

Almost-Certain Eventualities and Abstract Probabilities in Quantitative Temporal Logic

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Abstract

‘Almost-certain eventualities’ are liveness properties that hold with probability 1. ‘Abstract probabilities’ are probabilities in transition systems about which we know only that they are neither 0 nor 1.

Vardi [17] showed that almost-certain properties in linear temporal logic depend only on abstract probabilities, rather than on the probabilities’ precise values; we discuss the extent to which a similar result holds in quantitative temporal logic [9,10], and we show how to specialise the logic to those cases. The aim is to provide a simpler calculus than the full logic, one that is in a certain sense complete for proving almost-certain eventualities from abstract-probabilistic assumptions.

We consider briefly the complexity of the specialised logic.

1 Introduction

Liveness properties of ‘standard’, that is non-probabilistic transition systems rely only on the connectivity of the system (considered as a graph). The same is true in probabilistic systems, up to a point: ‘almost-certain eventualities’ depend only on ‘abstract probabilities’, not on precise probabilistic values.

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A typical *eventuality* is loop termination, for example, expressed in temporal logic by the formula $\Diamond[\neg G]$ where G is the loop guard; it is *almost certain* iff it occurs with probability 1. Over the state space $\{H, T\}$ the ‘coin-flipping’ system

$$s := H \quad p \oplus \quad s := T ,$$

in which $p \oplus$ represents probabilistic choice, satisfies both $\Diamond[s=H]$ and $\Diamond[s=T]$ almost certainly: no matter where the system is started, the state s will eventually be H , and will eventually be T , provided $0 < p < 1$.

An *abstract probability* is one which — like p above — is known only to be neither 0 nor 1: beyond that, its precise value is immaterial for the conclusions that are to be drawn.

In this paper we concentrate on our quantitative extension $qM\mu$ [9,10] of the modal μ -calculus [6]; the extension in many cases acts as a probabilistic μ -calculus or even as a probabilistic temporal logic. (It can go beyond those, however, dealing directly with more general aspects like expected complexity [7].)

Our principal contribution here is that the quantitative calculus can be specialised to a form of almost-certain eventualities and abstract probabilities, and that results are obtained that are similar to the ‘traditional’ probabilistic calculi: one does not need precise numeric values for the probabilistic transitions in the underlying system if one is interested only in almost-certain conclusions.

Our second contribution is to give a complexity bound for evaluation of almost certainties.

In the remainder of this section we describe the transition systems with which we will be concerned. Sections 2 and 3 review the existing calculi, in both their Boolean (traditional) and quantitative (our numeric extension) form; in Sections 4 and 5 we present our principal results. Complexity is considered in Section 7.

1.1 Standard transition systems and the μ -calculus

We say that a system is *standard* if it is not probabilistic or, if it is probabilistic, when its probabilities are all either 0 or 1. Standard transition systems over a state space S support a *modal μ -calculus* [6] for reasoning about their behaviour; expressions in the calculus denote Boolean-valued *predicates* (equivalently subsets of S), which are sets of states that can be shown with the calculus to lead to certain behaviours of the transition system.

The transition system can be given as elements of a state-to-state relation \mathcal{T} : if $(s, s') \in \mathcal{T}$ then moving from state s to state s' is a possible transition; and if both (s, s') and (s, s'') are in \mathcal{T} , for $s' \neq s''$, then in a move from s the choice between s' and s'' can be resolved either ‘demonically’ or ‘angelically’ depending on one’s application.

The μ -calculus can be specialised to a form of temporal logic by defining

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// State space is  $Bool \times \mathbb{N}$ .
var  $b: Bool; n: \mathbb{N}$ ;

// Transition is ‘enabled’ only when  $b$  holds;
// otherwise it acts as skip.
 $b \rightarrow b := False \text{ }_{1/n^2} \oplus n := n + 1$ 

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Fig. 1. A probabilistic transition system.³

temporal operators, like eventually \Diamond above, within the calculus and then using only those, in effect a subset of the full language.

1.2 Probabilistic transition systems and $qM\mu$

Probabilistic transition systems support a ‘quantitative’ modal μ -calculus — which we call $qM\mu$ — whose expressions are real- rather than Boolean-valued over S ; the expressions denote ‘expected values’ of random variables over probabilistic distributions on the state space. The transitions exhibit *probabilistic* nondeterminism as well as potentially the other two kinds.

As in the standard μ -calculus, temporal operators can be defined within $qM\mu$: the result is a *quantitative* temporal logic which we call qTL [9,10].

The standard μ -calculus embeds into $qM\mu$ by taking predicates, or their equivalent subsets, to the corresponding characteristic functions; as a consequence, standard temporal logic embeds similarly into qTL .

For example, consider the probabilistic system of Fig. 1. If b holds and $n > 0$, then b is eventually *False* only with probability $1/n$ — that is, the eventuality $\Diamond[\neg b]$ depends on n ’s initial value — and in qTL (details below) we would simply say that $\Diamond[\neg b] = 1/n$ in all states that satisfy $b \wedge n > 0$. Clearly the $1/n$ result depends on the precise value $1/n^2$ given in the transition: that is, the proof of $\Diamond[\neg b] = 1/n$ in the calculus would involve quantitative reasoning based on that specific probability. (We give a proof in Sec. 3.2.)

On the other hand, in the probabilistic system

$$(1) \quad b \rightarrow b := False \text{ }_p \oplus n := n + 1$$

$\neg b$ is reached with probability 1, provided p is bounded away from zero.⁴

³ We use a UNITY-like [2] pseudo-code to describe the transitions.

⁴ By “bounded away from zero” we mean that if we allow p to be some function $p.(b, n)$ of the state, then we require the existence of a fixed $\varepsilon > 0$ such that $p.(b, n) \geq \varepsilon$ for all b, n . When S is finite, however, it is sufficient to say “ p is non-zero”; and if p is some constant (*e.g.*, is $1/2$), then “ p is non-zero” is sufficient whether S is finite or not.

Note that in Fig. 1 the probability $1/n^2$ is everywhere non-zero, but it is not bounded away

We say in that case that eventually $\neg b$ occurs *almost certainly* over *abstract probability* p and, given that p 's precise value is irrelevant for that conclusion, we could write the system (as Rao does for example [15])

$$(2) \quad b \rightarrow b := \text{False} \mid n := n + 1 ,$$

with the additional implication however that the probability is abstract for both alternatives — that is, the implicit p indicated by \mid is bounded away from both zero and one.

In the sequel we show that in qTL , at least for finite state spaces, the truth of almost-certain eventualities depends only on abstract probabilities, never on their precise values; and we show how to specialise the calculus so that it can act directly over transition systems described as at (2).

2 Summary of the μ -calculi

In this section we give a brief description of both the standard [6] and quantitative [9,10] μ -calculi.

2.1 The standard calculus

Consider a transition system $\mathcal{T}: S \leftrightarrow S$ over a state space S . The standard modal μ -calculus comprises (expressions denoting) predicates of the form shown in Fig. 2, allowing propositional operators, least- and greatest fixed-points, and an implicit ‘next-time’ reference \circ to the effect of taking one step in \mathcal{T} , with demonic resolution of any branching.

As an example, consider the transition system of Fig. 3. We have

$a \in \circ\{c, d, e\}$ one step from a is guaranteed to reach $\{c, d, e\}$.

$a \notin \circ\{c, d\}$ one step from a might go to e .

$a \notin \circ\{a\}$ one step from a cannot reach a at all.

$b \in \circ\{b\}$ no explicit step is interpreted as **skip**.

As an illustration of conjunctivity (3) we have for example

$$\begin{aligned} & \circ(\{b, c, d, e\} \wedge \{c, d, e, f\}).a \\ \equiv & \circ\{c, d, e\}.a && \text{propositional } \wedge \\ \equiv & \text{True} && a \in \circ\{c, d, e\} \text{ by inspection of } \mathcal{T} \\ \equiv & \text{True} \wedge \text{True} \\ \equiv & \circ\{b, c, d, e\}.a \wedge \circ\{c, d, e, f\}.a . && \text{by inspection of } \mathcal{T} \end{aligned}$$

from zero.

