

Journal of Physics A: Mathematical and Theoretical



PAPER

Unitary categorical symmetries

OPEN ACCESS

RECEIVED
30 November 2025

REVISED
28 April 2026

ACCEPTED FOR PUBLICATION
11 May 2026

PUBLISHED
21 May 2026

Original content from this work may be used under the terms of the [Creative Commons Attribution 4.0 licence](https://creativecommons.org/licenses/by/4.0/).

Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.



Thomas Bartsch*

Mathematical Institute, University of Oxford, Oxford, United Kingdom

* Author to whom any correspondence should be addressed.

E-mail: thomas.bartsch@maths.ox.ac.uk

Keywords: generalised symmetries, quantum algebra, category theory

Abstract

Global invertible symmetries act unitarily on local observables or states of a quantum system. In this note, we aim to generalise this statement to non-invertible symmetries by considering unitary actions of higher fusion category symmetries \mathcal{C} on twisted sector local operators. We propose that the latter transform in $*$ -representations of the tube algebra associated to \mathcal{C} , which we introduce and classify using the notion of higher S -matrices of higher braided fusion categories.

1. Introduction

According to Wigner’s theorem [1], invertible global symmetries act unitarily (or anti-unitarily) on local observables or states of a quantum system. It is natural to ask how this generalises to the action of non-invertible symmetries described by (higher) fusion categories \mathcal{C} [2–6]. In this note, we try to answer this question in two steps:

1. We consider the *tube algebra* $\text{Tube}(\mathcal{C})$ [7–9] associated to \mathcal{C} and show that it canonically possesses the structure of a C^* -algebra, provided that the symmetry category \mathcal{C} is unitary in an appropriate sense.
2. We propose that twisted sector local operators transform in $*$ -representations of $\text{Tube}(\mathcal{C})$, which we construct and classify using the *Symmetry TFT* [10–12] and its associated *higher S-matrices* [13–15].

While the above is well known in spacetime dimension $D = 2$ [16–18], we provide a general construction for arbitrary $D \geq 2$ including several examples in $D = 2, 3$.

1.1. Background

In Euclidean (Wick-rotated) quantum field theory, unitarity manifests itself in the principle of *reflection positivity* [19–22]. Concretely, upon fixing an affine hyperplane Π in D -dimensional spacetime ($D \geq 2$), *reflection* states that reflecting the operator content of a correlation function about Π is equivalent to complex conjugating the correlation function,

$$\left\langle \left[\begin{array}{c} \Pi \\ \cdot \\ \mathcal{O} \\ \cdot \\ \mathcal{O} \end{array} \right]^* \right\rangle = \left\langle \left[\begin{array}{c} \mathcal{O}^\dagger \\ \cdot \\ \mathcal{O} \\ \cdot \\ \Pi \end{array} \right] \right\rangle. \tag{1}$$

Here, \mathcal{O}^\dagger denotes the operator that is obtained by reflecting an operator \mathcal{O} about Π . *Positivity* states that

$$\left\langle \left[\begin{array}{c} \mathcal{O}^\dagger \\ \cdot \\ \Pi \\ \cdot \\ \mathcal{O} \end{array} \right] \right\rangle \geq 0 \tag{2}$$

for all operators \mathcal{O} , with equality if and only if $\mathcal{O} = 0$.

Now suppose that a given quantum field theory \mathfrak{T} admits finite generalised symmetries described a fusion $(D - 1)$ -category \mathcal{C} (also called the *symmetry category* of \mathfrak{T} in what follows). Objects A, B of the latter correspond to codimension-one topological defects, morphisms φ, ψ to codimension-two topological interfaces between defects, 2-morphisms Θ to codimension-three topological junctions between interfaces, and so on and so forth:

$$\begin{array}{c} \varphi \\ \circlearrowleft \\ A \Downarrow \Theta B \\ \circlearrowright \\ \psi \end{array} \leftrightarrow \begin{array}{c} \psi \\ \circlearrowleft \\ A \square \Theta B \\ \circlearrowright \\ \varphi \end{array} . \tag{3}$$

Given a twisted sector local operator \mathcal{O} that is attached to a genuine topological line defect $\mu \in \Omega^{D-2}(\mathcal{C})$ (where $\Omega(\mathcal{C}) := \text{End}_{\mathcal{C}}(\mathbf{1})$ denotes the endomorphism space of the monoidal unit $\mathbf{1} \in \mathcal{C}$ and $\Omega^n(\mathcal{C}) := \Omega(\Omega^{n-1}(\mathcal{C}))$ is defined inductively for $n = 1, \dots, D - 1$), symmetry defects $A \in \mathcal{C}$ can act on \mathcal{O} via linking,

$$\begin{array}{c} A \\ \circlearrowleft \\ \nu \leftarrow \mu \leftarrow \Phi \\ \circlearrowright \\ O \end{array} , \tag{4}$$

mapping \mathcal{O} to a local operator in a potentially different twisted sector $\nu \in \Omega^{D-2}(\mathcal{C})$. Configurations of the above type then generate the *tube algebra*² $\text{Tube}(\mathcal{C})$ of \mathcal{C} , which is a finite-dimensional complex associative algebra whose elements we denote by

$$\langle \left\langle \nu \left[\frac{A}{\Phi} \right] \mu \right\rangle \rangle \in \text{Tube}(\mathcal{C}) \tag{5}$$

in what follows (see [18] for a discussion of the tube algebra and its physical applications in two dimensions and [9] for generalisations to higher D). Using this, the unit of $\text{Tube}(\mathcal{C})$ can be written as³

$$\mathbb{1} = \sum_{[\mu] \in \pi_{D-2}(\mathcal{C})} \langle \left\langle \mu \left[\frac{\mathbf{1}}{\text{Id}_{\mu}} \right] \mu \right\rangle \rangle , \tag{6}$$

where $\mathbf{1} \in \mathcal{C}$ denotes the monoidal unit in \mathcal{C} . Twisted sector local operators \mathcal{O} then transform in linear representations of the tube algebra, which can be constructed using an operator-state map⁴ as follows: By computing the half-space correlation function

$$|\mathcal{O}\rangle := \left| \left\langle \left[\frac{A}{\Phi} \right] \mu \right\rangle \right\rangle , \tag{7}$$

we obtain a state $|\mathcal{O}\rangle \in \mathcal{H}_{\mu}$ in the μ -twisted Hilbert space of the theory, which is acted upon by elements of the tube algebra via

$$R\left(\left\langle \left[\frac{A}{\Phi} \right] \mu \right\rangle \right) |\mathcal{O}\rangle := \left| \left\langle \left[\frac{A}{\Phi} \right] \nu \left[\frac{A}{\Phi} \right] \mu \right\rangle \right\rangle . \tag{8}$$

² In the language of [9], the above is the S^{D-1} -tube algebra associated to the linking of local operators with codimension-one symmetry defects placed on a $(D - 1)$ -sphere. For the purposes of this note, we will simply call it the *tube algebra* in what follows.

³ Here, we denote by $\pi_0(\mathcal{B})$ the set of connected components of a (higher) fusion category \mathcal{C} , i.e. the set of simple objects in \mathcal{C} modulo the existence of a non-zero morphism between them. Furthermore, we set $\pi_n(\mathcal{C}) := \pi_0(\Omega^n(\mathcal{C}))$. We often call $\pi_1(\mathcal{C})$ the *fundamental hypergroup* of \mathcal{C} .

⁴ For generic \mathfrak{T} , this map will be non-surjective. By abuse of notation, we will henceforth use \mathcal{H}_{μ} to denote the subsector of the μ -twisted Hilbert space that is given by the image of the operator-state map. Furthermore, since we are interested in the action of finite symmetries on these spaces, we will without loss of generality assume the \mathcal{H}_{μ} to be finite-dimensional.

This induces linear maps $R(\langle \nu \left[\frac{A}{\Phi} \mu \right] \rangle) : \mathcal{H}_\mu \rightarrow \mathcal{H}_\nu$, which assemble into a linear representation

$$R : \text{Tube}(\mathcal{C}) \rightarrow \text{End}(\mathcal{H}) \tag{9}$$

of $\text{Tube}(\mathcal{C})$ on the total Hilbert space $\mathcal{H} = \bigoplus_\mu \mathcal{H}_\mu$. In order for this representation to be compatible with reflection positivity, we require that

$$\begin{aligned} \langle \tilde{\mathcal{O}} | R(\langle \nu \left[\frac{A}{\Phi} \mu \right] \rangle) | \mathcal{O} \rangle^* &= \left\langle \tilde{\mathcal{O}}^\dagger \left[\begin{array}{c} \Pi \\ \nu \left[\frac{A}{\Phi} \mu \right] \end{array} \right] \mathcal{O} \right\rangle^* \\ &= \left\langle \mathcal{O}^\dagger \left[\begin{array}{c} A^\vee \\ \mu \left[\frac{\Pi}{\Phi^\dagger} \nu \right] \end{array} \right] \tilde{\mathcal{O}} \right\rangle = \left\langle \mathcal{O}^\dagger \left[\begin{array}{c} \Pi \\ \mu \left[\frac{A^\vee}{\Phi^\dagger} \nu \right] \end{array} \right] \tilde{\mathcal{O}} \right\rangle \\ &= \langle \mathcal{O} | R(\langle \mu \left[\frac{A^\vee}{\Phi^\dagger} \nu \right] \rangle) | \tilde{\mathcal{O}} \rangle \end{aligned} \tag{10}$$

for all \mathcal{O} and $\tilde{\mathcal{O}}$, where we used that reflection about Π maps the codimension-one symmetry A defect to its orientation reversal A^\vee . As a result, we see that

$$R(\langle \nu \left[\frac{A}{\Phi} \mu \right] \rangle)^\dagger \stackrel{!}{=} R(\langle \nu \left[\frac{A}{\Phi} \mu \right] \rangle^*) \tag{11}$$

where the prescription

$$\langle \nu \left[\frac{A}{\Phi} \mu \right] \rangle^* := \langle \mu \left[\frac{A^\vee}{\Phi^\dagger} \nu \right] \rangle \tag{12}$$

defines an antilinear involution $* : \text{Tube}(\mathcal{C}) \rightarrow \text{Tube}(\mathcal{C})$ on the tube algebra (i.e. $*^2 = \text{id}$). Since for a simple $\mu \in \Omega^{D-2}(\mathcal{C})$, the assignment

$$F(\langle \nu \left[\frac{A}{\Phi} \mu \right] \rangle) := \delta_{A,1} \cdot \delta_{\mu,\nu} \Phi \tag{13}$$

induces a faithful positive functional $F : \text{Tube}(\mathcal{C}) \rightarrow \mathbb{C}$ (where we used that $\Phi \in (D-1)\text{-End}_{\mathcal{C}}(\mu) \cong \mathbb{C}$ can be identified with a scalar since μ is simple), this shows that $\text{Tube}(\mathcal{C})$ is a C^* -algebra [17]. Condition (11) then means R is a $*$ -representation of $\text{Tube}(\mathcal{C})$, which forms a natural generalisation of the notion of a unitary representation of a finite group. The aim of this note is to classify all irreducible $*$ -representations of the tube algebra for a given symmetry category \mathcal{C} .

1.2. Overview

In a unitary quantum field theory with finite internal symmetry group G , local operators transform in unitary representations of G , which are group homomorphisms

$$R : G \rightarrow U(\mathcal{H}) \tag{14}$$

from G into the unitary group of a Hilbert space \mathcal{H} . Equivalently, by linear continuation, we may view R as a $*$ -representation of the group algebra $\mathbb{C}[G]$ of G ,

$$R : \mathbb{C}[G] \rightarrow \text{End}(\mathcal{H}) \tag{15}$$

where the involution $*$ on $\mathbb{C}[G]$ is the antilinear continuation of the inversion map $g \mapsto g^{-1}$.

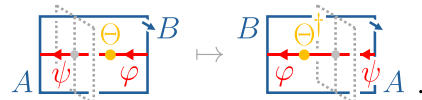
As motivated in the previous subsection, upon replacing the finite group G with a generalised (higher) fusion category symmetry \mathcal{C} , the natural generalisation of (15) is a $*$ -representation of the tube

algebra of \mathcal{C} on the Hilbert space \mathcal{H} of (genuine and twisted sector) local operators, where the involution $*$ on $\text{Tube}(\mathcal{C})$ is as in (12). In particular, this requires the symmetry category \mathcal{C} to be equipped with the following:

1. **Duals:** For each symmetry defect $A \in \mathcal{C}$ there exists a *dual defect* $A^\vee \in \mathcal{C}$ that corresponds to the orientation reversal of A obtained by ‘bending’ the topological defect A around,

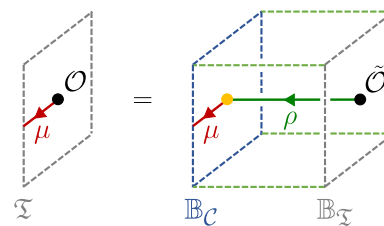

(16)

2. **Daggers:** For each local junction Θ in \mathcal{C} , there exists its *dagger* Θ^\dagger which corresponds to the reflection of Θ about a fixed hyperplane⁵,

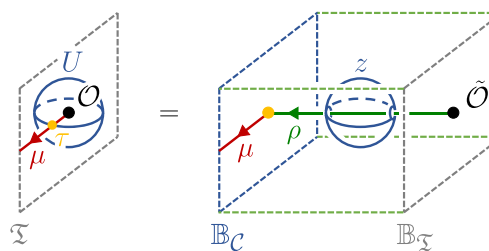

(17)

Furthermore, we require the above structures to be compatible with one another in an appropriate sense in order for the involution $*$ on $\text{Tube}(\mathcal{C})$ to be well-defined.

