Ring-Theoretic Properties of Augmented Iwasawa Algebras

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Abstract

Let p be a prime, G be a p-adic Lie group and k be a field of characteristic p. The augmented Iwasawa algebra kG is a noncommutative algebra extending the construction of the Iwasawa algebra of a compact p-adic Lie group. In this thesis we investigate the ring-theoretic properties of augmented Iwasawa algebras.

We prove that a split-semisimple group over a p-adic field F has a coherent augmented Iwasawa algebra if and only if its root system is of rank one, and deduce that $kGL_n(F)$ is coherent precisely when n is at most two. We also characterise when certain solvable p-adic Lie groups have a coherent augmented Iwasawa algebra. When kG is a coherent ring, the category of finitely-presented smooth representations of G over k is abelian. We give a module-theoretic characterisation of the latter property, and hence give examples of p-adic Lie groups for which it fails. These results are achieved by developing the theory of augmented Iwasawa algebras in detail, including their relation to smooth and admissible representations of G.

The Iwasawa algebra of a compact p-adic Lie group is always an Auslander-Gorenstein ring. Thus finitely-generated modules over Iwasawa algebras have a canonical dimension. Let G be a compact p-adic Lie group whose Lie algebra is isomorphic to a split simple F-Lie algebra. We prove that whenever F is a non-trivial finite extension of the p-adic numbers, kG has no modules of canonical dimension one. Consequently kG has a module of canonical dimension one only if the Lie algebra of G is isomorphic to $\mathfrak{sl}_2(\mathbb{Q}_p)$. As a corollary we obtain a new upper bound on the Krull dimension of kG.

When G is an arbitrary p-adic Lie group, any smooth admissible representation V is the dual of a kG-module M, finitely-generated over the Iwasawa algebra of a compact open subgroup of G. The canonical dimension of M measures the rate of growth of invariant subspaces of V. We prove a canonical dimension-theoretic criterion for V to be a finite length representation. We deduce that the finite length smooth admissible representations of $GL_2(\mathbb{Q}_p)$ with central character are precisely those whose duals have canonical dimension at most one. Finally, let F be a non-trivial finite extension of the p-adic numbers, $G = GL_n(F)$, and V be a smooth admissible representation of G with central character. A combination of our results shows that V has finite length if its dual has canonical dimension two.

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1 Introduction

Notation. Throughout this thesis, p denotes a prime number. We denote the finite field with p elements by $\mathbb{F}_p$, the ring of p-adic integers by $\mathbb{Z}_p$, and the field of p-adic numbers by $\mathbb{Q}_p$. Let F be a finite field extension of the p-adic numbers, unless otherwise stated, and let $\mathcal{O}_F$ be its ring of integers.

When G is a profinite group and we wish to study its representations over a coefficient ring k, it is natural to replace the usual group algebra $k[G]$ with the *completed group algebra* kG. Since G is an inverse limit of finite groups, we define the completed group algebra to be an inverse limit of finite group algebras. In particular, the completed group algebra of a finite group is the usual group algebra. Studying the completed group algebra is preferable to the usual group algebra because the latter construction essentially forgets the topological group structure on G. On the other hand, the pseudocompact topology on kG ([Bru66]) algebraically encodes the topology on G into the algebra, and matches up continuous representations of G with kG-modules.

Among all profinite groups, the subclass of compact p-adic Lie groups is a very special and interesting one. For example, every closed subgroup of a compact p-adic Lie group G is topologically finitely-generated. In particular every ascending chain of closed subgroups in G must terminate. Consequently, kG must satisfy the ascending chain condition on the corresponding ideals. In fact, the ascending chain condition holds for all ideals due to the fundamental work of Lazard in [Laz65]:

Theorem 1.1 (Theorem 2.2, Theorem 3.32)**:**

> Let G be a compact p-adic Lie group, and k be either a field of characteristic p or a complete discrete valuation ring with residue characteristic p. Then the completed group algebra kG is a Noetherian ring.

When k is a field of characteristic p or a discrete valuation ring of residue characteristic p, for example $\mathbb{F}_p$ or $\mathbb{Z}_p$, we call the completed group algebra kG an *Iwasawa algebra*. When $k = \mathbb{Z}_p$ and $G = (\mathbb{Z}_p, +)$, the Iwasawa algebra $\mathbb{Z}_p[[T]]$ plays an important role in the subject of Iwasawa theory, so called due to Iwasawa's study of infinite towers of number fields. However, in this thesis we focus on noncommutative Iwasawa algebras, giving rise to questions of a rather different nature.

For the remainder of this introduction, let k be a field of characteristic p.

Much of the motivation in our work is due to a wish to understand better the representation theory of p-adic Lie groups over the field k. In turn, this study plays a large part in the *mod p Langlands programme*, which predicts a correspondence between (certain) smooth admissible representations of $GL_n(F)$ and (certain) n-dimensional representations of the absolute Galois group of F.

Kohlhaase has defined in [Koh17] a generalisation of the Iwasawa algebra for a non-compact p-adic Lie group G, called the augmented Iwasawa algebra, also denoted kG. This terminology recognises that an augmented representation of G, defined by Emerton in [Eme10], is exactly a module over the augmented Iwasawa algebra of G, see Corollary 3.31. Augmented Iwasawa algebras arise naturally in the study of smooth representations of G carried out in Section 4. For example, any smooth representation of G has a unique kG-module structure, Corollary 3.18.

We can also characterise the duals of smooth admissible representations of G, as follows.

Theorem 1.2 (Corollary 4.19)**:**

> Let V be a smooth representation of G and $H \leq G$ be a compact open subgroup. The dual $V^\vee = \mathrm{Hom}_k(V, k)$ is a kG-module, and V is admissible if and only if $V^\vee$ is finitely-generated as a kH-module.

Combining Theorem 1.2 with Theorem 1.1, it follows that the smooth admissible representations of any p-adic Lie group G form an abelian category, Corollary 4.26. Theorem 1.1 plays a crucial role in the proof because a ring is Noetherian precisely when its category of finitely-generated modules is abelian.

These results spur us to understand the ring-theoretic properties of augmented Iwasawa algebras. The most obvious question to tackle is whether augmented Iwasawa algebras are Noetherian. However, a non-compact p-adic Lie group G will typically contain a non-terminating ascending chain of closed subgroups. Thus in contrast to Theorem 1.1, the augmented Iwasawa algebra of G is almost never Noetherian.

One might hope that augmented Iwasawa algebras are coherent rings, meaning that every finitely-generated ideal is also finitely-presented. Equivalently, a ring is coherent precisely when its category of finitely-presented modules is an abelian category. Shotton has shown in [Sho20] that the augmented Iwasawa algebra of $SL_2(F)$ is coherent, and we see in subsection 7.3 that the argument naturally extends to $GL_2(F)$. We also show that the augmented Iwasawa algebra of any unipotent group is coherent, see Corollary 7.2.

However, one of the main purposes of this thesis is to prove that augmented Iwasawa algebras are rarely coherent. We first state the result for semisimple groups over F.

Theorem A:

Let $\mathbb{G}$ be a split-semisimple affine group scheme defined over F. Let $G = \mathbb{G}(F)$. Then the augmented Iwasawa algebra kG is coherent if and only if the rank of the root system of $\mathbb{G}$ is 1.

That is, any split-semisimple $G = \mathbb{G}(F)$ has non-coherent augmented Iwasawa algebra unless $\mathbb{G} = SL_2$ or PGL_2. From Theorem A we can deduce the corresponding result for general linear groups.

Corollary B:

The augmented Iwasawa algebra of $GL_n(F)$ is coherent if and only if $n \leq 2$.

In order to prove the above results, we consider solvable p-adic Lie groups. In this case the augmented Iwasawa algebra may be constructed from Iwasawa algebras via ring-theoretic tools such as skew polynomial rings and Ore localisation, see Section 6. By applying a result on the coherence of skew polynomial rings, subsection 6.4, alongside careful study of two minimal counterexamples in Section 8, we obtain a classification of coherence for a useful, relatively wide, class of solvable p-adic Lie groups.

Deferring the full details until Section 9, we consider upper-triangular solvable groups $T \ltimes U \leq GL_n(F)$ where U has F-Lie algebra $\mathfrak{u}$ and set of roots Φ. We define a group homomorphism $f_G : T \to \mathbb{Z}^\Phi$ which records the p-adic valuations of the action of T on $\mathfrak{u}$. The homomorphism f_G must satisfy a restrictive condition for $k(T \ltimes U)$ to be coherent.

Theorem C:

Let $G = T \ltimes U$ be an upper-triangular subgroup of $GL_n(F)$ as described above, with set of roots Φ and group homomorphism $f_G : T \to \mathbb{Z}^\Phi$. The augmented Iwasawa algebra kG is a coherent ring if and only if the image $f_G(T) \leq \mathbb{Z}^\Phi$ is a cyclic subgroup, generated by an element of $(\mathbb{Z}_{\geq 0})^\Phi$.

An important case of the above appears when T is a split F-torus, in which case $T \ltimes U$ is a split-solvable group over F.

Corollary D:

Let $\mathbb{G}$ be a finite-dimensional split-solvable affine group scheme defined over F, and $G = \mathbb{G}(F)$. If the root system of $\mathbb{G}$ has rank 2 or greater, then kG is not coherent.

Crucially, the class of groups in Corollary D contains Borel subgroups of split-semisimple groups. Theorem A and Corollary B are then a consequence of Corollary D combined with a crucial faithful flatness result, proved in Section 5. Namely, Theorem 5.13 states that whenever $G_1 \leq G$ is a closed subgroup of a p-adic Lie group, the augmented Iwasawa algebra kG is a faithfully flat kG_1-module. Since the property of coherence descends along faithfully flat maps, Proposition 6.13, our results follow by embedding solvable groups into semisimple groups and general linear groups.

Shotton has shown in [Sho20] that the category of finitely-presented smooth representations of G is an abelian category if the augmented Iwasawa algebra of G is coherent. Our results raise the question of whether this category remains abelian in the absence of the augmented Iwasawa algebra being coherent. Emerton, Gee, and Hellmann have conjectured that this holds for the general linear groups in [EGH]. I do not know the answer in this case, or for any other reductive or semisimple groups. However, in Section 10 we give a module-theoretic criterion, which allows us to answer the question in the negative for some solvable p-adic Lie groups. We also reprove the main result of [Sho20] by a novel argument, reducing to a Borel subgroup and using Theorem C, see Theorem 10.23.

See subsection 9.1 for the full statement of Theorem C, and subsections 9.2-9.4 for its proof. Then, Corollary D and Theorem A are deduced in subsection 9.5. Corollary B follows directly in subsection 9.6.

Recall from Theorem 1.2 that the dual of any smooth admissible representation of a p-adic Lie group is a finitely-generated module over an Iwasawa algebra. Because Iwasawa algebras are Noetherian, we already know that the finitely-generated modules form an abelian category, but the Iwasawa algebra has further useful ring-theoretic properties with which to study these modules. The following was first shown in [Ven02].

Theorem 1.3 (Theorem 2.2)**:**

Let G be a compact p-adic Lie group. The Iwasawa algebra kG is an Auslander-Gorenstein ring.

Auslander-Gorenstein noncommutative rings are a generalisation of Gorenstein commutative rings (Noetherian of finite Krull and injective dimension) by the addition of the *Auslander condition*. See, for example, [Lev92]. In particular, associated to every finitely-generated module M over an Auslander-Gorenstein ring R is an integer known as the canonical dimension, $\mathrm{Cdim}(M)$.

The canonical dimension is homologically defined, see Definition 2.4. However when M is a module over an Iwasawa algebra kG, it is fruitful to notice that the canonical dimension is the dimension of an affine variety associated to M, known as the characteristic variety, see Theorem 2.10. We therefore are inclined to view $\mathrm{Cdim}(M)$ as a measure of the "size" of M, expressed as an integer lying between zero and $\dim G$.

In fact, the canonical dimension directly measures "growth" in admissible representations. Let V be a (smooth) admissible representation of an arbitrary p-adic Lie group G with compact open subgroup $H \leq G$. The invariant subspaces V^K form an increasing chain of finite-dimensional vector spaces as K varies along a decreasing chain of compact open subgroups. On the other hand, Theorem 1.2 tells us that $M = V^\vee$ is a finitely-generated module over the Iwasawa algebra kH, thus has a canonical dimension. Emerton and Paškūnas have proved a formula connecting these two phenomena in [EP20]: the asymptotic behaviour of the dimensions $\dim_k V^K$ is exponential, with growth rate $\mathrm{Cdim}(M)$, see Theorem 12.1. The canonical dimension also appears, although not by name, in conjectures on completed (co)homology [CE12] and in the work of Gee and Newton, [GN20].

More recently, and partially motivated by the works mentioned above, the canonical dimension has been studied by Breuil, Herzig, Hu, Morra, Schraen, and Wang. Their primary interest has been in representations associated with the mod p cohomology of Shimura varieties, which are believed to realise a mod p (local) Langlands correspondence. In [BHH$^+$23] and [HW22], the canonical dimensions associated to such representations of $GL_2(F)$, when F is unramified, are determined and found to be equal to the degree of the field extension $|F : \mathbb{Q}_p|$. In [BHH$^+$], the authors make numerous conjectures towards a mod p Langlands correspondence, and prove many of their conjectures due to the results of [BHH$^+$23] and [HW22].

We take a complementary perspective to that of the above authors: rather than computing the canonical dimension for certain representations, we fix a group and investigate, for a given integer, whether a representation exists with canonical dimension equal to that value.

This approach was taken by Ardakov and Wadsley in studying the $\mathbb{Q}_p$-Iwasawa algebras (close cousins of those studied in this thesis). In [AW13] it is shown that the canonical dimension of a finitely-generated module for $\mathbb{Q}_pG$ is bounded below by half the dimension of a coadjoint orbit. Similar results are believed to hold for complex representations and in characteristics different from p, [Vig, section 16]. However, for characteristic p representations, little has been discovered in the fifteen to twenty years since [Ard04] and [Wad07].

We consider compact p-adic Lie groups G whose Lie algebras $\mathcal{L}(G)$ – see Definition 2.6 – are simple and split over F. Examples include $G = SL_n(\mathcal{O}_F)$, as well as others corresponding to root systems of type B, C, D, E, F, G. These compact p-adic Lie groups may be thought of as "almost simple" – the only closed normal subgroups are either finite or of finite index. We show that the canonical dimension of a module over the Iwasawa algebra cannot be 1 in many cases.

Theorem E:

Let $\mathfrak{g}$ be a finite-dimensional F-Lie algebra which is simple and split over F. Let G be a compact p-adic Lie group with associated Lie algebra $\mathcal{L}(G) \cong \mathfrak{g}$. If $|F : \mathbb{Q}_p| > 1$, then the Iwasawa algebra kG has no finitely-generated modules of canonical dimension one.

From this, we deduce corollaries in two different directions. A restriction on the canonical dimension of modules over a noncommutative ring gives an upper bound on the Krull(-Gabriel-Rentschler) dimension of the ring, Proposition 11.22. Therefore Theorem E implies a non-trivial upper bound on the Krull dimension of kG.

Corollary F:

Let $\mathfrak{g}$ be a finite-dimensional F-Lie algebra which is simple and split over F. Let G be a compact p-adic Lie group with associated Lie algebra $\mathcal{L}(G) \cong \mathfrak{g}$. If $|F : \mathbb{Q}_p| > 1$, the Krull dimension of kG is at most $\dim G - 1$.

In [Bru66] it was proved that the global dimension of kG is $\dim G$ (or infinity). Thus Corollary F shows that the Iwasawa algebra has Krull dimension strictly less than its global dimension. This generalises a result of Ardakov in [Ard04]. Moreover Corollary F implies that a more precise conjecture of Ardakov and Brown holds for the group $SL_2(\mathcal{O}_F)$, when F is a quadratic extension of the p-adic numbers, see Question 4.30 and Corollary 11.24.

Let us mention that although Theorem E and Corollary F generalise results proved when $F = \mathbb{Q}_p$ in [Ard04], the methods used here are somewhat different. We explicitly rely on having a non-trivial finite field extension of $\mathbb{Q}_p$, and use group structures most clearly found in $SL_2(\mathcal{O}_F)$, which Ardakov's methods cannot apply to.

Theorem E is a lower bound on the canonical dimension of infinite-dimensional modules for an Iwasawa algebra. This may be viewed as a first step towards a Bernstein's inequality for representations of p-adic Lie groups. In fact, this perspective gives our second application of Theorem E, in the representation theory of non-compact p-adic Lie groups.

We say that an infinite-dimensional representation V is *holonomic* when the canonical dimension $\mathrm{Cdim}(V^\vee)$ is minimal among infinite-dimensional admissible representations of G (Definition 12.4). We prove a result showing that such a representation is often of finite length.

Theorem G:

Let G be a p-adic Lie group. Suppose G acts trivially on all of its finite-dimensional smooth representations over k. Every holonomic smooth admissible representation of G over k is of finite length.

For example, split-simple p-adic Lie groups such as $SL_n(F)$ satisfy the condition of Theorem G. One consequence is a characterisation, purely in terms of the canonical dimension, of the admissible finite length representations of $GL_2(\mathbb{Q}_p)$ with central character.

Corollary H:

The finite length smooth admissible representations V of $GL_2(\mathbb{Q}_p)$ with central character are precisely those with $\mathrm{Cdim}(V^\vee) \leq 1$.

In the case $GL_n(F) \neq GL_2(\mathbb{Q}_p)$, we apply Theorem E to Theorem G and deduce that any admissible representation with central character has finite length if its dual has canonical dimension two.

Corollary J:

Let V be a smooth admissible representation of $GL_n(F)$ with central character. If $|F : \mathbb{Q}_p| > 1$ or $n > 2$, and $\mathrm{Cdim}(V^\vee) = 2$, then V is of finite length.

Therefore, the admissible representations of $GL_2(\mathbb{Q}_{p^2})$ considered in [BHH$^+$23] and [HW22] are necessarily of finite length, as conjectured in [BHH$^+$] – see subsection 12.2.

Theorem E is proved in subsection 11.4, and Corollary F is then deduced in subsection 11.6. The proof of Theorem G may be found in subsection 12.1, with Corollary H and Corollary J proved in subsection 12.2.

NB. Much of the content of this thesis has appeared in [Tim23] and [Tim]. The reader looking for a more streamlined account may like to look there. I would like to thank the anonymous referee whose careful reading greatly improved [Tim23]. Nicolas Dupré and Marie-France Vignéras kindly informed me of typos in [Tim], which have been corrected in this thesis. Benjamin Schraen offered helpful comments on an earlier version of that article. Finally, I am grateful for the comments Dan Ciubotaru, Karin Erdmann and Nikolay Nikolov made on early versions of the articles during the transfer and confirmation of status processes at Oxford.

2 Augmented Iwasawa algebras

2.1 p-adic Lie groups

We recall definitions and basic facts about six important types of topological groups.

A profinite group is a topological group isomorphic to an inverse limit of finite discrete groups, or equivalently, a compact Hausdorff totally disconnected topological group. A locally profinite group is a topological group with an open profinite subgroup, or equivalently, a locally compact Hausdorff totally disconnected topological group.

A p-adic Lie group (also known as a p-adic analytic group) is a topological group locally homeomorphic to $\mathbb{Z}_p^d$ for some $d \geq 0$. Then d is known as the dimension of the group. A compact p-adic Lie group is a p-adic Lie group which is compact, or equivalently, a profinite group which is a p-adic Lie group. Moreover, a topological group is a p-adic Lie group if and only if it has an open subgroup which is a compact p-adic Lie group.

A pro-p group is a topological group isomorphic to an inverse limit of finite p-groups. A uniform pro-p group [DDMS03, Definition 4.1] is a topologically finitely-generated pro-p group with conditions on its p-th power and commutator subgroups. A topological group is a p-adic Lie group if and only if it has an open uniform pro-p subgroup. A topological group is a compact p-adic Lie group if and only if it contains a uniform pro-p subgroup of finite index.

The following diagram summarises the inclusions between the six classes of groups discussed.

$$\begin{array}{ccccc}
\text{pro-}p\text{ groups} & \subseteq & \text{profinite groups} & \subseteq & \begin{array}{c}\text{locally profinite}\\ \text{groups}\end{array} \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\begin{array}{c}\text{uniform pro-}p\\ \text{groups}\end{array} & \subseteq & \begin{array}{c}\text{compact }p\text{-adic}\\ \text{Lie groups}\end{array} & \subseteq & \begin{array}{c}p\text{-adic}\\ \text{Lie groups}\end{array}
\end{array}$$

In this thesis we will be particularly interested in (compact) p-adic Lie groups, with uniform pro-p groups playing a useful technical role. However, many of our results are stated in the greater generality of locally profinite groups.

2.2 The Iwasawa algebra

When G is a profinite group, we can define its completed group ring as an inverse limit of group rings of finite groups.

Definition 2.1:

Let G be a profinite group and R be a commutative ring. The completed group ring is

$$RG = \varprojlim_{U \trianglelefteq_o G} R\left[{}^{G}\!/\!_{U}\right],$$

where the limit is taken over the inverse system of open normal subgroups of G, with respect to reverse inclusion.

In the case when p is a prime, G is a compact p-adic Lie group, and k is a field of characteristic p, the completed group ring kG is known as the (mod p) Iwasawa algebra of G. In this case, the noncommutative ring kG has very nice properties, which we summarise.

Theorem 2.2:

Let k be a field of characteristic p and G be a compact p-adic Lie group. The Iwasawa algebra kG is a left and right Noetherian ring, which is Auslander-Gorenstein of injective dimension $\dim G$. When G has no p-torsion elements, kG is moreover Auslander-regular.

Proof:

(The case $k = \mathbb{F}_p$ follows from Corollary 7.25 of [DDMS03], Theorem J of [AB07], and Theorem 3.26 of [Ven02]. See also Lemma 6.2 of [Ard12].)

We assume that G is a uniform pro-p group, and show that the Iwasawa algebra kG has a filtration such that $\operatorname{gr} kG$ is a polynomial ring over k in $\dim G$ variables. When $k = \mathbb{F}_p$, section 7.4 of [DDMS03] shows that under the I-adic filtration, where $I = \mathbb{F}_p[G](G-1)$, the completion $\widehat{\mathbb{F}_p[G]} = \mathbb{F}_pG$, and

$$\operatorname{gr} \mathbb{F}_p[G] \cong \operatorname{gr} \mathbb{F}_pG \cong \mathbb{F}_p[X_1, \ldots, X_d],$$

where d is the dimension of G, [DDMS03, Theorem 7.24].

When k is an arbitrary field extension of $\mathbb{F}_p$, we consider the I_k-adic filtration on the group algebra $k[G]$, where

$$I_k = k \otimes_{\mathbb{F}_p} I = k[G](G-1).$$

Clearly ${I_k}^n = k \otimes_{\mathbb{F}_p} I^n$, thus the associated graded ring is

$$\operatorname{gr} k[G] = \operatorname{gr}\left(k \otimes_{\mathbb{F}_p} \mathbb{F}_p[G]\right) = k \otimes_{\mathbb{F}_p} \operatorname{gr} \mathbb{F}_p[G] \cong k[X_1, \ldots, X_d].$$

Since G is uniform pro-p, it is straightforward that $I_k^{p^n} = k[G](G^{p^n} - 1)$ with quotient ring $k[G/G^{p^n}]$, where $G^{p^n} = \{g^{p^n} \mid g \in G\}$. The inverse system $(G^{p^n})_{n\geq 0}$ is cofinal in all open normal subgroups of G, and therefore

$$\widehat{k[G]} = \varprojlim_{n\geq 0} k[G]/I_k^n = \varprojlim_{U \trianglelefteq_o G} k[G/U] = kG.$$

It follows that kG has a complete filtration with associated graded

$$\operatorname{gr} kG = \operatorname{gr} k[G] \cong k[X_1, \ldots, X_d].$$

The polynomial ring $k[X_1, \ldots, X_d]$ is (left and right) Noetherian, and Auslander-regular (of global dimension d). Since the filtration on kG is complete, it follows from [LVO96, Proposition II.2.2.1, Theorem III.2.2.5] that kG has the same properties.

Now suppose G is an arbitrary compact p-adic Lie group. Let G' be an open normal uniform pro-p subgroup. By the arguments of section 2.3 of [AB06], kG is a crossed product with a finite group over the Noetherian ring kG',

$$kG \cong kG' * (G/G'),$$

hence kG is Noetherian. Since kG' is Auslander-regular, Lemma 5.4 of [AB07] shows that kG is Auslander-Gorenstein. By Corollary 5.4 of [AB07], the injective dimension of kG is equal to the injective dimension of kG', which in turn equals the global dimension of kG' by Remark 6.4 of [Ven02]. Thus Theorem 4.1 of [Bru66] implies kG has injective dimension $\dim G' = \dim G$, and if G has no p-torsion elements, that kG is Auslander-regular. □

These properties are easy to check directly when G is abelian.

Example 2.3

Let $G = \mathbb{Z}_p$ and k be a field of characteristic p. The open normal subgroups of G are the $p^n\mathbb{Z}_p$ for $n \geq 0$, with corresponding quotients $\mathbb{Z}/p^n\mathbb{Z}$. Therefore

$$k\mathbb{Z}_p = \varprojlim_{n\geq 0} k\left[{}^{\mathbb{Z}}\!/\!_{p^n\mathbb{Z}}\right] \cong \varprojlim_{n\geq 0} {}^{k[X]}\!/\!_{(X^{p^n})} \cong k[[X]],$$

where $1 + X$ corresponds to the group element $1 + p^n\mathbb{Z} \in \mathbb{Z}/p^n\mathbb{Z}$ for each n, and hence corresponds to $1 \in \mathbb{Z}_p$ in $k\mathbb{Z}_p$. Notice, $(1 + X)^p = 1 + X^p$ since k has characteristic p.

Similar calculations also show that

$$k\mathbb{Z}_p^d \cong \varprojlim_{n\geq 0} k[X_1, \ldots, X_d]\big/(X_1^{p^n}, \ldots, X_d^{p^n}) \cong k[[X_1, \ldots, X_d]],$$

for any $d \geq 0$. This ring is clearly Noetherian, and it is straightforward to check that it is Auslander-regular of injective dimension d.

However, our overriding focus in this thesis will be on non-abelian p-adic Lie groups, for which explicit descriptions of the Iwasawa algebras are much harder to come by, necessitating Theorem 2.2 as a crucial building block.

In fact, Theorem 2.2 may be seen as the source from which the principal problems and results of this thesis flow; in Sections 5 to 9 we determine whether a generalisation of the Noetherian property holds when G is no longer compact, whilst in Sections 11 and 12 we use the Auslander-Gorenstein property of the Iwasawa algebra to study the modules for kG.

2.3 Canonical dimension and the Lie algebra

We turn to the second of the two problems mentioned above. In this subsection, we assume that k is a field of characteristic p. Since kG is Auslander-Gorenstein, Definition 4.5 of [Lev92] gives us the definition of canonical dimension below.

Definition 2.4:

Let G be a compact p-adic Lie group and M be a finitely-generated left kG-module. The canonical dimension of M is

$$\mathrm{Cdim}(M) = \dim G - \min\{j \geq 0 \mid \mathrm{Ext}^j_{kG}(M, kG) \neq 0\}.$$

Many properties of canonical dimension are proved in section 4 of [Lev92], and a summary can be found in subsection 5.3 of [AB06]. We will frequently work with an alternative characterisation of canonical dimension, see Theorem 2.10. Recall also, the definition of Krull dimension from section 6.2 of [MR01]. For R a ring, we denote the Krull dimension of a left R-module M by $\mathcal{K}(M)$. The (left) Krull dimension of the ring is the Krull dimension of R viewed as a free left R-module, denoted $\mathcal{K}(R)$.

Proposition 2.5:

Let G_2 be a compact p-adic Lie group and $G_1 \trianglelefteq_o G_2$ be an open normal subgroup. Then,

$$\{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kG_1)\} = \{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kG_2)\},$$

and the Krull dimensions $\mathcal{K}(kG_1) = \mathcal{K}(kG_2)$.

Proof:

By the arguments of section 2.3 of [AB06], we have that kG_2 is isomorphic to a crossed product,

$$kG_2 \cong kG_1 * (G_2/G_1).$$

Thus kG_2 is a finitely-generated free kG_1-module, and Corollary 6.5.3 of [MR01] implies that $\mathcal{K}(kG_1) = \mathcal{K}(kG_2)$.

Moreover, Lemma 5.4 of [AB07] implies that for any finitely-generated kG_2-module M,

$$\mathrm{Ext}^n_{kG_2}(M, kG_2) \cong \mathrm{Ext}^n_{kG_1}(M, kG_1),$$

and thus $\mathrm{Cdim}_{kG_2}(M) = \mathrm{Cdim}_{kG_1}(M)$. So

$$\{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kG_1)\} \supseteq \{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kG_2)\}.$$

Conversely, let N be a finitely-generated kG_1-module, and let $N = kG_2 \otimes_{kG_1} M$. By Theorem 10.74 of [Rot09],

$$\mathrm{Ext}^n_{kG_2}(N, kG_2) \cong \mathrm{Ext}^n_{kG_1}(M, kG_2) \cong \bigoplus_{j=1}^{m} \mathrm{Ext}^n_{kG_1}(M, kG_1),$$

because kG_2 is a free kG_1-module of rank $m = |G_2/G_1|$. Thus $\mathrm{Ext}^n_{kG_2}(N, kG_2)$ is zero precisely when $\mathrm{Ext}^n_{kG_1}(M, kG_1)$ is zero, from which it follows that $\mathrm{Cdim}_{kG_2}(N) = \mathrm{Cdim}_{kG_1}(M)$. Therefore

$$\{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kG_1)\} = \{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kG_2)\},$$

as required. $\square$

Recall from section 4.5 of [DDMS03] that any uniform pro-p group G has an associated $\mathbb{Z}_p$-Lie algebra L_G, which has underlying set G. We now extend the definition to any compact p-adic Lie group, and the scalars to $\mathbb{Q}_p$.

Definition 2.6:

Let G be a compact p-adic Lie group, and $H \trianglelefteq_o G$ be an open normal uniform pro-p subgroup. The $\mathbb{Q}_p$-Lie algebra $\mathcal{L}(G) = \mathcal{L}(H) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_H$ with the obvious $\mathbb{Q}_p$-linear Lie bracket.

Any compact p-adic Lie group has an open normal uniform pro-p subgroup, by Corollary 8.34 of [DDMS03]. Since the open subgroups of G all have finite index, it is easy to check that $\mathcal{L}(G)$ does not depend on choice of H up to isomorphism. We now show that the canonical dimension of modules for the Iwasawa algebra, and its Krull dimension, are determined only by the Lie algebra of G.

Corollary 2.7:

Let G_1, G_2 be compact p-adic Lie groups with $\mathcal{L}(G_1) \cong \mathcal{L}(G_2)$. Then

$$\{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kG_1)\} = \{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kG_2)\},$$

and the Krull dimensions $\mathcal{K}(kG_1) = \mathcal{K}(kG_2)$.

Proof:

Let $K_1 \trianglelefteq G_1$, $K_2 \trianglelefteq G_2$ be open normal uniform pro-p subgroups of G_1, G_2. Let ϕ be an isomorphism of $\mathbb{Q}_p$-Lie algebras,

$$\phi : \mathcal{L}(G_1) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_{K_1} \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_{K_2} = \mathcal{L}(G_2).$$

Then, $\phi(L_{K_1})$, L_{K_2} are $\mathbb{Z}_p$-Lie lattices in $\mathcal{L}(G_2)$, so for some $n \geq 0$,

$$p^n \phi(L_{K_1}) = \phi(p^n L_{K_1}) \leq L_{K_2}.$$

Now, recall from Theorem 9.10 of [DDMS03] that the assignment $G \mapsto L_G$ is an equivalence of categories between uniform pro-p groups and powerful $\mathbb{Z}_p$-Lie algebras. K_1 is uniform pro-p, so L_{K_1} is a powerful Lie algebra, hence $p^n L_{K_1}$ and $\phi(p^n L_{K_1})$ are powerful. Then $\phi(p^n L_{K_1})$ corresponds to a open uniform pro-p subgroup $K \leq G_2$. The $\mathbb{Z}_p$-Lie algebra $p^n L_{K_1}$ corresponds to $K_1^{p^n} = \{x^{p^n} \mid x \in K_1\} \leq G_1$, and thus the restriction of ϕ gives us an isomorphism of compact p-adic Lie groups $\psi : K_1^{p^n} \to K$. Now, let

$$H = \bigcap_{g \in G_2/K} gKg^{-1},$$

so H is an open normal subgroup of G_2. It follows that H is also normal in K. So, we have open normal subgroups $K_1^{p^n} \trianglelefteq_o G_1$, $H \trianglelefteq_o K = \psi(K_1^{p^n})$, and $H \trianglelefteq_o G_2$. By Proposition 2.5, it follows that

$$\begin{aligned}\{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kG_1)\} &= \{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kK_1^{p^n})\} \\ &= \{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kK)\} \\ &= \{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kH)\} \\ &= \{\mathrm{Cdim}(M) \mid M \in \mathrm{mod}(kG_2)\},\end{aligned}$$

and $\mathcal{K}(kG_1) = \mathcal{K}(kK_1^{p^n}) = \mathcal{K}(kK) = \mathcal{K}(kH) = \mathcal{K}(kG_2)$. □

Alternatively to Definition 2.4, we may understand the canonical dimension as the dimension of an affine variety known as the *characteristic variety*. When G is a uniform pro-p group, kG is a non-commutative local ring with unique maximal ideal $J = \mathrm{Ker}\,(kG \to k)$.

Proposition 2.8:

Let G be a uniform pro-p group. The J-adic filtration on kG is complete, and the associated graded ring is isomorphic to a commutative polynomial ring in $\dim G$ variables over k.

Proof:

Let G be as in the proof of Theorem 2.2, where it is shown that kG has a complete filtration given by the ideals $J_n = \mathrm{Ker}\,(kG \to k[G]/I_k^n)$, with associated graded ring

$$\mathrm{gr}\, kG \cong k[X_1, \ldots, X_{\dim G}].$$

Thus it is enough to show that $J_n = J^n$ for all $n \geq 0$. Clearly $J = J_1$, and a direct computation shows that $(J^m + J_n)/J_n = (J_m + J_n)/J_n$ for all $m, n \geq 0$. Therefore the closure of J^m in kG is J_m (in the induced pseudocompact topology on kG, see [Bru66]). But kG is Noetherian, hence Corollary 22.4 of [Sch11] implies all ideals are closed, and the result follows. □

The characteristic variety of a module is given by the *characteristic ideal* in the commutative ring $\mathrm{gr}\, kG$ coming from the J-adic filtration.

Definition 2.9:

Let G be a uniform pro-p group, M be a finitely-generated kG-module. The characteristic ideal of M is the radical ideal

$$J(M) = \sqrt{\mathrm{Ann}(\mathrm{gr}\, M)} \subseteq \mathrm{gr}\, kG,$$

where $\mathrm{Ann}(\mathrm{gr}\, M) = \{x \in \mathrm{gr}\, kG \mid x \cdot m = 0\ \forall m \in \mathrm{gr}\, M\}$ is the annihilator of the graded module $\mathrm{gr}\, M$.

Note that $J(M)$ is an ideal of $\mathrm{gr}\, kG$, whilst the augmentation ideal J is a two-sided ideal of kG.

Theorem 2.10:

Let G be a uniform pro-p group, M be a finitely-generated kG-module. The canonical dimension of M is given by

$$\mathrm{Cdim}(M) = \mathcal{K}\left({}^{\mathrm{gr}\, kG}\!\diagup_{J(M)}\right).$$

The proof of Theorem 2.10 is essentially given in section 5.3 of [AB06], where it is shown that $\mathrm{Cdim}(M) = \mathcal{K}(\mathrm{gr}\, M)$, which is easily seen to be equal to $\mathcal{K}\left(\mathrm{gr}\, kG/J(M)\right)$. This integer can be naturally seen as the dimension of the variety associated to $J(M)$, which we do not define. We will frequently use Theorem 2.10 rather than Definition 2.4 when discussing the canonical dimension of a kG-module. However, we now leave the study of canonical dimension and Krull dimension, returning to it in Section 11.

2.4 Augmented Iwasawa algebras

We now define the rings mentioned in the title of this thesis. Recall that the completed group ring of a profinite group was defined over arbitrary coefficient rings in Definition 2.1. Kohlhaase has extended the definition to any locally profinite group, see section 1, page 6 of [Koh17]. See also section 6.1 of [Ard21] for a generalised construction. We summarise the definition as the following proposition, which is Proposition 3.2 of [Sho20]. Recall that a locally profinite group is defined to be a topological group with an open profinite subgroup.

Proposition 2.11:

Let G be a locally profinite group, R be a commutative ring. Let $K \leq G$ be an open profinite subgroup, with completed group ring RK. Then there is a unique R-algebra structure on

$$RG = RK \otimes_{R[K]} R[G]$$

such that the natural maps $R[G] \to RG$ and $RK \to RG$ are R-algebra homomorphisms. This R-algebra is independent of the choice of K, up to canonical isomorphism.

This means that RG is generated as an R-algebra by RK and the elements of G.

In the case that G is a p-adic Lie group and k is a field of characteristic p, we call the k-algebra kG the (mod p) augmented Iwasawa algebra of G. If G is compact, then the Iwasawa algebra and the augmented Iwasawa algebra are isomorphic rings, so our notation is unambiguous. The nomenclature "augmented" comes from the fact that the augmented representations given in Definition 2.1.5 of [Eme10] are modules over the augmented Iwasawa algebra, see Corollary 3.31.

Example 2.12

Let $G = \mathbb{Q}_p^\times$ and k be a field of characteristic p. Then G has an open profinite subgroup $\mathbb{Z}_p^\times$, and $\mathbb{Q}_p^\times/\mathbb{Z}_p^\times \cong \mathbb{Z}$. It follows that

$$k\mathbb{Q}_p^\times \cong k\mathbb{Z}_p^\times[X, X^{-1}] \cong k[[t]][Y][X, X^{-1}]/(Y^{p-1}),$$

so in fact $k\mathbb{Q}_p^\times$ is a Noetherian ring. Similarly $kF^\times \cong k\mathcal{O}_F^\times[X, X^{-1}]$ is Noetherian if F is a finite extension of $\mathbb{Q}_p$.

In contrast to Theorem 2.2, it is uncommon for the augmented Iwasawa algebra kG to be Noetherian if G is a non-compact p-adic Lie group (see also Example 6.8).

3 Universal properties

In this section, we prove a number of results which assist in understanding and working with augmented Iwasawa algebras and their modules.

In general, it is useful to consider augmented Iwasawa algebras defined over a range of coefficient rings. For example, as well as fields of characteristic p, we may be interested in coefficients such as rings of integers $\mathcal{O}_F$ of a p-adic field F, profinite commutative rings such as $\hat{\mathbb{Z}}$, or the complete local Noetherian $\mathcal{O}_F$-algebras considered in [Eme10] and [Sho20]. The following definition, see the introduction of [Bru66], encompasses all of these cases.

Definition 3.1:

A pseudocompact ring R is a complete Hausdorff topological ring which has a neighbourhood basis of 0 consisting of two-sided ideals I with R/I an Artinian ring.

Throughout the remainder of this section we will assume that k is a commutative pseudocompact ring. Later on, we will specialise to the case of k being a field of characteristic p (as in subsection 2.3), with the discrete topology.

Definition 3.2:

A pseudocompact k-algebra is a complete Hausdorff topological ring A, with a continuous ring homomorphism from k to the centre of A, and such that A has a neighbourhood basis of 0 consisting of two-sided ideals I with A/I a finite-length k-module.

The completed group algebra of a profinite group is naturally a pseudocompact k-algebra, see section 4 of [Bru66].

3.1 Universal property of completed group algebras

The completed group algebra kG of a profinite group satisfies the following property, which is proved over profinite coefficient rings in Proposition 7.1.2 of [Wil98]. Throughout this section, the units $A^\times$ of a topological ring A are given the subspace topology from A, although this need not make $A^\times$ a topological group.

Proposition 3.3:

Let G be a profinite group, k a pseudocompact commutative ring, and A be a pseudocompact k-algebra. Let $\phi : G \to A^\times$ be a continuous group homomorphism to the units of A. There is a unique continuous k-algebra homomorphism $\tilde{\phi} : kG \to A$ extending ϕ.

Proof:

Let $\mathcal{B}$ be a neighbourhood basis of 0 consisting of two-sided ideals I with A/I a finite-length k-module. For such an ideal $I \in \mathcal{B}$, composition of ϕ with the natural map $A^\times \to (A/I)^\times$ gives a continuous group homomorphism

$$\phi_I : G \to (A/I)^\times.$$

Because I is open, $N_I = \operatorname{Ker} \phi_I$ is an open normal subgroup of G, and by the universal property of group algebras, there is a unique k-algebra homomorphism

$$\tilde{\phi}'_I : k[G/N_I] \to A/I$$

extending the natural group homomorphism $\phi'_I : G/N_I \to (A/I)^\times$. To check that $\tilde{\phi}'_I$ is continuous, note that the homomorphism from k to the centre of A that makes A a k-algebra and $\tilde{\phi}'_I$ agree on the restriction

$$i_I : k \to A/I.$$

Because I is open, A/I is discrete and $\alpha_I = \operatorname{Ker} i_I$ is an open ideal of k. Then $k[G/N_I]\alpha_I \subseteq k[G/N_I]$ is open because G/N_I is finite, and $\operatorname{Ker} \tilde{\phi}'_I$ contains $k[G/N_I]\alpha_I$, hence is open. It follows that $\tilde{\phi}'_I$ is continuous.
By composing with the natural homomorphisms $kG \to k[G/N_I]$, we obtain a system of continuous k-algebra homomorphisms

$$\tilde{\phi}_I : kG \to A/I,$$

which are compatible under composition with the maps $A/I \to A/J$ for $I, J \in \mathcal{B}$ with $I \subseteq J$, because $N_I \leq N_J$. Since A is complete, it follows from the universal property of inverse limit that there is a continuous k-algebra homomorphism

$$\tilde{\phi} : kG \to A$$

extending the $\tilde{\phi}_I$, thus extending ϕ. Moreover, there is a unique k-algebra homomorphism $k[G] \to A$ extending ϕ, thus $\tilde{\phi}$ is unique because $k[G]$ is dense in kG. $\square$

This property is enough to define the completed group algebra kG up to canonical isomorphism of pseudocompact k-algebras. Moreover, associating the completed group algebra to a profinite group gives a functor.

Proposition 3.4:

The mapping F given on objects by $F(G) = kG$, and on morphisms by $F(\phi) = \tilde{\phi}$, is a functor from the category of profinite groups (with continuous group homomorphisms) to the category of pseudocompact k-algebras. The functor preserves injectivity and surjectivity of morphisms.

Proof:

Let $\phi : G_1 \to G_2$ be a continuous group homomorphism. By Proposition 3.3, there is a unique extension of $\phi : G_1 \to kG_2^\times$ to a continuous k-algebra homomorphism $\tilde{\phi} : kG_1 \to kG_2$, so F is well-defined. Clearly if $G_2 = G_1$ and $\phi = id_{G_1}$, then $\tilde{\phi} = id_{kG_1}$. If $\psi : G_2 \to G_3$ is another continuous group homomorphism, then $\tilde{\psi} \circ \tilde{\phi} : kG_1 \to kG_3$ extends $\psi \circ \phi$, and therefore is equal to $\widetilde{\psi \circ \phi}$ by uniqueness. Therefore F is a functor.

Now, $\tilde{\phi}$ is constructed by taking the inverse limit of the k-algebra homomorphisms

$$\phi_U : k[G_1/\phi^{-1}(U)] \to k[G_2/U],$$

as U ranges over the open normal subgroups of G_2. If ϕ is injective then ϕ_U is injective for all U, so $\tilde{\phi}$ is injective by left-exactness of the inverse limit.

If ϕ is surjective, the ϕ_U are all surjective. The transition maps that define the inverse limit on each side are all surjective, meaning the strong Mittag-Leffler condition is satisfied and so $\tilde{\phi}$ is surjective, see Definition 4.8.3 and Theorem 4.8.5 of [EH76]. □

Propositions 3.3 and 3.4 can be used to prove basic facts about maps between completed group algebras. Motivated by this, we will generalise this universal property to augmented Iwasawa algebras and to particular modules.

3.2 Profinite modules for an Iwasawa algebra

In this subsection, let H be a profinite group, and recall that k is pseudocompact.

Definition 3.5:

A profinite H-space is a compact Hausdorff totally disconnected topological space X with a continuous action $H \times X \to X$, and *finitely many* H-orbits.

Proposition 1.1.7 of [Wil98] tells us that the topological space X is the inverse limit of its finite discrete quotient spaces, justifying the terminology here.

We impose the condition of having finitely many orbits because we will be most interested in the case where H acts transitively on X. (When a profinite group acts on a profinite space continuously with infinitely many orbits, the notion of *locally profinite* in Definition 3.33 may instead be applied.)

We now show that the topology on X can be defined by an inverse limit of finite H-spaces.

Lemma 3.6:

In the category of topological spaces,

$$X \cong \varprojlim_{U \trianglelefteq_o H} \mathrm{Orb}_U(X),$$

where $\mathrm{Orb}_U(X)$ is the space of U-orbits in X. Moreover, the isomorphism is H-equivariant with respect to the natural H-actions.

Proof:

We show that $\{U \cdot x \mid U \trianglelefteq_o H, x \in X\}$ is a basis for the topology on X.
Let U be an open normal subgroup of H. Let $Z \subseteq X$ be an H-orbit, and consider its U-orbits $\mathrm{Orb}_U(Z)$. Because H acts transitively on Z, the finite group H/U acts transitively on $\mathrm{Orb}_U(Z)$, and therefore $\mathrm{Orb}_U(Z)$ is finite. Because X has finitely many H-orbits, it follows that $\mathrm{Orb}_U(X)$ is finite. Now, U is profinite and hence compact. Thus for any $x \in X$, the U-orbit $U \cdot x$ is a continuous image of a compact set, so is compact. But X is Hausdorff, and therefore $U \cdot x$ is a closed subset of X. Because the U-orbits are closed and give a finite partition of X, it follows that any U-orbit is also open.
Now, let $Y \subseteq X$ be an open subset and $x \in Y$. Let $f : H \times \{x\} \to X$ be the restriction of the map defining the H-action. Then $f^{-1}(Y)$ is open, and hence $V = \{h \in H \mid h \cdot y \in Y\} \subseteq H$ is open. But V contains the identity of H, and therefore V contains an open subgroup of H, hence contains an open normal

subgroup U because H is profinite. Then $U \cdot x \subseteq Y$. It follows that Y is a union of open sets of this form, and so $\{U \cdot x \mid U \trianglelefteq_o H, x \in X\}$ forms a basis of X. Moreover, suppose $Y \subseteq X$ is both open and closed. Then Y is compact, so it is a finite union of open sets in the above basis,

$$Y = \bigcup_{j=1}^{n} U_j \cdot y_j.$$

Let $U = \bigcap_{j=1}^{n} U_j \trianglelefteq_o H$. Then Y is a union of finitely many U-orbits, which must all be disjoint.

Now suppose we have a partition of X into subsets that are both open and closed,

$$X = \bigcup_{j=1}^{n} Y_j.$$

By the above, there exist $U_j \trianglelefteq_o H$ such that Y_j is a union of U_j-orbits. Letting $U = \bigcap_{j=1}^{n} U_j$, we have that each Y_j is a disjoint union of U-orbits, and $U \trianglelefteq_o H$.

By Proposition 1.1.7 of [Wil98], X is the inverse limit of its finite discrete quotient spaces $X/\sim$. Given such a quotient space, let $x_1, \ldots, x_n$ be representatives of the equivalence classes of $\sim$, and $Y_j = \Pi^{-1}(\{\Pi(x_j)\})$, where $\Pi : X \to X/\sim$ is the quotient map. Since $X/\sim$ is discrete, each Y_j is both open and closed, so by the above paragraph there exists $U \trianglelefteq_o H$ such that each Y_j is a disjoint union of U-orbits. The orbit space $\mathrm{Orb}_U(X)$ is a quotient space of X which is finite, and discrete, since orbits are open. Thus we have a commuting diagram of continuous maps

$$\begin{tikzcd}
 & \mathrm{Orb}_U(X) \arrow[d, "q_U"] \\
X \arrow[ur, "\Pi_U"] \arrow[r, "\Pi"] & X/\sim
\end{tikzcd}$$

where Π_U, q_U are the natural quotient maps. So the inverse system of orbit spaces $\mathrm{Orb}_U(X)$ is cofinal in the inverse system of all discrete finite quotients, so by Proposition 1.1.7 of [Wil98], we have that $X \cong \varprojlim_{U \trianglelefteq_o H} \mathrm{Orb}_U(X)$.

Moreover, each of the natural maps $\Pi_U : X \to \mathrm{Orb}_U(X)$, $\Pi_{UV} : \mathrm{Orb}_V(X) \to \mathrm{Orb}_U(X)$, for $V \leq U$ open normal subgroups of H, are easily seen to be H-equivariant. Hence the above topological isomorphism is H-equivariant with respect to the natural H-actions. $\square$

The above description of a profinite H-space X motivates us to define a "completed module" associated to X.

Definition 3.7:

Let X be a profinite H-space. The completed module of X is the (left) kH-module

$$kX = \varprojlim_{U \trianglelefteq_o H} k[\mathrm{Orb}_U(X)].$$

The k-module kX is indeed a kH-module because each $k[\mathrm{Orb}_U(X)]$ is a left $k[H/U]$-module, kH is the inverse limit of these k-algebras, and all morphisms are H-equivariant. Moreover, the $k[H]$-module $k[X]$ naturally embeds into kX, and is easily seen to be dense by definition of the inverse limit topology.

Definition 3.8:

Let A be a pseudocompact k-algebra, M be a (left) topological A-module. M is pseudocompact if and only if it is an inverse limit of discrete A-modules of finite length.

Note that pseudocompact A-modules are automatically Hausdorff, by considering the construction of an inverse limit as a subspace of a product, for example.

