Aspects of Supersymmetry and Gauge Theories



Melissa Duncan

Merton College

University of Oxford

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Contents

1 Introduction **8**

1.1 Motivations 8

1.2 Outline 10

1.3 Original Contribution 10

2 Witt Construction of Supersymmetry Algebras **12**

2.1 Some Preliminaries 12

2.1.1 Witt Constructions 13

2.1.2 Relevant Spinor Modules 14

2.2 Constructing $\mathfrak{so}(p,q)$ Superalgebras 16

2.2.1 The Super-Poincaré Algebra 16

2.2.2 General Construction for $\mathfrak{so}(p,q)$ 17

2.3 Including Additional Symmetries 21

2.3.1 Extended Supersymmetry 21

2.3.2 Internal Symmetries 22

2.3.3 Reality Conditions 23

3 Vacuum Properties of SUSY dS Space **26**

3.1 The Unruh Effect 26

3.2 Algebraic QFT 27

3.3 Unruh Effect in dS space 31

3.4 Towards Graded Tomita-Takesaki Theory 32

3.4.1 Some Useful Structures 33

3.4.2 Modular Hilbert Algebras with Krein Structure 34

4 Lie 2-Algebras and Gauge Anomalies 36

4.1 Review of Categorification of Lie Algebras 36

4.1.1 The Idea of Categorification 36

4.1.2 Categorified Lie Groups and Lie Algebras 38

4.1.3 Relationship to Crossed Modules 41

4.1.4 Definition of Lie Algebra Cohomology 42

4.2 Lie 2-Algebras and Witt Constructions 43

4.2.1 Defining a 3-Cocycle 43

4.3 Applications to Gauge Anomalies 45

4.3.1 Basic Extension Diagrams 46

4.3.2 Anomaly Examples 49

5 Vacuum Geometry of SUSY Quiver Gauge Theories 51

5.1 Preliminaries on Quiver Gauge Theories 53

5.1.1 Quiver Diagrams 53

5.1.2 Quiver Theories 53

5.1.3 The Action 54

5.2 Parameterising the Vacuum 55

5.2.1 Algebraic Varieties 55

5.2.2 The Vacuum as a Moduli Space 56

5.2.3 Note on Disconnected Quivers 57

5.3 Outline of Algorithms 58

5.3.1 Generating Quivers 58

5.3.2 Generating Gauge Invariants and Superpotentials 59

5.3.3 Computing the Moduli Space 59

5.3.4 Some Illustrative Examples 60

5.4 Computational Results 64

5.4.1 Two Node Quivers 66

5.4.2 Three Node Quivers 68

5.4.3 Larger Data Sets 70

6 Conclusions and Outlook **73**

Abstract

In this thesis we consider generalisations of symmetry structures, and their applications to quantum field theories and associated vacuum spaces. The construction of supersymmetry algebras is described in abstract notation. We then consider applications of a further generalisation of symmetry structures, namely categorified groups and algebras. In particular, the links between gauge anomalies as 3-cocyle obstructions and Lie 2-algebras are discussed. Finally, we detail algorithms we have implemented to generate statistics describing the vacuum spaces for supersymmetric non-toric Calabi-Yau quiver gauge theories. A selection of results are presented for low order quivers.

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Chapter 1

Introduction

There are two common themes underlying this thesis: supersymmetry and gauge symmetries. Whilst the sections of this thesis can be treated as self-contained, all the topics arose from a general intention to investigate the structure of symmetries relevant to quantum field theories.

1.1 Motivations

In constructing a quantum field theory one generally begins with a collection of symmetry principles. In this thesis we are primarily concerned with generalisations of the standard notion of symmetry as a Lie group or Lie algebra. There are two directions one can take from here: the first is to extend symmetries to supersymmetries by developing graded versions of Lie groups and Lie algebras. Alternatively, categorisation of concepts such as a group or algebra lead to notions of Lie n-groups or Lie n-algebras. In a loose sense these objects introduce the idea of having two layers of symmetries, or the idea that the group or algebra structure is only satisfied up to particular transformations.

Supersymmetries first appeared in connection with field theories when attempts were made to generalise symmetries for realistic models satisfying various physical conditions. According to the Coleman-Mandula theorem [1], internal symmetries can only be trivially linked to the spacetime Poincare group. That is, any physically

relevant symmetry group must be in the form of a direct product of these two groups. Haag, Lopuszanski and Sohnius [2] found a way around this no-go theorem, showing that there is a non-trivial extension for symmetries if we replace the usual Lie group describing symmetries with a Lie supergroup. Although this theorem does not apply directly in the absence of an abelian translation subgroup, there are similar arguments that show supersymmetry needs to be introduced in the same way to extend other spacetime symmetry groups [3].

The study of categorised groups and algebras is more recent. Categorised versions of groups and Lie algebras have been introduced in detail in a series of papers by Baez and collaborators [4, 5] purely using the tools of category theory. These algebras have since been found to have numerous connections to quantum field theories. Perhaps the most developed direction is the idea of higher gauge theory, in which gauge groups are replaced by their categorised versions. Most notably there are 2-group versions of electromagnetism [6].

Whilst there are a plethora of ways to choose the symmetry principles and field content from which to develop a model, constructing models to describe particular low energy physics at the outset is often difficult. In a third direction, we investigate the signatures of setups of supersymmetric gauge theories that appear in the classical vacuum space.

Of interest are a particular set of supersymmetric gauge theories known as quiver theories. These are standard gauge theories, where the group is simply a direct product of Lie groups which arise in the low energy limit of string theory models. So far in the literature the classification has focussed on toric quiver theories which are relevant to brane tilings. In this preliminary study we collect statistics for the broader class of non-toric quiver theories.

1.2 Outline

This thesis consists of four sections. The first explores in some detail the extension of $\mathfrak{so}(p,q)$ Lie algebras to superalgebras. The Witt construction provides a standard way of combining a Lie algebra and a module into a new algebra when certain conditions are satisfied. We highlight the fact that the superalgebra construction is a generalisation of this construction. We briefly discuss extended supersymmetry and reality conditions in this language.

In the second chapter, we take a slight detour and discuss the links between vacuum properties of spaces and the structure of symmetry algebras. In particular we detail the Unruh effect in de Sitter space and mention some initial steps made towards extending the Unruh effect to superspaces.

In the third chapter, we explore some aspects of categorified Lie algebras. After introducing the basic definitions for these objects we draw links between these and Lie algebra extensions through exact sequences. Gauge anomalies have been described as obstructions to Lie algebra extensions. We note how a gauge anomaly can be characterised by a particular Lie 2-algebra equivalence.

Finally, we initiate a study into the vacuum moduli spaces of quiver gauge theories using computational algebraic geometry. We outline the basic concepts behind our calculations and describe the algorithms used in detail. In the final sections we present a selection of results from our current dataset.

1.3 Original Contribution

The abstract notation used here in chapter 2 for the construction of superalgebras is well known [7]. However the link to the usual Witt construction and relevance of the Casimir elements mentioned on page 21 is highlighted. We believe there is value in presenting extra structures placed on the superalgebras also in this language.

Although links between crossed modules and Lie 2-groups are well explored in the literature, viewing the presence of gauge anomalies in the language of categorified

gauge theory in section 4.3.2 is a new idea.

Finally, the vacuum space statistics of quiver gauge theories collected in section 5.4 is original and is currently being developed into a publication.

Chapter 2

Witt Construction of Supersymmetry Algebras

The structure of supersymmetric extensions to classical Lie algebras has been calculated explicitly in most cases relevant to physics. However, it is important to construct the minimal extensions in a formal manner so as the general features of these algebras are obvious. We view the construction as an explicit generalisation of a Witt construction [8] which states conditions for combining a direct sum of two Lie algebras into a new Lie algebra for the graded case. After completing this work it was discovered that superalgebras were already discussed in this general notation in the literature [7]. However, we include in addition a link to Casimir elements and mention reality conditions and extended algebras in our formalism.

2.1 Some Preliminaries

We define a super Lie algebra by placing a $\mathbb{Z}/2$-grading on a Lie algebra. Hence, a super Lie algebra is a $\mathbb{Z}/2$-graded vector space $\mathfrak{g}$ with bilinear bracket $[,] : \mathfrak{g} \oplus \mathfrak{g} \mapsto \mathfrak{g}$. The bracket is even so we have $\forall a, b \in \mathfrak{g}$ that

$$[a, b] = (-1)^{p(a)p(b)}[b, a]$$

where $p(a) = 0$ or 1 for a belonging to the even and odd subspaces respectively. Also, the bracket must satisfy a graded version of the usual Jacobi identity so $\forall a, b, c \in \mathfrak{g}$ we must have

$$(-1)^{p(a)p(c)}[a,[b,c]] + (-1)^{p(a)p(b)}[b,[c,a]] + (-1)^{p(b)p(c)}[c,[a,b]] = 0$$

The super Lie algebra generates a Lie supergroup G which has the structure of a supermanifold equipped with a smooth map $G \times G \mapsto G$.

The supermanifold is a generalisation of a manifold for which the local coordinate rings are isomorphic to $C^\infty(\mathbb{R}^p) \otimes (\eta_1, ..\eta_q)$ for (p, q) dimensions where the η_i are Grassman variables. In analogy with the usual structure, the superalgebra corresponds to left invariant fields on the supergroup.

Quantum particles are classified as irreducible representations of the relevant symmetry group. Hence, particles with supersymmetry correspond to representations of the relevant Lie supergroups. In the Poincaré case [9], the superalgebra can be viewed as a Clifford algebra for each momentum representation. Hence, the representation breaks into a finite sum of irreducible representations of the Poincare group referred to as supermultiplets, which are essentially a boson and fermion pair. However, more generally for $\mathfrak{so}(p, q)$, this is no longer true and the structure of the superalgeba representations must be investigated in more detail.

2.1.1 Witt Constructions

The Witt construction describes how to construct a new Lie algebra from the direct sum of a semisimple Lie algebra $\mathfrak{g}_0$ and a module $\mathfrak{g}_1$ when particular conditions are satisfied. It requires the existence of a symmetric invariant form on the algebra which we take to be the Killing form K and another on the module which we denote by ψ. To construct a Lie structure on $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ we define the adjoint action of $\mathfrak{g}_0 \oplus 1$ on $1 \oplus \mathfrak{g}_1$ to be the module action. The difficulty arises in the definition of the bracket on the module. Witt determined that a bracket on $\mathfrak{g}_1$ elements is explicitly given by the solution to the equation

$$K([A, B], X)) = \psi(A, X.B) \tag{2.1}$$

where $A, B \in \mathfrak{g}_1$, $X \in \mathfrak{g}_0$ and $X.B$ denotes the module action of X on B. In order for this construction to work for a nontrivial module there are certain constraints on the Casimirs of $\mathfrak{g}_o$ [10]. We note that these additional restraints are irrelevant if $\dim(\mathfrak{g}_1) < 4$.

We generalise this construction for superalgebras by taking $\mathfrak{g}_1$ to be a spinor module and exploiting the existence of an antisymmetric invariant form on the module. We then investigate what conditions on the structure of the spinor modules are required for the sum to satisfy the super Jacobi identity. In most cases we would like a real spinor algebra in order to construct physical theories and so investigate compatible involutions.

2.1.2 Relevant Spinor Modules

We consider the spinor representations of the $\mathfrak{so}(p,q)$, that is, the representations of the double cover $\mathrm{Spin}(p,q)$ where $p+q=n$. First we recall the Clifford algebra $Cl_{p,q}(\mathbb{C})$ contains $\mathrm{Spin}(p,q)$ as a subgroup, and is determined by the defining relation

$$v^2 = g(v,v) \tag{2.2}$$

where $v \in Cl_{p,q}(\mathbb{C})$ and g is a metric with the signature (p,q). The structure of the Clifford algebra representations are different depending on whether n is even or odd so we discuss these cases separately.

We recall that for n even we have a representation on $\mathbb{C}^{n/2}$ which decomposes into two irreducible representations on $\mathbb{C}^{n/2-1}$ referred to as Weyl spinors. However for the odd case there is a single representation on $\mathbb{C}^{(n-1)/2}$ that no longer decomposes nicely as in the even case. We note the even subalgebra of $Cl_n(\mathbb{C})$ is isomorphic to $Cl_{n-1}(\mathbb{C})$ and so we can construct the representation of Spin algebras with n odd from cases of n even.

Since we shall construct a map from the symmetric subspace of the module we consider the Clebsch-Gordon decomposition of the $\mathrm{Spin}(p,q)$ modules. If n is even with a Weyl spinor module $\mathbb{C}^n$ we have

$$\mathbb{C}^n \otimes \mathbb{C}^n = \oplus_{i=0}^{n} \Lambda^i \mathbb{C}^n \tag{2.3}$$

And if n is odd with spinor module $\mathbb{C}^n$ the decomposition is given by

$$\mathbb{C}^n \otimes \mathbb{C}^n = \oplus_{i=0}^{(n-1)/2} \Lambda^{2i+1}\mathbb{C}^n \tag{2.4}$$

Some important decompositions that arise in the discussion of the superalgebras are the following. Denoting the representation on $\mathbb{C}^n$ by $\mathbf{n}$ in the standard fashion, we have for the dS and AdS ($n = 5$) cases the decomposition

$$\mathbf{4} \otimes_{\mathbf{S}} \mathbf{4} = \mathbf{1} \oplus \mathbf{5} \oplus \mathbf{10} \tag{2.5}$$

In the case of the conformal algebra ($n = 6$), we calculate the decomposition of the tensor product of the combination of the module $\mathbf{4}$ and its conjugate $\mathbf{4}^*$ to obtain

$$(\mathbf{4} \oplus \mathbf{4}^*) \otimes_{\mathbf{S}} (\mathbf{4} \oplus \mathbf{4}^*) = \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{10}^* \oplus \mathbf{15} \tag{2.6}$$

Also of interest for our later construction is the existence of forms s on the various spinor representations for different dimensions. There is a detailed analysis in [7] but we simply state below the relevant results detailing the symmetry of these forms.

n	Symmetry of s
4	Skew-Symmetric
3,5	Skew-Symmetric
2,6	Skew-Symmetric[1]
1,7	Symmetric

As an explicit example we construct the representation of $\mathrm{Spin}(1,4)$ from that of $\mathrm{Spin}(1,3)$. We take our representation space for $\mathrm{Spin}(1,3)$ to be the Dirac gamma matrices [11] which form a Clifford algebra with the property

$$\{\gamma_a, \gamma_b\} = 2g_{ab}$$

The operators given by

$$M_{ab} = \frac{1}{2}\Gamma_{ab}, \quad \text{where} \quad \Gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b] \tag{2.7}$$

[1]This is a special case where the form only exists on the Dirac spinor representation

are a representation of $\mathfrak{so}(1,3)$. We now introduce a further four operators

$$M_{a5} = \frac{1}{2}\Gamma_{a5} = \pm\frac{i}{2}\gamma_a$$

The Γ_{a5} are easily shown to satisfy

$$\frac{1}{2}[\Gamma_{ab}, \Gamma_{c5}] = \eta_{bc}\Gamma_{a5} - \eta_{ac}\Gamma_{b5}$$
$$\text{and} \quad \frac{1}{2}[\Gamma_{a5}, \Gamma_{c5}] = -\Gamma_{ac}$$

Hence we have a representation of $\mathfrak{so}(1,4)$.