A convenient tool to characterise and classify irreducible $*$ -representations of the tube algebra is the so-called *sandwich construction* [10–12]. In this picture, a D -dimensional theory \mathfrak{T} with generalised symmetry \mathcal{C} is viewed as an interval compactification of a $(D + 1)$ -dimensional topological theory (called the *Symmetry TFT*) with canonical topological boundary condition $\mathbb{B}_{\mathcal{C}}$ on the left and a physical (typically non-topological) boundary condition $\mathbb{B}_{\mathfrak{T}}$ on the right. Twisted sector local operators \mathcal{O} in \mathfrak{T} then correspond to junctions between a left boundary line μ and a bulk line ρ stretched between the two boundaries of the $(D + 1)$ -dimensional bulk:


(18)

Now suppose that we link a twisted sector local operator \mathcal{O} with a symmetry defect $U \in \mathcal{C}$ that can be pulled into the $(D + 1)$ -dimensional bulk:


(19)

Since the latter hosts a Turaev-Viro TFT [24–27] based on the symmetry category \mathcal{C} (or higher-dimensional analogues thereof [28]), this necessitates U to form part of a D -tuple⁶ $z = (U, \tau, \dots) \in \mathcal{Z}(\mathcal{C})$ in the Drinfeld centre of \mathcal{C} . In particular, the datum τ corresponds to a half-braiding for U , which allows us to commute U past any other topological defect in \mathcal{C} . For brevity, we denote the corresponding

⁵ In general, for an (∞, n) -category with all adjoints, different \dagger -structures are parametrised by different choices of subgroup $\mathfrak{G} \subset \text{Aut}(\text{AdjCat}_{(\infty, n)}) \cong \text{PL}(n)$ [23]. In our case, we choose $\mathfrak{G} = \mathbb{Z}_2$ implementing involutory reflections only at the top level of morphisms. In this way, we inductively view a \dagger - n -category as an n -category enriched in \dagger - $(n - 1)$ -categories.

⁶ Here, the ‘...’ refers to higher coherence data associated to the object U and the half-braiding τ (see e.g. [29] for $D = 3$).

tube algebra element by

$$\langle \mu \left[\begin{array}{c} U \\ \tau_{U,\mu} \end{array} \right] \mu \rangle =: \langle \mu \left[\begin{array}{c} z \\ \mu \end{array} \right] \mu \rangle \tag{20}$$

in what follows. This has the property that

$$\langle \nu \left[\begin{array}{c} A \\ \Phi \end{array} \right] \mu \rangle \circ \langle \mu \left[\begin{array}{c} z \\ \mu \end{array} \right] \mu \rangle = \langle \nu \left[\begin{array}{c} z \\ \nu \end{array} \right] \nu \rangle \circ \langle \nu \left[\begin{array}{c} A \\ \Phi \end{array} \right] \mu \rangle \tag{21}$$

for all $\langle \nu \left[\begin{array}{c} A \\ \Phi \end{array} \right] \mu \rangle \in \text{Tube}(\mathcal{C})$, with involution given by⁷

$$\langle \mu \left[\begin{array}{c} z \\ \mu \end{array} \right] \mu \rangle^* = \langle \mu \left[\begin{array}{c} z^\vee \\ \mu \end{array} \right] \mu \rangle. \tag{22}$$

From (19), it is then clear that the action of $\langle \mu \left[\begin{array}{c} z \\ \mu \end{array} \right] \mu \rangle$ on \mathcal{O} is completely determined by the linking of the topological defects z and ρ in the $(D + 1)$ -dimensional bulk, i.e.

$$R\left(\langle \mu \left[\begin{array}{c} z \\ \mu \end{array} \right] \mu \rangle\right) |\mathcal{O}\rangle = d_z \cdot S_{z,\rho} \cdot |\mathcal{O}\rangle, \tag{23}$$

where we abbreviated $d_z := \dim(z)$ and defined the multiplicative factor⁸

$$S_{z,\rho} := \frac{1}{d_z \cdot d_\rho} \cdot \left(\text{linking diagram of } z \text{ and } \rho \right) \in \mathbb{C} \tag{24}$$

indexed by $z \in \mathcal{Z}(\mathcal{C})$ and $\rho \in \Omega^{D-2}(\mathcal{Z}(\mathcal{C}))$. The collection of these factors then descends to a well-defined pairing [13, 14]

$$S: \pi_0(\mathcal{Z}(\mathcal{C})) \times \pi_{D-2}(\mathcal{Z}(\mathcal{C})) \rightarrow \mathbb{C}, \tag{25}$$

which we call the *S-matrix* of $\mathcal{Z}(\mathcal{C})$ in what follows⁹. This matrix has the following properties:

① It is invertible, so in particular we have that

$$|\pi_0(\mathcal{Z}(\mathcal{C}))| = |\pi_{D-2}(\mathcal{Z}(\mathcal{C}))|. \tag{27}$$

② It satisfies $S_{z^\vee,\rho} = (S_{z,\rho})^*$ for all z and ρ .

③ It satisfies the Verlinde formula

$$S_{x,\rho} \cdot S_{y,\rho} = \sum_{[z] \in \pi_0} N_{xy}^z \cdot S_{z,\rho} \tag{28}$$

for some unique coefficients $N_{xy}^z \in \mathbb{C}$.

⁷ Here, we implicitly assume that $z = (U, \tau, \dots)$ lies in the *unitary Drinfeld centre* $\mathcal{Z}^\dagger(\mathcal{C})$, meaning that the top components associated to the half-braiding τ are unitary: $(\tau_{U,\mu})^\dagger = (\tau_{U,\mu})^{-1}$ for all $\mu \in \Omega^{D-2}(\mathcal{C})$. While $\mathcal{Z}^\dagger(\mathcal{C}) \cong \mathcal{Z}(\mathcal{C})$ in $D = 2$ [30], we expect this to hold also in $D > 2$.

⁸ Note that this picture schematically represents the linking of an S^{D-1} and an S^1 in a $(D + 1)$ -dimensional ambient space.

⁹ For a general braided fusion n -category \mathcal{B} , there exists a whole collection of (generalised) *S-matrices* [13, 14]

$$S^{(k)}: \pi_k(\mathcal{B}) \times \pi_{n-1-k}(\mathcal{B}) \rightarrow \mathbb{C}, \tag{26}$$

where $k = 0, \dots, n - 1$. For $n = 1$, this reduces to the ordinary *S-matrix* for braided fusion categories [31]. In this note, we consider the case $n = D - 1$ and $k = 0$ with $\mathcal{B} = \mathcal{Z}(\mathcal{C})$ the Drinfeld centre of the symmetry $(D - 1)$ -category \mathcal{C} .

From a physical perspective, the (generically non-integer) coefficients N_{xy}^z capture the tube algebra products

$$\langle \mu \square_x^\mu \rangle \circ \langle \mu \square_y^\mu \rangle = \sum_{[z] \in \pi_0} \frac{d_x \cdot d_y}{d_z} \cdot N_{xy}^z \cdot \langle \mu \square_z^\mu \rangle. \tag{29}$$

From a mathematical perspective, they define a hypergroup structure on the set $\pi_0(\mathcal{Z}(\mathcal{C}))$ of connected components of the Drinfeld centre of \mathcal{C} [14]:

Definition ([14, 32–34]). A \mathbb{C} -hypergroup is a finite set X with an involution $\vee : X \rightarrow X$, a distinguished element $e \in X$ and a collection $\{N_{xy}^z \in \mathbb{C}\}_{x,y,z \in X}$ such that

1. setting $x \cdot y := \sum_z N_{xy}^z \cdot z$ defines an associative algebra structure on $\mathbb{C}[X]$,
2. the distinguished element $e \in X$ acts as a unit for the above algebra,
3. the continuation of \vee to $\mathbb{C}[X]$ defines an algebra anti-isomorphism,
4. for all $x, y \in X$, we have $N_{xy}^e \neq 0 \Leftrightarrow y = x^\vee$,
5. for all $x, y \in X$, we have $\sum_z N_{xy}^z = 1$.

For the purposes of this note, the utility of the S-matrix stems from the fact that it captures diagonal actions of the tube algebra on twisted sector local operators as in (23). In fact, the latter are sufficient to determine the action of the entire tube algebra (for $D = 2$, this was shown by the authors of [18], and the following discussion is a straight-up generalisation of their arguments to $D > 2$): Given a simple left boundary line $\mu \in \Omega^{D-2}(\mathcal{C})$ and a simple line $\rho \in \Omega^{D-2}(\mathcal{Z}(\mathcal{C}))$ in the Symmetry TFT, we can define an associated tube algebra element

$$e_\rho^\mu := \sum_{[z] \in \pi_0} \frac{1}{d_z} \cdot (S^{-1})_{\rho,z} \cdot \langle \mu \square_z^\mu \rangle, \tag{30}$$

where we used the invertibility ① of the S-matrix. Using the Verlinde formula ③ and the composition rule (29), one can then check that the above elements satisfy

$$e_\rho^\mu \circ e_\sigma^\nu = \delta_{\rho,\sigma} \cdot \delta^{\mu,\nu} \cdot e_\rho^\mu. \tag{31}$$

Furthermore, using (22) and ② one finds $(e_\rho^\mu)^* = e_\rho^\mu$. As a result, using (21) we see that

$$e_\rho := \sum_{[\mu] \in \pi_{D-2}} e_\rho^\mu \tag{32}$$

defines a self-adjoint central idempotent (projector) in $\text{Tube}(\mathcal{C})$ for every simple $\rho \in \Omega^{D-2}(\mathcal{Z}(\mathcal{C}))$, which is *minimal* in the sense that $e_\rho \neq e_1 + e_2$ for two non-zero central idempotents e_1 and e_2 . Since the set of minimal projectors is in 1:1-correspondence with the irreducible $*$ -representations of the tube algebra, this establishes an equivalence (see [35–37] for $D = 2$)

$$\text{Rep}^\dagger(\text{Tube}(\mathcal{C})) \cong \Omega^{D-2}(\mathcal{Z}^\dagger(\mathcal{C})), \tag{33}$$

where the right hand side denotes the category of $*$ -representations of $\text{Tube}(\mathcal{C})$ and intertwiners between them.

The aim of this note is to use the equivalence (33) to classify unitary actions of (higher) fusion category symmetries on twisted sector local operators in two and three spacetime dimensions. We provide an explicit construction of the tube algebra and its canonical $*$ -structure as well as a variety of examples in sections 2 ($D = 2$) and 3 ($D = 3$), respectively.

As a by-product, we find that the S-matrix associated to the Drinfeld centre of a finite 2-group symmetry [38, 39] $\mathcal{G} = A[1] \rtimes G$ in three dimensions is given by

$$S_{[x,\chi],[a,\rho]} = \frac{1}{\dim(\rho) \cdot |G|} \cdot \sum_{\substack{g \in G: \\ g_x \in G_a}} \langle \tau_x(\lambda)(g), a \rangle \cdot \text{Tr}_\rho(g_x) \cdot \chi(a^g), \tag{34}$$

where the index labels of the matrix are given by

1. pairs (x, χ) consisting of a representative of a conjugacy class $[x] \in \text{Cl}(G)$ and a character $\chi \in (A_x)^\vee$ on the subgroup $A_x \subset A$ of x -invariant 1-form defects,
2. pairs (a, ρ) consisting of a representative of a G -orbit $[a] \in A/G$ in A and an irreducible representation ρ of the stabiliser subgroup $G_a \subset G$ of a ,


and $\tau(\lambda)$ denotes the *transgression* of a mixed 't Hooft anomaly¹⁰ $[\lambda] \in H^2(G, A^\vee)$ between G and A .

2. Two dimensions

In this section, we review the construction of the tube algebra $\text{Tube}(\mathcal{C})$ associated to a fusion category \mathcal{C} based on [16–18]. We describe its canonical $*$ -structure and classify irreducible $*$ -representations in several examples of invertible and non-invertible symmetries.

