $f \simeq g$ when $f \preceq g$ and $g \preceq f$.

The following standard lemma will be needed to control the size of the discrepancy r that arises when we lift an equality $u = v$ in Γ_{d-1} to an equality of the form $u = va'_d$ in Γ_d .

Lemma 2.1 *If two words u, v in the letters t, a_1, \dots, a_{d-1} define the same element of Γ_{d-1} , then $u = va'_d$ in Γ_d , where $|r| \leq \text{Area}_{\Gamma_{d-1}}(uv^{-1})$.*

Proof By definition, $\text{Area}_{\Gamma_{d-1}}(uv^{-1})$ is the least integer A for which there is an equality

$$uv^{-1} = \prod_{i=1}^A \theta_i^{-1} \rho_i \theta_i$$

in the free group $F = F(t, a_1, \dots, a_{d-1})$ with each $\rho_i^{\pm 1}$ one of the defining relations of Γ_{d-1} . We modify the righthand side of this expression by replacing each instance of $[a_{d-1}, t]$ with $[a_{d-1}, t]a_d$, leaving the other relations untouched. Let ρ'_i be the resulting relations. Then, in $F \times \langle a_d \rangle$ we have the equality

$$uv^{-1} = a_d^{-r} \prod_{i=1}^A \theta_i^{-1} \rho'_i \theta_i,$$

where $-r$ is the number of letters a_d added during the editing of the ρ_i . The righthand side has image a_d^r under the surjection $F \times \langle a_d \rangle \rightarrow \Gamma_d$ that is implicit in the notation. Thus $u = va'_d$ in Γ_d , with $|r| \leq A$. □

The following well-known result will be needed in connection with the above lemma.

Theorem 2.2 ([3]) *The Dehn function of Γ_d is $\simeq n^{d+1}$.*

Positive powers of the matrices for ϕ and ϕ^{-1} can be calculated using the binomial theorem, writing each as $I + N$ and expanding $(I + N)^n$. For ϕ , the entry in column i on the bottom row for ϕ^n is $\epsilon(i, n) = \binom{n}{d-i}$, while the corresponding entry $\epsilon(i, -n)$ for ϕ^{-n} is a signed sum on binomial coefficients. The following crude estimate will suffice for our purposes.

Lemma 2.3 *There is a constant ϵ_d such that $|\epsilon(i, m)| \leq \epsilon_d |m|^{d-i}$ for all $m \in \mathbb{Z}$ and $i = 1, \dots, d - 1$.*

The significance for us of the integers $\epsilon(i, n)$ is that for $i = 1, \dots, d - 1$, if $\phi_{d-1}^n(a_i) \in \Gamma_{d-1}$ and $\phi_d^n(a_i) \in \Gamma_d$ are written as words in the letters a_1, \dots, a_d , then in Γ_d we have

$$\phi_d^n(a_i) = \phi_{d-1}^n(a_i) a_d^{\epsilon(i, n)}. \tag{2}$$

3 Normal forms and distance

The essential result for us concerning navigating Γ_d metrically is:

Theorem 3.1 *For $i = 1, \dots, d$, the function $n \mapsto d_{\Gamma_d}(1, a_i^n)$ is Lipschitz equivalent to $n^{1/i}$.*

This theorem played an important role in [6] where it was proved with an argument from [10, 11]. This argument takes place in the Malcev completion of Γ_d , which is the nilpotent Lie group $G_d = \mathbb{R}^d \rtimes \mathbb{R}$ where Γ_d is a lattice. The Lie algebra of G_d , which is graded, has presentation

$$(X_1, \dots, X_d, T : [X_i, X_j] = 0 = [X_d, T], [T, X_i] = X_{i-1}, i = 1, \dots, d - 1).$$

This Lie algebra is generated under brackets by $\{X_1, T\}$, so there is a Carnot-Carathéodory metric on G_d : this is defined by fixing an inner product on the plane $\langle X_1, T \rangle$ so that $\{X_1, T\}$ is orthonormal, transporting this plane around G_d by left translation, then defining the distance $d_{cc}(x, y)$ between two points $x, y \in G_d$ to be the infimum of the lengths of piecewise smooth curves from x to y that are almost everywhere tangent to this field of 2-planes, with the length of tangent vectors measured using the transported inner product.

The Lie algebra of G_d admits the 1-parameter family of automorphisms $m_s : X_i \mapsto s^i X_i, T \mapsto s.T$ for $s \neq 0$. These automorphisms induce (via the exponential map) homotheties of the Carnot-Carathéodory metric on G_d . For all $s \in \mathbb{R}$ we have $d_{cc}(1, \exp(sX_i)) = s^{1/i}$. The restriction of d_{cc} to the lattice Γ_d is Lipschitz equivalent to the word metric on Γ_d and $a_i^n = \exp(nX_i)$.

Here we give an alternative, more combinatorial proof of Theorem 3.1. This proof invokes Hilbert’s solution to Waring’s problem, but is otherwise elementary. Lemma 3.2 and Proposition 3.3 below will establish the complementary upper and lower bounds for the theorem.

