Hyperbolic Problems in Fluids and Relativity

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This thesis is dedicated to my parents,
whose unconditional love has carried me through everything.

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Declaration

I declare that the contents of this thesis are, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known, nor has any part of this thesis been submitted for a degree at another university. The work in Chapter 4 is a joint work with Gui-Qiang Chen and has been published in the Archive for Rational Mechanics and Analysis [14].

Abstract

In this thesis, we present a collection of newly obtained results concerning the existence of vanishing viscosity solutions to the one-dimensional compressible Euler equations of gas dynamics, with and without geometric structure. We demonstrate the existence of such vanishing viscosity solutions, which we show to be entropy solutions, to the transonic nozzle problem and spherically symmetric Euler equations in Chapter 4, in both cases under the simple and natural assumption of relative finite-energy.

In Chapter 5, we show that the viscous solutions of the one-dimensional compressible Navier-Stokes equations converge, as the viscosity tends to zero, to an entropy solution of the Euler equations, again under the assumption of relative finite-energy. In so doing, we develop a compactness framework for the solutions and approximate solutions to the Euler equations under the assumption of a physical pressure law.

Finally, in Chapter 6, we consider the Euler equations in special relativity, and show the existence of bounded entropy solutions to these equations. In the process, we also construct fundamental solutions to the entropy equations and develop a compactness framework for the solutions and approximate solutions to the relativistic Euler equations.

Contents

Chapter 1

Introduction

1.1 The Euler equations

The Euler equations of compressible fluid dynamics, a system of partial differential equations, have been a source of mathematical and physical interest since the formulation of the conservation of mass equation by Euler in the 1750s. The equations are the conservation of mass, momentum and energy for the motion of a compressible fluid, and form an archetypal system for the theory of (hyperbolic) conservation laws.

This thesis is concerned with the existence of solutions to these equations in both classical Newtonian space-time and the Minkowski space-time of special relativity, in both cases under the assumption that the flow is isentropic (i.e. that the entropy remains constant - hence the system is reduced to the conservation of mass and momentum only) and admits certain symmetries, allowing for the reduction of the equations to a system in one spatial dimension, with or without source terms.

The isentropic Euler equations for a compressible fluid in $\mathbb{R}^n$ are a system of $n+1$ conservation laws,

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \mathbf{m} = 0, \\ \partial_t \mathbf{m} + \operatorname{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right) + \nabla_x p = 0, \end{cases}$$

where $\rho : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is the density of the fluid (and hence $\rho \geq 0$), $\mathbf{m} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is its momentum, and the scalar function $p : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is the pressure. We write $\mathbb{R}_+ = (0, \infty)$ throughout.

We consider the Cauchy problem for the Euler equations by imposing initial data:

$$(\rho, \mathbf{m})|_{t=0} = (\rho_0, \mathbf{m}_0).$$

To close the system, the pressure is assumed to be determined by an equation of state. In this thesis, we make the usual assumption of a barotropic gas, so that the pressure is determined directly from the density, i.e. $p = p(\rho)$. A frequently used pressure law (equation of state) for a barotropic fluid is that of a gamma-law gas, or polytropic gas, given by the relation

$$p(\rho) = \kappa\rho^{\gamma},$$

for constants $\gamma \geq 1$ and $\kappa > 0$. In the case $\gamma = 1$, the fluid is said to be isothermal. We note that, by scaling the equations, the constant $\kappa > 0$ may be freely chosen. This pressure is often considered to be a good model for the pressure law for physical gases when the density is small (and hence in the context of bounded solutions). Indeed, one expects physically that γ should be in the range $(1, 3]$, with $\gamma = 5/3$ for a monatomic gas. However, at high densities, one expects that the pressure will grow linearly with the density. We will be able to handle such physical pressures (approximately a γ-law for small density and of linear growth for high density) in certain contexts, and throughout this thesis will make precise the assumptions on the pressure in each section.

Under certain symmetries of the solution, the system may be reduced to a system in one spatial dimension. Such symmetries include:

- Planar symmetry, where the solution depends only on the time t and a single Cartesian spatial coordinate x;
- Spherical symmetry, where the solution depends only on t and the radial distance $r = |x|$ to a fixed origin;
- Transonic nozzle flow, where we suppose that the fluid is constrained to a pipe of varying cross-sectional area $A(x)$, where the solution depends only on t and x.

Most of the current literature concerning one-dimensional Euler equations concerns the simplest case, i.e. planar symmetry. Inspired by this work, in this thesis, we consider first the more general situations of transonic nozzle flows and spherical symmetry and give a new result of existence of solutions to these systems in Chapter 4. We then return to the planar symmetric case in Chapter 5 to analyse the vanishing viscosity limit from the Navier-Stokes equations, a problem which falls outside the scope of the usual methods for the planar symmetric case. We prove that, under the assumption of a physical pressure law, as described above, the solutions of the Navier-Stokes equations can be shown to converge to the corresponding solutions of the Euler equations under reasonable assumptions on the initial data. Such convergence was previously known only in the case of a gamma-law gas and so this result represents a new extension. Finally, in Chapter 6, we provide a new existence result for the relativistic Euler equations, the counterpart of the planar symmetric equations in the setting of special relativity, and show that the obtained solutions converge to solutions of the classical Euler equations in the so-called Newtonian limit, i.e. as the speed of light converges to infinity. A summary of these results and their statements is given in §1.1.1–§1.1.3, but before we give these details, we first provide an introduction to the existing literature for the planar symmetric case.

In the case of planar symmetry, the isentropic compressible Euler equations become

$$\begin{cases} \partial_t\rho + \partial_x m = 0, \\ \partial_t m + \partial_x\big(\frac{m^2}{\rho} + p(\rho)\big) = 0. \end{cases} \tag{1.1.1}$$

We introduce initial data in order to consider the Cauchy problem for system (1.1.1):

$$(\rho, m)(0,x) = (\rho_0, m_0)(x).$$

For details on the analysis and history of this system, we refer the reader to [4, 8, 12, 16, 18, 24, 26, 52, 53, 61, 67, 72] and Chapter 2 of this thesis. In the rest of this section, we provide a preliminary discussion based on these references.

By writing u for the velocity, so that $m = \rho u$, the Euler equations read

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = 0. \end{cases} \tag{1.1.2}$$

As the Euler equations form a non-linear system of conservation laws (which is strictly hyperbolic when $p'(\rho) > 0$), we expect that solutions will form discontinuities in finite time even for smooth initial data. We therefore look not just for weak (distributional) solutions, but for entropy solutions, defined in the following.

We first write the Euler equations as a system of conservation laws,

$$U_t + F(U)_x = 0,$$

where $U = (\rho, m)^\top$, $F(U) = (m, \frac{m^2}{\rho} + p(\rho))^\top$. This system is strictly hyperbolic when $p'(\rho) > 0$ and the characteristic fields are genuinely non-linear when $\rho p''(\rho) + 2p'(\rho) > 0$. Throughout this thesis, we will always assume $p'(\rho) > 0$ for $\rho > 0$.

A pair of functions $(\eta, q) : \mathbb{R}^2_+ \to \mathbb{R}^2$ is called an entropy/entropy-flux pair (or entropy pair for short) for a system of hyperbolic conservation laws of this form if

$$\nabla q(U) = \nabla \eta(U) \nabla F(U).$$

For the Euler equations, an entropy pair (η, q) is called a *weak entropy pair* if $\eta(\rho, m)$ vanishes on the set $\{\rho = 0\}$. An *entropy solution* of the Euler equations is then a distributional solution of system (1.1.1) satisfying

$$\eta(\rho, m)_t + q(\rho, m)_x \leq 0$$

in the sense of distributions for all weak entropy pairs (η, q) such that $\eta(\rho, m)$ is convex.

The first result concerning the existence of entropy solutions to the Cauchy problem for the one-dimensional system of isentropic Euler equations (1.1.1) for a γ-law gas was given in 1983 by DiPerna [24] via the introduction of the compensated compactness method. The existence of entropy solutions for the interval $\gamma \in (1, \frac{5}{3}]$ was established in Chen [3] and Ding, Chen and Luo [20] through the introduction of sophisticated estimates, enabling the application of the compensated compactness method with entropy analysis techniques, involving fractional derivatives and the Hilbert transform. With a new key insight that the entropy/entropy-flux pairs in the reduction of Young measures governed by the Tartar commutation relation can be replaced by the entropy/entropy-flux kernels, a reduction technique was developed to establish the existence of entropy solutions by Lions, Perthame and Tadmor [52] for the case $\gamma \geq 3$, and by Lions, Perthame and Souganidis [53] for the case $\gamma \in (\frac{5}{3}, 3)$.

Convergence of various approximation schemes to entropy solutions of the isentropic Euler equations with general L^∞ initial data have been shown. Chen and Ding, Chen and Luo showed convergence of the Lax-Friedrichs scheme [3, 4, 20] and the Godunov scheme [21]; DiPerna [24] and Lions, Perthame and Souganidis [53] showed convergence of the vanishing viscosity method.

In Chen and LeFloch [8], the existence of entropy solutions to the Euler equations for a general class of pressure laws was demonstrated for bounded initial data, requiring the pressure to be such that the system is strictly hyperbolic with genuinely non-linear characteristic fields, as well as the assumption

$$p(\rho) = \kappa \rho^\gamma (1 + P(\rho)), \tag{1.1.3}$$

where $P(\rho) \to 0$ as $\rho \to 0$ quantifiably, so that the pressure is approximately a γ-law.

The broad approach taken was to construct a sequence of approximate solutions (via, for example, the introduction of artificial viscosity), and then to apply the theory of compensated compactness to pass to a strong limit of the approximate solutions. This application of compensated compactness relied heavily on the existence of an *invariant region* for the equations, giving uniform bounds on the approximate solutions in the space L^∞. In the settings under consideration in this thesis, these uniform bounds will no longer hold. Rather, we will look at solutions of relative finite-energy, which we now define.

The mechanical energy of the system is defined by

$$\eta^*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho),$$

where the internal energy $e(\rho)$ is related to the pressure via the relation

$$p(\rho) = \rho^2 e'(\rho).$$

The total energy of the system is then the spatial integral of the mechanical energy. Clearly, any solution satisfying the end-state conditions $\lim_{x\to\pm\infty}(\rho, u) = (\rho_\pm, u_\pm)$ cannot satisfy the condition of finite energy unless the end-states are trivial. We therefore work instead with the related concept of relative finite-energy. In order to define the energy relative to the end-states $(\rho_\pm, m_\pm) = (\rho_\pm, \rho_\pm u_\pm)$, we follow [12] in defining smooth, monotone functions $(\bar{\rho}(x), \bar{u}(x))$ such that, for some $L_0 > 1$,

$$(\bar{\rho}(x), \bar{u}(x)) = \begin{cases} (\rho_+, u_+), & \text{for } x \geq L_0, \\ (\rho_-, u_-), & \text{for } x \leq -L_0. \end{cases} \tag{1.1.4}$$

We define the relative mechanical energy with respect to $(\bar{\rho}(x), \bar{m}(x)) = (\bar{\rho}(x), \bar{\rho}(x)\bar{u}(x))$ as

$$\begin{aligned} \overline{\eta^*}(\rho, m) &= \eta^*(\rho, m) - \eta^*(\bar{\rho}, \bar{m}) - \nabla\eta^*(\bar{\rho}, \bar{m}) \cdot (\rho - \bar{\rho}, m - \bar{m}) \\ &= \frac{1}{2}\rho|u - \bar{u}|^2 + e^*(\rho, \bar{\rho}) \geq 0, \end{aligned} \tag{1.1.5}$$

where $e^*(\rho, \bar{\rho}) = \rho e(\rho) - \bar{\rho}e(\bar{\rho}) - (\bar{\rho}e'(\bar{\rho}) + e(\bar{\rho}))(\rho - \bar{\rho}) \geq 0$. Now the relative total mechanical energy for (1.1.1) with respect to the end-states $(\rho_\pm, m_\pm)$ through $(\bar{\rho}, \bar{m})$ is

$$E[\rho, m](t) := \int_{-\infty}^{\infty} \overline{\eta^*}(\rho(t, x), m(t, x))\, dx \geq 0.$$

We say that a pair of functions $(\rho, m) : \mathbb{R}^2_+ \to \mathbb{R}^2$ with $\rho \geq 0$ is of *relative finite-energy* with respect to the end-states $(\rho_\pm, m_\pm)$ if $E[\rho, m](t) < \infty$ for almost every $t > 0$. This concept is a central one in the first original result of this thesis, to which we now turn.

1.1.1 Euler equations with geometric effects

The first principal result of this thesis establishes the existence of entropy solutions to the Euler equations with geometric structure given by the symmetry considerations above. In particular, we consider the Euler equations for transonic nozzle flows with general cross-sectional areas and related compressible flows of geometric structure including spherically symmetric flows. These flows arise when one considers a compressible fluid contained in a long nozzle or tube such that the density ρ and velocity u of the fluid only depend on a single spatial coordinate, such as the distance along the nozzle or a radial coordinate, which we call x. We write $A(x) > 0$ for the cross-sectional area of the nozzle at point x. Under these symmetry assumptions, the equations become the one-dimensional Euler equations with geometric source terms:

$$\begin{cases} \rho_t + m_x + \frac{A'(x)}{A(x)} m = 0, \\ m_t + \big(\frac{m^2}{\rho} + p(\rho)\big)_x + \frac{A'(x)}{A(x)}\frac{m^2}{\rho} = 0, \end{cases} \tag{1.1.6}$$

for $(t,x) \in \mathbb{R}^2_+$. Further details on the derivation of the equations can be found in [16]. We make precise assumptions on the cross-sectional area of the nozzle below in Theorem 1.1.1, but for now $A = A(x)$ is simply a given C^2–function $A : \mathbb{R} \to \mathbb{R}_+$. We wish to consider a broad class of such functions, allowing for nozzles that open up at one end with unbounded area, and nozzles that pinch off at an end with area tending to zero ($A(x) \to 0$ as $x \to \pm\infty$) as well as the case of spherical symmetry. In the spherically symmetric case, we have $A(x) = \omega_n x^n$ and $x \geq 0$, where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

We focus on the Cauchy problem for system (1.1.6):

$$(\rho, m)|_{t=0} = (\rho_0, m_0), \tag{1.1.7}$$

where the initial data $(\rho_0, m_0)(x)$ have end-point states $(\rho_\pm, m_\pm)$ as $x \to \pm\infty$, for constants $\rho_\pm \geq 0$ and $m_\pm \in \mathbb{R}$. The end-states are allowed to be different, i.e. $(\rho_+, m_+) \neq (\rho_-, m_-)$ in general. This corresponds to allowing fluid to flow into or out of the nozzle at the ends at different rates and with different densities. As the equations are no longer in conservation form, they do not admit the uniform L^∞ bounds employed in the resolution of the Cauchy problem for (1.1.1), preventing us from simply adapting the methods for that system. We therefore move to the finite energy method.

The first main result of this thesis is to provide a robust framework that gives the existence of entropy solutions to system (1.1.6) under general assumptions on the function $A(x)$. This framework relies in turn on the compensated compactness framework of Chen and Perepelitsa [12] for polytropic gases, i.e. $p(\rho) = \kappa\rho^\gamma$ with $\gamma > 1$. We therefore assume throughout this section that we have such a pressure.

Theorem 1.1.1 (Entropy Solutions in a Transonic Nozzle)**.** *Suppose that $p(\rho) = \kappa\rho^\gamma$ with $\gamma \in (1,\infty)$ and $\kappa = \frac{(\gamma-1)^2}{4\gamma}$ and let the cross-sectional area function $A(x) > 0$ be a C^2–function satisfying*

$$\|\frac{A'}{A}\|_{L^\infty(\mathbb{R})} + \|A'\|_{L^1(-\infty,0)} < \infty \quad \textit{or} \quad \|\frac{A'}{A}\|_{L^\infty(\mathbb{R})} + \|A'\|_{L^1(0,\infty)} < \infty. \tag{1.1.8}$$

Assume that $(\rho_0, m_0) \in (L^1_{loc}(\mathbb{R}))^2$ with $\rho_0(x) \geq 0$ is of relative finite-energy with respect to the end-states $(\rho_\pm, m_\pm)$. Then there exists a global-in-time relative finite-energy entropy solution $(\rho, m)(t, x)$ of the transonic nozzle problem (1.1.6)–(1.1.7) such that

$$(\rho, m) \in L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+) \quad \textit{for } p \in [1, \gamma+1) \textit{ and } q \in [1, \tfrac{3(\gamma+1)}{\gamma+3}). \tag{1.1.9}$$

Remark 1.1.2. Condition (1.1.8) allows for the nozzles to open up at one end (with an unbounded cross-sectional area), and to shrink at both ends with rate $|x|^{-\alpha}$ for any $\alpha > 0$ (closed ends). In particular, a positive lower or upper bound of the cross-sectional area function is not required.

We obtain the entropy solutions presented in the above theorem as a vanishing viscosity limit of appropriately designed approximate solutions. We therefore develop a viscosity method to construct such approximate solutions and derive uniform estimates for the approximate solutions so that they satisfy the requirements of the compensated compactness framework for polytropic gases in [12]. Applying this framework then yields the strong convergence of the approximate solutions to a limit, which we show to be an entropy solution of (1.1.6), as desired. Moreover, as a consequence of the approach and techniques developed throughout Chapter 4, we prove the following result for the spherically symmetric Euler equations, including the previously unsolved case, $\gamma > 3$.

Theorem 1.1.3 (Entropy Solutions in Spherical Symmetry)**.** *Let $p(\rho) = \kappa\rho^\gamma$, $\gamma > 1$, and let (ρ_0, m_0) be initial data with $(\rho_0, m_0) \in (L^1_{loc}(\mathbb{R}_+))^2$ such that $\rho_0(x) \geq 0$ and*

$$E_*[\rho_0, m_0] := \int_0^\infty \eta^*(\rho_0, m_0) x^{n-1}\, dx < \infty.$$

Then, for the whole range $\gamma \in (1,\infty)$, there exists a global entropy solution $(\rho, m)(t, x)$ of the spherically symmetric Euler equations with $A(x) = \omega_n x^{n-1}$, $x \geq 0$, where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$, such that $(\rho, m) \in L^p_{loc}((\mathbb{R}_+)^2) \times L^q_{loc}((\mathbb{R}_+)^2)$ for $p \in [1, \gamma+1)$ and $q \in [1, \frac{3(\gamma+1)}{\gamma+3})$, and

$$E_*[\rho, m](t) := \int_0^\infty \eta^*(\rho, m) x^{n-1} dx \leq E_*[\rho_0, m_0] < \infty. \tag{1.1.10}$$

1.1.2 Vanishing physical viscosity limit

An alternative proposed admissibility criterion to the entropy condition for identifying "physical" solutions to the Euler equations is the vanishing viscosity limit from the Navier-Stokes equations. Especially in higher dimensions, where it is now known that the entropy solutions of the Euler equations need not be unique, this criterion has been widely suggested as a possible alternative. According to this condition, a weak solution of the Euler equations, system (1.1.1), is admissible if and only if it can be obtained as a vanishing viscosity limit of solutions to the Navier-Stokes equations:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = \varepsilon u_{xx}. \end{cases} \tag{1.1.11}$$

The main result of the second part of this thesis demonstrates the existence of such admissible solutions under the assumption of a physical pressure law, that is, a pressure that grows linearly with respect to the density as the density grows large, but is asymptotically a gamma-law gas as the density approaches the vacuum state. As with the transonic nozzle problem, the solutions to the Navier-Stokes equations do not admit uniform L^∞ bounds, and so we work again in the relative finite-energy setting. In [12], Chen and Perepelitsa worked in this finite-energy setting to prove that for polytropic gases, where $p(\rho) = \kappa\rho^\gamma$, the solutions of the Navier-Stokes equations converge to entropy solutions of the Euler equations as desired. We extend this result in Chapter 5 to cover the more general case of physically relevant pressure laws just described.

To make explicit the assumptions on the pressure described above, we require that there exists a constant $\rho_* > 0$ such that the pressure satisfies the following conditions:

(i) For $0 \le \rho \le \rho_*$, the pressure is an approximate γ-law for $\gamma \in (1,3)$,

$$p(\rho) = \kappa\rho^\gamma(1 + P(\rho)), \tag{1.1.12}$$

where $|P^{(n)}(\rho)| \le C\rho^{\gamma-1-n}$ for $n = 1,2,3$, where $C > 0$ may depend on ρ_*;

(ii) For $\rho \ge \rho_*$, we have that $p(\rho) = c_*\rho$ for a constant $c_* > 0$;

(iii) For all $\rho > 0$, the conditions of strict hyperbolicity and genuine non-linearity hold:

$$p'(\rho) > 0, \qquad \rho p''(\rho) + 2p'(\rho) > 0.$$

Under these assumptions, the main theorem is:

Theorem 1.1.4 (Vanishing Physical Viscosity Limit). *Let (ρ_0, u_0) be initial data of relative finite-energy, $\rho_0(x) \geq 0$ almost everywhere. Then there exists smooth approximate initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$ for all $\varepsilon > 0$ such that the solutions $(\rho^\varepsilon, u^\varepsilon)$ to (1.1.11) with initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$ converge, up to a subsequence, as $\varepsilon \to 0$, $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \to (\rho, \rho u)$, where $(\rho, \rho u)$ is a relative finite-energy entropy solution of the Euler equations (1.1.1) with initial data $(\rho_0, \rho_0 u_0)$. The convergence is taken almost everywhere and in $L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+)$ for $p \in [1, 2)$ and $q \in [1, 3/2)$.*

Although the general strategy for showing the strong convergence of the solutions is, as in the previous section, to employ the theory of compensated compactness, the fact that we are no longer working with an exact gamma-law gas forces us to adapt and extend the reduction argument of [12] significantly, as this argument was heavily reliant on the precise structure of the entropy kernel for the polytropic gas. We therefore give a new result providing a representation formula for the entropy kernel under the assumption of linear pressure. This theorem provides a general representation formula for the entropies of the Euler equations of isothermal gases, $p(\rho) = \rho$. A rough statement of this theorem, which will be made precise in Chapter 5, Theorem 5.2.2, is the following.

Theorem 1.1.5 (Isothermal Entropy Kernels). *Any weak entropy function $\eta(\rho, m)$ of the isothermal Euler equations may be generated by convolution with two fundamental solutions of the isothermal entropy equation:*

$$\eta(\rho, \rho u) = \int_{\mathbb{R}} \eta_\rho(1, s)\chi^\sharp(\rho, u - s) + \eta(1, s)\chi^\flat(\rho, u - s)\, ds.$$

Here $\chi^\sharp(\rho, u)$ and $\chi^\flat(\rho, u)$ are the solutions of

$$\left\{ \begin{array}{cc} \chi^\sharp_{\rho\rho} - \frac{1}{\rho^2}\chi^\sharp_{uu} = 0, & \chi^\flat_{\rho\rho} - \frac{1}{\rho^2}\chi^\flat_{uu} = 0, \\ \chi^\sharp|_{\rho=1} = 0, & \chi^\flat|_{\rho=1} = \delta_{u=0}, \\ \chi^\sharp_\rho|_{\rho=1} = \delta_{u=0}, & \chi^\flat_\rho|_{\rho=1} = 0. \end{array} \right\} \tag{1.1.13}$$

Moreover, the functions $\chi^\sharp$ and $\chi^\flat$ are explicitly determined.

This new representation formula for the entropies allows us to gain the uniform higher integrability estimates on the approximate solutions that we require. It is also essential for the passage to the limit, as the representation formula gives sufficient control over the entropies for us to derive a new convergence framework in this more general setting.

1.1.3 Relativistic Euler equations

In the final part of this thesis, we take the first step towards understanding the relativistic version of the Euler equations. The isentropic relativistic Euler equations of the conservation of baryon number and momentum form a natural generalisation of the Euler equations for classical fluid flow (i.e. system (1.1.1)) to the setting of special relativity. These equations describe the motion of inviscid fluids when the effects predicted by the theory of special relativity are taken into account and, under the assumption of planar symmetry, are given by

$$\begin{cases} \partial_t\big(\frac{n}{\sqrt{1-u^2/c^2}}\big) + \partial_x\big(\frac{nu}{\sqrt{1-u^2/c^2}}\big) = 0, \\ \partial_t\big(\frac{(\rho+p/c^2)u}{1-u^2/c^2}\big) + \partial_x\big(\frac{(\rho+p/c^2)u^2}{1-u^2/c^2} + p\big) = 0, \end{cases} \tag{1.1.14}$$

where ρ, p and u represent the mass-energy density, pressure, and the particle speed, respectively, n is the proper number density of baryons, and c is the speed of light. The baryon number is determined from the density by

$$n = n(\rho) = e^{\int_0^\rho \frac{ds}{s+\varepsilon p(s)}}.$$

These equations were derived in [68], which also provides a derivation of the Rankine-Hugoniot jump conditions across a shock and a discussion of possible models for the pressure. As for the classical Euler equations, to close the system, we impose an equation of state of a barotropic gas, $p = p(\rho)$.

This system can be considered to be a simple model problem for the full problem of compressible fluids in general relativity. Indeed, the Minkowski space of special relativity is the simplest solution to the vacuum Einstein equations. In general, for the Einstein-Euler equations, one would expect black holes and other non-trivial background spaces to influence the evolution of the fluid. These problems are, of course, significantly more challenging, but the problem described here is a natural starting point.

The condition of strict hyperbolicity is again met if $p'(\rho) > 0$ and the characteristic fields are both genuinely non-linear if

$$\rho p''(\rho) + 2p'(\rho) + \frac{1}{c^2}(p(\rho)p''(\rho) - 2p'(\rho)^2) > 0. \tag{1.1.15}$$

Typically, these two criteria will be met only for $\rho > 0$ and so, as for the classical Euler equations, we may not simply apply the theory of hyperbolic conservation laws directly.

The final contribution of this thesis is to solve the Cauchy problem for the system (1.1.14) when the speed of light is large by demonstrating the existence of a bounded entropy solution given bounded initial data.

Theorem 1.1.6 (Entropy Solutions in Special Relativity)**.** *Let* (ρ_0, u_0) *be measurable and bounded initial data satisfying*

$$0 \le \rho_0(x) \le M_0, \quad |u_0(x)| \le M_0 < c \text{ for almost every } x \in \mathbb{R},$$

where $M_0 > 0$*, and suppose that the pressure* $p(\rho)$ *satisfies* $p'(\rho) > 0$ *for* $\rho > 0$*, (1.1.3) and (1.1.15). Then there exists* $c_0 > 0$ *such that, if* $c \ge c_0$*, there exists an entropy solution* (ρ, u) *of (1.1.14) so that, for some* $M > 0$*,*

$$0 \le \rho(t,x) \le M, \quad |u(t,x)| \le M < c \text{ for almost every } (t,x) \in \mathbb{R}^2_+.$$

To prove this theorem, we develop a compactness framework for approximate solutions, derived as a consequence of a careful analysis of the approximate solutions via the entropy functions of the system. In order to do this, we characterise the weak entropies. For this, it is convenient to use the alternative coordinates

$$k(\rho) = \int_0^\rho \frac{\sqrt{p'(s)}}{s + p(s)/c^2}\, ds, \quad v(u) = \frac{c}{2}\log\Big(\frac{1+u/c}{1-u/c}\Big).$$

Working in these coordinates, we prove the existence of a fundamental solution of the entropy equation, generating all of the weak entropy functions by convolution with this kernel.

Theorem 1.1.7 (Relativistic Weak Entropy Kernel)**.** *All weak entropies* $\eta(\rho, v)$ *of the relativistic Euler equations may be written as the convolution of an entropy kernel* $\chi(\rho, v)$ *with a test function* $\psi(s)$*,*

$$\eta(\rho, v) = \int_{\mathbb{R}} \psi(s)\chi(\rho, v - s)\, ds. \tag{1.1.16}$$

The entropy kernel admits the expansion

$$\chi(\rho, v) = a_1(\rho)[k(\rho)^2 - v^2]_+^\lambda + a_2(\rho)[k(\rho)^2 - v^2]_+^{\lambda+1} + g(\rho, v), \tag{1.1.17}$$

where $\lambda > 0$*, the coefficients* $a_1(\rho)$ *and* $a_2(\rho)$ *are explicitly determined, and the remainder function* $g(\rho, v)$ *is of strictly higher Hölder regularity than the other terms.*

Formally, in the Newtonian limit (i.e. $c \to \infty$), system (1.1.14) reduces to the classical isentropic Euler equations for compressible fluids (1.1.2). We therefore prove also that the entropy solutions to the relativistic Euler equations (1.1.14) converge to an entropy solution of the classical Euler equations in a suitable topology as $c \to \infty$. Our analysis of the entropy kernel is sufficient to control the convergence of a sequence of solutions to the relativistic equations as $c \to \infty$, resulting in the following theorem.

Theorem 1.1.8 (Newtonian Limit)**.** *Suppose that* $(\rho_0, u_0) \in (L^\infty(\mathbb{R}))^2$ *with* $\rho_0(x) \geq 0$, $\varepsilon_0 > 0$, *and that, for* $\varepsilon \in (0, \varepsilon_0]$, $(\rho^\varepsilon, u^\varepsilon)$ *is an entropy solution to* (1.1.14) *with light speed* $c = \frac{1}{\sqrt{\varepsilon}}$ *and initial data* $(\rho_0^\varepsilon, u_0^\varepsilon) \in (L^\infty(\mathbb{R}))^2$ *with* $\rho_0^\varepsilon \geq 0$ *such that*

$$\sqrt{p'(\rho_0^\varepsilon(x))},\ |u_0^\varepsilon(x)| < c \text{ for } \varepsilon \in (0, \varepsilon_0] \text{ and almost every } x \in \mathbb{R}.$$

Suppose that there exists $M > 0$, *independent of* ε, *such that*

$$0 \leq \rho^\varepsilon(t,x) \leq M, \quad |v^\varepsilon(u^\varepsilon(t,x))| \leq M \text{ for almost every } (t,x) \in \mathbb{R}^2_+,$$

where $v^\varepsilon(u^\varepsilon) := \frac{1}{2\sqrt{\varepsilon}} \log\left(\frac{1+\sqrt{\varepsilon}u^\varepsilon}{1-\sqrt{\varepsilon}u^\varepsilon}\right)$, *and that* $(\rho_0^\varepsilon, v^\varepsilon(u_0^\varepsilon)) \rightharpoonup (\rho_0, v_0)$ *as* $\varepsilon \to 0$ *in the sense of distributions.*

Then, up to a subsequence, $(\rho^\varepsilon, v^\varepsilon) \to (\rho, v)$ *as* $\varepsilon \to 0$ *almost everywhere and in* $L^r_{loc}(\mathbb{R}^2_+)$ *for all* $r \in [1, \infty)$, *where* (ρ, v) *is an entropy solution of the classical Euler equations* (1.1.2) *with initial data* (ρ_0, v_0) *and such that*

$$0 \leq \rho(t,x) \leq M, \quad |v(t,x)| \leq M \text{ for almost every } (t,x) \in \mathbb{R}^2_+.$$

1.2 Outline of the thesis

The structure of the thesis is as follows. In Chapter 2, we give an introduction to the theory of hyperbolic conservation laws, setting out and motivating much of the terminology and tool-kit for the detailed study of the Euler equations to follow. In the final sections of the chapter, we provide further introductory and background material concerning each of the three main problems solved in the sequel. This chapter is followed by a chapter of analytic preliminaries, setting out the main tools that we use from the theory of compensated compactness as well as material concerning the theory of Young measures that we make frequent recourse to.

Following these preliminaries, we turn to the first chapter of original material, Chapter 4. In this chapter, we give the proof of the first of our main theorems, Theorem 1.1.1, by constructing the artificial viscosity solutions to the transonic nozzle problem (1.1.6) and demonstrating the uniform estimates necessary to apply the compensated compactness framework. We also apply our framework in this chapter to prove the second theorem, Theorem 1.1.3, concerning the solutions of the spherically symmetric problem.

Chapter 5 is then concerned with the vanishing physical viscosity limit of the Navier-Stokes equations. We provide a more detailed version and proof of Theorem 1.1.5 for the entropies for pressure laws with linear growth as well as uniform estimates on the solutions of the one-dimensional Navier-Stokes equations. We then show how these estimates and the structure of the entropy functions can be used to construct a compensated compactness framework and deduce the strong convergence of the solutions to the Navier-Stokes equations to an entropy solution of the Euler equations, concluding the proof of Theorem 1.1.4.

In Chapter 6, we prove the existence of the bounded entropy solutions to the relativistic Euler equations (1.1.14) with large speed of light, as stated in Theorem 1.1.6. This again follows from a construction of approximate solutions, and we use the derived asymptotic expansions of the entropy kernels given in Theorem 1.1.7 to deduce a compactness framework for such approximate solutions, allowing us to pass to a strong limit and obtain the desired entropy solution. Finally, we employ the analysis of the entropy kernels again to show the strong convergence of the entropy solutions of the relativistic Euler equations to a solution of the classical Euler equations in the Newtonian limit, proving the final theorem, Theorem 1.1.8.

Chapter 2

The Euler Equations

The study of hyperbolic conservation laws dates back to the formulation of the Euler equations by Euler in 1757, [26], with later significant contributions by Stokes [67] and Riemann [61]. From the 20th century onwards, this theory began to be systematised, especially following the work of Lax, [44], and a rigorous framework for the analysis of solutions to these systems was developed. In this chapter, we outline some of the main obstacles to the analyst, and explain many of the essential concepts and techniques for the formulation and solution of the Euler equations and hyperbolic conservation laws more generally. Useful references for the theory of hyperbolic conservation laws are [18, 27, 46].

2.1 The one-dimensional isentropic Euler equations

We recall that the isentropic compressible Euler equations are given by

$$\begin{cases} \partial_t \rho + \partial_x m = 0, & (t,x) \in \mathbb{R}^2_+, \\ \partial_t m + \partial_x \big(\frac{m^2}{\rho} + p(\rho) \big) = 0, & (t,x) \in \mathbb{R}^2_+. \end{cases} \tag{2.1.1}$$

A general system of conservation laws (in one space dimension), on the other hand, is a system of equations of the form

$$\begin{cases} U_t + F(U)_x = 0, & (t,x) \in \mathbb{R}^2_+, \\ U(0,x) = U_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1.2}$$

where $U : \mathbb{R}^2_+ \to \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$. We consider the Cauchy problem for the system.

Definition 2.1.1. The system (2.1.2) is said to be strictly hyperbolic if and only if for all $U \in \mathbb{R}^n$, the matrix $\nabla F(U)$ has n real and distinct eigenvalues

$$\lambda_1(U) < \cdots < \lambda_n(U).$$

We henceforth write r_j and l_j for the normalised right and left eigenvectors associated to the eigenvalues λ_j for $j = 1, \dots, n$.

The Euler equations, (2.1.1), may now be formulated as a system of conservation laws by setting $U = (\rho, m)^\top$, $F(U) = (m, \frac{m^2}{\rho} + p(\rho))^\top$. Then

$$\nabla F(U) = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + p'(\rho) & \frac{2m}{\rho} \end{pmatrix}.$$

This has eigenvalues

$$\lambda_1 = \frac{m}{\rho} - \sqrt{p'(\rho)}, \qquad \lambda_2 = \frac{m}{\rho} + \sqrt{p'(\rho)},$$

and eigenvectors

$$r_1 = \begin{pmatrix} 1 \\ \frac{m}{\rho} - \sqrt{p'(\rho)} \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ \frac{m}{\rho} + \sqrt{p'(\rho)} \end{pmatrix},$$

and so the Euler equations are strictly hyperbolic provided $p'(\rho) > 0$.

In addition to the assumption of strict hyperbolicity, we wish for the characteristic fields to satisfy the condition of genuine non-linearity, introduced by Lax in [44].

Definition 2.1.2. Consider the system of conservation laws (2.1.2). For each $k = 1, \dots, n$, the k-th field is said to be genuinely non-linear provided, for all $U \in \mathbb{R}^n$,

$$\nabla \lambda_k(U) \cdot r_k(U) \neq 0.$$

In the case of the Euler equations, we find that the system is genuinely non-linear provided

$$\rho p''(\rho) + 2p'(\rho) > 0.$$

As it is well known that, in general, classical solutions of systems of conservation laws form discontinuities in finite time (see [18, 27, 46] for simple examples), we look first for measurable solutions in the sense of distributions, i.e. such that for all test functions $\phi \in C_0^\infty(\overline{\mathbb{R}^2_+}; \mathbb{R}^n)$,

$$\int_0^\infty \int_{\mathbb{R}} U(t,x) \cdot \phi_t(t,x) + F(U(t,x)) \cdot \phi_x(t,x) \, dx \, dt + \int_{\mathbb{R}} U_0(x) \cdot \phi(0,x) \, dx = 0. \quad (2.1.3)$$

Such solutions are said to be generalised or weak solutions.

Unfortunately, such weak solutions cannot be expected to be unique. To isolate the "correct" solution of (2.1.2), we therefore impose an admissibility condition. Throughout this thesis, the admissibility condition of choice is the entropy condition.

Definition 2.1.3. An entropy/entropy-flux pair (or entropy pair for short) is a pair of C^2–functions $(\eta, q) : \mathbb{R}^n \to \mathbb{R}^2$ such that, for all $U \in \mathbb{R}^n$,

$$\nabla q(U) = \nabla \eta(U) \nabla F(U). \tag{2.1.4}$$

An entropy solution of (2.1.2) is a generalised solution satisfying the entropy inequality:

$$\eta(U)_t + q(U)_x \leq 0 \text{ in } \mathcal{D}'(\mathbb{R}^2_+) \tag{2.1.5}$$

for all entropy pairs with η convex, i.e. for all such entropy pairs (η, q),

$$\int_0^\infty \int_{\mathbb{R}} \eta(U)\phi_t + q(U)\phi_x \, dx \, dt \geq 0 \text{ for all } \phi \in C_0^\infty(\mathbb{R}^2_+) \text{ such that } \phi \geq 0.$$

For the Euler equations, a weak entropy is an entropy $\eta(\rho, m)$ such that $\eta = 0$ when $\rho = 0$. An entropy solution of the Euler equations is then defined to be a weak solution satisfying the entropy inequality for all convex weak entropy pairs.

In the case of a polytropic gas, $p(\rho) = \kappa\rho^\gamma$, the weak entropy pairs (η, q) of the Euler equations may be generated from test functions $\psi \in C^2(\mathbb{R})$ by

$$\eta^\psi(\rho, m) = \rho \int_{-\infty}^\infty \psi(\frac{m}{\rho} + \rho^\theta s)[1 - s^2]_+^{\frac{3-\gamma}{2(\gamma-1)}} \, ds, \tag{2.1.6}$$

$$q^\psi(\rho, m) = \rho \int_{-\infty}^\infty (\frac{m}{\rho} + \theta\rho^\theta s)\psi(\frac{m}{\rho} + \rho^\theta s)[1 - s^2]_+^{\frac{3-\gamma}{2(\gamma-1)}} \, ds, \tag{2.1.7}$$

where $\theta = \frac{\gamma-1}{2}$, and $\psi : \mathbb{R} \to \mathbb{R}$ is said to generate the entropy pair (η^ψ, q^ψ) (*cf.* [24, 52]).

In the case that we have a system of two conservation laws, which is the case in particular for the isentropic Euler equations, we may take advantage of a construction of Riemann, derived in [61].

Definition 2.1.4. A function $w^j : \mathbb{R}^2 \to \mathbb{R}$ is a j-th Riemann invariant for $j \in \{1, 2\}$ if, for all $U \in \mathbb{R}^2$,

$$\nabla w^j(U) \cdot r_j(U) = 0.$$

We find Riemann invariants for the Euler equations (1.1.1) given by

$$w(\rho, m) = \frac{m}{\rho} + k(\rho), \qquad z(\rho, m) = \frac{m}{\rho} - k(\rho),$$

where

$$k(\rho) = \int_0^\rho \frac{\sqrt{p'(s)}}{s} \, ds.$$

2.2 A theorem of Tartar for scalar conservation laws

In the case of a one-dimensional scalar conservation law, the existence of entropy solutions via the method of compensated compactness was shown by Tartar in [70]. We give here a version of the statement due to Chen [4]. Tartar's proof is instructive as it gives us our first insight into how the theory of compensated compactness (described in more detail in §3.4) may be applied to demonstrate the convergence of viscosity solutions to entropy solutions of hyperbolic conservation laws.

We consider the Cauchy problem for a scalar conservation law for a function $u : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} u_t + f(u)_x = 0, & (t,x) \in \mathbb{R}^2_+, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.2.1}$$

Theorem 2.2.1. *Suppose the initial data $u_0 \in W^{1,\infty}(\mathbb{R})$ and $f \in C^2(\mathbb{R})$. Then the Cauchy problem above has a weak solution $u \in L^\infty(\mathbb{R}^2_+)$. If, moreover, the flux satisfies $f''(\lambda) \neq 0$ for almost every $\lambda \in [-\|u_0\|_{L^\infty}, \|u_0\|_{L^\infty}]$, then u is an entropy solution.*

The strategy of the proof is as follows. One first constructs approximate solutions to the scalar conservation law by adding artificial viscosity and mollifying the initial data:

$$\begin{cases} u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}, \\ u^\varepsilon|_{t=0} = u^\varepsilon_0(x) = \int_{\mathbb{R}} \eta^\varepsilon(x-y) u_0(y)\, dy, \end{cases} \tag{2.2.2}$$

where η^ε is a standard mollifier. This yields, by standard parabolic theory, a sequence of uniformly bounded smooth functions as $\varepsilon \to 0$. Passing to a weak limit, $u^\varepsilon \overset{*}{\rightharpoonup} u$ in $L^\infty(\mathbb{R}^2_+)$, one may extract a Young measure $\nu_{t,x}$ generated by this sequence (see §3.1 for the theory of Young measures).

One then shows, for certain entropy pairs (η, q), that the entropy dissipation measure

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{ is (pre-)compact in } H^{-1}_{loc}(\mathbb{R}^2_+).$$

From this compactness of the entropy dissipation measures, one then applies the div-curl lemma, Lemma 3.4.1, to deduce the so-called Tartar commutation relation for the Young measure. This relation, for entropy pairs (η_1, q_1) and (η_2, q_2), takes the form

$$\langle \nu_{t,x}, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu_{t,x}, \eta_1 \rangle \langle \nu_{t,x}, q_2 \rangle - \langle \nu_{t,x}, \eta_2 \rangle \langle \nu_{t,x}, q_1 \rangle.$$

The commutation relation is then sufficient to show that the Young measure is always concentrated at a point, $\nu_{t,x} = \delta_{u(t,x)}$ for almost every $(t,x) \in \mathbb{R}^2_+$, and hence conclude that the sequence $u^\varepsilon \to u$ strongly in $L^r_{loc}(\mathbb{R}^2_+)$, $1 \le r < \infty$, and almost everywhere. This strong convergence allows us to pass to the limit in the approximate equations and obtain an entropy solution.

2.3 Existence of entropy solutions to the Euler equations

Motivated by the pioneering papers of DiPerna, [23, 24], a sequence of authors showed the existence of entropy solutions to the Cauchy problem for the Euler equations for a range of pressure laws $p = p(\rho)$. In [24], DiPerna proved an existence result for entropy solutions in the case of the γ-law gas, $p(\rho) = \kappa\rho^\gamma$, for certain special exponents, namely $\gamma = 1 + 2/N$, $N \geq 5$, odd. Chen, [3], and Ding, Chen and Luo, [20], completed the argument of [24] and extended the result to the range $\gamma \in (1, 5/3]$. Lions, Perthame and Tadmor, [52], dealt with the case $\gamma \geq 3$ through the introduction of the kinetic formulation, before Lions, Perthame and Souganidis, [53], simplified the proof for the whole interval in extending the result to cover all $\gamma \in (1, 3)$. Finally, Chen and LeFloch, [8, 9], considered the case of a more general pressure law, under the principal assumptions of strict hyperbolicity and genuine non-linearity away from the vacuum. Precise formulations of their assumptions can be found in the two afore-mentioned papers. It should be noted that one expects both strict hyperbolicity and genuine non-linearity to fail at the vacuum, i.e. when the density $\rho = 0$, the eigenvalues of the flux function associated to system (1.1.1) coincide.

The scheme of the proof initially set out by DiPerna, [23, 24], can be summarised as follows. The proof follows the broad outline set out in the proof of Tartar for the scalar conservation law discussed in the previous section. First, one constructs a sequence of approximate solutions $(\rho^\varepsilon, u^\varepsilon)$ to the Euler equations by an approximation method, for example by using a finite difference scheme such as the Lax-Friedrichs scheme or the introduction of artificial viscosity. Once this is done and uniform bounds have been obtained in L^∞ by establishing an invariant region for the approximate solutions, one may extract a subsequence $\varepsilon_k \to 0$ such that

$$(\rho^{\varepsilon_k}, u^{\varepsilon_k}) \overset{*}{\rightharpoonup} (\rho, u),$$

weak-star in L^∞, generating a Young measure $\nu_{t,x}$ (see §3.1 for the theory of Young measures). The result then follows if one can show that this Young measure reduces to a Dirac mass at a point or has support contained in the vacuum region. If this is the case, we have that in the (ρ, m) coordinates, $\nu_{t,x}$ reduces to a point mass for almost every (t, x), and the weak-star convergence can be improved to almost everywhere convergence as well as strong convergence in $L^r_{loc}(\mathbb{R}^2_+)$ for $r \in [1, \infty)$. With this strong convergence, one may pass to the limit in the weak formulations of both the equation and the entropy condition.

Further to the result discussed in §2.2, Tartar conjectured, [70], that the commutation relation for the Young measure could still be derived for a system of conservation laws by exploiting the method of compensated compactness (in particular, by applying the div-curl lemma). Tartar moreover conjectured that this relation could again be seen to represent an imbalance of regularity that could be resolved only if the Young measure were concentrated at a point, as was the case for the scalar conservation law.

DiPerna observed that the commutation relation could be obtained for the Euler equations by establishing the compactness of the entropy dissipation measures in H^{-1}_{loc}, as in §2.2, for certain pairs of entropy functions, exploiting a compact embedding observed by Murat, [58]. However, this compactness holds only for entropies that vanish at the vacuum, the weak entropies, and so one may use only these weak entropies in the reduction argument. The commutation relation available was therefore

$$\langle \nu_{t,x}, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu_{t,x}, \eta_1 \rangle \langle \nu_{t,x}, q_2 \rangle - \langle \nu_{t,x}, \eta_2 \rangle \langle \nu_{t,x}, q_1 \rangle$$

for any two weak entropy pairs (η_1, q_1) and (η_2, q_2). That the compactness can only be shown to hold for the weak entropies provides the motivation for restricting the admissible entropy functions to the weak entropies in the definition of entropy solution to the Euler equations.

Using the Lax progressing entropy waves [45], DiPerna gave an analysis of the Young measure that provided the reduction. In [53], it was observed that in the commutation relation, the fundamental kernels used in (2.1.6) for the weak entropy functions could be substituted in place of explicit weak entropy functions. The authors observed that certain cancellations arose from the explicit form of the kernels that implied an imbalance of regularity in the commutation relation that could only be resolved if the Young measure were supported at a point. They then made use of fractional order derivative operators to make this imbalance clear and so derive the reduction.

Chen and LeFloch, [8, 9], extended this method further by showing that in the case of a more general pressure law, an expansion could be obtained for the fundamental solution of the entropy equation that provided sufficient information on the asymptotic behaviour of the kernels near to the vacuum to follow through the reduction argument using the fractional order derivative operators. In particular, they showed that even without the explicit form of the kernel available in the γ-law case, the same cancellation of singularities occurs in the expressions arising in the commutation relation, allowing for the imbalance of regularity to be shown.

2.4 Euler equations with geometric effects

In this section, we consider the Euler equations under the symmetry assumptions discussed in the Introduction. We recall that these symmetry assumptions allow us to reduce the Euler equations to a one-dimensional system of equations, this time of conservation laws with geometric source terms. In particular, we consider the Euler equations for transonic nozzle flows with general cross-sectional areas and related compressible flows of geometric structure including spherically symmetric flows. The system we obtain is the system of one-dimensional Euler equations with geometric source terms:

$$\begin{cases} \rho_t + m_x + \frac{A'(x)}{A(x)} m = 0, \\ m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x + \frac{A'(x)}{A(x)}\frac{m^2}{\rho} = 0, \end{cases} \tag{2.4.1}$$

for $(t, x) \in \mathbb{R}^2_+$. These flows form an important class of flows within fluid dynamics, having been a source of study for many decades. A derivation of the equations may be found in [16], and a discussion of the system can be found there also.

Although we will make precise conditions on the cross-sectional area of the nozzle later, for now we simply consider $A = A(x)$ to be a given C^2–function $A : \mathbb{R} \to \mathbb{R}_+$. We wish to consider a broad class of such functions, allowing for nozzles that open up at one end with unbounded area, and nozzles that pinch off at an end with area going to 0 ($A(x) \to 0$ as $x \to \pm\infty$).

The derivation of the spherically symmetric flows is very similar, and can also be found in [16]. In this case, one obtains $A(x) = \omega_n x^{n-1}$, where ω_n is the area of the unit sphere.

The crucial difference between system (2.4.1) and the usual one-dimensional Euler equations for the analyst is that the geometric source terms mean that the equations are no longer in conservation form. The effect of this should not be understated. As we have seen above in §2.3, with the one-dimensional equations in conservation form, the equations (and approximations to them via artificial viscosity or numerical methods) admit an *invariant region,* giving uniform bounds in L^∞ on approximate solutions, and making this a suitable class in which to search for entropy solutions. The principle fails, however, for these geometrically influenced equations. Instead of the usual invariant region approach, we are forced to consider a wider class of functions, those with finite energy, as described in the Introduction.

As in the case of the usual one-dimensional Euler equations, the strategy for constructing entropy solutions to the system (2.4.1) will be to construct a sequence of approximate solutions and pass to a Young measure limit before performing a reduction argument on the support of the Young measure to deduce strong convergence of the approximate solutions. With only energy bounds (and some improved, uniform higher integrability), we must also allow for the possibility that the Young measure so derived could have unbounded support in phase space. To handle these additional difficulties, we employ the Young measure framework and reduction argument of [12, 48], described in detail in §3.3 and §3.5.

This framework relies heavily on the assumption that the pressure law for the gas is exactly that of a polytropic gas (a gamma-law). For this reason, we assume that the equation of state is:

$$p(\rho) = \kappa\rho^\gamma, \qquad \gamma > 1,$$

where $\kappa = \frac{(\gamma-1)^2}{4\gamma}$ by scaling (without loss of generality).

We recall that we are focusing on the Cauchy problem for system (2.4.1):

$$(\rho, m)|_{t=0} = (\rho_0, m_0), \tag{2.4.2}$$

where the initial data $(\rho_0, m_0)(x)$ have end-point states $(\rho_\pm, m_\pm)$ at $x = \pm\infty$, for constants $\rho_\pm \geq 0$ and $m_\pm \in \mathbb{R}$. The end-states are allowed to be different, i.e. $(\rho_+, m_+) \neq (\rho_-, m_-)$ in general. This corresponds to allowing fluid to flow in or out of the nozzle at the ends at different rates and with different densities. However, if these end-states are non-trivial, the total energy of the system will not remain bounded. We will therefore work with the *relative* energy of the system, that is, the energy calculated with respect to a fixed background state, as defined in the Introduction.

The Cauchy problem (2.4.1)–(2.4.2) modelling transonic nozzle flow through a variable-area duct has been studied in detail by numerous authors, see for example [16, 25, 32, 34, 54, 55, 56, 72]. The existence of global solutions of this problem was first obtained in [54] by means of a Glimm scheme (that is, a random choice method, *cf.* [33]) incorporating steady-state building blocks, provided that the initial data have small total variation and are bounded away from both sonic and vacuum states. Some numerical methods were introduced to compute transient gas flows in a de Laval nozzle in [32, 34]. A mathematical analysis of the qualitative behavior of non-linear waves for nozzle flow was given in [56]. On the other hand, the general transonic nozzle problem, such that the initial data are allowed to be arbitrarily large with different end-states and only relative finite-energy and that the cross-sectional areas are allowed to tend to either zero (closed ends) or infinity (unbounded ends) as $x \to \pm\infty$, is poorly understood in the literature. Indeed, the existence theory for all $\gamma \in (1,\infty)$ was previously still open, since the solution may blow up at the closed ends. This problem of existence of solutions is resolved in Chapter 4 for the first time, under the assumption that

$$\|\frac{A'}{A}\|_{L^\infty(\mathbb{R})} + \|A'\|_{L^1(-\infty,0)} < \infty \;\text{ or }\; \|\frac{A'}{A}\|_{L^\infty(\mathbb{R})} + \|A'\|_{L^1(0,\infty)} < \infty. \tag{2.4.3}$$

As we have seen, the case that $A : \mathbb{R}_+ \to \mathbb{R}_+$ with $A(x) = \omega_n x^{n-1}, \omega_n > 0$, for $x \in \mathbb{R}_+$ corresponds to the equations for the isentropic Euler equations with spherical symmetry, which does not satisfy condition (2.4.3). The study of spherically symmetric solutions dates back to the 1950s, and is motivated by many important physical problems such as flow in a jet engine inlet manifold and stellar dynamics including gaseous stars and supernovae formation (*cf.* [16, 35, 50, 62, 65, 72]). A central feature is the strengthening of waves as they move radially inward, especially near the origin. Substantial evidence indicates that spherically symmetric solutions of the compressible Euler equations may blow up near the origin in finite time in some situations. A long-standing, open, fundamental problem is whether a concentration could form at the origin, that is, whether the density could become a delta measure at the origin, especially when a focusing spherical shock moves inward towards the origin (*cf.* [16, 62, 72]). The first existence result for global entropy solutions including the origin was established in Chen [5] for a class of L^∞ Cauchy data of arbitrarily large amplitude, which model outgoing blast waves and large-time asymptotic solutions; some further extensions of such Cauchy initial data for global entropy solutions in L^∞ can be found in Huang-Li-Yuan [41] and the references therein. Also see Slemrod [65] for the resolution of the spherical piston problem for isentropic gas dynamics via a self-similar viscous limit. In Chen-Perepelitsa [13], under the assumption that $\gamma \in (1,3]$, the existence of globally defined spherically symmetric entropy solutions was established via the vanishing viscosity approximation.

In Chapter 4, after demonstrating the existence of the entropy solutions to the transonic nozzle problem, (2.4.1), we will make a direct application of the approach and techniques developed in this chapter for the general transonic nozzle problem to the spherical symmetry problem, successfully removing the restriction that $\gamma \in (1,3]$ and establishing the existence of globally defined finite-energy entropy solutions for the general case of large initial data for the full range $\gamma > 1$, including the unsolved case $\gamma > 3$. This implies that there exist globally defined spherically symmetric solutions with large initial data of finite-energy for the whole range $\gamma \in (1,\infty)$, which do not form concentrations at the origin.

2.5 Vanishing viscosity limit of Navier-Stokes equations

The compressible, isentropic Navier-Stokes equations may be written as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = \varepsilon u_{xx}, \end{cases} \tag{2.5.1}$$

where, as usual, $\rho \geq 0$ is the density of the fluid, u is its velocity and $p = p(\rho)$ is the pressure. The positive constant $\varepsilon > 0$ is the viscosity of the fluid. We complement this system of equations with initial data

$$(\rho, u)|_{t=0} = (\rho_0, u_0), \tag{2.5.2}$$

where, as in the previous section, the initial data should be of relative finite-energy with (possibly non-trivial) end-states $(\rho_\pm, u_\pm)$ as $x \to \pm\infty$.

As was discussed in the Introduction, the vanishing viscosity limit of the Navier-Stokes equations forms an admissibility criterion for the compressible Euler equations. The question of whether or not the solutions to the Navier-Stokes equations could be shown to converge to a solution of the Euler equations, however, remained open for a long time. The first significant result in this direction is that of Chen-Perepelitsa [12], who showed that in the case of a γ-law gas, this limit may be taken rigorously and that the obtained solution of the Euler equations satisfies an entropy condition.

The existence of weak solutions (and smooth solutions) with positive density for the Navier-Stokes equations (2.5.1) was proved by Hoff, [37, 38]. Earlier, Gilbarg [31] analysed the vanishing viscosity limit from the Navier-Stokes equations to the Euler equations, showing the convergence of a sequence of solutions of system (2.5.1) to a simple shock wave solution of the Euler equations. An analysis of the inviscid limit to such simple shock solutions was undertaken by Hoff and Liu [39], see also [36]. More recently, the vanishing viscosity and capillarity limits of the Navier-Stokes-Korteweg system have been studied by Germain and LeFloch [30] for certain γ-law gases.

The Navier-Stokes equations satisfy a priori bounds on the velocity and density depending on the viscosity; indeed, for smooth initial data with the density bounded away from zero, the solutions can be shown to be smooth also, due to a parabolic type regularising effect. Unfortunately, as in the case of the transonic nozzle problem discussed in the previous section, the bounds in spaces such as L^∞ depend on the viscosity parameter, that is, they are not uniform with respect to ε. For this reason, the vanishing physical viscosity problem must be handled in the framework of relative finite-energy solutions.

Rather than consider the polytropic gases, that is, those with a γ-law pressure, we consider in Chapter 5 a more physical class of pressure functions, as described in the Introduction and recalled here. We assume that $p'(\rho) > 0$ and $\rho p''(\rho) + 2p'(\rho) > 0$ for $\rho > 0$ and that, for some $\gamma \in (1,3)$,

$$p(\rho) = \kappa\rho^{\gamma}\left(1 + P(\rho)\right), \quad P^{(n)}(\rho) = O\big(\rho^{\gamma-1-n}\big) \text{ as } \rho \to 0 \text{ for } 0 \leq n \leq 3, \tag{2.5.3}$$

where $\kappa > 0$. For the large density regime, we assume that there exists a constant $\rho_* > 0$ such that

$$p(\rho) = c_*\rho \text{ for } \rho \geq \rho_*. \tag{2.5.4}$$

Without loss of generality, we may assume that $c_* = 1$ and that $\rho_* = 1$ also.

Our strategy for passing to the limit in the equations is then to employ the theory of compensated compactness to find a Young measure solution of the Euler equations that is constrained by the Tartar commutation relation. This requires uniform integrability bounds on the solutions of the Navier-Stokes equations, independent of the viscosity $\varepsilon > 0$. In particular, we show that the relative energy of the system remains uniformly bounded and that the density satisfies uniform integrability properties. In order to deduce the entropy condition for the limit solution, we require higher integrability for the velocity also. To gain such an estimate, we construct an entropy pair of precisely controlled growth rate. To achieve this, in Chapter 5, we construct new entropy kernels for the isothermal Euler equations, giving an accurate representation formula for the entropies under a physical pressure law, which we use to obtain the desired integrability.

The main tool we make use of from the theory of compensated compactness is the div-curl lemma. To employ the div-curl lemma in the context of the one-dimensional compressible Euler equations, one would usually attempt to prove that the weak entropy dissipation measures $\eta(\rho^\varepsilon, m^\varepsilon)_t + q(\rho^\varepsilon, m^\varepsilon)_x$ associated to a sequence of approximate solutions $(\rho^\varepsilon, m^\varepsilon)$ is (pre)-compact in the space $H^{-1}_{loc}(\mathbb{R}^2_+)$. Unfortunately, in the case we are considering, the uniform estimate that we will obtain,

$$\int_0^T \int_K \rho p(\rho)\, dx\, dt \leq M,$$

is sufficient only to obtain ρ uniformly bounded in $L^2_{loc}(\mathbb{R}^2_+)$, and hence that the entropy dissipation measures are uniformly bounded in $H^{-1}_{loc}(\mathbb{R}^2_+)$, but not necessarily compact in this space. We therefore show only the weaker condition of $W^{-1,q}_{loc}(\mathbb{R}^2_+)$ compactness of the entropy dissipation measures, relying later on additional properties of equi-integrability for the product of an entropy with an entropy flux to apply the extension of the div-curl lemma in [15].

2.6 Euler equations in special relativity

The final model under consideration in this thesis is that of the compressible Euler equations in special relativity. As stated in the Introduction, the isentropic relativistic Euler equations of the conservation laws of baryon number and momentum are

$$\begin{cases} \partial_t\big(\frac{n}{\sqrt{1-u^2/c^2}}\big) + \partial_x\big(\frac{nu}{\sqrt{1-u^2/c^2}}\big) = 0, \\ \partial_t\big(\frac{(\rho+p/c^2)u}{1-u^2/c^2}\big) + \partial_x\big(\frac{(\rho+p/c^2)u^2}{1-u^2/c^2} + p\big) = 0, \end{cases} \tag{2.6.1}$$

where ρ, p and u represent the mass-energy density, pressure, and the particle speed, respectively, n is the proper number density of baryons, and c is the speed of light. Henceforth, we will write $\varepsilon = \frac{1}{c^2}$ for notational convenience.

The equations are derived from the Einstein equations of special relativity in [68], which also provides a derivation of the Rankine-Hugoniot jump conditions across a shock and a discussion of possible models for the pressure. As for the classical Euler equations and as discussed in §2.3, to close the system, we impose an equation of state, relating the pressure to the density. Having done so, as in the case of the classical equations, the properties of strict hyperbolicity and genuine non-linearity for the system will depend on the choice of pressure function. We therefore work with a pressure law $p = p(\rho)$, a C^3–function such that $p'(\rho) > 0$ for $\rho > 0$. In addition to providing an *a priori* bound on the velocity of a fluid (the speed of light in the space-time), the theory of special relativity predicts certain additional non-linear behaviour.

As the problem of global existence of entropy solutions for the classical equations has been resolved, at least in one spatial dimension, as described in §2.3, we here investigate the analogous problem for the relativistic equations, with a view to better understanding relativistic effects in fluid problems. In particular, in Chapter 6, we prove the existence of solutions to the Cauchy problem for these equations with arbitrarily large (but bounded) initial data under the assumption that the speed of light is large.

Before continuing, a few remarks concerning the baryon number are in order. The proper number density of baryons is determined by the first law of thermodynamics:

$$\theta dS = \frac{d\rho}{n} - \frac{\rho + p}{n^2} dn,$$

where θ is the temperature and S is the entropy per baryon. In particular, for an isentropic fluid (i.e. S is constant, the situation under consideration here), we have

$$\frac{dn}{n} = \frac{d\rho}{\rho + \varepsilon p},$$

and so

$$n = n(\rho) = n_0 e^{\int_0^\rho \frac{ds}{s+\varepsilon p(s)}}.$$

By rescaling the first equation in (2.6.1) if necessary, we may assume that $n_0 = 1$. By way of comparison with the classical Euler equations, we observe that in the Newtonian limit, that is, as $\varepsilon \to 0$, equivalently the speed of light $c \to \infty$, $n(\rho)$ converges locally uniformly to ρ.

As mentioned above, our approach to the relativistic equations is motivated by the successful strategies employed in resolving the Cauchy problem for the one-dimensional isentropic Euler equations in the classical setting. To motivate this comparison, we observe that, formally, in the Newtonian limit ($\varepsilon \to 0$), system (2.6.1) reduces to the classical isentropic Euler equations for compressible fluids:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = 0. \end{cases} \tag{2.6.2}$$

This formal observation raises the question of whether or not the limit here can be taken rigorously: do the entropy solutions to the relativistic Euler equations (2.6.1) converge to an entropy solution of the classical Euler equations as $\varepsilon \to 0$? One of the main contributions of Chapter 6 is to give an affirmative answer to this question under very general assumptions.

To place the relativistic Euler equations in the framework of hyperbolic conservation laws, we introduce here some additional notation. We consider system (2.6.1) as being a system of conservation laws by setting

$$\begin{aligned} U &= \left(\frac{n}{\sqrt{1-\varepsilon u^2}}, \frac{(\rho + \varepsilon p)u}{1 - \varepsilon u^2} \right)^\top, \\ F(U) &= \left(\frac{nu}{\sqrt{1-\varepsilon u^2}}, \frac{\rho u^2 + p}{1 - \varepsilon u^2} \right)^\top, \end{aligned} \tag{2.6.3}$$

so that system (2.6.1) takes on the form

$$\partial_t U + \partial_x F(U) = 0.$$

In particular, the condition of strict hyperbolicity is met if $p'(\rho) > 0$ and the characteristic fields are both genuinely nonlinear if

$$\rho p''(\rho) + 2p'(\rho) + \varepsilon(p(\rho)p''(\rho) - 2p'(\rho)^2) > 0. \tag{2.6.4}$$

Typically, these two criteria will be met only for $\rho > 0$ and so, as for the classical Euler equations, we may not simply apply the theory of hyperbolic conservation laws directly.

We remark that an alternative 2×2 system of conservation laws predicted by the theory of special relativity, and also sometimes called relativistic Euler equations in the literature (e.g. in [10]), is the following:

$$\begin{cases} \partial_t\big(\rho + \frac{\varepsilon(\rho+\varepsilon p)u^2}{1-\varepsilon u^2}\big) + \partial_x\big(\frac{(\rho+\varepsilon p)u}{1-\varepsilon u^2}\big) = 0, \\ \partial_t\big(\frac{(\rho+\varepsilon p)u}{1-\varepsilon u^2}\big) + \partial_x\big(\frac{(\rho+\varepsilon p)u^2}{1-\varepsilon u^2} + p(\rho)\big) = 0. \end{cases} \tag{2.6.5}$$

Many of the properties of the two systems translate from one to the other. Indeed, the governing entropy equation for system (2.6.5) is the same as that for (2.6.1). We therefore obtain results for the alternative system which are analogous to those for the system (2.6.1).

The first global existence result for the relativistic system of Euler equations was obtained by Smoller and Temple [66] in the case of an isothermal flow ($p(\rho) = \kappa\rho$) under the assumption of small total variation of initial data. In this setting, the authors were able to use the Glimm scheme to create a convergent sequence of approximate solutions by the random choice method. Subsequently, Ding and Li [22] again employed the Glimm scheme to obtain, for initial data of small total variation, global existence of entropy solutions to the relativistic piston problem for the isentropic Euler equations. They were also able to show that in the Newtonian limit, the relativistic solutions that they obtain converge to the entropy solution of the classical piston problem for the Euler equations. Liang [51] studied the formation of shocks and structure of simple waves, based on the work of Taub [68]. In [40], Hsu, Lin and Makino have obtained existence of entropy solutions with large data for a special class of pressure laws under the assumption of large speed of light (or, equivalently, small data), while other large data results have been obtained by Chen and Li [11], showing the existence and stability of entropy solutions to the Riemann problem for this system, and the same properties have been shown by the same authors for system (2.6.5) in [10]. For the variant relativistic Euler system of conservation of momentum and energy, Li, Feng and Wang [49] were also able to employ the Glimm scheme to show the existence of entropy solutions for a class of large initial data. All of these results make heavy restrictions on the type of pressure law that can be considered as well as, for many of them, conditions on smallness of total variation. We significantly weaken the requirements for existence of entropy solutions in Chapter 6 as well as providing a robust framework for analysing the Newtonian limit.

Chapter 3

Analytic preliminaries

3.1 Young measures

In each of the chapters to follow, we will need to consider the limit of a sequence of functions, designed as approximate solutions to a system of equations. Although the functions will satisfy uniform bounds (and hence will admit a weak limit), as was the case in §2.2, we will need to work rather harder to demonstrate that this limit is strong in order to pass to the limit in the non-linear terms in our equations. As was used in the proof of existence of solutions to a scalar conservation law, our primary tool for analysing weak convergence is the Young measure, introduced first by L.C. Young [73, 74]. The Young measure provides a family of probability measures that characterise the limiting oscillations of a weakly converging sequence, and so give a partial characterisation of the failure of weak convergence to be strong.

3.1.1 The fundamental theorem for Young measures

We state first a fairly classical version of the fundamental theorem for Young measures, here in the form given by Ball in [2]. The proof given in [2] relies on a duality method, exploiting the duality relation between the spaces $C_0(\mathbb{R}^m)$ of continuous functions on $\mathbb{R}^m$ that vanish at infinity, and the space $\mathcal{M}(\mathbb{R}^m)$ of bounded Radon measures on $\mathbb{R}^m$, via the duality relation

$$\langle \nu, f \rangle = \int_{\mathbb{R}^m} f(u)\, d\nu(u)$$

for functions $f \in C_0(\mathbb{R}^m)$ and measures $\nu \in \mathcal{M}(\mathbb{R}^m)$.

Theorem 3.1.1 (Fundamental Theorem for Young Measures, [2])**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set, let $K \subset \mathbb{R}^m$ be closed, and let $u^k : \Omega \to \mathbb{R}^m$ be a sequence of measurable functions such that $u^k \to K$ in measure.*

Then there exists a subsequence (still labelled u^k) and a family of measures $\{\nu_x\}_{x\in\Omega}$ depending measurably on x with $\nu_x \in \mathcal{M}(\mathbb{R}^m)$, the space of finite Radon measures, such that

(i) $\|\nu_x\|_{\mathcal{M}} := \int_{\mathbb{R}^m} d\nu_x \leq 1$ *for almost every* $x \in \Omega$,

(ii) *supp* $\nu_x \subset K$ *for almost every* $x \in \Omega$,

(iii) $f(u^k) \overset{*}{\rightharpoonup} \langle \nu_x, f\rangle = \int_{\mathbb{R}^m} f(z)\, dz$ *in* $L^\infty(\Omega)$ *for all* $f \in C_0(\mathbb{R}^m)$.

If, moreover,

$$\lim_{\lambda\to\infty} \sup_k \left|\{x \in \Omega \cap B_R \,:\, |u^k(x)| \geq \lambda\}\right| = 0$$

for every $R > 0$, then $\|\nu_x\|_{\mathcal{M}} = 1$ for almost every $x \in \Omega$ and, given a measurable subset $A \subset \Omega$,

$$f(u^k) \rightharpoonup \langle \nu_x, f\rangle \text{ in } L^1(A)$$

for every $f \in C(\mathbb{R}^m)$ such that the sequence $f(u^k)$ is sequentially weakly relatively compact in $L^1(A)$.

Remark 3.1.2. (i) In the scenario described in this theorem, we say that u^k generates the Young measure $\{\nu_x\}_{x\in\Omega}$.

(ii) If the sequence u^k is bounded uniformly in $L^\infty(\Omega; \mathbb{R}^m)$, then $f(u^k)$ is uniformly bounded for all $f \in C(\mathbb{R}^m)$, and hence the associated Young measure is such that, up to a subsequence,

$$f(u^k) \overset{*}{\rightharpoonup} \langle \nu_x, f\rangle \text{ in } L^\infty(\Omega) \text{ for all such } f.$$

As mentioned in the introduction to this section, the Young measure offers us a tool for measuring oscillations in a weak limit that prevent the limit from being strong. In essence, the support of the Young measure ν_x represents the set over which the sequence $u^k(x)$ is oscillating. This is quantified in the next proposition, which tells us when we may use the Young measure to deduce that weak convergence is also strong.

Proposition 3.1.3. *The (sub-)sequence u^k generating the Young measure $\{\nu_x\}_{x\in\Omega}$ converges in measure to a function $u : \Omega \to K$ if and only if $\nu_x = \delta_{u(x)}$ for almost every $x \in \Omega$.*

3.1.2 Young measures for maps into compact metric spaces

In general, we will not have uniform L^∞ bounds available for our sequences of approximate solutions, nor will we have available the equi-integrability that would allow an application of the above theorem. Instead, we will work on a compactification of our phase space, that is, we will consider our sequence of approximate solutions to take values in a compact metric space (see the following section for results concerning compactifications). This allows us to employ the Young meaure theory developed by Alberti and Müller in [1] for sequences of maps into compact metric spaces.

The setting is the following: let $\Omega \subset \mathbb{R}^n$ be an open set and (K, d) a compact metric space. We identify the space of finite real Borel measures on K, $\mathcal{M}(K)$, with the dual of the continuous functions, that is $\mathcal{M}(K) \cong \big(C(K)\big)^*$. Then we may consider the space $L^\infty_w\big(\Omega; \mathcal{M}(K)\big)$ of weak-star measurable maps $\nu : \Omega \to \mathcal{M}(K)$ that are essentially bounded (in fact, the space of equivalence classes of such maps that agree almost everywhere). We recall that such a map ν is weak-star measurable if the composition map $x \mapsto \langle \nu_x, g\rangle$ is a measurable map on Ω for every function $g \in C(K)$. The space $L^\infty_w\big(\Omega; \mathcal{M}(K)\big)$ is isometrically isomorphic to the dual of $L^1\big(\Omega; C(K)\big)$ via the duality pairing

$$\langle \nu, \phi\rangle_{L^\infty, L^1} := \int_\Omega \langle \nu_x, \phi_x\rangle_{\mathcal{M}, C}\, dx.$$

We then say that a map $\nu \in L^\infty_w\big(\Omega; \mathcal{M}(K)\big)$ is a (K-valued) Young measure if the measure ν_x is a probability measure for almost every $x \in \Omega$.

Theorem 3.1.4 ([1, Theorem 2.4]). *Let $u^k : \Omega \to K$ be a sequence of measurable maps. Then there exists a subsequence (not relabelled) that generates a Young measure ν such that*

(i) If $f : \Omega \times K \to \mathbb{R}$ is a measurable function that is continuous with respect to the second variable and $|f(x, z)| \leq h(x)$ for some $h \in L^1(\Omega)$, then

$$\int_\Omega f\big(x, u^k(x)\big)\, dx \to \int_\Omega \langle \nu_x, f_x\rangle\, dx.$$

(ii) The maps u^k converge in measure to a function $u : \Omega \to K$ if and only if $\nu_x = \delta_{u(x)}$ for almost every $x \in \Omega$.

3.2 Compactifications

We collect here the necessary facts concerning compactifications of topological spaces for future reference. All of the material in this section is taken from [63, Sections 1.1 and 1.5].

We recall that a completely regular topological space (U, τ) is a Hausdorff space such that for any point $u \in U$ and neighbourhood N of u, there exists a continuous map $f : U \to \mathbb{R}$ such that $f(u) = 0$ and $f(U \setminus N) = 1$.

Definition 3.2.1. Let (U, τ) be a completely regular topological space. A pair $(\mathcal{U}, \iota)$ is a compactification of U if the map $\iota : U \to \mathcal{U}$ is a dense embedding of U into $\mathcal{U}$ which is a homeomorphism onto its image.

As we wish to construct compactifications for use with the Young measure theory described above, we need to understand if and how the continuous functions over a compactification relate to the continuous functions over the original space U.

A linear subspace $\mathcal{R} \subset C^0(U)$ of the bounded, continuous functions (with the usual supremum norm) is said to be a ring if $f_1, f_2 \in \mathcal{R}$ implies that the pointwise product $f_1 f_2 \in \mathcal{R}$ also. A ring $\mathcal{R}$ is called complete if it is closed as a subspace of $C^0(U)$ and if for every closed subset $A \subset U$ and point $u \in U \setminus A$, there exists a function $f \in \mathcal{R}$ such that $f = 1$ on A and $f(u) = 0$.

Theorem 3.2.2 ([63, Proposition 1.5.3])**.** *Let $\mathcal{R}$ be a complete subring of $C^0(U)$ containing the constant functions. Then there exists a compactification $(\mathcal{U}, \iota)$ of U such that the space of continuous functions on $\mathcal{U}$, $C(\mathcal{U})$, is isometrically isomorphic to $\mathcal{R}$.*

The topology of $\mathcal{U}$ is the weak-star topology induced by the continuous functions, that is, $u_n \to u$ in $\mathcal{U}$ if and only if

$$\lim_{n\to\infty} f(u_n) \to f(u) \text{ for all } f \in C(\mathcal{U}).$$

Note that if the space U is separable under this (induced) topology, then $\mathcal{U}$ is also separable.

This theorem allows us to construct a compactification of a given space (in practice, this will be our phase space) such that we can characterise the continuous functions completely. In particular, we will know, up to isometric isomorphism, what are the admissible test functions for our Young measure, constructed as in Theorem 3.1.4.

Remark 3.2.3. To apply the theory of Young measures to our compactification, we must have that the space $\mathcal{U}$ is metrizable. We recall from [64, Section 3.8] that a compact topological space is metrizable if there is a sequence of continuous functions that separates points. As the ring $\mathcal{R}$ was assumed to be complete, it separates points in U and so as $\mathcal{R} \cong C(\mathcal{U})$ and U is dense in the compact space $\mathcal{U}$, $C(\mathcal{U})$ separates points in $\mathcal{U}$. Moreover, the space of bounded, continuous functions on a compact space is separable. As $\mathcal{U}$ is compact by construction, we therefore have that $C(\mathcal{U})$ is separable, and hence contains a sequence of functions that separates points in $\mathcal{U}$. Thus $\mathcal{U}$ is metrizable.

3.3 The Young measure construction for finite-energy gas dynamics

We now explain how the above considerations may be used to construct a Young measure that is adapted for the analysis of a sequence of approximate solutions to the Euler equations without uniform L^∞ bounds. The construction here follows that of [48, Section 2.3].