Lemma 3.9:

Let X be a profinite H-space. Then kX is a pseudocompact kH-module.

Proof:

Recall that X has finitely many H-orbits. Thus for each $U \trianglelefteq_o H$, the kH-module $k[\mathrm{Orb}_U(X)]$ is a free k-module of finite rank. Let $\mathcal{B}$ be a basis of open ideals I of k with k/I Artinian. Then

$$kX = \varprojlim_{U \trianglelefteq_o H} k[\mathrm{Orb}_U(X)] \cong \varprojlim_{U \trianglelefteq_o H, I \in \mathcal{B}} (k/I)[\mathrm{Orb}_U(X)],$$

and each $(k/I)[\mathrm{Orb}_U(X)]$ is of finite length as a k-module, hence is a (discrete) kH-module of finite length. Therefore kX is a pseudocompact kH-module. $\square$

Lemma 3.10:

1. For X a profinite H-space, $kX \cong \displaystyle\bigoplus_{Z \in \mathrm{Orb}_H(X)} kZ$.
2. Let $J \leq H$ be an open subgroup, and $\mathrm{Res}^H_J X$ be X considered as a profinite

J-space. There is an isomorphism of kJ-modules $i_{J,H} : k(\mathrm{Res}^H_J X) \to \mathrm{Res}^{kH}_{kJ} kX$.

Proof:

1. For each $U \trianglelefteq_o H$, we have that

$$\mathrm{Orb}_U(X) = \bigsqcup_{Z \in \mathrm{Orb}_H(X)} \mathrm{Orb}_U(Z),$$

so there is a $k[H/U]$-module isomorphism,

$$k[\mathrm{Orb}_U(X)] \cong \bigoplus_{Z \in \mathrm{Orb}_H(X)} k[\mathrm{Orb}_U(Z)].$$

By assumption, X has finitely many H-orbits, so this direct sum is also a direct product. Since direct products commute with limits, taking inverse limits gives the result.

2. Since J is open, it has finite index in H, and X has finitely many H-orbits, so X has finitely many J-orbits. So X is indeed a profinite J-space. For a fixed $V \trianglelefteq_o H$ with $V \leq J$, the inverse system $(V \cap U)_{U \trianglelefteq_o H}$ is cofinal in both $(U)_{U \trianglelefteq_o H}$ and $(U')_{U' \trianglelefteq_o J}$. By considering the natural $k[J/V \cap U]$-module homomorphism

$$k[\mathrm{Orb}_{V\cap U}(\mathrm{Res}^H_J X)] \to \mathrm{Res}^{k[H/V\cap U]}_{k[J/V\cap U]} k[\mathrm{Orb}_{V\cap U}(X)],$$

and taking inverse limits, we obtain the desired isomorphism. □

We now prove a universal property of these modules which is analogous to Proposition 3.3.

Proposition 3.11:

Let X be a profinite H-space and M be a pseudocompact kH-module. Let $\phi : X \to M$ be a continuous H-equivariant map. There is a unique extension of ϕ to a continuous kH-module homomorphism $\tilde{\phi} : kX \to M$.

Proof:

Since M is a pseudocompact kH-module, let $M = \varprojlim_{i \in I} M_i$ where each M_i is a discrete finite-length kH-module. The image of the natural H-equivariant continuous map $\phi_i : X \to M_i$ is both discrete and compact, hence finite. Let $U_i \trianglelefteq H$ be the kernel of the natural continuous group homomorphism

$$H \to \mathrm{Sym}(\phi_i(X)),$$

so U_i is an open normal subgroup of H. It follows that ϕ_i has a reduction $\bar{\phi}_i : \mathrm{Orb}_{U_i}(X) \to M_i$. This naturally extends to a continuous $k[H/U_i]$-module homomorphism $\bar{\phi}_i{}' : k[\mathrm{Orb}_{U_i}(X)] \to M_i$, giving a continuous kH-module homomorphism $\tilde{\phi}_i : kX \to M_i$. By construction, the $\tilde{\phi}_i$ agree under the transition morphisms on the M_i. Thus we define

$$\tilde{\phi} : kX \to M = \varprojlim_{i \in I} M_i$$

to be the continuous kH-module homomorphism constructed from the $\tilde{\phi}_i$.
To show uniqueness, note that there is a unique k-module homomorphism $k[X] \to M$ extending ϕ. Since $k[X]$ is dense in kX, it follows there is a unique continuous extension $kX \to M$. $\square$

As for completed group algebras, it follows that the construction of the completed module gives a functor.

Proposition 3.12:
The mapping F given on objects by $F(X) = kX$, and on morphisms by $F(\phi) = \tilde{\phi}$, is a functor from the category of profinite H-spaces (with continuous H-equivariant maps) to the category of pseudocompact kH-modules. The functor preserves injectivity and surjectivity of morphisms.

Proof:
Let $\phi : X_1 \to X_2$ be a continuous H-equivariant map. By Proposition 3.11, there is a unique extension of $\phi : X_1 \to kX_2$ to a continuous kH-module homomorphism $\tilde{\phi} : kX_1 \to kX_2$, so F is well-defined. Clearly if $X_2 = X_1$ and $\phi = id_{X_1}$, then $\tilde{\phi} = id_{kX_1}$. If $\psi : X_2 \to X_3$ is another continuous H-equivariant map, then $\tilde{\psi} \circ \tilde{\phi} : kX_1 \to kX_3$ extends $\psi \circ \phi$, and therefore is equal to $\widetilde{\psi \circ \phi}$ by uniqueness. Therefore F is a functor.
Now, $\tilde{\phi}$ is constructed by taking the inverse limit of the kH-algebra homomorphisms

$$\phi_U : k[\mathrm{Orb}_U(X_1)] \to k[\mathrm{Orb}_U(X_2)],$$

as U ranges over the open normal subgroups of H. If ϕ is injective then ϕ_U is injective for all U, so $\tilde{\phi}$ is injective by left-exactness of the inverse limit.
If ϕ is surjective, the ϕ_U are all surjective. The transition maps that define the

inverse limit on each side are all surjective, meaning the strong Mittag-Leffler condition is satisfied and so $\tilde{\phi}$ is surjective, see Definition 4.8.3 and Theorem 4.8.5 of [EH76]. □

3.3 Universal property of augmented Iwasawa algebras

Motivated by subsections 3.1 and 3.2, we now prove a similar universal property for the augmented Iwasawa algebra. First, we must define an appropriate topology.

Definition 3.13:

Let G be a locally profinite group, with open profinite subgroup H. Then

$$kG = \bigoplus_{g \in H\backslash G} kH \otimes g.$$

The topology on kG is given by the direct sum topology for the topological abelian groups $kH \otimes g$, where the topology on $kH \otimes g$ is defined by requiring the natural right multiplication map $kH \to kH \otimes g$ to be a homeomorphism.

A description of the direct sum topology on abelian topological groups can be found in [CD03], Proposition 5, and a more special case in section 5.E of [Sch02].
Then, we can show kG has the following properties.

Proposition 3.14:

1. The topology on kG is independent of choice of open profinite subgroup $H \leq G$, and is Hausdorff.
2. The natural map $G \to kG$ is a homeomorphism onto its image.
3. If $K \leq G$ is an open profinite subgroup, then K lies in the group of units of a pseudocompact k-subalgebra of kG.
4. The subring $k[G]$ is dense in kG.

Proof:

1. Let $H' \leq H \leq G$ be open profinite subgroups of G. We first show that

$$\bigoplus_{h \in H'\backslash H} kH' \otimes h \cong kH$$

is a topological isomorphism, where the the left hand side has the direct sum topology and the right hand side the inverse limit topology.

Now, the inverse system $X = \{U \mid U \trianglelefteq_o H, U \trianglelefteq_o H'\}$ (under reverse inclusion) is cofinal in both $\{U \trianglelefteq_o H\}$ and $\{U \trianglelefteq_o H'\}$. This follows from the fact that if U is normal in H', then the group

$$V = \bigcap_{h \in H} hUh^{-1}$$

is normal in H, and is a finite intersection of conjugates of U, since H' has finite index in H. Therefore we have topological isomorphisms

$$kH \cong \varprojlim_{U \in X} k\left[{}^{H}\!/\!_{U}\right], \quad kH' \cong \varprojlim_{U \in X} k\left[{}^{H'}\!/\!_{U}\right].$$

Then, we have topological isomorphisms

$$\bigoplus_{h \in H' \backslash H} kH' \otimes h \cong \bigoplus_{h \in H' \backslash H} \left(\varprojlim_{U \in X} k\left[{}^{H'}\!/\!_{U}\right] \otimes h\right) \cong \varprojlim_{U \in X} \left(\bigoplus_{h \in H' \backslash H} k\left[{}^{H'}\!/\!_{U}\right] \otimes h\right),$$

since the direct sums are finite. Then consider the natural isomorphism

$$\bigoplus_{h \in H' \backslash H} k\left[{}^{H'}\!/\!_{U}\right] \otimes h \cong k\left[{}^{H}\!/\!_{U}\right].$$

The left hand side has the direct sum topology, which coincides with the product topology because the direct sum is finite. Since $k[H'/U]$, $k[H/U]$ are free k-modules of finite rank with the corresponding product topology, it follows that the above isomorphism is a topological isomorphism. Thus

$$\bigoplus_{h \in H' \backslash H} kH' \otimes h \cong \varprojlim_{U \in X} k\left[{}^{H}\!/\!_{U}\right] \cong kH$$

is a topological isomorphism.

It then follows that the topologies on kG defined by H', H are equivalent, due to the isomorphism

$$\bigoplus_{g \in H \backslash G} kH \otimes g \cong \bigoplus_{g \in H \backslash G} \bigoplus_{h \in H' \backslash H} kH' \otimes hg \cong \bigoplus_{g \in H' \backslash G} kH' \otimes g.$$

For arbitrary $H, H' \leq G$ open profinite, we obtain an isomorphism by considering $H \cap H' \leq G$ and applying the result twice.

The topology on kG is Hausdorff since the topology on each $kH \otimes g$ is Hausdorff.

2. First we show this result in the profinite case: let H be a profinite group, and

consider the natural map $i = i_H : H \to kH$. The algebra kH has the inverse limit topology with projection maps

$$\Pi_{U,I} : kH \to \varprojlim_{U \trianglelefteq_o H, I \in \mathcal{B}} (k/I)[H/U].$$

Here $\mathcal{B}$ is an open basis of ideals of k with each k/I Artinian. The map i is continuous if and only if $\phi_{U,I} = \Pi_{U,I} \circ i$ is continuous for all U, I. Now, the image of $\phi_{U,I} : H \to (k/I)[H/U]$ is H/U, a finite set. Thus for any subset $X \subseteq (k/I)[H/U]$, the inverse image $\phi_{U,I}^{-1}(X)$ is a union of finitely many cosets of U in H, which is open. Hence, $\phi_{U,I}$ is continuous. So, i is continuous.
Moreover, i is clearly injective, whilst H is compact and kH Hausdorff. Thus $i : H \to i(H)$ is a homeomorphism.
Now, let G be a locally profinite group, and consider the natural map

$$i_G : G \to kG = \bigoplus_{g \in H\backslash G} kH \otimes g.$$

Clearly $i_G(x) = i_H(xg^{-1}) \otimes g$ when $x \in Hg$, for $g \in H\backslash G$. So, for each $g \in H\backslash G$, the restriction

$$j_g = i_G|_{Hg} : Hg \to i_H(H) \otimes g$$

is a homeomorphism. Clearly, the cosets Hg form a disjoint open cover of G, whilst the sets $i_H(H) \otimes g$ are also disjoint. We show that each $i_H(H) \otimes g$ is also open. For a fixed $g' \in H\backslash G$, let $U_{g'} = kH$. When $g' \notin Hg$, let $U_g \subseteq kH$ be a proper two-sided ideal of kH – which exists since kH is a pseudocompact ring. In this case, $U_g \cap i_H(H)$ is empty. Let

$$U = \bigoplus_{g \in H\backslash G} U_g \otimes g \leq kG,$$

so U is an open subgroup of kG. Therefore $U \cap i_G(G) = i_H(H) \otimes g'$ is open in $i_G(G)$. Thus $\{i_H(H) \otimes g \mid g \in H\backslash G\}$ is a disjoint open cover of $i_G(G)$.
The restrictions $j_g = i_G|_{Hg}$ above give homeomorphisms between corresponding parts of the disjoint open covers of G and of $i_G(G)$, from which it follows straightforwardly that i_G is a homeomorphism onto its image.
3. This is trivial, as $K \leq (kK)^{\times}$ and kK is a pseudocompact subalgebra of $kG = kK \otimes_{k[K]} k[G]$.
4. We have that

$$k[G] = \bigoplus_{g \in H\backslash G} k[H] \otimes g \leq kG,$$

and $k[H]$ is dense in kH. If C is a closed additive subgroup containing $k[G]$, then for all $g \in H \backslash G$, $C \cap (kH \otimes g)$ is closed in $kH \otimes g$, but contains $k[H] \otimes g$. Hence $C \cap (kH \otimes g) = kH \otimes g$, and hence $C = kG$. Thus, $k[G]$ is dense in kG. $\square$

To prove a universal property for kG similar to that of Proposition 3.3, we will need a class of algebras to replace "pseudocompact k-algebras", and which contains all algebras kG where G is locally profinite. For this reason, and using the proposition just proved, we define the following class of algebras.

Definition 3.15:

Let A be a topological k-algebra, that is, a topological ring with a continuous homomorphism from k to the centre of A, and assume A is Hausdorff. Let $G \leq A^\times$ be a subgroup of the units of A. The subgroup G makes A Iwasawa-like if and only if the subspace topology makes G a topological group (that is, the inversion map on G is continuous), and there is an open profinite subgroup $K \leq G$ and a pseudocompact k-subalgebra $B \leq A$ such that K is a subgroup of $B^\times$.

The careful reader may notice that if A is Iwasawa-like via G, it is also Iwasawa-like via an open profinite subgroup $K \leq G$. This redundancy in the definition is removed in Definition 3.21, for the purposes of Corollary 3.23. For the remainder of our results, however, it is more convenient to work with Definition 3.15.

If G is a locally profinite group, the subgroup $G \leq (kG)^\times$ makes kG an Iwasawa-like topological k-algebra, by the second and third statements of Proposition 3.14. The ring of k-linear endomorphisms of a smooth representation is also Iwasawa-like.

Proposition 3.16:

Let V be a smooth representation of G. The ring of k-linear endomorphisms of V is naturally an Iwasawa-like algebra.

This is proved in subsection 4.2. See also Definition 4.1 for our notion of smooth representation. We can prove the following universal property for augmented Iwasawa algebras.

Proposition 3.17:

Let G be a locally profinite group. Let A be an Iwasawa-like topological k-algebra, via subgroup $L \leq A^\times$. Let $\phi : G \to L$ be a continuous group homomorphism. There is a unique continuous k-algebra homomorphism $\tilde{\phi} : kG \to A$ extending ϕ.

An application of Proposition 3.17 to the result of Proposition 3.16 gives the following result on smooth representations.

Corollary 3.18:

Let V be a smooth representation of G. There is a unique kG-module structure extending the $k[G]$-module action on V.

The proof of Corollary 3.18 can also be found in subsection 4.2. We now prove Proposition 3.17, which requires the following weaker statement.

Lemma 3.19:

Let H be a profinite group. Let A be an Iwasawa-like topological k-algebra, via subgroup $L \leq A^\times$. Let $\phi : H \to L$ be a continuous group homomorphism. There is a unique continuous k-algebra homomorphism $\tilde{\phi} : kH \to A$ extending ϕ.

Proof of Proposition 3.17:

For any open profinite $H \leq G$, the restriction of $\phi : G \to L$ to $\phi|_H : H \to L$ is a continuous group homomorphism, and hence by Lemma 3.19, there is a unique continuous k-algebra homomorphism $\psi_H : kH \to A$ extending $\phi|_H$. There is also a unique k-algebra homomorphism $\psi_G : k[G] \to A$ extending ϕ, by the universal property of the usual group algebra. Then, the restrictions $\psi_G|_{k[H]} = \psi_H|_{k[H]}$, and $\psi_H|_{kJ} = \psi_J$ if $J \leq H$ are open profinite subgroups of G.

For any open profinite subgroup $H \leq G$, define the following k-bilinear map:

$$\phi'_H : kH \times k[G] \to A, \quad \phi'_H(x, y) = \psi_H(x)\psi_G(y).$$

The map ϕ'_H is also $(kH, k[G])$-bilinear, where $kH, k[G]$ act on A via ψ_H, ψ_G. Moreover if $r \in k[H]$, then because ψ_G, ψ_H are ring homomorphisms and agree on $k[H]$,

$$\begin{aligned}\phi'_H(xr,y) &= \psi_H(xr)\psi_G(y)\\ &= \psi_H(x)\psi_H(r)\psi_G(y)\\ &= \psi_H(x)\psi_G(r)\psi_G(y)\\ &= \psi_H(x)\psi_G(ry)\\ &= \phi'_H(x,ry).\end{aligned}$$

Thus, using the universal property of the tensor product, ϕ'_H induces a homomorphism of $(kH, k[G])$-bimodules

$$\tilde{\phi}_H : kH \otimes_{k[H]} k[G] \to A,$$

where $\tilde{\phi}_H(x \otimes y) = \psi_H(x)\psi_G(y)$.

We now show that the homomorphisms $\tilde{\phi}_H$ for varying H agree, under appropriate identifications.

Let $J \leq H$ be open profinite subgroups of G. Let $i'_{J,H} : kJ \to kH$ be the natural inclusion of their completed group algebras, and define

$$i_{J,H} : kJ \otimes_{k[J]} k[G] \to kH \otimes_{k[H]} k[G], \quad i_{J,H}(x \otimes z) = i'_{J,H}(x) \otimes z.$$

We also define a map $\Pi_{J,H}$ in the opposite direction, in the following way. Let

$$\alpha : kH \to kJ \otimes_{k[J]} k[H]$$

be the natural $(kJ, k[H])$-bimodule isomorphism with $\alpha \circ i'_{J,H} = id_{kJ} \otimes 1$. Let s be the natural isomorphism of $(kJ, k[G])$-bimodules,

$$s : (kJ \otimes_{k[J]} k[H]) \otimes_{k[H]} k[G] \to kJ \otimes_{k[J]} (k[H] \otimes_{k[H]} k[G]),$$

and β be the natural $(k[H], k[G])$-bimodule isomorphism

$$\beta : k[H] \otimes_{k[H]} k[G] \to k[G].$$

Then we have the maps

$$\begin{aligned}kH \otimes_{k[H]} k[G] &\xrightarrow{\alpha\otimes 1} (kJ \otimes_{k[J]} k[H]) \otimes_{k[H]} k[G]\\ &\xrightarrow{s} kJ \otimes_{k[J]} (k[H] \otimes_{k[H]} k[G])\\ &\xrightarrow{1\otimes\beta} kJ \otimes_{k[J]} k[G],\end{aligned}$$

so let

$$\Pi_{J,H} = (1 \otimes \beta) \circ s \circ (\alpha \otimes 1).$$

Then $\Pi_{J,H}$ is an isomorphism since β, s, α are, and

$$\begin{aligned}
\Pi_{J,H} \circ i_{J,H}(x \otimes z) &= \Pi_{J,H}(i'_{J,H}(x) \otimes z) \\
&= (1 \otimes \beta) \circ s(\alpha(i'_{J,H}(x) \otimes z)) \\
&= (1 \otimes \beta) \circ s((x \otimes 1) \otimes z) \\
&= 1 \otimes \beta(x \otimes (1 \otimes z)) \\
&= x \otimes z
\end{aligned}$$

for all $x \in kJ, z \in k[G]$. Thus $i_{J,H}$, $\Pi_{J,H}$ are mutually inverse isomorphisms. We will also write $i_{H,J} = \Pi_{J,H}$.

Now, the following diagram,

$$\begin{tikzcd}
kH \otimes_{k[H]} k[G] \arrow[r, "\tilde{\phi}_H"] & A \\
kJ \otimes_{k[J]} k[G] \arrow[u, "i_{J,H}"] \arrow[ur, "\tilde{\phi}_J"'] &
\end{tikzcd}$$

commutes, because for all $x \in kJ, z \in k[G]$,

$$\tilde{\phi}_H \circ i_{J,H}(x \otimes z) = \tilde{\phi}_H(i'_{J,H}(x) \otimes z) = \psi_H(i'_{J,H}(x))\psi_G(z) = \psi_J(x)\psi_G(z) = \tilde{\phi}_J(x \otimes z).$$

This uses that $\psi_H \circ i'_{J,H} = \psi_J$, by uniqueness of the extensions ψ_J, ψ_H. So, if $J \leq H$, then $\tilde{\phi}_H \circ i_{J,H} = \tilde{\phi}_J$, and so $\tilde{\phi}_H = \tilde{\phi}_J \circ i_{J,H}^{-1} = \tilde{\phi}_J \circ i_{H,J}$ also. For arbitrary open profinite subgroups $H, H' \leq G$, define $i_{H,H'} = i_{H \cap H', H'} \circ i_{H, H \cap H'}$. It then follows that $\tilde{\phi}_H \circ i_{H',H} = \tilde{\phi}_{H'}$.

We now use these identifications to show that $\tilde{\phi}_H$ is a ring homomorphism, for any $H \leq G$ open profinite. Let $g \in G$, $x \in kH$. We consider the product $(1 \otimes g)(x \otimes 1) \in kG = kH \otimes_{k[H]} k[G]$. The multiplication on kG is such that

$$(1 \otimes g)(x \otimes 1) = gx = i_{K,H}(gxg^{-1} \otimes g),$$

where $K = gHg^{-1}$. Thus

$$\begin{aligned}
\tilde{\phi}_H\big((1 \otimes g)(x \otimes 1)\big) &= \tilde{\phi}_H \circ i_{K,H}(gxg^{-1} \otimes g) \\
&= \tilde{\phi}_K(gxg^{-1} \otimes g) \\
&= \psi_K(gxg^{-1})\psi_G(g).
\end{aligned}$$

It remains to show that $\psi_K(gxg^{-1}) = \psi_G(g)\psi_H(x)\psi_G(g)^{-1}$. Let

$$d : kH \to A, \quad d(t) = \psi_K(gtg^{-1}) - \psi_G(g)\psi_H(t)\psi_G(g)^{-1}.$$

Now, ψ_K, ψ_H are continuous and multiplication in A is continuous, so d is a continuous function. If $t \in k[H]$, then $gtg^{-1} \in k[G]$, and

$$\psi_K(gtg^{-1}) = \psi_G(gtg^{-1}) = \psi_G(g)\psi_G(t)\psi_G(g)^{-1} = \psi_G(g)\psi_H(t)\psi_G(g)^{-1},$$

so $d(t) = 0$. Thus $d(k[H]) = \{0\}$, but $k[H]$ is dense in kH and A is Hausdorff, so $\{0\}$ is closed and $d(kH) = \{0\}$. Thus,

$$\tilde{\phi}_H\big((1 \otimes g)(x \otimes 1)\big) = \psi_G(g)\psi_H(x)\psi_G(g)^{-1}\psi_G(g) = \psi_G(g)\psi_H(x).$$

Because $\tilde{\phi}_H$ is a $(kH, k[G])$-bimodule homomorphism, for all $z \in kH \otimes_{k[H]} k[G]$,

$$\tilde{\phi}_H(zg) = \tilde{\phi}_H(z)\psi_G(g), \quad \tilde{\phi}_H(yz) = \psi_H(y)\tilde{\phi}_H(z).$$

So, let $x_1, x_2 \in kH$ and $g_1, g_2 \in G$. Then

$$\begin{aligned}\tilde{\phi}_H\big((x_1 \otimes g_1)(x_2 \otimes g_2)\big) &= \psi_H(x_1)\tilde{\phi}_H\big((1 \otimes g_1)(x_2 \otimes 1)\big)\psi_G(g_2)\\ &= \psi_H(x_1)\psi_G(g_1)\psi_H(x_2)\psi_G(g_2)\\ &= \tilde{\phi}_H(x_1 \otimes g_1)\tilde{\phi}_H(x_2 \otimes g_2).\end{aligned}$$

Therefore $\tilde{\phi}_H$ is a ring (and hence k-algebra) homomorphism. Now, ψ_H is continuous, so $\tilde{\phi}_H$ is continuous on $kH \otimes g$, for any $g \in H\backslash G$. Because

$$\tilde{\phi}_H = \bigoplus_{g \in H\backslash G} \tilde{\phi}_H|_{kH \otimes g} : \bigoplus_{g \in H\backslash G} kH \otimes g \to \bigoplus_{g \in H\backslash G} kH \otimes g,$$

it follows that $\tilde{\phi}_H$ is a continuous k-algebra homomorphism extending $\phi : G \to A^\times$.

To show uniqueness, recall from Proposition 3.14 that $k[G]$ is dense in kG. Thus any two continuous maps $kG \to A$ agreeing on $k[G]$ are equal, since A is Hausdorff. But $\psi_G : k[G] \to A$ is the unique k-algebra homomorphism extending ϕ to $k[G]$, therefore $\tilde{\phi}$ is the unique continuous k-algebra homomorphism extending ϕ. $\square$

We now prove Lemma 3.19.

Proof of Lemma 3.19:

We have a continuous group homomorphism $\phi : H \to L \leq A^\times$. Since L makes A Iwasawa-like, let $K \leq L$ be an open profinite subgroup, and $B \leq A$ be a pseudocompact k-subalgebra containing K in its units. Then, $\phi(H) \cap K$ is an open subgroup of $\phi(H)$, and so $\phi^{-1}(\phi(H) \cap K)$ is an open subgroup of H. Let $K' \leq H$ be an open normal subgroup of H such that $K' \leq \phi^{-1}(\phi(H) \cap K)$, which exists because H is profinite. So, we have a continuous group homomorphism

$$\phi|_{K'} : K' \to K \leq B^\times.$$

By the universal property in Proposition 3.3, $\phi|_{K'}$ extends uniquely to a continuous k-algebra homomorphism $\psi : kK' \to B$.

Using the universal property of ordinary group algebras, let $f : k[H] \to A$ be the unique k-algebra extension of ϕ. By uniqueness of these maps, we have that $f|_{k[K']} = \psi|_{k[K']}$. Thus, define

$$\tilde{\phi} : kH \to A, \quad \tilde{\phi}(x \otimes h) = \psi(x) f(h),$$

where we identify $kH \cong kK' \otimes_{k[K']} k[H]$, and transport the ring structure.

Then if $x_1, x_2 \in kK'$ and $h_1, h_2 \in H$,

$$\begin{aligned}
\tilde{\phi}\big((x_1 \otimes h_1)(x_2 \otimes h_2)\big) &= \tilde{\phi}\big((x_1 \otimes 1)(h_1 x_2 h_1^{-1} \otimes h_1 h_2)\big) \\
&= \tilde{\phi}(x_1 h_1 x_2 h_1^{-1} \otimes h_1 h_2) \\
&= \psi(x_1 h_1 x_2 h_1^{-1}) f(h_1 h_2) \\
&= \psi(x_1) \psi(h_1 x_2 h_1^{-1}) f(h_1) f(h_2).
\end{aligned}$$

Because $k[K']$ is dense in kK', very similar reasoning as in the proof of Proposition 3.17 shows that

$$\psi(h_1 x_2 h_1^{-1}) = f(h_1) \psi(x_2) f(h_1)^{-1},$$

and so

$$\tilde{\phi}\big((x_1 \otimes h_1)(x_2 \otimes h_2)\big) = \psi(x_1) f(h_1) \psi(x_2) f(h_2) = \tilde{\phi}(x_1 \otimes h_1) \tilde{\phi}(x_2 \otimes h_2).$$

Thus $\tilde{\phi}$ is a k-algebra homomorphism that extends ϕ. Because ψ is continuous and kH can be identified with $kK \otimes_{k[K]} k[H]$, identical reasoning as in the proof of Proposition 3.17 shows that $\tilde{\phi}$ is continuous.

Since $k[H]$ is dense in kH and f is the unique k-algebra homomorphism $k[H] \to A$ extending ϕ, it follows that $\tilde{\phi}$ must be the unique continuous k-algebra homomorphism $kH \to A$ extending ϕ. $\square$

It follows from Proposition 3.17 that construction of the augmented Iwasawa algebra gives a functor.

Proposition 3.20:

The mapping F given on objects by $F(G) = kG$ and on morphisms by $F(\phi) = \tilde{\phi}$, is a functor from the category of locally profinite groups (with continuous group homomorphisms) to the category of Iwasawa-like topological k-algebras (with continuous k-algebra homomorphisms). The functor preserves surjectivity of morphisms and injectivity of closed morphisms.

Proof:

That F is a well-defined functor follows directly from Proposition 3.17, using an identical argument as found in the proof of Proposition 3.4.

Let $\phi : G_1 \to G_2$ be a continuous group homomorphism between locally profinite groups.

Suppose ϕ is injective and a closed map. Then $\phi(G_1)$ is a closed subgroup of G_2. Let $H_2 \leq G_2$ be an open profinite subgroup, and $H_1 = \phi^{-1}(H_2)$. Then H_1 is an open subgroup of G_1, and is isomorphic to $\phi(H_1) = \phi(G_1) \cap H_2$ as a topological group, hence is profinite. Now, $\tilde{\phi}|_{kH_1} = \widetilde{\phi|_{H_1}}$ is injective by Proposition 3.4, and the natural map induced by ϕ on cosets,

$$H_1\backslash G_1 \to H_2\backslash G_2, \quad H_1 g \mapsto H_2\phi(g),$$

is injective. Now, $\tilde{\phi}$ is given by

$$\tilde{\phi} : \bigoplus_{g \in H_1\backslash G_1} kH_1 \otimes g \to \bigoplus_{g \in H_2\backslash G_2} kH_2 \otimes g, \quad \tilde{\phi}(x \otimes g) = \tilde{\phi}|_{kH_1}(x) \otimes \phi(g),$$

therefore $\tilde{\phi}$ is injective.

Suppose instead ϕ is a surjective continuous group homomorphism. Let $H_1 \leq G_1$ be an open profinite subgroup. Because surjective maps between topological groups are open, $\phi(H_1) \leq G_2$ is an open profinite subgroup. By Proposition 3.4, $\tilde{\phi}|_{kH_1} = \widetilde{\phi|_{H_1}}$ is surjective, and so

$$\tilde{\phi} : \bigoplus_{g \in H_1\backslash G_1} kH_1 \otimes g \to \bigoplus_{g \in \phi(H_1)\backslash G_2} k\phi(H_1) \otimes g$$

is surjective. □

Proposition 3.20, and its proof, implies that if $G' \leq G$ is a closed subgroup, then kG' is identified with the subring of kG,

$$\bigoplus_{g \in H \backslash HG'} k(G' \cap H) \otimes g,$$

where $H\backslash HG'$ denotes (a set of representatives for) the set $\{Hg \mid g \in G'\} \subseteq H\backslash G$, for $H \leq G$ open profinite. The proposition also implies that if L is a quotient of G, kL is a quotient ring of kG.

In fact, we can improve the functor $G \mapsto kG$ to a left-adjoint, and hence show it commutes with colimits.

Definition 3.21:

The category $\mathcal{LP}$ is the full subcategory of the category of topological groups, whose objects are locally profinite groups.

The category $\mathcal{IL}_k$ is the category with objects (A, L) where A is an Iwasawa-like algebra via $L \subseteq A^\times$, and morphisms $(A_1, L_1) \to (A_2, L_2)$ given by a continuous k-algebra homomorphism $f : A_1 \to A_2$ such that $f(L_1) \subseteq L_2$.

Proposition 3.22:

Let $F : \mathcal{LP} \to \mathcal{IL}_k$ be the functor given on objects by $F(G) = (kG, G)$ and on morphisms by $F(\phi) = \tilde{\phi}$. Let $H : \mathcal{IL}_k \to \mathcal{LP}$ be the functor given on objects by $H\big((A, L)\big) = L$ and on morphisms $\psi : (A_1, L_1) \to (A_2, L_2)$ by $H(\psi) = \psi|_{L_1} : L_1 \to L_2$. Then F is a left adjoint to H.

Proof:

Let G be a locally profinite group, and A be an Iwasawa-like algebra via $L \subseteq A^\times$, making (A, L) an object of $\mathcal{IL}_k$. For $\phi : G \to L$ a morphism of locally profinite groups, it is clear from Proposition 3.17 that $H(F(\phi)) = \tilde{\phi}|_G = \phi$. For $\psi : (kG, G) \to (A, L)$ a morphism of Iwasawa-like algebras, $H(\psi) = \psi|_G : G \to L$ is a continuous group homomorphism since ψ is continuous. Then $F(\psi|_G) = \widetilde{\psi|_G}$ is a continuous k-algebra homomorphism $\widetilde{\psi|_G} : kG \to A$, extending $\psi|_G$. By uniqueness in Proposition 3.17, $\widetilde{\psi|_G} = \psi$. Thus, we have a bijection

$$\mathrm{Hom}_{\mathcal{IL}_k}\big(F(G), (A, L)\big) \cong \mathrm{Hom}_{\mathcal{LP}}\big(G, H(A, L)\big),$$

and which is clearly natural in G and (A, L). So, F is left adjoint to H (respectively, H is right adjoint to F). $\square$

Because F is a left adjoint, F commutes with colimits. As a consequence, so does construction of the augmented Iwasawa algebra.

Corollary 3.23:

If $G = \mathrm{colim}_{b\in\mathcal{B}}(H_b)$ is a colimit in the category of locally profinite groups (and G is locally profinite), then $kG \cong \mathrm{colim}_{b\in\mathcal{B}}(kH_b)$.

Here we give G the colimit topology, defined below.

Definition 3.24:

Let $G = \mathrm{colim}_{b\in\mathcal{B}}(H_b)$ be a colimit of topological groups. The colimit topology on G is the coarsest topology such that whenever $(f_b : G \to G')_{b\in\mathcal{B}}$ are compatible continuous group homomorphisms, there is a unique continuous group homomorphism $f : G \to G'$ extending the f_b.

This leads to a small subtlety: it may be the case that the group structure of a locally profinite group can be realised as a colimit in the category of locally profinite groups, but with its original locally profinite topology not agreeing with the colimit topology. Fortunately, in our applications, this situation does not arise.

Lemma 3.25:

Let $(H_b)_{b\in\mathcal{B}}$ be a diagram of topological groups. Let G be the abstract group corresponding to the colimit. Suppose G has a topology τ making it a topological group, such that there exists $c \in \mathcal{B}$ with $H_c \to (G, \tau)$ an open map, and for all $b \in \mathcal{B}$, the map $H_b \to (G, \tau)$ is continuous. Then the colimit topology on G agrees with τ.

Proof:

Let τ_c denote the colimit topology. For each $b \in \mathcal{B}$, we have a commutative diagram

$$\begin{tikzcd}
(G, \tau_c) \arrow[r, "\alpha"'] & (G, \tau) \arrow[l, bend right, "\beta"'] \\
H_b \arrow[u, "i_b"] \arrow[ur, "j_b"'] &
\end{tikzcd},$$

where α, β are the identity maps, the continuous group homomorphisms i_b come from the colimit $(H_b)_{b\in\mathcal{B}}$, and the j_b are continuous by assumption on τ. Since $\alpha \circ i_b = j_b$ is continuous for all b, it follows from the universal property of the colimit topology that α is continuous.

Conversely, if $U \leq (G, \tau_c)$ is an open subgroup, $i_c^{-1}(U) = j_c^{-1}(\beta^{-1}(U))$ is open since i_c is continuous. But since $j_c : H_c \to (G, \tau)$ is an open map between topological groups, it is straightforward to show that $V \leq G_2$ is open precisely when $j_c^{-1}(V) \leq H_c$ is open – note V is a union of cosets of $j_c(j_c^{-1}(V))$. Hence, $\beta^{-1}(U)$ is open.

Thus β is continuous with continuous inverse, hence a homeomorphism; so τ and the colimit topology agree. □

In particular, Lemma 3.25 implies that if a topological group is abstractly the colimit of closed subgroups, at least one of which is open, then the colimit topology agrees with the original topology. We can now give two main applications of Corollary 3.23.

Example 3.26

Suppose a locally profinite group G is a direct limit of open subgroups, $G = \varinjlim_{a \in A} G_a$. The colimit topology on G agrees with the original topology by Lemma 3.25, so by Corollary 3.23, the augmented Iwasawa algebra $kG \cong \varinjlim_{a \in A} kG_a$.

Example 3.27

The above is particularly useful when the G_a are profinite or compact p-adic Lie groups. The additive group $G = \mathbb{Q}_p$ is the direct limit of open subgroups $p^{-n}\mathbb{Z}_p$, and therefore the augmented Iwasawa algebra of $\mathbb{Q}_p$ is

$$k\mathbb{Q}_p = \varinjlim_{n \geq 0} k(p^{-n}\mathbb{Z}_p).$$

See also Example 6.8.

Example 3.28

Suppose a locally profinite group $G = G_1 *_H G_2$ is an amalgamated product of two closed subgroups $G_1, G_2 \leq G$, over their intersection $H = G_1 \cap G_2$. If $H \leq G$ is open, then by Lemma 3.25 and Corollary 3.23, the augmented Iwasawa algebra is given by

$$kG \cong kG_1 *_{kH} kG_2.$$

For an application see subsection 7.3.

3.4 Modules for an augmented Iwasawa algebra

In this subsection, let G be a locally profinite group. We define modules associated to locally profinite G-spaces and prove a similar universal property for certain kG-modules as in subsection 3.2.

Before doing this, it will be useful to prove the characterisation of kG-modules as being the augmented representations defined in [Eme10].

Proposition 3.29:

Let M be a $k[G]$-module equipped with a kH-module structure for some open profinite subgroup $H \leq G$ such that the two induced $k[H]$-actions coincide. Suppose that M has a Hausdorff topology such that each element of G acts continuously on M, and the action $kH \times M \to M$ is continuous. Then M has a unique kG-module structure which extends the action of $k[G]$ and kH.

Proof:

Let $\alpha_H : kH \times M \to M$ denote the kH-module action on M and $\beta : k[G] \times M \to M$ denote the $k[G]$-module structure. The actions agree on $k[H]$, that is, $\alpha_H|_{k[H]\times M} = \beta|_{k[H]\times M}$. Then the map

$$\phi'_H : kH \times k[G] \times M \to M, \quad \phi'_H(x, y, m) = \alpha_H(x, \beta(y, m))$$

is a $k[H]$-balanced map and so we get an induced map

$$\phi_H : kH \otimes_{k[H]} k[G] \times M \to M, \quad \phi_H(x \otimes y, m) = \alpha_H(x, \beta(y, m)).$$

Note ϕ_H extends α_H and β, and is continuous on $(kH \otimes g) \times M$ for all $g \in G$, thus is continuous.

Let $g \in G$ and $m \in M$. Consider the function

$$d : kH \to M, \quad d(x) = \phi_H(gxg^{-1}, \beta(g, m)) - \beta(g, \alpha_H(x, m)).$$

Now, α_H is continuous, G acts continuously on kH by conjugation, and g acts continuously on M (via β). Therefore d is a continuous function. If $x \in k[H]$, we have that $x, gxg^{-1} \in k[G]$, so it follows by the module axioms for $k[G]$ that $d(x) = 0$. But, $k[H]$ is dense in kH, therefore as M is Hausdorff, we have that $d(kH) = \{0\}$.

It follows that

$$\phi_H(gxg^{-1}, \beta(g, m)) = \beta(g, \alpha_H(x, m)),$$

that is,

$$\phi_H(gxg^{-1}, \phi_H(g,m)) = \phi_H(g, \phi_H(x,m)),$$

for all $x \in kH$, $g \in G$, $m \in M$. Moreover, by definition ϕ_H has the property that

$$\phi_H(x'zg', m) = \phi_H(x', \phi_H(z, \phi_H(g', m))),$$

for any $x' \in kH$, $z \in kG$, $g' \in G$. Combining these statements, it is straightforward to show

$$\phi_H((x_1 \otimes g_1)(x_2 \otimes g_2), m) = \phi_H(x_1(g_1x_2g_1^{-1})g_1g_2, m) = \phi_H(x_1 \otimes g_1, \phi_H(x_2 \otimes g_2, m)),$$

for any $x_1, x_2 \in kH$, $g_1, g_2 \in G$. By k-linearity, it follows that for any $z_1, z_2 \in kG$, we have the module axiom

$$\phi_H(z_1, \phi_H(z_2, m)) = \phi_H(z_1z_2, m).$$

Thus ϕ_H defines a module action of $kG = kH \otimes_{k[H]} k[G]$ on M. This extends the actions of kH and of $k[G]$, and moreover must be the unique such action because these subrings generate kG. □

Here we had to make a topological assumption, that M is Hausdorff. If kH is Noetherian, this assumption may be dispensed with, as follows.

Lemma 3.30:

Let A be a left Noetherian pseudocompact k-algebra and M be a finitely-generated left A-module. Then, there exists a unique topology on M such that the action $A \times M \to M$ is continuous. Moreover, M with this topology is a pseudocompact module.

Proof:

Since A is Noetherian, any surjective A-module homomorphism $A^s \to M$ has finitely-generated kernel, and hence there is an exact sequence

$$A^t \xrightarrow{f} A^s \xrightarrow{q} M \to 0.$$

Now we show that any A-module homomorphism $f : A^t \to A^s$ is continuous. By properties of the product topology, it is enough to show this in the case $s = 1$. Let $\mathcal{B}$ be a neighbourhood basis of open two-sided ideals as in Definition 3.2, and $I \in \mathcal{B}$. Then the submodules $(I^t)_{I \in \mathcal{B}}$ form a neighbourhood basis in A^t. Since I

is two-sided, it is straightforward that $I^t \subseteq f^{-1}(I)$, hence $f^{-1}(I)$ is open. So, f is continuous as required.

By Theorem 22.3 of [Sch11], the image $N = f(A^t) \leq A^s$ is closed and pseudocompact. In fact, since A^s is a Noetherian A-module, every submodule is finitely-generated and therefore closed. It follows that the homomorphism $q : A^s \to M$ must be continuous whenever M has a topology that makes it a topological A-module. But, q is a surjective continuous map of topological groups, hence is an open map, and therefore M must have the quotient topology induced by $M \cong A^s/N$. Moreover, the quotient topology induced on M by the natural isomorphism $M \cong A^s/N$ is pseudocompact, again by Theorem 22.3 of [Sch11]. $\square$

We call the unique topology determined by Lemma 3.30 the *canonical* topology, which is Hausdorff since it is pseudocompact. A similar statement to Lemma 3.30 is proved for k profinite in Proposition 2.1.3 of [Eme10].

Corollary 3.31:

Let M be a $k[G]$-module equipped with a kH-module structure for some open profinite subgroup $H \leq G$ such that the two induced $k[H]$-actions coincide. If kH is a Noetherian ring, then M has a unique kG-module structure that extends the action of $k[G]$ and of kH.

Proof:

Since k is pseudocompact and H is profinite, kH is a pseudocompact k-algebra, by section 4 of [Bru66]. Since kH is Noetherian, Lemma 3.30 implies that every finitely-generated submodule of M is pseudocompact with the canonical topology. We give M the final topology corresponding to the inclusions $N \to M$ of all finitely-generated submodules N of M. That is, $U \subseteq M$ is open if and only if $U \cap N \subseteq N$ is open for every finitely-generated submodule N. We show that the topology on M satisfies the assumptions of Proposition 3.29.

Firstly, the topology on M is Hausdorff, since it is Hausdorff on any finitely-generated (pseudocompact) submodule N.

For $N \leq M$ a finitely-generated kH-submodule, the kH-action $kH \times N \to N$ is continuous, again by Lemma 3.30. That is, letting $\alpha_H : kH \times M \to M$ denote the kH-module action, the restriction $\alpha_H|_{kH \times N}$ is continuous for any finitely-generated N. It is also straightforward to check that a subset $U' \subseteq kH \times M$ is open if and only if $U' \cap (kH \times N) \subseteq kH \times N$ is open for all N, using the fact that the inclusions $N \to N'$ of one finitely-generated submodule to another are open continuous maps, so for any open subset $V \subseteq N$ of any finitely-generated submodule, V is open in M.

Now, let $U \subseteq M$ be an open subset of M. Then

$$\alpha_H^{-1}(U) \cap (kH \times N) = (\alpha_H|_{kH\times N})^{-1}(U \cap N)$$

is open in $kH \times N$, for any finitely-generated submodule N. Therefore $\alpha_H^{-1}(U)$ is open, by the above. So α_H is continuous.

Let $g \in G$. The action of g on M induces a natural bijection $g : N \to gN$ for any finitely-generated submodule N. We show it is a homeomorphism.

Now, gN is a finitely-generated $k(gHg^{-1} \cap H)$-submodule of M, and N is a finitely-generated $k(H \cap g^{-1}Hg)$-submodule of M. The canonical topology on N, gN is determined by these respective module structures. The map $g : N \to gN$ naturally induces a bijective correspondence between $k(H \cap g^{-1}Hg)$-submodules of N and $k(gHg^{-1} \cap H)$-submodules of gN. The open submodules of N, gN are those of finite index, and since the action of g gives a bijection between N and gN, it must therefore induce a bijective correspondence between open submodules, $(U \leq N) \mapsto (gU \leq gN)$. Thus, $g : N \to gN$ is a homeomorphism.

Now, let $U \subseteq M$ be open, and N be a finitely-generated kH-submodule of M. Then $U \cap (kH \cdot gN)$ is open in $kH \cdot gN$ and hence $U \cap gN$ is open in gN. Then $g^{-1}(U) \cap N = g^{-1}(U \cap gN)$ is open, by the above. Thus by definition $g^{-1}(U)$ is open. So the action $g : M \to M$ is continuous for any $g \in G$.

Thus, M satisfies the properties of Proposition 3.29. Therefore M has a unique kG-module structure extending the $k[G]$-module and kH-module actions. □

A very important situation in which Corollary 3.31 holds is when G is a p-adic Lie group and k is a field of characteristic p; in this case H is a compact p-adic Lie group and kH is Noetherian by Theorem 2.2. The same result holds for certain complete discrete valuation rings k.

Theorem 3.32:

Let H be a compact p-adic Lie group, and k be a complete discrete valuation ring with residue field of characteristic p. Then kH is a Noetherian ring.

Proof:

Let $\pi \in k$ be a uniformiser, so that $k/\pi k = \mathbb{F}$ is a field of characteristic p. Since k is π-adically complete,

$$kH = \varprojlim_{U \trianglelefteq_o H} k\left[{}^{H}\!\big/\!{}_{U}\right] = \varprojlim_{U \trianglelefteq_o H, n \geq 0} \left({}^{k}\!\big/\!{}_{\pi^n k}\right)\left[{}^{H}\!\big/\!{}_{U}\right] = \varprojlim_{n \geq 0} \left({}^{k}\!\big/\!{}_{\pi^n k}\right) H$$

because we may commute the inverse limits. It follows that kH has a complete descending filtration where the associated graded is a polynomial ring,

$$\operatorname{gr} kH \cong \bigoplus_{n \geq 0} \left({}^{\pi^n k}\!\big/\!{}_{\pi^{n+1} k}\right) H \cong \mathbb{F}H[X],$$

the variable X corresponding to the degree one element $\pi + \pi^2 k$. Since $\mathbb{F}H$ is Noetherian by Theorem 2.2, Hilbert's basis theorem implies $\operatorname{gr} kH$ is Noetherian. It follows from [LVO96, Proposition II.2.2.1] that kH is Noetherian. $\square$

We now generalise Definition 3.5 to locally profinite groups and spaces.

Definition 3.33:

A locally profinite G-space is a locally compact Hausdorff totally disconnected topological space X with a continuous G-action $G \times X \to X$.

Notice that for H an open profinite subgroup, the H-orbits of a locally profinite G-space X are compact, Hausdorff and totally disconnected, and (trivially) each has finitely many H-orbits, thus are profinite H-spaces. Unlike subsection 3.2, we make no assumption on the number of orbits of G on X here. (In particular we are free to take G and X to be profinite, with infinitely many G-orbits.)

Definition 3.34:

Let X be a locally profinite G-space, H be an open profinite subgroup of G. The augmented completed module of X is

$$kX = \bigoplus_{Z \in \mathrm{Orb}_H(X)} kZ,$$

with the direct sum topology, where each kZ has its natural pseudocompact topology as in Lemma 3.9.

We now show that this definition gives a kG-module which does not depend on choice of H.

Proposition 3.35:

Let X be a locally profinite G-space. For any $H \leq G$ an open profinite subgroup, there is a kG-action extending the left kH-module action on $(kX)_H = \bigoplus_{Z \in \mathrm{Orb}_H(X)} kZ$. Moreover, each of these kG-modules are naturally isomorphic and the topologies agree.

Proof:

Let $J \leq H$ be open profinite subgroups of G. For Z any H-orbit, there is a kJ-module isomorphism

$$i^Z_{J,H} : \bigoplus_{Z' \in \mathrm{Orb}_J(Z)} kZ' \to kZ,$$

using both parts of Lemma 3.10. Thus, define the kJ-module isomorphism

$$j_{J,H} = \bigoplus_{Z \in \mathrm{Orb}_H(X)} i^Z_{J,H} : (kX)_J \to (kX)_H.$$

We can extend this definition to arbitrary $J, H \leq G$ by

$$j_{J,H} = j_{J \cap H, H} \circ j^{-1}_{J \cap H, J}.$$

We define a kG-module action on $(kX)_H$, by first defining a G-action on kX. For Z an H-orbit of X and $g \in G$, we have the natural continuous action $g : Z \to gZ$. We can consider both Z and gZ as profinite H-spaces by letting H act on gZ via the isomorphism $H \cong gHg^{-1}$. Thus we obtain a (bijective) map $\tilde{g} : kZ \to k(gZ)$, by Proposition 3.11. Note that gZ is a profinite gHg^{-1}-space, so linearly extending, we obtain a (continuous) map

$$\tilde{g}_H : (kX)_H \to (kX)_{gHg^{-1}}.$$

Define the action of $g \in G$ on $m \in (kX)_H$ by $g \cdot m = j_{gHg^{-1},H}(\tilde{g}_H(m))$. It is easy to check that

$$\tilde{g}_H(j_{J,H}(m)) = j_{gJg^{-1}, gHg^{-1}}(\tilde{g}_J(m))$$

for any open profinite subgroups $J, H \leq G$ (by first considering the case $J \leq H$). Then for any $g, g' \in G$ and $m \in (kX)_H$, it follows that $(gg') \cdot m = g \cdot (g' \cdot m)$, because $\tilde{g}_{g'H(g')^{-1}} \circ \tilde{g'}_H = \widetilde{gg'}_H$ by Proposition 3.12 and because $j_{H_2,H_3} \circ j_{H_1,H_2} = j_{H_1,H_3}$ for any open profinite subgroups $H_1, H_2, H_3 \leq G$. So, the action of G on $(kX)_H$ is a group action.