A *standard* μ -calculus expression \mathcal{E} is of the form

P	predicate over S , typed $S \rightarrow \text{Bool}$ or equivalently $\mathbb{P}S$
$\mathcal{E} \text{ op } \mathcal{E}$	for propositional operators op
$\circ \mathcal{E}$	‘next-time’ \mathcal{E}
$\mu \mathbb{E}$	least fixed-point of predicate-to-predicate function \mathbb{E}
$\nu \mathbb{E}$	greatest fixed-point of predicate-to-predicate function \mathbb{E}

Notes:

- For state s in S and predicate \mathcal{E} we write $\mathcal{E}.s$ for the value of \mathcal{E} at s , and we say that s *satisfies* \mathcal{E} , or \mathcal{E} *holds at* s , whenever that value is *True*. When \mathcal{E} is given explicitly as a subset S' of S , we can write $s \in S'$ for $S'.s$.
- In this paper we interpret the \circ operator *demonically* with respect to the underlying transition system \mathcal{T} , so that s satisfies $\circ \mathcal{E}$ precisely when *for all* s' we have $(s, s') \in \mathcal{T} \Rightarrow \mathcal{E}.s'$.
- Predicate-to-predicate functions \mathbb{E} are sometimes called *predicate transformers*. We apply μ and ν only to \mathbb{E} that are \Rightarrow -monotonic. Note that operator \circ is monotonic by construction.
- The next-time operator \circ is interpreted demonically, and is assumed (beyond monotonicity) to satisfy the *conjunctivity* property

$$(3) \quad \circ(P \wedge Q) \quad \equiv \quad \circ P \wedge \circ Q ,$$

for all predicates P, Q .

Fig. 2. The standard modal μ -calculus

The transition system \mathcal{T} is

$$s = a \quad \rightarrow \quad s := c \sqcap s := d \sqcap s := e .$$

The state space S is $\{a, b, c, d, e, f\}$, and \sqcap represents choice (interpreted demonically by \circ). For convenience we write the system using a programming-language like syntax, in which for example $s = a$ denotes the predicate $\{a\}$ and $s := c$ denotes the single transition $S \times \{c\}$.

The overall system is thus the relation $\mathcal{T} := \{a\} \times \{c, d, e\}$.

Fig. 3. Standard transition system

2.2 The quantitative calculus $qM\mu$

Consider a probabilistic transition system over a state space S , this time of the form $S \rightarrow \mathbb{P}\bar{S}$ in which initial states are taken to *sets* (\mathbb{P}) of *distributions* ($\bar{\cdot}$) over S .⁵ (Discrete) distributions \bar{S} over S are maps from S into the unit

⁵ Note for comparison with the standard case that $S \leftrightarrow S$ is just $S \rightarrow \mathbb{P}S$, so that we have merely changed the final ‘set of points’ S to the set of discrete distributions \bar{S} (into which S can be embedded).

A *quantitative* μ -calculus expression \mathcal{E} is of the form

A	expectation over S , typed $S \rightarrow \mathbb{R}_{\geq}$
$\mathcal{E} \text{ op } \mathcal{E}$	for \mathbb{R}_{\geq} -closed operators op (extended pointwise)
$\circ \mathcal{E}$	‘next-time’ \mathcal{E}
$\mu \mathbb{E}$	least fixed-point of expectation transformer \mathbb{E}
$\nu \mathbb{E}$	greatest fixed-point of expectation transformer \mathbb{E}

Notes:

- For state s in S we write $\mathcal{E}.s$ for the value of \mathcal{E} at s . For predicate P we write $[P]$ for its characteristic function, which embeds it into the quantitative model: thus $[P].s = 1$ iff $s \in P$.
- The \circ operator is interpreted over \mathcal{T} , and we assume here that it is demonic and *probabilistic* so that expression $\circ \mathcal{E}$ is the least (over the demonic non-determinism) expected value (over the probabilistic nondeterminism) of \mathcal{E} after the computational step. That is $\circ \mathcal{E}.s$ is the minimum over all distributions D with $(s, D) \in \mathcal{T}$ of the expected value $\text{Exp}_D \mathcal{E}$ of \mathcal{E} over distribution D .
- Note that the special case $\circ[P].s$ gives the (demonically least) probability that one step from s will reach a state satisfying P , since the probability assigned an event P by a (state) distribution is equal to the expected value of its characteristic function $[P]$ over that same distribution: thus $\text{Exp}_D[P] = \text{Prob}_D P$.
- We write \Rightarrow for “is everywhere no more than”, and \Leftarrow, \equiv similarly.
- Operator \circ is assumed to satisfy the new property of *sublinearity* [12], that is

$$(4) \quad \circ(aA + bB \ominus c) \Leftarrow a(\circ A) + b(\circ B) \ominus c$$

where $a, b, c \geq 0$ are scalars, juxtaposition is multiplication and A, B are expectations; *truncated subtraction* \ominus is defined

$$x \ominus y := (x - y) \sqcup 0$$

with lower syntactic precedence than $+$.

- Note that we write c both for the scalar and for the constant ‘everywhere- c ’ function.

Fig. 4. The standard modal μ -calculus

interval $[0, 1]$ of probabilities, and sum to 1 over the space.

The quantitative modal μ -calculus comprises \mathbb{R}_{\geq} -valued functions of the form shown in Fig. 4, called *expectations*, and by analogy with the standard case we allow arithmetic operators, least- and greatest fixed-points, and an implicit reference \circ to (the now demonic/probabilistic) \mathcal{T} .

$$s = a \quad \rightarrow \quad s := c \text{ }_{2/3} \oplus (s := d \square s := e)$$

The state space S is again $\{a, b, c, d, e, f\}$, and ${}_p \oplus$ represents probabilistic choice taking the left (resp. right) operand with probability p (resp. $1-p$).

The transition system here is

$$\{(a, \langle 0, 0, 2/3, 1/3, 0, 0 \rangle), (a, \langle 0, 0, 2/3, 0, 1/3, 0 \rangle)\},$$

where $\langle \dots \rangle$ lists the component probabilities of a discrete distribution over the space $a \dots f$.

Fig. 5. Probabilistic and demonic transition system

As an example, consider the transition system of Fig. 5. We have

- $\circ\{c, d, e\}.a = 1$ one step from a is guaranteed to reach $\{c, d, e\}$.
- $\circ\{c, d\}.a = 2/3$ when the probabilistic choice resolves to the right, the demonic choice will avoid d .
- $\circ\{a\}.a = 0$ one step from a cannot reach a at all.
- $\circ\{b\}.b = 1$ no explicit step is interpreted as **skip**.

(To avoid the clutter of $[\{c, d, e\}]$ for example, we have omitted the embedding brackets $[\cdot]$ (see notes of Fig. 4) when they occur around set comprehensions.)

For an illustration of sublinearity (Property 4, Fig. 4), consider the special case in which its scalars a, b, c are all 1. We define $x \& y := x + y \ominus 1$, and note that sublinearity then gives us *&-subdistribution through* \circ : for all expectations A, B we have

$$(5) \quad \circ(A \& B) \quad \Leftarrow \quad \circ A \& \circ B .$$

Operator $\&$ is useful because it both generalises Boolean conjunction⁶ and specialises sublinearity: it is our ‘best quantitative approximation’ to conjunctivity (3).

In the system of Fig. 4, because we have for example that $\{c\} \equiv \{c, d\} \& \{c, e\}$, we can check (5) by verifying that

$$\begin{aligned} & \circ\{c\}.a \\ \equiv & \quad 2/3 && \text{inspection of } \mathcal{T} \\ \Leftarrow & \quad 1/3 \\ \equiv & \quad 2/3 \& 2/3 && \text{definition of } \& \\ \equiv & \quad \circ\{c, d\}.a \& \circ\{c, e\}.a . && \text{inspection of } \mathcal{T} \end{aligned}$$

Note that we have only an inequality,⁷ whereas in the standard case (conjunctivity) we have equality.

⁶ That is, we have $[P] \wedge [Q] \equiv [P] \& [Q]$ for all predicates P, Q .

⁷ The inequality is because $\circ\{c, d\}.a \equiv \circ\{c, e\}.a \equiv 2/3$ is true of other transition systems

Consequences of sublinearity include (by simple arithmetic [12, Sec. 7 pp. 340ff]) the following properties for all expectations A, B :

monotonicity If $A \Rightarrow B$ then $\circ A \Rightarrow \circ B$.

feasibility $\circ A \Rightarrow \sqcup A$.

scaling For $c \geq 0$ we have $\circ(cA) \equiv c(\circ A)$.

bounded up-continuity Provided S is finite, the set of expectations \mathcal{A} is up-directed and $\sqcup \mathcal{A}$ is bounded above, we have

$$\circ(\sqcup \mathcal{A}) \equiv (\sqcup \mathcal{A} : \mathcal{A} \cdot \circ A) .$$

down-continuity Provided S is finite and the set of expectations \mathcal{A} is down-directed, we have

$$\circ(\sqcap \mathcal{A}) \equiv (\sqcap \mathcal{A} : \mathcal{A} \cdot \circ A) .$$

3 Specialisations to the temporal calculi

The modal calculi act as *temporal* calculi if one identifies specific types of expression for concepts like (among others) ‘eventually’, ‘always’ and ‘unless’ [4]. When based on the standard calculus, they give absolute (*i.e.*, true or false) judgements; in the quantitative case, the judgements are probabilistic.