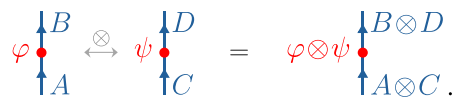
2.1. Preliminaries

In two dimensions, the finite (bosonic) generalised symmetries of a quantum field theory are described by a fusion category \mathcal{C} , whose objects $A, B \in \mathcal{C}$ correspond to topological line defects and whose morphisms $\varphi : A \rightarrow B$ correspond to topological junctions between lines:



$$\varphi \tag{35}$$


The monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ captures the fusion of topological lines and their junctions:



$$\tag{36}$$

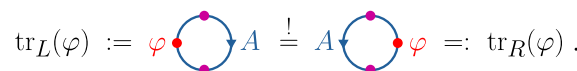
Moreover, we assume \mathcal{C} to be equipped with the following additional structures:

1. **Dual structure:** A *dual structure* on \mathcal{C} allows us to bend topological line defects around in the sense that for each $A \in \mathcal{C}$ there exists a *dual object* $A^\vee \in \mathcal{C}$ together with evaluation and coevaluation morphisms



$$\tag{37}$$

satisfying suitable *zig-zag relations* [40]. The assignment $A \mapsto A^\vee$ then extends to an op-monoidal functor¹¹ $\vee : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$. A *pivotal structure* on \mathcal{C} is a natural isomorphism $\xi : \vee^2 \Rightarrow \text{id}_{\mathcal{C}}$. The latter is called *spherical* if the associated left and right traces of endomorphisms $\varphi \in \text{End}_{\mathcal{C}}(A)$ agree, i.e.



$$\tag{38}$$

¹⁰ Here, we denote by $A^\vee := \text{Hom}(A, U(1))$ the *Pontryagin dual group* of the finite abelian group A .

¹¹ Here, \mathcal{C}^{op} is the category with the same objects as \mathcal{C} and morphisms given by $\text{Hom}_{(\mathcal{C}^{\text{op}})}(x, y) = \text{Hom}_{\mathcal{C}}(y, x)$ for all $x, y \in \mathcal{C}$.

In this case, we define the dimension of an object $A \in \mathcal{C}$ by $\dim(A) := \text{tr}_{L/R}(\text{id}_A) \in \mathbb{C}$.

2. **Dagger structure:** A \dagger -structure on \mathcal{C} allows us to reflect topological junctions in the sense that there exists a functor $\dagger : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ that acts as the identity on objects and anti-linearly on morphisms via

$$\begin{array}{c} B \\ \uparrow \\ \bullet \\ \uparrow \\ A \end{array} \xrightarrow{\dagger} \begin{array}{c} A \\ \uparrow \\ \bullet \\ \uparrow \\ B \end{array} \quad (39)$$

such that $\dagger^2 = \text{id}_{\mathcal{C}}$ and $\text{End}_{\mathcal{C}}(A)$ is a C^* -algebra for all $A \in \mathcal{C}$. We assume this \dagger -structure to be compatible with the monoidal structure on \mathcal{C} in the sense that $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a \dagger -functor (i.e. $\dagger \circ \otimes = \otimes \circ \dagger$) with unitary coherence data¹². In this case, we call \mathcal{C} a *unitary fusion category* [41].

We assume the above structures to be compatible with one another in the sense that $\vee : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ is a \dagger -functor with unitary coherence data. In this case,

$$\begin{array}{c} A \\ \uparrow \\ \bullet \\ \uparrow \\ (A^\vee)^\vee \end{array} \quad := \quad \begin{array}{c} \text{ev}_{A^\vee} \\ \uparrow \\ (A^\vee)^\vee \end{array} \quad \begin{array}{c} A^\vee \\ \uparrow \\ \bullet \\ \uparrow \\ A \end{array} \quad \text{ev}_A \quad (40)$$

defines a canonical pivotal structure on \mathcal{C} . A dual structure of this type is called a *unitary dual structure*. There exists a unique unitary dual structure \vee on \mathcal{C} such that (40) is spherical [42]. In the following, we assume \mathcal{C} to be a unitary fusion category equipped with the unique unitary dual structure such that the associated canonical pivotal structure (40) is spherical.

2.2. Tube algebra

Given the symmetry category \mathcal{C} , we can associate to it its *tube algebra* $\text{Tube}(\mathcal{C})$, which captures the linking action of topological line defects on twisted sector local operators in two dimensions. As a vector space, the tube algebra decomposes into a direct sum [16, 43]

$$\text{Tube}(\mathcal{C}) = \bigoplus_{[X], [Y] \in \pi_0(\mathcal{C})} \text{Tube}(\mathcal{C})_{X,Y}, \quad (41)$$

where the summand associated to simple lines $X, Y \in \mathcal{C}$ is given by the quotient space

$$\text{Tube}(\mathcal{C})_{X,Y} = \bigoplus_{A \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(A \otimes X, Y \otimes A) / \sim \quad (42)$$

of local intersection morphisms

$$\begin{array}{c} Y \\ \uparrow \\ \bullet \\ \uparrow \\ A \end{array} \quad \begin{array}{c} X \\ \uparrow \\ \bullet \\ \uparrow \\ A \end{array} \quad \varphi \quad (43)$$

subjected to the equivalence relation generated by

$$\begin{array}{c} Y \\ \uparrow \\ \bullet \\ \uparrow \\ A \end{array} \quad \begin{array}{c} B \\ \uparrow \\ \bullet \\ \uparrow \\ X \end{array} \quad \eta \quad \sim \quad \begin{array}{c} Y \\ \uparrow \\ \bullet \\ \uparrow \\ B \end{array} \quad \begin{array}{c} A \\ \uparrow \\ \bullet \\ \uparrow \\ X \end{array} \quad \gamma \quad (44)$$

¹² We collectively refer to any natural coherence isomorphisms as *coherence data* in what follows. In the above case, the coherence data associated to the product \otimes is given by the associator and left and right unitor isomorphisms.

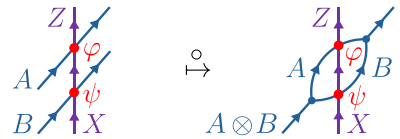
We denote the equivalence class of morphisms (43) by

$$\langle \frac{Y^A X}{\varphi} \rangle \in \text{Tube}(\mathcal{C})_{X,Y} . \tag{45}$$

As a vector space, $\text{Tube}(\mathcal{C})_{X,Y}$ is isomorphic to

$$\text{Tube}(\mathcal{C})_{X,Y} \cong \bigoplus_{[A] \in \pi_0(\mathcal{C})} \text{Hom}_{\mathcal{C}}(A \otimes X, Y \otimes A) , \tag{46}$$

which shows that the tube algebra is finite-dimensional. The algebra multiplication is induced by



$$\tag{47}$$

which we denote by

$$\langle \frac{Z^A Y'}{\varphi} \rangle \circ \langle \frac{Y^B X}{\psi} \rangle = \delta_{Y,Y'} \cdot \langle \frac{Z^{A \otimes B} X}{\varphi \circ \psi} \rangle \tag{48}$$

on the generators of $\text{Tube}(\mathcal{C})$. The tube algebra further admits an antilinear involution $*$: $\text{Tube}(\mathcal{C}) \rightarrow \text{Tube}(\mathcal{C})$, which is induced by [17]



$$\tag{49}$$

and which we denote by

$$\langle \frac{Y^A X}{\varphi} \rangle^* = \langle \frac{X^{A^\vee} Y}{\varphi^\dagger} \rangle \tag{50}$$

on the generators of the tube algebra.

2.3. Examples

We conclude this section with examples of unitary fusion category symmetries and $*$ -representations of their associated tube algebras. We discuss anomalous invertible group-like symmetries as well as non-invertible symmetries of Tambara–Yamagami and Fibonacci type.

2.3.1. Group symmetry

We first consider the symmetry category $\mathcal{C} = \text{Hilb}_G^\omega$ corresponding to a finite group G with 't Hooft anomaly $[\omega] \in H^3(G, U(1))$. Simple objects in \mathcal{C} are given by group elements $g \in G$ that fuse according to the group law of G with associator

$$(g \cdot h) \cdot k \xrightarrow{\omega(g,h,k) \cdot \text{id}_{ghk}} g \cdot (h \cdot k) \tag{51}$$

The duals and dimensions of simple objects are given by $g^\vee = g^{-1}$ and $\dim(g) = 1$, respectively. This symmetry category is unitary for all G and ω .

Tube algebra: As discussed in [8, 9], the tube algebra of $\mathcal{C} = \text{Hilb}_G^\omega$ is the twisted Drinfeld double [44–46]

$$\text{Tube}(\text{Hilb}_G^\omega) = \mathcal{D}^\omega(G) . \tag{52}$$

The latter has generators $\langle \frac{g}{x} \mid x \rangle$ (where $g, x \in G$ and we denote by $\varepsilon_x := gxg^{-1}$ the conjugation action of g on x), whose algebra multiplication is given by

$$\langle \frac{g}{x} \mid x \rangle \circ \langle \frac{h}{y} \mid y \rangle = \delta_{x,hy} \cdot \tau_y(g,h) \cdot \langle \frac{g^h}{y} \mid y \rangle. \tag{53}$$

Here, we defined the multiplicative phase

$$\tau_x(\omega)(g,h) := \frac{\omega(g,h,x) \cdot \omega(g^h x, g, h)}{\omega(g, hx, h)} \tag{54}$$

known as the *transgression* of the 't Hooft anomaly ω . As a result of the cocycle condition for ω , it satisfies

$$(d\tau)_x(g,h,k) := \frac{\tau_x(h,k) \cdot \tau_x(g,hk)}{\tau_x(gh,k) \cdot \tau_x(g,h)} = 1, \tag{55}$$

which ensures that the algebra multiplication (53) is associative. Using (49), the $*$ -structure on $\mathcal{D}^\omega(G)$ can be computed to be

$$\langle \frac{g}{x} \mid x \rangle^* = \mu_x(g) \cdot \langle \frac{g^{-1}}{x} \mid x \rangle, \tag{56}$$

where we defined the multiplicative phase

$$\mu_x(g) := \tau_x^*(g^{-1}, g). \tag{57}$$

As a consequence of (55), it satisfies

$$\mu_x(g^{-1}) = \mu_{(x^g)}(g) \quad \text{and} \quad d\mu = \widehat{\tau}/\tau, \tag{58}$$

where we defined the 2-cocycle

$$\widehat{\tau}_x(g,h) := \tau_{(gh,x)}^*(h^{-1}, g^{-1}). \tag{59}$$

This ensures that the $*$ -structure (56) is involutory and compatible with the algebra multiplication.

Tube representations: In order to classify irreducible $*$ -representations R of $\mathcal{D}^\omega(G)$, we use the equivalence (33) as well as the fact that [36, 47]

$$\mathcal{Z}(\text{Hilb}_G^\omega) = \bigoplus_{[x] \in \text{Cl}(G)} \text{Rep}^{\tau_x(\omega)}(G_x), \tag{60}$$

which shows that we can label each such R by a pair (x, ρ) consisting of

1. a representative x of a conjugacy class $[x] \in \text{Cl}(G)$,
2. an irreducible representation ρ of the centraliser G_x of x with projective 2-cocycle $\tau_x(\omega) \in Z^2(G_x, U(1))$.

The associated irreducible representation $R = R_{(x,\rho)}$ of $\mathcal{D}^\omega(G)$ can be constructed via induction [48]: To this end, we fix for each $y \in [x]$ a representative $r_y \in G$ such that $(r_y)y = x$ (with $r_x := e$). Using these, we can define

$$g_y := r_{(gy)} \cdot g \cdot r_y^{-1} \in G_x \tag{61}$$

for all $g \in G$ and $y \in [x]$. If we denote by \mathcal{V} the Hilbert space underlying the (projective) representation ρ of G_x , then $R_{(x,\rho)}$ acts on the Hilbert space \mathcal{H} with G -grading

$$\mathcal{H}_y = \begin{cases} \mathcal{V} & \text{if } y \in [x] \\ 0 & \text{otherwise} \end{cases} \tag{62}$$

via the induced tube action

$$R_{(x,\rho)}\left(\langle \begin{array}{c} g_y \\ \hline y \end{array} \rangle\right) := \kappa_y(g) \cdot \rho(g_y), \tag{63}$$

where we defined the multiplicative phase

$$\kappa_y(g) := \frac{\tau_y(r_{(g_y)}, g)}{\tau_y(g_y, r_y)}. \tag{64}$$

As a consequence of (55), it satisfies

$$\frac{\kappa_y(h) \cdot \kappa_{(h_y)}(g)}{\kappa_y(gh)} = \frac{\tau_y(g, h)}{\tau_x(g_{(h_y)}, h_y)}, \tag{65}$$

which ensures that $R_{(x,\rho)}$ respects the algebra multiplication of $\mathcal{D}^\omega(G)$. It is straightforward to check that $R_{(x,\rho)}$ is a $*$ -representation of $\mathcal{D}^\omega(G)$ if and only if ρ is a unitary projective representation of G_x . Since every finite-dimensional representation of a finite group is equivalent to a unitary one, this shows that all irreducible representations of $\mathcal{D}^\omega(G)$ are $*$ -representations as expected.

