

Shared appreciation mortgages: property  
derivatives and unconventional loan interest  
charges

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MSc in Mathematical and Computational  
Finance, University of Oxford

Terry Boon <terry@tkb.me.uk>

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# Chapter 1

## Introduction

Home ownership is a significant proportion of household wealth in the United Kingdom. In the late 1990s and 2000s, there has been rapid growth in house prices, and a number of retail products exist which allow homeowners to withdraw equity from their homes. Some of these – including shared appreciation mortgages (SAMs) and home reversion plans – entail a transfer, from the homeowner to the lender, of exposure to changes in house prices. Products which transfer such risk can be considered to be derivatives on the underlying property.

There appears to have been little research in the academic literature into the pricing of such products. This pricing requires a model of house price movements – the housing market has a number of significant differences from equity or fixed income markets, including a failure to satisfy the efficient market hypothesis (returns on property demonstrate short-term positive autocorrelation), and the lack of facility to hold a fractional or short position in an individual underlying property. In addition, the expiry time of these products is frequently not fixed, but is triggered by the homeowner’s selling or moving out of the property; this introduces American-style features, where the homeowner can decide when the derivative is to be exercised.

In this dissertation, we develop an approach to pricing property derivatives. We start, in Chapter 2, by describing a variety of products, and define particular examples of shared appreciation mortgages and home reversion plans to consider in detail. In Chapter 3 we construct models for movements in property prices, and calibrate and validate these using UK data for the period 1983-2006; we also consider stochastic models for short interest rates. In Chapter 4 we use these models to obtain prices for property derivatives where the expiry time is fixed. In doing this, we first assume that the market *can* be treated as complete (through the availability of a tradeable asset – such as an instrument based on the house price index – to serve as a proxy for the price of the underlying asset), and then consider the case of an incomplete market, where such a tradeable asset is *not* available. In Chapter 5, we extend this to property derivatives where the expiry of the contract is not fixed, and is at least partly under the control of the homeowner. To do this, we examine the two broad approaches found in

the literature to modelling mortgage prepayment in order to price mortgages and mortgage-backed securities (MBSs) – the option-based approach, and the reduced-form approach. We suggest that for these products the reduced-form approach is more appropriate, and we examine how this could be applied to the pricing of property derivatives. Finally, Chapter 6 summarises our findings and conclusions.

## Chapter 2

# Property derivatives

### 2.1 Overview

Home ownership makes up a significant proportion of household wealth in the United Kingdom. In 2006, the household sector's holdings of residential buildings was reported to be £3,696 billion: 45% of total household sector wealth. Owner-occupancy is widespread: in 2006, 40% of households owned their residence with a mortgage, and 32% owned it outright. [19]

Homeowners are prepared to withdraw equity from their homes: in 2005/06 almost 5 per cent of owner-occupiers in England had withdrawn equity from their home within the previous three years, the average amount being released by each homeowner being £33,300, a significant amount for a typical household. A variety of such equity-release products are available on the retail market. Some are conventional mortgages or other loans secured on the property, in which the lender has no exposure to the value of the property; however, some products – including (using the Financial Services Authority's terminology) shared appreciation mortgages, home reversion plans, and lifetime mortgages which incorporate a no-negative-equity guarantee – do transfer at least some exposure to house price movement from the homeowner to the lender. [12]

Other shared equity arrangements exist to allow borrowers to purchase a share of a home. Examples of such arrangements are the UK government's Open Market HomeBuy and New Build HomeBuy schemes, particularly aimed at public sector key workers and social housing tenants. [9]

### 2.2 Contracts

#### 2.2.1 Overview of contracts

Shiller and Weiss identify, in [20], a variety of types of contract which enable homeowners to transfer an interest in their home to a lender. In this dissertation, we consider two types of such contracts in detail: *shared appreciation mortgages*,

and *home reversion plans* (which Shiller and Weiss term a “sale of remainder interest” contracts). Parts of the contract descriptions below are drawn from [20] and [12].

**Shared appreciation mortgages (SAMs)** In a shared appreciation mortgage, the lender makes a loan to a homeowner at a reduced rate of interest, in return for a share in any increase in the property’s value. In one typical arrangement, the lender provides a loan to the homeowner at the initiation of the contract, at a zero rate of interest. No repayments are made during the term of the loan – but at the expiry of the contract, the lender receives a proportion (fixed at initiation) of the *increase (appreciation)* in the property’s value since initiation (if any), as well as repayment of the original principal.

**Home reversion plans** In a home reversion plan, the lender purchases a share (perhaps 100%) of the homeowner’s property from the homeowner, at a discount to the current valuation of that share – the homeowner typically receives between 35% and 65% of the share’s value [1]. Again, the homeowner receives this principal at initiation (and can continue to live in the property, typically at peppercorn rent), and the lender receives no repayments during the term of loan. At the expiry of the contract, the lender receives a share of the property’s value at that time according to the proportion which they had purchased upon initiation.

**Common features: older target market, and non-fixed expiry times** Shared appreciation mortgages and home reversion plans are typically marketed to older consumers who own their homes outright. They do not usually have a fixed expiry time, but instead expire upon the sale of the property on which it is based, or upon the homeowner’s death or permanent move out of the property (e.g. into residential care). The homeowner can therefore (at least partially) influence when the contract expires, while the lender has no control over this after the contract has been initiated.

**Other products** We also describe here three other products which we will not consider in detail but which have some relevant features of interest.

*Lifetime mortgages* are designed for a similar market sector to SAMs and home reversion plans. The homeowner retains ownership of the property, but obtains a loan secured on the it from the lender. Interest charges may be paid during the term of the loan, or rolled up into the outstanding loan balance. At expiry, the proceeds of the property sale can be used to repay the outstanding loan. It is common (but not universal) for lifetime mortgages to include a “no-negative-equity” guarantee, so the outstanding loan balance upon expiry is capped at the property’s value at that time: such a guarantee therefore transfers some risk of a fall in property prices from the homeowner (or their estate) to the lender. With this type of guarantee, the product is similar to the “reverse mortgage” of [20].

In contrast, *shared equity mortgages* are typically marketed to first-time buyers who might otherwise be unable to afford to buy a home. As described in [20], there are three parties to the mortgage contract: the homeowner, an (equity) investor, and a mortgage lender – in effect, both the homeowner (using finance provided by the mortgage lender) and the investor buy shares in the home. For example, under the MyChoiceHomeBuy product of the Open Market HomeBuy scheme, an equity loan of 15-50% of the purchase price is provided by a partnership of housing associations, while a conventional mortgage is used to finance the remainder. The homeowner may be permitted to increase their share of the property (effectively purchasing it from the equity loan provider) during the course of the contract. A shared equity mortgage, being an arrangement where a lender owns a share of the property, is similar to a home reversion plan.

*Home equity insurance* is a product which is proposed by Shiller and Weiss in (among others) [21]. Unlike the loan products described above, this is a type of insurance policy, which homeowners might use in order to protect against the risk of a fall in their home’s value. Shiller and Weiss suggest that it could form an add-on to the homeowner’s insurance policy or mortgage. So far, home equity insurance is a theoretical product; these policies are not currently available on the retail market.

**Treatment as derivatives** All these contracts can be considered as derivatives on the value of the property, where the payoff is received by the lender upon the contract’s expiry and its value is a function of the property’s value at that time. The treatment of the products as derivatives on the property value provides a “fair” price for the lender to pay to the homeowner for the payoff obtained from the property.

Variations on such contracts may be developed, although those described above will illustrate the essential features, and can also be used to construct more complex arrangements. For example, if the lender is also entitled to payments (such as interest charges) *during* the contract, then these can be valued separately (for example as a fixed income instrument). Conversely, if the homeowner does not receive a lump sum upon initiation, but instead receives an entitlement to a regular income, then the entitlement can be valued as an annuity granted to the homeowner.

### 2.2.2 Formal definition of SAM and home reversion contracts and payoffs

We consider a contract initiated at time  $t = 0$ , and with expiry at time  $t = T$ . Let  $N$  denote the principal paid by the lender to the homeowner at  $t = 0$ , and  $H_t$  denote the value of the property at time  $t$ . The payoffs for these contracts are then defined as follows:

**Shared appreciation mortgage** Let  $\theta_{SA}$  be the proportion of the appreciation to which the lender is entitled. Then the payoff which the lender receives

upon expiry is  $N + \theta_{SA}(H_T - H_0)^+$ .

It will also be convenient to let  $\kappa_{SA} = N/(\theta_{SA}H_0)$ ; this is a measure of the discount at which the lender has purchased its share in the property's appreciation, and allows all cashflows to be expressed in terms of property prices. With this notation, the payoff received by the lender is  $\theta_{SA}(\kappa_{SA}H_0 + (H_T - H_0)^+)$ .

For SAMs sold in the late 1990s in the UK, a typical value of  $\kappa_{SA}$  was  $1/3$ : for example, a homeowner could borrow 25% of the current value of their property, in return for paying 75% of any *increase* in the property's value upon expiry (e.g. when they wished to sell the property) in addition to repayment of the principal.

**Home reversion plan** Let  $\theta_{HR}$  be the proportion of the appreciation to which the lender is entitled. Then the payoff which the lender receives upon expiry is simply  $\theta_{HR}H_T$ . Similarly, we let  $\kappa_{HR} = N/(\theta_{HR}H_0)$  denote the discount at which the lender has purchased its share in the property.