3.1 Standard temporal logic

We define some typical temporal operators in Fig. 6. The role of conjunctivity (3) here is that it allows high-level proofs of temporal properties *without* referring directly to the underlying transition system. For example, one such property is the *eventually-until lemma*⁸

$$(6) \quad P \triangleright (P \wedge Q) \wedge \Diamond Q \Rightarrow \Diamond(P \wedge Q) ,$$

which states that if P holds up to *and including* a possible step at which Q holds, and Q eventually does hold, then in fact $P \wedge Q$ eventually holds.⁹ We give the straightforward proof of that (Lem. A.1) as an example of the use of conjunctivity.

3.2 Quantitative temporal logic qTL

From here on we restrict our expectations to the range $[0, 1]$ rather than \mathbb{R}_{\geq} , using only operators for which $[0, 1]$ is closed. (Note that feasibility above gives the closure of \circ itself.)

over S as well; one of those is for example

$$s = a \rightarrow s := c_{1/3} \oplus (s := d_{1/2} \oplus s := e) ,$$

for which $\circ\{c\}.a$ is in fact as low as $1/3$. It can be shown that sublinearity gives the highest estimate possible under those general circumstances: it is only as “pessimistic” as necessary.

⁸ Compare the *PSP* lemma of UNITY [2].

⁹ Here for uniformity we use \Rightarrow for ‘entails’, which is consistent with its quantitative definition since $P \vdash Q$ iff $[P] \Rightarrow [Q]$.

“eventually P ”

$$\Diamond P := (\mu X \cdot P \vee \circ X) \quad \text{If sufficiently many steps are taken, then } P \text{ will hold.}$$

“always P ”

$$\Box P := (\nu X \cdot P \wedge \circ X) \quad \text{No matter how many steps are taken } P \text{ will continue to hold.}$$

“ P unless Q ”

$$P \triangleright Q := (\nu X \cdot Q \vee (P \wedge \circ X)) \quad \text{No matter how many steps are taken, } P \text{ will continue to hold, unless a state is reached in which } Q \text{ holds.}$$

We write “ $:=$ ” for “is defined to be”.

Fig. 6. Definition of some standard temporal operators in the modal μ -calculus

$$\text{“eventually } A\text{”} \quad \Diamond A := (\mu X \cdot A \sqcup \circ X)$$

$$\text{“always } A\text{”} \quad \Box A := (\nu X \cdot A \sqcap \circ X)$$

$$\text{“} A \text{ unless } B\text{”} \quad A \triangleright B := (\nu X \cdot B \sqcup (A \sqcap \circ X))$$

In qTL we restrict expectations to the range $[0, 1]$ instead of \mathbb{R}_{\geq} .

Fig. 7. Definition of the quantitative temporal operators for qTL in the quantitative modal μ -calculus

In the quantitative case, we define the temporal operators as in Fig. 7.

Consider “ $\Diamond A$ ” however: for general expectation A it is not helpful to interpret it as “the probability that eventually A is established”, because the meaning of “establish A ” is itself unclear if A is a number. So what does $\Diamond A$ mean? (Similar remarks apply of course to the other temporal operators.)

It can be shown that the special case $\Diamond[P]$ is indeed the probability of

eventually establishing P .¹⁰ More generally [9] the expression $\Diamond A$ is

the supremum, over all strategies that determine in each state whether to take another transition or to stop, of the expected value of A when the strategy says “stop”; the strategy “never stop” gives 0 by definition.

The situation with the other operators is similar.¹¹

Again (the generalisation of) conjunctivity plays an important role in high-level reasoning. Using $\&$ -subdistribution, for example, we can prove a generalisation of (6); it is the quantitative eventually-until lemma

$$(7) \quad A \triangleright (A \& B) \ \& \ \Diamond B \quad \Rightarrow \quad \Diamond(A \& B),$$

which we prove as Lem. A.2.

As an example of probabilistic eventualities, we return to the system Fig. 1. We write out expectations as expressions over the program variables b, n , and calculate $\Diamond[\neg b]$ directly (and unimaginatively) from the least-fixed-point limit implied by its definition (Fig. 7).

$$\text{term 0: } 0 \qquad \qquad \qquad \perp \equiv 0$$

$$\begin{aligned} \text{term 1: } & [\neg b] \sqcup \circ 0 & \text{definition } \Diamond: \text{ term } k+1 &= [\neg b] \sqcup \circ(\text{term } k) \\ \equiv & [\neg b] & \circ 0 &\equiv 0 \text{ by feasibility} \end{aligned}$$

$$\begin{aligned} \text{term 2: } & [\neg b] \sqcup \circ[\neg b] \\ \equiv & [\neg b] \sqcup ([\neg b] \sqcup [b]/n^2) & \text{inspection of } \mathcal{T} \\ \equiv & [\neg b] \sqcup [b]/n^2 \end{aligned}$$

$$\begin{aligned} \text{term 3: } & [\neg b] \sqcup \circ([\neg b] \sqcup [b]/n^2) \\ \equiv & [\neg b] \sqcup [b](1/n^2 + (1 - 1/n^2)(1/(n+1)^2)) \\ \equiv & [\neg b] \sqcup 2[b]/n(n+1) \end{aligned}$$

$$\begin{aligned} \text{term 4: } & [\neg b] \sqcup \circ([\neg b] \sqcup 2[b]/n(n+1)) \\ \equiv & [\neg b] \sqcup 3[b]/n(n+2) \end{aligned}$$

\vdots

¹⁰ As is usual, we mean by that probability the measure, in the Borel algebra of ‘cones’ within the tree of possible executions, of the set of paths along which P eventually occurs.

¹¹ Again we have agreement with the standard case, since if P is guaranteed to hold eventually then the strategy “stop when P holds” will establish *True*; if P is not guaranteed to hold eventually then, given any strategy, demonic choice could force either a “stop” in a state where P is false or an infinite run, also giving *False* (by definition).

term k : $[\neg b] \sqcup (k-1)[b]/n(n+k-2)$, induction

so that we have

$$\begin{aligned}
 & \diamond[\neg b] \\
 \equiv & \text{terms ascending, so } \sqcup_k \text{ agrees with } \lim_{k \rightarrow \infty} \\
 & \lim_{k \rightarrow \infty} [\neg b] \sqcup (k-1)[b]/n(n+k-2) \\
 \equiv & [\neg b] \sqcup [b]/n \qquad \lim_{k \rightarrow \infty} (k-1)/(n+k-2) = 1 \\
 \equiv & 1/n \text{ if } b \text{ else } 1 . \qquad \text{arithmetic}
 \end{aligned}$$

That is, termination is certain if $\neg b$ holds (at the start), and occurs with probability $1/n$ if it does not.

4 Abstract reasoning in qTL

We have now completed our review of the existing calculi, and turn to our present contribution.

At the end of Sec. 3, we gave a calculation of $\circ[\neg b]$ for the system of Fig. 1. In System (1) following it, a similar calculation would be ¹²

term 0: 0 $\perp \equiv 0$

term 1: $[\neg b] \sqcup \circ 0$ term $k+1 = [\neg b] \sqcup \circ(\text{term } k)$
 $\equiv [\neg b]$

term 2: $[\neg b] \sqcup \circ[\neg b]$
 $\equiv [\neg b] \sqcup p[b]$

term 3: $[\neg b] \sqcup \circ([\neg b] \sqcup p[b])$
 $\equiv [\neg b] \sqcup p[b](1 + (1-p))$

term 4: $[\neg b] \sqcup \circ([\neg b] \sqcup p[b](1 + (1-p)))$
 $\equiv [\neg b] \sqcup p[b](1 + (1-p) + (1-p)^2)$

¹² This heavy-handed ‘limit’ approach is not the only way to calculate $\diamond[\neg b]$ here: an alternative is to show from the definitions that

$$\diamond[\neg b] \equiv p + (1-p)\diamond[\neg b]$$

holds for this system, whence rearrangement and dividing by p gives us $\diamond[\neg b] \equiv 1$. But the point about explicit treatment of p remains.

⋮

term ∞ : $[\neg b] \sqcup p \sum_{k=0}^{\infty} (1-p)^k [b]$, induction

whence we conclude that $\Diamond[\neg b] \equiv [\neg b] \sqcup p[b]/p \equiv 1$ provided p is not 0.

Our aim is simply to show that in abstract systems like (1) it is possible to avoid explicit numeric calculations like the above.

