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THE WELFARE EFFECTS OF THIRD-DEGREE PRICE DISCRIMINATION WITH NON-LINEAR DEMAND FUNCTIONS

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The welfare effects of third-degree price discrimination

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Abstract

The welfare effects of third-degree price discrimination are known to be negative when demand functions are linear, marginal cost is constant and all markets are served. This paper shows that discrimination lowers welfare for a more general class of demand functions. Demand varies across markets with additive and multiplicative shift factors. Total welfare (defined as consumer surplus plus profits) with discrimination is lower that with uniform pricing when the density function of consumer valuations satisfies a weak version of concavity that encompasses logconcavity. Most standard demand functions, including linear, quadratic, probit, logit, exponential and iso-elastic ones, satisfy this assumption, which is also a weak sufficient condition for existence.

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1. Introduction

In 1999 Coca-Cola admitted that it was developing a vending machine that would raise the price of a Coke when the external temperature increased above a certain level. The subsequent negative publicity, however, caused Coca-Cola to abandon any plans it might have had to introduce this type of machine. This paper addresses the general question: what are the welfare effects of allowing a monopolist to practise third-degree price discrimination rather than requiring it to set a uniform price in all markets? This question has a history that goes back to Joan Robinson (1933). A firm practises third-degree price discrimination when it classifies customers into separate markets using observed exogenous characteristics such as the customers' locations or ages (or the external temperature at the time of purchase) and sets different prices in these markets. When price discrimination is allowed the firm earns larger profits and consumers gain or lose individually depending on whether the discriminatory price in their own market is below or above the uniform price. It is usually argued that the net effect of allowing discrimination on total welfare, defined as aggregate consumer surplus plus profits, can go either way. In one case the sign of the effect is known. If all demand functions are linear, marginal cost is constant and all markets are served with uniform pricing then total welfare is lower with price discrimination than with uniform pricing. This paper presents a more general result. Suppose that demand in a market is $Q = a + bq(P)$ where $a \geq 0$ and $b > 0$, $q(P)$ is the underlying demand function and P denotes the price. Linear demand is a special case. The parameters a and b vary across markets, so the firm wants to discriminate, but the function $q(P)$ is the same. The main result is that for a defined class of underlying demand functions, including all the ones that are typically used in theoretical and econometric models of imperfect competition, discrimination is worse for social welfare than uniform pricing as long as all markets are served with uniform pricing.

A natural interpretation of the demand structure is that a represents the number of committed customers in the market who buy whatever the price, while b is the number of price-sensitive customers. Alternatively a and b may be taste parameters that shift demand, for example the external temperature affects the demands for soft drinks and for heating. David Friedman (1987) presents a model of the demand for heating where the external temperature enters additively in the direct demand

function. Similarly it is natural to model the demand for chilled carbonated drinks with an additive shift factor that represents the external temperature.

The outline of the argument is as follows. First, the discriminatory monopoly price is characterized by a function $P^*(a, b)$. As the number of committed consumers, a, increases, the discriminatory price rises. Second, because profits are linear in a and b the optimal uniform price is $P^*(E[a], E[b])$ where $E[.]$ is the expectations operator. Third, total welfare is defined in the standard way as consumer surplus plus producer surplus. Expected social welfare with uniform pricing equals welfare with price discrimination when $a = E[a]$ and $b = E[b]$. Jensen's inequality then implies that total welfare with uniform pricing is above that with discrimination if the discriminatory welfare function is strictly concave in a and b. Finally, an assumption about the distribution of tastes of the price-sensitive consumers is shown to be sufficient for global concavity of the welfare function and thus for discriminatory welfare to be lower.

The distributional assumption imposes a weak form of concavity on the density of consumer valuations and is a version of that used by Andrew Caplin and Barry Nalebuff (1991a) to prove existence of equilibrium for a differentiated-products oligopoly. All logconcave densities are covered. Mark Bagnoli and Ted Bergstrom (2004) show that many common distributions are logconcave, such as the uniform, normal, exponential, Gumbel and logistic. In addition, following Caplin and Nalebuff (1991a), the class of densities that meets the distributional assumption is wider still and includes those that are not logconcave such as the F, t and Pareto ones (with the latter generating iso-elastic demand). Intuitively the assumption means that the price-sensitive consumers are not too concentrated in both tails of the distribution. Logconcavity itself implies that the density function is unimodal.

The main focus of welfare analysis in the literature has been on the effect of price discrimination on total output. Price discrimination has the undesirable effect of ensuring that marginal utilities differ between consumers and thus output is distributed inefficiently, but this negative effect may be offset if total output is higher with discrimination. Hal Varian (1985), building on the analysis of Richard Schmalensee (1981), shows that a necessary condition for discrimination to raise welfare above the uniform-pricing level is that total output increases. Marius Schwartz (1990) extended Varian's condition to cover decreasing marginal costs. Robinson (1933) and Arthur Pigou (1929) proved that when demands are linear,

marginal cost is constant and all markets are served total output is constant and thus total welfare with price discrimination is lower than with uniform pricing.

It has proved difficult to say much about the effect of discrimination on total output beyond the linear case. The most general analysis is by Jun-ji Shih, Chao-cheng Mai and Jung-chao Liu (1988), who use the mean-value theorem to characterize the difference in output in the two cases and show that the short-cuts used by Robinson (1933) and Edgar Edwards (1950) to sign the output effect do not always work. To some extent the concentration on the output effect has proved to be unproductive except for the special case of linear demand functions. The output test is bypassed in this paper because the structure of the demand functions allows a direct welfare comparison. I do, though, discuss the output test briefly and show that the welfare result does not depend on output with discrimination being no higher than with uniform pricing. It turns out that Joan Robinson's output test works for a subset of the demand functions in the class considered here. When Robinson's test is valid total output is higher (lower) with discrimination if demand is concave (convex). Linear demand is both concave and convex and thus total output is constant.