Moreover, each isomorphism $j_{J,H}$ is G-equivariant. The isomorphisms of Lemma 3.10 are easily seen to be topological isomorphisms, so each $i^Z_{J,H}$, therefore each $j_{J,H}$, is also a topological isomorphism. Hence, every $g \in G$ acts continuously on $(kX)_H$, and $(kX)_H$ is Hausdorff because each kZ is Hausdorff. The G-action is obviously k-linear, and agrees with the H-action induced from the natural kH-action. Thus we have a $k[G]$-module action on $(kX)_H$ which agrees with the kH-module action on $k[H]$. By Proposition 3.29, it follows that there is a unique kG-module action on $(kX)_H$ extending these actions.

Since each $j_{J,H}$ is a topological kJ-module isomorphism, and is G-equivariant, it follows that $(kX)_J \cong (kX)_H$ as topological kG-modules whenever $J \leq H$, and this obviously extends to arbitrary open profinite subgroups J, H. $\square$

Example 3.36

Let $G = \mathbb{Q}_p^\times$ and $X = \mathbb{Q}_p$ with G acting by multiplication. A compact open subgroup of $\mathbb{Q}_p^\times$ is $H = \mathbb{Z}_p^\times$. The H-orbits in X are $\{p^n\mathbb{Z}_p^\times \mid n \in \mathbb{Z}\} \cup \{\{0\}\}$, and so

$$kX = \bigoplus_{n\in\mathbb{Z}} k\mathbb{Z}_p^\times \oplus k$$

as a $k\mathbb{Z}_p^\times$-module, where k is the trivial module. The element $p^m \in \mathbb{Q}_p^\times$ acts on the infinite direct sum by the natural "shift by m" among the isomorphic summands, and acts trivially on k. Since $k\mathbb{Q}_p^\times \cong k\mathbb{Z}_p^\times[X, X^{-1}]$, we have a $k\mathbb{Q}_p^\times$-module isomorphism

$$k\mathbb{Q}_p \cong k\mathbb{Q}_p^\times \oplus k.$$

Example 3.37

Let G be a locally profinite group, X be a locally profinite G-space such that X has the discrete topology. For $H \leq G$ any open profinite subgroup, any H-orbit $Z \subseteq X$ is compact, hence finite, so $kZ = k[Z]$. It follows that the augmented completed module

$$kX = k[X],$$

is the usual permutation module.

We now prove a universal property analogous to those in the previous three subsections. Similarly to subsection 3.3, this requires us to define a class of modules which generalises "pseudocompact kH-modules", but which may look unnatural at first sight.

Definition 3.38:

Let M be a Hausdorff topological left kG-module. Let $L \subseteq M$ be a subset closed under the action of G. The subset L makes M an Iwasawa-like kG-module if and only if there is an open profinite subgroup $H \leq G$ such that every H-orbit in L is contained in a pseudocompact kH-submodule of M.

For any locally profinite G-set X the kG-module kX is Iwasawa-like via X, because for any open profinite $H \leq G$ and any H-orbit Z in X, we have that $kZ \leq kX$ is a pseudocompact kH-submodule of kX.
In this case, any open profinite subgroup $H \leq G$ may be considered, and we now show this holds for a general Iwasawa-like module.

Lemma 3.39:

Let M be an Iwasawa-like kG-module via $L \subseteq M$. For any open profinite subgroup $J \leq G$, every J-orbit in L is contained in a pseudocompact kJ-submodule of M.

Proof:

Let $H \leq G$ be an open profinite subgroup such that every H-orbit of L is contained in a pseudocompact kH-submodule. Clearly if J is a subgroup of H, this property also holds for J.
Let $J \leq G$ be open profinite with $H \leq J$. Without loss of generality, we may assume H is small enough such that H is normal in J. Let Y be a J-orbit of L and $Z \subseteq Y$ be an H-orbit. By our assumption on H, there is a pseudocompact kH-submodule N such that $Z \subseteq N$. Then, because H is normal in J, we have a kH-submodule

$$P = \sum_{j \in J} jN = \sum_{j \in J/H} jN \leq M$$

which must contain

$$Z' = \bigcup_{j \in J} jZ$$

Moreover, the G-action on M restricts to a J-action on P extending the H-action. Because $kJ = kH \otimes_{k[H]} k[J]$, it follows that P is a kJ-module. Moreover, where $N = \varprojlim_{i \in I} N/N_i$ is an inverse limit of finite-length kH-modules, then $P = \varprojlim_{i \in I} \sum_{j \in J/H} j(N/N_i)$ is an inverse limit of finite-length kJ-submodules, so P is a pseudocompact kJ-submodule.

So if $H \leq G$ has the property of Definition 3.38, so does any open profinite subgroup contained in or containing H. Thus, by considering the inclusions $J \cap H \leq J$ and $J \cap H \leq H$, the statement follows. $\square$

That is, for any kG-module M and subset $L \subseteq M$ closed under the action of G, the condition of Definition 3.38 holds for a particular open profinite subgroup $H \leq G$ if and only if it holds for all open profinite subgroups.

Proposition 3.40:

Let X be a locally profinite G-space, M be an Iwasawa-like kG-module via $L \subseteq M$, and $\phi : X \to L$ be a continuous G-equivariant map. There is a unique continuous kG-module homomorphism $\tilde{\phi} : kX \to M$ extending ϕ.

Proof:

By Lemma 3.39, any open profinite subgroup of G verifies the condition of Definition 3.38, of M being Iwasawa-like via L. Let $H \leq G$ be open profinite, and let Z be an H-orbit of X.

Then $\phi(Z)$ is an H-orbit of L, so there exists a pseudocompact kH-submodule $M_Z \leq M$ containing $\phi(Z)$. By Proposition 3.11, the continuous H-equivariant map $\phi|_Z : Z \to M_Z$ extends to a continuous kH-module homomorphism $\psi_Z : kZ \to M_Z$. Define

$$\tilde{\phi}_H = \sum_{Z \in \mathrm{Orb}_H(X)} \psi_Z : kX \to M.$$

Clearly $\tilde{\phi}_H$ is a kH-module homomorphism. We show that $\tilde{\phi}_H$ is G-equivariant. Recall the definition of the isomorphisms $j_{J,H}$ from the proof of Proposition 3.35. By Lemma 3.39, Proposition 3.12, and the second part of Lemma 3.10, $\tilde{\phi}_H \circ j_{J,H} = \tilde{\phi}_J$ for any open profinite subgroups $H, J \leq G$.

Now, let $Z \in \mathrm{Orb}_H(X)$, $m \in kZ$, and $g \in G$. Then

$$\tilde{\phi}_H(g \cdot m) = \tilde{\phi}_{gHg^{-1}} \circ j_{H,gHg^{-1}}(g \cdot m) = \psi_{gZ}(gm).$$

If $m \in k[Z]$ then $\psi_{gZ}(gm) = g\psi_Z(m)$ because ϕ is G-equivariant and the ψ are k-linear. Because the ψ are continuous, $k[Z]$ is dense in kZ, and M is Hausdorff, it follows that $\psi_{gZ}(gm) = g\psi_Z(m)$ for all $m \in kZ$. Thus, by linearity, $\tilde{\phi}_H(g \cdot m) = g\tilde{\phi}_H(m)$, for all $m \in kX$.

So $\tilde{\phi}_H$ is G-equivariant, and is a kH-module homomorphism by Proposition 3.11,

hence is a kG-module homomorphism, which is continuous since each ψ_Z is continuous.
Finally, $k[X]$ is dense in kX by similar reasoning to the fourth part of Proposition 3.14, using that each $k[Z]$ is dense in kZ, and

$$k[X] = \bigoplus_{Z \in \mathrm{Orb}_H(X)} k[Z].$$

Now, M is Hausdorff, and clearly ϕ has a unique k-linear extension to $k[X]$. Therefore $\tilde{\phi} = \tilde{\phi}_H : kX \to M$ must be the unique continuous kG-module homomorphism extending ϕ. $\square$

Proposition 3.41:
The mapping F given on objects by $F(X) = kX$ and on morphisms by $F(\phi) = \tilde{\phi}$, is a functor from the category of locally profinite G-spaces (with continuous G-equivariant maps) to the category of Iwasawa-like topological kG-modules (with continuous kG-module homomorphisms). The functor preserves surjectivity of morphisms and injectivity of morphisms.

Proof:
By Proposition 3.40, and identical reasoning to that given in the proof of Proposition 3.12, F is a functor.
Let $\phi : X \to Y$ be a continuous G-equivariant map between locally profinite G-spaces, and let $H \leq G$ be an open profinite subgroup. If Z is an H-orbit of X, then $\phi(Z)$ is an H-orbit of Y.
If ϕ is injective, then each $\tilde{\phi}|_{kZ} = \widetilde{\phi|_Z}$ is injective by Proposition 3.12, and so $\tilde{\phi}$ is injective since it is the direct sum of such maps.
If ϕ is surjective, then for all H-orbits $Z' \subseteq Y$, there is an H-orbit $Z \subseteq X$ such that $\phi(Z) = Z'$. Again by Proposition 3.12, each $\tilde{\phi}|_{kZ}$ is surjective, so has image $kZ' \leq kY$. So $\tilde{\phi}(kX)$ contains the (direct) sum of all such kZ', which is equal to kY. Thus $\tilde{\phi}$ is surjective. $\square$

Notice that a potential ambiguity appears given the two main constructions detailed in this section. Namely, when G is a locally profinite group, G can be considered as a locally profinite G-space, or simply as a locally profinite group, and the resulting constructions of the object kG may differ, *a priori*. However, in fact the constructions agree. Let G, X be locally profinite groups and $\phi : G \to X$ be a continuous group homomorphism. We then have a group action of G on X given by

$$\phi \times id_X : G \times X \to X, \quad (g, x) \mapsto \phi(g)x.$$

This is easily seen to be a continuous group action, and hence by Proposition 3.35, we obtain a continuous kG-module action on kX, which we denote

$$\widetilde{\phi \times id_X} : kG \times kX \to kX.$$

Alternatively, by Proposition 3.17, ϕ extends to a continuous k-algebra homomorphism

$$\tilde{\phi} : kG \to kX,$$

and this gives a continuous kG-module action on kX via

$$\tilde{\phi} \times id_{kX} : kG \times kX \to kX, \quad (y, x) \mapsto \tilde{\phi}(y)x.$$

Now, $\widetilde{\phi \times id_X}$ and $\tilde{\phi} \times id_{kX}$ agree on $k[G] \times k[X]$. Thus, they must agree on $kG \times kX$ because $k[G] \times k[X]$ is dense and kX is Hausdorff.
In particular, the construction of the module kG from the left multiplication action of G on itself is the same as constructing the augmented Iwasawa algebra kG and considering the rank one free left module.

3.5 Properties of augmented Iwasawa algebras

We now deduce some basic properties of augmented Iwasawa algebras from the universal property of Proposition 3.17.

Proposition 3.42:

Let G be a locally profinite group. Then kG is isomorphic to its opposite ring $(kG)^{op}$.

Proof:

Let A be an Iwasawa-like k-algebra, via the subgroup $L \leq A^\times$. The opposite algebra A^{op} has the same underlying set as A, but reversed multiplication. Then, A^{op} is an Iwasawa-like algebra, via the subgroup $L^{op} \leq (A^{op})^\times$, where L^{op} is the set L, with reversed group operation $g \cdot h = hg$.
Let $\phi : G \to L$ be a continuous group homomorphism. Then, define $\phi' : G \to L^{op}$ by $\phi'(g) = \phi(g)^{-1}$. Then, ϕ' is also a continuous group algebra homomorphism,

and so by the universal property of Proposition 3.17, ϕ' extends to a continuous k-algebra homomorphism $\tilde{\phi'} : kG \to A^{op}$. Taking opposite algebras obtains a continuous k-algebra homomorphism

$$\tilde{\phi} = \tilde{\phi'}^{op} : (kG)^{op} \to A.$$

Now, $(kG)^{op}$ is an Iwasawa-like algebra, via the subgroup $G^{op} \leq ((kG)^{op})^{\times}$. This is because the topology on $(kG)^{op}$ is the same as the topology on kG, and because $(kK)^{op}$ is a pseudocompact subalgebra of $(kG)^{op}$ for any open profinite $K \leq G$. The map $G \to G^{op}, g \mapsto g^{-1}$ is an isomorphism of topological groups. Thus we have an injection $j : G \to (kG)^{op}$, and $\tilde{\phi} \circ j = \phi$. So, $\tilde{\phi}$ is a continuous k-algebra homomorphism extending ϕ. It also must be the unique such map, because the values on $k[G^{op}] = k[G]$ are determined by the homomorphism property, and $k[G^{op}]$ is dense in $(kG)^{op}$.

So, j embeds the group G into the units of the Iwasawa-like algebra $(kG)^{op}$, such that the universal property of Proposition 3.17 is satisfied. Therefore, $(kG)^{op}$ is (canonically) isomorphic to kG. $\square$

As a consequence, kG has a left property if and only if it has the corresponding right property. For example, kG is left coherent (see Definition 6.1) if and only if kG is right coherent, Thus we will often refer to augmented Iwasawa algebras as being coherent or not coherent, similarly for Noetherian, without a prefix "left" or "right". It also follows that $kG \cong k[G] \otimes_{k[H]} kH$ as a $(k[G], kH)$-bimodule.

To prove statements about quotients of augmented Iwasawa algebras, it is necessary to define the augmentation ideal of a subgroup.

Definition 3.43:

Let G be a locally profinite group, $H \leq G$ be a closed subgroup. Consider the locally profinite G-space G/H where G acts by left multiplication, the G-equivariant quotient map $q_H : G \to G/H$, and its unique extension $\tilde{q_H} : kG \to k(G/H)$ to a continuous kG-module homomorphism, given by Proposition 3.40. The augmentation ideal of H in G is $\epsilon_G(H) = \operatorname{Ker} \tilde{q_H}$.

Notice that $\epsilon_G(H)$ is a closed ideal because $\tilde{q_H}$ is continuous and $k(G/H)$ is Hausdorff. By Proposition 3.41, $\tilde{q_H}$ is surjective, giving a natural isomorphism of left kG-modules $kG/\epsilon_G(H) \cong k(G/H)$ for any closed subgroup $H \leq G$. We may occasionally write $\epsilon(H) = \epsilon_G(H)$, when G is clear from context.

Proposition 3.44:

Let G be a locally profinite group, N a closed normal subgroup. There is a natural surjective ring homomorphism $kG \to k(G/N)$, with kernel the augmentation ideal $\epsilon_G(N)$.

Proof:

Note that the G-equivariant quotient map $q_N : G \to G/N$ is a continuous group homomorphism. We can extend this to a continuous k-algebra homomorphism $kG \to k(G/N)$ and to a continuous kG-module homomorphism $kG \to k(G/N)$, by Propositions 3.17 and 3.40. By the remark at the end of subsection 3.4, these homomorphisms may be identified, so we have a natural continuous ring homomorphism $kG \to k(G/N)$ with kernel the (two-sided) ideal $\operatorname{Ker} \tilde{q_N} = \epsilon_G(N)$. $\square$

We now prove some results on augmentation ideals which will be of later use.

Lemma 3.45:

Let $G' \leq G$ be a closed subgroup. The augmentation ideal $\epsilon_G(G') = kG \otimes_{kG'} \epsilon_{G'}(G')$.

Proof:

Consider the trivial kG'-module $k \cong kG'/\epsilon_{G'}(G')$. We show that $kG \otimes_{kG'} k$ is isomorphic to $k(G/G')$. Suppose $\phi : G/G' \to M$ is a G-equivariant map to an Iwasawa-like kG-module as in Proposition 3.40. There is a unique extension of the composition $\phi \circ q_{G'} : G \to M$ to $\widetilde{\phi \circ q_{G'}} : kG \to M$. Now, $(\phi \circ q_{G'})|_{G'}$ is constant, and therefore $\widetilde{\phi \circ q_{G'}}|_{kG'} : kG' \to M$ factors through a kG'-module homomorphism $\theta : k \to M$. Then

$$f : kG \times k \to M, \quad f(x, y) = x\theta(y),$$

is a kG'-balanced k-bilinear map, and $f(x, 1) = x\widetilde{\phi \circ q_{G'}}(1) = \widetilde{\phi \circ q_{G'}}(x)$. By the universal property of tensor product, there is a unique extension $\tilde{f} : kG \otimes_{kG'} k \to M$, which extends the original map ϕ, where $gG' \in G/G'$ is identified as $g \otimes 1$. Any other extension of ϕ to $kG \otimes_{kG'} k \to M$ must restrict to $\widetilde{\phi \circ q_{G'}}$ on kG and θ on k, so $\tilde{f}$ is unique. Therefore $kG \otimes_{kG'} k$ satisfies the universal property of Proposition 3.40, and we have a natural isomorphism $k(G/G') \cong kG \otimes_{kG'} k$.

By Proposition 5.10 (which does not depend on this lemma), kG is a flat right kG'-module, so there is a commutative diagram with exact rows as follows.

$$\begin{tikzcd}
0 \arrow[r] & \epsilon_G(G') \arrow[r] & kG \arrow[r] & k(G/G') \arrow[r] & 0 \\
0 \arrow[r] & kG \otimes_{kG'} \epsilon_{G'}(G') \arrow[r] \arrow[u] & kG \arrow[r] \arrow[u, "\cong"'] & kG \otimes_{kG'} k \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}$$

Therefore the natural (inclusion) map $kG \otimes_{kG'} \epsilon_{G'}(G') \to \epsilon_G(G')$ is an isomorphism. □

Clearly, the augmentation ideal of a closed subgroup H must contain the set $\{h - 1 \mid h \in H\}$. In the cases of most interest, this set is a generating set.

Lemma 3.46:

Let G be a locally profinite group. Suppose $G' \leq G$ is a closed subgroup such that there is an open profinite $H \leq G'$ with kH Noetherian. Then $\epsilon_G(G') = kG\{g' - 1 \mid g' \in G'\}$.

Proof:

By Lemma 3.45, it is enough to prove this in the case $G' = G$. Since $H \leq G$ is open, we have that G/H is discrete and $k(G/H) \cong k[G/H]$. Considering the chain of surjective homomorphisms

$$kG \to k(G/H) \to k = kG/\epsilon_G(G),$$

it follows that $\epsilon_G(G) = \epsilon_G(H) + kG\{g - 1 \mid g \in G/H\}$.
Now consider the case $G = G' = H$. Since kH is Noetherian, every left ideal of kH is finitely-generated, hence closed by Corollary 22.4 of [Sch11]. From the definition of the (inverse limit) topology on kH, the ideal $kH\{h - 1 \mid h \in H\}$ is dense in $\epsilon_H(H)$, but must be closed. So $\epsilon_H(H) = kH\{h - 1 \mid h \in H\}$.
Thus, by Lemma 3.45,

$$\begin{aligned}
\epsilon_G(H) &= kG \otimes_{kH} \epsilon_H(H) \\
&= kG \otimes_{kH} (kH\{h - 1 \mid h \in H\}) \\
&= kG\{h - 1 \mid h \in H\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\epsilon_G(G) &= kG\{h - 1 \mid h \in H\} + kG\{g - 1 \mid g \in G/H\} \\
&\subseteq kG\{g - 1 \mid g \in G\}.
\end{aligned}$$

The reverse inclusion is obvious, completing the proof. □

If k is a field of characteristic p or a complete discrete valuation ring with characteristic p residue field, then Lemma 3.46 holds whenever G is a p-adic Lie group, by Theorems 2.2 and 3.32

Lemma 3.47:

Let G be a p-adic Lie group, k be as above, and $G' \leq G$ be a closed subgroup. Let σ be a ring endomorphism of kG which restricts to a group homomorphism $G \to G$. Then $kG\sigma(\epsilon_G(G')) = \epsilon_G(\sigma(G'))$.

Proof:

By Lemma 3.46, any augmentation ideal $\epsilon_G(G')$ is generated abstractly by $\{g-1 \mid g \in G'\}$. The left ideal $kG\sigma(\epsilon_G(G'))$ clearly contains $\{\sigma(g) - 1 \mid g \in G'\}$, hence contains $\epsilon_G(\sigma(G'))$. Conversely, if $x = \sum_{g \in G'} c_g(g - 1) \in \epsilon_G(G')$, then $\sigma(x) = \sum_{g \in G'} \sigma(c_g)(\sigma(g) - 1) \in \epsilon_G(\sigma(G'))$, thus $\epsilon_G(\sigma(G'))$ must contain $kG\sigma(\epsilon_G(G'))$. □

4 Representation theory of p-adic Lie groups

In this section we introduce some important types of representations of p-adic Lie groups, namely the smooth and admissible representations, and explain how they may be viewed in relation to modules over the (augmented) Iwasawa algebra.

4.1 Smooth and admissible representations

In fact, we will give definitions of smooth and admissible representations over locally profinite groups.

Definition 4.1:

Let G be a locally profinite group, and k be a commutative pseudocompact ring. A smooth representation of G over k is a $k[G]$-module V such that for any $v \in V$, there exists an open profinite subgroup $H \leq G$ and an open ideal $I \subseteq k$, such that $H \cdot v = \{v\}$ and $I \cdot v = \{0\}$.

This definition naturally generalises Definitions 2.2.1 and 2.2.5 of [Eme10]. Moreover, if k is a field (with the discrete topology), we obtain the usual definition of smooth; every vector of V has an open subgroup of G fixing it.

Definition 4.2:

Let G be a locally profinite group, k be a commutative pseudocompact ring and V be a smooth representation of G over k. V is admissible if and only if for every open profinite subgroup $H \leq G$ and open ideal $I \subseteq k$, the k-submodule

$$V^{H,I} = \big\{v \in V \mid H \cdot v = \{v\}, I \cdot v = \{0\}\big\}$$

has finite length.

When k is discrete and $I = 0$, we will write $V^H = V^{H,0}$. In the case k is a field, a smooth representation V is admissible precisely when every V^H is a finite-dimensional k-vector space. Definition 4.2 generalises Definition 2.2.9 of [Eme10].

Example 4.3

Let G be a locally profinite group and X be a locally profinite G-space which has the discrete topology. The augmented completed module of X is $kX = k[X]$ by

Example 3.37. Since X is discrete, the stabiliser of any $x \in X$ is an open subgroup of G, hence contains an open profinite subgroup $H_x \leq G$. Given an element

$$y = \sum_{x \in X} c_x x \in k[X],$$

the finite intersection

$$H = \bigcap_{\substack{x \in X \\ c_x \neq 0}} H_x$$

is an open profinite subgroup of G, and fixes y. Thus if k is discrete, $kX = k[X]$ is a smooth representation of G.

4.2 Smooth representations are augmented Iwasawa modules

We now give a proof that any smooth representation is a module for the augmented Iwasawa algebra kG, using Proposition 3.17. Showing that any smooth representation carries a kG-module structure requires working with finite-length k-submodules, necessitating the following lemmas.

Lemma 4.4:

Let V be a smooth representation of G over k and $M \leq V$ be a k-submodule of V. The following are equivalent:

- M is a finitely-generated $k[H]$-module for some open profinite subgroup $H \leq G$.
- M is a finite-length $k[H]$-module for some open profinite subgroup $H \leq G$.
- M is a finitely-generated k-module.
- M is a finite-length k-module.

Lemma 4.5:

If M is a finite-length k-module, then $\mathrm{End}_k(M)$ is a finite length k-module.

With input from these lemmas, we show that the endomorphism ring of any smooth representation is an Iwasawa-like algebra, and deduce that there is a natural and unique kG-module structure on any smooth representation of G.

Proposition 3.16:

Let V be a smooth representation of G. The ring of k-linear endomorphisms of V is naturally an Iwasawa-like algebra.

Proof:

Let $A = \operatorname{End}_k(V)$ be the ring of k-linear endomorphisms of V. For any M a finite-length k-submodule of V, define the left ideal of A,

$$I^M = \{f \in A \mid f(M) = 0\}.$$

The collection of such ideals, $\{I^M \mid M \text{ a finite-length } k\text{-submodule of } V\}$, is closed under finite intersections. Thus we give A the linear topology with an open neighbourhood basis of 0 given by this collection, and we now show that A is a topological ring. We show that the multiplication of A, which is given by composition of functions,

$$c : A \times A \to A, \quad c(f, h) = f \circ h,$$

is continuous. Given M a finite-length k-submodule of V, consider

$$c^{-1}(I^M) = \{(f, h) \in A \times A \mid f(h(M)) = 0\}.$$

Note that for any $h \in A$, if $f \in I^{h(M)}$ and $h' \in I^M$, then $(f, h+h') \in c^{-1}(I^M)$, so

$$I^{h(M)} \times (h + I^M) \subseteq c^{-1}(I^M).$$

Moreover, $c^{-1}(I^M) = \{(f, h) \in A \times A \mid f \in I^{h(M)}\}$. Thus we have a chain of inclusions

$$c^{-1}(I^M) \subseteq \bigcup_{h \in A} I^{h(M)} \times \{h\} \subseteq \bigcup_{h \in A} I^{h(M)} \times (h + I^M) \subseteq c^{-1}(I^M),$$

in which equality must hold throughout. It follows $c^{-1}(I_M)$ is open in $A \times A$. Thus c is continuous and A is a topological ring. Also, A is Hausdorff since by Lemma 4.4, V is the union of its finite-length k-submodules, so the intersection of all the I^M is zero.

Moreover, $A^\times$ is a topological group under the subspace topology. To show this, let $a \in A^\times$, M be a finite-length k-submodule of V, and consider the open subset $U_{a,M} = (a + I^M) \cap A^\times \subseteq A^\times$. Then

$$U_{a,M}^{-1} = \{b \in A^\times \mid b(a + f) = 1 \text{ for some } f \in I^M\} = \{b \in A^\times \mid (1 - ba)(M) = 0\},$$

with inclusion of the third term in the second given by considering the choice $f = b^{-1}(1 - ba)$. It follows that

$$U_{a,M}^{-1} = \{b \in A^\times \mid (a^{-1} - b)(a(M)) = 0\} = (a^{-1} + I^{a(M)}) \cap A^\times$$

is open, and so the inversion map is continuous and $A^\times$ is a topological group. Now, for any open profinite $H \leq G$, define the subring of A,

$$B_H = \{f \in A \mid f(N) \subseteq N \text{ for all finite-length } k[H]\text{-submodules } N \leq V\}.$$

Then B_H is a topological ring, which we show is a pseudocompact k-algebra. An open neighbourhood basis of $0 \in B_H$ is given by the collection of ideals of the form $I^M \cap B_H$ for M a finite-length k-submodule of V. But $I^{k[H]M} \cap B_H \subseteq I^M \cap B_H$, and $k[H]M$ is a finitely-generated $k[H]$-module, hence is a finite-length $k[H]$-module and k-module by Lemma 4.4. Thus an open neighbourhood basis of $0 \in B_H$ is given by the collection of ideals,

$$\{I_H^M = I^M \cap B_H \mid M \text{ is a finite-length } k[H]\text{-submodule of } V\}.$$

Now, I_H^M is the kernel of the natural k-algebra homomorphism from B_H to $\mathrm{End}_k(M)$ given by restriction. This gives a natural k-algebra homomorphism

$$\psi_H : B_H \to \varprojlim_M \mathrm{End}_k(M) = \{(f_M)_M \mid f_M|_N = f_N \text{ if } N \leq M\} \subseteq \prod_M \mathrm{End}_k(M).$$

Now, ψ_H is injective because V is the union of its finitely-generated, hence by Lemma 4.4, finite-length $k[H]$-submodules. Given an element $(f_M)_M$ of the inverse limit, we can define an element of B_H by

$$f : V = \bigcup_M M \to V, \quad f(x) = f_M(x) \text{ if } x \in M,$$

and then $\psi_H(f) = (f_M)_M$. So, ψ_H is an isomorphism.

By Lemma 4.5, $\mathrm{End}_k(M)$ is a finite-length k-module for any finite-length k-submodule $M \leq V$. Therefore, B_H is a pseudocompact k-algebra under the subspace topology induced from A.

Let $\phi : G \to A^\times$ be the natural group homomorphism given by $\phi(g)(v) = g \cdot v$. Because $A^\times$ is a topological group under the subspace topology from A, the continuity of ϕ can be checked on a basis of open neighbourhoods of $1 \in A^\times$. Let $U_M = U_{1,M} \subseteq A^\times$ for M a finite-length k-submodule of V. Then

$$\phi^{-1}(U_M) = \{g \in G \mid (g - 1)(M) = 0\} = \{g \in G \mid g \cdot m = m, \forall m \in M\}.$$

Let S be a finite generating set of M over k, and for each $s \in S$, choose an open profinite subgroup $H_s \leq G$ that fixes s. Then the open profinite subgroup

$$H = \bigcap_{s \in S} H_s$$

fixes all elements of M, because the actions of G and of k commute. So $H \leq \phi^{-1}(U_M)$, which implies $\phi^{-1}(U_M)$ is an open subgroup of G. Thus ϕ is continuous.

Thus, $\phi : G \to \phi(G)$ is a surjective continuous map of topological groups, hence is an open map. Therefore $\phi(G)$ is a locally profinite group with open profinite subgroup $\phi(H)$. Moreover, $\phi(H)$ is contained in the units of B_H, which is a pseudocompact subalgebra of A. This shows that A is an Iwasawa-like algebra, via the subgroup $\phi(G) \leq A^\times$. □

Corollary 3.18:

Let V be a smooth representation of G. There is a unique kG-module structure extending the $k[G]$-module action on V.

Proof:

Recall from the proof of Proposition 3.16 the natural group homomorphism

$$\phi : G \to \mathrm{End}_k(V)^\times,$$

given by $\phi(g)(v) = g \cdot v$. In that proof we showed that $\mathrm{End}_k(V)$ is Iwasawa-like via $\phi(G)$, and that ϕ is continuous. By Proposition 3.17, there is a unique continuous k-algebra homomorphism $\tilde{\phi} : kG \to \mathrm{End}_k(V)$ extending ϕ. This defines a kG-module structure via $x \cdot v = \tilde{\phi}(x)(v)$ for all $x \in kG$, $v \in V$. Hence, there is a unique kG-module structure on the smooth representation V extending the $k[G]$-module structure. □

We now give the straightforward proofs of Lemmas 4.4 and 4.5.

Proof of Lemma 4.4:

Suppose M is a finitely-generated $k[H]$-module, and let S be a finite set of generators. For each $s \in S$, there exists an open normal subgroup $K_s \trianglelefteq_o H$

that fixes s, and an open ideal $I_s \subseteq k$ that annihilates s. Thus M is a finitely-generated $(k/I)[H/K]$-module, where

$$I = \bigcap_{s \in S} I_s, \quad K = \bigcap_{s \in S} K_s.$$

Then, k/I is Artinian and H/K is a finite group, so $(k/I)[H/K]$ is an Artinian ring, thus M has finite length as a $k[H]$-module. Moreover, $(k/I)[H/K]$ is finitely-generated as a k-module, so M is finitely-generated as a k-module.
If M is any finitely-generated k-module, there exist open ideals $I_s \subseteq k$ annihilating each of the elements of a finite generating set S for M. Then M is a finitely-generated k/I-module, and k/I is Artinian (where I is as above), so M is of finite length as a k-module.
Thus, it remains to show that any finite-length k-submodule M is a finitely-generated $k[H]$-submodule for some open profinite subgroup $H \leq G$. Clearly such an M is finitely-generated, let $m_1, \ldots, m_n$ be generators for M as a k-module. Because V is smooth, there exist open profinite subgroups H_j such that H_j fixes m_j, and so the open profinite subgroup

$$H = \bigcap_{j=1}^{n} H_j$$

fixes each m_j, and so M is naturally a finitely-generated $k[H]$-submodule of V. □

Proof of Lemma 4.5:

We show that for any finite-length k-modules M, N, that the k-module $\mathrm{Hom}_k(M, N)$ is of finite length, inducting on the length of M. If M has length 1, then M is simple and $M \cong k/I$ for some maximal ideal $I \trianglelefteq k$. So $\mathrm{Hom}_k(M, N)$ is isomorphic to the submodule $\{x \in N \mid Ix = 0\}$ of N, and hence is of finite length. If M is of length greater than 1, let $M' \leq M$ be a submodule such that M/M' is simple. Then $\mathrm{Hom}_k(M/M', N)$ is of finite length, and M' has length strictly less than that of M, so $\mathrm{Hom}_k(M', N)$ is of finite length by induction. By considering the exact sequence

$$0 \to \mathrm{Hom}_k(M', N) \to \mathrm{Hom}_k(M, N) \to \mathrm{Hom}_k(M/M', N),$$

it follows that $\mathrm{Hom}_k(M, N)$ is a finite-length k-module. Putting $M = N$ gives the result. □

4.3 The dual of a smooth representation

We now turn our attention to the mod p representation theory of p-adic Lie groups. Let G be a p-adic Lie group, and for the rest of this section, let k be a field of characteristic p (given the discrete topology as usual).

We now define a duality between the category of smooth representations and a new category. We will see that the augmented Iwasawa algebra and its module category kG-Mod play a role on "both sides" of the duality. The results in this subsection are essentially taken from section 1 of [Koh17] and section 2 of [Eme10].

Definition 4.6:

Let V be a k-vector space. The k-linear dual of V is $V^\vee = \mathrm{Hom}_k(V, k)$.

Definition 4.7:

Let V be a k-vector space with the discrete topology. Let $(V_i)_{i\in I}$ be a directed system of finite-dimensional subspaces such that $\varinjlim_{i\in I} V_i = V$. We give $V^\vee$ the inverse limit topology given by

$$V^\vee \cong \varprojlim_{i\in I} V_i^\vee,$$

where each $V_i^\vee$ is discrete.

It is straightforward to check that the topology given on $V^\vee$ does not depend on the choice of directed system, and thus we may take the directed system to comprise all finite-dimensional subspaces of V, under inclusion.

Lemma 4.8:

When V is a discrete k-vector space, $V^\vee$ is a pseudocompact k-module.

Proof:

This is straightforward: each $V_i^\vee$ is a finite-dimensional, hence Artinian, k-module, thus $V^\vee \cong \varprojlim_{i\in I} V_i^\vee$ is pseudocompact. $\square$

Definition 4.9:

Let M be a topological k-vector space. The continuous k-linear dual of M is

$$M^* = \{f \in M^\vee \mid f \text{ is continuous}\},$$

which we give the discrete topology.

Lemma 4.10:

Let V be a k-vector space with the discrete topology and M be a pseudocompact k-module. Then $(V^\vee)^* = V$ and $(M^*)^\vee = M$.

Proof:

Let $e_V : V \to (V^\vee)^*$ be the k-linear map given by

$$e_V(v)(f) = f(v)$$

for any $v \in V$, $f \in V^\vee$. Firstly we show the map is well defined onto the codomain. Notice that for any $v \in V$, $e_V(v) : V^\vee \to k$ has kernel

$$\operatorname{Ker} e_V(v) = \{f \in V^\vee \mid f(v) = 0\} = \operatorname{Ker}(V^\vee \to (kv)^\vee).$$

Since kv is finite-dimensional, it follows that $\operatorname{Ker} e_V(v)$ is open in $V^\vee$ and $e_V(v)$ is continuous. Next, since $V^\vee$ is the full linear dual, it is straightforward that e_V is injective. Finally, let $e : V^\vee \to k$ be a continuous k-linear map. For each finite-dimensional subspace $W \leq V$, let $\pi_W : V^\vee \to W^\vee$ be the projection map. Since $\operatorname{Ker} e \leq V^\vee$ is an open subspace of $V^\vee$, there exists a finite-dimensional W such that $e(\operatorname{Ker} \pi_W) = 0$. Thus there exists a k-linear map $\bar{e} : W^\vee \to k$ such that

$$e = \bar{e} \circ \pi_W.$$

But since W is finite-dimensional, $e_W : W \to (W^\vee)^\vee$ is an isomorphism and there exists a unique $v \in W$ such that $\bar{e} = e_W(v)$. Then $e = e_V(v)$.
So, e_V is surjective, hence is a k-linear isomorphism $V \to (V^\vee)^*$. Both spaces are discrete so e_V is a topological isomorphism.
Similarly, let $E_M : M \to (M^*)^\vee$ be the k-linear map given by

$$E_M(m)(f) = f(m)$$

for any $m \in M$, $f \in M^*$. Suppose $m \in M$ is such that $E_M(m) = 0$, that is, $f(m) = 0$ for every continuous k-linear map $f : M \to k$. It follows that m must lie in the kernel of any continuous k-linear map from M to a finite-dimensional k-vector space, but this forces $m = 0$ since

$$M = \varprojlim_{W \in \mathcal{B}} W$$

is an inverse limit of finite-dimensional k-vector spaces. So, E_M is injective.
To show E_M is surjective, let $E \in (M^*)^\vee$, and note for each W, the natural projection map

$$p_W : M \to W$$

induces an injective map

$$i_W : W^* \to M^*,$$

and since W is finite-dimensional, there exists a unique $x_W \in W$ such that

$$E \circ i_W = E_W(x_W).$$

The x_W must be compatible under the inverse system by uniqueness, and hence there exists $x \in M$ such that $E = E_M(x)$. So, E_M is surjective. It is then straightforward to check that the $W \in \mathcal{B}$ induce the pseudocompact topology on $(M^*)^\vee$, and that E_M is continuous, hence also a topological isomorphism. □

We now introduce the action of our p-adic Lie group G.

Definition 4.11:

Let V be a representation of G over k. We give $V^\vee$ the G-action by $(g \cdot f)(v) = f(g^{-1}v)$ for all $g \in G, f \in V^\vee, v \in V$.

Definition 4.12:

Let M be a pseudocompact k-module with an action of G by continuous automorphisms. We give M^* the G-action by $(g \cdot f)(m) = f(g^{-1}m)$ for all $g \in G, f \in M^*, m \in M$.

Lemma 4.13:

Let V be a smooth representation of G and $H \leq G$ be a compact open subgroup. Then $V^\vee$ has a (unique) kG-module structure which extends the $k[G]$-module structure. Moreover this makes $V^\vee$ a topological kH-module.

Proof:

Since V is a smooth representation (and k is a discrete field), we have that V is a direct limit of invariant subspaces

$$V = \varinjlim_{U \trianglelefteq_o H} V^U,$$

and therefore

$$V^\vee = \varprojlim_{U \trianglelefteq_o H} (V^U)^\vee.$$

Since each V^U is a $k[H]$-submodule, and thus a $k[H/U]$-module, each $(V^U)^\vee$ is a $k[H/U]$-module, from which it follows that $V^\vee$ is a kH-module. The kH-module action extends the $k[H]$-module action induced by the $k[G]$-module action defined above. Since H is a compact p-adic Lie group, kH is Noetherian by Theorem 2.2, thus Corollary 3.31 shows that $V^\vee$ has a uniquely determined kG-module structure.

Then, for any $W \leq V$ a finite-dimensional subspace, since V is smooth there exists $U \trianglelefteq_o H$ such that $W \leq V^U$. Composing the action map with the continuous map $V^\vee \to W^\vee$, we obtain the map

$$\phi_W : kH \times V^\vee \to W^\vee,$$

and $\phi_W^{-1}(0)$ must contain $\epsilon_H(U) \times V^\vee$. Since $W^\vee$ is discrete, it follows that ϕ_W is continuous. This holds for all W, and thus the characteristic property of the inverse limit (pseudocompact) topology on $V^\vee$ implies that

$$kH \times V^\vee \to V^\vee$$

is continuous – that is, $V^\vee$ is a topological kH-module. □

Lemma 4.14:

Let M be a pseudocompact k-module, such that M has a kG-module structure making it a topological kH-module for some compact open subgroup $H \leq G$. Then M^* is a smooth representation of G.

Proof:

Let $f \in M^*$ and $\alpha : kH \times M \to M$ be the continuous map giving the kH-module structure. Since $f : M \to k$ is continuous, the composition

$$f \circ \alpha : kH \times M \to k$$

is continuous. It follows that there exists an open ideal $I' \subseteq kH$ and an open submodule $N \leq M$ such that

$$(f \circ \alpha)(I' \times N) = \{0\}.$$

Since N is an open submodule of M and M is a pseudocompact k-module, the quotient M/N is a finite-dimensional k-vector space. Let $m_1, \ldots, m_n \in M$ be such that

$$M = N + km_1 + \cdots + km_n.$$

Since the restriction

$$(f \circ \alpha)|_{kH \times \{m_j\}} : kH \times \{m_j\} \to k$$

is continuous, there exists an open ideal $I_j \subseteq kH$ such that

$$(f \circ \alpha)(I_j \times km_j) = \{0\},$$

for each j. Let

$$I = I' \cap \bigcap_{j=1}^{n} I_j \subseteq kH.$$

Then I is an open ideal of kH, and by construction

$$(f \circ \alpha)(I \times M) = \{0\}.$$

That is, for all $x \in I$, $m \in M$, we have $f(x \cdot m) = 0$. Since I is open, there exists an open subgroup $U \leq H$ such that $\epsilon_H(U) \subseteq I$, from which it follows that $u \cdot f = f$ for all $u \in U$. The subgroup U is also open in G. Therefore M^* is a smooth representation of G (since k is discrete). $\square$

In fact, a (k-pseudocompact) kG-module which is a topological kH-module is automatically topological.

Proposition 4.15:

Let M be a kG-module which is pseudocompact as a k-module. Let $H \leq G$ be a compact open subgroup. Then M is a topological kG-module if and only if M is a topological kH-module.

Proof:

Here M has the topology given by the property of being a pseudocompact k-module. If M is a topological kG-module, the restriction of the continuous kG-action

$$\alpha_G : kG \times M \to M,$$

to $kH \times M$ is continuous, thus M is a topological kH-module.
Conversely, suppose α_H is continuous. Let the pseudocompact topology on M be given by the inverse limit of finite-dimensional k-vector spaces

$$M = \varprojlim_W W,$$

with projection maps $\pi_W : M \to W$. To show that α_G is continuous, it is enough to show that

$$\phi_W = \pi_W \circ \alpha_G : kG \times M \to W$$

is continuous (for all W). By a similar argument to Lemma 4.14, we show that there exists an open (two-sided) ideal $I \subseteq kH$ such that

$$\phi_W(I \times M) = \{0\}.$$

Namely, since W is discrete, there are $I_1 \trianglelefteq kH$, $N \leq M$ open such that $\phi_W(I_1 \times N) = 0$. Since M/N is finite-dimensional, there is an open $I_2 \trianglelefteq kH$ such that

$$\phi_W(I_1 \cap I_2 \times M) = \{0\},$$

and $I = I_1 \cap I_2$ is open. Let $I' = IkG$ be the right ideal of kG generated by I. Then I' contains

$$\bigoplus_{g \in H\backslash G} I \otimes g \subseteq kG,$$

and therefore is open in kG. Since $\phi_W(I' \times M) = \{0\}$, it follows that $\phi_W^{-1}(\{0\})$ is open in $kG \times M$. Since W is discrete, ϕ_W is continuous. So, α_G is continuous as required. $\square$

Combining Proposition 4.15 with Lemma 4.14, Lemma 4.13 and Lemma 4.10, we obtain a succinct dual characterisation of smooth representations.

Corollary 4.16:

Let V be a representation of G over k. V is smooth if and only if the k-pseudocompact module $V^\vee$ is a topological kG-module.

4.4 The dual of an admissible representation

In Corollary 4.16 we have given a module-theoretic condition on $V^\vee$ for V to be a smooth representation. We continue this approach for the admissible representations.

Lemma 4.17:

Let V be a smooth admissible representation of G over k, and $H \leq G$ be a compact open subgroup. Then $V^\vee$ is a finitely-generated kH-module.

Proof:

Recall from Lemma 4.13 that V has a kG-module structure such that V is a topological kH-module. Let $K \trianglelefteq_o H$ be an open pro-p subgroup. Since V is admissible, the invariant subspace V^U is finite-dimensional for any $U \trianglelefteq_o K$. Thus we have a topological isomorphism of pseudocompact k-modules

$$V^\vee \cong \varprojlim_{U \trianglelefteq_o K} \left(V^U\right)^\vee.$$

Since kK is Noetherian, $\epsilon_K(K) \subseteq kK$ is finitely-generated, from which it follows that $\epsilon_K(K)V^\vee$ is both dense and closed inside, hence equal to, the kernel of the continuous kH-module homomorphism

$$V^\vee \to (V^K)^\vee.$$

Therefore, the quotient

$$V^\vee / \epsilon_K(K)V^\vee \cong (V^K)^\vee$$

is finite-dimensional over k. Now, $\epsilon_K(K)$ is the Jacobson radical of the Iwasawa algebra kK by Proposition 19.7 of [Sch11]. Thus the topological Nakayama lemma, Corollary 1.5 of [Bru66], implies $V^\vee$ is a finitely-generated kK-module. In particular V is a finitely-generated kH-module. □

Lemma 4.18:

Let M be a kG-module and $H \leq G$ be a compact open subgroup. If M is a finitely-generated topological kH-module, then M^* is a smooth admissible representation of G, where M has the canonical topology of Lemma 3.30.

Proof:

Notice that since M is a finitely-generated kH-module, the canonical topology makes it a pseudocompact kH-module and a pseudocompact k-module. By Lemma 4.14, M^* is a smooth representation. To show that M^* is admissible, it is enough to show that $(M^*)^U$ is a finite-dimensional k-vector space for any $U \trianglelefteq_o H$. Given U, we have a surjective continuous kH-module homomorphism

$$M \to {}^{M}\!\diagup_{\epsilon_H(U)M},$$

which gives an injective map of smooth H-representations,

$$\left({}^{M}\!\diagup_{\epsilon_H(U)M}\right)^* \to M^*,$$

noticing that $\epsilon_H(U)M$ is closed in M. Since taking invariants is left exact, we have an injective map

$$\left({}^{M}\!\diagup_{\epsilon_H(U)M}\right)^* \to (M^*)^U,$$

but this map is also surjective since if $f \in (M^*)^U$, the continuity of f implies that $f(\epsilon_H(U)M) = 0$. Now, since M is finitely-generated, $M/\epsilon_H(U)M$ is a finitely-generated $k[H/U]$-module, thus is finite-dimensional over k. It follows that $(M^*)^U$ is finite-dimensional. So M^* is admissible. □

The combination of Lemma 4.17, Lemma 4.18 and Lemma 4.10 implies the following.

Corollary 4.19:

Let V be a smooth representation of G and $H \leq G$ be a compact open subgroup. The representation V is admissible if and only if the kG-module $V^\vee$ is finitely-generated as a kH-module.

4.5 Categories of modules and representations

In this subsection we write down category-theoretic versions of the statements proved in the above subsections, and outline some open questions which motivate the research carried out in this thesis.

First we summarise the results of subsections 4.2, 4.3 and 4.4 in category-theoretic language.

Recall that G is a p-adic Lie group and k is a field of characteristic p throughout this section.

Definition 4.20:

The category $\mathrm{Rep}_k^{\mathrm{sm}}(G)$ is the full subcategory of $k[G]$-modules consisting of the smooth representations.

Definition 4.21:

The category $\mathrm{Rep}_k^{\mathrm{ad}}(G)$ is the full subcategory of $\mathrm{Rep}_k^{\mathrm{sm}}(G)$ consisting of the smooth admissible representations.

Naturally, we call $\mathrm{Rep}_k^{\mathrm{sm}}(G)$ the *category of smooth representations* and $\mathrm{Rep}_k^{\mathrm{ad}}(G)$ the *category of (smooth) admissible representations.*

Definition 4.22:

The category $\mathrm{Mod}_k^{\mathrm{pc}}(G)$ has objects topological kG-modules whose topology is k-pseudocompact, with morphisms being continuous kG-module homomorphisms.

Definition 4.23:

The category $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ is the full subcategory of kG-modules which are finitely-generated kH-modules for some compact open subgroup $H \leq G$.

We call $\mathrm{Mod}_k^{\mathrm{pc}}(G)$ the *category of pseudocompact kG-modules* and $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ the *category of coadmissible kG-modules.* Notice that kG is not (in general) a pseudocompact k-algebra, thus the nomenclature of $\mathrm{Mod}_k^{\mathrm{pc}}(G)$ should not be taken literally. However, in the case that G is compact, it is true that $\mathrm{Mod}_k^{\mathrm{pc}}(G)$ is the category of pseudocompact kG-modules in the sense of Definition 3.8.

Corollary 4.24:

The functors $(\ )^\vee$, $(\ )^*$ are quasi-inverse anti-equivalences of categories between

$$\mathrm{Rep}_k^{\mathrm{sm}}(G) \longleftrightarrow \mathrm{Mod}_k^{\mathrm{pc}}(G)$$

and

$$\mathrm{Rep}_k^{\mathrm{ad}}(G) \longleftrightarrow \mathrm{Mod}_k^{\mathrm{sm}}(G).$$

Proof:

By an anti-equivalence of categories $\mathcal{C} \to \mathcal{D}$ we mean an equivalence of categories $\mathcal{C} \to \mathcal{D}^{op}$. It is clear that $(\ )^\vee$, $(\ )^*$ are functors. Lemma 4.10 along with Corollary 4.16 and Corollary 4.19 show that $(\ )^\vee$, $(\ )^*$ are essentially surjective. Because k is a field, $(\ )^\vee$ is exact from which it follows that $(\ )^\vee$ is faithful. If $\phi : W^\vee \to V^\vee$ is a morphism, and e_V, e_W are as inLemma 4.10, we define

$$\psi = e_W^{-1} \circ \phi^* \circ e_V,$$

and then $\phi = \psi^\vee$. So, $(\ )^\vee$ is also full. Similar reasoning applies to $(\ )^*$. $\square$

Corollary 4.25:

The category of smooth representations $\mathrm{Rep}_k^{\mathrm{sm}}(G)$ and the category of pseudo-compact kG-modules $\mathrm{Mod}_k^{\mathrm{pc}}(G)$ are abelian categories.