We find we can take the usual symplectic form $s = C$ referred to as charge conjugation from the even case that satisfies $C = C^{-1} = -C^T$.

2.2 Constructing $\mathfrak{so}(p,q)$ Superalgebras

Initially we outline the structure of the super-Poincaré algebra. In fact, only the abelian translation subgroup is extended to a superalgebra. The super-Poincaré algebra then taken to be the semi-direct product of the Lorentz cover group $\mathfrak{sl}(2,\mathbb{C})$ with the supertranslation group. However, in the more general case we must extend the entire group to a graded algebra as the subgroup is no longer commutative.

We place particular emphasis on the dS and AdS superalgebras. Nieuwenhuizen et al [12] construct the super extensions of both $\mathfrak{so}(2,3)$ and $\mathfrak{so}(4,1)$ directly in terms of a standard set of generators and discuss the relevant internal symmetry groups. Differences arise between the dS and AdS superalgebras when introducing an involution on the superalgebra. There are restrictions on the possible internal symmetry groups unlike in the case of the Poincaré algebra: the internal symmetry group is restricted to $O(p,q)$ with $p+q=N$ for the AdS case and $O^*(N) = O\left(\frac{N}{2}, \mathbb{H}\right)$ for the dS case.

2.2.1 The Super-Poincaré Algebra

The existence of the abelian subgroup of translations enables one to construct a minimal superalgebra where we may simply extend the smaller translation subgroup

via a Witt construction rather than the entire algebra. Alternatively we interpret this special case as taking a module with a trivial action of $\mathfrak{sl}(2,\mathbb{C})$. This derivation is a summary of that in [7].

We start by considering a spinor module for the translation subgroup n_0 that we denote by n_1. The super-Poincaré algebra is defined to be the algebra

$$\mathfrak{sl}(2,\mathbb{C}) \ltimes (n_0 \oplus n_1)$$

where $n_0 \oplus n_1$ is a super Lie algebra. We must determine the conditions on the structure of $n_0 \oplus n_1$ to ensure the Jacobi identity is satisfied. Since n_0 is abelian, the identity $[n_0, n_0, n_0]$ is satisfied. Also by taking n_0 to be central in $n_0 \oplus n_1$ and noting we must have $[n_1, n_1] \in n_0$ it is easy to check the Jacobi identity is valid for $[n_0, n_0, n_1]$ and $[n_1, n_1, n_1]$. Finally we consider a suitable definition of the bracket on n_1 such that the case $[n_0, n_1, n_1]$ is also satisfied.

First we note that n_0 and n_0^* are isomorphic to $(\mathbb{R}^4, +)$.

So we have a symmetric invariant form ψ on n_0 simply given by the usual vector space duality where

$$\psi(n_0^*, n_0) = n_0^*(n_0) \tag{2.8}$$

We recall that the spinor module can be constructed from the Clifford algebra $Cl_{1,3}(\mathbb{C})$. For the minimal supersymmetry algebra we take n_1 to be a Majorana spinor module which is an element of the real representation $\pi : n_0 \mapsto GL(\mathbb{R}^4)$. This representation will be naturally equipped with an invariant symplectic form which is often referred to as the charge conjugation C so we define our map $\mathfrak{g}_1 \otimes_S \mathfrak{g}_1 \mapsto \mathfrak{g}_0$ via the condition

$$([A, B])^*(X) = 2C(A, \pi(X)B)$$

where $A, B \in n_1$ and $X \in n_0$. We now generalise this to construct the $\mathfrak{so}(p,q)$ algebras in a similar manner.

2.2.2 General Construction for $\mathfrak{so}(p,q)$

We take the Lie algebra $\mathfrak{g}_0$ to be $\mathfrak{so}(p,q)$ and construct the Lie superalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_1$ is a $\mathrm{Spin}(p,q)$-module. Calculations are easily extended to include the

presence of an internal symmetry group and also extended supersymmetry. In order to keep notation conistent with the standard notation for supersymmetry algebra, we consider a general basis for $\mathfrak{g}_1$ defined by the map Q. The action of $\mathfrak{g}_0$ on the module $\mathfrak{g}_1$ is defined by

$$X.Q(n) = Q(\pi(X)n)$$

for $X \in \mathfrak{g}_0$ and $n \in \mathfrak{g}_1$. In order to define a Lie algebra bracket on the direct sum we then set

$$[X \oplus Q(m), Y \oplus Q(n)] = ([X, Y] + [Q(m), Q(n)]) \oplus (Q(\pi(X)n) - Q(\pi(Y)m))$$

for $X, Y \in \mathfrak{g}_0$ and $Q(m), Q(n) \in \mathfrak{g}_1$. The term $[Q(m), Q(n)]$ must now be appropriately defined so that the above bracket satisfies the Jacobi identity. We first note that the Lie algebra $\mathfrak{g}_0$ is equipped with a Killing form that we denote by

$$K(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$$

This form is by definition bilinear and symmetric. In our case, since the algebra is semi-simple it is also non-degenerate and is unique up to scalar multiples. Also from discussion in section (2.1.2) we see we have a skew-symmetric form if we take the appropriate module. Drawing on previous examples we require that $[Q(m), Q(n)]$ is the solution to

$$K([Q(m), Q(n)], X) = s(m, \pi(X)n)$$

where the invariance and symmetry of s implies

$$s(m, X.n) = -s(X.m, n).$$

so the bracket is appropriately symmetric. It remains to show that this construction ensures that the Jacobi identity is satisfied. One again we can consider four cases separately

1. $[\mathfrak{g}_0, \mathfrak{g}_0, \mathfrak{g}_0]$: Since $\mathfrak{g}_0$ is a Lie algebra this is automatically satisfied.

2. $[\mathfrak{g}_1, \mathfrak{g}_0, \mathfrak{g}_0]$:

$$\begin{aligned}
&[Q(m), [X, Y]] + [X, [Y, Q(m)]] + [Y, [Q(m), X]] \\
&= -Q(\pi([X, Y])m) + Q([\pi(X), \pi(Y)], m) \\
&= 0
\end{aligned}$$

since π as a representation of $\mathfrak{g}_0$.

3. $[\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_1]$

$$\begin{aligned}
&K([X, [Q(m), Q(n)]] + [Q(m), [Q(n), X]] - [Q(n), [X, Q(m)]], Y) \\
&= -K([Q(m), Q(n)], [X, Y]) - K(Q(m), Q(\pi(X)n)], Y) - K([Q(n), Q(\pi(X)m)], Y) \\
&= -s(m, \pi([X, Y])n) - s(m, \pi(Y)\pi(X)n) + s(\pi(Y)\pi(X)m, n) \\
&= 0
\end{aligned}$$

since $s(\pi(Y)\pi(X)m, n) = s(m, \pi(X)\pi(Y)n)$. As the Killing form is non-degenerate this implies the Jacobi identity is satisfied in this case.

4. $[\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1]$ Since the form s is non-degenerate we can write the Jacobi identity as

$$\begin{aligned}
0 &= -s([m, [n, o]] + [n, [o, m]] + [o, [m, n]], \alpha) \\
&= K([Q(m), Q(\alpha)], [Q(n), Q(o)]) + K([Q(n), Q(\alpha)], [Q(o), Q(m)]) \\
&\quad + K([Q(o), Q(\alpha)], [Q(m), Q(n)]) \\
&= s(n, \pi([m, \alpha])o) + s(m, (\pi([o, \alpha]) + \pi([n, \alpha]))o) \qquad (2.9)
\end{aligned}$$

So the final case places a restriction on the representation π.

We consider the last case in detail to determine more clearly what features we require of the representation π. Defining an orthonormal basis with respect to the Killing form $\{X_\lambda\}$ we can expand the commutator on the module in this basis to obtain explicitly

$$[Q(m), Q(n)] = \sum_\lambda s(m, \pi(\lambda)n))X_\lambda$$

Thus we can write

$$K([Q(m),Q(\alpha)],[Q(n),Q(o)]) = \tilde{s}(m\otimes n, \mu(\alpha\otimes o))$$

where $\tilde{s}$ is the non-degenerate form on $\mathfrak{g}_1\otimes\mathfrak{g}_1$ defined by

$$\tilde{s}(a\otimes b, c\otimes d) := s(a,c)s(b,d)$$

and the operator μ is

$$\mu = \sum_\lambda \pi(X_\lambda)\otimes\pi(X_\lambda).$$

Using the above the condition (2.9) can be rewritten in the form

$$0 = \tilde{s}(m\otimes n - n\otimes m + n\otimes o, \mu((o+n))\otimes\alpha)), \quad \forall\alpha,m,n,o\in g_1 \tag{2.10}$$

Using a change of basis and careful rearrangement this condition reduces to the much simpler equation

$$0 = \tilde{s}(m\otimes n, \mu(n\otimes n)), \quad \forall\alpha,m,n\in\mathfrak{g}_1 \tag{2.11}$$

So (2.10) indicates we are actually placing a constraint on the form of the operator μ, that is, the action of $\mathfrak{g}_0$ on the tensor product space $\mathfrak{g}_1\otimes\mathfrak{g}_1$. In fact we can write

$$\mu = \frac{1}{2}\sum_\lambda (1\otimes\pi(X_\lambda)+\pi(X_\lambda)\otimes 1)^2 - (1\otimes\pi(X_\lambda)^2+\pi(X_\lambda)^2\otimes 1)$$

If we take $\mathfrak{g}_1$ to be an irreducible module then we have $\sum_\lambda \pi(X_\lambda)^2\in\mathbb{C}I$ so

$$\sum_\lambda (1\otimes\pi(X_\lambda)^2+\pi(X_\lambda)^2\otimes 1) = \mathbb{C}I$$

Also we note that

$$\sum_\lambda (1\otimes\pi(X_\lambda)+\pi(X_\lambda)\otimes 1)^2 = \sum_\lambda \pi(\Delta(X_\lambda)^2) = \pi(C_\otimes)$$

where $C_\otimes$ is the Casimir operator on $\mathfrak{g}_1\otimes\mathfrak{g}_1$. The tensor product will not in general be irreducible but we have a decomposition into a sum of irreducible modules. Let P_i denote the projection of $\mathfrak{g}_1\otimes\mathfrak{g}_1$ onto the ith submodule in the Clebsch-Gordon decomposition

$$\mathfrak{g}_1\otimes\mathfrak{g}_1 = \bigoplus_i \mathfrak{g}_i \tag{2.12}$$

and C_i the Casimir associated with that module. The final condition becomes

$$0 = \sum_i C_i \tilde{s}(m \otimes n, P_i(n \otimes n)), \quad \forall n, m \in \mathfrak{g}_1 \tag{2.13}$$

The only relevant modules will be those that are symmetric subspaces of $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ since all must be either symmetric or anti-symmetric. Clearly in the cases where there is single symmetric subspace in the decomposition this is satisfied trivially.

From section (2.1.2) we see that the de Sitter algebras $\mathfrak{so}(p,q)$ with $n = 5$ satisfy this final condition as there is a single symmetric subspace in the decomposition of $\mathbf{4} \otimes \mathbf{4}$. We note when constructing the conformal algebra $\mathfrak{so}(2,4)$ with the spinor module $\mathbf{4} \oplus \mathbf{4}^*$ (that of Wess and Zumino) there is more than one symmetric irreducible submodule and so we cannot satisfy the above condition. In this case in order to construct a superalgebra we must include an internal symmetry group (referred to as R-symmetry).

2.3 Including Additional Symmetries

2.3.1 Extended Supersymmetry

So far we have only considered the case where $\mathfrak{g}_1$ is an irreducible spinor module, that is, minimal or $N = 1$ supersymmetry. In this section, we extend the superalgebra to cases where the module is the direct sum of multiple copies of the irreducible module. We construct the algebra

$$\mathfrak{g}_S = \mathfrak{g}_0 \oplus_{i=1}^N \mathfrak{g}_1 = \mathfrak{g}_0 \oplus (\mathfrak{g}_1 \otimes \mathbb{C}^N)$$

Once again our arguments from section (2.2.2) carry over with $\mathfrak{g}_1$ replaced by $\mathbb{C}^N \otimes \mathfrak{g}_1$. If we take representations on $\mathbb{C}^N$ and $\mathfrak{g}_1$ to be defined by maps ρ_1 and ρ_2 respectively then the representation of $\mathfrak{g}_0$ on the tensor product space is

$$\pi(X)(m \otimes u) = (\rho_1(X)m) \otimes u + m \otimes (\rho_2(X)u)$$

for $X \in \mathfrak{g}_1, m \in \mathfrak{g}_1$ and $u \in \mathbb{C}^N$. We can write the symplectic form as the product of forms on the $\mathbb{C}^N$ and $\mathfrak{g}_1$ spaces via the identification

$$s(m \otimes u, n \otimes v) = s_1(m,n)s_2(u,v)$$

where s_2 must be symmetric as we take s_1 to be our usual symplectic form. We take the action on $\mathbb{C}^n$ to be trivial so $\rho_2(X) = I$ and the form $s_2(u, v) = 1$ so it is not significant in the structure. However, the extended supersymmetry plays a significant role when the internal symmetry group is introduced.

We note here the case $N = 1/2$ supersymmetry appearing in the literature refers to where half of the $\mathfrak{g}_1$ is twisted/deformed.

2.3.2 Internal Symmetries

We denote the internal symmetry algebra by T and require that $[T, \mathfrak{g}_0] = 0$. So we now repeat our construction on the algebra

$$\mathfrak{g}_S = (T \oplus \mathfrak{g}_0) \oplus (\mathfrak{g}_1 \otimes \mathbb{C}^N)$$

Note that the same construction from the previous section follows by simply replacing $\mathfrak{g}_0$ with $T \oplus \mathfrak{g}_0$. The condition on the action on the module in (2.9) can be written in a more specific form. We separate the commutator on the module into two parts so that

$$[Q(m), Q(n)] = [Q(m), Q(n)]_{\mathfrak{g}_0} + [Q(m), Q(n)]_T$$

Whilst the action of $\mathfrak{g}_0$ on $\mathbb{C}^N$ remains trivial as in the previous section, we take the internal symmetries to have a non-trivial action on $\mathbb{C}^N$. Due to commutativity with $\mathfrak{g}_0$ the internal symmetries have a trivial action on $\mathfrak{g}_1$. So we have a representation π of the even subalgebra on $\mathfrak{g}_1 \otimes \mathbb{C}^N$

$$\begin{aligned} \pi(\mathfrak{g}_0 \oplus T) &= \pi(\mathfrak{g}_0) \oplus \pi(T) \\ &\cong \rho_1(\mathfrak{g}_0) \otimes I + I \otimes \rho_2(T) \end{aligned}$$

The term for the symplectic form becomes

$$\begin{aligned} &s(m_1 \otimes m_2, \pi(\mathfrak{g}_0 \oplus T)(n_1 \otimes n_2)) \\ &= s_1(m_1, \rho_1(\mathfrak{g}_0)n_1)s_2(m_2, \rho_2(T)n_2) \end{aligned}$$

for $m = m_1 \otimes m_2$ and $n = n_1 \otimes n_2$ where $n_1, m_1 \in \mathfrak{g}_1$ and $n_2, m_2 \in \mathbb{C}^N$. Since we know our selected s_1 is anti-symmetric we must have s_2 symmetric. Since $\mathfrak{g}_0$ and T

are both ideals with trivial intersection, they are orthogonal subspaces with respect to the Killing form. Hence the bracket is given by the solutions to the condition

$$K([Q(m),Q(n)]_{\mathfrak{g}_0},X)=s_1(m_1,\rho_1(X)n_1)s_2(m_2,n_2) \tag{2.14}$$

and also

$$K([Q(m),Q(n)]_T,T)=s_1(m_1,n_1)s_2(m_2,\rho_2(T)n_2) \tag{2.15}$$

It remains to consider the final condition which in terms of the forms s_1 and s_2 becomes

$$0=\tilde{s}_1(m_1\otimes n_1,\mu(n_1\otimes n_1)))s_2(m_2,n_2)s_2(n_2,n_2) \tag{2.16}$$

where

$$\mu=\sum_\lambda \rho_1(X_\lambda)\otimes\rho_1(X_\lambda).$$

So the final condition with internal symmetries included is the same as that when we considered the minimal supersymmetric extension.