The main technical result will be that the *floor* $\lfloor \cdot \rfloor$ and *ceiling* $\lceil \cdot \rceil$ operators can abstract from the ‘intermediate’ values lying strictly between 0 and 1: in finite state spaces we prove

$$\lfloor \Diamond[P] \rfloor \equiv \lfloor \lceil \Diamond[P] \rceil \triangleright [P] \rfloor ,$$

whose left-hand side is 1 if $\Diamond[P]$ is almost certain, and 0 otherwise; and the constructions $\lfloor \cdot \triangleright \cdot \rfloor$ and $\lceil \Diamond \cdot \rceil$ used in the right-hand side will be shown to depend only on abstract probabilities.

We begin with a general discussion.

4.1 ‘Almost-certain’ is special for probabilistic systems

We place our work in context by recalling the following facts from finite-state Markov process theory, but in our notation. Let S be the finite state space.

- Operator \circ is a transition function over S . If we write state predicates P as $\{0, 1\}$ -valued column vectors of height $\#S$, then \circ (if it contains no nondeterministic choice) can be seen as a Markov matrix, and $\circ[P]$ is post-multiplication of \circ by the column vector representing P : each element $\circ[P].s$ of the product $\circ[P]$ gives the probability of reaching P from s .

More generally, for expectation A as a column vector we have $\circ A$ as post-multiplication, and each element $\circ A.s$ of the product gives the expected final value of A when taking a transition from s .

- State s' is reachable from state s iff $\circ^n \{s'\}.s > 0$ for some finite n (the number of transitions taken).
- A subset P of S is closed (with respect to \circ) iff $[P] \Rightarrow \circ[P]$.
- The probability of reaching P in one step from s — call it $\circ_1.P.s$ — is $\circ[P].s$.
- The probability of reaching P for the first time at the n^{th} step, for $n > 1$, is $\circ_n.P := \circ([\neg P] \sqcap \circ_{n-1}.P)$.
- The probability of eventually reaching a subset P from state s , say $\circ_\infty.P.s$, is

$$\sum_{n>0} \circ_n.P.s ,$$

which is also known as *the first-passage probability from s to P* .

- $\circ_\infty.\{s\}.s$ is the probability of eventual return to s .

In that notation we can state the following theorem for Markov processes:

Let \circ represent a Markov matrix, let S be a finite state space and s a state; and let C be the set of reachable states from s . Then

$$\circ_{\infty}.\{s\}.s = 1 \text{ iff } p[C] \Rightarrow \circ_{\infty}.\{s\} \text{ for some } p > 0. \text{ }^{13}$$

The important thing to note about this result is that p is only specified to be greater than 0. Equivalently, only the connectivity of the Markov process is important, rather than the actual values of the probabilities — which is why the proof rule for $\circ_{\infty}.\{s\}.s$ is so simple.

We regard the result as a form of completeness, because it states that the connectivity information is sufficient to establish the eventuality.

Our aim is to demonstrate that for probabilistic and demonic programs, a simpler calculus is all that is needed to prove (eventuality) properties with probability 1: as for standard programs only the “connectivity” of the program is important and not the actual probabilistic values. We will prove a similar completeness result for general eventuality properties. For many probabilistic programs that will provide a sufficient proof rule, since probability 1 (or not) is all that is of interest.

Other recent work on the special properties of “probability 1” events in programs includes results of Rao [15], Pnueli/Zuck [13] and Hart/Sharir/Pnueli [16]. Their completeness results in some cases assume various kinds of fairness.

4.2 Relevant properties of our temporal operators

We concentrate on next-time \circ , eventually \Diamond and unless \triangleright . The following properties can be proved directly from the operators’ definitions [10] or — in some cases — have been given above.

Lemma 4.1 Properties of next-time — *For all expectations A, B ,*

- (i) $\circ A \ \& \ \circ B \Rightarrow \circ(A \ \& \ B)$.
- (ii) *If $A \Rightarrow A'$, then $\circ A \Rightarrow \circ A'$.*
- (iii) $\circ 1 \equiv 1$.

Lemma 4.2 Properties of eventually — *For all expectations A, B ,*

- (i) $A \Rightarrow \Diamond A$.¹⁴
- (ii) $\circ \Diamond A \Rightarrow \Diamond A$.
- (iii) *If $B \sqcup \circ A \Rightarrow A$, then $\Diamond B \Rightarrow A$.*
- (iv) *If $A \Rightarrow A'$, then $\Diamond A \Rightarrow \Diamond A'$.*

¹³Note that $p[C].s'$ is just $(p \text{ if } s' \in C \text{ else } 0)$, so that — after applying both sides to s' — the inequality $p[C] \Rightarrow \circ_{\infty}.\{s\}$ says that for all $s' \in C$ the first-passage probability $\circ_{\infty}.\{s\}.s'$ from s' to s is at least p .

¹⁴Note how this follows from our intuitive ‘strategic’ explanation earlier of $\Diamond A$: since the strategy “stop right now” is guaranteed to return at least A , the value of $\Diamond A$ can never be less than that.

Lemma 4.3 Properties of unless — *For all expectations A, B ,*

- (i) $B \Rightarrow A \triangleright B \Rightarrow A \sqcup B$.
- (ii) *If $C \Rightarrow B \sqcup (A \sqcap \circ C)$, then $C \Rightarrow A \triangleright B$.*
- (iii) $A \triangleright B \equiv B \sqcup (A \sqcap \circ (A \triangleright B))$.
- (iv) *If $A \Rightarrow A'$ and $B \Rightarrow B'$, then $A \triangleright B \Rightarrow A' \triangleright B'$.*

From these we have a form of completeness, based on the fact that the above properties determine the action of their respective operators.

Theorem 4.4 Standard completeness — *If \circ, P, Q are interpreted over a finite state (standard) transition system, then the above properties are sufficient to calculate $\Diamond[P]$ and $[P] \triangleright [Q]$ — only the transitions must be specified.*

Although for probabilistic programs the same idea of finding the least solution to an equation remains valid (and is in that sense complete¹⁵), even for finite-state programs discovering the actual real number values can still be rather tortuous, as we saw above. Indeed that is always going to be the case for non-(0-1) properties.

We desire a completeness property like Thm. 4.4 for abstract probabilistic programs — the idea is that if we only specify the transitions, merely indicating when they are probabilistic, then we only need use standard techniques, without having to introduce all the complications of the full quantitative calculus.

Our first task is to show how to extract information “with probability 1”.

From this point we assume that the transition system is probabilistic, and that the state space is finite. Recall our restriction in qTL to expectations that take values only in the unit interval $[0, 1]$ rather than in the more general range \mathbb{R}_{\geq} .

4.3 Floor and ceiling for ‘almost certain’

Our principal tool will be the ceiling $\lceil \cdot \rceil$ and floor $\lfloor \cdot \rfloor$ operators (both taking expectations to expectations), defined

ceiling $\lceil A \rceil.s := \lceil A.s \rceil$, that is $[A.s \neq 0]$

floor $\lfloor A \rfloor.s := \lfloor A.s \rfloor$, that is $[A.s = 1]$

With them we can write “almost certainly $\Diamond[P]$ ” as $\lfloor \Diamond[P] \rfloor$, and our aim is to calculate that from the ‘connectivity’ — the *abstract* probabilistic properties — of \circ .

4.4 Floor and ceiling for the ‘connectivity’ of \circ

We also use ceiling and floor to extract the connectivity (rather than the particular values of) the probabilistic transitions \circ . With them we define two

¹⁵ ... provided we replace Lem. 4.1 Property (i) with full sublinearity.

‘derived’ transition operators, one converting probabilities to angelic choice, and the other converting them to demonic.

Definition 4.5 The *angelic* and *demonic* projections of \circ are defined

$$\begin{aligned} \text{angelic projection } \circ_a A &:= \lceil \circ A \rceil \\ \text{demonic projection } \circ_d A &:= \lfloor \circ A \rfloor \end{aligned}$$

For example, if $\circ[P].s > 0$ then there is a non-zero probabilistic transition from s into P , which revealed by the fact $\circ_a[P].s = 1$. That means for example that

$$(s := H_p \oplus s := T)_a = s := H \sqcup s := T ,$$

provided $0 < p < 1$, where we are abusing notation to compare \circ_a for the transition system on the left with \circ for the system on the right. The operator \sqcup is angelic choice.