## 2.3 Illustration of a shared appreciation mortgage

To illustrate how a shared appreciation mortgage operates, we consider the following example:

A property is valued at £100,000, when the homeowner takes out a shared appreciation mortgage of £25,000 (25% of the property's value). Under the arrangement, the lender is entitled to 75% of appreciation from initiation until the expiry of the contract (so  $\theta_{SA} = 0.75$  and  $\kappa_{SA} = 1/3$ ), but receives no repayments before expiry. After 10 years, the homeowner sells the property and the contract expires.

We compute the share of the appreciation to which the lender is entitled, the sale proceeds received by the homeowner, and the implied annual interest rate which the lender has paid for the SAM.

| Average annual growth rate | Sale value (£) | Lender's share (£) | Homeowner's balance (£) | Implied interest rate |
|----------------------------|----------------|--------------------|-------------------------|-----------------------|
| -2%                        | 81,707         | 0                  | 81,707                  | 0%                    |
| 0%                         | 100,000        | 0                  | 100,000                 | 0%                    |
| 2%                         | 121,899        | 16,424             | 105,475                 | 5%                    |
| 5%                         | 162,890        | 47,167             | 115,722                 | 11%                   |
| 10%                        | 259,374        | 119,531            | 139,844                 | 19%                   |

For a given growth rate, the implied interest rate increases if the expiry occurs sooner: for example, at 5% annual growth, a sale after only 5 years would result in the lender's share of appreciation being £20,721, an implied rate of 13% interest (compared to the 11% after 10 years).

If there is a period of rapid house inflation after the SAM has been taken out, then the lender's shared appreciation will be large. For example, if the annual rate of growth were 10% then, upon sale after 10 years, the homeowner would

receive £139,844 – but if they wanted to buy a similar home at that point, then the market price for such a property would be approximately £260,000 (if house prices in the market tended to rise uniformly), leaving a significant gap. Some consumers who took out SAMs in the late 1990s found themselves in a similar situation – following the strong growth in property prices after that time, they would have to pay a large amount to their lender if they were to sell, and would be in a disadvantageous position for buying another home.

## Chapter 3

# House price and interest rate dynamics

### 3.1 House price dynamics – a selective review of the literature

The dynamics of the residential property market is a thriving area of research: the majority of this work is to be found in the economics and econometrics literature, rather than in that of mathematical finance.

Case and Shiller's widely cited 1989 study [7] of residential property prices in Atlanta, Chicago, Dallas, and San Francisco/Oakland during the period 1970-1986 identifies persistence in the rate of change in an inflation-adjusted (real) house price index (i.e. positive autocorrelation in one-year house price increases). This demonstrates that the housing market does not satisfy the efficient market hypothesis. Further analysis by Case and Shiller in [8] confirms this, and also identifies (weaker) negative autocorrelation in house price increases at 2-, 3-, and 4-year lags; of other variables considered, only construction costs, change in per capita real income, and the change in the adult population were found to be significant.

In [11], Englund and Ioannides's study of house price dynamics across 15 OECD countries (including the UK) identified a similar highly significant positive autocorrelation effect at a one-year lag, and evidence of negative autocorrelation at lags of up to 5 years. Englund and Ioannides also find the contemporaneous GDP growth rate and change in real interest rates significantly related to house prices.

Meen [16] examines differences between dynamics of real house prices in the US and the UK. As Case and Shiller identified in [8], real income (now defined as real personal disposable income) was found to be significantly related to UK real house prices: real interest rates were also found to be related to real house prices, but less strongly. He notes that construction costs are rarely included in

UK house price models, and that annual new housing build in the UK is only approximately 1% of the housing stock; this suggests that construction costs may not be a significant component of house prices in this country.

We use these results to inform the construction of an initial basic model for house price dynamics in Section 3.3 (based on general inflation rates and interest rates and capturing the identified autocorrelative effects), and an extended model incorporating economic growth in Section 3.4.

## 3.2 Interest rates and general inflation

We require a model for the interest rate, both for the effect which it has upon house price dynamics and, as in the pricing of other instruments, to allow the discounting of cashflows to present values. We will use a short-rate model, and define a bank account asset with price  $B_t$ , and an interest rate process  $r_t$  such that  $dB_t = r_t B_t dt$ , so that  $B_t = B_0 \exp\left(\int_0^t r_\tau d\tau\right)$  and (for convenience)  $B_0 = 1$ .

We follow the approach, generally adopted in the literature, of modelling house prices after they have been adjusted for general inflation in consumer prices. We describe inflation-adjusted prices and growth rates as *real*, while those which are not so adjusted are termed *nominal*.

To model the market in real terms, we define the process  $C_t$  to be the value of an appropriate *consumer price index* (CPI), and  $i_t$  to be the general inflation rate, so  $dC_t = i_t C_t dt$ .

We use a tilde to denote prices measured in real terms – for example,  $\tilde{H}_t = H_t/C_t$  – and rates of change adjusted for inflation – for example,  $\tilde{r}_t = r_t - i_t$ .

For these processes, and other processes used in this dissertation, we may suppress the subscript  $t$  where this does not result in ambiguity.

## 3.3 A basic model for the HPI

Based on the findings identified in the literature, we initially seek a basic model where real returns on housing have the following characteristics:

- positive short-term autocorrelation (e.g. if prices have recently risen, then their next step is also likely to be upwards)
- (weaker) negative long-term autocorrelation (consistent with a mean-reverting characteristic)
- negative correlation with real interest rates.

To produce a model with these characteristics, we define the following auxiliary processes:

- An exponentially weighted moving average (EWMA) of the real HPI:

$$\tilde{A}_t = \xi \int_{-\infty}^t e^{\xi(t-\tau)} \tilde{H}_\tau d\tau$$

and this EWMA expressed in nominal terms:

$$A_t = C_t \tilde{A}_t$$

for some weight  $\xi > 0$ . These processes can be used to model the short-term autocorrelation of returns: if  $\tilde{H}_t$  has been increasing (or, conversely, decreasing) in the recent past then  $\tilde{H}_t - \tilde{A}_t$  will tend to be positive (or, correspondingly, negative).

Applying the Leibniz rule for differentiation of an integral, we find

$$d\tilde{A} = \xi(\tilde{H} - \tilde{A})dt$$

and then the Itô product rule applied to  $A = \tilde{A}C$  gives

$$dA = [\xi(H - A) + iH] dt.$$

- A predictor of the real HPI based on an assumption of a constant rate of growth

$$\tilde{M}_t = \tilde{M}_0 e^{\tilde{m}t}$$

for some constant  $\tilde{M}_0$  and  $\tilde{m}$  chosen so that  $\tilde{M}_t$  is a reasonable predictor of  $\tilde{H}_t$ ; and this predictor expressed in nominal terms:

$$M_t = C_t \tilde{M}_t.$$

These processes can be used to model a mean-reverting effect for the real HPI: if  $\tilde{H}_t$  is above (or, conversely, below) the predicted value, then  $\tilde{H}_t - \tilde{M}_t$  will be positive (or, correspondingly, negative).

These processes have dynamics

$$d\tilde{M} = \tilde{m}\tilde{M}dt \quad \text{and} \quad dM = (\tilde{m} + i)Mdt = mMdt$$

where we define  $m = \tilde{m} + i$ .

Note that  $A_t$  and  $M_t$  are based on the *real* HPI and then converted to nominal terms at time  $t$ , rather than being EWMA's and constant-growth predictors of the nominal HPI.

Using these processes, we consider a model of the form

$$d\tilde{H} = \left[ \alpha(\tilde{H} - \tilde{A}) + \beta(\tilde{H} - \tilde{M}) + \gamma\tilde{r}\tilde{H} + \tilde{\mu}\tilde{H} \right] dt + \sigma\tilde{H}dW^{(1)} \quad (3.1)$$

for some constants  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\tilde{\mu}$ , and  $\sigma$ , and a standard Brownian motion process  $W_t^{(1)}$  (the superscript <sup>(1)</sup> being used to distinguish it from the Brownian motion to be used in the stochastic model for interest rates). Based on the

characteristics to be modelled, we expect  $\alpha > 0$  (positive short-term autocorrelation of returns),  $\beta < 0$  (mean-reversion), and  $\gamma < 0$  (negative correlation with real interest rates). Without loss of generality we assume  $\sigma > 0$ ; we have no expectations at this stage about the sign of  $\tilde{\mu}$ .

Based on this, we can apply the Itô product rule to  $H_t = C_t \tilde{H}_t$  (and drop subscripts where there is no ambiguity):

$$\begin{aligned} dH &= \tilde{H}dC + C d\tilde{H} + d\tilde{H}dC \\ \therefore dH &= [\alpha(H - A) + \beta(H - M) + \gamma(r - i)H + \mu H] dt + \sigma H dW^{(1)} \end{aligned} \quad (3.2)$$

where we define a nominal drift coefficient  $\mu = \tilde{\mu} + i$ .

### 3.3.1 Validating and calibrating the basic model

We now use historic data to validate whether this basic model provides a reasonable model for property values. Calculations and plots were produced using the R programming language and environment [17].

#### 3.3.1.1 Obtaining historic data, and computing time series for validation and calibration

The following time series, indexed by month  $[k]$ , are available on a monthly basis:

$t^{[k]}$ : The time at month  $k$ .