2.3.3 Reality Conditions

We introduce a reality structure on the supersymmetry algebra and note how it places restrictions on the structure of the internal symmetry group. These conditions highlight the fundamental difference in the structures of the supersymmetry algebras for $\mathfrak{so}(1,4)$ and $\mathfrak{so}(2,3)$.

We begin by introducing the natural involution * on $\mathfrak{g}_0$. This structure on $\mathfrak{g}_0$ is extended to an involution on $\mathfrak{g}_0\oplus\mathfrak{g}_1$ so that it is consistent with the superalgebra structure by a particular choice of an involution on the spinor module that we also denote by *. To avoid complications with notation we will no longer distinguish between spinor representations by omitting the map Q. In this way we find there are two conditions that must be satisfied

1. Using the fact that the Killing form is invariant under automorphisms we must have

$$K([m,n],X^*)=s^C(m,\pi(X^*)n) \tag{2.17}$$

where s^C is the transformed symplectic form under conjugation. The superalgebra structure thus holds if s is invariant under conjugation, that is, the involution on $\mathfrak{g}_1$ belongs to $Sp(2n, \mathbb{C})$ where n is the dimension of the spinor representation.

2. We also require that

$$(\pi(X)n)^* = \pi(X^*)n^* \tag{2.18}$$

which is simply the statement that the involution map $\phi : m \mapsto m^*$ is an intertwiner between the module action and its conjugate. If π is an irreducible representation then we know that the involution is unique by Schur's lemma.

In case of $\mathfrak{so}(2,3)$ we have a real representation so the intertwiner must be the identity map. This clearly belongs to $Sp(2n, \mathbb{C})$ so condition (2.17) is also satisfied in this case. However for $\mathfrak{so}(1,4)$ we need to introduce internal symmetries in order to have a consistent involution.

If we introduce extended supersymmetry and internal symmetries the involution must now be extended to the space $\mathfrak{g}_1 \otimes \mathbb{C}^N$ where

$$(m_1 \otimes m_2)^* = m_1^* \otimes m_2^*$$

Following the techniques of the previous case we again have two conditions

1. Once again the invariance of the Killing form implies

$$\begin{aligned} K([m,n], X^* \oplus T^*) &= s_1^C(m_1, \rho_1(X^*)n_1)s_2^C(m_2, \rho_2(T^*)n_2) \\ &= s_1(m_1, \rho_1(X^*).n_1)s_2(m_2, \rho_2(T^*)n_2) \end{aligned}$$

so we have

$$s_1^C = \pm s_1 \quad \text{and} \quad s_2^C = \pm s_2$$

2. The second condition remains that the involution is an intertwiner between the representation of $\mathfrak{g}_0$ and the conjugate representation

$$(\pi(X)n)^* = \pi(X^*)n^* \tag{2.19}$$

Now restrictions are placed on the form of the internal symmetry algebra by the form of the involution since the involution determines properties of the form s_2.

Chapter 3

Vacuum Properties of SUSY dS Space

In this chapter, we make a digression and outline some attempts towards investigating the effect of supersymmetry on the vacuum in de Sitter space. We first discuss the Unruh Effect in de Sitter space in terms of Tomita-Takesaki Theory. This is an abstract approach which is closely linked to the structure of the symmetry algebra. The first section is an overview of the original derivation in [13]. We then discuss the approach we took using Krein spaces in an attempt to develop a graded version.

3.1 The Unruh Effect

The Unruh effect demonstrates that the concept of particles in field theory is observer dependent as the vacuum appears to a uniformly accelerating observer to be a thermal equilibrium state. This behaviour is found to be model independent and necessitates a reinterpretation of the vacuum state in field theories. It is similar to the near-horizon form of Hawking radiation and is strongly associated to the geometry of the relevant spacetime via effects on the analyticity properties of the vacuum and other elementary states.

In Minkowski space, one can define Rindler coordinates associated with an accelerating observer that only cover a region we refer to as a Rindler wedge. In other

words, there is a cosmological horizon associated with Minkowski geometry. With an acceleration given by ρ^{-1} an observer will see the vacuum at the inverse temperature of $\beta = 2\pi\rho$. In de Sitter space, there is an equivalent region for which the temperature has been calculated for Langrangian field theories by various authors. However, it is more insightful to calculate the temperature from a more general framework with as few assumptions as possible.

The first step towards generalisation from Langrangian field theories is the Wightman formalism, where fields are replaced by operator-valued-distributions that satisfy a set of axioms. These include a spectrum condition for flat space which essentially ensures positivity of the energy. In other spacetimes such as de Sitter where such a condition no longer makes sense, it is replaced with **minimal analyticity conditions**. Using these conditions one can prove the Bisognano-Wichman Theorem which states the modular automorphisms and a modular conjugation for the wedge regions. The Unruh effect can be shown to be the geometric interpretation of a special case of this theorem and the temperature is calculated to be $2\pi R$ where R is the radius parameter associated with the space.

We approach the problem using the formalism of the algebraic quantum field theory introduced by Haag, Araki, and Kastler. This formalism merges equivalent field theory models and also includes theories that are not generated by pointlike fields. Using Tomita-Takesaki theory one can determine the modular conjugation and automorphisms.

3.2 Algebraic QFT

Algebraic quantum field theory arose from the need to construct field theory from a rigorous mathematical framework and has more recently been complemented by the study of modular Hilbert algebras by Tomita. We construct an algebraic quantum field theory on a spacetime[1] $\mathcal{M}$. Consider open, bounded regions of our spacetime

[1]At this stage by spacetime we include any space with a Riemannian metric that allows definition of spacelike separation.

$\mathcal{O} \subset \mathcal{M}$ and the mapping

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}). \tag{3.1}$$

which assigns to each region a W^*-algebra[2] referred to as the **local algebra of observables**. The collection of these algebras for all regions forms a net of local algebras $\{\mathcal{A}(\mathcal{O})\}$ and the **global algebra** $\mathcal{A}$ is taken as the W^*-inductive limit of the local algebras. The global algebra is of a fixed type and the information is encoded in the relationships between local algebras, that is, the map of (3.1). Isomorphisms between nets give equivalences of field theories in this framework. In fact nets can be constructed from a few local algebras.

We require the local algebras satisfy several conditions as follows

1. The algebra must be local, that is

   $$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$$

   elementwise where $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated.

2. We also assume **isotony**, that is,

   $$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \tag{3.2}$$

   if $\mathcal{O}_1 \subset \mathcal{O}_2$.

3. A **state** in this context is a positive linear functional ω on the algebra with unit C^*-norm

   $$\omega : \mathcal{A} \to \mathbb{C}$$

   We assume **stability**: the existence of a distinguished state describing a stable elementary system, that is, a unique vacuum.

4. A condition of **covariance** is also included and the theory of covariance under a general group has close ties with that of $*$-dynamical systems (see [14]). In general, for a topological symmetry group G and group homomorphism

   $$\alpha : G \to \mathrm{Aut}(\mathcal{A})$$

[2] a unital *-algebra weakly closed in the operator topology

and a continuous unitary representation $U(g)$ we require

$$U(g)\pi(A)U^*(g) = \pi(\alpha_g A)$$

G is generally taken to be the translation subgroup $\mathbb{R}^d$ in the case of flat spacetime. The spectrum condition then means we require the spectrum of translations to be contained in the light cone. However for our purposes we take the group G to be the entire group dS_4

5. Finally, we assume for our calculations a **weak additivity** property for every $\mathcal{O}$

$$\mathcal{A} = \bigvee_{g \in G} \mathcal{A}(g\mathcal{O})$$

where the product $\mathcal{A}_1 \vee \mathcal{A}_2$ denotes the algebra generated by $\mathcal{A}_1$ and $\mathcal{A}_2$. This holds automatically in the case of G=$SO(1,4)$.

The abstract theory can be linked back to a Wightman field theory if we define the local algebra to be the W^*-algebra generated by the observables $\theta(x)$. So we have

$$\mathcal{A}(\mathcal{O}) = \{\theta(x) : x \in \mathcal{O}\}''$$

and the fields can then be recovered from the algebra via

$$\{\theta(x)\} = \bigcap_{\mathcal{O} \ni x} \overline{\mathcal{A}(\mathcal{O})}$$

where the completion allows for unbounded operators. However, the algebras are more general as may describe objects that cannot be expressed as Wightman fields.

The concept of the vacuum in this abstract framework needs to be introduced. In the Minkowski case, a **vacuum representation** of $\mathcal{A}$ is a nondegenerate representation on Hilbert space $\mathcal{H}$ where

- There is a strongly continuous representation of the translation group $U(g)$.
- The spectrum of $U(g)$ is in the forward light cone.
- The representation is covariant under the translations.

- There is a vector Ω invariant under translations.

For dS we take the vacuum to be covariant under the identity component of dS_4. We find that we do not need to impose a spectral condition in order to prove the Bisognano-Wichman property.

Modular theory provides useful theorems relating to AQFT and requires the definition of **thermal equilibrium states** that are distinguished states satisfying the KMS condition [15]. A thermal state of inverse temperature β

$$\phi(t) = (\Omega, B\sigma_t(A)\Omega)$$

where $A, B \in \mathcal{A}$ and $\sigma_t(\mathcal{A})$ is a 1-parameter automorphism group of $\mathcal{A}$ that can be analytically continued into the strip $\{z \in \mathbb{C} | 0 < \Im(z) < \beta\}$ such that

$$\phi(t + \beta i) = (\Omega, \sigma_t(A)B\Omega)$$

well in local regions. Borchers show that locally satisfies the Reeh-Schlieder than under weak closure the global algebra of observables satisfies this also.

The **Reeh-Schlieder Theorem** states that Ω is a cyclic and separating vector for any $\mathcal{A}(\mathcal{O})$. Hence, the vectors $\mathcal{O}$ are dense in $\mathcal{H}$. When this theorem holds true it is possible to apply Tomita-Takesaki theory which allows the construction modular automorphisms from a polar decomposition of an involution. Tomita-Takesaki theory states that for a W^*-algebra $\mathcal{A}$ acting on the Hilbert space $\mathcal{H}$ with Ω a cyclic and separating vector, we have a map for all A in domain of $\Delta^{1/2}$

$$S : A\Omega \mapsto A^*\Omega = J\Delta^{1/2}$$

where

- J is the **modular conjugation** $J = J^* = J^{-1}$ such that

$$AdJ\mathcal{A} = \mathcal{A}\prime \quad \text{and} \quad AdJ\Delta = \Delta^{\frac{1}{2}} \tag{3.3}$$

- Δ is the positive self-adjoint and invertible **modular operator**

$$\sigma^t(A) := Ad\Delta^{it}A, \quad \forall t \in \mathbb{R} \tag{3.4}$$

These imply that the state $(\Omega, B\sigma^t(A)\Omega)$ satisfies the KMS condition. This uniquely characterises the modular group, that is, the group of automorphisms satisfying this condition is the modular group associated to $(\mathcal{M}, \Omega)$.

3.3 Unruh Effect in dS space

The Unruh effect in dS space can be derived in a similar manner as for the case of flat spacetime. Once again, we look for the relationship between the translations and boosts on a wedge region. The operators $\{M_{0j}, M_{jk}\}$ preserve the wedge $\mathcal{W}_j$. From the commutation relations of dS_4 we have the commutators

$$[M_{0j}, M_{0k}] = M_{jk}$$
$$\text{and} \quad [M_{jk}, M_{0k}] = M_{j0}$$

Using the Baker-Campbell-Hausdorff lemma we find the boosts are related to rotations on the wedge by

$$e^{isM_{0j}} e^{itM_{0k}} e^{-isM_{0j}} = e^{it(ch(s)M_{0k} + sh(s)M_{jk})} \tag{3.5}$$

We have an algebra $\mathcal{W}_1$ acting on a Hilbert space $\mathcal{H}$. The boosts generated by M_{01} form the unique 1-parameter Lie subgroup that preserves the wedge. So the modular group is

$$\{e^{-itM_{01}} | t \in \mathbb{R}\}$$

From Tomita-Takesaki theory we also have a modular conjugation $J_{\mathcal{W}_1}$ associated with the wedge that satisfies

$$J_{\mathcal{W}_1} \mathcal{A}(\mathcal{W}_1) J_{\mathcal{W}_1} = \mathcal{A}(\mathcal{W}_1)'$$

Consider a vector in our Hilbert space $\mathcal{H}$ given by

$$e^{-itM_{01}} A\Omega = J_{\mathcal{W}_1} A^* \Omega \tag{3.6}$$

where $A \in \mathcal{W}_1$. We can write the operator A as

$$A = e^{isM_{0j}} B e^{-isM_{0j}}$$

for sufficiently small $s \in \mathbb{R}$ and a $B \in \mathcal{O}_\infty$ where $\mathcal{O}_1 \subset \mathcal{W}_1$ and all de Sitter transformations of this region given s remain in the wedge. So using the expression (3.5) to substitute for the operator A in (3.3) we obtain

$$e^{-itM_{01}} A\Omega = e^{is(ch(t)M_{0k}+sh(t)M_{jk})} e^{-itM_{01}} B\Omega \tag{3.7}$$

Alternatively we note

$$e^{-itM_{01}} A\Omega = J_{\mathcal{W}_1} A^*\Omega \tag{3.8}$$

since $S = J_{\mathcal{W}_1} e^{-itM_{01}}$. Now take the inner product of (3.7) and (3.8) with a vector ϕ

$$(J_{\mathcal{W}_1} A^*\Omega, \phi) = (e^{it(ch(s)M_{0k}+sh(s)M_jk)} e^{-itM_{01}} B\Omega, \phi)$$
$$(J_{\mathcal{W}_1} e^{-isM_{0j}} B^*\Omega, \phi) = (e^{-itM_{01}} B\Omega, e^{-it(ch(s)M_{0k}+sh(s)M_{jk})}\phi)$$

Now using the KMS-property of the state Ω, the map

$$s \mapsto e^{-isM_{0j}} B^*\Omega$$

can be analytically continued into $\{z \in \mathbb{C} : 0 \geq \mathrm{Im}(z) \geq -\frac{\beta}{2}\}$ with continuous boundary values. Similarly, we can analytically extend s on the right hand side into $\{z \in \mathbb{C} : |z| \leq a\}$ for $|z| \leq \frac{\beta}{2}$. So we have

$$(J_{\mathcal{W}_1} A^*\Omega, \phi) = (e^{it(ch(s)M_{0k}+sh(s)M_jk)} e^{-itM_{01}} B\Omega, \phi)$$
$$(J_{\mathcal{W}_1} e^{-iM_{0j}} B^*\Omega, \phi) = (e^{-itM_{01}} B\Omega, e^{-it(ch(z)M_{0k}+sh(z)M_jk)}\phi)$$

Evaluating at $z = \beta/2$ gives the result. It is important to note that it is the modular conjugations that fix the thermal temperature associated with the wedge $\mathcal{W}$.