Proof:

If V is a smooth representation, then any $k[G]$-submodule or quotient is also smooth. If V_1, V_2 are smooth representations, then for any compact open subgroup $U \leq G$, the invariants

$$(V_1 \oplus V_2)^U = V_1^U \oplus V_2^U.$$

Since direct sums commute with direct limits it follows that $V_1 \oplus V_2$ is smooth. It follows that $\mathrm{Rep}_k^{\mathrm{sm}}(G)$ is an abelian category. Thus $\mathrm{Mod}_k^{\mathrm{pc}}(G)$ is also abelian since it is anti-equivalent to $\mathrm{Rep}_k^{\mathrm{sm}}(G)$ by Corollary 4.24. $\square$

Corollary 4.26:

The category of admissible representations $\mathrm{Rep}_k^{\mathrm{ad}}(G)$ and the category of coadmissible kG-modules $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ are abelian categories.

Proof:

Let $H \leq G$ be a compact open subgroup. Because k is a field of characteristic p and H is a compact p-adic Lie group, the Iwasawa algebra kH is a Noetherian ring by Theorem 2.2. Because kH is Noetherian, when

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of kG-modules, M_i is finitely-generated as a kH-module if and only if M_j is a finitely-generated kH-module for both $j \in \{1,2,3\}\backslash\{i\}$. So, $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ is an abelian category. Then $\mathrm{Rep}_k^{\mathrm{ad}}(G)$ is also abelian since it is an anti-equivalent category by Corollary 4.24. $\square$

Now, there are functors between the four categories mentioned above, and the category of all kG-modules kG-Mod, as follows.

Clearly the forgetful functor

$$\mathrm{Rep}_k^{\mathrm{ad}}(G) \to \mathrm{Rep}_k^{\mathrm{sm}}(G)$$

is an inclusion. Next, Corollary 3.18 implies there is a functor

$$\mathrm{Rep}_k^{\mathrm{sm}}(G) \to kG\text{-Mod}$$

given by taking each smooth representation to its unique kG-module. This is also an inclusion.
Now, where $H \leq G$ is a compact open subgroup, any finitely-generated kH-module has a unique (pseudocompact) topology making it a topological kH-module, by Lemma 3.30. This determines an inclusion functor

$$\mathrm{Mod}_k^{\mathrm{fg}}(G) \to \mathrm{Mod}_k^{\mathrm{pc}}(G)$$

which equips each module with its canonical topology. Finally, there is a forgetful functor

$$\mathrm{Mod}_k^{\mathrm{pc}}(G) \to kG\text{-Mod}$$

which forgets the topology, taking each object to its underlying kG-module. The composition

$$\mathrm{Mod}_k^{\mathrm{fg}}(G) \to \mathrm{Mod}_k^{\mathrm{pc}}(G) \to kG\text{-Mod}$$

is equal to the natural inclusion functor from $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ to kG-Mod, which has essential image lying in the finitely-generated modules kG-mod.

Including the anti-equivalences $(\ )^\vee$ and $(\ )^*$ of Corollary 4.24, we obtain the following commutative diagram, a version of which appears in section 2.2 of [Eme10].

$$\begin{tikzcd}
kG\text{-Mod} & \mathrm{Rep}_k^{\mathrm{sm}}(G) \arrow[l, hook', "\text{Corollary 3.18}"'] \arrow[r, shift left, "\vee"] & \mathrm{Mod}_k^{\mathrm{pc}}(G) \arrow[l, shift left, "*"] \arrow[r, "\text{forget}"] & kG\text{-Mod} \\
& \mathrm{Rep}_k^{\mathrm{ad}}(G) \arrow[u, hook] \arrow[r, shift left, "\vee"] & \mathrm{Mod}_k^{\mathrm{fg}}(G) \arrow[u, hook, "\text{Lemma 3.30}"'] \arrow[l, shift left, "*"] \arrow[r, hook] & kG\text{-mod} \arrow[u, hook]
\end{tikzcd}$$

We now ask some category-theoretic questions that motivate the results of this thesis.

Question 4.27:
Is the category of smooth representations $\mathrm{Rep}_k^{\mathrm{sm}}(G)$ a *locally coherent* category?

See, for example, [Kra97] for the definition of a locally coherent abelian category. It is known that $\mathrm{Rep}_k^{\mathrm{sm}}(GL_2(\mathbb{Q}_p))$ is a *locally Noetherian* category, by the work of Ollivier in [Oll09]. Emerton, Gee and Hellmann conjecture that the answer to Question 4.27 is always "yes" when $G = GL_n(F)$, [EGH, Conjecture 6.1.4]. Question 4.27 is equivalent to the following.

Question 4.28:
Let $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}}$ be the full subcategory of smooth representations which are finitely-presented – see Definition 10.3. Is $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}}$ an abelian category?

In Section 10 we see that Question 4.28 has a positive answer for $SL_2(F)$ and $GL_2(F)$, but not for the solvable p-adic Lie groups studied in Section 8.

In [Gab62], the Krull dimension of a (locally Noetherian) abelian category is defined. Since the Iwasawa algebra kH of a compact open subgroup $H \leq G$ is always Noetherian, the category $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ is a locally Noetherian abelian category. Moreover, the Krull dimension of $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ is finite, since it is bounded above by $\mathcal{K}(kH)$ (itself finite by [Ard04, Theorem A]).

Question 4.29:
What is the Krull dimension of $\mathrm{Mod}_k^{\mathrm{fg}}(G)$?

Since $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ is dual to the category of admissible representations, an answer to this question gives a quantitative notion of size to $\mathrm{Rep}_k^{\mathrm{ad}}(G)$. The results of Section 12 are a first step towards answering Question 4.29.

If G is a compact p-adic Lie group, then the category $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ is equal to the category of finitely-generated modules kG-mod. Then, the Krull dimension of the category $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ equals the Krull dimension $\mathcal{K}(kG)$. The following question is thus a special case of Question 4.29, and first appeared as Question D of [AB06].

Question 4.30:

When G is a compact p-adic Lie group and $\mathfrak{g}$ is the Lie algebra of G, is $\mathcal{K}(kG) = \lambda(\mathfrak{g})$?

Here λ denotes the length of a Lie algebra, as defined in Corollary 11.23. The results of subsection 11.6 give affirmative answers to certain cases of Question 4.30.

It would perhaps be remiss not to mention the below (less precise and very difficult) open questions.

Question 4.31:

What are the irreducible smooth (admissible) representations of $GL_n(F)$? Similarly, can we classify the finite length objects of $\mathrm{Mod}_k^{\mathrm{fg}}(GL_n(F))$ and $\mathrm{Mod}_k^{\mathrm{pc}}(GL_n(F))$?

Question 4.31 currently only has a satisfactory answer for $GL_2(\mathbb{Q}_p)$, [Bre03]. In all other cases, the irreducible representations known as *supersingular* are extremely challenging to understand. In Section 12, we contribute by proving that all modules of *minimal-positive* dimension in $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ are of finite length.

Finally, the below related question may be seen as a concise but vague overview of the (mod p) Langlands programme.

Question 4.32:

How does $\mathrm{Rep}_k^{\mathrm{ad}}(GL_n(F))$ relate to n-dimensional representations of the absolute Galois group $\mathrm{Gal}\left(\overline{F}/F\right)$?

Again, this is currently satisfactorily understood for the group $GL_2(\mathbb{Q}_p)$ only.

5 Faithful flatness for augmented Iwasawa algebras

In this section we show that the mod p augmented Iwasawa algebra of any p-adic Lie group is faithfully flat over the augmented Iwasawa algebra of any closed subgroup.

5.1 The compact case

Let k be a commutative pseudocompact ring throughout this section.

Theorem 5.1:

Let G be a profinite group, and $H \leq G$ be a closed subgroup. Then kG is a flat right kH-module.

Proof:

In Lemma 4.5 of [Bru66], Brumer proved that kG is a projective object in the category of pseudocompact kH-modules. This does not imply that kG is a projective kH-module in the usual sense, so we cannot immediately deduce flatness. However, Lemma 2.1 of [Bru66] implies that the completed tensor product $kG\widehat{\otimes}_{kH}$ is an exact functor and $kG\widehat{\otimes}_{kH}M = kG\otimes_{kH}M$ for any finitely-generated left kH-module M. It follows that if $I' \subseteq kH$ is a finitely-generated left ideal, applying $kG\otimes_{kH}$ to the short exact sequence

$$0 \to I' \to kH \to kH/I' \to 0$$

gives a short exact sequence

$$0 \to kG \otimes_{kH} I' \to kG \to kG \otimes_{kH} (kH/I') \to 0.$$

Let $I \subseteq kH$ be an arbitrary ideal of kH, and let $I = \varinjlim_{\alpha\in A} I_\alpha$ be the direct limit of the finitely-generated ideals it contains. By the above, there is a short exact sequence

$$0 \to kG \otimes_{kH} I_\alpha \to kG \to kG \otimes_{kH} (kH/I_\alpha) \to 0,$$

for each $\alpha \in A$. By Theorem 2.6.15 of [Wei94], direct limits of kH-modules are exact, so we have a short exact sequence

$$0 \to \varinjlim_{\alpha\in A}(kG \otimes_{kH} I_\alpha) \to kG \to \varinjlim_{\alpha\in A}(kG \otimes_{kH} (kH/I_\alpha)) \to 0.$$

Since tensor products commute with direct limits, we have that

$$0 \to kG \otimes_{kH} I \to kG \to kG \otimes_{kH} (kH/I) \to 0$$

is exact. Therefore $\mathrm{Tor}_1^{kH}(kG, kH/I) = 0$ for any left ideal $I \subseteq kH$, so by Proposition 3.2.4 of [Wei94], kG is a flat right kH-module. $\square$

In this section we will use a version of Mackey's restriction formula to generalise this result and deduce various corollaries.

5.2 Mackey's restriction formula

If G is a finite group, Mackey's restriction formula tells us how to compute restrictions of a representation induced from a subgroup of G.

Theorem 5.2 (Mackey's restriction formula)**:**

Let K be any field, G be a finite group, $H_1, H_2 \leq G$ be subgroups. Let W be a left $K[H_2]$-module. Then

$$\mathrm{Res}_{H_1}^G \mathrm{Ind}_{H_2}^G W \cong \bigoplus_{g \in H_1 \backslash G/H_2} \mathrm{Ind}_{g(H_2)g^{-1} \cap H_1}^{H_1} W_g,$$

as left $K[H_1]$-modules.

Here W_g is the K-vector space of symbols $g \otimes W$, with action $ghg^{-1} \cdot (g \otimes w) = g \otimes hw$ for all $h \in H_2$. Ind_A^B means the functor $K[B] \otimes_{K[A]}$. See [Ser77], Proposition 22, for a proof of this result.

In the next subsection, we show that an analogous statement to Mackey's restriction formula for finite groups holds for locally profinite groups. To be precise, we will prove the following theorem.

Theorem 5.3:

Let G be a locally profinite group. Let $G_1 \leq G$ be a closed subgroup, and $H \leq G$ be an open profinite subgroup. Let M be a left kG_1-module. Then

$$\mathrm{Res}_H^G \mathrm{Ind}_{G_1}^G M \cong \bigoplus_{g \in H \backslash G/G_1} \mathrm{Ind}_{gG_1g^{-1} \cap H}^H M_g,$$

as left kH-modules.

Here Ind_A^B means the functor $kB \otimes_{kA}$, whilst Res_A^B means the kA-module restriction of a kB-module. We interpret M_g similarly to the finite group case, as the k-vector space of symbols $g \otimes M$, with action $gxg^{-1} \cdot (g \otimes m) = g \otimes xm$ for all $x \in kG_1$, noticing that $k(gG_1g^{-1})$ is equal to $g(kG_1)g^{-1}$ as subrings of kG.

5.3 Proof of the restriction formula

In this subsection let us fix G a locally profinite group, $G_1 \leq G$ a closed subgroup, and $H \leq G$ an open profinite subgroup.
Consider the functors which make up each side of the Mackey formula:

Definition 5.4:

The functors $\mathrm{Mack}_1, \mathrm{Mack}_2 : kG_1\text{-Mod} \to kH\text{-Mod}$ are

$$\mathrm{Mack}_1 = \mathrm{Res}_H^G \mathrm{Ind}_{G_1}^G, \quad \mathrm{Mack}_2 = \bigoplus_{g \in H \backslash G / G_1} \mathrm{Ind}_{gG_1g^{-1} \cap H}^H (\ )_g.$$

Definition 5.5:

For $g \in G$, $\mathrm{Mack}_{2,g} : kG_1\text{-Mod} \to kH\text{-Mod}$ is the functor $\mathrm{Ind}_{gG_1g^{-1} \cap H}^H (\ )_g$.

The statement of Theorem 5.3 is then that $\mathrm{Mack}_1(M) \cong \mathrm{Mack}_2(M)$ for any M. We will in fact show that there is a canonical such isomorphism, defined next. Recall from Proposition 2.11 that the augmented Iwasawa algebra $kG \cong kH \otimes_{k[H]} k[G]$ as a left kH-module.

Definition 5.6:

Let M be a left kG_1-module. We define the kH-module homomorphism

$$\psi_M : \mathrm{Mack}_2(M) \to \mathrm{Mack}_1(M)$$

to be

$$\psi_M = \bigoplus_{g \in H \backslash G / G_1} \psi_{M,g}$$

where

$$\psi_{M,g} : \mathrm{Ind}_{gG_1g^{-1} \cap H}^H M_g \to \mathrm{Res}_H^G \mathrm{Ind}_{G_1}^G M$$

is given by

$$\psi_{M,g}(r \otimes_{k(gG_1g^{-1} \cap H)} (g \otimes m)) = (r \otimes_{k[H]} g) \otimes_{kG_1} m,$$

for all $r \in kH, m \in M$.

Proposition 5.7:

The kH-module homomorphism $\psi_{kG_1} : \mathrm{Mack}_2(kG_1) \to \mathrm{Mack}_1(kG_1)$ is an isomorphism.

Proof:

Throughout, let $M = kG_1$, a left kG_1-module under left multiplication. Fix a set $X \subseteq G$ of representatives for the double cosets $H\backslash G/G_1$.

Let $g \in X$. The left $k(gG_1g^{-1})$-module M_g is given by the symbols $g \otimes M = g \otimes kG_1$, and so as a left module may be considered as $g \otimes kG_1 = k(gG_1g^{-1}) \otimes g$. Then, because $gG_1g^{-1} \cap H$ is an open profinite subgroup of gG_1g^{-1},

$$k(gG_1g^{-1}) = k(gG_1g^{-1} \cap H) \otimes_{k[gG_1g^{-1}\cap H]} k[gG_1g^{-1}],$$

as a left $k(gG_1g^{-1} \cap H)$-module. Therefore

$$\begin{aligned}\mathrm{Ind}^H_{gG_1g^{-1}\cap H} k(gG_1g^{-1}) &= kH \otimes_{k(gG_1g^{-1}\cap H)} k(gG_1g^{-1} \cap H) \otimes_{k[gG_1g^{-1}\cap H]} k[gG_1g^{-1}] \\ &= kH \otimes_{k[gG_1g^{-1}\cap H]} k[gG_1g^{-1}] \\ &= \bigoplus_{a \in (gG_1g^{-1}\cap H)\backslash gG_1g^{-1}} kH \otimes a.\end{aligned}$$

Hence, as $M_g = g \otimes kG_1 = k(gG_1g^{-1}) \otimes g$,

$$\mathrm{Mack}_{2,g}(M) = \mathrm{Ind}^H_{gG_1g^{-1}\cap H} M_g = \bigoplus_{a \in (gG_1g^{-1}\cap H)\backslash gG_1g^{-1}} kH \otimes ag.$$

Note that each $kH \otimes ag$ is a free rank 1 left kH-module.

For each $g \in X$, choose a set of representatives $Y_g \subseteq gG_1g^{-1}$ for the right cosets $(gG_1g^{-1} \cap H)\backslash gG_1g^{-1}$. It then follows that

$$\mathrm{Mack}_2(M) = \bigoplus_{g \in H\backslash G/G_1} \mathrm{Ind}^H_{gG_1g^{-1}\cap H} M_g = \bigoplus_{(g,a)\in S} kH \otimes ag,$$

where S is the set

$$S = \{(g, a) \in G \times G \mid g \in X, a \in Y_g\}.$$

Then, $\mathrm{Mack}_1(M)$ can be straightforwardly seen to be

$$\mathrm{Mack}_1(M) = \mathrm{Res}^G_H \mathrm{Ind}^G_{G_1} kG_1 = \mathrm{Res}^G_H(kH \otimes_{k[H]} k[G]) = \bigoplus_{b \in H\backslash G} kH \otimes b.$$

We thus have the homomorphisms, for each $g \in X$,

$$\psi_{M,g} : \bigoplus_{a \in Y_g} kH \otimes ag \to \bigoplus_{b \in H\backslash G} kH \otimes b,$$

and their sum

$$\psi_M : \bigoplus_{(g,a) \in S} kH \otimes ag \to \bigoplus_{b \in H\backslash G} kH \otimes b.$$

Let $g \in X$, and $a \in Y_g$. Let $r \in kH$. Using the identifications described above, we have

$$\psi_{M,g}(r \otimes ag) = \psi_{M,g}(r \otimes_{k(gG_1g^{-1} \cap H)} (g \otimes g^{-1}ag)) = (r \otimes_{k[H]} g) \otimes_{kG_1} g^{-1}ag.$$

Then $r \otimes_{k[H]} g \in kG$ and $g^{-1}ag \in kG_1$, so their tensor product lies in kG, and we have

$$(r \otimes_{k[H]} g) \otimes_{kG_1} g^{-1}ag = r \otimes_{k[H]} gg^{-1}ag = r \otimes_{k[H]} ag \in kH \otimes ag.$$

So ψ_M is the identity when restricted to $kH \otimes ag \to kH \otimes ag \leq \mathrm{Mack}_1(M)$.
Thus, to show that ψ_M is an isomorphism, it is enough to check that $Z = \{ag \mid (a, g) \in S\} \subseteq G$ is a set of right coset representatives for H in G, that is, Z is a set of representatives for $H\backslash G$.
Because Y_g is a set of representatives for $(gG_1g^{-1} \cap H)\backslash gG_1g^{-1}$, it is easy to see that $g^{-1}Y_gg$ is a set of representatives for $(G_1 \cap g^{-1}Hg)\backslash G_1$. By Exercise I.8 of [Lan02] (interchanging left and right), it follows that

$$\{gy \mid g \in X, b \in g^{-1}Y_gg\} = \{ag \mid g \in X, a \in Y_g\} = Z$$

is a set of representatives for $H\backslash G$. Therefore ψ_M is an isomorphism. $\square$

Next we extend the result of Proposition 5.7 to arbitrary free modules.

Lemma 5.8:
Let M be a left kG_1-module, and $M' = \bigoplus_J M$, for some index set J. If ψ_M is an isomorphism, then $\psi_{M'}$ is an isomorphism.

Proof:

Because $\mathrm{Mack}_1, \mathrm{Mack}_2$ are functors composed of direct sums of tensor products, and because tensor products commute with arbitrary direct sums, we have that

$$\mathrm{Mack}_1(M') = \mathrm{Mack}_1\left(\bigoplus_J M\right) = \bigoplus_J \mathrm{Mack}_1(M),$$
$$\mathrm{Mack}_2(M') = \mathrm{Mack}_2\left(\bigoplus_J M\right) = \bigoplus_J \mathrm{Mack}_2(M),$$

and

$$\psi_{M'} = \psi_{\oplus_J M} = \bigoplus_J \psi_M : \mathrm{Mack}_2(M') \to \mathrm{Mack}_1(M').$$

It follows that if ψ_M is an isomorphism, so is $\psi_{M'}$. $\square$

Corollary 5.9:

If F is a free kG_1-module, then ψ_F is an isomorphism.

We can now extend to arbitrary modules, providing a proof of Theorem 5.3.

Proof of Theorem 5.3:

Let M be a left kG_1-module. Then M has a free presentation

$$F_1 \to F_0 \to M \to 0.$$

Now, each of the functors $\mathrm{Mack}_1, \mathrm{Mack}_2$ are (direct sums of) tensor products, and hence are right exact functors. Therefore we have exact sequences

$$\mathrm{Mack}_1(F_1) \to \mathrm{Mack}_1(F_0) \to \mathrm{Mack}_1(M) \to 0$$

and

$$\mathrm{Mack}_2(F_1) \to \mathrm{Mack}_2(F_0) \to \mathrm{Mack}_2(M) \to 0.$$

Moreover, the homomorphisms $\psi_M, \psi_{F_0}, \psi_{F_1}$ give us the following commutative diagram, which has exact rows.

$$\begin{tikzcd}
\mathrm{Mack}_1(F_1) \arrow[r] & \mathrm{Mack}_1(F_0) \arrow[r] & \mathrm{Mack}_1(M) \arrow[r] & 0 \arrow[r] & 0 \\
\mathrm{Mack}_2(F_1) \arrow[r] \arrow[u, "\psi_{F_1}"] & \mathrm{Mack}_2(F_0) \arrow[r] \arrow[u, "\psi_{F_0}"] & \mathrm{Mack}_2(M) \arrow[r] \arrow[u, "\psi_M"] & 0 \arrow[r] \arrow[u, "\psi_0"] & 0 \arrow[u, "\psi_0"]
\end{tikzcd}$$

By Corollary 5.9, ψ_{F_1}, ψ_{F_0} are isomorphisms and ψ_0 is clearly an isomorphism. Thus by the five lemma, ψ_M must be an isomorphism. So, $\mathrm{Mack}_1(M) \cong \mathrm{Mack}_2(M)$. $\square$

5.4 Flatness of augmented Iwasawa algebras

The formula in Theorem 5.3 now allows us to prove results similar to Theorem 5.1, for locally profinite groups.

Proposition 5.10:

Let G be a locally profinite group, $G_1 \leq G$ be a closed subgroup. Then kG is a flat right kG_1-module.

Proof:

To show that kG is a flat kG_1-module, we show that for any injection $i : M \to N$ of kG_1-modules, the induced map

$$j : kG \otimes_{kG_1} M \to kG \otimes_{kG_1} N$$

is an injection. Let H be an open profinite subgroup of G.
Note that $j = \mathrm{Mack}_1(i)$. For each $g \in H\backslash G/G_1$, let $j_g = \mathrm{Mack}_{2,g}(i)$, so that

$$j_g : \mathrm{Ind}^H_{gG_1g^{-1}\cap H} M_g \to \mathrm{Ind}^H_{gG_1g^{-1}\cap H} N_g.$$

Since gG_1g^{-1} is a closed subgroup of G, we have that $gG_1g^{-1} \cap H$ is a closed subgroup of H, and thus by Theorem 5.1, $\mathrm{Ind}^H_{gG_1g^{-1}\cap H}$ is a (left) exact functor, as is $(\ )_g$. It follows that j_g is injective.
By the proof of Theorem 5.3, we have that

$$j = \mathrm{Mack}_1(i) = \psi_N \circ \mathrm{Mack}_2(i) \circ \psi_M^{-1}.$$

Then

$$\mathrm{Mack}_2(i) = \bigoplus_{g\in H\backslash G/G_1} \mathrm{Mack}_{2,g}(i) = \bigoplus_{g\in H\backslash G/G_1} j_g$$

is injective because each j_g is injective. Because ψ_M, ψ_N are isomorphisms, j must be injective. $\square$

5.5 Faithful flatness of augmented Iwasawa algebras

In this subsection we restrict k to be a field of characteristic p or a complete discrete valuation ring of residue characteristic p.

Theorem 5.11:

Let G be a locally pro-p group, $G_1 \leq G$ be a closed subgroup. Then kG is a faithfully flat right kG_1-module.

Given Proposition 5.10, clearly only the faithful part of the theorem requires proving, and to do this we first show it for pro-p groups.

Lemma 5.12:

Let U be a pro-p group, $H \leq U$ be a closed subgroup. Then kU is a faithfully flat right kH-module.

Proof:

We show that $kU \otimes_{kH} M \neq 0$ for any non-zero left kH-module. Because kU is a flat right kH-module by Theorem 5.1, it is enough to show this for finitely-generated modules M. Now, there is an obvious projection map from $kU \otimes_{kH} M$ onto

$$kU \otimes_{kH} M \big/ \epsilon_U(U)(kU \otimes_{kH} M) \cong kU \big/ \epsilon_U(U) \otimes_{kH} M \cong k \otimes_{kH} M \cong M \big/ \epsilon_H(H)M.$$

But H is pro-p, since it is a closed subgroup of a pro-p group. Now, $\epsilon_H(H)$ is the Jacobson radical of kH, by (the proof of) Proposition 19.7 of [Sch11]. By Nakayama's lemma, $M/\epsilon_H(H)M$ is non-zero, hence $kU \otimes_{kH} M \neq 0$. $\square$

We can now prove the result for all locally pro-p groups.

Proof of Theorem 5.11:

Let M be a non-zero left kG_1-module. Let $H \leq G$ be an open pro-p subgroup. Then by Theorem 5.3,

$$\mathrm{Res}^G_H \mathrm{Ind}^G_{G_1} M \cong \bigoplus_{g \in H \backslash G / G_1} \mathrm{Ind}^H_{gG_1g^{-1} \cap H} M_g.$$

But taking $g = e \in G$, we have that $\mathrm{Ind}^H_{H \cap G_1} M$ is identified with a submodule of $\mathrm{Ind}^G_{G_1} M$, and is non-zero by Lemma 5.12. Thus $kG \otimes_{kG_1} M \neq 0$. So, kG is a faithfully flat right kG_1-module, by Proposition 5.10. $\square$

Any p-adic Lie group is locally pro-p, so the following result is immediate.

Theorem 5.13:

Let G be a p-adic Lie group, $G_1 \leq G$ be a closed subgroup. Then kG is a faithfully flat right kG_1-module.

6 Ring-theoretic definitions and results

6.1 Coherent rings

We recall the definition of a coherent ring.

Definition 6.1:

Let R be a ring. R is left coherent if and only if every finitely-generated left ideal of R is finitely-presented.

Note that any left Noetherian ring is left coherent. We can also define the notion of a coherent module.

Definition 6.2:

Let R be a ring. A left R-module M is coherent if and only if M is finitely-generated, and every finitely-generated submodule of M is finitely-presented.

A ring R is left coherent if and only if it is coherent as a left R-module. A core motivation for considering coherent rings in the context of representation theory comes from the following proposition, see [Gla89], Theorems 2.5.1 and 2.1.2. (Technically only commutative rings are considered, but the proof works identically for non-commutative rings.)

Proposition 6.3:

Let R be a ring. Then, R is left coherent if and only if the category of finitely-presented left R-modules is an abelian subcategory of the category of left R-modules.

Shotton makes use of Proposition 6.3 to deduce that the finitely-presented smooth mod p representations of $SL_2(F)$ form an abelian category, see Theorems 1.2 and 4.5 of [Sho20].

6.2 Skew polynomial rings

See Chapter 2 of [GW04] for an extensive treatment of the theory of skew polynomial rings.

Definition 6.4:

Let R be a ring, and $\sigma_X : R \to R$ be a ring endomorphism of R. The skew polynomial ring $R[X; \sigma_X]$ is the free left R-module $\bigoplus_{j \geq 0} RX^j$, given a ring structure by addition being R-module addition, and multiplication given by

$$r_1 X^{n_1} \cdot r_2 X^{n_2} = r_1 \sigma_X^{n_1}(r_2) X^{n_1+n_2},$$

extended suitably.

In this thesis we consider only the skew polynomial rings of the above type. Discussion of the more general theory can be found be in [GW04], Chapter 2, page 32. We will often omit the endomorphism from our notation and write simply $R[X; \sigma_X] = R[X]$.

6.3 Extensions of coherent rings

It will be important to us to be able to transfer the property of coherence when extending a ring in some fashion. The following was proved by Harris in Corollary 1.2 of [Har66].

Lemma 6.5:

Let R be a left coherent ring, and $\phi : R \to S$ be a ring homomorphism such that S is finitely-presented as a left R-module. Then S is a left coherent ring.

One consequence of this lemma is an analogue, for coherent rings, of the fact that any quotient of a Noetherian ring is Noetherian. See Theorem 2 of [Har67].

Lemma 6.6:

Let R be a left coherent ring, $I \subseteq R$ be a two-sided ideal. Suppose that I is finitely-generated as a left ideal. Then the ring R/I is left coherent.

The next result can be found as Theorem 2.3.3 of [Gla89] – the proof is identical for commutative or non-commutative rings.

Lemma 6.7:

Let R be a ring such that $R = \varinjlim_{a \in A} R_a$ is a directed colimit of some left coherent subrings R_a, ordered by inclusion, and such that R is a flat right R_a-module for all $a \in A$. Then R is left coherent.

In particular, a direct limit of Noetherian rings as in Lemma 6.7 will form a coherent ring. For example the polynomial ring $\mathbb{Z}[X_1, X_2, X_3, \dots]$ in an infinite number of variables is coherent.

Example 6.8

Let $G = \mathbb{Q}_p$. Then G is the direct limit of its compact open subgroups $H_n = p^{-n}\mathbb{Z}_p$, and so the augmented Iwasawa algebra of G is a direct limit,

$$kG = \varinjlim_{n \geq 0} kH_n,$$

see also Example 3.27. If k is a field of characteristic p, then kH_n is a left Noetherian ring, hence left coherent. By Proposition 5.10, kG is a flat right kH_n-module, since $H_n \leq G$ is a closed subgroup. So by Lemma 6.7, kG is a left coherent ring. Note, kG is not Noetherian since the ideals $\epsilon_G(H)$ form a non-terminating ascending chain. Since $H_n \cong \mathbb{Z}_p$, we have $kG \cong \varinjlim_{n \geq 0} k[[X^{\frac{1}{p^n}}]]$, which we write as $kG \cong k[[X]]^{\frac{1}{p^\infty}}$.

Example 6.9

More generally, if $G = \mathbb{Q}_p^m$, then

$$kG = \varinjlim_{n \geq 0} k(p^{-n}\mathbb{Z}_p^m) \cong \varinjlim_{n \geq 0} k[[X_1^{\frac{1}{p^n}}, \dots, X_m^{\frac{1}{p^n}}]],$$

which we write as $kG \cong k[[X_1, \dots, X_m]]^{\frac{1}{p^\infty}}$, and this ring is also coherent.

The following lemma was first proved, for commutative rings only, by Harris in [Har66]. For the convenience of the reader, we have replicated this proof, with the appropriate qualifications needed in the non-commutative case.

Lemma 6.10:

Let R be a left coherent ring. Let $X \subseteq R$ be a left denominator set (a left reversible Ore set). Let M be a left coherent R-module. Then $X^{-1}M$ is left coherent as a $X^{-1}R$-module.

Proof:

By Proposition 10.12 and Corollary 10.13 of [GW04], $X^{-1}R \otimes_R A \cong X^{-1}A$ for

any R-module A, and $X^{-1}R$ is a flat right R-module.
Let $N' \leq X^{-1}M$ be a finitely-generated $X^{-1}R$-module. Since

$$X^{-1}M \cong X^{-1}R \otimes_R M,$$

and by Lemma 10.2 of [GW04], the generators of N' can be written with a common left denominator, hence there exist $m_1, \dots m_r \in M$ such that

$$N' = X^{-1}R \cdot 1^{-1}m_1 + \cdots + X^{-1}R \cdot 1^{-1}m_r.$$

Let

$$N = Rm_1 + \cdots + Rm_r \leq M.$$

Let $i : N \to M$ be the natural R-module injection. Tensoring, we have the induced map

$$j : X^{-1}R \otimes_R N \to X^{-1}R \otimes_R M,$$

and because $X^{-1}R$ is flat as a right R-module, $j : X^{-1}N \to X^{-1}M$ is injective. Clearly $j(X^{-1}N) = N'$, so $X^{-1}N \cong N'$ as $X^{-1}R$-modules.
Now, since M is coherent and N is finitely-generated, N is finitely-presented. Let

$$0 \to K \to R^n \to N \to 0$$

be an exact sequence of R-modules with K finitely-generated. Again, because $X^{-1}R$ is flat,

$$0 \to X^{-1}R \otimes_R K \to (X^{-1}R)^n \to X^{-1}N \to 0$$

is exact. Then $X^{-1}R \otimes_R K$ is finitely-generated, so $N' \cong X^{-1}N$ is finitely-presented. So $X^{-1}M$ is a coherent $X^{-1}R$-module. $\square$

Corollary 6.11:

Let R be a left coherent ring, and $X \subseteq R$ be a left denominator set. If R is left coherent, then $X^{-1}R$ is left coherent.

A less common way of extending rings is to take their amalgamated product. Åberg showed that certain amalgamated products of coherent rings remain coherent, see Theorem 12 of [Åbe82].

Theorem 6.12:

Let A be a left Noetherian ring and B, C be left coherent rings. Let $D = B *_A C$ be the amalgamated product along injective ring homomorphisms $A \to B$, $A \to C$. If D is a flat right module over each of A, B, C, then D is left coherent.

Åberg's intended application of Theorem 6.12 was to group algebras; we utilise it for augmented Iwasawa algebras in Corollary 7.11.
Finally, see Theorem 2 and Corollary 2.1 of [Har66] for a proof of the following result.

Proposition 6.13:

Let $\phi : R \to S$ be a ring homomorphism such that S is a faithfully flat right R-module, via ϕ. If S is left coherent, then R is left coherent.

Proposition 6.13 plays a crucial part in the results of this thesis; for its application in our situation, see Proposition 7.1.

6.4 Coherence of a skew polynomial ring

The Hilbert basis theorem (which originates with Hilbert's 1890 paper [Hil90]) tells us that if R is a left Noetherian ring, then the polynomial ring $R[X]$ is also left Noetherian ([GW04, Theorem 1.9]).

Example 6.14

Let F be a finite field extension of $\mathbb{Q}_p$ and $F^\times$ be its group of units. Then $F^\times = \mathcal{O}_F^\times \times \pi^{\mathbb{Z}}$ where $\pi \in \mathcal{O}_F$ is a choice of uniformiser. Since $\mathcal{O}_F^\times$ is a compact p-adic Lie group, $k\mathcal{O}_F^\times$ is a Noetherian ring by Theorem 2.2. It follows from the Hilbert basis theorem that $k\mathcal{O}_F^\times[X]$ is Noetherian. Then the localisation

$$kF^\times \cong k\mathcal{O}_F^\times[X, X^{-1}]$$

is also a Noetherian ring, by [Kem11, Corollary 6.4].

There is a corresponding result for a nice class of skew polynomial rings, which we call the non-commutative Hilbert basis theorem.

Theorem 6.15 (non-commutative Hilbert basis theorem)**:**

Let R be a ring. Let $\sigma_X : R \to R$ be a ring automorphism. If R is left Noetherian, then the skew polynomial ring $R[X; \sigma_X]$ is left Noetherian. If R is right Noetherian, then $R[X; \sigma_X]$ is right Noetherian.

A proof is given in [GW04], Theorem 1.14. Note that in Theorem 6.15, it is enough in fact to assume that σ_X is a surjective endomorphism, since any surjective endomorphism of a Noetherian ring is also injective.
If we have a non-surjective ring endomorphism, then Theorem 6.15 can fail, see Exercise 2P of [GW04], page 38. However, with appropriate hypotheses on a non-surjective endomorphism, we can conclude that the skew polynomial ring is left coherent.

Theorem 6.16:

Let A be a left Noetherian ring, and $\sigma_F : A \to A$ be an injective ring endomorphism, such that A is flat as a right $\sigma_F(A)$-module. Then, the skew polynomial ring $A[F;\sigma_F]$ is left coherent.

This result is due to Emerton in [Eme]. For the reader's convenience, we provide a proof of Theorem 6.16 in the next subsection.

6.5 Coherence of a skew polynomial ring, proof

In this subsection we prove Theorem 6.16. We will often use the convention of omitting the endomorphism, so writing $A[F]$ for $A[F;\sigma_F]$.

Let A be a left Noetherian ring and $\sigma_F : A \to A$ be an injective ring endomorphism. A is naturally a right $\sigma_F(A)$-module, under right multiplication, and we suppose that this right module is flat. We write $R = A[F] = A[F;\sigma_F]$ for the corresponding skew polynomial ring.

The ring R is a free left A-module on the basis $\{1, F, F^2, \dots\}$. So

$$R = A[F] = \bigoplus_{j=0}^{\infty} AF^j.$$

We define an increasing filtration on R given by F-degree,

$$R^{\leq k} = \bigoplus_{j=0}^{k} AF^j.$$

This also defines a filtration on the free module R^n, and any left submodule $M \leq R^n$. Let

$$J = RF = \bigoplus_{j=1}^{\infty} AF^j \subseteq R,$$

so J is the left ideal generated by F. Note that J is also a right ideal of R.

Lemma 6.17:

The ideal J is a flat right R-module.

Proof:

Notice that the endomorphism σ_F can be naturally extended to R by defining $\sigma_F(F) = F$, and then $\sigma_F(R) = \sigma_F(A)[F]$. Then, there is an (R, R)-bimodule isomorphism

$$J = RF \cong R \otimes_{\sigma_F(R)} (FR), \quad rF \mapsto r \otimes F \cdot 1, \quad r\sigma_F(s)F \leftarrow\!\shortmid r \otimes Fs.$$

Moreover, for any left R-module M, there is a natural left A-module isomorphism,

$$R\otimes_{\sigma_F(R)} M \cong A\otimes_{\sigma_F(A)} M, \quad \Big(\sum_{j=0}^{\infty} a_j F^j\Big)\otimes m \mapsto \sum_{j=0}^{\infty} a_j \otimes (F^j m), \quad a\otimes m \leftarrow\!\shortmid a\otimes m.$$

Because A is a flat right $\sigma_F(A)$-module, it follows that R is a flat right $\sigma_F(R)$-module. Also, FR is a free right R-module because σ_F is injective, and so the functor

$$J\otimes_R = R \otimes_{\sigma_F(R)} (FR)\otimes_R$$

is a composition of two exact functors, hence is exact. So J is a flat right R-module. □

Proving that R is coherent relies on describing the finitely-generated submodules of R^n.

Lemma 6.18:

Let $M \leq R^n$ be a left R-submodule of R^n. Then, M is finitely-generated as an R-module if and only if ${}^{M}\!/\!{}_{JM}$ is finitely-generated as an A-module.

This lemma allows us to show that R is coherent.

Proof of Theorem 6.16:

Let $I \subseteq R$ be a finitely-generated left ideal of R. Then we have a short exact sequence

$$0 \to M \to R^n \to I \to 0.$$

Now, I is finitely-presented if and only if M is finitely-generated, by Lemma 2.1.1 of [Gla89]. By Lemma 6.18, M is finitely-generated if and only if $M/JM =$

$R/J \otimes_R M$ is finitely generated as an A-module. Applying $(R/J)\otimes_R$ to the short exact sequence, we obtain the exact sequence

$$0 \to \mathrm{Tor}_1^R(R/J, I) \to M/JM \to A^n \to I/JI \to 0.$$

Since A is Noetherian, any submodule of A^n is finitely-generated. Thus M/JM is finitely generated as an A-module if and only if $\mathrm{Tor}_1^R(R/J, I)$ is. Now, by dimension shifting,

$$\mathrm{Tor}_1^R(R/J, I) = \mathrm{Tor}_2^R(R/J, R/I) = \mathrm{Tor}_1^R(J, R/I).$$

But J is a flat right R-module by Lemma 6.17. Thus $\mathrm{Tor}_1^R(J, R/I) = 0$. Hence M is finitely-generated and I is finitely-presented.
So every finitely-generated left ideal of R is finitely-presented – therefore R is left coherent. □

It remains, therefore, to prove Lemma 6.18. This will rely on the following description of filtrations.

Lemma 6.19:
Let M be a left submodule of R^n. Then $(JM)^{\leq k} = AFM^{\leq k-1}$ for all $k \geq 1$.

Proof:
Note $JM = RFM = AFM$ because M is an $A[F]$-module. Let $m \in (JM)^{\leq k}$. Then there are $x_1, x_2, \ldots, x_d \in A$ and $m_1, m_2, \ldots, m_d \in M$ such that

$$m = \sum_{j=1}^{d} x_j F m_j.$$

Let $K \geq k$ be such that $m_1, m_2, \ldots, m_d \in M^{\leq K}$. Denote the coefficient of F^r in the ith component of an element $z \in R^n$ by $z^{(i,r)} \in A$. For any $r \in \{k, \ldots, K\}$, $i \in \{1, \ldots, n\}$,

$$\sum_{j=1}^{d} x_j F m_j^{(i,r)} F^r = \sum_{j=1}^{d} x_j \sigma_F(m_j^{(i,r)}) F^{r+1} = 0,$$

so

$$\sum_{j=1}^{d} x_j \sigma_F(m_j^{(i,r)}) = 0.$$

Now, A is a flat right $\sigma_F(A)$-module. By Lemma 3.65 of [Rot09], there exist $b_{qj} = \sigma_F(a_{qj}) \in \sigma_F(A)$, $y_q \in A$ such that for all $j \in \{1, 2, \ldots, d\}$,

$$x_j = \sum_{q=1}^{N} y_q b_{qj},$$

and for all $q \in \{1, 2, \ldots, N\}$,

$$\sum_{j=1}^{d} b_{qj} \sigma_F\big(m_j^{(i,r)}\big) = \sigma_F \left(\sum_{j=1}^{d} a_{qj} m_j^{(i,r)} \right) = 0.$$

Thus

$$m = \sum_{j=1}^{d} x_j F m_j = \sum_{j=1}^{d} \sum_{q=1}^{N} y_q b_{qj} F m_j = \sum_{q=1}^{N} y_q F \left(\sum_{j=1}^{d} a_{qj} m_j \right).$$

We define

$$m'_q = \sum_{j=1}^{d} a_{qj} m_j \in M,$$

for all $q \in \{1, 2, \ldots, N\}$. Then, by the properties of the b_{qj} and because σ_F is injective,

$${m'_q}^{(i,r)} = \sum_{j=1}^{d} a_{qj} m_j^{(i,r)} = 0,$$

for all $i \in \{1, 2, \ldots, n\}$ and $r \in \{k, \ldots, K\}$. Also, $m'_q \in \sum_{j=1}^{d} A m_j \leq M^{\leq K}$, and therefore $m'_q \in M^{\leq k-1}$, for all q. Hence

$$m = \sum_{q=1}^{N} y_q F m'_q \in AFM^{\leq k-1}.$$

Therefore, $(JM)^{\leq k} \subseteq AFM^{\leq k-1}$, and the reverse inclusion is obvious. $\square$

Now we can prove Lemma 6.18.

Proof of Lemma 6.18:

Let M be a submodule of R^n. If M is finitely generated, there is a surjection $R^N \to M$. Applying $(R/J) \otimes_R$, we obtain a surjection $A^N \to M/JM$, and hence M/JM is finitely generated as an A-module.

Conversely, suppose that M/JM is finitely generated as an A-module. This means that there is a finite set $S \subseteq M$, such that

$$M = AS + JM.$$

Then, since S is finite, there exists $d \in \mathbb{N}$ such that $S \subseteq M^{\leq d}$. Then

$$M = M^{\leq d} + JM.$$

We claim that $M = RM^{\leq d}$. If $k \leq d$ then trivially $M^{\leq k} \subseteq RM^{\leq d}$.
Let $k > d$, and $m \in M^{\leq k}$. Let $y \in M^{\leq d}$ and $z \in JM$ such that $m = y + z$. Then $z = m - y \in M^{\leq k}$ and $z \in JM$, so $z \in (JM)^{\leq k} = AFM^{\leq k-1}$, by Lemma 6.19. By induction, we may assume $M^{\leq k-1} \subseteq RM^{\leq d}$, and therefore $z \in RM^{\leq d}$. So $m = y + z \in RM^{\leq d}$. Hence $M^{\leq k} \subseteq RM^{\leq d}$, for all k. Therefore $M = RM^{\leq d}$.
Now, $M^{\leq d}$ is a submodule of the finitely-generated A-module $(R^n)^{\leq d}$, hence is finitely generated as an A-module, since A is Noetherian. Therefore $M = RM^{\leq d}$ is finitely-generated as an R-module. □

We conclude that $A[F]$ is a left coherent ring.

6.6 Two-variable skew polynomial rings

In this subsection we present a generalisation of Theorem 6.16, concerning skew polynomial rings over a Noetherian base ring with two commuting variables. It will be seen in Section 9 that such a ring need not be coherent, however, we show that it nevertheless satisfies the pleasant property of *2-coherence*.

Definition 6.20:

Let R be a ring, M be a (left) R-module, and $n \geq -1$ be an integer. We say that M is FP_n if there exists an exact sequence

$$R^{m_n} \to \cdots \to R^{m_1} \to R^{m_0} \to M \to 0.$$

Thus the finitely-generated and finitely-presented R-modules are precisely the FP_0 and FP_1 modules, respectively. Our convention is that any R-module is FP_{-1}.

Definition 6.21:

Let R be a ring, $n \geq 0$ be an integer. R is (left) n-coherent if every FP_n R-module M is FP_{n+1}.

Thus, a ring is left 1-coherent if and only if it is left coherent, and left 0-coherent if and only if it is left Noetherian.

Theorem 6.22:

Let A be a left Noetherian ring, and $\sigma_E, \sigma_F : A \to A$ be commuting injective ring endomorphisms such that A is flat as both a right $\sigma_E(A)$-module and as a right $\sigma_F(A)$-module. The skew polynomial ring $A[E, F; \sigma_E, \sigma_F]$ is left 2-coherent.

We may view $R = A[E, F; \sigma_E, \sigma_F]$ as either of the iterated skew polynomial rings $R = A[E; \sigma_E][F; \sigma_F]$ or $R = A[F; \sigma_F][E; \sigma_E]$ by extending the endomorphisms such that $\sigma_F(E) = E$, $\sigma_E(F) = F$.
Let J be the left ideal of R generated by the two skew variables, $J = RE + RF$.

Lemma 6.23:

The ideals RE, RF and REF are flat right R-modules, and there is a short exact sequence

$$0 \to REF \to RE \oplus RF \to J \to 0$$

which is a flat resolution for J as a right R-module.

Proof:

When $\sigma : A \to A$ is a ring endomorphism, we may extend it to a ring endomorphism of R by letting $\sigma(E) = E, \sigma(F) = F$. Then for any left R-module M, there is a left A-module isomorphism

$$R \otimes_{\sigma(R)} M \cong A \otimes_{\sigma(A)} M,$$
$$\Big(\sum_{j,k=0}^{\infty} a_{jk} E^j F^k\Big) \otimes m \mapsto \sum_{j,k=0}^{\infty} a_{jk} \otimes (E^j F^k m),$$
$$a \otimes m \leftarrow\!\shortmid a \otimes m.$$

Now, if $X = E^j F^k \in R$ is a monomial and $\sigma = \sigma_E^j \sigma_F^k$, we have an (R, R)-bimodule isomorphism

$$RX \cong R \otimes_{\sigma(R)} XR, \quad rX \mapsto r \otimes X \cdot 1, \quad r\sigma(s)X \leftarrow\!\shortmid r \otimes Xs.$$

Thus

$$RX \otimes_R = R \otimes_{\sigma(R)} XR \otimes_R .$$

Since A is a flat right module over $\sigma_E(A), \sigma_F(A)$, A is a flat right $\sigma(A)$-module, thus $R \otimes_{\sigma(R)}$ is an exact functor, as is $XR \otimes_R$, hence $RX \otimes_R$ is exact and RX is flat. Putting $X = E, F, EF$ shows that RE, RF, REF are flat right R-modules. Define $\phi : RE \oplus RF \to J$ by $\phi(x,y) = x + y$. Let $\psi : REF \to RE \oplus RF$, $\psi(z) = (z, -z)$. Clearly $\phi \circ \psi = 0$, and ψ is injective. If $(x,y) \in \operatorname{Ker}\phi$, then $y = -x$ and $x \in RE \cap RF = REF$. Thus ψ, ϕ define the exact sequence of right R-modules in the statement. □

Lemma 6.24:

Let $R = A[E, F]$. Let $I \leq R^m$ be a finitely-generated left submodule of R^m, with presentation

$$0 \to K \to R^n \to I \to 0.$$

Then $REK = (RE)^n \cap K$.

Proof:

It is clear that $REK \leq (RE)^n \cap K$. The quotient

$$\begin{aligned} {}^{(RE)^n \cap K}\!\!\diagup_{REK} &= \operatorname{Ker}\left({}^{K}\!\diagup_{REK} \to {}^{R^n}\!\diagup_{(RE)^n}\right) \\ &= \operatorname{Ker}\left({}^{R}\!\diagup_{RE} \otimes_R K \to {}^{R}\!\diagup_{RE} \otimes_R R^n\right) \\ &\cong \operatorname{Tor}_1^R\left({}^{R}\!\diagup_{RE}, I\right) \end{aligned}$$

because $R^n/K \cong I$. Dimension shifting, we have

$$\operatorname{Tor}_1^R\left({}^{R}\!\diagup_{RE}, I\right) = \operatorname{Tor}_2^R\left({}^{R}\!\diagup_{RE}, {}^{R^m}\!\diagup_{I}\right) = \operatorname{Tor}_1^R\left(RE, {}^{R^m}\!\diagup_{I}\right) = 0,$$

because RE is a flat right R-module by Lemma 6.23. The result follows. □

An identical result holds, of course, for RFK. It follows that K/REK is naturally a submodule of $(R/RE)^n = A[F]^n$, and similarly $K/RFK \leq A[E]^n$.

Proposition 6.25:

Let $I \leq R^m$ be a finitely-generated left submodule, and $K \leq R^n$ be as in Lemma 6.24. Then, K is finitely-generated as an R-module if and only if ${}^{K}\!\diagup_{REK}$ is finitely-generated as an $A[F]$-module.