$H^{[k]}$ : The Halifax House Price Index (All Houses (All Buyers), Seasonally Adjusted – Monthly Data) is available on a monthly basis from January 1983. This data is published by HBOS [3].

$C^{[k]}$ : The Consumer Price Index (CPI) is calculated monthly by the UK Office of National Statistics (ONS). Pre-1996 CPI is estimated by the ONS. We use the CPI rather than the Retail Price Index (RPI) because the RPI already includes elements of housing costs. This data was obtained from Global Financial Data [4].

$r^{[k]}$ : UK overnight interbank rate (sterling), captured daily since January 1975. When LIBOR overnight rates have been recorded, that rate is used. This data was obtained from Global Financial Data. To obtain a monthly time series, the overnight rates for each month were averaged [2].

From these, we compute the other series required by the model:

$i^{[k]}$ : The continuously compounded annual general inflation rate  $i^{[k]} = \log(C^{[k+12]}/C^{[k]})$ . We use this definition instead of the single-timestep forward difference  $\log(C^{[k+1]}/C^{[k]})$  because we find that there is a significant seasonal component to general inflation, which is eliminated by computing year-on-year increases.

**Expressions in real terms:** Similar to the definition of the inflation-adjusted processes, the real HPI series is given by  $\tilde{H}^{[k]} = H^{[k]}/C^{[k]}$ , and the real short interest rate by  $\tilde{r}^{[k]} = r^{[k]} - i^{[k]}$ .

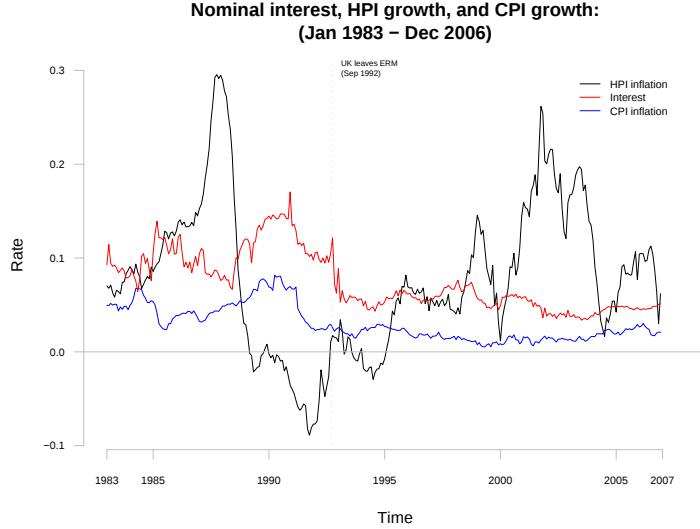
$\Delta\tilde{H}^{[k]}$ : The one-step forward difference in the real HPI,  $\Delta\tilde{H}^{[k]} = \tilde{H}^{[k+1]} - \tilde{H}^{[k]}$

$\tilde{M}^{[k]}$  and  $M^{[k]}$ : Constants  $\tilde{M}_0$  and  $\tilde{m}$  are required such that  $\tilde{M}_t = \tilde{M}_0 e^{\tilde{m}t}$  is a reasonable long-term predictor of real property values. We obtain these by performing a linear regression of  $\log(\tilde{H}^{[k]})$  against  $t^{[k]}$ ; the intercept (at  $t = 0$ ) and gradient of the best-fit line provide  $\tilde{M}_0$  and  $\tilde{m}$  respectively. Then define the real-value predictor  $\tilde{M}^{[k]} = \tilde{M}_0 e^{\tilde{m}t^{[k]}}$  and nominal-value predictor  $M^{[k]} = C^{[k]} \tilde{M}^{[k]}$ .

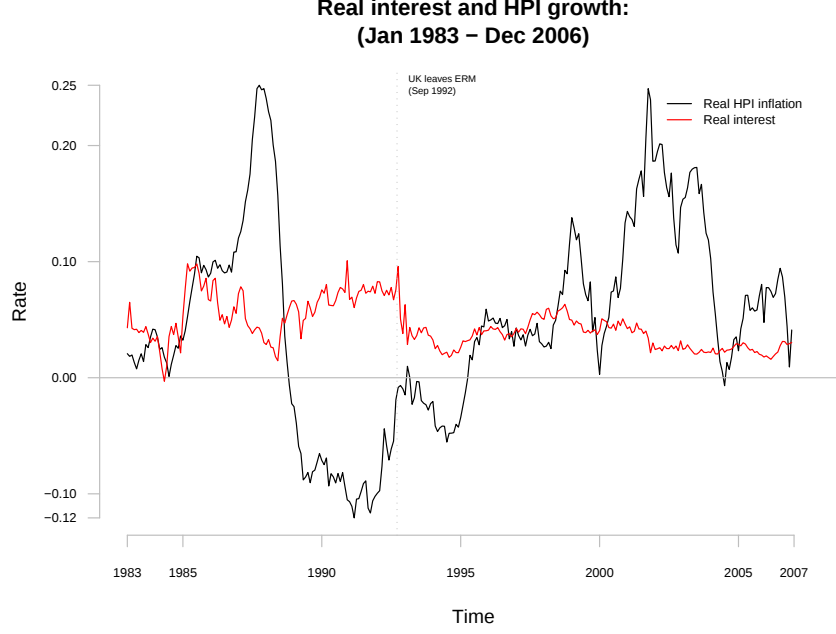
$\tilde{A}^{[k]}$  and  $A^{[k]}$ : The discrete approximation to the real-terms EWMA given by  $\tilde{A}^{[0]} = \tilde{H}^{[0]}$  and  $\tilde{A}^{[k+1]} = (1 - \Xi)\tilde{A}^{[k]} + \Xi\tilde{H}^{[k+1]}$ , where  $\Xi$  is expressed in terms of the continuous EWMA weight  $\xi$ . The nominal-terms EWMA is then given by  $A^{[k]} = C^{[k]} \tilde{A}^{[k]}$ .

### 3.3.1.2 Examination of historical data

We obtain plots of nominal interest rates, house price inflation, and general inflation:



and real interest rates and HPI growth:



The plots show the “Lawson Boom” in the late 1980s, where house prices underwent rapid growth; the period of decline in house prices in the early 1990s associated with a recession in the UK; and the ongoing regime of lower interest rates following the UK’s departure from the European Exchange Rate Mechanism (ERM) in September 1992.

These time series plots will differ from time series data obtained from some other sources. In this model, we are considering general inflation and house price inflation rates as the one-year *forward* differences, so the plots of these will be shifted forward by one year compared to the backward changes generally used to report year-on-year inflation. In addition, we are using continuously compounded annual rates instead of the generally reported simple one-year proportional change; the difference arising from this is negligible.

### 3.3.1.3 Calibration and validation of the deterministic component of the basic model

We apply the Euler-Maruyama discretisation to (3.1), using a timestep of one month ( $\Delta t = 1/12$ ), to obtain

$$\Delta \tilde{H}^{[k]} = \left[ \alpha(\tilde{H}^{[k]} - \tilde{A}^{[k]}) + \beta(\tilde{H}^{[k]} - \tilde{M}^{[k]}) + \gamma \tilde{r} \tilde{H}^{[k]} + \tilde{\mu} \tilde{H}^{[k]} \right] \Delta t + \sigma \tilde{H}^{[k]} \Delta W^{[k]}$$

or

$$\frac{\Delta \tilde{H}^{[k]}}{\tilde{H}^{[k]}} = \left[ \alpha \left( \frac{\tilde{H}^{[k]} - \tilde{A}^{[k]}}{\tilde{H}^{[k]}} \right) + \beta \left( \frac{\tilde{H}^{[k]} - \tilde{M}^{[k]}}{\tilde{H}^{[k]}} \right) + \gamma \tilde{r} + \tilde{\mu} \right] \Delta t + \sigma \Delta W^{[k]}$$

where the  $\Delta W^{[k]}$  are independently identically distributed random variables:  $\Delta W^{[k]} \sim N(0, \Delta t)$ . We perform a linear regression with the following variables:

**Regressors:**  $\left( \frac{H^{[k]} - A^{[k]}}{H^{[k]}} \Delta t \right); \left( \frac{H^{[k]} - M^{[k]}}{H^{[k]}} \Delta t \right); r^{[k]} \Delta t;$

**Dependent variable:**  $\frac{\Delta H^{[k]}}{H^{[k]}}$ .

The resulting fitted model, based on the data for the period 1983-2006, is as follows:

| Variable                                           | Coefficient                | Std error | t-value | p-value                |
|----------------------------------------------------|----------------------------|-----------|---------|------------------------|
| constant                                           | $\hat{\mu} = 0.003308$     | 0.001739  | 1.902   | 0.0582                 |
| $\left( \frac{H^{[k]} - A^{[k]}}{H^{[k]}} \right)$ | $\hat{\alpha} = 1.853315$  | 0.218789  | 8.471   | $1.35 \times 10^{-15}$ |
| $\left( \frac{H^{[k]} - M^{[k]}}{H^{[k]}} \right)$ | $\hat{\beta} = -0.081805$  | 0.039172  | -2.088  | 0.0377                 |
| $\tilde{r}$                                        | $\hat{\gamma} = -0.056630$ | 0.032588  | -1.738  | 0.0833                 |

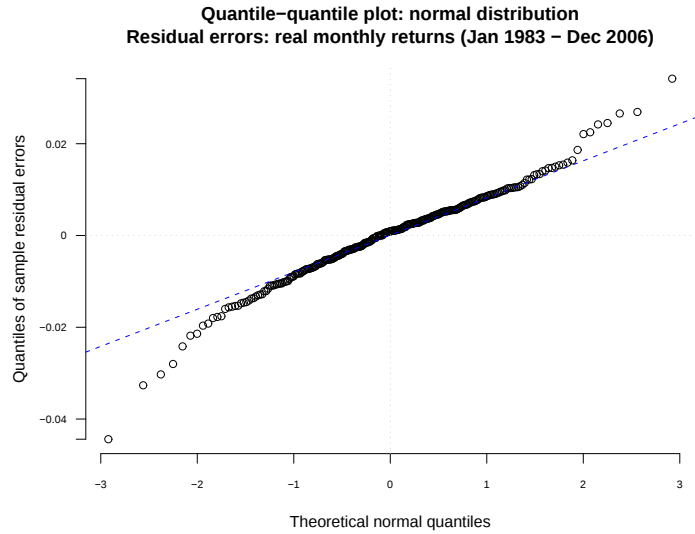
The coefficient of determination statistic,  $R^2$ , is 0.2742. The adjusted  $R_{adj}^2$  value (accounting for the number of predictors) is 0.2665, so our model has substantial explanatory power.