3.4 Towards Graded Tomita-Takesaki Theory

We ask whether we can create some kind of graded Hilbert structure that ensures the existence of a graded KMS condition in analogy to the way the Hilbert algebra guarantees a KMS structure. We introduce some of the structure which may be required to answer this question and note our obstruction to proceeding further.

3.4.1 Some Useful Structures

A Hilbert superspace is a complex $\mathbb{Z}_2$-graded vector space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0$ and $\mathcal{H}_1$ are both Hilbert spaces, equipped with a super Hermitian form $\langle\ ,\ \rangle$ given by

$$\begin{aligned}\langle \mathcal{H}_0, \mathcal{H}_1 \rangle &= 0 \\ \langle \mathcal{H}_0, \mathcal{H}_0 \rangle &= (\mathcal{H}_0, \mathcal{H}_0)_0 \\ \langle \mathcal{H}_1, \mathcal{H}_1 \rangle &= i(\mathcal{H}_1, \mathcal{H}_1)_1\end{aligned}$$

where $(\ ,\ )_0$ and $(\ ,\ )_1$ are the inner products on $\mathcal{H}_0$ and $\mathcal{H}_1$ respectively.

If an operator T has an adjoint T^* in the Hilbert superspace then it has a super adjoint given by

$$\langle Tx, y \rangle = (-1)^{\mathrm{p}(T)\mathrm{p}(x)} \langle x, T^\dagger y \rangle$$

Equipped with an endomorphism J satisfying $J^3 = J$ the superspace $\mathcal{H}$ can be written as a Hilbert space with an inner product defined by the quotient of the space with

$$(x, y) = \langle x, Jy \rangle, \quad x, y \in \mathcal{H}$$

We define a trace on a C*-algebra to be any functional Tr that has the property

$$\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$$

and similarly a supertrace is a functional on a $\mathbb{Z}_2$-graded algebra satisfying

$$\mathrm{STr}(ab) = (-1)^{\mathrm{p}(a)\mathrm{p}(b)} \mathrm{STr}(ba)$$

Given bounded operators on a $\mathbb{Z}_2$-graded Hilbert space with grading operator Γ we can define a regularised supertrace through

$$\mathrm{STr}(a) = \mathrm{Tr}(\Gamma a)$$

Using the supertrace we define an indefinite form $\langle\ ,\ \rangle$ via

$$\langle a, b \rangle = \mathrm{STr}(a^* b)$$

A Krein space is a vector space $\mathcal{K}$ equipped with an indefinite Hermitian sesquilinear form $[\ ,\ ]$ such that the space decomposes as

$$\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1 \tag{3.9}$$

where $\mathcal{K}_0$ and $\mathcal{K}_1$ are orthogonal Hilbert spaces where the indefnite product is positive and negative respectively.

A functional is called self-adjoint iff $\omega(a^*) = \omega(a)^*$ and graded self-adjoint iff $\omega(a^*) = (-1)^{\mathrm{p}(a)}\omega(a)^*$. A super-KMS functional ω is a functional defined on a $\mathbb{Z}_2$ graded C*-algebra with an automorphism α_t of $\mathcal{A}$ and a grading operator Γ such that

$$\omega(ab) = \omega(a^\gamma \alpha_{it}(b))$$

and the functional is both even and invariant under α_t.

A supertrace $\mathrm{STr}(a) = \mathrm{Tr}(\Gamma a)$ on a Hilbert space $\mathcal{H}$ gives rise to a super-KMS state with the correct grading on the algebra $a^\gamma = \Gamma a \Gamma$ iff the Hilbert representation is the graded one.

this sign factor can be absorbed into the involution. That is, we can have a graded involution ie super hermitian form and self-adjoint form or an involution and non-self-adjoint form. If it is self-adjoint this form is an inner product however if we take involution is graded we have a superhermitian form.

3.4.2 Modular Hilbert Algebras with Krein Structure

Consider a graded *-algebra $\mathcal{A}$ that is equipped with an indefinite Hermitian sesquilinear form $\langle\ ,\ \rangle$ where

- Every $\gamma \in \mathcal{A}$ gives rise to a bounded linear operator $\rho(\gamma)\alpha$ where ρ denotes the left (right) regular representation on $\mathcal{A}$.
- The involution on $\mathcal{A}$ is preclosed, that is, is closeable on $\mathcal{A}$.
- The subalgebra spanned by $\rho(\gamma)\alpha$ is dense in $\mathcal{A}$.
- The involution satisfies $\langle \rho(\gamma)\alpha, \beta\rangle = \langle \alpha, \rho(\gamma^*)\beta\rangle$.

We assume that the algebra admits the structure of a Krein space so there is a decomposition of the algebra

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- \tag{3.10}$$

where $\mathcal{A}_+$ and $\mathcal{A}_-$ are equipped with inner products. The Krein operator $\Gamma = P_+ - P_-$ projects onto these spaces. This is not necessarily the same as the grading on the algebra itself.

Define S_K as the closure of the involution on the algebra. This will be an antilinear operator by definition as an involution. It will have an adjoint on the Krein space defined by

$$\langle S_K(x), y\rangle = \langle S_K^\dagger(y), x)$$

By a theorem in [16] we discover that to proceed with a polar decomposition of the operator S_K as required, we must restrict ourselves to normal operators thus this does not seem the way to proceed in the graded case.

Chapter 4

Lie 2-Algebras and Gauge Anomalies

This chapter is concerned with extensions of Lie algebras and Lie groups in a different sense - the merging of the structure of a category with that of a Lie group or algebra in the process of categorification. We show how the structures arising in the Witt construction give a method for constructing particular types of Lie 2-algebras and Lie 2-superalgebras. This was recently explored for some very specific examples related to the Poincaré superalgebra [17, 18] and closely relates to the structures discussed in chapter 2.

Secondly we highlight a link between Lie 2-algebras and their equivalences, and the traditional perspective on certain Jacobi gauge anomalies which utilizes combinations of exact sequences. In particular, we look at the connection to the discussion in the paper by Carey and collaborators [19] of the chiral QCD anomaly.

4.1 Review of Categorification of Lie Algebras

4.1.1 The Idea of Categorification

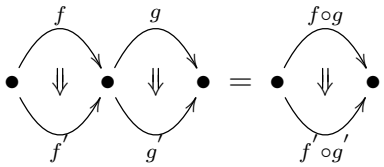
Categorification refers to the process of replacing set-theoretic concepts by their category-theoretic analogs. That is we replace sets with categories, where the ele-

ments and relations become objects and morphisms respectively. We replace functions with functors on this new category and so forth.

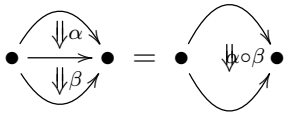
As a demonstration of this idea it is useful to describe categorification of categories themselves. A **2-category** is a category in which it is the Hom-sets that are replaced by categories. This is alternatively referred to as an enriched category over Cat. Thus we have the following ingredients for the 2-category $\mathcal{C}$:

- The class $\mathrm{Obj}(\mathcal{C})$ referred to as 0-cells
- The categories $\mathrm{Hom}(A, B)$ for each $A, B \in \mathrm{Obj}(\mathcal{C})$ consisting of
  - The objects $\mathrm{Obj}(\mathrm{Hom}(A, B))$ referred to as 1-cells
  - The morphism $\mathrm{Hom}(\mathrm{Hom}(A, B))$ referred to as 2-cells

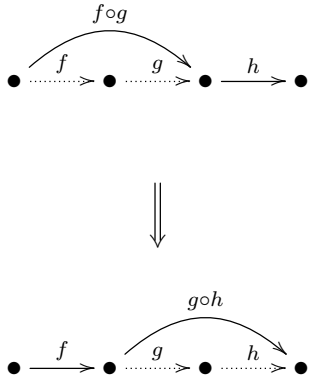
The usual properties of a category fully define the structure. Both the 1-cells and 2-cells must be associative and unital. We denote a 1-cell morphism by a horizontal arrow and a 2-cell morphism by a vertical arrow. Since the 2-cells must be morphisms of the underlying category, they are functors of the Hom-sets. This gives the following composition laws



and



What we have described is a **strict** 2-category. This notion can be weakened into that of a **bicategory** where the associativity of the 1-cells is required only up to 2-cell morphisms. Graphically we have the weaker requirement that



We can define n-categories by a recursive construction by enriching an $(n-1)$-category over Cat at each step. A similar process is used to produce detailed definitions of both the strict and weaker categorised versions of Lie groups and algebras [4, 5].

4.1.2 Categorified Lie Groups and Lie Algebras

Following [4] we can replace the underlying set of a group with a category to define a 2-group. Since we are working with algebras, we will not be concerned with the methods used to define smooth 2-groups hence Lie 2-groups in particular. As in usual Lie theory there are equivalent infinitesimal notions of Lie 2-algebras.

In a 2-group the multiplication operation on the category G,

$$m : G \times G \to G \tag{4.1}$$

is a bifunctor acting on the objects and morphisms of G. As for the 2-category, there is some freedom in the definition of the group laws satisfied by (4.1). We weaken the notion of a group such that the laws hold up to natural transformations. In applications where the groups describe symmetries of a system, the extra 2-cells can be thought of describing symmetries of the symmetries themselves.

Since the following work is more concerned with algebras, we detail the definitions related to Lie 2-algebras [5]. We start with the underlying vector space. A 2-vector space is a category in Vect that consists of a vector space of objects V_0 and a vector

space of morphisms V_1, where the category source, target and identity maps are linear. In order to define a Lie-2-algebra we equip this 2-vector space with a bracket

$$[.,.] : V \times V \to V \tag{4.2}$$

which is bilinear and skew-symmetric. We then require that the Jacobi identity is satisfied up to natural isomorphism. So we have a natural transformation referred to as the Jacobiator which we denote by

$$J_{x,y,x} : [[x,y],z] \to [x,[y,z]] + [[x,z],y] \tag{4.3}$$

which is trilinear and skew-symmetric. This satisfies a consistency condition known as the Jacobiator identity for which we refer the reader to [5].

Lie 2-algebras are equivalent to objects known as 2-term algebras which makes more evident the close relationship to chain complexes and crossed modules we are dealing with. We define a 2-term L_∞-algebra [5] as an object consisting of four components, namely

- two vector spaces V_0 and V_1
- a linear map $d : V_1 \to V_0$
- the bilinear totally antisymmetric maps $l_2 : V_i \times V_j \to V_{i+j}$ where $0 \leq i + j \leq 1$
- a trilinear totally antisymmetric map $l_3 : V_0 \times V_0 \times V_0 \to V_1$

Given $x, y, z \in V_0$ and $h, k \in V_1$ and denoting the maps l_2 by a Lie bracket, we require

that these objects satisfy the conditions

$$[x, y] = -[y, x] \tag{4.4}$$

$$[x, h] = -[h, x] \tag{4.5}$$

$$[h, k] = 0 \tag{4.6}$$

$$d([x, h]) = [x, dh] \tag{4.7}$$

$$d(l_3(x, y, x)) = -[[x, y], z] + [y, [x, z] + [x, [y, z]] \tag{4.8}$$

$$l_3(dh, x, y) = -[[x, y], h] + [[x, h], y] + [x, [y, h]] \tag{4.9}$$

$$\begin{aligned}[l_3(w, x, y), z]+[l_3(w, y, z), x] + l_3([w, y], [x, z) + l_3([x, z], w, y) \\ = [l_3(w, x, z), y] + [l_3(x, y, z), w] + l_3([w, x], y, z)+ \\ l_3([w, z], x, y) + l_3([x, y], w, z) + l_3([y, z], w, x)\end{aligned} \tag{4.10}$$

So we see the Jacobi on V_0 now only holds up to chain homotopy in (4.8).

There is a more convenient way of presenting this structure. It is shown in [5] that 2-term Lie algebras can be fully represented via the data $(\mathfrak{g}, V, \rho, l_3)$ where $\mathfrak{g}$ is a Lie algebra, V is a vector space, ρ a representation of $\mathfrak{g}$ on V and l_3 is an element of $C^3(\mathfrak{g}, V)$. The proof is straightforward so is not recorded here: we take $V_0 = \mathfrak{g}$, $V_1 = V$ and the action of the representation is given by the map $l_2 : V_0 \times V_1 \to V_1$.

For our future discussions we define the subclasses of strict and skeletal algebras among 2-term algebras. These myriad of results are discussed in detail in [5] and summarised in table 4.1. A skeletal Lie 2-algebra is the 2-term algebra in which the map d vanishes. The strict version of a Lie 2-algebra is by definition taking $l_3 = 0$. All Lie 2-algebras are equivalent to a skeletal one, and can be classified by the cohomology class of l_3. We note in particular the case of an algebra that is both skeletal and strict. This object is simply a Lie algebra $\mathfrak{g}$ together with a representation on a vector space V.

Finally after this bombardment of definitions we give a standard example of a Lie 2-algebra. Given the Lie algebra $\mathfrak{g}$ equipped with a Killing form $K : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ we take $V_0 = \mathfrak{g}$ and $\mathfrak{g}_1 = \mathbb{R}$. When $l_3 = K(x, [y, z])$ we have the skeletal algebra

$$\mathfrak{g} \xleftarrow{d=0} \mathbb{R}$$

Table 4.1: A Dictionary for Lie 2-Algebras

Lie 2-Algebras	**Quadruples**
Skeletal Lie 2-algebras $V_0 \xleftarrow[d=0]{} V_1$	Quadruples $(\mathfrak{g}, V, \rho, l_3)$
Isomorphism classes of skeletal Lie 2-algebras $V_0 \xleftarrow[d=0]{} V_1$	Quadruples $(\mathfrak{g}, V, \rho, [l_3])$
Strict skeletal Lie 2-algebras $V_0 \xleftarrow[d]{} V_1$	Quadruples $(\mathfrak{g}, V, \rho, 0)$

4.1.3 Relationship to Crossed Modules

We note that a strict 2-group is equivalent to a group crossed module and this is often used as the language for dealing with 2-groups. A crossed module consists of a pair of groups G and H where there is an action of G on H which we denote by α and also a homomorphism $d : H \to G$. These maps must satisfy the equivariance condition

$$d(\alpha(g)h) = gd(h)g^{-1}$$

and also obey the Peiffer identity

$$\alpha(d(h_1))h_2 = h_1h_2h_1^{-1}.$$

Examples include the normal subgroup of a group with the conjugation action. Also the group itself is a special case of the crossed module when we take the trivial module and the zero homomorphism.