We will push the letters $t^{\pm 1}$ to the left when writing elements of Γ_d in the standard normal form for polycyclic groups.

Lemma 3.2 *Every element $g \in \Gamma_d$ can be expressed uniquely as a word in normal form $t^r a_1^{p_1} a_2^{p_2} \dots a_d^{p_d}$, and if $d(1, g) \leq n$ then $|r| \leq n$ and $|p_i| \leq n^i$ for $i = 1, \dots, d$.*

Proof The only non-trivial assertion is the one about the size of the exponents. We proceed by induction on n . Given a word $w = b_1 \dots b_n \beta$ of length $n + 1$, we apply the inductive hypothesis to write $w' = b_1 \dots b_n$ in normal form, then shuffle β past the letters of this normal form to write w in normal form. If $\beta = a_j^{\pm 1}$ then this shuffling leaves r and p_i unchanged for $i \neq j$ and alters p_j by ± 1 . If $\beta = t$ then this shuffling leaves p_1 unchanged and for $i \geq 2$ increases p_i by p_{i-1} , so the observation $n^i + n^{i-1} < (n + 1)^i = n^i + in^{i-1} + \dots$ finishes the induction. If $\beta = t^{-1}$ then the shuffling replaces p_i by

$$(-1)^{i+1}(p_1 - p_2 + \dots + (-1)^{i+1} p_i),$$

whose absolute value is, by induction, less than $n + n^2 + \dots + n^i$, which is less than $(n + 1)^i$. □

Our next result provides a converse to the preceding lemma.

Proposition 3.3 *There is a constant $C_d \geq 1$ such that for every $g \in \Gamma_d$ with normal form $t^r a_1^{p_1} a_2^{p_2} \dots a_d^{p_d}$, where $|r| \leq n$ and $|p_i| \leq n^i$, we have $d(1, g) \leq C_d n$.*

The following special case of the proposition contains the main idea in the proof; it also facilitates a simple induction to prove the general case.

Lemma 3.4 *For every $d \geq 1$ there is a constant D_d such that for all positive integers $0 < p \leq n^d$ we have $d(1, a_d^p) \leq D_d n$ in Γ_d .*

Proof We proceed by induction on d . The base case is $\Gamma_1 = \mathbb{Z}^2$, where the assertion is trivial.

The Hilbert-Waring Theorem provides us with an integer M_d such that every positive integer p can be written as a sum $p = k_1^d + \dots + k_m^d$ with $k_i \in \mathbb{N}$ and $m \leq M_d$. Then $\sum_{i=1}^m k_i \leq M_d n$, because $p \leq n^d$ implies that $k_i \leq n$ for all i . In Γ_d , for any $k \in \mathbb{N}$ we have the equation

$$a_{d-1}^{-k^{d-1}} t^{-k} a_{d-1}^{k^{d-1}} t^k = a_d^{k^d}.$$

By induction, for $i = 1, \dots, m$, there is a word w_i in the letters t, a_1, \dots, a_{d-1} of length at most $D_{d-1} k_i$ such that $w_i = a_{d-1}^{k_i^{d-1}}$ in Γ_{d-1} . Recalling that Γ_{d-1} is obtained from Γ_d by killing the central subgroup $\langle a_d \rangle$, we have $w_i = a_{d-1}^{k_i^{d-1}} a_d^{l_i}$ in Γ_d for some $l_i \in \mathbb{Z}$. As a_d is central, in Γ_d we have

$$w_i^{-1} t^{-k_i} w_i t^{k_i} = a_{d-1}^{-k_i^{d-1}} t^{-k_i} a_{d-1}^{k_i^{d-1}} t^{k_i} = a_d^{k_i^d}.$$

Hence, in Γ_d ,

$$a_d^p = \prod_{i=1}^m w_i^{-1} t^{-k_i} w_i t^{k_i},$$

and the righthand side is a word of length at most $2(D_{d-1} + 1) \sum_{i=1}^m k_i \leq 2(D_{d-1} + 1)M_d n$. Taking $D_d = 2(D_{d-1} + 1)M_d$ completes the induction. \square

Proof of Proposition 3.3 We proceed by induction on d . Again, the base case $d = 1$ is trivial. Assume that the proposition is true in Γ_{d-1} . Given $g = t^r a_1^{p_1} a_2^{p_2} \dots a_d^{p_d} \in \Gamma_d$, with $|r| \leq n$ and each $|p_i| \leq n^i$, we know by induction that there is a word $W = W(t, a_1, \dots, a_{d-1})$ of length at most $C_{d-1}n$ such that in Γ_{d-1} we have

$$W = t^r a_1^{p_1} a_2^{p_2} \dots a_{d-1}^{p_{d-1}}.$$

In Γ_d , then, for a unique $q \in \mathbb{Z}$ we have

$$W = t^r a_1^{p_1} a_2^{p_2} \dots a_{d-1}^{p_{d-1}} a_d^q,$$

and by Lemma 3.2 we know $|q| \leq |W|^d \leq C_{d-1}^d n^d$. So, $|pd - q| \leq |pd| + |q| \leq (1 + C_{d-1}^d)n^d$, and then Lemma 3.4 leads to $d(1, a_d^{p_d - q}) \leq (1 + C_{d-1}^d)D_d n$. Therefore, by the triangle inequality in Γ_d , noting that $g = W a_d^{p_d - q}$,

$$d(1, g) \leq d(1, W) + d(1, a_d^{p_d - q}) \leq C_{d-1}n + D_d(1 + C_{d-1}^d)n,$$

and defining $C_d = C_{d-1} + D_d(1 + C_{d-1}^d)$ completes the induction.