On the other hand $\circ_d[P].s = 1$ iff all the transitions from s (whether probabilistic or not) end up in P , so that we have

$$(s := H_p \oplus s := T)_d = s := H \sqcap s := T ,$$

again provided $0 < p < 1$. Clearly \circ_a and \circ_d depend only on the connectivity, since they discard all numeric information; it is not difficult to show that they determine the connectivity.¹⁶

4.5 Properties of \circ_a and \circ_d

Before proceeding to almost-eventually properties, we need the following technical results for our connectivity operators.

Lemma 4.6 Some properties of \circ_d — *Projection \circ_d in effect replaces probabilistic by demonic choice: it is conjunctive over predicates, monotonic in general, and distributes through greatest fixed-points:*

$$\text{conjunctive } \circ_d([P] \& [Q]) \equiv \circ_d[P] \& \circ_d[Q].$$

$$\text{monotonic } \text{If } A \Rightarrow A' \text{ then } \circ_d A \Rightarrow \circ_d A'.$$

gfp-distributive If $\mathbb{E}.\circ$ is an expression containing the operator \circ , and $\mathbb{F}.\circ_d$ similarly, and they together satisfy $\lfloor \mathbb{E}.\circ.X \rfloor \equiv \mathbb{F}.\circ_d.\lfloor X \rfloor$ for all expectations X , then

$$\lfloor \nu \mathbb{E}.\circ \rfloor \equiv \nu(\mathbb{F}.\circ_d) .$$

Proof. Lemmas A.3 and A.5. □

Lemma 4.7 Some properties of \circ_a — *Projection \circ_a in effect replaces probabilistic by angelic choice: it is monotonic, and distributes through least fixed-points:*

$$\text{monotonic } \text{If } A \Rightarrow A' \text{ then } \circ_a A \Rightarrow \circ_a A'.$$

¹⁶ For a purely probabilistic or purely demonic system, either \circ_a or \circ_d would be sufficient on its own; only for a mixture of the two forms of choice does one need both operators.

lfp-distributive If $\mathbb{E}.\circ$ is an expression containing the operator \circ , and $\mathbb{F}.\circ_a$ similarly, and they together satisfy $\llbracket \mathbb{E}.\circ.X \rrbracket \equiv \mathbb{F}.\circ_a.\llbracket X \rrbracket$ for all expectations X , then

$$\llbracket \mu \mathbb{E}.\circ \rrbracket \equiv \mu(\mathbb{F}.\circ_a) .$$

Proof. Lemmas A.4 and A.6. □

4.6 Almost-certainly is related to connectivity

We can now show that some almost-certainly properties — though not yet the one we want — depend only on the connectivity of \circ , as captured by \circ_a and \circ_d .

Lemma 4.8 $\llbracket A \triangleright B \rrbracket$ and $\llbracket \Diamond A \rrbracket$ can be calculated from the connectivity \circ_a , \circ_d of \circ , and do not depend on the actual values of the probabilistic transitions.

Proof. $A \triangleright B$ is a greatest fixed-point, and so the result follows from gfp-distribution of \circ_d (Lem. 4.6) once we notice from Lem. A.3 that

$$\llbracket B \sqcup (A \sqcap \circ X) \rrbracket \equiv \llbracket B \rrbracket \sqcup (\llbracket A \rrbracket \sqcap \circ_d \llbracket X \rrbracket) .$$

We treat $\Diamond A$ similarly (lfp-distribution (Lem. 4.7) and Lem. A.4).¹⁷ □

Unfortunately however, our aim is to calculate $\llbracket \Diamond A \rrbracket$ (not $\llbracket \Diamond A \rrbracket$), and indeed $\llbracket \cdot \rrbracket$ does not distribute through *least* fixed-points. For consider \circ interpreted as

$$s := H \quad 1/2 \oplus \quad s := T ,$$

and compare $\llbracket \Diamond \{H\} \rrbracket \equiv 1$ and $(\mu X \cdot \llbracket \{H\} \rrbracket \sqcup \circ_d X) \equiv \{H\}$.

it will turn out that we can reach $\llbracket \Diamond A \rrbracket$ indirectly, via $\llbracket \Diamond \cdot \rrbracket$, at least when A is standard; for that we begin with the following lemma:

Lemma 4.9 For all expectations A and transition systems \circ we have

$$\Diamond A \Rightarrow \llbracket \Diamond A \rrbracket \triangleright A .$$

Proof. We show that $A \sqcup \circ(\llbracket \Diamond A \rrbracket \triangleright A) \Rightarrow \llbracket \Diamond A \rrbracket \triangleright A$, which allows us to apply Property (iii) of Lem. 4.2:

$$\begin{aligned} & A \sqcup \circ(\llbracket \Diamond A \rrbracket \triangleright A) \Rightarrow \llbracket \Diamond A \rrbracket \triangleright A \\ \text{iff} \quad & A \sqcup \circ(\llbracket \Diamond A \rrbracket \triangleright A) \Rightarrow A \sqcup (\llbracket \Diamond A \rrbracket \sqcap \circ(\llbracket \Diamond A \rrbracket \triangleright A)) \quad \text{definition } \triangleright \\ \text{iff} \quad & \circ(\llbracket \Diamond A \rrbracket \triangleright A) \Rightarrow \llbracket \Diamond A \rrbracket \quad \text{arithmetic; } A \Rightarrow \llbracket \Diamond A \rrbracket \text{ (Lem. 4.2)} \\ \text{iff} \quad & \circ(\llbracket \Diamond A \rrbracket \triangleright A) \Rightarrow \llbracket A \rrbracket \sqcup \llbracket \circ \Diamond A \rrbracket \quad \text{definition } \Diamond A; \text{ arithmetic} \\ \text{if} \quad & \llbracket \Diamond A \rrbracket \triangleright A \Rightarrow \llbracket \Diamond A \rrbracket \sqcup A \Rightarrow \llbracket \Diamond A \rrbracket \quad \text{(Lemmas 4.2, 4.3); monotonicity } \circ \end{aligned}$$

¹⁷With the obvious definitions we could write just

$$\llbracket A \triangleright B \rrbracket \equiv \llbracket A \rrbracket \triangleright_d \llbracket B \rrbracket \quad \text{and} \quad \llbracket \Diamond A \rrbracket \equiv \Diamond_a \llbracket A \rrbracket .$$

$$\circ[\Diamond A] \Rightarrow [\circ\Diamond A] ,$$

which is a consequence of Lem. A.4. □

Lem. 4.9 gives us trivially a connectivity-calculable upper bound on $[\Diamond A]$:

Lemma 4.10 Upper bound for almost-certain eventuality

$$[\Diamond A] \Rightarrow [[\Diamond A] \triangleright A] .$$

Proof. Lem. 4.9 and the monotonicity of $[\cdot]$. □

The right hand side is calculable from the connectivity of \circ , because by Lem. 4.8 we know that $[\Diamond A]$ is so calculable, and by Lem. 4.8 (again) so is $[[\Diamond A] \triangleright A]$.

In the next section we show that we achieve equality when A is standard.

5 0-1 laws and temporal logic

In this section we show how the introduction of a 0-1 law (or axiom) is all that is needed to show that $[\Diamond[P]]$ does indeed rely only on connectivity.¹⁸ We gave an example of the 0-1 law for purely probabilistic programs; the idea has been extended to probabilistic/demonic programs [5,11] using the notation and ideas of temporal logic.

Lemma 5.1 0-1 Law — *For any expectation A , predicate P and probability $p > 0$, if $p(A \triangleright [P]) \Rightarrow \Diamond[P]$ then in fact $A \triangleright [P] \Rightarrow \Diamond[P]$.*

Proof. *The full proof — allowing demonic nondeterminism and possibly-aborting transitions — is beyond the scope of this paper; but it is a simple consequence of 0-1 results on the probabilistic treatment of loops [11, Lem. 6.1 p10]. As an illustration, however, we give a proof entirely in qTL (Thm. A.7) for the restricted case of non-demonic and terminating transitions.* □

The above law is valid for all state spaces: but for finite state spaces it has a much more compact formulation.

Lemma 5.2 0-1 Law (finite state spaces) — *In finite state models, Lem. 5.1 is equivalent to*

$$(8) \quad [\Diamond[P]] \triangleright [P] \Rightarrow \Diamond[P] .$$

Proof. *Suppose the interpretation of \circ is over a finite state space. That means that Lem. 5.1 is equivalent to the following, in which we have eliminated*

¹⁸It is only now that we must make some restrictions to predicates, rather than general expectations, which is why we write $[P]$ rather than A .

the abstract p .¹⁹

$$(9) \quad \text{if } A \triangleright [P] \Rightarrow [\Diamond[P]] \quad \text{then} \quad A \triangleright [P] \Rightarrow \Diamond[P]$$

We now show that (8) holds iff (9) holds.