A differential crossed module consists of a pair of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ where there is a derivation action of $\mathfrak{g}$ on $\mathfrak{h}$ which denote by α. There is also an algebra homomorphism $d : \mathfrak{h} \to \mathfrak{g}$ where the two conditions from the group case become

$$d(\alpha(g)h) = [g, d(h)]$$

and

$$\alpha(dh_1)h_2 = [h_1, h_2]$$

Just as for crossed modules, differential crossed modules can be described by categorical objects. Specifically from a differential crossed module we construct a

strict Lie 2-algebra and vice versa. As shown in [5], a crossed module gives a Lie 2-algebra with objects $c_0 = \mathfrak{g}$ and morphisms $c_1 = \mathfrak{g} \rtimes \mathfrak{h}$ where the semi-direct product is defined using the action α. We have target and source maps $t(\mathfrak{h}, \mathfrak{g}) = t(\mathfrak{h})$ and $s(\mathfrak{h}, \mathfrak{g}) = \mathfrak{g}$.

4.1.4 Definition of Lie Algebra Cohomology

We remind ourselves of the Chevalley-Eilenberg cohomology for Lie algebras [20]. Let $\mathfrak{g}$ be a Lie algebra and M a module for the algebra. We define cochains

$$C^n(\mathfrak{g}, M) = \{\omega : \Lambda^n \mathfrak{g} \mapsto M\}$$

which belong to the complex

$$0 \xrightarrow{d} C^0(\mathfrak{g}, M) \to C^{n+1}(\mathfrak{g}, M) \to \cdots$$

where the coboundary operator $d : C^{n+1}(\mathfrak{g}, M) \to C^n(\mathfrak{g}, M)$ defined on the basis $\{X_i\}$ via

$$\begin{aligned} d\omega(X_1, \ldots, X_j, \ldots X_{n+1}) = &\sum_{q=1}^{n+1} (-1)^{n+1-q} \omega(X_1, \ldots, \hat{X}_q, \ldots, X_{n+1}) X_q \\ &+ \sum_{\substack{r=1 \\ r>q}}^{i+1} \omega(X_1, \ldots, \hat{X}_q, \ldots \hat{X}_r, \ldots, X_{n+1}, [X_q, X_r]) \end{aligned}$$

The condition that $d^2 = 0$ is equivalent to the Jacobi identity.

The n-cochains for which $d\omega = 0$ are **closed** and referred to as an n-**cocycles**. When $\omega = d\beta$ the cochain is said to be **exact** and is an n-**coboundary**. We denote the space of n-cocycles by $Z^n(\mathfrak{g}, M)$ and the n-coboundaries by $B^n(\mathfrak{g}, M)$. The cohomology group $H^n(\mathfrak{g}, M)$ is given by the quotient of closed by the exact forms, that is

$$H^n(\mathfrak{g}, M) = Z^n(\mathfrak{g}, M)/B^n(\mathfrak{g}, M)$$

The zeroth cohomology group is by definition the invariants of the lie algebra on M. The 2nd cohomology group classifies the central extensions of the Lie algebra. The 3rd cohomology group classifies crossed module extensions. We note in the case where we take the trivial module Cartan's theorem shows that the de Rham and the Chevalley-Eilenberg cohomology coincide.

4.2 Lie 2-Algebras and Witt Constructions

Recent papers discuss how categorified Lie algebras and Lie groups can be constructed using cocycles on the algebra or group respectively. We already considered as an example of a Lie 2-algebra the extension by the usual Killing form in section 4.1.2.

Baez and Huerta [17, 18] explore the existence of cocycles on the supertranslation algebras arising from the four normed division algebras. With these cocycles new super Lie 2-algebras are constructed as an extension of the supertranslation algebra with particular relevance to String Theory. They also integrate to Lie 2-supergroups [21].

Inspired by these papers and in light of our discussion in chapter 2, we discuss how we can define Lie algebra cocycles from the ingredients involved in the Witt construction including for graded cases. We show that in general for any Witt construction of an abelian (super-)algebra we may obtain a 3-cocycle in similar way which allows an extension to a Lie 2-algebra.

A 3-cocycle $\phi \in C^3(\mathfrak{g}, M)$ is a map $\phi : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to M$ that satisfies linearity, skew-symmetry and $d\phi = 0$. Taking the trivial module $M = \mathbb{R}$ the cocycle condition becomes

$$d\phi(X,Y,Z,W) = \phi([X,Y],Z,W) - \phi([X,Z],Y,W) + \phi([Y,Z],X,W) + \\ \phi([X,W],Y,Z) - \phi([Y,W],X,Z) + \phi([Z,W],X,Y) = 0 \qquad (4.11)$$

We now define a 3-cocycle using the same ingredients as were involved in the Witt construction.

4.2.1 Defining a 3-Cocycle

Consider an abelian lie algebra $\mathfrak{g}$. Let M be a $\mathfrak{g}$-module with an invariant symmetric quadratic form B. Denote the direct sum $\mathfrak{g} \oplus M$ by h. Define $\beta : h \otimes h \otimes h \to \mathbb{R}$ by

the functions

$$\begin{aligned}
\beta(X,Y,Z) &= B(X,[Y,Z]) \quad \text{for} \quad X,Y,Z \in h \\
\beta(X,m,n) &= B([m,n],X) := s(m,\gamma(X)n) \quad \text{for} \quad X \in h \quad \text{and} \quad m,n \in M \\
\beta(X,Y,n) &= \beta(m,n,p) = 0
\end{aligned}$$

extended as linear and skew-symmetric under permutations of pairs of entries. Here s is a symmetric invariant form on M and $\gamma : \mathfrak{g} \otimes M \to$ is an equivariant map. We calculate the conditions under which this defines a cocycle for the algebra constructed on $\mathfrak{g} \oplus M$.

We investigate when the cocycle condition (4.11) is satisfied by splitting the problem into the following cases:

- All $\mathfrak{g}$: Since we have that $\beta([X,Y],Z,W) = B([X,Y],[Z,W])$ it is clear that with careful selection of terms we may write

$$d\beta(X,Y,Z,W) = 2B(J[X,Y,Z],W) \tag{4.12}$$

  where $J[X,Y,Z]$ denotes the usual Jacobi expansion. This is zero due to the fact that $\mathfrak{g}$ is a Lie algebra.

- 1 element in M: Since all terms are of the form $\beta(X,Y,[Z,n])$ this gives zero.

- 2 elements in M: We use the fact that $\beta([n,m],X,Y) = B([n,m],[X,Y]) = s(n,\gamma([X,Y])m])$ we arrive at

$$s(n,[X,\gamma(y)m] - [Y,\gamma(x)m] - \gamma([X,Y])m) = 0 \tag{4.13}$$

  As before the second set of terms is a repeat of this combination under the transformation $m \leftrightarrow n$.

- 3 elements in M: Since all entries are in M this is trivially zero.

- All in M: We have

$$s(m,\gamma([l,n])p) + s(n,\gamma([l,m])p] + s(l,\gamma([n,m])p) = 0 \tag{4.14}$$

From the above we have to satisfy the remaining equations (4.13) and (4.14). Clearly if we take $\mathfrak{g}$ to be abelian and the bracket on $[\mathfrak{g}, M]$ trivial we easily satisfy (4.13). It remains to be seen how this condition can be viewed for the case of non-trivial actions.

We note taking $\gamma = \alpha$ allows (4.13) to be satisfied for non-abelian algebras and hence the construction of a Lie 2-algebra. This reduces to the case of the standard extension of a Lie algebra mentioned in section 4.1.2.

Once again we only require a slight modification for Lie superalgebras. Take a Lie algebra $\mathfrak{g}$ and let M be a spinor $\mathfrak{g}$-module with an invariant symmetric form B and invariant skew-symmetric form s. Take the direct sum $\mathfrak{h} = \mathfrak{g} \oplus M$ and define $\gamma : h \otimes h \otimes h \to \mathbb{R}$ in the same way.

Repeating the calculations gives two conditions for β to be a cocycle as before:

$$s(n, [X, \gamma(y)m] - [Y, \gamma(x)m] - \gamma([X, Y])m) = 0 \tag{4.15}$$

$$s(m, \gamma([a, a])a) = 0 \tag{4.16}$$

where we have used the argument from chapter 2 to simplify the last expression.

Given the 3-cocycles in both cases we can construct a skeletal Lie 2-(super)algebra of the form

$$\mathfrak{h} \xleftarrow{d=0} \mathbb{R}$$

where $l_3 = \beta$.

4.3 Applications to Gauge Anomalies

An anomaly arises where a classical symmetry is not preserved under quantization. Various anomalies arising from the canonical Hamiltonian quantization of gauge theories in QFT are described using cocycles [22, 19]. A particular example was given by Jo [23] as the quantization of a chiral fermion coupled to a QCD gauge field. What was viewed as the failure of the Jacobi identity can be alternatively viewed as a 3-cocycle obstruction to a particular extension of the Lie algebra or corresponding group.

We highlight the link between Lie 2-algebras and the description of gauge anomalies in terms of 3-cocycle obstructions. As we discussed in section 4.1.2, skeletal Lie 2-algebras are classified by 3-cocycles. Working from the other direction, gauge anomalies can be investigated by considering 3-cocyles arising from the splicing of various exact sequences together. These structures lend themselves to a description in terms of crossed modules. We begin by introducing such structures in general before describing the specific arrangement for discussing gauge anomalies.

The purpose of this chapter is to highlight that the fact that the existence of gauge anomalies can be restated as the fact that a particular Lie 2-algebra we construct is not equivalent to a strict Lie 2-algebra. In this sense we can view the quantization as defining a more general 2-group of gauge symmetries rather than a group.

4.3.1 Basic Extension Diagrams

The importance of spliced extension diagrams or prolongation problems with reference to gauge theories is discussed. Most of the concepts can be found in an introductory textbook on the topic [24].

We outline in generality the types of extensions relevant to our discussion of gauge anomalies. First we take the exact sequence of groups

$$1 \rightarrow N \rightarrow F \rightarrow G \rightarrow 1$$

so we are looking at the extension of a group G by F. Now we consider taking a central extension of N by an abelian group A. These types of constructions are sometimes referred to as crossed extensions. Combining the two sequences gives the familiar diagram shown below. The question we are addressing is whether we can construct the diagram on the right - we ask whether the extension of N can induce

or be prolonged to an extension of the group F to $\hat{F}$.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & A & & & & \\ & & & & \downarrow & & & & \\ & & & & \hat{N} & & & & \\ & & & & \downarrow & & & & \\ 1 & \to & N & \to & F & \to & G & \to & 1 \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array} \qquad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & A & & & & \\ & & & & \downarrow & & & & \\ 1 & \to & \hat{N} & \to & \hat{F} & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & N & \to & F & \to & G & \to & 1 \\ & & \downarrow & & & & & & \\ & & 1 & & & & & & \end{array}$$

We show that this diagram simply defines a crossed module. Since N is a normal subgroup we have an action α of F on N defined by conjugation. We now take a section $\nu : N \to \hat{N}$ which is defined by the group cocycle $\sigma : N \times N \to A$. This section allows us to extend the action of F to the extended group $\hat{N}$. Considering the spliced exact sequence

$$1 \to A \to \hat{N} \overset{\phi}{\hookrightarrow} F \to G \to 1$$

we can define a crossed module.

We know that central extensions of a group are classified by the second cohomology group and quantify the obstruction of the abelian group from a product. When this is trivial there is a homomorphism that splits the sequence. As mentioned earlier crossed module extensions are classified by the third cohomology group $H^3(G, A)$. In the case of crossed modules the cocycle represents an obstruction to the extension $\hat{F}$. In order for the extension to exist the 3-cocycle must be trivial.

It is a simple exercise to calculate this 3-cocycle. We lift the cocycle $\omega : G \times G \to N$defining a section from G to F into $\hat{F}$ through the vertical section ν. So we define $\hat{\omega} : G \times G \to \hat{N}$ which is a cochain but not necessarily a cocycle. Applying the coboundary operator for cochains $C^2(G, \hat{N})$ gives the 3-cocycle $d\hat{\omega} : G \times G \times G \to \hat{N}$. Lifting the cocycle to A via a new section gives a map $\tau : G \times G \times G \to A$ which can be shown to satisfy the 3-cocycle condition [24].

When this cocycle is trivial we can form the extended group $\hat{F}$ which belongs in the sequence

$$1 \to \hat{N} \to \hat{F} \to G \to 1$$

The section of G in $\hat{F}$ is defined by the cocycle $W(x,y) = \hat{\omega}(x,y)\rho(x,y)^{-1}$ where $\tau = d\rho$. The proof [19] is a straightforward but lengthy calculation and not pertinent to our discussion.

In the infinitesimal setting there is a similar construction. Many of the derivations are analogous to the group case and so not included. This time we have a Lie algebra exact sequence

$$0 \to \mathfrak{n} \to \mathfrak{f} \to \mathfrak{g} \to 0$$

and consider a central extension of $\mathfrak{n}$ by the abelian Lie algebra $\mathfrak{a}$. So we are interested in the spliced sequences

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathfrak{a} & & & & \\ & & & & \downarrow & & & & \\ & & & & \hat{\mathfrak{n}} & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & \mathfrak{n} & \to & \mathfrak{f} & \to & \mathfrak{g} & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array} \qquad \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathfrak{a} & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & \hat{\mathfrak{n}} & \to & \hat{\mathfrak{f}} & \to & \mathfrak{g} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \mathfrak{n} & \to & \mathfrak{f} & \to & \mathfrak{g} & \to & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

Since $\mathfrak{n}$ is an ideal in $\mathfrak{f}$ we have no issues constructing a differential crossed module in the same way as before. This is again characterised by a 3-cocycle $\tau : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to a$.

We can define two different Lie 2-algebras associated with this diagram. Given the 3-cocycle τ and the induced representation of the algebra $\mathfrak{g}$ on a we define a skeletal semi-strict Lie 2-algebra. Expressed as a 2-term Lie algebra we can take

$$\mathfrak{g} \xleftarrow{d=0} \mathfrak{a} \tag{4.17}$$

where $l_3(x,y,z) = \tau(x,y,x)$ for all $x,y,z \in \mathfrak{g}$. Alternatively we can construct the strict Lie 2-algebra directly from the differential crossed module .

In terms of the Lie 2-algebras we can view the obstruction to the extension in terms of Lie algebra equivalences. For example the extension can be constructed if and only if the algebra (4.17) is equivalent to the strict skeletal Lie 2-algebra which consists of $\mathfrak{g}$ and the representation on a only. It would be interesting to further explore the links between the various Lie 2-algebras in the category Lie2Alg.

4.3.2 Anomaly Examples

We briefly describe the Lie 2-algebra structure associated with physical examples explored in the series of papers by Carey and collaborators. We note the language of Lie 2-algebras allows one to view the existence of an anomaly as the existence of a higher order gauge theory.

Usually gauge anomalies are presented as a breakdown of the Jacobi identity. Instead following [19] we can view them as an obstruction to their implementation as operators preserving a common dense domain of a Hilbert space. For example we take the operators Γ defined by the exact sequence

$$0 \to \Delta \to \Gamma \to \Omega \to 0 \tag{4.18}$$

where Δ is an ideal. We wish to implement gauge operators as derivations on a field algebra $\mathcal{A}$ acting on the Hilbert space $\mathcal{H}$. Assume we have a larger algebra of operators $\mathcal{Y}$ and that $\mathcal{V} \subset \mathcal{Y}$ implements the adjoint action of the algebra Δ. So we have the sequence

$$0 \to \mathcal{W} \to \mathcal{V} \to \Delta \to 0$$

where $\mathcal{V} = \{u \in \mathcal{Y} | ad|_{\mathcal{A}} u \in \Delta\}$. We want to construct an algebra of operators that implements Γ such that $\mathcal{V}$ is an ideal of these operators.

