A CONTINUOUS-PARAMETER KATZNELSON–TZAFRIRI THEOREM FOR ALGEBRAS OF ANALYTIC FUNCTIONS

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Abstract. We prove a continuous-parameter version of the recent theorem of Katznelson–Tzafiri type for power-bounded operators which have a bounded calculus for analytic Besov functions. We also show that the result can be extended to some operators which have functional calculi with respect to some larger algebras.

1. Introduction

In 1986, Katznelson and Tzafriri [13] proved a theorem concerning asymptotics of the discrete semigroup $(T^n)_{n\geq 0}$ for a power-bounded operator T on a complex Banach space X. They showed that $\lim_{n\to\infty} \|T^n(I-T)\| = 0$ if $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$. More generally, they considered functions in the Wiener algebra $W^+(\mathbb{D})$ of the form $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where $\sum_{k=0}^{\infty} |a_k| < \infty$ and $|z| \leq 1$. Let $f(T) = \sum_{k=0}^{\infty} a_k T^k$. Let $W(\mathbb{T})$ be the space of all functions on $\mathbb{T}$ of the form $g(z) = \sum_{k=-\infty}^{\infty} b_k z^k$, where $\|g\|_{W(\mathbb{T})} := \sum_{k=-\infty}^{\infty} |b_k| < \infty$. It was shown in [13] that $\lim_{n\to\infty} \|T^n f(T)\| = 0$ if $f \in W^+(\mathbb{D})$ and f is of spectral synthesis in $W(\mathbb{T})$ with respect to $\sigma(T) \cap \mathbb{T}$. This assumption means that there exist functions $(g_k)_{k\geq 1}$ in $W(\mathbb{T})$ such that each g_k vanishes on a neighbourhood U_k of $\sigma(T) \cap \mathbb{T}$ in $\mathbb{T}$ and $\lim_{k\to\infty} \|g_k - f\|_{W(\mathbb{T})} = 0$.

These theorems have had a variety of applications. For a selection of them, see Section 4 of the recent survey article [6]. One drawback of the more general theorem is the assumption of spectral synthesis, which is stronger than simply assuming that f vanishes on $\sigma(T) \cap \mathbb{T}$. The theorem is not true in general if the weaker assumption is used, but the two assumptions are equivalent if $\sigma(T) \cap \mathbb{T}$ is countable, for example. If X is a Hilbert space, then the weaker assumption is sufficient [14]. The survey article [6] covers these and other variants of the theorem, mainly in the discrete case.

The following theorem is an analogue of the original theorem for bounded C_0-semigroups, and it was proved in [11] and [19]. Their proofs were quite different from the proofs in [13] and from each other. For a discussion of

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other proofs, see [6, Section 3.1]. In this paper we shall closely follow the approach used in [19].

Theorem 1.1. *Let $-A$ be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Let f be the Laplace transform of a function in $L^1(\mathbb{R}_+)$, and assume that f is of spectral synthesis in $L^1(\mathbb{R})$ with respect to $\sigma(A)\cap i\mathbb{R}$. Then $\lim_{t\to\infty}\|T(t)f(A)\| = 0$.*

Some variants of this continuous-parameter version of the Katznelson–Tzafriri theorem have been obtained. If X is a Hilbert space, the result is true with the weaker assumption that f vanishes on $\sigma(A)\cap i\mathbb{R}$ [18]. Several extensions to more general semigroups of operators have appeared in [18] and [21]. Some papers have considered Banach algebras other than the Laplace transforms of integrable functions. An example in the discrete case is a Banach algebra $\mathcal{B}(\mathbb{D})$ of analytic Besov functions on the unit disc $\mathbb{D}$, and Peller showed in [17] that any power-bounded operator on a Hilbert space has a bounded $\mathcal{B}(\mathbb{D})$-calculus. In the continuous-parameter case, the theory of functional calculus for the corresponding algebra $\mathcal{B}$ of analytic Besov functions on the right half-plane $\mathbb{C}_+$ and bounded C_0-semigroups has recently been developed in [3] and [4].

In [7], we proved a version of the Katznelson–Tzafriri theorem in the discrete case where T is assumed to have a bounded functional calculus with respect to the Banach algebra $\mathcal{B}(\mathbb{D})$. It applies to functions $f\in\mathcal{B}(\mathbb{D})$ which vanish on $\sigma(T)\cap\mathbb{T}$, so it includes not only some functions outside $W^+(\mathbb{D})$, but also functions in $W^+(\mathbb{D})$ which are not of spectral synthesis with respect to $\sigma(T)\cap\mathbb{T}$. When we wrote [7], we did not see how to obtain a corresponding result in the continuous-parameter case. Subsequently we have been able to prove the result stated in Theorem 1.2 below, by using an approximation argument based on Arveson's theory of spectral subspaces for C_0-groups of isometries; see [18] for a related argument. We also rely crucially on results by Cojuhari and Gomilko [8] concerning a certain integral resolvent condition which characterises those operators that admit a bounded $\mathcal{B}$-calculus.