To this end, we define some notation. First, we set

$$\mathbb{H} = \{(\rho, u) \in \mathbb{R}^2 : \rho > 0\}$$

and consider on $\mathbb{H}$ the set of continuous functions

$$\bar{C}(\mathbb{H}) = \left\{ \phi \in C(\bar{\mathbb{H}}) \,\middle|\, \begin{array}{l} \phi(\rho, u) \text{ is constant on } \{\rho = 0\} \text{ and such that} \\ \text{the function } (\rho, u) \mapsto \lim_{s \to \infty} \phi(s\rho, su) \in C(\mathbb{S}^1 \cap \bar{\mathbb{H}}) \end{array} \right\} \tag{3.3.1}$$

where $\mathbb{S}^1$ is the unit circle.

As $\bar{C}(\mathbb{H})$ is a complete sub-ring of the continuous functions on $\mathbb{H}$ containing the constant functions, by Theorem 3.2.2, there exists a compactification $\overline{\mathcal{H}}$ of $\mathbb{H}$ such that $C(\overline{\mathcal{H}}) \cong \bar{C}(\mathbb{H})$, where $\cong$ denotes isometric isomorphism. The topology of $\overline{\mathcal{H}}$ is the weak-star topology induced by $C(\overline{\mathcal{H}})$, i.e. a sequence $v_n \in \overline{\mathcal{H}}$ converges to $v \in \overline{\mathcal{H}}$ if and only if $\phi(v_n) \to \phi(v)$ for all $\phi \in C(\overline{\mathcal{H}})$. Considering the induced topology on $\bar{\mathbb{H}}$, this topology has the obvious advantage of not distinguishing points in the vacuum set. We write V for the weak-star closure of the set $\{\rho = 0\}$ and set

$$\mathcal{H} = \mathbb{H} \cup V.$$

To see that the topology on $\overline{\mathcal{H}}$ is metrizable, we appeal to Remark 3.2.3.

With the compactification $\overline{\mathcal{H}}$, we apply the fundamental theorem of Young measures for maps into compact metric spaces, Theorem 3.1.4. That is, given a sequence of functions $(\rho^\varepsilon, u^\varepsilon) : \mathbb{R}^2_+ \to \overline{\mathcal{H}}$, there exists a subsequence generating a Young measure $\nu_{t,x}$ in the sense that, for any $\phi \in C(\overline{\mathcal{H}})$,

$$\phi(\rho^\varepsilon, u^\varepsilon) \xrightharpoonup{*} \int_{\overline{\mathcal{H}}} \phi(\rho, u)\, d\nu_{t,x} \text{ in } L^\infty(\mathbb{R}^2_+).$$

Moreover, moving to the (ρ, m) coordinates, where $m = \rho u$, we have that $(\rho^\varepsilon, m^\varepsilon)$ converges in measure (and hence almost everywhere up to subsequence) to some functions $(\rho(t,x), m(t,x))$ if and only if $\nu_{t,x} = \delta_{(\rho(t,x), m(t,x))}$ for almost every $(t,x) \in \mathbb{R}^2_+$.

3.4 Compensated compactness

The theory of compensated compactness stems from the seminal work of Murat and Tartar in the late 1970s and early 1980s, [57, 58, 59, 69, 70, 71]. In very rough terms, the theory attempts to find sufficient conditions on sequences of weakly convergent functions that allows the passage of the weak limit through a non-linear combination of the functions. Such conditions are usually assumptions on the compactness of combinations of derivatives of the functions in a very weak function space, such as a Sobolev space of negative order.

3.4.1 The Div-Curl lemma

The cornerstone of the compensated compactness theory is the div-curl lemma. Developed first by Murat in [57], a classical version of this lemma is the following.

Lemma 3.4.1 (Div-Curl Lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set, $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and suppose that v^ε and w^ε are sequences of vector fields such that*

$$v^\varepsilon \rightharpoonup v \text{ in } L^p(\Omega; \mathbb{R}^n), \qquad w^\varepsilon \rightharpoonup w \text{ in } L^q(\Omega; \mathbb{R}^n) \quad \text{as } \varepsilon \to 0.$$

Suppose also that

$$\operatorname{div} v^\varepsilon \text{ is (pre-)compact in } W^{-1,p}(\Omega), \quad \operatorname{curl} w^\varepsilon \text{ is (pre-)compact in } W^{-1,q}(\Omega; \mathbb{R}^{n\times n}).$$

Then the product $v^\varepsilon \cdot w^\varepsilon$ converges in distribution,

$$v^\varepsilon \cdot w^\varepsilon \rightharpoonup v \cdot w \text{ in } \mathcal{D}'.$$

A simple proof of this lemma in the case $p = q = 2$ can be found in [28, Section 5.B, Theorem 4]. This proof easily extends to the case stated here.

An improvement of the div-curl lemma in the case when the available compactness is weaker was given in [15]. In this situation, one has only the weaker compactness of the sequences in the space $W^{-1,1}$, and so the additional assumption is made that the product $v^\varepsilon \cdot w^\varepsilon$ is equi-integrable. By $W^{-1,1}(\Omega)$, we refer to the dual of $W_0^{1,\infty}(\Omega)$.

Lemma 3.4.2 ([15, Theorem]). *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and Lipschitz, $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and suppose that v^ε, w^ε are bounded sequences in $L^p(\Omega; \mathbb{R}^n)$ and $L^q(\Omega; \mathbb{R}^n)$ respectively and that*

$$v^\varepsilon \rightharpoonup v \text{ in } L^p(\Omega; \mathbb{R}^n),$$
$$w^\varepsilon \rightharpoonup w \text{ in } L^q(\Omega; \mathbb{R}^n),$$

and that, moreover,

$$\operatorname{div} v^\varepsilon \text{ and } \operatorname{curl} w^\varepsilon \text{ are (pre)-compact in } W_{loc}^{-1,1}(\Omega),\ W_{loc}^{-1,1}(\Omega; \mathbb{R}^{n\times n}) \text{ respectively.}$$

If, in addition, the sequence $v^\varepsilon \cdot w^\varepsilon$ is equi-integrable, then

$$v^\varepsilon \cdot w^\varepsilon \rightharpoonup v \cdot w \text{ in } \mathcal{D}'.$$

3.4.2 Compensated compactness interpolation and embedding theorems

Another of the useful tools of compensated compactness is the so-called compensated compactness interpolation theorem:

Theorem 3.4.3. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary. Suppose that $S \subset W_{loc}^{-1,p}(\Omega) \cap W_{loc}^{-1,q}(\Omega)$ is a set such that S is compact in $W_{loc}^{-1,q}(\Omega)$ and S is uniformly bounded in $W_{loc}^{-1,p}(\Omega)$ where $1 < q < p < \infty$. Then, for any r such that $q \leq r < p$, S is compact in $W_{loc}^{-1,r}(\Omega)$.*

The proof may be found in, for example, [4]. We use also the following lemma of Murat.

Theorem 3.4.4 ([58]). *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set, and suppose $1 < r < \infty$. Suppose that a sequence u^ε in $W^{-1,r}(\Omega)$ satisfies*

$$\sup_\varepsilon \|u^\varepsilon\|_{W^{-1,r}(\Omega)} < +\infty$$

and

$$u^\varepsilon \geq 0 \text{ in } W^{-1,r}(\Omega).$$

Then $\{u^\varepsilon\}_{\varepsilon>0}$ is (pre-)compact in $W^{-1,p}(\Omega)$ for all $1 < p < r$.

3.5 The reduction framework of Chen-Perepelitsa

In the case of a gamma-law gas, Chen and Perepelitsa have created a framework for analysing the Young measure generated by a sequence of approximate solutions to the Euler equations under fairly general integrability assumptions. In this section, we outline the assumptions and give a general statement of the main result of this framework. The proofs may be found, along with further discussion, in [12].

The set up for the problem is as follows: Let $(\rho^\varepsilon, u^\varepsilon) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, $\varepsilon > 0$, be a sequence of functions with $\rho^\varepsilon > 0$, satisfying the following bounds. Given a compact set $K \subset \mathbb{R}$, there exists a constant $M > 0$ depending on K but independent of ε such that

$$\int_0^T \int_K (\rho^\varepsilon)^{\gamma+1} \, dx \, dt + \int_0^T \int_K \big(\rho^\varepsilon |u^\varepsilon|^3 + (\rho^\varepsilon)^{\gamma+\theta}\big) \, dx \, dt \leq M, \tag{3.5.1}$$

where $\gamma \in (1, \infty)$ and $\theta = \frac{\gamma-1}{2}$. Suppose moreover that the entropy dissipation measures

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \text{ are compact in } H^{-1}_{loc}(\mathbb{R}^2_+) \tag{3.5.2}$$

for all weak entropy pairs (η^ψ, q^ψ) of the isentropic Euler equations generated by a test function $\psi \in C^2_c(\mathbb{R})$ of compact support as in (2.1.6), where $m^\varepsilon = \rho^\varepsilon u^\varepsilon$.

Applying the construction of §3.3, we deduce the existence of a Young measure $\nu_{t,x}$ defined on the compactification $\overline{\mathcal{H}}$ of phase space with notation as in that section, so that for all $\phi \in C(\overline{\mathcal{H}})$,

$$\phi(\rho^\varepsilon, u^\varepsilon) \overset{*}{\rightharpoonup} \int_{\overline{\mathcal{H}}} \phi(\rho, u) \, d\nu_{t,x} \text{ in } L^\infty(\mathbb{R}^2_+) \text{ as } \varepsilon \to 0.$$

The main result of the framework of [12] can now be summarised as follows.

Theorem 3.5.1. *Suppose $(\rho^\varepsilon, u^\varepsilon)$ is a sequence of functions satisfying the assumptions (3.5.1) and (3.5.2). Then the Young measure $\nu_{t,x}$ in phase coordinates (ρ, m) reduces to a Dirac mass at almost every $(t, x) \in \mathbb{R}^2_+$, that is, there exist measurable functions $(\rho, m) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ such that for almost every $(t, x) \in \mathbb{R}^2_+$,*

$$\nu_{t,x} = \delta_{(\rho(t,x), m(t,x))}.$$

Moreover, the convergence may be taken to be in $L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+)$ for $p \in [1, \gamma+1)$ and $q \in [1, \frac{3(\gamma+1)}{\gamma+3})$.

This theorem is the content of [12, Proposition 7.3] in the case $\gamma \in (1, 3)$ and [12, Proposition 6.3] in the case $\gamma \geq 3$. The range of exponents q may be seen as $|m|^q = \rho^{\frac{q}{3}} |u|^q \rho^{\frac{2q}{3}} \leq M\big(\rho |u|^3 + \rho^{\gamma+1}\big)$, precisely when $q = \frac{3(\gamma+1)}{\gamma+3}$.

Chapter 4

Euler equations with geometric effects

4.1 Introduction

This chapter is concerned with establishing the existence of entropy solutions to the Euler equations with geometric structure. We consider the Euler equations for transonic nozzle flows with general cross-sectional area and related compressible flows of geometric structure including spherical symmetry. As discussed in §2.4, these equations take the form of the one-dimensional Euler equations with geometric source terms:

$$\begin{cases} \rho_t + m_x + \frac{A'(x)}{A(x)} m = 0, \\ m_t + \big(\frac{m^2}{\rho} + p(\rho)\big)_x + \frac{A'(x)}{A(x)} \frac{m^2}{\rho} = 0, \end{cases} \tag{4.1.1}$$

for $(t, x) \in \mathbb{R}^2_+$. In this model, $\rho \geq 0$ is the density and the velocity is $u = \frac{m}{\rho} : \mathbb{R} \to \mathbb{R}$. We make precise assumptions on the cross-sectional area of the nozzle $A(x)$ later, but for now we consider $A : \mathbb{R} \to \mathbb{R}_+$ to be a given C^2–function. We consider a broad class of such functions, allowing for nozzles that open up at one end with unbounded area, and nozzles that pinch off at an end with area going to zero.

We consider the Cauchy problem for system (4.1.1):

$$(\rho, m)|_{t=0} = (\rho_0, m_0), \tag{4.1.2}$$

where the initial data $(\rho_0, m_0)(x)$ have end-point states $(\rho_\pm, m_\pm)$ at $x = \pm\infty$, for constants $\rho_\pm \geq 0$ and $m_\pm \in \mathbb{R}$. Throughout, we assume the equation of state of a polytropic gas, that is,

$$p(\rho) = \kappa \rho^\gamma, \qquad \text{where } \gamma > 1 \text{ and } \kappa = \frac{(\gamma-1)^2}{4\gamma}.$$

In this chapter, we develop a framework with which we demonstrate the existence of relative finite-energy entropy solutions to (4.1.1)–(4.1.2) under general assumptions on the area function $A(x)$ and the end-states of the fluid. As a corollary of this method, we also extend the existing results for the spherically symmetric equations. We recall that the background to the problem and some relevant techniques were discussed in §2.4.

The structure of the chapter is as follows: in §4.2, we recall the properties of weak entropy functions of system (4.1.1), introduce the definition of entropy solutions, and present the main theorems, Theorems 4.2.3–4.2.5.

In §4.3–§4.5, we develop the viscosity method to construct the afore-mentioned approximate solutions and demonstrate their convergence to entropy solutions of the Cauchy problem (4.1.1)–(4.1.2), first under the following assumptions on the cross-sectional area function $A(x)$:

$$0 < A_0 \leq A(x) \leq A_1 < \infty, \tag{4.1.3}$$

$$\|A\|_{C^2(\mathbb{R})} + \|A'\|_{L^1(\mathbb{R})} \leq A_2 < \infty. \tag{4.1.4}$$

For the general case, we make the assumption that the corresponding cross-sectional area function $A(x) > 0$ is a C^2–function satisfying either

$$\|\frac{A'}{A}\|_{L^\infty(\mathbb{R})} + \|A'\|_{L^1(-\infty,0)} < \infty \tag{4.1.5}$$

or

$$\|\frac{A'}{A}\|_{L^\infty(\mathbb{R})} + \|A'\|_{L^1(0,\infty)} < \infty. \tag{4.1.6}$$

In particular, this general class of nozzles includes the de Laval nozzles with closed ends. In §4.6, we establish the existence of globally defined entropy solutions to the Euler equations (4.1.1) for compressible fluid flows in transonic nozzles with general cross-sectional areas satisfying (4.1.5) or (4.1.6). To achieve this, we develop a viscosity method to construct globally defined approximate solutions and establish several essential uniform estimates for the whole interval of adiabatic exponents $\gamma \in (1, \infty)$, so that the viscosity approximate solutions satisfy the general L^p compensated compactness framework established in Chen-Perepelitsa [12] as described in §3.5. This viscosity method is carefully designed to incorporate artificial viscosity terms with the natural Dirichlet boundary conditions to ensure the uniform estimates for all $\gamma \in (1, \infty)$. Then these uniform estimates lead to both the convergence of the approximate solutions and the existence of globally defined entropy solutions of problem (4.1.1)–(4.1.2) for initial data of relative finite-energy for all $\gamma \in (1, \infty)$, provided that the cross-sectional area function $A(x)$ satisfies (4.1.5) or (4.1.6).

The construction of the approximate solutions in §4.3 employs the standard theory of quasilinear parabolic systems as detailed in [43]. In §4.4, we derive *uniform* estimates, independent of the viscosity coefficient ε, on the approximate solutions, so that they satisfy the requirements of the general L^p compensated compactness framework of §3.5 to pass to the limit as $\varepsilon \to 0$, and hence deduce, in §4.5, the existence of globally defined entropy solutions of the Cauchy problem (4.1.1)–(4.1.2) with general large initial data of different end-states and relative finite-energy in the sense of Theorem 4.2.3.

In §4.6, we describe how the additional assumptions on $A(x)$ in (4.1.3)–(4.1.4) can be removed to obtain the results in Theorem 4.2.3 for the general case (4.1.5) or (4.1.6).

Finally, in §4.7, we show how the approach and techniques developed in §4.3–§4.6 can be extended to the general setting of spherically symmetric solutions to the Euler equations, leading to the second main theorem of this chapter, Theorem 4.2.5, for the whole range of adiabatic exponents $\gamma \in (1, \infty)$, including the unsolved case: $\gamma > 3$.

4.2 Entropy solutions and main theorems

In this section, we discuss the properties of weak entropy functions of system (4.1.1), introduce the definition of entropy solutions, and present the main theorems of this chapter, Theorems 4.2.3 and 4.2.5.

4.2.1 Entropy

We recall that an entropy-entropy flux pair (or entropy pair, for simplicity) is a pair of functions $(\eta, q) : \mathbb{R}^2_+ \to \mathbb{R}^2$ such that

$$\nabla q(\rho, m) = \nabla \eta(\rho, m) \nabla \begin{pmatrix} m \\ \frac{m^2}{\rho} + p(\rho) \end{pmatrix}, \tag{4.2.1}$$

where ∇ is the gradient with respect to the conservative variables (ρ, m). For example, the mechanical energy and its flux form an entropy pair:

$$\eta^*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + \frac{\kappa}{\gamma - 1}\rho^\gamma, \tag{4.2.2}$$

$$q^*(\rho, m) = \frac{1}{2}\frac{m^3}{\rho^2} + \frac{\kappa\gamma}{\gamma - 1}m\rho^{\gamma-1}. \tag{4.2.3}$$

One of the important properties of this entropy pair for the purposes of our analysis is that it provides a convex, positive entropy. Indeed, one may easily calculate and find that, writing $h(\rho) = \rho e(\rho) = \frac{\kappa\rho^\gamma}{\gamma-1}$, so that $p(\rho) = \rho^2 e'(\rho)$,

$$h''(\rho)|\rho_x^2| + \rho|u_x|^2 \le (\rho_x, m_x)\nabla^2\eta^*(\rho_x, m_x)^\top, \tag{4.2.4}$$

compare [13, Proof of Proposition 2.1]. Thus, by integrating the additional conservation law obtained from the pair, we gain *a priori* bounds on solutions. This observation is the core idea of the first main energy estimate, Proposition 4.3.2 below, which demonstrates that relative finite-energy initial data leads to relative finite-energy solutions without employing invariant region theory.

We recall from (2.1.6) that any weak entropy pair for system (4.1.1) can be expressed as

$$\eta^\psi(\rho, m) = \rho \int_{-\infty}^{\infty} \psi(\frac{m}{\rho} + \rho^\theta s)[1 - s^2]_+^{\frac{3-\gamma}{2(\gamma-1)}}\, ds, \tag{4.2.5}$$

$$q^\psi(\rho, m) = \rho \int_{-\infty}^{\infty} (\frac{m}{\rho} + \theta\rho^\theta s)\psi(\frac{m}{\rho} + \rho^\theta s)[1 - s^2]_+^{\frac{3-\gamma}{2(\gamma-1)}}\, ds, \tag{4.2.6}$$

where $\theta = \frac{\gamma-1}{2}$, and $\psi : \mathbb{R} \to \mathbb{R}$ is said to generate the entropy pair (η^ψ, q^ψ).

4.2.2 Entropy solutions

Now we introduce the definition of entropy solutions of the Cauchy problem (4.1.1)–(4.1.2). As a result of our restriction to the L^p framework, we do not require the solution to satisfy the entropy inequality (4.2.8) for all weak entropies, but ask it only for those weak entropies satisfying a sub-quadratic growth condition. The reason for this is simple: we have only energy bounds for the solution, and so entropies of higher order growth need not even be locally integrable.

Definition 4.2.1. An *entropy solution* of the Cauchy problem (4.1.1)–(4.1.2) is a pair $(\rho, m) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ such that

(i) For any $\phi \in C_c^\infty(\overline{\mathbb{R}^2_+})$,

$$\begin{aligned} &\int_{\mathbb{R}^2_+} (\rho\phi_t + m\phi_x)A(x)\, dx\, dt + \int_{\mathbb{R}} \rho_0(x)\phi(0, x)A(x)\, dx = 0, \\ &\int_{\mathbb{R}^2_+} \big(m\phi_t + \frac{m^2}{\rho}\phi_x + p(\rho)(\phi_x + \frac{A'}{A}\phi)\big)A(x)\, dx\, dt + \int_{\mathbb{R}} m_0(x)\phi(0, x)A(x)\, dx = 0; \end{aligned} \tag{4.2.7}$$

(ii) For any convex $\psi(s)$, with sub-quadratic growth at infinity, generating the entropy pair (η^ψ, q^ψ),

$$\big(\eta^\psi A(x)\big)_t + \big(q^\psi A(x)\big)_x + A'(x)\big(m\eta^\psi_\rho + \frac{m^2}{\rho}\eta^\psi_m - q^\psi\big) \le 0 \tag{4.2.8}$$

in the sense of distributions.

Recalling the definition of the relative mechanical energy from the Introduction, we define smooth, monotone functions $(\bar{\rho}(x), \bar{u}(x))$ such that, for some $L_0 > 1$,

$$(\bar{\rho}(x), \bar{u}(x)) = \begin{cases} (\rho_+, u_+), & x \ge L_0, \\ (\rho_-, u_-), & x \le -L_0. \end{cases} \tag{4.2.9}$$

By the Galilean invariance of the system, without loss of generality, we may assume either $u_- = 0$ or $u_+ = 0$. We then define the relative mechanical energy with respect to $(\bar{\rho}(x), \bar{m}(x)) = (\bar{\rho}(x), \bar{\rho}(x)\bar{u}(x))$:

$$\begin{aligned} \overline{\eta^*}(\rho, m) &= \eta^*(\rho, m) - \eta^*(\bar{\rho}, \bar{m}) - \nabla\eta^*(\bar{\rho}, \bar{m}) \cdot (\rho - \bar{\rho}, m - \bar{m}) \\ &= \frac{1}{2}\rho|u - \bar{u}|^2 + \overline{h}(\rho, \bar{\rho}) \ge 0, \end{aligned} \tag{4.2.10}$$

where $\overline{h}(\rho, \bar{\rho}) = h(\rho) - h(\bar{\rho}) - h'(\bar{\rho})(\rho - \bar{\rho})$.

Definition 4.2.2. The *relative total mechanical energy* for (4.1.1) with respect to the end-states $(\rho_\pm, m_\pm)$ through $(\bar{\rho}, \bar{m})$ is

$$E[\rho, m](t) := \int_{-\infty}^{\infty} \overline{\eta^*}(\rho(t,x), m(t,x))A(x)\, dx \ge 0.$$

We say that a pair (ρ, m) is of *relative finite-energy* with respect to the end-states $(\rho_\pm, m_\pm)$ if $E[\rho, m](t) < \infty$ for almost every $t > 0$.

4.2.3 Main theorems

We now present the two main theorems of this chapter.

Theorem 4.2.3 (Entropy Solutions of the Transonic Nozzle Problem)**.** *Let the cross-sectional area function $A(x) > 0$ be a C^2–function satisfying (4.1.5) or (4.1.6). Assume that $(\rho_0, m_0) \in L^1_{loc}(\mathbb{R}; \mathbb{R}^2)$ with $\rho_0(x) \ge 0$ is of relative finite-energy with respect to the prescribed end-states $(\rho_\pm, m_\pm)$. Then there exists a global finite-energy entropy solution $(\rho, m)(t,x)$ of the transonic nozzle problem (4.1.1)–(4.1.2) in the sense of Definitions 4.2.1–4.2.2 such that*

$$(\rho, m) \in L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+) \qquad \textit{for } p \in [1, \gamma+1) \textit{ and } q \in [1, \tfrac{3(\gamma+1)}{\gamma+3}). \tag{4.2.11}$$

Remark 4.2.4. Condition (4.1.5) or (4.1.6) allows for the nozzles to open up at one end (with an unbounded cross-sectional area), and to shrink at both ends with rate $|x|^{-\alpha}$ for any $\alpha > 0$ (closed ends). In particular, a positive lower bound, or upper bound, of the cross-sectional area function is not required.

The existence of entropy solutions presented in the above theorem is established as a vanishing viscosity limit of appropriately designed approximate solutions. We therefore first develop the viscosity method to construct such approximate solutions and then derive the uniform estimates of the approximate solutions so that they satisfy the requirements of the L^p compensated compactness framework in §3.5. Applying this framework then yields the strong convergence of the approximate solutions to a limit, which we show to be an entropy solution of (4.1.1), as desired. Moreover, as a consequence of the approach and techniques developed throughout the following sections, we are able to prove the following result for the spherically symmetric Euler equations, including the previously unsolved case: $\gamma > 3$.

Theorem 4.2.5 (Entropy Solutions of the Spherically Symmetric Problem)**.** *Suppose that $(\rho_0, m_0) \in (L^1_{loc}(\mathbb{R}_+))^2$ are finite-energy initial data such that $\rho_0(x) \geq 0$ and*

$$E_*[\rho_0, m_0] := \int_0^\infty \eta^*(\rho_0, m_0) x^{n-1}\, dx < \infty.$$

Then, for the whole range $\gamma \in (1, \infty)$, there exists a global entropy solution $(\rho, m)(t, x)$ of the spherically symmetric Euler equations with $A(x) = \omega_n x^{n-1}, x \geq 0$, where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$, such that $(\rho, m) \in L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+)$ for $p \in [1, \gamma + 1)$ and $q \in [1, \frac{3(\gamma+1)}{\gamma+3})$, and

$$E_*[\rho, m](t) := \int_0^\infty \eta^*(\rho, m) x^{n-1} dx \leq E_*[\rho_0, m_0] < \infty. \tag{4.2.12}$$

4.3 Viscosity approximate solutions

We now develop the viscosity method to construct the viscosity approximate solutions of the Cauchy problem (4.1.1)–(4.1.2) and make some necessary estimates. Throughout this section and §4.4–§4.5, we first assume that there exist $A_0, A_1, A_2 > 0$ such that conditions (4.1.3)–(4.1.4) hold for the cross-sectional area function $A(x)$ for simplicity of presentation.

For fixed $\varepsilon > 0$, we consider the following approximate equations for $(t,x) \in \mathbb{R}_+ \times (a,b)$:

$$\begin{cases} \rho^\varepsilon_t + m^\varepsilon_x + \frac{A'(x)}{A(x)} m^\varepsilon = \varepsilon\big(\rho^\varepsilon_{xx} + \frac{A'(x)}{A(x)}\rho^\varepsilon_x\big), \\ m^\varepsilon_t + \big(\frac{(m^\varepsilon)^2}{\rho^\varepsilon} + p_\delta(\rho^\varepsilon)\big)_x + \frac{A'(x)}{A(x)}\frac{(m^\varepsilon)^2}{\rho^\varepsilon} = \varepsilon\big(m^\varepsilon_x + \frac{A'(x)}{A(x)} m^\varepsilon\big)_x, \end{cases} \tag{4.3.1}$$

where

$$p_\delta(\rho) = \kappa\rho^\gamma + \delta\rho^2, \qquad \delta = \delta(\varepsilon) > 0 \text{ with } \delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0. \tag{4.3.2}$$

Here $a = a(\varepsilon)$ and $b = b(\varepsilon)$ are chosen such that

$$a(\varepsilon) < -L_0, \qquad a(\varepsilon) \to -\infty \quad \text{as } \varepsilon \to 0, \tag{4.3.3}$$

$$b(\varepsilon) > L_0, \qquad b(\varepsilon) \to \infty \quad \text{as } \varepsilon \to 0 \tag{4.3.4}$$

for L_0 in (4.2.9), independent of ε. We can take $a(\varepsilon)$ and $b(\varepsilon)$ to diverge at the same rate as $\varepsilon \to 0$, but this rate is not arbitrary. However, we have the freedom to design the approximating problems in a convenient manner so that this rate can be chosen carefully in §4.4–§4.5.

The choice of viscous terms is designed to allow both for the application of standard parabolic theory and also to be compatible with the entropy structure of the equations. The main problem for establishing the existence of the approximate solutions is the need to gain *a priori* bounds on the approximate solutions that bound solutions away from the singularities of the coefficients in the equations. In particular, we must demonstrate both upper bounds for ρ and m, and also a lower bound for ρ. By restricting the domain of the approximate problems, we allow for the imposition of boundary data and the possibility of a weak maximum principle that gives these desired upper bounds. To control the second issue, that of the lower bound for ρ, the additional term $\delta\rho^2$ is introduced into the pressure function. This term prevents cavitation (the formation of a vacuum state) in the approximate solutions when $\varepsilon > 0$.

We pose approximate initial and Dirichlet boundary data:

$$\begin{cases} (\rho, m)|_{t=0} = (\rho^\varepsilon_0, m^\varepsilon_0)(x), & x \in (a,b), \\ (\rho, m)|_{x=a} = (\rho^\varepsilon_-, m^\varepsilon_-), & t \ge 0, \\ (\rho, m)|_{x=b} = (\rho^\varepsilon_+, m^\varepsilon_+), & t \ge 0, \end{cases} \tag{4.3.5}$$

where $\rho^\varepsilon_0 > 0$, $\rho^\varepsilon_\pm > 0$, and $(\rho^\varepsilon_\pm, m^\varepsilon_\pm) \to (\rho_\pm, m_\pm)$ as $\varepsilon \to 0$. The imposition that the end-states for ρ are strictly positive is to prevent the possibility of cavitation. One of the motivations for imposing the Dirichlet boundary data for the approximate solutions is to allow for the use of the weak maximum principle for obtaining the L^∞ estimate for the approximate solutions for the whole range $\gamma \in (1,\infty)$; see Lemma 4.3.4.

With this carefully designed initial-boundary value problem (4.3.1)–(4.3.5), the next goal of this section is now to establish the existence of globally defined approximate solutions. Throughout this section, for $\beta \in (0,1)$, $C^{2+\beta}([a,b])$ and $C^{2+\beta,1+\frac{\beta}{2}}(Q_T)$ denote the usual Hölder and parabolic Hölder spaces on the interval $[a,b]$ and the parabolic cylinder $Q_T := [0,T] \times [a,b]$, respectively, as defined in [43].

Theorem 4.3.1 (Artifical Viscosity Solutions)**.** *For $\varepsilon > 0$, let $(\rho_0^\varepsilon, m_0^\varepsilon) \in \big(C^{2+\beta}([a,b])\big)^2$ be a sequence of functions such that*

(i) $\inf_{a \le x \le b} \rho_0^\varepsilon(x) > 0$;

(ii) $(\rho_0^\varepsilon, m_0^\varepsilon)$ *satisfies* (4.3.5) *and the compatibility conditions at* $x = a(\varepsilon)$ *and* $b(\varepsilon)$:

$$\big(A(x) m_0^\varepsilon\big)_x = \varepsilon\big(A(x)\rho_{0,x}^\varepsilon\big)_x, \qquad \big(A(x)\frac{(m_0^\varepsilon)^2}{\rho_0^\varepsilon}\big)_x + A(x) p_\delta(\rho_0^\varepsilon)_x = \varepsilon\big(A(x) m_0^\varepsilon\big)_x;$$

(iii) $\int_a^b \big(\frac{(m_0^\varepsilon)^2}{2\rho_0^\varepsilon} + \frac{\kappa(\rho_0^\varepsilon)^\gamma}{\gamma-1}\big) A(x)\, dx < \infty$;

(iv) $(\delta(\varepsilon), a(\varepsilon), b(\varepsilon))$ *satisfy* (4.3.2)–(4.3.4).

Then there exists a unique global solution $(\rho^\varepsilon, m^\varepsilon)(t,x)$ of problem (4.3.1)–(4.3.5) for $\gamma \in (1,\infty)$ such that $(\rho^\varepsilon, m^\varepsilon) \in \big(C^{2+\beta,1+\frac{\beta}{2}}(Q_T)\big)^2$ with $\inf_{Q_T} \rho^\varepsilon(t,x) > 0$ for all $T > 0$.

We assume from now on that $a = a(\varepsilon)$ and $b = b(\varepsilon)$ are constants depending on $\varepsilon > 0$ such that

$$\big(1 + \big|\big(\frac{A'}{A}\big)'\big|\big)\varepsilon|a-b| \le M, \tag{4.3.6}$$

where $M > 0$ is a constant, independent of ε. For simplicity of presentation, we drop the explicit ε–dependence of all functions in this section, since the results hold for each fixed $\varepsilon > 0$. As we have modified the pressure function, we find that the associated mechanical energy needs adjusting likewise. To this end, we write:

$$h_\delta(\rho) = \rho e_\delta(\rho), \qquad e_\delta(\rho) = \int_0^\rho \frac{p_\delta(s)}{s^2}\, ds, \tag{4.3.7}$$

taking the place of $h(\rho)$ defined above and used in the mechanical energy.

By the standard theory of quasilinear parabolic systems detailed in [43], in order to establish Theorem 4.3.1, it suffices to show that, for a generic $C^{2,1}(Q_T)$ solution of (4.3.1), the following *a priori* bound holds:

$$\big\|\big(\rho, \frac{1}{\rho}, \frac{m}{\rho}\big)\big\|_{L^\infty(Q_T)} < \infty,$$

where the bound depends on ε, T, and the initial-boundary data. Then any solution of the equations with the prescribed regularity remains in a bounded region away from the singularities of the coefficients of the quasilinear parabolic system (4.3.1). The results in §5 and Theorem 7.1 in §7 of [43] then lead to the conclusion of Theorem 4.3.1.

The remainder of this section consists of the energy estimates necessary to deduce these bounds. Our fundamental energy estimate concerns the control of the relative mechanical energy for system (4.3.1). We define a modified entropy pair:

$$\eta_\delta^* = \frac{1}{2}\frac{m^2}{\rho} + h_\delta(\rho), \qquad q_\delta^* = \frac{1}{2}\frac{m^3}{\rho^2} + m h_\delta'(\rho),$$

and further modify these to obtain a modified relative entropy pair,

$$\begin{aligned}\overline{\eta_\delta^*}(\rho, m) &= \eta_\delta^*(\rho, m) - \eta_\delta^*(\bar\rho, \bar m) - \nabla\eta_\delta^*(\bar\rho, \bar m)\cdot(\rho-\bar\rho, m-\bar m)\\ &= \frac{1}{2}\rho|u-\bar u|^2 + \overline{h_\delta}(\rho,\bar\rho) \geq 0,\end{aligned}$$

where $\overline{h_\delta}(\rho,\bar\rho) = h_\delta(\rho) - h_\delta(\bar\rho) - h_\delta'(\bar\rho)(\rho-\bar\rho)$.

Then the *total relative mechanical energy* for (4.3.1) is

$$E[\rho, m](t) := \int_a^b \overline{\eta_\delta^*}(\rho(t,x), m(t,x))A(x)\,dx \geq 0.$$

Now we obtain our first main energy estimate.

Proposition 4.3.2. *Let*

$$E_0 := \sup_{\varepsilon>0}\int_a^b \overline{\eta_\delta^*}(\rho_0^\varepsilon(x), m_0^\varepsilon(x))A(x)\,dx < \infty.$$

Then there exists $M>0$, independent of ε, such that, for all $\varepsilon>0$,

$$\begin{aligned}&\sup_{t\in[0,T]}\int_a^b \big(\frac{1}{2}\rho|u-\bar u|^2 + \overline{h_\delta}(\rho,\bar\rho)\big)A(x)\,dx\\ &+\varepsilon\int_{Q_T}\Big(h_\delta''(\rho)\rho_x^2 + \rho u_x^2 + \big|\big(\frac{A'(x)}{A(x)}\big)'\rho u(u-\bar u)\big|\Big)A(x)\,dx\,dt \leq M(E_0+1).\end{aligned} \tag{4.3.8}$$

Moreover, for any $t\in[0,T]$, $|\{x:\rho(t,x)>\frac{3}{2}\breve\rho\}|\leq c_2E_0$, where $\breve\rho = \max\{\rho_-,\rho_+\}$ and $c_2>0$ may depend on T.

Before we prove Proposition 4.3.2, we note that there exists a constant $c_1>0$, independent of ε, depending only on $\bar\rho$ and γ such that $\overline{h_\delta}(\rho,\bar\rho)\geq c_1\rho(\rho^\theta-\bar\rho^\theta)^2$.

Proof. Multiplying the first equation in (4.3.1) by $\big(\overline{\eta_\delta^*}\big)_\rho A(x)$, the second equation by $\big(\overline{\eta_\delta^*}\big)_m A(x)$, and then adding them together, we have, after a short calculation,

$$
\begin{aligned}
&\big(\overline{\eta_\delta^*}A(x)\big)_t+\Big(\big(q_\delta^*-m(\eta_\delta^*)_\rho(\bar\rho,\bar m)-(\tfrac{m^2}{\rho}+p_\delta(\rho))(\eta_\delta^*)_m(\bar\rho,\bar m)\big)A(x)\Big)_x\\
&\quad+\big((\eta_\delta^*)_\rho(\bar\rho,\bar m)\big)_x mA(x)+\big((\eta_\delta^*)_m(\bar\rho,\bar m)\big)_x\big(\tfrac{m^2}{\rho}+p_\delta(\rho)\big)A(x)+p_\delta(\rho)(\eta_\delta^*)_m(\bar\rho,\bar m)A'(x)\\
&=\varepsilon(A(x)\rho_x)_x(\overline{\eta_\delta^*})_\rho+\varepsilon A(x)\big(A(x)^{-1}(A(x)m)_x\big)_x(\overline{\eta_\delta^*})_m.
\end{aligned}
\tag{4.3.9}
$$

Observe that, at the end-points a and b,

$$
q_\delta^*-m(\eta_\delta^*)_\rho(\bar\rho,\bar m)-\big(\frac{m^2}{\rho}+p_\delta(\rho)\big)(\eta_\delta^*)_m(\bar\rho,\bar m)=-\frac{m_\pm}{\rho_\pm}p_\delta(\rho_\pm).
$$

Then, by integrating (4.3.9) over (a,b),

$$
\begin{aligned}
&\frac{dE}{dt}-\frac{m}{\rho}p_\delta(\rho)A\Big|_a^b+\int_a^b\Big(m\big((\eta_\delta^*)_\rho(\bar\rho,\bar m)\big)_x+\big((\eta_\delta^*)_m(\bar\rho,\bar m)\big)_x\big(\tfrac{m^2}{\rho}+p_\delta(\rho)\big)\\
&\qquad\qquad+\frac{A'(x)}{A(x)}(\eta_\delta^*)_m(\bar\rho,\bar m)p_\delta(\rho)-\varepsilon\big(\frac{A'(x)}{A(x)}\big)'m\big((\eta_\delta^*)_m-(\eta_\delta^*)_m(\bar\rho,\bar m)\big)\Big)A(x)\,dx\\
&\quad=-\varepsilon\int_a^b(\rho_x,m_x)\nabla^2\overline{\eta_\delta^*}\,(\rho_x,m_x)^\top A(x)\,dx\\
&\qquad+\varepsilon\int_a^b\big(\rho_x((\eta_\delta^*)_\rho(\bar\rho,\bar m))_x+m_x((\eta_\delta^*)_m(\bar\rho,\bar m))_x\big)A(x)\,dx.
\end{aligned}
$$

Note that $\rho+p_\delta(\rho)\le M\big(\overline{h_\delta}(\rho,\bar\rho)+1\big)$, $\rho u^2\le M(\rho|u-\bar u|^2+\rho+1)$, and $\|A'\|_{L^1(-\infty,L_0)}\le M$ and that $(\bar\rho_x,\bar m_x)$ are supported in $(-L_0,L_0)$. Moreover, for $x>L_0$, as we have used the Galilean invariance to impose that $u_+=0$ without loss of generality, $\bar m=0$ and $(\eta_\delta^*)_m(\bar\rho,\bar m)=0$ for $x>L_0$. Therefore, we have

$$
\begin{aligned}
&\int_a^b\Big(m\big((\eta_\delta^*)_\rho(\bar\rho,\bar m)\big)_x+\big((\eta_\delta^*)_m(\bar\rho,\bar m)\big)_x\big(\tfrac{m^2}{\rho}+p_\delta(\rho)\big)+\frac{A'(x)}{A(x)}(\eta_\delta^*)_m(\bar\rho,\bar m)p_\delta(\rho)\Big)A(x)dx\\
&\le M(E+1).
\end{aligned}
$$

Moreover, integrating by parts gives

$$
\begin{aligned}
&\varepsilon\int_a^b\big(\rho_x((\eta_\delta^*)_\rho(\bar\rho,\bar m))_x+m_x((\eta_\delta^*)_m(\bar\rho,\bar m))_x\big)A(x)\,dx\\
&\quad=-\varepsilon\int_a^b\big(\rho((\eta_\delta^*)_\rho(\bar\rho,\bar m))_{xx}+m((\eta_\delta^*)_m(\bar\rho,\bar m))_{xx}\big)A(x)\,dx\\
&\qquad-\varepsilon\int_a^b\big(\rho((\eta_\delta^*)_\rho(\bar\rho,\bar m))_x+m((\eta_\delta^*)_m(\bar\rho,\bar m))_x\big)A'(x)\,dx\\
&\quad\le M\varepsilon\big(E+L_0+\|A'\|_{L^1(-L_0,L_0)}\big),
\end{aligned}
$$

where we have used again that $\rho+\rho u\le M\big(\rho|u-\bar u|^2+\overline{h_\delta}(\rho,\bar\rho)+1\big)$ and that $\bar\rho$ and $\bar m$ are bounded with bounded and compactly supported derivatives. Finally, we have

$$
\int_a^b\varepsilon\Big|\big(\frac{A'}{A}\big)'m\big((\eta_\delta^*)_m-(\eta_\delta^*)_m(\bar\rho,\bar m)\big)\Big|A(x)\,dx\le\varepsilon ME+\varepsilon M|b-a|\big(1+\big|\big(\frac{A'}{A}\big)'\big|\big).
$$

Since $(\rho_x, m_x)\nabla^2\overline{\eta^*_\delta}(\rho_x, m_x)^\top$ dominates $h''_\delta(\rho)|\rho_x|^2 + \rho|u_x|^2$ as observed in (4.2.4), we may combine all of these inequalities to obtain

$$\frac{dE}{dt} + \varepsilon \int_a^b \big((2\delta + \kappa\gamma\rho^{\gamma-2})|\rho_x|^2 + \rho|u_x|^2\big)A(x)\,dx \le M(E+1).$$

Hence, by the Gronwall inequality, we have

$$E(T) + \varepsilon \int_{Q_T} \big((2\delta + \kappa\gamma\rho^{\gamma-2})|\rho_x|^2 + \rho|u_x|^2\big)A(x)\,dx \le M(T)(E_0+1). \tag{4.3.10}$$

In particular, we obtain

$$\sup_{t\in[0,T]} \int_a^b \big(\rho(u-\bar{u})^2 + \overline{h_\delta}(\rho,\bar{\rho})\big)A(x)\,dx \le M(E_0+1). \tag{4.3.11}$$

Since $A(x) \ge A_0 > 0$, and $\overline{h_\delta}(\rho,\bar{\rho}) \ge 0$ is quadratic in $(\rho - \bar{\rho})$ for ρ near $\bar{\rho}$ and grows as $\rho^{\max\{\gamma,2\}}$ for large ρ, we conclude

$$|\{\rho(t,\cdot) > \frac{3}{2}\check{\rho}\}| \le c_2 E_0,$$

where $\check{\rho} = \max\{\rho_-, \rho_+\}$. □

The following lemma is a simple consequence of the fundamental theorem of calculus and the main energy estimate, Proposition 4.3.2; compare [13, Lemma 2.1].

Lemma 4.3.3. *There exists $C = C(\varepsilon, T, E_0) > 0$ such that*

$$\int_0^T \|\rho(t,\cdot)\|^{2\max\{2,\gamma\}}_{L^\infty(a,b)}dt \le C.$$

Proof. If $\big|\{\rho(t,\cdot) > \frac{3}{2}\check{\rho}\}\big| = 0$, then we are done. If not, then for $x \in (a,b)$, let x_0 be the point of (a,b) closest to x such that $\rho(t,x_0) = \frac{3}{2}\check{\rho}$. Clearly $|x - x_0| \le C(E_0, c_2)$. Then

$$\begin{aligned}
|\rho^\gamma(t,x) - \rho^\gamma(t,x_0)| &\le \gamma \left| \int_{x_0}^x \rho^{\gamma-1}(t,y)\rho_y(t,y)\,dy \right| \\
&\le C \left| \int_{x_0}^x \rho^\gamma(t,y)A(y)\,dy \right|^{\frac{1}{2}} \left| \int_a^b \rho^{\gamma-2}(t,y)|\rho_y(t,y)|^2 A(y)\,dy \right|^{\frac{1}{2}} \\
&\le \big(C(E_0) + C|x-x_0|\big) \left| \int_a^b \rho^{\gamma-2}(t,y)|\rho_y(t,y)|^2 A(y)\,dy \right|^{\frac{1}{2}}.
\end{aligned}$$

Hence, by (4.3.10),

$$\int_0^T \|\rho(t,\cdot)\|^{2\gamma}_{L^\infty(a,b)} \le C.$$

Repeating this argument with ρ^2 instead of ρ^γ, we conclude. □

From now on, the constants labelled C may depend on ε, whereas the constants labelled M are independent of ε.

The next aim is to demonstrate upper bounds for the density and velocity. Our strategy for doing this is to exploit the parabolic structure of the problem to obtain a maximum principle. We make use of the fact that the Riemann invariants for the system satisfy heat equations of their own with a much better structure. The following maximum principle result for the Riemann invariants then gives *a priori* control on $\|(\rho, u)\|_{L^\infty}$ in terms of the Dirichlet boundary data and initial data.

Lemma 4.3.4. *There exists $C = C(\varepsilon, T, E_0)$ such that, for any $t \in [0, T]$,*

$$\|(\rho, u)\|_{L^\infty(Q_t)} \le C\big(\|u_0 + R(\rho_0)\|_{L^\infty(a,b)} + \|u_0 - R(\rho_0)\|_{L^\infty(a,b)}\big) + C(\rho_\pm, u_\pm),$$

where

$$R(\rho) := \int_0^\rho \frac{\sqrt{p_\delta'(s)}}{s}\, ds.$$

Proof. We recall that the eigenvalues of the Euler equations with pressure function $p_\delta(\rho)$ are

$$\lambda_1 := u - \sqrt{p_\delta'(\rho)}, \qquad \lambda_2 := u + \sqrt{p_\delta'(\rho)},$$

and the corresponding Riemann invariants are

$$w := u + R(\rho), \qquad z := u - R(\rho).$$

We first note that these Riemann invariants are quasiconvex:

$$\nabla^\perp w \nabla^2 w (\nabla^\perp w)^\top \ge 0, \qquad -\nabla^\perp z \nabla^2 z (\nabla^\perp z)^\top \ge 0.$$

Following [13, Lemma 2.2], we multiply the first equation in (4.3.1) by w_ρ, multiply the second by w_m, and then add them together to obtain, after a short calculation,

$$w_t + \big(\lambda_2 - \varepsilon\frac{A'}{A}\big)w_x - \varepsilon w_{xx} = -\varepsilon(\rho_x, m_x)\nabla^2 w(\rho_x, m_x)^\top - u\sqrt{p_\delta'}\frac{A'}{A} + \varepsilon\big(\frac{A'}{A}\big)'u. \tag{4.3.12}$$

We exploit the quasi-convexity property of the Riemann invariants to derive a parabolic inequality for w allowing the application of a maximum principle by decomposing

$$(\rho_x, m_x) = \alpha\nabla w + \beta\nabla^\perp w,$$

where

$$\alpha = \frac{w_x}{|\nabla w|^2}, \qquad \beta = \frac{\rho_x w_m - m_x w_\rho}{|\nabla w|^2}.$$

With this notation, we re-write (4.3.12) above as

$$w_t + \lambda w_x - \varepsilon w_{xx} = -\varepsilon\beta^2 \nabla^\perp w \nabla^2 w (\nabla^\perp w)^\top - u\sqrt{p_\delta'}\frac{A'}{A} + \varepsilon(\frac{A'}{A})' u,$$

where

$$\lambda := \lambda_2 - \varepsilon\frac{A'}{A} + \varepsilon\alpha\frac{\nabla w \nabla^2 w (\nabla w)^\top}{|\nabla w|^2} + 2\varepsilon\beta\frac{\nabla^\perp w \nabla^2 w(\nabla w)^\top}{|\nabla w|^2}.$$

Then, using the quasiconvexity property of w, we find that

$$\tilde{w}_t + \lambda\tilde{w}_x - \varepsilon\tilde{w}_{xx} \leq 0$$

for $\tilde{w}$ defined by

$$\tilde{w}(t,x) := w(t,x) - \int_0^t \|u\sqrt{p_\delta'}\frac{A'}{A} - \varepsilon(\frac{A'}{A})' u\|_{L^\infty(a,b)}\, d\tau.$$

The maximum principle for parabolic equations gives us

$$\max_{Q_t} \tilde{w} \leq \max\{w_0, \max_{[0,t]\times\{a,b\}} \tilde{w}\}.$$

Now we have

$$\begin{aligned}\max_{[0,t]\times\{a,b\}} \tilde{w} \leq{}& \max_{(a,b)} w_0 \\ &+ \check{\rho}\, C\int_0^t \big(1 + \|\rho(\tau,\cdot)\|_{L^\infty(a,b)}^{\frac{1}{2}\max\{1,\gamma-1\}}\big)\|u(\tau,\cdot)\|_{L^\infty(a,b)}\, d\tau,\end{aligned}$$

where C depends on $\sup_{(a,b)}\big\{\big|\frac{A'}{A}\big|, \big|\big(\frac{A'}{A}\big)'\big|\big\}$. Thus, we obtain

$$\begin{aligned}\max_{Q_t} w \leq{}& \max_{(a,b)} w_0 + \max_{\pm} w(\rho_\pm, u_\pm) \\ &+ C\int_0^t \big(1 + \|\rho(\tau,\cdot)\|_{L^\infty(a,b)}^{\frac{1}{2}\max\{1,\gamma-1\}}\big)\|u(\tau,\cdot)\|_{L^\infty(a,b)}\, d\tau.\end{aligned}$$

Similarly,

$$\begin{aligned}\max_{Q_t}(-z) \leq{}& \max_{(a,b)}(-z_0) + \max_{\pm} z(\rho_\pm, u_\pm) \\ &+ C\int_0^t \big(1 + \|\rho(\tau,\cdot)\|_{L^\infty(a,b)}^{\frac{1}{2}\max\{1,\gamma-1\}}\big)\|u(\tau,\cdot)\|_{L^\infty(a,b)}\, d\tau.\end{aligned}$$

Noting that w_0 satisfies the corresponding boundary condition to (4.3.5) and that $\rho \geq 0$, we see that

$$\max_{Q_t} |u| \leq \max_{(a,b)} |w_0| + \max_{(a,b)} |z_0| + C\int_0^t \big(1 + \|\rho(\tau,\cdot)\|_{L^\infty(a,b)}^{\frac{1}{2}\max\{1,\gamma-1\}}\big)\|u(\tau,\cdot)\|_{L^\infty(a,b)}\, d\tau.$$

Since $\max\{1,\gamma-1\}<2\gamma$, we obtain that, by the Hölder inequality and Lemma 4.3.3,

$$
\begin{aligned}
\max_{Q_t}|u|^2 \le & C\big(\max_{(a,b)}|w_0|+\max_{(a,b)}|z_0|\big)^2 \\
& + C\Big(\int_0^t \big(1+\|\rho(\tau,\cdot)\|_{L^\infty(a,b)}^{\max\{1,\gamma-1\}}\big)\,d\tau\Big)\Big(\int_0^t \|u(\tau,\cdot)\|_{L^\infty(a,b)}^2\,d\tau\Big) \\
\le & C\big(\max_{(a,b)}|w_0|+\max_{(a,b)}|z_0|\big)^2 + C\int_0^t \|u(\tau,\cdot)\|_{L^\infty(a,b)}^2\,d\tau.
\end{aligned}
$$

We then conclude the estimate for $\|u\|_{L^\infty(Q_t)}$ by using the Gronwall inequality.

The estimate for ρ is now a direct consequence of the estimates of the Riemann invariants and the estimate of u above. In particular, we have

$$
\max_{Q_t} R(\rho) = \tfrac{1}{2}\max_{Q_t}(w-z) \le C + C\int_0^t \big(1+\|\rho(\tau,\cdot)\|_{L^\infty(a,b)}^{\frac{1}{2}\max\{1,\gamma-1\}}\big)\,d\tau \le C.
$$

□

This maximum principle argument gives us the first bounds that we require to apply the standard parabolic theory and deduce the existence of the approximate solutions. In order to obtain the remaining bound, that is the lower bound on ρ, we first prove higher order energy estimates.

Lemma 4.3.5. *There exists $C=C(\varepsilon,\|(\rho_0,u_0)\|_{L^\infty(a,b)},\|(\rho_0,m_0)\|_{H^1(a,b)},T,\gamma)>0$ such that*

$$
\sup_{t\in[0,T]}\int_a^b \big(|\rho_x|^2+|m_x|^2\big)dx + \int_{Q_T}\big(|\rho_{xx}|^2+|m_{xx}|^2\big)\,dx\,dt \le C.
$$

Proof. We make frequent use of the upper bounds on (ρ,u) from Lemma 4.3.4.

Multiplying the first equation in (4.3.1) by ρ_{xx} and the second by m_{xx}, we obtain

$$
\begin{aligned}
&(\rho_t\rho_x)_x - \frac{1}{2}\big(|\rho_x|^2\big)_t + (m_t m_x)_x - \frac{1}{2}\big(|m_x|^2\big)_t - \varepsilon\big(|\rho_{xx}|^2+|m_{xx}|^2\big) \\
&= -m_x\rho_{xx} - (\rho u^2+p_\delta)_x m_{xx} - \frac{A'}{A}m\rho_{xx} - \frac{A'}{A}\big(\rho u^2 m_{xx} - \varepsilon\rho_x\rho_{xx}\big) + \varepsilon\big(\frac{A'}{A}m\big)_x m_{xx}.
\end{aligned}
$$

Integrating over Q_T and recalling that ρ and m are constant on $x=a$ and b, we have

$$
\begin{aligned}
&\frac{1}{2}\int_a^b \big(|\rho_x|^2+|m_x|^2\big)\Big|_0^T\,dx + \varepsilon\int_{Q_T}\big(|\rho_{xx}|^2+|m_{xx}|^2\big)\,dx\,dt \\
&= \int_{Q_T}\big(m_x\rho_{xx}+\frac{A'}{A}m\rho_{xx}\big)\,dx\,dt + \int_{Q_T}(\rho u^2+p_\delta)_x m_{xx}\,dx\,dt \\
&\quad + \int_{Q_T}\frac{A'}{A}\big(\rho u^2 m_{xx} - \varepsilon\rho_x\rho_{xx}\big)\,dx\,dt - \varepsilon\int_{Q_T}\big(\frac{A'}{A}m\big)_x m_{xx}\,dx\,dt.
\end{aligned}
$$

Thus, using the upper bounds for ρ and u in L^∞ from the maximum principle, we have

$$\begin{aligned}
&\int_a^b \big(|\rho_x(T,x)|^2 + |m_x(T,x)|^2\big)\,dx + \varepsilon\int_{Q_T}\big(|\rho_{xx}|^2 + |m_{xx}|^2\big)\,dx\,dt\\
&\le \frac{\varepsilon}{2}\int_{Q_T}\big(|\rho_{xx}|^2 + |m_{xx}|^2\big)\,dx\,dt + \int_a^b \big(|\rho_{0,x}(x)|^2 + |m_{0,x}(x)|^2\big)\,dx\\
&\quad + C\int_{Q_T}\big(|\rho_x|^2 + |m_x|^2\big)\,dx\,dt + C.
\end{aligned}$$

Absorbing the first term into the left, we conclude the expected estimate. $\square$

It now remains only to demonstrate an *a priori* upper bound for ρ^{-1}. To this end, we define

$$\phi(\rho) = \begin{cases} \frac{1}{\rho} - \frac{1}{\tilde\rho} + \frac{\rho-\tilde\rho}{\tilde\rho^2}, & \rho \le \tilde\rho,\\ 0, & \rho > \tilde\rho, \end{cases}$$

for some $\tilde\rho > 0$. Note that

$$\phi'(\rho) = \big(-\frac{1}{\rho^2} + \frac{1}{\tilde\rho^2}\big)\mathbb{1}_{\{\rho<\tilde\rho\}}, \qquad \phi''(\rho) = \frac{2}{\rho^3}\mathbb{1}_{\{\rho<\tilde\rho\}}.$$

Lemma 4.3.6. *There exists $C > 0$ depending on $\|\phi(\rho_0)\|_{L^1(a,b)}$ and the other parameters of the problem such that*

$$\sup_{t\in[0,T]}\int_a^b \phi(\rho(t,\cdot))\,dx + \int_{Q_T}\frac{|\rho_x|^2}{\rho^3}\,dx\,dt \le C.$$

Proof. Multiplying the first equation of (4.3.1) by $\phi'(\rho)$, we have

$$\begin{aligned}
&\phi_t + (u\phi)_x - \varepsilon\phi_{xx} + 2\varepsilon\frac{\rho_x^2}{\rho^3}\mathbb{1}_{\{\rho<\tilde\rho\}}\\
&= 2\big(\frac{1}{\rho} - \frac{1}{\tilde\rho}\big)u_x\mathbb{1}_{\{\rho<\tilde\rho\}} + \frac{A'}{A}\rho u\big(\frac{1}{\rho^2} - \frac{1}{\tilde\rho^2}\big)\mathbb{1}_{\{\rho<\tilde\rho\}} + \varepsilon\frac{A'}{A}\rho_x\big(\frac{1}{\tilde\rho^2} - \frac{1}{\rho^2}\big)\mathbb{1}_{\{\rho<\tilde\rho\}}.
\end{aligned}$$

Integrating in (t,x) and using the boundary conditions with $\tilde\rho < \min_{x\in(a,b)}\rho_0(x)$ chosen, we obtain

$$\begin{aligned}
&\int_a^b \phi(\rho)\,dx + \varepsilon\int_{Q_T\cap\{\rho<\tilde\rho\}}\frac{|\rho_x|^2}{\rho^3}\,dx\,dt\\
&\le \Big|\int_{Q_T\cap\{\rho<\tilde\rho\}} 2\big(\frac{1}{\rho} - \frac{1}{\tilde\rho}\big)u_x\,dx\,dt\Big| + \Big|\int_{Q_T\cap\{\rho<\tilde\rho\}}\frac{A'}{A}\rho u\big(\frac{1}{\rho^2} - \frac{1}{\tilde\rho^2}\big)\,dx\,dt\Big|\\
&\quad + \Big|\int_{Q_T\cap\{\rho<\tilde\rho\}}\varepsilon\frac{A'}{A}\rho_x\big(\frac{1}{\tilde\rho^2} - \frac{1}{\rho^2}\big)\,dx\,dt\Big|\\
&\le \frac{\varepsilon}{2}\int_{Q_T\cap\{\rho<\tilde\rho\}}\frac{|\rho_x|^2}{\rho^3}\,dx\,dt + C\Big(1 + \int_{Q_T}\phi(\rho)\,dx\,dt\Big)
\end{aligned}$$

and conclude the estimate by the Gronwall inequality, where we have integrated by parts and applied the Young inequality in the first term after the first inequality above. $\square$

For $x \in (a,b)$, we take x_0 to be the closest point to x such that $\rho(t,x_0) = \tilde{\rho}$, where $\tilde{\rho} = \min_{x\in(a,b)} \rho_0(x)$; otherwise, we already have an upper bound on ρ^{-1}. By writing

$$|\rho(t,x)^{-1}| \le \Big|\int_{x_0}^{x} \frac{\rho_y(t,y)}{\rho^2(t,y)}\,dy\Big| + |(\tilde{\rho})^{-1}|,$$

we obtain

$$\begin{aligned}\int_0^T \|\rho(t,\cdot)^{-1}\|_{L^\infty(a,b)}\,dt \le& C\Big(1+\Big(\int_{Q_T}\frac{|\rho_x|^2}{\rho^3}\,dx\,dt\Big)^{\frac{1}{2}}\Big(\Big(\int_{Q_T}\phi(\rho)\,dx\,dt\Big)^{\frac{1}{2}}+1\Big)\Big)\\ \le& C\Big(1+\Big(\int_{Q_T}\frac{|\rho_x|^2}{\rho^3}\,dx\,dt\Big)^{\frac{1}{2}}\Big). \qquad (4.3.13)\end{aligned}$$

Finally, we use the previous lemmas to show the lower bound on ρ.

Lemma 4.3.7. *There exists $C > 0$, depending on ε, T, and the other parameters of the problem, such that*

$$C^{-1} \le \rho(t,x) \le C, \qquad \int_0^T \big\|\big(\frac{m_x}{\rho}, \frac{\rho_x}{\rho}, u_x\big)(t,\cdot)\big\|_{L^\infty(a,b)}\,dt \le C.$$

Proof. By the Sobolev embedding and (4.3.13),

$$\begin{aligned}&\int_0^T \Big\|\frac{m_x(t,\cdot)}{\rho(t,\cdot)}\Big\|_{L^\infty(a,b)}\,dt\\ &\le \int_0^T \|m_x(t,\cdot)\|_{L^\infty(a,b)}\|\rho^{-1}(t,\cdot)\|_{L^\infty(a,b)}\,dt\\ &\le C\int_0^T \Big(\int_a^b |m_{xx}|^2\,dx\Big)^{\frac{1}{2}}\Big(1+\Big(\int_a^b \frac{|\rho_x|^2}{\rho^3}\,dx\Big)^{\frac{1}{2}}\Big(\int_a^b \phi(\rho)\,dx\Big)^{\frac{1}{2}}\Big)\,dt\\ &\le C.\end{aligned}$$

The estimate for $\frac{\rho_x}{\rho}$ is obtained in the same way.

For u_x, note first that $u_x = \frac{m_x}{\rho} - \frac{u\rho_x}{\rho}$ and argue similarly to the above. Finally, let $v = \frac{1}{\rho}$. Then, by (4.3.1), we calculate

$$v_t + \big(u - \varepsilon\frac{A'}{A}\big)v_x - \varepsilon v_{xx} = \frac{\rho u_x}{\rho^2} - \frac{2\varepsilon\rho_x^2}{\rho^3} + \frac{A'}{A}\frac{u}{\rho} \le \big(u_x + \frac{A'}{A}u\big)v.$$

By the maximum principle,

$$\max_{Q_T} v \le C\max\big\{\|v_0\|_{L^\infty}, \rho_-^{-1}, \rho_+^{-1}\big\}e^{\int_0^T \|(u_x,u)(\tau,\cdot)\|_{L^\infty(a,b)}d\tau} \le C\|v_0\|_{L^\infty(a,b)}.$$

$\square$

This completes the proof of Theorem 4.3.1, establishing the existence of the approximate solutions.

4.4 Uniform estimates for the approximate solutions

As mentioned earlier, the purpose of this section and §4.5–§4.6 is to establish the proof of Theorem 4.2.3 by making uniform estimates on the approximate solutions, independent of ε. These uniform estimates allow us to pass to a Young measure limit and apply the compensated compactness and reduction framework of §3.5 to deduce strong convergence of the approximate solutions to a limit solving system (4.1.1). The crucial higher integrability bound of Lemma 4.4.5 is sufficient to pass to the limit in the entropy equation for all entropies generated by test functions of sub-quadratic growth. Throughout this section, the universal constant $M > 0$ is independent of ε, and we assume that the area function $A(x) \in C^2$ satisfies (4.1.3) and (4.1.4).

The first uniform estimate concerns the local higher integrability of the density.

Lemma 4.4.1. *Let $K \subset \mathbb{R}$ be a compact set such that $K \subset (a, b)$. Then, for $T > 0$, there exists a constant $M = M(K, T)$, independent of ε, such that*

$$\int_0^T \int_K \left(\rho^{\gamma+1} + \delta\rho^3\right) dx\, dt \le M. \tag{4.4.1}$$

The proof of this lemma is similar to [13, Lemma 3.3], compare also [12, Lemma 3.3].

Proof. Let $\omega(x) \ge 0$ be a smooth, compactly supported function on (a, b). Multiplying the approximate momentum equation in (4.3.1) by $\omega(x)$, we observe

$$\begin{aligned}&(\rho u\omega)_t + ((\rho u^2 + p_\delta)\omega)_x + \frac{A'}{A}\rho u^2\omega - \varepsilon\big(\omega\big(m_x + \frac{A'}{A}m\big)\big)_x\\ &\qquad = \big(\rho u^2 + p_\delta - \varepsilon\big(m_x + \frac{A'}{A}m\big)\big)\omega_x.\end{aligned}$$

Hence,

$$\Big(\int_x^b \rho u\omega\, dy\Big)_t + \int_x^b \frac{A'}{A}\rho u^2\omega\, dy + \varepsilon\omega\big(m_x + \frac{A'}{A}m\big) = (\rho u^2 + p_\delta)\omega + f_1,$$

where

$$f_1 = \int_x^b \big(\rho u^2 + p_\delta - \varepsilon\big(m_y + \frac{A'}{A}m\big)\big)\omega_y\, dy.$$

Multiplying by ρ and using the approximate continuity equation in (4.3.1), we obtain

$$\begin{aligned}&\Big(\rho\int_x^b \rho u\omega\, dy\Big)_t + \big((\rho u)_x + \frac{A'}{A}\rho u - \varepsilon\big(\rho_{xx} + \frac{A'}{A}\rho_x\big)\big)\int_x^b \rho u\omega\, dy\\ &\quad + \rho\int_x^b \frac{A'}{A}\rho u^2\omega\, dy + \varepsilon\rho\omega\big(m_x + \frac{A'}{A}m\big)\\ &= (\rho^2u^2 + p_\delta\rho)\omega + \rho f_1,\end{aligned}$$

so we have that

$$\Big(\rho\int_x^b \rho u\omega\,dy\Big)_t+\Big(\rho u\int_x^b \rho u\omega\,dy\Big)_x+\varepsilon\Big(-(\rho_{xx}+\frac{A'}{A}\rho_x)\int_x^b \rho u\omega\,dy\Big)$$
$$+\varepsilon\rho\omega\big(m_x+\frac{A'}{A}m\big)=\rho p_\delta\omega+f_2,$$

where we set

$$f_2=\rho f_1-\frac{A'}{A}\rho u\int_x^b \rho u\omega\,dy-\rho\int_x^b \frac{A'}{A}\rho u^2\omega\,dy.$$

We note that

$$\begin{aligned}
&-(\rho_{xx}+\frac{A'}{A}\rho_x)\int_x^b \rho u\omega\,dy+\rho\omega(m_x+\frac{A'}{A}m)\\
&=-\Big(\rho_x\int_x^b \rho u\omega\,dy\Big)_x-\rho_x\rho u\omega-\Big(\frac{A'}{A}\rho\int_x^b \rho u\omega\,dy\Big)_x\\
&\quad-\frac{A'}{A}\rho^2u\omega+(\frac{A'}{A})'\rho\int_x^b \rho u\omega\,dy+\rho^2u_x\omega+\rho u\rho_x\omega+\rho^2u\frac{A'}{A}\omega\\
&=-\Big(\rho\int_x^b \rho u\omega\,dy\Big)_{xx}-(\rho^2u\omega)_x-\Big(\frac{A'}{A}\rho\int_x^b \rho u\omega\,dy\Big)_x\\
&\quad+(\frac{A'}{A})'\rho\int_x^b \rho u\omega\,dy+\rho^2u_x\omega.
\end{aligned}$$

Thus we obtain

$$\Big(\rho\int_x^b \rho u\omega\,dy\Big)_t+\Big(\rho u\int_x^b \rho u\omega\,dy\Big)_x-\varepsilon\Big(\rho\int_x^b \rho u\omega\,dy\Big)_{xx}-\varepsilon(\rho^2u\omega)_x$$
$$-\varepsilon\Big(\frac{A'}{A}\rho\int_x^b \rho u\omega\,dy\Big)_x+\varepsilon\rho^2u_x\omega=p_\delta\rho\omega+f_3,$$

where

$$f_3=f_2-\varepsilon(\frac{A'}{A})'\rho\int_x^b \rho u\omega\,dy.$$

Multiplying again by $\omega(x)$, we obtain

$$\Big(\rho\omega\int_x^b \rho u\omega\,dy\Big)_t+\Big(\rho u\omega\int_x^b \rho u\omega\,dy\Big)_x-\varepsilon\Big(\omega\big(\rho\int_x^b \rho u\omega\,dy\big)_x\Big)_x$$
$$+\varepsilon\Big(\rho\omega_x\int_x^b \rho u\omega\,dy\Big)_x-\varepsilon(\rho^2u\omega^2)_x+\varepsilon\rho^2u\omega\omega_x$$
$$-\varepsilon\Big(\frac{A'}{A}\rho\omega\int_x^b \rho u\omega\,dy\Big)_x+\varepsilon\rho^2u_x\omega^2=p_\delta\rho\omega^2+f_4,$$

where

$$f_4=\omega f_3+\rho u\omega_x\int_x^b \rho u\omega\,dy-\frac{A'}{A}\rho\omega_x\int_x^b \rho u\omega\,dy+\varepsilon\omega_{xx}\rho\int_x^b \rho u\omega\,dy.$$

Integrating over Q_T, we now find

$$
\begin{aligned}
&\int_{Q_T} (\delta\rho^3 + \kappa\rho^{\gamma+1})\omega^2\, dx\, dt \\
&\quad = \varepsilon \int_{Q_T} \rho^2 u\omega\omega_x + \rho^2 u_x \omega^2\, dx\, dt + \int_a^b \Big(\rho\omega \int_x^b \rho u\omega\, dy\Big)\Big|_0^T dx - \int_{Q_T} f_4\, dx\, dt \\
&\quad \le \frac{\varepsilon}{2} \int_{Q_T} \rho^3\omega^2\, dx\, dt + \varepsilon M \int_{Q_T} (\rho|u_x|^2\omega^2 + \rho u^2|\omega_x|^2)\, dx\, dt \\
&\qquad + \int_a^b \Big(\rho\omega \int_x^b \rho u\omega\, dy\Big)\Big|_0^T dx - \int_{Q_T} f_4\, dx\, dt.
\end{aligned}
$$

Now we may bound

$$
\begin{aligned}
\Big|\int_x^b \rho u\omega\, dy\Big| &\le \int_{\operatorname{supp}\omega} (\rho + \rho u^2) A(x)\, dx \\
&\le M(\operatorname{supp}\omega, E_0),
\end{aligned}
$$

by Proposition 4.3.2, and so

$$
\Big|\int_a^b \Big(\rho\omega \int_x^b \rho u\omega\, dy\Big)\Big|_0^T dx\Big| \le M.
$$

Similarly,

$$
\Big|\int_{Q_T} \rho u\omega_x \int_x^b \rho u\omega\, dy\Big| \le M(\operatorname{supp}\omega, T).
$$

Also, we make the following estimates using Proposition 4.3.2:

$$
\begin{aligned}
\Big|\int_{Q_T} \varepsilon\big(\frac{A'}{A}\big)'\rho\omega \int_x^b \rho u\omega\, dy\, dx\, dt\Big| &\le M(\operatorname{supp}\omega), \\
\Big|\int_{Q_T} \frac{A'}{A}\rho u\omega \int_x^b \rho u\omega\, dy\, dx\, dt\Big| &\le M(\operatorname{supp}\omega), \\
\Big|\int_{Q_T} \rho\omega \int_x^b \frac{A'}{A}\rho u^2\omega\, dy\, dx\, dt\Big| &\le M,
\end{aligned}
$$

$$
\Big|\int_{Q_T} \rho\omega \int_x^b \big(\rho u^2 + p_\delta - \varepsilon\big(m_y + \frac{A'}{A}m\big)\big)\omega_y\, dy\, dx\, dt\Big| \le M + \int_{Q_T} \rho\omega\Big|\int_x^b \varepsilon m_y\omega_y\, dy\Big| dx\, dt,
$$

and

$$
\int_{Q_T} \rho\omega\Big|\int_x^b \varepsilon m_y\omega_y\, dy\Big| dx\, dt \le M + \varepsilon \int_{Q_T} \rho^2|u|\omega|\omega_x|\, dx\, dt \le M_\Delta + \Delta\varepsilon \int_{Q_T} \rho^3\omega^2\, dx\, dt
$$

by integrating by parts, where $\Delta > 0$ is to be chosen small.

Thus, combining the above estimates, we find

$$
\int_{Q_T} (\delta\rho^3 + \kappa\rho^{\gamma+1})\omega^2\, dx\, dt \le \Delta\varepsilon \int_{Q_T} \rho^3\omega^2\, dx\, dt + M. \tag{4.4.2}
$$

Claim. There exists $M = M(\operatorname{supp}\omega, T, E_0)$ such that

$$\varepsilon \int_{Q_T} \rho^3 \omega^2 \, dx \, dt \leq M + M\varepsilon \int_{Q_T} \rho^{\gamma+1} \omega^2 \, dx \, dt.$$

If $\gamma \geq 2$, the claim is obvious. Let $\gamma < \beta \leq 3$. Then

$$\begin{aligned}
&\varepsilon \int_{Q_T} \rho^\beta \omega^2 \, dx \, dt \\
&\quad \leq \varepsilon \int_0^T \sup_{\operatorname{supp}\omega} \big(\rho^{\beta-\gamma}\omega^2\big)\Big|_t \Big(\int_{\operatorname{supp}\omega} \rho^\gamma \, dx \Big) \, dt \\
&\quad \leq \varepsilon M \int_0^T \int_{\operatorname{supp}\omega} \big(\big|(\rho^{\beta-\gamma})_x \omega^2\big| + \rho^{\beta-\gamma}\omega|\omega_x| \big) \, dx \, dt \\
&\quad = \varepsilon M \int_{Q_T \cap \operatorname{supp}\omega} \big(\rho^{\beta-\gamma-\frac{\gamma}{2}} \big|(\rho^{\frac{\gamma}{2}})_x\big| \omega^2 + \rho^{\beta-\gamma}\omega|\omega_x| \big) \, dx \, dt \\
&\quad \leq \varepsilon M \Big(\int_{Q_T} \big|(\rho^{\frac{\gamma}{2}})_x\big|^2 \omega^2 \, dx \, dt + \int_{Q_T} \rho^{2\beta-3\gamma}\omega^2 \, dx \, dt \Big) \\
&\qquad + \varepsilon M \int_{Q_T} \rho^{\beta-\gamma}\omega|\omega_x| \, dx \, dt \\
&\quad \leq \varepsilon M \Big(\int_{Q_T} \rho^{\gamma-2}|\rho_x|^2 \omega^2 \, dx \, dt + \int_{Q_T} \rho^{2\beta-3\gamma}\omega^2 \, dx \, dt \\
&\qquad\qquad + \int_{Q_T} \rho^{2\beta-3\gamma}\omega^2 \, dx \, dt + \int_{Q_T} \rho^\gamma \omega_x^2 \, dx \, dt \Big).
\end{aligned}$$

Thus

$$\varepsilon \int_{Q_T} \rho^\beta \omega^2 \, dx \, dt \leq M \Big(1 + \varepsilon \int_{Q_T} \rho^{2\beta-3\gamma}\omega^2 \, dx \, dt \Big).$$

If $2\beta - 3\gamma \leq \gamma + 1$, we are done. If not, we iterate the above argument (noting that $2\beta - 3\gamma < \beta$) to obtain

$$\varepsilon \int_{Q_T} \rho^\beta \omega^2 \, dx \, dt \leq M_n \Big(1 + \varepsilon \int_{Q_T} \rho^{\beta_n \gamma}\omega^2 \, dx \, dt \Big),$$

where $\beta_n = 2^n\beta - 3\gamma(2^{n-1} - 1) \leq \gamma + 1$ for some $n \in \mathbb{N}$. This proves the claim.

So, from the claim and (4.4.2),

$$\int_{Q_T} (\kappa\rho^{\gamma+1} + \delta\rho^2)\omega^2 \, dx \, dt \leq M + \varepsilon\Delta M \int_{Q_T} \rho^{\gamma+1}\omega^2 \, dx \, dt.$$

Taking $\Delta > 0$ sufficiently small, we conclude

$$\int_{Q_T} (\kappa\rho^{\gamma+1} + \delta\rho^2)\omega^2 \, dx \, dt \leq M(\operatorname{supp}\omega, T, E_0).$$

$\square$

The other key estimate concerns a local higher integrability property for the velocity:

$$\int_0^T \int_K \big(\rho|u|^3 + \rho^{\gamma+\theta}\big)\, dx\, dt \le M.$$

The proof of this estimate relies on the observation that a careful choice of the generating function $\psi = \psi(s)$ yields an entropy flux function of strictly higher growth rate than its associated weak entropy function. This observation was first made in [52] and is contained in the next result, which has been taken and adapted from [12, 13].