2.3.2. Tambara–Yamagami symmetry

As a second example, let us consider a symmetry category $\mathcal{C} = \text{TY}_A^{\chi,s}$ of Tambara–Yamagami type [49], which is specified by the following pieces of data:

1. A finite abelian group A ,
2. a non-deg. symmetric bicharacter $\chi : A \times A \rightarrow U(1)$,
3. a square-root s of $1/|A|$.

The simple objects of \mathcal{C} consist of group elements $a \in A$ together with a non-invertible defect m , which are subject to the fusion rules

$$a \otimes b = a \cdot b, \quad a \otimes m = m \otimes a = m, \quad m \otimes m = \bigoplus_{a \in A} a. \tag{66}$$

The non-trivial components of the associator are

$$\begin{aligned} (a \otimes m) \otimes b &\underset{= m}{\xrightarrow{\chi(a,b) \cdot \text{id}_m}} a \otimes (m \otimes b), \\ (m \otimes a) \otimes m &\underset{= \bigoplus_{b \in A} b}{\xrightarrow{\bigoplus_b \chi(a,b) \cdot \text{id}_b}} m \otimes (a \otimes m), \\ (m \otimes m) \otimes m &\underset{= \bigoplus_{a \in A} m}{\xrightarrow{s \cdot \bigoplus_{a,b} \chi^*(a,b) \cdot \text{id}_m}} m \otimes (m \otimes m). \end{aligned} \tag{67}$$

The duals of simple objects are $a^\vee = a^{-1}$ and $m^\vee = m$. Their dimensions are $\dim(a) = 1$ and $\dim(m) = \sqrt{|A|}$. This symmetry category is unitary for all A , χ and s .

Tube algebra: The tube algebra of $\mathcal{C} = \text{TY}_A^{\chi,s}$ has the following types of generators:

$$\langle \begin{array}{c} x \\ \hline a \end{array} \begin{array}{c} a \\ \hline x \end{array} \rangle, \quad \langle \begin{array}{c} y \\ \hline m \end{array} \begin{array}{c} m \\ \hline x \end{array} \rangle, \quad \langle \begin{array}{c} m \\ \hline m \end{array} \begin{array}{c} m \\ \hline a \end{array} \rangle, \quad \langle \begin{array}{c} m \\ \hline a \end{array} \begin{array}{c} a \\ \hline m \end{array} \rangle, \tag{68}$$

where $a, x, y \in A$. Using (47), their algebra multiplication can be determined to be

$$\begin{aligned}
 \langle \frac{x}{|} \frac{a}{|} \frac{x}{|} \rangle \circ \langle \frac{x}{|} \frac{b}{|} \frac{x}{|} \rangle &= \langle \frac{x}{|} \frac{ab}{|} \frac{x}{|} \rangle, \\
 \langle \frac{y}{|} \frac{m}{|} \frac{x}{|} \rangle \circ \langle \frac{x}{|} \frac{a}{|} \frac{x}{|} \rangle &= \chi(a, y) \cdot \langle \frac{y}{|} \frac{m}{|} \frac{x}{|} \rangle, \\
 \langle \frac{y}{|} \frac{a}{|} \frac{y}{|} \rangle \circ \langle \frac{y}{|} \frac{m}{|} \frac{x}{|} \rangle &= \chi(a, x) \cdot \langle \frac{y}{|} \frac{m}{|} \frac{x}{|} \rangle, \\
 \langle \frac{z}{|} \frac{m}{|} \frac{y}{|} \rangle \circ \langle \frac{y}{|} \frac{m}{|} \frac{x}{|} \rangle &= \delta_{x,z} \cdot \chi^*(x, y) \\
 &\quad \cdot \sum_a \chi^*(a, y) \cdot \langle \frac{x}{|} \frac{a}{|} \frac{x}{|} \rangle, \\
 \langle \frac{m}{|} \frac{a}{|} \frac{m}{|} \rangle \circ \langle \frac{m}{|} \frac{b}{|} \frac{m}{|} \rangle &= \chi^*(a, b) \cdot \langle \frac{m}{|} \frac{ab}{|} \frac{m}{|} \rangle, \\
 \langle \frac{m}{|} \frac{a}{|} \frac{m}{|} \rangle \circ \langle \frac{m}{|} \frac{m}{|} \frac{m}{|} \rangle &= \langle \frac{m}{|} \frac{m}{|} \frac{m}{|} \rangle \circ \langle \frac{m}{|} \frac{a}{|} \frac{m}{|} \rangle \\
 &= \chi(a, ab) \cdot \langle \frac{m}{|} \frac{m}{|} \frac{m}{|} \rangle, \\
 \langle \frac{m}{|} \frac{m}{|} \frac{m}{|} \rangle \circ \langle \frac{m}{|} \frac{m}{|} \frac{m}{|} \rangle &= \frac{s}{|A|} \cdot \chi(a, b) \\
 &\quad \cdot \sum_c \chi^*(ab, c) \cdot \langle \frac{m}{|} \frac{c}{|} \frac{m}{|} \rangle.
 \end{aligned} \tag{69}$$

Using (49), the $*$ -structure can be computed to be

$$\begin{aligned}
 \langle \frac{x}{|} \frac{a}{|} \frac{x}{|} \rangle^* &= \langle \frac{x}{|} \frac{a^{-1}}{|} \frac{x}{|} \rangle, \\
 \langle \frac{y}{|} \frac{m}{|} \frac{x}{|} \rangle^* &= \chi(x, y) \cdot \langle \frac{x}{|} \frac{m}{|} \frac{y}{|} \rangle, \\
 \langle \frac{m}{|} \frac{a}{|} \frac{m}{|} \rangle^* &= \chi^*(a, a) \cdot \langle \frac{m}{|} \frac{a^{-1}}{|} \frac{m}{|} \rangle, \\
 \langle \frac{m}{|} \frac{m}{|} \frac{m}{|} \rangle^* &= s \cdot \sum_b \chi^*(a, b) \cdot \langle \frac{m}{|} \frac{m}{|} \frac{m}{|} \rangle.
 \end{aligned} \tag{70}$$

Associativity of the algebra multiplication as well as its compatibility with the $*$ -structure then holds as a consequence of the character identity

$$\frac{1}{|A|} \cdot \sum_{a \in A} \chi(a, b) = \delta_{b,e}. \tag{71}$$

Tube representations: Using the equivalence (33), we can find the irreducible $*$ -representations of $\text{Tube}(\text{TY}_A^{\chi, s})$ from the classification of simple objects in the Drinfeld centre of $\text{TY}_A^{\chi, s}$ [50]. Concretely, there are $\frac{1}{2}|A| \cdot (|A| + 7)$ irreducible representations of $\text{Tube}(\text{TY}_A^{\chi, s})$, which can be grouped into the following three categories:

- There are $2 \cdot |A|$ one-dimensional representations R_x^\pm labelled by elements $x \in A$, which act on the non-trivial twisted sector $\mathcal{H}_x \cong \mathbb{C}$ via

$$\begin{aligned}
 R_x^\pm \left(\langle \frac{x}{|} \frac{a}{|} \frac{x}{|} \rangle \right) &= \chi(a, x), \\
 R_x^\pm \left(\langle \frac{x}{|} \frac{m}{|} \frac{x}{|} \rangle \right) &= \frac{1}{s} \cdot (\pm \Delta_x),
 \end{aligned} \tag{72}$$

where Δ_x is a square-root of $\chi^*(x, x) \in U(1)$.

- There are $2 \cdot |A|$ one-dimensional representations R_ρ^\pm labelled by antiderivatives¹³ ρ of χ , which act on the non-trivial twisted sector $\mathcal{H}_m \cong \mathbb{C}$ via

$$\begin{aligned} R_\rho^\pm \left(\left\langle \frac{m}{a} \middle| m \right\rangle \right) &= \rho^*(a), \\ R_\rho^\pm \left(\left\langle \frac{m}{a} \middle| \frac{m}{a} \right\rangle \right) &= s \cdot (\pm \Delta_\rho) \cdot \rho(a^{-1}), \end{aligned} \tag{73}$$

where Δ_ρ is a square-root of $s \cdot \sum_c \rho^*(c) \in U(1)$.

- There are $\frac{1}{2}|A| \cdot (|A| - 1)$ two-dimensional representations $R_{x,y}$ labelled by distinct elements $x, y \in A$, which act on the twisted sectors $\mathcal{H}_x \cong \mathcal{H}_y \cong \mathbb{C}$ via

$$\begin{aligned} R_{x,y} \left(\left\langle \frac{x}{a} \middle| x \right\rangle \right) &= \begin{pmatrix} \chi(a, y) & 0 \\ 0 & 0 \end{pmatrix}, \\ R_{x,y} \left(\left\langle \frac{y}{a} \middle| y \right\rangle \right) &= \begin{pmatrix} 0 & 0 \\ 0 & \chi(a, x) \end{pmatrix}, \\ R_{x,y} \left(\left\langle \frac{y}{a} \middle| x \right\rangle \right) &= \frac{1}{s} \cdot \begin{pmatrix} 0 & 0 \\ \chi^*(x, y) & 0 \end{pmatrix}, \\ R_{x,y} \left(\left\langle \frac{x}{a} \middle| y \right\rangle \right) &= \frac{1}{s} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{74}$$

It is easy to check that all of the above are $*$ -representations as expected. In the case $A = \mathbb{Z}_2$ and $s = 1/\sqrt{2}$, this reproduces the Ising symmetry category and its known action on twisted sector local operators in the two-dimensional critical Ising CFT [51, 52].

2.3.3. Fibonacci symmetry

As a last example, we consider a symmetry category \mathcal{C} with only two simple objects, 1 and τ , whose fusion rules are given by

$$\tau \otimes \tau = 1 \oplus \tau. \tag{75}$$

The pentagon equation then admits the following solution for the associator [53],

$$= 1 \oplus \tau^{\oplus 2} \xrightarrow{(\tau \otimes \tau) \otimes \tau} \begin{pmatrix} \text{id}_1 & 0 \\ 0 & A \cdot \text{id}_\tau \end{pmatrix} \xrightarrow{\tau \otimes (\tau \otimes \tau)} = 1 \oplus \tau^{\oplus 2}, \tag{76}$$

where the self-inverse (2×2) -matrix A is given by

$$A = \begin{pmatrix} -a & 1/\lambda \\ -a\lambda & a \end{pmatrix}. \tag{77}$$

Here, $a \in \mathbb{R}$ is one of the two solutions of the quadratic equation $a^2 = a + 1$ given by

$$a_\pm = \frac{1}{2} \left(1 \pm \sqrt{5} \right), \tag{78}$$

where $a_+ \equiv -1/a_- \approx 1.62$ is the golden ratio. Furthermore, $\lambda \in \mathbb{C}^\times$ is a gauge parameter that describes a family of equivalent fusion categories for fixed a . Up to equivalence, there are hence only two distinct fusion categories with fusion rules (75) – the *Fibonacci* and the *Yang-Lee category*—which we denote by Fib^+ and Fib^- and which correspond to choosing $a = a_-$ and $a = a_+$, respectively. The reason for this

¹³ An antiderivative of χ is a map $\rho : A \rightarrow U(1)$ such that $(d\rho)(a, b) := \rho(a) \cdot \rho(b) \cdot \rho^*(ab) \equiv \chi(a, b)$ for all $a, b \in A$. Since χ is symmetric, such a ρ always exists and the set of all antiderivatives forms a torsor over $A^\vee = \text{Hom}(A, U(1))$. This shows that there are $|A^\vee| = |A|$ antiderivatives of χ .

(perhaps confusing) notation is that the dimension of the non-invertible self-dual object τ in each case is given by

$$\dim(\tau) = -\frac{1}{a} \equiv \begin{cases} a_+ > 0 & \text{for Fib}^+ \\ a_- < 0 & \text{for Fib}^- \end{cases} \tag{79}$$

Since the matrix A is unitary if and only if

$$|\lambda|^2 = -\frac{1}{a}, \tag{80}$$

we see that only Fib^+ can be made unitary, whereas Fib^- represents its non-unitary counterpart.