Proof:

Clearly if K is finitely generated, K/REK is a finitely-generated $A[F]$-module. Conversely, suppose K/REK is finitely-generated, as an $A[F]$-module. Consider the natural ascending filtration by E-degree on R, R^n, and K, denoted by $(\ )^{\leq d}$. By identical reasoning as in Lemma 6.19, we can show that

$$(REK)^{\leq k} = (AEK)^{\leq k} = AEK^{\leq k-1}$$

for any $k \geq 1$ (this does not require any Noetherianity assumption). Then, copying the proof of Lemma 6.18, it follows that K is generated by $K^{\leq d}$ as an R-module, for some $d \geq 0$.

It is then sufficient to prove that $K^{\leq d}$ is a finitely-generated $A[F]$-module. Note that $K^{\leq d}$ is an $A[F]$-submodule of $(R^n)^{\leq d}$. By definition of the filtration by E-degree, $(R^n)^{\leq d} \cong A[F]^{n(d+1)}$ is a finitely-generated free left $A[F]$-module. Hence by Lemma 6.18, $K^{\leq d}$ is finitely-generated if and only if $K^{\leq d}/AFK^{\leq d}$ is.

We now show that the inclusion of $A[F]$-modules

$$AFK^{\leq d} \leq AF(R^n)^{\leq d} \cap K^{\leq d},$$

is an equality, by a similar method to Lemma 6.24. The quotient

$$\begin{aligned} {AF(R^{\leq d})^n \cap K^{\leq d}}\big/{AFK^{\leq d}} &= \operatorname{Ker}\left({K^{\leq d}}\big/{AFK^{\leq d}} \to {R^{\leq d}}\big/{AFR^{\leq d}}\right) \\ &= \operatorname{Tor}_1^{A[F]}\left({A[F]}\big/{A[F]F}, {R^{\leq d}}\big/{K^{\leq d}}\right) \end{aligned}$$

because $R^{\leq d}$ is a free, hence flat, left $A[F]$-module. Now let

$$M = {(R^n)^{\leq d}}\big/{K^{\leq d}} = {(R^n)^{\leq d}}\big/{R^{\leq d} \cap K} \cong {(R^n)^{\leq d} + K}\big/{K} \leq {R^n}\big/{K} = I.$$

Because I is a submodule of R^m, there is a natural injective $A[F]$-module homomorphism $M \to R^m$. Since R^m is a free, hence flat, $A[F]$-module, dimension shifting shows that

$$\begin{aligned} \operatorname{Tor}_1^{A[F]}\left({A[F]}\big/{A[F]F}, M\right) &= \operatorname{Tor}_2^{A[F]}\left({A[F]}\big/{A[F]F}, {R^m}\big/{M}\right) \\ &= \operatorname{Tor}_1^{A[F]}\left(A[F]F, {R^m}\big/{M}\right), \end{aligned}$$

but Lemma 6.17 implies the final Tor group is zero. Hence, we have equality

$$AFK^{\leq d} = AF(R^n)^{\leq d} \cap K^{\leq d},$$

from which it follows that $K^{\leq d}/AFK^{\leq d}$ is naturally an A-submodule of

$${(R^n)^{\leq d}}\big/{AF(R^n)^{\leq d}} \cong A^{n(d+1)}.$$

Since A is Noetherian, $K^{\leq d}/AFK^{\leq d}$ must be a finitely-generated A-module, from which it follows that $K^{\leq d}$ is a finitely-generated $A[F]$-module, and therefore K is a finitely-generated R-module. □

Using Proposition 6.25, we can give a necessary and sufficient condition for a finitely-generated ideal of $A[E, F]$ to be finitely-presented.

Theorem 6.26:

Let $R = A[E, F]$. Let $I \leq R^m$ be a finitely-generated left R-submodule. Let $J = RE + RF$. Then, I is a finitely-presented R-module if and only if $\mathrm{Tor}_1^R(R/J, I)$ is finitely-generated as an A-module.

Proof:

Let I have presentation

$$0 \to K \to R^n \to I \to 0.$$

Then I is a finitely-presented R-module precisely when K is finitely-generated as an R-module, that is, when K/REK is finitely-generated as an $A[F]$-module, by Proposition 6.25. But by Lemma 6.24, $M = K/REK$ is naturally an $A[F]$-submodule of $A[F]^n$, and

$$M \diagup AFM = K \diagup AEK + AFK = K \diagup JK.$$

So by Lemma 6.18, I is finitely-presented if and only if K/JK is a finitely-generated A-module. But tensoring the presentation of I with $R/J \otimes_R$, we have the exact sequence

$$0 \to \mathrm{Tor}_1^R(R/J, I) \to K/JK \to A^n \to I/JI \to 0.$$

Since A is Noetherian, K/JK is a finitely-generated A-module precisely when $\mathrm{Tor}_1^R(R/J, I)$ is, giving the result. □

We can now generalise Theorem 6.16.

Proof of Theorem 6.22:

It is enough to show that any finitely-presented submodule of R^m is FP_2.

Let $I \leq R^m$ be a finitely-presented left R-submodule. Let

$$0 \to K \to R^n \to I \to 0.$$

Then K is finitely-generated, and it is enough to show that K is also finitely-presented. By Theorem 6.26, this is equivalent to showing that $\mathrm{Tor}_1^R(R/J, K)$ is a finitely-generated A-module. By dimension shifting,

$$\mathrm{Tor}_1^R(R/J, K) \cong \mathrm{Tor}_2^R(R/J, I) \cong \mathrm{Tor}_3^R(R/J, R^m/I) \cong \mathrm{Tor}_2^R(J, R^m/I).$$

But by Lemma 6.23, J has a flat resolution with only two non-zero terms, hence $\mathrm{Tor}_2^R(J, M) = 0$ for any R-module M. Thus $\mathrm{Tor}_1^R(R/J, K) = 0$, and the result follows. □

We will see in subsection 7.2 that Theorem 6.16, along with Corollary 6.11, allows us to prove that certain augmented Iwasawa algebras are coherent. Unfortunately we do not know if Corollary 6.11 has a generalisation to 2-coherent rings. This prevents any immediate application of Theorem 6.22 to the study of augmented Iwasawa algebras.

7 Coherent augmented Iwasawa algebras

One of the main results of this thesis is that many augmented Iwasawa algebras are not coherent. However, in this section we build on the results presented in earlier sections to give examples of augmented Iwasawa algebras that are coherent.

We can combine Proposition 6.13 with Theorem 5.13 to deduce the following. Throughout this section, let k be either a field of characteristic p, or a complete discrete valuation ring with residue field of characteristic p.

Proposition 7.1:

Let G be a p-adic Lie group, H be a closed subgroup of G. If kG is a coherent ring, then kH is a coherent ring.

Notice that the contrapositive to this proposition says that kG cannot be coherent if even one of its closed subgroups has a non-coherent augmented Iwasawa algebra.

7.1 Unipotent groups

We can use the results above to show that unipotent p-adic Lie groups always have a coherent augmented Iwasawa algebra.

Corollary 7.2:

Let F be a finite field extension of $\mathbb{Q}_p$, $\mathbb{U}_n$ be the affine group scheme of upper unitriangular matrices in $\mathbb{GL}_n$. Let U be a closed subgroup of $\mathbb{U}_n(F)$ with the p-adic topology. Then the augmented Iwasawa algebra kU is a coherent ring.

Proof:

First we show that $\mathbb{U}_n(F)$ is a direct limit of compact subgroups.

Let $\pi \in F$ be a uniformiser, and v_F be the discrete valuation on F. For each $j \in \mathbb{Z}_{\geq 0}$, let

$$G_j = \begin{pmatrix} 1 & \pi^{-j}\mathcal{O}_F & \pi^{-2j}\mathcal{O}_F & \dots & \pi^{-(n-1)j}\mathcal{O}_F \\ & 1 & \pi^{-j}\mathcal{O}_F & \ddots & \vdots \\ & & \ddots & \ddots & \pi^{-2j}\mathcal{O}_F \\ & & & 1 & \pi^{-j}\mathcal{O}_F \\ & & & & 1 \end{pmatrix} \subseteq \mathbb{U}_n(F).$$

Then it is easy to check that G_j is a subgroup of $\mathbb{U}_n(F)$, and clearly it is compact and closed. We also have that $G_j \leq G_{j+1}$ for all j. If $A \in \mathbb{U}_n(F)$, let $N = -\min\{v_F(A_{ij}) \mid 1 \leq i \leq j \leq n\}$. Then $A \in G_N$. So, $\mathbb{U}_n(F)$ is the direct limit of the G_j.

Then $k\mathbb{U}_n(F) = \varinjlim_{j\geq 0} kG_j$ by Corollary 3.23. Because the G_j are compact, each kG_j is a left Noetherian (by Theorem 2.2 or Theorem 3.32), hence left coherent, subring of $k\mathbb{U}_n(F)$. Because the G_j are closed, $k\mathbb{U}_n(F)$ is a flat right kG_j-module for all $j \geq 0$, by Proposition 5.10. Thus $k\mathbb{U}_n(F)$ is a direct limit of left coherent subrings, satisfying the properties in Lemma 6.7, hence is left coherent.

Let $U \leq \mathbb{U}_n(F)$ be a closed subgroup. Then by Proposition 7.1, kU is also left coherent. □

7.2 A solvable example

Recall that in Corollary 7.2, we showed that all unipotent p-adic Lie groups have a coherent augmented Iwasawa algebra. In this subsection, we give an example of a solvable, non-unipotent p-adic Lie group with coherent augmented Iwasawa algebra. The reasoning here is a precursor to that for Theorem 9.8.

Let G be the subgroup of $GL_2(\mathbb{Q}_p)$,

$$G = \begin{pmatrix} p^{\mathbb{Z}} & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} p^m & x \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z}, x \in \mathbb{Q}_p \right\}.$$

In this subsection we show that the augmented Iwasawa algebra kG is a coherent ring.

Lemma 7.3:

The augmented Iwasawa algebra kG is a skew-Laurent polynomial ring

$$kG = B[F, F^{-1}; \sigma_F],$$

where B is the subring

$$B = kH = k\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix} = \varinjlim_{n\geq 0} k[[t^{\frac{1}{p^n}}]] = k[[t]]^{\frac{1}{p^\infty}},$$

and F corresponds to the group element $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in G$. Then σ_F is the k-algebra automorphism of B given by $\sigma_F(f(t^a)) = f(t^{pa})$.

This ring structure is determined by first noting $B = kH$ is naturally a subring of kG, by Proposition 3.20. Then, we note kG is generated as a k-algebra by kH and coset representatives for G/H.

Let A be the Iwasawa algebra of the compact subgroup $H_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$, so $A = k[[t]] \leq B$. Note that $FgF^{-1} = g^p$ for all $g \in H_0$ (this corresponds to multiplication by p on $\mathbb{Z}_p$). Thus σ_F restricts to a ring endomorphism σ'_F of A, given by $\sigma'_F(f(t)) = f(t^p)$. Thus the subring of kG generated by A and F is a skew polynomial ring $A[F] = A[F; \sigma'_F]$.

Proposition 7.4:

The skew polynomial ring $A[F]$ is left coherent.

Proof:

Note that $A = k[[t]]$ is left Noetherian. Because $\sigma'_F = \sigma_F|_A$ and σ_F is an automorphism, σ'_F is injective. Then, $\sigma'_F(A) = k[[t^p]]$, and so A is free of rank p as a $\sigma'_F(A)$-module, hence flat. Therefore $A[F]$ is left coherent, by Theorem 6.16. □

From this we can use facts about coherent rings to show that kG is coherent.

Corollary 7.5:

The augmented Iwasawa algebra kG is coherent.

Proof:

For each $n \geq 0$, let $A_n = kH_n$, where $H_n = \begin{pmatrix} 1 & p^{-n}\mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ is a compact subgroup. Then, $A_n = k[[t^{\frac{1}{p^n}}]]$, and by similar reasoning to that above, $A_n[F]$ is a skew polynomial ring. Then, $\sigma^n_F|_{A_n}$ is a ring isomorphism onto A, and commutes with the action of F, therefore $A_n[F] \cong A[F]$ is left coherent by Proposition 7.4.

Now, $H = \varinjlim_{n\geq 0} H_n$, therefore $B = kH = \varinjlim_{n\geq 0} A_n$. Now, each H_n is a closed subgroup of H, therefore B is a flat right A_n-module by Proposition 5.10. It follows that $B[F]$ is a flat right $A_n[F]$-module, since there is a natural isomorphism of left B-modules,

$$B[F] \otimes_{A_n[F]} M \cong B \otimes_{A_n} M, \quad \Big(\sum_{j=0}^{\infty} b_j F^j\Big) \otimes m \mapsto \sum_{j=0}^{\infty} b_j \otimes (F^j m), \quad b \otimes m \leftarrow b \otimes m,$$

for any left $A_n[F]$-module M. Because $B\otimes_{A_n}$ is an exact functor, so is $B[F]\otimes_{A_n[F]}$. Thus

$$B[F] = \varinjlim_{n\geq 0} A_n[F]$$

is left coherent by Lemma 6.7. Then, $kG = X^{-1}B[F]$, where $X = \{F^n \mid n \geq 0\}$ is a left denominator set, therefore kG is left coherent by Corollary 6.11. □

7.3 The groups $SL_2(F)$ and $GL_2(F)$

In this subsection we show that the augmented Iwasawa algebras $kSL_2(F)$, $kGL_2(F)$ are coherent when F is any finite extension of $\mathbb{Q}_p$. We reuse much of the notation from section 5 of [Sho20], where the first of these results was shown. In particular, let $G = GL_2(F)$ and $G' = SL_2(F)$ throughout the remainder of the subsection.

Definition 7.6:

Let

$$K_1 = GL_2(\mathcal{O}_F), \quad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \in G, \quad K_2 = \alpha K_1 \alpha^{-1},$$

and

$$K_1' = K_1 \cap G' = SL_2(\mathcal{O}_F), \quad K_2' = K_2 \cap G' = \alpha K_1' \alpha^{-1}.$$

Let

$$G^0 = \operatorname{Ker}\big(v_F \circ \det : GL_2(F) \to \mathbb{Z}\big),$$

where $\det : GL_2(F) \to F^\times$ is the matrix determinant. Let

$$G^+ = (v_F \circ \det)^{-1}(2\mathbb{Z}),$$

and

$$z = \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix} \in G.$$

Let

$$H_1 = \langle K_1, z \rangle, \quad H_2 = \langle K_2, z \rangle$$

be the (closed) subgroups generated by K_1, K_2 and z.

Theorem 7.7:

The p-adic Lie groups G' and G^0 are amalgamated products of subgroups,

$$G' \cong K_1' *_{K_1' \cap K_2'} K_2', \quad G^0 \cong K_1 *_{K_1 \cap K_2} K_2.$$

Proof:

This follows from Theorem II.3 of [Ser80], and the topologies are uniquely determined by Lemma 3.25. □

In fact the larger group G^+ is also an amalgamated product of open subgroups, but for this we need to prove supplementary statements.

Proposition 7.8:

Let $H = B *_A C$ be an amalgamated product of groups, D be a group. The amalgamated product of groups

$$(B \times D) *_{(A \times D)} (C \times D) \cong H \times D.$$

Proof:

We show that $H \times D$ satisfies the universal property of being a pushout.
Let H be the amalgamated product of B, C over A via the group homomorphisms $f : A \to B$, $g : A \to C$, giving the following commutative diagram.

$$\begin{tikzcd}
H & B \arrow[l, "j_B"'] \\
C \arrow[u, "j_C"'] & A \arrow[l, "g"] \arrow[u, "f"]
\end{tikzcd}$$

Let $F = f \times \mathrm{id}_D, G = g \times \mathrm{id}_D, J_B = j_B \times \mathrm{id}_D, J_C = j_C \times \mathrm{id}_D$, so we have

$$\begin{tikzcd}
H \times D & B \times D \arrow[l, "J_B"'] \\
C \times D \arrow[u, "J_C"'] & A \times D \arrow[l, "G"] \arrow[u, "F"]
\end{tikzcd}$$

and let $i_A : A \to A \times D$ be the natural inclusion, similarly defining i_B, i_C, i_H.
Now, let Q be a group, with group homomorphisms $\theta : B \times D \to Q$, $\phi : C \times D \to Q$ such that $\theta \circ F = \phi \circ G$, as shown in the following diagram.

$$\begin{tikzcd}
Q & & \\
& H \times D \arrow[ul, dashed, "u"'] & B \times D \arrow[l, "J_B"'] \arrow[ull, "\theta"'] \\
& C \times D \arrow[u, "J_C"'] \arrow[uul, "\phi"] & A \times D \arrow[l, "G"] \arrow[u, "F"]
\end{tikzcd}$$

We show that there exists a unique group homomorphism $u : H \times D \to Q$ such that the diagram commutes.

Notice that $F \circ i_A = i_B \circ f$, $J_B \circ i_B = i_D \circ j_B$, similarly for G, J_C, and that $\theta \circ i_B = \theta|_B$, $\phi \circ i_C = \phi|_C$. Thus we have a commutative diagram,

$$
\begin{tikzcd}
Q & & \\
& H \arrow[ul, dashed, "v"'] & B \arrow[l, "j_B"] \arrow[ull, "\theta|_B"'] \\
& C \arrow[u, "j_C"'] \arrow[uul, "\phi|_C"] & A \arrow[l, "g"'] \arrow[u, "f"]
\end{tikzcd}
$$

and there exists a unique $v : H \to Q$ such that $v \circ j_B = \theta|_B$, $v \circ j_C = \phi|_C$, by the universal property of pushout for H. Also, $\theta \circ F = \phi \circ G$, so $\theta|_D = \phi|_D$, and all elements of $\theta(D) = \phi(D)$ commute with all elements of $v \circ j_B(B), v \circ j_C(C)$. Since H is generated by $j_B(B), j_C(C)$, it follows that $\theta(D)$ commutes with $v(H)$. Then by the property of direct product, there is a unique group homomorphism $u : H \times D \to Q$ satisfying $u \circ i_H = v$ and $u|_D = \theta|_D$. Now,

$$u \circ J_B \circ i_B = u \circ i_H \circ j_B = v \circ j_B = \theta|_B, \quad u|_D = \theta|_D,$$

so $u \circ J_B = \theta$, and

$$u \circ J_C \circ i_C = u \circ i_H \circ j_C = v \circ j_C = \phi|_C, \quad u|_D = \phi|_D,$$

so $u \circ J_C = \phi$.

Let $u' : H \times D \to Q$ be any group homomorphism such that $u' \circ J_B = \theta$ and $u' \circ J_C = \phi$. Then $u'|_D = \theta|_D = \phi|_D = u|_D$. Also

$$(u' \circ i_H) \circ j_B = u' \circ (i_H \circ j_B) = u' \circ J_B \circ i_B = \theta|_B = v \circ j_B,$$

and similarly $(u' \circ i_H) \circ j_C = v \circ j_C$. Since H is generated by $j_B(B), j_C(C)$, it follows $u' \circ i_H = v$. Therefore $u' = u$.

Therefore, there is a unique group homomorphism $u : H \times D \to Q$ making the diagram commute. So $H \times D$ satisfies the universal property of pushout, as required. $\square$

When X is a topological group and D is a discrete group, we make $X \times D$ a topological group by requiring the natural group homomorphism $X \to X \times D$ to be continuous and an open map.

Corollary 7.9:

Let H be a topological group and $A, B, C \leq H$ be open subgroups such that $H = B *_A C$ as a group. Let D be a discrete group. There is an isomorphism of topological groups

$$H \times D \cong (B \times D) *_{(A \times D)} (C \times D).$$

Proof:

Since $A, B, C \leq H$ are open, the colimit topology on $H = B *_A C$ agrees with the original topology on H by Lemma 3.25. Also, $A \times D \leq H \times D$ is open, thus by Proposition 7.8 and another application of Lemma 3.25, we have isomorphisms of topological groups,

$$H \times D \cong (B *_A C) \times D \cong (B \times D) *_{(A \times D)} (C \times D),$$

where the last term has the colimit topology. □

We now show that G^+ is an amalgamated product.

Corollary 7.10:

The p-adic Lie group $G^+ \cong H_1 *_{H_1 \cap H_2} H_2$ as topological groups.

Proof:

Notice that $G^+ = \langle G^0, z \rangle \cong G^0 \times \langle z \rangle$, where $\langle z \rangle \cong \mathbb{Z}$ is discrete. Also $H_1 \cong K_1 \times \langle z \rangle$ and $H_2 \cong K_2 \times \langle z \rangle$. By Corollary 7.9 and Theorem 7.7, we have an isomorphism of topological groups,

$$G^+ \cong G^0 \times \mathbb{Z} \cong (K_1 *_{K_1 \cap K_2} K_2) \times \mathbb{Z} \cong H_1 *_{H_1 \cap H_2} H_2,$$

since K_1, K_2 are open in G_0. □

Corollary 7.11:

The augmented Iwasawa algebras kG', kG^0, kG^+ are coherent rings.

Proof:

Each of the groups G', G^0, G^+ are an amalgamated product of open subgroups. Moreover, the open subgroups have Noetherian augmented Iwasawa algebras.

Since K_1, K_2, $K_1 \cap K_2$, K_1', K_2', $K_1' \cap K_2'$ are compact p-adic Lie groups, their Iwasawa algebras are Noetherian (Theorem 2.2). Then,

$$H_1 \cong K_1 \times \mathbb{Z}, \quad H_2 \cong K_2 \times \mathbb{Z}, \quad H_1 \cap H_2 \cong (K_1 \cap K_2) \times \mathbb{Z},$$

are non-compact p-adic Lie groups, but $k(K \times \mathbb{Z}) \cong kK[X, X^{-1}]$ is a Noetherian ring whenever kK is Noetherian, by the usual Hilbert basis theorem (see also Theorem 6.15).

Let $H = G'$, G^0 or G^+, and let $H = B *_A C$ be the amalgamated product given in Theorem 7.7 or Theorem 7.10. By Corollary 3.23, we have that $kH = kB *_{kA} kC$, and by Proposition 5.10, the augmented Iwasawa algebra kH is flat over the Noetherian subalgebras kA, kB, kC. Therefore, Theorem 12 of [Åbe82] implies that kH is a coherent ring. □

We conclude that the degree two special linear and general linear groups have coherent augmented Iwasawa algebras.

Theorem 7.12:

The augmented Iwasawa algebras $kSL_2(F)$, $kGL_2(F)$ are coherent rings.

Proof:

The case of $G' = SL_2(F)$ is treated in Corollary 7.11. For $G = GL_2(F)$, notice that $G^+ \leq G$ is an open subgroup of index 2. Where $H \leq G^+$ is a compact open subgroup, H is also open in G, and we have

$$kG = \bigoplus_{g \in H \backslash G} kH \otimes g = \bigoplus_{\substack{g \in H \backslash G^+ \\ g' \in G^+ \backslash G}} kH \otimes gg' = \bigoplus_{g \in G^+ \backslash G} kG^+ g.$$

So, kG is a free kG^+-module of rank 2. Hence, kG is a finitely-presented kG^+-module, and kG^+ is coherent by Corollary 7.11. It follows from Lemma 6.5 that kG is coherent. □

We will see in Section 10 that Theorem 7.12 implies that the categories of finitely-presented smooth representations of $SL_2(F)$ or $GL_2(F)$ are abelian categories.

8 Non-coherence for two groups

In this section we show that the augmented Iwasawa algebras of two particular p-adic Lie groups are not coherent. These examples are crucial in proving our characterisation of when augmented Iwasawa algebras are coherent in Theorem C.
Throughout, let F be a finite extension of $\mathbb{Q}_p$ of degree n, with discrete valuation $v_F : F^\times \to \mathbb{Z}$, and ring of integers $\mathcal{O}_F$. Let k be a field of characteristic p.

8.1 Statements

Theorem 8.1:

Let $\mathbb{U}_3'$ be the affine group scheme of upper unitriangular matrices in GL_3 with $(1,2)$-entry zero, and

$$U_3' = \mathbb{U}_3'(F) = \begin{pmatrix} 1 & 0 & F \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $u, v \in F^\times$ be such that $v_F(u) > 0, v_F(v) < 0$, and

$$t' = \begin{pmatrix} u & & \\ & v & \\ & & 1 \end{pmatrix} \in GL_3(F).$$

Let $G_{3,t'} = \langle t', U_3' \rangle \leq GL_3(F)$. Then $kG_{3,t'}$ is not coherent.

Theorem 8.2:

Let $\mathbb{U}_3$ be the affine group scheme of upper unitriangular matrices in GL_3. Let

$$U_3 = \mathbb{U}_3(F) = \begin{pmatrix} 1 & F & F \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $u, v \in F^\times$ be such that $v_F(u) > 0, v_F(v) < 0$, and

$$t = \begin{pmatrix} u & & \\ & 1 & \\ & & v^{-1} \end{pmatrix} \in GL_3(F).$$

Let $H_{3,t} = \langle t, U_3 \rangle \leq GL_3(F)$. Then $kH_{3,t}$ is not coherent.

8.2 Overview of the proof

We provide a summary of the arguments used to prove Theorem 8.1 and 8.2.

We first show Theorem 8.1. We wish to show that the augmented Iwasawa algebra of the p-adic Lie group $G_{3,t'} = \langle t', U_3' \rangle$ is not (left) coherent.

- First, we choose a closed subgroup $G_{3,g} = \langle g, U_3' \rangle \leq G_{3,t'}$, where g is a power of t', the choice made to simplify later calculations.

By Proposition 7.1, it is enough to show that $kG_{3,g}$ is not left coherent.

- We choose a particular finitely-generated left ideal $I_g \subseteq kG_{3,g}$. We will show it is not finitely-presented.

- Next, we find the subgroup $G_{3,D,E} = \langle D, E, U_3' \rangle \leq GL_3(F)$, where $D, E \in GL_3(F)$ are particular diagonal elements such that $DE^{-1} = g$.

- Then $G_{3,g}$ is a closed subgroup of $G_{3,D,E}$, and $kG_{3,D,E}$ is a (faithfully) flat right $kG_{3,g}$-module, by Theorem 5.13.

By the following lemma, to show I_g is not finitely-presented, it will be enough to show that the corresponding left ideal of $kG_{3,D,E}$, which is $I_{D,E} = kG_{3,D,E} \otimes_{kG_{3,g}} I_g$, is not finitely-presented.

Lemma 8.3:

Let $R \leq S$ be rings, and I be a finitely-generated left ideal of R. If I is finitely-presented as an R-module, then $S \otimes_R I$ is finitely-presented as an S-module. Moreover, if S is a flat right R-module, then $S \otimes_R I$ is naturally identified with a left ideal of S.

Proof:

The first part follows because $S\otimes_R$ is a right exact functor. If S is a flat right R-module, then tensoring the natural injection of left R-modules $I \to R$ gives an injection $S \otimes_R I \to S$, of left S-modules. $\square$

- Consider the compact unipotent group $\mathbb{U}_3'(\mathcal{O}_F) \leq G_{3,D,E}$, and let $A = k\mathbb{U}_3'(\mathcal{O}_F)$ be its Iwasawa algebra. The subring of $kG_{3,D,E}$ generated by A and the elements D, E is a skew polynomial ring $A[D, E]$.

- Then $kG_{3,D,E}$ is a flat right $A[D,E]$-module, and $I_{D,E}$ can be generated by finitely many elements of $A[D,E]$.

This gives a finitely-generated left ideal $I_A \subseteq A[D,E]$ such that

$$I_{D,E} = kG_{3,D,E} \otimes_{A[D,E]} I_A.$$

- We then compute an infinite set of relations S for I_A, over the ring $A[D,E]$.

The set of relations S for I_A bijectively corresponds to a set of relations X for the ideal $I_{D,E}$.

- We show that no finite subset of X can be a set of relations for $I_{D,E}$. This proves that $I_{D,E}$ cannot be finitely-presented, by the following lemma.

Lemma 8.4:

Let R be a ring, M be a left R-module, X be a generating set of M. Suppose that M is finitely-generated. Then M is generated by a finite subset of X.

Proof:

Let M have finite generating set Y. Each $y \in Y$ can be written as a finite R-linear combination of elements in X, let

$$y = r_{1y}x_{1y} + r_{2y}x_{2y} + \cdots + r_{a_y y}x_{a_y y}, \qquad a_y \in \mathbb{N}, r_{iy} \in R, x_{iy} \in X.$$

Let $X' = \{x_{iy} \mid y \in Y, 1 \leq i \leq a_y\} \subseteq X$. Then, X' is a finite subset of X, and Y is contained in the module generated by X', so X' must generate M. $\square$

As $I_{D,E}$ is not finitely-presented, it then follows that I_g is not finitely-presented, hence $kG_{3,g}$ is not left coherent, so $kG_{3,t'}$ is not coherent, proving Theorem 8.1.

We then deduce Theorem 8.2 as follows.

We wish to show that $H_{3,t} = \langle t, U_3 \rangle$ is not left coherent.

- We note that $Z(U_3)$ is a normal subgroup of $H_{3,t}$, and that the quotient $H_{3,t}/Z(U_3) \cong G_{3,t'}$.

Then, the augmentation ideal $J = \epsilon(Z(U_3)) \subseteq kH_{3,t}$ is two-sided with quotient $kG_{3,t'}$.

- Because $I_g \subseteq kG_{3,g}$ is not finitely-presented, the corresponding left ideal

  $$I' = kG_{3,t'} \otimes_{kG_{3,g}} I_g \subseteq kG_{3,t'}$$

  is not finitely-presented, by faithful flatness.

- We find a finitely-generated left ideal $I \subseteq kH_{3,t}$ such that I contains J, we have an isomorphism $I/J \cong I'$, and $JI = J$.

From this, we can show that since I' is not finitely-presented, I is not finitely-presented. Therefore $kH_{3,t}$ is not left coherent, and Theorem 8.2 is proved.

8.3 Proof of Theorem 8.1: change of rings

As in the statement of Theorem 8.1, let $\mathbb{U}_3'$ be the affine group scheme of upper unitriangular matrices in GL_3 with $(1,2)$-entry zero, and

$$U_3' = \mathbb{U}_3'(F) = \begin{pmatrix} 1 & 0 & F \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix}.$$

Then let $u, v \in F^\times$ be such that $v_F(u) > 0, v_F(v) < 0$, and

$$t' = \begin{pmatrix} u & & \\ & v & \\ & & 1 \end{pmatrix} \in GL_3(F).$$

We define $G_{3,t'} = \langle t', U_3' \rangle \leq GL_3(F)$.

Define $n' = v_F(p)$. The field F is a finite extension of $\mathbb{Q}_p$, so $\mathbb{Z}_p$ is a subring of $\mathcal{O}_F$, therefore $v_F|_{\mathbb{Q}_p^\times} = n' v_{\mathbb{Q}_p}$. It follows that

$$u^{n'} = p^{n_u} u_0, \quad v^{n'} = (p^{n_v} v_0)^{-1},$$

for some positive integers n_u, n_v and elements $u_0, v_0 \in \mathcal{O}_F^\times$, and we define

$$g = (t')^{n'} = \begin{pmatrix} p^{n_u} u_0 & & \\ & (p^{n_v} v_0)^{-1} & \\ & & 1 \end{pmatrix}.$$

Definition 8.5:

$G_{3,g} = \langle g, U_3' \rangle \leq G_{3,t'}$.

We describe the corresponding augmented Iwasawa algebra.

Now, $\mathcal{O}_F$ is a free $\mathbb{Z}_p$-module of rank n, so fix a basis $\{x_0, \dots, x_{n-1}\}$ with $x_0 = 1$. Then the Iwasawa algebra of $\mathcal{O}_F$ is

$$k\mathcal{O}_F = k[[X_0, \dots, X_{n-1}]],$$

where $1 + X_i = x_i \in \mathcal{O}_F$. Then the augmented Iwasawa algebra of F is

$$kF = \varinjlim_{r \geq 0} k[[X_0^{\frac{1}{p^r}}, \dots, X_{n-1}^{\frac{1}{p^r}}]] = k[[X_0, \dots, X_{n-1}]]^{\frac{1}{p^\infty}},$$

where $1 + X_i^{\frac{1}{p^r}} = p^{-r} x_i \in p^{-r}\mathcal{O}_F \leq F$.

Now, $U_3' \cong F \oplus F$, and thus we write the augmented Iwasawa algebra as

$$kU_3' = k[[s_0, \dots, s_{n-1}, t_0, \dots, t_{n-1}]]^{\frac{1}{p^\infty}},$$

where

$$1 + s_i = \begin{pmatrix} 1 & 0 & x_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 1 + t_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x_i \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the augmented Iwasawa algebra of $G_{3,g}$ is a skew-Laurent ring $kU_3'[g, g^{-1}]$. We define a finitely-generated left ideal of this ring.

Definition 8.6:

I_g is the left ideal of $kG_{3,g}$ generated by $\{s_0, \dots, s_{n-1}, t_0, \dots, t_{n-1}, g - 1\}$.

We will show that I_g is not finitely-presented, by considering the ideal it generates in a larger ring. This larger ring is given by considering a group containing $G_{3,g}$, splitting the element g into its diagonal components.

Definition 8.7:

The group $G_{3,D,E} = \langle D, E, U_3' \rangle \leq GL_3(F)$, where

$$D = \begin{pmatrix} p^{n_u} u_0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & & \\ & p^{n_v} v_0 & \\ & & 1 \end{pmatrix} \in GL_3(F).$$

Notice that $g = DE^{-1}$, and so $G_{3,g}$ is a closed subgroup of $G_{3,D,E}$. We have that $kG_{3,D,E}$ is a skew-Laurent ring in two variables, that is,

$$kG_{3,D,E} = kU_3'[D, D^{-1}, E, E^{-1}] = kU_3'[D, D^{-1}, E, E^{-1}; \sigma_D, \sigma_E],$$

where σ_D, σ_E are the automorphisms of kU_3' given by $\sigma_D(x) = DxD^{-1}, \sigma_E(x) = ExE^{-1}$. Note that the group elements D, E commute, therefore σ_D, σ_E are commuting endomorphisms.

Definition 8.8:

The left ideal $I_{D,E} = kG_{3,D,E} \otimes_{kG_{3,g}} I_g \subseteq kG_{3,D,E}$.

By Lemma 8.3, if I_g is finitely-presented, then so is $I_{D,E}$. Since D, E are units of $kG_{3,D,E}$, we have that $I_{D,E}$ is generated by the set $\{s_0, \ldots, s_{n-1}, t_0, \ldots, t_{n-1}, D - E\}$. Let the natural presentation corresponding to these generators be

$$0 \to K_{D,E} \to (kG_{3,D,E})^{2n+1} \to I_{D,E} \to 0.$$

We will determine a set of generators for $K_{D,E}$, and then demonstrate that this set cannot be reduced to a finite one. To determine a set of generators for $K_{D,E}$, we will restrict the calculations to a skew polynomial subring of $kG_{3,D,E}$.

Definition 8.9:

The ring $A = k\mathbb{U}_3'(\mathcal{O}_F) = k\begin{pmatrix} 1 & 0 & \mathcal{O}_F \\ 0 & 1 & \mathcal{O}_F \\ 0 & 0 & 1 \end{pmatrix}$.

Then A is the Noetherian subring $k[[s_0, \ldots, s_{n-1}, t_0, \ldots, t_{n-1}]] \leq kG_{3,D,E}$. Now, it can be easily seen that $\sigma_D(A), \sigma_E(A) \subseteq A$, therefore $\sigma_D|_A, \sigma_E|_A$ are (injective) ring endomorphisms of A, and so the subring of $kG_{3,D,E}$ generated by D, E and A is a skew polynomial ring $A[D, E] = A[D, E; \sigma_D|_A, \sigma_E|_A]$. Also, $kG_{3,D,E}$ is flat over this subring. To prove this requires a minor result about localisations.

Lemma 8.10:

Let R be a ring, $X \subseteq R$ be a left denominator set. Let M be a (R, R)-bimodule such that M is a flat right R-module. Then $X^{-1}M$ is a flat right R-module.

Proof:

By Proposition 10.12 of [GW04], the functor $X^{-1}M\otimes_R$ is equal to the composition of functors $X^{-1}R \otimes_R M\otimes_R$. Since M is a flat right R-module, $M\otimes_R$ is an exact functor. By Corollary 10.13 of [GW04], $X^{-1}R$ is a flat right R-module, so $X^{-1}R\otimes_R$ is an exact functor. Thus $X^{-1}M\otimes_R$ is an exact functor, hence $X^{-1}M$ is a flat right R-module. $\square$

Proposition 8.11:

The augmented Iwasawa algebra $kG_{3,D,E}$ is a flat right $A[D, E]$-module.

Proof:

Because $\mathbb{U}_3'(\mathcal{O}_F) \leq U_3'$ is a closed subgroup, kU_3' is a flat A-module by Theorem 5.13. For any left $A[D,E]$-module M, there is a natural isomorphism of left kU_3'-modules,

$$kU_3'[D,E] \otimes_{A[D,E]} M \cong kU_3' \otimes_A M,$$
$$\Big(\sum_{i,j\geq 0} b_{ij} D^i E^j\Big) \otimes m \mapsto \sum_{i,j\geq 0} b_{ij} \otimes (D^i E^j m),$$
$$b \otimes m \hookleftarrow b \otimes m.$$

Since $kU_3' \otimes_A$ is exact, so is $kU_3'[D,E] \otimes_{A[D,E]}$, thus $kU_3'[D,E]$ is a flat right $A[D,E]$-module. By Lemma 8.10, the localisation $X^{-1}kU_3'[D,E]$ is also a flat right $A[D,E]$-module, where $X = \{D^a E^b \mid a, b \geq 0\}$ is a left denominator set. But $kG_{3,D,E} \cong X^{-1}kU_3'[D,E]$. $\square$

Now, notice that $I_{D,E}$ is generated by elements which all lie in $A[D,E]$. So we can consider the corresponding left ideal of $A[D,E]$.

Definition 8.12:

I_A is the left ideal of $A[D,E]$ generated by $\{s_0, \ldots, s_{n-1}, t_0, \ldots, t_{n-1}, D - E\}$.

Then, we have that $kG_{3,D,E} \otimes_{A[D,E]} I_A = I_{D,E}$, since $kG_{3,D,E}$ is a flat $A[D,E]$-module. Let the natural presentation of I_A be

$$0 \to K_A \to A[D,E]^{2n+1} \to I_A \to 0.$$

Again by flatness, it follows that $kG_{3,D,E} \otimes_{A[D,E]} K_A = K_{D,E}$. So any set of generators of K_A gives a corresponding set of generators of $K_{D,E}$. Let

$$\Pi_{D,E} : K_{D,E} \to (kG_{3,D,E})^{2n}, \quad \Pi_A : K_A \to A[D,E]^{2n}$$

be the maps giving projection onto the first $2n$ coordinates, and let

$$L_{D,E} = \mathrm{Im}\ \Pi_{D,E}, \quad L_A = \mathrm{Im}\ \Pi_A.$$

Then $\Pi_{D,E}, \Pi_A$ are injective, since $D - E$ is not a zero divisor in $kG_{3,D,E}$. So $K_{D,E} \cong L_{D,E}$ and $K_A \cong L_A$. It is clear that $kG_{3,D,E} \otimes_{A[D,E]} \Pi_A = \Pi_{D,E}$, and thus $kG_{3,D,E} \otimes_{A[D,E]} L_A = L_{D,E}$ by Proposition 8.11. So any set of generators of L_A is in bijection with a set of generators of $L_{D,E}$, which in turn corresponds to a set of generators for $K_{D,E}$. In the next subsection we calculate a set of generators for L_A, and use this information to show that in fact $K_{D,E}$ cannot be finitely-generated.

8.4 Proof of Theorem 8.1: calculation of generators

The following proposition gives a set of generators for L_A. This information allows us to prove Theorem 8.1, see subsection 8.5.

Proposition 8.13:

The left $A[D, E]$-module L_A has a generating set $S = S_1 \cup S_2 \cup S_3$, where S_1, S_2, S_3 are as follows.

There exist elements $b_{ji}, c_{ji} \in A$ for $i, j \in \{0, \dots, n-1\}$, such that

$$b_j = (-b_{j0}E, -b_{j1}E, \dots, D - b_{jj}E, \dots, -b_{j(n-1)}E, 0^n) \in L_A,$$

and

$$c_j = (0^n, -c_{j0}D, -c_{j1}D, \dots, E - c_{jj}D, \dots, -c_{j(n-1)}D) \in L_A,$$

for all $j \in \{0, \dots, n-1\}$. Then

$$S_1 = \{b_j \mid j \in \{0, \dots, n-1\}\},$$

and

$$S_2 = \{c_j \mid j \in \{0, \dots, n-1\}\}.$$

Then,

$$S_3 = \{(0 \dots, t_j E^m, \dots, 0, 0, \dots, -s_i D^m, \dots, 0) \mid m \in \mathbb{Z}_{\geq 0}, 0 \leq i, j \leq n-1\},$$

where $t_j E^m$ is in the $(i+1)$st place, and $-s_i D^m$ is in the $(n+j+1)$st place, of the $2n$-tuple.

The calculation of these generators depends on understanding the action of D, E on certain augmentation ideals.

Definition 8.14:

The affine group scheme $\mathbb{U}'_{1,3}$ is the group scheme of matrices in $\mathbb{U}'_3$ with $(2, 3)$-entry zero.

The affine group scheme $\mathbb{U}'_{2,3}$ is the group scheme of matrices in $\mathbb{U}'_3$ with $(1, 3)$-entry zero.

Lemma 8.15:

Let $m \in \mathbb{Z}$. The subring of $kU_3'(F)$ corresponding to the Iwasawa algebra of

$$\mathbb{U}_{1,3}'(p^m\mathcal{O}_F) = \begin{pmatrix} 1 & 0 & p^m\mathcal{O}_F \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the subring $k[[s_0^{p^m}, \ldots, s_{n-1}^{p^m}]]$. The corresponding augmentation ideal $\epsilon_{\mathbb{U}_{1,3}'(p^m\mathcal{O}_F)}(\mathbb{U}_{1,3}'(p^m\mathcal{O}_F))$ is generated by $\{s_0^{p^m}, \ldots, s_{n-1}^{p^m}\}$.

An identical result holds for $\mathbb{U}_{2,3}'(p^m\mathcal{O}_F)$.

This lemma is clear from the description of kF in subsection 8.3. It follows, by the description of augmentation ideals in Lemma 3.46, that if $\mathbb{U}_{1,3}'(p^m\mathcal{O}_F)$ is a subgroup of any p-adic Lie group G, then $\epsilon_G(\mathbb{U}_{1,3}'(p^m\mathcal{O}_F))$ is the left ideal generated by $\{s_0^{p^m}, \ldots, s_{n-1}^{p^m}\}$.

We can now prove Proposition 8.13, which we do in two parts.

Lemma 8.16:

There exist elements $b_{ji}, c_{ji} \in A$ for $i, j \in \{0, \ldots, n-1\}$, such that

$$b_j = (-b_{j0}E, -b_{j1}E, \ldots, D - b_{jj}E, \ldots, -b_{j(n-1)}E, 0^n) \in L_A,$$

and

$$c_j = (0^n, -c_{j0}D, -c_{j1}D, \ldots, E - c_{jj}D, \ldots, -c_{j(n-1)}D) \in L_A,$$

for all $j \in \{0, \ldots, n-1\}$.

Proof:

Note that L_A is the set of tuples $(\lambda_0, \ldots, \lambda_{n-1}, \mu_0, \ldots, \mu_{n-1}) \in A[D,E]^{2n}$ such that

$$\sum_{j\geq 0}^{n-1} \lambda_j s_j + \mu_j t_j \in A[D,E](D-E).$$

Let $j \in \{0, \ldots, n-1\}$. Then

$$Ds_j = \sigma_D(s_j)D = \sigma_D(s_j)E + \sigma_D(s_j)(D-E),$$

so

$$Ds_j - \sigma_D(s_j)E \in A[D,E](D-E).$$

Now, notice that $s_j \in \epsilon_{\mathbb{U}'_3(\mathcal{O}_F)}(\mathbb{U}'_{1,3}(\mathcal{O}_F))$ and that σ_D is a ring endomorphism of $A = k\mathbb{U}'_3(\mathcal{O}_F)$ corresponding to a group endomorphism of $\mathbb{U}'_3(\mathcal{O}_F)$. Moreover, the image of $\mathbb{U}'_{1,3}(\mathcal{O}_F)$ under this group homomorphism is $\mathbb{U}'_{1,3}(p^{n_u}\mathcal{O}_F)$. By Lemma 3.47, it follows that

$$\sigma_D(s_j) \in \epsilon_{\mathbb{U}'_3(\mathcal{O}_F)}(\sigma_D(\mathbb{U}'_{1,3}(\mathcal{O}_F))) = \epsilon_{\mathbb{U}'_3(\mathcal{O}_F)}(\mathbb{U}'_{1,3}(p^{n_u}\mathcal{O}_F)).$$

So by Lemma 8.15,

$$\sigma_D(s_j) \in As_0^{p^{nu}} + \cdots + As_{n-1}^{p^{nu}} \subseteq As_0 + \cdots + As_{n-1}.$$

So, there exist $b_{j0}, b_{j1}, \ldots, b_{j(n-1)} \in A$ such that $\sigma_D(s_j) = \sum_{0 \le i \le n-1} b_{ji}s_i$, thus

$$\begin{aligned} Ds_j - \sigma_D(s_j)E &= Ds_j - \sum_{0 \le i \le n-1} b_{ji}s_iE \\ &= Ds_j - \sum_{0 \le i \le n-1} b_{ji}Es_i \\ &= (D - b_{jj}E)s_j + \sum_{0 \le i \le n-1, i \ne j} -b_{ji}Es_i \in A[D,E](D-E). \end{aligned}$$

So $b_j = (-b_{j0}E, -b_{j1}E, \ldots, D - b_{jj}E, \ldots, -b_{j(n-1)}E, 0^n) \in L_A$, for all j.
Identical reasoning with the elements E, t_j similarly gives the tuples $c_j \in L_A$. $\square$

Proof of Proposition 8.13:

We can put a grading by total (D,E)-degree on the ring $A[D,E]$. Under this grading, each of the elements $s_0, \ldots, s_{n-1}, t_0, \ldots, t_{n-1}$ is homogeneous of degree zero, and $D - E$ is homogeneous of degree 1. Therefore every element of L_A will be a sum of elements $(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1})$ where each x_j, y_j is homogeneous of the same degree.
Let L'_A be the $A[D,E]$-submodule generated by $S_1 \cup S_2$, as given by Lemma 8.16. We have that for any j,

$$b_j = (0, \ldots, D, \ldots, 0, 0^n) - (b_{j0}E, b_{j1}E, \ldots, b_{j(n-1)}E, 0^n) \in L'_A,$$

so for any j,

$$(0, \ldots, D, \ldots, 0, 0^n) \in L'_A + (A[E]^n, 0^n).$$

Now, because $DA[E] = \sigma_D(A)[E]D \subseteq A[E]D$, left multiplying by D this equation gives that for any j,

$$(0, \ldots, D^2, \ldots, 0, 0^n) \in L'_A + (A[E]^nD, 0^n) = L'_A + (A[E]^n, 0^n),$$

since L'_A is a left $A[D,E]$-module. Continuing by induction we see that

$$(0,\dots,D^m,\dots,0,0^n) \in L'_A + (A[E]^n, 0^n)$$

for any $m \geq 0$. Identical reasoning applies to show that

$$(0^n, 0, \dots, E^m, \dots, 0,) \in L'_A + (0^n, A[D]^n).$$

It follows that if $x, y \in A[D,E]^n$ and $(x,y) \in L_A$, then $(x,y) + L'_A = (x',y') + L'_A$ for some $x' \in A[E]^n$ and $y' \in A[D]^n$. (Informally, the relations above give a way to replace any Ds appearing in the first n places with polynomials in E, and vice versa for the last n places.)
Then $(x',y') \in L_A$ and is the sum of its homogeneous parts, which also lie in L_A, as discussed above. Therefore, L_A/L'_A is generated by the image of homogeneous elements $(x'',y'') \in L_A$, with $x'' \in (AE^m)^n, y'' \in (AD^m)^n$ for some m.
Now, for $x'_j, y'_j \in A$,

$$(x'_0E^m, \dots, x'_{n-1}E^m, y'_0D^m, \dots, y'_{n-1}D^m) \in L_A$$

if and only if

$$\sum_{j=0}^{n-1} x'_jE^m s_j + y'_jD^m t_j = \sum_{j=0}^{n-1} x'_j s_j E^m + y'_j t_j D^m \in A[D,E](D-E)$$

if and only if

$$\sum_{j=0}^{n-1} x'_j s_j + y'_j t_j = 0,$$

since $D^m - E^m \in A[D,E](D-E)$. Then, a generating set for

$$\Big\{(x'_0, \dots, x'_{n-1}, y'_0, \dots, y'_{n-1}) \in A^{2n} \mid \sum_{j=0}^{n-1} x'_j s_j + y'_j t_j = 0\Big\}$$

is

$$S'_3 = \{(0\dots, t_j, \dots, 0, 0, \dots, -s_i, \dots, 0) \mid 0 \leq i,j \leq n-1\},$$

and thus a generating set for ${}^{L_A}\!/_{L'_A}$ is the image of

$$S_3 = \{(0\dots, t_jE^m, \dots, 0, 0, \dots, -s_iD^m, \dots, 0) \mid m \geq 0, 0 \leq i,j \leq n-1\}.$$

It follows that L_A is generated by $S = S_1 \cup S_2 \cup S_3$, as required. $\square$

8.5 Proof of Theorem 8.1: conclusion

We can now finish proving Theorem 8.1. We have a set S of generators for L_A, and hence for $L_{D,E} \cong K_{D,E}$. Using the set S we show that a certain quotient of $L_{D,E}$ is not finitely-generated, implying $K_{D,E}$ cannot be finitely-generated, and the theorem follows.