Lemma 4.4.2. *Let $(\eta^{\#}, q^{\#})$ be the entropy pair generated by $\psi_{\#}(s) = \frac{1}{2}s|s|$. Then, for $\check{\psi}(s) = \psi_{\#}(s-u_-)$, the entropy pair generated by $\check{\psi}$, which we denote $(\check{\eta}, \check{q})$, can be written, for some constant α, as*

$$\begin{aligned}
&\check{\eta}(\rho, m) = \eta^{\#}(\rho, m - \rho u_-),\\
&\check{q}(\rho, m) = q^{\#}(\rho, m - \rho u_-) + u_- \eta^{\#}(\rho, m - \rho u_-),\\
&\check{\eta}(\rho, m) = \alpha \rho^{\theta+1}(u - u_-) + r_2(\rho, \rho(u - u_-)),\\
&|r_2(\rho, \rho(u - u_-))| \le M\rho|u - u_-|^2.
\end{aligned}$$

Moreover, with the notation:

$$\begin{aligned}
&\tilde{\eta}(\rho, m) := \check{\eta}(\rho, m) - \nabla\check{\eta}(\rho_-, m_-) \cdot (\rho - \rho_-, m - m_-),\\
&\tilde{q}(\rho, m) := \check{q}(\rho, m) - \nabla\check{\eta}(\rho_-, m_-) \cdot (m, \frac{m^2}{\rho} + p),
\end{aligned}$$

the following inequalities and identities hold:

$$\begin{aligned}
&|\tilde{\eta}(\rho, m)| \le M\big(\rho|u - u_-|^2 + \rho(\rho^\theta - (\rho_-)^\theta)^2\big),\\
&\tilde{q}(\rho, m) \ge \frac{1}{M}\big(\rho|u - u_-|^3 + \rho^{\gamma+\theta}\big) - M\big(\rho + \rho|u - u_-|^2 + \rho^\gamma\big),\\
&\tilde{q}(\rho_-, m_-) < 0,\\
&\big| - \check{q} + m\check{\eta}_\rho + \frac{m^2}{\rho}\check{\eta}_m\big| \le M\tilde{q} + M,\\
&\tilde{\eta}_m = \alpha(\rho^\theta - (\rho_-)^\theta) + (u - u_-)r(\rho, u), \quad \textit{where} \quad |r(\rho, u)| \le M,\\
&|m\tilde{\eta}_m| \le M\big(\rho(u - u_-)^2 + \rho(\rho^\theta - (\rho_-)^\theta)^2 + \rho\big),\\
&|(\tilde{\eta}_m)_\rho| \le M\rho^{\theta-1}, \qquad |(\tilde{\eta}_m)_u| \le M,\\
&|\tilde{\eta}_m| \le M\big(|u - u_-| + |\rho^\theta - (\rho_-)^\theta|\big).
\end{aligned}$$

The proof of this lemma is entirely standard, simply by computing from the representation formula (4.2.5)–(4.2.6) for the entropy and entropy flux.

Before proving the higher integrability estimate, we need the following technical lemma.

Lemma 4.4.3. *For any compact set $K \subset (a, b)$, there exists $\beta > 2$ such that*

$$\varepsilon \int_0^T \int_a^x \rho^3(t,y) A(y)\, dy\, dt \le M|a|^\beta \qquad \text{for any } x \in K, \tag{4.4.3}$$

where $M = M(K)$ is independent of $\varepsilon > 0$.

Proof. The proof is divided into three cases.

1. When $\gamma \in (1, 2)$,

$$\begin{aligned}
\varepsilon \int_0^T \int_a^x \rho^3 A\, dy\, dt &\le M\varepsilon|a| \int_0^T \sup_{(a,x)} \rho^{3-\gamma}\, dt \\
&\le M\varepsilon|a|\Big(1 + \int_0^T \int_a^x \rho^{3-\frac{3\gamma}{2}} |(\rho^{\frac{\gamma}{2}})_y|\, dy\, dt\Big) \\
&\le M\varepsilon|a|\Big(1 + \int_0^T \int_a^x \rho^{6-3\gamma} A^{-1}\, dy\, dt + \int_0^T \int_a^x \rho^{\gamma-2} |\rho_y|^2 A\, dy\, dt\Big) \\
&\le M\varepsilon|a|\Big(1 + \int_0^T \int_a^x \rho^{6-3\gamma} A^{2-\gamma} A^{\gamma-3}\, dy\, dt\Big) \\
&\le M\varepsilon|a|\Big(1 + \int_0^T \int_a^x \rho^3 \Big(\frac{A}{2(|a|+1)M} + M|a|^{\beta_1(\gamma)} A^{\frac{\gamma-3}{\gamma-1}}\Big)\, dy\, dt\Big),
\end{aligned}$$

where we have used Young's inequality. Then

$$\frac{\varepsilon}{2} \int_0^T \int_a^x \rho^3\, dy\, dt \le M\varepsilon|a|^{\beta(\gamma)},$$

where $\beta(\gamma) = \beta_1(\gamma) + 1$.

2. Now we consider the case $\gamma \in [2, 3]$. Note that $\int_a^x \rho A\, dy \le M(E_0) + M|a| \le M|a|$. Then

$$\begin{aligned}
\int_0^T \int_a^x \rho^3 A\, dy\, dt &\le \int_0^T \sup_{(a,x)} \rho^2(t,\cdot) \Big(\int_a^x \rho A\, dy\Big)\, dt \\
&\le M|a|\Big(1 + \int_0^T \int_a^x \rho|\rho_y|\, dy\, dt\Big) \\
&= M|a|\Big(1 + \int_0^T \int_a^x \rho^{2-\frac{\gamma}{2}} \rho^{\frac{\gamma-2}{2}} |\rho_y|\, dy\, dt\Big) \\
&\le M|a|\Big(1 + \frac{1}{\varepsilon} + \int_0^T \int_a^x \rho^{4-\gamma} A^{-1}\, dy\, dt\Big) \\
&\le M|a|\Big(1 + \varepsilon^{-1} + \int_0^T \int_a^x \big(\rho^\gamma A + A^{-\frac{4}{2\gamma-4}}\big)\, dy\, dt\Big) \\
&\le M|a|\big(1 + \varepsilon^{-1} + |a|\big),
\end{aligned}$$

where we have written $A^{-1} = A^{\frac{4-\gamma}{\gamma}} A^{-\frac{4}{\gamma}}$ and applied Young's inequality with exponents $\frac{\gamma}{4-\gamma}$ and $\frac{\gamma}{2\gamma-4}$.

3. Finally, for $\gamma > 3$, the estimate follows directly from the main energy estimate, Proposition 4.3.2. $\square$

Lemma 4.4.4. *Let (η^ψ, q^ψ) be an entropy pair such that the generating function ψ satisfies*

$$\sup_s |\psi''(s)| < \infty.$$

Then, for any $(\rho, m) \in \mathbb{R}^2_+$, $\xi \in \mathbb{R}^2$,

$$|\xi^\top \nabla^2 \eta^\psi(\rho, m)\xi| \leq M_\psi \xi^\top \nabla^2 \eta^*(\rho, m)\xi,$$

where M_ψ depends only on ψ and γ.

The proof is direct; see [13]. Also recall that the mechanical energy $\eta^*(\rho, m)$ is convex.

The next lemma concerns the desired higher integrability property for the velocity and, as mentioned, is the other crucial estimate for applying the compensated compactness framework of §3.5.

Lemma 4.4.5. *Let $K \subset (a, b)$ be compact. Then there exist a constant $M = M(K, T)$ and an exponent $\beta > 2$, both independent of ε, such that*

$$\int_0^T \int_K \left(\rho|u|^3 + \rho^{\gamma+\theta}\right) dx\, dt \leq M\Big(1 + \frac{|a|^\beta \delta}{\varepsilon} + \varepsilon|a|\Big).$$

Proof. We prove the property for the nozzles satisfying condition (4.1.5), since the other case (4.1.6) can be handled by corresponding similar arguments.

We multiply the continuity equation in (4.3.1) by $\tilde{\eta}_\rho A$ and the momentum equation by $\tilde{\eta}_m A$ to obtain, after a short calculation, that

$$\begin{aligned}
&(\tilde{\eta}A)_t + (\tilde{q}A)_x + A'\big(-\check{q} + m\check{\eta}_\rho + \frac{m^2}{\rho}\check{\eta}_m\big) + A'\check{\eta}_m(\rho_-, m_-)p(\rho) \\
&= \varepsilon\big(\rho_{xx} + \frac{A'}{A}\rho_x\big)\tilde{\eta}_\rho A + \varepsilon\big(m_x + \frac{A'}{A}m\big)_x\tilde{\eta}_m A - (\delta\rho^2)_x\tilde{\eta}_m A.
\end{aligned} \tag{4.4.4}$$

Integrating both sides of (4.4.4), we find

$$\begin{aligned}
\int_0^T A(x)\tilde{q}(x)\, dt = &\int_0^T A(a)\tilde{q}(a)\, dt - \int_a^x (\tilde{\eta}(T, y) - \tilde{\eta}(0, y))A(y)\, dy \\
&- \int_0^T \int_a^x A'(-\check{q} + m\check{\eta}_\rho + \frac{m^2}{\rho}\check{\eta}_m)\, dy\, dt \\
&+ \varepsilon \int_0^T \int_a^x A(\rho_{yy} + \frac{A'}{A}\rho_y)\tilde{\eta}_\rho + A(m_y + \frac{A'}{A}m)_y\tilde{\eta}_m\, dy\, dt \\
&- \delta \int_0^T \int_a^x A(\rho^2)_y\tilde{\eta}_m\, dy\, dt - \int_0^T \int_a^x A'\check{\eta}_m(\rho_-, m_-)p(\rho)\, dy\, dt.
\end{aligned} \tag{4.4.5}$$

Now we employ the crucial observation that the first term, which is poorly controlled in absolute value, is actually negative by Lemma 4.4.2, and so may be neglected. Considering the fourth integral on the right hand side, we integrate by parts to see

$$\begin{aligned}
&\varepsilon\int_0^T\int_a^x\big[(\rho_{yy}+\frac{A'}{A}\rho_y)\tilde{\eta}_\rho+(m_{yy}+\frac{A'}{A}m)_y\tilde{\eta}_m\big]A\,dy\,dt\\
&=\varepsilon\int_0^T A(y)\big(\tilde{\eta}(t,x)\big)_x\,dt-\varepsilon\int_0^T\int_a^x A(y)\big(\rho_y(\tilde{\eta}_\rho)_y+m_y(\tilde{\eta}_m)_y\big)\,dy\,dt\\
&\quad+\varepsilon\int_0^T\int_a^x\big(\frac{A'}{A}\big)_y m\tilde{\eta}_m A(y)\,dy\,dt,
\end{aligned}\tag{4.4.6}$$

as $\tilde{\eta}_\rho=\tilde{\eta}_m=0$ at $x=a$ by construction. Likewise, integrating by parts in the fifth integral,

$$\begin{aligned}
&\delta\int_0^T\int_a^x(\rho^2)_y\tilde{\eta}_m A(y)\,dy\,dt\\
&=-\delta\int_0^T\int_a^x\big(\rho^2(\tilde{\eta}_m)_\rho\rho_y+(\tilde{\eta}_m)_u u_y)A(y)+\rho^2\tilde{\eta}_m A'(y)\big)\,dy\,dt\\
&+\int_0^T\rho^2\tilde{\eta}_m A(x)\,dt.
\end{aligned}\tag{4.4.7}$$

Combining (4.4.4)–(4.4.7) and applying the bound for $-\check{q}+m\check{\eta}_\rho+\frac{m^2}{\rho}\check{\eta}_m$ from Lemma 4.4.2, we obtain

$$\begin{aligned}
\int_0^T\tilde{q}(x)A(x)\,dt\le I(x)&+M\int_0^T\int_a^x|A'(y)|(\tilde{q}(t,y)+1)\,dy\,dt\\
&+\varepsilon\int_0^T\big(A(x)\tilde{\eta}(t,x)\big)_x\,dt-\delta\int_0^T\rho^2\tilde{\eta}_m A(x)\,dt,
\end{aligned}\tag{4.4.8}$$

where

$$\begin{aligned}
I(x)=&M\int_a^x A(x)|\tilde{\eta}(T,y)-\tilde{\eta}(0,y)|\,dy\,dt+\varepsilon\int_0^T\int_a^x\big|A\big(\frac{A'}{A}\big)_y m\tilde{\eta}_m\big|\,dy\,dt\\
&+\varepsilon\int_0^T\int_a^x\big|A\rho_y(\tilde{\eta}_\rho)_y+Am_y(\tilde{\eta}_m)_y\big|\,dy\,dt+\delta\int_0^T\int_a^x\big|A'\rho^2\tilde{\eta}_m\big|\,dy\,dt\\
&+\delta\int_0^T\int_a^x\big|A\rho^2\big((\tilde{\eta}_m)_\rho\rho_y+(\tilde{\eta}_m)_u u_y\big)\big|\,dy\,dt+\int_0^T\int_a^x\big|A'\check{\eta}_m(\rho_-,m_-)p(\rho)\big|\,dy\,dt\\
&+\varepsilon\int_0^T|A'||\tilde{\eta}(t,x)|\,dt\\
=&I_1+\cdots+I_7.
\end{aligned}$$

Note that I is increasing in x and $|\frac{A'}{A}|\le M$. By the Gronwall inequality,

$$\begin{aligned}
\int_0^T A\tilde{q}(x)\,dx\le I(x)e^{\int_a^x|A'|\,dy}&+\varepsilon\int_0^T\big(A\tilde{\eta}(t,x)\big)_x\,dt-\int_0^T\delta\rho^2\tilde{\eta}_m A\,dt\\
&+\int_a^x\Big(\varepsilon\big(\int_0^T A\tilde{\eta}(t,y)\,dt\big)_y-\int_0^T\delta\rho^2\tilde{\eta}_m A\,dt\Big)|A'|e^{\int_y^x|A'|\,ds}\,dy.
\end{aligned}\tag{4.4.9}$$

We take a smooth cut-off function $\omega(x)$ with compact support in K such that $\omega(x) \leq 1$. Observe first that by the bound $|\tilde{\eta}(\rho, u)| \leq M(\rho|u-u_-|^2 + \rho(\rho^\theta - (\rho_-)^\theta)^2)$, we obtain, from Proposition 4.3.2,

$$\int_a^x \tilde{\eta}(t,y)A(y)\,dy\,dt \leq M(E_0+1) \qquad \text{for any } x \in \operatorname{supp}\omega. \tag{4.4.10}$$

Thus, we integrate (4.4.9) against $\omega(x)$ and estimate the terms on the right as follows:

$$\begin{aligned}
&\Big|\int_K \omega \int_a^x \varepsilon\Big(\int_0^T A\tilde{\eta}(t,y)\,dt\Big)_y |A'| e^{\int_y^x |A'|\,ds}\,dy\,dx\Big| \qquad (4.4.11)\\
&\qquad \leq \varepsilon M(\|A'\|_{L^1}, \|A''\|_{L^\infty}) \int_K \omega \int_0^T \int_a^x A(y)\tilde{\eta}(t,y)\,dy\,dt\,dx + M \leq \varepsilon M|a|,\\
&\Big|\int_a^x \Big(\int_0^T \delta\rho^2\tilde{\eta}_m A\,dt\Big)|A'|e^{\int_x^y |A'|\,ds}\,dy\Big| \leq M\int_0^T\int_a^x \delta\rho^3 A\,dy\,dt \leq M\frac{|a|^\beta\delta}{\varepsilon,}, \qquad (4.4.12)\\
&\Big|\varepsilon\int_K \omega \int_0^T \big(A\tilde{\eta}(t,x)\big)_x\,dt\,dx\Big| = \Big|\varepsilon\int_0^T\int_K \omega_x A\tilde{\eta}(t,x)\,dx\,dt\Big| \leq M, \qquad (4.4.13)
\end{aligned}$$

where we have used Lemma 4.4.3.

Therefore, the lower bound $\check{q} \geq \frac{1}{M}(\rho|u-u_-|^3 + \rho^{\gamma+\theta}) - M(\rho + \rho|u-u_-|^2 + \rho^\gamma)$ yields that, after multiplying by ω and integrating in x, we have

$$\begin{aligned}
&\int_K\int_0^T \omega A\big(\rho|u-u_-|^3 + \rho^{\gamma+\theta}\big)\,dt\,dx\\
&\quad \leq \int_K \omega(x)I(x)e^{\int_a^x |A'|\,ds}\,dx + M\int_K \omega\int_0^T \big(\rho|u-u_-|^2 + \rho^\gamma + \rho\big)\,dt\,dx\\
&\quad + M\big(1 + \frac{|a|^\beta\delta}{\varepsilon} + \varepsilon|a|\big).
\end{aligned}$$

For the term involving $I(x)$, we have seen $\int_K |I_1|\,dx \leq M$ by (4.4.10). Now we can use the inequality for $|m\tilde{\eta}_m|$ from Lemma 4.4.2 to obtain

$$\begin{aligned}
\int_K \omega|I_2|\,dx &\leq M\Big|\int_K \omega\varepsilon\int_0^T\int_a^x \big|A\big(\frac{A'}{A}\big)_y m\tilde{\eta}_m\big|\,dy\,dt\,dx\Big|\\
&\leq M\int_K \omega\varepsilon\int_0^T\int_a^x \big|A\big(\frac{A'}{A}\big)_y(\rho + \rho(u-u_-)^2 + \rho(\rho^\theta - (\rho_-)^\theta)^2)\big|\,dy\,dt\,dx\\
&\leq M\int_K \omega\varepsilon\int_0^T\int_a^x \big|A\big(\frac{A'}{A}\big)_y(1 + \overline{\eta_\delta^*})\big|\,dy\,dt\,dx\\
&\leq \varepsilon M\big\|A\big(\frac{A'}{A}\big)'\big\|_{L^\infty(a,x)}|a| \leq \varepsilon M|a|,
\end{aligned}$$

where we have used that $x \in K$ and M depends on K.

Moreover, by Proposition 4.3.2 and Lemma 4.4.4, we have

$$\begin{aligned}|I_3| &= \varepsilon \int_0^T \int_a^x |A\rho_y(\tilde{\eta}_\rho)_y + Am_y(\tilde{\eta}_m)_y|\,dy\,dt\\ &= \varepsilon \int_0^T \int_a^x A|(\rho_y, m_y)\nabla^2\tilde{\eta}(\rho_y, m_y)^\top|\,dy\,dt\\ &\le M_{\check{\psi}}\varepsilon \int_0^T \int_a^x A(\rho_y, m_y)\nabla^2\overline{\eta^*}(\rho_y, m_y)^\top\,dy\,dt\\ &\le ME_0,\end{aligned}$$

From Lemma 4.4.2,

$$\begin{aligned}|I_4| &\le M\delta \int_0^T \int_a^x |A'|\rho^2\big(\alpha|\rho^\theta - (\rho_-)^\theta| + M|u-u_-|\big)\,dy\,dt\\ &\le \delta M \int_0^T \int_a^x A(y)\big(\rho^3 + \rho(\rho^\theta - (\rho_-)^\theta)^2 + \rho|u-u_-|^2\big)\,dy\,dt\\ &\le \delta\Big(M + M\int_0^T \int_a^x A\rho^3\,dy\,dt\Big) \le \big(M + M\frac{|a|^\beta\delta}{\varepsilon}\big).\end{aligned}$$

Next, also by Lemmas 4.4.2 – 4.4.3, we obtain

$$\begin{aligned}|I_5| &= \delta \int_0^T \int_a^x \big|A\rho^2\big((\tilde{\eta}_m)_\rho\rho_y + (\tilde{\eta}_m)_u u_y\big)\big|\,dy\,dt\\ &\le \delta \int_0^T \int_a^x A\rho^2\big(M\rho^{\theta-1}|\rho_y| + |u_y|\big)\,dy\,dt\\ &\le \delta M\Big(\int_0^T \int_a^x A\rho^3\,dy\,dt\Big)^{\frac12}\Big(\int_0^T \int_a^x A\rho^{\gamma-2}|\rho_y|^2\,dy\,dt\Big)^{\frac12}\\ &\quad + \delta M\Big(\int_0^T \int_a^x A\rho^3\,dy\,dt\Big)^{\frac12}\Big(\int_0^T \int_a^x A\rho|u_y|^2\,dy\,dt\Big)^{\frac12}\\ &\le \frac{\delta}{\sqrt{\varepsilon}}M\Big(\int_0^T \int_a^x A\rho^3\,dy\,dt\Big)^{\frac12}\\ &\le M\frac{|a|^\beta\delta}{\varepsilon}.\end{aligned}$$

Since $|\frac{A'}{A}|$ is uniformly bounded, we get the bounds

$$\begin{aligned}|I_6| &= \int_0^T \int_a^x |A'\check{\eta}_m(\rho_-, m_-)p(\rho)|\,dy \le M\int_0^T \int_a^x (A\overline{h_\delta}(\rho, \bar{\rho}) + A')\,dy\\ &\le MT(E_0 + \|A'\|_{L^1(-\infty,\,\sup(K))}),\end{aligned}$$

and

$$\int_K |I_7|\,dx = \int_K \omega \int_0^T |A'||\tilde{\eta}|\,dt\,dx \le M\int_0^T \int_{\operatorname{supp}\omega} A(y)|\tilde{\eta}(t,x)|\,dx\,dt \le M.$$

These estimates for I_j, $j = 1, \dots, 7$, show that

$$\int_K \omega(x)I(x)e^{\int_a^x |A'|\,ds}\,dx \le M,$$

and hence we conclude the proof. $\square$

Once we have shown the above estimates to be uniform, we can apply the compensated compactness techniques from §3.5. However, as we modified the initial data for the approximate problems (both to give us smooth data and to avoid the problem of cavitation), we need to make certain assumptions on the convergence of the approximate initial data as well as on the quantities appearing in the bounds for Lemmas 4.4.1 and 4.4.5. We therefore require our approximate data to satisfy the following:

(i) $(\rho_0^\varepsilon, m_0^\varepsilon) \to (\rho_0, m_0)$ for almost every $x \in \mathbb{R}$ as $\varepsilon \to 0$, where we take $(\rho_0^\varepsilon, m_0^\varepsilon)$ to be the zero extension of $(\rho_0^\varepsilon, m_0^\varepsilon)$ outside (a, b);

(ii) $\int_a^b \overline{\eta_\delta^*}(\rho_0^\varepsilon(x), m_0^\varepsilon(x))A(x)\,dx \to \int_{\mathbb{R}} \overline{\eta^*}(\rho_0(x), m_0(x))A(x)\,dx$ as $\varepsilon \to 0$;

(iii) $\frac{\delta|a|^\beta}{\varepsilon} \leq M$;

(iv) $\varepsilon|b-a| \leq M$;

where $M < \infty$ is independent of $\varepsilon \in (0, 1]$.

With these properties of the approximate data, the bounds in the above lemmas become uniform in ε.

In order to apply the reduction technique of §3.5, we need to deduce the compactness of the entropy dissipation measures, $\eta(\rho^\varepsilon, m^\varepsilon)_t + q(\rho^\varepsilon, m^\varepsilon)_x$, in the space $H^{-1}_{loc}(\mathbb{R}^2_+)$. This is the content of the following Proposition.

Proposition 4.4.6. *Let (η, q) be the entropy pair generated by $\psi \in C_c^2(\mathbb{R})$. Then the entropy dissipation measures*:

$$\eta(\rho^\varepsilon, m^\varepsilon)_t + q(\rho^\varepsilon, m^\varepsilon)_x,$$

lie in a compact subset of H^{-1}_{loc}.

Proof. We divide the proof into six steps.

1. We first recall the following fact from Lemma 2.1 in [12]: For a C^2 function $\psi : \mathbb{R} \to \mathbb{R}$ of compact support, the associated entropy pair (η, q) satisfies

$$\begin{aligned}
&|\eta(\rho, m)| \leq M_\psi \rho \\
&|q(\rho, m)| \leq M_\psi \rho \quad \text{for } \gamma \in (1, 3], \qquad |q(\rho, m)| \leq M_\psi \rho \max\{1, \rho^\theta\} \quad \text{for } \gamma > 3, \\
&|\eta_m(\rho, m)| + |\rho \eta_{mm}(\rho, m)| \leq M_\psi, \\
&|\eta_\rho(\rho, m) + u\eta_m(\rho, m)| \leq M_\psi(1 + \rho^\theta), \\
&|\partial_u \eta_m(\rho, \rho u)| + |\rho^{1-\theta} \partial_\rho \eta_m(\rho, \rho u)| \leq M_\psi.
\end{aligned}$$

2. We write $\eta^\varepsilon = \eta(\rho^\varepsilon, m^\varepsilon)$ and $q^\varepsilon = q(\rho^\varepsilon, m^\varepsilon)$ with $m^\varepsilon = \rho^\varepsilon u^\varepsilon$. Then

$$\begin{aligned}\eta_t^\varepsilon + q_x^\varepsilon =& - \frac{A'}{A}\rho^\varepsilon u^\varepsilon\big(\eta_\rho^\varepsilon + u^\varepsilon \eta_m^\varepsilon\big) + \varepsilon\big(\frac{A'}{A}\rho_x^\varepsilon \eta_\rho^\varepsilon + \big(\frac{A'}{A}m^\varepsilon\big)_x \eta_m^\varepsilon\big)\\ &- \varepsilon\big(\rho_x^\varepsilon(\eta_\rho^\varepsilon)_x + m_x^\varepsilon(\eta_m^\varepsilon)_x\big) + \varepsilon\eta_{xx}^\varepsilon - \big(\delta(\rho^\varepsilon)^2\big)_x \eta_m^\varepsilon\\ =&: I_1^\varepsilon + \cdots + I_5^\varepsilon.\end{aligned} \tag{4.4.14}$$

We prove that this is a sum of terms bounded uniformly in $L^1(0,T;L^1_{loc}(\mathbb{R}))$ and terms that are compact in $W^{-1,q}_{loc}(\mathbb{R}^2_+)$ for some $q>1$. Then the compact Sobolev embedding of L^1 into $W^{-1,q}$ (locally in $\mathbb{R}^2_+$) gives the compactness of the entropy dissipation measures in some $W^{-1,q_1}_{loc}(\mathbb{R}^2_+)$.

3. To this end, first observe that

$$|I_1^\varepsilon(t,x)| \le M\big|\frac{A'}{A}\big|\rho^\varepsilon\big(1+(\rho^\varepsilon)^\theta\big)|u^\varepsilon| \le \ M\big|\frac{A'}{A}\big|\big(\rho^\varepsilon|u^\varepsilon|^2 + \rho^\varepsilon + (\rho^\varepsilon)^\gamma\big)$$

is uniformly bounded in $L^1(0,T;L^1_{loc}(\mathbb{R}))$.

Now we see

$$\begin{aligned}I_2^\varepsilon =& \varepsilon\big(\frac{A'}{A}\eta_x^\varepsilon + \big(\frac{A'}{A}\big)_x m^\varepsilon \eta_m^\varepsilon\big) = \varepsilon\big(\frac{A'}{A}\big)_x (m\eta_m^\varepsilon - \eta^\varepsilon) + \varepsilon\big(\frac{A'}{A}\eta^\varepsilon\big)_x\\ =&: I_{2a}^\varepsilon + I_{2b}^\varepsilon.\end{aligned}$$

One can easily check that $|\eta^\varepsilon - m^\varepsilon\eta_m^\varepsilon| \le M\big(\rho^\varepsilon + \rho^\varepsilon|u^\varepsilon|^2\big)$. Thus, $I_{2a}^\varepsilon \to 0$ in $L^1_{loc}(\mathbb{R}^2_+)$ as $\varepsilon\to 0$. Next, for $\omega \in C_c^\infty(\mathbb{R}^2_+)$,

$$\begin{aligned}\varepsilon\big|\int_{\operatorname{supp}\omega} I_{2b}^\varepsilon \omega(t,x)\,dx\,dt\big| =& \varepsilon\big|\int_{\operatorname{supp}\omega}\frac{A'}{A}\eta^\varepsilon\omega_x\,dx\,dt\big|\\ &\le \varepsilon M(\operatorname{supp}\omega)\|\rho^\varepsilon\|_{L^{\gamma+1}(\operatorname{supp}\omega)}\|\omega\|_{H^1(\mathbb{R}^2_+)}.\end{aligned}$$

Hence, $I_{2b}^\varepsilon \to 0$ in $H^{-1}_{loc}(\mathbb{R}^2_+)$ as $\varepsilon\to 0$.

Moreover,

$$|I_3^\varepsilon| = \ \varepsilon|\langle \nabla^2\eta(\rho^\varepsilon,m^\varepsilon)(\rho_x^\varepsilon,m_x^\varepsilon),(\rho_x^\varepsilon,m_x^\varepsilon)\rangle| \le \ M_\psi\varepsilon\langle\nabla^2\overline{\eta^*}(\rho^\varepsilon,m^\varepsilon)(\rho_x^\varepsilon,m_x^\varepsilon),(\rho_x^\varepsilon,m_x^\varepsilon)\rangle,$$

which is uniformly bounded in $L^1(0,T;L^1_{loc}(\mathbb{R}))$ by the main energy estimate, Proposition 4.3.2.

4. The next step is to show that $I_4^\varepsilon \to 0$ in $W^{-1,q}_{loc}$ for some $q>1$.

Claim. Let $K\subset\mathbb{R}$ be compact, $0<\Delta<1$. Then

$$\int_0^T\int_K \varepsilon^{\frac{3}{2}}|\rho_x^\varepsilon|^2\,dx\,dt \le M(T,\operatorname{supp}\omega)(\Delta+\varepsilon+\varepsilon\Delta^{4-\gamma}+\frac{\varepsilon^2}{\Delta}).$$

Hence

$$\int_0^T \int_K \varepsilon^{\frac{3}{2}} |\rho_x^\varepsilon|^2 \, dx \, dt \to 0,$$

and

$$\varepsilon \eta_x^\varepsilon \to 0 \qquad \text{in } L^q(0,T;L^q_{loc}(\mathbb{R})) \text{ for some } q \in (1,2).$$

We now drop the superscript ε and prove the claim. Define

$$\phi(\rho) = \begin{cases} \frac{\rho^2}{2}, & \rho < \Delta, \\ \frac{\Delta^2}{2} + \Delta(\rho - \Delta), & \rho \geq \Delta, \end{cases}$$

so that $\phi''(\rho) = \mathbb{1}_{\{\rho<\Delta\}}$, and

$$\rho\phi'(\rho) - \phi(\rho) = \begin{cases} \frac{\rho^2}{2}, & \rho < \Delta, \\ \frac{\Delta^2}{2}, & \rho \geq \Delta. \end{cases}$$

From the approximate continuity equation in (4.3.1),

$$\begin{aligned}
&(\phi\omega^2)_t + (\phi u\omega^2)_x - 2\phi u\omega_x\omega \\
&= \phi'(\rho)\rho_t\omega^2 + \phi'(\rho)\rho_x u\omega^2 + \phi(\rho)u_x\omega^2 \\
&= -\phi'(\rho)\rho_x u\omega^2 - \phi'(\rho)\rho u_x\omega^2 + \phi'(\rho)\rho_x u\omega^2 + \phi(\rho)u_x\omega^2 \\
&\quad - \frac{A'}{A}\phi'(\rho)\rho u\omega^2 + \varepsilon\phi'(\rho)\big(\rho_{xx} + \frac{A'}{A}\rho_x\big)\omega^2 \\
&= -\frac{A'}{A}\rho u\omega^2 \min\{\rho, \Delta\} - \frac{1}{2}u_x\omega^2\big(\rho^2\mathbb{1}_{\{\rho<\Delta\}} + \Delta^2\mathbb{1}_{\{\rho>\Delta\}}\big) \\
&\quad + \varepsilon(\phi'\omega^2\rho_x)_x - \varepsilon\omega^2|\rho_x|^2\mathbb{1}_{\{\rho<\Delta\}} - 2\varepsilon\min\{\rho,\Delta\}\omega_x\rho_x\omega + \varepsilon\frac{A'}{A}\min\{\rho,\Delta\}\rho_x\omega^2.
\end{aligned}$$

Thus, by integrating, we have

$$\begin{aligned}
&\int_0^T \int \varepsilon\omega^2|\rho_x|^2\mathbb{1}_{\{\rho<\Delta\}} \, dx \, dt \\
&\qquad = -\int \phi\omega^2\Big|_0^T dx + 2\int_0^T \int \phi u\omega_x\omega \, dx \, dt \\
&\qquad\quad - \frac{1}{2}\int_0^T \int (\rho^2\mathbb{1}_{\{\rho<\Delta\}} + \Delta^2\mathbb{1}_{\{\rho>\Delta\}})\omega^2 u_x \, dx \, dt \\
&\qquad\quad - \int_0^T \int \frac{A'}{A}\rho u\omega^2 \min\{\rho,\Delta\} \, dx \, dt - 2\int_0^T \int \varepsilon\min\{\rho,\Delta\}\omega_x\omega\rho_x \, dx \, dt \\
&\qquad\quad + \int_0^T \int \varepsilon\frac{A'}{A}\min\{\rho,\Delta\}\rho_x\omega^2 \, dx \, dt \\
&\qquad =: J_1 + \cdots + J_6.
\end{aligned}$$

The first four terms, $J_1, ..., J_4$, are bounded by using the uniform energy estimates, giving a bound of $M\frac{\Delta}{\sqrt{\varepsilon}}$. We therefore focus on the last two terms, J_5 and J_6.

$$\begin{aligned}
|J_5| \leq& 2\sqrt{\varepsilon}\int_0^T\int \sqrt{\varepsilon}\rho\mathbb{1}_{\{\rho<\Delta\}}|\omega_x|\omega|\rho_x|\,dx\,dt + 2\sqrt{\varepsilon}\int_0^T\int \sqrt{\varepsilon}\Delta\mathbb{1}_{\{\rho>\Delta\}}|\omega_x|\omega|\rho_x|\,dx\,dt\\
&\leq \frac{\varepsilon}{4}\int_0^T\int |\rho_x|^2\mathbb{1}_{\{\rho<\Delta|\}}\omega^2\,dx\,dt + M\varepsilon\int_0^T\int \Delta^2|\omega_x|^2\,dx\,dt\\
&\quad + M\sqrt{\varepsilon}\int_0^T\int \sqrt{\varepsilon}|\rho_x|\rho^{\frac{\gamma-2}{2}}\mathbb{1}_{\{\rho>\Delta\}}\rho^{\frac{2-\gamma}{2}}\Delta|\omega_x|\omega\,dx\,dt\\
&\leq \frac{\varepsilon}{4}\int_0^T\int |\rho_x|^2\mathbb{1}_{\{\rho<\Delta|\}}\omega^2\,dx\,dt + \varepsilon\Delta^2 M\\
&\quad + M\sqrt{\varepsilon}\int_0^T\int \varepsilon\rho^{\gamma-2}|\rho_x|^2\omega^2\,dx\,dt + M\sqrt{\varepsilon}\int_0^T\int \mathbb{1}_{\{\rho>\Delta\}}\rho^{2-\gamma}\Delta^2\omega_x^2\,dx\,dt\\
&\leq \frac{\varepsilon}{4}\int_0^T\int |\rho_x|^2\mathbb{1}_{\{\rho<\Delta\}}\omega^2\,dx\,dt + \varepsilon\Delta^2 M + \sqrt{\varepsilon}M\\
&\quad + M\sqrt{\varepsilon}\int_0^T\int \Delta\mathbb{1}_{\{\Delta<\rho\}}\rho^{3-\gamma}\omega_x^2\,dx\,dt\\
&\leq \frac{\varepsilon}{4}\int_0^T\int |\rho_x|^2\mathbb{1}_{\{\rho<\Delta\}}\omega^2\,dx\,dt + \sqrt{\varepsilon}M + \sqrt{\varepsilon}\Delta^{4-\gamma}M,
\end{aligned}$$

where we have used that $\rho^{3-\gamma} \leq \rho^{\gamma+1}$ and Lemma 4.4.1 when $\gamma \in (1,3]$, and $\rho^{3-\gamma} \leq \Delta^{3-\gamma}$ on $\{\Delta < \rho\}$ when $\gamma > 3$.

Next, using a similar argument to that of J_5 and Proposition 4.3.2, we have

$$\begin{aligned}
|J_6| \leq& \varepsilon\int_0^T\int |\frac{A'}{A}|\omega^2\rho\mathbb{1}_{\{\rho<\Delta\}}|\rho_x|dx\,dt + \varepsilon\int_0^T\int |\frac{A'}{A}|\omega^2\Delta\mathbb{1}_{\{\rho>\Delta\}}|\rho_x|\,dx\,dt\\
&\leq \frac{\varepsilon}{4}\int_0^T\int |\rho_x|^2\omega^2\mathbb{1}_{\{\rho<\Delta\}}\,dx\,dt + M\varepsilon\Delta^2\|\frac{A'}{A}\|_{L^\infty(\operatorname{supp}\omega)}|\operatorname{supp}\omega\times[0,T]|\\
&\quad + M\sqrt{\varepsilon}\int_0^T\int \sqrt{\varepsilon}|\frac{A'}{A}||\rho_x|\rho^{\frac{\gamma-2}{2}}\mathbb{1}_{\{\Delta<\rho\}}\rho^{\frac{2-\gamma}{2}}\Delta\omega^2\,dx\,dt\\
&\leq \frac{\varepsilon}{4}\int_0^T\int |\rho_x|^2\omega^2\mathbb{1}_{\{\rho<\Delta\}}\,dx\,dt + \sqrt{\varepsilon}\Delta M + \sqrt{\varepsilon}\Delta^{4-\gamma}M.
\end{aligned}$$

Thus,

$$\int_0^T\int \varepsilon\omega^2|\rho_x|^2\mathbb{1}_{\{\rho<\Delta\}}\,dx\,dt \leq M\big(\sqrt{\varepsilon}(1+\Delta^{4-\gamma}) + \frac{\Delta}{\sqrt{\varepsilon}}\big).$$

Moreover, we have

$$\begin{aligned}
&\int_0^T\int \varepsilon\omega^2|\rho_x|^2\mathbb{1}_{\{\rho>\Delta\}}\,dx\,dt\\
&\leq M\sqrt{\varepsilon}\int_0^T\int \varepsilon\rho^{\gamma-2}|\rho_x|^2\omega^2\mathbb{1}_{\{\Delta<\rho\}}\,dx\,dt + M\sqrt{\varepsilon}\int_0^T\int \varepsilon\mathbb{1}_{\{\Delta<\rho\}}\rho^{2-\gamma}\omega^2dx\,dt\\
&\leq \sqrt{\varepsilon}M + \sqrt{\varepsilon}M\int_0^T\int \varepsilon\mathbb{1}_{\{\Delta<\rho\}}\omega^2(1+\rho^{\gamma+1})\,dx\,dt \leq \sqrt{\varepsilon}M + \frac{\varepsilon^{\frac{3}{2}}M}{\Delta}.
\end{aligned}$$

Therefore, we conclude

$$\int_0^T \int \varepsilon^{\frac{3}{2}} |\rho_x|^2 \omega^2 \, dx \, dt \le M(\Delta + \varepsilon + \varepsilon \Delta^{4-\gamma} + \frac{\varepsilon^2}{\Delta}),$$

so that

$$\int_0^T \int \varepsilon^{\frac{3}{2}} |\rho_x|^2 \omega^2 \, dx \, dt \to 0 \text{ as } \varepsilon \to 0.$$

For the second part of the claim, note that

$$|\eta_x| \le M\big(|\rho_x||\eta_\rho + u\eta_m| + \rho|u_x\eta_m|\big) \le M\big(|\rho_x|(1+\rho^\theta) + \rho|u_x|\big).$$

Let $q \in (1,2)$ be such that $\frac{q}{2-q} = \gamma + 1$. Then

$$\begin{aligned}
\int_0^T \int_K \varepsilon^q |\eta_x|^q \, dx \, dt &\le M \int_0^T \int_K \varepsilon^q |\rho_x|^q \, dx \, dt + \int_0^T \int_K \varepsilon^q \big||\rho_x|\rho^\theta + \rho|u_x|\big|^q \, dx \, dt \\
&\le \Delta + \frac{M}{\Delta} \int_0^T \int_K \varepsilon^2 |\rho_x|^2 \, dx \, dt \\
&\quad + M \int_0^T \int_K \varepsilon^q \rho^{\frac{q}{2}} \big(|\rho^{\frac{\gamma-2}{2}} \rho_x|^q + |\sqrt{\rho} u_x|^q\big) \, dx \, dt \\
&\le \Delta + \frac{M\sqrt{\varepsilon}}{\Delta} \int_0^T \int_K \varepsilon^{\frac{3}{2}} |\rho_x|^2 \, dx \, dt + \varepsilon^{q-1} M \int_0^T \int_K \varepsilon \rho^{\frac{q}{2-q}} \, dx \, dt \\
&\quad + \varepsilon^{q-1} M \int_0^T \int_K \varepsilon \big(\rho^{\gamma-2}|\rho_x|^2 + \rho|u_x|^2\big) \, dx \, dt \\
&\le \Delta + \frac{M\sqrt{\varepsilon}}{\Delta} \int_0^T \int_K \varepsilon^{\frac{3}{2}} |\rho_x|^2 \, dx \, dt + \varepsilon^{q-1} M.
\end{aligned}$$

Thus, $I_4^\varepsilon = \varepsilon \eta_{xx}^\varepsilon \to 0$ in $W_{loc}^{-1,q}$.

5. Consider finally $I_5^\varepsilon = \big(\delta(\rho^\varepsilon)^2\big)_x \eta_m^\varepsilon$. In the case that $\gamma \le 3$,

$$\begin{aligned}
\int_0^T \int_K |I_5^\varepsilon| \, dx \, dt &\le M_\psi \int_0^T \int_K \delta \rho |\rho_x| \, dx \, dt \\
&\le M \int_0^T \int_K \big(\varepsilon \rho^{\gamma-2}|\rho_x|^2 + \frac{\delta^2}{\varepsilon}\rho^{4-\gamma}\big) \, dx \, dt \\
&\le M\Big(\int_0^T \int_K \frac{\delta^2}{\varepsilon}\rho^3 \, dx \, dt + \frac{\delta^2}{\varepsilon} + 1\Big) \\
&\le M\Big(\frac{\delta^2}{\varepsilon} + \frac{\delta}{\varepsilon} + 1\Big),
\end{aligned}$$

which is uniformly bounded in ε.

If $\gamma > 3$, consider the above on the domains: $\{\rho < 1\}$ and $\{\rho > 1\}$. On the latter domain, we argue as above to obtain

$$\int_0^T \int_{K \cap \{\rho > 1\}} |I_5^\varepsilon| \le M,$$

so that

$$\begin{aligned}\int_0^T\int_K |I_5^\varepsilon|\,dx\,dt &\le M + M_\psi\int_0^T\int_{K\cap\{\rho<1\}} \delta\rho|\rho_x|\,dx\,dt\\ &\le M + M(T|K|)^{\frac12}\Big(\int_0^T\int_{K\cap\{\rho<1\}} \delta^2\rho^2\rho_x^2\,dx\,dt\Big)^{\frac12}\\ &\le M + M\Big(\int_0^T\int_K \varepsilon\delta\rho_x^2\,dx\,dt\Big)^{\frac12}.\end{aligned}$$

Thus, I_5^ε is uniformly bounded in $L^1(0,T;L^1_{loc}(\mathbb{R}))$ by Proposition 4.3.2.

6. In Steps 1–5 above, we have shown that $\eta(\rho^\varepsilon,m^\varepsilon)_t+q(\rho^\varepsilon,m^\varepsilon)_x$ can be written as

$$\eta(\rho^\varepsilon,m^\varepsilon)_t+q(\rho^\varepsilon,m^\varepsilon)_x = f^\varepsilon + g^\varepsilon,$$

where f^ε is uniformly bounded in $L^1(0,T;L^1_{loc}(\mathbb{R}))$, and $g^\varepsilon\to 0$ in $W^{-1,q}_{loc}(\mathbb{R}^2_+)$ for some $q\in(1,2)$. Thus, there exists $q_1\in(1,2)$ such that

$$\eta(\rho^\varepsilon,m^\varepsilon)_t+q(\rho^\varepsilon,m^\varepsilon)_x \qquad \text{is pre-compact in } W^{-1,q_1}_{loc}(\mathbb{R}^2_+).$$

We also know from Lemmas 4.4.1 and 4.4.5 that $\eta(\rho^\varepsilon,m^\varepsilon)$ and $q(\rho^\varepsilon,m^\varepsilon)$ are uniformly bounded in $L^{q_2}_{loc}(\mathbb{R}^2_+)$, where $q_2=\gamma+1>2$ when $\gamma\in(1,3]$ and $q_2=\frac{\gamma+\theta}{1+\theta}>2$ when $\gamma>3$. Then, by the compensated compactness interpolation theorem, Theorem 3.4.3,

$$\eta(\rho^\varepsilon,m^\varepsilon)_t+q(\rho^\varepsilon,m^\varepsilon)_x \qquad \text{is pre-compact in } W^{-1,2}_{loc}(\mathbb{R}^2_+).$$

□

4.5 Convergence of the approximate solutions to an entropy solution

Proposition 4.4.6, combined with the uniform estimates above, implies that the sequence of approximate solutions satisfies the compensated compactness framework laid out in §3.5. With this framework, specifically Theorem 3.5.1, we conclude that

$$(\rho^\varepsilon,m^\varepsilon)\to(\rho,m) \qquad \text{for almost every } (t,x)\in\mathbb{R}^2_+,$$

and also in $L^p_{loc}(\mathbb{R}^2_+)\times L^q_{loc}(\mathbb{R}^2_+)$ for $p\in[1,\gamma+1)$ and $q\in[1,\frac{3(\gamma+1)}{\gamma+3})$. This restriction may be seen as $|m|^q=\rho^{\frac q3}|u|^q\rho^{\frac{2q}{3}}\le\rho|u|^3+\rho^{\gamma+1}$ precisely when $q=\frac{3(\gamma+1)}{\gamma+3}$.

The uniform estimates also give us the convergence of the mechanical energy as $\varepsilon\to0$:

$$\eta^*(\rho^\varepsilon,m^\varepsilon)\to\eta^*(\rho,m) \qquad \text{in } L^1_{loc}(\mathbb{R}^2_+).$$

Moreover, from (4.3.11), we have

$$\int_{t_1}^{t_2}\int_{\mathbb{R}} \eta^*(\rho,m)(t,x)A(x)\,dx\,dt \le M(t_2-t_1)\int_{\mathbb{R}} \eta^*(\rho_0,m_0)(x)A(x)\,dx + M,$$

so that, for almost every $t \ge 0$,

$$\int_{\mathbb{R}} \eta^*(\rho,m)(t,x)A(x)\,dx \le M\int_{\mathbb{R}} \eta^*(\rho_0,m_0)(x)A(x)\,dx + M.$$

This implies that no concentration of the density ρ is formed at any point in finite time.

We also see that, for any convex $\psi(s)$ with sub-quadratic growth, the uniform energy estimates of Lemmas 4.4.1 and 4.4.5 and the main energy estimate, Proposition 4.3.2, give that the sequences

$$\eta^\psi(\rho^\varepsilon,m^\varepsilon),\ \ q^\psi(\rho^\varepsilon,m^\varepsilon),\ \ m^\varepsilon\partial_\rho\eta^\psi(\rho^\varepsilon,m^\varepsilon)$$

are equi-integrable. Thus, given such a function ψ, we multiply (4.4.14) by $A(x)$ and integrate against a smooth, compactly supported test function in $\mathbb{R}^2_+$, and then pass to the limit to obtain the entropy inequality (4.2.8).

It remains only to check the limit in the equations in the sense of distributions. This is straightforward by multiplying the equations of (4.3.1) by a test function and integrating by parts before passing to the limit and using the uniform bounds above. Note that in the momentum equation, the term $\delta\rho^2$ vanishes in the limit as $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. This completes the proof of the main theorem of this chapter, Theorem 4.2.3, under the stronger assumptions (4.1.3)–(4.1.4).

4.6 Transonic nozzles with general cross-sectional area

As stated in Theorem 4.2.3, the approach laid out above applies to much more general situations, for example an infinitely narrowing nozzle with closed ends or an expanding nozzle with unbounded ends. In this section, we remove the additional assumptions placed on $A(x)$ in §4.3–§4.5 to conclude the proof of Theorem 4.2.3 for transonic nozzles with general cross-sectional area function $A(x)$ satisfying (4.1.5) or (4.1.6).

More precisely, we now relax the lower bound on A to $A(x) > 0$ and allow A' to be only in $L^1(-\infty,0)$ or $L^1(0,\infty)$, allowing in particular that the cross-sectional area function may converge to 0 as $x \to \pm\infty$ and to ∞ as $x \to -\infty$ or ∞. For ease of reference, we restate the main theorem, Theorem 4.2.3, as follows:

Theorem 4.6.1 (Transonic Nozzle with General Cross-Sectional Area). *Assume that $(\rho_0, m_0) \in (L^1_{loc}(\mathbb{R}))^2$ with $\rho_0 \geq 0$ is of relative finite-energy and that the cross-sectional area function $A \in C^2$ satisfies (4.1.5) or (4.1.6). Then there exists a sequence of approximate solutions $(\rho^\varepsilon, m^\varepsilon)(t,x)$ solving (4.3.1)–(4.3.5) for some initial data $(\rho_0^\varepsilon, m_0^\varepsilon)$ such that $(\rho_0^\varepsilon, m_0^\varepsilon) \to (\rho_0, m_0)$ for almost every $x \in \mathbb{R}$ as $\varepsilon \to 0$, with $(\rho_0^\varepsilon, m_0^\varepsilon)$ taken to be the zero extension of $(\rho_0^\varepsilon, m_0^\varepsilon)$ outside (a,b), and $(\rho^\varepsilon, m^\varepsilon) \to (\rho, m)$ for almost every $(t,x) \in \mathbb{R}^2_+$ and in $L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+)$ for $p \in [1, \gamma+1)$ and $q \in [1, \frac{3(\gamma+1)}{\gamma+3})$ as $\varepsilon \to 0$, such that $(\rho, m)(t,x)$ is a global finite-energy entropy solution of the transonic nozzle problem (4.1.1)–(4.1.2) for end-states $(\rho_\pm, m_\pm)$.*

First we recall that, without loss of generality, we assume $u_+ = 0$ by Galilean invariance of the equations.

As we continue to impose the Dirichlet boundary conditions for the approximate problems (4.3.1), the existence of the approximate solutions is obtained as before (as once we restrict to the interval (a,b), we recover $0 < A_0(\varepsilon) \leq A(x) \leq A_1(\varepsilon) < \infty$). However, we need to demonstrate more carefully the uniform energy estimates of Proposition 4.3.2 and of §4.4 that enable us to employ the framework in §3.5 to pass to the vanishing viscosity limit and obtain an entropy solution of the transonic nozzle problem. In order to make these estimates uniform with respect to ε, we choose $\delta(\varepsilon)$, $a(\varepsilon)$, and $b(\varepsilon)$ such that the following hold for a constant M independent of ε:

- $\varepsilon|b-a| \leq M$,
- $\varepsilon\|\big(\frac{A'}{A}\big)'\|_{L^\infty(a,b)}\|A\|_{L^\infty(a,b)}|b-a| \leq M$,
- $\varepsilon\|A''\|_{L^\infty(a,b)} \leq M$,
- $\delta\varepsilon^{-1}\|A\|_{L^\infty(a,b)}|a|^{\beta(\gamma)}\big\|A^{\frac{\gamma-3}{\gamma-1}}\big\|_{L^\infty(a,b)} \leq M$,
- $\delta\varepsilon^{-1}\|A\|_{L^\infty(a,b)}|a| \leq M$,
- $\delta\|A\|_{L^\infty(a,b)}|a|^2\big\|A^{-\frac{4}{2\gamma-4}}\big\|_{L^\infty(a,b)} \leq M$.

Such choices of the parameters can be achieved as long as the cross-sectional area function $A \in C^2$ satisfies (4.1.5) or (4.1.6).

4.6.1 Uniform estimates

As mentioned, to establish Theorem 4.6.1 for more general nozzles, the first step is to obtain the uniform energy estimate, which is the subject of the following proposition.

Proposition 4.6.2. *Let*

$$E_0 := \sup_{\varepsilon>0} \int_a^b \overline{\eta_\delta^*}(\rho_0^\varepsilon(x), m_0^\varepsilon(x))A(x)\,dx < \infty.$$

Then there exists $M>0$, independent of $\varepsilon>0$, such that, for all $\varepsilon>0$,

$$\sup_{t\in[0,T]} \int_a^b \big(\tfrac{1}{2}\rho|u-\bar{u}|^2 + \overline{h_\delta}(\rho,\bar{\rho})\big)A(x)\,dx \tag{4.6.1}$$

$$+\,\varepsilon \int_{Q_T} \big(h_\delta''(\rho)\rho_x^2 + \rho u_x^2 + \big|\big(\frac{A'(x)}{A(x)}\big)'\rho u(u-\bar{u})\big|\big)A(x)\,dx\,dt$$

$$\le M(E_0+1). \tag{4.6.2}$$

Proof. We prove estimate (4.6.2) for the nozzles satisfying condition (4.1.5), as the case (4.1.6) can be handled analogously. Exactly as in the proof of Proposition 4.3.2,

$$\begin{aligned}\frac{dE}{dt} - \frac{m}{\rho}p_\delta(\rho)A\Big|_a^b + \int_a^b \Big(& m\big((\eta_\delta^*)_\rho(\bar{\rho},\bar{m})\big)_x A(x) + \big((\eta_\delta^*)_m(\bar{\rho},\bar{m})\big)_x\big(\frac{m^2}{\rho} + p_\delta(\rho)\big)A(x) \\ &+ (\eta_\delta^*)_m(\bar{\rho},\bar{m})A'(x)p_\delta(\rho) + \varepsilon\big(\frac{A'(x)}{A(x)}\big)' m\big((\eta_\delta^*)_m - (\eta_\delta^*)_m(\bar{\rho},\bar{m})\big)\Big)dx\end{aligned}$$

$$\begin{aligned}= -\varepsilon &\int_a^b (\rho_x, m_x)\nabla^2\overline{\eta_\delta^*}(\rho_x,m_x)^\top A(x)\,dx \\ &+\varepsilon \int_a^b \big(\rho_x((\eta_\delta^*)_\rho(\bar{\rho},\bar{m}))_x + m_x((\eta_\delta^*)_m(\bar{\rho},\bar{m}))_x\big)A(x)\,dx.\end{aligned}$$

Note that $u_+ = 0$ and that $A(a) \le M$ as $\|A'\|_{L^1(-\infty,0)} \le M$. We observe now that

$$\begin{aligned}&\varepsilon \int_a^b \big(\rho_x((\eta_\delta^*)_\rho(\bar{\rho},\bar{m}))_x + m_x((\eta_\delta^*)_m(\bar{\rho},\bar{m}))_x\big)A(x)\,dx \\ &\le -\varepsilon \int_{-L_0}^{L_0} \big((\rho((\eta_\delta^*)_\rho)(\bar{\rho},\bar{m}))_x + m((\eta_\delta^*)_m(\bar{\rho},\bar{m}))_x\big)A'(x)\,dx \\ &\quad - \varepsilon \int_{-L_0}^{L_0} \big(\rho((\eta_\delta^*)_\rho(\bar{\rho},\bar{m}))_{xx} + m((\eta_\delta^*)_m(\bar{\rho},\bar{m}))_{xx}\big)A(x)\,dx \\ &\quad + \varepsilon\big(\rho((\eta_\delta^*)_\rho(\bar{\rho},\bar{m}))_x + m((\eta_\delta^*)_m(\bar{\rho},\bar{m}))_x\big)A(x)\big|_{x=a}^{x=b} \\ &\le \varepsilon M(\bar{\rho},\bar{m})\big(E + \|A'\|_{L^1(-L_0,L_0)} + \|A\|_{L^\infty(-L_0,L_0)}L_0\big) \\ &\le M(E+1),\end{aligned}$$

since $(\bar{\rho},\bar{m})$ are constant outside $(-L_0,L_0)$ and $|a(\varepsilon)|>L_0$ and $|b(\varepsilon)|>L_0$, so that the boundary term vanishes.

Similarly, we have

$$\int_a^b m\big((\eta_\delta^*)_\rho(\bar\rho,\bar m)\big)_x A(x)\,dx \le M\|A\|_{L^\infty(-L_0,L_0)}L_0(E+1),$$
$$\int_a^b \big((\eta_\delta^*)_m(\bar\rho,\bar m)\big)_x\big(\frac{m^2}{\rho}+p_\delta(\rho)\big)A(x)\,dx \le M\|A\|_{L^\infty(-L_0,L_0)}L_0(E+1).$$

Using the uniform bound $|\frac{A'}{A}|\le M$, and $(\eta_\delta^*)_m(\bar\rho,\bar m)=0$ for $x>L_0$, we obtain

$$\begin{aligned}\int_a^b p_\delta(\rho)(\eta_\delta^*)_m(\bar\rho,\bar m)A'(x)\,dx &\le M\int_a^{L_0}\big(A(x)\overline{h_\delta}(\rho,\bar\rho)+|A'(x)|\big)\,dx\\ &\le M\big(E+\|A'\|_{L^1(-\infty,L_0)}\big).\end{aligned}$$

Finally, we make the estimate:

$$\begin{aligned}&\int_a^b \varepsilon\big|\big(\frac{A'(x)}{A(x)}\big)'m\big((\eta_\delta^*)_m-(\eta_\delta^*)_m(\bar\rho,\bar m)\big)\big|\,dx\\ &\le \varepsilon M\|\big(\frac{A'}{A}\big)'\|_{L^\infty(a,b)}\int_a^b\big(\rho(u-\bar u)^2+\rho\bar u^2\big)A(x)\,dx\\ &\le \varepsilon M\|\big(\frac{A'}{A}\big)'\|_{L^\infty(a,b)}\big(E+\|A\|_{L^\infty(a,b)}|b-a|\big)\\ &\le M(E+1).\end{aligned}$$

With these estimates, we use the Gronwall inequality to obtain the result as before. □

We must now consider the key estimate of Lemma 4.4.5. Before that, we first consider Lemma 4.4.3. In the first line of the proof, we make the initial estimate:

$$\varepsilon\int_0^T\int_a^x \rho^3A(y)\,dy\,dt \le \varepsilon|a|M\|A\|_{L^\infty(a,b)}\int_0^T\sup_{(a,x)}\rho^{3-\gamma}\,dt.$$

Continuing with the proof as before, we find that, by assumption,

$$\varepsilon\int_0^T\int_a^x \rho^3\,dy\,dt \le M\varepsilon\|A\|_{L^\infty(a,b)}\big(|a|^{\beta(\gamma)}\|A^{\frac{\gamma-3}{\gamma-1}}\|_{L^\infty(a,b)}+1\big)\le M\frac{\varepsilon}{\delta}.$$

The other case, $\gamma\in[2,3]$, requires a similar modification, leading to

$$\varepsilon\int_0^T\int_a^x \rho^3A\,dy\,dt \le M\|A\|_{L^\infty(a,b)}|a|\big(\varepsilon+1+\varepsilon|a|\|A^{-\frac{4}{2\gamma-4}}\|_{L^\infty(a,b)}\big)\le M\frac{\varepsilon}{\delta}.$$

Finally, we examine the proof of Lemma 4.4.5. Turning attention to estimates (4.4.11)–(4.4.13), we see that they remain uniform under the assumption that $\varepsilon\|A''\|_{L^\infty(a,b)}\le M$ and the uniform bound of Lemma 4.4.3.

We need to make the bounds on all of the terms I_j, $j=1,\dots,7$, uniform in ε in order to conclude the estimate. Note first that I_1 and I_3 need no adjustment. We have seen that

$$\int_K \omega|I_2|\,dx \le M\varepsilon|a|\big\|A\big(\frac{A'}{A}\big)'\big\|_{L^\infty(a,b)}\le M,$$

by assumption. The bounds for I_4 and I_6 are seen to be uniform, due to the uniform bound of Lemma 4.4.3 and $|\frac{A'}{A}| \leq M$. Examining the estimate for I_5 shows that

$$|I_5| \leq \frac{\delta}{\sqrt{\varepsilon}} M\Big(\int_0^T \int_a^x \rho^3(t,y)A(y)\,dy\,dt\Big)^{\frac{1}{2}} \leq M \qquad \text{uniformly in } \varepsilon.$$

Finally, we have

$$\int_K |I_7|\,dx = \big\|\frac{A'}{A}\big\|_{L^\infty} \int_0^T \int_K \omega(x)|\tilde{\eta}(t,x)|A(x)\,dx\,dt \leq M,$$

and so we are done.

4.6.2 Compactness of the approximate solutions

With the uniform estimates discussed above, the only thing remaining to prove is Proposition 4.4.6. Its proof is seen to follow as before by using the uniform bound $|\frac{A'}{A}| \leq M$ and by observing that, in each of terms $J_1, j = 1, \dots, 6$, all the integrals are taken over the support of the test function ω, so that the factors of A, A^{-1} etc. may be introduced freely. This concludes the proof of Theorem 4.2.3.

4.7 Spherically symmetric solutions of the Euler equations

In this section, we explain how the approach and techniques developed in §4.3– §4.6 above can be generalized to construct global spherically symmetric solutions to the multidimensional Euler equations for compressible fluid flows, as considered by Chen-Perepelitsa in [13]. This situation corresponds to the case that $A(x) = \omega_n x^{n-1} : \mathbb{R}_+ \to \mathbb{R}_+$, where ω_n is the surface area of the n-dimensional unit sphere.

In [13], the possibility of a blow-up of the density at the origin is allowed by imposing a Neumann boundary condition at the left boundary for the approximate solutions. By the Galilean invariance, the velocity can be taken to vanish at $x = \infty$, so the end-states $(\rho_+, u_+) = (\bar{\rho}, 0)$ at $b(\varepsilon)$ are imposed, where $\bar{\rho} \to 0$ as $\varepsilon \to 0$. Then the solutions of this approximate problem were constructed under the assumption that $\gamma \in (1, 3]$, while the convergence of the approximate solutions to the limit is shown for all $\gamma > 1$ in [13].

The following theorem gives the extension of this result for all the physical interval $\gamma > 1$, especially including the unsolved case $\gamma > 3$, which leads to Theorem 4.2.5.

Theorem 4.7.1. *Let $(\rho_0, m_0) \in (L^1_{loc}(\mathbb{R}_+))^2$ be finite-energy initial data such that $\rho_0 \geq 0$. Then there exists a global finite-energy entropy solution of the spherically symmetric Euler equations*:

$$\begin{cases} \rho_t + m_x + \frac{n-1}{x} m = 0, & (t,x) \in (\mathbb{R}_+)^2, \\ m_t + \big(\frac{m^2}{\rho} + p(\rho)\big)_x + \frac{n-1}{x}\frac{m^2}{\rho} = 0, & (t,x) \in (\mathbb{R}_+)^2, \\ (\rho, m)|_{t=0} = (\rho_0, m_0), & x \in \mathbb{R}_+, \end{cases}$$

where $p(\rho) = \kappa\rho^\gamma$ with $\gamma > 1$. This global entropy solution is the vanishing viscosity limit of the approximate solutions of the following system:

$$\begin{cases} \rho^\varepsilon_t + m^\varepsilon_t + \frac{n-1}{x} m^\varepsilon = \varepsilon x^{-(n-1)}\big(x^{n-1}\rho^\varepsilon_x\big)_x, & (t,x) \in \mathbb{R}_+ \times (a(\varepsilon), b(\varepsilon)), \\ m^\varepsilon_t + \big(\frac{(m^\varepsilon)^2}{\rho^\varepsilon} + p_\delta(\rho^\varepsilon)\big)_x + \frac{n-1}{x}\frac{(m^\varepsilon)^2}{\rho^\varepsilon} = \varepsilon\big(m^\varepsilon_x + \frac{n-1}{x}m^\varepsilon\big)_x, & (t,x) \in \mathbb{R}_+ \times (a(\varepsilon), b(\varepsilon)), \\ (\rho^\varepsilon, m^\varepsilon)|_{t=0} = (\rho^\varepsilon_0, m^\varepsilon_0), & x \in (a(\varepsilon), b(\varepsilon)), \end{cases} \tag{4.7.1}$$

with appropriately chosen boundary conditions; *that is, as $\varepsilon \to 0$,*

$$(\rho^\varepsilon, m^\varepsilon) \to (\rho, m) \qquad \text{for almost every } (t,x) \in (\mathbb{R}_+)^2$$

and in $L^p_{loc}\big((\mathbb{R}_+)^2\big) \times L^q_{loc}\big((\mathbb{R}_+)^2\big)$ for $p \in [1, \gamma+1)$ and $q \in [1, \frac{3(\gamma+1)}{\gamma+3})$. Here $p_\delta(\rho) = \kappa\rho^\gamma + \delta\rho^2$, $a(\varepsilon) \to 0$ and $b(\varepsilon) \to \infty$ as $\varepsilon \to 0$, and $(\rho^\varepsilon_0, m^\varepsilon_0) \to (\rho_0, m_0)$ for almost every $x \in \mathbb{R}$ as $\varepsilon \to 0$, where $(\rho^\varepsilon_0, m^\varepsilon_0)$ are taken to be the zero extension of $(\rho^\varepsilon_0, m^\varepsilon_0)$ outside $(a(\varepsilon), b(\varepsilon))$.

To overcome the obstacle encountered in [13] for the unsolved case $\gamma > 3$, we develop two approaches here, thereby providing two methods for re-solving the problem for the spherically symmetric solutions for the whole range $\gamma \in (1, \infty)$. In §4.7.1, instead of taking the Neumann data for ρ at $a(\varepsilon)$ as in [13], we choose Dirichlet data at both ends. One of the motivations is the fact that the boundary data are allowed not to be preserved in the limit (as boundary layers) by choosing $(\rho, u)|_{x=a} = (\bar{\rho}, 0)$ for the same $\bar{\rho}$ as at the other end-point in the construction of the approximate solutions. This allows us to use the above construction from §4.3 (in particular the maximum principle for the approximate equations) to demonstrate the existence of the approximate solutions.

In §4.7.2, we consider the same problem but with the Neumann boundary data initiated in [13]. We are able to demonstrate how higher order *a priori* energy estimates may be obtained for the approximate solutions even in this situation to conclude the existence of the approximate solutions for all $\gamma > 1$, especially including the unsolved case $\gamma > 3$, concluding the scheme set out in [13].

4.7.1 Dirichlet boundary conditions for the density

We first demonstrate how the imposition of the Dirichlet boundary conditions for the approximate system (4.7.1) may be used to obtain the existence of globally defined, spherically symmetric entropy solutions of the compressible Euler equations. We therefore assign the Dirichlet boundary conditions to system (4.7.1):

$$(\rho, m)|_{x=a} = (\bar{\rho}, 0), \qquad (\rho, m)|_{x=b} = (\bar{\rho}, 0), \tag{4.7.2}$$

where $\bar{\rho} = \bar{\rho}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Once again, the main point to check is that the key energy estimate holds for the approximate solutions. Indeed, the imposition of the Dirichlet data (rather than the Neumann data) enables us to apply the framework in §4.3 directly to construct the approximate solutions. Moreover, by choosing the same boundary data at the two endpoints, we may take the monotone reference functions $(\bar{\rho}(x), \bar{u}(x))$ to be the constant state $(\bar{\rho}, 0)$. Then the relative mechanical entropy becomes

$$\overline{\eta_\delta^*}(\rho, m) = \eta_\delta^*(\rho, m) - \eta_\delta^*(\bar{\rho}, 0) - (\eta_\delta^*)_\rho(\bar{\rho}, 0)(\rho - \bar{\rho}).$$

Proposition 4.7.2. *Let*

$$E_0 := \sup_{\varepsilon > 0} \int_a^b \overline{\eta_\delta^*}(\rho_0^\varepsilon(x), m_0^\varepsilon(x)) A(x)\, dx < \infty.$$

Then there exists $M > 0$, independent of ε, such that, for all $\varepsilon > 0$, any solution $(\rho, m) \in \big(C^{2,1}(Q_T)\big)^2$ of (4.7.1) satisfies, for $m = \rho u$,

$$\begin{aligned}&\sup_{t \in [0,T]} \int_a^b \big(\tfrac{1}{2}\rho u^2 + \overline{h_\delta}(\rho, \bar{\rho})\big) x^{n-1}\, dx \\ &+ \varepsilon \int_{Q_T} \big(h_\delta''(\rho)\rho_x^2 + \rho u_x^2 + (n-1)\frac{\rho u^2}{x^2}\big) x^{n-1}\, dx\, dt \le M E_0.\end{aligned} \tag{4.7.3}$$

Proof. With the above choice of $A(x) = \omega_n x^{n-1}$, equation (4.3.9) is simplified as

$$\begin{aligned}&(\overline{\eta_\delta^*} x^{n-1})_t + \big((q_\delta^* - (\eta_\delta^*)_\rho(\bar{\rho}, 0)\big)_x + \varepsilon(n-1) m (\eta_\delta^*)_m x^{n-3} \\ &= \varepsilon(\rho_x x^{n-1})_x \big((\eta_\delta^*)_\rho - (\eta_\delta^*)_\rho(\bar{\rho}, 0)\big) + \varepsilon(m_x x^{n-1})_x (\eta_\delta^*)_m.\end{aligned}$$

Integrating this over the parabolic cylinder Q_T and then integrating by parts with the new Dirichlet boundary conditions implies

$$\int_a^b \overline{\eta_\delta^*} x^{n-1}\, dx + \varepsilon \int_{Q_T} \big((\rho_x, m_x)\nabla^2 \overline{\eta_\delta^*}(\rho_x, m_x)^\top + \frac{n-1}{x^2}\frac{m^2}{\rho}\big) x^{n-1}\, dx\, dt = E_0,$$

as desired. Then we conclude as in Proposition 4.3.2 by the convexity of $\bar{\eta}_\delta^*$. $\square$

With this estimate, we are now in the situation of §4.3 of this chapter. We can therefore proceed with the maximum principle estimate of Lemma 4.3.4 to obtain *a priori* upper bounds on the density and velocity of the fluid, and continue to deduce the existence of solutions $(\rho^\varepsilon, m^\varepsilon)$ of the initial-boundary value problem (4.7.1)–(4.7.2), exactly as in Theorem 4.3.1:

Lemma 4.7.3. *For $\varepsilon > 0$, let $(\rho_0^\varepsilon, m_0^\varepsilon) \in \big(C^{2+\beta}([a,b])\big)^2$ be a sequence of functions such that*

(i) $\inf_{a\le x\le b} \rho_0^\varepsilon(x) > 0$;

(ii) $(\rho_0^\varepsilon, m_0^\varepsilon)$ *satisfies* (4.7.2) *and the compatibility conditions*:

$$
\begin{aligned}
&\big(x^{n-1}m_0^\varepsilon\big)_x\big|_{x=a} = 0,\\
&m_{0,x}^\varepsilon\big|_{x=b} = \varepsilon x^{-(n-1)}\big(x^{n-1}\rho_{0,x}^\varepsilon\big)_x\big|_{x=b},\\
&\Big(\frac{(m_0^\varepsilon)^2}{\rho_0^\varepsilon} + p_\delta(\rho_0^\varepsilon)\Big)_x\Big|_{x=b} = \varepsilon x^{-(n-1)}\big(x^{n-1}m_0^\varepsilon\big)_x\big|_{x=b};
\end{aligned}
$$

(iii) $\int_a^b \big(\frac{(m_0^\varepsilon)^2}{2\rho_0^\varepsilon} + \frac{\kappa(\rho_0^\varepsilon)^\gamma}{\gamma-1}\big)x^{n-1}\,dx < \infty$.

Then there exists a unique global solution $(\rho^\varepsilon, m^\varepsilon)$ of (4.7.1)–(4.7.2) for $\gamma \in (1,\infty)$ such that $(\rho^\varepsilon, m^\varepsilon) \in \big(C^{2+\beta,1+\frac{\beta}{2}}(Q_T)\big)^2$ with $\inf_{Q_T}\rho^\varepsilon(t,x) > 0$ for all $T > 0$.

To conclude the proof of Theorem 4.7.1, we note that, as remarked in [13], the arguments to derive the uniform estimates and convergence of approximate solutions in [13] do not depend on the choice of $\gamma \in (1,3]$, but rather work for all $\gamma \in (1,\infty)$. In addition, we now observe that, in obtaining these uniform estimates, the integrals were considered only over compact regions $K \subset \mathbb{R}_+$, or over intervals (x,b) with $x \in K$ for a compact region $K \subset \mathbb{R}_+$. Therefore, our modification of the boundary condition at $a(\varepsilon)$ in the above does not affect these estimates at all. In particular, we have the following lemma:

Lemma 4.7.4. *Suppose that $(\rho_0^\varepsilon, m_0^\varepsilon)$ satisfy the assumptions of Lemma 4.7.3 and that*

$$
\sup_\varepsilon \int_a^b \Big(\frac{(m_0^\varepsilon)^2}{2\rho_0^\varepsilon} + \frac{\kappa(\rho_0^\varepsilon)^\gamma}{\gamma-1}\Big)x^{n-1}\,dx < \infty.
$$

Let $\delta(\varepsilon), a(\varepsilon), b(\varepsilon)$, and $\bar\rho(\varepsilon)$ satisfy the following relation:

$$
\bar\rho^\gamma b^n + \frac{\delta}{\varepsilon}b^n \le M,
$$

where M is a constant independent of ε. Then, for any compact region $K \subset \mathbb{R}_+$, there exists $M > 0$ independent of ε such that

$$\int_0^T \int_K \big((\rho^\varepsilon)^{\gamma+1} + \delta(\rho^\varepsilon)^3 + \rho^\varepsilon|u^\varepsilon|^3 + (\rho^\varepsilon)^{\gamma+\theta}\big)\,dx\,dt \le M.$$

In addition, if $\psi(s)$ is any smooth, compactly supported test function on $\mathbb{R}$, then the associated entropy pair (η^ψ, q^ψ) satisfies

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \qquad \text{are compact in } H^{-1}_{loc}.$$

The convergence of the approximate solutions to an entropy solution of the spherically symmetric Euler equations now follows from the standard argument in [13], as in §3.5. This completes the proof of Theorem 4.2.5.

4.7.2 Neumann boundary conditions for the density

Finally, we consider the approximate solutions of system (4.7.1) with the boundary data given by

$$(\rho_x, m)|_{x=a} = (0,0), \qquad (\rho, m)|_{x=b} = (\bar\rho, 0). \tag{4.7.4}$$

This choice of Neumann data for the density at $x = a$ has the advantage that it allows the density of the approximate solutions to become very large near the origin, a motivation for its choice in [13].

From [13, Proposition 2.1], we see that the mechanical energy is uniformly bounded:

Lemma 4.7.5. *Suppose that (ρ, u) is a $C^{2,1}(Q_T)$ solution of problem (4.7.1) and (4.7.4). Then, for each $\varepsilon > 0$,*

$$\begin{aligned}&\sup_{[0,T]} \int_a^b \big(\frac{1}{2}\rho u^2 + \overline{h_\delta}(\rho, \bar\rho)\big) x^{n-1}\,dx \\ &+ \varepsilon \int_{Q_T} \big(h_\delta''(\rho)|\rho_x|^2 + \rho|u_x|^2 + (n-1)\frac{\rho u^2}{x^2}\big) x^{n-1}\,dx\,dt \le E_0.\end{aligned}$$

Observe now that $\overline{h_\delta}(\rho, \bar\rho)$ grows, for ρ large, as ρ^γ. In order to close the estimates of [13, Lemma 2.3] for all $\gamma > 1$, we require an additional higher order integrability of the density ρ. To this end, we take the test function $\psi(s) = s^4$ and consider the entropy η^ψ generated from it. We observe that, by a simple calculation, η^ψ satisfies

$$\rho u^4 + \rho^{2\gamma-1} \le M\eta^\psi(\rho, m) \qquad \text{for all } (\rho, m) \in \mathbb{R}^2_+, \tag{4.7.5}$$

where the constant M depends only on $\gamma > 1$.

Lemma 4.7.6. *Let*

$$\mathcal{E}_0 := \int_a^b \eta^\psi(\rho_0^\varepsilon, m_0^\varepsilon) x^{n-1}\, dx < \infty.$$

Then solutions $(\rho^\varepsilon, m^\varepsilon)$ *of* (4.7.1) *and* (4.7.4) *satisfy that, for any* $T > 0$,

$$\sup_{[0,T]} \int_a^b \eta^\psi(\rho^\varepsilon, m^\varepsilon) x^{n-1}\, dx \le \mathcal{E}_0. \tag{4.7.6}$$

In particular, there exists $C = C(\varepsilon)$ *such that*

$$\sup_{[0,T]} \int_a^b \rho^{2\gamma-1} x^{n-1}\, dx \le C. \tag{4.7.7}$$

The proof of this lemma follows from the same calculation as that of [13, Proposition 2.1], but with the new entropy η^ψ in place of η^*. Observe that η^ψ is also convex.