Lower Bound for Theorem 1.1

The key observation here is that an arbitrary element $t^r x \in \Gamma_d = \mathbb{Z}^d \rtimes_{\phi} \mathbb{Z}$ with $x \in \mathbb{Z}^d \rtimes 1$ conjugates a_{d-1} to $a_{d-1} a_d^r$, with x playing no role. It follows that for each positive integer $r > 0$, the unique shortest element of Γ_d conjugating a_{d-1} to $a_{d-1} a_d^r$ is t^r .

In Lemma 3.4 we established the existence of a constant D_d such that $a_d^{n^d}$ lies in the ball of radius $D_d n$ about $1 \in \Gamma_d$, hence $d(1, a_{d-1} a_d^{n^d}) \leq D_d n + 1$. As t^{n^d} is the unique shortest element conjugating a_{d-1} to $a_{d-1} a_d^{n^d}$, we deduce

$$CL_{\Gamma_d}(D_d n + 1) \geq n^d,$$

hence $n^d \leq CL_{\Gamma_d}(n)$.

4 Centralisers in Γ_d

Picking up a theme from the last argument of the previous section, we note that $g \in \Gamma_d$ will conjugate γ to an element of the form γa_d^* if and only if the image of g in Γ_{d-1} centralises the image of γ . This highlights our need to understand centralisers $C(\gamma)$ in the groups Γ_d .

Notation. We will need to discuss the subgroup $\langle a_1, \dots, a_d \rangle = \mathbb{Z}^d \rtimes 1 < \Gamma_d$ repeatedly in what follows, and it will be important to distinguish it from an abstract copy of \mathbb{Z}^d . For this reason, we shall henceforth denote it by \mathbb{A}_d .

Proposition 4.1 *Given $\gamma \in \Gamma_d$, if $\gamma = \gamma_0^p a_d^q$ with $p > 0$ maximal, then*

- (1) $C(\gamma) = \langle \gamma_0, a_d \rangle \cong \mathbb{Z}^2$ if $\gamma \notin \mathbb{A}_d$,
- (2) $C(\gamma) = \mathbb{A}_d$ if $\gamma \in \mathbb{A}_d \setminus \langle a_d \rangle$,
- (3) $C(\gamma) = \Gamma_d$ if $\gamma \in \langle a_d \rangle$.

Proof We again proceed by induction on d , viewing Γ_{d-1} as the quotient of Γ_d by $\langle a_d \rangle$ and writing $\bar{\gamma}$ for the image of γ in Γ_{d-1} . The base step $\Gamma_1 \cong \mathbb{Z}^2$ is trivial (with case (2) vacuous).

Cases (2) and (3) are easily verified, so in the inductive step we concentrate on the case $\gamma = \gamma_0^p a_d^q$ with $\gamma_0 \notin \mathbb{A}_d$.

Some preliminary observations are in order: if $\bar{\gamma} = g^m$ in Γ_{d-1} , then for any preimage \tilde{g} in Γ_d of g we have $\tilde{g}^m = \gamma a_d^r$ for some r , so $m \leq p$, by the maximality of p , and if $m = p$ then $g = \bar{\gamma}_0$ by the uniqueness of roots in torsion-free nilpotent groups. Thus $\bar{\gamma}_0$ is the unique maximal root of $\bar{\gamma}$ in Γ_{d-1} (even though γ_0 itself is only uniquely defined up to multiplication by a power of a_d).

By induction, $C(\bar{\gamma}) < \Gamma_{d-1}$ is a free abelian group of rank 2 that contains $\langle a_{d-1} \rangle$. The image of $C(\gamma)$ in $C(\bar{\gamma})$ intersects $\langle a_{d-1} \rangle$ trivially, because either γ or γ^{-1} is xt^{-j} with $x \in \mathbb{A}_d$ and $j > 0$, and in Γ_d (n.b. Γ_d rather than Γ_{d-1})

$$\tilde{a}_{d-1}^{-k} x t^{-j} \tilde{a}_{d-1}^k = a_d^{kj} x t^{-j} \neq x t^{-j}$$

for any preimage $\tilde{a}_{d-1} = a_{d-1} a_d^*$ in Γ_d of $a_{d-1} \in \Gamma_{d-1}$. Thus the image of $C(\gamma)$ in $C(\bar{\gamma})$ is a cyclic group. This cyclic group contains $\bar{\gamma}$, so by the considerations of the previous paragraph, it is contained in $\langle \bar{\gamma}_0 \rangle$. As γ_0 commutes with γ , the image is exactly $\langle \bar{\gamma}_0 \rangle$.