I concentrate here on the case where price discrimination does not open up new markets. If instead price discrimination opens up new markets then the welfare effects are likely to be positive, and indeed weak Pareto improvements can be achieved (Jerry Hausman and Jeff Mackie-Mason, 1988). An assumption is made that rules out the possibility of discrimination opening up new markets. The model covers only the standard case of a pure monopolist. In practice price discrimination is also common in industries with competition, such as airlines, and Coca-Cola itself is subject to significant competitive pressure. See Mark Armstrong and John Vickers (2001) for an analysis of competitive price discrimination.

The structure of the paper is as follows. Section 2 contains an example with linear demand functions that illustrates the argument. Section 3 covers logconcavity and the more general concept of ρ-concavity. Section 4 contains the positive analysis of the model of price discrimination and uniform pricing. The welfare analysis is in Section 5 and Section 6 discusses the total output test briefly. Section 7 concludes.

2. A linear demand example

An example with linear demand functions is presented here. It is already known that the welfare effects of discrimination are negative when demand functions are linear and no new markets are opened up. The purpose of the example is to illustrate the features of the more general argument that follows. Indirect utility is $V = -AP + P^2/2$ (plus exogenous income) where P is the price and A is a positive taste parameter whose variation across markets provides a motive for price discrimination. The demand function derived from the utility function using Roy's Identity is $Q = A - P$. The firm has a constant marginal cost that is normalized to zero and profits are $P(A - P)$. With discrimination the price that maximizes profits for a particular value of A is $P^* = A/2$. Since A is assumed to vary prices differ across markets. When the firm is required to set a uniform price, P_u, its problem is the same as choosing P_u to maximize expected profits $P_u\left(E[A] - P_u\right)$ where expectations are taken over A. The solution is $P_u = E[A]/2$, which is the same as the optimal discriminatory price when $A = E[A]$, i.e. $P_u = P^*(E[A])$. Every market is served with uniform pricing as long as the lowest value of A exceeds $E[A]/2$.

Social welfare is the sum of utility and profits, so $W = V + PQ = -P^2/2$. Welfare with the uniform price is $W_U = -(E[A])^2/8$. With discrimination welfare for a particular value of A is $W_D = -A^2/8$, a concave function of A. The difference between expected welfare with discrimination and with uniform pricing is $E[W_D] - E[W_U] = -\left\{E\left[A^2\right] - \left(E[A]\right)^2\right\}/8$, which is negative because the term in curly brackets is the variance of A. The important feature of the welfare comparison is that it is the difference between the expected value of a concave function and the function of the expected value, which is negative. The rest of the paper shows that this result applies much more widely and in particular there is no need to restrict the underlying demand function to be linear. For a large set of underlying demand functions welfare with discrimination is a concave function of the shift factors and this implies that discriminatory welfare is lower than welfare with uniform pricing. The next section introduces the concepts that enable concavity of the welfare function to be shown more generally.

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3. Logconcavity and ρ-concavity

A function, $f(x)$, on a convex domain that maps into the positive real numbers is logconcave (logconvex) if and only if $\ln f(x)$ is concave (convex). If $f(x)$ is twice differentiable, as is assumed, then logconcavity holds if and only if $ff''/[f']^2 \leq 1$ for $f' \neq 0$. A function that is concave ($f'' \leq 0$) is thus also logconcave. The relationship between logconvexity and convexity goes the other way: a logconvex function is convex since $ff''/[f']^2 \geq 1$ implies $f'' > 0$, i.e. strict convexity. Mark Yuying An (1998) discusses logconcavity and logconvexity.

The concept of ρ-concavity may be used to parameterize the degree of concavity and nests logconcavity, standard concavity and quasi-concavity. See Simon Anderson and Régis Renault (2003) for an analysis of surplus bounds for Cournot competition that uses ρ-concavity extensively. The function $f(x)$ is ρ-concave for $\rho \neq 0$ if f^{ρ}/ρ is concave, while $\rho = 0$ corresponds to (exact) logconcavity. The standard definition of concavity has $\rho = 1$. Logconvexity holds if $\rho \leq 0$ and $\rho = -\infty$ implies that the function is quasi-concave. The function $f(x)$ is interpreted as a probability density function. With a uniform distribution $f(x)$ is a constant and $\rho = \infty$. If a function is ρ-concave for a particular value ρ^* then it is also ρ-concave for all $\rho < \rho^*$. For example a logconcave function ($\rho \geq 0$) is also ρ-concave for all $\rho < 0$.

Suppose that $f(x)$ is strictly positive on its domain and $f' \neq 0$. The second derivative of f^{ρ}/ρ is $(\rho - 1)f^{\rho-2}[f']^2 + f^{\rho-1}f'' \leq 0$. This implies an upper bound for the degree of ρ-concavity:

$$(1) \qquad\qquad \rho \leq 1 - \frac{ff''}{[f']^2}.$$

For many densities the ρ-value equals the upper bound. For example the exponential density $\gamma e^{-\gamma x}$ for $\gamma > 0$ is just logconcave and thus $\rho = 0$, which equals the upper bound. Examples of distributions with differing degrees of ρ-concavity are presented in the next section.

A useful feature of logconcave densities is that their integrals inherit similar properties. If $f(x)$ is a logconcave density and $F(x)$ is its distribution function then both

F(*x*) and 1 – *F*(*x*) are logconcave (see Bagnoli and Bergstrom, 2004, Theorems 1 and 3). Furthermore the integrals of ρ-concave densities can inherit ρ-concavity properties. Caplin and Nalebuff (1991a) use the Prékopa-Borell theorem, an aggregation theorem for ρ-concave densities, and their approach is used here.