We are now in a position to see that the existence of the approximate solutions for the spherically symmetric Euler equations in [13, Theorem 2.1] (under the assumption of the Neumann boundary data for density ρ) can be extended to the full range of γ-law gases. It suffices to show only Lemma 2.3 in [13], since the rest of the argument follows as in the case $\gamma \in (1,3]$ considered already in [13]. We therefore show for the whole range $\gamma \in (1,\infty)$ that there exists C, depending on $\varepsilon > 0$, such that the approximate solutions $(\rho^\varepsilon, m^\varepsilon)$ satisfy

$$\sup_{[0,T]} \int_a^b \left(|\rho_x^\varepsilon|^2 + |m_x^\varepsilon|^2\right) dx + \int_{Q_T} \left(|\rho_{xx}^\varepsilon|^2 + |m_{xx}^\varepsilon|^2\right) dx\, dt \le C.$$

Proof. We proceed as in the proof of [13, Lemma 2.3] and Lemma 4.3.5. Throughout this proof, $\Delta > 0$ is a constant which will be taken small at the end of the argument. Multiplying the first equation in (4.7.1) by ρ_{xx} and the second by m_{xx}, we obtain

$$\begin{aligned}
&(\rho_t\rho_x)_x - \frac{1}{2}\partial_t\left(|\rho_x|^2\right) + (m_t m_x)_x - \frac{1}{2}\partial_t\left(|m_x|^2\right) - \varepsilon\left(|\rho_{xx}|^2 + |m_{xx}|^2\right)\\
&= - m_x\rho_{xx} - (\rho u^2 + p_\delta)_x m_{xx} - \frac{1}{x} m\rho_{xx} - \frac{1}{x}\rho u^2 m_{xx} + \varepsilon\frac{1}{x}\rho_x\rho_{xx} + \varepsilon\left(\frac{1}{x}m\right)_x m_{xx}.
\end{aligned}$$

Integrating over Q_t and recalling that ρ, m are constant on $x = a, b$,

$$\begin{aligned}
&\frac{1}{2}\int_a^b \left(|\rho_x(t,x)|^2 + |m_x(t,x)|^2\right) dx + \varepsilon\int_{Q_t}\left(|\rho_{xx}|^2 + |m_{xx}|^2\right) dx\, d\tau\\
&= \frac{1}{2}\int_a^b \left(|\rho_{0,x}|^2 + |m_{0,x}|^2\right) dx + \int_{Q_t}\big(m_x\rho_{xx} + \frac{n-1}{x} m\rho_{xx}\big)\, dx\, d\tau\\
&\quad + \int_{Q_t} (\rho u^2 + p_\delta)_x m_{xx}\, dx\, d\tau + (n-1)\int_{Q_t}\big(\frac{1}{x}\rho u^2 m_{xx} - \frac{\varepsilon}{x}\rho_x\rho_{xx}\big)\, dx\, d\tau\\
&\quad - (n-1)\varepsilon\int_{Q_t}\left(\frac{m}{x}\right)_x m_{xx}\, dx\, d\tau.
\end{aligned}$$

We first give the estimate of the critical term $(\rho u^2)_x m_{xx} = u^2\rho_x m_{xx} + 2\rho u u_x m_{xx}$. Applying Young's inequality, we obtain that, for $\Delta > 0$ to be chosen later,

$$\int_{Q_t} |u^2\rho_x m_{xx}|\,dx\,d\tau \le \Delta \int_{Q_t} |m_{xx}|^2\,dx\,d\tau \\ + C_\Delta \int_0^t \Big(\|u(\tau,\cdot)\|_{L^\infty}^4 \int_a^b h''_\delta(\rho)|\rho_x(\tau,x)|^2\,dx\Big)\,d\tau.$$

From the maximum principle lemma of [13], Lemma 2.2,

$$\|u(\tau,\cdot)\|_{L^\infty}^4 \le C\Big(1 + \|\rho\|_{L^\infty(Q_\tau)}^{2\max\{1,\gamma-1\}}\Big),$$

which yields

$$\int_{Q_t} |u^2\rho_x m_{xx}|\,dx\,d\tau \le \Delta \int_{Q_t} |m_{xx}|^2\,dx\,d\tau \\ +C_\Delta \int_0^t \Big(\big(1 + \sup_{s\in[0,\tau]} \|\rho(s,\cdot)\|_{L^\infty}^{2\max\{1,\gamma-1\}}\big) \int_a^b h''_\delta(\rho)|\rho_x(\tau,x)|^2\,dx\Big)\,d\tau.$$

Now we can use the improved estimate (4.7.7) and the argument of Lemma 4.3.3 to show that

$$\|\rho^{2\gamma+1}\|_{L^\infty(a,b)} \le C\Big(1 + \int_a^b |\rho_y|^2 dy\Big).$$

We therefore obtain

$$\int_{Q_t} |u^2\rho_x m_{xx}|\,dx\,d\tau \le \Delta \int_{Q_t} |m_{xx}|^2\,dx\,d\tau \\ + C_\Delta \int_0^t \Big(\big(1 + \sup_{s\in[0,\tau]} \int_a^b |\rho_x(s,x)|^2\,dx\big) \int_a^b h''_\delta(\rho)|\rho_x(\tau,x)|^2\,dx\Big)\,d\tau$$

for the whole range $\gamma \in (1,\infty)$, especially including $\gamma > 3$. Moreover,

$$\begin{aligned}\int_{Q_t} |\rho u u_x m_{xx}|\,dx\,d\tau \le\, & \Delta \int_{Q_t} |m_{xx}|^2\,dx\,d\tau \\ &+C_\Delta \int_0^t \big(\|\rho u^2(\tau,\cdot)\|_{L^\infty} \int_a^b \rho(\tau,x)|u_x(\tau,x)|^2\,dx\big)\,d\tau \\ \le\, & \Delta \int_{Q_t} |m_{xx}|^2\,dx\,d\tau \\ &+C_\Delta \int_0^t \Big(\big(1 + \sup_{s\in[0,\tau]} \int_a^b |\rho_x(s,x)|^2\,dx\big) \int_a^b \rho(\tau,x)|u_x(\tau,x)|^2\,dx\Big)d\tau.\end{aligned}$$

Considering the pressure term,

$$\begin{aligned}\Big|\int_{Q_t} (p_\delta)_x m_{xx}\,dx\,d\tau\Big| &\le \Delta \int_{Q_t} |m_{xx}|^2\,dx\,d\tau + C_\Delta \int_{Q_t} \big(2\delta\rho + \kappa\gamma\rho^{\gamma-1}\big)^2 |\rho_x|^2\,dx\,d\tau \\ &\le \Delta \int_{Q_t} |m_{xx}|^2\,dx\,d\tau + C_\Delta \int_0^t \big(1 + \|\rho(\tau,\cdot)\|_{L^\infty}^{2\gamma}\big) \int_a^b |\rho_x|^2\,dx\,d\tau,\end{aligned}$$

where we have used that $h_\delta'' = \kappa\gamma\rho^{\gamma-2} + 2\delta$, the maximum principle [13, Lemma 2.2], and finally Lemma 4.3.2.

Combining these,

$$\begin{aligned}\Big|\int_{Q_t} (\rho u^2 + p_\delta)_x m_{xx}\, dx\, d\tau\Big| \leq \Delta \int_{Q_t} |m_{xx}|^2\, dx\, d\tau \\ + C_\Delta \int_0^t \Phi_1(\tau)\big(1 + \sup_{s\in[0,\tau]} \int_a^b |\rho_x(s,x)|^2\, dx\big)\, d\tau,\end{aligned}$$

where $\Phi_1(\tau) = \int_a^b \big(h_\delta''(\rho)|\rho_x(\tau,x)|^2 + \rho(\tau,x)|u_x(\tau,x)|^2\big)\, dx$. For the other terms, we observe that on the interval (a,b), x, x^{-1} and their derivatives remain bounded. Thus we see, for example,

$$\begin{aligned}\Big|\int_{Q_t} \frac{1}{x}\rho u^2 m_{xx}\, dx\, d\tau\Big| &\leq \Delta \int_{Q_t} |m_{xx}|^2\, dx\, d\tau + C_\Delta \int_0^t \big(\|\rho u^2(\tau,\cdot)\|_{L^\infty} \int_a^b \rho u^2(\tau,x)\, dx\big)\, d\tau \\ &\leq \Delta \int_{Q_t} |m_{xx}|^2\, dx\, d\tau + C_\Delta \int_0^t \big(1 + \sup_{s\in[0,\tau]} \int_a^b |\rho_x(s,x)|^2\, dx\big)\, d\tau.\end{aligned}$$

Remaining terms may be dealt with likewise. Thus, combining all of these estimates,

$$\begin{aligned}\sup_{\tau\in[0,t]} &\int_a^b \big(|\rho_x(\tau,x)|^2 + |m_x(\tau,x)|^2\big)\, dx + \varepsilon \int_{Q_\tau} \big(|\rho_{xx}|^2 + |m_{xx}|^2\big)\, dx\, d\tau \\ &\leq \Delta \int_{Q_\tau} \big(|\rho_{xx}|^2 + |m_{xx}|^2\big)\, dx\, d\tau + \int_a^b \big(|\rho_{0,x}(x)|^2 + |m_{0,x}(x)|^2\big)\, dx \\ &\quad + C_\Delta \int_0^t \big(1 + \Phi(\tau)\big)\big(1 + \sup_{[s\in[0,\tau]} \int_a^b \Big(|\rho_x(s,x)|^2 + |m_x(s,x)|^2\Big)\, dx\, d\tau,\end{aligned}$$

with $\Phi(\tau) = \Phi_1(\tau) + \|(\rho(\tau,\cdot)\|_{L^\infty}^{2\max\{2,\gamma\}}$. We may then take Δ small to absorb the first term onto the left and conclude by a Gronwall argument. $\square$

This result now allows us to conclude the argument of Theorem 2.1 in [13] as it is done there. As remarked previously, this is sufficient to apply the results of §3 in [13] to conclude Theorem 4.7.1, and hence Theorem 4.2.5.

Chapter 5

Vanishing viscosity limit of the Navier-Stokes equations

5.1 Introduction

In what we have considered so far, we have been concerned with the vanishing viscosity limit of sequences of approximate solutions to the Euler equations generated by the introduction of artificial viscosity to the equations. In this chapter, we move to the more physically appropriate situation of the vanishing physical viscosity limit, that is, the limit of the solutions to the one-dimensional compressible, isentropic Navier-Stokes equations. We develop a framework that allows us to pass to the limit in these equations under the assumption of a general, physically meaningful pressure law.

We recall that the compressible, isentropic Navier-Stokes equations may be written as:

$$\begin{cases} \rho_t + (\rho u)_x = 0, & (t,x) \in \mathbb{R}^2_+, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = \varepsilon u_{xx}, & (t,x) \in \mathbb{R}^2_+, \end{cases} \tag{5.1.1}$$

where $\rho \geq 0$ is the density of a fluid, u is its velocity and p is the pressure, determined by an equation of state of the form $p = p(\rho)$. The positive constant $\varepsilon > 0$ is the viscosity of the fluid. We complement this system of equations with initial data

$$(\rho, u)|_{t=0} = (\rho_0, u_0), \tag{5.1.2}$$

where, as in the previous chapter, the initial data should be of relative finite-energy with (possibly non-trivial) end-states $(\rho_\pm, u_\pm)$ at $x = \pm\infty$.

Rather than consider the polytropic gases, that is, those with a γ-law pressure, we consider instead a more physical class of pressure functions such that the pressure is asymptotically a γ-law in the low density regime, but grows linearly with the density ρ as ρ grows large. In particular, we assume the conditions of strict hyperbolicity and genuine non-linearity of the characteristic fields for $\rho > 0$,

$$p'(\rho) > 0 \quad \text{and} \quad \rho p''(\rho) + 2p'(\rho) > 0 \quad \text{for } \rho > 0, \tag{5.1.3}$$

and that, for some $\gamma \in (1,3)$,

$$p(\rho) = \kappa\rho^\gamma \left(1 + P(\rho)\right), \quad P^{(n)}(\rho) = O\big(\rho^{2\theta - n}\big) \text{ as } \rho \to 0 \text{ for } 0 \leq n \leq 3, \tag{5.1.4}$$

where $\kappa > 0$ and $\theta = \frac{\gamma-1}{2}$. For large densities, we assume there exists $\rho_* > 0$ such that

$$p(\rho) = c_* \rho \text{ for } \rho \geq \rho_*. \tag{5.1.5}$$

Without loss of generality, we assume that $c_* = 1$ and that $\rho_* = 1$ also.

Formally, as $\varepsilon \to 0$, solutions of (5.1.1) converge to solutions of the Euler equations,

$$\begin{cases} \rho_t + m_x = 0, & (t,x) \in \mathbb{R}^2_+, \\ m_t + \big(\frac{m^2}{\rho} + p(\rho)\big)_x = 0, & (t,x) \in \mathbb{R}^2_+, \end{cases} \tag{5.1.6}$$

where $m = \rho u$.

We recall again the notion of finite energy solutions, discussed in the Introduction and analogous to that in Chapter 4. We begin by recalling the mechanical entropy pair,

$$\eta^*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho),$$
$$q^*(\rho, m) = \frac{1}{2}\frac{m^3}{\rho^2} + me(\rho) + \rho m e'(\rho),$$

where the internal energy $e(\rho)$ is related to the pressure via the relation

$$p(\rho) = \rho^2 e'(\rho).$$

We choose fixed, smooth, monotone functions $(\bar{\rho}(x), \bar{u}(x))$ such that, for some $L_0 > 1$,

$$(\bar{\rho}(x), \bar{u}(x)) = \begin{cases} (\rho_+, u_+), & x \geq L_0, \\ (\rho_-, u_-), & x \leq -L_0. \end{cases}$$

The relative mechanical energy with respect to $(\bar{\rho}(x), \bar{m}(x)) = (\bar{\rho}(x), \bar{\rho}(x)\bar{u}(x))$ is then

$$\begin{aligned} \overline{\eta^*}(\rho, m) &= \eta^*(\rho, m) - \eta^*(\bar{\rho}, \bar{m}) - \nabla\eta^*(\bar{\rho}, \bar{m}) \cdot (\rho - \bar{\rho}, m - \bar{m}) \\ &= \frac{1}{2}\rho|u - \bar{u}|^2 + e^*(\rho, \bar{\rho}) \geq 0, \end{aligned}$$

where $e^*(\rho, \bar{\rho}) = \rho e(\rho) - \bar{\rho} e(\bar{\rho}) - (\bar{\rho} e'(\bar{\rho}) + e(\bar{\rho}))(\rho - \bar{\rho}) \geq 0$.

The total relative mechanical energy is now defined by

$$E[\rho, u](t) := \int_{\mathbb{R}} \overline{\eta^*}(\rho, \rho u)(t,x)\, dx. \tag{5.1.7}$$

We recall that an entropy/entropy-flux pair (or entropy pair, for simplicity) is a pair of functions $(\eta, q) : \mathbb{R}^2_+ \to \mathbb{R}^2$ such that

$$\nabla q(\rho, m) = \nabla\eta(\rho, m)\nabla \begin{pmatrix} m \\ \frac{m^2}{\rho} + p(\rho) \end{pmatrix},$$

where ∇ is the gradient with respect to the conservative variables (ρ, m). Eliminating q by matching derivatives, $q_{\rho m} = q_{m\rho}$, now leads to the *entropy equation*,

$$\eta_{\rho\rho} - \frac{p'(\rho)}{\rho^2}\eta_{uu} = 0.$$

Moreover, recalling [4, 8, 52] for example, any weak entropy (an entropy η vanishing at $\rho = 0$) may be generated by the convolution of a test function $\psi(s) \in C^2(\mathbb{R})$ with a fundamental solution $\chi(\rho, u - s)$ of the entropy equation, that is,

$$\eta^\psi(\rho, u) = \int_{\mathbb{R}} \psi(s)\chi(\rho, u - s)\, ds,$$

with a corresponding entropy flux generated from an entropy flux kernel $\sigma(\rho, u, s)$,

$$q^\psi(\rho, u) = \int_{\mathbb{R}} \psi(s)\sigma(\rho, u, s)\, ds.$$

Definition 5.1.1. Given initial data (ρ_0, u_0) of relative finite-energy, $E[\rho_0, u_0] \le E_0$, we say that a pair of functions $(\rho, u) : \mathbb{R}^2_+ \to \mathbb{R}^2$ with $\rho \ge 0$ is a relative finite-energy entropy solution of the Euler equations (5.1.6) if:

(i) There exists a constant $M(E_0, t)$, monotone increasing and continuous with respect to t, such that

$$E[\rho, u](t) \le M(E_0, t) \text{ for almost every } t \ge 0;$$

(ii) For any $\phi \in C_c^\infty(\overline{\mathbb{R}^2_+})$,

$$\begin{aligned} &\int_{\mathbb{R}^2_+} \big(\rho\phi_t + m\phi_x\big)\, dx\, dt + \int_{\mathbb{R}} \rho_0(x)\phi(0, x)\, dx = 0, \\ &\int_{\mathbb{R}^2_+} \big(m\phi_t + \big(\frac{m^2}{\rho} + p(\rho)\big)\phi_x\big)\, dx\, dt + \int_{\mathbb{R}} m_0(x)\phi(0, x)\, dx = 0; \end{aligned} \tag{5.1.8}$$

(iii) There exists a bounded Radon measure $\mu(t, x, s)$ on $\mathbb{R}^2_+ \times \mathbb{R}$ such that

$$\mu(U \times \mathbb{R}) \ge 0 \qquad \text{for any open set } U \subset \mathbb{R}^2_+,$$

and the corresponding entropy kernel and its flux satisfy

$$\partial_t\chi(\rho(t, x), u(t, x), s) + \partial_x\sigma(\rho(t, x), u(t, x), s) = \partial_s^2\mu(t, x, s), \tag{5.1.9}$$

in the sense of distributions on $\mathbb{R}^2_+ \times \mathbb{R}$.

The main theorem of this chapter, working in the framework of relative finite-energy solutions to the Euler equations, is the following.

Theorem 5.1.2 (Vanishing Viscosity Limit of the Navier-Stokes Equations)**.** *Suppose that $(\rho_0, u_0) \in L^1_{loc}(\mathbb{R};\mathbb{R}^2)$ with $\rho_0 \geq 0$ is initial data of relative finite-energy,*

$$E[\rho_0, u_0] = \int_{\mathbb{R}} \overline{\eta^*}(\rho_0, \rho_0 u_0)\, dx \leq E_0 < \infty,$$

and suppose that the pressure function $p(\rho)$ satisfies (5.1.3)–(5.1.5).

Then there exists a sequence of regularised initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$ such that the unique, smooth solutions $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ to (5.1.1) with this initial data converge almost everywhere to a relative finite-energy entropy solution (ρ, m) of the Euler equations (5.1.6) with initial data $(\rho_0, m_0) = (\rho_0, \rho_0 u_0)$. The convergence may be taken also in $L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+)$, where $p \in [1, 2)$ and $q \in [1, 3/2)$.

In order to prove this theorem, we require three main ingredients. The first is uniform estimates on solutions to the Navier-Stokes equations (5.1.1). Without such estimates, we cannot hope to pass to the limit as the viscosity tends to zero. It is well-known that system (5.1.1) does not admit an invariant region in the sense of Chueh-Conley-Smoller [17], and so we work in the regime of finite-energy solutions, relying on the fact, which we prove in §5.3, that the solutions to (5.1.1) admit uniform energy estimates.

In order to obtain the crucial estimate of higher integrability of the velocity, we require a thorough understanding of the entropy pairs of the system with the general pressure law $p(\rho)$. We especially need a good understanding of the behaviour of the entropy pairs for large densities and velocities, and so in §5.2, we prove the existence of a new representation formula for such entropies. We use this new representation to obtain estimates on entropy pairs generated by compactly supported functions as well as constructing a special entropy pair in order to obtain a higher integrability estimate for the velocity. In §5.2, we also provide a detailed analysis of the singularities of the entropy and entropy flux kernels used for the representation formulae.

This understanding of the entropy kernels is crucial for the third and final ingredient in the proof. We use the general framework of §3.3 to construct a Young measure generated by the solutions in §5.4, and then apply the generalised div-curl lemma, Lemma 3.4.2, to show that this measure is constrained by the Tartar commutation relation. Finally, in §5.5, we show that this relation represents an imbalance of regularity, which we make precise using the analysis of the entropy kernels developed in the earlier sections of the chapter to conclude that the Young measure is always supported at a single point.

5.2 Entropy and entropy flux kernels

The purpose of this section is to analyse the structure of weak entropies to the isentropic Euler equations with pressure law as discussed above and to derive a fundamental solution of the entropy equation to generate such entropies. We recall first of all that the entropy kernel $\chi = \chi(\rho, u, s)$ satisfies the following equation:

$$\begin{cases} \chi_{\rho\rho} - k'(\rho)^2 \chi_{uu} = 0; \\ \chi|_{\rho=0} = 0; \\ \chi_\rho|_{\rho=0} = \delta_{u=s}; \end{cases}$$

where

$$k(\rho) = \int_0^\rho \frac{\sqrt{p'(s)}}{s}\, ds$$

is as defined in §2.3. We note that this equation is invariant under Galilean transformations, and so $\chi(\rho, u, s) = \chi(\rho, u - s, 0) = \chi(\rho, 0, s - u)$. We therefore write, in a slight abuse of notation, $\chi = \chi(\rho, u - s)$. It was shown in [8, 9] that for pressure laws such as we are considering, this equation is well-posed, giving rise to a solution of the form

$$\chi(\rho, u - s) = a_\sharp(\rho) G_\lambda(\rho, u - s) + a_\flat(\rho) G_{\lambda+1}(\rho, u - s) + g_1(\rho, u - s), \tag{5.2.1}$$

where $G_\lambda(\rho, u - s) = [k(\rho)^2 - (u - s)^2]_+^\lambda$, $\lambda = \frac{3-\gamma}{2(\gamma-1)} > 0$, and where the remainder function g_1 and its fractional derivative $\partial_u^{\lambda+1} g_1$ are Hölder continuous and, for any fixed $\rho_{\max} > 0$, satisfy

$$|g_1(\rho, u - s)| \leq C(\rho_{\max})[k(\rho)^2 - (u - s)^2]_+^{\lambda+1+\alpha_0} \quad \text{for } 0 \leq \rho \leq \rho_{\max}$$

and some $\alpha_0 \in (0, 1)$. Moreover, there exists a constant $M_\lambda > 0$ such that the coefficients $a_\sharp(\rho)$ and $a_\flat(\rho)$ satisfy, for $0 \leq \rho \leq \rho_{\max}$,

$$\begin{aligned} a_\sharp(\rho) &= M_\lambda k(\rho)^{-\lambda} k'(\rho)^{-\frac{1}{2}} > 0; \\ a_\sharp(\rho) &+ |a_\flat(\rho)| \leq C(\rho_{\max}). \end{aligned}$$

The associated entropy flux kernel $\sigma(\rho, u, s)$ satisfies the equation

$$\begin{cases} (\sigma - u\chi)_{\rho\rho} - k'(\rho)^2 (\sigma - u\chi)_{uu} = \frac{p''(\rho)}{\rho} \chi_u; \\ (\sigma - u\chi)|_{\rho=0} = 0; \\ (\sigma - u\chi)_\rho|_{\rho=0} = 0; \end{cases}$$

where we recall from [8] that the difference $\sigma - u\chi$ satisfies the same Galilean invariance property as χ. This function is given by the expansion

$$(\sigma - u\chi)(\rho, u - s) = -(u - s)\big(b_\sharp(\rho) G_\lambda(\rho, u - s) + b_\flat(\rho) G_{\lambda+1}(\rho, u - s)\big) + g_2(\rho, u - s). \tag{5.2.2}$$

The coefficients $b_\sharp(\rho)$ and $b_\flat(\rho)$ satisfy

$$\begin{aligned} & b_\sharp(\rho) = M_\lambda \rho k(\rho)^{-\lambda-1} k'(\rho)^{\frac{1}{2}} = \frac{\rho k'(\rho)}{k(\rho)} a_\sharp(\rho) > 0; \\ & b_\sharp(\rho) + |b_\flat(\rho)| \leq C(\rho_{\max}). \end{aligned}$$

Again, the remainder function g_2 and its fractional derivative $\partial_u^{\lambda+1} g_2$ are Hölder continuous and satisfy, for $0 \leq \rho \leq \rho_{\max}$,

$$|g_2(\rho, u-s)| \leq C(\rho_{\max})[k(\rho)^2 - (u-s)^2]_+^{\lambda+1+\alpha_0}$$

for some $\alpha_0 \in (0,1)$.

Proposition 5.2.1 ([8, Proposition 2.4])**.** *The coefficients in the expansions of the entropy kernels satisfy*

$$D(\rho) = a_\sharp(\rho) b_\sharp(\rho) - k(\rho)^2 \big(a_\sharp(\rho) b_\flat(\rho) - a_\flat(\rho) b_\sharp(\rho) \big) > 0.$$

As we cannot rely on an invariant region to give *a priori* bounds in L^∞ for the solutions to the Navier-Stokes equations, we need to understand better the behaviour of the entropies as $\rho \to \infty$. In order to obtain good control over the entropy kernels at high density, we search for new expansions for the kernels in this region as convolutions with fundamental solutions of the entropy equations for the isothermal gas dynamics (i.e. $p(\rho) = \rho$). It was noted in [47] that such fundamental solutions should properly be based not around the vacuum, but around the line $\{\rho = 1\}$. We therefore look for solutions of

$$\left\{ \begin{array}{cc} \chi^\sharp_{\rho\rho} - \frac{1}{\rho^2}\chi^\sharp_{uu} = 0; & \chi^\flat_{\rho\rho} - \frac{1}{\rho^2}\chi^\flat_{uu} = 0; \\ \chi^\sharp|_{\rho=1} = 0; & \chi^\flat|_{\rho=1} = \delta_{u=0}; \\ \chi^\sharp_\rho|_{\rho=1} = \delta_{u=0}; & \chi^\flat_\rho|_{\rho=1} = 0. \end{array} \right\}$$

We then obtain from such kernels an expansion of the form

$$\chi(\rho, u) = \chi_\rho(1, u) * \chi^\sharp(\rho, u) + \chi(1, u) * \chi^\flat(\rho, u).$$

5.2.1 Entropy kernels for isothermal gas dynamics

We recall that for $\rho \geq 1$, the pressure is given by $p(\rho) = \rho$. It can easily be seen that in the (ρ, u) coordinates, any weak entropy function for the Euler equations with this isothermal pressure must satisfy the singular wave equation

$$\begin{cases} \eta_{\rho\rho} - \frac{1}{\rho^2}\eta_{uu} = 0, \\ \eta|_{\rho=0} = 0. \end{cases}$$

Changing coordinates to (R,u) coordinates, where $R=\log\rho$, the wave equation is

$$\mathbf{L}(\eta) := \eta_{RR} - \eta_{uu} - \eta_R = 0.$$

Following [47], we let $f=f(m)$ be the function given by

$$f(m) := \sum_{n=0}^{\infty} (\frac{A^n}{n!})^2 (-1)^n m^n,$$

where $A^2 = \frac{1}{16}$, so that

$$mf'' + f' + A^2 f = 0,$$

$f(0)=1,\ f'(0) = -A^2$, and

$$f(-y^2) = \sum_{n=0}^{\infty} \left(\frac{A^n y^n}{n!}\right)^2 = I_0\left(\frac{y}{2}\right),$$

where I_0 is the modified Bessel function of the first kind.

Theorem 5.2.2 (Isothermal Entropy Kernels)**.** *The function*

$$\chi^\sharp(R,u-s) := \frac{1}{2}\mathrm{sgn}(R) e^{\frac{R}{2}} f(|u-s|^2 - R^2)\mathbb{1}_{|u-s|<|R|} \tag{5.2.3}$$

and measure (considered for fixed R as a measure in $u-s$)

$$\begin{aligned} \chi^\flat(R,u-s) = &\frac{1}{2}e^{R/2}\big(\delta_{u-s=R} + \delta_{u-s=-R}\big) \\ &- \frac{1}{2}e^{R/2}\big(\frac{1}{2}f((u-s)^2 - R^2) + 2Rf'((u-s)^2-R^2)\big)\mathbb{1}_{|u-s|<|R|} \end{aligned} \tag{5.2.4}$$

satisfy

$$\begin{Bmatrix} \chi^\sharp_{RR} - \chi^\sharp_{uu} - \chi^\sharp_R = 0; & \chi^\flat_{RR} - \chi^\flat_{uu} - \chi^\flat_R = 0; \\ \lim_{R\to 0}\chi^\sharp(R,\cdot) = 0; & \lim_{R\to 0}\chi^\flat(R,\cdot) = \delta_{u=s}; \\ \lim_{R\to 0}\chi^\sharp_R(R,\cdot) = \delta_{u=s}; & \lim_{R\to 0}\chi^\flat_R(R,\cdot) = 0; \\ \lim_{R\to -\infty}\chi^\sharp(R,\cdot) = 0; & \lim_{R\to -\infty}\chi^\flat(R,\cdot) = 0; \end{Bmatrix} \tag{5.2.5}$$

respectively in the sense of distributions. Moreover, the measure $\chi^\flat(R,u-s)$ may also be expressed as $\chi^\flat(R,u-s) = \chi^\sharp_R(R,u-s) - \chi^\sharp(R,u-s)$.

Proof. It suffices by Galilean invariance to solve only in the case $s=0$. We write, for any function $g=g(R,u)$, $\bar{g} = \mathrm{sgn}(R)g(R,u)\mathbb{1}_{|u|<|R|}$. In particular,

$$\bar{f}(u^2-R^2) = \mathrm{sgn}(R)f(u^2-R^2)\mathbb{1}_{|u|<|R|}.$$

Then we have

$$\mathbf{L}(\chi^\sharp) = e^{\frac{R}{2}}\Big(\bar{f}_{RR} - \bar{f}_{uu} - \frac{\bar{f}}{4}\Big).$$

Let $\phi \in \mathcal{D}(\mathbb{R})$ be a test function. Then we calculate

$$\begin{aligned}
\langle \bar{f}_{RR}, \phi\rangle =&\langle \bar{f}, \phi_{RR}\rangle = \iint_{R>|u|} f\phi_{RR}\, dR\, du - \iint_{R<-|u|} f\phi_{RR}\, dR\, du \\
=&\iint_{R>|u|} \big((\phi_R f)_R + 2Rf'\phi_R\big)\, dR\, du - \iint_{R<-|u|} \big((\phi_R f)_R + 2Rf'\phi_R\big)\, dR\, du \\
=&\int_{\mathbb{R}} \big(-f(0)\phi_R(|u|,u) - f(0)\phi_R(-|u|,u)\big)\, du \\
&+ \iint_{R>|u|} \big((2Rf'\phi)_R - \phi(2f' - 4R^2 f'')\big)\, dR\, du \\
&- \iint_{R<-|u|} \big((2Rf'\phi)_R - \phi(2f' - 4R^2 f'')\big)\, dR\, du \\
=&-\int_{\mathbb{R}} \big(\phi_R(|u|,u) + \phi_R(-|u|,u)\big)\, du - 2f'(0)\int_{\mathbb{R}} |u|\big(\phi(|u|,u) - \phi(-|u|,u)\big)\, du \\
&+ \iint_{|R|>|u|} \phi\, \mathrm{sgn}(R)(4R^2 f'' - 2f')\, dR\, du \\
=&\langle \overline{4R^2 f'' - 2f'}, \phi\rangle + \langle J_1, \phi\rangle,
\end{aligned}$$

where

$$\begin{aligned}
\langle J_1, \phi\rangle =&-\int_{-\infty}^{0} \big(\phi_R(-u,u) + \phi_R(u,u)\big)\, du - \int_{0}^{\infty} \big(\phi_R(u,u) + \phi_R(-u,u)\big)\, du \\
&+ 2f'(0)\left(\int_{-\infty}^{0} u\big(\phi(-u,u) - \phi(u,u)\big)\, du + \int_{0}^{\infty} u\big(-\phi(u,u) + \phi(-u,u)\big)\, du\right) \\
=&-\int_{\mathbb{R}} \big(\phi_R(-u,u) + \phi_R(u,u)\big)\, du + 2f'(0)\int_{\mathbb{R}} -u\big(\phi(u,-u) + \phi(u,u)\big)\, du.
\end{aligned}$$

Also,

$$\begin{aligned}
\langle \bar{f}_{uu}, \phi\rangle =&\int\int_{|u|<|R|} f\mathrm{sgn}(R)\phi_{uu}\, du\, dR \\
=&\int_{0}^{\infty}\int_{-R}^{R} f\phi_{uu}\, du\, dR - \int_{-\infty}^{0}\int_{-|R|}^{|R|} f\phi_{uu}\, du\, dR \\
=&\int_{0}^{\infty}\int_{-R}^{R} \big((\phi_u f)_u - 2u\phi_u f'\big)\, du\, dR - \int_{-\infty}^{0}\int_{-|R|}^{|R|} \big((\phi_u f)_u - 2u\phi_u f'\big)\, du\, dR \\
=&\int_{0}^{\infty} f(0)\big(\phi_u(R,R) - \phi_u(R,-R)\big)\, dR \\
&- \int_{-\infty}^{0} f(0)\big(\phi_u(R,-R) - \phi_u(R,R)\big)\, dR \\
&+ \int_{0}^{\infty}\int_{-R}^{R} \big(\phi(2f' + 4u^2 f'') - 2(uf'\phi)_u\big)\, du\, dR \\
&- \int_{-\infty}^{0}\int_{-|R|}^{|R|} \big(\phi(2f' + 4u^2 f'') - 2(uf'\phi)_u\big)\, du\, dR \\
=&\langle \overline{2f' + 4u^2 f''}, \phi\rangle + \langle J_2, \phi\rangle,
\end{aligned}$$

where

$$\begin{aligned}
\langle J_2, \phi\rangle &= \int_{\mathbb{R}} \big(\phi_u(R,R) - \phi_u(R,-R)\big)\, dR \\
&\quad - 2f'(0)\int_0^\infty R\big(\phi(R,R) + R\phi(R,-R)\big)\, dR \\
&\quad + 2f'(0)\int_{-\infty}^0 \big(-R\phi(R,-R) - R\phi(R,R)\big)\, dR \\
&= \int_{\mathbb{R}} \big(\phi_u(R,R) - \phi_u(R,-R)\big)\, dR \\
&\quad - 2f'(0)\int_{\mathbb{R}} \big(R\phi(R,R) + R\phi(R,-R)\big)\, dR.
\end{aligned}$$

Now we observe that

$$\begin{aligned}
\frac{d}{dR}\phi(R,R) &= \phi_u(R,R) + \phi_R(R,R), \\
\frac{d}{dR}\phi(R,-R) &= -\phi_u(R,-R) + \phi_R(R,-R),
\end{aligned}$$

so that

$$\begin{aligned}
\langle J_2, \phi\rangle = &- \int_{\mathbb{R}} \big(\phi_R(R,R) + \phi_R(R,-R)\big)\, dR \\
&- 2f'(0)\int_{\mathbb{R}} \big(R\phi(R,R) + R\phi(R,-R)\big)\, dR.
\end{aligned}$$

Then we have that

$$\bar{f}_{RR} - \bar{f}_{uu} - \frac{\bar{f}}{4} = J_1 - J_2 = 0$$

in terms of distributions.

Now we fix $\phi = \phi(u), \psi = \psi(R) \in \mathcal{D}(\mathbb{R})$. Then we may calculate,

$$\begin{aligned}
2\langle \chi_R^\sharp, \phi\psi\rangle &= -\int_{\mathbb{R}} \phi(u) \int_{|R|>|u|} \operatorname{sgn}(R) e^{\frac{R}{2}} f(u^2 - R^2)\psi_R \, dR\, du \\
&= -\int_{\mathbb{R}} \phi(u) \Big(\int_{|u|}^\infty e^{\frac{R}{2}} f(u^2 - R^2)\psi_R\, dR - \int_{-\infty}^{-|u|} e^{\frac{R}{2}} f(u^2-R^2)\psi_R\, dR\Big)\, du \\
&= -\int_{\mathbb{R}} \phi(u)\Big(- e^{\frac{|u|}{2}} f(0)\psi(|u|) - \int_{|u|}^\infty \psi(R) e^{\frac{R}{2}}\Big(\frac{f}{2} - 2Rf'\Big) dR\Big)\, du \\
&\quad + \int_{\mathbb{R}} \phi(u)\Big(e^{\frac{-|u|}{2}} f(0)\psi(-|u|) - \int_{-\infty}^{-|u|} \psi(R) e^{\frac{R}{2}}\Big(\frac{f}{2} - 2Rf'\Big) dR\Big)\, du \\
&= \int_{\mathbb{R}} \phi(u)\Big(e^{-\frac{|u|}{2}}\psi(-|u|) + e^{\frac{|u|}{2}}\psi(|u|)\Big)\, du \\
&\quad + \int\int_{|R|>|u|} \operatorname{sgn}(R)\phi(u)\psi(R) e^{\frac{R}{2}}\Big(\frac{f}{2} - 2Rf'\Big)\, du\, dR.
\end{aligned}$$

We note that

$$
\begin{aligned}
\int_{\mathbb{R}} \phi(u) e^{-\frac{|u|}{2}} \psi(-|u|)\, du &= \int_{-\infty}^{0} \phi(u) e^{\frac{u}{2}} \psi(u)\, du + \int_{0}^{\infty} \phi(u) e^{-\frac{u}{2}} \psi(-u)\, du \\
&= \int_{-\infty}^{0} \phi(R) e^{\frac{R}{2}} \psi(R)\, dR + \int_{-\infty}^{0} \phi(-R) e^{\frac{R}{2}} \psi(R)\, dR,
\end{aligned}
$$

and also

$$
\int_{\mathbb{R}} \phi(u) e^{\frac{|u|}{2}} \psi(|u|)\, du = \int_{0}^{\infty} (\phi(R) + \phi(-R)) e^{\frac{R}{2}} \psi(R)\, dR.
$$

Hence, dropping ψ, we have obtained

$$
\begin{aligned}
\langle \chi^\sharp_R(R,\cdot), \phi(\cdot)\rangle = &\int_{|u|<|R|} \frac{1}{2}\mathrm{sgn}(R)\phi(u) e^{\frac{R}{2}} (\frac{f}{2} - 2Rf')\, du \\
&+ \frac{1}{2} e^{\frac{R}{2}} (\phi(R) + \phi(-R)),
\end{aligned}
$$

which may alternatively be written as

$$
\begin{aligned}
\chi^\sharp_R(R,u) = &\frac{1}{2}\mathrm{sgn}(R) e^{\frac{R}{2}} \Big(\frac{f(u^2 - R^2)}{2} - 2Rf'(u^2 - R^2)\Big) \mathbb{1}_{|u|<|R|} \\
&+ \frac{1}{2} e^{\frac{R}{2}} (\delta_{u=R} + \delta_{u=-R}).
\end{aligned}
$$

Now we observe that if we let $\chi^\flat(R, u - s) = \chi^\sharp_R(R, u - s) - \chi^\sharp(R, u - s)$, then $\chi^\flat_R = \chi^\sharp_{uu} = 0$ on the line $\{R = 0\}$. Moreover, $\chi^\flat(0, u) = \delta_{u=0}$ and, as $\mathbf{L}$ is a linear, constant coefficient operator, $\chi^\flat$ still satisfies the PDE $\mathbf{L}(\chi^\flat) = 0$. □

Remark 5.2.3. It is interesting to note that with these definitions, we now have two *independent* weak entropy kernels for the isothermal gas dynamics, in that both $\chi^\sharp$ and $\chi^\flat$ vanish at the vacuum, $\{\rho = 0\}$. This is in stark contrast with the case of the isentropic Euler equations, where we expect a weak entropy kernel and a strong entropy kernel. Moreover, one may check from the expressions given for these kernels that they produce singularities in their derivatives of a much higher order than those corresponding to the γ-law gas for $\gamma > 1$. Indeed, the derivatives of the kernel behave roughly as $\frac{1}{\rho\sqrt{|\log \rho|}}$ worse than their polytropic counterparts near the vacuum.

To generate the entropy flux associated to an entropy for the isothermal gas dynamics, we associate an entropy flux kernel to each of the entropy kernels produced above. To do this, we observe the following relations. Let (η, q) be an entropy pair. Then the function $Q = q - u\eta$ satisfies

$$
Q_R = \eta_u, \qquad Q_u = \eta_R - \eta. \tag{5.2.6}
$$

Theorem 5.2.4 (Isothermal Entropy Flux Kernels)**.** *The entropy flux kernels $\sigma^\sharp$ and $\sigma^\flat$ associated to $\chi^\sharp$ and $\chi^\flat$ respectively can be decomposed, for $R > 0$, as*

$$\sigma^\sharp(R, u-s) = u\chi^\sharp(R, u-s) + h^\sharp(R, u-s),$$
$$\sigma^\flat(R, u-s) = u\chi^\flat(R, u-s) + h^\flat(R, u-s),$$

where, with the notation $a \vee b = \max\{a, b\}$,

$$\begin{aligned}
h^\sharp(R, u-s) &= \frac{1}{2}\mathrm{sgn}(u-s) + \frac{\partial}{\partial u}\int_0^R \chi^\sharp(r, u-s)\,dr \\
&= \frac{1}{2}\mathrm{sgn}(u-s) - \frac{1}{2}\mathrm{sgn}(u-s)e^{|u-s|/2}\mathbb{1}_{|u-s|<R} \\
&\quad + \int_{|u-s|}^{R\vee|u-s|}(u-s)e^{r/2}f'((u-s)^2 - r^2)\,dr,
\end{aligned}$$

and

$$\begin{aligned}
h^\flat(R, u-s) =&\, \chi^\sharp_u(R, u-s) - h^\sharp(R, u-s) \\
=&\, \frac{1}{2}e^{R/2}\big(\delta_{u-s=-R} - \delta_{u-s=R}\big) + e^{R/2}(u-s)f'((u-s)^2 - R^2)\mathbb{1}_{|u-s|<R} \\
&- \frac{1}{2}\mathrm{sgn}(u-s) + \frac{1}{2}\mathrm{sgn}(u-s)e^{|u-s|/2}\mathbb{1}_{|u-s|<R} \\
&- \int_{|u-s|}^{R\vee|u-s|}(u-s)e^{r/2}f'((u-s)^2 - r^2)\,dr.
\end{aligned}$$

Proof. By relying on the Galilean invariance of the problem for $\sigma - u\chi$, we see that it suffices to prove the identities in the case $s = 0$.

For the weak kernel, we see that

$$\begin{aligned}
h^\sharp(R, u) &= \int_0^R \chi^\sharp_u(r, u)\,dr + \int_{-\infty}^u \big(\chi^\sharp_R(0, z) - \chi^\sharp(0, z)\big)\,dz \\
&= \frac{\partial}{\partial u}\int_0^R \chi^\sharp(r, u)\,dr + \mathbb{1}_{u>0},
\end{aligned}$$

by using the initial data for $\chi^\sharp$ on the line $\{R = 0\}$. Observing that the kernel need only be defined up to addition of a constant, we normalise the second term to $\frac{1}{2}\mathrm{sgn}(u)$. Considering now the first term,

$$\begin{aligned}
\int_0^R \chi^\sharp_u(r, u)\,dr &= \int_{|u|}^{R\vee|u|}\frac{1}{2}e^{r/2}\frac{\partial}{\partial u}f(u^2 - r^2)\,dr \\
&= \int_{|u|}^{R\vee|u|} ue^{r/2}f'(u^2 - r^2)\,dr - \frac{1}{2}\mathrm{sgn}(u)e^{|u|/2} + \frac{1}{2}\mathrm{sgn}(u)e^{|u|/2}\mathbb{1}_{|u|>R} \\
&= \int_{|u|}^{R\vee|u|} ue^{r/2}f'(u^2 - r^2)\,dr - \frac{1}{2}\mathrm{sgn}(u)e^{|u|/2}\mathbb{1}_{|u|<R}.
\end{aligned}$$

For the other kernel, we argue likewise to see

$$
\begin{aligned}
h^\flat(R,u) &= \int_0^R \chi^\flat_u(r,u)\,dr + \int_{-\infty}^u \big(\chi^\flat_R(0,z) - \chi^\flat(0,z)\big)\,dz \\
&= \frac{\partial}{\partial u}\int_0^R \big(\chi^\sharp_r(r,u) - \chi^\sharp(r,u)\big)\,dr - \mathbb{1}_{u>0} \\
&= \chi^\sharp_u(R,u) - h^\sharp(R,u)
\end{aligned}
$$

up to addition of a constant, again using the initial conditions for $\chi^\flat$ on $\{R=0\}$. We now conclude by observing that

$$
\chi^\sharp_u(R,u) = \frac{1}{2}e^{R/2}\big(\delta_{u=-R} - \delta_{u=R}\big) + e^{R/2}uf'(u^2 - R^2)\mathbb{1}_{|u|<R}.
$$

□

By construction, the entropy flux kernels $\sigma^\sharp(\rho,u,s)$ and $\sigma^\flat(\rho,u,s)$ generate isothermal entropy flux functions in the region $\rho > 1$ associated to the isothermal entropies generated from $\chi^\sharp(\rho,u-s)$ and $\chi^\flat(\rho,u-s)$. In particular, the isothermal entropy

$$
\eta^\psi(\rho,u) = \int_{\mathbb{R}} \psi(s)\chi^\sharp(\rho,u-s)\,ds
$$

has an associated entropy flux given by

$$
q^\psi(\rho,u) = \int_{\mathbb{R}} \psi(s)\sigma^\sharp(\rho,u,s)\,ds,
$$

and likewise for the entropies and entropy fluxes generated by $\chi^\flat$ and $\sigma^\flat$.

Given two generating functions, ψ_1,ψ_2, we now introduce the notation

$$
\eta^{\psi_1,\psi_2}(\rho,u) = \int_{\mathbb{R}} \psi_1(s)\chi^\sharp(\rho,u-s) + \psi_2(s)\chi^\flat(\rho,u-s)\,ds
$$

for the entropy generated from these kernels with "initial data" $\eta^{\psi_1,\psi_2}_\rho(1,u) = \psi_1(u)$ and $\eta^{\psi_1,\psi_2}(1,u) = \psi_2(u)$. This entropy comes with a corresponding flux,

$$
q^{\psi_1,\psi_2}(\rho,u) = \int_{\mathbb{R}} \psi_1(s)\sigma^\sharp(\rho,u,s) + \psi_2(s)\sigma^\flat(\rho,u,s)\,ds.
$$

Using this representation, we see that, for $\rho > 1$, the entropy and entropy flux kernels for the physical pressure law can be written as

$$
\chi(\rho,u) = \eta^{\chi_\rho(1,\cdot),\chi(1,\cdot)}(\rho,u), \qquad \sigma(\rho,u,0) = q^{\chi_\rho(1,\cdot),\chi(1,\cdot)}(\rho,u),
$$

i.e.

$$
\begin{aligned}
\chi(\rho,u) &= \int_{\mathbb{R}} \Big(\chi_\rho(1,s)\chi^\sharp(\rho,u-s) + \chi(1,s)\chi^\flat(\rho,u-s)\Big)\,ds \\
\sigma(\rho,u,0) &= \int_{\mathbb{R}} \Big(\chi_\rho(1,s)\sigma^\sharp(\rho,u,s) + \chi(1,s)\sigma^\flat(\rho,u,s)\Big)\,ds.
\end{aligned}
$$

Remark 5.2.5. Arguing from the entropy equation, it is easy to show that

$$\int_{\mathbb{R}} \big(\rho\chi_\rho(\rho, u-s) - \chi(\rho, u-s)\big)\, ds = 0 \tag{5.2.7}$$

for all (ρ, u). Indeed, mollifying χ with a standard mollifier ϕ^δ, $\chi^\delta = \chi * \phi^\delta$, we check that

$$\frac{\partial}{\partial\rho}\int_{\mathbb{R}} (\rho\chi^\delta_\rho - \chi^\delta)(\rho, u-s)\, ds = \int_{\mathbb{R}} \rho\chi^\delta_{\rho\rho}(\rho, u-s)\, ds = \int_{\mathbb{R}} \rho k'(\rho)^2\chi^\delta_{uu}(\rho, u-s)\, ds = 0,$$

hence

$$\int_{\mathbb{R}} \big(\rho\chi^\delta_\rho(\rho, u-s) - \chi^\delta(\rho, u-s)\big)\, ds = c_\delta,$$

for some constant $c_\delta \in \mathbb{R}$. Passing $\delta \to 0$ and using the initial data for $\chi(\rho, u-s)$, we verify (5.2.7).

The previous theorem gives explicit formulae for the entropy flux kernels $\sigma^\sharp$ and $\sigma^\flat$ for the isothermal gas dynamics. However, in practice, it will be convenient for us to exploit a further property of these kernels: for isothermal entropy pairs (η, q), the difference $Q = q - u\eta$ is also an entropy. Indeed, by (5.2.6), we see

$$Q_{RR} - Q_{uu} - Q_R = 0,$$

and hence we may write

$$Q(R, u) = Q_R(1, u) * \chi^\sharp(\rho, u) + Q(1, u) * \chi^\flat(\rho, u).$$

In particular, returning to the entropy and entropy flux kernels for the general pressure law and the usual (ρ, u) coordinates, we obtain the following theorem.

Theorem 5.2.6 (Entropy Kernel Expansions)**.** *The entropy kernel and entropy flux kernel may be written, for $\rho > 1$, as*

$$\begin{aligned} \chi(\rho, u-s) &= \chi_\rho(1, u-s) * \chi^\sharp(\rho, u-s) + \chi(1, u-s) * \chi^\flat(\rho, u-s), \\ (\sigma - u\chi)(\rho, u-s) &= (\sigma - u\chi)_\rho(1, u-s) * \chi^\sharp(\rho, u-s) \\ &\quad + (\sigma - u\chi)(1, u-s) * \chi^\flat(\rho, u-s), \end{aligned} \tag{5.2.8}$$

where

$$\chi^\sharp(\rho, u-s) = \frac{1}{2}\sqrt{\rho}\, I_0\Big(\frac{\sqrt{(\log\rho)^2 - |u-s|^2}}{2}\Big)\mathbb{1}_{|u-s|<\log\rho} \tag{5.2.9}$$

and

$$\begin{aligned} \chi^\flat(\rho, u-s) = {} & \frac{1}{2}\sqrt{\rho}\big(\delta_{u-s=\log\rho} + \delta_{u-s=-\log\rho}\big) \\ & - \frac{1}{4}\sqrt{\rho}\, I_0\Big(\frac{\sqrt{(\log\rho)^2 - |u-s|^2}}{2}\Big)\mathbb{1}_{|u-s|<\log\rho} \\ & + \frac{1}{4}\sqrt{\rho}\frac{\log\rho}{\sqrt{(\log\rho)^2 - |u-s|^2}} I_1\Big(\frac{\sqrt{(\log\rho)^2 - |u-s|^2}}{2}\Big)\mathbb{1}_{|u-s|<\log\rho}. \end{aligned} \tag{5.2.10}$$

5.2.2 Singularities of the entropy kernels

We observe that the entropy kernels $\chi(\rho, u-s)$ and $\sigma(\rho, u, s)$ are supported in the set $\mathcal{K} := \{(u-s)^2 \leq k(\rho)^2\}$ and are smooth in the interior of $\mathcal{K}$ as well as Hölder continuous with exponent λ up to the boundary of $\mathcal{K}$. In what follows, we need a careful analysis of the structure of the singularities of the fractional derivatives $\partial_s^{\lambda+1}\chi$ and $\partial_s^{\lambda+1}\sigma$.

Recall that the α-th fractional derivative of a compactly supported function $g(s)$ is defined to be

$$\partial_s^{\alpha} g(s) = g(s) * \Gamma(-\alpha)[s]_+^{-\alpha-1},$$

where Γ is the usual gamma function.

Define the function $f_\lambda(s) = [1-s^2]_+^{\lambda}$. Then we have from [48, Proposition 3.4] and [53, Lemma I.2] that

$$\begin{aligned}
\partial_s^{\lambda} f_\lambda(s) =& A_1^\lambda(H(s+1)+H(s-1)) + A_2^\lambda(\mathrm{Ci}(s+1)-\mathrm{Ci}(s-1)) + r(s),\\
\partial_s^{\lambda+1} f_\lambda(s) =& A_1^\lambda(\delta(s+1)+\delta(s-1)) + A_2^\lambda(\mathrm{PV}(s+1)-\mathrm{PV}(s-1))\\
&+ A_3^\lambda(H(s+1)-H(s-1)) + A_4^\lambda(\mathrm{Ci}(s+1)+\mathrm{Ci}(s-1)) + q(s),
\end{aligned}$$

where δ is the Dirac mass, PV the principal value distribution, H the Heaviside function, Ci is the Cosine integral, and the $A_j^\lambda \in \mathbb{C}$ are constants. Then we get the following expression for the fractional derivative of $\chi(\rho, u-s)$:

$$\partial_s^{\lambda+1}\chi(\rho,u-s) = a_\sharp(\rho)\partial_s^{\lambda+1}G_\lambda(\rho,u-s) + a_\flat(\rho)\partial_s^{\lambda+1}G_{\lambda+1}(\rho,u-s) + \partial_s^{\lambda+1}g_1(\rho,u-s). \tag{5.2.11}$$

Now, we observe that $G_\lambda(\rho, u-s) = k(\rho)^{2\lambda} f\big(\frac{s-u}{k(\rho)}\big)$ and recall that δ and PV are both homogeneous of degree -1. We use the chain rule to calculate

$$\begin{aligned}
\partial_s^{\lambda} G_\lambda(\rho,u-s) =& k(\rho)^\lambda\Big(A_1^\lambda\big(H(s-u+k(\rho))+H(s-u-k(\rho))\big)\\
&+ A_2^\lambda\big(\mathrm{Ci}(s-u+k(\rho))-\mathrm{Ci}(s-u-k(\rho))\big)\Big)\\
&+ k(\rho)^\lambda r\big(\frac{s-u}{k(\rho)}\big)
\end{aligned}\tag{5.2.12}$$

and

$$\begin{aligned}
\partial_s^{\lambda+1} G_\lambda(\rho,u-s) =& k(\rho)^\lambda\Big(A_1^\lambda\big(\delta(s-u+k(\rho))+\delta(s-u-k(\rho))\big)\\
&+ A_2^\lambda\big(\mathrm{PV}(s-u+k(\rho))-\mathrm{PV}(s-u-k(\rho))\big)\Big)\\
&+ k(\rho)^{\lambda-1}\Big(A_3^\lambda\big(H(s-u+k(\rho))-H(s-u-k(\rho))\big)\\
&+ A_4^\lambda\big(\mathrm{Ci}(s-u+k(\rho))+\mathrm{Ci}(s-u-k(\rho))\big)\Big)\\
&+ k(\rho)^{\lambda-1}\big(-A_4^\lambda\log k(\rho)^2 + q\big(\frac{s-u}{k(\rho)}\big)\big).
\end{aligned}\tag{5.2.13}$$

Moreover,

$$\begin{aligned}\partial_\rho\partial_s^\lambda G_\lambda(\rho,u-s) = k(\rho)^\lambda k'(\rho)\Big(&A_1^\lambda\big(\delta(s-u+k(\rho))-\delta(s-u-k(\rho))\big)\\&+A_2^\lambda\big(\mathrm{PV}(s-u+k(\rho))+\mathrm{PV}(s-u-k(\rho))\big)\Big)\\+k(\rho)^{\lambda-1}k'(\rho)\Big(&A_1^\lambda\big(H(s-u+k(\rho))+H(s-u-k(\rho))\big)\\&+A_2^\lambda\big(\mathrm{Ci}(s-u+k(\rho))-\mathrm{Ci}(s-u-k(\rho))\big)\Big).\end{aligned}\tag{5.2.14}$$

We will require also the expressions for the derivatives of the entropy flux kernel, $\sigma(\rho,u,s)$. Using the expansion (5.2.2) and the identity $\partial_s^{\lambda+1}(sg) = s\partial_s^{\lambda+1}g+(\lambda+1)\partial_s^\lambda g$ for a generic function $g(s)$, we calculate

$$\begin{aligned}\partial_s^{\lambda+1}(\sigma-u\chi)(\rho,u-s) &= (s-u)\partial_s^{\lambda+1}\big(b_\sharp(\rho)G_\lambda(\rho,u-s)+b_\flat(\rho)G_{\lambda+1}(\rho,u-s)\big)\\&+(\lambda+1)\partial_s^\lambda\big(b_\sharp(\rho)G_\lambda(\rho,u-s)+b_\flat(\rho)G_{\lambda+1}(\rho,u-s)\big)+\partial_s^{\lambda+1}g_2(\rho,u-s).\end{aligned}\tag{5.2.15}$$

Next, we consider, for $\rho\geq 1$,

$$\chi(\rho,u) = \chi^\sharp(\rho,\cdot)*\chi_\rho(1,\cdot)+\chi^\flat(\rho,\cdot)*\chi(1,\cdot).$$

Thus, distributing derivatives across the convolution,

$$\partial_s^{\lambda+1}\chi(\rho,u-s) = \chi_s^\sharp(\rho,\cdot)*\partial_s^\lambda\chi_\rho(1,\cdot)+\chi^\flat(\rho,\cdot)*\partial_s^{\lambda+1}\chi(1,\cdot).$$

From (5.2.11)-(5.2.15), we obtain the structure of these expressions as sums of measures and distributions with coefficients depending on ρ. By considering these new expressions for the region $\{\rho\geq 1\}$, however, we are able to demonstrate control on the growth of these coefficients. This is the content of the following lemma.

Lemma 5.2.7. *The expressions $\partial_s^{\lambda+1}\chi(\rho,u-s)$ and $\partial_s^{\lambda+1}\sigma(\rho,u,s)$ admit the following expansions:*

$$\begin{aligned}\partial_s^{\lambda+1}\chi(\rho,u-s) =& \sum_\pm\Big(A_{1,\pm}(\rho)\delta(s-u\pm k(\rho))+A_{2,\pm}(\rho)H(s-u\pm k(\rho))\\&\quad+A_{3,\pm}(\rho)\mathrm{PV}(s-u\pm k(\rho))+A_{4,\pm}(\rho)\mathrm{Ci}(s-u\pm k(\rho))\Big)\\&+r_\chi(\rho,u-s),\\\partial_s^{\lambda+1}(\sigma-u\chi)(\rho,u-s) =& \sum_\pm(s-u)\big(B_{1,\pm}(\rho)\delta(s-u\pm k(\rho))+B_{2,\pm}(\rho)H(s-u\pm k(\rho))\\&\quad+B_{3,\pm}(\rho)\mathrm{PV}(s-u\pm k(\rho))+B_{4,\pm}(\rho)\mathrm{Ci}(s-u\pm k(\rho))\big)\\&+\sum_\pm\tilde B_{2,\pm}(\rho)H(s-u\pm k(\rho))+\tilde B_{4,\pm}(\rho)\mathrm{Ci}(s-u\pm k(\rho))\\&+r_\sigma(\rho,u,s).\end{aligned}\tag{5.2.16}$$

Moreover, the coefficients $A_{j,\pm}(\rho), B_{j,\pm}(\rho), \tilde{B}_{j,\pm}(\rho)$ all satisfy the following bound:

$$\sum_{\substack{j=1,\dots,4,\\ \pm}} \big(|A_{j,\pm}(\rho)| + |B_{j,\pm}(\rho)|\big) + \sum_{\substack{j=2,4,\\ \pm}} |\tilde{B}_{j,\pm}(\rho)| \leq C\sqrt{\rho}\log\rho, \tag{5.2.17}$$

where C is independent of ρ, u, s. The remainders r_χ and r_σ are compactly supported, Hölder continuous functions such that

$$|r_\chi(\rho, u-s)| + |r_\sigma(\rho, u-s)| \leq C\rho,$$

where C is independent of ρ, u, s.

Proof. We recall, for $\rho \geq 1$, that

$$\chi(\rho, u-s) = \chi_\rho(1, u-s) * \chi^\sharp(\rho, u-s) + \chi(1, u-s) * \chi^\flat(\rho, u-s).$$

Thus we may distribute derivatives across the convolution as follows:

$$\partial_s^{\lambda+1}\chi(\rho, u-s) = \partial_s^\lambda \chi_\rho(1, u-s) * \partial_s \chi^\sharp(\rho, u-s) + \partial_s^{\lambda+1}\chi(1, u-s) * \chi^\flat(\rho, u-s). \tag{5.2.18}$$

From the expansion (5.2.11), we see

$$\partial_s^{\lambda+1}\chi(1, u-s) = a_\sharp(1)\partial_s^{\lambda+1}G_\lambda(1, u-s) + a_\flat(1)\partial_s^{\lambda+1}G_{\lambda+1}(1, u-s) + \partial_s^{\lambda+1}g_1(1, u-s),$$

which is clearly independent of ρ. The expressions $\partial_s^{\lambda+1}G_{\lambda+1}$ and $\partial_s^{\lambda+1}G_\lambda$ are given in (5.2.12) and (5.2.13).

One easily sees also that

$$\partial_\rho G_\lambda(\rho, u-s) = 2\lambda k(\rho)k'(\rho)G_{\lambda-1}(\rho, u-s),$$

hence

$$\begin{aligned}
\partial_s^\lambda \chi_\rho(1, u-s) = {} & 2\lambda k(1)k'(1)a_\sharp(1)\partial_s^\lambda G_{\lambda-1}(1, u-s) + a_\sharp'(1)\partial_s^\lambda G_\lambda(1, u-s) \\
& + 2(\lambda+1)k(1)k'(1)a_\flat(1)\partial_s^\lambda G_\lambda(1, u-s) + a_\flat'(1)\partial_s^\lambda G_{\lambda+1}(1, u-s) \\
& + \partial_s^\lambda \partial_\rho g_1(1, u-s).
\end{aligned}$$

Returning to consider (5.2.18), we recall the expressions from Theorem 5.2.6,

$$\begin{aligned}
\partial_s \chi^\sharp(\rho, u-s) = {} & \frac{1}{2}\sqrt{\rho}\big(\delta_{u-s=\log\rho} - \delta_{u-s=-\log\rho}\big) \\
& - \sqrt{\rho}(u-s)f'\big((u-s)^2 - (\log\rho)^2\big)\mathbb{1}_{|u-s|<\log\rho}, \\
\chi^\flat(\rho, u-s) = {} & \frac{1}{2}\sqrt{\rho}\big(\delta_{u-s=\log\rho} + \delta_{u-s=-\log\rho}\big) - \frac{1}{2}\sqrt{\rho}\Big(\frac{1}{2}f\big((u-s)^2 - (\log\rho)^2\big) \\
& + 2\log\rho f'\big((u-s)^2 - (\log\rho)^2\big)\Big)\mathbb{1}_{|u-s|<\log\rho}.
\end{aligned}$$

We observe that, up to the addition of a compactly supported Lipschitz function, these are sums of Dirac masses and Heaviside functions. For example, as $f(0) = 1$,

$$f\big((u-s)^2 - (\log\rho)^2\big)\mathbb{1}_{|u-s|<\log\rho} = H(u-s+\log\rho) - H(u-s-\log\rho) + R(\rho, u-s),$$

where $R(\rho, u-s)$ is a Lipschitz continuous function such that $\operatorname{supp} R = \{|u-s| \le \log\rho\}$ and $|R(\rho, u-s)| \le C\rho$, for C independent of ρ, u, s.

Thus, in expanding (5.2.18), we obtain terms as convolutions of $\sqrt{\rho}\delta(u-s\pm\log\rho)$, $\sqrt{\rho}H(u-s\pm\log\rho)$, $\sqrt{\rho}\log\rho H(u-s\pm\log\rho)$ and terms of higher regularity and compact support with distributions ψ such that

$$\psi \in \{\delta(u-s\pm k(1)), H(u-s\pm k(1)), \mathrm{PV}(u-s\pm k(1)), \mathrm{Ci}(u-s\pm k(1))\}.$$

From these considerations, we obtain the bounds of (5.2.17) on the coefficients of the distributional terms.

Finally, we see that the terms of higher regularity are all at least Hölder continuous and bounded (up to a constant) by ρ. □

Remark 5.2.8. In a similar way to the above, we see that $\chi(\rho, u-s)$ is α-Hölder continuous with respect to s with any exponent $\alpha \in [0, \lambda]$. Moreover, we may estimate,

$$\|\chi(\rho, u-\cdot)\|_{C^\alpha(\mathbb{R})} \le \begin{cases} C\rho^{(2\lambda-\alpha)\theta}, & \text{for } \rho \le 1, \\ C\frac{\rho}{\sqrt{\log\rho}}, & \text{for } \rho > 1. \end{cases} \tag{5.2.19}$$

5.3 Uniform estimates for the Navier-Stokes equations

Now that we have gained a better understanding of the entropies of the Euler equations, we begin the analysis of the vanishing viscosity limit of the Navier-Stokes equations, (5.1.1). We first recall the following theorem of Hoff [38].

Theorem 5.3.1 (Existence of Solutions for Navier-Stokes Equations, [38])**.** *Suppose that the initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$ are smooth and satisfy*

$$\rho_0^\varepsilon \in L^\infty(\mathbb{R}), \quad \operatorname*{ess\,inf}_{\mathbb{R}} \rho_0^\varepsilon > 0,$$
$$\rho_0^\varepsilon - \bar\rho,\ u_0^\varepsilon - \bar u \in L^2(\mathbb{R}).$$

Then the initial value problem (5.1.1) admits a unique, global, smooth solution $(\rho^\varepsilon, u^\varepsilon)$ satisfying also the integrability $u^\varepsilon(t,\cdot) \in H^1(\mathbb{R})$ for all $t > 0$. Moreover, there exists a constant $c_\varepsilon(t) > 0$ depending on ε, t and the initial data, such that for all $(t,x) \in \mathbb{R}^2_+$, $\rho(t,x) \ge c_\varepsilon(t) > 0$. Finally, $\lim_{x\to\pm\infty}(\rho^\varepsilon(t,x), u^\varepsilon(t,x)) = (\rho_\pm, u_\pm)$ for all $t \ge 0$.

Of crucial importance is the lower bound on the density. This guarantees that the singularities of the viscous equations are avoided for all times, provided they are avoided initially (though we do, of course, expect cavitation to occur in the vanishing viscosity limit).

5.3.1 Initial estimates

We now proceed by making several uniform estimates, that is, estimates that are independent of $\varepsilon \in (0, \varepsilon_0]$ for some fixed $\varepsilon_0 > 0$, on the physical viscosity solutions. Throughout this section, we will denote constants independent of ε by M, while constants depending on ε will be called C. The pair (ρ, u) will always be the smooth solution of (5.1.1) guaranteed by the theorem above (we drop the explicit dependence of the functions on ε for notational simplicity).

The first of our uniform estimates is the now standard estimate on the relative mechanical energy.

Lemma 5.3.2. *Let $E[\rho_0, u_0] \le E_0 < \infty$ for some constant $E_0 > 0$ independent of ε and suppose that (ρ, u) is the smooth solution of the Cauchy problem (5.1.1) with initial data (ρ_0, u_0). Then, for any $T > 0$, there exists a constant $M > 0$, independent of ε but depending on $E_0, T, \bar{\rho}, \bar{u}$ such that*

$$\sup_{t\in[0,T]} E[\rho, u](t) + \int_0^T \int_{\mathbb{R}} \varepsilon |u_x|^2 \, dx \, dt \le M. \tag{5.3.1}$$

Although the proof of this lemma is by now standard (see, for example, [12, Lemma 3.1]), we include it here for completeness.

Proof. We calculate directly that

$$\frac{dE}{dt} = \frac{d}{dt}\int \eta^*(\rho, m)\, dx - \frac{d}{dt}\int \eta^*(\bar{\rho}, \bar{m})\, dx - \int \nabla\eta^*(\bar{\rho}, \bar{m}) \cdot (\rho_t, m_t)\, dx.$$

As the reference functions $\bar{\rho}$ and $\bar{m}$ are independent of t, it is clear that the second term on the right vanishes identically. On the other hand, multiplying the first equation in (5.1.1) by $\eta^*_\rho(\rho, m)$ and the second equation by $\eta^*_m(\rho, m)$, we obtain

$$\eta^*(\rho, m)_t + q^*(\rho, m)_x = \varepsilon \eta^*_m(\rho, m) u_{xx},$$

and hence, as $\eta^*_m(\rho, m) = u$,

$$\frac{d}{dt}E[\rho, u](t) = q^*(\rho_-, m_-) - q^*(\rho_+, m_+) - \varepsilon\int |u_x|^2\, dx - \int \nabla\eta^*(\bar{\rho}, \bar{m}) \cdot (\rho_t, m_t)\, dx.$$

From the equation (5.1.1), we may re-write the final term here and bound it by

$$\begin{aligned}\Big|\int \nabla\eta^*(\bar\rho,\bar m)\cdot(\rho_t,m_t)\,dx\Big| &=\Big|-\int \nabla\eta^*(\bar\rho,\bar m)\cdot(m_x,(\rho u^2+p(\rho))_x-\varepsilon u_{xx})\,dx\Big|\\ &\leq \int \big|(\nabla\eta^*(\bar\rho,\bar m))_x\cdot(m,\rho u^2+p(\rho)-\varepsilon u_x)\big|\,dx\\ &\quad+\big|\nabla\eta^*(\bar\rho,\bar m)\cdot(m,\rho u^2+p(\rho)-\varepsilon u_x)\big|_{x=-\infty}^{x=+\infty}\big|\\ &\leq\frac{\varepsilon}{2}\int|u_x|^2\,dx+M\int\rho|u-\bar u|^2\,dx\\ &\quad+M\Big(1+\int_{-L_0}^{L_0}(\rho+p(\rho))\,dx\Big),\end{aligned}$$

as the functions $\bar\rho$ and $\bar m$ are constant outside the interval $[-L_0,L_0]$.

Finally, we recall that $(\rho+p(\rho))\leq Me^*(\rho,\bar\rho)$ to derive

$$\frac{d}{dt}E[\rho,u](t)+\frac{\varepsilon}{2}\int|u_x|^2\,dx\leq M(E+1),$$

and conclude by Gronwall's inequality. □

The second uniform estimate for the viscous solutions concerns the spatial derivative of the density. Again, the proof may be found in [12, Lemma 3.2], based on an argument of [42].

Lemma 5.3.3 ([12], Lemma 3.2)**.** *Let (ρ_0,u_0) be initial data such that*

$$\varepsilon^2\int\frac{|\rho_{0,x}(x)|^2}{\rho_0(x)^3}\,dx\leq E_1<\infty,$$

for a constant $E_1>0$ independent of ε. Then, for any $T>0$, there exists a constant M, depending on $E_0,E_1,\bar\rho,\bar u,T$ such that

$$\varepsilon^2\int\frac{|\rho_x(T,x)|^2}{\rho(T,x)^3}\,dx+\varepsilon\int_0^T\int\frac{p'(\rho)}{\rho^2}|\rho_x|^2\,dx\,dt\leq M. \tag{5.3.2}$$

The proof is taken from [12, Lemma 3.2], adjusted only to allow for the more general pressure law.

Proof. Let $v=\frac{1}{\rho}$. The conservation of mass equation in (5.1.1) becomes

$$v_t+uv_x=vu_x.$$

Taking a derivative in x, we obtain

$$v_{xt}+(uv_x)_x=(vu_x)_x,$$

which we multiply by $2v_x$ to find

$$(|v_x|^2)_t + u(|v_x|^2)_x + 2u_x|v_x|^2 = 2v_x(vu_x)_x.$$

Multiplying this by ρ and applying again the conservation of mass equation,

$$(\rho|v_x|^2)_t + (\rho u|v_x|^2)_x = 2v_x u_{xx}. \tag{5.3.3}$$

Now we apply the conservation of momentum equation and the equation above for v_{xt} to deduce

$$\begin{aligned}
2v_x u_{xx} &= \frac{2}{\varepsilon}v_x(p_x + (\rho u)_t + (\rho u^2)_x) \\
&= \frac{2}{\varepsilon}v_x p_x + \frac{2}{\varepsilon}\big((\rho(u-\bar{u})v_x)_t - (\bar{u}(\log\rho)_x)_t + J\big),
\end{aligned}$$

where $J = -\rho u(vu_x)_x + \rho u(uv_x)_x + v_x(\rho u^2)_x$. Integrating J by parts in space, we find

$$\begin{aligned}
\int J\,dx &= \int (vu_x(\rho u)_x - uv_x(\rho u)_x + v_x(u(\rho u)_x + \rho u u_x)\,dx \\
&= \int (vu_x(\rho u)_x + \rho u v_x u_x)\,dx = \int |u_x|^2\,dx.
\end{aligned}$$

Moreover,

$$v_x p_x = -\frac{p'(\rho)}{\rho^2}|\rho_x|^2.$$

We integrate (5.3.3) in space-time and use the preceding considerations to find

$$\begin{aligned}
&\varepsilon^2\int\frac{|\rho_x(t,x)|^2}{\rho(t,x)^3}\,dx + \varepsilon\int_0^t\int\frac{p'(\rho)}{\rho^2}|\rho_x|^2\,dx\,d\tau \\
&= -2\varepsilon\int\frac{\rho_x(t,x)(u(t,x)-\bar{u}(x))}{\rho(t,x)}\,dx - 2\varepsilon\int\bar{u}(x)(\log\rho)_x(t,x)\,dx \\
&\quad + 2\varepsilon\int_0^t\int|u_x|^2\,dx\,d\tau + 2\varepsilon\int\frac{\rho_{0,x}(x)(u_0(x)-\bar{u}(x))}{\rho_0(x)}\,dx \\
&\quad + 2\varepsilon\int\bar{u}(x)(\log\rho_0)_x(x)\,dx + \varepsilon^2\int\frac{|\rho_{0,x}(x)|^2}{\rho_0(x)^3}\,dx.
\end{aligned}$$

We control the first and fourth terms on the right by observing

$$2\varepsilon\int\frac{|\rho_x(u-\bar{u})|}{\rho}\,dx \le \frac{\varepsilon^2}{4}\int\frac{|\rho_x|^2}{\rho^3}\,dx + 8\int\rho|u-\bar{u}|^2\,dx \le \frac{\varepsilon^2}{4}\int\frac{|\rho_x|^2}{\rho^3}\,dx + 16E[\rho,u].$$

For the second term, we note

$$\varepsilon\int\bar{u}(\log\rho)_x\,dx = -\varepsilon\int_{A_1}\bar{u}_x\log\rho\,dx - \varepsilon\int_{A_1^c}\bar{u}_x\log\rho\,dx + \varepsilon(u^+\log\rho^+ - u^-\log\rho^-),$$

where

$$A_1 = \Big\{x\,:\,\rho(t,x)\le\frac{\check{\rho}}{2}\Big\},$$

and $\check{\rho} = \min\{\rho_-, \rho_+\}$. As $|\log \rho| \leq M\rho$ on A_1^c and $\bar{u}_x$ is compactly supported, we obtain

$$\Big|\varepsilon \int_{A_1^c} \bar{u}_x \log \rho \, dx\Big| \leq M\Big(1 + \int e^*(\rho, \bar{\rho})\, dx\Big).$$

On A_1 (if the set is non-empty), then

$$\Big|\varepsilon \int_{A_1} \bar{u}_x \log \rho \, dx\Big| \leq M\varepsilon \sup_{x\in A_1} |\log \rho(t,x)| \leq M\varepsilon \sup_{x\in A_1} \frac{1}{\sqrt{\rho(t,x)}}.$$

Moreover, we estimate the measure of A_1 by

$$|A_1| \leq \frac{M}{e^*(\check{\rho}/2, \check{\rho})} =: d(t).$$

Now, for any $(t,x) \in A_1$, there exists $x_0(t,x)$ such that $|x - x_0| \leq d(t)$ and $\rho(t, x_0) = \frac{\check{\rho}}{2}$. Thus,

$$\begin{aligned}
\varepsilon \sup_{x\in A_1} \frac{1}{\sqrt{\rho(t,x)}} &\leq \varepsilon \sup_{x\in A_1} \Big|\frac{1}{\sqrt{\rho(t,x)}} - \frac{1}{\sqrt{\rho(t,x_0)}}\Big| + \frac{\varepsilon}{\sqrt{\check{\rho}/2}} \\
&\leq \varepsilon \int_{x_0-d(t)}^{x_0+d(t)} \Big|\Big(\frac{1}{\sqrt{\rho(t,x)}}\Big)_x\Big|\, dx + \frac{\varepsilon}{\sqrt{\check{\rho}/2}} \\
&\leq \Big(\frac{\varepsilon^2}{2} \int \frac{|\rho_x|^2}{\rho^3}\, dx\Big)^{\frac{1}{2}} \sqrt{d(t)} + \frac{\varepsilon}{\sqrt{\check{\rho}/2}} \\
&\leq \frac{\varepsilon^2}{4} \int \frac{|\rho_x|^2}{\rho^3}\, dx + M.
\end{aligned}$$

Thus, we obtain

$$2\varepsilon\Big|\int \bar{u}(\log \rho)_x \, dx\Big| \leq \frac{\varepsilon^2}{4} \int \frac{|\rho_x|^2}{\rho^3}\, dx + M.$$

Combining this with the earlier estimates, we find

$$\begin{aligned}
\varepsilon^2 \int \frac{\rho_x(t,x)|^2}{\rho(t,x)^3}\, dx + \varepsilon \int_0^t \frac{p'(\rho)}{\rho^2} |\rho_x|^2 \, dx\, d\tau& \\
\leq \frac{\varepsilon^2}{2} \int \frac{\rho_x(t,x)|^2}{\rho(t,x)^3}\, dx + 2\varepsilon^2 \int \frac{|\rho_{0,x}(x)|^2}{\rho_0(x)^3}\, dx + M,&
\end{aligned}$$

as desired. □

The final estimate of this section is again taken from [12], this time Lemma 3.3.

Lemma 5.3.4. *Let $E_0[\rho_0, u_0] \leq E_0 < \infty$ with E_0 independent of ε and let $K \subset \mathbb{R}$ be compact. Then, for any $T > 0$, there exists a constant $M = M(E_0, K, \bar{\rho}, \bar{u}, T) > 0$ such that*

$$\int_0^T \int_K \rho p(\rho)\, dx\, dt \leq M. \tag{5.3.4}$$

The proof is as in [12], but compare also Lemma 4.4.1.