Consider in particular the chiral QCD anomaly [23] in (3+1) dimensions which arises when one uses equal time commutators to quantise the theory. Without recourse to the details in the reference, we have a Yang-Mills gauge field A and its conjugate field E. Jo calculates that the commutators for the smeared gauge fields in the quantised theory are

$$[A(f), A(g)] = 0 \tag{4.19}$$

$$[E(f), A(g)] = -i(f, g) \tag{4.20}$$

$$[E(f), E(g)] = i\alpha(f, g) \tag{4.21}$$

where (f, g) is the usual inner product on the space of functions $f : \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^n$. The function α defines a cochain on these functions.

The algebras Ω and Δ are taken to be abelian Lie algebras corresponding to the smeared fields. The algebra Δ is embedded in Γ via the cocycle $\sigma(f,g) = (f,g)$. We implement the E fields as operators satisfying the commutation relation (4.21) and find a 3-cocycle obstruction.

Even though the operators cannot be implemented consistently on a dense domain we still have the structure of a strict Lie 2-algebra to exploit. The associated Lie 2-algebra has objects Δ and morphisms of the form $\Delta \rtimes \mathcal{V}$.

Chapter 5

Vacuum Geometry of SUSY Quiver Gauge Theories

The development of new unified theories in physics typically takes a top-down approach. Higher energy theories exhibit testable signatures at lower energy scales, and can often be excluded using current experimental capabilities. Identifying the features of a theory from which particular characteristics in the low energy limit appear is in general a non-trivial task. A new scheme outlined in [25] proposes focussing on the geometry of the classical vacuum space as a selection principle for higher energy completions.

The geometric features of the vacuum space are explored in the case of Minimally Supersymmetric Standard Model (MSSM) with a view to obtaining hints of higher energy models. Here we restrict ourselves to particular models occurring in string theory as there is concrete connection between the structure of the classical vacuum space and the original theory. Hence we have hints at how to construct theories for desired low energy outcomes.

The most notable demonstration of the significance of the vacuum space geometry is perhaps in models of D-branes probing singularities. In such models, the D-branes reside at the singularity of a local Calabi-Yau manifold. The geometry of the compactified dimensions affects the gauge theory resulting from the D-brane configuration. An often quoted example is that of the D3-brane world-volume theory probing the

Calabi-Yau 3-fold singularity referred to as the conifold. This model at low energies has a vacuum moduli space which is exactly the conifold. This result applies more generally: theories of D-brane configurations probing any Calabi-Yau singularity have a vacuum moduli space $\mathcal{M}$ with the same geometry as the space transverse to the D-branes.

The low energy theories arising from these models of D-branes at a toric singularity are always a $\mathcal{N} = 1$ SUSY gauge theory consisting of bifundamental and adjoint matter. As the particle content of the theory can be encoded in a quiver they are referred to as **quiver gauge theories.** We take these quiver theories themselves as the fundamental object of our investigations and explore the landscape of vacua as a form of classification.

The literature focuses on the subclass of toric quiver gauge theories, where the underlying vacuum space is a toric variety. Quiver gauge theories subject to an additional combinatorial condition have links to brane tilings and hence also play a role in the AdS/CFT correspondence. Attempts are being made to look at non-toric examples in detail. A recent paper [26] explores new examples of non-toric quiver gauge theories as deformations from the toric case and discusses deformations of the dual CFT. We conduct a systematic and statistical investigation of the vacuum geometry for the broader set of Calabi-Yau quiver gauge theories.

Features of the action of an $\mathcal{N} = 1$ supersymmetric gauge theory allow the vacuum space to be viewed naturally through algebraic geometry. Such an algorithm has been applied to the vacuum geometry for the sectors of the MSSM [25]. Using the same approach, the statistics in section 5.4 were generated using a combination of Mathematica and the Macaulay2 software packages.

5.1 Preliminaries on Quiver Gauge Theories

5.1.1 Quiver Diagrams

Quivers provide a convenient representation of the particle content of gauge theories [27]. A **quiver diagram**, Q, is a finite directed multigraph where each node represents a compact gauge factor G_i of the total gauge group $\prod_i G_i$. Nodes are customarily labelled by the index i of these gauge factors. The edges represent bi-fundamental fields, those which transform under $(G_a, \overline{G_b})$ where G_a and G_b are the head and tail gauge groups respectively. We denote these fields by X^i_{ab} for fields charged under a and b respectively whilst multiplicities are indexed by i. Self-loops are fields transforming under the usual adjoint representation Ad_{G_i} that we denote by ϕ_i.

The gauge invariant terms are constructed by contracting the gauge indices between fields hence they correspond to closed paths in the quiver. A generating set for these invariants is easily obtained from the quiver without adjoints by taking the simple cycles. However, when self-loops are present, additional terms including the adjoint fields must be included among the gauge invariants. Henceforth we shall refer to minimal loops in reference to this expanded set of cycles.

5.1.2 Quiver Theories

A **quiver theory** is a pair (Q, W) where W is the superpotential defining the theory. The superpotential is a formal polynomial in the edges of the quiver where each monomial term is gauge invariant. Hence the superpotential takes the form

$$W = \sum_n a_n \mathrm{Tr}(X_{ij} X_{jk} .... X_{mi}) \tag{5.1}$$

where in general $a_n \in \mathbb{C}$.

Allowing for fractional branes one can choose the ranks of the nodes freely and the fields are matrices. In the following we consider gauge groups of the form $\prod_i U(1)$ describing a single brane-probe where all the representations are one dimensional.

This introduces a significant redundancy in the number of possible superpotentials for each quiver diagram due to the commutativity of the fields.

We consider **Calabi-Yau** quivers: these are where the vacuum space is Calabi-Yau. A sufficient condition is that the theory is anomaly-free, a requirement independent of the superpotential. It can be stated as a restriction on the quiver diagram: a quiver theory is Calabi-Yau if all nodes have indegree equal to the outdegree.

A toric quiver theory usually can be defined as a theory in which each field appears exactly twice in the superpotential in separate terms with opposite sign. We relax this condition in our investigations and consider any polynomials in the gauge invariants as possible superpotentials.

5.1.3 The Action

We consider a general gauge theory with $\mathcal{N} = 1$ supersymmetry. In the presence of a (compact) gauge group G we have chiral superfields Φ_i transforming under G and a vector superfield V transforming under the Lie algebra of G.

The standard action in terms of superspace coordinates is

$$S = \int d^4x \left[\int d^4\theta \Phi_i^\dagger e^V \Phi_i + \frac{1}{4g^2} \left(\int d^2\theta \mathrm{tr}[W_\alpha W^\alpha] + \int d^2\theta W(\Phi) + h.c. \right) \right] \quad (5.2)$$

where $W_\alpha = i\bar{D}^2 e^{-V} D_\alpha e^V$ is the gauge field strength and $W(\Phi_i)$ is the superpotential which is a holomorphic function of the chiral superfields. We note that any quadratic terms in the superpotential can be integrated out so we only consider monomials of cubic and higher order.

In the usual way, we calculate the conditions on the classical vacuum state in the Wess-Zumino gauge. It reduces to two types of conditions on the scalar components ϕ_i of the chiral superfields Φ_i. We must satisfy the following:

$$\frac{\partial W(\phi_i)}{\partial \phi_i} = 0 \quad \text{(F-flatness conditions)} \quad (5.3)$$

$$\text{and} \quad \sum_i \phi_i^\dagger T^A \phi_i = 0 \quad \text{(D-flatness conditions)} \quad (5.4)$$

The space of solutions to these flatness conditions can be parameterised by an algebraic variety.

5.2 Parameterising the Vacuum

5.2.1 Algebraic Varieties

Since we will be viewing the vacuum space in terms of algebraic geometry we briefly introduce some necessary terminology. The fundamental objects of study in algebraic geometry are algebraic sets, that is, sets of solutions to polynomial equations. Let K be an algebraically closed field. We introduce an **affine variety**, X, as the locus of points in the affine space A^K on which a set of polynomials vanish. So we have

$$X = \{x \in A^K | f_i(x) = 0, \quad \forall i\} \tag{5.5}$$

Henceforth we shall set $K = \mathbb{C}$ as relevant to this work. It is well known that we can describe any submanifold of A^K with a finite set of polynomials.

The geometric properties of an affine variety can be encoded in its coordinate ring. Associated with $\mathbb{C}^n$ we have the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. We take the ideal generated by the set of polynomials defining X which we denote by $I(X)$ for brevity. Since $I(X)$ is an ideal we take the quotient $\mathbb{C}[x_1, \ldots, x_n]/I(X)$ as our primary object associated with the variety. Tools of algebraic geometry allow one to extract geometric data from this ring.

For example, the **dimension** of the space has numerous equivalent definitions including the usual Krull dimension of the quotient ring. The **degree** is closely related to the number of intersection points of the variety with a straight line but the detailed definition is not required for our analysis.

It is possible to define regular maps $\Phi : X \to Y \subset \mathbb{C}^m$ between varieties which are given by m polynomials on the points of X whose images lie in Y. These induce maps on the polynomial rings (often written again using the same symbol) which is actually the pullback $\Phi^* : \mathbb{C}[x_1, \ldots, x_m]/I(Y) \to \mathbb{C}[x_1, \ldots, x_n]/I(X)$ defined by $\Phi^*(f) = f(\Phi)$.

We note given two varieties X and Y we can define a product $X \times Y$ which is also a variety. Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be varieties defined by the sets of polynomials

$\{f_i(x)\}$ and $\{g_j(y)\}$ respectively. We have

$$X \times Y = \{(x, y) \in \mathbb{C}^{n+m} | f_i(x) = g_j(y) = 0 \quad \forall i, j\} \tag{5.6}$$

The quotient ring associated with this variety is simply

$$\mathbb{C}[x_1, \ldots, x_{n+m}]/(I(X) + I(Y))$$

We note some further structure relevant to later discussion. Consider a regular map on a product variety $D : X \times Y \to \mathbb{C}^{n+m}$ that is a Cartesian product of functions $D = (D_1, D_2)$ where $D_1 : X \to \mathbb{C}^n$ and $D_2 : X \to \mathbb{C}^m$. The image of this map is clearly the product variety

$$\Im(D_1) \times \Im(D_2) \subset \mathbb{C}^n \times \mathbb{C}^m \tag{5.7}$$

Equivalently the coordinate ring map we defined earlier has the domain $\mathbb{C}[x_1, \ldots, x_{n+m}]/(I(\Im(D_1)) + I(\Im(D_2)))$.

5.2.2 The Vacuum as a Moduli Space

Exploiting the extra gauge invariance of the action (5.2) allows one to write the space of vacuum solutions as a symplectic quotient and hence an algebraic variety. For a more detailed discussion of the main points we refer the reader to the original paper containing these results [28].

We can introduce a less restrictive gauge in which the invariance under the complexified gauge group G^c remains. The F-terms are covariant under the imaginary part of the gauge group since the superpotential is a holomorphic function of terms with a gauge index. Hence under this transformation of a solution the F-flatness conditions remained satisfied.

For every F-term solution there is only one D-term solution. Under the imaginary part of the gauge group we can rotate the solution on a D-orbit which contains exactly one solution to the D-terms. Hence we have the fact that the vacuum moduli space is given by the symplectic quotient

$$\mathcal{M} = \mathcal{F}//G^c \tag{5.8}$$

where $\mathcal{F}$ is the F-term solution manifold.

Now using some standard results we relate this to a variety. First we notice $\mathcal{F}$ can be naturally viewed as an affine variety since it corresponds to points at which the polynomial conditions of (5.3) vanish in $\mathbb{C}^n$, where n refers to the number of fields. So we have the polynomial ring $\mathcal{F} \cong \mathbb{C}[\phi_1, \ldots, \phi_n]/\langle \partial_i W \rangle$. Lutty and Taylor [28] highlight the fact that given $\mathcal{F}$ as a variety there is a bijection between (5.8) and the variety defined by the ring of gauge invariant elements in the ring $\mathcal{F}$.

So taking the minimal generating set of gauge invariants $D = \{r_1, \ldots, r_m\}$ we need the image of this map in $S = \mathbb{C}^m$. The vacuum moduli space is

$$\mathcal{M} \backsimeq \Im\left(\mathcal{F} \xrightarrow{D} S\right) \tag{5.9}$$

Given the original ingredients of the gauge invariants or minimal loops of the quiver and F-flatness conditions we can calculate features of the vacuum moduli space (VMS). This technique is sometimes referred to as a version of the **Forward Algorithm**.

We utilise Macaulay2 functions to calculate the ideal of the ring map associated to (5.9), namely $\ker(S \xrightarrow{D} \mathcal{F})$. This gives the ideal in S defining the vacuum space. There are inbuilt functions to calculate the dimension, degree and Hilbert polynomial of the associated variety in the standard sense.

5.2.3 Note on Disconnected Quivers

We consider a disconnected quiver diagram with connected components Q_j and associated fields $\{\Phi_j\}$. The sets of minimal loops and hence gauge invariants can be partitioned into sets associated with each connected component. Thus, any superpotential takes the form $W(\Phi_i) = \sum_j W_j(\Phi_j)$. The algebraic variety is thus a product $\mathcal{F} = \prod_j \mathcal{F}_j$ and the map D defined by the gauge invariants is a Cartesian product of the maps on each $\mathcal{F}_j$.

By our earlier discussion of these types of maps in (5.7), we know the image of the gauge invariants calculated using (5.9) will be a product of the images of the $\mathcal{F}_j$ in $S_j \subset S$. Thus we present results only for connected quiver theories.

5.3 Outline of Algorithms

Our strategy is to use the following procedure to generate the statistics in a systematic manner:

1. Given a pair of integers (n, e) generate every unique quiver (including disconnected quivers) with n nodes and e edges and satisfying the Calabi-Yau condition.

2. For each quiver generate all the possible minimal loops of the quiver corresponding to the gauge invariants.

3. Find all possible superpotentials W with coefficients taken from a particular integer set whilst removing the extra redundancy in the superpotentials.

4. For each superpotential calculate the dimension, degree and Hilbert series corresponding to the algebraic variety describing the vacuum space $\mathcal{M}$.

In these initial investigations we restrict ourselves to the lower order quiver diagrams for which we are able to obtain data using a standard modern computer. Routines were tested by checking the results for known quivers with small (n, e). We outline in further detail the algorithms involved in each of these steps.

5.3.1 Generating Quivers

Generating all possible edge sets given n nodes by brute force methods is computationally intensive. Since we are interested in Calabi-Yau quivers only, we exploit the fact that the Calabi-Yau condition implies that the quiver is composed of edge-disjoint cycles. We identify all possible combinations of cycle lengths by generating all tuples $\{c_1, \ldots, c_k\}$ that satisfy $\sum_k kc_k = e$, where c_k is the number of cycles of length k.

Within realisations of these cycle combinations $\{c_1, \ldots, c_k\}$ we must include only graphs up to isomorphism. Unfortunately the algorithms involved in checking if two graphs are isomorphic are the main obstruction to generating quivers for large (n, e)

efficiently, since there are no known simple invariants to compare. We optimise this process by adding cycles sequentially. We consider all possible configurations with the additional cycle added and filter out the isomorphic graphs at this stage. This significantly reduces the number of calls to the graph isomorphism algorithms.