We now state the main result of our paper for operators which have a bounded $\mathcal{B}$-calculus. The assumption that f vanishes at infinity is natural in the result because infinity can be thought of as being an invisible part of $\sigma(A)$ when A is unbounded; indeed without this assumption the theorem would be false if $\sigma(A)\cap i\mathbb{R}$ is empty, $(T(t))_{t\geq 0}$ is not exponentially stable, and f is a constant function. In the statement, the operator $f(A)$ is defined by the $\mathcal{B}$-calculus for A.

Theorem 1.2. *Let $-A$ be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ and assume that A has a bounded $\mathcal{B}$-calculus. Let $f \in \mathcal{B}$, and assume that f vanishes on $\sigma(A) \cap i\mathbb{R}$ and $\lim_{z\in\mathbb{C}_+,\,|z|\to\infty} f(z) = 0$. Then*

$$\lim_{t\to\infty} \|T(t)f(A)\| = 0.$$

Since Laplace transforms of functions in $L^1(\mathbb{R}_+)$ lie in $\mathcal{B}$ and vanish at infinity, this result extends Theorem 1.1 in the case when A admits a bounded $\mathcal{B}$-calculus, both by enlarging the class of functions to which it applies and by weakening the spectral synthesis condition to the condition that f vanishes on $\sigma(A) \cap i\mathbb{R}$ (which is necessary for the conclusion of the theorem). Furthermore, as the negative generator of every bounded C_0-semigroup on a Hilbert space admits a bounded $\mathcal{B}$-calculus (see [3, Section 4]), we obtain the following result as an immediate consequence of Theorem 1.2.

Corollary 1.3. *Let $-A$ be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space. Let $f \in \mathcal{B}$, and assume that f vanishes on $\sigma(A) \cap i\mathbb{R}$ and $\lim_{z\in\mathbb{C}_+,\,|z|\to\infty} f(z) = 0$. Then $\lim_{t\to\infty} \|T(t)f(A)\| = 0$.*

This extends the implication (i) $\Longrightarrow$ (iii) of [18, Theorem 3.1] in the case when $S = \mathbb{R}_+$ to a larger class of functions f; see also [14].

In the next section, we recall relevant facts for understanding the statement and proof of Theorem 1.2. We originally considered only the algebra $\mathcal{B}$, but an anonymous referee suggested that we might be able to prove Theorem 1.2 for algebras other than $\mathcal{B}$ by similar methods. After some time, we proved a version of the result for some algebras larger than $\mathcal{B}$, and that version is set out in Theorem 2.1. The proof is given in Section 3.

The papers [1] and [2] have identified two function algebras which are larger than $\mathcal{B}$, and to which Theorem 2.1 can be extended in a slightly modified form. These algebras are briefly described in Section 4.

2. The setting and the main result

Let $\mathcal{B}$ be the space of all holomorphic functions f on $\mathbb{C}_+$ such that

$$\|f\|_{\mathcal{B}_0} := \int_0^\infty \sup_{\beta\in\mathbb{R}} |f'(\alpha + i\beta)|\, d\alpha < \infty.$$

These functions are bounded and uniformly continuous on $\mathbb{C}_+$, and $\mathcal{B}$ is a Banach algebra in the norm

$$\|f\|_{\mathcal{B}} := \|f\|_{\mathcal{B}_0} + \|f\|_\infty,$$

where $\|\cdot\|_\infty$ is the supremum norm on $H^\infty(\mathbb{C}_+)$. We will consider any function $f \in \mathcal{B}$ to be defined and uniformly continuous on $\overline{\mathbb{C}_+}$. For details about $\mathcal{B}$, see [3].

We will also consider the following subalgebras of $\mathcal{B}$, adopting the notation used in [3], [4] and [5]:

$$\mathcal{L}L^1 := \left\{ g : g \in L^1(\mathbb{R}_+) \right\}, \quad \text{where is the Laplace transform,}$$
$$\mathcal{B}_{00} := \left\{ f \in \mathcal{B} : \lim_{|\beta|\to\infty} f(i\beta) = 0 \right\} = \left\{ f \in \mathcal{B} : \lim_{z\in\mathbb{C}_+,\,|z|\to\infty} f(z) = 0 \right\}.$$

The algebra $\mathcal{L}L^1$ is dense in $\mathcal{B}_{00}$ in the $\mathcal{B}$-norm [4, Theorem 4.4], and $\mathcal{B}_{00}$ is a closed subalgebra of $\mathcal{B}$, so $\mathcal{B}_{00}$ is the closure of $\mathcal{L}L^1$ in $\mathcal{B}$. The norms $\|\cdot\|_{\mathcal{B}_0}$ and $\|\cdot\|_{\mathcal{B}}$ are equivalent on $\mathcal{B}_{00}$, but $\|\cdot\|_{\mathcal{B}_0}$ is not submultiplicative. Note that the space $\mathcal{B}_{00}$ is denoted by $\mathcal{B} \cap C_0(\overline{\mathbb{C}_+})$ in [4].