The preimage in Γ_d of $\langle \bar{\gamma}_0 \rangle$ is $\langle \gamma_0, a_d \rangle$, which centralises γ . Thus $C(\gamma) = \langle \gamma_0, a_d \rangle$ and the induction is complete. □

5 The lengths of roots in Γ_d

The emergence of γ_0 in Proposition 4.1 throws up a need to understand roots in the groups Γ_d . In particular, when it comes to estimating the lengths of minimal conjugating elements, we will need to bound the lengths of roots of elements in Γ_d .

The bounds in the following proposition are not optimal, but they will suffice for our purposes and a sharpening would involve an unwelcome increase in notation.

Proposition 5.1 *There is a constant $k_d > 0$ so that for all $p, n > 0$ and all $\gamma \in \Gamma_d$ with $d(1, \gamma) \leq n$, if $h^p = \gamma$ then*

$$h = t^{m_0} a_1^{m_1} \cdots a_d^{m_d}$$

with $|m_0| \leq n/p$ and $|m_i| \leq k_d n^i / p$ for $i = 1, \dots, d$.

For the most part, we shall only need the following easy consequence of Propositions 3.3 and 5.1.

Corollary 5.2 *There is a constant K_d so that if $h^p = \gamma$ in Γ_d and $d(1, \gamma) \leq n$, then $d(1, h) \leq K_d n$.*

Before proving Proposition 5.1, we consider the case $d = 2$, which admits a more direct argument.

Example 5.3 In the 3-dimensional Heisenberg group $\Gamma_2 = \langle a_1, a_2, t \rangle$ we have

$$(a_1 t)^n = t^n a_1^n a_2^{\sigma(n)},$$

where

$$\sigma(n) = 1 + \cdots + n = n(n + 1)/2 = \text{Area}_{(a_1, t)}(u^n v^{-1})$$

with $u \equiv a_1 t$ and $v \equiv t^n a_1^n$. One can see that $\sigma(n)$ really is $\text{Area}_{(a_1, t)}(u^n v^{-1})$ either by considering the van Kampen enclosed by the loop $u^n v^{-1}$ in the (a_1, t) -plane tessellated by squares, or else by arguing algebraically: we reduce $u^n v^{-1} \equiv (a_1 t)^n a_1^{-n} t^{-n}$ to the empty word by starting in the middle and replacing the subword $a_1 t a_1^{-1}$ by t at a cost of applying 1 commutator relation, then $a_1 t^2 a_1^{-1}$ by t^2 at an additional cost of 2, etc., until finally $a_1 t^n a_1^{-1}$ is replaced by t^n at a cost of n , after which we can freely reduce $t^n t^{-n}$ to the empty word.

If n is odd, then

$$(a_1 t a_2^{-(n+1)/2})^n = t^n a_1^n.$$

Thus $h_n := a_1 t a_2^{-(n+1)/2}$ is the unique n -th root of $g := t^n a_1^n$ in Γ_2 . Projecting to $\Gamma_1 = \mathbb{Z}^2$, it is easy to see that $d(1, t^n a_1^n) = 2n$ in Γ_2 . On the other hand, $d(1, h_n)$ is approximately \sqrt{n} , even though $h_n^n = g$.

We now consider the more general situation where $\gamma \in \Gamma_2 \setminus \langle a_1, a_2 \rangle$ is a p -th power, say $\gamma = h^p$. Counting occurrences of t , we see that $|p| \leq d(1, \gamma)$. Our goal is to estimate the length of h . Suppose $n = d(1, \gamma)$ and fix a geodesic word w representing γ . Let \bar{w} be the word obtained from w by deleting all occurrences of a_2 and note that this word equals the image of γ in Γ_1 . We write the image of h in $\Gamma_1 = \mathbb{Z}^2$ as a geodesic word v in the free abelian group on $\{a_1, t\}$. Replacing γ and h by their inverses if necessary, we may assume that only positive powers of t arise in v . In $\Gamma_1 = \mathbb{Z}^2$ we know that $v^p = \bar{w}$ implies $|v| = d(1, \bar{w})/p$, so $|v| \leq |w|/p = n/p$. As in Lemma 2.1, in Γ_2 we have

$$v^p = w a_2^r,$$

where $|r| \leq \text{Area}_{\Gamma_1}(v^{-p} \bar{w})$, which is less than n^2 , by the quadratic isoperimetric inequality in \mathbb{Z}^2 . (In more detail, $v^{-p} \bar{w}$ is a non-empty word τ in a_1 and t of length at most $2n$ that equals the identity in Γ_1 and any such τ has a subword which freely reduces to $a_1^{\pm 1} t^l a_1^{\mp 1}$ or $t^{\pm 1} a_1^l t^{\mp 1}$ for some $|l| \leq n/2$; this subword can be replaced by t^l or a_1^l , respectively, at a cost

of applying at most $n/2$ commutator relations, from which it follows that $\text{Area}(\tau) < n^2$, by an induction on length.) But $v = ha_2^j$ in Γ_2 for some $j \in \mathbb{Z}$, so

$$wa_2^r = v^p = h^p a_2^{jp} = \gamma a_2^{jp} = wa_2^{jp}.$$

Therefore $r = jp$.