Consider the left module $V = {}^{kG_{3,D,E}}\!/\!_{kG_{3,D,E}(D-E)}$. Note that V is a cyclic $kG_{3,D,E}$-module, and is a free left kU_3'-module, with basis

$$\{z_m = E^m + kG_{3,D,E}(D-E) \mid m \in \mathbb{Z}\}.$$

Let $q : L_{D,E} \to V$ be the left $kG_{3,D,E}$-module homomorphism given by

$$q(\lambda_0, \ldots, \lambda_{n-1}, \mu_0, \ldots, \mu_{n-1}) = \sum_{j=0}^{n-1} \lambda_j s_j + kG_{3,D,E}(D-E).$$

Let $M_{D,E} = \text{Im } q$, so $M_{D,E}$ is a submodule of V. We will prove the following.

Proposition 8.17:
The left $kG_{3,D,E}$-module $M_{D,E}$ is not finitely-generated.

From this we deduce the theorem.

Proof of Theorem 8.1:
Suppose $kG_{3,t'}$ is left coherent. Then, since $G_{3,g} \leq G_{3,t'}$ is a closed subgroup, $kG_{3,g}$ is left coherent by Proposition 7.1. Now, I_g is a finitely-generated left ideal of $kG_{3,g}$, and so it is also finitely-presented. Then by Lemma 8.3, it follows that the left ideal $I_{D,E}$ of $kG_{3,D,E}$ is finitely-presented, and therefore its relation module $K_{D,E}$ is finitely-generated. Then, $K_{D,E} \cong L_{D,E}$, and $M_{D,E}$ is a quotient of $L_{D,E}$, hence is finitely-generated. But this is a contradiction, by Proposition 8.17. So $kG_{3,t'}$ is not left coherent, and so also not right coherent. □

Now we prove the proposition.

Lemma 8.18:
The set $T = \{s_i t_j z_m \mid i, j \in \{0, \ldots, n-1\}, m \geq 0\}$ generates $M_{D,E}$ as a left $kG_{3,D,E}$-module.

Proof:

We have a generating set $S = S_1 \cup S_2 \cup S_3$ for $L_{D,E}$, and a surjection $q : L_{D,E} \to M_{D,E}$, so $M_{D,E}$ is generated by $q(S)$. It is easy to compute that $q(S_1) = q(S_2) = \{0\}$ and $q(S_3) = T$. $\square$

Proof of Proposition 8.17:

Suppose, for a contradiction, that $M_{D,E}$ is finitely-generated. In Lemma 8.18, we have an infinite generating set T for $M_{D,E}$. By Lemma 8.4, a finite subset of T generates $M_{D,E}$, in particular there exists a non-negative integer N such that

$$T_N = \{s_i t_j z_m \mid i, j \in \{0, \dots, n-1\}, 0 \leq m \leq N\}$$

generates $M_{D,E}$.

Then $s_0 t_0 z_{N+1} \in M_{D,E} = \langle T_N \rangle$. Now, $kG_{3,D,E} = kU_3'[D, D^{-1}, E, E^{-1}]$, and so there exist $N' \in \mathbb{N}$, $c_{abijm} \in kU_3'$, such that

$$s_0 t_0 z_{N+1} = \sum_{m=0}^{N} \sum_{-N' \leq a,b \leq N'} \sum_{i,j \in \{0,\dots,n-1\}} c_{abijm} D^a E^b s_i t_j z_m.$$

Now, $D^a E^b z_m = z_{m+a+b}$ for any m, a, b, so we have that

$$\begin{aligned} s_0 t_0 z_{N+1} &= \sum_{m=0}^{N} \sum_{-N' \leq a,b \leq N'} \sum_{i,j \in \{0,\dots,n-1\}} c_{abijm} \sigma_D^a \sigma_E^b (s_i t_j) D^a E^b z_m \\ &= \sum_{m=0}^{N} \sum_{-N' \leq a,b \leq N'} \sum_{i,j \in \{0,\dots,n-1\}} c_{abijm} \sigma_D^a \sigma_E^b (s_i t_j) z_{m+a+b}. \end{aligned}$$

Because $\{z_m \mid m \in \mathbb{Z}\}$ is a kU_3'-basis of V, we can compare the coefficients of z_{N+1} to obtain

$$s_0 t_0 = \sum_{\substack{-N' \leq a,b \leq N' \\ 1 \leq a+b \leq N+1}} \sum_{i,j \in \{0,\dots,n-1\}} c_{abij(N+1-a-b)} \sigma_D^a \sigma_E^b (s_i t_j).$$

Therefore $s_0 t_0$ lies in the ideal of kU_3' generated by

$$\{\sigma_D^a \sigma_E^b (s_i t_j) \mid a + b \geq 1,\ 0 \leq i, j \leq n-1\}.$$

Then, $\sigma_D^a \sigma_E^b (s_i t_j) = \sigma_D^a (s_i) \sigma_E^b (t_j)$ since D, t_j and E, s_i commute. Now,

$$s_i \in \epsilon_{U_3'}(\mathbb{U}_{1,3}'(\mathcal{O}_F)), \quad t_j \in \epsilon_{U_3'}(\mathbb{U}_{2,3}'(\mathcal{O}_F)),$$

as seen in Lemma 8.15, and

$$\sigma_D(\mathbb{U}'_{1,3}(\mathcal{O}_F)) = \mathbb{U}'_{1,3}(p^{n_u}\mathcal{O}_F), \quad \sigma_E(\mathbb{U}'_{2,3}(\mathcal{O}_F)) = \mathbb{U}'_{2,3}(p^{n_v}\mathcal{O}_F).$$

So, again by Lemma 8.15, the element $\sigma_D^a(s_i)\sigma_E^b(t_j)$ lies in the ideal of kU'_3,

$$J_{ab} = (s_0^{p^{an_u}}, \ldots, s_{n-1}^{p^{an_u}})(t_0^{p^{bn_v}}, \ldots, t_{n-1}^{p^{bn_v}}).$$

It follows that s_0t_0 lies in the ideal

$$\sum_{\substack{a,b\in\mathbb{Z},\\ a+b\geq 1}} J_{ab} = \sum_{\substack{a,b\in\mathbb{Z},\\ a\geq 1, b\geq 1-a}} J_{ab} + \sum_{\substack{a,b\in\mathbb{Z},\\ b\geq 1, a\geq 1-b}} J_{ab}.$$

Since $J_{ab} \subseteq J_{a'b'} \Leftrightarrow a \geq a', b \geq b'$, it follows that s_0t_0 lies in a finite sum of ideals,

$$s_0t_0 \in \sum_{a\geq 1} J_{ab'} + \sum_{b\geq 1} J_{a'b} = J_{1b'} + J_{a'1}$$

for some integers $a', b' \in \mathbb{Z}$. So

$$s_0t_0 \in (s_0^{p^{n_u}}, \ldots, s_{n-1}^{p^{n_u}})(t_0^{p^{b'n_v}}, \ldots, t_{n-1}^{p^{b'n_v}}) + (s_0^{p^{a'n_u}}, \ldots, s_{n-1}^{p^{a'n_u}})(t_0^{p^{n_v}}, \ldots, t_{n-1}^{p^{n_v}}).$$

But, since $n_u, n_v > 0$, this cannot occur, thus we have a contradiction. So $M_{D,E}$ is not finitely-generated. $\square$

8.6 Proof of Theorem 8.2

We recall the objects in the statement of Theorem 8.2. Let $\mathbb{U}_3$ be the affine group scheme of upper unitriangular matrices in GL_3, and $U_3 = \mathbb{U}_3(F)$, so

$$U_3 = \begin{pmatrix} 1 & F & F \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix}.$$

As in the above subsections, let $u, v \in F^\times$ be such that $v_F(u) > 0, v_F(v) < 0$. Then let

$$t = \begin{pmatrix} u & & \\ & 1 & \\ & & v^{-1} \end{pmatrix} \in GL_3(F).$$

We define $H_{3,t} = \langle t, U_3 \rangle \leq GL_3(F)$.

Note that

$$Z(U_3) = \begin{pmatrix} 1 & 0 & F \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a normal subgroup of $H_{3,t}$. Moreover, the quotient group is

$$H_{3,t}/Z(U_3) \cong G_{3,t'},$$

as seen in Theorem 8.1, where $t' = \mathrm{diag}(u, v, 1)$. Then, by Proposition 3.44, we have an isomorphism of augmented Iwasawa algebras,

$$\phi : kH_{3,t}/J \xrightarrow{\cong} kG_{3,t'},$$

where $J = \epsilon_{H_{3,t}}(Z(U_3))$ is the augmentation ideal of $Z(U_3)$, which is two-sided.
Let us define notation for the elements of the augmented Iwasawa algebras. We have that

$$U_3 = \varinjlim_{r\geq 0} \begin{pmatrix} 1 & p^{-r}\mathcal{O}_F & p^{-2r}O_F \\ 0 & 1 & p^{-r}\mathcal{O}_F \\ 0 & 0 & 1 \end{pmatrix},$$

and so we write the augmented Iwasawa algebra as

$$\begin{aligned} kU_3 &= \varinjlim_{r\geq 0} k[[s_0^{\frac{1}{p^r}}, \dots, s_{n-1}^{\frac{1}{p^r}}, t_0^{\frac{1}{p^r}}, \dots, t_{n-1}^{\frac{1}{p^r}}, w_0^{\frac{1}{p^{2r}}}, \dots, w_{n-1}^{\frac{1}{p^{2r}}}]] \\ &= k[[s_0, \dots, s_{n-1}, t_0, \dots, t_{n-1}, w_0, \dots, w_{n-1}]]^{\frac{1}{p^\infty}}, \end{aligned}$$

where

$$1 + s_i = \begin{pmatrix} 1 & x_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 1 + t_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x_i \\ 0 & 0 & 1 \end{pmatrix}, \quad 1 + w_i = \begin{pmatrix} 1 & 0 & x_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that kU_3 is non-commutative – it is a direct limit of non-commutative power series rings – but the w_j commute with any s_i, t_i.
The augmentation ideal of $Z(U_3)$ is generated by $\{w_0^{\frac{1}{p^r}}, \dots, w_{n-1}^{\frac{1}{p^r}} \mid r \geq 0\}$.
Then $kH_{3,t} = kU_3[t, t^{-1}]$ is a skew-Laurent ring over kU_3, and $\phi(t) = t'$, $\phi(s_i) = s_i, \phi(t_j) = t_j$ with the labelling described above. Let $y = t^{n'}$, where n' is the positive integer defined at the beginning of subsection 8.3, so that $\phi(y) = g = (t')^{n'} \in G_{3,t'}$.

Consider the left ideal of $kH_{3,t}$,

$$I = kH_{3,t}s_0 + \cdots + kH_{3,t}s_{n-1} + kH_{3,t}t_0 + \cdots + kH_{3,t}t_{n-1} + kH_{3,t}(1 - y).$$

Let I' be the left ideal of $kG_{3,t'}$ given by

$$I' = kG_{3,t'} \otimes_{kG_{3,g}} I_g,$$

that is explicitly,

$$I' = kG_{3,t'}s_0 + \cdots + kG_{3,t'}s_{n-1} + kG_{3,t'}t_0 + \cdots + kG_{3,t'}t_{n-1} + kG_{3,t'}(1-g).$$

Lemma 8.19:

There is containment of ideals $J \subseteq I$, and I/J corresponds to the left ideal I' under the isomorphism $kH_{3,t}/J \cong kG_{3,t'}$.

Proof:

For all $x \in I$, $x - x(1-y) = xy \in I$, so $y^{-1}xy \in I$. Similarly, $y^{-1}(1-y) = y^{-1} - 1 \in I$, and so $y(x + x(y^{-1} - 1)) = yxy^{-1} \in I$. Write σ_y for the ring automorphism of kU_3 given by $\sigma_y(x) = yxy^{-1}$.

Then, by almost identical calculations as for D, E and $g = DE^{-1} \in kG_{3,t'}$, we can compute how σ_y acts on the elements, and ideals, of kU_3. We have that

$$\sigma_y(kU_3t_0 + \cdots + kU_3t_{n-1}) = kU_3t_0^{p^{-n_v}} + \cdots + kU_3t_{n-1}^{p^{-n_v}}.$$

Therefore $t_0^{p^{-r}} \in I$ for all $r \geq 0$. Then, a matrix calculation gives that

$$(1+s_j)(1+t_0^{p^{-r}})(1+s_j)^{-1}(1+t_0^{p^{-r}})^{-1} = 1 + w_j^{p^{-r}}.$$

(This uses that the element x_0 of the $\mathbb{Z}_p$-basis for $\mathcal{O}_F$ is chosen to be $x_0 = 1$.) We can rearrange to find that

$$s_jt_0^{p^{-r}} - t_0^{p^{-r}}s_j = (1+s_j)(1+t_0^{p^{-r}})w_j^{p^{-r}}.$$

Since $1+s_j, 1+t_0^{p^{-r}}$ are units in kU_3, it follows that $w_j^{p^{-r}} \in I$, for any j, r.

Thus, I contains J, which is generated by such elements.

It is clear that g corresponds to y and the s_j, t_j correspond under the isomorphism $kH_{3,t}/J \cong kG_{3,t'}$, therefore $I/J = I'$. $\square$

Lemma 8.20:

The product of ideals $JI = J$.

Proof:

Recall J is two-sided by Proposition 3.44 as $Z(U_3)$ is normal in $H_{3,t}$. It is clear that $J^p = J$, and hence $J = J^2$. Since $J \subseteq I$, we have $J = J^2 \subseteq JI \subseteq J \cap I = J$, hence $J = JI$. $\square$

We can then prove Theorem 8.2.

Proof of Theorem 8.2:

Consider the finitely-generated left ideal $I \subseteq kH_{3,t}$. We show that I is not finitely-presented.

For a contradiction, suppose that I is finitely-presented. Then we have a short exact sequence

$$0 \to K \to (kH_{3,t})^{2n+1} \to I \to 0$$

with K finitely-generated. We then apply the right exact functor $(kH_{3,t}/J)\otimes_{kH_{3,t}}$. Note that $(kH_{3,t}/J)\otimes_{kH_{3,t}} I = I/JI = I/J \cong I'$. Thus we have an exact sequence

$$K/JK \to (kG_{3,t'})^{2n+1} \to I' \to 0.$$

Now K is a finitely-generated $kH_{3,t}$-module, so K/JK is a finitely-generated $kG_{3,t'}$-module, and hence I' is a finitely-presented $kG_{3,t'}$-module.

But, $I' = kG_{3,t'} \otimes_{kG_{3,g}} I_g$, and I_g is not finitely-presented (this follows from the proof of Theorem 8.1). Since $kG_{3,t'}$ is faithfully flat as a right $kG_{3,g}$-module, I' cannot be finitely-presented, a contradiction.

So, the finitely-generated left ideal I is not finitely-presented, and hence $kH_{3,t}$ is not left coherent. $\square$

9 Characterisations of coherence

In this section, we define a class of solvable p-adic Lie groups G and give a characterisation of when kG is a coherent ring – Theorem C. By analysing solvable subgroups, we deduce a characterisation of when a split-semisimple group or a general linear group has a coherent augmented Iwasawa algebra – Theorem A and Corollary B.

9.1 The class of solvable groups

We define the class of solvable groups considered in Theorem C.

Let $n \in \mathbb{N}$. Let $\mathbb{U}_n$ be the affine group scheme of upper unitriangular matrices in GL_n over F. Let $\mathbb{U}$ be a closed affine group subscheme of $\mathbb{U}_n$, defined and split over F. Let $U = \mathbb{U}(F)$.
Let T be a closed subgroup – in the p-adic topology – of the diagonal elements $\mathbb{D}_n(F) \leq GL_n(F)$, such that T normalises U. Let $G = \langle T, U\rangle \cong T \ltimes U$, so G is a p-adic Lie group.
We define a set of roots of U with respect to T, similarly to section 8.17 of [Bor91].
Let $\mathfrak{u} = \mathrm{Lie}(U)$. Since T acts on U via conjugation, T acts on $\mathfrak{u}$. Let

$$X(T) = \{\beta : T \to F^{\times} \mid \beta \text{ a continuous group homomorphism}\}$$

be the character group of T. For each $\beta \in X(T)$, define the subspace, called a weight space,

$$\mathfrak{u}_\beta = \{v \in \mathfrak{u} \mid t \cdot v = \beta(t)v \ \forall t \in T\} \leq \mathfrak{u}.$$

The $\mathfrak{u}_\beta$ are finite-dimensional subspaces of $\mathfrak{u}$, and may have dimension larger than 1.
We define the set of weights

$$\Phi = \{\beta \in X(T) \mid \mathfrak{u}_\beta \neq 0\}.$$

Note that we allow the trivial character to be an element of Φ.
Let $\mathbb{T}_{\mathbb{U}}$ be the normaliser of $\mathbb{U}$ in $\mathbb{D}_n$, so T is a subgroup of $\mathbb{T}_{\mathbb{U}}(F)$. By Proposition 8.4 of [Bor91], $\mathbb{T}_{\mathbb{U}}(F)$ acts diagonalisably on $\mathfrak{u}$, and hence so does T.
Thus $\mathfrak{u}$ is the direct sum of its weight spaces:

$$\mathfrak{u} = \bigoplus_{\beta \in \Phi} \mathfrak{u}_\beta.$$

The $\mathfrak{u}_\beta$ correspond to root subgroups $U_\beta \leq U$.
For $\beta_1, \beta_2 \in X(T)$, $v_1 \in \mathfrak{u}_{\beta_1}, v_2 \in \mathfrak{u}_{\beta_2}$, we have that $t \cdot [v_1, v_2] = [t \cdot v_1, t \cdot v_2] =$

$\beta_1(t)\beta_2(t)[v_1, v_2]$ for all $t \in T$. We use additive notation $\beta_1+\beta_2$ to denote the character $t \mapsto \beta_1(t)\beta_2(t)$. Thus, we have that

$$[\mathfrak{u}_{\beta_1}, \mathfrak{u}_{\beta_2}] \leq \mathfrak{u}_{\beta_1+\beta_2}.$$

We define the group homomorphism f_G by

$$f_G : T \to \mathbb{Z}^{\Phi}, \quad f_G(t) = \big(v_F(\beta(t))\big)_{\beta\in\Phi},$$

where $v_F : F^{\times} \to \mathbb{Z}$ is the discrete valuation on F. For each β, we will write $n_\beta(t) = v_F(\beta(t))$, so that $f_G(t) = (n_\beta(t))_{\beta\in\Phi}$. Note that if $\beta_1, \beta_2 \in \Phi$ are such that $\beta_1 + \beta_2 \in \Phi$, then for all $t \in T$,

$$n_{\beta_1+\beta_2}(t) = v_F(\beta_1(t)\beta_2(t)) = v_F(\beta_1(t)) + v_F(\beta_2(t)) = n_{\beta_1}(t) + n_{\beta_2}(t).$$

In the following three subsections, we prove Theorem C.

Theorem C:

Let $G = T \ltimes U$ be an upper-triangular subgroup of $GL_n(F)$ as described above, with set of roots Φ and group homomorphism $f_G : T \to \mathbb{Z}^{\Phi}$. The augmented Iwasawa algebra kG is a coherent ring if and only if the image $f_G(T) \leq \mathbb{Z}^{\Phi}$ is a cyclic subgroup, generated by an element of $(\mathbb{Z}_{\geq 0})^{\Phi}$.

9.2 Non-coherence: The case of cyclic torus

In this subsection, assume that T is cyclic and choose a fixed generator $t_0 \in T$. For convenience, we will write $n_\beta = n_\beta(t_0)$ in this subsection.

The next two lemmas are basic Lie algebra facts we need for examining the structure of $\mathfrak{u} = \mathrm{Lie}(U)$.

Lemma 9.1:

Let $\mathfrak{g}$ be a Lie algebra over a field K, and

$$\mathfrak{g} = \mathfrak{g}^1 \geq \mathfrak{g}^2 \geq \mathfrak{g}^3 \geq \dots$$

be its lower central series. Let S be a generating set for $\mathfrak{g}$. Then, for any $m \in \mathbb{N}$, the quotient Lie algebra $\mathfrak{g}^m/\mathfrak{g}^{m+1}$ is generated by the image of the m-fold commutators

$$S^m = \{[s_1, [s_2, \dots, s_m]] \mid s_1, s_2, \dots, s_m \in S\}.$$

Proof:

Let $\mathfrak{h}^m$ be the Lie subalgebra of $\mathfrak{g}$ generated by S^m. We proceed by induction. The case $m = 1$ is clear. If $m \geq 1$, then

$$\mathfrak{g}^{m+1} = [\mathfrak{g}, \mathfrak{g}^m] = [\mathfrak{g}, \mathfrak{h}^m + \mathfrak{g}^{m+1}] = [\mathfrak{g}, \mathfrak{h}^m] + \mathfrak{g}^{m+2},$$

by induction. Then $[\mathfrak{g}, \mathfrak{h}^m]$ is the Lie subalgebra generated by $\cup_{r\geq m+1} S^k$. If $r > m+1$, then $S^r \subseteq \mathfrak{g}^{m+2}$, and so

$$[\mathfrak{g}, \mathfrak{h}^m] + \mathfrak{g}^{m+2} = \mathfrak{h}^{m+1} + \mathfrak{g}^{m+2},$$

therefore $\mathfrak{g}^{m+1}/\mathfrak{g}^{m+2}$ is generated by (the image of) $\mathfrak{h}^m$, hence by S^m, as required. $\square$

Lemma 9.2:

Let $\mathfrak{g}$ be a non-abelian nilpotent Lie algebra over a field K, generated by two distinct 1-dimensional subspaces $\mathfrak{h}_1, \mathfrak{h}_2$. Let $\mathfrak{z} \leq \mathfrak{g}^2$ be a 1-dimensional subspace. Let $\mathfrak{g}'$ be the subalgebra generated by $\mathfrak{h}_1, \mathfrak{z}$. Then $\mathfrak{g}'$ is a proper subalgebra of $\mathfrak{g}$.

Proof:

Let $x \in \mathfrak{h}_1, y \in \mathfrak{h}_2, z \in \mathfrak{z}$ be non-zero. Suppose that $\mathfrak{g} = \mathfrak{g}'$. Then $\mathfrak{g} = \mathfrak{g}'$ is generated by x, z, so $\mathfrak{g}^2/\mathfrak{g}^3$ is generated by $[x, z]$, by Lemma 9.1. But $z \in \mathfrak{g}^2$, so $[x, z] \in \mathfrak{g}^3$, hence $\mathfrak{g}^2 = \mathfrak{g}^3$. Because $\mathfrak{g}$ is nilpotent, $\mathfrak{g}^2 = 0$, hence $z = 0$, a contradiction. So $\mathfrak{g}' \neq \mathfrak{g}$. $\square$

Recall the p-adic Lie groups $G_{3,t'}, H_{3,t}$ from Theorems 8.1 and 8.2. The following proposition characterises when these groups appear as closed subgroups of G.

Proposition 9.3:

Suppose there exists $\alpha, \beta \in \Phi$ such that $n_\alpha > 0, n_\beta < 0$. Then G contains a closed subgroup of the form $G_{3,t'}$ or $H_{3,t}$.

Proof:

Let $\mathfrak{u}'_\alpha \leq \mathfrak{u}_\alpha$ and $\mathfrak{u}'_\beta \leq \mathfrak{u}_\beta$ be 1-dimensional subspaces. The subalgebra $\mathfrak{v}$ generated by these subspaces corresponds to a closed subgroup $U' \leq U$ that is normalised by $T = \langle t_0 \rangle$, since T fixes $\mathfrak{u}_\alpha$ and $\mathfrak{u}_\beta$. So, it is enough to prove that U' contains a closed subgroup of the form $G_{3,t'}$ or $H_{3,t}$. That is, we may without loss of generality assume that $\mathfrak{u}$ is generated by $\mathfrak{u}_\alpha, \mathfrak{u}_\beta$, and that these spaces are 1-dimensional.

We proceed by induction on $\dim U = \dim \mathfrak{u}$. Clearly $\dim U \geq 2$, and if $\dim U = 2$, then $\mathfrak{u}_\alpha, \mathfrak{u}_\beta$ must commute, implying G is of the form $G_{3,t'}$.

If $\dim U > 2$, notice that if we can find $\alpha_0, \beta_0 \in \Phi$ with $n_{\alpha_0} > 0, n_{\beta_0} < 0$, and subspaces $\mathfrak{u}'_{\alpha_0} \leq \mathfrak{u}_{\alpha_0}$, $\mathfrak{u}'_{\beta_0} \leq \mathfrak{u}_{\beta_0}$ such that $\mathfrak{u}'_{\alpha_0}, \mathfrak{u}'_{\beta_0}$ generate a proper subalgebra $\mathfrak{v} \leq \mathfrak{u}$, we are done by induction. This follows because $\mathfrak{v}$ corresponds to a proper closed subgroup $V \leq U$ with V normalised by T, and satisfying the hypothesis of the proposition.

Let $\dim U > 2$. Then $\mathfrak{u}_\alpha, \mathfrak{u}_\beta$ do not commute. Let $\delta = \alpha + \beta$, and

$$\mathfrak{u}'_\delta = [\mathfrak{u}_\alpha, \mathfrak{u}_\beta] \leq \mathfrak{u}_\delta.$$

If $\mathfrak{u}'_\delta$ commutes with both $\mathfrak{u}_\alpha$ and $\mathfrak{u}_\beta$, then G must be of the form $H_{3,t}$.

If not, suppose $n_\alpha + n_\beta = n_\delta > 0$. Let $\mathfrak{v}$ be the Lie subalgebra of $\mathfrak{u}$ generated by $\mathfrak{u}'_\delta, \mathfrak{u}_\beta$. As $\mathfrak{u}'_\delta \leq \mathfrak{u}^2$, by Lemma 9.2, $\mathfrak{v}$ is a proper subalgebra and $n_\delta > 0, n_\beta < 0$, so we are done by induction. If $n_\alpha + n_\beta = n_\delta < 0$, the argument is very similar.

Suppose that $n_\alpha + n_\beta = n_\delta = 0$, and suppose that $[\mathfrak{u}_\alpha, \mathfrak{u}'_\delta] \neq 0$. Let $\gamma = \alpha + \delta$, and $[\mathfrak{u}_\alpha, \mathfrak{u}'_\delta] = \mathfrak{u}'_\gamma \leq \mathfrak{u}_\gamma$. Then $n_\gamma = n_\alpha + n_\delta = n_\alpha > 0$. Let $\mathfrak{v}$ be the subalgebra generated by $\mathfrak{u}_\beta, \mathfrak{u}'_\gamma$. Then $\mathfrak{u}'_\gamma \leq \mathfrak{u}^3 \leq \mathfrak{u}^2$, so by Lemma 9.2, the subalgebra $\mathfrak{v}$ is proper. Also $n_\gamma > 0, n_\beta < 0$, so we are done by induction. If $[\mathfrak{u}_\alpha, \mathfrak{u}'_\delta] = 0$, then $[\mathfrak{u}_\beta, \mathfrak{u}'_\delta] \neq 0$ and the argument is very similar. □

Since $kG_{3,t'}, kH_{3,t}$ are not coherent, by Theorems 8.1 and 8.2, if the hypothesis of the above proposition is satisfied, then kG is not coherent, by Proposition 7.1.

Corollary 9.4:

Suppose $f_G(t_0) \notin (\mathbb{Z}_{\geq 0})^\Phi \cup (\mathbb{Z}_{\leq 0})^\Phi$, and let $G = \langle T, U \rangle = \langle t_0, U \rangle$. Then the augmented Iwasawa algebra kG is not coherent.

9.3 Non-coherence for solvable groups

Let G be as in subsection 9.1. We show that the class of groups G for which kG is coherent is quite small.

Theorem 9.5:

The augmented Iwasawa algebra kG is coherent only if $f_G(T) \leq \mathbb{Z}^\Phi$ is cyclic and generated by an element in $(\mathbb{Z}_{\geq 0})^{\Phi}$.

The proof of this theorem is a deduction from Corollary 9.4, using some information about subgroups of $\mathbb{Z}^N$.

Lemma 9.6:

Let $N \in \mathbb{N}$. Let $x, y \in (\mathbb{Z}_{\geq 0})^N$. Suppose that the submodule

$$A = \mathbb{Z}x + \mathbb{Z}y \leq \mathbb{Z}^N$$

is contained in $(\mathbb{Z}_{\geq 0})^N \cup (\mathbb{Z}_{\leq 0})^N$. Then A is generated by one element.

Proof:

Note that if $x = 0$ or $y = 0$ we are done so assume $x, y \neq 0$. If there exist $i, j \in \{1, 2, \ldots, N\}$ such that $x_i > y_i$ but $x_j < y_j$, then $x - y \notin (\mathbb{Z}_{\geq 0})^N \cup (\mathbb{Z}_{\leq 0})^N$, a contradiction. So, without loss of generality, assume that $x_i \geq y_i$ for all $i \in \{1, 2, \ldots, N\}$.
If there exists $i \in \{1, 2, \ldots, N\}$ such that $x_i = y_i = 0$, then we may consider A as a submodule of $\mathbb{Z}^{N-1}$ by removing the ith component. Moreover, if there exists $i \in \{1, 2, \ldots, N\}$ such that $y_i = 0$ and $x_i > 0$, let $j \in \{1, 2, \ldots, N\}$ be such that $y_j > 0$. Then, there exists $m \in \mathbb{N}$ large enough such that $x_j - my_j < 0$. But $x_i - my_i = x_i > 0$, thus $x - my \notin (\mathbb{Z}_{\geq 0})^N \cup (\mathbb{Z}_{\leq 0})^N$, a contradiction. Therefore, we may assume that $x_i \geq y_i > 0$ for all $i \in \{1, 2, \ldots N\}$.
For each i, let $q_i = x_i y_i^{-1} \in \mathbb{Q}$. Let $q = \min\{q_i \mid i \in \{1, 2, \ldots N\}\}$, and by reordering, assume that $q = q_1$ without loss of generality. Let $q = \dfrac{a}{b}$ for coprime $a, b \in \mathbb{N}$, and $z = bx - ay \in A$. Suppose $z \neq 0$. Then $z_1 = 0$ – but $y_1 > 0$, so we have a contradiction as seen above. Thus, $z = 0$.
So, $bx = ay$. By Euclid's algorithm, $\exists r, s \in \mathbb{Z}$ such that $ra - sb = 1$, since a, b are coprime. Then

$$\begin{aligned} x &= (ra - sb)x = rax - sbx = rax - say = a(rx - sy), \\ y &= (ra - sb)y = ray - sby = rbx - sby = b(rx - sy), \end{aligned}$$

and so A is generated by $rx - sy$. $\square$

Proof of Theorem 9.5:

Suppose kG is coherent. Note that for any $t \in T$, $\langle t, U \rangle$ is a closed subgroup of

G, so by Proposition 7.1, $k\langle t, U\rangle$ is coherent. Thus, by Corollary 9.4, $f_G(t) \in (\mathbb{Z}_{\geq 0})^\Phi \cup (\mathbb{Z}_{\leq 0})^\Phi$ for all $t \in T$. That is,

$$f_G(T) \subseteq (\mathbb{Z}_{\geq 0})^\Phi \cup (\mathbb{Z}_{\leq 0})^\Phi.$$

Now, $f_G(T)$ is finitely-generated, thus by Lemma 9.6 and induction, $f_G(T)$ is a cyclic subgroup of $\mathbb{Z}^\Phi$, and must be generated by an element of $(\mathbb{Z}_{\geq 0})^\Phi$. □

9.4 Coherence for solvable groups

In this subsection we show that the converse to Theorem 9.5 also holds.
First, we examine the structure of the subgroups of $(F^\times)^n$.

Lemma 9.7:

Let $T \leq (F^\times)^n$ be a closed subgroup. Then,

$$T \cong \mathbb{Z}^d \times (T \cap (\mathcal{O}_F^\times)^n)$$

as p-adic Lie groups, and $T \cap (\mathcal{O}_F^\times)^n$ is compact.

Proof:

Note that $(F^\times)^n = (\pi^\mathbb{Z} \times \mathcal{O}_F^\times)^n$, where $\pi \in F$ is a uniformiser. So let $q : T \to (\pi^\mathbb{Z})^n$ be the natural projection restricted to T. Then $\operatorname{Im} q$ is (isomorphic to) a subgroup of $\mathbb{Z}^n$, hence is isomorphic to $\mathbb{Z}^d$ for some $d \leq n$. Then, $\operatorname{Ker} q = T \cap (\mathcal{O}_F^\times)^n$. Thus we have a short exact sequence of abelian groups

$$0 \to T \cap (\mathcal{O}_F^\times)^n \to T \to \mathbb{Z}^d \to 0,$$

which must split because $\mathbb{Z}^d$ is a free $\mathbb{Z}$-module. So $T \cong \mathbb{Z}^d \times (T \cap (\mathcal{O}_F^\times)^n)$, and $T \cap (\mathcal{O}_F^\times)^n$ is a closed subgroup of the compact group $(\mathcal{O}_F^\times)^n$, hence $T \cap (\mathcal{O}_F^\times)^n$ is compact. □

Theorem 9.8:

If $f_G(T) \leq \mathbb{Z}^\Phi$ is cyclic and generated by an element of $(\mathbb{Z}_{\geq 0})^\Phi$, then the augmented Iwasawa algebra kG is coherent.

Proof:

By Lemma 9.7, $T = T_0 \times T_1$ where $T_0 \cong \mathbb{Z}^d$ and T_1 is a compact p-adic Lie group, with T_1 a subgroup of the group of diagonal matrices in $GL_n(\mathcal{O}_F)$. Because U is a subgroup of the upper unitriangular matrices in $GL_n(F)$, we can show easily that $T_1 \leq \operatorname{Ker} f_G$. Thus $f_G(T_0) = f_G(T)$ is cyclic. If $f_G(T_0)$ is non-trivial, let $\{t_1, \ldots, t_{d-1}\} \subseteq T_0$ generate $\operatorname{Ker}(f_G|_{T_0})$ and $t_0 \in T_0$ be such that $f_G(t_0)$ generates $f_G(T_0)$. If $f_G(T_0)$ is trivial, let $\{t_0, t_1, \ldots, t_{d-1}\}$ be any set of generators for T_0. Then $T_0 = \langle t_0 \rangle \times \langle t_1, \ldots, t_{d-1} \rangle \cong \mathbb{Z} \times \mathbb{Z}^{d-1}$, and $\langle t_1, \ldots, t_{d-1} \rangle \leq \operatorname{Ker} f_G$.

Next, choose $\mathcal{O}_F$-Lie subalgebras of $\mathfrak{u}$,

$$\mathfrak{v}_0 \leq \mathfrak{v}_1 \leq \mathfrak{v}_2 \leq \ldots$$

of the form

$$\mathfrak{v}_m = \bigoplus_{\beta \in \Phi} \mathfrak{v}_{\beta,m}$$

where $\mathfrak{v}_{\beta,m} \leq \mathfrak{u}_\beta$ is an $\mathcal{O}_F$-lattice, and such that $\mathfrak{u} = \varinjlim_{m \geq 0} \mathfrak{v}_m$. These correspond to compact open subgroups $V_0 \leq V_1 \leq V_2 \leq \ldots$ of U, and we have that V_{m-1} is an open (hence closed) subgroup of V_m for all $m \geq 1$. Then,

$$t_0 \cdot \mathfrak{v}_m = \bigoplus_{\beta \in \Phi} t_0 \cdot \mathfrak{v}_{\beta,m} = \bigoplus_{\beta \in \Phi} \pi^{n_\beta(t_0)} \mathfrak{v}_{\beta,m},$$

where $\pi \in F$ is the uniformiser of F. Thus $t_0 \cdot \mathfrak{v}_m$ has finite index in $\mathfrak{v}_m$, and so $t_0 V_m t_0^{-1}$ is a finite index subgroup of V_m. Thus $t_0 V_m t_0^{-1} \leq V_m$ is an open subgroup (since it is also closed). Moreover, $t_0 \cdot \mathfrak{v}_m$ is an $\mathcal{O}_F$-Lie ideal of $\mathfrak{v}_m$, thus $t_0 V_m t_0^{-1}$ is an open normal subgroup of V_m.

Let $V'_m = \langle T_1, V_m \rangle = T_1 \ltimes V_m$ for each $m \geq 0$. Let $\mathfrak{v}'_m$ be the corresponding Lie algebra. Because $t_0 \cdot \mathfrak{v}_m$ is a finite index Lie ideal of $\mathfrak{v}_m$, it follows that $t_0 \cdot \mathfrak{v}'_m$ is a finite index Lie ideal of $\mathfrak{v}'_m$, for all m. Thus $t_0 V'_m t_0^{-1}$ is a finite index, closed, normal subgroup of V'_m, and hence is also an open normal subgroup. By Corollary 19.4 of [Sch11], it follows that the Iwasawa algebra kV'_m is a free right $k(t_0 V'_m t_0^{-1})$-module.

Let $V''_m = \langle V'_m, t_1, \ldots, t_{d-1} \rangle$ for each $m \geq 0$. Now, each of the $t_1, \ldots t_{d-1}$ commute with T_1, and act, via conjugation, as an automorphism on V_m. Thus the $t_1, \ldots, t_{d-1}$ act as automorphisms of V'_m, and therefore as ring automorphisms of the Iwasawa algebra kV'_m. So, the augmented Iwasawa algebra of V''_m is a skew Laurent polynomial ring, $kV''_m = kV'_m[t_1, t_1^{-1}, \ldots, t_{d-1}, t_{d-1}^{-1}]$.

By the non-commutative Hilbert basis theorem, Theorem 6.15, kV''_m is a Noetherian ring for each m. Let

$$\sigma_{t_0} : kV''_m \to kV''_m, \quad \sigma_{t_0}(x) = t_0 x t_0^{-1}$$

be the ring endomorphism given by conjugation by t_0. Then σ_{t_0} is injective. Note σ_{t_0} fixes the $t_1, \dots, t_{d-1}$. The image of σ_{t_0} is

$$\sigma_{t_0}(kV''_m) = k(t_0 V''_m t_0^{-1}) = k(t_0 V'_m t_0^{-1})[t_1, t_1^{-1}, \dots, t_{d-1}, t_{d-1}^{-1}].$$

Thus kV''_m is a free, hence flat, right $\sigma_{t_0}(kV''_m)$-module.

By Theorem 6.16, it follows that the skew polynomial ring $kV''_m[t_0; \sigma_{t_0}]$ is a left coherent ring, for all $m \geq 0$. Now, since V_m is a closed subgroup of V_{m+1}, it follows that V''_m is a closed subgroup of V''_{m+1}, therefore kV''_{m+1} is a flat right kV''_m-module, by Proposition 5.10. Hence $kV''_{m+1}[t_0]$ is a flat right $kV''_m[t_0]$-module, for all $m \geq 0$. Thus, by Lemma 6.7, the direct limit

$$\varinjlim_{m\geq 0} kV''_m[t_0] = \left(\varinjlim_{m\geq 0} kV''_m \right) [t_0] = kG'[t_0]$$

is a left coherent ring, where $G' = \langle t_1, \dots t_{d-1}, T_1, U\rangle$. Then, the set $\{1, t_0, t_0^2, \dots\}$ is a left denominator set in $kG'[t_0]$, and

$$kG = kG'[t_0, t_0^{-1}] = \{1, t_0, t_0^2, \dots\}^{-1} kG'[t_0].$$

Therefore, by Corollary 6.11, kG is a left coherent ring. □

We can conclude from Theorem 9.5 and Theorem 9.8 a necessary and sufficient criterion for kG to be coherent.

Corollary 9.9 (Theorem C)**:**

The augmented Iwasawa algebra kG is coherent if and only if $f_G(T) \leq \mathbb{Z}^\Phi$ is a cyclic subgroup generated by an element of $(\mathbb{Z}_{\geq 0})^\Phi$.

9.5 Coherence for algebraic groups

We now prove Corollary D, deducing Theorem A and Corollary B.

Proposition 9.10:

Let $\mathbb{U}$ be an affine group subscheme of the upper unitriangular matrices $\mathbb{U}_n$, defined and split over F. Let $\mathbb{T}$ be a split affine group subscheme of the diagonal group $\mathbb{D}_n \leq GL_n$ such that $\mathbb{T}$ normalises $\mathbb{U}$, and let $\mathbb{G} = \mathbb{T} \ltimes \mathbb{U} \leq GL_n$. Let $G = \mathbb{G}(F) = T \ltimes U$, where $T = \mathbb{T}(F)$, $U = \mathbb{U}(F)$.
If the rank of the root system of U with respect to T is at least two, then kG is not a coherent ring.

Proof:

The character group of $\mathbb{T}$ is $X(\mathbb{T}) = \{$morphisms of group schemes $\mathbb{T} \to \mathbb{G}_m\}$. Let $\mathbb{T} \cong \mathbb{G}_m^d$. Then $X(\mathbb{T}) \cong \mathbb{Z}^d$, and $T \cong (F^\times)^d$, so

$$X(T) = \{\alpha_{(j_1,\dots,j_d)} \mid (j_1, \dots, j_d) \in \mathbb{Z}^d\} \cong \mathbb{Z}^d,$$

where

$$\alpha_{(j_1,\dots,j_d)} : T \to F^\times, \quad \alpha_{(j_1,\dots,j_d)}(t_1, \dots, t_d) = t_1^{j_1} \dots t_d^{j_d}.$$

We have that $\Phi = \{\beta \in X(T) \mid u_\beta \neq 0\}$. For $\beta \in \Phi$, let $(j_1^\beta, \dots, j_d^\beta) \in \mathbb{Z}^d$ such that $\beta = \alpha_{(j_1^\beta,\dots,j_d^\beta)}$.
Let $t = (\lambda_1, \dots, \lambda_d) \in T$, and let $n_i = v_F(\lambda_i)$ for each $i \in \{1, 2, \dots, d\}$. Then

$$v_F(\beta(t)) = v_F(\lambda_1^{j_1^\beta} \dots \lambda_d^{j_d^\beta}) = j_1^\beta n_1 + \dots + j_d^\beta n_d.$$

Let M be the $(|\Phi| \times d)$-matrix with coefficients $M_{\beta i} = j_i^\beta$. Recall from subsection 9.1 the group homomorphism $f_G : T \to \mathbb{Z}^\Phi$. Then, f_G is given by

$$f_G(t) = M \cdot \begin{pmatrix} n_1 \\ \dots \\ n_d \end{pmatrix},$$

where $n_i = v_F(\lambda_i)$. So, the image $f_G(T) \leq \mathbb{Z}^\Phi$ is given by the $\mathbb{Z}$-span of the columns of M.
If kG is coherent, then by Theorem C, $f_G(T)$ is cyclic, so $\mathbb{Q} \otimes_{\mathbb{Z}} f_G(T) \leq \mathbb{Q}^\Phi$ is either trivial or a 1-dimensional vector space. Hence the $\mathbb{Q}$-rank of M is at most 1, and so the vectors $\{(j_1^\beta, \dots, j_d^\beta) \mid \beta \in \Phi\}$ are contained in a 1-dimensional vector subspace of $\mathbb{Q}^d$. So, the rank of the root system of U with respect to T is at most 1.
Hence, if the rank of the root system of U with respect to T is 2 or greater, kG is not coherent. $\square$

We then deduce Corollary D.

Corollary D:

Let $\mathbb{G}$ be a finite-dimensional split-solvable affine group scheme defined over F, and $G = \mathbb{G}(F)$. If the root system of $\mathbb{G}$ has rank 2 or greater, then kG is not coherent.

This follows directly from Proposition 9.10 because the class of groups considered is the same. We can then show that nearly all split-semisimple groups over F have a non-coherent augmented Iwasawa algebra.

Corollary 9.11:

Let $\mathbb{G}$ be a split-semisimple affine group scheme defined over F, with root system of rank at least 2. Let $G = \mathbb{G}(F)$. Then the augmented Iwasawa algebra kG is not coherent.

Proof:

This follows by considering the Borel subgroup $\mathbb{B} \leq \mathbb{G}$, with $B = \mathbb{B}(F) \leq G$. We have that $\mathbb{B}$ is a split-solvable affine group scheme with a root system of rank at least 2, thus $\mathbb{B}$ is as in the statement of Corollary D, and hence kB is not coherent. But B is a closed subgroup of G, so by Proposition 7.1, kG is not coherent. $\square$

Theorem A:

Let $\mathbb{G}$ be a split-semisimple affine group scheme defined over F. Let $G = \mathbb{G}(F)$. Then the augmented Iwasawa algebra kG is coherent if and only if the rank of the root system of $\mathbb{G}$ is 1.

Proof:

The "only if" part of the statement follows from Corollary 9.11.

If the root system of $\mathbb{G}$ has rank 1, then $\mathbb{G} = SL_2$ or PGL_2, by the classification of split-semisimple linear algebraic groups, see for example Table 9.2 of [MT11].

If $\mathbb{G} = SL_2$, then Corollary 4.4 of [Sho20] shows $kG = kSL_2(F)$ is coherent.

If $\mathbb{G} = PGL_2$, then G is isomorphic to a quotient of $SL_2(F)$, namely $G \cong {SL_2(F)}/{H}$, where $H = \{I, -I\}$ is the centre of $SL_2(F)$. Then, by Proposition 3.44, the augmented Iwasawa algebra $kG \cong kSL_2(F)/\epsilon(H)$. The two-sided ideal $\epsilon(H)$ is certainly finitely-generated as a left ideal, because H is finite. Thus, kG is coherent by Lemma 6.6, since $kSL_2(F)$ is coherent. $\square$

9.6 Coherence for the general linear group

Since $kSL_n(F)$ is not coherent for $n \geq 3$ by Theorem A, and the special linear group is a closed subgroup of the general linear group, Proposition 7.1 implies the following.

Theorem 9.12:

If $n \geq 3$, then the augmented Iwasawa algebra $kGL_n(F)$ is not coherent.

Consequently we know precisely when the augmented Iwasawa algebra of each general linear group is coherent.

Corollary B:

The augmented Iwasawa algebra of $GL_n(F)$ is coherent if and only if $n \leq 2$.

10 The category of finitely-presented smooth representations

Let k be a field of characteristic p throughout this section.

10.1 Finitely-presented representations

Schraen, Hu, Vignéras and Wu have previously studied finitely-presented representations of p-adic Lie groups, particularly those of $GL_2(F)$. See [Sch15], [Hu12], [Vig11], and [Wu21] respectively. Recently, Shotton has proved a link between coherence and finitely-presented smooth representations – refer to Definition 1.1 and the proof of Theorem 4.5 in [Sho20].

Theorem 10.1:

Let G be a p-adic Lie group. If kG is a coherent ring, then the category of finitely-presented smooth representations of G over k is an abelian category.

In fact, it was recently conjectured in [EGH] that the latter statement holds whenever $G = GL_n(F)$. To make sense of this, it is of course necessary to explain what we mean by a finitely-presented smooth representation. We give this definition in the language of this thesis, but see also Definitions 2.1-2.3 of [Sho20]. Recall that any smooth representation of G is a kG-module, Corollary 3.18.

Definition 10.2:

Let V be a smooth representation of G over k. V is a finitely-generated smooth representation if there exists a compact open subgroup $H \leq G$ and a surjective kG-module homomorphism

$$kG \otimes_{kH} U \to V$$

for some finite-dimensional smooth representation U of H over k.

Definition 10.3:

Let V be a smooth representation of G over k. V is a finitely-presented smooth representation if there exists $H \leq G$ compact open and a surjective kG-module homomorphism

$$\phi : kG \otimes_{kH} U \to V$$

for some finite-dimensional smooth representation U of H over k, such that $\operatorname{Ker} \phi$ is a finitely-generated smooth representation.

Notice that $kG\otimes_{kH} = k[G]\otimes_{k[H]}$ is equal to the compact induction functor on smooth representations.

Proposition 10.4:

Let G be a p-adic Lie group, $H \leq G$ be a compact open subgroup, and U be a finite-dimensional smooth representation of H. Then $kG \otimes_{kH} U$ is a finitely-presented smooth representation and finitely-presented kG-module.

Proof:

Since U is a smooth representation of H, every vector $u \in U$ is fixed by some open subgroup $H_u \leq H$. Every element $v \in kG \otimes_{kH} U$ may be written in the form

$$v = \sum_{j=1}^{n} g_j \otimes u_j,$$

for $g_j \in G$, $u_j \in U$. Then v is fixed by the open subgroup of G,

$$\bigcap_{j=1}^{n} g_j H_{u_j} {g_j}^{-1},$$

so $kG \otimes_{kH} U$ is smooth, and trivially is a finitely-presented smooth representation. Because U is finite-dimensional, U is a finitely-generated kH-module. Since kH is Noetherian (Theorem 2.2), U is a finitely-presented kH-module. Then Proposition 5.10 implies $kG \otimes_{kH} U$ is a finitely-presented kG-module. $\square$

Taking $U = k$ to be the trivial representation in the above implies that

$$kG \otimes_{kH} k = {}^{kG}\!/_{\epsilon_G(H)} \cong k\left({}^{G}\!/_{H}\right)$$

is a finitely-presented smooth representation whenever $H \leq G$ is a compact open subgroup (see also Example 4.3).

Lemma 10.5:

Let V be a smooth representation of G over k. V is a finitely-presented smooth representation if and only if V is a finitely-presented kG-module.

Proof:

First note that if V is a finitely-generated smooth representation, then a consequence of Proposition 10.4 is that V is a finitely-generated kG-module. Conversely if V is a finitely-generated kG-module, let $v_1, \ldots, v_n$ generate V. Since

V is smooth there exists a compact open subgroup $H \leq G$ that fixes every v_j. Thus the kernel of the natural map $kG^n \to V$ contains the submodule $\epsilon_G(H)^n$. Hence there is a surjection

$$kG \otimes_{kH} U = (kG/\epsilon_G(H))^n \to V,$$

where $U = k^n$ is the trivial n-dimensional representation of H, so V is a finitely-generated smooth representation.

Let V be a finitely-generated smooth representation, and

$$\phi : kG \otimes_{kH} U \to V$$

be any kG-module surjection. Now, $kG \otimes_{kH} U$ is a finitely-presented kG-module by Proposition 10.4. By Theorem 2.1.2 of [Gla89], V is a finitely-presented kG-module if and only if $\operatorname{Ker} \phi$ is a finitely-generated kG-module, if and only if (by the above) $\operatorname{Ker} \phi$ is a finitely-generated representation. The result follows. □

Due to Lemma 10.5, we may say a smooth representation V is "finitely-presented" without any ambiguity.