Based on the p-values for these coefficients, and our considerations when constructing the model, we retain all regressors in our model. We also note that the signs of these estimators of the coefficients  $\alpha$ ,  $\beta$ , and  $\gamma$  are as expected from the construction of the model. The most significant regressor is that of  $(H^{[k]} - A^{[k]}/H^{[k]})$ , which drives the positive short-term autocorrelation in returns.

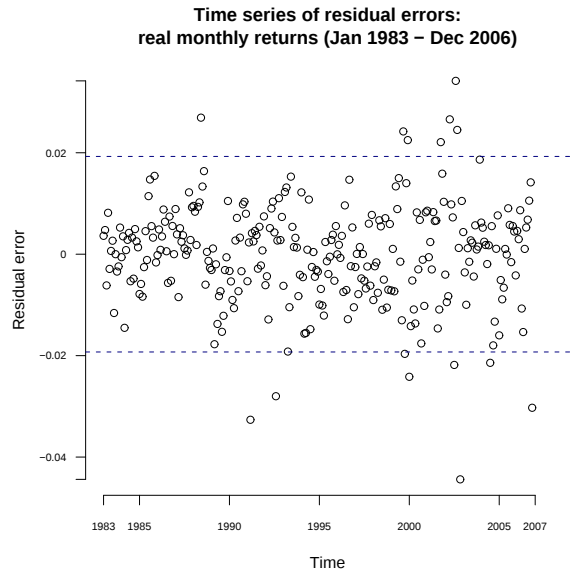
### 3.3.1.4 Characteristics of the residual errors

We must also examine the residual errors to assess whether it is reasonable to model these as increments of a Brownian motion.

**Normal distribution** A quantile-quantile plot of the residual errors against those of a theoretical normal distribution shows that the distribution *is* approximately normal. The direction of deviation from the theoretical line at the tails is consistent with an empirical distribution with higher kurtosis (“fatter tails”) than a normal distribution. This effect is particularly strong at the left-hand tail, which would be consistent with house price dynamics where property values occasionally undergo a large downward jump (e.g. a property price crash).



**Homoscedasticity** We have assumed that the variance  $\sigma^2$  of the residual errors is constant (i.e. that the residuals are homoscedastic). To check whether this assumption is valid, we plot the time series of residual errors, and a 95% confidence interval computed on the assumption that the variance is constant:



The distribution of the residual errors indicates that the variance *may* be increasing with time, but the assumption of homoscedasticity is not grossly violated. (However, future work could examine a model where the stochastic

term  $\sigma \tilde{H} dW$  in 3.1 is replaced with one of the form  $\sigma \tilde{H}^\nu dW$  for some  $\nu > 1$ .)

We also apply the Breusch-Pagan test to the model. This is a statistical test of homoscedasticity which can be applied to a linear regression model, and is based on considering the squared residual errors as a dependent variable, and regressing this series on the regressors used to construct the original model. We apply this test at a 95% level of significance:

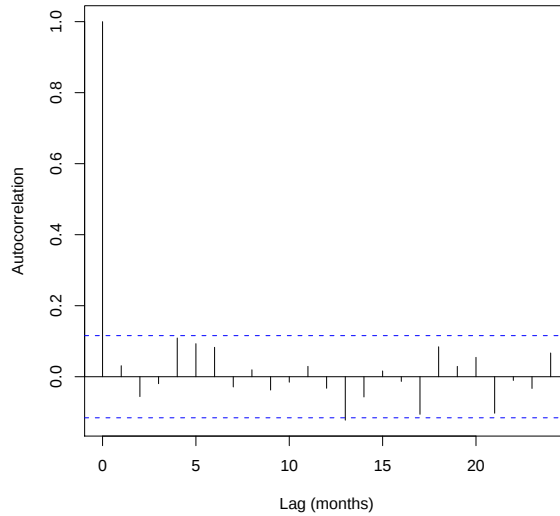
**Null hypothesis:** The variance of the residual errors are not explained by a linear regression on the regressors (so the test statistic follows a  $\chi^2$  distribution with 3 degrees of freedom, corresponding to the 3 regressors in the model);

**Alternative hypothesis:** The variance of the residual errors *is* explained by a linear regression on the regressors.

The resulting Breusch-Pagan statistic is 12.8446, with a corresponding p-value of 0.004985. We therefore reject the null hypothesis, confirming the evidence of heteroscedasticity found by our inspection of the plot.

**Lack of autocorrelation** Finally, we plot the autocorrelation function of the series of residual errors – along with the 95% acceptance interval for the correlation coefficients – to assess whether there are any periodic dependencies between them:

**Autocorrelation of residual errors:  
real monthly returns (Jan 1983 – Dec 2006)**



The only autocorrelation coefficient which falls outside the 95% confidence interval is at 13 months, while the other large autocorrelations are at 4, 5, 17, and 21 months. These do not have a natural interpretation in terms of the periodic/seasonal behaviour of property values (as they might if large correlation

was found at 12 or 24 months), and we conclude that this does not indicate the presence of significant autocorrelation.

We also apply the Durbin-Watson test to the residuals, at a 95% level of significance, to test whether successive errors exhibit correlation:

**Null hypothesis:** Successive residuals are uncorrelated (i.e. have zero correlation);

**Alternative hypothesis:** Successive residuals are correlated (i.e. have non-zero correlation).

The resulting Durbin-Watson statistic is 1.9045, with a corresponding p-value of 0.3250. We therefore do not reject the null hypothesis that the successive residuals are uncorrelated.

### 3.3.1.5 Computation of sample volatility

Having concluded that it is reasonable to consider the residual errors as increments of a Brownian motion, we now calculate the resulting volatility. The residual errors  $\epsilon^{[k]}$  are given by  $\sigma\Delta W^{[k]}$  where  $\Delta W^{[k]} \sim N(0, \Delta t)$ , so  $\epsilon^{[k]} \sim N(0, \sigma^2/12)$ . The sample standard deviation is 0.00984, so we estimate the volatility  $\sigma = 0.00984 \times \sqrt{12} = 0.0340$ .

## 3.4 Extending the basic model: using GDP growth as a predictor for the HPI

Other variables identified in the literature as closely related to the rate of HPI growth are the rate of gross domestic product (GDP) growth and real personal income. Since growth in GDP and income are closely related, we will examine the effect of introducing GDP growth as an additional regressor. To incorporate this into our model, we define the following processes:

$g_t$ : The instantaneous rate of nominal GDP growth

$\tilde{g}_t$ : The instantaneous rate of real GDP growth, defined by  $\tilde{g}_t = g_t - i_t$ .

We then extend (3.1) and consider a model of the form

$$d\tilde{H} = \left[ \alpha(\tilde{H} - \tilde{A}) + \beta(\tilde{H} - \tilde{M}) + \gamma\tilde{r}\tilde{H} + \delta\tilde{g}\tilde{H} + \tilde{\mu}\tilde{H} \right] dt + \sigma\tilde{H}dW^{(1)} \quad (3.3)$$

To validate and calibrate this model based on historical data, we now require a time series for  $\tilde{g}_t$ . GDP information is published on a quarterly basis, so we will use quarterly forms of all relevant time series (so the index  $[k]$  now identifies the quarter); we compute them as follows:

$G^{[k]}$ : The nominal Gross Domestic Product (GDP) is calculated and published quarterly, and includes production for the previous year (four quarters). This data was obtained from Global Financial Data.

$g^{[k]}$ : We compute the instantaneous rate of annual GDP growth  $g^{[k]} = \log(G^{[k+4]}/G^{[k]})$ .

We use this definition instead of the single-timestep forward difference  $\log(G^{[k+1]}/G^{[k]})$  to eliminate the seasonal component of GDP growth and the double-counting of quarterly GDP.

$\tilde{g}^{[k]}$ : Finally, we compute the instantaneous rate of real annual GDP growth by defining  $\tilde{g}^{[k]} = g^{[k]} - i^{[k]}$ .