Proof. We let $\omega(x) \in C_c^\infty(\mathbb{R})$ be a test function with $0 \le \omega(x) \le 1$. Multiplying the second equation in (5.1.1) by $\omega(x)$ and integrating in space from $-\infty$ to x, we obtain

$$\rho u^2\omega + p\omega = \varepsilon u_x\omega - \Big(\int_{-\infty}^x \rho u\omega\,dy\Big)_t + \int_{-\infty}^x ((\rho u^2 + p)\omega_x - \varepsilon u_x\omega_x)\,dy.$$

Multiplying this again with $\rho\omega$, we apply the first equation from (5.1.1) to get

$$\begin{aligned}
\rho p\omega^2 =& - \rho^2u^2\omega^2 + \varepsilon\rho u_x\omega^2 - \Big(\rho\omega\int_{-\infty}^x \rho u\omega\,dy\Big)_t\\
&- (\rho u)_x\omega\int_{-\infty}^x \rho u\omega\,dy + \rho\omega\int_{-\infty}^x ((\rho u^2+p)\omega_x - \varepsilon u_x\omega_x)\,dy\\
=& \,\varepsilon\rho u_x\omega^2 - \Big(\rho\omega\int_{-\infty}^x \rho u\omega\,dy\Big)_t - \Big(\rho u\omega\int_{-\infty}^x \rho u\omega\,dy\Big)_x\\
&+ \rho u\omega_x\int_{-\infty}^x \rho u\omega\,dy + \rho\omega\int_{-\infty}^x ((\rho u^2+p)\omega_x - \varepsilon u_x\omega_x)\,dy.
\end{aligned}$$

Integrating this equation over the compact region $(0,T)\times\operatorname{supp}\omega$, we obtain

$$\begin{aligned}
\int_0^T\!\!\int \rho p\omega^2\,dy\,dt =& \,\varepsilon\int_0^T\!\!\int \rho u_x\omega^2\,dy\,dt - \int \rho\omega\Big(\int_{-\infty}^x \rho u\omega\,dy\Big)dx\\
&+ \int \rho_0\omega\Big(\int_{-\infty}^x \rho_0u_0\omega\,dy\Big)dx + r_1(T),
\end{aligned}\tag{5.3.5}$$

where

$$\begin{aligned}
r_1(T) =& \int_0^T\!\!\int \rho u\omega_x\Big(\int_{-\infty}^x \rho u\omega\,dy\Big)dx\,dt\\
&+ \int_0^T\!\!\int \rho\omega\Big(\int_{-\infty}^x ((\rho u^2+p)\omega_x - \varepsilon u_x\omega_x)\,dy\Big)dx\,dt.
\end{aligned}$$

Applying Hölder's inequality, for any $\Delta > 0$,

$$\begin{aligned}
\varepsilon\int_0^T\!\!\int \rho u_x\omega^2\,dx\,dt \le& \frac{\varepsilon^2}{\Delta}\int_0^T\!\!\int |u_x|^2\,dx\,dt + \Delta\int_0^T\!\!\int \rho^2\omega^4\,dx\,dt\\
\le& M + M\Delta\int_0^T\!\!\int \rho p(\rho)\omega^2\,dx\,dt,
\end{aligned}$$

for $\varepsilon \in (0,\varepsilon_0]$. We now apply Hölder's inequality and Lemma 5.3.2 to deduce

$$\begin{aligned}
\Big|\int_{-\infty}^x \rho u\omega\,dy\Big| \le& \int_{\operatorname{supp}\omega} |\rho u|\,dy \le \Big(\int_{\operatorname{supp}\omega}\rho\,dy\Big)^{\frac12}\Big(\int_{\operatorname{supp}\omega}\rho u^2\,dy\Big)^{\frac12}\\
\le& M\Big(\int_{\operatorname{supp}\omega}(1+e^*(\rho,\bar\rho))\,dy\Big)^{\frac12}\Big(\int_{\operatorname{supp}\omega}\rho u^2\,dy\Big)^{\frac12} \le M.
\end{aligned}$$

We may therefore make the estimate

$$\Big|\int \rho\omega\Big(\int_{-\infty}^x \rho u\omega\,dy\Big)dx\Big| \le M,$$

again using that $\rho \le M(1+e^*(\rho,\bar\rho))$.

Similarly, we have

$$\Big|\int_0^T \int \rho u\omega_x \Big(\int_{-\infty}^x \rho u\omega\, dy\Big) dx\, dt\Big| + \Big|\int_0^T \int \rho\omega \Big(\int_{-\infty}^x (\rho u^2 + p)\omega_x\, dy\Big) dx\, dt\Big|$$
$$+ \Big|\int_0^T \int \rho\omega \Big(\int_{-\infty}^x \varepsilon u_x \omega_x\, dy\Big) dx\, dt\Big| \le M.$$

Combining all of these estimates for the terms on the right hand side of (5.3.5), we conclude that

$$\int_0^T \int \rho p(\rho)\omega^2\, dx\, dt \le M\Delta \int_0^T \int \rho p(\rho)\omega^2\, dx\, dt + M, \tag{5.3.6}$$

and so, by taking $\Delta > 0$ sufficiently small, we conclude the proof. □

5.3.2 Higher integrability for the velocity

The final uniform estimate that we require is a higher integrability property for the velocity. Indeed, to pass to the limit in the non-linear term ρu^2 in the equations, we wish to control the quantity $\rho|u|^3$, at least locally in space-time. The basic technique for obtaining this bound is as follows: we construct an entropy pair $(\eta^\sharp, q^\sharp)$ (which we generate by a test function of quadratic growth at infinity) satisfying the rough bounds

$$q^\sharp(\rho, m) \ge M^{-1}\rho|u|^3, \quad |\eta^\sharp(\rho, m)| \le M\eta^*(\rho, m).$$

We then argue from the entropy equation for $\eta^\sharp$ and $q^\sharp$,

$$\eta^\sharp(\rho^\varepsilon, m^\varepsilon)_t + q^\sharp(\rho^\varepsilon, m^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}\eta^\sharp_m(\rho^\varepsilon, m^\varepsilon),$$

to relate $q^\sharp(\rho, m)$ to $\int^x \eta^\sharp(\rho, m)\, dx$. Controlling the latter via the main energy estimate, we may integrate again over a compact region of space-time and bound the integral of $q^\sharp$, and hence of $\rho|u|^3$. We begin with the construction of this entropy pair.

Lemma 5.3.5. *Let $\psi^\sharp(s) = \frac{1}{2}s|s|$. Then the associated entropy pair $(\eta^\sharp, q^\sharp)$ satisfies the following bounds for all $\rho \ge \rho_*$, where $\rho_* > 1$ is fixed:*

$$\begin{aligned} &|\eta^\sharp(\rho, m)| \le M\eta^*(\rho, m), && q^\sharp(\rho, m) \ge M^{-1}\rho|u|^3 - M\big(\rho|u|^2 + \rho(\log\rho)^3\big),\\ &|\eta^\sharp_m(\rho, m)| \le M(|u| + \sqrt{\log\rho}), && |\rho\eta^\sharp_{mm}(\rho, m)| \le M, \end{aligned} \tag{5.3.7}$$

and, if we consider the function $\eta^\sharp_m(\rho, m)$ as a function of ρ and u, the partial derivatives may be bounded by

$$|\eta^\sharp_{mu}(\rho, \rho u)| \le M\frac{1}{\sqrt{\log\rho}}, \qquad |\eta^\sharp_{m\rho}(\rho, \rho u)| \le M\frac{1}{\rho\log\rho}. \tag{5.3.8}$$

Moreover, on the complement region $\rho \leq \rho_*$, we have

$$|\eta^\sharp(\rho,m)| \leq M\eta^*(\rho,m), \qquad q^\sharp(\rho,m) \geq M^{-1}\big(\rho|u|^3+\rho^{\gamma+\theta}\big) - M\big(\rho|u|^2+\rho^\gamma\big),$$
$$|\eta^\sharp_m(\rho,m)| \leq M(|u|+\rho^\theta), \qquad |\rho\eta^\sharp_{mm}(\rho,m)| \leq M, \tag{5.3.9}$$

and, if we consider the function $\eta^\sharp_m(\rho,m)$ as a function of ρ and u, the partial derivatives may be bounded by

$$|\eta^\sharp_{mu}(\rho,\rho u)| \leq M, \qquad |\eta^\sharp_{m\rho}(\rho,\rho u)| \leq M\rho^{\theta-1}. \tag{5.3.10}$$

Proof. The bounds for the low density region are standard, following as in [12, 52]. Indeed, on this region, the expansions (5.2.1) and (5.2.2) tell us that the leading order terms dominate the behaviour, allowing for arguments analogous to those for the polytropic case considered in [12].

We therefore focus on the region $\rho \geq \rho_* > 1$. Examining first the entropy $\eta^\sharp(\rho,m)$, we have that

$$\begin{aligned}\eta^\sharp(\rho,m) &= \int_{\mathbb{R}} \chi(\rho,u-s)\frac{1}{2}s|s|\,ds\\ &= \int_{\mathbb{R}} \big(\chi_\rho(1,u-s)*\chi^\sharp(\rho,u-s)+\chi(1,u-s)*\chi^\flat(\rho,u-s)\big)\frac{1}{2}s|s|\,ds.\end{aligned}$$

We recall the expansions for $\chi^\sharp$ and $\chi^\flat$ of Theorem 5.2.6 and decompose the entropy kernel into three terms as

$$\chi(\rho,u-s) = K_1(\rho,u-s)+K_2(\rho,u-s)+K_3(\rho,u-s),$$

where

$$\begin{aligned}K_1(\rho,u) &= \frac{\sqrt{\rho}}{2}\int_{\mathbb{R}} I_0\Big(\frac{\sqrt{(\log\rho)^2-(u-t)^2}}{2}\Big)\mathbb{1}_{|u-t|<\log\rho}\chi_\rho(1,t)\,dt,\\ K_2(\rho,u) &= \frac{\sqrt{\rho}}{2}\big(\chi(1,u-\log\rho)+\chi(1,u+\log\rho)\big),\\ K_3(\rho,u) &= \frac{\sqrt{\rho}}{4}\int_{\mathbb{R}}\Big(\frac{\log\rho}{\sqrt{(\log\rho)^2-(u-t)^2}}I_1\Big(\frac{\sqrt{(\log\rho)^2-(u-t)^2}}{2}\Big)\\ &\qquad\qquad - I_0\Big(\frac{\sqrt{(\log\rho)^2-(u-t)^2}}{2}\Big)\Big)\mathbb{1}_{|u-t|<\log\rho}\chi(1,t)\,dt.\end{aligned}$$

For simplicity of presentation, we demonstrate the bound for the first term, noting that the term K_2 is much more straightforward and that the K_3 term may be controlled analogously to K_1 using properties of modified Bessel functions.

We therefore consider, in the contribution of K_1 to $|\eta^\sharp(\rho, m)|$, the following identity:

$$
\begin{aligned}
&\int_{\mathbb{R}} I_0\Big(\frac{\sqrt{(\log\rho)^2-(u-s-t)^2}}{2}\Big)s|s|\mathbb{1}_{|u-s-t|<\log\rho}\,ds\\
&=\int_{\mathbb{R}} I_0\Big(\frac{\sqrt{(\log\rho)^2-z^2}}{2}\Big)(z+u-t)|z+u-t|\mathbb{1}_{|z|<\log\rho}\,dz\\
&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} I_0\Big(\frac{\log\rho}{2}\cos\theta\Big)\log\rho\cos\theta(\log\rho\sin\theta+u-t)|\log\rho\sin\theta+u-t|\,d\theta.
\end{aligned}
$$

Supposing without loss of generality that $u > 2k(1)$ (so that $u-t>0$ for all $t \in \operatorname{supp}\chi_\rho(1,\cdot)$), we exploit the partial cancellation of odd and even terms in the region $u-t<\log\rho$ to obtain that the contribution to $\eta^\sharp(\rho,u)$ from K_1 may be written as

$$
\begin{aligned}
&\int_{\mathbb{R}} K_1(\rho,u-s)\frac{1}{2}s|s|\,ds\\
&=\frac{\sqrt{\rho}}{4}\int_{\mathbb{R}}\chi_\rho(1,t)\int_{\mathbb{R}} I_0\Big(\frac{\sqrt{(\log\rho)^2-(u-s-t)^2}}{2}\Big)s|s|\mathbb{1}_{|u-s-t|<\log\rho}\,ds\,dt\\
&=\frac{\sqrt{\rho}}{4}\int_{-k(1)}^{k(1)}\chi_\rho(1,t)\int_{-\arcsin\left(\frac{u-t}{\log\rho}\right)}^{\frac{\pi}{2}} I_0\Big(\frac{\log\rho}{2}\cos\theta\Big)\log\rho\cos\theta(\log\rho\sin\theta+u-t)^2\,d\theta\,dt\\
&-\frac{\sqrt{\rho}}{4}\int_{-k(1)}^{k(1)}\chi_\rho(1,t)\int_{-\frac{\pi}{2}}^{-\arcsin\left(\frac{u-t}{\log\rho}\right)} I_0\Big(\frac{\log\rho}{2}\cos\theta\Big)\log\rho\cos\theta(\log\rho\sin\theta+u-t)^2\,d\theta\,dt\\
&=\frac{\sqrt{\rho}}{4}\int_{-k(1)}^{k(1)}\chi_\rho(1,t)\Big(4\int_{\arcsin\left(\frac{u-t}{\log\rho}\right)}^{\frac{\pi}{2}} I_0\Big(\frac{\log\rho}{2}\cos\theta\Big)\log\rho\cos\theta\log\rho\sin\theta(u-t)\,d\theta\\
&\qquad+\int_{-\arcsin\left(\frac{u-t}{\log\rho}\right)}^{\arcsin\left(\frac{u-t}{\log\rho}\right)} I_0\Big(\frac{\log\rho}{2}\cos\theta\Big)\log\rho\cos\theta\big((\log\rho)^2\sin^2\theta+(u-t)^2\big)\,d\theta\Big)dt.
\end{aligned}
\tag{5.3.11}
$$

Observe that in the region $u-t>\log\rho$, this expression becomes much simpler due to the complete cancellations of odd and even terms in the integral.

We now estimate this expression using integral properties of the modified Bessel function to calculate these integrals. For example,

$$
\int_0^{\frac{\pi}{2}} I_0\Big(\frac{R}{2}\cos\theta\Big)\cos\theta\,d\theta=\frac{2\sinh(R/2)}{R}.
$$

We may therefore bound a typical term as

$$
\begin{aligned}
&\frac{\sqrt{\rho}}{4}\int_{-k(1)}^{k(1)}\chi_\rho(1,t)\int_{-\arcsin\left(\frac{u-t}{\log\rho}\right)}^{\arcsin\left(\frac{u-t}{\log\rho}\right)} I_0\Big(\frac{\log\rho}{2}\cos\theta\Big)\log\rho\cos\theta(u-t)^2\,d\theta\,dt\\
&\le\frac{\sqrt{\rho}u^2\log\rho}{4}\int_{-k(1)}^{k(1)}\chi_\rho(1,t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} I_0\Big(\frac{\log\rho}{2}\cos\theta\Big)\cos\theta\,d\theta\,dt\\
&\le\frac{\sqrt{\rho}u^2\log\rho}{4}\frac{2(\rho-1)}{\sqrt{\rho}\log\rho}\int_{-k(1)}^{k(1)}\chi_\rho(1,t)\,dt\le M\rho u^2.
\end{aligned}
$$

We hence obtain the desired bound of

$$|\eta^\sharp(\rho,m)| \leq M(\rho|u|^2 + \rho\log\rho) \leq M\eta^*(\rho,m).$$

The most important quantity to control is the entropy flux, $q^\sharp(\rho,m)$. We recall that we may write, for $\rho > 1$,

$$q^\sharp(\rho,m) = u\eta^\sharp(\rho,m) + h^\sharp(\rho,m),$$

where $h^\sharp(\rho,m)$ is also an entropy in this region. The function $h^\sharp$ is thus generated from the entropy kernel $\chi(\rho,u-s)$. As the generating function is again of quadratic growth at infinity the previous estimate easily extends to give a bound

$$h^\sharp(\rho,m) \geq -M(\rho|u|^2 + \rho\log\rho).$$

We therefore focus on the term $u\eta^\sharp(\rho,m)$. In analogy to the decomposition and calculation above in (5.3.11), we may write $u\eta^\sharp$ as a sum of terms carrying factors of $|u|$, u^2 and $|u|^3$. As the former two terms are easier to control, we focus on the last group.

Assuming again that $u - 2k(1) \geq 0$, so that $u - t \geq 0$ always, we calculate, this time including all of the terms from the entropy kernel, that the u^3 contribution is

$$\begin{aligned}
\frac{\sqrt{\rho}u^3\log\rho}{2}\bigg(&\int_{-k(1)}^{k(1)}\big(\chi_\rho(1,t) - \frac{1}{2}\chi(1,t)\big)\Big(\int_0^{\frac{\pi}{2}} I_0\Big(\frac{\log\rho}{2}\cos\theta\Big)\cos\theta\,d\theta\Big)\,dt \\
&- \int_{u-\log\rho}^{k(1)}\big(\chi_\rho(1,t) - \frac{1}{2}\chi(1,t)\big)\Big(\int_{\arcsin\left(\frac{u-t}{\log\rho}\right)}^{\frac{\pi}{2}} I_0\Big(\frac{\log\rho}{2}\cos\theta\Big)\cos\theta\,d\theta\Big)\,dt \\
&+ \frac{1}{2}\int_{-k(1)}^{k(1)}\chi(1,t)\int_0^{\frac{\pi}{2}} I_1\Big(\frac{\log\rho}{2}\cos\theta\Big)\,d\theta\,dt \\
&- \frac{1}{2}\int_{u-\log\rho}^{k(1)}\chi(1,t)\int_{\arcsin\left(\frac{u-t}{\log\rho}\right)}^{\frac{\pi}{2}} I_1\Big(\frac{\log\rho}{2}\cos\theta\Big)\,d\theta\,dt\bigg),
\end{aligned}$$

where the second and fourth lines of this expression vanish identically if $|u| \geq \log\rho + k(1)$, allowing exact calculation of the integrals:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} I_0\Big(\frac{\log\rho}{2}\cos\theta\Big)\cos\theta\,d\theta &= \frac{\rho-1}{\sqrt{\rho}\log\rho}, \\
\int_0^{\frac{\pi}{2}} I_1\Big(\frac{\log\rho}{2}\cos\theta\Big)\,d\theta &= \frac{\rho - 2\sqrt{\rho}+1}{\sqrt{\rho}\log\rho}.
\end{aligned}$$

Thus we obtain a lower bound of $M^{-1}\rho|u|^3$ (we recall from (5.2.7) that $\int\chi(1,t)\,dt = \int\chi_\rho(1,t)\,dt$). On the other hand, if $|u| < \log\rho + k(1)$, we see that still $q^\sharp(\rho,m) \geq 0$. Moreover, on this region, it is sufficient to write

$$q^\sharp(\rho,m) \geq \rho|u|^3 - \rho|u|^3 \geq \rho|u|^3 - \rho(\log\rho)^3.$$

This concludes the proof of the lower bound for the entropy flux.

For the sake of concision, we omit the proofs of the derivative bounds, as the calculations are lengthy and use no new techniques, relying simply on the decompositions and estimates used already. $\square$

As advertised at the beginning of this subsection, we use this construction to obtain the higher integrability of the velocity. Before continuing, we observe that the lower bound on $q^\sharp$ includes the term $-\rho(\log\rho)^3$. We make no claim that this is an optimal bound. However, as we seek only to bound $q^\sharp(\rho,m)$ locally, the local integrability of the quantity $\rho p(\rho)$ is more than sufficient to control this extra term.

In order to make the uniform estimate, we modify the entropy pair $(\eta^\sharp, q^\sharp)$ suitably to allow for integration in space. We define

$$\begin{aligned}\check{\eta}(\rho,m) &= \eta^\sharp(\rho, m-\rho u_-),\\ \check{q}(\rho,m) &= q^\sharp(\rho, m-\rho u_-) + u_-\eta^\sharp(\rho, m-\rho u_-).\end{aligned} \tag{5.3.12}$$

We may then expand $\check{\eta}(\rho,m)$ as

$$\check{\eta}(\rho,m) = \eta^\sharp_m(\rho,0)\rho(u-u_-) + \check{r}(\rho,m),$$

where

$$|\check{r}(\rho,m)| \leq M\rho|u-u_-|^2.$$

To see this, we first note that by evenness of $\chi(\rho,-s)$ in s, we have that

$$\eta^\sharp(\rho,0) = \frac{1}{2}\int s|s|\chi(\rho,-s)\,ds = 0.$$

Thus, taking a Taylor expansion in m around $m=0$ for $\eta^\sharp(\rho,m)$, we find

$$\eta^\sharp(\rho,m) = \eta^\sharp_m(\rho,0)m + \eta^\sharp_{mm}(\rho,\xi)\frac{m^2}{2},$$

for some ξ between 0 and m. By the bound $|\eta^\sharp_{mm}| \leq M\rho^{-1}$ of Lemma 5.3.5, we obtain the desired expansion.

Lemma 5.3.6. *Suppose in addition to the assumptions of Lemmas 5.3.2–5.3.4 that*

$$\int_{\mathbb{R}} \rho_0(x)|u_0(x)-\bar{u}(x)|\,dx \leq M_0 < \infty,$$

where M_0 is independent of ε. Then, for any compact subset $K\subset\mathbb{R}$, there exists a constant $M>0$, depending on K but not on ε, such that

$$\int_0^T\int_K \rho|u|^3\,dx\,dt \leq M. \tag{5.3.13}$$

Proof. Multiplying the first equation in (5.1.1) by $\check{\eta}_\rho$ and the second equation by $\check{\eta}_m$ and summing, we obtain

$$\check{\eta}(\rho,m)_t + \check{q}(\rho,m)_x = \varepsilon u_{xx}\check{\eta}_m(\rho,m).$$

Integrating this in space from $-\infty$ to x and integrating by parts, we find

$$\begin{aligned}\int_{-\infty}^{x} \check{\eta}(\rho,m)_t\,dy + q^\sharp(\rho,\rho(u-u_-)) + u_-\eta^\sharp(\rho,\rho(u-u_-)) - q^\sharp(\rho_-,0)\\ -\varepsilon u_x\check{\eta}_m(\rho,m) + \int_{-\infty}^{x}\big(\varepsilon u_x^2\check{\eta}_{mu}(\rho,m) + \varepsilon\rho_x u_x\check{\eta}_{m\rho}(\rho,m)\big)\,dy = 0.\end{aligned}$$

We integrate this equation over the region $(0,T)\times K$ to obtain

$$\begin{aligned}\int_0^T\int_K q^\sharp(\rho,\rho(u-u_-))\,dx\,dt =& T|K|q^\sharp(\rho_-,0) - u_-\int_0^T\int_K \eta^\sharp(\rho,\rho(u-u_-))\,dx\,dt\\ &+\int_K\int_{-\infty}^{x}\check{\eta}(\rho_0,m_0)\,dy\,dx - \int_K\int_{-\infty}^{x}\check{\eta}(\rho,m)\,dy\,dx\\ &+\varepsilon\int_0^T\int_K u_x\check{\eta}_m(\rho,m)\,dx\,dt\\ &-\int_0^T\int_K\int_{-\infty}^{x}\big(\varepsilon u_x^2\check{\eta}_{mu} + \varepsilon\rho_x u_x\check{\eta}_{m\rho}\big)\,dy\,dx\,dt\\ =& I_1+\cdots+I_6.\end{aligned}$$

Clearly, $|I_1|\le M$. Moreover, applying Lemma 5.3.2 and Lemma 5.3.3 with Hölder's inequality and the bounds for $|\check{\eta}_{mu}|$ and $|\check{\eta}_{m\rho}|$ of Lemma 5.3.5, we also get $|I_6|\le M$. For I_2, we recall that $|\eta^\sharp(\rho,m)|\le\eta^*(\rho,m)$ to obtain a bound of

$$|I_2| \le M\int_0^T\int_K\big(\rho|u-u_-|^2+\rho e(\rho)\big)\,dx\,dt \le M\Big(1+\int_0^T E[\rho,u](t)\,dt\Big)\le M.$$

We consider next I_3 and I_4. We begin by bounding

$$\begin{aligned}\Big|\int_{-\infty}^{x}\check{\eta}(\rho,m)\,dy\Big| \le& \Big|\int_{-\infty}^{x}\check{\eta}(\rho,m)-\eta_m^\sharp(\rho,0)\rho(u-u_-)\,dy\Big|\\ &+\Big|\int_{-\infty}^{x}\eta_m^\sharp(\rho,0)\rho(u-u_-)\,dy\Big|,\\ \le& \Big|\int_{-\infty}^{x}\check{r}(\rho,m)\,dy\Big| + \Big|\int_{-\infty}^{x}\eta_m^\sharp(\rho_-,0)\rho(u-u_-)\,dy\Big|\\ &+\Big|\int_{-\infty}^{x}\big(\eta_m^\sharp(\rho,0)-\eta_m^\sharp(\rho_-,0)\big)\rho(u-u_-)\,dy\Big|\\ \le& M\int_{-\infty}^{x}\rho|u-u_-|^2\,dy + |\check{\eta}_m(\rho_-,0)|\Big|\int_{-\infty}^{x}\rho(u-u_-)\,dy\Big|\\ &+\int_{-\infty}^{x}\rho\big|\eta_m^\sharp(\rho,0)-\eta_m^\sharp(\rho_-,0)\big|^2\,dy\end{aligned}\tag{5.3.14}$$

from the bound $|\check{r}(\rho,m)|\le M\rho|u-u_-|^2$ and the Cauchy-Schwarz inequality.

We next bound the last two terms on the right-hand side. To begin with, we observe that by integrating the momentum equation in time and space and applying the conservation of mass equation, we have

$$\begin{aligned}\int_{-\infty}^{x} \rho(t,y)(u(t,y)-u_-)\,dy = &\int_{-\infty}^{x} \rho_0(u_0-\bar{u})\,dy + \int_{-\infty}^{x} \rho_0(\bar{u}-u_-)\,dy \\ &- \int_0^T \big(\rho u^2 + p(\rho) - p(\rho_-) - \rho u u_-\big)\,dt + \varepsilon \int_0^T u_x\,dt,\end{aligned}$$

which, after integrating over K, is easily bounded using the assumption on the initial momentum and the main energy estimate, Lemma 5.3.2.

Moreover, it is not hard to show that

$$\rho\big|\eta_m^\sharp(\rho,0) - \eta_m^\sharp(\rho_-,0)\big|^2 \le M e^*(\rho,\bar{\rho})$$

from the estimates of Lemma 5.3.5.

We thus conclude that I_3 and I_4 are bounded as

$$\int_K \Big| \int_{-\infty}^{x} \check{\eta}(\rho,m)\,dy\Big|\,dx \le M.$$

It remains only to prove that I_5 is well-controlled. First, from Lemma 5.3.5, we bound

$$\begin{aligned}|I_5| &\le \varepsilon M \int_0^T \int_K \big(|u||u_x| + \big(\rho^\theta \mathbb{1}_{\rho\le 1} + \sqrt{\log\rho}\,\mathbb{1}_{\rho>1}\big)|u_x|\big)\,dx\,dt \\ &\le \varepsilon M \int_0^T \int_K \big(|u|^2 + |u_x|^2 + \log\rho + 1\big)\,dx\,dt \\ &\le M\Big(1 + \varepsilon \int_0^T \int_K |u|^2\,dx\,dt\Big).\end{aligned}$$

To control the last term in this expression, we follow the argument of [12, Lemma 3.4]. We therefore begin by observing that there is a positive, non-decreasing function of time $M(t)$ such that

$$\int_{\{\rho(t,x)\le\bar{\rho}/2\}} e^*(\rho(t,x),\bar{\rho}(x))\,dx \le M(t)$$

and thus

$$|\{x\in\mathbb{R} : \rho(t,x) \le \check{\rho}/2\}| \le \frac{M(t)}{e^*(\check{\rho}/2,\check{\rho})},$$

where $\check{\rho}$ is defined as $\min\{\rho_-,\rho_+\}$.

Without loss of generality, we suppose that K contains an interval $[a,b]$ of length at least $2M(t)/e^*(\frac{\check{\rho}}{2},\check{\rho})$. Thus there exists a measurable set $A(t)\subset(a,b)$ such that $\rho \ge \frac{\check{\rho}}{2}$ on A where A has measure at least $M(t)/e^*(\frac{\check{\rho}}{2},\check{\rho})$. We then write

$$u_A(t) := \frac{1}{|A|}\int_A u(t,x)\,dx.$$

Then we observe that for any $x \in [a,b]$, at fixed t, we have

$$|u(t,x)| \leq |u_A(t)| + \int_K |u_x|\,dx.$$

It follows straightforwardly that

$$\begin{aligned}
|u_A(t)| \leq \frac{1}{|A|}\int_A |u(t,x)|\,dx &\leq \frac{1}{|A|}\sqrt{\frac{2}{\check{\rho}}}\int_A \sqrt{\rho(t,x)}|u(t,x)|\,dx, \\
&\leq \sqrt{\frac{2}{\check{\rho}|A|}}\left(\int_K \rho(t,x)|u(t,x)|^2\,dx\right)^{1/2}, \\
&\leq \sqrt{\frac{2e^*(\check{\rho}/2,\check{\rho})}{\check{\rho}M(t)}}M(E_0,K).
\end{aligned}$$

So, we have

$$\varepsilon M\int_0^T\int_K |u|^2\,dx\,dt \leq M\left(\varepsilon\int_0^T\int_{\mathbb{R}} |u_x|^2\,dx\,dt + \int_0^T\int_K |u_A(t)|^2\,dx\,dt\right) \leq M.$$

Hence we have $|I_5| \leq M$, and so

$$\int_0^T\int_K q^\sharp(\rho, m - \rho u_-)\,dx\,dt \leq M,$$

from which the conclusion of the lemma is immediate. □

Remark 5.3.7. As we have made successive assumptions in the statements of the uniform estimates for the solutions, we collect these assumptions here for future reference.

- The initial data must be of finite-energy:

$$\sup_\varepsilon E[\rho_0^\varepsilon, u_0^\varepsilon] \leq E_0 < \infty;$$

- The initial density must satisfy a weighted derivative bound

$$\sup_\varepsilon \varepsilon^2 \int \frac{|\rho_{0,x}^\varepsilon(x)|^2}{\rho_0^\varepsilon(x)^3}\,dx \leq E_1 < \infty;$$

- The relative initial momentum should be finite:

$$\sup_\varepsilon \int \rho_0^\varepsilon(x)|u_0^\varepsilon(x) - \bar{u}(x)|\,dx \leq M_0 < \infty.$$

All three of these conditions and the additional condition $\rho_0^\varepsilon \geq c_0^\varepsilon$ may be guaranteed by cutting the initial data of the problem by $\max\{\rho_0, \varepsilon^{\frac{1}{2}}\}$ and then mollifying at a suitable scale.

5.3.3 Entropies generated by compactly supported functions

In order to analyse the vanishing viscosity limit of the solutions of the Navier-Stokes equations, (5.1.1), we need to employ a large class of entropies to gain information on the Young measure solution obtained in the limit. As noted in [12], a suitable class is given by those entropy pairs that are generated by test functions of compact support. In this section, we therefore collect various properties of such entropy pairs.

Lemma 5.3.8. *Let $\psi \in C_c^2(\mathbb{R})$ be a compactly supported test function such that the support of ψ is contained in an interval $[z_*, w_*]$. Then the corresponding entropy pair (η^ψ, q^ψ) satisfies the following:*

$$\mathrm{supp}\,\eta^\psi,\ \mathrm{supp}\,q^\psi \subset \{(\rho, u) : w(\rho, u) \geq z_*,\ z(\rho, u) \leq w_*\},$$

where $w(\rho, u)$ and $z(\rho, u)$ are the Riemann invariants defined in §2.3. Moreover, for any $\rho_* > 1$,

(i)

$$\begin{aligned} |\eta^\psi(\rho, m)| &\leq \begin{cases} C_\psi \rho, & \rho \leq \rho_*, \\ C_\psi \frac{\rho}{\sqrt{\log \rho}}, & \rho > \rho_*, \end{cases} \\ |q^\psi(\rho, m)| &\leq C_\psi \rho; \end{aligned} \tag{5.3.15}$$

(ii)

$$|\eta_m^\psi(\rho, m)| + |\rho \eta_{mm}^\psi(\rho, m)| \leq \begin{cases} C_\psi, & \rho \leq \rho_*, \\ C_\psi \frac{1}{\sqrt{\log \rho}}, & \rho > \rho_*; \end{cases} \tag{5.3.16}$$

(iii) Considering η_m^ψ as a function of ρ and u,

$$\begin{aligned} |\eta_{mu}^\psi(\rho, \rho u)| &\leq \begin{cases} C_\psi, & \rho \leq \rho_*, \\ C_\psi \frac{1}{\sqrt{\log \rho}}, & \rho > \rho_*, \end{cases} \\ |\rho \eta_{m\rho}^\psi(\rho, \rho u)| &\leq \begin{cases} C_\psi \rho^\theta, & \rho \leq \rho_*, \\ C_\psi \frac{1}{\log \rho}, & \rho > \rho_*, \end{cases} \end{aligned} \tag{5.3.17}$$

where, for example, $\eta_{mu}^\psi(\rho, \rho u) = \partial_u \eta_m^\psi(\rho, \rho u)$. In particular,

$$|\eta_{m\rho}^\psi(\rho, \rho u)| \leq C_\psi \frac{\sqrt{p'(\rho)}}{\rho}.$$

Proof. Working from the expansions (5.2.1)-(5.2.2), the desired bounds in the region $\rho \leq \rho_*$ all follow from the arguments in [12, Lemma 2.1]. For the high density region, the estimates follow from the new expansions of Theorem 5.2.6. For the sake of brevity, we omit the straightforward but tedious calculation. □

5.3.4 Compactness of the entropy dissipation measures

We can now use the uniform estimates above to obtain a compactness property for the entropy dissipation measures associated to weak entropies generated from compactly supported functions and the solutions of the Navier-Stokes equations (5.1.1). This compactness, obtained in the negative order Sobolev space $W^{-1,q}_{loc}$, will be used later to apply the theory of compensated compactness to our sequence of approximate solutions.

Proposition 5.3.9. *Let $\psi \in C^2_c(\mathbb{R})$ generate the weak entropy pair (η^ψ, q^ψ) and suppose that $(\rho^\varepsilon,\, u^\varepsilon)$ is a sequence of solutions to the Navier-Stokes equations (5.1.1) satisfying the assumptions in Remark 5.3.7. Then the weak entropy dissipation measures*

$$\eta^\psi(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)_t + q^\psi(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)_x \text{ are (pre)-compact in } W^{-1,q}_{loc}(\mathbb{R}^2_+) \tag{5.3.18}$$

for any $q \in (1, 2)$.

Proof. We write $m^\varepsilon = \rho^\varepsilon u^\varepsilon$ throughout this proof. Multiplying the first equation in (5.1.1) by $\eta^\psi_\rho(\rho^\varepsilon, m^\varepsilon)$ and the second equation by $\eta^\psi_m(\rho^\varepsilon, m^\varepsilon)$ and summing, we obtain

$$\begin{aligned}\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x =& \,\varepsilon\big(\eta^\psi_m(\rho^\varepsilon, m^\varepsilon)u^\varepsilon_x\big)_x \\ &- \varepsilon\eta^\psi_{mu}(\rho^\varepsilon, m^\varepsilon)|u^\varepsilon_x|^2 - \varepsilon\eta^\psi_{m\rho}(\rho^\varepsilon, m^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x,\end{aligned} \tag{5.3.19}$$

where we use the notation from Lemma 5.3.8 for η^ψ_{mu} and $\eta^\psi_{m\rho}$.

We recall from Lemma 5.3.8 that

$$|\eta^\psi_{mu}(\rho^\varepsilon, m^\varepsilon)| \le M, \qquad |\eta^\psi_{m\rho}(\rho^\varepsilon, m^\varepsilon)| \le M\frac{\sqrt{p'(\rho^\varepsilon)}}{\rho^\varepsilon}.$$

Thus we may estimate, for any compact set $K \subset \mathbb{R}$,

$$\begin{aligned}\int_0^T \int_K |\varepsilon\eta^\psi_{mu}(\rho^\varepsilon, m^\varepsilon)|u^\varepsilon_x|^2|\,dx\,dt &\le \varepsilon M \int_0^T \int_K |u^\varepsilon_x|^2\,dx\,dt \le M,\\ \int_0^T \int_K |\varepsilon\eta^\psi_{m\rho}(\rho^\varepsilon, m^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x|\,dx\,dt &\le \big\|\varepsilon\eta^\psi_{m\rho}(\rho^\varepsilon, m^\varepsilon)\rho^\varepsilon_x\big\|_{L^2}\|\varepsilon u^\varepsilon_x\|_{L^2}\\ &\le M\Big\|\varepsilon\frac{\sqrt{p'(\rho^\varepsilon)}}{\rho^\varepsilon}\rho^\varepsilon_x\Big\|_{L^2}\\ &\le M,\end{aligned}$$

where we have used the estimates of Lemma 5.3.2 and Lemma 5.3.3. Thus

$$-\varepsilon\eta^\psi_{mu}(\rho^\varepsilon, m^\varepsilon)|u^\varepsilon_x|^2 - \varepsilon\eta^\psi_{m\rho}(\rho^\varepsilon, m^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x$$

is uniformly bounded in $L^1_{loc}(\mathbb{R}^2_+)$ and hence is (pre)-compact in $W^{-1,q}_{loc}(\mathbb{R}^2_+)$ for any $q \in (1, 2)$ by the standard Rellich-Kondrachov embedding.

For the final term, $\varepsilon\big(\eta^{\psi}_m(\rho^\varepsilon, m^\varepsilon)u^\varepsilon_x\big)_x$, we recall that $|\eta^{\psi}_m(\rho^\varepsilon, m^\varepsilon)| \le M$, and hence

$$\|\varepsilon\eta^{\psi}_m(\rho^\varepsilon, m^\varepsilon)u^\varepsilon_x\|_{L^2} \le M\sqrt{\varepsilon} \to 0,$$

i.e. $\big(\varepsilon\eta^{\psi}_m(\rho^\varepsilon, m^\varepsilon)u^\varepsilon_x\big)_x \to 0$ in $W^{-1,2}_{loc}(\mathbb{R}^2_+)$. Thus, considering the sum of this term with the terms previously considered, we obtain that

$$\eta^{\psi}(\rho^\varepsilon, m^\varepsilon)_t + q^{\psi}(\rho^\varepsilon, m^\varepsilon)_x \text{ is (pre)-compact in } W^{-1,q}_{loc}(\mathbb{R}^2_+)$$

as desired. □

5.4 Convergence to a Young measure solution

As the estimates available to us do not offer uniform control over the velocity in a suitable L^p space, we may not directly apply the fundamental theorem of Young measures as classically stated. Instead, we follow the construction outlined in §3.3 to construct a measure-valued solution defined on a compactification of our phase space.

To this end, we recall some notation from §3.3. First, we set

$$\mathbb{H} = \{(\rho, u) \in \mathbb{R}^2 : \rho > 0\}$$

and consider on $\mathbb{H}$ the set of continuous functions

$$\bar{C}(\mathbb{H}) = \left\{ \phi \in C(\bar{\mathbb{H}}) \;\middle|\; \begin{array}{l} \phi(\rho, u) \text{ is constant on } \{\rho = 0\} \text{ and such that} \\ \text{the function } (\rho, u) \mapsto \lim_{s\to\infty} \phi(s\rho, su) \in C(\mathbb{S}^1 \cap \bar{\mathbb{H}}) \end{array} \right\} \quad (5.4.1)$$

where $\mathbb{S}^1$ is the unit circle. We recall from [12, 48] that this space allows us to deal with both the difficulty at the vacuum and the difficulty at large densities.

As $\bar{C}(\mathbb{H})$ is a complete sub-ring of the continuous functions on $\mathbb{H}$ containing the constant functions, by Theorem 3.2.2, there exists a compactification $\overline{\mathcal{H}}$ of $\mathbb{H}$ such that $C(\overline{\mathcal{H}}) \cong \bar{C}(\mathbb{H})$, where $\cong$ denotes isometric isomorphism. The topology of $\overline{\mathcal{H}}$ is the weak-star topology induced by $C(\overline{\mathcal{H}})$, i.e. a sequence $v_n \in \overline{\mathcal{H}}$ converges to $v \in \overline{\mathcal{H}}$ if and only if $\phi(v_n) \to \phi(v)$ for all $\phi \in C(\overline{\mathcal{H}})$. From §3.3, we know that this topology is both separable and metrizable. Moreover, considering the induced topology on $\bar{\mathbb{H}}$, this topology has the obvious advantage of not distinguishing points in the vacuum set. We write V for the weak-star closure of the set $\{\rho = 0\}$ and set

$$\mathcal{H} = \mathbb{H} \cup V.$$

Now that we have the metrizable compactification $\overline{\mathcal{H}}$, we recall the fundamental theorem of Young measures stated in Theorem 3.1.4 for maps into compact metric spaces. This theorem gives the existence of a Young measure $\nu_{t,x}$ satisfying the following. Given a sequence of functions $(\rho^\varepsilon, u^\varepsilon) : \mathbb{R}^2_+ \to \overline{\mathcal{H}}$, there exists a subsequence generating a Young measure $\nu_{t,x} \in \mathrm{Prob}(\overline{\mathcal{H}})$ for almost every (t,x) in the sense that, for any $\phi \in C(\overline{\mathcal{H}})$,

$$\phi(\rho^\varepsilon, u^\varepsilon) \overset{*}{\rightharpoonup} \int_{\overline{\mathcal{H}}} \phi(\rho, u)\, d\nu_{t,x} \text{ in } L^\infty(\mathbb{R}^2_+).$$

Moreover, moving to the (ρ, m) coordinates, we have that $(\rho^\varepsilon, m^\varepsilon) \to (\rho, m)$ in measure (and hence almost everywhere up to subsequence) if and only if $\nu_{t,x} = \delta_{(\rho(t,x), m(t,x))}$ for almost every $(t,x) \in \mathbb{R}^2_+$.

In what follows, we will often suppress the subscript t,x for $\nu_{t,x}$ where there is no risk of confusion.

Lemma 5.4.1. *Let $\nu_{t,x}$ be a Young measure generated by a sequence of solutions to (5.1.1) satisfying the assumptions of Remark 5.3.7. Then, for almost every $(t,x) \in \mathbb{R}^2_+$, the measure $\nu_{t,x} \in \mathrm{Prob}(\mathcal{H})$.*

Proof. We argue as in [48, Proposition 2.3]. For $R > 0$, we fix a radial function $\phi_R \in C(\bar{\mathbb{H}})$ with $0 \le \phi_R \le 1$ and such that $\phi_R = 1$ in $\bar{\mathbb{H}} \setminus B_{2R}(0)$, $\phi_R = 0$ in $\bar{\mathbb{H}} \cap B_R(0)$. We choose also a function $\varphi \in C_c(\mathbb{S}^1 \cap \mathbb{H})$, which we extend to all of $\mathbb{H}$ as a homogeneous function of order 0. Then $\Phi_R = \varphi\phi_R \in \bar{C}(\mathbb{H}) \cong C(\overline{\mathcal{H}})$, and so for any compact subset $K \subset \mathbb{R}$,

$$\begin{aligned}
\Big| \int_0^T \int_K \Big(\int_{\overline{\mathcal{H}}} \Phi_R \, d\nu_{t,x} \Big) dx\, dt \Big| &= \lim_{\varepsilon \to 0} \Big| \int_0^T \int_K \Phi_R(\rho^\varepsilon, u^\varepsilon)\, dx\, dt \Big| \\
&\le \sup_\varepsilon |\{(t,x) \in (0,T) \times K \,:\, \rho^\varepsilon(t,x) \ge c_\varphi R\}| \\
&\le \sup_\varepsilon \frac{1}{(c_\varphi R)^2} \int_0^T \int_K \rho p(\rho)\, dx\, dt \to 0 \text{ as } R \to \infty.
\end{aligned}$$

Thus $\nu_{t,x}$ must be supported in $\mathcal{H}$. $\square$

Proposition 5.4.2. (i) *The Young measure generated by the sequence of solutions to (5.1.1) satisfying the assumptions of Remark 5.3.7 satisfies the following additional integrability property:*

$$\int_{\mathbb{H}} (\rho p(\rho) + \rho |u|^3)\, d\nu_{t,x} \in L^1_{loc}(\mathbb{R}^2_+).$$

(ii) *Moreover, the space of admissible test functions for the Young measure may be extended as follows:*
Let $\phi \in C(\bar{\mathbb{H}})$ *be a function vanishing on the set* $\partial\mathbb{H}$*. Suppose that there exists* $a > 0$ *such that* $supp\,\phi \subset \{w(\rho, u) \geq -a,\ z(\rho, u) \leq a\}$*. If, in addition,* ϕ *satisfies the growth bound*

$$\lim_{\rho\to\infty} \frac{|\phi(\rho, u)|}{\rho^2} = 0 \tag{5.4.2}$$

uniformly in u*, then* ϕ *is integrable with respect to* $\nu_{t,x}$ *for almost all* $(t, x) \in \mathbb{R}^2_+$ *and*

$$\phi(\rho^\varepsilon, u^\varepsilon) \rightharpoonup \int_{\mathbb{H}} \phi\, d\nu_{t,x} \ \textit{in}\ L^1_{loc}(\mathbb{R}^2_+).$$

The proof follows the proof of [12, Proposition 5.1] with a small adjustment in part *(ii)* to allow for the different growth rate of the function ϕ.

Proof. We begin by defining a cut-off function $\omega_h(\rho, u) \geq 0$ such that ω_h equals 1 on the set

$$\{(\rho, u)\ :\ k(\rho) \in [\frac{1}{h}, h],\ |u| \leq h\},$$

and vanishes outside the set

$$\{(\rho, u)\ :\ k(\rho) \in [\frac{1}{2h}, 2h],\ |u| \leq 2h\}.$$

Then the functions $(\rho^\varepsilon p(\rho^\varepsilon) + \rho^\varepsilon |u^\varepsilon|^3)\omega_h(\rho^\varepsilon, u^\varepsilon)$ are in $\bar{C}(\mathbb{H})$, so that we may take the limit

$$\begin{aligned}\lim_{\varepsilon\to0} \int_{[0,T]\times K} &(\rho^\varepsilon p(\rho^\varepsilon) + \rho^\varepsilon |u^\varepsilon|^3)\omega_h(\rho^\varepsilon, u^\varepsilon)\, dx\, dt \\ &= \int_{[0,T]\times K} \Big(\int_{\mathbb{H}} (\rho p(\rho) + \rho |u|^3)\omega_h(\rho, u)\, d\nu_{t,x} \Big) dx\, dt,\end{aligned}$$

for any compact subset $K \subset \mathbb{R}$. By the uniform estimates of Lemmas 5.3.4 and 5.3.6,

$$\int_{[0,T]\times K} (\rho^\varepsilon p(\rho^\varepsilon) + \rho^\varepsilon |u^\varepsilon|^3)\omega_h(\rho^\varepsilon, u^\varepsilon)\, dx\, dt \leq M,$$

uniformly in ε and h. Thus by the monotone convergence theorem,

$$\lim_{h\to\infty} \int_{\mathbb{H}} (\rho p(\rho) + \rho |u|^3)\omega_h(\rho, u)\, d\nu_{t,x} = \int_{\mathbb{H}} (\rho p(\rho) + \rho |u|^3)\, d\nu_{t,x},$$

where the limit is integrable locally in (t, x) and hence finite for almost every $(t, x) \in \mathbb{R}^2_+$.

To prove the second part of the lemma, we note that for such functions ϕ, the product $\phi\, \omega_h \in \bar{C}(\mathbb{H})$, where ω_h is as before. Thus the product $\phi\, \omega_h$ is $\nu_{t,x}$-integrable for almost every (t, x). Applying the dominated convergence theorem and part (i), we deduce

$$\lim_{h\to\infty} \int_{\mathbb{H}} \phi\, \omega_h\, d\nu_{t,x} = \int_{\mathbb{H}} \phi\, d\nu_{t,x} \text{ for almost every } (t, x) \in \mathbb{R}^2_+$$

and

$$\lim_{h\to\infty}\int_{[0,T]\times K}\int_{\mathbb{H}}\phi\,\omega_h\,d\nu_{t,x}\,dx\,dt=\int_{[0,T]\times K}\int_{\mathbb{H}}\phi\,d\nu_{t,x}\,dx\,dt.$$

Now we apply the definition of the Young measure to see that

$$\lim_{h\to\infty}\lim_{\varepsilon\to 0}\int_{[0,T]\times K}\phi(\rho^\varepsilon,u^\varepsilon)\,\omega_h(\rho^\varepsilon,u^\varepsilon)\,dx\,dt=\int_{[0,T]\times K}\int_{\mathbb{H}}\phi\,d\nu_{t,x}\,dx\,dt. \tag{5.4.3}$$

If we can interchange these limits, then we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{[0,T]\times K}\phi(\rho^\varepsilon,u^\varepsilon)\,dx\,dt=\lim_{\varepsilon\to 0}\lim_{h\to\infty}\int_{[0,T]\times K}\phi(\rho^\varepsilon,u^\varepsilon)\omega_h(\rho^\varepsilon,u^\varepsilon)\,dx\,dt\\ &=\lim_{h\to\infty}\lim_{\varepsilon\to 0}\int_{[0,T]\times K}\phi(\rho^\varepsilon,u^\varepsilon)\omega_h(\rho^\varepsilon,u^\varepsilon)\,dx\,dt=\int_{[0,T]\times K}\int_{\mathbb{H}}\phi\,d\nu_{t,x}\,dx\,dt,\end{aligned}$$

and so we are done. To show that we may interchange these limits, we prove that

$$\int_{[0,T]\times K}\phi(\rho^\varepsilon,u^\varepsilon)\omega_h(\rho^\varepsilon,u^\varepsilon)\,dx\,dt\to\int_{[0,T]\times K}\phi(\rho^\varepsilon,u^\varepsilon)\,dx\,dt \text{ as } h\to\infty \tag{5.4.4}$$

uniformly in ε for $\varepsilon\in(0,\varepsilon_0]$.

To prove (5.4.4), we take $h_1<h_2$ and consider the difference

$$\int_{[0,T]\times K}\phi(\rho^\varepsilon,u^\varepsilon)\big(\omega_{h_1}(\rho^\varepsilon,u^\varepsilon)-\omega_{h_2}(\rho^\varepsilon,u^\varepsilon)\big)\,dx\,dt.$$

We first notice that

$$\operatorname{supp}(\omega_{h_1}-\omega_{h_2})\subset\Big(\Big\{\frac{1}{h_1}\le k(\rho)\le h_1,\,|u|\le h_1\Big\}\Big)^c,$$

and

$$\sup_{\{0\le k(\rho)\le 1/h_1\}}|\phi(\rho,u)|\le m_{h_1}\to 0 \text{ as } h_1\to\infty.$$

Also, if $(\rho,u)\in\operatorname{supp}\phi\cap\operatorname{supp}(\omega_{h_1}-\omega_{h_2})$, then for h_1 sufficiently large, $k(\rho)\ge\frac{1}{2}h_1$ or $k(\rho)\le\frac{1}{h_1}$. Moreover, by (5.4.2), for any $\Delta>0$, there exists $M_\Delta>0$ such that

$$|\phi(\rho,u)|\le M_\Delta+\Delta\rho p(\rho).$$

Thus we estimate, for h_1 large,

$$\begin{aligned}&\Big|\int_{[0,T]\times K}\phi(\rho^\varepsilon,u^\varepsilon)\big(\omega_{h_1}(\rho^\varepsilon,u^\varepsilon)-\omega_{h_2}(\rho^\varepsilon,u^\varepsilon)\big)\,dx\,dt\Big|\\ &\qquad\le T|K|m_{h_1}+M_\Delta|\{(t,x)\in(0,T)\times K\,:\,k(\rho^\varepsilon(t,x))\ge h_1/2\}|\\ &\qquad\quad+\Delta\int_{[0,T]\times K}\rho^\varepsilon(t,x)p(\rho^\varepsilon(t,x))\,dx\,dt.\end{aligned}$$

By Chebyshev's inequality, we bound

$$|\{(t,x)\in(0,T)\times K\,:\,k(\rho^\varepsilon(t,x))\ge h_1/2\}|\le Me^{-h_1}\int_{[0,T]\times K}\rho^\varepsilon(t,x)p(\rho^\varepsilon(t,x))\,dx\,dt.$$

Thus the uniform estimate of Lemma 5.3.4 gives

$$\Big| \int_{[0,T]\times K} \phi(\rho^\varepsilon, u^\varepsilon)\big(\omega_{h_1}(\rho^\varepsilon, u^\varepsilon) - \omega_{h_2}(\rho^\varepsilon, u^\varepsilon)\big)\, dx\, dt \Big| \leq T|K| m_{h_1} + M_\Delta e^{-h_1} + M\Delta,$$

independent of ε. This proves the uniform convergence in (5.4.4). □

From henceforth, for an admissible function $f(\rho, u)$, we write

$$\overline{f} = \int_{\mathcal{H}} f\, d\nu_{t,x}$$

when there is no confusion over the point $(t, x) \in \mathbb{R}^2_+$.

To gain the strong compactness of the sequence of approximate solutions $(\rho^\varepsilon, u^\varepsilon)$, we need to show that the Young measure $\nu_{t,x}$ is concentrated at a single point. As discussed previously, this will be shown as a consequence of the Tartar commutation relation, which we now prove.

Proposition 5.4.3. *Let $\nu_{t,x}$ be the Young measure generated by the sequence $(\rho^\varepsilon, u^\varepsilon)$ of solutions to (5.1.1) satisfying the assumptions of Remark 5.3.7. Then $\nu_{t,x}$ is constrained by the Tartar commutation relation,*

$$\overline{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)} = \overline{\chi(s_1)}\,\overline{\sigma(s_2)} - \overline{\chi(s_2)}\,\overline{\sigma(s_1)} \tag{5.4.5}$$

for all $s_1, s_2 \in \mathbb{R}$.

The main tool for proving this result is the div-curl lemma, Lemma 3.4.2, for sequences with divergence and curl compact in $W^{-1,1}_{loc}$.

Proof of Proposition 5.4.3. We recall from Lemma 5.3.8 that if $\psi \in C^2_c(\mathbb{R})$, then

$$|\eta^\psi(\rho, m)| \leq M\rho\big(\rho^{-\theta}\mathbb{1}_{\rho\leq\rho_*} + (\log\rho)^{-\frac{1}{2}}\mathbb{1}_{\rho>\rho_*}\big),$$
$$|q^\psi(\rho, m)| \leq M\rho,$$

and the supports of η^ψ, q^ψ are contained in sets of the form $\{w(\rho,u) \geq z_*,\ z(\rho,u) \leq w_*\}$ by Lemma 5.3.8. We choose two test functions, $\psi_1, \psi_2 \in C^2_c(\mathbb{R})$ and consider the sequences of vector fields

$$v^\varepsilon = (\eta^{\psi_1}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon), q^{\psi_1}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)), \quad w^\varepsilon = (q^{\psi_2}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon), -\eta^{\psi_2}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)).$$

From Lemma 5.3.4 and the bound just stated, it is clear that both v^ε and w^ε are uniformly bounded sequences in L^2_{loc}. Moreover, by Proposition 5.4.2(ii),

$$v^\varepsilon \rightharpoonup (\overline{\eta^{\psi_1}}, \overline{q^{\psi_1}}) \text{ in } L^2_{loc},$$
$$w^\varepsilon \rightharpoonup (\overline{q^{\psi_2}}, -\overline{\eta^{\psi_2}}) \text{ in } L^2_{loc}.$$

From Proposition 5.3.9, we obtain immediately that the entropy dissipation measures

$$\eta^{\psi_1}(\rho^\varepsilon,\rho^\varepsilon u^\varepsilon)_t + q^{\psi_1}(\rho^\varepsilon,\rho^\varepsilon u^\varepsilon)_x = \operatorname{div} v^\varepsilon,$$
$$\eta^{\psi_2}(\rho^\varepsilon,\rho^\varepsilon u^\varepsilon)_t + q^{\psi_2}(\rho^\varepsilon,\rho^\varepsilon u^\varepsilon)_x = \operatorname{curl} w^\varepsilon,$$

are compact in $W^{-1,q}_{loc}$ for all $q\in[1,2)$, in particular in $W^{-1,1}_{loc}$. Finally, we see that the product $v^\varepsilon\cdot w^\varepsilon$ satisfies the bound

$$|v^\varepsilon\cdot w^\varepsilon| \le M(\rho^\varepsilon)^2\big((\rho^\varepsilon)^{-\theta}\mathbb{1}_{(\rho^\varepsilon)\le\rho_*} + (\log(\rho^\varepsilon))^{-\frac12}\mathbb{1}_{(\rho^\varepsilon)>\rho_*}\big),$$

hence is equi-integrable as a result of the uniform bound of Lemma 5.3.4.

Thus, by the div-curl lemma, Lemma 3.4.2, we may pass to the limit in the product to obtain

$$v^\varepsilon\cdot w^\varepsilon \rightharpoonup \overline{\eta^{\psi_1}}\,\overline{q^{\psi_2}} - \overline{\eta^{\psi_2}}\,\overline{q^{\psi_1}} \quad \text{in } \mathcal{D}'.$$

On the other hand, passing to the Young measure limit directly in the product (observe that the previously described bounds on the entropies are sufficient to allow the product as a test function for the Young measure by Proposition 5.4.2(ii)),

$$v^\varepsilon\cdot w^\varepsilon \rightharpoonup \overline{\eta^{\psi_1}q^{\psi_2} - \eta^{\psi_2}q^{\psi_1}} \quad \text{in } L^1_{loc}.$$

By uniqueness of limits, we therefore obtain

$$\overline{\eta^{\psi_1}q^{\psi_2} - \eta^{\psi_2}q^{\psi_1}} = \overline{\eta^{\psi_1}}\,\overline{q^{\psi_2}} - \overline{\eta^{\psi_2}}\,\overline{q^{\psi_1}}. \tag{5.4.6}$$

Arguing by density of the test functions $\psi_1,\psi_2\in C^2_c(\mathbb{R})$ as in [12, 52], we obtain the equivalent relation for the entropy and entropy flux kernels, that is

$$\overline{\chi(s_1)\sigma(s_2)-\chi(s_2)\sigma(s_1)} = \overline{\chi(s_1)}\,\overline{\sigma(s_2)} - \overline{\chi(s_2)}\,\overline{\sigma(s_1)}, \tag{5.4.7}$$

for $s_1,s_2\in\mathbb{R}$ as follows: we deduce from the commutation relation (5.4.6), that

$$\begin{aligned}
&\int\Big(\int_{\mathbb{R}}\psi_1(s_1)\chi(\rho,v-s_1)\,ds_1\int_{\mathbb{R}}\psi_2(s_2)\sigma(\rho,v,s_2)\,ds_2\\
&\qquad-\int_{\mathbb{R}}\psi_2(s_2)\chi(\rho,v-s_2)\,ds_2\int_{\mathbb{R}}\psi_1(s_1)\sigma(\rho,v,s_1)\,ds_1\Big)d\nu_{t,x}(\rho,v)\\
&=\int\int_{\mathbb{R}}\psi_1(s_1)\chi(\rho,v-s_1)\,ds_1\,d\nu_{t,x}(\rho,v)\int\int_{\mathbb{R}}\psi_2(s_2)\sigma(\tilde\rho,\tilde v,s_2)\,ds_2\,d\nu_{t,x}(\tilde\rho,\tilde v)\\
&\quad-\int\int_{\mathbb{R}}\psi_2(s_2)\chi(\rho,v-s_2)\,ds_2\,d\nu_{t,x}(\rho,v)\int\int_{\mathbb{R}}\psi_1(s_1)\sigma(\tilde\rho,\tilde v,s_1)\,ds_1\,d\nu_{t,x}(\tilde\rho,\tilde v)
\end{aligned}$$

which, by Fubini's theorem, implies

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{\chi(s_1)\sigma(s_2)-\chi(s_2)\sigma(s_1)}\psi_1(s_1)\psi_2(s_2)\,ds_1\,ds_2\\
&\qquad=\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{\chi(s_1)}\,\overline{\sigma(s_2)}-\overline{\chi(s_2)}\,\overline{\sigma(s_1)}\psi_1(s_1)\psi_2(s_2)\,ds_1\,ds_2,
\end{aligned}$$

and we conclude via density of the test functions. $\square$

5.5 Reduction framework for the Young measure

In this section, we analyse the Young measure generated by the solutions of the Navier-Stokes equations (5.1.1) and which we showed in the previous section was constrained by the Tartar commutation relation. Our goal is to show that the support of the Young measure is either contained in the vacuum region V or at a single point in $\mathbb{H}$.

Our strategy for analysing the Young measure goes back to the work of [52]. The basic idea is that we may apply fractional derivative operators to the commutation relation in order to demonstrate that the relation represents an imbalance of regularity. This leads to the main theorem of this section.

Theorem 5.5.1. *Let $\nu \in \mathrm{Prob}(\mathcal{H})$ be a probability measure such that the function $(\rho, u) \mapsto \rho^2 \in L^1(\mathcal{H}, \nu)$ and*

$$\overline{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)} = \overline{\chi(s_1)}\,\overline{\sigma(s_2)} - \overline{\chi(s_2)}\,\overline{\sigma(s_1)} \tag{5.5.1}$$

for all $s_1, s_2 \in \mathbb{R}$.

Then either ν is supported in V or the support of ν is a single point in $\mathbb{H}$.

To prove this, we will require the following auxiliary lemma, the proof of which is postponed until after the proof of the theorem.

Lemma 5.5.2. *(i) The function $s \mapsto \overline{\chi(s)}$ is a $C^\alpha(\mathbb{R})$ map for all $\alpha \in [0, \lambda]$.*
(ii) The set

$$\mathbb{S} := \{s \in \mathbb{R} : \overline{\chi(s)} > 0\} \tag{5.5.2}$$

is open and may be equivalently expressed as a union of intervals,

$$\mathbb{S} = \bigcup_{(\rho,u)\in supp\,\nu} (u - k(\rho), u + k(\rho)). \tag{5.5.3}$$

We recall from §2.3 that $w(\rho, u) = u + k(\rho)$, $z(\rho, u) = u - k(\rho)$ are Riemann invariants.

Remark 5.5.3. In the case that the Riemann invariants are convex, that is, that $k''(\rho) \leq 0$, equivalently $2p'(\rho) - \rho p''(\rho) > 0$, we may deduce the stronger claim that the set $\mathbb{S}$ is open and connected. Hence, in this case, we may write

$$\mathbb{S} = (s_-, s_+) \tag{5.5.4}$$

for some numbers $s_- \leq s_+$ in the extended real line $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ (the case $s_- = s_+$ corresponds to the case that $\mathbb{S} = \emptyset$).

Proof of Theorem 5.5.1. We begin by taking $s_1, s_2, s_3 \in \mathbb{R}$. Multiplying the commutation relation for s_1, s_2 by $\overline{\chi(s_3)}$, we obtain

$$\overline{\chi(s_3)}\,\overline{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)} = \overline{\chi(s_3)}\,\overline{\chi(s_1)}\,\overline{\sigma(s_2)} - \overline{\chi(s_3)}\,\overline{\chi(s_2)}\,\overline{\sigma(s_1)}.$$

Cyclically permuting the s_j, and summing the obtained relations, we observe that the right hand sides cancel, leaving us with

$$\begin{aligned}\overline{\chi(s_1)}\,\overline{\chi(s_2)\sigma(s_3) - \chi(s_3)\sigma(s_2)} = &\overline{\chi(s_3)}\,\overline{\chi(s_2)\sigma(s_1) - \chi(s_1)\sigma(s_2)} \\ &- \overline{\chi(s_2)}\,\overline{\chi(s_3)\sigma(s_1) - \chi(s_1)\sigma(s_3)}.\end{aligned}$$

Before mollifying this expression, we first apply the fractional derivative operators $P_2 = \partial_{s_2}^{\lambda+1}$ and $P_3 = \partial_{s_3}^{\lambda+1}$ in the sense of distributions to obtain

$$\begin{aligned}\overline{\chi(s_1)}\,\overline{P_2\chi(s_2)P_3\sigma(s_3) - P_3\chi(s_3)P_2\sigma(s_2)} = &\overline{P_3\chi(s_3)}\,\overline{P_2\chi(s_2)\sigma(s_1) - \chi(s_1)P_2\sigma(s_2)} \\ &- \overline{P_2\chi(s_2)}\,\overline{P_3\chi(s_3)\sigma(s_1) - \chi(s_1)P_3\sigma(s_3)},\end{aligned} \tag{5.5.5}$$

where, for example, the distribution $\overline{P_2\chi(s_2)}$ is defined as acting on test functions $\psi \in \mathcal{D}(\mathbb{R})$ by

$$\langle \overline{P_2\chi(s_2)}, \psi\rangle = -\int_{\mathbb{R}} \overline{\partial_{s_2}^{\lambda}\chi(s_2)}\psi'(s_2)\,ds_2.$$

We take two standard (but different) mollifying kernels, $\phi_2, \phi_3 \in C_c^\infty(-1,1)$ such that $\int_{\mathbb{R}} \phi_j(s_j)\,ds_j = 1$ and $\phi_j \geq 0$ for $j = 2, 3$. For $\delta > 0$, we define $\phi_j^\delta(s_j) = \delta^{-1}\phi_j(s_j/\delta)$ in the usual way. We choose ϕ_2, ϕ_3 such that the quantity

$$Y(\phi_2, \phi_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{s_2} \phi_2(s_2)\phi_3(s_3) - \phi_2(s_3)\phi_3(s_2)\,ds_3\,ds_2 > 0.$$

We now mollify and take $s_2, s_3 \to s_1$, i.e. we integrate (5.5.5) against the function $\phi_2^\delta(s_1 - s_2)\phi_3^\delta(s_1 - s_3)$ with respect to s_2 and s_3. This yields

$$\begin{aligned}\overline{\chi(s_1)}\,\overline{P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta} = &\overline{P_3\chi_3^\delta}\,\overline{P_2\chi_2^\delta\sigma_1 - \chi_1 P_2\sigma_2^\delta} \\ &- \overline{P_2\chi_2^\delta}\,\overline{P_3\chi_3^\delta\sigma_1 - \chi_1 P_3\sigma_3^\delta}\end{aligned} \tag{5.5.6}$$

with the obvious notation, where, for example,

$$\overline{P_2\chi_2^\delta} = \overline{P_2\chi_2} * \phi_2^\delta(s_1) = \int \overline{\partial_{s_2}^{\lambda}\chi(s_2)}\delta^{-2}\phi_2'(\frac{s_1 - s_2}{\delta})\,ds_2.$$

We now claim the following two lemmas, to be proved later.

Lemma 5.5.4. *For any test function $\psi \in \mathcal{D}(\mathbb{R})$,*

$$\begin{aligned}\lim_{\delta\to 0}\int_{\mathbb{R}} &\overline{\chi(s_1)}\,\overline{P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta}(s_1)\psi(s_1)\,ds_1 \\ &= Y(\phi_2, \phi_3)\int_{\mathcal{H}} Z(\rho)\sum_{\pm}(K^\pm)^2\overline{\chi(u \pm k(\rho))}\psi(u \pm k(\rho))\,d\nu(\rho, u),\end{aligned}$$

where $Z(\rho) = (\lambda+1)M_\lambda^{-2}k(\rho)^{2\lambda}D(\rho) > 0$ for $\rho > 0$, and $D(\rho)$ is the coefficient defined in Proposition 5.2.1.

Lemma 5.5.5. *For any test function $\psi \in \mathcal{D}(\mathbb{R})$,*

$$\lim_{\delta\to 0}\int_{\mathbb{R}} \overline{P_3\chi_3^\delta}\,\overline{P_2\chi_2^\delta\sigma_1 - \chi_1 P_2\sigma_2^\delta}\psi(s_1)\,ds_1 = \lim_{\delta\to 0}\int_{\mathbb{R}} \overline{P_2\chi_2^\delta}\,\overline{P_3\chi_3^\delta\sigma_1 - \chi_1 P_3\sigma_3^\delta}\psi(s_1)\,ds_1.$$

With these lemmas, we multiply (5.5.6) by $\psi(s_1) \in \mathcal{D}(\mathbb{R})$, integrate in s_1 and pass to the limit to obtain

$$Y(\phi_2,\phi_3)\int_{\mathcal{H}} Z(\rho)\sum_{\pm}(K^{\pm})^2\overline{\chi(u\pm k(\rho))}\psi(u\pm k(\rho))\,d\nu(\rho,u) = 0. \tag{5.5.7}$$

As the coefficient $Z(\rho) > 0$ for $\rho > 0$, $Y(\phi_2,\phi_3) \neq 0$ by construction, $\overline{\chi(s)} \geq 0$ for all $s \in \mathbb{R}$ and $\psi(s)$ is an arbitrary test function, this implies that

$$\int_{\mathcal{H}} Z(\rho)\overline{\chi(u+k(\rho))}\,d\nu(\rho,u) = 0, \tag{5.5.8}$$

$$\int_{\mathcal{H}} Z(\rho)\overline{\chi(u-k(\rho))}\,d\nu(\rho,u) = 0. \tag{5.5.9}$$

Recalling now the set $\mathbb{S}$ from Lemma 5.5.2, we note that in the case that $\mathbb{S} = \emptyset$, we immediately have that $\overline{\chi(s)} = 0$ for all $s \in \mathbb{R}$, and hence $\operatorname{supp}\nu \subset V$.

On the other hand, if $\mathbb{S} \neq \emptyset$, then, as it is a union of intervals, we may write

$$\mathbb{S} = \bigcup_k (z_k, w_k),$$

for (possibly countably many) numbers z_k, w_k in the extended real line $\mathbb{R}\cup\{-\infty\}\cup\{+\infty\}$ such that $z_k < w_k < z_{k+1}$ for all k. Thus we obtain that

$$\operatorname{supp}\nu \subset \bigcup_k \{(\rho,u)\in\mathbb{H} \,:\, z_k \leq z(\rho,u) < w(\rho,u) \leq w_k\} \cup V.$$

Observe that if z_k and w_k are both finite, then as $k(\rho)$ is a strictly monotone increasing and unbounded function of ρ, the set $\{(\rho,u) \,:\, z_k \leq z(\rho,u) < w(\rho,u) \leq w_k\}$ is bounded.

Now we deduce from (5.5.8) that

$$\operatorname{supp}\nu \cap \{(\rho,u)\in\mathbb{H} : w(\rho,u) \in (z_k,w_k)\} = \emptyset \text{ for all } k.$$

Likewise, from (5.5.9),

$$\operatorname{supp}\nu \cap \{(\rho,u)\in\mathbb{H} : z(\rho,u) \in (z_k,w_k)\} = \emptyset \text{ for all } k.$$

Thus the support of the measure ν must be contained in the vacuum V and an at most countable union of points $P_k = (\rho(w_k,z_k), u(w_k,z_k))$:

$$\operatorname{supp}\nu \subset V \cup \bigcup_{\substack{k:\\ z_k,w_k\in\mathbb{R}}} (\rho(w_k,z_k), u(w_k,z_k)).$$

Observe that, by construction, the points P_k are such that if $P_k \in \operatorname{supp}\chi(s)$, then $P_{k'} \notin \operatorname{supp}\chi(s)$ for all $k' \neq k$. Indeed, recall $\operatorname{supp}\chi(s) = \{w(\rho,u) \geq s,\, z(\rho,u) \leq s\}$, so if $P_k \in \operatorname{supp}\chi(s)$, necessarily $z_k \leq s \leq w_k$. Hence if $P_{k'} \in \operatorname{supp}\chi(s)$ also, we must have $z_{k'} \leq s \leq w_{k'}$. But as the points P_k are ordered such that $w_{k-1} < z_k \leq w_k < z_{k+1}$, this is impossible for any $P_{k'}$ with $k' \neq k$.

Thus we may analyse the measure ν at each point P_k individually via the commutation relation (5.5.1) as follows. We write

$$\nu = \nu_V + \sum_k \alpha_k \delta_{P_k},$$

where all $\alpha_k \in [0,1]$ and the measure ν_V is supported in the vacuum V.

Take $s_1, s_2 \in \mathbb{R}$ such that $P_k \in \operatorname{supp}\chi(s_1)\chi(s_2)$. Then, from the commutation relation (5.5.1), we obtain

$$(\alpha_k - \alpha_k^2)\big(\chi(P_k,s_1)\sigma(P_k,s_2) - \chi(P_k,s_2)\sigma(P_k,s_1)\big) = 0.$$

Taking s_1 and s_2 such that the second factor in this expression is non-zero, we deduce that $\alpha_k \in \{0,1\}$ for all k, and hence conclude the proof of the theorem. □

Proof of Lemma 5.5.2. We first demonstrate the Hölder continuity of the function $\overline{\chi(s)}$. Indeed, from Remark 5.2.8, we may control the Hölder semi-norm by

$$\sup_{s_1 \neq s_2} \frac{|\chi(s_1) - \chi(s_2)|}{|s_1 - s_2|^\alpha} \leq C \max\{\rho^{(2\lambda-\alpha)\theta}, \frac{\rho}{\sqrt{\log\rho}}\}.$$

We may therefore bound

$$\begin{aligned}
\sup_{s_1 \neq s_2} \frac{|\overline{\chi(s_1)} - \overline{\chi(s_2)}|}{|s_1 - s_2|^\alpha} &\leq \sup_{s_1 \neq s_2} \int_{\mathcal{H}} \frac{|\chi(s_1) - \chi(s_2)|}{|s_1 - s_2|^\alpha}\, d\nu \\
&\leq C \int_{\mathcal{H}} \max\{\rho^{(2\lambda-\alpha)\theta}, \frac{\rho}{\sqrt{\log\rho}}\}\, d\nu \\
&\leq C \int_{\mathcal{H}} 1 + \rho p(\rho)\, d\nu \\
&\leq C.
\end{aligned}$$

For the second part of the lemma, we suppose $(\rho_*, u_*) \in \operatorname{supp}\nu$ and $\rho_* > 0$ (if $\rho_* = 0$, observe that the interval $(u_* - k(\rho_*), u_* + k(\rho_*))$ is empty). For convenience, we work in (w,z) coordinates, considering ν to be a measure on the phase space $\{w \geq z\}$ in the obvious way. We write $w_* = w(\rho_*, u_*)$, $z_* = z(\rho_*, u_*)$. Then, for all radii $r > 0$ such that $B_r(w_*, z_*) \subset \mathbb{H}$,

$$\overline{\chi(s)} \geq \int_{B_r(w_*,z_*)} \chi(\rho,u,s)\, d\nu(\rho,u) > 0 \text{ for all } s \text{ such that } \chi(s) > 0 \text{ on } B_r(w_*,z_*).$$

As $\operatorname{supp}\chi(s) = \{z \le s \le w\}$, we have that $\chi(s) > 0$ on $B_r(w_*, z_*)$ for $s \in (z_*+r, w_*-r)$. Thus $(z_* + r, w_* - r) \subset \mathbb{S}$. As $r > 0$ was arbitrary, we deduce

$$(u_* - k(\rho_*), u_* + k(\rho_*)) \subset \mathbb{S} \text{ for all } (\rho_*, u_*) \in \operatorname{supp}\nu.$$

On the other hand, if $\overline{\chi(s)} > 0$, we must have

$$\int_{\mathbb{H}} \chi(s)\, d\nu > 0,$$

as $\chi(s) = 0$ on V. Thus $\nu(\{(\rho, u) \in \mathbb{H} : u - k(\rho) < s < u + k(\rho)\}) > 0$, so there exists $(\rho, u) \in \operatorname{supp}\nu$ such that $s \in (u - k(\rho), u + k(\rho))$. This concludes the proof of the lemma. $\square$

In order to conclude this section, we give the proofs of the two main technical lemmas, Lemma 5.5.4 and Lemma 5.5.5. These lemmas exploit the properties of *cancellation of singularities* shown in [8] and the fact that the limit of a regularised product of a measure and a BV function depends on the choice of regularisation, *cf.* [19]. However, as emphasised above, the arguments of [8] do not carry over to our setting as we do not have uniform bounds on the density and velocity. We therefore exploit the representation formulae for the entropy and entropy flux kernels obtained above in §5.2 for the high density region to gain uniform control on the products in order to pass to the limit. We first recall the following estimates from [48].