Let $-A$ be the generator of a bounded C_0-semigroup on a Banach space X, so that $\sigma(A) \subseteq \overline{\mathbb{C}_+}$. We say that A *satisfies the* (GSF) *condition* if, for all $x \in X$ and $x^* \in X^*$,

$$\sup_{\alpha>0} \alpha \int_{\mathbb{R}} \left| \langle (\alpha + i\beta + A)^{-2} x, x^* \rangle \right| \, d\beta < \infty. \tag{2.1}$$

The Closed Graph Theorem then implies that there exists a finite constant γ_A such that the supremum above is bounded by $\gamma_A \|x\| \|x^*\|$ for all $x \in X$ and $x^* \in X^*$.

Let $L(X)$ be the Banach algebra of bounded linear operators on X. For $\lambda \in \mathbb{C}$, we will let r_λ denote the function $r_\lambda(z) := (\lambda + z)^{-1}$ with appropriate domain in $\mathbb{C}$. Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{B}$ is continuously included in $\mathcal{A}$ and $\mathcal{A}$ is continuously included in $H^\infty(\mathbb{C}_+)$. We say that A *has a bounded* $\mathcal{A}$*-calculus* if there is a bounded algebra homomorphism $\Phi : \mathcal{A} \to L(X)$ such that $\Phi(r_\lambda) = (\lambda + A)^{-1}$ for some (or equivalently, all) $\lambda \in \mathbb{C}_+$.

It is shown in [3, Theorem 4.4] and [4, Theorem 6.1] that A satisfies the (GSF) condition if and only if A has a bounded $\mathcal{B}$-calculus. Moreover, A satisfies the (GSF) condition if Φ is defined and bounded only on $\mathcal{B}_{00}$, as all the functions $G_{\alpha,\varphi}$ in the proof of [4, Theorem 6.1] belong to $\mathcal{B}_{00}$. In addition, any bounded $\mathcal{B}$-calculus for A is unique [4, Theorem 6.2], and we will denote it by $\Phi^A_{\mathcal{B}}$.

In the case when $\mathcal{A} = \mathcal{B}$, the following result coincides with Theorem 1.2, since $\mathcal{B}_{00}$ is the closure of $\mathcal{L}L^1$ in $\mathcal{B}$. In general, the closure of $\mathcal{L}L^1$ in $\mathcal{A}$ is contained in $\{f \in \mathcal{A} : \lim_{z\in\mathbb{C}_+,\,|z|\to\infty} f(z) = 0\}$, but equality may not hold.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{B}$ is continuously included in $\mathcal{A}$ and $\mathcal{A}$ is continuously included in $H^\infty(\mathbb{C}_+)$. Let $-A$ be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ and assume that A has a* *bounded $\mathcal{A}$-calculus. Let f be in the closure of $\mathcal{L}L^1$ in $\mathcal{A}$, and assume that f vanishes on $\sigma(A) \cap i\mathbb{R}$. Then* $\lim_{t\to\infty} \|T(t)f(A)\| = 0$.

To prove Theorem 2.1, we will need to consider the case of a *skew-hermitian* operator Z on a Banach space Y, so that $-Z$ generates a C_0-group of isometries on Y. Then $\sigma(Z) \subseteq i\mathbb{R}$. If Z has a bounded $\mathcal{B}$-calculus, then Z also has a bounded $C_0(i\mathbb{R})$-calculus, that is, a bounded algebra homomorphism from $C_0(i\mathbb{R})$ to $L(X)$ mapping r_λ to $(\lambda + Z)^{-1}$ for some (or all) $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. The converse also holds. These statements can be seen in [4, Theorem 6.5], or by using a combination of [4, Theorem 6.1], [8, Lemma 3.4] and [10, Theorem 3.6]. The density of the algebra generated by the resolvent functions $\{r_\lambda : \lambda \in \mathbb{C} \setminus i\mathbb{R}\}$ in $C_0(i\mathbb{R})$ shows that any bounded $C_0(i\mathbb{R})$-calculus for Z is unique, and we denote it by $\Phi^Z_{C_0}$. If $f \in \mathcal{B}_{00}$, then $\Phi^Z_{\mathcal{B}}(f)$ and $\Phi^Z_{C_0}(f)$ are both defined (here we use f to denote the boundary function of f on $i\mathbb{R}$). The proof of [4, Theorem 6.5] did not show explicitly that the two definitions coincide. This compatibility can be deduced from the paragraph following the proof of Theorem 6.5 and equation (6.4) in [4]. They establish a multiplier formula for $\Phi^Z_{\mathcal{B}}$, namely