Tube algebra: The tube algebra associated to Fib^\pm is spanned by seven generators

$$\begin{aligned} &\langle \begin{array}{c} 1 \\ \hline 1 \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle, \quad \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle, \quad \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle, \quad \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle, \\ &\langle \begin{array}{c} \tau \\ \hline 1 \end{array} \begin{array}{c} 1 \\ \hline \tau \end{array} \rangle, \quad \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle, \quad \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle, \end{aligned} \tag{81}$$

whose algebra multiplication can be computed to be

$$\begin{aligned} \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle \circ \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle &= \langle \begin{array}{c} 1 \\ \hline 1 \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle + \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle, \\ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle \circ \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle &= a \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle, \\ \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \circ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle &= \langle \begin{array}{c} 1 \\ \hline 1 \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle + a \cdot \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle, \\ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \circ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle &= -a \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle, \\ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \circ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle &= a^2 \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle, \\ \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle \circ \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle &= a \cdot \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle, \\ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle \circ \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle &= -a \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle + \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \\ &\quad + a^2 \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle, \\ \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \circ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle &= -a \cdot \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle, \\ \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \circ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle &= a^2 \cdot \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle, \\ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \circ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle &= -a^3 \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle + a^2 \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle, \\ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \circ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle &= -a^2 \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle + \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \\ &\quad + a^3 \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle, \\ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \circ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle &= \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \circ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle \\ &= a^2 \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle - a^2 \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle. \end{aligned} \tag{82}$$

Here, $a = a_-$ for Fib^+ and $a = a_+$ for Fib^- . Using (49), the $*$ -structure¹⁴ can be computed to be

$$\begin{aligned} \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle^* &= \langle \begin{array}{c} 1 \\ \hline 1 \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle, \\ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle^* &= -\frac{1}{a} \cdot \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle, \\ \langle \begin{array}{c} 1 \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle^* &= -a \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle, \\ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle^* &= -a \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \rangle - a \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle, \\ \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle^* &= \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle + a \cdot \langle \begin{array}{c} \tau \\ \hline \tau \end{array} \begin{array}{c} \tau \\ \hline \tau \end{array} \rangle, \end{aligned} \tag{83}$$

¹⁴ Although Fib^- is not unitary, (83) still defines an antilinear involution on $\text{Tube}(\text{Fib}^-)$. However, this $*$ -structure is no longer positive w.r.t. the canonical linear functional F as in (13).

where again $a = a_-$ for Fib^+ and $a = a_+$ for Fib^- . Associativity of the algebra multiplication as well as its compatibility with the $*$ -structure can be checked to hold as a consequence of the identity $a^2 = a + 1$.

Tube representations: There is a total of four irreducible representations of $\text{Tube}(\text{Fib}^\pm)$, which can be described as follows (here, $a = a_\pm$ for Fib^\mp):

- There is a one-dimensional representation R_1 acting on the twisted sector $\mathcal{H}_1 \cong \mathbb{C}$ via

$$R_1\left(\left\langle \begin{array}{c} 1 \\ \tau \\ 1 \end{array} \right\rangle\right) = -\frac{1}{a}. \tag{84}$$

This is a $*$ -representation for both Fib^\pm .

- There are two one-dimensional representations R_τ^\pm acting on the twisted sector $\mathcal{H}_\tau \cong \mathbb{C}$ via

$$\begin{aligned} R_\tau^\pm\left(\left\langle \begin{array}{c} \tau \\ \tau \\ 1 \end{array} \right\rangle\right) &= x_\pm, \\ R_\tau^\pm\left(\left\langle \begin{array}{c} \tau \\ 1 \\ \tau \end{array} \right\rangle\right) &= \frac{1}{a} \cdot \left(1 - \frac{x_\pm}{a}\right), \end{aligned} \tag{85}$$

where x_\pm are the two solutions of $x^2 + x + a^2 = 0$,

$$x_\pm = -\frac{1}{2} \pm i \cdot \sqrt{a + \frac{3}{4}} \tag{86}$$

which are related by $x_- = (x_+)^*$ and $x_+ \cdot x_- = a^2$. These are $*$ -representation for both Fib^\pm .

- There is one two-dimensional representation $R_{1,\tau}$ acting on the twisted sectors $\mathcal{H}_1 \cong \mathcal{H}_\tau \cong \mathbb{C}$ via

$$\begin{aligned} R_{1,\tau}\left(\left\langle \begin{array}{c} 1 \\ \tau \\ 1 \end{array} \right\rangle\right) &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \\ R_{1,\tau}\left(\left\langle \begin{array}{c} \tau \\ \tau \\ 1 \end{array} \right\rangle\right) &= \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}, \\ R_{1,\tau}\left(\left\langle \begin{array}{c} \tau \\ 1 \\ \tau \end{array} \right\rangle\right) &= \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix}, \\ R_{1,\tau}\left(\left\langle \begin{array}{c} \tau \\ \tau \\ 1 \end{array} \right\rangle\right) &= \lambda \cdot \begin{pmatrix} 0 & 0 \\ 1 + ia & 0 \end{pmatrix}, \\ R_{1,\tau}\left(\left\langle \begin{array}{c} 1 \\ \tau \\ \tau \end{array} \right\rangle\right) &= \frac{1}{\lambda} \cdot \begin{pmatrix} 0 & 1 - ia \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{87}$$

Here, $\lambda \in \mathbb{C}^\times$ is a gauge parameter that describes equivalent representations of $\text{Tube}(\text{Fib}^\pm)$. One can check that these are $*$ -representations if and only if

$$|\lambda|^2 = -\frac{1}{a}, \tag{88}$$

which admits a solution only when $a = a_-$ (in which case $-1/a_- = a_+ > 0$). Consequently, we see that the non-unitary Yang-Lee category Fib^- has a tube representation which is not a $*$ -representation.

3. Three dimensions

In this section, we review the construction of the tube algebra $\text{Tube}(\mathcal{C})$ associated to a fusion 2-category \mathcal{C} as introduced in [9]. We describe its canonical $*$ -structure and classify irreducible $*$ -representations in several examples related to higher group symmetries.

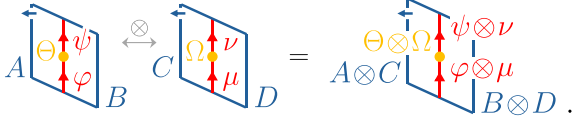
3.1. Preliminaries

In three dimensions, the finite (bosonic) generalised symmetries of a quantum field theory are described by a fusion 2-category \mathcal{C} [28], whose objects $A, B \in \mathcal{C}$ are topological surface defects, 1-morphisms $\varphi, \psi : A \rightarrow B$ are topological line interfaces and 2-morphisms $\Theta : \varphi \Rightarrow \psi$ are topological junctions:



$$A \boxed{\begin{array}{c} \psi \\ \Theta \\ \varphi \end{array}} B. \tag{89}$$

The monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ captures the fusion of topological surfaces, their interfaces and junctions:



$$A \boxed{\begin{array}{c} \psi \\ \Theta \\ \varphi \end{array}} B \otimes C \boxed{\begin{array}{c} \nu \\ \Omega \\ \mu \end{array}} D = A \otimes C \boxed{\begin{array}{c} \psi \otimes \nu \\ \Theta \otimes \Omega \\ \varphi \otimes \mu \end{array}} B \otimes D. \tag{90}$$

We further assume the following structures on \mathcal{C} :

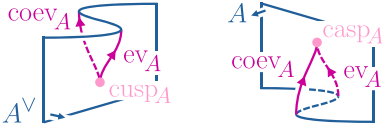
1. **Dual structure:** A dual structure on \mathcal{C} allows us to bend both topological surfaces and lines around. Concretely, it consists of the following pieces of data:

- A choice of dual object $A^\vee \in \mathcal{C}$ for every object $A \in \mathcal{C}$ together with evaluation and coevaluation 1-morphisms



$$A^\vee \text{ (cylinder)} \xrightarrow{\text{ev}_A} A \quad \text{coev}_A \text{ (cylinder)} \xrightarrow{A^\vee} A \tag{91}$$

and cusp and casp 2-isomorphisms



$$A^\vee \text{ (cusp)} \xrightarrow{\text{cusp}_A} A \quad \text{casp}_A \text{ (casp)} \xrightarrow{A^\vee} A \tag{92}$$

satisfying swallowtail relations [28]. The assignment $A \mapsto A^\vee$ then extends to a 1op-monoidal 2-functor $\vee : \mathcal{C} \rightarrow \mathcal{C}^{1\text{op}}$. A pivotal structure on \mathcal{C} is a natural isomorphism $\xi : \vee^2 \Rightarrow \text{id}_{\mathcal{C}}$.

- A choice of an adjoint 1-morphism $\widehat{\varphi} : B \rightarrow A$ for every 1-morphism $\varphi : A \rightarrow B$ in \mathcal{C} together with unit and counit 2-morphisms



$$A \boxed{\begin{array}{c} \text{un}_\varphi \\ \widehat{\varphi} \end{array}} B \quad A \boxed{\begin{array}{c} \text{coun}_\varphi \\ \widehat{\varphi} \end{array}} B \tag{93}$$

satisfying suitable snake or zig-zag relations. The assignment $\varphi \mapsto \widehat{\varphi}$ then extends to a monoidal 2-functor¹⁵ $\wedge : \mathcal{C} \rightarrow \mathcal{C}^{2\text{op}}$ acting as the identity on objects. A 2-pivotal structure on \mathcal{C} is a natural isomorphism $\Xi : \wedge^2 \Rightarrow \text{id}_{\mathcal{C}}$ with trivial component 1-morphisms. Using this, the left and

¹⁵ We denote by $\mathcal{C}^{2\text{op}}$ the 2-category with the same objects and 1-morphisms as \mathcal{C} and 2-morphisms given by $2\text{Hom}_{(\mathcal{C}^{2\text{op}})}(\varphi, \psi) = 2\text{Hom}_{\mathcal{C}}(\psi, \varphi)$ for all 1-morphisms φ and ψ .

right traces of a 2-endomorphism $\Theta \in 2\text{End}_{\mathcal{C}}(\varphi)$ of a 1-morphism $\varphi : A \rightarrow B$ are defined by¹⁶

$$\begin{aligned} \text{tr}_L(\Theta) &:= \text{Diagram 1} , \\ \text{tr}_R(\Theta) &:= \text{Diagram 2} . \end{aligned} \tag{94}$$

A dual structure on \mathcal{C} is called *spherical* if it is equipped with pivotal structures such that the associated front and back traces of 2-endomorphisms $\Theta \in 2\text{End}_{\mathcal{C}}(\text{id}_A)$ agree [28], i.e.

$$\text{Tr}_F(\Theta) := \text{Diagram 3} \stackrel{!}{=} \text{Diagram 4} =: \text{Tr}_B(\Theta) . \tag{95}$$

In this case, we define the dimension of an object $A \in \mathcal{C}$ by $\text{dim}(A) := \text{Tr}_{F/B}(\text{Id}_{\text{id}_A})$. Similarly, we set $\text{dim}(\varphi) := \text{Tr}_{F/B}(\text{tr}_{L/R}(\text{Id}_{\varphi}))$ for any morphism φ ¹⁷.

- 2. **Dagger structure:** A \dagger -structure on \mathcal{C} allows us to reflect topological junctions about a fixed hyperplane. Concretely, there is a 2-functor $\dagger : \mathcal{C} \rightarrow \mathcal{C}^{2\text{op}}$ that acts as the identity on objects and 1-morphisms and anti-linearly on 2-morphisms via

$$\text{Diagram 5} \xrightarrow{\dagger} \text{Diagram 6} \tag{96}$$

such that $\dagger^2 = \text{id}_{\mathcal{C}}$ and $2\text{End}_{\mathcal{C}}(\varphi)$ is a C^* -algebra for all 1-morphisms φ in \mathcal{C} . Furthermore, we assume all higher coherence data¹⁸ associated to \mathcal{C} as a 2-category to be unitary. This turns \mathcal{C} into a \dagger -2-category in the sense of [54, 55]. We assume the \dagger -structure to be compatible with the monoidal structure on \mathcal{C} in the sense that $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a \dagger -2-functor¹⁹ with unitary higher coherence data.

As before, we assume the above structures to be compatible with one another in the sense that both $\vee : \mathcal{C} \rightarrow \mathcal{C}^{1\text{op}}$ and $\wedge : \mathcal{C} \rightarrow \mathcal{C}^{2\text{op}}$ are \dagger -2-functors with unitary higher coherence data. In this case,

$$\text{Diagram 7} := \text{Diagram 8} \tag{97}$$

defines a canonical 2-pivotal structure on \mathcal{C} . Furthermore, we assume the pivotal structure ξ to have unitary higher coherence data. We call a dual structure of this type a *unitary dual structure*. In what follows, we assume \mathcal{C} to be a unitary fusion 2-category equipped with a spherical unitary dual structure.