(8) implies (9) Suppose that $A \triangleright [P] \Rightarrow [\Diamond[P]]$. It follows from Lem. 4.3 (ii) that $A \triangleright [P] \Rightarrow [\Diamond[P]] \triangleright [P]$, because

$$\begin{aligned} & [P] \sqcup ([\Diamond[P]] \sqcap \circ(A \triangleright [P])) \\ \Leftarrow & [P] \sqcup (A \triangleright [P] \sqcap \circ(A \triangleright [P])) && \text{assumption} \\ \equiv & A \triangleright [P] , && \text{by cases on } P.s \end{aligned}$$

whence our assumption (8) gives us $A \triangleright [P] \Rightarrow \Diamond[P]$, as desired overall.

(9) implies (8) From Lems. 4.3(i) and 4.2(i) we have

$$[\Diamond[P]] \triangleright [P] \Rightarrow [\Diamond[P]] \sqcup [P] \equiv [\Diamond[P]] ,$$

hence we have immediately from (9) that

$$[\Diamond[P]] \triangleright [P] \Rightarrow \Diamond[P] .$$

□

Corollary 5.3 For finite models, $\Diamond[P] \equiv [\Diamond[P]] \triangleright [P]$.

Proof. In finite models we may use the second form Lem. 5.2 of the 0-1 law; the result then follows from Lemma 4.9. □

Cor. 5.3 is the key to showing that for probability-1 properties, connectivity is sufficient.

Theorem 5.4 Completeness for probability-1 eventualities — If \circ is interpreted over a finite-state probabilistic system, and P is a state predicate, then $[\Diamond[P]]$ is determined by \circ_a and \circ_d , the probabilistic/demonic connectivity of \circ .

Proof. Cor. 5.3 gives us that $\Diamond[P] \equiv [\Diamond[P]] \triangleright [P]$, from which we have

$$(10) \quad [\Diamond[P]] \equiv [[\Diamond[P]] \triangleright [P]] .$$

Since $[\cdot \triangleright \cdot]$ and $[\Diamond \cdot]$ depend only on the connectivity, the result follows. □

6 Example

Consider the abstract system

$$s := H \quad | \quad s := T ,$$

¹⁹ To see that (9) does not hold for infinite state spaces, consider this system over $S := \mathbb{Z}$ that defines a random walker on the integers:

$$s := s + 1 \quad \text{2/3} \oplus \quad s := s - 1 .$$

Observe that $[\Diamond[s \leq 0]] \equiv 1 \equiv [s > 0] \triangleright [s \leq 0]$, but that $\Diamond[s \leq 0]$ is not equal to 1.

A qTL formula Φ is of the form

A	explicitly given numeric function over S , typed $S \rightarrow [0, 1]$
$\Phi \sqcap \Phi$	minimum, generalising \wedge
$\Phi \sqcup \Phi$	maximum, generalising \vee
$\circ\Phi$	next-time
$\diamond\Phi$	eventually
$\square\Phi$	always
$\Phi \triangleright \Phi$	unless ,

and the interpretation of the formula is via the quantitative μ -calculus, as given earlier in Fig. 7. The formula is said to be *almost certain* at a state s of a given transition system if it evaluates to 1 at s .

Fig. 8. Quantitative temporal logic formulae and interpretation

where the “|” represents an abstract $_p\oplus$ with $0 < p < 1$. The probabilistic connectivity is given by

angelic $\circ_a[P] \equiv [P \neq \{\}]$, because there is a non-zero probability of establishing any non-empty predicate over $\{H, T\}$.

demonic $\circ_d[P] \equiv [P \equiv \{H, T\}]$, because there is a non-zero probability of avoiding any non-total predicate over $\{H, T\}$.

Now we look at the almost-certain eventuality $[\diamond\{H\}]$; we have

$$\begin{aligned}
 & [\diamond\{H\}] \\
 \equiv & \quad [[\diamond\{H\}] \triangleright \{H\}] && \text{Cor. 5.3} \\
 \equiv & \quad [\circ_a\{H\} \triangleright \{H\}] && \text{Lem. 4.8} \\
 \equiv & \quad [1 \triangleright \{H\}] && \text{inspection: choice | is abstract} \\
 \equiv & \quad 1 . && \text{Lems. 4.3(ii), 4.1(iii)}
 \end{aligned}$$

7 Complexity analysis

We now look briefly at the time-complexity of evaluating almost-certainties in qTL : the precise language and its interpretation is set out in Fig. 8; and our result is that the complexity of evaluating $[\Phi]$ over transition system \mathcal{T} is linear in the number of temporal operators in Φ and in the number of transitions in \mathcal{T} . We outline a proof of that in this section.

Throughout the following we will use the specific formula Φ_0 , defined

$$\diamond(A \sqcup (B \triangleright \square C)) ,$$

as a running example: we want to evaluate $[\Phi_0]$.

“angelic-eventually A ” $\Diamond_a A := (\mu X \cdot A \sqcup \circ_a X)$

“demonic-always A ” $\Box_d A := (\nu X \cdot A \sqcap \circ_d X)$

“ A demonic-unless B ” $A \triangleright_d B := (\nu X \cdot B \sqcup (A \sqcap \circ_d X))$

These operators are analogues of the $\exists\circ, \forall\Box$ etc. of conventional (probabilistic) temporal logic.

Fig. 9. Angelic/demonic temporal operators

7.1 Propagate $\lfloor \cdot \rfloor$ inwards

Recalling Lem. 4.8, define angelic/demonic versions of the temporal operators as in Fig. 9. (Compare Fig. 7.)

To distribute $\lfloor \cdot \rfloor$ inwards we use the equalities set out in this lemma:²⁰

Lemma 7.1

$$(11) \quad \lfloor \circ A \rfloor \equiv \circ_d \lfloor A \rfloor$$

$$(12) \quad \lfloor \Diamond A \rfloor \equiv \lfloor \Diamond \lfloor A \rfloor \rfloor$$

$$(13) \quad \lfloor \Box A \rfloor \equiv \Box_d \lfloor A \rfloor$$

$$(14) \quad \lfloor A \triangleright B \rfloor \equiv \lfloor A \rfloor \triangleright_d \lfloor B \rfloor$$

Proof. Only (12) needs comment. Its proof relies on the fact that \Diamond , like \circ , is semi-sublinear [8]; given that, its proof mimics that of Lem. A.3.

Note that we do not use \Diamond_d (with the obvious definition) in this case, because we cannot: recall the remarks following Lem. 4.8. \square

Using our lemma with Φ_0 , we have

$$\begin{aligned} & \lfloor \Diamond(A \sqcup (B \triangleright \Box C)) \rfloor \\ & \equiv \lfloor \Diamond(\lfloor A \rfloor \sqcup \lfloor B \rfloor \triangleright_d (\Box_d \lfloor C \rfloor)) \rfloor, \end{aligned}$$

in which all explicit expectations (A, B, C) have been made standard $(\lfloor A \rfloor, \lfloor B \rfloor, \lfloor C \rfloor)$.

7.2 Convert \Diamond 's to standard operators

The procedure of the previous section eliminated all properly probabilistic modal operators, replacing them with demonic versions, except for \Diamond . To deal with \Diamond we use our main result Cor. 5.3 which, combined with the above and Lem. 4.8, allows us to state that

$$(15) \quad \lfloor \Diamond A \rfloor \equiv (\Diamond_a A) \triangleright_d A,$$

²⁰ Some of these have been defined/stated elsewhere in this report; we repeat them here for the convenience of having them all together.

provided A is standard. Since the inward propagation of the previous section has made all sub-formulae standard, indeed (15) applies: in the case of Φ_0 we can continue

$$\begin{aligned} & \llbracket \Diamond(\llbracket A \rrbracket \sqcup \llbracket B \rrbracket \triangleright_d (\Box_d \llbracket C \rrbracket)) \rrbracket \\ & \equiv (\Diamond_a X) \triangleright_d X, \end{aligned}$$

where $X := \llbracket A \rrbracket \sqcup \llbracket B \rrbracket \triangleright_d (\Box_d \llbracket C \rrbracket)$.

We use the ‘where’-clause to remember that X has been duplicated, so that we need calculate it only once.

7.3 Evaluate $\llbracket \Phi \rrbracket$

The two translations of the previous sections transform Φ into an expression containing only \circ_d , \Diamond_a , \Box_d and \triangleright_d . The number of those operators is no more than twice the number of operators in the original formula, provided the duplication inherent in (15) is properly noted. Thus our result will follow if we can establish that evaluation of each of those operators is linear in the size of the transition system. We discuss that briefly for each operator in turn; in each case A, B are standard.