4. The model of price discrimination and uniform pricing

4.1 Utility and demand

For convenience it is assumed that consumers have unit demands, though the results apply more generally. There are two types of consumer in each market. There are *a* consumers who value the good at $\bar{v} > c$ where *c* is the constant marginal cost. Each of these consumers buys one unit as long as the price, *P*, is not above $\bar{v}$. In equilibrium the price always satisfies this condition so these are called the committed consumers. The second group of consumers is price-sensitive and their valuations are bounded above by $\bar{v}$. A consumer in this class has a valuation of *v* and has indirect utility of *v* – *P* if one unit is bought and 0 if there is no purchase. The valuation *v* is a continuous random variable with distribution function *F*(*v*) and density *f*(*v*) on finite support $[\underline{v}, \bar{v}]$ where $\underline{v} \geq 0$. All those with $v \geq P$ buy. There are *b* price-sensitive consumers and their total demand is *b*[1 – *F*(*P*)]. Define $q(P) \equiv 1 - F(P)$ as the underlying demand function. Total demand in the given market is $Q = a + bq(P)$. The slope of underlying demand, defined as a positive number, is $-q' = f(P)$. A possible interpretation of the maximum valuation, $\bar{v}$, is that it is the price of a competitively produced alternative product.

The aggregate surplus of the committed consumers is $a[\bar{v} - P]$. Without loss of generality the term $a\bar{v}$ may be dropped since the committed consumers will always purchase. Aggregate indirect utility of all consumers is then

$$(2) \qquad S = -aP + b\int_P^{\bar{v}} (v - P) f(v) dv = -aP + bs(P),$$

where *s*(*P*) is the integral and $s'(P) = -[1 - F(P)] = -q(P)$. *S* satisfies Roy's identity since $S'(P) = -a - bq(P) = -Q$ and the marginal utility of income is implicitly unity.

Identical results hold when individual consumers have continuous demand functions. The utility function is quasi-linear and is $U(Q, x) = bu((Q - a)/b) + x =$

$bu(q) + x$ where x is the consumption of other goods and the sub-utility function $u(q)$ is concave and four-times differentiable. In this interpretation a and b are variables such as the external temperature that shift preferences.

4.2 Demand curvature and concavity measures

The shape of demand is important in the analysis of price discrimination (see David Malueg, 1993, for example). Two measures of demand curvature are used. Both apply as long as the price is below $\bar{v}$. The first is the curvature of direct demand or, equivalently, the elasticity of the slope of direct demand

$$(3) \qquad \alpha \equiv \frac{-PQ''}{Q'} = \frac{-Pq''(P)}{q'(P)} = \frac{-Pf'(P)}{f(P)}$$

which uses $Q' = bq'$ and $-q' = f$. This measure is analogous to relative risk aversion. Because demand slopes down α has the same sign as q'' so it is positive when demand is strictly convex. A feature of α is that neither shift factor directly affects it. Instead a and b affect α indirectly through their impact on the price and the effect (if any) of the price on α.

The second measure of curvature relates to inverse demand. The inverse demand function is written generally as $P(Q)$, and its convexity is measured by $R \equiv - QP''/P'$. R is also the elasticity of the slope of inverse demand. Since $P(Q)$ is an inverse function the following relationships hold: $P'(Q) = 1/Q'(P)$, $P''(Q) = - Q''/[Q']^3$ and hence $R = QQ''/[Q']^2$. The price elasticity of demand is defined as $\eta \equiv - PQ'/Q$ and thus $R = \alpha/\eta$. For the demand function used here $R = \{qq''/[q']^2\}(Q/bq)$. The sufficient second-order condition for the pricing problem turns out to be $R < 2$. Because $Q/bq \geq 1$, with equality when $a = 0$, a necessary condition for the second-order condition to hold is that $qq''/[q']^2 < 2$.

Assumption A1. *The probability density of the valuations of the price-sensitive consumers, $f(v)$, is ρ-concave for $\rho > - 0.5$.*

A1 is a version of Caplin and Nalebuff's Assumption A2 (1991a, p 30) that is used to prove existence of a price-setting equilibrium with firms selling differentiated

products. Because the model here is a monopoly one the inequality in A1 is strict (Caplin and Nalebuff have a weak inequality). All logconcave distributions, for which $\rho \geq 0$, are included in A1. Caplin and Nalebuff (1991a, b) and Bagnoli and Bergstrom (2004) discuss logconcave distributions. The logconcave class (with associated demand functions in brackets where the names differ) includes: the uniform distribution (linear demand), normal (probit), exponential, logistic (logit), and Gumbel or Type I extreme value (used for the multinomial logit model). The power function distribution function is $F(v) = v^\gamma$ and its density, $f(v) = \gamma v^{\gamma-1}$, is logconcave for $\gamma \geq 1$. The Weibull density, $f(v) = \gamma v^{\gamma-1} \exp\{-v^\gamma\}$, is logconcave for $\gamma \geq 1$.

A1 includes distributions that are moderately logconvex, i.e. which satisfy $0 > \rho > -0.5$. Caplin and Nalebuff (1991b) discuss such distributions and provide the ρ values. Both the t and F densities are in this class (subject to restrictions on the degrees of freedom parameters). For the t distribution $\rho = -1/(1 + m)$ where m is the degrees of freedom parameter. Thus A1 holds when $m \geq 2$. For the F distribution $\rho = -1/(1 + m_2/2)$ where m_2 is the second degrees of freedom parameter. This means that the F distribution satisfies A1 when $m_2 > 2$, which is required for the mean to exist, since $E[v] = m_2/(m_2 - 2)$. For demand analysis an important distribution is the Pareto, which has distribution function $F(v) = 1 - v^{-\gamma}$ for $\gamma > 0$ and $v \geq 1$. This generates an iso-elastic underlying demand function with the price elasticity of $q(P)$, i.e. $-Pq'/q$, equal to γ. The density, $f(v) = \gamma v^{-(1+\gamma)}$, is $-1/(\gamma+1)$-concave, and thus A1 is satisfied for $\gamma > 1$. In other words when the price elasticity of underlying demand exceeds unity A1 holds. This is the condition necessary for existence of an interior monopoly solution. With this distribution demand is convex and $\alpha = 1 + \gamma$.