3.2. Tube algebra

Given the symmetry 2-category \mathcal{C} , we can associate to it its *tube algebra* $\text{Tube}(\mathcal{C})$ [9], which captures the linking action of topological surface defects on twisted sector local operators in three dimensions. As a vector space, the tube algebra decomposes into a direct sum

$$\text{Tube}(\mathcal{C}) = \bigoplus_{[\mu],[\nu] \in \pi_1(\mathcal{C})} \text{Tube}(\mathcal{C})_{\mu,\nu} \tag{98}$$

¹⁶ Note that $\text{tr}_L(\Theta) \in 2\text{End}_{\mathcal{C}}(\text{id}_A)$ whereas $\text{tr}_R(\Theta) \in 2\text{End}_{\mathcal{C}}(\text{id}_B)$.

¹⁷ Here, we use the fact that $\text{Tr}_F(\text{tr}_L(\Theta)) = \text{Tr}_F(\text{tr}_R(\Theta))$ for any 2-endomorphism Θ and similarly for the back trace [28].

¹⁸ We collectively refer to any natural coherence 2-isomorphisms as *higher coherence data* in what follows. In the above case, the higher coherence data associated to \mathcal{C} as a 2-category is given by the associator and unitors for the composition of 1-morphisms.

¹⁹ A 2-functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ between two \dagger -2-categories \mathcal{C} and \mathcal{C}' is called a \dagger -2-functor if $\dagger' \circ F = F^{2\text{op}} \circ \dagger$ and all associated higher coherence data is unitary.

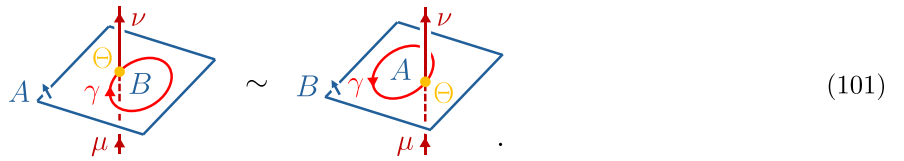
where the summand associated to simple lines $\mu, \nu \in \Omega\mathcal{C}$ is given by the quotient space²⁰

$$\text{Tube}(\mathcal{C})_{\mu, \nu} = \bigoplus_{A \in \mathcal{C}} 2\text{Hom}_{\mathcal{C}}(A \otimes \mu, \nu \otimes A) / \sim \tag{99}$$

of local intersection 2-morphisms



subjected to the equivalence relation generated by²¹



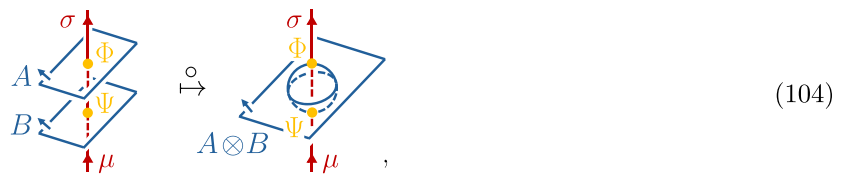
We denote the equivalence class of 2-morphisms (100) by

$$\text{Tube}(\mathcal{C})_{\mu, \nu} \cong \bigoplus_{[A] \in \pi_0(\mathcal{C})} 2\text{Hom}_{\mathcal{C}}(A \otimes \mu, \nu \otimes A) / \sim, \tag{103}$$

As a vector space, $\text{Tube}(\mathcal{C})_{\mu, \nu}$ is isomorphic to

$$\text{Tube}(\mathcal{C})_{\mu, \nu} \cong \bigoplus_{[A] \in \pi_0(\mathcal{C})} 2\text{Hom}_{\mathcal{C}}(A \otimes \mu, \nu \otimes A) / \sim, \tag{103}$$

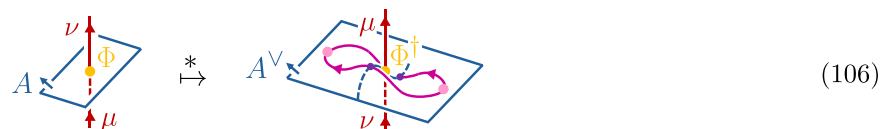
which shows that the tube algebra is finite-dimensional. The algebra multiplication is induced by



which we denote by

$$\langle \sigma \left[\begin{array}{c} A \\ \hline \nu' \\ \hline \Phi \end{array} \right] \rangle \circ \langle \nu \left[\begin{array}{c} B \\ \hline \mu \\ \hline \Psi \end{array} \right] \rangle = \delta_{\nu, \nu'} \cdot \langle \sigma \left[\begin{array}{c} A \otimes B \\ \hline \mu \\ \hline \Phi \circ \Psi \end{array} \right] \rangle \tag{105}$$

on the generators of $\text{Tube}(\mathcal{C})$. The tube algebra further admits an antilinear involution $*$: $\text{Tube}(\mathcal{C}) \rightarrow \text{Tube}(\mathcal{C})$, which is induced by²²



²⁰ For brevity, we often denote the identity 1-morphism of an object $A \in \mathcal{C}$ by the same letter A . Similarly for the identity 2-morphism of a 1-morphism φ in \mathcal{C} .

²¹ Here, we implicitly make use of the adjoint structure on \mathcal{C} .

²² Here, we implicitly make use of the dual structure on \mathcal{C} .

and which we denote by

$$\langle \nu \left[\begin{smallmatrix} A \\ \Phi \end{smallmatrix} \mu \right] \rangle^* = \langle \mu \left[\begin{smallmatrix} A^\vee \\ \Phi^\dagger \end{smallmatrix} \nu \right] \rangle \tag{107}$$

on the generators of the tube algebra.

3.3. Examples

We conclude this section with examples of unitary fusion 2-category symmetries and $*$ -representations of their associated tube algebras in the context of higher group symmetries in three dimensions.

3.3.1. Group symmetry

Let us first consider the fusion 2-category²³ $\mathcal{C} = 2\text{Hilb}_G^\pi$ corresponding to a finite group symmetry G with 't Hooft anomaly $[\pi] \in H^4(G, U(1))$. In this case, the associated tube algebra is simply the group algebra $\mathbb{C}[G]$ of G with $*$ -structure given by inversion [9]. In particular, $*$ -representations of the tube algebra are unitary representations of G . We now aim to rederive this result using higher S -matrices.

Drinfeld centre: As shown in [29], the Drinfeld centre of \mathcal{C} (as a linear 2-category) decomposes as

$$\mathcal{Z}(2\text{Hilb}_G^\pi) = \bigoplus_{[x] \in \text{Cl}(G)} 2\text{Rep}^{\tau_x(\pi)}(G_x), \tag{108}$$

so that the simple objects can be labelled by pairs (x, σ) consisting of a representative $x \in G$ of a conjugacy class $[x] \in \text{Cl}(G)$ together with an irreducible 2-representation [57–59] σ of the centraliser G_x of x with projective 3-cocycle $\tau_x(\pi) \in Z^3(G_x, U(1))$. Here, we defined the *transgression* of the 't Hooft anomaly π by

$$\tau_x(\pi)(g, h, k) := \frac{\pi(g, h, k, x) \cdot \pi(g, {}^{hk}x, h, k)}{\pi(g, h, {}^kx, k) \cdot \pi(g^{hk}x, g, h, k)}. \tag{109}$$

The monoidal unit of $\mathcal{Z}(\mathcal{C})$ is given by $\mathbf{1} = (e, \mathbf{1})$, where $e \in G$ is the identity element and $\mathbf{1}$ is the trivial 2-representation of G . In order to compute the connected components, we use the fact that

$$2\text{Rep}^\alpha(H) = \text{Mod}(\text{Hilb}_H^\alpha) \tag{110}$$

for any finite group H and 3-cocycle $\alpha \in Z^3(H, U(1))$, which, since $\text{Mod}(\mathcal{B})$ is connected for any \mathcal{B} [15, 28], shows that the connected components of $\mathcal{Z}(\mathcal{C})$ are in 1:1-correspondence with the conjugacy classes of G , i.e.

$$\pi_0(\mathcal{Z}(2\text{Hilb}_G^\pi)) = \text{Cl}(G). \tag{111}$$

Moreover, using that gauging a G -symmetry in three dimensions leads to topological Wilson lines [60–64],

$$\text{End}_{2\text{Rep}(G)}(\mathbf{1}) = \text{Rep}(G), \tag{112}$$

we see that the fundamental hypergroup of $\mathcal{Z}(\mathcal{C})$ is

$$\pi_1(\mathcal{Z}(2\text{Hilb}_G^\pi)) = \text{Irr}(\text{Rep}(G)). \tag{113}$$

In particular, since the number of conjugacy classes equals the number of isomorphism classes of irreducible representations, we see that $|\pi_0(\mathcal{Z}(\mathcal{C}))| = |\pi_1(\mathcal{Z}(\mathcal{C}))|$.

S-matrix: The S -matrix of $\mathcal{Z}(\mathcal{C})$ is the square matrix

$$S: \text{Cl}(G) \times \text{Irr}(\text{Rep}(G)) \rightarrow \mathbb{C} \tag{114}$$

²³ Here, we denote by 2Hilb the 2-category of 2-Hilbert spaces in the sense of [56], viewed as a \dagger -2-category in the sense of [54, 55].

that computes the character table of G [14], i.e.

$$S_{[x],[\rho]} = \frac{\text{Tr}(\rho(x))}{\dim(\rho)}. \tag{115}$$

In order to simplify notation, we set

$$n := |\text{Cl}(G)| = |\text{Irr}(\text{Rep})(G)| \tag{116}$$

and fix for each $i \in \{1, \dots, n\}$ a representative $x_i \in G$ of the corresponding conjugacy class $[x_i] \in \text{Cl}(G)$ as well as a representative $\rho_i \in \text{Rep}(G)$ of the corresponding isomorphism class $[\rho_i] \in \text{Irr}(\text{Rep}(G))$ of irreducible representations of G . Furthermore, we denote by $G_i := G_{x_i}$ the centraliser of x_i and by $\chi_i := \text{Tr}(\rho_i(\cdot))$ the character associated to the irreducible representation ρ_i (whose dimension we denote by $d_i := \dim(\rho_i)$). Using this, the S -matrix is the $(n \times n)$ -matrix with entries

$$S_{ij} := S_{[x_i],[\rho_j]} \equiv \frac{\chi_j(x_i)}{d_j}. \tag{117}$$

As a consequence of the properties of characters such as the character orthogonality relations

$$\sum_{k=1}^n \frac{1}{|G_k|} \cdot \chi_i^*(x_k) \cdot \chi_j(x_k) = \delta_{ij}, \tag{118}$$

$$\frac{1}{|G_i|} \cdot \sum_{k=1}^n \chi_k^*(x_i) \cdot \chi_k(x_j) = \delta_{ij}, \tag{119}$$

the S -matrix has the following properties:

- ① It is invertible with inverse given by

$$(S^{-1})_{ij} = \frac{d_i}{|G_j|} \cdot \chi_i^*(x_j). \tag{120}$$

- ② It satisfies $S_{i^\vee, j} = (S_{ij})^*$, where $i^\vee \in \{1, \dots, n\}$ is such that $[x_{i^\vee}] = [(x_i)^{-1}] \in \text{Cl}(G)$.

- ③ It satisfies the Verlinde formula

$$S_{i\ell} \cdot S_{j\ell} = \sum_{k=1}^n N_{ij}^k \cdot S_{k\ell}, \tag{121}$$

where the hypergroup coefficients $N_{ij}^k \in \mathbb{C}$ are

$$N_{ij}^k = \frac{1}{|G_k|} \cdot \sum_{p=1}^n \frac{1}{d_p} \cdot \chi_p(g_i) \cdot \chi_p(g_j) \cdot \chi_p^*(g_k). \tag{122}$$

Tube idempotents: As described in section 1.2, the S -matrix allows us to establish an equivalence between the category of $*$ -representations of the tube algebra and

$$\Omega(\mathcal{Z}(2\text{Hilb}_G^\pi)) = \text{Rep}(G) \tag{123}$$

via minimal central idempotents. Concretely, using (32), the minimal central idempotent associated to an irreducible representation $\rho \in \text{Rep}(G)$ is computed to be

$$e_\rho = \frac{\dim(\rho)}{|G|} \cdot \sum_{g \in G} \chi_\rho^*(g) \cdot g, \tag{124}$$

which reproduces the well-known formula for minimal central idempotents in the group algebra $\mathbb{C}[G]$ [65].

3.3.2. 2-group symmetry

As a second example, let us consider $\mathcal{C} = 2\text{Hilb}_G^\lambda$ corresponding to a finite anomalous 2-group symmetry [38, 39] $\mathcal{G} = A[1] \rtimes_\alpha G$ that is specified by the following data:

1. a finite 0-form symmetry group G ,
2. a finite abelian 1-form symmetry group A ,
3. a group action²⁴ $\triangleright : G \rightarrow \text{Aut}(A)$,
4. a Postnikov class $[\alpha] \in H_\triangleright^3(G, A)$,
5. a mixed 't Hooft anomaly $[\lambda] \in H_\triangleright^2(G, A^\vee)$.