Considering only connected quivers reduces the redundant data generated as these can be built out of smaller quivers as discussed in (5.2.3). Although there are some more efficient approaches for generating connected quivers [29], for the purposes of this work the more general algorithm is sufficient giving quivers with $n, e \leq 10$.

5.3.2 Generating Gauge Invariants and Superpotentials

The set of gauge invariants D is found by generating the minimal loops of the quiver. For each vertex, a recursive algorithm traces all possible paths beginning at the vertex until any vertex in the path is repeated (excluding the cases when the repeated vertex is a loop). Higher order loops are then generated by taking all possible combinations of the set of gauge invariants.

Superpotentials generated from a set of terms by selecting coefficients given a coefficient set C taken to be $\{-1, 0, 1\}$ in our study. Clearly the number of superpotential terms grows exponentially with the number of invariants. Given x gauge invariants of three or more fields, we have $f(x) = (3^x - 1)/2$ nontrivial superpotentials. For large numbers of gauge invariants we take random samples of possible superpotentials. In general we took a sample size $s = 100$ although realistically our algorithms could be pushed to $s = 400$ to give tighter error bounds.

5.3.3 Computing the Moduli Space

For each quiver theory generated, the calculations involving algebraic varieties were performed using the Macaulay2 software. We note that for the purposes of computation the field $\mathbb{C}$ was replaced with the more manageable $\mathbb{Z}/101\mathbb{Z}$. We found that for large numbers of superpotential terms (hence F-terms) algorithms became difficult to run on a standard computer. Fixing the number of superpotential terms is required

to extend analysis to higher order theories.

5.3.4 Some Illustrative Examples

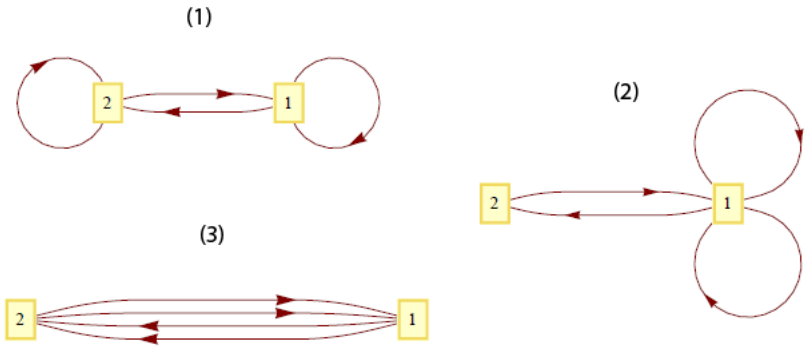


Figure 5.1: Unique Connected Quivers for $(n, e) = (2, 4)$

As a simple example consider the case of two nodes and four edges. We have three inequivalent quivers presented in figure 5.1. For each we generate the minimal loops. The first two diagrams have only a single cycle giving the gauge invariant $\{X_{12}X_{21}\}$. However including the adjoint fields in each case gives an additional three invariants in the generating set. Diagram (3) has a set of four operators $\{X^1_{12}X^1_{21}, X^1_{12}X^2_{21}, X^2_{12}X^1_{21}, X^2_{12}X^2_{21}\}$ which generate the invariants.

Considering terms that are first order in the gauge invariants only, we obtain the superpotentials

$$W_1 = a_1\phi_1 X_{12}X_{21} + a_2\phi_2 X_{12}X_{21} + a_3\phi_1\phi_2 X_{12}X_{21}$$
$$\text{and} \quad W_2 = a_1\phi^1_1 X_{12}X_{21} + a_2\phi^2_1 X_{12}X_{21} + a_3\phi^1_1\phi^2_1 X_{12}X_{21}$$

where W_i corresponds to diagram (i) and $a_j \in \{-1, 0, 1\}$. The VMS details for these models are given in table 5.1.

We note the extra redundancy due to the commutativity of the fields. The position of adjoints in the quiver is only relevant in so far as determining simple cycles the

loop can be combined with. The invariants and hence the VMS for quivers (1) and (2) are identical as the structure does not distinguish between the fields ϕ_1 and ϕ_2.

Restricting to first order terms in gauge invariants for (3) gives a trivial superpotential as we require monomials of degree three or higher. We note including products of invariants to second order we obtain the set of models with

$$W_3 = a_1 X_{12}^1 X_{21}^1 X_{12}^2 X_{21}^2 + a_2 X_{12}^1 X_{21}^1 X_{12}^1 X_{21}^2 + a_3 X_{21}^1 X_{12}^1 X_{21}^2 X_{12}^1 + a_4 X_{12}^1 X_{21}^1 X_{12}^1 X_{21}^1 + a_5 X_{12}^2 X_{21}^2 X_{12}^2 X_{21}^2 + a_6 X_{21}^2 X_{12}^2 X_{21}^2 X_{12}^1$$

Increasing the number of edges rapidly produces theories with large numbers of invariants. Hence we can still obtain interesting results whilst we strict ourselves to superpotentials to first order in the invariants. For example the quiver with $(n, e) = (3, 7)$ shown in figure 5.2 has the five simple loops

$$\{X_{12}X_{21}, X_{13}X_{31}, X_{23}X_{32}, X_{13}X_{32}X_{21}, X_{12}X_{23}X_{31}\}$$

The set of invariants including terms with ϕ_3 has an additional four terms

$$\{X_{13}\phi_3 X_{31}, X_{23}\phi_3 X_{32}, X_{13}\phi_3 X_{32}X_{21}, X_{12}X_{23}\phi_3 X_{31}\}.$$

Taking the full set of gauge invariants and coefficients in the set $\{-1, 0, 1\}$ we obtain a total of 364 non-trivial superpotentials. Details of the VMS for a sample of theories (with $s = 100$) for this quiver are given in figure 5.3.

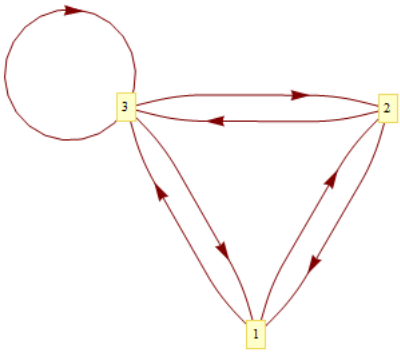


Figure 5.2: A Connected Quiver for $(n, e) = (3, 7)$

Table 5.1: VMS Data for Quivers (1) and (2) of Figure 5.1

Superpotentials	$\mathrm{Dim}(\mathcal{M})$	$\mathrm{Deg}(\mathcal{M})$
$\phi_1{}^1\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1$	1	1
$\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1+\phi_1{}^1\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1$	0	1
$\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1-\phi_1{}^1\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1$	1	1
$\phi_1{}^1X_{1,2}{}^1X_{2,1}{}^1$	1	1
$\phi_1{}^1X_{1,2}{}^1X_{2,1}{}^1+\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1+\phi_1{}^1\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1$	0	1
$\phi_1{}^1X_{1,2}{}^1X_{2,1}{}^1+\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1-\phi_1{}^1\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1$	1	1
$\phi_1{}^1X_{1,2}{}^1X_{2,1}{}^1-\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1$	1	1
$\phi_1{}^1X_{1,2}{}^1X_{2,1}{}^1-\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1+\phi_1{}^1\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1$	0	1
$\phi_1{}^1X_{1,2}{}^1X_{2,1}{}^1-\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1-\phi_1{}^1\phi_1{}^2X_{1,2}{}^1X_{2,1}{}^1$	0	1

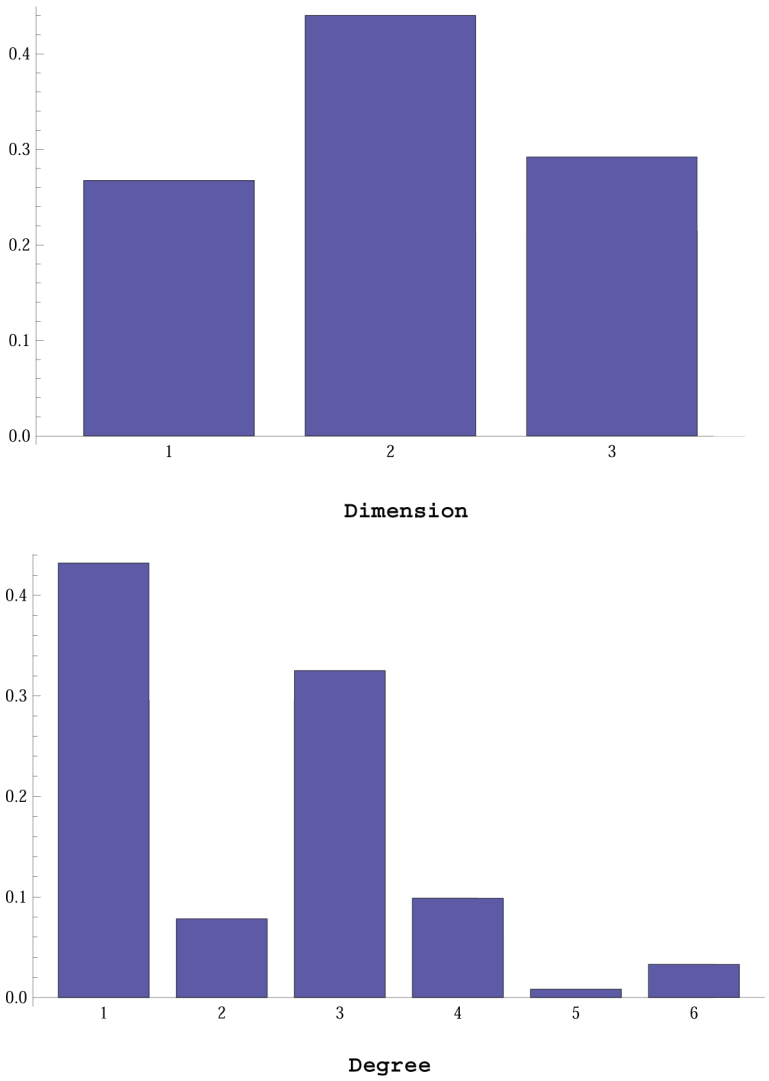


Figure 5.3: Proportion of Quiver Theories Plotted Against VMS Degree and Dimension Respectively for Quiver in Figure 5.2

5.4 Computational Results

We outline results for quiver theories where superpotentials are constructed from terms to first order in the gauge invariants, that is, terms corresponding to the minimal loops in the quiver as discussed in section 5.3.2. Table 5.2 details the number of connected quivers generated for various pairs (n, e). The maximum possible number of gauge invariants and possible superpotential terms are listed in table 5.3. Even for these low examples it is clear we have begun to reach the limits of efficient calculations in Macaulay2 due to the large number of terms we can include in the superpotential.

With the algorithms in place we have calculated the VMS data for each quiver generated. We present a short selection of these results along with the associated quivers in sections 5.4.1 and 5.4.2. Sample sizes $s = 100$ are used unless otherwise stated. We collate the statistics in general under the pairs (n, e) rather than individual quivers in section 5.4.3 to give a clearer picture of the vacuum landscape.

n	e					
	3	4	5	6	7	8
2	1	3	3	6	6	10
3	1	2	5	15	25	49
4	0	1	2	13	37	122
5	0	0	1	3	19	101
6	0	0	0	1	3	35

Table 5.2: Counts of Connected Quivers for Various (n, e)

n	e					
	3	4	5	6	7	8
2	2/1	4/3	8/7	32/31		
3	1/1	2/2	4/4	8/8		
4	0	1/1	2/2	4/4	8/8	
5	0	0	1/1	2/2	4/4	8/8
6	0	0	0	1/1	2/2	4/4

Table 5.3: Counts of Invariants / Terms for Various (n, e)

5.4.1 Two Node Quivers

We list some low order examples of quivers with $n = 2$ along with some features. We give the number of gauge invariants and the number of possible non-toric superpotentials. Where statistics were calculated for a random sample of models rather than all possible theories we indicate. The third column contains tables showing the distribution of the vacuum spaces according to the dimension or degree of the moduli space.