Some distributions, such as the normal and t, allow for negative values of v. Fortunately Caplin and Nalebuff (1991b, p 31) state that "a truncation of the density causes no additional difficulties. A2[1] includes all truncations of the above distributions provided only that the support set is convex." It should be remarked that although A1 covers a very large number of distributions it does not apply to all: a density that does not satisfy A1 is the lognormal. Similarly A1 does not include the power function density when demand is strictly convex. The power function density is $1/(\gamma - 1)$-concave. When $\gamma < 1$ demand is strictly convex, with $\alpha = 1 - \gamma > 0$, and the

value of ρ is below -1. Note that for two distributions, the power function and the Pareto, the value of α is constant.[1]

A1 is used to prove the existence and uniqueness of an interior monopoly pricing solution. Existence and uniqueness hold if the profit function is quasi-concave. When $a = 0$ (there are no committed consumers) a sufficient condition for quasi-concavity of the profit function is that underlying demand, $q = 1 - F$, is strictly -1-concave. Theorem 1 from Caplin and Nalebuff (1991a) can be used here. This is an application of the Prékopa-Borell Theorem on the inheritance properties of integrals of ρ-concave densities. The theorem states that if the density is ρ-concave, $\rho \geq -1$, then the integral

$$\int_{\{v \geq P\}} f(v)dv = 1 - F(P) \equiv q(P)$$

is $\rho/(1 + \rho)$-concave. To illustrate suppose that v has a Pareto distribution with density $f(v) = \gamma v^{-(1+\gamma)}$. The density is $-1/(\gamma+1)$-concave and underlying demand, $P^{-\gamma}$, is $-1/\gamma$-concave, where $-1/\gamma = \rho/(1 + \rho)$. The theorem can be applied immediately.

Proposition 1. *When A1 holds the underlying demand function in each market is strictly -1-concave.*

Proof. By Theorem 1 of Caplin and Nalebuff (1991a) underlying demand is $\rho/(1 + \rho)$-concave. Suppose that $\rho/(1 + \rho) \leq -1$ and thus $(2\rho + 1)/(1 + \rho) \leq 0$. A1 implies that both the numerator and the denominator of the left-hand side of this expression are positive so there is a contradiction. Q.E.D.

An implication of Proposition 1 is that $qq''/[q']^2 < 2$. The upper bound on the degree of ρ-concavity of q is found by applying (1) and is $\rho/(1+\rho) \leq 1 - qq''/[q']^2$. Since $\rho/(1 + \rho) > -1$ by Proposition 1 the result follows immediately. Thus the necessary condition for the second-order condition holds. When $a = 0$ existence and uniqueness of an interior solution, i.e. a price that satisfies $\bar{v} > P > c$, are guaranteed, in other

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[1] These two distributions are related. If x has a Pareto distribution then $y \equiv 1/x$ has a power function distribution.

words A1 is also sufficient. For positive *a*, however, existence of an interior solution is more problematic. Since the focus of this paper is on welfare analysis when all markets are served a further assumption is made that ensures that all pricing problems have interior solutions.

With a logconcave density the underlying demand function may be convex or concave. The uniform distribution generates linear demand, which is both convex and concave. The exponential demand curve is convex. The power function density with $\gamma \geq 1$ gives a concave demand function with $\alpha = 1 - \gamma \leq 0$. When the density is logconvex, however, an additional consequence of Caplin and Nalebuff's Theorem 1 is the following lemma.

Lemma. *If the density is ρ-concave, $0 \geq \rho > -1$, then both underlying demand and the demand function are strictly convex.*

Proof. By Caplin and Nalebuff's Theorem 1, underlying demand is $\rho/(1 + \rho)$-concave. Since $\rho/(1 + \rho) \leq 0$ $q(P)$ is logconvex and thus is also strictly convex. Since Q is a positive affine transformation of q it is also strictly convex. Q.E.D.

Example 1. The exponential density, $f(v) = \gamma e^{-\gamma v}$ (for $\gamma > 0$), is logconvex as well as logconcave. The underlying demand function is $1 - F(P) = e^{-\gamma P}$ which is strictly convex with $\alpha = \gamma P > 0$.

Example 2. The Pareto density $f(v) = \gamma v^{-(1+\gamma)}$ is $-1/(1 + \gamma)$-concave. The underlying demand function is $q = P^{-\gamma}$ which is iso-elastic and convex with $\alpha = 1 + \gamma > 0$.

The lemma plays an important role in the theorem on welfare. In particular it rules out the possibility that ρ is negative and demand is concave.

4.3 Monopoly pricing with discrimination
How does the monopolist set its price? The monopolist faces a continuum of markets characterized by different values of *a* and *b*. The shift parameters are random variables with a joint distribution and expected values $E[a]$ and $E[b]$. The support is

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the rectangle $[0,\bar{a}] \times [\underline{b},\bar{b}]$, with a varying between zero and an upper bound of $\bar{a}$ while $\underline{b} > 0$. In order to provide a reason for price discrimination a takes a strictly positive value in at least one market and is not proportional to b. If a were proportional to b, or $a = 0$ everywhere, then the price elasticity of demand would be constant at a given price and there would be no motive for discrimination. An important point in the support of the distribution is the south-east corner of the rectangle $(\bar{a}, \underline{b})$, which is the point with the highest value of the ratio of committed to price-sensitive consumers.

The discriminating monopolist chooses the price for each pair (a, b), i.e. for each market, to maximize profits $\Pi = (P - c)[a + bq(P)]$ subject to the constraint that $P \leq \bar{v}$. This is a standard third-degree price discrimination problem. The firm can check that a consumer belongs to a particular market but cannot discriminate further. It cannot tell whether any particular consumer is committed or not – if it could then it would extract all the surpluses of committed customers by charging them a price equal to $\bar{v}$ and would set a (lower) price for the price-sensitive consumers that would be the same in all markets. Marginal cost, c, is assumed to be strictly positive but the main results also hold for zero marginal cost.