Tube algebra: As demonstrated in [9], the tube algebra associated to 2Hilb_G^λ is the twisted groupoid algebra $\mathbb{C}^{\varepsilon(\lambda)}[A//G]$ with generators $\langle \frac{g}{\square} \frac{g}{\square} a \rangle$ ($g \in G, a \in A$) ($g \in G, a \in A$) that multiply according to

$$\langle \frac{g}{\square} \frac{g}{\square} a \rangle \circ \langle \frac{h}{\square} \frac{h}{\square} b \rangle = \delta_{a, hb} \cdot \varepsilon_b(g, h) \cdot \langle \frac{gh}{\square} \frac{gh}{\square} b \rangle, \tag{125}$$

where we defined the multiplicative phase

$$\varepsilon_a(\lambda)(g, h) = \langle \lambda(g, h), {}^{g^h}a \rangle. \tag{126}$$

As a result of the cocycle condition for λ , it obeys

$$(d\varepsilon)_a(g, h, k) := \frac{\varepsilon_a(h, k) \cdot \varepsilon_a(g, hk)}{\varepsilon_a(gh, k) \cdot \varepsilon_{(k_a)}(g, h)} = 1, \tag{127}$$

which ensures that the algebra multiplication in (125) is associative. Using (106), the $*$ -structure can be computed to be

$$\langle \frac{g}{\square} \frac{g}{\square} a \rangle^* = \mu_a(g) \cdot \langle \frac{a}{\square} \frac{g^{-1}}{\square} a \rangle, \tag{128}$$

where we defined the multiplicative phase

$$\mu_a(g) := \varepsilon_a^*(g^{-1}, g). \tag{129}$$

As a consequence of (127), it satisfies

$$\mu_a(g^{-1}) = \mu_{(a^g)}(g) \quad \text{and} \quad d\mu = \widehat{\varepsilon}/\varepsilon, \tag{130}$$

where we defined the 2-cocycle

$$\widehat{\varepsilon}_a(g, h) := \varepsilon_{(g^h a)}^*(h^{-1}, g^{-1}). \tag{131}$$

This ensures that the $*$ -structure (128) is involutory and compatible with the algebra multiplication. As in the case of an ordinary group, we now aim to construct $*$ -re-presentations of this algebra using higher S -matrices.

Drinfeld centre: In order to determine the Drinfeld centre of $\mathcal{C} = 2\text{Hilb}_G^\lambda$, we use the fact that we can gauge the 1-form symmetry A to obtain a pure 0-form symmetry \widehat{G} given by the group extension [60–64, 66]

$$\widehat{G} := A^\vee \rtimes_\lambda G, \tag{132}$$

²⁴ We will henceforth denote the action of a group element $g \in G$ on $a \in A$ by $g \triangleright a =: {}^g a$. Furthermore, we set $a^g := (g^{-1}) \triangleright a$.

which as a set is $A^\vee \times G$ with multiplication defined by

$$(\mu, g) \cdot (\nu, h) := (\mu \cdot {}^g\nu \cdot \lambda(g, h), g \cdot h). \tag{133}$$

This symmetry then has an 't Hooft anomaly parametrised by the 4-cocycle $\langle \cdot, \alpha \rangle \in Z^4(\widehat{G}, \mathbb{C}^\times)$ defined by

$$\langle \cdot, \alpha \rangle[(\mu, g), (\nu, h), (\varphi, k), (\psi, \ell)] := \langle {}^{ghk}\psi, \alpha(g, h, k) \rangle. \tag{134}$$

Using the gauge invariance of the Drinfeld centre,

$$\mathcal{Z}(2\text{Hilb}_G^\lambda) = \mathcal{Z}(2\text{Hilb}_{\widehat{G}}^{\langle \cdot, \alpha \rangle}), \tag{135}$$

we can then trace back the centre of the 2-group symmetry \mathcal{G} to the computation of the centre of an ordinary group symmetry as in (108). Concretely, the homotopy components of $\mathcal{Z}(2\text{Hilb}_G^\lambda)$ can be described as follows:

- The connected components of $\mathcal{Z}(2\text{Hilb}_G^\lambda)$ are in one-to-one correspondence with the conjugacy classes of $\widehat{G} = A^\vee \rtimes_\lambda G$. To describe the latter, we note that

$${}^{(\mu, g)}(\chi, x) = \left({}^g\chi \cdot \left(\mu / {}^{(g_x)}\mu \right) \cdot \tau_x(g), {}^g x \right) \tag{136}$$

for any $(\mu, g), (\chi, x) \in \widehat{G}$, where we defined the *transgression* of the 't Hooft anomaly λ by

$$\tau_x(\lambda)(g) := \frac{\lambda(g, x)}{\lambda(gx, g)}. \tag{137}$$

As a consequence of the twisted 2-cocycle condition obeyed by λ , it satisfies

$$(d\tau)_x(g, h) := \frac{{}^g[\tau_x(h)] \cdot \tau_{(h_x)}(g)}{\tau_x(gh)} = \frac{\lambda(g, h)}{{}^{(gh_x)}\lambda(g, h)}. \tag{138}$$

Upon defining the subgroup $A_x := \{a \in A \mid {}^x a = a\}$ and using the canonical identification

$$\frac{A^\vee}{\{\mu / {}^x\mu \mid \mu \in A^\vee\}} \cong (A_x)^\vee, \tag{139}$$

this shows that we can label the connected components of $\mathcal{Z}(2\text{Hilb}_G^\lambda)$ by equivalence classes of pairs (x, χ) consisting of a group element $x \in G$ and a character $\chi \in (A_x)^\vee$, where two such pairs (x, χ) and (x', χ') are considered equivalent if there exists a $g \in G$ such that

$$x' = {}^g x, \quad \chi' = {}^g \chi \cdot \tau_x(g). \tag{140}$$

We will denote the equivalence class of (x, χ) by $[x, \chi]$ in what follows.

- The fundamental hypergroup of $\mathcal{Z}(2\text{Hilb}_G^\lambda)$ is given by the set of isomorphism classes of irreducible representations of $\widehat{G} = A^\vee \rtimes_\lambda G$. By induction, these can be labelled by pairs (a, ρ) consisting of
 1. a group element $a \in A$, viewed as a character on the Pontryagin dual group A^\vee ,
 2. an irreducible representation ρ of the stabiliser $G_a := \{g \in G \mid {}^g a = a\}$ of a with projective 2-co-cycle $\langle \lambda, a \rangle \in Z^2(G_a, U(1))$.

The corresponding irreducible representation $\widehat{\rho}$ of \widehat{G} is then given by

$$\widehat{\rho} = \text{Ind}_{G_a}^{\widehat{G}}(a \otimes \rho), \tag{141}$$

where we defined $\widehat{G}_a := A^\vee \rtimes_\lambda G_a$. Two such pairs (a, ρ) and (a', ρ') are considered equivalent if $\widehat{\rho}$ and $\widehat{\rho}'$ are equivalent as representations of \widehat{G} . More concretely, (a, ρ) and (a', ρ') are equivalent if there exists a $g \in G$ such that

$$a' = {}^g a, \quad \rho' \cong \langle \sigma_g, {}^g a \rangle \otimes {}^g \rho, \tag{142}$$

where we defined the multiplicative factor

$$\sigma_g(\lambda)(h) := \frac{\lambda(h, g)}{\lambda(g, h^g)}. \tag{143}$$

As a result of the 2-cocycle condition for λ , it obeys

$$(d\sigma)_g(h, k) := \frac{h[\sigma_g(k)] \cdot \sigma_g(h)}{\sigma_g(hk)} = \frac{\lambda(h, k)}{g\lambda(h^g, k^g)}, \tag{144}$$

which ensures that (142) defines an equivalence relation. We will denote the equivalence class of (a, ρ) by $[a, \rho]$ in what follows.

S-matrix: Using the above, the S-matrix associated to $\mathcal{Z}(2\text{Hilb}_G^\lambda)$ is the square matrix that is indexed by equivalence classes $[x, \chi]$ and $[a, \rho]$ with entries

$$S_{[x, \chi], [a, \rho]} = \frac{\text{Tr}_{\widehat{\rho}}(\widehat{x})}{\dim(\widehat{\rho})}, \tag{145}$$

where $\widehat{\rho} = \text{Ind}_{\widehat{G}_a}^{\widehat{G}}(a \otimes \rho) \in \text{Rep}(\widehat{G})$ and $\widehat{g} = (\chi, x) \in \widehat{G}$. Using the character formula for induced representations,

$$\text{Tr}_{\widehat{\rho}}(\widehat{x}) = \frac{1}{|\widehat{G}_a|} \cdot \sum_{\substack{\widehat{g} \in \widehat{G}: \\ \widehat{x} \in \widehat{G}_a}} \text{Tr}_{a \otimes \rho}(\widehat{g}\widehat{x}), \tag{146}$$

as well as the character orthogonality relation

$$\frac{1}{|A^\vee|} \cdot \sum_{\mu \in A^\vee} \mu^*(b) \cdot \mu(c) = \delta_{b,c}, \tag{147}$$

we can then compute that the S-matrix is given by

$$S_{[x, \chi], [a, \rho]} = \frac{1}{\dim(\rho) \cdot |G|} \cdot \sum_{\substack{g \in G: \\ \widehat{x} \in G_a}} \langle \tau_x(g), a \rangle \cdot \text{Tr}_\rho(gx) \cdot \chi(a^g). \tag{148}$$

In particular, it satisfies properties ①–③, as is clear from its expression as a character table (145).

Tube representations: Using the above expression for the S-matrix and (32), we can compute the minimal central idempotents in $\mathbb{C}^{\varepsilon(\lambda)}[A//G]$ to be

$$e_{(a, \rho)} = \frac{\dim(\rho)}{|G_a|^2} \cdot \sum_{\substack{x, g \in G: \\ gx \in G_a}} \langle \tau_x(g), a \rangle^* \cdot \text{Tr}_\rho^*(gx) \cdot \left\langle \frac{g_a}{G} \frac{x}{G} \frac{g_a}{G} \right\rangle, \tag{149}$$

where, as before, the label (a, ρ) consists of

1. a group element $a \in A$,
2. an irreducible representation ρ of G_a with projective 2-cocycle $\varepsilon_a(\lambda) \in Z^2(G_a, U(1))$.

In particular, since there is a one-to-one correspondence between minimal central idempotents and irreducible representations of the tube algebra, we see that the latter are labelled by the same data (a, ρ) . Concretely, the irreducible representation $R_{(a, \rho)}$ corresponding to (a, ρ) can be constructed via induction: To this end, we fix for each $b \in [a]$ in the G -orbit $[a] \subset A$ a representative $r_b \in G$ such that $(r_b)b = a$ (with $r_a = e$). Using these, we can define

$$g_b := r_{(sb)} \cdot g \cdot r_b^{-1} \in G_a \tag{150}$$

for all $g \in G$ and $b \in [a]$. If we denote by \mathcal{V} the Hilbert space underlying the (projective) representation ρ of G_a , then $R_{(a,\rho)}$ acts on the Hilbert space \mathcal{H} with A -grading

$$\mathcal{H}_b = \begin{cases} \mathcal{V} & \text{if } b \in [a] \\ 0 & \text{otherwise} \end{cases} \tag{151}$$

via the induced tube action

$$R_{(a,\rho)}\left(\left\langle \begin{matrix} g_b \\ \boxed{\uparrow} \\ b \end{matrix} \right\rangle\right) := \kappa_b(g) \cdot \rho(g_b), \tag{152}$$

where we defined the multiplicative phase

$$\kappa_b(g) := \frac{\varepsilon_b(r_{(sb)}, g)}{\varepsilon_b(g_b, r_b)}. \tag{153}$$

As a consequence of (127), it satisfies

$$\frac{\kappa_b(h) \cdot \kappa_{(h_b)}(g)}{\kappa_b(gh)} = \frac{\varepsilon_b(g, h)}{\varepsilon_a(g_{(h_b)}, h_b)}, \tag{154}$$

which ensures that $R_{(a,\rho)}$ respects the algebra multiplication. As in the two-dimensional case, it is straightforward to check that $R_{(a,\rho)}$ is a $*$ -representation if and only if ρ is a unitary projective representation of G_a . Since every finite-dimensional representation of a finite group is equivalent to a unitary one, this shows that all irreducible representations of the tube algebra are $*$ -representations as expected.