$$\Phi^Z_{\mathcal{B}}(f)\Phi^Z_{C_0}(g) = \lim_{n\to\infty} \Phi^Z_{C_0}(fe_n)\Phi^Z_{C_0}(g) \tag{2.2}$$

in the strong operator topology, where $g \in C_0(i\mathbb{R})$ and $(e_n)_{n\geq 1}$ is a bounded approximate unit in $C_0(i\mathbb{R})$. The same formula holds when $\Phi^Z_{\mathcal{B}}(f)$ is replaced by $\Phi^Z_{C_0}(f)$. The compatibility of the two calculi follows; see also Remark 3.3.

If Z satisfies the (GSF) condition, then $-Z$ also satisfies the (GSF) condition. This can be seen from [8, Lemma 3.4]. It corresponds to the fact that if Z has a bounded $C_0(i\mathbb{R})$-calculus, then so does $-Z$, with $\Phi^{-Z}_{C_0}(f) = \Phi^Z_{C_0}(\tilde{f})$, where $\tilde{f}(i\beta) = f(-i\beta)$ for $f \in C_0(i\mathbb{R})$ and $\beta \in \mathbb{R}$.

3. The proof of Theorem 2.1

The first step in the proof is the observation that it suffices to prove the result under the additional assumption that $\mathcal{B}$ is dense in $\mathcal{A}$. We make this assumption without any loss of generality, as $\mathcal{A}$ can be replaced by the closure of $\mathcal{B}$ in $\mathcal{A}$. This assumption implies that all functions in $\mathcal{A}$ are bounded and uniformly continuous on $\overline{\mathbb{C}_+}$ and any bounded $\mathcal{A}$-calculus for an operator A is unique, by the uniqueness of any $\mathcal{B}$-calculus and the density of $\mathcal{B}$ in $\mathcal{A}$. Throughout this section, we take $\mathcal{A}$ to be a fixed Banach algebra of this form. It may be advantageous to take $\mathcal{A} = \mathcal{B}$ on first reading of the proof.

Next we consider the case of a skew-hermitian operator Z which has a bounded $\mathcal{B}$-calculus. Then Z also has a bounded $\mathcal{A}$-calculus $\Phi^Z_{\mathcal{A}}$, given by the multiplier formula (2.2) with $\mathcal{B}$ replaced by $\mathcal{A}$. This calculus is an extension of the $\mathcal{B}$-calculus for Z, and it is compatible with the $C_0(i\mathbb{R})$-calculus for Z.

Now assume that Z is a bounded skew-hermitian operator with a bounded $\mathcal{B}$-calculus, and let K be a compact subset of $i\mathbb{R}$ such that $\sigma(Z)$ is contained in the interior of K in $i\mathbb{R}$. The following lemma shows that there is a bounded $C(K)$-calculus for Z with properties similar to those of a single polynomially bounded isometry; see [16, Lemma 1.1]. A *bounded $C(K)$-calculus* for Z is a bounded algebra homomorphism mapping the constant function 1 to the identity operator I and the identity function to Z, or equivalently mapping r_λ to $(\lambda + Z)^{-1}$ for all $\lambda \in \mathbb{C} \setminus K$. Such a calculus is necessarily unique by the density of the polynomials in $C(K)$.

Lemma 3.1. *Let Z be a bounded skew-hermitian operator on a Banach space Y, and assume that Z has a bounded $\mathcal{B}$-calculus. Let $\mathcal{A}$ be a Banach algebra as described above, and let K be a compact subset of $i\mathbb{R}$ such that $\sigma(Z)$ is contained in the interior of K in $i\mathbb{R}$. There is a unique bounded $C(K)$-calculus Ψ for Z. It has the following properties:*

(a) *If $f \in C_0(i\mathbb{R})$ and g is the restriction of f to K, then $\Psi(g) = \Phi^Z_{C_0}(f)$.*
(b) *If $f \in \mathcal{A}$ and g is the restriction of f to K, then $\Psi(g) = \Phi^Z_{\mathcal{A}}(f)$.*
(c) *If $g \in C(K)$, then $\sigma(\Psi(g)) = g(\sigma(Z))$.*
(d) *If $g \in C(K)$ and g vanishes on $\sigma(Z)$, then $\Psi(g) = 0$.*