Demand is $Q = a + bq(P)$ and technically at $P = \bar{v}$ this relationship is a correspondence (though it is a function for all other feasible prices). Assume initially that all pricing solutions are interior and unique. An assumption is made subsequently that justifies this. The first-order condition for the monopoly pricing problem is

(4) $\qquad Q(P^*) + (P^* - c)Q'(P^*) = a + bq(P^*) + (P^* - c)bq'(P^*) = 0,$

which implicitly defines the optimal price, P^*, as a function of a and b written as $P^*(a, b)$. The second-order condition for the pricing problem is

$$2bq'(P^*) + (P^* - c)bq''(P^*) < 0.$$

Using the first-order condition and the definition of α in (3) the second-order condition can be written as $R < 2$ or $2\eta - \alpha > 0$.

The assumption that ensures that attention can be focussed on interior solutions is:

Assumption A2. *The profit-maximizing price when the ratio of committed to price-sensitive customers is at its highest possible level is in the interior of the set of feasible prices, i.e.* $\bar{v} > P^*(\bar{a}, \underline{b}) > c$.

Before examining the role of A2 it helps to return to A1. A1 ensures that when $a = 0$ a unique solution to the first-order condition (4) exists. With A1 the function $\pi \equiv (P - c)q(P)$, which is profits per price-sensitive customer, is quasi-concave. This is Caplin and Nalebuff's existence result. In the neighborhood of the optimal price, $P^*(0, b)$, marginal profitability (per price-sensitive customer), $\pi_P = q + (P - c)q'$, is downward sloping so the second-order condition automatically holds. Figure 1 illustrates. From the first-order condition (4) the optimal price when $a > 0$ is found by setting $a/b + \pi_P = 0$. With A2 the optimal price when the ratio of committed to price-sensitive customers is at its maximum, $P^*(\bar{a}, \underline{b})$, is shown in Figure 1. For all other possible values of a/b the price is in between $P^*(0, b)$ and $P^*(\bar{a}, \underline{b})$. A final point to note is that A2 guarantees that for all (a, b) pairs the second-order condition holds, i.e. $R < 2$.

Figure 1

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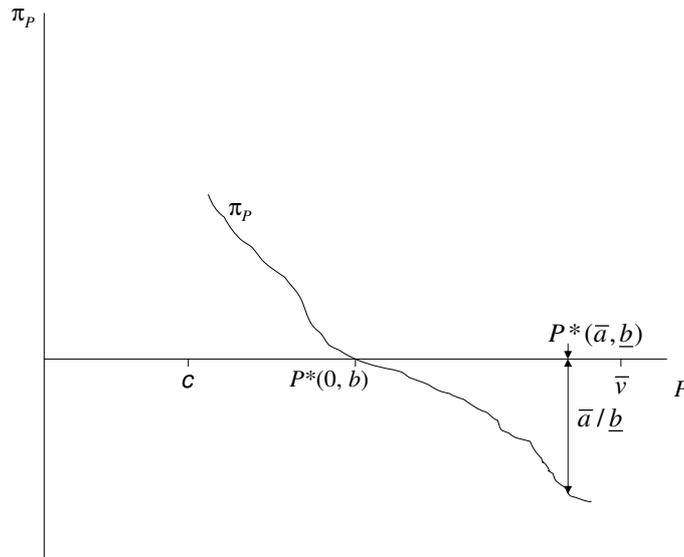
Without A2 a large enough ratio of committed to price-sensitive customers would induce the firm to exclude the price-sensitive customers and set $p = \bar{v}$. The validity of A2 can be checked. Two examples follow. First, let $q = 1 - P$ and thus $\bar{v} = 1$. Ignoring the upper bound on P the profit-maximizing price is $P^* = ((a/b) + 1 + c)/2$. A2 applies if $\bar{a}/\underline{b} < 1 - c$. Second, let underlying demand be iso-elastic, i.e. $q = P^{-\gamma}$ for $\gamma > 1$. The profit-maximizing price for $a = 0$ is $P^*(0, b) = \gamma c/(\gamma - 1)$. For this demand function π_P is decreasing when $P < (1 + \gamma)c/(\gamma - 1) \equiv \hat{P}$ but rises with P thereafter. A necessary condition for A2 is that $\bar{v} < \hat{P}$. Sufficiency can be checked numerically. Suppose that $\gamma = 2$, $c = 0.1$ and $\bar{a}/\underline{b} = 3.2$. With these coefficients $\hat{P} = 0.3$ and $P^*(\bar{a}, \underline{b}) = 0.25$, so with $0.25 < \bar{v} < 0.3$ A2 holds.

The price elasticity of demand is $\eta \equiv -PQ'/Q$ and equation (4) can be written in the familiar mark-up or Lerner index form $(P^* - c)/P^* = 1/\eta$, so $P^* = \eta c/(\eta - 1)$. Simple monopoly theory implies that demand must be elastic ($\eta > 1$) at the chosen price when marginal cost is positive. If this were not the case then a rise in price would reduce costs and raise revenue and the initial price would not maximize profits. Both the price-cost mark-up and the price are inversely related to the price elasticity of demand so any shift in demand that reduces the price elasticity of demand will cause the firm to increase the price.

Suppose that a and b are both multiplied by the same positive factor. By inspection of the first-order condition (4) there is no effect on the optimal price, so $P^*(a, b)$ is homogeneous of degree zero. This is just the well-known result that a multiplicative shift of the whole demand function does not alter the price elasticity of demand and thus does not affect the optimal monopoly price (given constant marginal cost). As has already been seen in Figure 1 the price depends on the ratio a/b.

More can now be said about the feasible values of ρ. Suppose that the distribution is Pareto with the elasticity of underlying demand, γ, satisfying $0 < \gamma \leq 1$. Because demand is inelastic (unit elastic for $\gamma = 1$) there is no interior monopoly solution. The ρ-value for the density, $-1/(1 + \gamma)$, lies in the range $[-0.5, -1)$. Note, however, that while A1 restricts ρ to exceed -0.5, and non-existence means cases where $\rho \in [-0.5, -1)$ can be ruled out, it is feasible for ρ to be strictly below -1. An example with the power function distribution is provided in Section 4.