For the other required series, we extract quarterly data from the monthly series computed above. We can then perform a linear regression with the following variables:

**Regressors:**  $\left(\frac{H^{[k]} - A^{[k]}}{H^{[k]}} \Delta t\right); \left(\frac{H^{[k]} - M^{[k]}}{H^{[k]}} \Delta t\right); \tilde{g}^{[k]} \Delta t; \tilde{r}^{[k]} \Delta t;$

**Dependent variable:**  $\frac{\Delta H^{[k]}}{H^{[k]}}$ .

The resulting fitted model, based on the data for the period 1983-2006, is as follows:

| Variable                                         | Coefficient                 | Std error | t-value | p-value |
|--------------------------------------------------|-----------------------------|-----------|---------|---------|
| constant                                         | $\hat{\mu} = -0.0003366$    | 0.0075294 | -0.045  | 0.96445 |
| $\left(\frac{H^{[k]} - A^{[k]}}{H^{[k]}}\right)$ | $\hat{\alpha} = 0.9632299$  | 0.3612058 | 2.667   | 0.00908 |
| $\left(\frac{H^{[k]} - M^{[k]}}{H^{[k]}}\right)$ | $\hat{\beta} = 0.0044696$   | 0.0549963 | 0.081   | 0.93541 |
| $\tilde{r}$                                      | $\hat{\gamma} = -1.0435585$ | 0.5207277 | -2.004  | 0.04807 |
| $\tilde{g}$                                      | $\hat{\delta} = 2.2595500$  | 0.7131674 | 3.168   | 0.00209 |

The coefficient of determination statistic,  $R^2$ , is 0.4531. The adjusted  $R_{adj}^2$  value (accounting for the number of predictors) is 0.4288, so this extended model (3.3) a significant improvement over the basic model (3.1).

We note that the signs of  $\hat{\alpha}$  and  $\hat{\gamma}$  are (as before) as expected from the construction of the model, as is the sign of  $\hat{\delta}$  (showing positive correlation between rates of GDP and house price growth). The sign of  $\hat{\beta}$  is now unexpectedly positive, but is not significantly different from zero. This may indicate that the mean-reverting characteristics of HPI growth are reflected by corresponding mean-reverting properties of GDP growth.

Although this extended model for house price dynamics has greater explanatory power than our original basic model, its use in pricing property derivatives would require the development of a model for GDP growth: this is a non-trivial problem in itself. Therefore, in our development of a pricing model for property derivatives in the rest of this dissertation, we use the basic model rather than this extended model – but we note here that, if a GDP model *were* available, the extended model could improve the pricing of those derivatives.

### 3.5 Modelling the short interest rate

We also require a model for the short interest rate  $r_t$ , because of its effect upon both the dynamics of the HPI and the discounting of payoffs to the time at

which they are priced. We let the price at time  $t$  of a zero-coupon bond (ZCB) with maturity  $T$  be denoted  $P^T(t)$ . (The presentation here draws from that of Brigo and Mercurio in [6].)

### 3.5.1 Deterministic/constant models

In this simple case,  $r_t$  is given by some deterministic function:  $r_t = r(t)$  for some function  $r$ . Here, the bond price is given by

$$P^T(t) = \exp\left(-\int_t^T r_u du\right)$$

### 3.5.2 Stochastic short-rate models

We now consider the case of stochastic short interest rates. Suppose that  $r_t$  satisfies (under the objective measure  $\mathbb{P}$ ) the SDE

$$dr_t = \zeta(t, r_t)dt + \varsigma(t, r_t)dW_t^{(2)} \quad (3.4)$$

where  $W_t^{(2)}$  is a Brownian motion, correlated with the property value's driving Brownian motion  $W_t^{(1)}$  so that  $dW_t^{(1)}dW_t^{(2)} = \rho dt$ .

Under such a model, we find that for any bond maturity  $T$ , the price of the bond at time  $t$  (denoted  $P^T(t, r)$ ) satisfies the term structure equation

$$\frac{\partial P^T}{\partial t} + (\zeta - \varsigma\lambda^P)\frac{\partial P^T}{\partial r} + \frac{1}{2}\varsigma^2\frac{\partial^2 P^T}{\partial r^2} - rP^T = 0 \quad (3.5)$$

where  $\lambda^P(t, r)$  represents the market price of risk for bonds, which is independent of the maturity of the bond.

The bond has payoff 1 at maturity  $t = T$ , so this enforces the boundary condition  $P^T(T, r) = 1$ .

We can now solve this equation to obtain the bond price  $P^T(t, r_t)$ . We apply the Feynman-Kac formula to obtain the solution

$$\begin{aligned} P(t, r_t) &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_u du\right) P^T(T, r_T) \mid r_t = r\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_u du\right) \mid r_t = r\right] \end{aligned}$$

where the dynamics of  $r_t$  under the risk-neutral measure  $\mathbb{Q}$  are defined by

$$dr_t = (\zeta(t, r_t) - \varsigma(t, r_t)\lambda^P(t, r_t))dt + \varsigma(t, r_t)dW_t^{(2)} \quad (3.6)$$

$$= \nu(t, r_t)dt + \varsigma(t, r_t)dW_t^{(2)} \quad (3.7)$$

where we define  $\nu(t, r) = \zeta(t, r) - \varsigma(t, r)\lambda^P(t, r)$  to be the risk-neutral drift of  $r_t$ .

It is the dynamics of  $r$  under  $\mathbb{Q}$ , rather than the objective measure, which are relevant for the pricing of instruments, and hence also relevant to the calibration of the model. We will therefore select a model for the risk-neutral dynamics, expressed in terms of  $\nu$  and  $\varsigma$  rather than considering the objective drift  $\zeta$  and explicit market price of risk  $\lambda^P$ .

### 3.5.2.1 Examples of short rate models

One simple short rate model is the Vasicek model:

$$dr_t = k[\vartheta - r_t]dt + \varsigma dW^{(2)} \quad (3.8)$$

for constants  $k > 0$ ,  $\vartheta$ , and  $\varsigma$ . This has the desirable features of reverting to a mean level  $\vartheta$ , where  $k$  indicates the strength of this mean-reverting effect, and of being analytically tractable. Less desirable is the non-zero probability for  $r_t$  to fall below 0. A variation which addresses this limitation is the Cox-Ingersoll-Ross (CIR) model

$$dr_t = k[\vartheta - r_t]dt + \varsigma\sqrt{r_t}dW^{(2)} \quad (3.9)$$

which retains the mean-reverting property but, for reasonable choices of the parameters, cannot result in negative interest rates.

The Vasicek and CIR models both suffer from another profound limitation: with only a small number of parameters, it is not generally possible to calibrate the parameters in such a way that the modelled bond prices (i.e. the yield curve) matches the bond prices observed in the market. One of the simplest models which overcomes this limitation is the Hull-White model for the short rate:

$$dr_t = [\vartheta(t) - ar_t]dt + \varsigma dW_t^{(2)} \quad (3.10)$$

where  $a$  and  $\varsigma$  are constants, but  $\vartheta$  is now a function of  $t$ . It is this time-dependence of  $\vartheta$  which allows, by suitable calibration of that function, the yield curve predicted by the model to match the observed yield curve.

## Chapter 4

# Pricing fixed-expiry property derivatives

In this chapter, we consider the pricing of property derivatives which have a fixed expiry time  $T$ , so neither party has the right to exercise the option before this expiry time. This is not typical of such derivatives – as mentioned earlier, the expiry is usually based on events either unpredictable or under the control of the homeowner – but it will allow examination of the characteristics of property derivatives.

The classic approach to pricing derivative contracts is based on the assumption of a number of properties of the market in which the relevant assets are traded (summarised, for example, in [14]): trading does not move the market prices; the market is liquid; market participants can hold a short position in the asset, or a position in fractional quantities; and there are no transaction costs.

These assumptions are clearly unrealistic with regard to an individual property. In recent years, however, a market has been developing in instruments based on a house price index (HPI). If the HPI's behaviour is sufficiently close to that of an underlying property, then these instruments could be used to hedge a property derivative.

We therefore consider two approaches to pricing these contracts. The first is the classic approach, based on the assumption that a tradeable instrument (such as one based on the HPI) is available which allows the effective replication of movements in the underlying property. The second assumes that there is not a tradeable asset following the property price, resulting in an incomplete market.

Under each approach, we let  $\mathcal{F}_t$  be the natural filtration generated by  $W_t^{(1)}$  and  $W_t^{(2)}$ .

## 4.1 Property derivative pricing in a complete market

We will first assume that there is an instrument (which we call an *HPI bond*), whose market exhibits the usual properties described above, and whose price movements replicate those of the underlying property.

### 4.1.1 Deterministic short rate and general inflation

We first consider the case where the short interest rate process  $r_t$  and the general inflation rate  $i_t$  are assumed to be deterministic (possibly constant) until expiry at  $T$ .