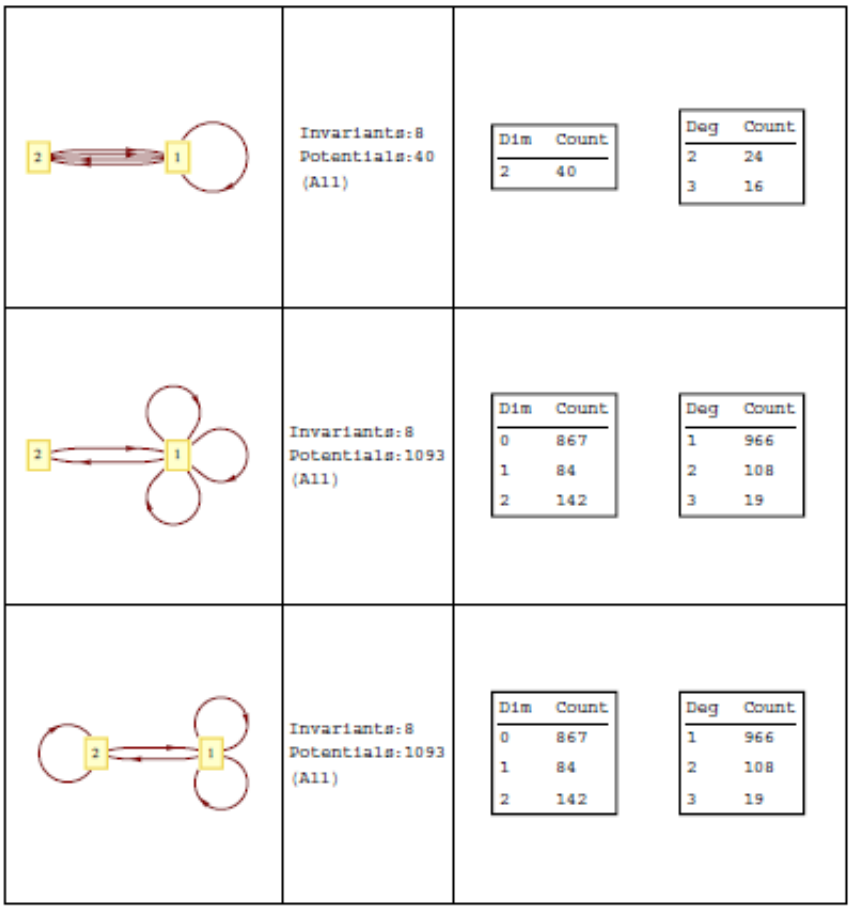


Figure 5.4: VMS Statistics for $(2, 5)$ Quivers

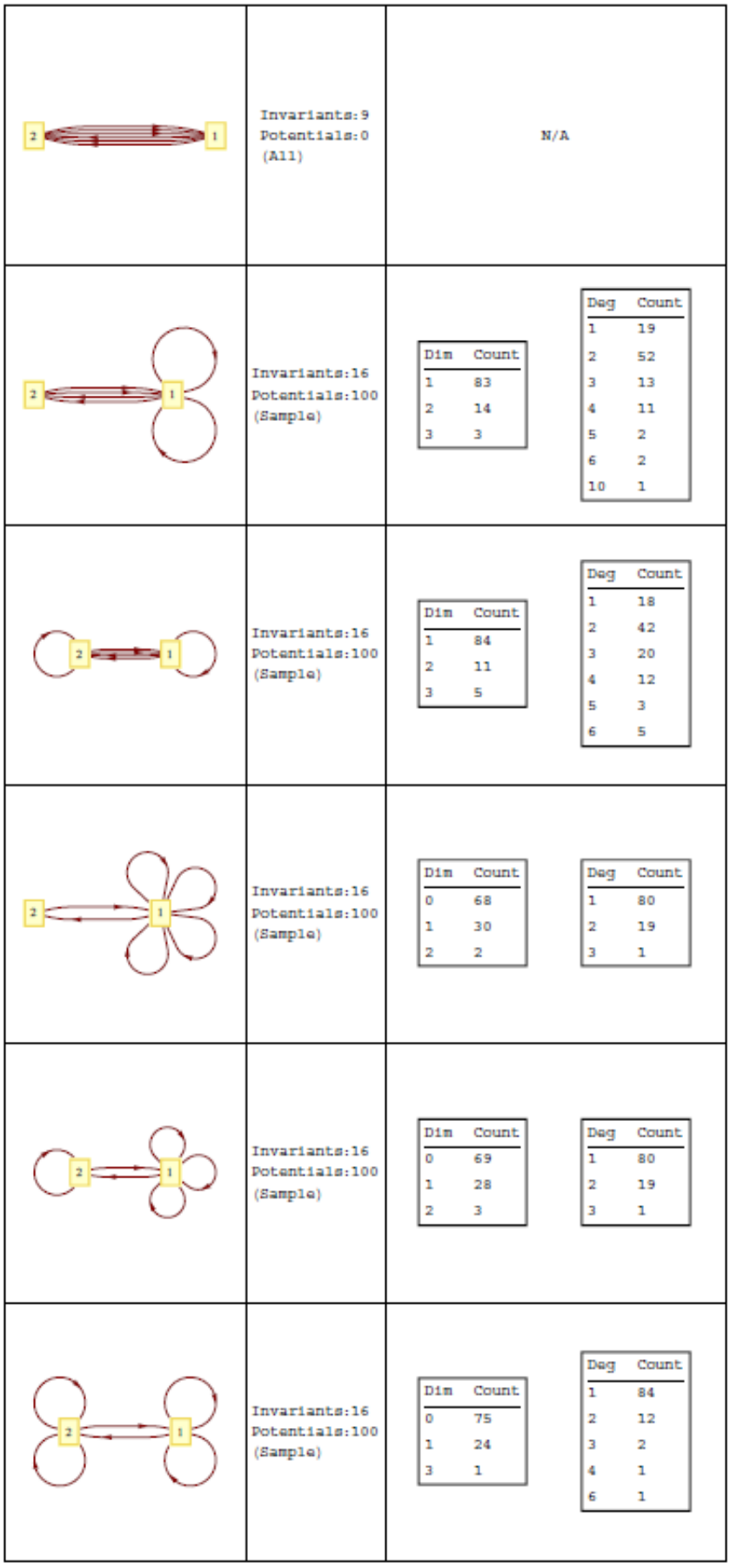


Figure 5.5: VMS Statistics for $(2, 6)$ Quivers

5.4.2 Three Node Quivers

We present data for quivers with $n = 3$ in the same format as in the previous section.



Figure 5.6: VMS Statistics for $(3, 3)$ Quivers

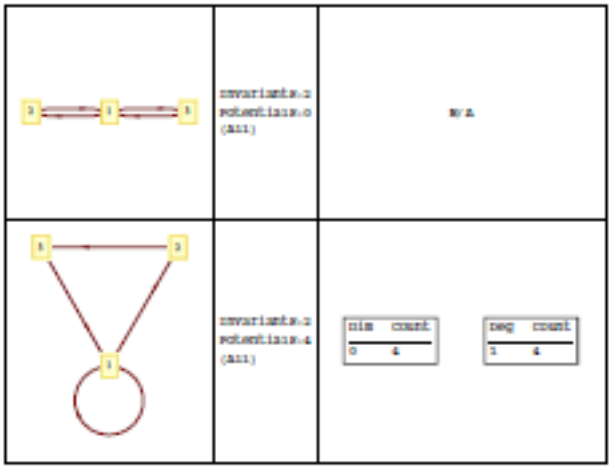


Figure 5.7: VMS Statistics for $(3, 4)$ Quivers

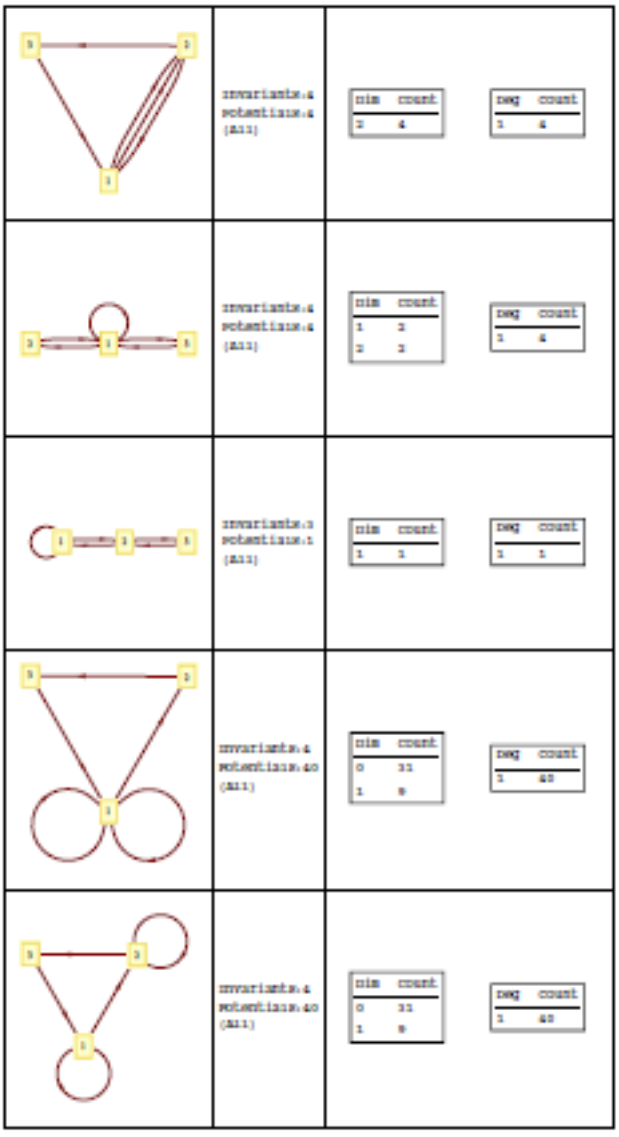


Figure 5.8: VMS Statistics for $(3, 5)$ Quivers

5.4.3 Larger Data Sets

Of particular interest are the statistics regarding the number of Calabi-Yau manifolds of complex dimension three for various quiver diagrams. We require large data sets to gain a clearer picture of the distribution of these spaces among the vacuum moduli of the quiver theories.

Presented in tables 5.4 and 5.5 is some further data generated in order to initiate the construction of such a picture. Statistics for the individual quivers are combined to give a list of the various types of vacuum spaces appearing given a pair (n, e). For each pair (n, e) we list the counts of the degree or dimension respectively.

We wish to give an idea of the distribution of such spaces across all the quiver gauge theories for a given (n, e). Thus, in cases where random samples were required, stratified sampling based on the number of theories per quiver was used in order to give an accurate distribution.

Table 5.4: Dimension of VMS for Pairs (n, e): Given a pair (n, e), we list combinations of values $a|b$ where a are integers corresponding to possible values of $\dim(\mathcal{M})$. If all possible vacuum spaces were computed then b is the number of theories with the given dimension VMS. Alternatively, a percentage listed at b indicates that a sample of the possible theories was investigated for each quiver and gives an estimated proportion of theories with the given dimension.

Nodes	Edges			
n	$e = n$	$e = n + 1$	$e = n + 2$	$e = n + 3$
2	N/A	0 | 1	0 | 16 1 | 10	0 | 81% 1 | 7% 2 | 13%
3	0 | 1	0 | 4 1 | 0	0 | 62 1 | 21 2 | 6	0 | 81% 1 | 10% 2 | 9% 3 | 0%
4	0 | 1	0 | 4 1 | 1	0 | 125 1 | 58 2 | 32	0 | 63% 1 | 23% 2 | 12% 3 | 2%
5	0 | 1	0 | 6 1 | 3	0 | 55% 1 | 29% 2 | 17%	0 | 56% 1 | 28% 2 | 13% 3 | 3%
6	0 | 1	0 | 6 1 | 3	0 | 286 1 | 207 2 | 121	0 | 49% 1 | 7% 2 | 17% 3 | 3%

Table 5.5: Degree of VMS for pairs (n, e): Given a pair (n, e), we list combinations of values $a|b$ where a are integers corresponding to possible values of $\deg(\mathcal{M})$. If all possible vacuum spaces were computed then b is the number of theories with the given degree VMS. Alternatively, a percentage listed at b indicates that a sample of the possible theories was investigated for each quiver and gives an estimated proportion of theories with the given dimension.

Nodes	Edges			
n	$e = n$	$e = n + 1$	$e = n + 2$	$e = n + 3$
2	N/A	1 | 1	1 | 26	1 | 87% 2 | 11% 3 | 2%
3	1 | 1	1 | 4	1 | 89	1 | 91% 2 | 8% 3 | 2%
4	1 | 1	1 | 5	1 | 123 2 | 2	1 | 88% 2 | 11% 3 | 1% 4 | 0%
5	1 | 1	1 | 9	1 | 99% 2 | 1%	1 | 88% 2 | 11% 3 | 1% 4 | 0%
6	1 | 1	1 | 9	1 | 596 2 | 18	1 | 85% 2 | 14% 3 | 0% 4 | 0%

Chapter 6

Conclusions and Outlook

We have studied extensions of symmetries relevant to physics in detail. We outlined in abstract notation the general structure of an extension of a $\mathfrak{so}(p,q)$ Lie algebra to a superalgebra. We then drew links between Lie 2-algebra equivalences in category theory and examples of gauge anomalies. Finally, we have investigated the landscape of vacuum spaces associated to non-toric quiver gauge theories.

The investigation of the links between Lie 2-algebras and gauge anomalies could be extended further. There are additional examples listed in the same series of papers as the one selected for this thesis [19] which could be explored. In particular, it would be interesting to look at the higher gauge theory associated with the Lie 2-algebra in the presence of the anomaly. This point of view may give an additional tool to investigate gauge anomalies.

Chapter 5 is currently being developed into a paper in collaboration with Yang-Hui He. Since the algorithms are set up it remains to generate statistics for larger numbers of quivers in order to gain a more complete picture. Using the more efficient method mentioned in [29] to generate connected quivers provides a next step to expand the results.

Bibliography

[1] Sidney Coleman and Jeffrey Mandula. All possible symmetries of the s matrix. *Phys. Rev.*, 159(5):1251–1256, July 1967.

[2] Rudolf Haag, Jan T. Łopuszański, and Martin Sohnius. All possible generators of supersymmetries of the S-matrix. *Nuclear Phys. B*, 88:257–274, 1975.

[3] Roberto Percacci. Mixing internal and spacetime transformations: some examples and counterexamples. *J. Phys.*, A41:335403, 2008.

[4] J. Baez and A. Lauda. Higher-Dimensional Algebra V: 2-Groups. *Theory and Applications of Categories*, 12:423–491, 2004.

[5] J. Baez and A. Crans. Higher-Dimensional Algebra VI: Lie 2-Algebras. *Theory and Applications of Categories*, 12:492–528, 2004.

[6] J. Baez and J. Huerta. An invitation to higher gauge theory. *General Relativity and Gravitation*, 43:2335–2392, 2011. 10.1007/s10714-010-1070-9.

[7] V. S. Varadarajan. *Supersymmetry for Mathematicians: an Introduction*, volume 11 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2004.

[8] E. Witt. Spiegelungsgruppen und aufzhlung halbeinfacher liescher ringe. *Abh. Math. Sem. Univ. Hamburg*, 14:289–337, 1985.

[9] Yu. A. Golfand and E. P. Likhtman. On extensions of the algebra of generators of the Poincaré group by bispinor generators. In *The many faces of the superworld*, pages 45–53. World Sci. Publ., River Edge, NJ, 2000.

[10] Z. Wan. *Lie algebras*. Pergamon Press, Oxford, New York, 1 edition, 1975.

[11] A. Zee. *Quantum Field Theory in a Nutshell*. Princeton University Press, Princeton, New Jersey, 2003.

[12] K. Pilch, P. van Nieuwenhuizen, and M. F. Sohnius. de sitter superalgebras and supergravity. *Comm. Math. Phys.*, 98(1):105–117, 1985.

[13] H. J. Borchers and D. Buchholz. Global properties of vacuum states in de Sitter space. *Ann. Inst. H. Poincaré Phys. Théor.*, 70(1):23–40, 1999.

[14] H. J. Borchers. Quantum field theory as dynamical system. In *Quantum theory and symmetries (Kraków, 2001)*, pages 43–54. World Sci. Publ., River Edge, NJ, 2002.

[15] R. Haag, N. M. Hugenholtz, and M. Winnink. On the equilibrium states in quantum statistical mechanics. *Comm. Math. Phys.*, 5:215–236, 1967.

[16] C. Mehl, A. C. M. Ran, and L. Rodman. Polar decompositions of normal operators in indefinite inner product spaces. 162:277–292, 2006.

[17] J. Baez and J. Huerta. Division Algebras and Supersymmetry I. 2009.

[18] John C. Baez and John Huerta. Division Algebras and Supersymmetry II. 2010.

[19] A. L. Carey, H. Grundling, C. A. Hurst, and E. Langmann. Realizing 3-cocycles as obstructions. *Journal of Mathematical Physics*, 36:2605–2620, June 1995.

[20] N. Jacobson. *Lie algebras*. Dover Publications Inc., New York, 1979. Republication of the 1962 original.

[21] J. Huerta. Division Algebras and Supersymmetry III. 2011.

[22] Alan L. Carey and M.K. Murray. Mathematical remarks on the cohomology of gauge groups and anomalies. *Mod. Phys. Lett.A*, 1994.

[23] S-G. Jo. Commutators in an anomalous non-abelian chiral gauge theory. *Physics Letters B*, 163(5-6):353 – 359, 1985.

[24] S. Maclane. *Homology*, volume 114 of *Grundlehren Mathematischen Wissenschaften*. Berlin, 1963.

[25] James Gray, Yang-Hui He, Vishnu Jejjala, and Brent D. Nelson. Vacuum geometry and the search for new physics. *Phys.Lett.*, B638:253–257, 2006.

[26] A. Butti, A. Zaffaroni, and D. Forcella. Deformations of conformal theories and non-toric quiver gauge theories. *JHEP*, 0702:081, 2007.

[27] Michael R. Douglas and Gregory W. Moore. D-branes, quivers, and ALE instantons. 1996.

[28] Markus A. Luty and Washington Taylor. Varieties of vacua in classical supersymmetric gauge theories. *Phys.Rev.*, D53:3399–3405, 1996.

[29] J. Hewlett and Yang-Hui He. Probing the Space of Toric Quiver Theories. *JHEP*, 1003:007, 2010.