4.4 Comparative statics and uniform pricing

The effect of an additive shift, or an increase in the number of committed customers, on the price is

(5)
$$\frac{\partial P}{\partial a} \equiv P_a = \frac{-1}{2bq' + (P - c)bq''} = \frac{P}{Q(2\eta - \alpha)},$$

which is strictly positive because the second-order condition holds. The subscript denotes a partial derivative.[2] It is intuitive that P_a is positive. At a given price an increase in a raises the quantity demanded while preserving the demand slope. Thus the price elasticity of demand falls and the firm wants to increase the mark-up and the price. The second version of (5) follows from the definitions of η and α, and the Lerner index, and is useful in the welfare analysis.

The price elasticity of demand in general depends on the price and the shift parameters,

$$\eta^*(P, a, b) = \frac{-PQ'}{Q} = \frac{-Pbq'(P)}{a + bq(P)}.$$

The elasticity may be written as a function of a and b alone, i.e. $\eta(a, b) \equiv \eta^*(P(a, b), a, b)$, and this function is also homogeneous of degree zero. To find the total effect on the price elasticity of a change in a, allowing for the fact that the price itself alters, differentiate $P(a, b) = \eta(a, b)c/(\eta(a, b) - 1)$ with respect to a to give

(6)
$$\frac{\partial \eta(a, b)}{\partial a} \equiv \eta_a = -\frac{(\eta - 1)^2 P_a}{c} < 0.$$

This expression is used in the welfare analysis.[3]

The simplest way to characterize the effect of the multiplicative shift factor, b, on the monopoly price is to exploit homogeneity. Since $P(a, b)$ is homogeneous of degree zero Euler's Theorem implies $P_b = -aP_a/b$. When a is strictly positive and

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[2] To save on notation from now on the star superscripts will usually be dropped.
[3] Note that equation (6) holds as long as c is strictly positive. If marginal cost is zero the price elasticity is always 1 as at this point revenue is maximized.

fixed a rise in b has a negative effect on the monopoly price because the price elasticity of demand increases. When $a = 0$, its lowest value, a rise in b has no effect on the elasticity because the whole demand curve is shifted multiplicatively.

The analysis of the firm's decision when it is required to set a uniform price, P_u, across all markets is now straightforward. The problem is to choose P_u to maximize expected (or total) profits $E[\Pi] = (P_u - c)\{E[a] + E[b]q(P_u)\}$. The first-order condition is

$$E[a] + E[b]q(P_u) + (P_u - c)E[b]q'(P_u) = 0.$$

Comparing this with (4) it can be seen that the optimal price is $P_u = P^*(E[a], E[b])$. To find the profit-maximizing uniform price the firm simply inserts the expected values of the shift parameters into the function for the discriminatory price. This follows from the linearity of the demand function in a and b and the constancy of marginal cost. This fact is important for the welfare analysis. A final point to note is that $P_u < P^*(\bar{a}, \underline{b})$. This is because the optimal uniform price is the discriminatory price evaluated at the center of gravity of the distribution of a and b. This inequality is strict as long as there is some variation in a or b. Thus given A2 with uniform pricing all markets are served: the same positive fraction of price-sensitive consumers buys the good in each market.

5. Social welfare and price discrimination

Social welfare in a given market is defined as the sum of consumer and producer surplus, and with discrimination the welfare function is

(7) $W(a,b) = -aP + bs(P) + (P-c)\{a + bq(P)\} = -ac + b\{s(P) + (P-c)q(P)\}$,

where the second version of equation (2) is used to represent consumer surplus and the arguments of $P(a, b)$ have been omitted. Again homogeneity properties can be exploited. Suppose that a and b increase proportionally so the price stays the same. Welfare also rises proportionally and thus $W(a, b)$ is homogeneous of degree 1. It follows that the partial derivatives, W_a and W_b, are homogeneous of degree 0. Euler's Theorem applied to the partial derivatives implies that $W_{bb} = (a/b)^2 W_{aa}$ which is either

0 (at $a = 0$) or has the same sign as W_{aa}. Attention can be focussed on the sign of W_{aa} with the sign of W_{bb} being the same when $a > 0$. Total welfare with discrimination is represented by the expected value of $W(a, b)$ with expectations taken over a and b, i.e. $E[W(a, b)]$.

Welfare in a particular market with uniform pricing is

$$(8) \qquad W(a,b,P_u) = -ac + b\{s(P_u) + (P_u - c)q(P_u)\}$$

where $P_u = P*(E[a], E[b])$. $W(a,b,P_u)$ is linear in a and b. Taking expectations gives total welfare with uniform pricing

$$(9) \qquad E[W(a,b,P_u)] = -E[a]c + E[b]\{s(P_u) + (P_u - c)q(P_u)\}.$$

Consider now welfare under discrimination when a and b equal their expected values, i.e. substitute $E[a]$ and $E[b]$ for a and b in (7) to give $W(E[a], E[b])$. This is the same as expected welfare with uniform pricing given in (9) because the price is the same. The difference between expected welfare with discrimination and with uniform pricing is then

$$(10) \qquad E[W(a,b)] - E[W(a,b,P_U)] = E[W(a,b)] - W(E[a], E[b]).$$

The second version of equation (10) is the key: the welfare difference is the difference between the expected value of the function and the function of the expected values. Jensen's inequality may be used to sign this. A concave function lies below its tangents, so a sufficient condition for the welfare difference in equation (10) to be negative is that $W(a, b)$ is strictly concave. To check concavity all that must be found is the sign of W_{aa} given the homogeneity of degree one of $W(a, b)$.