#### 4.1.1.1 Derivation of a PDE for the derivative price

Suppose that the price  $V$  of the derivative can be expressed as a suitably smooth function  $V = V(H, A, M, t)$ . Consider a portfolio  $\Pi$  consisting of +1 derivative and  $-\Delta$  HPI bond, so  $\Pi_t = V_t - \Delta_t H_t$ . Then

$$\begin{aligned} d\Pi &= dV - \Delta dH \\ &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial H} dH + \frac{\partial V}{\partial A} dA + \frac{\partial V}{\partial M} dM + \frac{1}{2} \left( \frac{\partial^2 V}{\partial H^2} (dH)^2 + \dots \right) - \Delta dH \\ &= \frac{\partial V}{\partial t} dt + \left( \frac{\partial V}{\partial H} - \Delta \right) ([\alpha(H - A) + \beta(H - M) + \gamma(r - i)H + \mu H] dt + \sigma H dW) \\ &\quad + \frac{\partial V}{\partial A} [\xi(H - A) + iH] dt + \frac{\partial V}{\partial M} mM dt + \frac{1}{2} \frac{\partial^2 V}{\partial H^2} \sigma^2 H^2 \end{aligned}$$

where  $\dots$  denotes the quadratic cross-products of  $dt$ ,  $dH$ ,  $dA$ , and  $dM$ : since only  $dH$  has a stochastic term, the terms other than  $(dH)^2$  are zero. To eliminate the non-deterministic term in  $dW$ , we choose a portfolio such that  $\Delta = \frac{\partial V}{\partial H}$ , so the above expression simplifies to

$$d\Pi = \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 H^2 \frac{\partial^2 V}{\partial H^2} + [\xi(H - A) + iH] \frac{\partial V}{\partial A} + mM \frac{\partial V}{\partial M} \right] dt. \quad (4.1)$$

But  $\Pi$  is now a riskless portfolio so, by the assumption of no arbitrage, it must grow at the risk-free rate:

$$d\Pi = r\Pi dt = \left( rV - rH \frac{\partial V}{\partial H} \right) dt. \quad (4.2)$$

So, by equating (4.1) and (4.2) and rearranging, we obtain an analogue of the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 H^2 \frac{\partial^2 V}{\partial H^2} + rH \frac{\partial V}{\partial H} + [\xi(H - A) + iH] \frac{\partial V}{\partial A} + mM \frac{\partial V}{\partial M} - rV = 0. \quad (4.3)$$

#### 4.1.1.2 Specification of boundary conditions

To price a property derivative, in addition to (4.3), we require the boundary conditions defined by the payoff of the instrument. This will typically depend only on  $H$ , the nominal property value upon expiry, expressed in nominal terms;  $A$  or  $M$  are only auxiliary processes introduced for modelling purposes, rather than being intrinsic to the contract. (Shiller and Weiss, describing home equity insurance products in [21], comment that these insurance policies would ideally be indexed for overall inflation – so, in this notation, the payoff would be based on the real property value  $\tilde{H}$  – but that most contracts of this type did not include such cost-of-living clauses.)

Using the notation of (2.2.2), our sample contracts have the following boundary conditions at  $t = T$ :

**Shared appreciation mortgage:**  $V(H_T, A_T, M_T, T) = \theta_{SA}(\kappa H_0 + (H_T - H_0)^+)$

**Home reversion plan:**  $V(H_T, A_T, M_T, T) = \theta_{HR} H_T$

#### 4.1.1.3 Pricing the derivative as a risk-neutral expectation

To obtain prices for property derivatives numerically, it will be more straightforward to express the prices as expectations under some probability measure and apply Monte Carlo techniques than it will be to directly solve the PDE. This is because the auxiliary processes  $A$  and  $M$  introduce complexity: although in this case of the complete market, we find that they are not relevant for typical payoff functions (which depend on the final value of  $H$  only), they turn out to be required for the incomplete market in Section 4.2. In addition, when we consider derivatives without fixed expiry times, we model the expiry using a hazard rate function, and it will then be natural to express the price as an expectation.

We can straightforwardly apply Feynman-Kac to (4.3) to obtain a solution:

$$\begin{aligned} V(H_t, A_t, M_t, t) &= \mathbb{E}^{\mathbb{Q}} \left[ V(H_T, A_T, M_T, T) \cdot \exp \left( - \int_t^T r_\tau d\tau \right) \mid \mathcal{F}_t \right] \\ &= e^{-\int_t^T r_\tau d\tau} \cdot \mathbb{E}^{\mathbb{Q}} [V(H_T, A_T, M_T, T) \mid H_t, A_t, M_t, t] \end{aligned} \quad (4.4)$$

where the dynamics of  $H_t$ ,  $A_t$ , and  $M_t$  under risk-neutral measure  $\mathbb{Q}$  are given by

$$\begin{aligned} dH &= rHdt + \sigma H dW^{(1)} \\ dA &= [\xi(H - A) + iH] dt \\ dM &= mMdt \end{aligned}$$

and we use the determinism of  $r_t$  to take the discount factor out of the expectation.

An alternative approach to this expectation is to note, from the dynamics defined above, that the discounted asset  $(V_t/B_t)$  is a  $\mathbb{Q}$ -martingale. ( $B_t$  is the

bank account asset defined in Section 3.2.) We can then express the property derivative price as an expectation as follows

$$\begin{aligned}
\frac{V(H_t, A_t, M_t, t)}{B_t} &= \mathbb{E}^{\mathbb{Q}} \left[ \frac{V(H_T, A_T, M_T, T)}{B_T} \mid \mathcal{F}_t \right] \\
\therefore V(H_t, A_t, M_t, t) &= B_t \cdot \mathbb{E}^{\mathbb{Q}} \left[ \frac{V(H_T, A_T, M_T, T)}{B_T} \mid H_t, A_t, M_t, t \right] \\
&= \left( \frac{e^{\left(\int_0^t r_\tau d\tau\right)}}{e^{\left(\int_0^T r_\tau d\tau\right)}} \right) \cdot \mathbb{E}^{\mathbb{Q}} [V(H_T, A_T, M_T, T) \mid H_t, A_t, M_t, t] \\
&= e^{\left(-\int_t^T r_\tau d\tau\right)} \cdot \mathbb{E}^{\mathbb{Q}} [V(H_T, A_T, M_T, T) \mid H_t, A_t, M_t, t] \quad (4.5)
\end{aligned}$$

which is the same as the price obtained by applying Feynman-Kac in (4.4). (As above, we can take  $B_T^{-1}$  outside the expectation here because we are assuming a deterministic short rate  $r$ .) The terminal expectation on the right-hand side of (4.4) is defined by the PDE's boundary conditions defined above in 4.1.1.2. Where the boundary condition on  $V(H_T, A_T, M_T, T)$  is specified only in terms of  $H_T$  (as is often the case), the dynamics of  $A_t$  and  $M_t$  therefore have no effect on the derivative price. This shows that, if an HPI bond is available to hedge the derivative, then the autocorrelative and mean-reverting characteristics, and the effect of real interest rates on house price drift, do not affect the derivative price.

#### 4.1.2 Stochastic short rates

We now consider the case where the short rate  $r_t$ , rather than being deterministic, satisfies the SDE (3.4). We will continue to treat the general inflation rate  $i_t$  as deterministic (possibly constant).

##### 4.1.2.1 Derivation of a PDE and boundary conditions for the property derivative price

To price the property derivative, we follow a similar argument to that followed for deterministic short rates, but now hedge with a (risky) bond, whose maturity  $S$  falls beyond the expiry  $T$  of the derivative, as well as the HPI bond as before. Suppose that the price  $V$  of the option can be expressed as a suitably smooth function  $V = V(H, A, M, r, t)$ . Consider a portfolio  $\Pi$  consisting of +1 option,  $-\Delta^{(1)}$  HPI bond, and  $-\Delta^{(2)}$  bond with expiry  $S$  (where  $S > T$ ), so then  $\Pi_t = V_t - \Delta_t^{(1)} H_t - \Delta_t^{(2)} P_t^S$ . Then

$$\begin{aligned}
d\Pi &= dV - \Delta^{(1)}dH - \Delta^{(2)}dP^S \\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial H}dH + \frac{\partial V}{\partial A}dA + \frac{\partial V}{\partial M}dM + \frac{\partial V}{\partial r}dr \\
&\quad + \frac{1}{2} \left( \frac{\partial^2 V}{\partial H^2}(dH)^2 + \frac{\partial^2 V}{\partial r^2}(dr)^2 + 2\frac{\partial^2 V}{\partial H \partial r}(dH dr) + \dots \right) - \\
&\quad - \Delta^{(1)}dH - \Delta^{(2)}dP^S \\
&= \frac{\partial V}{\partial t}dt + \left( \frac{\partial V}{\partial H} - \Delta^{(1)} \right) dH + \frac{\partial V}{\partial A} [\xi(H - A) + iH] dt + \frac{\partial V}{\partial M} mM dt \\
&\quad + \frac{\partial V}{\partial r} (\mu_r dt + \varsigma dW^{(2)}) \\
&\quad + \frac{1}{2} \left( \frac{\partial^2 V}{\partial H^2} \sigma^2 H^2 + \frac{\partial^2 V}{\partial r^2} \varsigma^2 + 2\frac{\partial^2 V}{\partial H \partial r} \sigma H \varsigma \rho \right) dt + \\
&\quad - \Delta^{(2)} \left( \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial r} (\mu_r dt + \varsigma dW^{(2)}) + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \varsigma^2 dt \right)
\end{aligned}$$

As before,  $\dots$  indicates the remaining quadratic cross-products of  $dt$ ,  $dH$ ,  $dA$ ,  $dM$ , and (newly)  $dr$ : now both  $dH$  and  $dr$  (only) have stochastic terms, but the others remain deterministic so have cross-products of 0. To hedge the portfolio and make its dynamics risk-free, we choose  $\Delta^{(1)}$  and  $\Delta^{(2)}$  to eliminate the stochastic terms:

$$\Delta^{(1)} = \frac{\partial V}{\partial H} \quad \text{and} \quad \Delta^{(2)} = \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r}$$