Lemma 5.5.6 ([48, Lemmas 3.8-3.9])**.** *Let $R \in C^\alpha(\mathbb{R})$ be a bounded, Hölder continuous function, $g \in C^\alpha_c(\mathbb{R})$ be such that $\operatorname{supp} g \subset B_{L-2}(0)$ for some $L > 2$.*
(i) Consider any pair of distributions $T_2, T_3 \in \mathcal{D}'(\mathbb{R})$ from the following table:

$$(T_2, T_3) = (\delta, Q_3), \quad (T_2, T_3) = (\mathrm{PV}, Q_3), \quad (T_2, T_3) = (Q_2, Q_3),$$

where $Q_2, Q_3 \in \{H, \mathrm{Ci}, R\}$. Then there exists a constant $C > 0$ such that

$$\sup_{\delta\in(0,1)} \Big| \int_{-\infty}^{\infty} g(s_1)\big(T_2(s_2 - u \pm k(\rho))T_3(s_3 - u \pm k(\rho))\big) * \phi_2^\delta * \phi_3^\delta(s_1)\, ds_1 \Big| \\ \le C\|g\|_{C^\alpha(\mathbb{R})}\big(1 + \|R\|_{C^\alpha(B_L(0))}\big)^2.$$

(ii) Consider now any pair of distributions from

$$(T_2, T_3) = \{\delta, \delta\}, \qquad (T_2, T_3) = \{\mathrm{PV}, \mathrm{PV}\}, \qquad (T_2, T_3) = \{Q_2, Q_3\}$$
$$(T_2, T_3) = \{\delta, \mathrm{PV}\}, \qquad (T_2, T_3) = \{\mathrm{PV}, Q_3\}, \qquad (T_2, T_3) = \{\delta, Q_3\},$$

where $Q_2, Q_3 \in \{H, \mathrm{Ci}, R\}$. Then there exists $C > 0$ such that

$$\sup_{\delta\in(0,1)} \Big| \int_{-\infty}^{\infty} g(s_1)\big((s_2 - s_3)T_2(s_2 - u \pm k(\rho))T_3(s_3 - u \pm k(\rho))\big) * \phi_2^\delta * \phi_3^\delta(s_1)\, ds_1 \Big| \\ \le C\|g\|_{C^\alpha(\mathbb{R})}\big(1 + \|R\|_{C^\alpha(B_L(0))}\big)^2.$$

Proof of Lemma 5.5.4. We begin by recalling from [8, Lemma 4.3] that the quantity

$$P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta \rightharpoonup Y(\phi_2,\phi_3)Z(\rho)\sum_\pm (K^\pm)^2\delta_{s_1=u\pm k(\rho)}$$

weakly-star in measures in s_1 and locally uniformly in (ρ,u) (actually uniformly on sets on which ρ is bounded), where

$$Y(\phi_2,\phi_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{s_2} \phi_2(s_2)\phi_3(s_3) - \phi_2(s_3)\phi_3(s_2)\,ds_3\,ds_2$$

and

$$Z(\rho) = (\lambda+1)M_\lambda^{-2}k(\rho)^{2\lambda}D(\rho).$$

In particular, we have the following uniform convergence on sets on which ρ is bounded:

$$\begin{aligned}\lim_{\delta\to 0}\int_{-\infty}^{\infty}&\overline{\chi(s_1)}(P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta)\psi(s_1)\,ds_1\\ &= Y(\phi_2,\phi_3)Z(\rho)\sum_\pm (K^\pm)^2\overline{\chi(u\pm k(\rho))}\psi(u\pm k(\rho)).\end{aligned}$$

We therefore find that, for any $\rho_*>0$,

$$\begin{aligned}\lim_{\delta\to 0}\int_{-\infty}^{\infty}&\overline{\chi(s_1)}\langle\nu,(P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta)\mathbb{1}_{\rho\le\rho_*}\rangle\psi(s_1)\,ds_1\\ &= \lim_{\delta\to 0}\langle\nu,\int_{-\infty}^{\infty}\overline{\chi(s_1)}(P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta)\psi(s_1)\,ds_1\mathbb{1}_{\rho\le\rho_*}\rangle\\ &= \langle\nu, Y(\phi_2,\phi_3)Z(\rho)\sum_\pm (K^\pm)^2\overline{\chi(u\pm k(\rho))}\psi(u\pm k(\rho))\mathbb{1}_{\rho\le\rho_*}\rangle\\ &= Y(\phi_2,\phi_3)\sum_\pm (K^\pm)^2\langle\nu, Z(\rho)\overline{\chi(u\pm k(\rho))}\psi(u\pm k(\rho))\mathbb{1}_{\rho\le\rho_*}\rangle.\end{aligned}$$

Considering now the complement region $\{\rho>\rho_*\}$ and supposing that $\rho_*>1$, we observe still that the pointwise limit of the regularised product is

$$\begin{aligned}\lim_{\delta\to 0}\int_{-\infty}^{\infty}&\overline{\chi(s_1)}(P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta)\psi(s_1)\,ds_1\\ &= Y(\phi_2,\phi_3)Z(\rho)\sum_\pm (K^\pm)^2\overline{\chi(u\pm k(\rho))}\psi(u\pm k(\rho))\end{aligned}$$

pointwise in (ρ,u) by the local uniform convergence just noted. It therefore suffices to show that

$$\Big|\int_{-\infty}^{\infty}\overline{\chi(s_1)}(P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta)\psi(s_1)\,ds_1\mathbb{1}_{\rho>\rho_*}\Big| \le C\rho^2 \tag{5.5.10}$$

for some $C>0$ independent of ρ, u and δ. Then if this is the case, we may apply the dominated convergence theorem to pass to the pointwise limit in the Young measure (as $\rho^2\in L^1(\mathcal{H},\nu)$).

We first observe that we may expand

$$P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta = P_2\chi_2^\delta P_3(\sigma_3^\delta - u\chi_3^\delta) - P_3\chi_3^\delta P_2(\sigma_2^\delta - u\chi_2^\delta).$$

Applying now Lemma 5.2.7, we see that this product consists of a sum of terms

$$A_{j,\pm}(\rho)B_{j,\pm}(\rho)(s_2 - s_3)T_2(s_2 - u \pm k(\rho))T_3(s_3 - u \pm k(\rho)),$$

where $T_2, T_3 \in \{\delta, \mathrm{PV}, H, \mathrm{Ci}\}$, and terms of the form

$$A_{j,\pm}(\rho)B_{j,\pm}(\rho)T_2(s_2 - u \pm k(\rho))T_3(s_3 - u \pm k(\rho)),$$

where $T_2 \in \{\delta, H, \mathrm{PV}, \mathrm{Ci}, r_\chi\}$ and $T_3 \in \{H, \mathrm{Ci}, r_\sigma\}$ and likewise with s_2 and s_3 reversed.

Applying the second bound of Lemma 5.5.6 yields

$$\begin{aligned}\Big|\int_{-\infty}^{\infty}\overline{\chi(s_1)}\psi(s_1)\big((s_2 - s_3)T_2(s_2 - u \pm k(\rho))T_3(s_3 - u \pm k(\rho))\big) * \phi_2^\delta * \phi_3^\delta(s_1)\,ds_1\Big| \\ \le C\|\overline{\chi}\psi\|_{C^\alpha(\mathbb{R})}\end{aligned} \tag{5.5.11}$$

for any pair $T_2, T_3 \in \{\delta, \mathrm{PV}, H, \mathrm{Ci}\}$. Likewise, the first bound of Lemma 5.5.6 gives

$$\begin{aligned}\Big|\int_{-\infty}^{\infty}\overline{\chi(s_1)}\psi(s_1)\big(T_2(s_2 - u \pm k(\rho))T_3(s_3 - u \pm k(\rho))\big) * \phi_2^\delta * \phi_3^\delta(s_1)\,ds_1\Big| \\ \le C\|\overline{\chi}\psi\|_{C^\alpha(\mathbb{R})}\big(1 + \|r_\chi\|_{C^\alpha_{s_1}(B_R)} + \|r_\sigma\|_{C^\alpha_{s_1}(B_R)}\big),\end{aligned} \tag{5.5.12}$$

for $T_2 \in \{\delta, H, \mathrm{PV}, \mathrm{Ci}, r_\chi\}$ and $T_3 \in \{H, \mathrm{Ci}, r_\sigma\}$. Here $R > 0$ is such that $\operatorname{supp}\psi(s) \subset B_{R-2}(0)$ and the terms involving r_χ and r_σ occur only if one of $T_2, T_3 \in \{r_\chi, r_\sigma\}$.

Applying (5.5.11)-(5.5.12), we therefore find

$$\begin{aligned}&\Big|\int_{-\infty}^{\infty}\overline{\chi(s_1)}(P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta)\psi(s_1)\,ds_1\Big| \\ &\le C\max_{j,\pm}\{|A_{j,\pm}B_{j,\pm}|, |A_{j,\pm}\tilde{B}_{j,\pm}|, |A_{j,\pm}|\|r_\chi\|_{C^\alpha_{s_1}(B_R)}, \big(|B_{j,\pm}| + |\tilde{B}_{j,\pm}|\big)\|r_\sigma\|_{C^\alpha_{s_1}(B_R)}\} \\ &\le C\rho^2\end{aligned}$$

by Lemma 5.2.7, proving the necessary claim, (5.5.10). □

Proof of Lemma 5.5.5. We begin by recalling from [8, Lemma 4.2] that there exists a Hölder continuous function $X(\rho, u, s_1)$ such that

$$\chi_1 P_j\sigma_j^\delta - P_j\chi_j^\delta\sigma_1 \to X(\rho, u, s_1) \text{ for } j = 2, 3,$$

uniformly on sets of the form $\{\rho \le \rho_*\}$ for fixed $\rho_* > 0$. In particular, the convergence may be taken to be uniform in s_1 if ρ, u are fixed.

We therefore observe that

$$
\begin{aligned}
&\int_{\mathbb{R}} \overline{P_2\chi_2^\delta}(\chi_1 P_3\sigma_3^\delta - P_3\chi_3^\delta\sigma_1)\psi(s_1)\,ds_1 \\
&\quad = \int_{\mathcal{H}}\int_{\mathbb{R}} P_2\chi_2^\delta(\tilde{\rho},\tilde{u},s_1)(\chi_1 P_3\sigma_3^\delta - P_3\chi_3^\delta\sigma_1)(\rho,u,s_1)\psi(s_1)\,ds_1\,d\nu(\tilde{\rho},\tilde{u}) \\
&\quad \to \int_{\mathcal{H}} \langle P_1\chi_1(\tilde{\rho},\tilde{u},\cdot), X(\rho,u,\cdot)\psi(\cdot)\rangle\,d\nu(\tilde{\rho},\tilde{u})
\end{aligned}
$$

as $\delta \to 0$, where the inner product $\langle\cdot,\cdot\rangle$ is, in a slight abuse of notation, the duality pairing of measures and continuous functions (we recall that the principal value distribution acts properly on Hölder continuous functions as a measure). To pass to the limit, we have used that $P_j\chi_j^\delta$, $j = 2, 3$, are measures in s_1 such that $\|P_j\chi_j^\delta(\rho,u,\cdot)\|_{\mathcal{M}} \leq C\rho$ in order to pass the limit inside the Young measure.

Claim. We may make the following bound, independent of δ:

$$
\Big| \int_{\mathbb{R}} \overline{P_2\chi_2^\delta}(\chi_1 P_3\sigma_3^\delta - P_3\chi_3^\delta\sigma_1)\psi(s_1)\,ds_1 \Big| \leq C(\rho^2+1). \tag{5.5.13}
$$

Assuming the bound of the claim, we may apply the dominated convergence theorem again to pass to the limit inside the Young measure with respect to (ρ, u), i.e. we obtain

$$
\begin{aligned}
&\lim_{\delta\to 0}\int_{\mathbb{R}} \overline{P_2\chi_2^\delta(s_1)}\,\overline{(\chi_1 P_3\sigma_3^\delta - P_3\chi_3^\delta\sigma_1)(s_1)}\psi(s_1)\,ds_1 \\
&\quad = \lim_{\delta\to 0}\int_{\mathcal{H}}\int_{\mathbb{R}} \overline{P_2\chi_2^\delta(s_1)}(\chi_1 P_3\sigma_3^\delta - P_3\chi_3^\delta\sigma_1)(\rho,u,s_1)\psi(s_1)\,ds_1\,d\nu(\rho,u) \\
&\quad = \int_{\mathcal{H}}\int_{\mathcal{H}} \langle P_1\chi_1(\tilde{\rho},\tilde{u},\cdot), X(\rho,u,\cdot)\psi(\cdot)\rangle\,d\nu(\tilde{\rho},\tilde{u})\,d\nu(\rho,u).
\end{aligned} \tag{5.5.14}
$$

As the limit is independent of the choice of mollifying functions ϕ_2 and ϕ_3, we may interchange the roles of s_2 and s_3 and so conclude the proof of the lemma.

It remains only to prove the bound (5.5.13) of the Claim.

Proof of claim. We begin by observing that for $j = 2, 3$, $\overline{P_j\chi_j^\delta(s_1)}$ is independent of ρ and u, as is $\psi(s_1)$. We therefore examine the function

$$
\chi_1 P_j\sigma_j^\delta - P_j\chi_j^\delta\sigma_1 = \chi_1 P_j(\sigma_j^\delta - u\chi_j^\delta) - (\sigma_1 - u\chi_1)P_j\chi_j^\delta.
$$

We recall from [8, Proof of Lemma 4.2] that this expression may be decomposed as a sum $E^{I,\delta} + E^{II,\delta} + E^{III,\delta}$, where

$$\begin{aligned} E^{I,\delta} =& \sum_{\pm} A^{I}_{\delta,\pm}(\rho) e^{I}_{\delta}(\rho, u - s_1)\big((s_1 - s_j)\delta(s_j - u \pm k(\rho))\big) * \phi^{\delta}_j(s_1) \\ &+ \sum_{\pm} A^{II}_{\mathrm{PV},\pm}(\rho) e^{I}_{\mathrm{PV}}(\rho, u - s_1)\big((s_1 - s_j)\mathrm{PV}(s_j - u \pm k(\rho))\big) * \phi^{\delta}_j(s_1), \\ E^{II,\delta} =& \sum_{\pm} A^{II}_{\delta,\pm}(\rho) e^{II}_{\delta}(\rho, u - s_1)\big((s_j - u)\delta(s_j - u \pm k(\rho))\big) * \phi^{\delta}_j(s_1) \\ &+ \sum_{\pm} A^{II}_{\mathrm{PV},\pm}(\rho) e^{II}_{\mathrm{PV}}(\rho, u - s_1)\big((s_j - u)\mathrm{PV}(s_j - u \pm k(\rho))\big) * \phi^{\delta}_j(s_1), \end{aligned} \tag{5.5.15}$$

and where $E^{III,\delta}$ is obtained by the mollification of Hölder continuous functions.

Considering now the expansions of Theorem 5.2.6 and Lemma 5.2.7, we find that the coefficients $A^{I,II}_{\delta,\mathrm{PV},\pm}$ may be bounded by

$$\sum_{\pm} |A^{I,II}_{\delta,\pm}(\rho)| + |A^{I,II}_{\mathrm{PV},\pm}(\rho)| \leq C\sqrt{\rho}\log\rho, \tag{5.5.16}$$

and the functions $e^{I}_{\delta,\mathrm{PV}}(\rho, u - s_1)$ and $e^{II}_{\delta,\mathrm{PV}}(\rho, u - s_1)$, which are respectively leading order and higher order terms in the expansions for χ and σ, satisfy

$$|e^{I}_{\delta,\mathrm{PV}}(\rho, u - s_1)| \leq C\rho, \tag{5.5.17}$$

$$|e^{II}_{\delta,\mathrm{PV}}(\rho, u - s_1)| + |\partial_{s_1} e^{II}_{\delta,\mathrm{PV}}(\rho, u - s_1)| \leq C\rho. \tag{5.5.18}$$

The second of these bounds is seen by the following consideration. The expansions (5.2.1) and (5.2.2) allow us to deduce, as in [8, Proof of Lemma 4.2], that the functions $e^{II}_{\delta,\mathrm{PV}}$ are at least Lipschitz continuous (in fact they have Hölder continuous derivatives). The convolution representation formula of (5.2.8) allows us now to deduce the desired bounds easily.

Moreover, an analogous argument gives that the Hölder continuous remainder term, $E^{III,\delta}$, is uniformly bounded, for $\rho \geq \rho_*$, by $C\rho^2$.

We first demonstrate the uniform bound on the function $E^{I,\delta}(\rho, u - s_1)$. This contains products of Dirac masses with Hölder continuous functions and products of principal value distributions with Hölder continuous functions. Considering a typical term of the first type, we use (5.5.16)-(5.5.17) to bound

$$\begin{aligned} &|A^{I}_{\delta,+}(\rho) e^{I}_{\delta}(\rho, u - s_1)(s_1 - (u + k))\phi^{\delta}_j(s_1 - (u + k))| \\ &\leq C\rho^{3/2}\log\rho |s_1 - (u + k)|\delta^{-1}\phi_j\big(\frac{s_1 - (u + k)}{\delta}\big) \leq C\rho^2, \end{aligned}$$

where we have used that $\operatorname{supp}\phi_j \subset (-1, 1)$.

A typical principal value term is bounded, using (5.5.16)-(5.5.17), by

$$
\begin{aligned}
&|A^I_{\mathrm{PV},+}(\rho)e^I_\delta(\rho,u-s_1)(s_j-s_1)\mathrm{PV}(s_j-(u+k))*\phi^\delta_j(s_1)| \\
&\le C\rho^{3/2}\log\rho\big(|(s_1-(u+k))\mathrm{PV}(s_j-(u+k))*\phi^\delta_j(s_1)| \\
&\qquad\qquad +|(s_j-(u+k))\mathrm{PV}(s_j-(u+k))*\phi^\delta_j(s_1)|\big).
\end{aligned}
$$

Observe that the second term is the mollified distributional constant function 1, and hence is bounded independent of $\delta>0$. For the first term, we note that, by definition,

$$
\begin{aligned}
|\mathrm{PV}*\phi^\delta_j(x)| &= \Big|\int_0^\infty \frac{\phi^\delta_j(x-y)-\phi^\delta_j(y-x)}{y}\,dy\Big| \\
&=\delta^{-1}\Big|\int_0^\infty y^{-1}\big(\phi_j\big(\frac{x-y}{\delta}\big)-\phi_j\big(\frac{y-x}{\delta}\big)\big)\,dy\Big|.
\end{aligned}
$$

Now if $|x|\le 2\delta$, this is controlled by

$$
|\mathrm{PV}*\phi^\delta_j(x)|\le\delta^{-1}\int_0^{4\delta}|y|^{-1}\big|\phi_j\big(\frac{x-y}{\delta}\big)-\phi_j\big(\frac{y-x}{\delta}\big)\big|\,dy\le C\delta\|\phi'\|_{L^\infty}\delta^{-2}=C\frac{1}{\delta}.
$$

On the other hand, if $|x|\ge 2\delta$, we obtain a bound of

$$
|\mathrm{PV}*\phi^\delta_j(x)|\le\delta^{-1}\int_{|x|-\delta}^{|x|+\delta}\frac{2\|\phi\|_{L^\infty}}{|x|-\delta}\,dy\le C\frac{1}{|x|},
$$

independent of $\delta>0$. Hence we obtain a bound of

$$
|(s_1-(u+k))\mathrm{PV}(s_j-(u+k))*\phi^\delta_j(s_1)|\le C\Big(\frac{1}{\delta}\cdot 2\delta+\frac{|s_1-(u+k)|}{|s_1-(u+k)|}\Big)\le C,
$$

independent of $\delta>0$, giving

$$
\begin{aligned}
&|A^I_{\mathrm{PV},+}(\rho)e^I_\delta(\rho,u-s_1)(s_j-s_1)\mathrm{PV}(s_j-(u+k))*\phi^\delta_j(s_1)| \\
&\le C\rho^{3/2}\log\rho\le C\rho^2.
\end{aligned}
$$

To bound $E^{II,\delta}$, we see that evaluating the terms produced by Dirac masses gives terms of the form

$$
|A^{II}_{\delta,+}(\rho)||e^{II}_{\delta,+}(\rho,u-s_1)|k(\rho)\delta^{-1}\phi_j\big(\frac{s_1-(u+k)}{\delta}\big)\le C\rho^{3/2}(\log\rho)^2\|\phi\|_{L^\infty}
$$

as $\operatorname{supp}\phi\subset(-1,1)$ and $e^{II}_{\delta,+}$ satisfies (5.5.18) and $\operatorname{supp}e^{II}_{\delta,+}\subset\{|u-s_1|\le k(\rho)\}$.

Moreover, for the final terms, we add and subtract $k(\rho)$ to the factor s_j-u to see

$$
\begin{aligned}
&|A^{II}_{\mathrm{PV},+}(\rho)e^{II}_{\mathrm{PV},+}(\rho,u-s_1)(s_j-u-k(\rho)+k(\rho))\mathrm{PV}(s_j-(u+k))*\phi^\delta_j(s_1)| \\
&\le C\rho^{3/2}\log\rho+C\rho^{1/2}k(\rho)\log\rho|e^{II}_{\mathrm{PV},+}(\rho,u-s_1)||\mathrm{PV}(s_j-(u+k))*\phi^\delta_j(s_1)| \\
&\le C\rho^{1/2}\log\rho\big(\rho+C\log\rho\cdot\rho\big) \\
&\le C\rho^2,
\end{aligned}
$$

where we have bounded the principal value as before and used the Lipschitz bound (5.5.18), concluding the proof of the claim.

This concludes the proof of the lemma. $\square$

5.6 Proof of the main result

With this reduction framework in hand, we have everything we need to demonstrate the strong convergence of the solutions of the Navier-Stokes equations to an entropy solution of the Euler equations in the sense of Definition 5.1.1, and therefore to prove Theorem 5.1.2.

To prove the theorem, we begin by adjusting the initial datum ρ_0 if necessary to ensure that it is strictly positive, that is, we take the cut-off $\max\{\rho_0, \varepsilon^{\frac{1}{2}}\}$. Mollifying this function and the initial velocity at a suitable scale, we obtain approximate initial data satisfying the assumptions of Remark 5.3.7. By the main theorem of [38], as given again in Theorem 5.3.1, we obtain a sequence of smooth solutions to the Navier-Stokes equations (5.1.1) which satisfy the estimates of §5.3. In particular, we may apply the construction of §5.4 to deduce the existence of a Young measure solution $\nu_{t,x}$ to the Euler equations, constrained by the Tartar commutation relation, as in Proposition 5.4.3. Applying now the reduction of support theorem, Theorem 5.5.1, we deduce that for almost every $(t,x) \in \mathbb{R}^2_+$, the Young measure $\nu_{t,x}$ is either a point mass or is supported in the vacuum set V. Moving now to the (ρ, m) coordinates, such a measure is exactly a Dirac mass at a point $(\rho(t,x), m(t,x))$. Thus we conclude that the approximate solutions converge $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \to (\rho, m)$ for almost every $(t,x) \in \mathbb{R}^2_+$ and also in $L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+)$ for $p \in [1,2)$ and $q \in [1, \frac{3}{2})$.

This strong convergence is, in particular, enough to pass to the limit in the relative energy,

$$\overline{\eta^*}(\rho^\varepsilon, m^\varepsilon) \to \overline{\eta^*}(\rho, m) \text{ in } L^1_{loc}(\mathbb{R}^2_+).$$

Hence we deduce that, for any $t_1 < t_2$,

$$\int_{t_1}^{t_2} \int_{\mathbb{R}} \overline{\eta^*}(\rho, m)(t,x)\, dx\, dt \leq M(t_2 - t_1) \int_{\mathbb{R}} \overline{\eta^*}(\rho_0, m_0)(x)\, dx + M,$$

so that, for almost every $t \geq 0$,

$$\int_{\mathbb{R}} \overline{\eta^*}(\rho, m)(t,x)\, dx \leq M \int_{\mathbb{R}} \overline{\eta^*}(\rho_0, m_0)(x)\, dx + M.$$

To verify part (iii) of Definition 5.1.1, we note from (5.2.7) that there exists a function $\mathcal{X}$ such that $\mathcal{X}_s(\rho, u - s) = \big(\rho \chi_\rho - \chi\big)(\rho, u - s)$ and observe from (5.3.19) that for any $\psi \in C^2_c(\mathbb{R})$,

$$\begin{aligned} \int_{\mathbb{R}} \big(\chi(\rho^\varepsilon, u^\varepsilon - s)_t + \sigma(\rho^\varepsilon, u^\varepsilon, s)_x\big)\psi(s)\, ds &= \varepsilon(\eta^\psi_m(\rho^\varepsilon, m^\varepsilon)_x)_x \\ &\quad - \varepsilon \int_{\mathbb{R}} \frac{1}{\rho^\varepsilon}\chi(\rho^\varepsilon)\psi''(s)|u^\varepsilon_x|^2 + \frac{1}{(\rho^\varepsilon)^2}\mathcal{X}(\rho^\varepsilon, u^\varepsilon - s)\psi''(s)\rho^\varepsilon_x u^\varepsilon_x\, ds. \end{aligned} \tag{5.6.1}$$

Passing $\varepsilon \to 0$ by the uniform energy estimates Lemma 5.3.2 and Lemma 5.3.3, we obtain (5.1.9) after noting that $\int_{\mathbb{R}} \mathcal{X}(\rho, u-s)\, ds = 0$ as $\mathcal{X}$ is odd and compactly supported in s for any fixed (ρ, u).

The limit functions (ρ, m) are easily seen to satisfy (5.1.8) from the strong convergence of $(\rho^\varepsilon, m^\varepsilon)$, and hence (ρ, m) satisfies all of the conditions of Definition 5.1.1, proving Theorem 5.1.2. □

5.7 Recovering the physical entropy inequality

The class of entropies generated by test functions of sub-quadratic growth does not include the physical entropy pair (η^*, q^*). Indeed, this pair is generated exactly by the test function $\psi(s) = \frac{1}{2}s^2$. Although our main theorem, Theorem 5.1.2, does not include this function, in the case that the initial data satisfies higher integrability estimates than just finite energy, we may obtain the entropy inequality for the physical entropy also. This is the purpose of this section.

One may check that the function

$$\eta^\dagger(\rho, m) = \frac{1}{12}\frac{m^4}{\rho^3} + \frac{e(\rho)}{\rho}m^2 + f(\rho),$$

where $f(\rho)$ solves

$$f''(\rho) = \frac{2p'(\rho)e(\rho)}{\rho}, \quad f(0) = f'(0) = 0,$$

is an entropy, with an associated entropy flux $q^\dagger(\rho, m)$.

We define, as usual,

$$\overline{\eta^\dagger}(\rho, m) = \eta^\dagger(\rho, m) - \eta^\dagger(\bar\rho, \bar m) - \nabla\eta^\dagger(\bar\rho, \bar m)\cdot(\rho - \bar\rho, m - \bar m).$$

Proposition 5.7.1. *Let*

$$\mathcal{E}_0 := \sup_{\varepsilon} \int_a^b \overline{\eta^\dagger}(\rho_0^\varepsilon, m_0^\varepsilon)\, dx < \infty.$$

Then the solutions $(\rho^\varepsilon, m^\varepsilon)$ of (5.1.1) satisfy that, for any $T > 0$,

$$\sup_{[0,T]} \int_{\mathbb{R}} \overline{\eta^\dagger}(\rho^\varepsilon, m^\varepsilon)\, dx + \varepsilon \int_0^T \int_{\mathbb{R}} \left(|u^\varepsilon u_x^\varepsilon|^2 + e(\rho)|u_x^\varepsilon|^2\right) dx\, dt \le M\mathcal{E}_0 + M. \tag{5.7.1}$$

In particular, for any compact set $K \subset \mathbb{R}$, there exists $M = M(K) > 0$ such that

$$\sup_{t\in[0,T]} \int_K \rho^\varepsilon(t,x)|u^\varepsilon(t,x)|^4\, dx \le M. \tag{5.7.2}$$

Proof. We drop the explicit dependence of the functions on ε for convenience during the proof. We calculate directly that

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\int \eta^\dagger(\rho,m)\,dx - \frac{d}{dt}\int \eta^\dagger(\bar\rho,\bar m)\,dx - \int \nabla\eta^\dagger(\bar\rho,\bar m)\cdot(\rho_t,m_t)\,dx.$$

As the reference functions $\bar\rho$ and $\bar m$ are independent of t, it is clear that the second term on the right vanishes identically. On the other hand, multiplying the first equation in (5.1.1) by $\eta^\dagger_\rho(\rho,m)$ and the second equation by $\eta^\dagger_m(\rho,m)$, we obtain

$$\eta^\dagger(\rho,m)_t + q^\dagger(\rho,m)_x = \varepsilon\eta^\dagger_m(\rho,m)u_{xx},$$

and hence, as $\eta^\dagger_m(\rho,m) = \frac{1}{3}u^3 + 2e(\rho)u$,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}[\rho,u](t) =& \, q^\dagger(\rho_-,m_-) - q^\dagger(\rho_+,m_+) - \int \nabla\eta^\dagger(\bar\rho,\bar m)\cdot(\rho_t,m_t)\,dx\\ &+\varepsilon\int\big(\frac{1}{3}u^3u_{xx} + 2e(\rho)uu_{xx}\big)\,dx.\end{aligned}$$

From (5.1.1), we may re-write the penultimate term here and bound it as in the proof of Lemma 5.3.2 by

$$\Big|\int \nabla\eta^\dagger(\bar\rho,\bar m)\cdot(\rho_t,m_t)\,dx\Big| \leq \frac{\varepsilon}{2}\int |u_x|^2\,dx + ME[\rho,u](t),$$

where we have used that $\bar\rho$ and $\bar m$ are constant outside the interval $[-L_0,L_0]$ and are bounded, as well as the main energy estimate, Lemma 5.3.2.

We integrate by parts to see

$$\varepsilon\int\big(\frac{1}{3}u^3u_{xx} + 2e(\rho)uu_{xx}\big)\,dx = -\,\varepsilon\int\big(u^2u_x^2 + 2e(\rho)u_x^2 + 2e'(\rho)u\rho_xu_x\big)\,dx.$$

To control the final term here, we note that, by the assumptions made on the pressure $p(\rho)$, we may always bound

$$\frac{p(\rho)^2}{\rho^4} \leq M\frac{p'(\rho)}{\rho^2}.$$

Hence we may bound, using the Cauchy-Young inequality,

$$\begin{aligned}\varepsilon\int 2e'(\rho)u\rho_xu_x\,dx =& \,\varepsilon\int\frac{p(\rho)}{\rho^2}u\rho_xu_x\,dx\\ \leq& \,\frac{\varepsilon}{2}\int u^2u_x^2\,dx + 4\varepsilon\int\frac{p(\rho)^2}{\rho^4}|\rho_x|^2\,dx\\ \leq& \,\frac{\varepsilon}{2}\int u^2u_x^2\,dx + \varepsilon M\int\frac{p'(\rho)}{\rho^2}|\rho_x|^2\,dx.\end{aligned}$$

In particular, we obtain that

$$\frac{d}{dt}\mathcal{E}[\rho,u](t) + \frac{\varepsilon}{2}\int\big(u^2u_x^2 + 2e(\rho)u_x^2\big)\,dx \leq M + \frac{\varepsilon}{2}\int|u_x|^2\,dx + \varepsilon M\int\frac{p'(\rho)}{\rho^2}|\rho_x|^2\,dx.$$

Integrating this inequality in time also, and applying Lemma 5.3.3,

$$\begin{aligned}\mathcal{E}[\rho,u](T)+\frac{\varepsilon}{2}\int_0^T\int\big(u^2u_x^2+2e(\rho)u_x^2\big)\,dx &\leq \mathcal{E}[\rho_0,u_0]+M+\varepsilon M\int_0^T\int\frac{p'(\rho)}{\rho^2}|\rho_x|^2\,dx\\ &\leq M\big(\mathcal{E}[\rho_0,u_0]+1\big),\end{aligned}$$

as required. □

The local higher integrability of the quantity $\rho|u|^4$ allows us to argue as in Proposition 5.4.2 to extend the admissible range of test functions for the Young measure to include those of sub-cubic growth, in particular allowing us to test with the physical entropy pair. Thus, if the initial data satisfies

$$\sup_{\varepsilon}\mathcal{E}[\rho_0^\varepsilon,u_0^\varepsilon]\leq E_2<\infty$$

in addition to the other conditions of the main theorem, Theorem 5.1.2, we obtain that the vanishing viscosity limit is an entropy solution of the Euler equations satisfying the entropy inequality also for the physical entropy.

Chapter 6

Relativistic Euler equations

6.1 Introduction

In this chapter, we demonstrate existence of entropy solutions to the relativistic Euler equations, which we recall are given by

$$\begin{cases} \partial_t\big(\frac{n}{\sqrt{1-u^2/c^2}}\big) + \partial_x\big(\frac{nu}{\sqrt{1-u^2/c^2}}\big) = 0, \\ \partial_t\big(\frac{(\rho+p/c^2)u}{1-u^2/c^2}\big) + \partial_x\big(\frac{(\rho+p/c^2)u^2}{1-u^2/c^2} + p\big) = 0, \end{cases} \tag{6.1.1}$$

for $(t,x) \in \mathbb{R}^2_+$, where ρ, p and u represent the mass-energy density, pressure, and the particle speed respectively, n is the proper number density of baryons, and c is the speed of light. Henceforth, we will write $\varepsilon = \frac{1}{c^2}$ for notational convenience.

Throughout this chapter, we will use the notation defined in §2.6 for the conserved variables, that is,

$$\begin{aligned} U &= \left(\frac{n}{\sqrt{1-\varepsilon u^2}}, \frac{(\rho+\varepsilon p)u}{1-\varepsilon u^2}\right)^\top, \\ F(U) &= \left(\frac{nu}{\sqrt{1-\varepsilon u^2}}, \frac{\rho u^2 + p}{1-\varepsilon u^2}\right)^\top, \end{aligned} \tag{6.1.2}$$

so that system (6.1.1) takes the form

$$\partial_t U + \partial_x F(U) = 0.$$

We recall that this system is strictly hyperbolic and genuinely non-linear provided

$$p'(\rho) > 0 \quad \text{and} \quad \rho p''(\rho) + 2p'(\rho) + \varepsilon(p(\rho)p''(\rho) - 2p'(\rho)^2) > 0. \tag{6.1.3}$$

The main aim of our analysis of this system is to demonstrate the existence of entropy solutions to the equations. The procedure that we undertake to establish existence of solutions to the relativistic equations (6.1.1) is similar to that outlined above for the classical Euler equations in §2.3. In particular, we construct a sequence of approximate solutions and apply the methods of compensated compactness to pass to the limit in the approximation. To apply these techniques, we require a thorough understanding of the entropy pairs for the system (6.1.1). We therefore begin by following the approach of [8, 9] in establishing the existence of fundamental kernels generating the admissible entropy pairs. In order to do this, we make an *ansatz* for the entropy kernel close to the vacuum to leading order and take an asymptotic expansion around this leading order term. This leaves us with an equation for the remainder that we solve via a constructive fixed point argument. We establish estimates on both the leading terms and the remainder to demonstrate their respective regularity properties.

As our strategy is to construct a sequence of approximate solutions to the equations and pass to a limit, we first obtain a measure valued solution in the form of a Young measure generated by the sequence of approximate solutions. As we cannot pass the weak convergence of the approximate solutions through the non-linear term of the equations or in the equations for the entropies, we improve the weak convergence to strong convergence. As in previous chapters and [8, 24], this is done by means of a reduction argument, showing that for almost every $(t, x) \in \mathbb{R}^2_+$, the Young measure $\nu_{t,x}$ is concentrated at a single point. Constructing a framework in which to do this is one of the main results of this chapter.

Finally, with this framework in hand, we construct the approximate solutions by means of a vanishing viscosity scheme and show that these approximate solutions satisfy the assumptions of the framework. This then leads to the first main theorem of this chapter:

Theorem 6.1.1 (Existence of Entropy Solutions to Relativistic Euler Equations)**.** *Let (ρ_0, u_0) be measurable and bounded initial data satisfying*

$$0 \le \rho_0(x) \le M_0, \quad |u_0(x)| \le M_0 < \frac{1}{\sqrt{\varepsilon}} \text{ for almost every } x \in \mathbb{R},$$

for some $M_0 > 0$ and suppose that the pressure $p(\rho)$ satisfies (6.1.3) for $\rho > 0$ and (6.1.10). Then there exists $\varepsilon_0 > 0$ such that, if $\varepsilon \le \varepsilon_0$, there exists an entropy solution (ρ, u) of (6.1.1) in the sense of Definition 6.1.6 such that

$$0 \le \rho(t, x) \le M, \quad |u(t, x)| \le M < \frac{1}{\sqrt{\varepsilon}} \text{ for almost every } (t, x) \in \mathbb{R}^2_+,$$

where $M > 0$ depends on the initial data.

We remark that this theorem requires an assumption that the speed of light is large, relative to the initial data. The reason for this may be seen in Proposition 6.4.6 and the proof of Theorem 6.5.2, where we can only conclude the reduction argument under this assumption.

Our analysis of the entropy functions for (6.1.1) and the associated compactness framework will also extend freely to the alternative system, discussed in §2.6,

$$\begin{cases} \partial_t\big(\rho + \frac{\varepsilon(\rho+\varepsilon p)u^2}{1-\varepsilon u^2}\big) + \partial_x\big(\frac{(\rho+\varepsilon p)u}{1-\varepsilon u^2}\big) = 0, \\ \partial_t\big(\frac{(\rho+\varepsilon p)u}{1-\varepsilon u^2}\big) + \partial_x\big(\frac{(\rho+\varepsilon p)u^2}{1-\varepsilon u^2} + p(\rho)\big) = 0. \end{cases} \tag{6.1.4}$$

We therefore obtain the following additional result.

Theorem 6.1.2 (Entropy Solutions for the Alternative System)**.** *Let (ρ_0, u_0) be measurable and bounded initial data satisfying the assumptions of Theorem 6.1.1. Then there exists $\varepsilon_0 > 0$ such that, if $\varepsilon \le \varepsilon_0$, there exists an entropy solution (ρ, u) of (6.1.4) satisfying*

$$0 \le \rho(t,x) \le M, \quad |u(t,x)| \le M < \frac{1}{\sqrt{\varepsilon}} \text{ for almost every } (t,x) \in \mathbb{R}^2_+,$$

and some $M > 0$ depending on the initial data.

Moreover, our analysis of the relativistic entropies is also sufficient to control the convergence of a sequence of solutions to the relativistic equations as $\varepsilon \to 0$. We recall that formally, as $\varepsilon \to 0$, system (6.1.1) converges to the classical Euler equations,

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0. \end{cases} \tag{6.1.5}$$

Theorem 6.1.3 (Convergence of the Newtonian Limit)**.** *Let $(\rho_0, u_0) \in (L^\infty(\mathbb{R}))^2$ with $\rho_0 \ge 0$ and suppose that for $\varepsilon \in (0,1)$, $(\rho^\varepsilon, u^\varepsilon)$ is an entropy solution to (6.1.1) with light speed $c = \frac{1}{\sqrt{\varepsilon}}$ and initial data $(\rho_0^\varepsilon, u_0^\varepsilon) \in (L^\infty(\mathbb{R}))^2$ with $\rho_0^\varepsilon \ge 0$ such that*

$$\sqrt{p'(\rho_0^\varepsilon(x))},\ |u_0^\varepsilon(x)| < c \text{ for all } \varepsilon > 0 \text{ and almost every } x \in \mathbb{R}.$$

Suppose that there exists $M > 0$, independent of ε, such that

$$0 \le \rho^\varepsilon(t,x) \le M, \quad |v^\varepsilon(u^\varepsilon(t,x))| \le M \text{ for almost every } (t,x) \in \mathbb{R}^2_+,$$

where $v^\varepsilon(u^\varepsilon) := \frac{1}{2\sqrt{\varepsilon}} \log\left(\frac{1+\sqrt{\varepsilon}u^\varepsilon}{1-\sqrt{\varepsilon}u^\varepsilon}\right)$, and $(\rho_0^\varepsilon, v^\varepsilon(u_0^\varepsilon)) \rightharpoonup (\rho_0, v_0)$ distributionally as $\varepsilon \to 0$.

Then, up to a subsequence, $(\rho^\varepsilon, v^\varepsilon) \to (\rho, v)$ almost everywhere and in $L^r_{loc}(\mathbb{R}^2_+)$ for all $r \in [1,\infty)$, where (ρ, v) is an entropy solution of the classical Euler equations (6.1.5) with initial data (ρ_0, v_0) and such that

$$0 \le \rho(t,x) \le M, \quad |v(t,x)| \le M \text{ for almost every } (t,x) \in \mathbb{R}^2_+.$$

6.1.1 Basic properties of the system

We consider some of the basic properties of system (6.1.1). Writing $U(\rho, u)$ as in (6.1.2) for the conserved variables, we calculate

$$\nabla_U F(U) = d_{(\rho,u)}F(U)(d_{(\rho,u)}U)^{-1} = \begin{pmatrix} \frac{\varepsilon(u^2-p')u}{1-p'u^2\varepsilon^2} & \frac{n(1-\varepsilon u^2)^{\frac{3}{2}}}{(\rho+\varepsilon p)(1-p'u^2\varepsilon^2)} \\ \frac{(p'-u^2)(\rho+\varepsilon p)\sqrt{1-\varepsilon u^2}}{n(1-p'u^2\varepsilon^2)} & \frac{(2-\varepsilon p'-\varepsilon u^2)u}{1-p'u^2\varepsilon^2} \end{pmatrix}, \quad (6.1.6)$$

d denoting differentiation in the (ρ, u) variables. This has eigenvalues

$$\lambda_1 := \frac{u - \sqrt{p'(\rho)}}{1 - \varepsilon u\sqrt{p'(\rho)}}, \quad \lambda_2 := \frac{u + \sqrt{p'(\rho)}}{1 + \varepsilon u\sqrt{p'(\rho)}},$$

and eigenvectors

$$r_1 = \begin{pmatrix} 1 \\ \frac{(\rho+\varepsilon p)(u-\sqrt{p'})}{n\sqrt{1-\varepsilon u^2}} \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ \frac{(\rho+\varepsilon p)(u+\sqrt{p'})}{n\sqrt{1-\varepsilon u^2}} \end{pmatrix}.$$

The sound speed in the fluid is given by $\sqrt{p'(\rho)}$, and so we henceforth assume that

$$\sqrt{p'(\rho)} < c.$$

We define $\rho_{\max}$ to be the smallest positive constant such that $\sqrt{p'(\rho_{\max})} = c$ if such a $\rho_{\max}$ exists, or ∞ else. Then, in the region $0 < \rho < \rho_{\max}$, $|u| < c$, we find $\lambda_2 - \lambda_1 > 0$, and hence the system is strictly hyperbolic.

We define Riemann invariants for the system,

$$w := v(u) + k(\rho), \quad z := v(u) - k(\rho),$$

where

$$v(u) := \frac{1}{2\sqrt{\varepsilon}} \log\left(\frac{1+\sqrt{\varepsilon}u}{1-\sqrt{\varepsilon}u}\right) \quad (6.1.7)$$

and

$$k(\rho) := \int_0^\rho \frac{\sqrt{p'(s)}}{s + \varepsilon p(s)} ds. \quad (6.1.8)$$

Note that the mapping $u \mapsto v(u)$ is a smooth, increasing bijection from $(-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}})$ to $\mathbb{R}$, and that $\rho \mapsto k(\rho)$ is a smooth, increasing bijection from $(0, \rho_{\max})$ onto its image. For the inverse of v, we will write u, that is,

$$u(v) := \frac{1}{\sqrt{\varepsilon}} \tanh(\sqrt{\varepsilon}v). \quad (6.1.9)$$

As mentioned previously, to close the system, we impose an equation of state, i.e. a pressure law. We assume, in addition to the condition $p'(\rho) > 0$ for $\rho > 0$, that the pressure behaves like a gamma-law close to the vacuum, while allowing for more general behaviour. To make this precise, we impose that, for some $\gamma \in (1,3)$,

$$p(\rho) = \kappa\rho^{\gamma}\left(1 + P(\rho)\right), \quad |P^{(n)}(\rho)| \leq C\rho^{2\theta - n}, \text{ for } 0 \leq n \leq 3, \tag{6.1.10}$$

where $\kappa = \frac{(\gamma-1)^2}{4\gamma}$ and $\theta = \frac{\gamma-1}{2}$.

We compare the function $k(\rho)$ to the equivalent function for the classical Euler equations equipped with a gamma-law pressure (compare [8]), for which we would have $k(\rho) = \rho^{\theta}$. With the assumption (6.1.10) on the pressure, we observe the following behaviour of the function $k(\rho)$ near the vacuum. For ease of reference, we state this as a lemma.

Lemma 6.1.4. *As $\rho \to 0$, the function $k(\rho)$ and its first derivative obey the following asymptotics:*

$$\begin{aligned} k(\rho) &= \rho^{\theta} + O(\rho^{3\theta}) \ \text{ as } \rho \to 0, \\ k'(\rho) &= \frac{\sqrt{p'(\rho)}}{\rho + \varepsilon p(\rho)} = \theta\rho^{\theta-1} + O(\rho^{3\theta-1}) \ \text{ as } \rho \to 0. \end{aligned} \tag{6.1.11}$$

Moreover, the derivatives $k^{(n)}(\rho)$, for $n = 2, 3$ can be expanded, as $\rho \to 0$, as

$$k''(\rho) = \theta(\theta-1)\rho^{\theta-2} + O(\rho^{3\theta-2}), \quad k^{(3)}(\rho) = \theta(\theta-1)(\theta-2)\rho^{\theta-3} + O(\rho^{3\theta-3}).$$

We define another exponent for use in the next section, $\lambda = \frac{3-\gamma}{2(\gamma-1)} > 0$, and note λ is related to θ by the equation $2\lambda\theta = 1 - \theta$.

An analysis of the alternative system (6.1.4) shows that it also has eigenvalues λ_1 and λ_2 as defined above, and Riemann invariants $w = v(u) + k(\rho)$ and $z = v(u) - k(\rho)$.

6.1.2 Entropy and entropy solutions

In order to analyse the limit of our approximate solutions to system (6.1.1) and show strong convergence of the sequence, we need to understand the structure and behaviour of the entropy pairs of the system. The purpose of this section is to provide the basis and framework for this analysis.

Definition 6.1.5. An *entropy/entropy-flux pair* (η, q) is a pair of C^1 functions satisfying the relation

$$\nabla q(U) = \nabla\eta(U)\nabla F(U).$$

A *weak entropy* is an entropy that vanishes at the vacuum $\{\rho = 0\}$.

We observe that an equivalent characterisation in Riemann invariant coordinates is given by

$$q_w = \lambda_2\, \eta_w, \qquad q_z = \lambda_1\, \eta_z.$$

In particular, as the eigenvalues and Riemann invariants of the two systems (6.1.1) and (6.1.4) coincide, we deduce that the two systems share entropy and entropy flux functions. We therefore restrict attention in the sequel to the system (6.1.1).

Definition 6.1.6. A pair (ρ, u) of bounded, measurable functions such that $0 \leq \rho < \rho_{\max}$, and $|u| < c = \frac{1}{\sqrt{\varepsilon}}$ is an *entropy solution* of (6.1.1) in $\mathbb{R}^2_+$ with initial data (ρ_0, u_0) if the following conditions hold. We write $U = U(\rho, u)$ for the conserved variables of (6.1.2).

i) For any $\phi \in C^1_0(\overline{\mathbb{R}^2_+}; \mathbb{R}^2)$,

$$\int_{\mathbb{R}^2_+} \left(U \cdot \partial_t \phi + F(U) \cdot \partial_x \phi\right)\, dx\, dt + \int_{-\infty}^{\infty} U_0(x) \cdot \phi(0, x)\, dx = 0;$$

ii) For any $\phi \in C^1_0(\overline{\mathbb{R}^2_+})$ with $\phi \geq 0$ and C^1 weak entropy pair (η, q) with η convex,

$$\int_{\mathbb{R}^2_+} \left(\eta(U)\partial_t \phi + q(U)\partial_x \phi\right)\, dx\, dt + \int_{-\infty}^{\infty} \eta(U_0(x))\phi(0, x)\, dx \geq 0.$$

An explicit entropy pair is given by

$$\eta^*(U) = \frac{\rho + p\varepsilon^2 u^2}{1 - \varepsilon u^2}, \qquad q^*(U) = \frac{(\rho + \varepsilon p)u}{1 - \varepsilon u^2}.$$

Now we may calculate that

$$\nabla^2 \eta^*(U) = \alpha_0(\rho, u) \begin{pmatrix} \frac{\varepsilon(\rho + \varepsilon p)(p' + u^2 + 2p'\varepsilon u^2)}{n(1 - \varepsilon u^2)} & \frac{-\varepsilon(1 + \varepsilon p')u}{\sqrt{1 - \varepsilon u^2}} \\ -\frac{\varepsilon(1 + \varepsilon p')u}{\sqrt{1 - \varepsilon u^2}} & \frac{\varepsilon n}{\rho + \varepsilon p} \end{pmatrix},$$

where $\alpha_0(\rho, u) = \frac{(1 - \varepsilon u^2)^2}{n(1 - p'\varepsilon^2 u^2)} > 0$. In particular, $\eta^*(U)$ is a convex entropy.

We begin our analysis of the entropy functions of (6.1.1) by constructing a fundamental solution of the *entropy equation*. This fundamental solution gives us a mechanism for generating weak entropies of the system via convolution with test functions, and so enables us to construct and use entropies in the sequel. In this section, we analyse the singularities of the entropy equation and give the definitions of the entropy and entropy flux kernels, along with the main theorems concerning their existence and regularity.

6.2 Entropy and entropy flux kernels

Let (η, q) be an entropy/entropy-flux pair. Then it follows from Definition 6.1.5 and (6.1.6) that

$$\begin{aligned} q_\rho &= \frac{u(1-\varepsilon p')}{1-p'\varepsilon^2 u^2}\eta_\rho + \frac{(1-\varepsilon u^2)^2 p'}{(\rho+\varepsilon p)(1-p'\varepsilon^2 u^2)}\eta_u, \\ q_u &= \frac{\rho+\varepsilon p}{1-p'\varepsilon^2 u^2}\eta_\rho + \frac{u(1-\varepsilon p')}{1-p'\varepsilon^2 u^2}\eta_u. \end{aligned} \tag{6.2.1}$$

Eliminating q and changing coordinates $u \mapsto v$ as in (6.1.7) yields

$$\eta_{\rho\rho} - k'(\rho)^2 \eta_{vv} + \varepsilon A(\rho, v)\eta_\rho + \varepsilon B(\rho, v) v \eta_v = 0, \tag{6.2.2}$$

where

$$\begin{aligned} A(\rho, v) &= \frac{2p'(\rho)}{\rho+\varepsilon p(\rho)} \frac{1-\varepsilon u^2\left(1-\frac{p''(\rho)(\rho+\varepsilon p)}{2p'(\rho)}\right)}{1-p'(\rho)\varepsilon^2 u^2}, \\ B(\rho, v) &= \frac{2up'(\rho)\left(1-\varepsilon p'(\rho)-\frac{p''(\rho)(\rho+\varepsilon p(\rho))}{2p'(\rho)}\right)}{v(u)(\rho+\varepsilon p(\rho))^2(1-p'(\rho)\varepsilon^2 u^2)}. \end{aligned}$$

To simplify notation, we will write throughout

$$\mathbf{L} := \partial_{\rho\rho} - k'(\rho)^2 \partial_{vv} + \varepsilon A(\rho, v)\partial_\rho + \varepsilon B(\rho, v) v \partial_v. \tag{6.2.3}$$

Definition 6.2.1. The *entropy kernel*, $\chi = \chi(\rho, v, s)$, by definition, satisfies the equation:

$$\begin{cases} \mathbf{L}\chi = \chi_{\rho\rho} - k'(\rho)^2\chi_{vv} + \varepsilon A(\rho, v)\chi_\rho + \varepsilon B(\rho, v) v \chi_v = 0, \\ \chi|_{\rho=0} = 0, \\ \chi_\rho|_{\rho=0} = \delta_{v=s}, \end{cases} \tag{6.2.4}$$

where $s \in \mathbb{R}$.

We recall that (6.1.1) is invariant under Lorentz transformations $t' = \frac{t-\varepsilon\tau x}{\sqrt{1-\varepsilon\tau^2}}$, $x' = \frac{x-\tau t}{\sqrt{1-\varepsilon\tau^2}}$, for $|\tau| < \frac{1}{\sqrt{\varepsilon}}$. Under such a transformation, the velocity u and the associated function v also transform as $u' = \frac{u-\tau}{1-\varepsilon\tau u}$, $v' := v(u') = v(u) - v(\tau)$. By invariance of the equations under these transformations, the entropy equation is also invariant under such Lorentz shifts. Thus, for $s = v(\tau)$, $\chi(\rho, v, s) = \chi(\rho, v-s, 0) = \chi(\rho, 0, s-v)$ and so it suffices to solve in the case $s = 0$. We therefore write, in a slight abuse of notation, $\chi(\rho, v-s) = \chi(\rho, v, s)$ henceforth.

The kernel provides a representation formula for weak entropies of the relativistic Euler equations, namely any entropy function may be written by convolution with a test function $\psi(s)$,

$$\eta^\psi(\rho, u) = \int_{\mathbb{R}} \psi(s)\chi(\rho, v(u)-s)\, ds.$$

Before we continue, it is worth making an aside at this point to compare the situation to the classical Euler equations, (6.1.5). For this system, the entropy equation is the simpler equation

$$\chi^*_{\rho\rho} - k'(\rho)^2 \chi^*_{vv} = 0. \tag{6.2.5}$$

For the gamma-law gas, we have $k'(\rho) = \theta\rho^{\theta-1}$, and (6.2.5) has fundamental solution

$$\chi^*(\rho, v) = M_\lambda [\rho^{2\theta} - v^2]^\lambda_+,$$

where $\lambda > 0$ is defined as in §6.1.1 and $M_\lambda > 0$ is a constant depending only on λ.

With this as a motivation and following [8, 9], we make an ansatz for the entropy kernel in the form

$$\chi(\rho, v) = a_1(\rho)[k(\rho)^2 - v^2]^\lambda_+ + a_2(\rho)[k(\rho)^2 - v^2]^{\lambda+1}_+ + g(\rho, v). \tag{6.2.6}$$

By the principle of finite propagation speed, we expect the remainder function $g(\rho, v)$ to have the same support as the first two terms.

In anticipation of the next theorem, we recall the definition of fractional derivative: for a function $f = f(s)$ of compact support, the fractional derivative of order $\mu > 0$ is

$$\partial^\mu_s f = \Gamma(-\mu) f * [s]^{-\mu-1}_+. \tag{6.2.7}$$

Henceforth, we suppose that we have a fixed upper bound for the density, $\rho \le \rho_M$ such that $\rho_M < \rho_{\max}$. All constants $C > 0$ will be independent of ρ, but may depend on ρ_M.

Theorem 6.2.2 (Relativistic Entropy Kernel). *The entropy kernel admits the expansion*

$$\chi(\rho, v) = a_1(\rho)[k(\rho)^2 - v^2]^\lambda_+ + a_2(\rho)[k(\rho)^2 - v^2]^{\lambda+1}_+ + g(\rho, v), \tag{6.2.8}$$

where the coefficients $a_1(\rho)$ and $a_2(\rho)$ are such that for $0 \le \rho \le \rho_M$,

$$a_1(\rho) = c_{*,\lambda} k(\rho)^{-\lambda} k'(\rho)^{-\frac{1}{2}} e^{\tilde{a}_1(\rho)} > 0, \tag{6.2.9}$$

for some constant $c_{*,\lambda} > 0$, and

$$a_1(\rho) + |a_2(\rho)| \le C. \tag{6.2.10}$$

Moreover, the remainder function $g(\rho, v)$ and its derivatives $\partial^\mu_v g(\rho, v)$ are Hölder continuous for $0 < \mu < \lambda + 2$. The remainder satisfies, for $0 < \beta < \mu$,

$$|\partial^\beta_v g(\rho, v)| \le C \rho^{1+\theta-2\mu\theta+\beta\theta} [k(\rho)^2 - v^2]^{\mu-\beta}_+.$$

By definition, to each entropy function is associated a corresponding entropy flux. These entropy flux functions are generated by another kernel, the *entropy flux kernel*, which we call $\sigma(\rho, v, s)$.

Definition 6.2.3. The entropy flux kernel is defined by

$$\begin{cases} \mathbf{L}\sigma = \sigma_{\rho\rho} - k'(\rho)^2\sigma_{vv} + \varepsilon A(\rho, v)\sigma_\rho + \varepsilon B(\rho, v)v\sigma_v = F(\rho, v), \\ \sigma|_{\rho=0} = 0, \\ \sigma_\rho|_{\rho=0} = \frac{u(1-\varepsilon p')}{1-p'\varepsilon^2 u^2}\delta_{v=s}, \end{cases} \tag{6.2.11}$$

where $F(\rho, v)$ is given explicitly later in (6.4.6).

The entropy function generated by the convolution of a test function $\psi(s)$ with the entropy kernel has a corresponding entropy flux given by

$$q(\rho, v) = \int_{\mathbb{R}} \sigma(\rho, v, s)\psi(s)\, ds.$$

As we saw for the entropy equation, the PDE in this equation is invariant under Lorentz transformations. However, it is also clear that the initial data is not. We therefore consider, instead of σ, the difference $\sigma - \frac{u(1-\varepsilon p')}{1-p'\varepsilon^2 u^2}\chi$. Writing $\tilde{u}(\rho, v) = \frac{u(1-\varepsilon p')}{1-p'\varepsilon^2 u^2}$, this difference satisfies the initial value problem

$$\begin{cases} \mathbf{L}(\sigma - \tilde{u}\chi) = \tilde{F}(\rho, v), \\ (\sigma - \tilde{u}\chi)|_{\rho=0} = 0, \\ (\sigma - \tilde{u}\chi)_\rho|_{\rho=0} = 0, \end{cases} \tag{6.2.12}$$

with $\tilde{F}(\rho, v)$ defined by

$$F(\rho, v) - \mathbf{L}(\tilde{u}\chi).$$

This equation is Lorentzian invariant, i.e.

$$(\sigma - \tilde{u}\chi)(\rho, v, s) = (\sigma - \tilde{u}\chi)(\rho, v - s, 0) = (\sigma - \tilde{u}\chi)(\rho, 0, s - v).$$

We therefore solve for the case $s = 0$.

Theorem 6.2.4 (Relativistic Entropy Flux Kernel)**.** *The entropy flux kernel admits the expansion*

$$(\sigma - \tilde{u}\chi)(\rho, v) = -v\big(b_1(\rho)[k(\rho)^2 - v^2]_+^\lambda + b_2(\rho)[k(\rho)^2 - v^2]_+^{\lambda+1}\big) + h(\rho, v), \tag{6.2.13}$$

where the coefficients $b_1(\rho)$ and $b_2(\rho)$ are given explicitly and, for $0 \le \rho \le \rho_M$,

$$\begin{aligned} &b_1(\rho) > 0, \\ &b_1(\rho) + |b_2(\rho)| \le C. \end{aligned} \tag{6.2.14}$$

Moreover, the remainder function $h(\rho, v)$ and its derivatives $\partial_v^\mu h(\rho, v)$ are Hölder continuous for $0 < \mu < \lambda + 2$. The remainder satisfies, for $0 < \beta < \mu$,

$$|\partial_v^\beta h(\rho, v)| \le C\rho^{1+\theta-2\mu\theta+\beta\theta}[k(\rho)^2 - v^2]_+^{\mu-\beta}.$$

6.3 Existence and regularity of the entropy kernel

This section is devoted to proving Theorem 6.2.2, i.e. showing the existence of the entropy kernel and examining its regularity properties.

As a preliminary observation, we note that by the principle of finite speed of propagation, the coefficients A and B may be redefined to be 0 outside the support of $\chi(\rho, v-s)$, which is the set $\{(v-s)^2 \leq k(\rho)^2\}$. Moreover, as A and B are functions of v^2, a simple Taylor expansion around $v^2 = k(\rho)^2$ gives:

Lemma 6.3.1. *We may write, in the equation for the entropy kernel,* (6.2.4),

$$A(\rho, v) = A_0(\rho)\mathbb{1}_{|v|\leq k(\rho)} + \varepsilon A_1(\rho)[k(\rho)^2 - v^2]_+ + \varepsilon^2 A_2(\rho, v)[k(\rho)^2 - v^2]_+^2,$$
$$B(\rho, v) = B_0(\rho)\mathbb{1}_{|v|\leq k(\rho)} + \varepsilon B_1(\rho)[k(\rho)^2 - v^2]_+ + \varepsilon^2 B_2(\rho, v)[k(\rho)^2 - v^2]_+^2,$$

where

$$|A_0(\rho)| + |A_1(\rho)| + |A_2(\rho, v)| \leq C\rho^{\gamma-2},$$
$$|B_0(\rho)| + |B_1(\rho)| \leq C\rho^{\gamma-3}.$$

Moreover, $|B(\rho, v)| \leq C\rho^{\gamma-3}$ *also.*

6.3.1 Coefficients for the entropy kernel

Having made an ansatz for the entropy kernel, we determine the coefficients $a_1(\rho)$ and $a_2(\rho)$. To do this, we substitute the ansatz (6.2.6) into the entropy equation (6.2.4) and examine the most singular terms. By choosing the coefficients such that these terms vanish, we are able to solve the equation for the higher order remainder.

We introduce some notation to simplify this process by defining, for $\nu \in \mathbb{R}$,

$$G_\nu(\rho, v) = [k(\rho)^2 - v^2]_+^\nu.$$

Then we have the following identites:

$$\partial_\rho G_\nu(\rho, v) = 2\lambda k(\rho)k'(\rho)G_{\nu-1}(\rho, v), \qquad \partial_v G_\nu(\rho, v) = -2\lambda v G_{\nu-1}(\rho, v).$$

Substituting (6.2.6) into (6.2.4) and grouping terms yields:

$$
\begin{aligned}
&\chi_{\rho\rho} - k'(\rho)^2\chi_{vv} + \varepsilon A\chi_\rho + \varepsilon Bv\chi_v \\
&= G_{\lambda-1}(\rho,v)\big(4\lambda kk'a_1' + 4\lambda^2k'^2a_1 + 2\lambda kk''a_1 + 2\lambda\varepsilon A_0kk'a_1 - 2\lambda\varepsilon B_0k^2a_1\big) \\
&\quad + G_\lambda(\rho,v)\big(a_1'' + 4(\lambda+1)kk'a_2' + 4(\lambda+1)^2k'^2a_2 + 2(\lambda+1)kk''a_2 \\
&\qquad + \varepsilon A_0a_1' + 2\lambda\varepsilon B_0a_1 + 2(\lambda+1)\varepsilon A_0kk'a_2 - 2(\lambda+1)\varepsilon B_0k^2a_2 \\
&\qquad + 2\lambda\varepsilon^2A_1kk'a_1 - 2\lambda\varepsilon^2B_1k^2a_1\big) \\
&\quad + G_{\lambda+1}(\rho,v)\big(a_2'' + 2(\lambda+1)\varepsilon B_0a_2 + \varepsilon(A-A_0)a_1' + 2\lambda\varepsilon^3A_2kk'a_1 + \varepsilon A_0a_2' \\
&\qquad + 2(\lambda+1)\varepsilon(A-A_0)kk'a_2 + 2(\lambda+1)\varepsilon B_0a_2 + 2\lambda\varepsilon^2B_1a_1 \\
&\qquad - 2\lambda\varepsilon^3B_2v^2a_1 - 2(\lambda+1)\varepsilon(B-B_0)v^2a_2\big) \\
&\quad + g_{\rho\rho} - k'(\rho)^2g_{vv} + \varepsilon Ag_\rho + \varepsilon Bvg_v.
\end{aligned}
$$

Thus we see that in order to cancel the highest order singularities, that is, to make the coefficient of $G_{\lambda-1}$ vanish, a_1 must solve

$$\frac{a_1'}{a_1} = -\frac{k''}{2k'} - \lambda\frac{k'}{k} - \frac{\varepsilon}{2}A_0 + \frac{\varepsilon}{2}\frac{k}{k'}B_0,$$

hence

$$a_1(\rho) = c_{*,\lambda}k(\rho)^{-\lambda}k'(\rho)^{-\frac{1}{2}}e^{\tilde{a}(\rho)},$$

where the constant $c_{*,\lambda} > 0$ is determined to satisfy the initial conditions and

$$\tilde{a}(\rho) := \frac{\varepsilon}{2}\int_0^\rho \big(-A_0(s) + \frac{k(s)}{k'(s)}B_0(s)\big)ds.$$

By defining $\alpha_1(\rho) := a_1(\rho)k(\rho)^{2\lambda+1}$, we obtain

$$\frac{\alpha_1'}{\alpha_1} = -\frac{k''}{2k'} + (\lambda+1)\frac{k'}{k} - \frac{\varepsilon}{2}A_0 + \frac{\varepsilon}{2}\frac{k}{k'}B_0. \tag{6.3.1}$$

Cancelling the next highest order singularities, we obtain the following equation for $a_2(\rho)$:

$$a_2' + a_2\big(\frac{k''}{2k'} + (\lambda+1)\frac{k'}{k} + \frac{\varepsilon}{2}A_0 - \frac{\varepsilon}{2}\frac{k}{k'}B_0\big) = -\frac{1}{4(\lambda+1)kk'}\tilde{W},$$

where

$$\tilde{W} = a_1'' + \varepsilon A_0a_1' + 2\lambda\varepsilon^2a_1(kk'A_1 - k^2B_1) + 2\lambda\varepsilon B_0a_1. \tag{6.3.2}$$

Defining $\alpha_2(\rho) := a_2(\rho)k(\rho)^{2\lambda+3}$,

$$
\begin{aligned}
&\alpha_2' + \alpha_2\big(\frac{k''}{2k'} - (\lambda+2)\frac{k'}{k} + \frac{\varepsilon}{2}A_0 - \frac{\varepsilon}{2}\frac{k}{k'}B_0\big) \\
&= -\frac{k}{4(\lambda+1)k'}\left[\alpha_1'' + \varepsilon A_0\alpha_1' - \varepsilon B_0\alpha_1 + 2\lambda\varepsilon^2\alpha_1(kk'A_1 - k^2B_1)\right] =: \Omega.
\end{aligned}
\tag{6.3.3}
$$

We take the less singular solution to this singular ODE given by

$$\alpha_2(\rho) = e^{\tilde{a}(\rho)} k(\rho)^{\lambda+2} k'(\rho)^{-\frac{1}{2}} \int_0^\rho e^{-\tilde{a}(\tau)} k(\tau)^{-\lambda-2} k'(\tau)^{\frac{1}{2}} \Omega(\tau)\, d\tau.$$

From the observations made in Lemma 6.1.4, we see that $k(\tau)^{-\lambda-2} k'(\tau)^{\frac{1}{2}} \Omega(\tau) = O\left(\tau^{\frac{\gamma-3}{2}}\right)$ as $\tau \to 0$, hence is integrable.

Then we conclude that

$$\begin{aligned} a_2(\rho) =& e^{\tilde{a}(\rho)} k(\rho)^{-\lambda-1} k'(\rho)^{-\frac{1}{2}} \int_0^\rho e^{-\tilde{a}(\tau)} k(\tau)^{-\lambda-2} k'(\tau)^{\frac{1}{2}} \Omega(\tau)\, d\tau \\ =& -\frac{1}{4(\lambda+1)} e^{\tilde{a}(\rho)} k(\rho)^{-\lambda-1} k'(\rho)^{-\frac{1}{2}} \int_0^\rho e^{-\tilde{a}(\tau)} k(\tau)^{\lambda} k'(\tau)^{-\frac{1}{2}} \tilde{W}(\tau)\, d\tau \end{aligned} \tag{6.3.4}$$

is well defined, where $\tilde{W}$ is defined in (6.3.2). Throughout, we will exploit the fact that there exist C_1 and C_2 depending on ρ_M such that $0 < C_1 \leq e^{\tilde{a}(\rho)} \leq C_2 < \infty$.

6.3.2 Proof of existence and regularity of the entropy kernel

Having determined the the coefficients of the expansion (6.2.6), we now need to solve for the remainder $g(\rho, v)$. It is again instructive at this point to consider the entropy equation in the classical situation, (6.2.5), studied in [8, 9]. In this situation, the coefficients of the entropy equation depend only on ρ, and so the equation lends itself to Fourier transform methods.

However, we observe that the coefficients of the first order terms in the relativistic entropy equation, that is, the functions A and B, are functions also of v as well as ρ. To avoid this obstacle, we therefore isolate the "principal part" of the operator (which will have coefficients depending only on ρ) and derive an equation in terms of this operator for the remainder $g(\rho, v)$ in (6.2.6).

We therefore define an operator $\tilde{\mathbf{L}}$ by

$$\tilde{\mathbf{L}} := \partial_{\rho\rho} - k'(\rho)^2 \partial_{vv} + \varepsilon A_0(\rho) \partial_\rho - \beta(\rho), \tag{6.3.5}$$

where

$$\beta(\rho) := \frac{\alpha_\sharp''(\rho)}{\alpha_\sharp(\rho)} + \varepsilon A_0(\rho) \frac{\alpha_\sharp'(\rho)}{\alpha_\sharp(\rho)},$$

and

$$\alpha_\sharp(\rho) = c_\sharp e^{-\frac{\varepsilon}{2} \int_0^\rho A_0(s)\, ds} k(\rho)^{\lambda+1} k'(\rho)^{-\frac{1}{2}},$$

where the constant $c_\sharp > 0$ will be chosen later. We observe that $\alpha_\sharp$ satisfies the equation

$$\frac{\alpha'_\sharp}{\alpha_\sharp} = -\frac{k''}{2k'} + \frac{(\lambda+1)k'}{k} - \frac{\varepsilon}{2}A_0.$$

By the asymptotics for $k(\rho)$ given in Lemma 6.1.4, we find $\beta(\rho)$ is $O(\rho^{\gamma-3})$ as $\rho \to 0$.

With the operator $\tilde{\mathbf{L}}$, we work in Fourier space. We therefore determine an equation for the expression $\mathcal{F}(\tilde{\mathbf{L}}g)(\rho,\xi)$, where $\mathcal{F}$ denotes the Fourier transform in the v variable. To calculate the Fourier transform of the function $[k(\rho)^2 - v^2]_+^\lambda$, we make use of the following.

For ease of notation, we write $f_\lambda(y) = [1-y^2]_+^\lambda$, so that

$$[k(\rho)^2 - v^2]_+^\lambda = k(\rho)^{2\lambda} f_\lambda\Big(\frac{v}{k(\rho)}\Big).$$

Recalling now from [29] that the Fourier transform of $f_\lambda(y)$ is

$$\widehat{f_\lambda}(\xi) = \sqrt{\pi}\Gamma(\lambda+1)2^{\lambda+\frac{1}{2}}|\xi|^{-\lambda-\frac{1}{2}} J_{\lambda+\frac{1}{2}}(\xi),$$

where J_ν is the Bessel function of first type of order ν, we see that

$$\mathcal{F}\big([k(\rho)^2 - v^2]_+^\lambda\big) = k(\rho)^{2\lambda+1}\widehat{f_\lambda}(k(\rho)\xi) = C\Big(\frac{k(\rho)}{\xi}\Big)^{\lambda+\frac{1}{2}} J_{\lambda+\frac{1}{2}}(k(\rho)\xi).$$

We are therefore able to derive an equation for $\tilde{\mathbf{L}}g$ in Fourier space.

Proposition 6.3.2. *The remainder $g(\rho,v)$ satisfies*

$$\begin{aligned}\mathcal{F}(\tilde{\mathbf{L}}g)(\rho,\xi) =& \mathcal{F}(\mathcal{S}(g))(\rho,\xi)\\ :=& -\mathcal{F}(\varepsilon(A-A_0)g_\rho) - \mathcal{F}(\varepsilon B v g_v) - \beta(\rho)\widehat{g} + H_0(\rho)\widehat{f}_{\lambda+1}(k(\rho)\xi) + \varepsilon^2 r(\rho,\xi),\end{aligned}$$

where $H_0(\rho) = O(\rho^{-1+2\theta})$ as $\rho \to 0$ and $r(\rho,\xi) = O\big(\rho^{-1+2\theta}(k(\rho)|\xi|)^{-\lambda-1-\alpha-\frac{1}{2}}\big)$ as $|\xi| \to \infty$, for some $\alpha > 0$.

In particular, the remainder $r(\rho,\xi)$ acts asymptotically like $\widehat{f}_{\lambda+1+\alpha}(k(\rho)\xi)$ as $\xi \to \infty$.

Proof. We write $X^1 = a_1(\rho)[k(\rho)^2 - v^2]_+^\lambda$, $X^2 = a_2(\rho)[k(\rho)^2 - v^2]_+^{\lambda+1}$. From the considerations above, we find $\mathcal{F}(X^1) = \alpha_1(\rho)\widehat{f_\lambda}(k(\rho)\xi)$, $\mathcal{F}(X^2) = \alpha_2(\rho)\widehat{f}_{\lambda+1}(k(\rho)\xi)$. Now rearranging the entropy equation, we obtain

$$\begin{aligned}\mathcal{F}(\tilde{\mathbf{L}}g)(\rho,\xi) =& -I - II - \varepsilon^3\mathcal{F}(A_2[k^2-v^2]_+^2 X_\rho^1) - \varepsilon\mathcal{F}((A-A_0)X_\rho^2)\\ &- \mathcal{F}(\varepsilon^3 B_2[k^2-v^2]_+^2 v X_v^1 + \varepsilon(B-B_0)vX_v^2)\\ &- \mathcal{F}(\varepsilon(A-A_0)g_\rho) - \mathcal{F}(\varepsilon B v g_v) - \beta(\rho)\widehat{g},\end{aligned}$$

where
$$I := \mathcal{F}(X^1_{\rho\rho}) + k'(\rho)^2\xi^2\mathcal{F}(X^1) + \mathcal{F}(\varepsilon(A_0 + \varepsilon A_1[k^2 - v^2]_+)X^1_\rho)$$
$$+ \mathcal{F}(\varepsilon(B_0 v + \varepsilon B_1[k^2 - v^2]_+ v)X^1_v)$$
and
$$II := \mathcal{F}(X^2_{\rho\rho}) + k'(\rho)^2\xi^2\mathcal{F}(X^2) + \varepsilon A_0\mathcal{F}(X^2_\rho) + \varepsilon\mathcal{F}(B_0 v X^2_v).$$
As $\widehat{f}'_\lambda(y) = -\frac{2\lambda+1}{y}\widehat{f}_\lambda + \frac{2\lambda}{y}\widehat{f}_{\lambda-1}$, we calculate
$$I = \widehat{f}_\lambda\big(\alpha_1'' + \varepsilon A_0\alpha_1' + 2\lambda\varepsilon^2 A_1 kk'\alpha_1 + 2\lambda\varepsilon B_0\alpha_1 - (2\lambda+1)\varepsilon B_0\alpha_1 - 2\lambda\varepsilon^2 B_1 k^2\alpha_1\big)$$
$$+ \widehat{f}_{\lambda+1}\big(\varepsilon^2 A_1 a_1' k^{2\lambda+3} + 2\lambda\varepsilon^2 B_1 k^2\alpha_1\big).$$
Similarly,
$$\begin{aligned} II = &- \widehat{f}_\lambda\big(\alpha_1'' + \varepsilon A_0\alpha_1' - \varepsilon B_0\alpha_1 + 2\lambda\varepsilon^2\alpha_1(kk'A_1 - k^2 B_1)\big) \\ &+ \widehat{f}_{\lambda+1}\big(\alpha_2'' + \varepsilon A_0\alpha_2' - \varepsilon B_0\alpha_2 \\ &\quad + \frac{2\lambda+3}{2(\lambda+1)}(\alpha_1'' + \varepsilon A_0\alpha_1' - \varepsilon B_0\alpha_1 + 2\lambda\varepsilon^2\alpha_1(kk'A_1 - k^2 B_1))\big), \end{aligned}$$
where we have used that the coefficients α_1 and α_2 satisfy (6.3.1) and (6.3.3). Then we see that
$$\begin{aligned} I + II = \widehat{f}_{\lambda+1}\big(&\alpha_2'' + \varepsilon A_0\alpha_2' - \varepsilon B_0\alpha_2 + \varepsilon^2 A_1 a_1' k^{2\lambda+3} + \varepsilon^2 2\lambda B_1 k^2\alpha_1 \\ &+ \frac{2\lambda+3}{2(\lambda+1)}(\alpha_1'' + \varepsilon A_0\alpha_1' - \varepsilon(2\lambda+1)B_0\alpha_1 + 2\lambda\varepsilon^2\alpha_1(kk'A_1 - k^2 B_1))\big). \end{aligned}$$
We treat $\mathcal{F}\big(-\varepsilon^3(A_2[k^2-v^2]^2_+X^1_\rho) - \varepsilon(A-A_0)X^2_\rho - \varepsilon^3 B_2[k^2-v^2]^2_+ vX^1_v + \varepsilon(B-B_0)X^2_v\big)$ as a higher order term and we call this term $\varepsilon^2 r(\rho,\xi)$. Thus we find
$$\begin{aligned} \mathcal{F}(\tilde{\mathbf{L}}g)(\rho,\xi) &= \mathcal{F}(\mathcal{S}(g)) \\ &:= -\mathcal{F}(\varepsilon(A-A_0)g_\rho) - \mathcal{F}(\varepsilon B v g_v) - \beta(\rho)\widehat{g} - H_0(\rho)\widehat{f}_{\lambda+1}(k(\rho)\xi) + \varepsilon^2 r(\rho,\xi), \end{aligned}$$
where
$$\begin{aligned} H_0(\rho) = \big(&\alpha_2'' + \varepsilon A_0\alpha_2' - \varepsilon B_0\alpha_2 + \varepsilon^2 A_1 a_1' k^{2\lambda+3} + \varepsilon^2 2\lambda B_1 k^2\alpha_1 \\ &+ \frac{2\lambda+3}{2(\lambda+1)}(\alpha_1'' + \varepsilon A_0\alpha_1' - \varepsilon(2\lambda+1)B_0\alpha_1 + 2\lambda\varepsilon^2\alpha_1(kk'A_1 - k^2 B_1))\big) \end{aligned}$$
as required. We note that $H_0(\rho) = O(\rho^{-1+2\theta})$ as $\rho \to 0$ from the limiting forms of $k(\rho)$ given in Lemma 6.1.4. Moreover, one can check that the remainder $r(\rho,\xi)$ acts as the Fourier transform of the product of a smooth (even Schwartz) function with $f_{\lambda+1+\alpha}$ for some $\alpha > 0$, and hence that
$$r(\rho,\xi) = O\big(\rho^{-1+2\theta}(k(\rho)|\xi|)^{-\lambda-1-\alpha-\frac{1}{2}}\big) \text{ as } \rho \to 0.$$
□

We note that taking the Fourier transform turns the partial differential operator $\tilde{\mathbf{L}}$ in ρ and v into an ordinary differential operator in ρ with parameter ξ. Indeed,

$$\mathcal{F}(\tilde{\mathbf{L}}g)(\rho,\xi) = \widehat{g}_{\rho\rho}(\rho,\xi) + k'(\rho)^2\xi^2\widehat{g}(\rho,\xi) + \varepsilon A_0(\rho)\widehat{g}_\rho(\rho,\xi) - \beta(\rho)\widehat{g}(\rho,\xi). \tag{6.3.6}$$

We wish to solve this ODE, at least formally, by the method of variation of parameters. In order to do this, we require the fundamental solutions of the operator $\mathcal{F}(\tilde{\mathbf{L}}\,\cdot)$. These fundamental solutions are, by definition, $\widehat{\chi}^\sharp$, $\widehat{\chi}^\flat$, satisfying:

$$\left\{\begin{matrix} \mathcal{F}(\tilde{\mathbf{L}}\chi^\sharp) = 0, & \mathcal{F}(\tilde{\mathbf{L}}\chi^\flat) = 0, \\ \widehat{\chi}^\sharp|_{\rho=0} = 0, & \widehat{\chi}^\flat|_{\rho=0} = 1, \\ \widehat{\chi}^\sharp_\rho|_{\rho=0} = 1, & \widehat{\chi}^\flat_\rho|_{\rho=0} = 0. \end{matrix}\right\}$$

We remark that inverting the Fourier transform gives

$$\left\{\begin{matrix} \tilde{\mathbf{L}}\chi^\sharp = 0, & \tilde{\mathbf{L}}\chi^\flat = 0, \\ \chi^\sharp|_{\rho=0} = 0, & \chi^\flat|_{\rho=0} = \delta_{v=0}, \\ \chi^\sharp_\rho|_{\rho=0} = \delta_{v=0}, & \chi^\flat_\rho|_{\rho=0} = 0. \end{matrix}\right\}$$

Lemma 6.3.3. *The fundamental solutions of the Fourier transformed equations are*

$$\begin{aligned} \widehat{\chi}^\sharp(\rho,\xi) &= \alpha_\sharp(\rho)(\xi k(\rho))^{-\nu} J_\nu(\xi k(\rho)), \\ \widehat{\chi}^\flat(\rho,\xi) &= \alpha_\sharp(\rho)\left(\frac{k(\rho)}{\xi}\right)^{-\nu} Y_\nu(\xi k(\rho)), \end{aligned} \tag{6.3.7}$$

where $\nu = \lambda + \frac{1}{2}$ and J_ν, Y_ν are the Bessel functions of order ν of first and second type respectively. The constant $c_\sharp > 0$ is chosen to satisfy the initial conditions.