Definition 10.6:

The category of finitely-presented smooth representations $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}}$ is the full subcategory of the category of smooth representations $\mathrm{Rep}_k^{\mathrm{sm}}(G)$ consisting of the finitely-presented objects.

We need the following technical fact.

Proposition 10.7:

Let G be a p-adic Lie group. The inclusion of $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}}$ into $\mathrm{Rep}_k^{\mathrm{sm}}(G)$ is exact.

Proof:

We will denote the inclusion functor $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}} \to \mathrm{Rep}_k^{\mathrm{sm}}(G)$ by $M \mapsto \underline{M}$.

Let $\phi : A \to B$ be a morphism in $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}}$, and suppose it has a kernel $j : K \to A$. We show that $\underline{j} : \underline{K} \to \underline{A}$ is the kernel of the morphism $\underline{\phi}$ in $\mathrm{Rep}_k^{\mathrm{sm}}(G)$.

For each $H \leq G$ a compact open subgroup, consider the functor

$$\mathrm{Hom}_{kG}(kG/\epsilon_G(H), \quad) : \mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}} \to k\text{-Mod}.$$

These functors are left exact because $kG/\epsilon_G(H) \in \mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}}$ by Proposition 10.4. Moreover, for any kG-module M, there is a natural isomorphism

$$\mathrm{Hom}_{kG}(kG/\epsilon_G(H), M) \cong M^H = \{x \in M \mid hx = x \text{ for all } h \in H\}.$$

Thus, the homomorphism $\underline{j}|_{\underline{K}^H} : \underline{K}^H \to \underline{A}^H$ is the kernel of $\underline{\phi}|_{\underline{A}^H} : \underline{A}^H \to \underline{B}^H$. We now show that $\underline{j}$ satisfies the universal property of the kernel of $\underline{\phi}$. Let $K' \in \mathrm{Rep}_k^{\mathrm{sm}}(G)$ be a smooth representation and $j' : K' \to \underline{A}$ be a kG-module homomorphism satisfying $\underline{\phi} \circ j' = 0$. Then for any compact open subgroup $H \leq G$, clearly $\underline{\phi}|_{\underline{A}^H} \circ j'|_{(K')^H} = 0$. So by the universal property for the kernel of $\underline{\phi}|_{\underline{A}^H}$, there is a unique morphism $u_H : (K')^H \to \underline{K}^H$ such that $\underline{j}|_{\underline{K}^H} \circ u_H = j'|_{(K')^H}$. If $H \leq H'$, then $u_H|_{(K')^{H'}} = u_{H'}$, by uniqueness. Moreover, K' is smooth, meaning that

$$K' = \bigcup_{\substack{\text{compact open} \\ H \leq G}} (K')^H.$$

Therefore we can define the morphism of smooth representations,

$$u : K' \to \underline{K}, \quad u(x) = u_H(x) \text{ if } x \in (K')^H,$$

and it follows that $\underline{j} \circ u = j'$ because this holds on each $(K')^H$ by definition of u_H. If v is another morphism with this property, then $v|_{(K')^H} = u_H$ by uniqueness of u_H, so $v = u$. Thus u is the unique such morphism. Therefore $j : \underline{K} \to \underline{A}$ satisfies the universal property of the kernel. Thus the inclusion functor $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}} \to \mathrm{Rep}_k^{\mathrm{sm}}(G)$ preserves kernels, so is left exact.
Furthermore, the cokernel of $\phi : A \to B$ in $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}}$ is $B/\phi(A)$ – notice B is finitely-presented and $\phi(A)$ is finitely-generated, so $B/\phi(A)$ is finitely-presented by Theorem 2.1.2 of [Gla89]. Thus the inclusion functor preserves cokernels, so is right exact. □

Applying Proposition 10.7 and Lemma 1.6.2 of [Wei94] to the inclusion of categories, and noticing that $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}}$ is always closed under cokernels taken in $\mathrm{Rep}_k^{\mathrm{sm}}(G)$ by Lemma 10.5 and Theorem 2.1.2 of [Gla89], we have the following.

Corollary 10.8:

The category $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}}$ of finitely-presented smooth representations is an abelian category if and only if it is closed under kernels taken in the category $\mathrm{Rep}_k^{\mathrm{sm}}(G)$ of smooth representations, that is, kG-module kernels.

We can now give a proof of Theorem 10.1.

Proof of Theorem 10.1:

Suppose kG is a coherent ring and let $\phi : V \to W$ be a morphism in $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}}$. By Lemma 10.5, V and W are finitely-presented kG-modules, so $\mathrm{Ker}\,\phi$ is finitely-presented by Proposition 6.3. By Corollary 10.8, $\mathrm{Rep}_k^{\mathrm{sm}}(G)^{\mathrm{fp}}$ is abelian. □

Combining Theorem 10.1 with Theorem 7.12 gives the following result.

Corollary 10.9:

Let $G = SL_2(F)$ or $GL_2(F)$. Then the category of finitely-presented smooth representations of G is abelian.

This is an improvement on Theorem 1.2 of [Sho20]. Moreover it answers the $GL_2(F)$ case of Conjecture 6.1.4 of [EGH] positively.

10.2 A module-theoretic characterisation

In Theorems 8.1 and 8.2, we have proved that not all p-adic Lie groups have a coherent augmented Iwasawa algebra. It is natural to ask whether all p-adic Lie groups must have the category of smooth mod p representations being abelian. We will show in this subsection that this is also not the case – again the counterexamples are the groups in Theorems 8.1 and 8.2. To do this, we improve the result of Theorem 10.1 to a necessary and sufficient module-theoretic condition.

Proposition 10.10:

Let G be a p-adic Lie group. The category of finitely-presented smooth representations of G is abelian if and only if $kG/\epsilon_G(H)$ is a coherent kG-module, for all compact open subgroups $H \leq G$.

Proof:

Suppose the category of finitely-presented smooth representations of G is abelian. Now, for any open compact subgroup H, the kG-module $kG/\epsilon_G(H)$ is a smooth finitely-presented representation of G, by Proposition 10.4. Because $kG/\epsilon_G(H)$ is smooth, any kG-submodule or quotient is also a smooth representation. If $N \leq kG/\epsilon_G(H)$ is a finitely-generated kG-submodule, we have a short exact sequence of smooth representations

$$0 \to N \to kG/\epsilon_G(H) \to (kG/\epsilon_G(H))/N \to 0,$$

where the third and fourth terms are finitely-presented, by Proposition 10.4 and Theorem 2.1.2 of [Gla89]. Hence, N must also be finitely-presented by Corollary 10.8. Thus $kG/\epsilon_G(H)$ is a coherent kG-module.

Conversely, suppose $kG/\epsilon_G(H)$ is a coherent kG-module for all open compact $H \leq G$. Let M be a finitely-presented smooth representation of G. Then, as in the proof of Lemma 10.5, there is a surjection

$$(kG/\epsilon_G(H))^n \to M,$$

for some compact open subgroup $H \leq G$. Then $(kG/\epsilon_G(H))^n$ is a coherent kG-module by our assumption. Because M is a finitely-presented quotient, it follows from Theorems 2.1.2 and 2.2.1 of [Gla89] that M is a coherent kG-module.

Therefore, the finitely-presented smooth representations of G are exactly the smooth representations which are coherent kG-modules. The kernel of any homomorphism between coherent kG-modules is itself a coherent kG-module by Theorem 2.2.1 of [Gla89]. Thus the kernel of any map between finitely-presented smooth representations must be a coherent, hence finitely-presented, representation. Therefore the category of finitely-presented smooth representations of G is an abelian category, by Corollary 10.8. □

Lemma 10.11:

Let G be a p-adic Lie group, $J \leq G$ be an open pro-p subgroup. If $kG/\epsilon_G(J)$ is a coherent kG-module, then $kG/\epsilon_G(H)$ is a coherent kG-module for any compact open subgroup $H \leq G$.

Proof:

Let $H_1 \leq H_2$ be open compact subgroups of G. Suppose that $kG/\epsilon_G(H_1)$ is a coherent kG-module. We have a short exact sequence

$$0 \to {}^{\epsilon_G(H_2)}\!\diagup_{\epsilon_G(H_1)} \to {}^{kG}\!\diagup_{\epsilon_G(H_1)} \to {}^{kG}\!\diagup_{\epsilon_G(H_2)} \to 0,$$

and $\epsilon_G(H_2)/\epsilon_G(H_1)$ is finitely-generated because H_2 is compact. Thus $\epsilon_G(H_2)/\epsilon_G(H_1)$ is coherent, hence so is $kG/\epsilon_G(H_2)$.

Now suppose $kG/\epsilon_G(H_2)$ is coherent, and that H_1 is normal and of p-power index in H_2. Denoting the trivial module for the group H_2/H_1 by k, we have that

$${}^{kG}\!\diagup_{\epsilon_G(H_2)} = kG \otimes_{kH_2} k,$$

and

$$kG/_{\epsilon_G(H_1)} = kG \otimes_{kH_2} k[H_2/H_1].$$

Now, because H_2/H_1 is a finite p-group and k is of characteristic p, the only irreducible $k[H_2/H_1]$-module is the trivial module k. Therefore the Jordan-Hölder series of $k[H_2/H_1]$ consists only of copies of k. That is, there exist $k[H_2/H_1]$-modules $M_1, \ldots, M_n$ and short exact sequences

$$\begin{aligned}
0 \to M_1 \to & k[H_2/H_1] \to k \to 0, \\
0 \to M_2 \to & M_1 \to k \to 0, \\
& \ldots \\
0 \to M_n \to & M_{n-1} \to k \to 0, \\
0 \to k \to & M_n \to k \to 0.
\end{aligned}$$

Now, $kG \otimes_{kH_2}$ is an exact functor by Proposition 5.10. Applying it to the final sequence, we obtain

$$0 \to kG \otimes_{kH_2} k \to kG \otimes_{kH_2} M_n \to kG \otimes_{kH_2} k \to 0.$$

But $kG \otimes_{kH_2} k$ is a coherent kG-module, and any extension of two coherent modules is coherent, by Theorem 2.2.1 of [Gla89]. Thus $kG \otimes_{kH_2} M_n$ is coherent. Repeating this argument, we find that $kG \otimes_{kH_2} M_j$ is coherent for all j, and so

$$kG \otimes_{kH_2} k[H_2/H_1] = kG/\epsilon_G(H_1)$$

is coherent.

We now apply the above reasoning, supposing that $kG/\epsilon_G(J)$ is a coherent kG-module. Now, $J \cap H$ is an open subgroup of J, and so there exists an open normal subgroup $N \trianglelefteq J$ contained in $J \cap H$. Because J is pro-p, N has p-power index in J. Thus, because $kG/\epsilon_G(J)$ is coherent, $kG/\epsilon_G(N)$ is coherent. Then, $N \leq H$ and $kG/\epsilon_G(N)$ is coherent, therefore $kG/\epsilon_G(H)$ is coherent. □

By combining Proposition 10.10 and Lemma 10.11, we can give a strengthening of Theorem 10.1.

Corollary 10.12:

Let G be a p-adic Lie group. The following are equivalent:

- The category of finitely-presented smooth representations of G is abelian.

- $kG/\epsilon_G(H)$ is a coherent kG-module, for all compact open subgroups $H \leq G$.
- $kG/\epsilon_G(J)$ is a coherent kG-module, for some open pro-p subgroup $J \leq G$.

Notice that if kG is a coherent ring, $kG/\epsilon_G(H)$ is a finitely-presented quotient by Proposition 10.4, hence a coherent kG-module by Theorems 2.1.2 and 2.2.1 of [Gla89]. Moreover, we can translate conditions on the modules $kG/\epsilon_G(H)$ into ideal-theoretic language.

Proposition 10.13:

Let G be a p-adic Lie group, $H \leq G$ be an open compact subgroup. The kG-module $kG/\epsilon_G(H)$ is coherent if and only if every finitely-generated ideal of kG containing $\epsilon_G(H)$ is finitely-presented.

Proof:

The finitely-generated submodules of $kG/\epsilon_G(H)$ are exactly $I/\epsilon_G(H)$ for $I \subseteq kG$ a finitely-generated left ideal containing $\epsilon_G(H)$. By considering the short exact sequence

$$0 \to \epsilon_G(H) \to I \to I/\epsilon_G(H) \to 0,$$

by Corollary 6.25 of [Rot09] and Theorem 2.1.2 of [Gla89], I is finitely-presented if and only if $I/\epsilon_G(H)$ is, since $\epsilon_G(H)$ is finitely-presented. The result follows. □

By Proposition 10.10, it follows that if kG has a finitely-generated ideal that contains the augmentation ideal of an open compact subgroup, but I is not finitely-presented, then the category of finitely-presented smooth representations of G is not abelian. This situation occurs with the two groups we have studied in Section Section 8.

Example 10.14

Recall the p-adic Lie group

$$G_{3,g} = \left\langle \begin{pmatrix} u^{n'} & & \\ & v^{n'} & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & F \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

and left ideal $I_g \subseteq kG_{3,g}$ from Definitions 8.5 and 8.6. By Lemma 8.15, I_g contains the augmentation ideal of the compact open subgroup $\mathbb{U}_3'(\mathcal{O}_F)$. The proof of Theorem 8.1 shows I_g is finitely-generated but not finitely-presented. Thus the category of finitely-presented smooth representations of $G_{3,g}$ is not an abelian category.

Example 10.15

Recall the p-adic Lie group

$$H_{3,t} = \left\langle \begin{pmatrix} u & & \\ & 1 & \\ & & v^{-1} \end{pmatrix}, \begin{pmatrix} 1 & F & F \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

and ideal $I \subseteq kH_{3,t}$, as defined in subsection 8.6. By Lemma 8.19, I contains the augmentation ideal of $\mathbb{U}_3(\mathcal{O}_F)$, and I is finitely-generated but not finitely-presented. So the category of finitely-presented smooth representations of $H_{3,t}$ is not abelian.

Note, if $G \leq G'$ is an open subgroup, then any compact open subgroup of G is compact open in G', and $kG' \otimes_{kG} (kG/\epsilon_G(H)) \cong kG'/\epsilon_{G'}(H)$. So by Corollary 10.12 and Theorem 5.13, if the category of finitely-presented smooth mod p representations of G is not abelian, neither is this category for G'. For example, we can apply this to $G = G_{3,g}$ and $G' = G_{3,t'}$ or $G_{3,D,E}$. However, if $G' \leq G$ is a closed subgroup, we cannot make such a deduction in general, since the functor $kG' \otimes_{kG}$ may not preserve smoothness.

10.3 Relation to the Borel subgroup

We have seen in subsection 7.3 that when $G = SL_2(F)$ or $GL_2(F)$, the augmented Iwasawa algebra kG is a coherent ring. It follows from Theorem 10.1 that the category of finitely presented smooth representations of G over k is abelian. This technique relies on the convenient fact, utilised by Shotton in [Sho20], that G is the amalgamated product of two open subgroups with Noetherian (augmented) Iwasawa algebra. However, this is a condition that appears rarely. In this subsection we give a more general method independent of Shotton's, which allows one to relate the category of finitely-presented smooth representations of a p-adic Lie group to that of a Borel subgroup, and thereby re-prove the results of subsection 7.3.

To motivate the next definition, notice that if $G = SL_2(F)$ or $GL_2(F)$ and B is the subgroup of upper triangular matrices, then we may identify the "flag variety" G/B with $\mathbb{P}^1(F) = \mathbb{P}^1(\mathcal{O}_F)$, which is compact.

Definition 10.16:

Let G be a p-adic Lie group, $B \leq G$ be a closed subgroup. We say B is a co-compact subgroup of G if the quotient topological space G/B is compact.

The *Iwasawa decomposition* shows that Borel subgroups of many interesting groups are cocompact.

Theorem 10.17 (Iwasawa decomposition)**:**

Let $\mathbb{G}$ be a finite-dimensional split-semisimple affine group scheme defined and split over F, or $\mathbb{G} = GL_n$. Let $G = \mathbb{G}(F)$. Let $\mathbb{B}$ be a Borel subgroup scheme of $\mathbb{G}$, and $B = \mathbb{B}(F) \leq G$. Let $K \leq G$ be a maximal compact subgroup of G. Then $G = KB = \{xb \mid x \in K, b \in B\}$. In particular B is a cocompact subgroup of G.

Proof:

Note that if $G = KB$, there is a surjective continuous map $K \to G/B$, hence G/B is compact. When $\mathbb{G}$ is a finite-dimensional split-semisimple affine group scheme, and $K \leq G$ is a maximal compact subgroup of G then K is also open, and the result follows from the theory of buildings described in [AB08], specifically Proposition 11.100. When $\mathbb{G} = GL_n$, the result is Proposition 4.5.2 of [Bum96]. □

More generally, one may replace B by any closed subgroup containing a Borel subgroup in the above theorem.

Lemma 10.18:

Let G be a p-adic Lie group and $B \leq G$ be a cocompact subgroup. For any compact open subgroup $H \leq G$, the restriction of the kG-module $k(G/H) = kG/\epsilon_G(H)$ is finitely-generated as a kB-module.

Proof:

Notice that since $H \leq G$ is open,

$$\{HgB/B \mid g \in H\backslash G/B\}$$

is a disjoint open cover of the compact topological space G/B. Hence, the double cosets $H\backslash G/B$ are a finite set. Thus, the double cosets $B\backslash G/H$ are also a finite set, from which it follows that the locally profinite B-space G/H has finitely many orbits. Hence there exists a surjective morphism of locally profinite B-spaces,

$$\phi : \coprod_{j=11}^{n} B \to {}^{G}\!/_{H},$$

for some $n \geq 1$. By Proposition 3.41,

$$\tilde{\phi} : kB^n \to k\left({}^{G}\!/_{H}\right)$$

is a surjective kB-module homomorphism, so $k(G/H)$ is a finitely-generated kB-module. $\square$

Lemma 10.19:

Let G be a p-adic Lie group and $B \leq G$ be a cocompact subgroup. A smooth representation V of G is finitely-generated as a kG-module if and only if it is finitely-generated as a kB-module.

Proof:

If V is finitely-generated as a kB-module, clearly it must be finitely-generated as a kG-module.

Conversely, suppose V is finitely-generated as a kG-module. Then there exists a surjection of kG-modules

$$q : kG^n \to V.$$

Let $e_1, \ldots, e_n$ be the standard generating elements of kG^n, and $v_j = q(e_j)$ for each j. Because V is smooth, there exist compact open subgroups $H_1, \ldots, H_n \leq G$ such that H_j acts trivially on v_j. Therefore q factors through a surjection of kG-modules

$$\bar{q} : \bigoplus_{j=1}^{n} k\left({}^{G}\!/_{H_j}\right) \to V.$$

By Lemma 10.18, the domain of $\bar{q}$ is finitely-generated as a kB-module, hence so is V. $\square$

The next two results replace the words "finitely-generated" in Lemma 10.19 with "finitely-presented".

Lemma 10.20:

Let $B \leq G$ be a cocompact subgroup. For any compact open subgroup $H \leq G$, the restriction of the kG-module $k(G/H) = kG/\epsilon_G(H)$ is finitely-presented as a kB-module.

Proof:

Because G/H has the discrete topology, and by the proof of Lemma 10.18, the locally profinite B-space G/H is isomorphic to the finite disjoint union

$$\coprod_{j=1}^{n} {}^{Bg_jH}\!\diagup_{H}$$

for some $g_1, \ldots, g_n \in G$. But, it is clear that we have an isomorphism of locally profinite B-spaces

$${}^{BgH}\!\diagup_{H} \cong {}^{B}\!\diagup_{B \cap gHg^{-1}}$$

for any $g \in G$. Since $gHg^{-1} \leq G$ is a compact open subgroup of G, it follows $B \cap gHg^{-1}$ is a compact open subgroup of B, therefore

$$k\left({}^{BgH}\!\diagup_{H}\right) \cong k\left({}^{B}\!\diagup_{B \cap gHg^{-1}}\right)$$

is a finitely-presented kB-module by Proposition 10.4. It follows that

$$k\left({}^{G}\!\diagup_{H}\right) \cong \bigoplus_{j=1}^{n} k\left({}^{Bg_jH}\!\diagup_{H}\right)$$

is a finitely-presented kB-module. $\square$

Proposition 10.21:

Let $B \leq G$ be a cocompact subgroup. A smooth representation V of G is finitely-presented as a kG-module if and only if it is finitely-presented as a kB-module.

Proof:

By Lemma 10.19, we may without loss of generality assume V is finitely-generated as a kG-module (equivalently, as a kB-module). By the proof of Lemma 10.19, there is a surjective kG-module homomorphism

$$\phi : W \to V$$

where W is a finite direct sum of modules of the form $k(G/H)$, H a compact open subgroup of G. By Proposition 10.4, W is a finitely-presented smooth representation of G, and by Lemma 10.20, is also a finitely-presented smooth representation of B. Let $U = \operatorname{Ker} \phi \leq W$, giving a short exact sequence

$$0 \to U \to W \to V \to 0.$$

By Theorem 2.1.2 of [Gla89], V is a finitely-presented kG-module if and only if U is a finitely-generated kG-module. But U is finitely-generated as a kG-module if and only if U is finitely-generated as a kB-module, by Lemma 10.19. This occurs precisely when V is a finitely-presented kB-module. □

We now prove the main result of this subsection.

Theorem 10.22:

Let G be a p-adic Lie group and $B \leq G$ be a cocompact subgroup. If the category of finitely-presented smooth representations of B over k is abelian, then the category of finitely-presented smooth representations of G over k is abelian.

Proof:

Suppose the category of finitely-presented smooth representations of B over k is abelian. Let V_1, V_2 be finitely-presented smooth representations of G and $\phi : V_1 \to V_2$ be a kG-module homomorphism. By Corollary 10.8 it is enough to show that the kernel of ϕ is a finitely-presented kG-module. Since V_1, V_2 are finitely-presented kG-modules, by Proposition 10.21, V_1, V_2 are finitely-presented kB-modules. Because the category of finitely-presented smooth representations of B over k is abelian, $\operatorname{Ker}\phi$ is a finitely-presented kB-module, hence, again by Proposition 10.21, is a finitely-presented kG-module, as required. □

As a consequence, we can give a proof that the categories of finitely-presented smooth representations of $SL_2(F)$, $GL_2(F)$ are abelian, independent of subsection 7.3.

Theorem 10.23:

Let F be a finite field extension of $\mathbb{Q}_p$ and $G = SL_2(F)$ or $G = GL_2(F)$. The category of finitely-presented smooth representations of G is abelian.

Proof:

Let $G = \mathbb{G}(F)$ where $\mathbb{G} = GL_2$ or SL_2, and $B = \mathbb{B}(F)$ be the Borel subgroup. Then $B \leq G$ is a cocompact subgroup by Theorem 10.17. Also, B is a solvable p-adic Lie group as in subsection 9.1, so we may apply Theorem C. The unipotent radical of B is 1-dimensional, thus the set of weights Φ contains a single element. Therefore, every subgroup of $\mathbb{Z}^\Phi \cong \mathbb{Z}$ is cyclic and must be generated by an element of $(\mathbb{Z}_{\geq 0})^\Phi$, so Corollary 9.9 implies kB must be a coherent ring.

By Theorem 10.1, the category of finitely-presented smooth representations of B over k is abelian, therefore Theorem 10.22 implies the category of finitely-presented smooth representations of G over k is abelian. $\square$

11 Canonical dimension of modules for Iwasawa algebras

We now return to the topic of modules over Iwasawa algebras of compact p-adic Lie groups, and their canonical dimensions, see subsection 2.3.

Throughout Sections 11 and 12, k denotes a field of characteristic p.

11.1 The group associated to a Lie algebra

In Definition 2.6, we have seen that to any compact p-adic Lie group, a finite-dimensional $\mathbb{Q}_p$-Lie algebra $\mathfrak{g}$ may be associated. Multiple compact p-adic Lie groups may share the same Lie algebra, thus in order to simplify later proofs, in this subsection we make a specific choice of group to represent each class.

Unless otherwise stated, isomorphisms of Lie algebras will be $\mathbb{Q}_p$-Lie algebra isomorphisms. Any F-Lie algebra is viewed as a $\mathbb{Q}_p$-Lie algebra via the usual restriction of scalars.

Let $\mathfrak{g}$ be a finite-dimensional F-Lie algebra which is simple and split over F.

We fix a uniform pro-p group $G_{\mathfrak{g}}$ with Lie algebra $\mathcal{L}(G_{\mathfrak{g}}) \cong \mathfrak{g}$. Corollary 2.7 will then show that proving Theorem E and Corollary F for $G_{\mathfrak{g}}$ implies these results for arbitrary G. We make the choice of $G_{\mathfrak{g}}$ in the following way – see also section 3.4 of [Ard04] for a similar construction.

Because $\mathfrak{g}$ is split simple over F, let $\mathfrak{g} = X_F$, where

$$X \in \{A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2\}$$

is an indecomposable root system. For any commutative ring R, there is a Chevalley basis for the Lie algebra X_R, and we can define the adjoint Chevalley group $X(R) \leq \mathrm{Aut}(X_R)$.

Consider the $\mathcal{O}_F$-Lie algebra $X_{\mathcal{O}_F}$, and notice that the subalgebra $pX_{\mathcal{O}_F}$ is a powerful $\mathcal{O}_F$-Lie algebra. Let Y be the uniform pro-p group constructed from $pX_{\mathcal{O}_F}$ using the Campbell-Hausdorff formula. Let $\mathrm{Ad} : Y \to GL(pX_{\mathcal{O}_F})$ be the group homomorphism given by $\mathrm{Ad}(g)(u) = gug^{-1}$.

Definition 11.1:

The compact p-adic Lie group $G_{\mathfrak{g}} = \mathrm{Ad}(Y)$ is the image of the group homomorphism Ad.

It is straightforward to show that $\operatorname{Ker}\mathrm{Ad} = Z(Y) = \{e\}$, so Ad is an injective homomorphism. It is also clear that $G = G_{\mathfrak{g}}$ has Lie algebra

$$\mathcal{L}(G) = \mathbb{Q}_p L_G \cong \mathbb{Q}_p L_Y = \mathbb{Q}_p p X_{\mathcal{O}_F} = X_F = \mathfrak{g}.$$

Lemma 11.2:

$G_{\mathfrak{g}}$ is a normal subgroup of the Chevalley group $X(\mathcal{O}_F)$.

Proof:

This is similar to the proof of Lemma 3.13 of [Ard04]. Let $\mathcal{B}$ be the Chevalley basis of $X_{\mathcal{O}_F}$. It is straightforward to show that $\mathrm{Ad}(pu\mathcal{O}_F) \subseteq X(\mathcal{O}_F)$ for all $u \in \mathcal{B}$. There exists a $\mathbb{Z}_p$-basis $\{pv_1, \ldots, pv_N\}$ of $pX_{\mathcal{O}_F}$ where each element is contained in some $pu\mathcal{O}_F$. Now, Y is equal to the product of the subgroups $pv_j\mathbb{Z}_p$ by Proposition 3.7 and Theorem 9.8 of [DDMS03], and $\mathrm{Ad}(pv_j\mathbb{Z}_p) \subseteq X(\mathcal{O}_F)$ for each j. So $G_{\mathfrak{g}} = \mathrm{Ad}(Y) \subseteq X(\mathcal{O}_F)$.

It is then straightforward to check that $G_{\mathfrak{g}}$ is normal in $X(\mathcal{O}_F)$, using the Steinberg relations. $\square$

From the above lemma it follows that the Chevalley group $X(\mathcal{O}_F)$ acts by automorphisms on $G_{\mathfrak{g}}$.

11.2 The microlocalised module

Recall the theory of algebraic microlocalisation from Chapter IV of [LVO96]. If $\bar{S} \subseteq \operatorname{gr} kG$ is a multiplicatively closed subset of homogeneous elements, we can extend kG to a microlocalised algebra $Q_{\bar{S}}(kG)$. Similarly, for any finitely-generated kG-module M, we can extend M to the microlocalisation $Q_{\bar{S}}(M)$.

We prove that a certain microlocalisation of M is isomorphic to a finite-dimensional vector space.

Proposition 11.3:

Let G be a uniform pro-p group, and M be a finitely-generated left kG-module. Let $g \in G$ be a non-trivial element, $H = \langle g \rangle \leq G$, and $\bar{S} = \{\sigma(g-1)^n \mid n \geq 0\}$.

Suppose that $\bar{S}^{-1}\left(\dfrac{\operatorname{gr} kG}{J(M)}\right)$ is finitely-generated over $\bar{S}^{-1}k[\sigma(g-1)]$.
Then the microlocalisation $Q_{\bar{S}}(M)$ is a finite-dimensional K-vector space, where K is the field $\operatorname{Frac}(kH) \cong \operatorname{Frac}(k[[g-1]]) = k((g-1))$.

Proof:

Because $J(M)$ is the radical of $\operatorname{Ann}(\operatorname{gr} M)$, and $\operatorname{gr} kG$ is a commutative Noetherian ring, there exists $r \in \mathbb{N}$ such that $J(M)^r$ annihilates $\operatorname{gr} M$. Because $\operatorname{gr} M$ is finitely-generated over $\operatorname{gr} kG$, it follows that there is a surjective homomorphism

$$\left(\frac{\operatorname{gr} kG}{J(M)^r}\right)^n \to \operatorname{gr} M,$$

hence we have a surjective homomorphism

$$\bar{S}^{-1}\left(\frac{\operatorname{gr} kG}{J(M)^r}\right)^n \to \bar{S}^{-1}\operatorname{gr} M,$$

for some $n \in \mathbb{N}$. Now, the module $\bar{S}^{-1}\left(\operatorname{gr} kG/J(M)^r\right)$ has a series of subquotients

$$\bar{S}^{-1}\left(\frac{\operatorname{gr} kG}{J(M)}\right), \bar{S}^{-1}\left(\frac{J(M)}{J(M)^2}\right), \ldots, \bar{S}^{-1}\left(\frac{J(M)^{r-1}}{J(M)^r}\right).$$

Each of these subquotients is finitely-generated over $\bar{S}^{-1}\operatorname{gr} kG$ and annihilated by $\bar{S}^{-1}J(M)$. Therefore each is a finitely-generated module for the ring $\bar{S}^{-1}\left(\operatorname{gr} kG/J(M)\right)$. Because $\bar{S}^{-1}\left(\operatorname{gr} kG/J(M)\right)$ is finitely generated as a $\bar{S}^{-1}k[\sigma(g-1)]$-module, it follows that each subquotient, and hence $\bar{S}^{-1}\left(\operatorname{gr} kG/J(M)^r\right)$ itself, is finitely-generated over $\bar{S}^{-1}k[\sigma(g-1)]$. Considering the surjective homomorphism above, it follows that $\bar{S}^{-1}\operatorname{gr} M$ is a finitely-generated $\bar{S}^{-1}k[\sigma(g-1)]$-module.
Now by Proposition 4.1.8 of [LVO96],

$$\bar{S}^{-1}k[\sigma(g-1)] = \bar{S}^{-1}\operatorname{gr} kH = \operatorname{gr} Q_{\bar{S}}(kH),$$

and we have shown $\bar{S}^{-1}\operatorname{gr} M \cong \operatorname{gr} Q_{\bar{S}}(M)$ is a finitely-generated $\operatorname{gr} Q_{\bar{S}}(kH)$-module. By Proposition 4.1.7 and Theorem 1.5.7 of [LVO96], the finite-generation can be lifted since $Q_{\bar{S}}(M)$ has a complete filtration, thus $Q_{\bar{S}}(M)$ is a finitely-generated $Q_{\bar{S}}(kH)$-module. But

$$Q_{\bar{S}}(kH) = \operatorname{Frac}(kH) = K$$

is a field, so the statement follows. □

We combine this with the following commutative algebra proposition, which is proved later.

Proposition 11.4:

Let G be a uniform pro-p group, M be a finitely-generated left kG-module of canonical dimension 1. Let $g \in G$ be such that $\sigma(g-1) \notin J(M)$. Then, $\bar{S}^{-1}\left(\dfrac{\operatorname{gr} kG}{J(M)}\right)$ is finitely-generated over the algebra $\bar{S}^{-1}k[\sigma(g-1)]$.

Recall the uniform pro-p group $G_{\mathfrak{g}}$ from Definition 11.1. We define some root subgroups of $G_{\mathfrak{g}}$.

Definition 11.5:

Let $\mathfrak{g} = X_F$ as in subsection 11.1, and $\alpha \in X$. The root subgroup corresponding to α is $A_\alpha = \operatorname{Ad}(p(X_{\mathcal{O}_F})_\alpha) \leq G_{\mathfrak{g}}$.

Corollary 11.6:

Suppose $|F : \mathbb{Q}_p| > 1$. Let M be a finitely-generated left $kG_{\mathfrak{g}}$-module of canonical dimension 1. Let $\alpha \in X$ be a root. Suppose there is a $g \in A_\alpha$ such that $\sigma(g-1) \notin J(M)$. Then the annihilator $\operatorname{Ann}_{kA_\alpha}(Q_{\bar{S}}(M))$ is non-zero.

Proof:

The root subgroup corresponding to α is an abelian uniform pro-p group,

$$A_\alpha \cong \mathcal{O}_F \cong \mathbb{Z}_p^{|F:\mathbb{Q}_p|}.$$

Since $|F : \mathbb{Q}_p| > 1$, let $h \in A_\alpha$ be non-trivial such that the closed subgroup of A_α generated by g, h is 2-dimensional. Because g, h commute, h acts as a K-linear map on $Q_{\bar{S}}(M)$. By Propositions 11.3 and 11.4, $Q_{\bar{S}}(M)$ is a finite-dimensional vector space over the field $K = k((g-1))$. The characteristic polynomial of h gives a non-zero element of $K[h]$ that annihilates $Q_{\bar{S}}(M)$. By clearing denominators, there exists a non-zero element of $k[[g-1]][h] \leq kA_\alpha$ annihilating $Q_{\bar{S}}(M)$. Thus the annihilator ideal is non-zero as claimed. $\square$

11.3 Γ-invariant ideals

In this subsection we recall and slightly extend a result of Ardakov, which in our situation classifies the prime ideals of the Iwasawa algebra kA_α of a root subgroup invariant under the action of a torus. Recall the following definition from subsection 8.1 of [Ard12].

Definition 11.7:

Let A be a free abelian pro-p group of finite rank, and Γ be a closed subgroup of the continuous automorphisms of A. Γ acts rationally irreducibly on A if and only if every non-trivial Γ-invariant closed subgroup of A is open.

Equivalently, Γ acts rationally irreducibly if and only if the $\mathbb{Q}_p$-Lie algebra $\mathcal{L}(A)$ is an irreducible module for the Lie algebra $\mathcal{L}(\Gamma)$. Ardakov has proved the following restriction on Γ-invariant prime ideals in this situation, see Corollary 8.1 of [Ard12].

Theorem 11.8:

Suppose Γ acts rationally irreducibly on A and

$$(h \cdot g)g^{-1} \in A^p = \{x^p \mid x \in A\}$$

for all $h \in \Gamma, g \in A$. The only non-zero Γ-invariant prime ideal of kA is the unique maximal ideal.

We extend this result slightly.

Proposition 11.9:

Suppose Γ acts rationally irreducibly on A and $(h \cdot g)g^{-1} \in A^p$ for all $h \in \Gamma, g \in A$. If I is a non-zero Γ-invariant ideal of kA, then $\sqrt{I}$ is maximal.

Proof:

Note that Γ acts continuously on kA by ring automorphisms. Let P be a minimal prime over I, and $\Gamma_P = \{h \in \Gamma \mid h \cdot P = P\}$. Since kA is Noetherian, let P have generators $x_1, \ldots, x_n \in kA$. Then

$$\Gamma_P = \bigcap_{i=1}^{n} \phi_i^{-1}(P)$$

where ϕ_i is the restriction of the action map to $\Gamma \cong \Gamma \times \{x_i\} \to kA$. Because P is closed in kA, each $\phi_i^{-1}(P)$ is closed. Hence Γ_P is a closed subgroup of Γ. Now, $h \cdot P$ is a minimal prime over $h \cdot I = I$ for any $h \in \Gamma$, and there are finitely many minimal primes over I, because kA is Noetherian. Therefore Γ_P must be a finite index closed subgroup of Γ, so is an open subgroup. Hence the inclusion map

$\Gamma_P \to \Gamma$ induces an isomorphism $\mathcal{L}(\Gamma_P) \cong \mathcal{L}(\Gamma)$, so Γ_P acts rationally irreducibly on A. Moreover, $(h \cdot g)g^{-1} \in A^p$ for all $h \in \Gamma_P$ because $\Gamma_P \leq \Gamma$.

By definition of Γ_P, P is a non-zero Γ_P-invariant prime ideal of kA. Thus by Theorem 11.8, P is the unique maximal ideal of kA. So, the only minimal prime over I is the unique maximal ideal, and hence the radical $\sqrt{I}$ is maximal. $\square$

We now apply this in the case that A and Γ are both subgroups of $G_{\mathfrak{g}}$. Recall $G_{\mathfrak{g}}$ is uniform pro-p and has $\mathbb{Z}_p$-Lie algebra $pX_{\mathcal{O}_F}$, which is a powerful $\mathcal{O}_F$-Lie algebra.

For the remainder of this section, fix Γ as follows.

Definition 11.10:

Let $p\mathfrak{t}' \leq pX_{\mathcal{O}_F}$ be the maximal torus of $pX_{\mathcal{O}_F}$ which corresponds to the root system X. The subgroup $\Gamma = \mathrm{Ad}(p\mathfrak{t}') \leq G_{\mathfrak{g}}$.

Then Γ is an abelian closed subgroup of $G_{\mathfrak{g}}$. Since $p\mathfrak{t}'$ normalises $p(X_{\mathcal{O}_F})_\alpha$ for any root $\alpha \in X$, it follows that Γ normalises any root subgroup A_α.

Lemma 11.11:

The group Γ acts rationally irreducibly, by conjugation, on A_α for any root $\alpha \in X$.

Proof:

We have that $\mathcal{L}(A_\alpha) = \mathfrak{g}_\alpha$ and $\mathcal{L}(\Gamma) = \mathfrak{t}$ is the maximal torus of $\mathfrak{g}$ corresponding to the root system X. Both can be considered as F-Lie algebras, with $\dim_F \mathfrak{g}_\alpha = 1$, and $\mathfrak{t}$ acts via scalars given by $\alpha : \mathfrak{t} \to F$. Therefore $\mathcal{L}(A_\alpha)$ is an irreducible $\mathcal{L}(\Gamma)$-module. So Γ acts rationally irreducibly. $\square$

Lemma 11.12:

For all $h \in \Gamma$, $g \in A_\alpha$, $hgh^{-1}g^{-1} \in {A_\alpha}^p$, for all roots $\alpha \in X$.

Proof:

This follows from the fact that $L_G = pX_{\mathcal{O}_F}$ is a powerful Lie algebra and that Γ normalises A_α. $\square$

Combining Lemmas 11.11 and 11.12 together with Proposition 11.9, we have the following.

Corollary 11.13:

Let $\alpha \in X$. If I is a non-zero Γ-invariant ideal of kA_α, then the radical of I is the augmentation ideal $\sqrt{I} = \epsilon_{A_\alpha}(A_\alpha)$.

11.4 The characteristic ideal

In this subsection we prove our first main result on canonical dimension, Theorem E. We do this by finding many elements which must lie in the characteristic ideal of a module of canonical dimension one.

Theorem 11.14:

Let M be a finitely-generated left $kG_{\mathfrak{g}}$-module of canonical dimension 1. For any root $\alpha \in X$ and any $g \in A_\alpha$, $\sigma(g-1) \in J(M)$.

Proof:

For a contradiction, suppose not. Then by Corollary 11.6, there is an $\alpha \in X$ and non-trivial $g \in A_\alpha$ such that the annihilator ideal

$$I = \mathrm{Ann}_{kA_\alpha}(Q_{\bar{S}}(M)) \trianglelefteq kA_\alpha$$

is non-zero, where $\bar{S} = \{\sigma(g-1)^n \mid n \geq 0\}$. Now, because the microlocalisation $Q_{\bar{S}}(M)$ has an action of kG, for any $x \in I$, $h \in \Gamma$, and $m \in Q_{\bar{S}}(M)$, we have

$$hxh^{-1} \cdot m = hx \cdot (h^{-1}m) = h \cdot 0 = 0,$$

therefore $hxh^{-1} \in I$. So, I is a Γ-invariant ideal of kA_α. By Corollary 11.13, $\sqrt{I} = \epsilon_{A_\alpha}(A_\alpha)$ contains $g-1$, so $(g-1)^n \in I$ for some $n \geq 1$. But, $Q_{\bar{S}}(M)$ is a vector space over $K = k((g-1))$ and is annihilated by a non-zero element of K, thus $Q_{\bar{S}}(M) = 0$. So $\bar{S}^{-1}\mathrm{gr}\, M = \mathrm{gr}\, Q_{\bar{S}}(M) = 0$, hence $\bar{S} \cap \mathrm{Ann}(\mathrm{gr}\, M) \neq \emptyset$. It follows that $\sigma(g-1) \in J(M)$, a contradiction. $\square$

Corollary 11.15:

Let M be a finitely-generated left $kG_{\mathfrak{g}}$-module of canonical dimension 1. For all continuous group automorphisms $\phi : G_{\mathfrak{g}} \to G_{\mathfrak{g}}$, all roots $\alpha \in X$ and all $g \in A_\alpha$, the element $\sigma(\phi(g)-1) \in J(M)$.

Proof:

For ease we write $G = G_{\mathfrak{g}}$. Any continuous group automorphism ϕ extends to a (filtered) ring automorphism $\widetilde{\phi} : kG \to kG$, which induces a (graded) ring automorphism $\bar{\phi} : \operatorname{gr} kG \to \operatorname{gr} kG$. We then define a kG-module M' on the set M via $x \cdot m = \widetilde{\phi}(x)m$. It is easy to see that $J(M') = \bar{\phi}^{-1}(J(M))$, and therefore M' is a module of canonical dimension 1. By Theorem 11.14, $\sigma(g-1) \in J(M')$ for any $\alpha \in X$, $g \in A_\alpha$. Hence $\bar{\phi}(\sigma(g-1)) = \sigma(\phi(g)-1) \in J(M)$. $\square$

We show that this forces the characteristic ideal $J(M)$ to be too large, by showing that $J(M)$ must contain the entirety of the first graded piece J/J^2 of $\operatorname{gr} kG_{\mathfrak{g}}$.

Lemma 11.16:

Let $G = G_{\mathfrak{g}}$. There is an isomorphism of $k[\mathrm{Aut}(G)]$-modules,

$$J/J^2 \cong k \otimes_{\mathbb{F}_p} \left(G/G^p\right) \cong k \otimes_{\mathbb{F}_p} \left(pX_{\mathcal{O}_F}/p^2X_{\mathcal{O}_F}\right),$$

where $\sigma(g-1) + J^2 \mapsto gG^p$ and $\mathrm{Ad}(pv)G^p \mapsto v + p^2X_{\mathcal{O}_F}$ for $g \in G$, $v \in X_{\mathcal{O}_F}$.

Proof:

It suffices to prove the statement when $k = \mathbb{F}_p$ (by then applying the appropriate tensor product).

In this case, the first isomorphism is given by Lemma 3.11 of [Ard04]. For the second isomorphism, Corollary 4.18 of [DDMS03] implies that the identity map $\phi : G \to L_G$ is an isomorphism of $\mathbb{Z}_p[\mathrm{Aut}(G)]$-modules. The image $\phi(G^p) = pL_G$, and therefore

$$\bar{\phi} : G/G^p \to L_G/pL_G$$

is an isomorphism of $\mathbb{F}_p[\mathrm{Aut}(G)]$-modules. Since $L_G = pX_{\mathcal{O}_F}$, the statement follows. $\square$

We can now prove Theorem E.

Theorem E:

Let $\mathfrak{g}$ be a finite-dimensional F-Lie algebra which is simple and split over F. Let G be a compact p-adic Lie group with associated Lie algebra $\mathcal{L}(G) \cong \mathfrak{g}$. If $|F : \mathbb{Q}_p| > 1$, then the Iwasawa algebra kG has no finitely-generated modules of canonical dimension one.

Proof:

By Corollary 2.7, we may assume that $G = G_{\mathfrak{g}}$ is the uniform pro-p group of Definition 11.1.

Suppose there exists a finitely-generated kG-module M with canonical dimension 1. We obtain a contradiction by showing that $J(M)$ contains J/J^2, as in this case $J(M)$ is the maximal graded ideal of $\operatorname{gr} kG$, which forces $\mathrm{Cdim}(M) = 0$.

By Lemma 11.16, let L be the image in $k \otimes_{\mathbb{F}_p} \left(pX_{\mathcal{O}_F}/p^2X_{\mathcal{O}_F}\right)$ of the degree 1 elements of $J(M)$. We adopt the notation of section 3.4 of [Ard04], so that $X_{\mathcal{O}_F}$ is spanned by $\{e_\alpha, h_\alpha \mid \alpha \in X\}$ and the Chevalley group $X(\mathcal{O}_F)$ is generated by elements $\{x_\alpha(t) \mid \alpha \in X, t \in \mathcal{O}_F\}$.

Since L is a k-vector space, it is enough to show that L contains the $\mathcal{O}_F/p\mathcal{O}_F$-Lie algebra $pX_{\mathcal{O}_F}/p^2X_{\mathcal{O}_F}$. By considering the group elements $\mathrm{Ad}(p\lambda e_\alpha) \in A_\alpha$, Lemma 11.16 and Theorem 11.14 imply that for any $\lambda \in \mathcal{O}_F$ and any $\alpha \in X$,

$$p\lambda e_\alpha + p^2X_{\mathcal{O}_F} \in L.$$

Moreover, by Lemma 11.2, the Chevalley group $X(\mathcal{O}_F)$ acts by automorphisms on G. Thus by Corollary 11.15,

$$x \cdot (p\lambda e_\alpha + p^2X_{\mathcal{O}_F}) \in L,$$

for any $x \in X(\mathcal{O}_F)$, $\lambda \in \mathcal{O}_F$ and $\alpha \in X$. Now, $x_{-\alpha}(1) \in X(\mathcal{O}_F)$ acts on $e_\alpha \in X_{\mathcal{O}_F}$ by

$$x_{-\alpha}(1) \cdot e_\alpha = e_\alpha + h_{-\alpha} - e_{-\alpha}.$$

Since the images of $p\lambda e_\alpha$, $p\lambda e_{-\alpha}$ lie in L, it follows that

$$p\lambda h_{-\alpha} + p^2X_{\mathcal{O}_F} = x_{-\alpha}(1) \cdot p\lambda e_\alpha + p\lambda e_{-\alpha} - p\lambda e_\alpha + p^2X_{\mathcal{O}_F} \in L,$$

for any $\lambda \in \mathcal{O}_F$. Since $X_{\mathcal{O}_F}$ is spanned over $\mathcal{O}_F$ by $\{e_\alpha, h_\alpha \mid \alpha \in X\}$, it follows that L contains $pX_{\mathcal{O}_F}/p^2X_{\mathcal{O}_F}$, as required. $\square$

11.5 Hilbert-Serre dimension

To finish the proof of Theorem E, we need only establish Proposition 11.4. We will actually prove a stronger statement, namely the following.

Proposition 11.17:

Let K be any field, and J be a graded ideal of the polynomial ring $S = K[X_1, \ldots, X_m]$, such that S/J has Krull dimension 1. Let $f \in S$ be a non-zero homogeneous element of non-zero degree such that $f \notin J$, and let $T = \{f^n \mid n \geq 0\}$. Then $T^{-1}(S/J)$ is finitely-generated over the subalgebra $T^{-1}K[f]$.

Of course putting $S = \operatorname{gr} kG$, $J = J(M)$ and $f = \sigma(g-1)$ gives Proposition 11.4. To prove the proposition we introduce the notion of the Hilbert function for S/J.

Definition 11.18:

Let $I \subseteq K[X_1, \ldots, X_n]$ be an ideal and $A = K[X_1, \ldots, X_n]/I$ be an ideal. We define the Hilbert function of I to be

$$h_I : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}, \quad h_I(r) = \dim_K(A_r),$$

where $A_r = \{x + I \in A \mid \deg(x) \leq r\}$ and the degree is given by the standard total degree filtration on $K[X_1, \ldots, X_n]$.

The Hilbert function is in fact eventually a polynomial, by the Hilbert-Serre theorem, see Corollary 11.10 and Lemma 11.12 of [Kem11].

Theorem 11.19 (Hilbert-Serre theorem)**:**

Let $I \subseteq K[X_1, \ldots, X_n]$ be an ideal. There is a polynomial $q = q_I \in \mathbb{Q}[X]$ such that for some $N \in \mathbb{Z}_{\geq 0}$, whenever $r \geq N$, the value of the Hilbert function $h_I(r) = q_I(r)$. Moreover if q, q' are two such polynomials, then their degrees are equal.

Definition 11.20:

Let $I \subseteq K[X_1, \ldots, X_n]$ be an ideal, and q_I be a polynomial as in Theorem 11.19. The Hilbert-Serre dimension of $A = K[X_1, \ldots, X_n]/I$ is the degree of q_I.

It transpires that the Hilbert-Serre dimension of a finitely-generated commutative algebra agrees with the Krull dimension, see Theorem 11.13 of [Kem11].

Theorem 11.21:

Let $I \subseteq K[X_1, \ldots, X_n]$ be an ideal, and $A = K[X_1, \ldots, X_n]/I$. The Hilbert-Serre dimension of A is equal to $\mathcal{K}(A)$.

We can use this alternative characterisation of Krull dimension to give a proof of Proposition 11.17.