Substituting these in yields the risk-free dynamics:

$$\begin{aligned}
d\Pi &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial A} [\xi(H - A) + iH] dt + \frac{\partial V}{\partial M} mM dt \\
&\quad + \frac{1}{2} \left( \frac{\partial^2 V}{\partial H^2} \sigma^2 H^2 + \frac{\partial^2 V}{\partial r^2} \varsigma^2 + 2\frac{\partial^2 V}{\partial H \partial r} \sigma H \varsigma \rho \right) dt \\
&\quad - \frac{\left( \frac{\partial V}{\partial r} \right)}{\left( \frac{\partial P}{\partial r} \right)} \left( \frac{\partial P}{\partial t} + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \varsigma^2 \right) dt
\end{aligned} \tag{4.6}$$

But since  $\Pi$  is risk-free, it must also satisfy

$$\begin{aligned}
d\Pi &= r\Pi dt \\
&= r \left( V - \frac{\partial V}{\partial H} H - \frac{\left( \frac{\partial V}{\partial r} \right)}{\left( \frac{\partial P}{\partial r} \right)} P \right) dt
\end{aligned} \tag{4.7}$$

so, by equating (4.6) and (4.7), we obtain

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 H^2 \frac{\partial^2 V}{\partial H^2} + rH \frac{\partial V}{\partial H} + [\xi(H - A) + iH] \frac{\partial V}{\partial A} + mM \frac{\partial V}{\partial M} - rV \\
+ \frac{1}{2} \frac{\partial^2 V}{\partial r^2} \varsigma^2 + \frac{\partial^2 V}{\partial H \partial r} \sigma H \varsigma \rho \\
+ \frac{\left( \frac{\partial V}{\partial r} \right)}{\left( \frac{\partial P}{\partial r} \right)} \left( rP - \frac{\partial P}{\partial t} - \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \varsigma^2 \right) = 0
\end{aligned}$$

Now we apply the term structure equation (3.5) to show the final bracketed term  $\left(rP - \frac{\partial P}{\partial t} - \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \varsigma^2\right) = \nu \frac{\partial P}{\partial r}$ , and observe the cancellation of  $\frac{\partial P}{\partial r}$ , to obtain an analogue of the Black-Scholes PDE:

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 H^2 \frac{\partial^2 V}{\partial H^2} + rH \frac{\partial V}{\partial H} + [\xi(H - A) + iH] \frac{\partial V}{\partial A} + mM \frac{\partial V}{\partial M} \\ + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} \varsigma^2 + \frac{\partial^2 V}{\partial H \partial r} \sigma H \varsigma \rho + \nu \frac{\partial V}{\partial r} - rV = 0 \end{aligned} \quad (4.8)$$

#### 4.1.2.2 Pricing the derivative as a risk-neutral expectation

The more direct approach is the second, martingale-based approach in 4.1.1.3. In this case of stochastic short rates, however, the expression  $B_T = \exp\left(\int_0^T r_\tau d\tau\right)$  cannot be simply taken out of the expectation. Following the argument which previously led to (4.5) now yields

$$V(H_t, A_t, M_t, r, t) = \mathbb{E}^{\mathbb{Q}} \left[ V(H_T, A_T, M_T, T) \cdot e^{(-\int_t^T r_\tau d\tau)} \mid H_t, A_t, M_t, t \right] \quad (4.9)$$

where the dynamics of  $H_t$ ,  $A_t$ ,  $M_t$ , and  $r_t$  under the risk-neutral measure  $\mathbb{Q}$  are

$$\begin{aligned} dH &= [(\alpha(H - A) + \beta(H - M) + \gamma(r - i)H + \mu H) - \omega \sigma H] dt + \sigma H dW^{(1)} \\ dA &= [\xi(H - A) + iH] dt \\ dM &= mM dt \\ dr &= \nu(t, r) dt + \varsigma(t, r) dW^{(2)}. \end{aligned}$$

Expressing the price of the property derivative in this form, and after selecting parameters for the house price dynamics model and the chosen short interest rate model, it is reasonably straightforward to compute the price numerically using Monte Carlo methods.

## 4.2 Pricing property derivatives without a market in the underlying property asset

We now consider the pricing of a property derivative in the absence of the HPI bond used in the above construction. This results in an incomplete market: it is not possible to replicate the contingent claim. We follow the approach similar to that described by Björk in [5], adapted to incorporate the additional auxiliary processes  $A_t$  and  $M_t$ .

### 4.2.1 The market price of risk for property derivatives

That approach is based around the construction of a *market price of risk* process, which we denote by  $\lambda^{(H)}(t)$ , and which is independent of the property derivative under consideration. The derivation of property derivative prices in

this incomplete market is then similar to that in the case of a complete market. The difference arises in the expression of the drift under the pricing measure: in the complete market this is based on the short rate and given by  $rH$ ; but in the incomplete market it is instead given by

$$\left[ (\alpha(H - A) + \beta(H - M) + \gamma(r - i)H + \mu H) - \lambda^{(H)}\sigma H \right].$$

This risk-neutral drift is not unique, but depends on the choice of  $\lambda^{(H)}(t)$ . As such, the model does not fix a *unique* arbitrage-free price for the property derivative. If we can observe a price for some tradeable derivative based on  $H$ , then this can be used as a benchmark to obtain the market's view of  $\lambda^{(H)}(t)$ : this will then force unique arbitrage-free prices for all other derivatives based on  $H$ . However, since the market in property derivatives does not yet appear well-developed, it may not be straightforward in practice to derive this market price of risk from it.

(An alternative approach, which we will not consider further here, is to identify an asset (or combination of assets) whose returns are *correlated* to the returns on property, but where the correlation is not perfect and so where the market is not complete. Examples of such assets may include shares in companies in the building and construction sector. The identified asset can then be used to hedge the property derivative, but the lack of perfect correlation introduces basis risk; one approach to constructing an optimal hedging strategy in this case is to consider the problem within a utility-maximisation framework.)

#### 4.2.2 Obtaining expressions for pricing property derivatives

For a specified market price of risk process, it is now straightforward to obtain expressions for pricing property derivatives.

##### 4.2.2.1 Deterministic short rates

In the case of deterministic short rates  $r_t$ , the derivative price  $V(H_t, A_t, M_t, t)$  satisfies the PDE

$$\begin{aligned} & \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 H^2 \frac{\partial^2 V}{\partial H^2} + [\xi(H - A) + iH] \frac{\partial V}{\partial A} + mM \frac{\partial V}{\partial M} - rV \\ & + \left[ (\alpha(H - A) + \beta(H - M) + \gamma(r - i)H + \mu H) - \lambda^{(H)}\sigma H \right] \frac{\partial V}{\partial H} = 0. \end{aligned} \quad (4.10)$$

where the boundary conditions are unchanged, as stated in 4.1.1.2.

To express the derivative price as an expectation, we construct a pricing measure  $\mathbb{S}(\lambda)$ , which depends on the chosen market price of risk process  $\lambda^{(H)}$ , and then express the derivative price as an expectation under  $\mathbb{S}(\lambda)$ :

$$\begin{aligned}
V(H_t, A_t, M_t, t) &= \mathbb{E}^{\mathbb{S}(\lambda)} \left[ V(H_T, A_T, M_T, T) \cdot \exp \left( - \int_t^T r_\tau d\tau \right) \mid \mathcal{F}_t \right] \\
&= e^{-\int_t^T r_\tau d\tau} \cdot \mathbb{E}^{\mathbb{S}(\lambda)} [V(H_T, A_T, M_T, T) \mid H_t, A_t, M_t, t]
\end{aligned}$$

where the dynamics of  $H_t$ ,  $A_t$ , and  $M_t$  under the pricing measure  $\mathbb{S}(\lambda)$  are

$$\begin{aligned}
dH &= \left[ (\alpha(H - A) + \beta(H - M) + \gamma(r - i)H + \mu H) - \lambda^{(H)} \sigma H \right] dt + \sigma H dW^{(1)} \\
dA &= [\xi(H - A) + iH] dt \\
dM &= mM dt
\end{aligned}$$

#### 4.2.2.2 Stochastic short rates

The derivation in the market with stochastic short rates is similar. Since the PDE construction is similar and less useful for numerical computation, we omit this, and include only the expression for pricing a property derivative as an expectation under the pricing measure  $\mathbb{S}(\lambda)$ :

$$V(H_t, A_t, M_t, t, r) = \mathbb{E}^{\mathbb{S}(\lambda)} \left[ V(H_T, A_T, M_T, T, r_T) \cdot e^{(-\int_t^T r_\tau d\tau)} \mid H_t, A_t, M_t, t, r_t \right]$$

where the dynamics of  $H_t$ ,  $A_t$ ,  $M_t$ , and  $r_t$  under the pricing measure  $\mathbb{S}(\lambda)$  are

$$\begin{aligned}
dH &= \left[ (\alpha(H - A) + \beta(H - M) + \gamma(r - i)H + \mu H) - \lambda^{(H)} \sigma H \right] dt + \sigma H dW^{(1)} \\
dA &= [\xi(H - A) + iH] dt \\
dM &= mM dt \\
dr &= \nu(t, r)dt + \varsigma(t, r)dW^{(2)}.
\end{aligned}$$

## Chapter 5

# Pricing property derivatives without fixed expiry times

As discussed in Chapter 2, property derivatives typically do *not* have a fixed expiry time, but instead expire upon the sale of the underlying property or the homeowner permanently moving out of the property. The lender therefore has no control over the expiry of the contract; the homeowner has the option to trigger the expiry of the contract, but circumstances outside the homeowner's control may force this expiry at a time not of their choosing. (For products aimed at older homeowners, the expiry may be occur when the homeowner dies or moves into long-term residential care.)