4. Discussion

In this note, we generalised the notion of unitary actions of global symmetry groups on local operators to the action of higher fusion category symmetries \mathcal{C} on twisted sector operators in arbitrary spacetime dimension. We motivated that the latter transform $*$ -representations of the tube algebra $\text{Tube}(\mathcal{C})$ associated to \mathcal{C} , which is expected to be a C^* -algebra provided that the symmetry category \mathcal{C} is unitary in an appropriate sense. In particular, \mathcal{C} needs to be equipped with a \dagger -structure that implements reflections of top level morphisms (local defects) in \mathcal{C} . We then classified all irreducible $*$ -representations of $\text{Tube}(\mathcal{C})$ using the Symmetry TFT and its associated higher S -matrices.

A natural question to ask is how the above generalises to the action of higher fusion category symmetries on extended observables. In the case of invertible symmetries, this was discussed in [67], where line operators in $D > 2$ were shown to transform in unitary 2-representations of a global 2-group symmetry. This required the usage of higher \dagger -structures that implement reflections of topological defects beyond the top level of morphisms. In the general case, we hence expect the symmetry category \mathcal{C} to be equipped with a higher \dagger -structure that is compatible with the higher duality and coherence data associated to \mathcal{C} [23]. As a result, the higher tube algebras of \mathcal{C} [9] are expected to inherit the structure of higher C^* -algebras, whose higher $*$ -representations capture the action of \mathcal{C} on extended observables. We leave the exploration of these structures to future work.

More generally, one can consider unitary actions of the higher fusion category symmetry \mathcal{C} on boundary operators sitting on topological boundary conditions for \mathcal{C} . In two dimensions, this was studied in [68, 69] (see also [70] for a Symmetry TFT perspective and [71–73] for earlier work on related aspects), where boundary changing local operators were shown to transform in $*$ -representations of a generalised tube (or strip) algebra associated to \mathcal{C} . Generalising these constructions to higher dimensions will be the subject of future work.

Acknowledgement

The author would like to thank Mathew Bullimore and Andrea Grigoletto for helpful discussions and useful comments on the draft. During the creation of this article, the author was supported by an EPSRC Doctoral Studentship award at Durham University, United Kingdom.

Data availability statement

No new data were created or analysed in this study.

ORCID iD

Thomas Bartsch  0000-0002-5993-5797

References

- [1] Wigner E P 1931 *Gruppentheorie und Ihre Anwendung auf die Quantenmechanik der Atomspektren* (Vieweg Verlag)
- [2] Etingof P, Nikshych D and Ostrik V 2002 On fusion categories (arXiv:math/0203060)
- [3] Gaiotto D, Kapustin A, Seiberg N and Willett B 2015 Generalized global symmetries *J. High Energy Phys.* **JHEP02(2015)172**
- [4] Bhardwaj L, Bottini L E, Schafer-Nameki S and Tiwari A 2023 Non-invertible higher-categorical symmetries *SciPost Phys.* **14** 007
- [5] Shao S H 2023 What's done cannot be undone: TASI lectures on non-invertible symmetries (arXiv:2308.00747)
- [6] Schafer-Nameki S 2024 ICTP lectures on (non-)invertible generalized symmetries *Phys. Rept.* **1063** 1–55
- [7] Ocneanu A 1994 *Chirality for Operator Algebras* (World Sci. Publ.) pp 39–63
- [8] Bullivant A and Delcamp C 2019 Tube algebras, excitations statistics and compactification in gauge models of topological phases *J. High Energy Phys.* **JHEP10(2019)216**
- [9] Bartsch T, Bullimore M and Grigoletto A 2023 Representation theory for categorical symmetries (arXiv:2305.17165)
- [10] Gaiotto D and Kulp J 2021 Orbifold groupoids *J. High Energy Phys.* **JHEP02(2021)132**
- [11] Apruzzi F, Bonetti F, García Etxebarria I n, Hosseini S S and Schafer-Nameki S 2023 Symmetry TFTs from String Theory *Commun. Math. Phys.* **402** 895–949
- [12] Freed D S, Moore G W and Teleman C 2022 Topological symmetry in quantum field theory (arXiv:2209.07471)
- [13] Johnson-Freyd T 2021 Higher S-matrices *Higher Structures and Field Theory Seminar* (available at: <http://categorified.net/S-matrices.pdf>)
- [14] Reutter D 2022 Higher S-matrices and higher modular categories *2022 Workshop on Global Categorical Symmetries* (Perimeter Institute) (available at: <https://drive.google.com/file/d/1n6uwJtcsMRRFMiX75cp79JcAb4kESsVM/view>)
- [15] Johnson-Freyd T and Reutter D 2023 Minimal nondegenerate extensions *J. Am. Math. Soc.* **37** 81–150
- [16] Neshveyev S and Yamashita M 2018 A Few Remarks on the Tube Algebra of a Monoidal Category *Proc. Edinburgh Math. Soc.* **61** 735–58
- [17] Izumi M 2000 The structure of sectors associated with Longo-Rehren inclusions. I: General theory *Commun. Math. Phys.* **213** 127–79
- [18] Lin Y-H, Okada M, Seifnashri S and Tachikawa Y 2023 Asymptotic density of states in 2d CFTs with non-invertible symmetries *J. High Energy Phys.* **JHEP03(2023)094**
- [19] Osterwalder K and Schrader R 1973 Axioms for Euclidean Green's Functions *Commun. Math. Phys.* **31** 83–112
- [20] Osterwalder K and Schrader R 1975 Axioms for Euclidean Green's Functions. 2 *Commun. Math. Phys.* **42** 281
- [21] Jaffe A 2018 Reflection positivity then and now (arXiv:1802.07880)
- [22] Freed D S and Hopkins M J 2021 Reflection positivity and invertible topological phases *Geom. Topol.* **25** 1165–330
- [23] Ferrer G *et al* 2024 Dagger n -categories (arXiv:2403.01651)
- [24] Turaev V G and Viro O Y 1992 State sum invariants of 3 manifolds and quantum 6j symbols *Topology* **31** 865–902
- [25] Turaev V G 1994 *Quantum Invariants of Knots and Three Manifolds* vol 18
- [26] Barrett J W and Westbury B W 1996 Invariants of piecewise linear three manifolds *Trans. Am. Math. Soc.* **348** 3997–4022
- [27] Barrett J W and Westbury B W 1999 Spherical categories *Adv. Math.* **143** 357–75
- [28] Douglas C L and Reutter D J 2018 Fusion 2-categories and a state-sum invariant for 4-manifolds (arXiv:1812.11933)
- [29] Kong L, Tian Y and Zhou S 2020 The center of monoidal 2-categories in 3+1D Dijkgraaf-Witten theory *Adv. Math.* **360** 106928
- [30] Galindo C 2014 On braided and ribbon unitary fusion categories *Can. Math. Bull.* **57** 506–10
- [31] Etingof P, Gelaki S, Nikshych D and Ostrik V 2017 *Tensor Categories (Mathematical Surveys and Monographs)* (American Mathematical Society)
- [32] Bloom W R and Heyer H 1995 *Harmonic Analysis of Probability Measures on Hypergroups* (De Gruyter)
- [33] Sunder V and Wildberger N 2003 Actions of finite hypergroups *J. Algebr. Comb.* **18** 135–51
- [34] Kaidi J, Tachikawa Y and Zhang H Y 2024 On a class of selection rules without group actions in field theory and string theory *SciPost Phys.* **17** 169
- [35] Popa S, Shlyakhtenko D and Vaes S 2017 Cohomology and L2-Betti numbers for subfactors and quasi-regular inclusions *Int. Math. Res. Not.* **2018** 2241–331
- [36] Gruen A and Morrison S 2021 Computing modular data for pointed fusion categories *Indiana Univ. Math. J.* **70** 561–93
- [37] Hardiman L and King A 2020 Decomposing the tube category *Glasgow Math. J.* **62** 441–58
- [38] Baez J C and Lauda A D 2004 Higher-dimensional algebra. V: 2-Groups *Theory Appl. Categories* **12** 423–91
- [39] Benini F, Córdova C and Hsin P-S 2019 On 2-group global symmetries and their anomalies *J. High Energy Phys.* **JHEP03(2019)118**
- [40] Bakalov B and Kirillov A 2001 Lectures on tensor categories and modular functors *Amer. Math. Soc. Univ. Lect. Ser.* **21** 01
- [41] Penneys D 2021 Unitary fusion categories *Lecture Notes* (available at: <https://people.math.osu.edu/penneys.2/8800/Notes/FusionCats.pdf>)
- [42] Penneys D 2018 Unitary dual functors for unitary multitensor categories, higher structures
- [43] Bhowmick J, Ghosh S K, Rakshit N and Yamashita M 2018 Tube representations and twisting of graded categories *Quantum Algebra*
- [44] Drinfeld V 1989 Quasi-Hopf algebras *Algebra i Analiz* **6** 114–48
- [45] Majid S 1991 Doubles of quasitriangular hopf algebras *Commun. Algebra* **19** 3061–73
- [46] Roche P, Pasquier V and Dijkgraaf R 1990 QuasiHopf algebras, group cohomology and orbifold models *Nucl. Phys. B Proc. Suppl.* **18** 60–72

- [47] Davydov A and Simmons D 2017 On lagrangian algebras in group-theoretical braided fusion categories *J. Algebra* **471** 149–75
- [48] Willerton S 2008 The twisted Drinfeld double of a finite group via gerbes and finite groupoids *Algebr. Geom. Topol.* **8** 1419–57
- [49] Tambara D and Yamagami S 1998 Tensor categories with fusion rules of self-duality for finite abelian groups *J. Algebra* **209** 692–707
- [50] Gelaki S, Naidu D and Nikshych D 2009 Centers of graded fusion categories (arXiv:0905.3117)
- [51] Chang C-M, Lin Y-H, Shao S-H, Wang Y and Yin X 2019 Topological defect lines and renormalization group flows in two dimensions *J. High Energy Phys.* [JHEP01\(2019\)026](#)
- [52] Thorngren R and Wang Y 2024 Fusion category symmetry. Part I. Anomaly in-flow and gapped phases *J. High Energy Phys.* [JHEP04\(2024\)132](#)
- [53] Davydov A and Booker T 2011 Commutative algebras in fibonacci categories (arXiv:1103.3537)
- [54] Longo R and Roberts J E 1997 A theory of dimension K *Theory* **11** 103–59
- [55] Chen Q, Hernández Palomares R, Jones C and Penneys D 2022 Q -system completion for C^* 2-categories *J. Funct. Anal.* **283** 109524
- [56] Baez J C 1997 Higher-dimensional algebra II: 2-Hilbert spaces *Adv. Math.* **127** 125–89
- [57] Ganter N and Kapranov M 2008 Representation and character theory in 2-categories *Adv. Math.* **217** 2268–300
- [58] Elgueta J 2007 Representation theory of 2-groups on Kapranov and Voevodsky’s 2-vector spaces *Adv. Math.* **213** 53–92
- [59] Osorno A M 2010 Explicit formulas for 2-characters *Topol. Appl.* **157** 369–77
- [60] Bartsch T, Bullimore M, Ferrari A E V and Pearson J 2024 Non-invertible symmetries and higher representation theory I *SciPost Phys.* **17** 015
- [61] Bhardwaj L, Schafer-Nameki S and Wu J 2022 universal non-invertible symmetries *Fortsch. Phys.* **70** 2200143
- [62] Bartsch T, Bullimore M, Ferrari A E V and Pearson J 2024 Non-invertible symmetries and higher representation theory II *SciPost Phys.* **17** 067
- [63] Bhardwaj L, Schafer-Nameki S and Tiwari A 2023 Unifying constructions of non-invertible symmetries *SciPost Phys.* **15** 122
- [64] Bhardwaj L, Bottini L E, Schafer-Nameki S and Tiwari A 2023 Non-invertible symmetry webs *SciPost Phys.* **15** 160
- [65] Janusz G J 1966 Primitive idempotents in group algebras *Proc. Am. Math. Soc.* **17** 520–3
- [66] Décoppet T D and Yu M 2023 Fiber 2-functors and Tambara–Yamagami fusion 2-categories (arXiv:2306.08117)
- [67] Bartsch T 2024 On unitary 2-group symmetries (arXiv:2411.05067)
- [68] Cordova C, Holfester N and Ohmori K 2024 Representation theory of solitons (arXiv:2408.11045)
- [69] Choi Y, Rayhaun B C and Zheng Y 2024 Generalized tube algebras, symmetry-resolved partition functions, and twisted boundary states (arXiv:2409.02159)
- [70] Bhardwaj L, Copetti C, Pajer D and Schafer-Nameki S 2024 Boundary SymTFT (arXiv:2409.02166)
- [71] Huang S-J and Cheng M 2023 Topological holography, quantum criticality, and boundary states (arXiv:2310.16878)
- [72] Copetti C 2024 Defect charges, gapped boundary conditions, and the symmetry TFT (arXiv:2408.01490)
- [73] Copetti C, Cordova L and Komatsu S 2025 S-matrix bootstrap and non-invertible symmetries *J. High Energy Phys.* [JHEP03\(2025\)204](#)