We saw earlier that $|v| \leq n/p$ and $|r| < n^2$, and from Proposition 3.3 we know $d(1, a_2^{-j}) \leq C_2 \sqrt{|j|} = C_2 \sqrt{|r/p|}$, which is less than $C_2 n/\sqrt{p}$. Thus, taking $v \in \langle a_1, t \rangle$ to be $v = t^{m_0} a_1^{m_1}$ with $|m_0| + |m_1| \leq n/p$, we have $h = va_2^{-j} = t^{m_0} a_1^{m_1} a_2^{-j}$ in the form required by Proposition 5.1, since $|j| \leq n^2/p$. Moreover,

$$d(1, h) \leq |v| + d(1, a_2^{-j}) \leq \frac{1}{p}(n + C_2 np^{1/2}) \leq C_2 \frac{1}{p}(n + np^{1/2}) \leq 2C_2 np^{-1/2}.$$

Proof of Proposition 5.1 We proceed by induction on d . The case $d = 1$ is trivial and the case $d = 2$ was covered in the preceding example. Assume true up to $d - 1$. We have $\gamma, h \in \Gamma_d$ with $h^p = \gamma$ and $d(1, \gamma) \leq n$. Let \bar{h} be the image of h in Γ_{d-1} . Then, by induction, $\bar{h} = t^{m_0} a_1^{m_1} \dots a_{d-1}^{m_{d-1}}$, in Γ_{d-1} , with the $|m_i|$ bounded as stated in the proposition. Let

$$h_0 := t^{m_0} a_1^{m_1} \dots a_{d-1}^{m_{d-1}} \in \Gamma_d.$$

Then $h_0 = ha_d^m$ for some $m \in \mathbb{Z}$, and if the normal form for $h_0^p \in \Gamma_d$ is

$$h_0^p = t^{pm_0} a_1^{s_1} \dots a_d^{s_d},$$

then the normal form for $\gamma = h^p = (h_0 a_d^{-m})^p$ is

$$\gamma = t^{pm_0} a_1^{s_1} \dots a_{d-1}^{s_{d-1}} a_d^{s_d - mp},$$

and from Lemma 3.2 we have $|s_d - mp| \leq n^d$. So, if we can prove that there is a constant α_d such that

$$|s_d| \leq \alpha_d n^d, \tag{3}$$

then we will have $|mp| \leq (\alpha_d + 1)n^d$ and defining $k_d := \max \{\alpha_d + 1, k_{d-1}\}$ will complete the induction.

To obtain the required bound on s_d , we transform h_0^p into normal form carefully, pushing all occurrences of $t^{\pm 1}$ to the left and then shuffling the resulting element of $\mathbb{A}_d \cong \mathbb{Z}^d$ into normal form. The pushing of the t letters amounts to using free identities $a_i t^{jm_0} = t^{jm_0} (t^{-jm_0} a_i t^{jm_0})$ and then evaluating $t^{-jm_0} a_i t^{jm_0}$ as $\phi_d^{jm_0}(a_i)$. At this point, it is important to recall from equation (2) that for each $i \leq d - 1$

$$\phi_d^{jm_0}(a_i) = \phi_{d-1}^{jm_0}(a_i) a_d^{\epsilon(i, jm_0)}.$$

(This is equally valid for the negative powers of ϕ arising from pushing t^{-1} if $m_0 < 0$.)

The result of shuffling h_0^p into normal form in Γ_{d-1} (producing the word $t^{pm_0} a_1^{s_1} \dots a_{d-1}^{s_{d-1}}$) differs from the result of doing it in Γ_d by a power of a_d that can be calculated by considering how many copies of each letter a_i in h_0^p we have to move each t^{m_0} past, adding up the number of a_d letters produced by each push. The total (ignoring cancellation due to signs if $m_0 < 0$) is

$$M = \sum_{j=1}^{p-1} \sum_{i=1}^{d-1} |m_i \epsilon(i, jm_0)|.$$

We estimate this (crudely) by using the inequality $|\epsilon(i, m)| \leq \epsilon_d m^{d-i}$ from Lemma 2.3, noting that $|jm_0|$ is less than $|pm_0|$, which is the absolute value of the exponent sum of t in γ , which is less than n . Thus

$$M \leq \sum_{j=1}^{p-1} \sum_{i=1}^{d-1} |m_i \epsilon_d (jm_0)^{d-i}| \leq \epsilon_d (p-1) \sum_{i=1}^{d-1} |m_i| n^{d-i},$$

and our induction assures us that $|pm_i| \leq k_{d-1}n^i$, so

$$|s_d| \leq M \leq \epsilon_d k_{d-1} \sum_{i=1}^{d-1} n^i n^{d-i} = \epsilon_d k_{d-1} \sum_{i=1}^{d-1} n^d.$$

Thus, to obtain the required bound (3), it suffices to set $\alpha_d := d\epsilon_d k_{d-1}$.

6 Quantifying Bézout’s Lemma

We will need an elementary observation concerning Bézout’s Lemma.