$\circ_d A$ — \circ_d treats the system as entirely demonic. Examine all states, and select only those all of whose outgoing transitions lead into A .

$\Diamond_a A$ — If the original system contains demonic (as well as probabilistic) choice, then the system will be treated as demonic/angelic by \Diamond_a — that is, although the probabilistic choice is made angelic, the pure demonic choice is retained. The operational behaviour for each complete transition is a ‘first-stage’ demonic choice of ‘half-transition’ followed by a ‘second-stage’ angelic choice of half-transition.

Start with the set of states A , and for each of its states follow all second-stage angelic half-transitions back, colouring their sources; if the source was uncoloured, continue on to follow back the first-stage half-transition, decrementing the ‘first-stage transition count’ of its originating state (prepared beforehand).

Having done that for all of A , go over the transitions again this time deleting all second-stage transitions followed, and adding all states whose first-stage count has become zero, in that case deleting the first-stage transitions as well.

Continue the process until no states are added; each transition will have been followed at most a constant number of times (amortised).

$\Box_d A$ — Treat it as $A \triangleright_d 0$.

$A \triangleright_d B$ — \triangleright_d treats the system as entirely demonic; we work with the complement. Start with the set of states $\neg A \wedge \neg B$, and for each transition leading backwards from there:

- if it leads into B , ignore it; and
- if it leads into $A \wedge \neg B$, add that state to the set.

In either case, delete the transition; and repeat the process with the added states, stopping the whole procedure when no new states are added. The result is the complement of the accumulated states; and in the process, each transition is considered at most once.

In the case of Φ_0 we carry out four calculations from the above, two within X and two outside it.

8 Conclusion

Abstract probabilities and 0-1 laws have long been recognised as important techniques for simplifying analysis in probabilistic systems. However the tendency has been to use formulations of those laws at the level of models [1,15,17] and not to integrate them formally as axioms of program logic, as is customary for other operational phenomena.

There are certainly difficulties in importing well-understood concepts directly from probability theory to a computational context, due to the complicating factor of nondeterminism: it is not present in classical probability theory. Many of those difficulties can be resolved using the probabilistic version of Dijkstra/Hoare-style program logic [12] which is intended to deal naturally with nondeterminism, probability and their interaction. In addition temporal logic provides a framework for handling properties of infinite (repeated) executions of programs — precisely the situation where 0-1 laws begin to bite. The resulting fragment of $qM\mu$ described in this paper, and used to define temporal operators, is thus ideal for studying probability, nondeterminism and 0-1 laws all together.

In $qM\mu$ we find, as in other works, that probabilistic choice when used specifically for “probability-1” properties can (to an extent) be interpreted angelically. But this is definitely not sound in all situations, and sometimes a demonic interpretation is necessary.

For example consider the formula $[\Diamond[s=2]]$ interpreted in the system defined by

$$s \neq 0 \quad \rightarrow \quad s := 0 \mid s := 1 \mid s := 2 .$$

Recall that “ \mid ” stands for some $_p\oplus$ with $0 < p < 1$.

A direct calculation shows that the probability of eventually reaching $s=2$ is strictly less than 1 (unless the system is initially in that state). But an angelic interpretation for \mid in $[\Diamond[s=2]]$ would give 1, and therefore must be unsound. To see that a demonic interpretation is also unsound consider the probability of eventually reaching $s=0$. Again a direct calculation shows that it is 1 irrespective of the initial state, whereas a demonic interpretation of \mid in the formula $[\Diamond[s=0]]$ gives 0 (except from $s=0$ initially).

Indeed finding an optimal balance between the two interpretations — in order to maintain soundness in all situations — is a major challenge. As we do, Rao [15] uses two interpretations of probabilistic choice, though he imposes

fairness on the execution sequences, which we do not. Others' work (Hart *et al.* [5], Vardi [17] use similar ideas, but are model- rather than logic-based.

The emphasis of our work has been to clarify exactly when each of the two interpretations of $|$ is appropriate for the interpretation of temporal formulae in probabilistic systems. Granting the 0-1 law the status of a logical axiom proved to be critical in doing so.

To summarise, we have shown that the demonic interpretation goes with greatest fixed-points (“always” and “unless”) and “=1” probabilities, and that the angelic goes with least fixed points (“eventually”) and “> 0” probabilities, finally leaving the 0-1 law standing out as the key idea underlying their combination in “=1” eventually properties.

The second contribution of this work is the complexity of the model checking problem. The result sketched in Sec. 7 shows that for the logic corresponding to “worst case” probabilistic CTL the complexity is linear in the size of the formula and the size of the underlying transition system. That matches the best known complexity for nonprobabilistic CTL interpreted over nonprobabilistic transition systems [3].

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A Proofs of lemmas

Lemma A.1 eventually-until — *For all predicates P, Q we have*

$$P \triangleright (P \wedge Q) \wedge \Diamond Q \quad \Rightarrow \quad \Diamond(P \wedge Q) .$$

Proof. Let L_n and R_n be the n^{th} terms respectively in the \sqcup -limits for the least fixed-points $\Diamond Q$ and $\Diamond(P \wedge Q)$. We show by induction that

$$P \triangleright (P \wedge Q) \wedge L_n \quad \Rightarrow \quad R_n$$

for all n .

Base case: $L_0 \equiv R_0 \equiv \text{False}$.

Inductive case:

$$P \triangleright (P \wedge Q) \wedge L_{n+1}$$

$$\begin{aligned} \equiv & (P \wedge Q) \vee (P \wedge \circ(P \triangleright (P \wedge Q))) && \text{definitions } \triangleright, \diamond \\ & \wedge Q \vee \circ L_n \end{aligned}$$

$$\begin{aligned} \equiv & (P \wedge Q) \wedge (Q \vee \circ L_n) && \text{propositional reasoning} \\ & \vee (P \wedge \circ(P \triangleright (P \wedge Q))) \wedge Q \\ & \vee (P \wedge \circ(P \triangleright (P \wedge Q))) \wedge \circ L_n \end{aligned}$$

$$\begin{aligned} \Rightarrow & P \wedge Q && \text{propositional reasoning; conjunctivity (3)} \\ & \vee P \wedge Q \\ & \vee \circ((P \triangleright (P \wedge Q)) \wedge L_n) \end{aligned}$$

$$\begin{aligned} \Rightarrow & P \wedge Q && \text{propositional reasoning; inductive hypothesis; monotonicity} \\ & \vee \circ R_n \end{aligned}$$

$$\equiv \diamond R_{n+1} . \quad \text{definition } \diamond$$

We complete the proof by observing that “ $(P \triangleright (P \wedge Q)) \wedge$ ” distributes through \sqcup -limits. \square

Lemma A.2 probabilistic eventually-until — For all expectations A, B we have

$$A \triangleright (A \& B) \& \diamond B \Rightarrow \diamond(A \& B) .$$

Proof. We follow the proof of Lem. A.1, but must be careful in two respects: first, that we generalise \wedge sometimes to \sqcap and sometimes to $\&$; and second that — unlike \wedge — the operator $\&$ is not idempotent. It is associative, however.

Let L_n and R_n be the n^{th} terms respectively in the \sqcup -limits for the least fixed-points $\diamond B$ and $\diamond(A \& B)$. We show by induction that

$$A \triangleright (A \& B) \& L_n \Rightarrow R_n$$

for all n .

Base case: $L_0 \equiv R_0 \equiv 0$.

Inductive case:

$$A \triangleright (A \& B) \& L_{n+1}$$

$$\begin{aligned} \equiv & (A \& B) \sqcup (A \sqcap \circ(A \triangleright (A \& B))) && \text{definitions } \triangleright, \diamond \\ & \& B \sqcup \circ L_n \end{aligned}$$

$$\begin{aligned}
 &\equiv (A \& B) \& (B \sqcup \circ L_n) && \text{arithmetic: } \sqcup, \sqcap \text{ distribute through } \& \\
 &\sqcup (A \sqcap \circ(A \triangleright (A \& B))) \& B \\
 &\sqcup (A \sqcap \circ(A \triangleright (A \& B))) \& \circ L_n
 \end{aligned}$$

$$\begin{aligned}
 &\Rightarrow A \& B && \text{arithmetic; } \&\text{-subadditivity (5)} \\
 &\sqcup A \& B \\
 &\sqcup \circ((A \triangleright (A \& B)) \& L_n)
 \end{aligned}$$

$$\begin{aligned}
 &\Rightarrow A \& B && \text{propositional reasoning; inductive hypothesis; monotonicity} \\
 &\sqcup \circ R_n
 \end{aligned}$$

$$\equiv \diamond R_{n+1} . \quad \text{definition } \diamond$$

We complete the proof by observing that “ $(A \triangleright (A \& B)) \&$ ” distributes through \sqcup -limits. \square

Lemma A.3 For all expectations A we have $\circ_d A \equiv \circ_d \lfloor A \rfloor$.