Proof. Since Z has a bounded $\mathcal{B}$-calculus, the (GSF) condition holds for Z and for $-Z$, so, for all $\alpha > 0$, $y \in Y$ and $y^* \in Y^*$,

$$\alpha \int_{\mathbb{R}} \left|\langle (\alpha + i\beta - Z)^{-2} y, y^* \rangle\right| \, d\beta \le \gamma_{-Z} \, \|y\| \, \|y^*\|.$$

For $\alpha \in (0, 1]$ and $\beta \in \mathbb{R}$, let

$$\begin{aligned} \Delta(\alpha, \beta, -Z) &:= (\alpha + i\beta - Z)^{-1} - (-\alpha + i\beta - Z)^{-1} \\ &= -2\alpha(\alpha + i\beta - Z)^{-1}(-\alpha + i\beta - Z)^{-1}. \end{aligned}$$

Fix $k > \|Z\|$, and let $K' = \{\beta \in \mathbb{R} : i\beta \in K\}$. Then

$$\begin{aligned} \|\Delta(\alpha, \beta, -Z)\| &\le \frac{2\alpha}{(|\beta| - \|Z\|)^2}, && \alpha \in (0,1],\ |\beta| > k, \\ \|\Delta(\alpha, \beta, -Z)\| &\le c, && \alpha \in (0,1],\ |\beta| \le k,\ \beta \in \mathbb{R} \setminus K', \\ \lim_{\alpha \to 0+} \|\Delta(\alpha, \beta, -Z)\| &= 0, && i\beta \notin \sigma(Z), \end{aligned}$$

for some constant c. The Dominated Convergence Theorem and these estimates imply that

$$\lim_{\alpha\to 0+}\int_{\mathbb{R}\setminus K'} \|\Delta(\alpha,\beta,-Z)\|\,d\beta = 0. \tag{3.1}$$

Let $f \in C_0(i\mathbb{R})$. It follows from [10, Theorem 3.6] and (3.1) that the $C_0(i\mathbb{R})$-calculus for Z is given by

$$\begin{aligned}\langle \Phi^Z_{C_0}(f)y, y^*\rangle &= \frac{1}{2\pi}\lim_{\alpha\to 0+}\int_{\mathbb{R}} f(i\beta)\langle\Delta(\alpha,\beta,-Z)y,y^*\rangle\,d\beta\\ &= \frac{1}{2\pi}\lim_{\alpha\to 0+}\int_{K'} f(i\beta)\langle\Delta(\alpha,\beta,-Z)y,y^*\rangle\,d\beta.\end{aligned}$$

From [8, Lemma 3.4] applied to $-iZ$, there is a constant C such that

$$\int_{\mathbb{R}} |\langle\Delta(\alpha,\beta,-Z)y,y^*\rangle|\,d\beta \le C\,\|y\|\,\|y^*\|, \qquad \alpha\in(0,1],\ y\in Y,\ y^*\in Y^*.$$

Then

$$\begin{aligned}\left|\langle \Phi^Z_{C_0}(f)y, y^*\rangle\right| &\le \frac{1}{2\pi}\int_{K'} \|f\|_{C(K)}\,|\langle\Delta(\alpha,\beta,-Z)y,y^*\rangle|\,d\beta\\ &\le \frac{C}{2\pi}\,\|f\|_{C(K)}\,\|y\|\,\|y^*\|.\end{aligned}$$

This implies that

$$\|\Phi^Z_{C_0}(f)\| \le \frac{C}{2\pi}\|f\|_{C(K)}. \tag{3.2}$$

We can now define a bounded $C(K)$-calculus for Z, as follows. Let $g\in C(K)$, and let $f\in C_0(i\mathbb{R})$ be any extension of g. Define $\Psi(g) := \Phi^Z_{C_0}(f)$. It follows from (3.2) that this definition is independent of the choice of f, and Ψ is a bounded algebra homomorphism from $C(K)$ to $L(Y)$ mapping r_λ to $(\lambda+Z)^{-1}$ for $\lambda\in\mathbb{C}\setminus K$. Thus it is a bounded $C(K)$-calculus for Z. Property (a) in Lemma 3.1 is immediate from the definition of Ψ.

The map from $\mathcal{A}$ to $L(Y)$ given by $f\mapsto\Psi(f|_K)$ is a bounded algebra homomorphism, and it is a bounded $\mathcal{A}$-calculus for Z. Now (b) follows from the uniqueness of the $\mathcal{A}$-calculus for Z; see the first paragraph in Section 3.