Lemma 6.1 *For positive integers A and B , if $e = \gcd(A, B)$ then there exist a non-negative integer $\mu < A$ and a positive integer $\lambda \leq B$ such that $\lambda A - \mu B = e$. If $A > 1$, one can assume μ is positive.*

Proof By Bézout’s Lemma, there exist non-zero integers a and b such that $aA + bB = e$. If $A = 1$, take $\lambda = 1$ and $\mu = 0$. If $A > 1$ and $B = kA$, take $\lambda = k + 1$ and $\mu = 1$. In all other cases, A does not divide e , so it does not divide b , and $b = kA - r$ with $0 < r < A$. Then $(kB + a)A - rB = e$ and

$$(kB + a) = (r/A)B + (e/A)$$

is a positive integer less than or equal to B , because r/A and e/A are both positive numbers less than 1. Let $\lambda = (kB + a)$ and let $\mu = r$. □

The simple estimate in the following corollary can be sharpened (cf. [4, Corollary 3.3]) but it is sufficient for our needs in this article.

Corollary 6.2 *Let A, B and m be integers and let $N = \max\{|A|, |B|\}$. If there is an integer solution (x, y) to the equation $Ax + By = m$, then there is an integer solution (x_0, y_0) with $\max\{|x_0|, |y_0|\} \leq |m| + N^2$.*

Proof If $N = 0$ the assertion is trivial, so suppose $|A| \geq 1$. Write $m = kA + r$ with $|k| \leq |m|$ and $0 \leq r < |A|$. As m lies in the subgroup of \mathbb{Z} generated by A and B , so does r . This subgroup is generated by $e = \gcd(A, B)$, so $r = r_0e$ for some $r_0 \leq r < |A|$. The lemma allows us to write $e = \lambda_0A + \mu_0B$ with $\max\{|\lambda_0|, |\mu_0|\} \leq N$. Then

$$m = (k + \lambda_0r_0)A + \mu_0B,$$

and $|k + \lambda_0r_0| \leq |k| + r|\lambda_0| \leq |m| + N^2$.

7 The proof of Theorem 1.1

At the end of Section 3 we proved that $CL_{\Gamma_d}(n) \geq n^d$, so the following proposition completes the proof of Theorem 1.1.

Proposition 7.1 *There is an integer B_d such that if $u, v \in \Gamma_d$ are conjugate and $\max\{d(1, u), d(1, v)\} \leq n$, then there exists $g \in \Gamma_d$ with $g^{-1}ug = v$ and $d(1, g) \leq B_d n^d$.*

We will again proceed by induction on d . If $d = 1$ then $\Gamma_d = \mathbb{Z}^2$ and the proposition is trivial. If $d = 2$ then Γ_d is the 3-dimensional integral Heisenberg group and since u and v are conjugate in Γ_2 , their images in $\Gamma_1 \cong \mathbb{Z}^2$ are equal. So $u = t^r a_1^\alpha a_2^\eta$ and $v = t^r a_1^\alpha a_2^\xi$ for some $r, \alpha, \eta, \xi \in \mathbb{Z}$, and there exist $s, \omega \in \mathbb{Z}$ such that $g = t^s a_1^\omega$ satisfies $ug = gv$ in Γ_d . Now in Γ_2 ,

$$ug = (t^r a_1^\alpha a_2^\eta)(t^s a_1^\omega) = t^{r+s} \phi^s(a_1^\alpha a_2^\eta) a_1^\omega = t^{r+s} a_1^{\alpha+\omega} a_2^{s\alpha+\eta}$$

and

$$gv = (t^s a_1^\omega)(t^r a_1^\alpha a_2^\xi) = t^{r+s} \phi^r(a_1^\omega) a_1^\alpha a_2^\xi = t^{r+s} a_1^{\alpha+\omega} a_2^{r\omega+\xi},$$

and these are equal if and only if $s\alpha + \eta = r\omega + \xi$. This condition can be expressed as $s\alpha - r\omega = \xi - \eta$, and if we regard this as an equation with variables s and ω , Corollary 6.2 implies that this equation has an integer solution with $|s|$ and $|\omega|$ both at most $|\xi - \eta| + \max\{|\alpha|^2, |r|^2\}$. Lemma 3.2 tells us that $|\alpha|$ and $|r|$ are at most n and $|\xi|$ and $|\eta|$ are at most n^2 . The existence of a suitable constant B_2 follows.

Assume now that $d \geq 3$ and that the existence of B_{d-1} has been established. Suppose that $u, v \in \Gamma_d$ are conjugate and $\max\{d(1, u), d(1, v)\} \leq n$.

Lemma 7.2 *Proposition 7.1 is true for elements $u, v \in \mathbb{A}_d = \mathbb{Z}^d \rtimes 1 < \Gamma_d$.*

Proof We first consider the case $v \in \langle a_{d-1}, a_d \rangle$. In this case, the conjugates of v all have the form va_d^{em} where e is the exponent sum of a_{d-1} in v , and the unique shortest element conjugating v to va_d^{em} is t^m . Also, $d(1, va_d^{em}) \leq n$ implies $d(1, a_d^{em}) \leq d(1, v) + d(1, va_d^{em}) \leq 2n$, hence $|m| \leq |em| \leq (2n)^d$ by Lemma 3.2.