Proof. We use sublinearity (Property 4 Fig. 4). For any $n \geq 0$ we have by arithmetic that

$$(A.1) \quad \lfloor A \rfloor \Rightarrow (n+1)A \ominus n$$

and, because the state space S is finite, there is some (large enough) n_A for which (A.1) is actually an equality. Now

$$\begin{aligned}
 &\circ_d \lfloor A \rfloor \\
 &\equiv \lfloor \circ((n_A+1)A \ominus n_A) \rfloor && \text{definition } \circ_d; \text{ choose } n_A \text{ large enough} \\
 &\Leftarrow \lfloor (n_A+1)\circ A \ominus n_A \rfloor && \text{sublinearity of } \circ \\
 &\Leftarrow \lfloor \lfloor \circ A \rfloor \rfloor && (A.1) \\
 &\equiv \circ_d A .
 \end{aligned}$$

The reverse inequality is immediate from monotonicity. \square

Lemma A.4 For all expectations A we have $\circ_a A \equiv \circ_a \lceil A \rceil$.

Proof. Again we use sublinearity (as scaling and feasibility; see end Sec. 2.2). For any $n \geq 0$ we have that

$$(A.2) \quad \lceil A \rceil \Leftarrow nA \sqcap 1$$

and, because the state space S is finite, there is some (large enough) n_A for

which (A.2) is actually an equality. Now

$$\begin{aligned}
 & \circ_a[A] \\
 \equiv & \quad [\circ(n_A A \sqcap 1)] && \text{definition } \circ_a; \text{ choose } n_A \text{ large enough} \\
 \Rightarrow & \quad [\circ(n_A A) \sqcap \circ 1] && \text{monotonicity} \\
 \Rightarrow & \quad [n_A(\circ A) \sqcap 1] && \text{scaling and feasibility of } \circ \\
 \Rightarrow & \quad [[\circ A]] && (A.2) \\
 \equiv & \quad \circ_a A .
 \end{aligned}$$

The reverse inequality is immediate from monotonicity. \square

Lemma A.5 If $\lfloor \mathcal{F}.X \rfloor \equiv \mathcal{G}.\lfloor X \rfloor$ for all expectations X , then

$$\lfloor \nu \mathcal{F} \rfloor \equiv \nu \mathcal{G} .$$

Proof. Because $\lfloor \cdot \rfloor$ distributes through infimum \sqcap , we prove by induction that

$$\lfloor \mathcal{F}^n.1 \rfloor \equiv \mathcal{G}^n.1 .$$

For the base case we require trivially that $\lfloor \mathcal{F}^0.1 \rfloor \equiv 1 \equiv \mathcal{G}^0.1$. For the induction we have

$$\begin{aligned}
 & \lfloor \mathcal{F}^{n+1}.1 \rfloor \\
 \equiv & \quad \lfloor \mathcal{F} . (\mathcal{F}^n.1) \rfloor \\
 \equiv & \quad \mathcal{G} . \lfloor \mathcal{F}^n.1 \rfloor && \text{assumption} \\
 \equiv & \quad \mathcal{G} . (\mathcal{G}^n.1) && \text{inductive hypothesis} \\
 \equiv & \quad \mathcal{G}^{n+1}.1 .
 \end{aligned}$$

\square

Lemma A.6 If $\lceil \mathcal{F}.X \rceil \equiv \mathcal{G}.\lceil X \rceil$ for all expectations X , then

$$\lceil \mu \mathcal{F} \rceil \equiv \mu \mathcal{G} .$$

Proof. Because $\lceil \cdot \rceil$ distributes through supremum \sqcup , we prove by induction that

$$\lceil \mathcal{F}^n.1 \rceil \equiv \mathcal{G}^n.1 .$$

For that the proof is analogous to Lem. A.5. \square

Theorem A.7 0-1 Law for deterministic/abort-free systems If for some probability p satisfying $0 < p \leq 1$ we have

$$(A.3) \quad p(I \triangleright [\neg G]) \Rightarrow \Diamond[\neg G] ,$$

then in fact we have

$$(A.4) \quad I \triangleright [\neg G] \Rightarrow \Diamond[\neg G] ,$$

provided \circ is deterministic and terminating.

Proof. We rely on four main ideas, based on thinking of I as a loop invariant and G as the loop guard. The first idea is that $I \triangleright [\neg G]$ is an invariant of any loop with guard G : if $I \triangleright [\neg G]$ holds²¹ initially, then it continues to hold up to and including loop termination, the point at which $\neg G$ is established.

The second idea is that invariance is preserved by scaling: if J is any invariant, then so is pJ for any scalar $0 \leq p$. That will tell us, from above, that $p(I \triangleright [\neg G])$ is invariant too.

The third idea is that $1 - \Diamond[\neg G]$ is invariant also, provided the system is deterministic and terminating. Its being invariant says “if $\neg G$ is not a guaranteed eventuality here, then taking a computational step won’t make it so”.

The fourth idea is that the sum of two invariants, provided that sum is well defined in the sense of lying between 0 and 1, is also an invariant.

Combining all those, we will be able to show that the complicated expression

$$(A.5) \quad J := p(I \triangleright [\neg G]) + (1 - \Diamond[\neg G])$$

is an invariant; but from it we’ll conclude that

$$(A.6) \quad p(I \triangleright [\neg G]) \Rightarrow p(\Diamond[\neg G]) ,$$

whence division by p will give us our desired conclusion (A.4). The only place we use our assumption (A.3) is to note that it ensures (trivially) that J is well defined (lies in $[0, 1]$); the only place we use $p > 0$ is in the division that takes us from (A.6) to (A.4).

We begin by noting that invariance of J conventionally means “if it holds now, then it continues to hold up until and including the step in which $\neg G$ becomes true, if $\neg G$ ever does become true”. That is, to say that J is invariant we require

$$(A.7) \quad J \Rightarrow J \triangleright (J \& [\neg G]) ,$$

where the extra $J\&$ ensures it remains true for the final step (‘as the loop exits’). But (A.7) follows from the simpler

$$(A.8) \quad J \& [G] \Rightarrow \circ J ,$$

which is just the way one reasons about loop invariants:²² to show that we calculate

$$\begin{aligned} & (J \& [\neg G]) \sqcup (J \sqcap \circ J) \\ \Leftarrow & (J \& [\neg G]) \sqcup (J \sqcap (J \& [G])) && \text{assumption (A.8)} \\ \equiv & J \& [\neg G] \sqcup J \& [G] && \text{arithmetic} \\ \equiv & J , \end{aligned}$$

²¹ We say “holds” even if I might not be standard: it assists the intuition when I is in fact standard; and the reasoning is sound in any case.

²² ... because (A.8) just says “invariant J is preserved by executing the loop body while the guard holds”.

whence we get (A.7) from Lem. 4.3 (ii). So we check that our particular J (A.5) satisfies (A.8) by calculating

$$\begin{aligned}
 & \circ(p(I \triangleright [\neg G]) + (1 - \diamond[\neg G])) \\
 \equiv & \quad \circ(p(I \triangleright [\neg G])) + \circ 1 - \circ \diamond[\neg G] && \circ \text{ deterministic} \\
 \Leftarrow & \quad \circ \text{ scaling and abort-free; } \circ \diamond[\neg G] \Rightarrow \diamond[\neg G] \\
 & p(\circ(I \triangleright [\neg G])) + 1 - \diamond[\neg G] \\
 \Leftarrow & \quad (I \triangleright [\neg G]) \& [G] \Rightarrow \circ(I \triangleright [\neg G]) \\
 & (p(I \triangleright [\neg G]) + 1 - \diamond[\neg G]) \& [G] .
 \end{aligned}$$

We have now shown that J satisfies (A.7).

To finish off, we put “ $\&\diamond[\neg G]$ ” on both sides of (A.7), and use Lem. A.2 to conclude by reasoning

$$\begin{aligned}
 & J \& \diamond[\neg G] \\
 \Rightarrow & \quad J \triangleright (J \& [\neg G]) \& \diamond[\neg G] && \text{from (A.7)} \\
 \Rightarrow & \quad \diamond(J \& [\neg G]) . && \text{Lem. A.2}
 \end{aligned}$$

Now by arithmetic $J \& \diamond[\neg G]$ is just $p(I \triangleright [\neg G])$, and $J \& [\neg G]$ is just $p[\neg G]$ whence — using scaling of \diamond — we end up with (A.6), as required. \square