The proofs of (c) and (d) are very similar to the proofs of [16, Lemma 1.1]. For (c), we may take any commutative Banach subalgebra $\mathcal{C}$ of $L(Y)$ containing I, Z, $\Psi(g)$ and all their resolvents. Then the spectra of Z and $\Psi(g)$ in $L(Y)$ coincide with their spectra in $\mathcal{C}$, so it suffices to show that $\chi(\Psi(g)) = g(\chi(Z))$ for every character χ of $\mathcal{C}$. We may approximate g uniformly on K by a sequence $(p_n)_{n\ge1}$ of polynomials, so $\Psi(g)=\lim_{n\to\infty}p_n(Z)$ in operator norm. Then

$$\chi(\Psi(g)) = \lim_{n\to\infty}\chi(p_n(Z)) = \lim_{n\to\infty}p_n(\chi(Z)) = g(\chi(Z)).$$

This establishes (c).

For (d), first assume that $g \in C(K)$ vanishes on a neighbourhood of $\sigma(Z)$ in K. Choose $h \in C(K)$ such that $h(\lambda) = 1$ for all $\lambda \in \sigma(Z)$ and $gh = 0$. By (c), $\sigma(\Psi(h)) = \{1\}$, so $\Psi(h)$ is invertible. Since $\Psi(g)\Psi(h) = 0$ it follows that $\Psi(g) = 0$.

Now assume only that $g \in C(K)$ vanishes on $\sigma(Z)$. Then there is a sequence $(g_n)_{n\geq 1}$ in $C(K)$ where each g_n vanishes on a neighbourhood U_n of $\sigma(Z)$ in K, and $\|g_n - g\|_{C(K)} \to 0$. Statement (d) follows from (3.2) and the paragraph above. ☐

Properties (a) and (b) in Lemma 3.1 establish that the functional calculus Ψ for a bounded skew-hermitian operator Z is compatible with the $C_0(i\mathbb{R})$-calculus and the $\mathcal{A}$-calculus for functions $f \in \mathcal{A}$. We will use the simple notation $f(Z)$ for the resulting operators in any of these calculi.

We now return to the case when Z is an unbounded skew-hermitian operator with a bounded $\mathcal{B}$-calculus.

Lemma 3.2. *Let Z be a skew-hermitian operator on a Banach space Y, and assume that Z has a bounded $\mathcal{B}$-calculus. Let $f \in \mathcal{A}$, and assume that f vanishes on $\sigma(Z)$. Then $f(Z) = 0$.*

Proof. We will apply the Arveson spectral theory for C_0-groups of isometries to the C_0-group $(V(t))_{t\in\mathbb{R}}$ generated by $-Z$ on Y; see [9, Section 8] for an account of the theory in this context.

For $k \in \mathbb{N}$, let Y_k be the spectral subspace corresponding to the interval $[-k, k]$. Then Y_k is a closed subspace of Y which is invariant under the operators $V(t)$, and the restrictions $V_k(t)$ of $V(t)$ to the subspace Y_k form a norm-continuous group of isometries on Y_k. Moreover, the negative generator Z_k is a bounded operator on Y_k, so Y_k is contained in the domain of Z, and $Zy = Z_k y$ for all $y \in Y_k$, and $\sigma(Z_k) \subseteq \sigma(Z) \cap i[-k, k]$; see [9, Theorems 8.19 and 8.27]. By restricting the operators $f(Z)$ to the subspace Y_k for $f \in C_0(i\mathbb{R})$, we obtain a bounded $C_0(i\mathbb{R})$-calculus for Z_k.

Now let $f \in \mathcal{A}$, and assume that f vanishes on $\sigma(Z)$. For each k, f vanishes on $\sigma(Z_k)$. Taking $K = i[-(k+1), k+1]$, Lemma 3.1 shows that $f(Z_k) = 0$. This implies that $f(Z)y = 0$ for all $y \in Y_k$, since $f(Z_k)$ is the restriction of $f(Z)$ to Y_k. Since $\bigcup_{k\in\mathbb{N}} Y_k$ is dense in Y (see [9, Lemma 8.12]), this implies that $f(Z) = 0$. ☐

Remark 3.3. Let Z be a skew-hermitian operator on Y, with a bounded $\mathcal{B}$-calculus, and let Y_k and Z_k be as in the proof of Lemma 3.2. Let $f \in \mathcal{A} \cap C_0(i\mathbb{R})$. Parts (a) and (b) of Lemma 3.1 show that $\Phi_{\mathcal{A}}^{Z_k}(f) = \Phi_{C_0}^{Z_k}(f)$. Hence

$\Phi^Z_{\mathcal{A}}(f)$ and $\Phi^Z_{C_0}(f)$ coincide on $\bigcup_{k\in\mathbb{N}} Y_k$, and then by continuity they coincide on Y. This is an alternative proof that the two calculi are compatible.