Next we suppose $v \in \mathbb{A}_d \setminus \langle a_{d-1}, a_d \rangle$. As $\langle a_{d-1}, a_d \rangle$ and \mathbb{A}_d are normal, u also lies in $\mathbb{A}_d \setminus \langle a_{d-1}, a_d \rangle$. The unique shortest element conjugating u to v is again a power of t . Proposition 4.1(2) tells us that this same power of t is the unique shortest element conjugating the image of u to the image of v in Γ_{d-1} , and by induction this has length less than $B_{d-1} n^{d-1}$. □

The generic case in Proposition 7.1. Continuing our proof of Proposition 7.1, we now consider the case where the normal forms of u and v contain a non-zero power of t , in other words $u, v \notin \mathbb{A}_d$. Let \bar{u} and \bar{v} be geodesic words representing the images of u and v in Γ_{d-1} . Note that $|\bar{u}| \leq d(1, u)$ and $|\bar{v}| \leq d(1, v)$. Therefore, by induction, there is a word w in the free group on $\{t, a_1, \dots, a_{d-1}\}$ of length at most $B_{d-1} n^{d-1}$ such that $w^{-1}\bar{u}w = \bar{v}$ in Γ_{d-1} .

Theorem 2.2 provides an integer A_{d-1} such that

$$\text{Area}_{\Gamma_{d-1}}(w^{-1}\bar{u}w\bar{v}^{-1}) \leq A_{d-1} |w^{-1}\bar{u}w\bar{v}^{-1}|^d \leq A_{d-1} (2n + 2B_{d-1}n^{d-1})^d \leq A'_{d-1} n^{d(d-1)},$$

where for convenience we define $A'_{d-1} := 2^{2d} A_{d-1} B_{d-1}^d$. It follows from Lemma 2.1 that $w^{-1}uw = va_d^\ell$ in Γ_d where $|\ell| \leq A'_{d-1} n^{d(d-1)}$.

Our inductive proof will be complete if we can prove the following lemma. For then, in the case $v = \gamma$ with $A = A'_{d-1}$, we have $(wz)^{-1}u(wz) = v$ with

$$d(1, wz) \leq |w| + d(1, z) \leq B_{d-1} n^{d-1} + A'_{d-1} E_d n^d$$

and we can take $B_d = B_{d-1} + A'_{d-1} E_d$.

Lemma 7.3 *For $d \geq 3$ and A a positive integer, there is a constant E_d such that whenever $\gamma \in \Gamma_d \setminus \mathbb{A}_d$ is conjugate to γa_d^ℓ with $d(1, \gamma) \leq n$ and $|\ell| \leq A n^{d(d-1)}$, then there exists $z \in \Gamma_d$ such that $z^{-1}\gamma z = \gamma a_d^\ell$ and $d(1, z) \leq A E_d n^d$.*

The proof of the lemma involves the homomorphisms $\zeta_g : C_{\Gamma_{d-1}}(g) \rightarrow \mathbb{Z}$ defined as follows.

Definition 7.4 (The homomorphisms ζ_g) Let $g \in \Gamma_{d-1}$. For each element of the centralizer $x \in C_{\Gamma_{d-1}}(g)$ and all preimages $\tilde{x}, \tilde{g} \in \Gamma_d$ we have $\tilde{x}^{-1}\tilde{g}\tilde{x} = \tilde{g}a_d^m$ in Γ_d , where $m \in \mathbb{Z}$ is independent of the choices of \tilde{x} and \tilde{g} because different choices differ by a power of a_d , which is central. Define $\zeta_g(x) := m$ and note that $\zeta_g : C_{\Gamma_{d-1}}(g) \rightarrow \mathbb{Z}$ is a homomorphism that vanishes on $\langle g \rangle$.

Proof The elements $c \in \Gamma_d$ that conjugate γ to an element of the form γa_d^ℓ are precisely those whose image $\bar{c} \in \Gamma_{d-1}$ centralise the image $\bar{\gamma}$ of γ and have $\zeta_{\bar{\gamma}}(\bar{c}) = \ell$. In the light of these comments, the lemma is a consequence of the following claim.

Claim 1: If $g \in \Gamma_{d-1} \setminus \mathbb{A}_{d-1}$ with $d(1, g) \leq n$ and $\ell \in \text{im } \zeta_g$ with $|\ell| \leq A n^{d(d-1)}$, then there exists $y \in C_{\Gamma_{d-1}}(g)$ with $d(1, y) < A E_d n^d$ and $\zeta_g(y) = \ell$.

Before proving Claim 1, we examine $\text{im } \zeta_g$. From Proposition 4.1(1) we know that $C_{\Gamma_{d-1}}(g) = \langle g_0, a_{d-1} \rangle \cong \mathbb{Z}^2$ where

$$g = g_0^p a_{d-1}^r$$

with $p > 0$ maximal. A count of the letters t gives $p \leq n$. Note that by moving the a_{d-1} letters into g_0 if necessary, we may assume $0 < r < p$.