Proof. This follows by an elementary calculation from the identities

$$\begin{aligned} \mathcal{C}'_\nu(y) &= \mathcal{C}_{\nu-1}(y) - \frac{\nu}{y}\mathcal{C}_\nu(y), \\ \mathcal{C}'_\nu(y) &= -\mathcal{C}_{\nu+1}(y) + \frac{\nu}{y}\mathcal{C}_\nu(y), \end{aligned}$$

for $\mathcal{C}_\nu(y) = J_\nu(y), Y_\nu(y)$. Then we are able to calculate (suppressing the argument $k(\rho)\xi$ of the function $\widehat{f}_\lambda$),

$$\begin{aligned} \widehat{\chi}^\sharp_{\rho\rho} + (k'(\rho)\xi)^2\widehat{\chi}^\sharp &= \alpha''_\sharp \widehat{f}_\lambda + 2\alpha'_\sharp\xi k'\widehat{f}_\lambda{}' + \alpha_\sharp\xi k''\widehat{f}_\lambda{}' + \alpha_\sharp(\xi k')^2\widehat{f}_\lambda{}'' + (k'\xi)^2\alpha_\sharp\widehat{f}_\lambda \\ &= \alpha''_\sharp \widehat{f}_\lambda + 2\alpha'_\sharp\xi k'\widehat{f}_\lambda{}' + \alpha_\sharp\xi k''\widehat{f}_\lambda{}' - \alpha_\sharp(\xi k')^2\frac{2\nu+1}{\xi k}\widehat{f}_\lambda{}' \\ &= \alpha''_\sharp \widehat{f}_\lambda + \widehat{f}_\lambda{}'\xi(2\alpha'_\sharp k' - \alpha_\sharp\frac{2(\lambda+1)}{k}(k')^2 + \alpha_\sharp k'') \\ &= \alpha''_\sharp \widehat{f}_\lambda - \alpha_\sharp k'\varepsilon A_0\xi\widehat{f}_\lambda{}' = -\varepsilon A_0\widehat{\chi}^\sharp_\rho + \beta(\rho)\widehat{\chi}^\sharp, \end{aligned}$$

as $\widehat{f}_\lambda(y) + \widehat{f}''_\lambda(y) = -\frac{2\nu+1}{y}\widehat{f}'_\lambda(y)$.
The calculation for $\widehat{\chi}^\flat$ is similar. □

We solve $\mathcal{F}(\tilde{\mathbf{L}}g)(\rho,\xi) = \mathcal{F}(\mathcal{S}(g))$ by the method of variation of parameters. First, we calculate the Wronskian of the fundamental solutions. Note (for example, from [60]) that the Wronskian

$$w(s,\xi) := Y_\nu(\xi k(s))J'_\nu(\xi k(s)) - J_\nu(\xi k(s))Y'_\nu(\xi k(s)) = \frac{2\nu}{\xi k(s)},$$

and so

$$\begin{aligned} W(s,\xi) &:= \big(\widehat{\chi}^\sharp(\xi k(s))\partial_s\widehat{\chi}^\flat(\xi k(s)) - \widehat{\chi}^\flat(\xi k(s))\partial_s\widehat{\chi}^\sharp(\xi k(s))\big)(s,\xi) \\ &= c_\sharp^2 e^{-\varepsilon\int_0^s A_0(\tau)d\tau}. \end{aligned}$$

The method of variation of parameters then gives that a particular solution to the ODE is given by:

$$\begin{aligned} \widehat{g}(\rho,\xi) &= \int_0^\rho \frac{\widehat{\chi}^\sharp(\rho,\xi)\widehat{\chi}^\flat(s,\xi) - \widehat{\chi}^\sharp(s,\xi)\widehat{\chi}^\flat(\rho,\xi)}{\widehat{\chi}_s^\sharp(s,\xi)\widehat{\chi}^\flat(s,\xi) - \widehat{\chi}^\sharp(s,\xi)\widehat{\chi}_s^\flat(s,\xi)}\mathcal{F}(\tilde{\mathbf{L}}g)(s,\xi)\,ds \\ &= \Big(\frac{k(\rho)}{k'(\rho)}\Big)^{\frac{1}{2}} e^{-\frac{\varepsilon}{2}\int_0^\rho A_0(\tau)\,d\tau}\int_0^\rho K(\rho,s;\xi)e^{\frac{\varepsilon}{2}\int_0^s A_0(\tau)\,d\tau}\Big(\frac{k(s)}{k'(s)}\Big)^{\frac{1}{2}}\mathcal{F}(\mathcal{S}(g))(s,\xi)\,ds, \end{aligned}$$

where

$$K(\rho,s;\xi) := Y_\nu(\xi k(\rho))J_\nu(\xi k(s)) - J_\nu(\xi k(\rho))Y_\nu(\xi k(s)).$$

We define a new integral kernel $\tilde{K}(\rho,s;\xi)$ by

$$\tilde{K}(\rho,s;\xi) = \Big(\frac{k(\rho)}{k'(\rho)}\Big)^{\frac{1}{2}}\Big(\frac{k(s)}{k'(s)}\Big)^{\frac{1}{2}}K(\rho,s;\xi)e^{-\frac{\varepsilon}{2}\int_0^\rho A_0(\tau)\,d\tau}e^{\frac{\varepsilon}{2}\int_0^s A_0(\tau)\,d\tau}. \tag{6.3.8}$$

We therefore look for a fixed point of

$$\widehat{g}(\rho,\xi) = \int_0^\rho \tilde{K}(\rho,s,\xi)\mathcal{F}(\mathcal{S}(g))\,ds.$$

We show via a fixed point argument that such a function $\widehat{g}(\rho,\xi)$ exists in Theorem 6.3.5. In preparation for the proof of this theorem, we make a few observations. To simplify notation and bounds later, we set

$$Q_{\pm\nu}(y) := \begin{cases} |y|^{\pm\nu} & \text{for } |y| \le 1, \\ |y|^{-\frac{1}{2}} & \text{for } |y| \ge 1, \end{cases}$$

$$R(y) := \begin{cases} 1 & \text{for } |y| \le 1, \\ 1/|y| & \text{for } |y| \ge 1. \end{cases}$$

Then $|J_\nu(y)| \le CQ_\nu(y)$, $|Y_\nu(y)| \le CQ_{-\nu}(y)$ for $y > 0$. So we may bound

$$|K(\rho,s;\xi)| \le CQ_\nu(\xi k(\rho))Q_\nu(\xi k(s))^{-1}R(\xi k(s)) \tag{6.3.9}$$

where $C > 0$ is independent of ρ, s and ξ for $0 \le s \le \rho \le \rho_M$, $\xi \in \mathbb{R}$.

The following lemma provides accurate estimates for various L^p and weighted L^p norms of the functions $K(\rho,s;\xi)$ and $\tilde{K}(\rho,s;\xi)$. These are a simple consequence of (6.3.9).

Lemma 6.3.4. *For $0 \leq s \leq \rho \leq \rho_M$ and $\xi \in \mathbb{R}$, the following identities and inequalities hold:*

$$\tilde{K}(\rho, \rho; \xi) = 0,$$
$$\tilde{K}(\rho, 0; \xi) = 0.$$

Also,

$$\begin{aligned}
\|K(\rho, s; \cdot)\|_{L^2} &\leq C\Big(\frac{k(\rho)}{k(s)}\Big)^{\nu} k(\rho)^{-\frac{1}{2}},\\
\|\tilde{K}(\rho, s; \cdot)\|_{L^2} &\leq C\rho^{1-\frac{\theta}{2}}.
\end{aligned}$$

Moreover, for $0 \leq \mu < \frac{1}{2}$,

$$\|\tilde{K}(\rho, s; \cdot)\|_{L^\infty} \leq C\rho, \tag{6.3.10}$$
$$\|\xi\tilde{K}(\rho, s; \xi)\|_{L^\infty_\xi} \leq C\rho^{1-\theta}, \tag{6.3.11}$$
$$\||\xi|^{\mu} K(\rho, s; \xi)\|_{L^2_\xi} \leq C\Big(\frac{k(\rho)}{k(s)}\Big)^{\nu} k(\rho)^{-\mu-\frac{1}{2}}, \tag{6.3.12}$$
$$\||\xi|^{\mu} \tilde{K}(\rho, s; \xi)\|_{L^2_\xi} \leq C\rho^{1-\frac{\theta}{2}-\mu\theta}. \tag{6.3.13}$$

Proof. These estimates are a consequence of (6.3.9). For example, we see the most complicated estimate (6.3.12) by bounding

$$\begin{aligned}
\frac{1}{2}\||\xi|^{\mu} K(\rho, s; \xi)\|^2_{L^2_\xi} \leq& \int_0^{\frac{1}{k(\rho)}} |\xi|^{2\mu}|K(\rho, s; \xi)|^2\, d\xi + \int_{\frac{1}{k(\rho)}}^{\frac{1}{k(s)}} |\xi|^{2\mu}|K(\rho, s; \xi)|^2\, d\xi\\
&+ \int_{\frac{1}{k(s)}}^{\infty} |\xi|^{2\mu}|K(\rho, s; \xi)|^2\, d\xi\\
\leq& C\int_0^{\frac{1}{k(\rho)}} |\xi|^{2\mu}\Big(\frac{k(\rho)}{k(s)}\Big)^{2\nu} d\xi + \int_{\frac{1}{k(\rho)}}^{\frac{1}{k(s)}} k(\rho)^{-1}k(s)^{-2\nu}|\xi|^{2\mu-2\nu-1}\, d\xi\\
&+ \int_{\frac{1}{k(s)}}^{\infty} (k(\rho)k(s))^{-1}|\xi|^{2\mu-2}\, d\xi\\
\leq& C\Big(\frac{k(\rho)}{k(s)}\Big)^{2}\nu k(\rho)^{-2\mu-1} + \Big(\frac{k(\rho)}{k(s)}\Big)^{2\nu} k(\rho)^{-2\mu-1} + k(\rho)^{-1}k(s)^{-2\mu}\\
\leq& C\Big(\frac{k(\rho)}{k(s)}\Big)^{2\nu} k(\rho)^{-2\mu-1}.
\end{aligned}$$

This implies the desired bound. □

With these bounds in hand, we are now in a position to state and prove the main theorem of this section. This theorem gives the existence of the remainder function $\widehat{g}(\rho, \xi)$, and therefore the existence of the entropy kernel itself.

Theorem 6.3.5 (Existence of the Entropy Kernel)**.** *For all $\gamma \in (1,3)$, there exists $\widehat{g} \in L^\infty(0,\rho_M; L^2(\mathbb{R}))$ that is a fixed point of*

$$\widehat{g}(\rho,\xi) = \int_0^\rho \tilde{K}(\rho,s,\xi)\mathcal{F}(\mathcal{S}(g))(s,\xi)\,ds. \tag{6.3.14}$$

The remainder $\widehat{g}(\rho,\xi)$ satisfies

$$\|\widehat{g}(\rho,\cdot)\|_{L^2} + \|\rho\widehat{g}_\rho(\rho,\cdot)\|_{L^2} \le C\rho^{1+\frac{3\theta}{2}} e^{\frac{\rho^{2\theta}}{2\theta}}. \tag{6.3.15}$$

Proof. To establish the existence of the kernel, we argue with a constructive fixed point scheme. Given the nth approximation $\widehat{g}^n(\rho,\xi)$, we construct $\widehat{g}^{n+1}(\rho,\xi)$ by setting

$$\begin{aligned}\widehat{g}^{n+1}(\rho,v) :=& \int_0^\rho \tilde{K}(\rho,s;\xi)\mathcal{F}\big(\mathcal{S}(g)\big)\,ds\\ =& \int_0^\rho \Big(\tilde{K}(\rho,s;\xi)\big(H_0(s)\widehat{f}_{\lambda+1}(k(s)\xi) + \varepsilon^2 r(s,\xi)\\ &\qquad - \mathcal{F}(\varepsilon(A-A_0)g^n_\rho - \varepsilon Bvg^n_v - \beta(s)g^n)\big)\Big)\,ds,\end{aligned}$$

and begin the procedure with $\widehat{g}^0(\rho,\xi) = 0$. To show that this scheme converges, we estimate the increment $\widehat{g}^{n+1} - \widehat{g}^n$. Set $\widehat{G}^{n+1} = \widehat{g}^{n+1} - \widehat{g}^n$. We then have, by linearity of the Fourier transform,

$$\widehat{G}^{n+1}(\rho,\xi) = \int_0^\rho \tilde{K}(\rho,s;\xi)\big(-\mathcal{F}(\varepsilon(A-A_0)G^n_s - \varepsilon BvG^n_v - \beta(s)G^n)\big)\,ds.$$

We first apply an integration by parts (recalling from Lemma 6.3.4 that $\tilde{K}(\rho,0;\xi) = \tilde{K}(\rho,\rho;\xi) = 0$) to see that

$$\begin{aligned}&\widehat{G}^{n+1}(\rho,\xi)\\ &= \int_0^\rho \tilde{K}(\rho,s;\xi)\Big(\mathcal{F}\big(-\varepsilon(A-A_0)G^n_s(s,v)\big) - \mathcal{F}(\varepsilon B(s,v)vG^n_v(s,v)) - \beta(s)\widehat{G}^n(s,v)\Big)\,ds\\ &= \int_0^\rho \tilde{K}(\rho,s;\xi)\Big(\mathcal{F}\big(\varepsilon(A-A_0)_sG^n - (\varepsilon BvG^n)_v + (\varepsilon Bv)_vG^n - \beta(s)G^n(s,\xi)\big)(s,\xi)\Big)\\ &\qquad + \tilde{K}_s(\rho,s;\xi)\mathcal{F}(\varepsilon(A-A_0)G^n)(s,\xi)\,ds.\end{aligned} \tag{6.3.16}$$

We recall that the Fourier transform of a product is the convolution of the Fourier transforms. We therefore use the bounds of Lemma 6.3.4 and Young's inequality for convolutions to estimate, for a typical term,

$$\begin{aligned}&\Big\|\int_0^\rho \tilde{K}(\rho,s;\xi)\mathcal{F}\big(\varepsilon(A-A_0)_sG^n\big)\,ds\Big\|_{L^2_\xi}\\ &\le \int_0^\rho \|\tilde{K}(\rho,s;\cdot)\|_{L^2}\|\mathcal{F}\big(\varepsilon(A-A_0)_s\big) * \mathcal{F}(G^n)\|_{L^\infty_\xi}\,ds\\ &\le \int_0^\rho \|\tilde{K}(\rho,s;\cdot)\|_{L^2}\|\varepsilon\mathcal{F}((A-A_0)_s)(s,\cdot)\|_{L^2}\|\widehat{G}^n(s,\cdot)\|_{L^2}\,ds\\ &\le C\varepsilon^2\int_0^\rho \rho^{1-\frac{\theta}{2}}s^{-2+\frac{5\theta}{2}}\|\widehat{G}^n(s,\cdot)\|_{L^2}\,ds,\end{aligned}$$

where we have estimated the L^2 norm of $\mathcal{F}((A-A_0)_s)$ by applying Plancherel's Theorem and estimating with the bounds of Lemma 6.3.1 and the compact support of the function to see

$$\|\varepsilon(A-A_0)_s\|^2_{L^2_v} \leq C\int_{-k(s)}^{k(s)} \varepsilon^4 s^{2\gamma-6}\,dv \leq C\varepsilon^4 s^{\theta+2\gamma-6} = C\varepsilon^2 s^{5\theta-4}.$$

Treating the other terms of (6.3.16) similarly, we find

$$\begin{aligned}
\|\widehat{G}^{n+1}(\rho,\cdot)\|_{L^2} \leq C\int_0^\rho \Big(&\|\tilde{K}(\rho,s;\cdot)\|_{L^2}\|\mathcal{F}(\varepsilon(A-A_0)_s)(s,\cdot)\|_{L^2}\|\widehat{G}^n(s,\cdot)\|_{L^2}\\
&+\|\tilde{K}_s(\rho,s,\cdot)\|_{L^2}\|\mathcal{F}(\varepsilon(A-A_0))(s,\cdot)\|_{L^2}\|\widehat{G}^n(s,\cdot)\|_{L^2}\\
&+\|\xi\tilde{K}(\rho,s;\xi)\|_{L^\infty_\xi}\|\mathcal{F}(\varepsilon BvG^n)(s,\xi)\|_{L^2_\xi}\\
&+\|\tilde{K}(\rho,s,\cdot)\|_{L^\infty}\|\mathcal{F}(\varepsilon\partial_v(Bv)G^n)(s,\cdot)\|_{L^2}\\
&+\|\tilde{K}(\rho,s;\cdot)\|_{L^\infty}|\beta(s)|\|\widehat{G}^n(s,\cdot)\|_{L^2}\Big)\,ds\\
\leq C\int_0^\rho \big(&\varepsilon^2\rho^{1-\frac{\theta}{2}}s^{-2+\frac{5\theta}{2}}+\varepsilon\rho^{1-\theta}s^{-2+3\theta}+\rho s^{-2+2\theta}\big)\|\widehat{G}^n(s,\cdot)\|_{L^2}\,ds.
\end{aligned}$$

In particular, we have obtained

$$\rho^{-1}\|\widehat{G}^{n+1}(\rho,\cdot)\|_{L^2} \leq C\int_0^\rho s^{-1+2\theta}s^{-1}\|\widehat{G}^n(s,\cdot)\|_{L^2}\,ds, \tag{6.3.17}$$

where $C>0$ is independent of ρ.

Before we consider $\widehat{G}^1(\rho,\xi)$, we note that, for $0<s\leq\rho$,

$$\begin{aligned}
&\Big\|Q_\nu(\xi k(\rho))Q_\nu(\xi k(s))^{-1}R(\xi k(s))\Big|\frac{J_{\nu+1}(\xi k(s))}{(\xi k(s))^{\nu+1}}\Big|\Big\|_{L^2_\xi}\\
&\leq C\Big(\int_0^{\frac{1}{k(\rho)}}\Big(\frac{k(\rho)}{k(s)}\Big)^{2\nu}d\xi+\int_{\frac{1}{k(\rho)}}^{\frac{1}{k(s)}}k(\rho)^{-1}k(s)^{-2\nu}|\xi|^{-2\nu-1}\,d\xi\\
&\quad+\int_{\frac{1}{k(s)}}^\infty k(\rho)^{-1}k(s)^{-2\nu-4}|\xi|^{-2\nu-5}\,d\xi\Big)^{\frac{1}{2}}\\
&\leq C\rho^{\frac{1-\theta}{2}}s^{-\frac{1}{2}}.
\end{aligned}\tag{6.3.18}$$

It follows from Proposition 6.3.2 that

$$\|\tilde{K}(\rho,s;\xi)\varepsilon^2 r(s,\xi)\|_{L^2_\xi} \leq \varepsilon^2 C\|\tilde{K}(\rho,s;\xi)H_0(s)\widehat{f}_{\lambda+1}(k(s)\xi)\|_{L^2_\xi}.$$

Hence we calculate, using the bound $|H_0(\rho)|\leq C\rho^{-1+2\theta}$ from Proposition 6.3.2 and (6.3.18),

$$\begin{aligned}
\|\widehat{G}^1(\rho,\cdot)\|_{L^2} &\leq C\Big\|\Big(\frac{k(\rho)}{k'(\rho)}\Big)^{\frac{1}{2}}\int_0^\rho K(\rho,s;\xi)\Big(\frac{k(s)}{k'(s)}\Big)^{\frac{1}{2}}H_0(s)\frac{J_{\nu+1}(\xi k(s))}{(\xi k(s))^{\nu+1}}\,ds\Big\|_{L^2_\xi}\\
&\leq C\Big\|\rho^{\frac{1}{2}}\int_0^\rho Q_\nu(\xi k(\rho))Q_\nu(\xi k(s))^{-1}R(\xi k(s))s^{-\frac{1}{2}+2\theta}\Big|\frac{J_{\nu+1}(\xi k(s))}{(\xi k(s))^{\nu+1}}\Big|\,ds\Big\|_{L^2_\xi}\\
&\leq C\rho^{1-\frac{\theta}{2}}\int_0^\rho s^{-1+2\theta}\,ds,
\end{aligned}$$

which implies

$$\rho^{-1}\|\widehat{G}^1(\rho,\cdot)\|_{L^2} \le C\rho^{\frac{3\theta}{2}}. \tag{6.3.19}$$

Thus, combining (6.3.17) and (6.3.19), we may bound iteratively

$$\begin{aligned}
\rho^{-1}\|\widehat{G}^{n+1}(\rho,\xi)\|_{L^2_\xi} &\le \int_0^\rho s_1^{-1+2\theta} s_1^{-1}\|\widehat{G}^n(s_1,\cdot)\|_{L^2}\,ds_1\\
&\le C\int_0^\rho\int_0^{s_1}\cdots\int_0^{s_{n-1}}(s_n\cdots s_1)^{-1+2\theta}s_n^{-1}\|\widehat{G}^1(s_n,\cdot)\|_{L^2}\,ds_n\cdots ds_1\\
&= \rho^{\frac{3\theta}{2}+2n\theta}\prod_{i=1}^n \frac{1}{\frac{3\theta}{2}+2i\theta}\\
&\le C\frac{\left(\frac{1}{2\theta}\right)^n}{n!}\rho^{\frac{3\theta}{2}+2n\theta}.
\end{aligned}$$

Hence we obtain

$$\|\widehat{G}^{n+1}(\rho,\xi)\|_{L^2_\xi} \le C\frac{\left(\frac{1}{2\theta}\right)^n}{n!}\rho^{1+\frac{3\theta}{2}+2n\theta}.$$

In particular, $\widehat{g}(\rho,\xi) := \lim_{n\to\infty}\widehat{g}^n(\rho,\xi) = \lim_{n\to\infty}\sum_{k=1}^n \widehat{G}^k(\rho,\xi)$ exists and satisfies

$$\|\widehat{g}(\rho,\cdot)\|_{L^2} \le C\sum_{n=0}^\infty \frac{\left(\frac{1}{2\theta}\right)^n}{n!}\rho^{1+\frac{3\theta}{2}+2n\theta} = C\rho^{1+\frac{3\theta}{2}}e^{\frac{\rho^{2\theta}}{2\theta}}.$$

It remains to show that the function $\widehat{g}$ obtained here is the desired fixed point. Although this argument is standard, we include it for completeness.

We define an operator $T_{\tilde{K}}$ acting on functions $f \in L^\infty(0,\rho_M;L^2(\mathbb{R}))$ by setting

$$\mathcal{F}(T_{\tilde{K}}f)(\rho,\xi) := \int_0^\rho \tilde{K}(\rho,s;\xi)\mathcal{F}(\mathcal{S}(f))(s,\xi)\,ds.$$

We want to show that $\mathcal{F}(T_{\tilde{K}}g) = \widehat{g}$. Note

$$\|\widehat{g}(\rho,\cdot)\|_{L^2} \le C\rho_M^{1+\frac{3\theta}{2}}e^{\frac{\rho_M^{2\theta}}{2\theta}},$$

hence $\widehat{g} \in L^\infty(0,\rho_M;L^2(\mathbb{R}))$.
Next, we observe

$$\|\widehat{g}-\mathcal{F}(T_{\tilde{K}}g)\|_{L^\infty(0,\rho_M;L^2)} \le \|\widehat{g}-\widehat{g}^{n+1}\|_{L^\infty(0,\rho_M;L^2)} + \|\mathcal{F}(T_{\tilde{K}}g^n)-\mathcal{F}(T_{\tilde{K}}g)\|_{L^\infty(0,\rho_M;L^2)},$$

as $\widehat{g}^{n+1} = \mathcal{F}(T_{\tilde{K}}g^n)$ by construction. The first term converges to 0 by construction as $n\to\infty$. By definition of $T_{\tilde{K}}$ and the fact that

$$g - g^n = \sum_{k=n+1}^\infty G^k,$$

we have

$$\begin{aligned}
&\|\mathcal{F}(T_{\tilde{K}}g^n) - \mathcal{F}(T_{\tilde{K}}g)\|_{L^\infty(0,\rho_M;L^2)} \\
&= \Big\| \int_0^\rho \tilde{K}(\rho,s;\xi)\mathcal{F}\big(\varepsilon(A-A_0)(g^n-g)_\rho + \varepsilon Bv(g^n-g)_v + \beta(s)(g^n-g)\big)ds\Big\|_{L^\infty(0,\rho_M;L^2)} \\
&= \Big\| \int_0^\rho \tilde{K}(\rho,s;\xi)\big(\mathcal{F}(\varepsilon(A-A_0)(\sum_{k=n+1}^\infty \widehat{G}^k)_\rho \\
&\qquad\qquad + \varepsilon Bv(\sum_{k=n+1}^\infty \widehat{G}^k)_v) + \mathcal{F}(\beta(s)(\sum_{k=n+1}^\infty \widehat{G}^k))\big)\, ds\Big\|_{L^\infty(0,\rho_M;L^2)} \\
&\le C\int_0^{\rho_M} s^{-1+2\theta}s^{-1}\big\|\sum_{k=n+1}^\infty \widehat{G}^k(s,\cdot)\big\|_{L^2}\, ds \to 0 \text{ as } n\to\infty,
\end{aligned}$$

where we have used the argument from the first part of the proof. Thus $\widehat{g} = \mathcal{F}(T_{\tilde{K}}g)$ as desired.

Finally, we remark that differentiating (6.3.14) with respect to ρ gives

$$\rho\widehat{g}_\rho(\rho,\xi) = \int_0^\rho \rho\tilde{K}_\rho(\rho,s,\xi)\mathcal{F}(\mathcal{S}(g))(s,\xi)\, ds,$$

which is of the same form as (6.3.14). An analogous argument therefore gives the second part of (6.3.15). This concludes the proof of Theorem 6.3.5. $\square$

Remark 6.3.6. A totally analogous argument yields $\xi\widehat{g}_\xi \in L^2_\xi$ with

$$\|\xi\widehat{g}_\xi\|_{L^2_\xi} \le C\rho^{1+\frac{3\theta}{2}}e^{\frac{\rho_M^{2\theta}}{2\theta}}.$$

Remark 6.3.7. From the proof of Theorem 6.3.5 and [9, Theorem 2.2], we deduce that, in (k,v) coordinates, we may bound $|\chi(k,v)-\chi^*(k,v)|$, where χ^* is the classical entropy kernel of [9, Theorem 2.2], by

$$|\chi(k,v)-\chi^*(k,v)| \le C\varepsilon\chi^*(k,v),$$

where C depends on ρ_M. Thus, for ε sufficiently small,

$$\chi(k,v) \ge \frac{1}{2}\chi^*(k,v) > 0$$

in the interior of its support, the set $\{|v|^2 \le k^2\}$. For details, we refer to Lemma 6.8.2.

Theorem 6.3.8 (Regularity of $g(\rho,v)$)**.** *The remainder $g = g(\rho,v)$ is such that $\partial_v^\mu g$ is Hölder continuous in (ρ,v) for $\rho>0$ for all μ with $0\le\mu<\lambda+2$. In addition, if $0<\beta<\mu$, we may bound, for all ρ with $0\le\rho\le\rho_M$, $v\in\mathbb{R}$,*

$$|\partial_v^\beta g(\rho,v)| \le C\rho^{1+\theta-2\mu\theta+\beta\theta}[k(\rho)^2-v^2]_+^{\mu-\beta}.$$

Proof. Recalling the definition of fractional derivative, (6.2.7), we see that the fractional derivative of order μ of the remainder function $g(\rho, v)$ may be bounded by

$$|\partial_v^\mu g(\rho, v)| \leq C \int_{\mathbb{R}} |\xi|^\mu |\widehat{g}(\rho, \xi)|\, d\xi.$$

For $0 \leq \mu < \frac{1}{2}$, we apply the same method as in the proof of Theorem 6.3.5 with the bounds of Lemma 6.3.4 to see that

$$\begin{aligned}
&\int_{\mathbb{R}} |\xi|^\mu |\widehat{g}(\rho, \xi)| \\
\leq & C \int_{\mathbb{R}} |\xi|^\mu \int_0^\rho \left| \tilde{K}(\rho, s; \xi) \Big(H_0(s) \Big| \frac{J_{\nu+1}(\xi k(s))}{(\xi k(s))^{\nu+1}} \Big| - \mathcal{F}\big(\varepsilon(A - A_0) g_s - \varepsilon B v g_v - \beta(s) g\big) \Big) \right| ds \\
\leq & C \int_0^\rho \| |\xi|^\mu \tilde{K}(\rho, s; \xi) \|_{L^2_\xi} \Big(\| H_0(s) \left| \frac{J_{\nu+1}(\xi k(s))}{(\xi k(s))^{\nu+1}} \right| \|_{L^2_\xi} + \varepsilon \| (A - A_0)_s \|_{L^\infty_v} \| \widehat{g} \|_{L^2_\xi} \\
& \qquad + |\beta(s)| \| \widehat{g} \|_{L^2_\xi} + \varepsilon \| B(s, v) \|_{L^\infty_v} (\| \widehat{g} \|_{L^2_\xi} + \| \xi \widehat{g}_\xi \|_{L^2_\xi}) \Big) \, ds \\
& + C \int_0^\rho \| |\xi|^\mu \tilde{K}_s(\rho, s; \xi) \|_{L^2_\xi} \| A - A_0 \|_{L^\infty_v} \| \widehat{g} \|_{L^2_\xi} \, ds \\
\leq & C \int_0^\rho \rho^{1 - \frac{\theta}{2} - \mu\theta} (s^{-2+2\theta} s^{1+2\theta+\frac{3\theta}{2}} + \varepsilon s^{-2+2\theta} s^{1+2\theta+\frac{3\theta}{2}} + s^{-1+\frac{3\theta}{2}}) \, ds \\
\leq & C \rho^{1+\theta-\mu\theta}.
\end{aligned}$$

Moreover, we observe that, for $0 \leq \mu < \frac{1}{2}$,

$$\int_{\mathbb{R}} |\xi|^\mu |\xi \widehat{g}_\xi(\rho, \xi)| \, d\xi \leq C \rho^{1+\theta-\mu\theta}.$$

To extend these inequalities to $\mu \geq \frac{1}{2}$, we note the following. For all $0 \leq \mu < \nu + \frac{3}{2}$,

$$\left\| |\xi|^{\mu-1} \left| \frac{J_{\nu+1}(\xi k(s))}{(\xi k(s))^{\nu+1}} \right| \right\|_{L^1_\xi} \leq s^{1+2\theta-\mu\theta}.$$

Distributing powers of ξ appropriately within the integral, it is then straightforward to estimate as above and obtain

$$|\partial_v^\mu g(\rho, v)| \leq C \rho^{1+\theta-\mu\theta}$$

for all $0 < \mu < \lambda + 2$. Similarly, we obtain

$$\rho |\partial_\rho \partial_v^\mu g(\rho, v)| \leq C \rho^{1+\theta-\mu\theta}$$

for μ in the same region. Hence by the standard embedding of the weighted Sobolev space $W^{1,p}_\rho \subset C^{0,\alpha}_\rho$, we have that $\partial_v^\mu g(\rho, v)$ is Hölder continuous.

To conclude the proof of Theorem 6.3.8, we observe that the function $[1 - z^2]_+^\mu$, for $z = \frac{v}{k(\rho)}$, is positive on the support of $g(\rho, v)$. We have

$$|\partial_z^\mu g(\rho, v)| = k(\rho)^\mu |\partial_v^\mu g(\rho, v)|.$$

Moreover, by the Hölder continuity just proved, we may bound

$$|\partial_z^\beta g| \leq \sup |\partial_z^\mu g|[1-z^2]_+^{\mu-\beta}.$$

We then calculate, for $0 < \beta < \mu$,

$$\begin{aligned}|\partial_v^\beta g(\rho,v)| &= Ck(\rho)^{-\beta}|\partial_z^\beta g(\rho,v)| \\ &\leq C\rho^{-\beta\theta}\sup_z |\partial_z^\mu g(\rho,v)|[1-z^2]_+^{\mu-\beta} \\ &\leq C\rho^{-\beta\theta+\mu\theta+1+\theta-\mu\theta}[1-z^2]_+^{\mu-\beta} \\ &\leq C\rho^{\beta\theta-\mu\theta+1+\theta-\mu\theta}[k(\rho)^2-v^2]_+^{\mu-\beta}.\end{aligned}$$

This concludes the proof of the theorem, and hence also of Theorem 6.2.2. □

6.4 Existence and regularity of the entropy flux kernel

Having proven Theorem 6.2.2, thereby demonstrating the existence of the entropy kernel, we now move onto the entropy flux kernel and Theorem 6.2.4.

To this end, we consider the expressions derived for general entropy-entropy flux pairs:

$$\begin{aligned} q_\rho &= \frac{u(1-\varepsilon p')}{1-p'\varepsilon^2u^2}\eta_\rho + \frac{p'(1-\varepsilon u^2)^2}{(1-p'\varepsilon^2u^2)(\rho+\varepsilon p)}\eta_u, \\ q_u &= \frac{\rho+\varepsilon p}{1-p'\varepsilon^2u^2}\eta_\rho + \frac{u(1-\varepsilon p')}{1-p'\varepsilon^2u^2}\eta_u. \end{aligned}$$

Changing variables to (ρ, v) and setting $\tilde{u} := \frac{u(1-\varepsilon p')}{1-p'\varepsilon^2u^2}$, $\tilde{\rho} := \frac{(\rho+\varepsilon p)(1-\varepsilon u^2)}{1-p'\varepsilon^2u^2}$, this becomes

$$\begin{aligned} q_\rho &= \tilde{u}\eta_\rho + \tilde{\rho}k'^2\eta_v, \\ q_v &= \tilde{\rho}\eta_\rho + \tilde{u}\eta_v. \end{aligned}$$

Lemma 6.4.1. *The functions $\tilde{\rho}(\rho,v)$ and $\tilde{u}(\rho,v)$ may be expanded as*

$$\tilde{\rho}(\rho,v) = \rho_0(\rho) + \varepsilon\rho_1(\rho)(k(\rho)^2-v^2) + \varepsilon^2\rho_2(\rho,v)(k(\rho)^2-v^2)^2, \tag{6.4.1}$$

$$\tilde{u}(\rho,v) = v\big(u_0(\rho) + \varepsilon u_1(\rho)(k(\rho)^2-v^2) + \varepsilon^2u_2(\rho,v)(k(\rho)^2-v^2)^2\big), \tag{6.4.2}$$

where

$$\rho_0(\rho) = \frac{(\rho+\varepsilon p)(1-\varepsilon u(k)^2)}{1-p'\varepsilon^2u(k)^2}, \tag{6.4.3}$$

$$u_0(\rho) = \frac{u(k)(1-\varepsilon p')}{k(1-p'\varepsilon^2u(k)^2)}, \tag{6.4.4}$$

and

$$\begin{aligned} &|\rho_1(\rho)| + |\rho_2(\rho,v)| \leq C(\rho+\varepsilon p(\rho)), \\ &|u_1(\rho)| + |u_2(\rho,v)| \leq C, \end{aligned} \tag{6.4.5}$$

where $C > 0$ does not depend on $\varepsilon > 0$.

The proof of this lemma is by taking a Taylor expansion in v^2 around $k(\rho)^2$, as in Lemma 6.3.1.

We now need to derive an equation for the entropy flux kernel. We recall the operator $\mathbf{L} = \partial_{\rho\rho} - k'^2\partial_{vv} + \varepsilon A\partial_\rho + \varepsilon Bv\partial_v$ and observe that, by definition, $\varepsilon A = \frac{\tilde{\rho}_\rho - \tilde{u}_v}{\tilde{\rho}}$, $\varepsilon Bv = \frac{\tilde{u}_\rho - k'^2\tilde{\rho}_v}{\tilde{\rho}}$. We may therefore calculate

$$\begin{aligned}\mathbf{L}(\sigma - \tilde{u}\chi) =& \sigma_{\rho\rho} - k'^2\sigma_{vv} + \varepsilon A\sigma_\rho + \varepsilon Bv\sigma_v - 2\tilde{u}_\rho\chi_\rho - \tilde{u}\chi_{\rho\rho} + 2k'^2\tilde{u}_v\chi_v \\ &+ k'^2\tilde{u}\chi_{vv} - \varepsilon A\tilde{u}\chi_\rho - \varepsilon Bv\tilde{u}\chi_v - \mathbf{L}(\tilde{u})\chi \\ =& -\tilde{u}_\rho\chi_\rho + 2k'k''\tilde{\rho}\chi_v + k'^2\tilde{\rho}_\rho\chi_v - k'^2\tilde{\rho}_v\chi_\rho + k'^2\tilde{u}_v\chi_v \\ &+ \varepsilon Ak'^2\tilde{\rho}\chi_v + \varepsilon Bv\tilde{\rho}\chi_\rho - \mathbf{L}(\tilde{u})\chi \\ =& 2k'k''\tilde{\rho}\chi_v + 2k'^2\tilde{\rho}_\rho\chi_v - 2k'^2\tilde{\rho}_v\chi_\rho - \mathbf{L}(\tilde{u})\chi \\ =& \tilde{F}(\rho, v)\end{aligned} \tag{6.4.6}$$

as claimed in (6.2.12). Setting $F(\rho, v) = \tilde{F}(\rho, v) + \mathbf{L}(\tilde{u}\chi)$, we derive (6.2.11).

6.4.1 Coefficients for entropy flux kernel

As we did with the entropy kernel in §6.3.1, we derive expressions for the coefficients of the entropy flux kernel, $b_1(\rho)$ and $b_2(\rho)$, in the present section.

Beginning by expanding $\tilde{F}(\rho, v)$ in coefficients of $G_\lambda(\rho, v)$ (recall from §6.3.1 that $G_\lambda(\rho, v) = [k(\rho)^2 - v^2]_+^\lambda$), we see that the most singular term is

$$\begin{aligned}&G_{\lambda-1}(\rho, v)\big(-4\lambda vk'k''\rho_0a_1 - 4\lambda vk'^2(\rho_0' + 2\varepsilon kk'\rho_1)a_1 + 8\lambda\varepsilon vkk'^3\rho_1a_1\big) \\ &\qquad = -4\lambda vk'(k'\rho_0)'a_1G_{\lambda-1}(\rho, v).\end{aligned}$$

To calculate the next most singular term, we need an expression for $\big(\mathbf{L}\tilde{u}\big)\chi$. Applying $\mathbf{L}$ to $\tilde{u}$ and expanding in powers of $(k(\rho)^2 - v^2)$ by Lemma 6.4.1 gives us that the lowest order term of $\mathbf{L}\tilde{u}$ is $v\omega(\rho)$, where

$$\omega(\rho) = u_0'' + 8\varepsilon k'^2u_1 + 2\varepsilon kk''u_1 + 4\varepsilon kk'u_1' + \varepsilon A_0u_0' + 2\varepsilon^2A_0kk'u_1 + \varepsilon B_0u_0 - 2\varepsilon^2B_0k^2u_1.$$

Hence, we calculate that the next most singular term in $\mathbf{L}(\sigma - \tilde{u}\chi)$ is

$$\begin{aligned}&G_\lambda(\rho, v)\big(-4(\lambda+1)vk'k''\rho_0a_2 - 4(\lambda+1)vk'^2(\rho_0' + 2\varepsilon kk'\rho_1)a_2 - 4\lambda\varepsilon vk'k''\rho_1a_1 \\ &\qquad - 4\lambda\varepsilon vk'^2\rho_1'a_1 + 4\varepsilon vk'^2\rho_1a_1' + 8(\lambda+1)\varepsilon vkk'^3\rho_1a_2 - v\omega a_1\big) \\ &= -vG_\lambda(\rho, v)\big(4(\lambda+1)k'(k'\rho_0)'a_2 + 4\lambda\varepsilon k'(k'\rho_1)'a_1 - 4\varepsilon k'^2\rho_1a_1' + \omega a_1\big).\end{aligned}$$

Thus we see that

$$\begin{aligned}
&\mathbf{L}(\sigma-\tilde{u}\chi)(\rho,v)\\
&\quad=-4\lambda v k'(k'\rho_0)'a_1G_{\lambda-1}(\rho,v)\\
&\qquad-vG_\lambda(\rho,v)\big(4(\lambda+1)k'(k'\rho_0)'a_2+4\lambda\varepsilon k'(k'\rho_1)'a_1-4\varepsilon k'^2\rho_1a_1'+\omega a_1\big)\\
&\qquad+v\tilde{f}(\rho,v),
\end{aligned}\tag{6.4.7}$$

where $\tilde{f}(\rho,v)$ is a more regular term satisfying the bound

$$|\tilde{f}(\rho,v)|\le C\rho^{2\theta-2}G_{\lambda+1}.$$

We note, moreover, from the asymptotics for $k(\rho)$ in Lemma 6.1.4 and the bounds for $a_1(\rho)$ and $a_2(\rho)$ in Theorem 6.2.2 that the coefficients of $G_{\lambda-1}(\rho,v)$ and $G_\lambda(\rho,v)$ also satisfy bounds of the form $C\rho^{2\theta-2}$.

We recall that we are looking for the entropy-flux kernel in the form

$$(\sigma-\tilde{u}\chi)(\rho,v)=-v\big(b_1(\rho)G_\lambda(\rho,v)+b_2(\rho)G_{\lambda+1}(\rho,v)\big)+h(\rho,v).$$

We write $b_i(\rho)=W_i(\rho)a_i(\rho)$ for $i=1,2$, in the following to emphasise the relation of the coefficients $b_i(\rho)$ to $a_i(\rho)$. Applying the operator $\mathbf{L}$ to this ansatz, we obtain

$$\begin{aligned}
&\mathbf{L}(\sigma-\tilde{u}\chi)\\
&=G_{\lambda-1}(\rho,v)\big(-4\lambda vW_1'a_1kk'-4\lambda vW_1a_1k'^2\big)\\
&\quad+G_\lambda(\rho,v)\big(-vW_1''a_1-2vW_1'a_1'-4(\lambda+1)vW_2'a_2kk'-4(\lambda+1)vW_2a_2k'^2\\
&\qquad\qquad+v(W_2-W_1)a_1''-\varepsilon A_0vW_1'a_1-\varepsilon B_0vW_1a_1+\varepsilon A_0v(W_2-W_1)a_1'\\
&\qquad\qquad+2\lambda\varepsilon B_0v(W_2-W_1)a_1-2\lambda\varepsilon^2vkk'A_1a_1(W_1-W_2)\\
&\qquad\qquad+2\lambda\varepsilon^2v(W_1-W_2)B_1a_1k^2\big)\\
&\quad+G_{\lambda+1}(\rho,v)\big(-vW_2''a_2-2vW_2'a_2'-vW_2a_2''-2\lambda\varepsilon^3A_2vW_1a_1kk'\\
&\qquad\qquad+2\lambda\varepsilon^3B_2v^3W_1a_1-\varepsilon(A-A_0)vW_1'a_1-\varepsilon(A-A_0)vW_1a_1'\\
&\qquad\qquad-2\varepsilon(A-A_0)(\lambda+1)vW_2a_2kk'-2\lambda\varepsilon(B-B_0)vW_1a_1\\
&\qquad\qquad+2(\lambda+1)\varepsilon(B-B_0)v^3W_2a_2-\varepsilon AvW_2'a_2-\varepsilon AvW_2a_2'\\
&\qquad\qquad-\varepsilon BvW_2a_2-2(\lambda+1)\varepsilon B_0vW_2a_2\big)\\
&\quad+\mathbf{L}(h).
\end{aligned}\tag{6.4.8}$$

Matching coefficients in the $G_{\lambda-1}(\rho,v)$ terms in (6.4.7) and (6.4.8), $W_1(\rho)$ must solve

$$(kW_1)'(\rho)=(k'\rho_0)'(\rho).$$

This has the solution

$$W_1(\rho)=\frac{k'(\rho)}{k(\rho)}\rho_0(\rho).\tag{6.4.9}$$

For the coefficient W_2, comparing the next most singular terms yields

$$W_2'(\rho) + \frac{k'(\rho)}{k(\rho)}W_2(\rho) - \frac{\tilde{W}(\rho)}{4(\lambda+1)k(\rho)k'(\rho)a_2(\rho)}W_2(\rho) = \frac{\tilde{\Omega}(\rho)}{4(\lambda+1)k(\rho)k'(\rho)a_2(\rho)},$$

where $\tilde{W}(\rho)$ is defined as in (6.3.2) and $\tilde{\Omega}$ is given by

$$\begin{aligned}\tilde{\Omega} = &- W_1''a_1 - 2W_1'a_1' - W_1\tilde{W} - \varepsilon A_0 W_1' a_1 - \varepsilon B_0 W_1 a_1 \\ &+ 4(\lambda+1)k'(k'\rho_0)'a_2 + 4\lambda\varepsilon k'(k'\rho_1)'a_1 - 4\varepsilon k'^2\rho_1 a_1' + \omega a_1.\end{aligned} \tag{6.4.10}$$

From the expressions for $W_1, \tilde{u}, \tilde{\rho}$ in (6.4.9) and Lemma 6.4.1, and applying once again Lemma 6.1.4, we see that $\tilde{\Omega}(\rho) = O\left(\rho^{2\theta-2}\right)$.

From the expression for $a_2(\rho)$, (6.3.4), we see that

$$-\frac{\tilde{W}(\rho)}{4(\lambda+1)k(\rho)k'(\rho)a_2(\rho)} = \frac{d}{d\rho}\Big(\log\int_0^\rho e^{-\tilde{a}(\tau)}k(\tau)^\lambda k'(\tau)^{-\frac{1}{2}}\tilde{W}(\tau)\,d\tau\Big).$$

Now we define an integrating factor

$$\begin{aligned}I(\rho) &= \exp\Big(\int_0^\rho \frac{k'(s)}{k(s)} + \frac{d}{ds}\Big(\log\int_0^s e^{-\tilde{a}(\tau)}k(\tau)^\lambda k'(\tau)^{-\frac{1}{2}}\tilde{W}(\tau)\,d\tau\Big)\,ds\Big)\\ &= k(\rho)\int_0^\rho e^{-\tilde{a}(\tau)}k(\tau)^\lambda k'(\tau)^{-\frac{1}{2}}\tilde{W}(\tau)\,d\tau.\end{aligned}$$

One may check $I(\rho) = O(\rho^\theta)$ as $\rho \to 0$. Thus $I(\rho)\frac{\tilde{\Omega}(\rho)}{4(\lambda+1)k(\rho)k'(\rho)a_2(\rho)} = O(\rho^{-1+\theta})$ as $\rho \to 0$, and hence is integrable. We therefore solve to find

$$W_2(\rho) = I(\rho)^{-1}\int_0^\rho I(s)\frac{\tilde{\Omega}}{4(\lambda+1)kk'a_2}(s)\,ds \tag{6.4.11}$$

and it is simple to verify that $|W_2(\rho)| \le C$, where C depends on ρ_M.

6.4.2 Proof of existence and regularity of the entropy flux kernel

As we did for the remainder in the expansion for the entropy kernel, we derive an equation for the quantity $\tilde{\mathbf{L}}h$, where $\tilde{\mathbf{L}}$ is defined as in (6.3.5).

Proposition 6.4.2. *The remainder $h(\rho, v)$ satisfies*

$$\begin{aligned}\mathcal{F}(\tilde{\mathbf{L}}h)(\rho,\xi) =& \mathcal{F}(\mathcal{T}(g))(\rho,\xi)\\ :=& - \mathcal{F}(\varepsilon(A-A_0)g_\rho) - \mathcal{F}(\varepsilon B v g_v) - \beta(\rho)\hat{g}\\ &+ H_1(\rho)k(\rho)\xi\widehat{f_{\lambda+2}}(k(\rho)\xi) + \varepsilon^2 s(\rho,\xi),\end{aligned}$$

where $H_1(\rho) = O(\rho^{-1+2\theta})$ as $\rho \to 0$ and $s(\rho,\xi) = O\big(\rho^{-1+2\theta}(k(\rho)|\xi|)^{-\lambda-1-\alpha-\frac{1}{2}}\big)$ as $\xi \to \infty$, for some $\alpha > 0$.

In particular, the remainder $s(\rho,\xi)$ acts asymptotically like $\widehat{f}_{\lambda+1+\alpha}(k(\rho)\xi)$ as $\xi \to \infty$.

Proof. We first observe that

$$\tilde{\mathbf{L}}h = \mathbf{L}h - \varepsilon(A - A_0)h_\rho - \varepsilon B v h_v - \beta(\rho)h$$

by definition. From (6.4.7) and (6.4.8), we see

$$\begin{aligned}
\mathbf{L}h =& \mathbf{L}(\sigma - \tilde{u}\chi) - \mathbf{L}\big(-v(b_1(\rho)G_\lambda(\rho,v) + b_2(\rho)G_{\lambda+1}(\rho,v))\big) \\
=& \tilde{F}(\rho,v) - \mathbf{L}\big(-v(b_1(\rho)G_\lambda(\rho,v) + b_2(\rho)G_{\lambda+1}(\rho,v))\big) \\
=& v\tilde{f}(\rho,v) \\
&- G_{\lambda+1}(\rho,v)\big(-vW_2''a_2 - 2vW_2'a_2' - vW_2a_2'' - 2\lambda\varepsilon^3 A_2 v W_1 a_1 k k' \\
&\qquad + 2\lambda\varepsilon^3 B_2 v^3 W_1 a_1 - \varepsilon(A - A_0)vW_1'a_1 - \varepsilon(A - A_0)vW_1 a_1' \\
&\qquad - 2\varepsilon(A - A_0)(\lambda+1)vW_2 a_2 k k' - 2\lambda\varepsilon(B - B_0)vW_1 a_1 \\
&\qquad + 2(\lambda+1)\varepsilon(B - B_0)v^3 W_2 a_2 - \varepsilon A v W_2' a_2 - \varepsilon A v W_2 a_2' \\
&\qquad - \varepsilon B v W_2 a_2 - 2(\lambda+1)\varepsilon B_0 v W_2 a_2\big).
\end{aligned}$$

Before taking a Fourier transform, we note that $\frac{\partial}{\partial y}\widehat{f}_{\lambda+1}(y) = -y\widehat{f}_{\lambda+2}(y)$ by properties of Bessel functions. Therefore

$$\begin{aligned}
\mathcal{F}(vG_{\lambda+1}(\rho,v))(\rho,\xi) =& k(\rho)^{2\lambda+2}\frac{\partial}{\partial\xi}\mathcal{F}\big(f_{\lambda+1}(v/k(\rho))\big) \\
=& k(\rho)^{2\lambda+3}\frac{\partial}{\partial\xi}\widehat{f}_{\lambda+1}(k(\rho)\xi) \\
=& k(\rho)^{2\lambda+4}\widehat{f}'_{\lambda+1}(k(\rho)\xi) \\
=& -k(\rho)^{2\lambda+4}(k(\rho)\xi)\widehat{f}_{\lambda+2}(k(\rho)\xi).
\end{aligned}$$

This implies that after taking Fourier transform:

$$\mathcal{F}(\mathbf{L}h) = H_1(\rho)k(\rho)\xi\widehat{f}_{\lambda+2}(k(\rho)\xi) + \varepsilon^2 s(\rho,\xi),$$

where $|H_1(\rho)| \le C\rho^{-1+2\theta}$ and $s(\rho,\xi) = O(\rho^{-1+2\theta}(k(\rho)|\xi|)^{-\lambda-1-\alpha-\frac{1}{2}})$ for some $\alpha > 0$, analogously to $r(\rho,\xi)$ determined in Proposition 6.3.2. Therefore, we obtain

$$\mathcal{F}(\tilde{\mathbf{L}}h) = \mathcal{T}(h),$$

where we have defined

$$\mathcal{T}(h) = H_1(\rho)k(\rho)\xi\widehat{f}_{\lambda+2}(k(\rho)\xi) + \varepsilon^2 s(\rho,\xi) - \mathcal{F}\big(\varepsilon(A - A_0)h_\rho + \varepsilon B v h_v + \beta(\rho)h\big).$$

□

Observing that

$$k(\rho)|\xi\widehat{f}_{\lambda+2}(k(\rho)\xi)| \leq \begin{cases} C & \text{if } k(\rho)|\xi| \leq 1, \\ C|k(\rho)\xi|^{-\lambda-\frac{3}{2}} & \text{if } k(\rho)|\xi| > 1, \end{cases}$$

i.e. $k(\rho)\xi\widehat{f}_{\lambda+2}(k(\rho)\xi)$ satisfies the same bounds as $\widehat{f}_{\lambda+1}(k(\rho)\xi)$, we may prove the following theorem analogously to Theorems 6.3.5 and 6.3.8.

Theorem 6.4.3 (Existence of the Entropy Flux Kernel)**.** *For $\gamma \in (1,3)$, there exists $\widehat{h} \in L^\infty(0,\rho_M; L^2(\mathbb{R}))$ that is a fixed point of*

$$\widehat{h}(\rho,\xi) = \int_0^\rho \tilde{K}(\rho,s,\xi)\mathcal{F}(\mathcal{T}(h))(s,\xi)\,ds. \tag{6.4.12}$$

The remainder $\widehat{h}(\rho,\xi)$ satisfies

$$\|\widehat{h}(\rho,\cdot)\|_{L^2} + \|\rho\widehat{h}_\rho(\rho,\cdot)\|_{L^2} \leq C\rho^{1+\frac{3\theta}{2}}e^{\frac{\rho^{2\theta}}{2\theta}}. \tag{6.4.13}$$

The remainder $h = h(\rho,v)$ is such that $\partial_v^\mu h$ is Hölder continuous in (ρ,v) for $\rho > 0$ for all μ with $0 \leq \mu < \lambda+2$. In addition, if $0 < \beta < \mu$, we may bound

$$|\partial_v^\beta h(\rho,v)| \leq C\rho^{1+\theta-2\mu\theta+\beta\theta}[k(\rho)^2 - v^2]_+^{\mu-\beta}.$$

In what follows, in particular in §6.5, we require not only an expansion for $\sigma - \tilde{u}\chi$, but also for $\sigma - \lambda_\pm\chi$, where $\lambda_\pm$ are the eigenvalues of the system.

Corollary 6.4.4. *The functions $\sigma - \lambda_\pm\chi$ satisfy the following expansion:*

$$(\sigma-\lambda_\pm\chi)(\rho,v,s) = (\sigma-\lambda_\pm\chi)(\rho,v-s,0) = (\mp k-(v-s))\rho_0\frac{k'}{k}\chi(\rho,v-s) + R(\rho,v-s),$$

where $|R(\rho,v-s)| \leq C|k(\rho)^2-(v-s)^2|\chi(\rho,v-s)$.

Proof. We begin by recalling that

$$\lambda_\pm = \tilde{u} \pm \tilde{\rho}k'(\rho).$$

To show that $\sigma - \lambda_\pm\chi$ remains invariant under Lorentzian transformation, it suffices to check that the function and its ρ derivative at $\rho = 0$ remain invariant. But this follows from the simple fact that $\big(\tilde{\rho}k'(\rho)\chi(\rho,v-s)\big)_\rho \to 0$ in the sense of measures as $\rho \to 0$. Considering without loss of generality the case for $s = 0$, we use the expansions for $\chi(\rho,v)$ and $(\sigma-\tilde{u}\chi)(\rho,v)$, (6.2.6) and (6.2.13), to obtain

$$\begin{aligned}(\sigma-\lambda_\pm\chi)(\rho,v,0) = {} & (\mp k - v)\rho_0\frac{k'}{k}a_1[k^2-v^2]_+^\lambda \\ & - (\pm\rho_1 a_1 k' \pm \rho_2(k^2-v^2)a_1k' \pm \tilde{\rho}a_2k' + vb_2)[k^2-v^2]_+^{\lambda+1} \\ & \mp \tilde{\rho}k'g(\rho,v) + h(\rho,v),\end{aligned}$$

from which the desired conclusion follows easily. ∎

In the reduction argument of the next section, we require an accurate analysis of the singularities of the entropy and entropy flux kernels constructed above. To this end, we here provide explicit formulae for the singularities in the fractional derivatives of order $\lambda+1$.

Proposition 6.4.5 (Explicit Singularities of the Entropy Kernels)**.** *The distributions $\partial_v^{\lambda+1}\chi$ and $\partial_v^{\lambda+1}\sigma$ satisfy*

$$\begin{aligned}\partial_v^{\lambda+1}\chi &= k'(\rho)^{-\frac{1}{2}}e^{\tilde{a}(\rho)}\sum_{\pm}K^{\pm}\delta_{v=\mp k(\rho)}+e^{I}(\rho,v),\\ \partial_v^{\lambda+1}(\sigma-\tilde{u}\chi) &= -\,v\rho_0(\rho)k(\rho)^{-1}k'(\rho)^{\frac{1}{2}}e^{\tilde{a}(\rho)}\sum_{\pm}K^{\pm}\delta_{v=\mp k(\rho)}+e^{II}(\rho,v),\end{aligned} \tag{6.4.14}$$

where $K^{\pm}$ are some constants and e^{I}, e^{II} are Hölder continuous functions in the interior of the support of the kernels such that

$$|e^{I}(\rho,v)| \le Ck(\rho)^{\lambda-1+2\alpha}[k(\rho)^2-v^2]_+^{-\alpha}(\rho,v), \tag{6.4.15}$$

$$|e^{II}(\rho,v)| \le Ck(\rho)^{\lambda+2\alpha}[k(\rho)^2-v^2]_+^{-\alpha}(\rho,v) \tag{6.4.16}$$

for all $\alpha\in(0,1]$.

Proof. Identities for the fractional derivatives $\partial_v^{\lambda+1}G_\lambda$ and $\partial_v^{\lambda}G_\lambda$ may be found in [8, Proof of Proposition 2.4]. The desired representations then follow from the expansions (6.2.6) and (6.2.13), exactly as in that proof. □

Finally, we record a property of the coefficients that we will require in the sequel.

Proposition 6.4.6. *We define a coefficient $D=D(\rho)$ as*

$$D(\rho) := a_1(\rho)b_1(\rho)-2k(\rho)^2(a_1(\rho)b_2(\rho)-a_2(\rho)b_1(\rho)).$$

Then there exists a finite $\rho_M\in(0,\rho_{\max}]$ such that, for $0\le\rho\le\rho_M$, we have that $D(\rho)>0$. Moreover, as $\varepsilon\to 0$, $\rho_M\to\infty$.

Proof. This follows from the formulae and bounds given for the coefficients above in Theorem 6.2.2 and Theorem 6.2.4. In particular, as $\lim_{\rho\to 0}a_1(\rho)b_1(\rho)>0$ and $|a_2(\rho)|+|b_2(\rho)|\le C$, we find $\lim_{\rho\to 0}D(\rho)>0$. By continuity, such a ρ_M can therefore be found. Moreover, one may check that $D(\rho)$ can be written as

$$\rho p''(\rho)+2p'(\rho)+\varepsilon(p(\rho)p''(\rho)-2p'(\rho)^2)+\varepsilon O(\rho^{\gamma-1}).$$

□

6.5 A relativistic compactness framework

As described in the introduction to this chapter, we seek to construct entropy solutions to system (6.1.1) by means of a sequence of approximate solutions. Thus the compactness properties of such a sequence are crucial to understanding and analysing the limit of the sequence. The principal aim of this section, therefore, is to show that a uniformly bounded sequence of functions satisfying a certain weak compactness property with respect to the entropies of the system can be shown to converge not just weakly, but strongly in L^p spaces. This strong convergence will allow us then to pass to a limit in the non-linear terms of the equation, and also in the associated entropy equations.

Theorem 6.5.1 (Compactness of Approximate Solutions)**.** *Let $(\rho^\delta, u^\delta) \in (L^\infty(\mathbb{R}^2_+))^2$ with $\rho^\delta \geq 0$ be functions such that the entropy dissipation measure*

$$\partial_t \eta(\rho^\delta, u^\delta) + \partial_x q(\rho^\delta, u^\delta) \text{ is compact in } H^{-1}_{loc}(\mathbb{R}^2_+), \tag{6.5.1}$$

for all weak entropy pairs (η, q), and that

$$0 \leq \rho^\delta(t,x) \leq \rho_M, \quad |u^\delta(t,x)| \leq C_0 < \varepsilon \text{ for almost every } (t,x) \in \mathbb{R}^2_+,$$

for ρ_M defined as in Proposition 6.4.6 and $C_0 > 0$ independent of $\delta > 0$. Then there exist measurable functions (ρ, u) such that

$$0 \leq \rho(t,x) \leq \rho_M, \quad |u(t,x)| \leq C_0 < \varepsilon \text{ for almost every } (t,x) \in \mathbb{R}^2_+,$$

where $C > 0$ depends on ε and the initial data, but is independent of T, and there is a subsequence (ρ^δ, u^δ) converging strongly to (ρ, u) in $L^r_{loc}(\mathbb{R}^2_+)$ for all $r \in [1, \infty)$.

The proof of this theorem rests on two main ingredients: the div-curl lemma, Lemma 3.4.1, and the following reduction result for probability measures constrained by the Tartar commutation relation, the proof of which we temporarily postpone.

Theorem 6.5.2 (Reduction of Support of the Young Measure)**.** *Suppose that $\nu(\rho, v)$ is a probability measure with bounded support contained in $\{0 \leq \rho \leq \rho_M, v \in \mathbb{R}\}$ satisfying the commutation relation*

$$\overline{\eta_1 q_2 - \eta_2 q_1} = \overline{\eta_1}\,\overline{q_2} - \overline{\eta_2}\,\overline{q_1} \tag{6.5.2}$$

for any two weak entropy pairs (η_1, q_1), (η_2, q_2), where we write

$$\overline{f} = \int f(\rho, v)\, d\nu(\rho, v) \text{ for any continuous function } f(\rho, v).$$

Then either ν is supported in the vacuum line $\{\rho = 0\}$, or the support of ν is a single point.

Proof of Theorem 6.5.1. For convenience, we work in (ρ, v) coordinates. As the sequence (ρ^δ, v^δ) is uniformly bounded in $L^\infty(\mathbb{R}^2_+)$ (where $v^\delta = v(u^\delta)$), we may extract a weakly-star convergent subsequence (not relabelled) $(\rho^\delta, v^\delta) \overset{*}{\rightharpoonup} (\rho, v)$ in $L^\infty(\mathbb{R}^2_+)$. By the fundamental theorem of Young measures, Theorem 3.1.1, we may associate, for each point $(t, x) \in \mathbb{R}^2_+$, a probability measure $\nu_{t,x}$ such that for any (t, x), $\operatorname{supp} \nu_{t,x} \subset \{0 \leq \rho \leq \rho_M, |v| \leq C\}$, and

$$f(\rho^\delta, v^\delta) \overset{*}{\rightharpoonup} \overline{f} = \int_{\mathbb{R}^2_+} f(\rho, v)\, d\nu_{t,x}(\rho, v) \text{ in } L^\infty(\mathbb{R}^2_+)$$

for any continuous function $f : \mathbb{R}^2_+ \to \mathbb{R}$. We now use the div-curl lemma, Lemma 3.4.1, to deduce a constraint on the Young measure ν. Take two weak entropy pairs, (η_1, q_1), (η_2, q_2). For simplicity of notation, we define $\eta_i^\delta = \eta_i(\rho^\delta(t, x), v^\delta(t, x))$, $q_i^\delta = q_i(\rho^\delta(t, x), v^\delta(t, x))$ for $i = 1, 2$. Then, by definition of the Young measure, we have that

$$\eta_1^\delta\, q_2^\delta - \eta_2^\delta\, q_1^\delta \overset{*}{\rightharpoonup} \overline{\eta_1 q_2 - \eta_2 q_1} \text{ in } L^\infty(\mathbb{R}^2_+) \text{ as } \delta \to 0.$$

On the other hand, by the assumption (6.5.1), we may apply the div-curl lemma to the sequences $w_1^\delta = (\eta_1^\delta, q_1^\delta)$, $w_2^\delta = (q_2^\delta, -\eta_2^\delta)$ to obtain

$$\eta_1^\delta\, q_2^\delta - \eta_2^\delta\, q_1^\delta \rightharpoonup \overline{\eta_1}\,\overline{q_2} - \overline{\eta_2}\,\overline{q_1}$$

in the sense of distributions as $\delta \to 0$ for almost every (t, x). Thus, by uniqueness of weak limits, for almost every (t, x), we have the Tartar commutation relation

$$\overline{\eta_1 q_2 - \eta_2 q_1} = \overline{\eta_1}\,\overline{q_2} - \overline{\eta_2}\,\overline{q_1}$$

for any two weak entropy pairs (η_1, q_1), (η_2, q_2).

We conclude by applying Theorem 6.5.2 to show that for almost every (t, x), we have that $\nu_{t,x}$ is either constrained to a point, $\nu_{t,x} = \delta_{(\rho(t,x), v(t,x))}$ or is contained in the vacuum line $\operatorname{supp} \nu_{t,x} \subset \{\rho = 0\}$. Moving to the conserved variables, this implies that the measure $\nu_{t,x}$ is a point mass almost everywhere. Thus we may apply Proposition 3.1.3 to conclude the strong convergence claimed in the theorem. $\square$

The rest of this section is devoted to the proof of Theorem 6.5.2. As a preliminary step, we extend a result of DiPerna for the classical Euler equations, [24], to the relativistic case. This lemma tells us that in the (w, z)-plane, the smallest triangle containing the support of the Young measure $\nu_{t,x}$ (considered as a measure in (w, z)) must have its vertex in the support of $\nu_{t,x}$. Note that the vacuum line $\{\rho = 0\}$ corresponds to the line $\{w = z\}$.

Lemma 6.5.3. *Let ν be a probability measure on the set $\{w \geq z\}$ with non-trivial support away from the vacuum line, i.e. $\operatorname{supp} \nu \cap \{w > z\} \neq \emptyset$ and suppose further that ν satisfies the commutation relation*

$$\overline{\eta_1 q_2 - \eta_2 q_1} = \overline{\eta_1}\,\overline{q_2} - \overline{\eta_2}\,\overline{q_1} \tag{6.5.3}$$

for all weak entropy pairs (η_1, q_1) and (η_2, q_2), where again $\overline{f} = \int f\, d\nu$. Let

$$\{(w, z) \,:\, z_{\min} \leq z \leq w \leq w_{\max}\}$$

be the smallest such triangle containing the support of ν in the (w, z)-plane. Then its vertex $(w_{\max}, z_{\min})$ belongs to $\operatorname{supp} \nu$.

Proof. Suppose that this is not the case. Then there exists $\alpha > 0$ such that

$$\operatorname{supp} \nu \cap \big([w_{\max} - \alpha, w_{\max}] \times [z_{\min}, z_{\min} + \alpha]\big) = \emptyset.$$

We recall that weak entropy pairs are generated by convolution of test functions against the weak entropy and entropy flux kernels. Thus, supposing (η_1, q_1) and (η_2, q_2) to be generated from C^2 test functions ψ_1 and ψ_2 respectively, we deduce from the commutation relation (6.5.2), that

$$\begin{aligned}
&\int \Big(\int_{\mathbb{R}} \psi_1(s_1) \chi(\rho, v - s_1)\, ds_1 \int_{\mathbb{R}} \psi_2(s_2) \sigma(\rho, v, s_2)\, ds_2 \\
&\qquad - \int_{\mathbb{R}} \psi_2(s_2) \chi(\rho, v - s_2)\, ds_2 \int_{\mathbb{R}} \psi_1(s_1) \sigma(\rho, v, s_1)\, ds_1 \Big) d\nu(\rho, v) \\
&= \int \int_{\mathbb{R}} \psi_1(s_1) \chi(\rho, v - s_1)\, ds_1\, d\nu(\rho, v) \int \int_{\mathbb{R}} \psi_2(s_2) \sigma(\tilde{\rho}, \tilde{v}, s_2)\, ds_2\, d\nu(\tilde{\rho}, \tilde{v}) \\
&- \int \int_{\mathbb{R}} \psi_2(s_2) \chi(\rho, v - s_2)\, ds_2\, d\nu(\rho, v) \int \int_{\mathbb{R}} \psi_1(s_1) \sigma(\tilde{\rho}, \tilde{v}, s_1)\, ds_1\, d\nu(\tilde{\rho}, \tilde{v})
\end{aligned}$$

which, by Fubini's theorem, implies, with obvious notation,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)} \psi_1(s_1) \psi_2(s_2)\, ds_1\, ds_2 \\
&\quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\chi(s_1)}\, \overline{\sigma(s_2)} - \overline{\chi(s_2)}\, \overline{\sigma(s_1)} \psi_1(s_1) \psi_2(s_2)\, ds_1\, ds_2
\end{aligned}$$

and hence we deduce, after slight rearrangement,

$$\frac{\overline{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)}}{\overline{\chi(s_1)}\,\overline{\chi(s_2)}} = \frac{\overline{\sigma(s_2)}}{\overline{\chi(s_2)}} - \frac{\overline{\sigma(s_1)}}{\overline{\chi(s_1)}} \quad \text{for all } s_1, s_2 \in \mathbb{R}. \tag{6.5.4}$$

Setting $s_- := z_{\min}$, $s_+ := w_{\max}$, we consider s_1, s_2 such that $0 < s_+ - s_2 < \alpha$, $0 < s_1 - s_- < \alpha$. Now we know from the expansions for $\chi(s)$ and $\sigma(s)$ given in

Theorems 6.2.2 and 6.2.4 that we have $\operatorname{supp}\chi(s) = \operatorname{supp}\sigma(s) = \{z \le s \le w\}$, so necessarily we have that $(w,z) \in \operatorname{supp}\chi(s_1)\sigma(s_2)$ gives that $(w,z) \notin \operatorname{supp}\nu$. Arguing in the same way for the term $\chi(s_2)\sigma(s_1)$, we see that the left hand side of (6.5.4) vanishes.

Now we recall from Corollary 6.4.4 that we may write

$$(\sigma - \lambda_\pm\chi)(\rho, v, s) = (\sigma - \lambda_\pm\chi)(\rho, v-s, 0) = (\mp k - (v-s))\rho_0\frac{k'}{k}\chi(\rho, v-s) + R(\rho, v-s),$$

where $|R(\rho, v-s)| \le C|k(\rho)^2 - (v-s)^2|\chi(\rho, v-s)$. Thus

$$\frac{\langle\sigma(s)\rangle}{\langle\chi(s)\rangle} = \frac{\langle\lambda_\pm\chi(s)\rangle}{\langle\chi(s)\rangle} + \frac{\langle(\mp k - (v-s))\rho_0\frac{k'}{k}\chi(s)\rangle}{\langle\chi(s)\rangle} + \frac{\langle R(s)\rangle}{\langle\chi(s)\rangle}. \tag{6.5.5}$$

We define the trace measures μ_+ and μ_- by

$$\frac{\overline{j\chi(s_2)}}{\overline{\chi(s_2)}} \to \langle\mu_+, j(w_{\max}, \cdot)\rangle := \int j(w_{\max}, z)\, d\mu_+(z) \text{ as } s_2 \to s_+$$

for all continuous $j = j(w,z)$;

$$\frac{\overline{j\chi(s_1)}}{\overline{\chi(s_1)}} \to \langle\mu_-, j(\cdot, z_{\min})\rangle := \int j(w, z_{\min})\, d\mu_-(w) \text{ as } s_1 \to s_-.$$

That these measures are well defined is by now standard. A proof may be found in [24]. We note that

$$\left|\frac{\overline{(-k - (v-s_2))\rho_0\frac{k'}{k}\chi(s_2)}}{\overline{\chi(s_2)}}\right| \le C \max_{(w,z)\in\operatorname{supp}\nu\cap\{w \ge s_2\}} |w - s_2| \to 0 \text{ as } s_2 \to s_+,$$

$$\left|\frac{\overline{(k - (v-s_1))\rho_0\frac{k'}{k}\chi(s_1)}}{\overline{\chi(s_1)}}\right| \le C \max_{(w,z)\in\operatorname{supp}\nu\cap\{z \le s_1\}} |z - s_1| \to 0 \text{ as } s_1 \to s_-.$$

Moreover,

$$\left|\frac{\overline{R(s)}}{\overline{\chi(s)}}\right| \le C \max_{(w,z)\in\operatorname{supp}\nu} [k^2 - (v-s)^2]_+ \le C \max_{\operatorname{supp}\nu} |w - s_2||z - s_1| \to 0$$

as s tends to either s_2 or s_1. Thus we deduce from (6.5.5) that

$$\langle\mu_+, \lambda_+\rangle - \langle\mu_-, \lambda_-\rangle = 0. \tag{6.5.6}$$

Now suppose that $(w_{\max}, z_1)$, $(w_2, z_{\min})$ are points on the edges of the triangle with $z_1 = v_1 - k(\rho_1)$, $w_2 = v_2 + k(\rho_2)$. Observe that necessarily, $v_1 \ge v_2$ with equality only at the vertex of the triangle. Now we calculate that

$$\begin{aligned}\lambda_+(w_{\max}, z_1) &- \lambda_-(w_2, z_{\min})\\ &= \frac{(u_1 - u_2)(1 + \varepsilon\sqrt{p'(\rho_1)}\sqrt{p'(\rho_2)}) + (\sqrt{p'(\rho_1)} + \sqrt{p'(\rho_2)})(1 - \varepsilon u_1 u_2)}{(1 - \varepsilon u_1\sqrt{p'(\rho_1)})(1 - \varepsilon u_2\sqrt{p'(\rho_2)})} > 0,\end{aligned}$$

as either $u_1 > u_2$ or $u_1 = u_2$ and $\rho_1 = \rho_2 > 0$.