We now prove Theorem 2.1. The structure of our argument is the same as that of [19], but we need the lemmas above to justify the crucial stage in the argument which enables us to cover functions outside $\mathcal{L}L^1$ and to avoid assumptions of spectral synthesis.

Proof of Theorem 2.1. If $i\mathbb{R} \subseteq \sigma(A)$, $f \in \mathcal{A}$, and f vanishes on $i\mathbb{R}$, then f vanishes on $\mathbb{C}_+$. So $f(A) = 0$, and the result holds trivially. Thus we may assume that $i\mathbb{R}$ is not contained in $\sigma(A)$. We will initially show that $\lim_{t\to\infty} T(t)f(A) = 0$ in the strong operator topology.

We use the limit isometric semigroup, as in [19, Theorem 3.2] or [20, Proposition 3.1]. There is a Banach space Y (which may be $\{0\}$), a bounded map $\pi : X \to Y$ with dense range such that $\|\pi(x)\|_Y = \limsup_{t\to\infty} \|T(t)x\|$ for $x \in X$, and a C_0-semigroup $(V(t))_{t\geq 0}$ of (not necessarily invertible) isometries on Y such that $V(t)\pi = \pi T(t)$ and the negative generator Z of $(V(t))_{t\geq 0}$ satisfies $\sigma(Z) \subseteq \sigma(A) \cap i\mathbb{R}$.

By a property of semigroups of isometries (see [15, Lemma, p. 38] or [12, pp. 419, 420]), the inequality $\|Zy - \lambda y\| \geq \operatorname{Re}\lambda \|y\|$ is valid for all $\lambda \in \mathbb{C}_+$ and $y \in Y$. Since $i\mathbb{R}$ is not contained in $\sigma(Z)$ and Z has no approximate eigenvalues in $\mathbb{C}_+$, it follows that $\sigma(Z) \cap \mathbb{C}_+$ is empty, and $\|(\lambda - Z)^{-1}\| \leq (\operatorname{Re}\lambda)^{-1}$ for $\lambda \in \mathbb{C}_+$. By the Hille-Yosida theorem, Z generates a C_0-semigroup, and hence $-Z$ generates a C_0-group of invertible isometries $(V(t))_{t\in\mathbb{R}}$ on Y.

For $g \in \mathcal{B}$, the operator $g(A) \in L(X)$ commutes with $T(t)$ for all $t \geq 0$, and so there is a unique operator $\Upsilon(f) \in L(Y)$, such that $\Upsilon(f)\pi = \pi f(A)$. Then Υ is a bounded algebra homomorphism of $\mathcal{B}$ into $L(Y)$ mapping r_λ to $(\lambda + Z)^{-1}$ for $\lambda \in \mathbb{C}_+$, so it is a bounded $\mathcal{B}$-calculus for Z.

Now assume that f is in the closure of $\mathcal{L}L^1$ in $\mathcal{A}$ and f vanishes on $\sigma(A) \cap i\mathbb{R}$. Then f vanishes on $\sigma(Z)$, and it follows from Lemma 3.2 that $\Upsilon(f) = 0$. Then $\pi f(A) = 0$, and hence $\lim_{t\to\infty} T(t)f(A) = 0$ in the strong operator topology.

It remains to lift this conclusion to the operator-norm topology. This can be achieved by using the method of [19, Theorem 3.2] and [20, Theorem 3.9]. Consider the induced bounded C_0-semigroup of left multiplications by $T(t)$ on the Banach space $\mathcal{X}$ of operators $S \in L(X)$ such that $t \mapsto T(t)S$ is norm-continuous on $\mathbb{R}_+$. Let H be the negative generator of this C_0-semigroup on $\mathcal{X}$. It is easily seen that $\sigma(H) \subseteq \sigma(A)$, and that the operators of left multiplication by $f(A)$, for $f \in \mathcal{A}$, form an $\mathcal{A}$-calculus for H. It is well known and elementary that $g(A) \in \mathcal{X}$ for all $g \in \mathcal{L}L^1$. Applying the result established in the paragraph above to the C_0-semigroup on $\mathcal{X}$ shows that $\lim_{t\to\infty} \|T(t)f(A)g(A)\| = 0$ for all $g \in \mathcal{L}L^1$.