The effect of an increase in a on welfare is

$$(11) \qquad W_a = -c - QP_a = -c - \frac{P}{2\eta - \alpha}.$$

The first version of (11) is derived by differentiating (7) and using the fact that $s'(P) = -q$ and the first-order condition (4). To interpret the first version of (11) note that an infinitesimal increase in a raises profits by $P - c$ (the indirect effect on profits caused by the induced price change is zero by the envelope theorem) and costs the new consumers P and the existing consumers their demand, Q, multiplied by the increase in price, P_a. The second version of (11) uses the comparative statics expression (5) to substitute for P_a.

Differentiating again while noting that $\alpha = \alpha(P(a, b))$ and using equation (6) for η_a gives

$$(12) \qquad W_{aa} = \frac{-P_a}{(2\eta - \alpha)^2}\left\{2\eta^2 + P\alpha_P - \alpha\right\}.$$

If the expression in curly brackets is positive everywhere then W_{aa} is negative, $W(a, b)$ is concave (by homogeneity) and total welfare is higher with uniform pricing than with discriminatory pricing.

Theorem. *Total welfare with price discrimination is below total welfare with uniform pricing if A1 and A2 hold.*

Proof. From (12) it is sufficient to show that $2\eta^2 + P\alpha_P - \alpha > 0$. Differentiating $\alpha \equiv - Pf'/f$ with respect to P gives $P\alpha_P - \alpha = [1 - ff''/(f')^2]\alpha^2 \geq \rho\alpha^2$ by (1). Divide $2\eta^2 + [1 - ff''/(f')^2]\alpha^2$ by η^2, so $2 + [1 - ff''/(f')^2]R^2 \geq 2 + \rho R^2$. First, consider logconcave densities with $\rho \geq 0$. It is immediate that $2 + \rho R^2 > 0$. Second, let $0 > \rho > - 0.5$. By the Lemma demand is strictly convex so $R > 0$. Since all pricing solutions are interior $R < 2$, so $R^2 < 4$. Thus $\rho R^2 > - 2$ and $2 + \rho R^2 > 0$ for negative ρ as well. Q.E.D.

The theorem requires $2 + \rho R^2$ to be signed. With logconcavity $\rho \geq 0$ so the expression is automatically positive. When ρ is negative, though, it might be possible for R to be positive and large enough that $2 + \rho R^2$ becomes negative. The Lemma, however, rules this out: when ρ is negative demand is strictly convex. Since R is bounded above by 2, $R^2 < 4$ and because A1 requires $\rho > - 0.5$ the result follows.

The intuition for the theorem is that when consumer valuations are not too concentrated in the tails, defined by A1, and the differences between markets are not too great that allowing price discrimination would cause new markets to be opened, social welfare is best served when the firm does not take account of the differences between markets. A surprising feature of the theorem is that global concavity of the welfare function can be proved under reasonable and quite general conditions.

The sign of $2\eta^2 + P\alpha_P - \alpha$ can be checked by more direct means. The following proposition provides sufficient conditions that are closely related to the more fundamental conditions of the theorem, but not identical.

Proposition 2. *Sufficient conditions for welfare to be lower with discrimination are:* (i) $P\alpha_P - \alpha \geq 0$; *or* (ii) $\alpha_P \geq 0$, *provided these properties hold everywhere and there are interior solutions in all cases.*

Proof. (i) Since $2\eta^2 > 0$ and $P\alpha_P - \alpha \geq 0$ the term in curly brackets in (12) is certainly positive. (ii) Since the second-order condition holds $2\eta - \alpha > 0$. Since $\eta \geq 1$ it follows that $2\eta^2 - \alpha \geq 2\eta - \alpha > 0$. If $\alpha_P \geq 0$ then $2\eta^2 - \alpha + P\alpha_P > 0$. Q.E.D.

Note that condition (i) is equivalent to logconcavity of minus one times the slope of underlying demand, i.e. $-q'(P) = f(P)$. The following examples show the value of Proposition 2. In both cases the distributions do *not* satisfy A1, so the Theorem does not apply, but condition (ii) holds so welfare is concave nevertheless.

Example 3. Suppose v has a power function distribution with $F(v) = v^\gamma$. The density is $1/(\gamma - 1)$-concave. This does not satisfy A1 for $0 < \gamma < 1$. Since α is constant condition (ii) applies directly.

Example 4. Let v be lognormally distributed with parameters μ and σ. The density,

$$f(v) = \frac{1}{v\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln(v) - \mu}{\sigma}\right)^2\right\},$$

does not satisfy A1. Differentiation yields $\alpha \equiv -Pf'(P)/f(P) = 1 + (\ln(P) - \mu)/\sigma^2$, so $\alpha_P > 0$.

When consumers have continuous demand functions, rather than unit demands, and the functional form of demand is known, the conditions in Proposition 2 are generally straightforward to check.

6. The output test

One of the points of this paper is that the output test can be bypassed and a direct welfare comparison made. Nevertheless it is interesting to see what happens to output in this model. It is also important to check that the theorem in the previous section is not just an indirect way of saying that total output with uniform pricing is weakly greater than with discrimination. The technique is similar to that used for the welfare comparison. Output in a particular market with discrimination is $Q(a,b) \equiv a + bq(P(a,b))$ and with the uniform price it is $Q_U \equiv a + bq(P_u)$. Expected output with the uniform price is $E[Q_U] = Q(E[a], E[b])$, i.e. output under discrimination when $a = E[a]$ and $b = E[b]$. Jensen's inequality implies that expected output with discrimination is above (below) that with the fixed price when $Q(a, b)$ is convex (concave), with equality when $Q(.)$ is linear. Note that this is not the same as the demand function being a convex or concave function of the price.