This then gives the desired contradiction to (6.5.6). $\square$

To prove Theorem 6.5.2, we demonstrate the existence of an *imbalance of regularity* in the commutation relation (6.5.2), following the methods of Tartar, [71], and DiPerna, [24], as developed in [52] and discussed in §2.2–2.3.

We write $P_j := \partial_{s_j}^{\lambda+1}$, $j = 2, 3$, for the fractional derivative operator, and define, for $j = 2, 3$, $\chi_j = \chi(\rho, v, s_j)$ and similarly for other terms. Then the distribution $\overline{P_2\chi(s_2)}$, for example, is defined as acting on test functions $\psi \in \mathcal{D}(\mathbb{R})$ by

$$\langle \overline{P_2\chi(s_2)}, \psi \rangle = -\int_{\mathbb{R}} \overline{\partial_{s_2}^{\lambda}\chi(s_2)}\psi'(s_2)\, ds_2.$$

We choose mollifiers $\phi_j \in C_c^{\infty}(\mathbb{R})$, $j = 2, 3$, so that $\phi_j(s_j) \geq 0$, $\int_{\mathbb{R}} \phi_j(s_j)\, ds_j = 1$, $\operatorname{supp} \phi_j(s_j) \subset (-1, 1)$ and set, for $\delta > 0$, $\phi_j^{\delta}(s_j) = \delta^{-1}\phi_j(s_j/\delta)$.

The strategy of the proof is first to apply the operators P_2 and P_3 to the commutation relation and then to mollify. To make clear the claimed imbalance of regularity, we make use of the fact that the limit of a mollified product of a measure with a BV function depends on the choice of mollifiers used, compare §5.5. Mollifying the entropy and entropy flux kernels, we obtain, for $j = 2, 3$, expressions of the form

$$\overline{P_j\chi_j^{\delta}} = \overline{P_j\chi_j} * \phi_j^{\delta}(s_1) = \int \overline{\partial_{s_j}^{\lambda}\chi(s_j)}\delta^{-2}\phi_j'\big(\frac{s_1 - s_j}{\delta}\big)\, ds_j. \tag{6.5.7}$$

Once we have differentiated and mollified the commutation relation, (6.5.2), we pass to the limit $\delta \to 0$, relying on properties of cancellation of singularities of the entropy and entropy flux kernels, as well as the previously mentioned dependence on the choice ϕ_2, ϕ_3. These properties are stated in the following two lemmas.

Lemma 6.5.4 (Cancellation of singularities I). *For $j = 2, 3$, the functions $\chi_1 P_j \sigma_j^{\delta} - \sigma_1 P_j \chi_j^{\delta}$ are Hölder continuous in (ρ, v, s_1), uniformly in δ. Also, there exists a continuous function $X_1 = X(\rho, v, s_1)$, independent of the choice of mollifying sequence, such that when $\delta \to 0$, $\chi_1 P_j \sigma_j^{\delta} - \sigma_1 P_j \chi_j^{\delta} \to X_1$ uniformly in (ρ, v, s_1).*

Lemma 6.5.5 (Cancellation of singularities II). *The functions $P_2\chi_2^{\delta}\, P_3\sigma_3^{\delta} - P_3\chi_3^{\delta}\, P_2\sigma_2^{\delta}$ are uniformly bounded measures such that, as $\delta \to 0$,*

$$P_2\chi_2^{\delta}\, P_3\sigma_3^{\delta} - P_3\chi_3^{\delta}\, P_2\sigma_2^{\delta} \rightharpoonup Y(\phi_2, \phi_3) Z(\rho) \sum_{\pm} (K^{\pm})^2 \delta_{s_1 = v \pm k(\rho)},$$

weakly-star in measures in s_1 and uniformly in (ρ, v), where

$$Y(\phi_2, \phi_3) = \int_{-1}^{1}\int_{s}^{1} (\phi_3(t - s - 1)\phi_2(t) - \phi_2(t - s - 1)\phi_3(t))\, dt\, ds,$$

and

$$Z(\rho) := M(\rho) D(\rho),$$

where $M(\rho) = (\lambda + 1)c_{*,\lambda}^{-2} k(\rho)^{2\lambda}$ for $\rho > 0$ and $D(\rho)$ is as in Proposition 6.4.6.

The proofs of these results are analogous to those of [8, Lemma 4.2–4.3]. For the sake of completeness, we include them here in the relativistic setting. They are based on the structure of the fractional derivatives of the kernels given in Proposition 6.4.5.

Proof of Lemma 6.5.4. As we only require the fine properties of the leading order term in each of the two expansions, we set here

$$\begin{aligned}\tilde{g}(\rho, v-s_1) =& \, a_2(\rho) G_{\lambda+1}(\rho, v-s_1) + g(\rho, v-s_1),\\ \tilde{h}(\rho, v-s_1) =& -(v-s_1) b_2(\rho) G_{\lambda+1}(\rho, v-s_1) + h(\rho, v-s_1).\end{aligned}$$

With this notation, and recalling that $b_1(\rho) = \rho_0(\rho)\frac{k'(\rho)}{k(\rho)} a_1(\rho)$, we expand as follows:

$$\begin{aligned}&\chi_1 P_j \sigma_j^\delta - \sigma_1 P_j \chi_j^\delta\\ &\quad = \chi_1 P_j(\sigma_j^\delta - \tilde{u}\chi_j^\delta) - (\sigma_1 - \tilde{u}\chi_1) P_j \chi_j^\delta\\ &\quad = \big(a_1 G_{\lambda,1} + \tilde{g}_1\big)\Big(\rho_0 \frac{k'}{k} e^{\tilde{a}} \sum_{\pm} K^{\pm}((s_j - v)\delta_{s_j = v\pm k}) * \phi_j^\delta + e_j^{II} * \phi_j^\delta\Big)\\ &\qquad - \big((s_1 - v) b_1 G_{\lambda,1} + \tilde{h}_1\big)\Big(k'^{-\frac{1}{2}} e^{\tilde{a}} \sum_{\pm} K^{\pm} \delta_{s_j = v\pm k} * \phi_j^\delta + e_j^I * \phi_j^\delta\Big).\end{aligned}$$

In order to analyse this expression, we break it into a sum of the most singular terms (that is, products of Hölder continuous functions with measures) and a remainder as follows. Defining

$$\begin{aligned}E^{I,\delta} :=& \, a_1 \rho_0 k^{-1} k'^{\frac{1}{2}} e^{\tilde{a}} G_{\lambda,1} \sum_{\pm} K^{\pm}((s_j - s_1)\delta_{s_j = v\pm k}) * \phi_j^\delta,\\ E^{II,\delta} :=& \, \rho_0 k^{-1} k'^{\frac{1}{2}} e^{\tilde{a}} \sum_{\pm} K^{\pm} \tilde{g}_1((s_j - s_1)\delta_{s_j = v\pm k}) * \phi_j^\delta\\ & - k'^{-\frac{1}{2}} e^{\tilde{a}} \sum_{\pm} K^{\pm} \tilde{h}_1 \delta_{s_j = v\pm k} * \phi_j^\delta,\\ E^{III,\delta} :=& (a_1 G_{\lambda,1} + \tilde{g}_1) e_j^{II} * \phi_j^\delta - ((s_1 - v) b_1 G_{\lambda,1} + \tilde{h}_1) e_j^I * \phi_j^\delta,\end{aligned}$$

we have

$$\chi_1 P_j \sigma_j^\delta - P_j \chi_j^\delta \sigma_1 = E^{I,\delta} + E^{II,\delta} + E^{III,\delta}.$$

Using the bounds given by Lemma 6.1.4 for $k(\rho)$, $k'(\rho)$, we bound $E^{I,\delta}$ by

$$\begin{aligned}|E^{I,\delta}(\rho, v, s_1)| \leq& \, C\rho^{\frac{1-\theta}{2}} [k - (v - s_1)]_+^\lambda [k + (v - s_1)]_+^\lambda \sum_{\pm} K^{\pm} |s_1 - v \mp k| \phi_j^\delta(s_1 - v \mp k)\\ \leq& \, C\rho^{\frac{1-\theta}{2}} \sum_{\pm} K^{\pm} |s_1 - v \mp k|^{\lambda+1} \delta^{-1} \phi_j\big(\frac{s_1 - v \mp k}{\delta}\big)\\ \leq& \, C\rho^{\frac{1-\theta}{2}} \delta^\lambda \to 0\end{aligned}$$

as $\delta \to 0$, locally uniformly in (ρ, v, s_1), where we have used that $\operatorname{supp}\phi_j \subset (-1, 1)$.

Next, for $E^{II,\delta}$, we make the bound

$$\begin{aligned} |E^{II,\delta}(\rho, v - s_1)| &\leq C\rho^{\frac{1-\theta}{2}}[k^2 - (v - s_1)^2]_+^{\lambda+1} \sum_{\pm} \phi_j^\delta(s_1 - v \mp k) \\ &\leq C\rho^{\frac{1-\theta}{2}}\delta^\lambda \to 0. \end{aligned}$$

Finally, we consider the remainder term, $E^{III,\delta}(\rho, v - s_1)$. Recalling the bounds (6.4.15) and (6.4.16), we observe that

$$|s_1 - v||e_j^I(\rho, v - s_1)| + |e^{II}(\rho, v - s_1)| \leq C\rho^{\frac{1-\theta}{2}+2\alpha\theta}G_{-\alpha}(\rho, v - s_1)$$

for $\alpha \in (0, 1)$. Clearly, this is not Hölder continuous up to the boundary of its support. However, in the region $\{|k^2 - (v - s_1)^2| \leq \Delta\}$ for $\Delta > 0$, we may bound

$$|e^{III,\delta}(\rho, v - s_1)| \leq C\rho^{\frac{1-\theta}{2}+2\alpha\theta}G_{\lambda-\alpha}(\rho, v - s_1) \leq \rho^{3\frac{1-\theta}{2}+2\alpha\theta}\Delta^{\lambda-\alpha},$$

which may be made arbitrarily small by taking $0 < \alpha < \min\{1, \lambda\}$ and Δ small. On the other hand, in the complement region, $\{|k^2 - (v - s_1)^2| > \Delta\}$, we have that G_λ, e_j^I and e_j^{II} are all uniformly Hölder continuous, and hence $E^{III,\delta}$ converges to a Hölder continuous limit on this set, independent of the choice of mollifying sequence. □

The proof of Lemma 6.5.5 rests on the observation of [52] (and, in greater generality of [19]) that the limit of a regularised product of a function of bounded variation with a measure depends on the choice of regularisation. In particular, for the case of the product of a Heaviside function and a Dirac mass, we have the following lemma.

Lemma 6.5.6. *For any $m_2, m_3 \in \mathbb{R}$, we have that*

$$\left(H_{s_2=m_2} * \phi_2^\delta\right)\left(\delta_{s_3=m_3} * \phi_3^\delta\right) \rightharpoonup \Omega^{\phi_2,\phi_3}(m_2, m_3)\delta_{s_1=m_3}$$

weak-star in measures, where

$$\Omega^{\phi_2,\phi_3}(m_2, m_3) := \begin{cases} 0, & \text{if } m_2 > m_3, \\ \int_{-1}^1 \int_s^1 \phi_2(t - s - 1)\phi_3(t)\,dt\,ds, & \text{if } m_2 = m_3, \\ 1, & \text{if } m_2 < m_3. \end{cases}$$

Proof. Observe that for any test function $\psi = \psi(s_1)$,

$$\begin{aligned} &\left(H_{s_2=m_2} * \phi_2^\delta\right)\left(\delta_{s_3=m_3} * \phi_3^\delta\right)(\psi) \\ &\qquad = \int_{\mathbb{R}}\int_{\mathbb{R}} H_{s_2=m_2}(s_2)\phi_2^\delta(s_1 - s_2)\,ds_2\,\phi_3^\delta(s_1 - m_3)\psi(s_1)\,ds_1 \\ &\qquad = \int_{\mathbb{R}}\int_{m_2}^\infty \phi_2^\delta(s_1 - s_2)\phi_3^\delta(s_1 - m_3)\psi(s_1)\,ds_2\,ds_1. \end{aligned}$$

The cases $m_2 > m_3$ and $m_2 < m_3$ are easy to verify. We therefore consider $m_2 = m_3$. Now we have that

$$
\begin{aligned}
&\left(H_{s_2=m_2} * \phi_2^\delta\right)\left(\delta_{s_3=m_3} * \phi_3^\delta\right)(\psi) \\
&\qquad = \int_{m_3-\delta}^{m_3+\delta}\int_{m_3}^{s_1+\delta} \frac{1}{\delta^2}\phi_2\left(\frac{s_1-s_2}{\delta}\right)\phi_3\left(\frac{s_1-m_3}{\delta}\right)\psi(s_1)\,ds_2\,ds_1 \\
&\qquad = \int_{m_3}^{m_3+2\delta}\int_{s_2-\varepsilon}^{m_3+\delta} \frac{1}{\delta^2}\phi_2\left(\frac{s_1-s_2}{\delta}\right)\phi_3\left(\frac{s_1-m_3}{\delta}\right)\psi(s_1)\,ds_1\,ds_2 \\
&\qquad = \int_{m_3}^{m_3+2\delta}\int_{\frac{s_2-m_3}{\delta}-1}^{1} \frac{1}{\delta}\phi_2\left(t+\frac{m_3-s_2}{\delta}\right)\phi_3(t)\psi(\delta t+m_3)\,dt\,ds_2 \\
&\qquad = \int_{-1}^{1}\int_{s}^{1} \phi_2(t-s-1)\phi_3(t)\psi(\delta t+m_3)\,dt\,ds,
\end{aligned}
$$

which is easily seen to converge to

$$
\psi(m_3)\int_{-1}^{1}\int_{s}^{1}\phi_2(t-s-1)\phi_3(t)\,dt\,ds
$$

as required. □

Proof of Lemma 6.5.5. We use the expansions of Theorem 6.2.2 and 6.2.4 to see that

$$
\begin{aligned}
&P_2\chi(\rho, v-s_2)P_3\sigma(\rho, v, s_3) - P_3\chi(\rho, v-s_3)P_2\sigma(\rho, v, s_2) \\
&\qquad = P_2\chi_2 P_3(\sigma_3 - \tilde{u}\chi_3) - P_3\chi_3 P_2(\sigma_2 - \tilde{u}\chi_2) \\
&\qquad = (a_1P_2G_{\lambda,2} + a_2P_2G_{\lambda+1,2} + P_2g_2)\big((s_3-v)(b_1P_3G_{\lambda,3} + b_2P_3G_{\lambda+1,3}) \\
&\qquad\qquad + P_3h_3 + (\lambda+1)b_1\partial_{s_3}^\lambda G_{\lambda,3} + (\lambda+1)b_2\partial_{s_3}^\lambda G_{\lambda+1,3}\big) \\
&\qquad\quad - (a_1P_3G_{\lambda,3} + a_2P_3G_{\lambda+1,3} + P_3g_3)\big((s_2-v)(b_1P_2G_{\lambda,2} + b_2P_2G_{\lambda+1,2}) \\
&\qquad\qquad + P_2h_2 + (\lambda+1)b_1\partial_{s_2}^\lambda G_{\lambda,2} + (\lambda+1)b_2\partial_{s_2}^\lambda G_{\lambda+1,2}\big) \\
&\qquad = E^I + E^{II} + E^{III},
\end{aligned}
$$

where we have decomposed the expression as

$$
\begin{aligned}
E^I &:= (s_3-s_2)a_1b_1P_2G_{\lambda,2}P_3G_{\lambda,3}, \\
E^{II} &:= a_1P_2G_{\lambda,2}\big((s_3-v)b_2P_3G_{\lambda+1,3} + (\lambda+1)b_1\partial_{s_3}^\lambda G_{\lambda,3}\big) \\
&\quad - a_1P_3G_{\lambda,3}\big((s_2-v)b_2P_2G_{\lambda+1,2} + (\lambda+1)b_1\partial_{s_2}^\lambda G_{\lambda,2}\big) \\
&\quad + a_2b_1\big(P_2G_{\lambda+1,2}(s_3-v)P_3G_{\lambda,3} - P_3G_{\lambda+1,3}(s_2-v)P_2G_{\lambda,2}\big),
\end{aligned}
$$

and E^{III} is the remainder.

We now take mollification using the mollifiers defined above. This yields

$$
P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta = \big(E^I + E^{II} + E^{III}\big) * \phi_2^\delta * \phi_3^\delta.
$$

We recall that as we are evaluating our mollified expressions at s_1 (compare (6.5.7)), this is now a function of (ρ, v, s_1) only. From symmetry considerations, the limit as $\delta \to 0$ of $E^{III,\delta}$ is 0 as this term contains only products of measures with Hölder continuous functions and more regular products. This convergence may be taken uniformly in (ρ, v) and weak-star in measures in s_1.

We consider next the most singular terms, arising in $E^{I,\delta}$. From Proposition 6.4.5, we see that this expression involves products of measures, products of measures with L^p functions, and products of L^p functions. Again by symmetry considerations, the last group of these terms vanishes in the limit $\delta \to 0$, and so we concern ourselves only with the first two. Observe first that a typical product of measures is of the form

$$(s_3 - s_2)a_1 b_1 k'(\rho)^{-\frac{1}{2}} e^{\tilde{a}(\rho)} \delta_{s_2 = v \pm k(\rho)} k'(\rho)^{-\frac{1}{2}} e^{\tilde{a}(\rho)} \delta_{s_3 = v \pm k(\rho)} * \phi_2^\delta * \phi_3^\delta.$$

Observe that if both Dirac masses are based at the same point, $s_j = v \pm k(\rho)$, then the factor $(s_3 - s_2)$ causes the expression to vanish. We hence need only consider the case that they are based at different points. The action of this expression as a measure acting on a continuous function $\psi(s_1)$ may then be written as

$$a_1 b_1 e^{2\tilde{a}} k'(\rho)^{-1} \int (w - z)\phi_2^\delta(s_1 - w)\phi_3^\delta(s_1 - z)\psi(s_1)\, ds_1.$$

Hence we bound

$$\begin{aligned} k'(\rho)^{-1} \Big| \int (w - z)\phi_2^\delta(s_1 - w)\phi_3^\delta(s_1 - z)\psi(s_1)\, ds_1 \Big| & \\ \leq C(w - z)^{1+2\lambda} \int \phi_2^\delta(s_1 - w)\phi_3^\delta(s_1 - z)|\psi(s_1)|\, ds_1 & \\ \leq C\delta^{2\lambda} \big(\frac{w - z}{\delta}\big)^{1+2\lambda} \int_{-1}^{1} \phi_2(s_1)\phi_3\big(s_1 + \frac{w - z}{\delta}\big)|\psi(w + \delta s_1)|\, ds_1 & \\ \leq C\delta^{2\lambda} \to 0, & \end{aligned}$$

where we have used that $\operatorname{supp}\phi_3 \subset (-1, 1)$. Arguing similarly for the other terms, we obtain

$$E^{I,\delta} \to 0 \text{ as } \delta \to 0$$

weak-star in measures in s_1 and uniformly in (ρ, v).

We now come to the most significant term, $E^{II,\delta}$. Replacing s_2 and s_3 with s_1 in this expression gives a remainder equipped with good factors of the form $s_j - s_1$, which may be shown to converge to 0 as above. Taking account of cancellations, we therefore consider the expression

$$\begin{aligned} \tilde{E}^{II,\delta} &= (\lambda + 1)a_1 b_1 \big(P_2 G_{\lambda,2} \partial_{s_3}^\lambda G_{\lambda,3} - P_3 G_{\lambda,3} \partial_{s_2}^\lambda G_{\lambda,2}\big) * \phi_2^\delta * \phi_3^\delta \\ &\quad + (s_1 - v)(a_1 b_2 - a_2 b_1)\big(P_2 G_{\lambda,2} P_3 G_{\lambda+1,3} - P_3 G_{\lambda,3} P_2 G_{\lambda+1,2}\big) * \phi_2^\delta * \phi_3^\delta. \end{aligned}$$

Observe that

$$\partial_s^{\lambda+1}G_{\lambda+1}(\rho, v-s) = [k^2-(v-s)^2]_+\partial_s^{\lambda+1}G_\lambda(\rho, v-s) - 2(\lambda+1)(s-v)\partial_s^\lambda G_\lambda(\rho, v-s),$$

and that the contributions from the first of these terms may be seen to converge to 0, we need only consider the contribution from

$$\begin{aligned}
&(\lambda+1)a_1b_1\big(P_2G_{\lambda,2}\partial_{s_3}^\lambda G_{\lambda,3} - P_3G_{\lambda,3}\partial_{s_2}^\lambda G_{\lambda,2}\big) * \phi_2^\delta * \phi_3^\delta \\
&\qquad - 2(\lambda+1)(s_1-v)^2(a_1b_2-a_2b_1)\big(P_2G_{\lambda,2}\partial_{s_3}^\lambda G_{\lambda,3} - P_3G_{\lambda,3}\partial_{s_2}^\lambda G_{\lambda,2}\big) * \phi_2^\delta * \phi_3^\delta \\
&\quad = (\lambda+1)(a_1b_1 - 2k^2(a_1b_2-a_2b_1))\big(P_2G_{\lambda,2}\partial_{s_3}^\lambda G_{\lambda,3} - P_3G_{\lambda,3}\partial_{s_2}^\lambda G_{\lambda,2}\big) * \phi_2^\delta * \phi_3^\delta \\
&\qquad + G_{err}
\end{aligned}$$

where G_{err} also converges to 0. Applying now the expansions for the explicit singularities calculated in Proposition 6.4.5 and Lemma 6.5.6, we conclude. □

With the above results, the proof of Theorem 6.5.2 follows exactly as in the proof of Theorem 4.2 of [8], included here for completeness. The proof is very similar to the proof of Theorem 5.5.1, though here because we have uniform bounds on ρ and v, the passage to the limit in the mollified sequences is far simpler.

Proof of Theorem 6.5.2. We begin by applying the commutation relation (6.5.2) with weak entropy pairs generated by test functions ψ_1, ψ_2 and use the density of test functions to derive the commutation relation directly for the entropy kernels themselves:

$$\overline{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)} = \overline{\chi(s_1)}\,\overline{\sigma(s_2)} - \overline{\chi(s_2)}\,\overline{\sigma(s_1)}$$

for any $s_1, s_2 \in \mathbb{R}$, as in the proof of Lemma 6.5.3 where, for example, $\overline{\chi(s_1)} = \int \chi(\rho, v, s_1)\, d\nu(\rho, v)$. We choose $s_1, s_2, s_3 \in \mathbb{R}$ and apply this identity to each of the pairs (s_2, s_3), (s_3, s_1) and (s_1, s_2). Multiplying these identities by $\overline{\chi(s_1)}$, $\overline{\chi(s_2)}$ and $\overline{\chi(s_3)}$ respectively and summing, we see that the right hand side vanishes (by an obvious symmetry), leaving us with

$$\begin{aligned}
\overline{\chi(s_1)}\;\overline{\chi(s_2)\sigma(s_3) - \chi(s_3)\sigma(s_2)} + \overline{\chi(s_2)}\;\overline{\chi(s_3)\sigma(s_1) - \chi(s_1)\sigma(s_3)}& \\
+ \overline{\chi(s_3)}\;\overline{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)} &= 0.
\end{aligned}$$

Applying now the operators P_2 and P_3 defined above, we obtain

$$\begin{aligned}
\overline{\chi(s_1)}\;\overline{P_2\chi(s_2)P_3\sigma(s_3) - P_3\chi(s_3)P_2\sigma(s_2)} + \overline{P_2\chi(s_2)}\;\overline{P_3\chi(s_3)\sigma(s_1) - \chi(s_1)P_3\sigma(s_3)}& \\
+ \overline{P_3\chi(s_3)}\;\overline{\chi(s_1)P_2\sigma(s_2) - P_2\chi(s_2)\sigma(s_1)} &= 0
\end{aligned}$$

distributionally in s_1, s_2, s_3.

We mollify this expression with the mollifiers ϕ_2, ϕ_3 as described above to obtain

$$\begin{aligned}\overline{\chi_1}\ \overline{P_2\chi_2^\delta P_3\sigma_3^\delta - P_3\chi_3^\delta P_2\sigma_2^\delta} + \overline{P_2\chi_2^\delta}\ \overline{P_3\chi_3^\delta\sigma_1 - \chi_1 P_3\sigma_3^\delta}\\ + \overline{P_3\chi_3^\delta}\ \overline{\chi_1 P_2\sigma_2^\delta - P_2\chi_2^\delta\sigma_1} = 0\end{aligned} \tag{6.5.8}$$

with obvious notation. Passing now $\delta \to 0$, we recall that as $P_j\chi_j$ is a bounded measure in s_j with coefficients uniformly bounded in ρ, v, we may pass

$$P_j\chi_j^\delta \to P_1\chi_1$$

weak-star with respect to measures in s_1 and uniformly with respect to ρ, v. Thus also

$$\overline{P_j\chi_j^\delta} \to \overline{P_1\chi_1}$$

weak-star in measures in s_1.

Considering now the latter two terms of (6.5.8), we may combine this convergence with the uniform convergence of Lemma 6.5.4 to deduce that

$$\overline{P_2\chi_2^\delta}\ \overline{P_3\chi_3^\delta\sigma_1 - \chi_1 P_3\sigma_3^\delta} + \overline{P_3\chi_3^\delta}\ \overline{\chi_1 P_2\sigma_2^\delta - P_2\chi_2^\delta\sigma_1} \to \overline{P_1\chi_1}\ \overline{X_1} - \overline{P_1\chi_1}\ \overline{X_1} = 0,$$

weak-star in measures in s_1. On the other hand, we may apply Lemma 6.5.5 to deduce that the first term of (6.5.8) converges weak-star in measures in s_1 to

$$\overline{\chi(s_1)}\ \overline{Y(\phi_2,\phi_3)Z(\rho)\sum_{\pm}(K^\pm)^2\delta_{s_1=v\pm k(\rho)}},$$

and so for any test function $\psi(s_1)$,

$$Y(\phi_2,\phi_3)\sum_{\pm}(K^\pm)^2\int\overline{\chi(v\pm k(\rho))}Z(\rho)\psi(v\pm k(\rho))\,d\nu(\rho,v) = 0.$$

By assumption, we take $Y(\phi_2,\phi_3) \neq 0$, so that

$$\sum_{\pm}\int\overline{\chi(v\pm k(\rho))}Z(\rho)\,d\nu(\rho,v) = 0.$$

As $Z(\rho) > 0$ for $\rho > 0$ by Proposition 6.4.6 (recall that $\rho \le \rho_M$), we deduce that

$$\operatorname{supp}\nu \cap \{(\rho,u)\,:\, z_{\min} < z(\rho,u) < w(\rho,u) < w_{\max}\} = \emptyset.$$

as, for all $s \in (z_{\min}, w_{\max})$, the function $\chi(s)$ (considered in (w,z) coordinates) contains the point $(w_{\max}, z_{\min})$ in the interior of its support and, by Lemma 6.5.3, the point $(w_{\max}, z_{\min}) \in \operatorname{supp}\nu$. Hence, for all $s \in (z_{\min}, w_{\max})$, we have $\overline{\chi(s)} > 0$. Thus the support of ν must be contained in the vacuum V and the point $(w_{\max}, z_{\min})$. Writing

$$\nu = \nu_V + \omega\delta_{(w_{\max},z_{\min})},$$

where ν_V is supported in the vacuum V and $\omega \in [0, 1]$, we deduce from the commutation relation that, for all $s_1, s_2 \in \mathbb{R}$,

$$(\omega - \omega^2)\big(\chi(w_{\max}, z_{\min}, s_1)\sigma(w_{\max}, z_{\min}, s_2) - \chi(w_{\max}, z_{\min}, s_2)\sigma(w_{\max}, z_{\min}, s_1)\big) = 0.$$

Choosing s_1, s_2 appropriately, we deduce that either $\omega = 0$ or $\omega = 1$. □

With this, we have completed the proof of Theorem 6.5.1. □

6.6 Construction of the artificial viscosity solutions

Now that we are equipped with a compactness framework, we provide a method for the construction of a sequence of approximate solutions satisfying the framework above. In this section we address this problem via the introduction of artificial viscosity. We therefore consider the system

$$\begin{cases} \partial_t\big(\frac{n}{\sqrt{1-u^2/c^2}}\big) + \partial_x\big(\frac{nu}{\sqrt{1-u^2/c^2}}\big) = \delta\partial_{xx}\big(\frac{n}{\sqrt{1-u^2/c^2}}\big), \\ \partial_t\big(\frac{(\rho+p/c^2)u}{1-u^2/c^2}\big) + \partial_x\big(\frac{(\rho+p/c^2)u^2}{1-u^2/c^2} + p\big) = \delta\partial_{xx}\big(\frac{(\rho+p/c^2)u}{1-u^2/c^2}\big), \end{cases} \tag{6.6.1}$$

where $\delta > 0$ is the viscosity parameter. The introduction of this artificial viscosity makes the system uniformly parabolic, and therefore it admits global, regular solutions. Moreover, we show that, as with the classical Euler equations, the artificial viscosity system admits an invariant region, which is precisely one of the conditions we require to apply our compactness framework.

Before we state the theorem giving existence of the solutions to this system, a few remarks on end-point states are in order. As we wish to allow for the possibility that the density and velocity do not vanish at infinity, we impose end-point states for the approximate solutions, $(\rho_\pm, u_\pm)$ such that $\rho_\pm \geq 0$, $|u_\pm| < c$. We introduce smooth, monotone functions $(\bar{\rho}(x), \bar{u}(x))$ such that $(\bar{\rho}(x), \bar{u}(x)) = (\rho_\pm, u_\pm)$ for $|x| \geq \pm 1$ and require that the approximate initial data satisfies $\rho_0^\delta - \bar{\rho}, u_0^\delta - \bar{u} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

The existence and uniform bounds of solutions for this system are as follows.

Theorem 6.6.1. *Let ρ_0^δ, u_0^δ be approximate initial data such that*

$$(\rho_0^\delta - \bar{\rho}, u_0^\delta - \bar{u}) \in C_0^\infty(\mathbb{R})^2 \tag{6.6.2}$$

and, for some $M_0 > 0$ independent of $\delta > 0$,

$$0 < \rho_0^\delta \leq M_0, \qquad |u_0^\delta| \leq M_0 < c. \tag{6.6.3}$$

Then there exist global solutions (ρ^δ, u^δ) *to system* (6.6.1) *such that*

$$(\rho^\delta(t,\cdot) - \bar{\rho}, u^\delta(t,\cdot) - \bar{u}) \in C^1 \cap H^1,$$

and

$$0 < \rho^\delta \leq M, \qquad |u^\delta| \leq M < c,$$

where $M > 0$ *is independent of* $\delta > 0$.

Such a theorem is a standard consequence of parabolic PDE theory. For the sake of completeness, we include a proof here. To demonstrate the existence of the invariant region for the regularised equations, we employ the Riemann invariants w and z. In terms of the Riemann invariants, the equations partially decouple, leaving us with heat equations for w and z that admit a maximum principle. This maximum principle and the ensuing invariant region is the content of the following lemma. Throughout this section, we drop the explicit dependence of functions on $\delta > 0$, which is assumed to be fixed and we allow constants to depend now on δ unless stated otherwise, as well as the other parameters of the problem.

Lemma 6.6.2. *Any* $C^{2,1}$ *solution* $(\rho(t,x), u(t,x))$ *to the system* (6.6.1) *admits the following uniform bounds, independent of* $\delta > 0$.

$$\|(k(\rho), v(u))\|_{L^\infty(\mathbb{R}^2_+)} \leq M(\|w_0\|_{L^\infty(\mathbb{R})} + \|z_0\|_{L^\infty(\mathbb{R})}),$$

where $w_0(x) = w(\rho_0, u_0)(x)$ *and* $z_0(x) = z(\rho_0, u_0)(x)$. *Thus we obtain that*

$$\|\rho\|_{L^\infty(\mathbb{R}^2_+)} \leq M(\|w_0\|_{L^\infty(\mathbb{R})} + \|z_0\|_{L^\infty(\mathbb{R})})$$

and

$$\|u\|_{L^\infty(\mathbb{R}^2_+)} \leq M(\|w_0\|_{L^\infty(\mathbb{R})} + \|z_0\|_{L^\infty(\mathbb{R})}) < \frac{1}{\sqrt{\varepsilon}}.$$

Proof. We calculate from the equations that

$$\partial_t w + \lambda_2 w_x - \delta w_{xx} = -\delta U_x \nabla^2 w \, U_x^\top,$$

where λ_2 is the second eigenvalue of ∇F and $U = U(\rho, u)$ is as in (6.1.2). We exploit the quasi-convexity property of the Riemann invariants in order to derive a parabolic inequality for w which allows the application of a maximum principle. Indeed, it is easy to see that

$$\nabla^\perp w \nabla^2 w (\nabla^\perp w)^\top \geq 0.$$

We therefore make the decomposition:

$$U_x = \alpha \nabla w + \beta \nabla^{\perp} w,$$

where

$$\alpha = \frac{U_x \cdot \nabla w}{|\nabla w|^2}, \qquad \beta = \frac{U_x \cdot \nabla^{\perp} w}{|\nabla w|^2}.$$

With this notation, we re-write the equation above as

$$w_t + \lambda w_x - \delta w_{xx} = -\delta \beta^2 \nabla^{\perp} w \nabla^2 w (\nabla^{\perp} w)^{\top},$$

where

$$\lambda := \lambda_2 + \delta\alpha \frac{\nabla w \nabla^2 w (\nabla w)^{\top}}{|\nabla w|^2} + 2\delta\beta \frac{\nabla^{\perp} w \nabla^2 w (\nabla w)^{\top}}{|\nabla w|^2}.$$

Then, using the quasi-convexity property of w, we find that

$$w_t + \lambda w_x - \delta w_{xx} \leq 0.$$

Likewise,

$$\partial_t z + \tilde{\lambda} z_x - \delta z_{xx} \geq 0,$$

where $\tilde{\lambda} = \lambda_1 + \delta\tilde{\alpha} \frac{\nabla z \nabla^2 z (\nabla z)^{\top}}{|\nabla z|^2} + 2\delta\tilde{\beta} \frac{\nabla^{\perp} z \nabla^2 z (\nabla z)^{\top}}{|\nabla z|^2}$. Therefore, applying the invariant region theory of Chueh, Conley and Smoller, [17], we obtain uniform bounds of the form

$$w(t,x) \leq \max_{\mathbb{R}} w_0(x), \qquad -z(t,x) \leq \max_{\mathbb{R}} -z_0(x).$$

Noting that $k(\rho) = \frac{w-z}{2}$, and that $k(\rho)$ is a strictly increasing function of ρ, this implies a uniform bound on k and hence on ρ. Moreover, we observe that in the region where $v(u) \geq 0$, we have that $v(u) \leq w$, and that in the region where $v(u) \leq 0$, we have that $|v(u)| \leq -z$. Therefore, we deduce a uniform bound on $|v(u)|$ also, and hence on $|u| \leq M < c$. □

To demonstrate the local existence of the solutions to the approximate equations, we argue via Banach's fixed point theorem. Writing

$$G(t,x) := \frac{1}{2\sqrt{\delta \pi t}} \exp\Big(-\frac{x^2}{4\delta t}\Big)$$

for the heat kernel, a solution $U = U(\rho, u)$ satisfies

$$U(t,x) = \int_{\mathbb{R}} G(t, x-y) U_0(y)\, dy + \int_0^t \int_{\mathbb{R}} \frac{\partial G(t-\tau, x-y)}{\partial y} F(U)\, dy\, d\tau. \tag{6.6.4}$$

The other a priori bound that we need is the a uniform lower bound on the approximate solutions, avoiding the vacuum region. This follows as a consequence of a lemma of DiPerna and Chen.

Lemma 6.6.3 ([6, Corrected Lemma]). *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a function such that $\phi(t) \geq 0$ and such that*

$$\begin{aligned}
&\phi(t) - \phi(s) \geq -c_1(t-s)^{\frac{1}{2}} \quad \text{for all } \ 0 \leq s < t \leq T < \infty, \\
&\int_0^T \phi^{-\alpha}(s)\, ds \leq c_2 \quad \text{for some } \alpha \in [2, \infty), \\
&\phi(0) \geq \delta > 0,
\end{aligned}$$

where c_1, c_2, T and δ are positive constants. Then there exists $c_3 > 0$ such that

$$\phi(t) \geq c_3 > 0 \text{ for all } t \in [0, T].$$

Now, given any solution of (6.6.1), we write $N = \frac{n(\rho)}{\sqrt{1-\varepsilon u^2}}$ and $\bar{N} = N(\bar{\rho}, \bar{u})$, and choose a function $h(N)$, strictly convex, such that $h(\bar{N}) = h'(\bar{N}) = 0$, and such that

$$h(N) = N^{-\alpha} \text{ for } N \in (0, \bar{N}/2).$$

Then

$$\begin{aligned}
h(N) &\leq c(N - \bar{N})^2 \text{ for } N \text{ near } \bar{N}, \\
N^2 h''(N) &\leq CN \text{ for } \frac{1}{2}\bar{N} \leq N \leq M, \\
N^2 h''(N) &\leq Ch(N) \text{ for } 0 < N \leq \frac{1}{2}\bar{N},
\end{aligned}$$

where $M > 0$. Hence if $|u| \leq M$, we have that $h''(N)N^2u^2 \leq C(Nu^2 + h)$. We multiply the first equation in (6.6.1) by $h'(N)$ to obtain

$$h(N)_t + (h'(N)Nu)_x - h''(N)N_x Nu = \delta h(N)_{xx} - \delta h''(N)N_x^2.$$

Integrating, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} h(t) - h(0)\, dx + \delta \int_0^t \int_{\mathbb{R}} h''(N)N_x^2\, dx\, d\tau &= \int_0^T \int_{\mathbb{R}} h''(N)N_x Nu\, dx\, d\tau \\
&\leq \int_0^t \int_{\mathbb{R}} \big(\frac{M}{\delta} h''(N)N^2u^2 + \frac{\delta}{M} h''(N)N^2N_x^2\big)\, dx\, d\tau.
\end{aligned}$$

Choosing h as above with $\alpha = 4$, we obtain that

$$\int_{\mathbb{R}} N^{-4}(t,x)\, dx + \int_0^T \int_{\mathbb{R}} (N^{-2})_x^2\, dx\, d\tau \leq \int_{\mathbb{R}} N_0^{-4}(x)\, dx \leq M.$$

Applying the Sobolev inequality, we deduce

$$\int_0^t \sup_x N(\tau, x)^{-2}\, d\tau \leq M(T),$$

i.e.

$$\int_0^t \big(\min_x N(\tau, x)\big)^{-2}\, d\tau \leq M(T).$$

We observe that the derivative

$$|G_x(t-\tau,x)| \le \frac{c_1}{t-\tau} \exp\Big(-\frac{(x-y)^2}{2(t-\tau)}\Big),$$

and hence

$$\int_{\mathbb{R}} |G_y(t-\tau,x-y)|\,dy \le c_2(t-\tau)^{-\frac{1}{2}}.$$

Thus, from the representation formula (6.6.4), we have

$$\begin{aligned}
N(t,x) &= \int_{\mathbb{R}} G(t,x-y)N_0(y)\,dy + \int_0^t \int_{\mathbb{R}} \frac{\partial G(t-\tau,x-y)}{\partial y} Nu\,dy\,d\tau \\
&\ge c\min N_0 - M\int_0^t \int_{\mathbb{R}} G_y(t-\tau,x-y)\,dy\,d\tau \\
&\ge c\min N_0 - Mt^{\frac{1}{2}}.
\end{aligned}$$

We therefore deduce that the assumptions of Lemma 6.6.3 are satisfied for the function

$$\phi(t) = \min_x N(t,x),$$

and hence the a priori bound

$$N(t,x) \ge c_3.$$

Lemma 6.6.4. *Let the initial data $U_0 = U(\rho_0,\rho_0)$ satisfy assumptions* (6.6.2) *and* (6.6.3)*. Then, for any $M > M_0$, there exists $T_0(M) > 0$ and a unique weak solution $U(t,x) \in C^{2,1}$ to the Cauchy problem* (6.6.1) *on $Q_{T_0} = [0,T_0)\times\mathbb{R}$ satisfying*

$$|U(t,x)| \le M, \qquad (\rho(t,\cdot)-\bar\rho, u(t,\cdot)-\bar u) \in C^1 \cap H^1.$$

Proof. We extend the pressure function $p(\rho)$ to a C^1 function $\tilde p$ defined on all of $\mathbb{R}$ such that $\tilde p(\rho)|_{\rho\ge 0} = p(\rho)$ and argue via contractivity of the operator

$$\begin{gathered}
P : C^{1,1}(Q_{T_0}) \times C^{1,1}(Q_{T_0}) \to C^{1,1}(Q_{T_0}) \times C^{1,1}(Q_{T_0}), \\
U(t,x) \mapsto P(U),
\end{gathered}$$

defined by

$$P(U) = \int_{\mathbb{R}} G(t,x-y)U_0(y)\,dy + \int_0^t \int_{\mathbb{R}} \frac{\partial G(t-\tau,x-y)}{\partial y} F(U)\,dy\,d\tau,$$

as in the standard theory for the heat equation. □

With the uniform estimates of Lemma 6.6.2 and the existence of a local solution, it is not hard to prove that the local solution may be extended globally.

Lemma 6.6.5. *The local solution of Lemma 6.6.4 may be extended to a global solution of system* (6.6.1)*.*

Proof. We must show that on a time interval $(0, T)$ for any $T > 0$, that we have a uniform estimate of the form

$$\|U\|_{C^{2,1}(Q_T)} \leq M$$

for some $M = M(T) > 0$. With this, we may apply the local existence result again to extend the locally defined solution on any set of the form $[0, T]$.

We observe that the derivative

$$|G_x(t-\tau, x)| \leq \frac{c_1}{t-\tau} \exp\Big(-\frac{(x-y)^2}{2(t-\tau)}\Big),$$

and hence

$$\int_{\mathbb{R}} |G_y(t-\tau, x-y)|\, dy \leq c_2(t-\tau)^{-\frac{1}{2}}.$$

Thus, from the representation formula (6.6.4), we may estimate the k-th derivatives of $U(t,x)$ by induction as

$$|\nabla^k U(t,x)| \leq C\|\nabla^k U_0\|_{L^\infty} + C\int_0^t \|\nabla^k U(t,\cdot)\|_{L^\infty}(t-\tau)^{-\frac{1}{2}}\, d\tau,$$

allowing us to employ Gronwall's inequality to deduce the estimate

$$\|\nabla^k U(t,\cdot)\|_{L^\infty} \leq C\|\nabla^k U_0\|_{L^\infty} \exp\Big(\int_0^t (t-\tau)^{-\frac{1}{2}}\, d\tau\Big) \leq C\|\nabla^k U_0\|_{L^\infty} e^{C\sqrt{t}}.$$

□

Now that we have established both the existence of the approximate solutions and the existence of the invariant region, it remains only to establish the H^{-1}_{loc} compactness of the entropy dissipation measures $\partial_t \eta(U^\delta) + \partial_x q(U^\delta)$. Before we can demonstrate this fact, however, we will require estimates that are *uniform in* δ on the derivatives ρ^δ_x and u^δ_x. These estimates arise as a result of the entropy structure of the equations and the relative entropy method.

We recall the definitions of the physical entropy and entropy flux pair from §6.1.2:

$$\eta^*(U) = \frac{\rho + p\varepsilon^2 u^2}{1-\varepsilon u^2}, \qquad q^*(U) = \frac{(\rho+\varepsilon p)u}{1-\varepsilon u^2},$$

and

$$\nabla^2\eta^*(U) = \alpha_0(\rho, u)\begin{pmatrix} \frac{\varepsilon(\rho+\varepsilon p)(p'+u^2+2p'\varepsilon u^2)}{n(1-\varepsilon u^2)} & \frac{-\varepsilon(1+\varepsilon p')u}{\sqrt{1-\varepsilon u^2}} \\ -\frac{\varepsilon(1+\varepsilon p')u}{\sqrt{1-\varepsilon u^2}} & \frac{\varepsilon n}{\rho+\varepsilon p} \end{pmatrix},$$

where $\alpha_0(\rho, u) = \frac{(1-\varepsilon u^2)^2}{n(1-p'\varepsilon^2 u^2)} > 0$. To allow for the non-trivial end-point states, we define a modified entropy pair (designed using the relative entropy method) by

$$\overline{\eta^*}(U) = \eta^*(U) - \eta^*(\bar{U}) - \nabla\eta^*(\bar{U}) \cdot (U - \bar{U}),$$

where $\bar{U} = U(\bar{\rho}, \bar{u})$. As η^* is convex, we have that $\overline{\eta^*}(U) \geq 0$, and observe that $\nabla^2\overline{\eta^*}(U) = \nabla^2\eta^*(U) \geq 0$. The modified entropy function has a modified entropy flux, given by

$$\overline{q^*}(U) = q^*(U) - q^*(\bar{U}) - \nabla\eta^*(\bar{U}) \cdot (F(U) - F(\bar{U})).$$

Lemma 6.6.6. *Suppose that*

$$\int_{\mathbb{R}} \overline{\eta^*}(U_0)(x)\, dx < \infty.$$

Then there exists $M > 0$, independent of δ, such that the solution $U^\delta(t, x)$ of (6.6.1) satisfies

$$\delta \int_0^T \int_{\mathbb{R}} \rho^{\gamma-2}|\rho_x|^2 + \rho|u_x|^2\, dx\, dt \leq M.$$

Proof. We multiply the equation

$$U_t + F(U)_x = \delta U_{xx}$$

by $\nabla\overline{\eta^*}(U^\delta)$ to obtain

$$\overline{\eta^*}(U)_t + \overline{q^*}(U)_x = \delta\big(\eta^*(U)\big)_{xx} - \delta(U_x)^\top \nabla^2\eta^*(U) U_x.$$

Integrating this equation and recalling the positivity of $\overline{\eta^*}$, we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \delta(U_x)^\top \nabla^2\eta^*(U) U_x\, dx\, dt &= \int_{\mathbb{R}} \overline{\eta^*}(U_0(x))\, dx - \int_{\mathbb{R}} \overline{\eta^*}(U(T,x))\, dx \\ &\leq \int_{\mathbb{R}} \overline{\eta^*}(U_0(x))\, dx \leq M.\end{aligned} \tag{6.6.5}$$

We now observe from a calculation that

$$\begin{aligned}U_x^\top \nabla^2\eta^* U_x = \frac{\alpha_0}{(1-\varepsilon u^2)^2}\Big(&\frac{\varepsilon n\rho_x^2 p'(1-p'\varepsilon^2u^2)}{\rho+\varepsilon p} + \frac{2\varepsilon^2 nup'\rho_x u_x(1-p'\varepsilon^2u^2)}{1-\varepsilon u^2} \\ &+ \varepsilon n u_x^2(\rho+\varepsilon p) + \frac{2\varepsilon^2 n u_x^2 u^2(\rho+\varepsilon p)(1-p'\varepsilon^2u^2)}{(1-\varepsilon u^2)^2} \\ &+ \frac{\varepsilon^3 u^2 n u_x^2(\rho+\varepsilon p)(p'(1-\varepsilon u^2)+u^2(1-\varepsilon p'))}{(1-\varepsilon u^2)^2}\Big).\end{aligned}$$

By completing the square on the first, second and final terms in this expression (observe that the other terms are positive already), we find that

$$U_x^\top \nabla^2\eta^* U_x \geq \varepsilon c_0(\rho+\varepsilon p)u_x^2,$$

and hence

$$\delta \int_0^T \int_{\mathbb{R}} \rho |u_x|^2 \, dx \, dt \leq M.$$

Returning now to (6.6.5), we apply the Cauchy-Young inequality and the uniform bounds on ρ and u to the second term to deduce that

$$\begin{aligned} U_x^\top \nabla^2 \eta^* U_x &\geq \frac{1}{2} \frac{\alpha_0}{(1 - \varepsilon u^2)^2} \frac{\varepsilon n \rho_x^2 p'(1 - p' \varepsilon^2 u^2)}{\rho + \varepsilon p} - M \rho u_x^2 \\ &\geq c_* \rho^{\gamma - 2} \rho_x^2 - M \rho u_x^2, \end{aligned}$$

and hence by applying the previous estimate to the second term here, we deduce that

$$\delta \int_0^T \int_{\mathbb{R}} \rho^{\gamma-2} |\rho_x|^2 \, dx \, dt \leq M$$

also. This completes the proof of the lemma. □

In the case $\gamma \leq 2$, this is sufficient to deduce the desired compactness of the entropy dissipation measures. However, when $\gamma > 2$, this bound is not enough. In fact, we must control the quantity

$$\delta \int_0^T \int_K |\rho_x|^2 \, dx \, dt$$

uniformly in $\delta > 0$ over compact sets $K \subset \mathbb{R}$. We state this additional control as another lemma.

Lemma 6.6.7. *On any compact subset $K \subset \mathbb{R}$, the solution $U = U(\rho, u)$ of (6.6.1) satisfies, for any $\Delta > 0$,*

$$\delta \int_0^T \int_{K \cap \{\frac{n(\rho)}{\sqrt{1 - \varepsilon u^2}} < \Delta\}} |\rho_x|^2 \, dx \, dt \leq M\Delta + M \frac{\Delta^2}{\delta} + M \Delta^{\frac{4-\gamma}{2}},$$

for some $M > 0$ independent of $\delta > 0$.

Proof. We write $N = \frac{n(\rho)}{\sqrt{1 - \varepsilon u^2}}$, so that the first equation in (6.6.1) becomes

$$N_t + (Nu)_x = \delta N_{xx}.$$

Now we set, for $\Delta > 0$ to be determined later,

$$\phi(N) := \begin{cases} \frac{1}{2} N^2, & N < \Delta, \\ \frac{1}{2} \Delta^2 + \Delta(N - \Delta), & N \geq \Delta. \end{cases}$$

Now $\phi'(N) = N \mathbb{1}_{N < \Delta} + \Delta \mathbb{1}_{N \geq \Delta} = \min\{N, \Delta\}$ and $\phi''(N) = \mathbb{1}_{N < \Delta}$. Multiplying the first equation in (6.6.1) by $\phi'(N) \omega^2(x)$, where $\omega \in C_c^\infty(\mathbb{R})$ is a spatial test function,

$$(\phi(N) \omega^2)_t + \phi'(N)(Nu)_x \omega^2 = \delta \phi'(N) N_{xx} \omega^2.$$

Thus, integrating by parts and writing $K = \operatorname{supp}\omega$, we find

$$\int_K \phi(N)\omega^2\,dx\Big|_0^T - \int_0^T\int_K \phi''(N)N_xNu\omega^2\,dx\,dt - \int_0^T\int_K 2\phi'(N)Nu\omega_x\omega\,dx\,dt$$
$$= -\delta\int_0^T\int_K \phi''(N)N_x^2\omega^2\,dx\,dt - 2\delta\int_0^T\int_K \phi'(N)N_x\omega_x\omega\,dx\,dt.$$

Rearranging this equation and recalling the expressions for $\phi'(N)$ and $\phi''(N)$ above, we find

$$\begin{aligned}
&\delta\int_0^T\int_{K\cap\{N<\Delta\}} N_x^2\omega^2\,dx\,dt\\
&\quad= \int_K \phi(N)\omega^2\,dx\Big|_0^T + \int_0^T\int_{K\cap\{N\le\Delta\}} N_xNu\omega^2\,dx\,dt\\
&\qquad+ 2\int_0^T\int_K \min\{N,\Delta\}Nu\omega_x\omega\,dx\,dt - 2\delta\int_0^T\int_{K\cap\{N<\Delta\}} NN_x\omega_x\omega\,dx\,dt\\
&\qquad- 2\delta\int_0^T\int_K \Delta N_x\omega_x\omega\,dx\,dt.
\end{aligned}$$

We recall now that we have a uniform bound on N and u as well as on ω and ω_x. We therefore note that $\phi(N)\le C\Delta$ and apply Hölder's inequality to obtain

$$\begin{aligned}
&\delta\int_0^T\int_{K\cap\{N<\Delta\}} N_x^2\omega^2\,dx\,dt\\
&\quad\le M\Delta + M\Delta\Big(\int_0^T\int_{K\cap\{N<\Delta\}} N_x^2\omega^2\,dx\,dt\Big)^{\frac{1}{2}} + M\sqrt{\delta}\Delta^{\frac{4-\gamma}{2}},
\end{aligned}$$

where we have used that

$$\delta\int_0^T\int_K \mathbb{1}_{N\ge\Delta}\Delta^2N_x^2\omega^2\,dx\,dt \le M\Delta^{4-\gamma}.$$

Thus

$$\delta\int_0^T\int_{K\cap\{N<\Delta\}} N_x^2\omega^2\,dx\,dt \le M\Delta + M\frac{\Delta^2}{\delta} + M\Delta^{\frac{4-\gamma}{2}}.$$

□

As a final step before considering the entropy dissipation measures, we note that a calculation based on the representation formula (6.2.6) shows that the Hessian of any entropy function for the relativistic Euler equations can be dominated by the Hessian of the physical entropy.

Lemma 6.6.8. *For any entropy/entropy-flux pair (η, q), the ordering*

$$|\nabla^2\eta| \le M\nabla^2\eta^*$$

holds, with matrices ordered in the usual way.

We now show the final result of this section.

Proposition 6.6.9. *Let $U^\delta = U^\delta(\rho^\delta, u^\delta)$ be a sequence of solutions to (6.6.1) with initial data U_0^δ. Then, for any weak entropy pair (η, q), the entropy dissipation measures*

$$\eta(U^\delta)_t + q(U^\delta)_x \text{ are compact in } H^{-1}_{loc}(\mathbb{R}^2_+).$$

Proof. Throughout the proof, we write $\eta^\delta = \eta(U^\delta)$ and $q^\delta = q(U^\delta)$. Multiplying the equation

$$U^\delta_t + F(U^\delta)_x = \delta U^\delta_{xx}$$

by $\nabla\eta(U^\delta)$, we find

$$\eta^\delta_t + q^\delta_x = \delta\eta^\delta_{xx} - \delta(U^\delta_x)^\top \nabla^2\eta^\delta U^\delta_x.$$

From Lemma 6.6.8, we may bound the last term in $L^1_{loc}(\mathbb{R}^2_+)$ by observing that for any compact $K \subset \mathbb{R}$,

$$\int_0^T\int_K \left|\delta(U^\delta_x)^\top\nabla^2\eta^\delta U^\delta_x\right| dx\,dt \leq \int_0^T\int_K \left|\delta(U^\delta_x)^\top\nabla^2\eta^{*,\delta} U^\delta_x\right| dx\,dt \leq M,$$

independent of $\delta > 0$. Applying the compact embedding of L^1_{loc} into $W^{-1,q}_{loc}$ with $1 < q < 2$, we find that this term is compact in $W^{-1,q}_{loc}$, $q < 2$.

For the other term on the right hand side, $\delta\eta^\delta_{xx}$, we recall that we may bound

$$|\eta_x(\rho, u)| \leq M(|\rho_x|(1+\rho^\theta) + \rho|u_x|).$$

Now if $\gamma \leq 2$, Lemma 6.6.6 implies that

$$\sqrt{\delta}\eta^\delta_x \text{ is uniformly bounded in } L^2(\mathbb{R}^2_+),$$

and hence that

$$\delta\eta^\delta_{xx} \text{ is compact in } W^{-1,2}_{loc}(\mathbb{R}^2_+).$$

In the case that $\gamma > 2$, we apply the estimate of Lemma 6.6.7 to deduce that on a compact set $K \subset \mathbb{R}$,

$$\begin{aligned}
\delta^2\int_0^T\int_K |\eta_x|^2\,dx\,dt &\leq M\delta^2\int_0^T\int_K |\rho_x|^2 + \rho|u_x|^2\,dx\,dt \\
&\leq M\delta^2\int_0^T\int_K |N_x|^2(\mathbb{1}_{N<\Delta} + \mathbb{1}_{N\geq\Delta}) + \rho|u_x|^2\,dx\,dt \\
&\leq M\big(\delta\Delta + \Delta^2 + \delta\Delta^{\frac{4-\gamma}{2}} + \delta\Delta^{2-\gamma} + \delta\big),
\end{aligned}$$

where $\Delta > 0$ is to be chosen now. In fact, we see that choosing $\Delta = \delta^\alpha$ with $\alpha \in (0, 1/(\gamma - 2))$ implies that this expression converges to 0 as $\delta \to 0$. Hence we see that

$$\delta\eta_{xx} \text{ is compact in } W^{-1,2}_{loc}(\mathbb{R}^2_+).$$

We have therefore seen that the entropy dissipation measure

$$\eta^\delta_t + q^\delta_x \text{ is compact in } W^{-1,q}_{loc}(\mathbb{R}^2_+),$$

for some $q \in (1, 2)$. On the other hand, as the approximate solutions (ρ^δ, u^δ) are uniformly bounded, we also have that

$$\eta^\delta_t + q^\delta_x \text{ is bounded uniformly in } W^{-1,\infty}_{loc}(\mathbb{R}^2_+).$$

Applying now the compensated compactness interpolation theorem, Theorem 3.4.3,

$$\eta^\delta_t + q^\delta_x \text{ is compact in } W^{-1,2}_{loc}(\mathbb{R}^2_+),$$

as desired. □

6.7 Proof of the main result

For clarity, we restate the main theorem of this chapter.

Theorem 6.7.1 (Existence of Entropy Solution). *Let $U_0 = U(\rho_0, u_0)$ be measurable and bounded initial data satisfying*

$$0 \le \rho_0(x) \le C_0, \quad |u_0(x)| \le C_0 < \frac{1}{\sqrt{\varepsilon}} \text{ for almost every } x \in \mathbb{R}$$

for a constant $C_0 > 0$ and suppose the pressure $p(\rho)$ satisfies (6.1.3) for $\rho > 0$ and (6.1.10).

Then there exists a sequence U^δ of artificial viscosity solutions to the approximate equations (6.6.1) and a constant $\varepsilon_0 > 0$ such that if $\varepsilon \le \varepsilon_0$, then the sequence U^δ converges to an entropy solution $U = U(\rho, u)$ of (6.1.1) in the sense of Definition 6.1.6 such that

$$0 \le \rho(t, x) \le C, \quad |u(t, x)| \le C < \frac{1}{\sqrt{\varepsilon}} \text{ for almost every } (t, x) \in \mathbb{R}^2_+,$$

where $C > 0$ depends on the initial data. The convergence may be taken to be almost everywhere and strong in $L^p_{loc}(\mathbb{R}^2_+)$ for all $p \in [1, \infty)$.

To prove the theorem, we simply observe that, after adding a small constant to the initial density, cutting off outside a compact set and then mollifying the initial data, we obtain initial data $(\rho_0^\delta, u_0^\delta)$ satisfying the assumptions of Theorem 6.6.1 and such that $(\rho_0^\delta, u_0^\delta) \to (\rho_0, u_0)$ for almost every $x \in \mathbb{R}$ and also in $L^p_{loc}(\mathbb{R})$ for all $p \in [1, \infty)$. The existence of the approximate solutions is then given by Theorem 6.6.1, and the results of §6.6 then imply that these approximate solutions are uniformly bounded and satisfy the condition

$$\eta(U^\delta)_t + q(U^\delta)_x \text{ is compact in } H^{-1}_{loc}(\mathbb{R}^2_+)$$

for all weak entropy pairs (η, q).

To find the constant ε_0, we take the constant $C_1 > 0$, where C_1 is the uniform bound on the approximate sequence of densities ρ^δ given by Lemma 6.6.2. We then use Proposition 6.4.6 to give $\varepsilon_0 > 0$ such that if $\varepsilon \le \varepsilon_0$, then the constant $\rho_M = \rho_M(\varepsilon) > 0$ is such that $C_1 \le \rho_M$.

This then implies that the sequence of approximate solutions (ρ^δ, u^δ) satisfies the assumptions of the compactness framework of §6.5. Thus Theorem 6.5.1 gives the strong convergence of the approximate solutions $(\rho^\delta, u^\delta) \to (\rho, u)$ in $L^p_{loc}(\mathbb{R}^2_+)$ for all $p \in [1, \infty)$.

To show that the limit functions provide an entropy solution of (6.1.1), we check from the weak formulation of the approximate solutions that for any test function $\phi \in C_c^\infty(\overline{\mathbb{R}^2_+}; \mathbb{R}^2)$,

$$\int_{\mathbb{R}^2_+} U^\delta \cdot \phi_t + F(U^\delta) \cdot \phi_x \, dx\, dt + \int_{\mathbb{R}} U_0^\delta \cdot \phi(0, x)\, dx = -\delta \int_{[0,T]\times\mathbb{R}} U^\delta \cdot \phi_{xx}\, dx\, dt.$$

As the sequence U^δ is uniformly bounded and converges almost everywhere, we may pass directly to the limit in this weak formulation and see that

$$\int_{\mathbb{R}^2_+} U \cdot \phi_t + F(U) \cdot \phi_x \, dx\, dt + \int_{\mathbb{R}} U_0 \cdot \phi(0, x)\, dx = 0,$$

as desired. To check the entropy condition, we take a convex, non-negative weak entropy η, multiply both sides of (6.6.1) by $\nabla\eta(U^\delta)$ and obtain

$$\eta(U^\delta)_t + q(U^\delta)_x = \delta\eta(U^\delta)_{xx} - \delta(U_x^\delta)^\top \nabla^2\eta(U^\delta) U_x^\delta.$$

Testing again with a non-negative function $\phi \in C_c^\infty(\overline{\mathbb{R}^2_+})$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2_+} \eta(U^\delta)\phi_t + q(U^\delta)\phi_x \, dx\, dt + \int_{\mathbb{R}} \eta(U_0^\delta)\phi(0,x)\, dx &= \delta \int_{\mathbb{R}^2_+} (U_x^\delta)^\top \nabla^2 \eta\, U_x^\delta \phi \, dx\, dt \\
&\quad - \delta \int_{\mathbb{R}^2_+} \eta(U^\delta)\phi_{xx}\, dx\, dt \\
&\geq - \delta \int_{\mathbb{R}^2_+} \eta(U^\delta)\phi_{xx}\, dx\, dt.
\end{aligned}
$$

By the strong convergence properties obtained, as $\delta \to 0$, this inequality converges to

$$
\int_{\mathbb{R}^2_+} \eta(U)\phi_t + q(U)\phi_x \, dx\, dt + \int_{\mathbb{R}} \eta(U_0)\phi(0,x)\, dx \geq 0.
$$

6.8 Convergence in the Newtonian limit

The final section of this chapter is devoted to proving the final theorem of this thesis, Theorem 6.1.3, concerning the Newtonian limit. Indeed, the construction and analysis of the relativistic entropy kernels set out above is sufficient to perform this limiting process rigorously. We are therefore able to show that a sequence of entropy solutions to the relativistic Euler equations converges strongly to an entropy solution of the classical Euler equations (up to subsequence) as the speed of light converges to infinity.

We first introduce some notation to make explicit the dependence on ε of quantities in this section. For $\varepsilon \in (0,1)$, we rewrite (6.1.1) as

$$
\partial_t U^\varepsilon + \partial_x F^\varepsilon(U^\varepsilon) = 0, \tag{6.8.1}
$$

where $U^\varepsilon(\rho^\varepsilon, u^\varepsilon) = \left(\frac{n(\rho^\varepsilon)}{\sqrt{1-\varepsilon(u^\varepsilon)^2}}, \frac{(\rho^\varepsilon + \varepsilon p(\rho^\varepsilon))u^\varepsilon}{1-\varepsilon(u^\varepsilon)^2}\right)^\top$, F^ε is the associated flux. We recall the definition of the function $v^\varepsilon = v^\varepsilon(u^\varepsilon)$ from §6.1.1,

$$
v^\varepsilon(u^\varepsilon) = \frac{1}{2\sqrt{\varepsilon}} \log\big(\frac{1+\sqrt{\varepsilon}u^\varepsilon}{1-\sqrt{\varepsilon}u^\varepsilon}\big).
$$

We write the classical equations, (6.1.5), as

$$
\partial_t U + \partial_x F(U) = 0, \tag{6.8.2}
$$

where $U = (\rho, m)^\top$, $F(U)$ is the associated flux, $F = (m, \frac{m^2}{\rho} + p(\rho))^\top$.

6.8.1 Proof of Theorem 6.1.3

By assumption, we have uniform bounds on $(\rho^\varepsilon, v^\varepsilon)$, and so we may extract a subsequence such that

$$(\rho^\varepsilon, v^\varepsilon) \overset{*}{\rightharpoonup} (\rho, v)$$

weakly-star in L^∞. To this sequence, we may associate a Young measure $\nu_{t,x}$, $(t,x) \in \mathbb{R}^2_+$. Our aim now is to show that we may apply a reduction argument analogous to that of Theorem 6.5.2. However, we observe that Theorem 6.5.2 holds only for fixed $\varepsilon > 0$, not for a sequence. We therefore recall the following theorem from [8].

Theorem 6.8.1 ([8], Theorem 4.2). *Let $\nu(\rho, v)$ be a probability measure with bounded support in $\{\rho \geq 0,\ v \in \mathbb{R}\}$ such that*

$$\langle \nu, \chi^*(s_1)\sigma^*(s_2) - \chi^*(s_2)\sigma^*(s_1)\rangle = \langle \nu, \chi^*(s_1)\rangle\langle \nu, \sigma^*(s_2)\rangle - \langle \nu, \chi^*(s_2)\rangle\langle \nu, \sigma^*(s_1)\rangle$$

for any $s_1, s_2 \in \mathbb{R}$, where $\chi^*(\rho, v - s)$ and $\sigma^*(\rho, v, s)$ are the entropy and entropy flux kernels of the classical Euler equations. Then the support of ν is either a single point or a subset of the vacuum line, $\{\rho = 0\}$.

Defining $m^\varepsilon := \rho^\varepsilon v^\varepsilon$, if we can show that

$$\eta(\rho^\varepsilon, m^\varepsilon)_t + q(\rho^\varepsilon, m^\varepsilon)_x \text{ is compact in } H^{-1}_{loc},$$

for any weak entropy pair of (η, q) of system (6.8.2), then we may apply the div-curl lemma to deduce the commutation relation and hence apply Theorem 6.8.1 to deduce the strong convergence of the sequence $(\rho^\varepsilon, v^\varepsilon)$.

Now we rewrite the entropy dissipation measures as

$$\begin{aligned} \eta(\rho^\varepsilon, \rho^\varepsilon v^\varepsilon)_t + q(\rho^\varepsilon, \rho^\varepsilon v^\varepsilon)_x =& (\eta(\rho^\varepsilon, \rho^\varepsilon v^\varepsilon) - \eta^\varepsilon(\rho^\varepsilon, \rho^\varepsilon v^\varepsilon))_t \\ &+ (q(\rho^\varepsilon, \rho^\varepsilon v^\varepsilon) - q^\varepsilon(\rho^\varepsilon, \rho^\varepsilon v^\varepsilon))_x \\ &+ \eta^\varepsilon(\rho^\varepsilon, \rho^\varepsilon v^\varepsilon)_t + q^\varepsilon(\rho^\varepsilon, \rho^\varepsilon v^\varepsilon)_x, \end{aligned} \tag{6.8.3}$$

where η, η^ε are weak entropies generated by the same test function ψ by convolution with the weak entropy kernels associated to (6.8.2), (6.8.1), respectively, and similarly for q, q^ε.

As, for each ε, U^ε is an entropy solution of (6.8.1), we have that for convex, non-negative entropies,

$$\eta^\varepsilon(U^\varepsilon)_t + q^\varepsilon(U^\varepsilon)_x \leq 0$$

in H^{-1}_{loc}. Hence, by Murat's lemma, Theorem 3.4.4, these entropy dissipation measures are compact in $W^{-1,q}_{loc}$ for any $q < 2$.

Likewise, for any concave, non-positive entropy, we have

$$\eta^\varepsilon(U^\varepsilon)_t + q^\varepsilon(U^\varepsilon)_x \text{ is compact in } W^{-1,q}_{loc} \text{ for any } q < 2.$$

On the other hand, this expression is clearly bounded in $W^{-1,\infty}_{loc}$ by the uniform bounds on U^ε. Thus the interpolation compactness theorem, Theorem 3.4.3, gives that

$$\eta^\varepsilon(U^\varepsilon)_t + q^\varepsilon(U^\varepsilon)_x \text{ is compact in } W^{-1,2}_{loc}$$

for any convex and non-negative or concave and non-positive entropy.

Considering now

$$(\eta(U^\varepsilon) - \eta^\varepsilon(U^\varepsilon))_t + (q(U^\varepsilon) - q^\varepsilon(U^\varepsilon))_x\,,$$

we note the following lemma, the proof of which will be postponed temporarily.

Lemma 6.8.2. *Let χ^ε denote the relativistic weak entropy kernel derived above in Theorem 6.2.2, χ^* the classical weak entropy kernel derived in [8, Theorem 2.2], and similarly for σ^ε, σ^* as entropy flux kernels. Then, for $(\rho, v) \in \{0 \le \rho \le \rho_M,\ |v| \in \mathbb{R}\}$,*

$$|\chi^\varepsilon(\rho, v) - \chi^*(\rho, v)| + |\sigma^\varepsilon(\rho, v, 0) - \sigma^*(\rho, v, 0)| \le M\varepsilon,$$

where M depends on ρ_M, but is independent of ε.

Thus, for any (ρ, v),

$$\begin{aligned}|\eta(\rho, v) - \eta^\varepsilon(\rho, v)| \le &\int_{|v-s|\le k(\rho)} |\psi(s)||\chi^*(\rho, v-s) - \chi^\varepsilon(\rho, v-s)|\,ds\\ \le &M\varepsilon \int_{|v-s|\le k(\rho)} |\psi(s)|\,ds,\end{aligned}$$

and similarly for $q(\rho, v)$.

Then, for any compact set $K \subset \mathbb{R}^2_+$, $\phi \in H^1_0(K)$, we have that

$$\begin{aligned}\langle(\eta(U^\varepsilon)\ &-\eta^\varepsilon(U^\varepsilon))_t + (q(U^\varepsilon) - q^\varepsilon(U^\varepsilon))_x\,, \phi\rangle_{H^{-1}\times H^1_0(K)}\\ &\le \left(\|\eta(U^\varepsilon) - \eta^\varepsilon(U^\varepsilon)\|_{L^2(K)} + \|q(U^\varepsilon) - q^\varepsilon(U^\varepsilon)\|_{L^2(K)}\right)\|\phi\|_{H^1_0(K)}\\ &\le \varepsilon M|K|^{\frac{1}{2}}\|\phi\|_{H^1_0(K)} \int_{|v^\varepsilon - s|\le k(\rho^\varepsilon)} |\psi(s)|\,ds \to 0,\end{aligned}$$

as $\varepsilon \to 0$ for any appropriate test function ψ, as U^ε is uniformly bounded. Thus the entropy dissipation measures (6.8.3) are compact in H^{-1}_{loc}. We have therefore shown compactness of the entropy dissipation measures (6.8.3) for all entropy pairs of the equations (6.8.2) such that the entropy is either convex and non-negative, or concave and non-positive. Applying the div-curl Lemma in the standard way, as in the proof of Theorem 6.5.1, we deduce the commutation relation

$$\langle \nu, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle$$

for all such weak entropy pairs.

We recall now that a weak entropy for the classical Euler equations, (6.8.2), is convex and non-negative (respectively concave and non-positive) if the generating test function is also convex and non-negative (respectively concave and non-positive). As the linear span of convex and non-positive functions and concave and non-negative functions is dense in our space of test functions, we argue by density as before to conclude that the commutation relation holds moreover for the kernels themselves, i.e.,

$$\langle \nu, \chi^*(s_1)\sigma^*(s_2) - \chi^*(s_2)\sigma^*(s_1) \rangle = \langle \nu, \chi^*(s_1) \rangle \langle \nu, \sigma^*(s_2) \rangle - \langle \nu, \chi^*(s_2) \rangle \langle \nu, \sigma^*(s_1) \rangle$$

for any $s_1, s_2 \in \mathbb{R}$.

We therefore apply Theorem 6.8.1 to deduce the strong convergence of the sequence $(\rho^\varepsilon, v^\varepsilon)$ almost everywhere and in $L^r_{loc}(\mathbb{R}^2_+)$ for all $r \in [1, \infty)$.

From this convergence, it is easy to see that (ρ, v) gives a distributional solution of (6.8.2). It remains to check that this solution is an entropy solution. By (6.8.3), to do this, we need only show that $(\eta(U^\varepsilon) - \eta^\varepsilon(U^\varepsilon))_t + (q(U^\varepsilon) - q^\varepsilon(U^\varepsilon))_x$ converges to 0 in distribution as $\varepsilon \to 0$, as the entropy inequality for the relativistic solutions then gives that the right hand side of (6.8.3) will, in the limit, be non-positive. But we have already shown convergence to 0 in H^{-1}_{loc}. Moreover, the limit of the left hand side of (6.8.3) is the desired entropy dissipation measure by continuity of η, q and strong convergence of $(\rho^\varepsilon, v^\varepsilon)$ in $L^r_{loc}(\mathbb{R}^2_+)$.

Proof of Lemma 6.8.2. It is clear that as $\varepsilon \to 0$,

$$k^\varepsilon(\rho) := \int_0^\rho \frac{\sqrt{p'(s)}}{s + \varepsilon p(s)}\, ds \to k(\rho) := \int_0^\rho \frac{\sqrt{p'(s)}}{s}\, ds$$

locally uniformly in ρ, and also from the expressions given in §6.3.1–§6.4.1 that the coefficients a_i, b_i, $i = 1, 2$, converge uniformly to their classical counterparts, as stated in [8, Theorem 2.2]. Thus we need only show that the remainders $g^\varepsilon(\rho, v)$, $h^\varepsilon(\rho, v)$

determined in §6.3.2 and §6.4.2 converge to the remainder terms g, h of the classical kernels.

We recall that g^ε is defined as the fixed point

$$\widehat{g^\varepsilon}(\rho,\xi) = \int_0^\rho \tilde{K}^\varepsilon(\rho,s;\xi)\mathcal{F}\big(\mathcal{S}^\varepsilon(g^\varepsilon)\big)(s,\xi)\,ds,$$

where $\tilde{K}^\varepsilon$ is defined in (6.3.8), $\mathcal{S}^\varepsilon$ is defined in (6.3.2), and g is defined as the fixed point of

$$\widehat{g}(\rho,\xi) = \int_0^\rho \tilde{K}(\rho,s;\xi)\mathcal{F}\big(\mathcal{S}(g)\big)(s,\xi)\,ds,$$

where exact expressions for $\tilde{K}$ and $\mathcal{S}$ can be found in [9].

We see then that the difference $\tilde{K^\varepsilon} - \tilde{K}$ satisfies

$$|\tilde{K^\varepsilon}(\rho,s;\xi) - \tilde{K}(\rho,s;\xi)| \le C\varepsilon\Big(\frac{k(\rho)}{k'(\rho)}\Big)^{\frac{1}{2}}\Big(\frac{k(s)}{k'(s)}\Big)^{\frac{1}{2}}Q_\nu(\xi k(\rho))Q_\nu(\xi k(s))^{-1}R(\xi k(s))$$

for $0 \le s \le \rho$ and $\xi \in \mathbb{R}$, and similarly

$$|\mathcal{F}\big(\mathcal{S}^\varepsilon(g^\varepsilon) - \mathcal{S}(g)\big)| \le C\varepsilon|\widehat{f}_{\lambda+1}(\xi k(\rho))|\rho^{-1+2\theta} + \varepsilon^2 r(\rho,\xi).$$

Using the following identity for the difference $\widehat{g^\varepsilon}(\rho,\xi) - \widehat{g}(\rho,\xi)$,

$$\begin{aligned}
\widehat{g^\varepsilon}(\rho,\xi) - \widehat{g}(\rho,\xi) &= \int_0^\rho \tilde{K^\varepsilon}(\rho,s;\xi)\mathcal{F}\big(\mathcal{S}^\varepsilon(g^\varepsilon)\big)(s,\xi)\,ds - \int_0^\rho \tilde{K}(\rho,s;\xi)\mathcal{F}\big(\mathcal{S}(g)\big)(s,\xi)\,ds \\
&= \int_0^\rho \big(\tilde{K^\varepsilon} - \tilde{K}\big)(\rho,s;\xi)\mathcal{F}\big(\mathcal{S}^\varepsilon(g^\varepsilon)\big)(s,\xi)\,ds \\
&\quad + \int_0^\rho \tilde{K}(\rho,s;\xi)\mathcal{F}\big(\mathcal{S}^\varepsilon(g^\varepsilon) - \mathcal{S}(g)\big)(s,\xi)\,ds,
\end{aligned}$$

we argue as in the proof of Theorem 6.3.8 to conclude. □

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