To price a property derivative without a fixed expiry, we require a model for the expiry time.

### 5.1 Models for mortgage prepayment

There appears to be little literature on the early exercise of property derivatives. However, there is extensive literature in a related problem: that of valuing mortgages, and hence mortgage-backed securities (MBSs), where the borrower has the option to repay the loan before maturity (referred to as *prepayment*). Gorovoy and Linetsky, in [13], provide an overview of this problem and approaches to its solution.

Considering the US market, they characterise a typical fully-amortising fixed-rate mortgage as an annuity with fixed monthly payments, but with the feature that it allows prepayment of the outstanding principal at any time prior to maturity without penalty. (For example, if interest rates were to fall after a borrower took out a mortgage, this feature would allow the borrower to refinance, repaying the original mortgage and taking out a new mortgage at this lower interest rate.) In order to value mortgages and MBSs, a model is required for the time at which a borrower will repay the principal: prepayments are described by Kalotay *et al.* in [15] as the dominant consideration when valuing

an MBS. (This is more relevant to the US mortgage market, where fixed-rate mortgages have been widely used, than that in the UK, where variable-rate mortgages are common.) There are two broad pricing approaches in the literature: the option-based approach, and the reduced-form approach.

The *option-based approach* treats a mortgage as an annuity with an embedded American-style option to repay. The mortgage is valued on the assumption that the borrower will exercise this option optimally: if the valuation of the existing mortgage were to exceed the outstanding principal, then they would refinance the loan at a lower effective rate of interest. This approach was pioneered by Dunn and McConnell (1981) in [10], and has been extended by, among others, Stanton [22] and Kalotay *et al.* [15]. It results in the usual challenges arising from American options, in identifying the optimal exercise strategy and the corresponding price.

The *reduced-form approach*, in contrast, does not assume that the borrower will act optimally, but treats the prepayment time as a random time based on an intensity or hazard rate. An early form of this approach was put forward by Schwartz and Torous (1989) [18]. It is based on the assumption that, in an infinitesimal interval, there is a probability of prepayment expressed as a function of the state of the economy. Historical data about prepayment rates can be used to calibrate the hazard rate function. This approach is similar to the reduced-form approach to pricing credit default swaps (CDSs), where a mortgage prepayment corresponds to a CDS default event.

## 5.2 Applying prepayment models to property derivatives

The expiry of property derivatives can reasonably be modelled in a similar way to the prepayment of a mortgage: both relate to a decision, partly or wholly under the control of the borrower, to trigger the expiry of the derivative. The approaches identified above can be adapted to these products.

The option-based approach has one clear advantage: it eliminates the possibility of arbitrage on the part of the homeowner, by pricing the product on the assumption that the homeowner will exercise the option optimally.

However, the assumption that a homeowner is in a position to follow an optimal exercise strategy is open to challenge on two grounds:

- A homeowner may be unlikely to have the financial sophistication to identify an optimal exercise strategy; and
- A homeowner may be forced by exogenous factors (e.g. ill-health, mortality, or a requirement to relocate) to exercise before it is optimal to do so, or (less likely) to refrain from exercising when it would be optimal to do so.

For these reasons, we primarily consider a reduced-form model.

### 5.3 Applying the reduced form model

We consider a reduced-form model for the case of the incomplete market: the complete market is similar. To set this model up, it will be useful to let

$\tau$  be the (random) time of exercise,

$\mathcal{G}_t$  be the natural filtration  $\mathcal{F}_t$  extended to include the events of whether exercise had occurred before time  $t$ :  $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\{\{\tau < u\}, u \leq t\}$ , and

$\bar{\mathbb{S}}(\lambda)$  be the pricing measure  $\mathbb{S}(\lambda)$  extended to the random variable  $\tau$ ; where there is no ambiguity, we will omit the indication of the market price of risk  $\lambda$ .

Now, following the approach to credit default swaps and filtration-change of [6], we can write down the price of the property derivative as an expectation under the pricing measure  $\bar{\mathbb{S}}(\lambda)$ :

$$\begin{aligned} V(H_t, A_t, M_t, t, r) &= \mathbb{E}^{\bar{\mathbb{S}}} \left[ V(H_\tau, A_\tau, M_\tau, T, r_\tau) \cdot e^{(-\int_t^T r_u du)} \mid \mathcal{G}_t \right] \\ &= \frac{1_{\{t < \tau\}}}{\bar{\mathbb{S}}\{t < \tau \mid \mathcal{F}_t\}} \mathbb{E}^{\bar{\mathbb{S}}} \left[ V(H_\tau, A_\tau, M_\tau, T, r_\tau) \cdot e^{(-\int_t^T r_u du)} \mid \mathcal{F}_t \right] \end{aligned}$$

where the dynamics of  $H_t$ ,  $A_t$ ,  $M_t$ , and  $r_t$  under the pricing measure  $\bar{\mathbb{S}}(\lambda)$  are

$$\begin{aligned} dH &= \left[ (\alpha(H - A) + \beta(H - M) + \gamma(r - i)H + \mu H) - \lambda^{(H)} \sigma H \right] dt + \sigma H dW^{(1)} \\ dA &= [\xi(H - A) + iH] dt \\ dM &= mM dt \\ dr &= \nu(t, r) dt + \varsigma(t, r) dW^{(2)} \end{aligned}$$

and the distribution of  $\tau$  is considered below.

We treat  $\tau$  as the first jump time of a Poisson process with some intensity process  $\eta_t$ . The Poisson process has well-known properties: the probability of a jump in a short interval is given by

$$\bar{\mathbb{S}}(\tau \in [t, t + \Delta t] \mid \mathcal{F}_t \text{ and } \tau > t) = \eta_t \Delta t + o(\Delta t) \quad (5.1)$$

and the unconditional survival probability is given by

$$\bar{\mathbb{S}}(\tau \geq t) = e^{(-\int_t^T \eta_u du)}.$$

The choice of  $\eta_t$  is not specified by the model. In the CDS market, the price of CDSs implies the market's view of the intensity process for credit default events, but there is no similarly liquid market in property derivatives. However, an empirical objective probability measure for expiry can be observed, based on data on the distribution of how long homeowners stay in their homes; this may help to inform the choice of intensity process for pricing.

To implement a Monte Carlo simulation which includes a non-fixed expiry, there are two approaches. If the intensity process  $\eta$  is independent of the other processes  $H$ ,  $A$ ,  $M$ , and  $r$ , then we can simulate the expiry time immediately at the beginning of the path generation. More generally, if the intensity process  $\eta$  may depend on those other processes, then we can use (5.1) to calculate, at the start of each timestep, the probability of a jump (and hence expiry) within that timestep and use a random number to simulate whether this occurs, at the cost of significantly more computational effort.

## Chapter 6

# Conclusion

The objective of this dissertation was to develop an approach for pricing retail financial products – usually structured as loans – which transfer exposure to movements in the price of a homeowner’s property from the homeowner to a bank or other financial institution. We reviewed the academic and industry literature to identify and characterise these products, and demonstrated how they could be viewed as derivatives where the property is the underlying asset. We formally defined two types of contract for further consideration.

The literature on house price dynamics identified characteristics relevant to the construction of a model for house prices, and ways in which the housing market differed from the equity markets, where the pricing of derivatives has been well-studied. We also obtained historical UK house price data for the period 1983-2006 – a period which included a variety of economic behaviour. Based on the identified characteristics, we developed a basic model for house prices based on real interest rates and the autocorrelative properties of house price movements, and found it to be a reasonable fit for our historical data. The fit was considerably improved when the model was extended to include GDP growth, but using this in pricing would introduce significant additional complexities, so we did not develop the use of this extended model further here.

Instead, we first used our basic model for house price dynamics to price property derivatives with a fixed expiry time. If the market could be treated as containing a tradeable asset replicating house price movements, then it was found to be complete, and unique prices for property derivatives were found (expressed either as the solution of a PDE analogous to the Black-Scholes equation or as a risk-neutral expectation). If we did not assume the existence of such a tradeable asset, then the market was incomplete; expressions for pricing property derivatives included a term representing the market price of risk, and it was noted that, in the absence of a well-developed market in property derivatives, that price may not be available.

These pricing methods were then extended to property derivatives *without* a fixed expiry time. The approaches considered were based on the extensive literature on modelling early prepayment of mortgages for the purposes of valuing

mortgage-backed securities. We suggested that, because of the difficulties which homeowners may have in *optimally* triggering the expiry of property derivative contracts, a reduced-form approach for modelling early exercise was more appropriate than an option-based approach; we demonstrated how such an approach could be applied to pricing property derivatives.

As we obtained expressions for pricing property derivatives, we also considered how these prices could be obtained explicitly with numerical methods. For derivatives with fixed expiry times, pricing could be performed either by numerical solution of a PDE (e.g. through finite-difference methods) or by estimating a risk-neutral expectation with Monte Carlo methods. When we progressed to non-fixed expiry times and the reduced-form approach, however, the lack of a fixed boundary condition at expiry made Monte Carlo methods the more natural for numerical solution.

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