Claim 2: If $g = g_0^p a_{d-1}^r = t^{pq} x$ with $x \in \mathbb{A}_{d-1}$ and $0 < r < p$, then $\text{im } \zeta_g < \mathbb{Z}$ is generated by $\zeta_g(a_{d-1}^{-1}) = pq$ and $\zeta_g(g_0) = rq$. Hence $\text{im } \zeta_g = qe\mathbb{Z}$, where $e = \text{gcd}(p, r)$.

To see the truth of Claim 2, recall that $\zeta_g(h)$ is independent of the choices of preimages that we take in Γ_d , so writing x as a geodesic word in the letters a_i , we can take $t^{pq}x$ as the preimage of g in Γ_d and calculate

$$-\zeta_g(a_{d-1}) = \zeta_g(a_{d-1}^{-1}) = pq \tag{4}$$

as follows

$$a_{d-1} t^{pq} x a_{d-1}^{-1} = t^{pq} \phi_d^{pq}(a_{d-1}) x a_{d-1}^{-1} = t^{pq} a_{d-1} a_d^{pq} x a_{d-1}^{-1} = t^{pq} x a_d^{pq},$$

and we can verify that $\zeta_g(g_0) = qr$ by fixing a preimage $\tilde{g}_0 = t^q y$ with $y \in \mathbb{A}_d$ then defining $\tilde{g} := \tilde{g}_0^p a_{d-1}^r$ and calculating

$$\tilde{g}_0^{-1} (\tilde{g}_0^p a_{d-1}^r) \tilde{g}_0 = \tilde{g}_0^p \tilde{g}_0^{-1} a_{d-1}^r \tilde{g}_0 = \tilde{g}_0^p y^{-1} \phi_d^q(a_{d-1}^r) y = \tilde{g}_0^p y^{-1} a_{d-1}^r a_d^{qr} y = \tilde{g}_0^p a_d^{qr}.$$

This proves Claim 2.

With Claim 2 in hand, our next goal is to find a small element of $C_{\Gamma_{d-1}}(g)$ whose image generates $\text{im } \zeta_g$. Lemma 6.1 provides us with positive integers $\lambda < p$ and $\mu \leq r < p$ such that $\lambda r - \mu p = e$. Then, for $g = g_0^p a_{d-1}^r$ we have

$$\zeta_g(g_0^\lambda a_{d-1}^\mu) = \lambda \zeta_g(g_0) - \mu \zeta_g(a_{d-1}^{-1}) = \lambda r q - \mu p q = qe. \tag{5}$$

Corollary 5.2, applied to the equality $g a_{d-1}^{-r} = g_0^p$ in Γ_{d-1} , gives us the first of the following inequalities and $r < p$ gives the second:

$$d(1, g_0) \leq K_{d-1} d(1, g a_{d-1}^{-r}) \leq K_{d-1} (d(1, g) + p). \tag{6}$$

So we estimate that in Γ_{d-1} we have

$$d(1, g_0^\lambda a_{d-1}^\mu) \leq \lambda d(1, g_0) + \mu \leq \lambda K_{d-1} (d(1, g) + p) + \mu \leq p(K_{d-1}(n + p) + 1). \tag{7}$$

The Final Argument. We are now ready to prove Claim 1. We have $q \neq 0$ because $g \notin \mathbb{A}_{d-1}$. So, given $\ell \in \text{im } \zeta_g$ with $|\ell| \leq A n^{d(d-1)}$, because $pq \in \text{im } \zeta_g = qe\mathbb{Z}$, we can write

$$\ell = Mpq + \theta qe$$

for integers θ and M with $0 \leq \theta < p$ and $|M| \leq A n^{d(d-1)}$. Thus, using (4) and (5), we have

$$\ell = M \zeta_g(a_{d-1}^{-1}) + \theta \zeta_g(g_0^\lambda a_{d-1}^\mu) = \zeta_g(a_{d-1}^{-M} (g_0^\lambda a_{d-1}^\mu)^\theta).$$

We are going to argue that the term

$$y := a_{d-1}^{-M} (g_0^\lambda a_{d-1}^\mu)^\theta$$

from the last bracket satisfies Claim 1.

From the triangle inequality in Γ_{d-1} we have, invoking Lemma 3.4 and estimate (7) in the second line, and using $|M| \leq A n^{d(d-1)}$ and $p \leq n$ in the third,

$$\begin{aligned} d(1, y) &= d(1, a_{d-1}^{-M} (g_0^\lambda a_{d-1}^\mu)^\theta) \leq d(1, a_{d-1}^M) + \theta d(1, g_0^\lambda a_{d-1}^\mu) \\ &\leq D_{d-1} |M|^{1/(d-1)} + p^2 (K_{d-1} (n + p) + 1) \\ &\leq D_{d-1} A n^d + n^2 (2K_{d-1} n + 1) \\ &\leq A E_d n^d \end{aligned}$$

provided $d \geq 3$ and we choose E_d to be sufficiently larger than D_{d-1} and K_{d-1} . This completes the proof of Claim 1, hence Lemma 7.3, Proposition 7.1 and Theorem 1.1. \square

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