Proof of Proposition 11.17:

Recall that we write $S = K[X_1, \ldots, X_m]$ and let $J \subseteq S$ be an ideal such that S/J has Krull dimension 1. For $f \in S$ a non-zero homogeneous element of positive degree with $f \notin J$, let

$$J' = \{x \in S \mid f^n x \in J \text{ for some } n \geq 0\} \subseteq S.$$

Then J' is an ideal of S containing J, and is graded. Thus, S/J' is a quotient of S/J, so has Krull dimension at most 1. Moreover, localising the natural quotient map gives an isomorphism of $T^{-1}S$-modules,

$$T^{-1}(S/J) \cong T^{-1}(S/J'),$$

due to the definition of J'.

If S/J' has Krull dimension zero, then it is a finite-dimensional K-algebra, and therefore must be finitely-generated as a $K[f]$-module. It follows that $T^{-1}(S/J')$ is finitely-generated as a $T^{-1}K[f]$-module, and hence so is $T^{-1}(S/J)$.

Thus, we assume from now on that the Krull dimension of S/J' is 1. By Theorem 11.21, the Hilbert-Serre dimension of S/J' is also 1. Now, the grading on S is inherited by S/J', and we write

$$S/J' = \bigoplus_{r \geq 0} (S/J')_r$$

for the graded pieces. Because the Hilbert-Serre dimension of S/J' is 1, it follows from Definition 11.18 and Theorem 11.19 that for some $N \in \mathbb{Z}_{\geq 0}$,

$$\dim_K (S/J')_r = \dim_K (S/J')_N$$

whenever $r \geq N$.

Now, notice that multiplication by f induces an injective map $S/J' \to S/J'$, because if $x \in S$ and $fx \in J'$, it follows from the definition of J' that $x \in J'$. Since f is a homogeneous element, let its degree be $d > 0$. Then, multiplication by f induces an injective K-linear map

$$(S/J')_r \to (S/J')_{r+d},$$

for all $r \geq 0$. When $r \geq N$, $\dim_K (S/J')_r = \dim_K (S/J')_{r+d}$, and therefore by the rank-nullity theorem, the above map must also be surjective. It follows that S/J' is generated as a $K[f]$-module by

$$\bigoplus_{r=0}^{N+d-1} (S/J')_r,$$

which is finite-dimensional over K. It follows that S/J' is a finitely-generated $K[f]$-module. Therefore, $T^{-1}(S/J) \cong T^{-1}(S/J')$ is a finitely-generated $T^{-1}K[f]$-module. □

We comment that although the results of this subsection may be generalised to higher dimensions, they do not necessarily combine to give sufficient restrictions on the characteristic variety attached to a module of canonical dimension greater than one. To give an imprecise illustration, a module of canonical dimension one has a characteristic variety which corresponds to a projective variety of dimension zero, whereas a module of canonical dimension at least two corresponds to a projective curve or higher-dimensional variety, which have a more complicated structure. This is the principal reason Theorem E does not straightforwardly generalise to modules of canonical dimension two or higher.

11.6 Krull dimension

Following our result limiting the possible canonical dimensions of modules for kG, we can bound the Krull dimension of the Iwasawa algebra above. This is a straightforward deduction from the following result for rings.

Proposition 11.22:

Let R be a complete filtered ring with R and $\operatorname{gr} R$ both Noetherian and of finite Krull dimension. Let $a, b \geq 0$ and suppose that for all finitely-generated R-modules M,

$$\mathcal{K}(\operatorname{gr} M) \notin \{a, a+1, \ldots, a+b-1\}.$$

Then $\mathcal{K}(R) \leq \mathcal{K}(\operatorname{gr} R) - b$.

Proof:

The proof is very similar to the proof of Theorem 3.12(ii) of [Ard04]. Let $\operatorname{Lat}(S)$ be the lattice of all (left) ideals of a ring S. There is an increasing map

$$\operatorname{gr}\colon \operatorname{Lat}(R) \to \operatorname{Lat}(\operatorname{gr} R),$$

given by putting the subspace filtration on left ideals of R.

Suppose $Y \subseteq X \subseteq kG$ are left ideals such that $M = X/Y$ is an a-critical module (see 6.2.9 of [MR01]). Give M the subquotient filtration from R. Because the filtration on R is complete and filtered with $\operatorname{gr} kG$ Noetherian, this is a good filtration, by Proposition 1.2.3 of Chapter II of [LVO96]. Then $\mathcal{K}(\operatorname{gr} M) = \mathcal{K}(\operatorname{gr} X/\operatorname{gr} Y) \geq a$, so in fact $\mathcal{K}(\operatorname{gr} M) \geq a + b$ by our assumption. The result then follows from the characterisation of Proposition 6.1.17 of [MR01] given in Lemma 2.3 of [Ard04]. □

Theorem E gives a restriction on the canonical dimension of modules for an Iwasawa algebra, hence by Theorem 2.10, we have a restriction as in Proposition 11.22. We can now deduce Corollary F.

Corollary F:

Let $\mathfrak{g}$ be a finite-dimensional F-Lie algebra which is simple and split over F. Let G be a compact p-adic Lie group with associated Lie algebra $\mathcal{L}(G) \cong \mathfrak{g}$. If $|F : \mathbb{Q}_p| > 1$, the Krull dimension of kG is at most $\dim G - 1$.

Proof:

By Corollary 2.7, we may assume that G is uniform pro-p. By Theorem 2.10 and Theorem E, it follows that $\mathcal{K}(\operatorname{gr} M) \neq 1$ for all finitely-generated kG-modules. Applying Proposition 11.22 with $R = kG$ and $a = b = 1$, the result follows, since $\operatorname{gr} kG$ is a polynomial ring in $\dim G$ variables. □

We can combine this with Theorems A and B of [Ard04] to obtain the following.

Corollary 11.23:

Let F be a finite extension of $\mathbb{Q}_p$ and $\mathfrak{g}$ be a finite-dimensional F-Lie algebra which is simple and split over F. Let G be a compact p-adic Lie group with associated Lie algebra $\mathcal{L}(G) \cong \mathfrak{g}$. If $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{Q}_p)$, then $\mathcal{K}(kG) = 3$. Otherwise,

$$\lambda(\mathfrak{g}) \leq \mathcal{K}(kG) \leq \dim \mathfrak{g} - 1,$$

where $\lambda(\mathfrak{g})$ is the maximum length of a chain of subalgebras of $\mathfrak{g}$, viewed as a $\mathbb{Q}_p$-Lie algebra.

In fact the lower bound of Corollary 11.23 holds for any compact p-adic Lie group G, and Ardakov and Brown have conjectured that $\mathcal{K}(kG) = \lambda(\mathfrak{g})$.
Corollary 11.23 means we can verify this in some cases. For example if $G = SL_3(\mathbb{Z}_p)$, then $\mathfrak{g} \cong \mathfrak{gl}_3(\mathbb{Q}_p)$, so $\dim_{\mathbb{Q}_p}(\mathfrak{g}) = 8$ and $\lambda(\mathfrak{g}) = 7$. It follows from Corollary 11.23 that $\mathcal{K}(kG) = 7$, see Corollary B of [Ard04].
Our new result allows us to compute the Krull dimension of the Iwasawa algebra of a 6-dimensional group.

Corollary 11.24:

Let F be a quadratic extension of $\mathbb{Q}_p$ and $G = SL_2(\mathcal{O}_F)$. Then $\mathcal{K}(kG) = 5$.

Proof:

We have that $\mathfrak{g} = \mathcal{L}(G) \cong \mathfrak{sl}_2(F)$, and $\dim_{\mathbb{Q}_p} \mathfrak{g} = 6$, $\lambda(\mathfrak{g}) = 5$. By Corollary F and [Ard04, Theorem A], it follows that $\mathcal{K}(kG) = 5$. $\square$

Finally, we can also deduce a bound on the Krull dimension of the Iwasawa algebras of the compact general linear groups.

Proposition 11.25:

Let $n \geq 2$ and F be any finite field extension of $\mathbb{Q}_p$. The Krull dimension

$$\mathcal{K}(kGL_n(\mathcal{O}_F)) = \mathcal{K}(kSL_n(\mathcal{O}_F)) + |F : \mathbb{Q}_p|.$$

Proof:

Let $G = GL_n(\mathcal{O}_F)$ and $Z = \{cI \mid c - 1 \in p\mathcal{O}_F\} \leq G$ where I is the identity matrix. Then Z is a closed central subgroup of G, and is uniform pro-p. We have

$$\mathcal{L}(G) = \mathfrak{gl}_n(F), \quad \mathcal{L}(Z) = \{cI \mid c \in F\},$$

and we have a direct sum decomposition into Lie ideals,

$$\mathfrak{gl}_n(F) = \mathfrak{sl}_n(F) \oplus \{cI \mid c \in F\}.$$

Therefore

$$\mathcal{L}(G/Z) = \mathcal{L}(G)/\mathcal{L}(Z) \cong \mathfrak{sl}_n(F).$$

Since $\mathcal{L}(SL_n(\mathcal{O}_F)) \cong \mathfrak{sl}_n(F)$, by Corollary 2.7 it follows that

$$\mathcal{K}(k(G/Z)) = \mathcal{K}(kSL_n(\mathcal{O}_F)).$$

Recall the augmentation ideals

$$\epsilon_G(N) = kG\epsilon_N(N) = \epsilon_N(N)kG$$

for closed normal subgroups $N \leq G$, by Lemma 3.46. Now, Z is a closed subgroup of the open uniform pro-p normal subgroup

$$H = \{X \in G \mid X - I \in p\mathrm{Mat}_n(\mathcal{O}_F)\} \leq G.$$

In fact, the augmentation ideal $\epsilon_G(H)$ lies in the Jacobson radical of kG. To see this, if a maximal left ideal $M \subseteq kG$ does not contain $\epsilon_G(H)$, then

$$M + \epsilon_G(H) = M + \epsilon_H(H)kG = kG.$$

Since kG is a finitely-generated kH-module and the Jacobson radical of kH is $\epsilon_H(H)$, Nakayama's lemma implies $M = kG$, a contradiction. Thus $\epsilon_G(Z) \subseteq \epsilon_G(H)$ is contained in the Jacobson radical of kG. Now, kZ is isomorphic to a commutative power series ring on $|F : \mathbb{Q}_p|$ elements, central in kG. Therefore the augmentation ideal $\epsilon_G(Z) = kG\epsilon_Z(Z)$ is generated by $|F : \mathbb{Q}_p|$ regular normal elements lying in the Jacobson radical of kG. By Corollary 1.4 of [Wal72], the Krull dimension of kG is given by

$$\mathcal{K}(kG) = \mathcal{K}(kG/\epsilon_G(Z)) + |F : \mathbb{Q}_p| = \mathcal{K}(k(G/Z)) + |F : \mathbb{Q}_p|.$$

The result then follows since $\mathcal{K}(k(G/Z)) = \mathcal{K}(kSL_n(\mathcal{O}_F))$. $\square$

Theorem 11.26:

Let $n \geq 2$ and F be any finite extension of $\mathbb{Q}_p$. If $n > 2$ or $F \neq \mathbb{Q}_p$, then

$$\mathcal{K}(kGL_n(\mathcal{O}_F)) \leq n^2|F : \mathbb{Q}_p| - 1.$$

Also, $\mathcal{K}(kGL_2(\mathbb{Z}_p)) = 4$, $\mathcal{K}(kGL_3(\mathbb{Z}_p)) = 8$, and $\mathcal{K}(kGL_2(\mathcal{O}_F)) = 7$ when F is a quadratic extension of $\mathbb{Q}_p$.

Proof:

By Corollary 11.23 and Proposition 11.25, if $n > 2$ or $F \neq \mathbb{Q}_p$,

$$\begin{aligned}\mathcal{K}((kGL_n(\mathcal{O}_F)) &= \mathcal{K}(kSL_n(\mathcal{O}_F) + |F : \mathbb{Q}_p| \\ &\leq \dim(\mathfrak{sl}_n(F)) - 1 + |F : \mathbb{Q}_p|.\end{aligned}$$

But the $\mathbb{Q}_p$-vector space dimension $\dim(\mathfrak{sl}_n(F)) = (n^2-1)|F : \mathbb{Q}_p|$, so the claimed inequality follows.
The equalities in specific cases follow by combining Proposition 11.25 with Corollary A of [Ard04], Corollary B of [Ard04], and Corollary 11.24, respectively. $\square$

12 Finite length admissible representations

In this section, let G be an arbitrary p-adic Lie group, and recall that k is assumed to be a field of characteristic p. We study some representation-theoretic consequences of our canonical dimension results.

12.1 Holonomic representations

Recall from subsection 4.5 that the category of smooth admissible representations of G is dual to the category $\mathrm{Mod}_k^{\mathrm{fg}}(G)$. For any compact open subgroup $H \leq G$, any module M in $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ is a finitely-generated kH-module, hence has a canonical dimension. The arguments of Proposition 2.5 and Corollary 2.7 show that the canonical dimension is invariant with respect to the choice of H, thus we unambiguously denote it by $\mathrm{Cdim}(M)$.

We are now in a position to give the third characterisation of canonical dimension appearing in this thesis, which was originally proved in Proposition 2.18 of [EP20].

Theorem 12.1:

Let G be a p-adic Lie group and V be a smooth admissible representation of G over k. Let K be an open uniform pro-p subgroup of G. There exists an integer $N \geq 0$ and real numbers $a, b > 0$ such that for all $n \geq N$,

$$a(p^n)^d \leq \dim_k V^{K_n} \leq b(p^n)^d,$$

where $K_n = \{h^{p^n} \mid h \in K\}$ are the p-power subgroups of K, and the canonical dimension $\mathrm{Cdim}(V^\vee) = d$.

Thus, we see that the canonical dimension $\mathrm{Cdim}(V^\vee)$ associated to an admissible representation V measures the growth rate of its invariant subspaces. In particular, V is finite-dimensional if and only if $\mathrm{Cdim}(V^\vee) = 0$.

When appropriate, we refer to $\mathrm{Cdim}(V^\vee)$ as the GK-dimension of V. We stress that this name is given only by analogy; Theorem 12.1 bears somewhat superficial similarity to the definitions of the usual Gelfand-Kirillov dimension. However, Iwasawa algebras are not Cohen-Macaulay with respect to the Gelfand-Kirillov dimension, and in all interesting cases the true Gelfand-Kirillov dimension of a kH-module will be infinite, as discussed in subsection 5.6 of [AB06]. It is however the case that $\mathrm{Cdim}(V^\vee)$

is equal to the Gelfand-Kirillov dimension of the graded module $\operatorname{gr} V^\vee$, see Proposition 5.4 of *loc. cit.*

Theorem 12.1 shows that the GK-dimension may be computed without reference to Iwasawa algebras, at least in principle. Of course, the results of this thesis would not have been obtained without this perspective, for example we have the following.

Theorem 12.2:

Let $\mathbb{G}$ be a split-simple affine group scheme defined over F, and $G = \mathbb{G}(F)$. Suppose $|F : \mathbb{Q}_p| > 1$. There are no admissible representations of G over k of GK-dimension 1.

Proof:

A compact open subgroup of G is $H = \mathbb{G}(\mathcal{O}_F)$. Then, $\mathcal{L}(H)$ is isomorphic to a finite-dimensional F-Lie algebra which is simple and split over F. By Theorem E, kH has no finitely-generated modules of canonical dimension 1. In particular, $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ has no objects of canonical dimension 1. Thus for every admissible representation V of G, the GK-dimension $\mathrm{Cdim}(V^\vee) \neq 1$. $\square$

An immediate corollary to Theorem 12.2 is that any infinite-dimensional admissible representation of $G = \mathbb{G}(F)$ has GK-dimension at least two, when F is a non-trivial extension of $\mathbb{Q}_p$. We expect that this lower bound will vary as G varies, and in general will be higher than two.

We now turn our attention to representations that satisfy such a lower bound. Whenever a module for the Weyl algebra (or more generally, a $\mathcal{D}$-module) satisfies the lower bound of Bernstein's inequality, it is called holonomic, and holonomic modules are always of finite length - see [Ehl87, Theorem 1.12, Definition 1.13, 1.16.1]. We now prove the analogous result for representations of p-adic Lie groups.

Definition 12.3:

Let M be a module in $\mathrm{Mod}_k^{\mathrm{fg}}(G)$. We say M has minimal-positive dimension if $\mathrm{Cdim}(M) > 0$ and any other module N of $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ has $\mathrm{Cdim}(N) \geq \mathrm{Cdim}(M)$ or $\mathrm{Cdim}(N) = 0$.

Definition 12.4:

Let V be an admissible representation of G. V is holonomic if and only if $V^\vee$ has minimal-positive dimension.

Notice that any admissible representation V with GK-dimension $\mathrm{Cdim}(V^\vee) = 1$ is automatically holonomic.

Theorem G:

Let G be a p-adic Lie group. Suppose G acts trivially on all of its finite-dimensional smooth representations over k. Every holonomic smooth admissible representation of G over k is of finite length.

In fact, we show the equivalent result that if $M \in \mathrm{Mod}_k^{\mathrm{fg}}(G)$ has minimal-positive dimension, then M is of finite length. Some assumption on the group G is necessary for this result to hold, as illustrated by the example of the free module of rank one over the Iwasawa algebra $k\mathbb{Z}_p = k[[T]]$. In the next section we will show that $SL_n(F)$ satisfies the hypothesis of Theorem G, see Corollary 12.14.

To use the condition of Theorem G, we consider the G-coinvariants of any module. Recalling subsection 3.5, we write $\epsilon(G)$ for the two-sided augmentation ideal of the augmented Iwasawa algebra kG.

Definition 12.5:

Let M be a left kG-module. The G-coinvariants of M are

$$M_G = M/\epsilon(G)M = kG/\epsilon(G) \otimes_{kG} M,$$

which is also a left kG-module.

Notice that both M_G and the submodule $\epsilon(G)M$ are in $\mathrm{Mod}_k^{\mathrm{fg}}(G)$, because M is finitely-generated over an Iwasawa algebra, which is a Noetherian ring. The module M_G may be characterised as the maximal trivial quotient of M.

Lemma 12.6:

Let M be a module in $\mathrm{Mod}_k^{\mathrm{fg}}(G)$. The quotient M_G is a finite-dimensional trivial kG-module (possibly zero-dimensional), and for any proper submodule $L \leq \epsilon(G)M$, the module M/L is not trivial.

Proof:

Any module M of $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ is a finitely-generated kG-module. Since $g-1 \in \epsilon(G)$ for all $g \in G$, it follows that every element of g acts trivially on M_G, from which it follows that M_G is a finitely-generated module over the quotient ring $kG/\epsilon(G) \cong k$. So M_G is finite-dimensional over k.

Now, the kG-module $\epsilon(G)M$ is generated by $\{(g-1)m \mid g \in G, m \in M\}$, so if L is a proper kG-submodule of $\epsilon(G)M$, there exists $g \in G$, $m \in M$ such that $(g-1)m \notin L$. Thus, G does not act trivially on the element $m + L \in M/L$, so M/L is not trivial. $\square$

Proposition 12.7:

Let G be as in Theorem G, and M be a module in $\mathrm{Mod}_k^{\mathrm{fg}}(G)$. Then $\epsilon(G)M$ has no non-zero trivial quotients.

Proof:

Because $\epsilon(G)M$ is finitely-generated, any trivial quotient $\epsilon(G)M/L$ is finite-dimensional. Since $M_G = M/\epsilon(G)M$ is finite-dimensional, it follows from the short exact sequence

$$0 \to \epsilon(G)M/L \to M/L \to M_G \to 0$$

that M/L is finite-dimensional. By the assumption on G, the dual of M/L is a trivial admissible representation. Hence M/L is a trivial kG-module, but by Lemma 12.6, $L = \epsilon(G)M$, thus $\epsilon(G)M/L = 0$. $\square$

We now introduce the notion of a critical module.

Definition 12.8:

We say that a module M in $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ is critical if for every non-zero submodule $N \leq M$, $\mathrm{Cdim}(M/N) < \mathrm{Cdim}(M)$.

Lemma 12.9:

Let G be as in Theorem G. Let M be a module in $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ with minimal-positive dimension. Suppose M is critical and has no non-zero trivial quotients. Then M is a simple object of $\mathrm{Mod}_k^{\mathrm{fg}}(G)$.

Proof:

Let $N \leq M$ be a non-zero proper submodule. Then $\mathrm{Cdim}(M/N) < \mathrm{Cdim}(M)$ because M is critical. Hence $\mathrm{Cdim}(M/N) = 0$ because M has minimal-positive dimension. Thus, by the assumption on G, M/N is a nonzero trivial module, a contradiction. So, M has no nonzero proper kG-submodules, making it a simple kG-module and a simple object in $\mathrm{Mod}_k^{\mathrm{fg}}(G)$. □

Corollary 12.10:

Let G be as in Theorem G, and M be a critical module in $\mathrm{Mod}_k^{\mathrm{fg}}(G)$ with minimal-positive dimension. Then M is of finite length.

Proof:

Proposition 4.5 of [Lev92] says that

$$\mathrm{Cdim}(M) = \max(\mathrm{Cdim}(\epsilon(G)M), \mathrm{Cdim}(M_G)).$$

But by Lemma 12.6, M_G is finite-dimensional so $\mathrm{Cdim}(M_G) = 0$, hence $\mathrm{Cdim}(\epsilon(G)M) = \mathrm{Cdim}(M)$. Therefore, $\epsilon(G)M$ is critical because M is critical and for any nonzero submodule $N \leq \epsilon(G)M$, we have the inequality $\mathrm{Cdim}(\epsilon(G)M/N) \leq \mathrm{Cdim}(M/N)$. By Proposition 12.7, $\epsilon(G)M$ has no non-trivial quotients, thus is simple by Lemma 12.9. It follows from the short exact sequence

$$0 \to \epsilon(G)M \to M \to M_G \to 0$$

that M is of finite length, since M_G is finite-dimensional. □

We now need an analogue of Proposition 6.2.20 of [MR01].

Proposition 12.11:

Any module $M \in \mathrm{Mod}_k^{\mathrm{fg}}(G)$ has a critical composition series. That is, there is a sequence of submodules

$$M = M_0 \geq M_1 \geq M_2 \geq \cdots \geq M_s = 0$$

such that M_j/M_{j+1} is critical for all j.

Proof:

Set $M = M_0$. We choose the M_j inductively, as follows. Given M_j, choose M_{j+1} to be a maximal submodule such that $\mathrm{Cdim}(M_j/M_{j+1}) = \mathrm{Cdim}(M_j)$ – a maximal such submodule exists because M_j is a finitely-generated submodule over the Noetherian ring kH, so is a Noetherian module. By construction, M_j/M_{j+1} is critical.

Thus we need only check that $M_s = 0$ for some $s \geq 0$. If not, then there is an infinite chain of modules

$$M_m \geq M_{m+1} \geq M_{m+2} \geq \dots$$

such that $\mathrm{Cdim}(M_j/M_{j+1}) = \mathrm{Cdim}(M_m)$ for all $j \geq m$. But by Proposition 4.5 of [Lev92] this cannot occur, a contradiction. $\Box$

Proof of Theorem G:

Given a holonomic representation V, its dual $M = V^\vee$ has minimal-positive dimension. It is enough to show that M has finite length, since the anti-equivalence of categories $\mathrm{Rep}_k^{\mathrm{ad}}(G) \leftrightarrow \mathrm{Mod}_k^{\mathrm{fg}}(G)$ preserves simple objects.

Let $d = \mathrm{Cdim}(M)$. By Proposition 12.11, there is a critical composition series

$$M = M_0 \geq M_1 \geq M_2 \geq \dots \geq M_s = 0.$$

Since M has minimal-positive dimension, it follows from Proposition 4.5 of [Lev92] that $\mathrm{Cdim}(M_j/M_{j+1})$ is equal either to d or to 0, for each j. By Corollary 12.10, any critical module of canonical dimension d is of finite length. On the other hand, any module of canonical dimension zero is finite-dimensional, hence also of finite length. Therefore, for every $j \in \{0, \dots, s-1\}$, the module M_j/M_{j+1} is of finite length, from which it follows that M is of finite length. $\Box$

12.2 Admissible representations of $GL_n(F)$

We can deduce interesting corollaries about certain admissible representations from Theorem G. First, we determine a large class of p-adic Lie groups whose finite-dimensional smooth representations are trivial – we expect this is well-known to the experts.

Proposition 12.12:

Let F be a finite field extension of $\mathbb{Q}_p$ and k be a field of characteristic p. All finite-dimensional smooth representations of the additive group F are trivial.

Proof:

Let $\bar{k}$ be an algebraic closure of k. Consider an n-dimensional smooth representation of F, $\phi : F \to GL_n(k)$. Let $x \in F$ and consider $\phi(x) \in GL_n(k)$. There exists $P \in GL_n(\bar{k})$ such that

$$A = P\phi(x)P^{-1}$$

is in Jordan normal form with Jordan blocks $J_{n_1,\lambda_1}, \dots, J_{n_m,\lambda_m}$. Here,

$$J_{r,\mu} = \mu I + U_r \in GL_r(\bar{k}),$$

and U_r is the matrix with entries $(U_r)_{ab} = 1$ if $b - a = 1$ and 0 otherwise. Note $U_r^s = 0$ if $s \geq r$.

Since ϕ is a smooth representation, $\operatorname{Ker}\phi$ is an open subgroup of F, and hence $p^{N_0}x \in \operatorname{Ker}\phi$ for some $N_0 \geq 0$. Then if $N \geq N_0$,

$$A^{p^N} = P\phi(x)^{p^N}P^{-1} = I.$$

If $N \geq N_0$ and furthermore $p^N \geq n$, then

$$J_{n_i,\lambda_i}^{p^N} = \lambda_i^{p^N} I + U_{n_i}^{p^N} = \lambda_i^{p^N} I = I,$$

for all $i \in \{1, \dots, m\}$. But $\bar{k}$ is a field of characteristic p, therefore $\lambda_i = 1$ for all i. So, A has Jordan blocks $J_{n_1,1}, \dots, J_{n_m,1}$.

Therefore, for any non-negative integer N such that $p^N \geq n$, the ith block of A^{p^N} is $I + U_{n_i}^{p^N} = I$, and so A^{p^N} is the identity matrix.

Thus whenever $p^N \geq n$, $\phi(x)^{p^N}$ must be the identity matrix for every $x \in F$. But, for any $y \in F$, there exists $x \in F$ such that $p^N x = y$. Then $\phi(y) = \phi(x)^{p^N}$ is the identity. So, ϕ is the trivial homomorphism, as required. □

Proposition 12.13:

Let X be an indecomposable root system, F be a finite extension of $\mathbb{Q}_p$. Let $G = X(F)$ be the adjoint Chevalley group over F. All finite-dimensional smooth representations of G are trivial.

Proof:

By definition of the Chevalley group $X(F)$, G is generated by subgroups

$$(U_\alpha = \{x_\alpha(t) \mid t \in F\})_{\alpha \in X},$$

and clearly $U_\alpha \cong F$ for all roots $\alpha \in X$. If V is a smooth finite-dimensional representation of G over k, then V is a smooth finite-dimensional representation of each U_α under restriction. By Proposition 12.12, V is a trivial representation for every U_α. Since the U_α generate G, it follows that V is a trivial representation of G. $\square$

By specialising the Chevalley group to be of type $X = A_n$ we immediately obtain a result for the special linear groups.

Corollary 12.14:

Let F be a finite field extension of $\mathbb{Q}_p$, $n \geq 2$ and $G = SL_n(F)$. All finite dimensional smooth representations of G are trivial.

The following application of Theorem G is then straightforward.

Corollary 12.15:

Let V be a smooth admissible representation of $SL_2(\mathbb{Q}_p)$ with $\mathrm{Cdim}(V^\vee) = 1$. Then V is of finite length.

Proof:

Since $\mathrm{Cdim}(V^\vee) = 1$, it is automatic that V is holonomic. By Corollary 12.14 and Theorem G, V is of finite length. $\square$

Of course, Corollary 12.15 holds with $SL_2(\mathbb{Q}_p)$ replaced by any simple Chevalley group over any extension F. However, Theorem E tells us that no representations of GK-dimension one exist in this case, making the statement vacuous.

Frequently in the context of the (mod p) Langlands programme we are most interested in admissible representations of $GL_n(F)$. However, we do not possess any useful results limiting the canonical dimension of modules for $kGL_n(\mathcal{O}_F)$ – note that many modules of small canonical dimension exist, by inflation along the homomorphism $kGL_n(\mathcal{O}_F) \to k\mathcal{O}_F^\times$ given by the determinant. Moreover, $GL_n(F)$ has non-trivial finite-dimensional smooth representations, so does not satisfy the hypothesis of Theorem G.

Fortunately, many interesting representations of $GL_n(F)$ have a central character, and this allows us to apply Theorems H and J by reducing to the case of special linear groups.

Definition 12.16:

Let V be a representation of a p-adic Lie group over the field k. V has a central character if and only if every element of the centre $Z(G)$ acts on V by a scalar in k.

Lemma 12.17:

Let V be a smooth admissible representation of $GL_n(F)$ with a central character. Then V is a smooth admissible representation of $SL_n(F)$ under restriction. Moreover $\mathrm{Cdim}_{GL_n(\mathcal{O}_F)}(V^\vee) = \mathrm{Cdim}_{SL_n(\mathcal{O}_F)}(V^\vee)$.

Proof:

Let Z be the centre of G. Consider the representation V restricted to Z, giving a smooth representation $\chi : Z \to GL(V)$. Because Z acts by scalars, $K = \operatorname{Ker}\chi$ is an open subgroup of Z.

Let H be an open subgroup of $SL_n(F)$. Then, because K is open in Z, HK is open in $SL_n(F)Z$, and hence is open in $GL_n(F)$. Because V is admissible, V^{HK} is finite-dimensional. But, K acts trivially on V by definition, so we have

$$V^{HK} = (V^K)^H = V^H.$$

Thus the fixed vectors of V under any open subgroup of $SL_n(F)$ are finite-dimensional, so V is an admissible representation of $SL_n(F)$.

Now let $N = GL_n(\mathcal{O}_F) \cap K$. Since K acts trivially on V, by Lemma 5.1.2 of [BHH$^+$23], the canonical dimension

$$\mathrm{Cdim}_{GL_n(\mathcal{O}_F)}(V^\vee) = \mathrm{Cdim}_{GL_n(\mathcal{O}_F)/N}(V^\vee).$$

But $SL_n(\mathcal{O}_F)N/N \trianglelefteq_o GL_n(\mathcal{O}_F)/N$ is an open normal subgroup, so the proof of Proposition 2.5 implies that

$$\mathrm{Cdim}_{GL_n(\mathcal{O}_F)/N}(V^\vee) = \mathrm{Cdim}_{SL_n(\mathcal{O}_F)N/N}(V^\vee).$$

Since $SL_n(\mathcal{O}_F)N/N$ is a quotient of $SL_n(\mathcal{O}_F)$, another application of Lemma 5.1.2 of [BHH+23] gives the result. $\square$

Theorem 12.18:

Let V be a smooth admissible representation of $GL_2(\mathbb{Q}_p)$, with central character, such that $\mathrm{Cdim}(V^\vee) = 1$. Then V is of finite length.

Proof:

Since V has a central character, Lemma 12.17 implies V is a smooth admissible representation of $SL_2(\mathbb{Q}_p)$ under restriction, and

$$\mathrm{Cdim}_{SL_2(\mathbb{Q}_p)}(V^\vee) = \mathrm{Cdim}_{GL_2(\mathbb{Q}_p)}(V^\vee) = 1.$$

By Corollary 12.15, V is a finite length representation of $SL_2(\mathbb{Q}_p)$, hence is of finite length as a representation of $GL_2(\mathbb{Q}_p)$. $\square$

The irreducible smooth admissible representations of $GL_2(\mathbb{Q}_p)$ were classified by Breuil in [Bre03], and all have a central character by [Ber12]. The work of Morra in [Mor13], which builds on that of Paškūnas in [Paš10], shows that these representations correspond to modules of canonical dimension either one or zero. From this we can characterise the finite-length representations of $GL_2(\mathbb{Q}_p)$ with central character.

Corollary H:

The finite length smooth admissible representations V of $GL_2(\mathbb{Q}_p)$ with central character are precisely those with $\mathrm{Cdim}(V^\vee) \leq 1$.

Proof:

By the results of Morra and Breuil, all irreducible admissible representations of $GL_2(\mathbb{Q}_p)$ have GK-dimension at most 1, since finite-dimensional representations have canonical dimension zero. Given any short exact sequence of admissible representations,

$$0 \to V_1 \to V_2 \to V_3 \to 0,$$

the equality $\mathrm{Cdim}(V_2^\vee) = \max(\mathrm{Cdim}(V_1^\vee), \mathrm{Cdim}(V_3^\vee))$ holds by Proposition 4.5 of [Lev92]. Therefore all finite length admissible representations of $GL_2(\mathbb{Q}_p)$ have GK-dimension at most 1.

Conversely, if V is a smooth admissible representation of $GL_2(\mathbb{Q}_p)$ of GK-dimension at most 1, then either V is finite-dimensional, hence clearly finite length, or $\mathrm{Cdim}(V^{\vee}) = 1$, so V is of finite length by Theorem 12.18. $\square$

In [HW] many smooth admissible representations of (the units of) quaternion algebras over $\mathbb{Q}_p$ are exhibited, which have canonical dimension equal to one. As commented by the authors, these representations are not of finite length, which can occur since the quaternion algebras do not satisfy the condition of Theorem G.

We now use our new results on canonical dimension to find representations of finite length over groups of higher dimension.

Theorem 12.19:

Let X be an indecomposable root system, and V be a smooth admissible representation of $X(F)$ where $|F : \mathbb{Q}_p| > 1$ or $X \neq A_1$. If $\mathrm{Cdim}(V^{\vee}) = 2$, then V is of finite length.

Proof:

If $|F : \mathbb{Q}_p| > 1$, then by Theorem 12.2, there are no admissible representations of $X(F)$ with GK-dimension 1. Hence if $\mathrm{Cdim}(V^{\vee}) = 2$, then V is holonomic.
If $F = \mathbb{Q}_p$ and $X \neq A_1$, then by Theorem 3.12 and Proposition 3.15 of [Ard04], combined with Corollary 2.7, the Iwasawa algebra $kX(\mathbb{Z}_p)$ has no modules of canonical dimension one. Since $X(\mathbb{Z}_p)$ is a compact open subgroup of $X(\mathbb{Q}_p)$, it follows that any admissible representation of $X(\mathbb{Q}_p)$ of GK-dimension 2 is holonomic.
In either case, Proposition 12.13 and Theorem G then imply V is of finite length.
$\square$

In particular we may specialise Theorem 12.19 to the case of $SL_n(F)$ – we now apply this result.

Corollary J:

Let V be a smooth admissible representation of $GL_n(F)$ with central character. If $|F : \mathbb{Q}_p| > 1$ or $n > 2$, and $\mathrm{Cdim}(V^{\vee}) = 2$, then V is of finite length.

Proof:

By Lemma 12.17, V is an admissible representation of $SL_n(F)$ under restriction and $\operatorname{Cdim}_{SL_n(F)}(V^\vee) = \operatorname{Cdim}_{GL_n(F)}(V^\vee) = 2$. By Theorem 12.19, V is a finite length representation of $SL_n(F)$, hence also of $GL_n(F)$. $\square$

Let us specialise to the case of the general linear group $GL_2(F)$. In [BHH$^+$], the following category of smooth admissible representations is considered.

Definition 12.20:

Let $G = GL_2(F)$ where F is a finite unramified extension of $\mathbb{Q}_p$. Consider the pro-p Iwahori subgroup $I_1 \leq GL_2(F)$ and its centre Z_1. Let $J \trianglelefteq \operatorname{gr} k(I_1/Z_1)$ be the graded ideal given by equation (117) on page 113 of [BHH$^+$]. We define the category $\mathcal{C}$ to be the full subcategory of $\operatorname{Rep}_k^{\mathrm{ad}}(G)$ whose objects V have a central character, such that $\operatorname{gr} V^\vee$ is annihilated by some power of J.

Because Z_1 is pro-p, its only one-dimensional smooth representations are trivial, hence $V^\vee$ is a $k(I_1/Z_1)$-module, so the above is well-defined. Note that the category $\mathcal{C}$ considered above is not quite the same as the category with that name in [BHH$^+$]. However, $\mathcal{C}$ is a full subcategory of that category, and the representations of most interest, namely those appearing in [BHH$^+$23, Theorem 1.1], lie in $\mathcal{C}$.

Corollary 12.21:

Let F be an unramified quadratic extension of $\mathbb{Q}_p$, and $G = GL_2(F)$. Let V be a smooth admissible representation in the category $\mathcal{C}$. Then V is of finite length.

Proof:

The quotient ring $\operatorname{gr} k(I_1/Z_1)/J$ is a commutative algebra generated by $2|F : \mathbb{Q}_p|$ elements, with Krull dimension equal to $|F : \mathbb{Q}_p|$. Thus, when $H \leq I_1/Z_1$ is an open uniform pro-p subgroup, $\operatorname{gr} kH/(\operatorname{gr} kH \cap J)$ is also a commutative algebra of Krull dimension $|F : \mathbb{Q}_p|$. Since V is in $\mathcal{C}$, we have

$$(\operatorname{gr} kH \cap J)^n \subseteq J^n \subseteq \operatorname{Ann}(\operatorname{gr} V^\vee)$$

for some $n \geq 1$. By Theorem 2.10,

$$\operatorname{Cdim}(V^\vee) \leq \mathcal{K}\left(\operatorname{gr} kH \big/ \operatorname{gr} kH \cap J\right) = |F : \mathbb{Q}_p|.$$

Since $|F : \mathbb{Q}_p| = 2$, Theorem 12.2 and Lemma 12.17 imply that the GK-dimension $\mathrm{Cdim}(V^\vee)$ is either 0 or 2. By Corollary J, and since representations of GK-dimension 0 are finite-dimensional, it follows V is of finite length. $\square$

It follows from Corollary 12.21 that the representations π considered in [BHH$^+$23, Theorem 1.1] are of finite length when F is a quadratic unramified extension of $\mathbb{Q}_p$. This improves on some of the results of [BHH$^+$]; when π is as in subsection 3.3.2 of *op. cit*, Corollary 12.21 implies that π is of finite length, removing the condition $\pi^{K_1} \cong D_0(\bar{\rho})$ needed to prove this (in the case $\bar{\rho}$ reducible split) in subsection 3.3.5 of *op. cit.* That is, Theorem 1.3.8 of [BHH$^+$] holds outside of the so-called "minimal case", if F is a quadratic unramified extension of $\mathbb{Q}_p$.

Theorem 1.1 of [HW22] states that certain representations $\pi(\bar{\rho})$ of $GL_2(F)$ (here $\bar{\rho}$ is non-semisimple) have GK-dimension $\mathrm{Cdim}(\pi(\bar{\rho})^\vee) = |F : \mathbb{Q}_p|$ when F is an unramified extension of $\mathbb{Q}_p$. Again by Corollary J, if F is a quadratic extension, then $\pi(\bar{\rho})$ is of finite length. Indeed, a stronger statement than this is shown in Theorem 1.7 of *op. cit.* by different methods.

References

[AB06] Konstantin Ardakov, Kenneth Brown, *Ring-theoretic properties of Iwasawa algebras: a survey*, Documenta Mathematica, volume Coates:pp. 7–33 (2006).

[AB07] Konstantin Ardakov, Kenneth Brown, *Primeness, semiprimeness and localisation in Iwasawa algebras*, Transactions of the American Mathematical Society, volume 359, no. 4:pp. 1499–1515 (2007).

[AB08] Peter Abramenko, Kenneth S. Brown, *Buildings: Theory and Applications*, number 248 in Graduate Texts in Mathematics, Springer (2008).

[Åbe82] Hans Åberg, *Coherence of Amalgamations*, Journal of Algebra, volume 78, no. 2:pp. 372–385 (1982).

[Ard04] Konstantin Ardakov, *Krull dimension of Iwasawa algebras*, Journal of Algebra, volume 280, no. 1:pp. 190–206 (2004).

[Ard12] Konstantin Ardakov, *Prime ideals in nilpotent Iwasawa algebras*, Inventiones mathematicae, volume 190, no. 2:pp. 439–503 (2012).

[Ard21] Konstantin Ardakov, *Equivariant $\mathcal{D}$-modules on rigid analytic spaces*, Astérisque, volume 423 (2021).

[AW13] Konstantin Ardakov, Simon Wadsley, *On irreducible representations of compact p-adic analytic groups*, Annals of Mathematics, volume 178:pp. 453–557 (2013).

[Ber12] Laurent Berger, *Central Characters for Smooth Irreducible Modular Representations of $GL_2(\mathbb{Q}_p)$*, Rendiconti del Seminario Matematico della Università di Padova, volume 128:pp. 1–6 (2012).

[BHH$^+$] Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra, Benjamin Schraen, *Conjectures and results on modular representations of $GL_n(K)$ for a p-adic field K*, arXiv preprint, arXiv:2102.06188.

[BHH$^+$23] Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra, Benjamin Schraen, *Gelfand-Kirillov dimension and mod p cohomology for GL_2*, Inventiones mathematicae, doi:10.1007/s00222-023-01202-8 (2023).

[Bor91] Armand Borel, *Linear Algebraic Groups*, number 126 in Graduate Texts in Mathematics, Springer-Verlag, second edition (1991).

[Bre03] Christophe Breuil, *Sur quelques représentations modulaires et p-adiques de $GL_2(\mathbb{Q}_p)$: I (On Some Modular Representations and p-adics of $GL_2(\mathbb{Q}_p)$: I)*, Compositio Mathematica, volume 138, no. 2:pp. 165–188 (2003).

[Bru66] Armand Brumer, *Pseudocompact algebras, profinite groups and class formations*, Bulletin of the American Mathematical Society, volume 72, no. 2:pp. 321–324 (1966).

[Bum96] Daniel Bump, *Automorphic Forms and Representations*, number 55 in Cambridge Studies in Advanced Mathematics, Cambridge University Press (1996).

[CD03] M.J. Chasco, X. Dominguez, *Topologies on the direct sum of topological Abelian groups*, Topology and its Applications, volume 133:pp. 209–223 (2003).

[CE12] Frank Calegari, Matthew Emerton, *Completed cohomology – a survey*, in *Non-abelian fundamental groups and Iwasawa theory*, number 393 in London Mathematical Society Lecture Note Series, Cambridge University Press, pp. 239–257 (2012).

[DDMS03] John D Dixon, Marcus PF Du Sautoy, Avinoam Mann, Dan Segal, *Analytic pro-p groups*, number 61 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, second edition (2003).

[EGH] Matthew Emerton, Toby Gee, Eugen Hellmann, *An introduction to the categorical p-adic Langlands program*, arXiv preprint arXiv:2210.01404.

[EH76] David A. Edwards, Harold M. Hastings, *Čech and Steenrod homotopy theories with applications to geometric topology*, number 542 in Lecture Notes in Mathematics, Springer-Verlag (1976).

[Ehl87] Fritz Ehlers, *The Weyl Algebra*, in *Algebraic D-modules*, number 2 in Perspectives in Mathematics, chapter V, Academic Press Inc., pp. 173–205 (1987).

[Eme] Matthew Emerton, *On a class of coherent rings, with applications to the smooth representation theory of $GL_2(\mathbb{Q}_p)$ in characteristic p*, Preprint available at `http://math.uchicago.edu/~emerton/pdffiles/frob.pdf`.

[Eme10] Matthew Emerton, *Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties*, Astérisque, volume 331:pp. 355–402 (2010).

[EP20] Matthew Emerton, Vytautas Paškūnas, *On the density of supercuspidal points of fixed regular weight in local deformation rings and global Hecke algebras*, Journal de l'École polytechnique—Mathématiques, volume 7:pp. 337–371 (2020).

[Gab62] Pierre Gabriel, *Des Catégories Abéliennes*, Bulletin de la Société Mathématique de France, volume 90:pp. 323–448 (1962).

[Gla89] Sarah Glaz, *Commutative Coherent Rings*, number 1371 in Lecture Notes in Mathematics, Springer-Verlag (1989).

[GN20] Toby Gee, James Newton, *Patching and the completed homology of locally symmetric spaces*, Journal of the Institute of Mathematics of Jussieu, volume 21, no. 2:pp. 395–498 (2020).

[GW04] K.R. Goodearl, R.B. Warfield, *An Introduction to Noncommutative Noetherian Rings*, number 61 in London Mathematical Society Student Texts, Cambridge University Press (2004).

[Har66] Morton E. Harris, *Some Results on Coherent Rings*, Proceedings of the American Mathematical Society, volume 17:pp. 474–479 (1966).

[Har67] Morton E. Harris, *Some Results on Coherent Rings II*, Glasgow Mathematical Journal, volume 8:pp. 123–126 (1967).

[Hil90] David Hilbert, *Ueber die Theorie der algebraischen Formen*, Mathematische Annalen, volume 36:pp. 473–534 (1890).

[Hu12] Yongquan Hu, *Diagrammes canoniques et représentations modulo p de $GL_2(F)$*, Journal of the Institute of Mathematics of Jussieu (2012).

[HW] Yongquan Hu, Haoran Wang, *On some mod p representations of quaternion algebra over $\mathbb{Q}_p$*, arXiv preprint, arXiv:2201.01464.

[HW22] Yongquan Hu, Haoran Wang, *On the mod p cohomology for GL_2: the non-semisimple case*, Cambridge Journal of Mathematics, volume 10, no. 2:pp. 261–431 (2022).

[Kem11] Gregor Kemper, *A Course in Commutative Algebra*, number 256 in Graduate Texts in Mathematics, Springer (2011).

[Koh17] Jan Kohlhaase, *Smooth duality in natural characteristic*, Advances in Mathematics, volume 317:pp. 1–49 (2017).

[Kra97] Henning Krause, *The spectrum of a locally coherent category*, Journal of Pure and Applied Algebra, volume 114:pp. 259–271 (1997).

[Lan02] Serge Lang, *Algebra*, number 211 in Graduate Texts in Mathematics, Springer-Verlag, third edition (2002).

[Laz65] Michel Lazard, *Groupes analytiques p-adiques*, Publications Mathématiques de l'IHÉS, volume 26:pp. 5–219 (1965).

[Lev92] Thierry Levasseur, *Some properties of non-commutative regular graded rings*, Glasgow Mathematical Journal, volume 34, no. 3:pp. 277–300 (1992).

[LVO96] Huishi Li, Freddy Van Ostaeyen, *Zariskian Filtrations*, number 2 in K-Monographs in Mathematics, Springer (1996).

[Mor13] Stefano Morra, *Invariant elements for p-modular representations of $GL_2(\mathbb{Q}_p)$*, Transactions of the American Mathematical Society, volume 365, no. 12:pp. 6625–6667 (2013).

[MR01] John C McConnell, James C Robson, *Noncommutative noetherian rings*, number 30 in Graduate Studies in Mathematics, American Mathematical Society, revised edition (2001).

[MT11] Gunter Malle, Donna Testerman, *Linear Algebraic Groups and Finite Groups of Lie Type*, number 133 in Cambridge Studies in Advanced Mathematics, Cambridge University Press (2011).

[Oll09] Rachel Ollivier, *Le foncteur des invariants sous l'action du pro-p-Iwahori de* $GL_2(F)$, Journal für die reine und angewandte Mathematik, volume 635:pp. 149–185 (2009).

[Paš10] Vytautas Paškūnas, *Extensions for supersingular representations of* $GL_2(\mathbb{Q}_p)$, Asterisque, volume 331:pp. 317–353 (2010).

[Rot09] Joseph J Rotman, *An introduction to homological algebra*, Springer, second edition (2009).

[Sch02] Peter Schneider, *Nonarchimedean Functional Analysis*, Springer Monographs in Mathematics, Springer (2002).

[Sch11] Peter Schneider, *p-Adic Lie Groups*, number 344 in Grundlehren der mathematischen Wissenschaften, Springer (2011).

[Sch15] Benjamin Schraen, *Sur la présentation des représentations supersingulières de* $GL_2(F)$, Journal für die Reine und Angewandte Mathematik, volume 704:pp. 187–208 (2015).

[Ser77] Jean-Pierre Serre, *Linear Representations of Finite Groups*, number 42 in Graduate Texts in Mathematics, Springer-Verlag, translated from the second French edition by Leonard L. Scott (1977).

[Ser80] Jean-Pierre Serre, *Trees*, Springer-Verlag, translated from the French by John Stillwell (1980).

[Sho20] Jack Shotton, *The category of finitely presented smooth mod p representations of* $GL_2(F)$, Documenta Mathematica, volume 25:pp. 143–157 (2020).

[Tim] James Timmins, *The canonical dimension of modules for Iwasawa algebras*, arXiv preprint, arXiv:2306.09696.

[Tim23] James Timmins, *Coherence of augmented Iwasawa algebras*, Advances in Mathematics, volume 417, no. 108916 (2023).

[Ven02] Otmar Venjakob, *On the structure theory of the Iwasawa algebra of a p-adic Lie group*, Journal of the European Mathematical Society, volume 4, no. 3:pp. 271–311 (2002).

[Vig] Marie-France Vignéras, *Representations of p-adic groups over coefficient rings*, arXiv preprint, arXiv:2205.02019.

[Vig11] Marie-France Vignéras, *Le foncteur de Colmez pour $GL(2, F)$*, in *Arithmetic Geometry and Automorphic Forms*, number 19 in Advanced Lectures in Mathematics, International Press, Somerville, MA, pp. 531–557 (2011).

[Wad07] Simon Wadsley, *A Bernstein-type inequality for localizations of Iwasawa algebras of Heisenberg pro-p groups*, Quarterly Journal of Mathematics, volume 58, no. 2:pp. 265–277 (2007).

[Wal72] R Walker, *Local rings and normalizing sets of elements*, Proceedings of the London Mathematical Society, volume 3, no. 1:pp. 27–45 (1972).

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, number 38 in Camridge Studies in Advanced Mathematics, Cambridge University Press (1994).

[Wil98] John S. Wilson, *Profinite Groups*, number 19 in London Mathematical Society Monographs, New Series, Clarendon Press (1998).

[Wu21] Zhixiang Wu, *A note on presentations of supersingular representations of $GL_2(F)$*, Manuscripta Mathematica, volume 165, no. 3:pp. 583–596 (2021).