Let $(e_n)_{n\geq 1}$ be a bounded approximate unit for $L^1(\mathbb{R}_+)$, and let $g_n = \mathcal{L}e_n$ for $n \geq 1$. Then $(g_n)_{n\geq 1}$ is a bounded approximate unit for $\mathcal{L}L^1$ and hence for the closure of $\mathcal{L}L^1$ in $\mathcal{A}$; see Section 2. Thus $\lim_{n\to\infty} \|fg_n - f\|_{\mathcal{B}} = 0$, and so $\lim_{n\to\infty} \|f(A)g_n(A) - f(A)\| = 0$. Since the semigroup $(T(t))_{t\geq 0}$ is bounded and $\lim_{t\to\infty} \|T(t)f(A)g_n(A)\| = 0$ for all $n \geq 1$, it follows that $\lim_{t\to\infty} \|T(t)f(A)\| = 0$. □

Remarks 3.4. (a) The above proof, without the final paragraph, shows that $\lim_{t\to\infty} T(t)f(A) = 0$ in the strong operator topology and

$$\lim_{t\to\infty} \|T(t)f(A)g(A)\| = 0$$

for all $g \in \mathcal{L}L^1$ even without the assumption that f vanishes at infinity.

(b) If $\sigma(A) \cap i\mathbb{R}$ is bounded then the application of Lemma 3.2 in the proof of Theorem 2.1 may be replaced by an application of Lemma 3.1.

(c) In Theorem 1.2 where $\mathcal{A} = \mathcal{B}$ and $f \in \mathcal{B}$, the condition that f vanishes on $\sigma(A) \cap i\mathbb{R}$ is necessary for the conclusion. Indeed, let $-A$ be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ and assume that A has a bounded $\mathcal{B}$-calculus. Let $f \in \mathcal{B}$, and suppose that $\lim_{t\to\infty} \|T(t)f(A)\| = 0$. Let $z \in \sigma(A) \cap i\mathbb{R}$. Then $e^{tz}f(z) \in \sigma(T(t)f(A))$ for all $t \geq 0$ by the spectral inclusion theorem for the $\mathcal{B}$-calculus [3, Theorem 4.17], and hence $|f(z)| \leq \|T(t)f(A)\| \to 0$ as $t \to \infty$. It follows that $f(z) = 0$, as required.

(d) In Theorem 2.1, we have assumed that $\mathcal{A}$ contains $\mathcal{B}$, and so contains the constant functions. It would suffice to assume that $\mathcal{A}$ contains the subalgebra $\mathcal{B}_0$ defined by

$$\mathcal{B}_0 := \left\{ f \in \mathcal{B} : \lim_{\operatorname{Re} z\to\infty} f(z) = 0 \right\},$$

as one may pass to the algebra obtained by adjoining the constant functions.

4. Examples

Here we briefly describe two Banach algebras introduced by Arnold and Le Merdy, to which Theorem 2.1 can be applied. In [1] they introduce a Banach algebra $\mathcal{A}(\mathbb{C}_+)$ which is continuously included in $H^\infty(\mathbb{C}_+)$, they identify a Banach subalgebra $\mathcal{A}_0(\mathbb{C}_+)$ and they show that it is the closure of the algebra $\mathcal{L}L^1$ in $\mathcal{A}(\mathbb{C}_+)$ [1, Lemma 3.14]. They also show that $\mathcal{B}_0$ is properly and continuously included in $\mathcal{A}(\mathbb{C}_+)$ [1, Proposition 5.2, Theorem 5.3]. (Readers should be aware that our spaces $\mathcal{B}_0$ and $\mathcal{B}_{00}$ are denoted by $\mathcal{B}(\mathbb{C}_+)$ and $\mathcal{B}_0(\mathbb{C}_+)$, respectively, in [1].) In finding the Banach algebra $\mathcal{A}(\mathbb{C}_+)$ with these properties, the authors were guided by Peller's work in the discrete case [17]. It is not easy to identify specific functions which are in $\mathcal{A}(\mathbb{C}_+)$ but not in $\mathcal{B}$.

In a further paper with an additional author [2], the authors introduce a Banach algebra $\mathcal{A}_S(\mathbb{C}_+)$ such that

$$\mathcal{A}(\mathbb{C}_+) \subsetneq \mathcal{A}_S(\mathbb{C}_+) \subsetneq H^\infty(\mathbb{C}_+),$$

with continuous inclusions. The closure of $\mathcal{L}L^1$ in $\mathcal{A}_S(\mathbb{C}_+)$ is an identified subalgebra $\mathcal{A}_{0,S}(\mathbb{C}_+)$. Again the definitions are complex, and finding explicit functions is not easy.

Let $-A$ be the generator of a bounded C_0-semigroup on a Hilbert space. The authors show in [1] that A has a (unique) bounded $\mathcal{A}(\mathbb{C}_+)$-calculus, and in [2] that A has a bounded $\mathcal{A}_S(\mathbb{C}_+)$-calculus. Hence Corollary 1.3 can be extended to the following.

Corollary 4.1. *Let $-A$ be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space. Let $f \in \mathcal{A}_{0,S}(\mathbb{C}_+)$, and assume that f vanishes on $\sigma(A) \cap i\mathbb{R}$. Then* $\lim_{t\to\infty} \|T(t)f(A)\| = 0$.

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