If a and b rise in the same proportion then the price is constant and Q rises proportionally, so $Q(a, b)$ is homogeneous of degree one in a and b and the sign of Q_{aa} is sufficient to establish whether $Q(.)$ is convex or concave. Partially differentiating $Q(a, b)$ and using the price derivative (5) gives the total effect of a on output

(13)
$$Q_a(a,b) = \frac{1-R}{2-R}.$$

R can be written as $R(a, b) \equiv R^*(P(a, b), a, b)$, which is homogeneous of degree zero so $R_b = -(a/b)R_a$. Differentiating equation (13) again gives

(14)
$$Q_{aa}(a,b) = \frac{-R_a}{(2-R)^2}.$$

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Proposition 3. *If inverse demand becomes less convex as the price rises – with convexity measured by R – then total output is higher with discrimination.*

Proof. When $R_a < 0$ an increase in a raises P and cuts R and equation (14) implies that $Q_{aa} > 0$. R_b and P_b have the opposite signs to R_a and P_a by homogeneity. A cut in b raises P and reduces R when $R_a < 0$. Q.E.D.

Example 5. Total output is the same with and without discrimination. R is constant in two cases. First, when underlying demand is linear $R = 0$. Second, suppose that the power function distribution applies with $\gamma > 0$ and $c = 0$. $\alpha = 1 - \gamma$ is constant. With zero marginal cost the discriminating firm maximizes revenue in each market and $\eta = 1$ always. Thus $R = \alpha/\eta$ is constant.

Example 6. Total output rises with discrimination. Suppose that valuations have a power function distribution with $\gamma > 1$ and that $c > 0$. With this distribution $\alpha = 1 - \gamma < 0$ and $R = (1 - \gamma)/\eta$. The elasticity falls as a rises by equation (6), so $R_a < 0$.

Example 7. Total output falls with discrimination. Suppose that demand is convex ($\alpha > 0$) and α is constant. There are two sub-cases. First, there is a power function distribution, so $\alpha = 1 - \gamma$, with $\gamma < 1$. Second, valuations have a Pareto distribution with $\gamma > 1$ so $\alpha = 1 + \gamma$. Marginal cost is positive. The elasticity falls as a rises and thus $R_a = -(\alpha/\eta^2)\eta_a > 0$.

Thus output can rise, fall or stay constant, but in all the examples the conditions of the welfare theorem apply. In Example 6 the necessary condition for welfare to rise with discrimination, that total output increases, is satisfied, but by the theorem the output increase is not sufficient to offset the negative effect of discrimination. Thus the theorem is more than a way of saying that total output does not rise with discrimination – it applies when output rises as well as when it falls. Common to all three examples is a constant α. When marginal cost is positive and α is constant total output rises (falls) with discrimination when demand is concave (convex). With demand that is both concave and convex, i.e. linear, total output is constant.

Robinson's criterion for output can be applied to the current model. In terms of the current morel she effectively claimed that if R falls as a rises while the price is fixed at P_u then total output is higher with discrimination.[4] In other words the Robinson criterion uses the partial derivative of $R^*(P_u, a, b)$ with respect to a. The complete version of the output test in Proposition 3, however, requires the impact of a on the discriminatory price to be taken into account via $R^*(P(a, b), a, b)$. In other words in equation (14) the total derivative of R evaluated at the discriminatory price is required: $R_a(a,b) \equiv R_P^* P_a + R_a^*$. Robinson's test works if and only if α is constant. The reason is simple. $R = \alpha/\eta$ and the partial derivative of the elasticity with respect to a (keeping the price constant) has the same sign as the total derivative in (6) that allows for the price change. With constant α the partial and total derivatives of $R = \alpha/\eta$ also have the same sign. It is perhaps not surprising that Robinson's test works only in special circumstances. She aimed to predict whether output would change with discrimination using only information about demands and curvatures at the uniform price.

7. Conclusions

This paper has shown that with standard demand functions that are subject to additive and multiplicative shifts price discrimination has undesirable consequences for welfare when all markets are served. The assumption about the distribution of tastes is a weak sufficient condition for existence. It is thus unlikely that relaxing the distributional assumption on its own will enable any cases to be found where price discrimination raises social welfare. Indeed in the examples of Proposition 2 welfare is lower with discrimination even though the distributional assumption does not hold. Of course the model is not completely general. Some densities do not satisfy the distributional assumption. The assumption of interior solutions ensures that all markets are served with uniform pricing. If discrimination opens new markets then welfare may be higher with discrimination. When the sources of the differences between markets are neither additive nor multiplicative shift factors the model is silent.

It should be noted, though, that the welfare accounting framework is valid generally. Effectively the model here is a special case of a more general framework

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[4] See Melvin Greenhut and Hiroshi Ohta (1976) for a clear description of Robinson's criterion.

that has two types of consumers in each market, with different mixtures of the types across the markets, so demand in a market is $aq_1(p) + bq_2(p)$. It remains the case that if social welfare is concave in a (or b) then welfare with discrimination is lower. It is not clear, however, that anything general can be said about the properties of the two underlying demand functions that would ensure global concavity or convexity of the welfare function. Numerical analysis of parametric families of common density functions shows that concavity holds in some cases (e.g. two Pareto distributions with similar elasticities), convexity in others (Pareto distributions with significantly different elasticities, quadratic demand functions) and that sometimes the welfare function is neither globally concave nor globally convex (e.g. with two exponential demand functions).

Two lessons about modelling techniques can be learned. First, the value of providing a source for the differences in demand that drive price discrimination has been shown. Instead of allowing the demand functions to vary arbitrarily across markets a plausible structure has been imposed that explains why the demand functions differ. Although there is a small loss of generality there are considerable gains in terms of the ability to make general statements about welfare. Second, the use of unit demand models with distributions of tastes has proved fruitful because it enables known results on the shapes of density functions to be applied. Such techniques are well known in the literatures on differentiated-products oligopoly and on auctions, but have not been used much to examine third-degree price discrimination (though Jeremy Bulow and John Roberts, 1989, explore the connections between third-degree price discrimination and auction theory).

There are possible implications for anti-trust policy. The theorem suggests that the presumption that third-degree price discrimination has ambiguous effects on welfare is not necessarily warranted. Under reasonable conditions price discrimination always reduces social welfare when all markets are served. It may be that price discrimination should only be allowed when it opens new markets.

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