Groups acting on graphs

By

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Trinity Term 1991

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ABSTRACT

In the first part of this thesis we investigate the automorphism groups of regular trees. In the second part we look at the action of the automorphism group of a locally finite graph on the ends of the graph. The two part are not directly related but trees play a fundamental role in both parts.

Let T_n be the regular tree of valency n. Put $G := \mathrm{Aut}(T_n)$ and let G_0 be the subgroup of G that is generated by the stabilisers of points. The main results of the first part are :

Theorem 4.1 *Suppose* $3 \leq n < \aleph_0$ *and* $\alpha \in T_n$. *Then* G_α *(the stabiliser of* α *in* G*) contains* $2^{2^{\aleph_0}}$ *subgroups of index less than* $2^{\aleph_0}$.

Theorem 4.2 *Suppose* $3 \leq n \leq \aleph_0$ *and* $H \leq G$ *with* $| G : H | < 2^{\aleph_0}$. *Then* $H = G$ *or* $H = G_0$ *or* H *fixes a point or* H *stabilises an edge.*

Theorem 4.3 *Let* $n = \aleph_0$ *and* $H \leq G$ *with* $| G : H | < 2^{\aleph_0}$. *Then* $H = G$ *or* $H = G_0$ *or there is a finite subtree* Φ *of* T_n *such that* $G_{(\Phi)} \leq H \leq G_{\{\Phi\}}$.

These are proved by finding a concrete description of the stabilisers of points in G, using wreath products, and also by making use of methods and results of Dixon, Neumann and Thomas [Bull. Lond. Math. Soc. 18, 580-586]. It is also shown how one is able to get short proofs of three earlier results about the automorphism groups of regular trees by using the methods used to prove these theorems.

In their book *Groups acting on graphs*, Warren Dicks and M. J. Dunwoody [Cambridge University Press, 1989] developed a powerful technique to construct trees from graphs. An end of a graph is an equivalence class of half-lines in the graph, with two half-lines, L_1 and L_2, being equivalent if and only if we can find the third half-line that contains infinitely many vertices of both L_1 and L_2.

In the second part we point out how one can, by using this technique, reduce questions about ends of graphs to questions about trees. This allows us both to prove several new results and also to give simple proofs of some known results concerning fixed points of group actions on the ends of a locally finite graph (see Chapter 10). An example of a new result is the classification of locally finite graphs with infinitely many ends, whose automorphism group acts transitively on the set of ends (Theorem 11.1).

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Madurinn einn er ei nema hálfur,
med ödrum er hann meiri en hann sjálfur.
Einar Benediktsson

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Contents

Preface

This thesis is about groups and graphs. If one has a group acting on a graph then there are two things one can do: use knowledge about the graph to gain information about the group, or one can use what one knows about the group and its action to draw conclusions about the graph. In this thesis we shall see examples of both approaches.

In the first part we look at the automorphism groups of regular trees. The main objective is to classify the subgroups that have index less than $2^{\aleph_0}$. In the second part we show how, using results from the book *Groups acting on graphs* by Warren Dicks and M. J. Dunwoody, one can reduce questions about graphs with infinitely many ends to questions about trees. Thus the main idea of this thesis is to use trees, which are very easily understood, to draw conclusions about some far more complex structures. For a more detailed description of individual results the reader is referred to the introductory chapters in each part.

I have written the two parts so that they can be read independently of each other. The same notation is used throughout with one notable exception: in Part 1 we think of an edge in a graph as an unordered pair of vertices, whereas in Part 2 we need to think of an edge as an ordered pair of vertices.

Part I

The automorphism groups of regular trees

Chapter 1

Introduction

In trying to understand certain infinite permutation groups it has proved to be useful to describe 'big' subgroups in terms of the action of the group. Here big is taken to mean having index less than $2^{\aleph_0}$. Our aim in this part is to investigate these 'big' subgroups in the case of automorphism groups of regular combinatorial trees.

Before we go on to describe our results we need first to develop some notation. Basically we will be using the notation for permutation groups used by Wielandt. The group action will be written on the right and composition in the group should be read from left to right. Let G be a group acting on some set Ω and $\Gamma \subseteq \Omega$. Then we let $G_{(\Gamma)}$ be the group of those elements of G that fix every point of Γ and denote by $G_{\{\Gamma\}}$ the group of those elements $g \in G$ such that $\Gamma g = \Gamma$. Call $G_{(\Gamma)}$ the pointwise stabiliser of Γ and $G_{\{\Gamma\}}$ the setwise stabiliser of Γ.

The first result about subgroups of small index was the following theorem of Peter M. Neumann, characterising the subgroups of small index in the symmetric group on a countable set :

Theorem *Let Ω be a countable set and $G \leq \mathrm{Sym}(\Omega)$. If G has index less than $2^{\aleph_0}$ then there is a finite subset Δ of Ω such that*

$$\mathrm{Sym}(\Omega)_{(\Delta)} \leq G \leq \mathrm{Sym}(\Omega)_{\{\Delta\}}.$$

A proof of this theorem and further discussion can be found in [3]. We will use some of the ideas in the proof in [3] to get our results. Other theorems giving analogous results about some other permutation groups have been found recently. Such results about automorphism groups of $\aleph_0$-categorical structures also hold considerable model theoretic interest.

A regular tree is a combinatorial tree in which every two vertices have the same number of adjacent vertices. Let T_n denote the regular tree in which every point has exactly n adjacent vertices. In this part of the thesis we intend to discuss the subgroups of small index in $\mathrm{Aut}(T_n)$ for $3 \leq n \leq \aleph_0$. One such group is the subgroup generated by the stabilisers of vertices in $\mathrm{Aut}(T_n)$. We adopt the convention that if $G := \mathrm{Aut}(T_n)$ then this subgroup is denoted by G_0. In Chapter 2 we go over some background material and prove that this subgroup has index 2 in $\mathrm{Aut}(T_n)$. In order to study the subgroups of small index we are led to study the structure and the subgroups of certain infinite wreath products. A description of the structure of these can be found in Chapter 3. In Chapter 4 we get onto the main subject of this part. In Section 4.1 we get the first result on the subgroups of small index; it is a negative one, showing that if n is finite then we can not expect to get a classification similar to the one above. This follows from the next theorem.

Theorem 4.1 *Let* $G := \mathrm{Aut}(T_n)$ *with* $3 \leq n < \aleph_0$ *and* $\alpha \in VT_n$. *Then* G_α *contains* $2^{2^{\aleph_0}}$ *subgroups of index less than* $2^{\aleph_0}$, *indeed, of index* 2.

The same result applies to the stabiliser of an edge. In spite of this there is some discipline among the subgroups of small index in $\mathrm{Aut}(T_n)$. For Proposition 4.1 in Section 4.2 gives a pretty good idea of what the subgroups of small index in the stabiliser of a vertex look like and, moreover, we have the following theorem:

Theorem 4.2 *Let* $3 \leq n \leq \aleph_0$ *and* $G := \mathrm{Aut}(T_n)$. *If* H *is a subgroup of* G *with index less than* $2^{\aleph_0}$ *then* $H = G$ *or* $H = G_0$ *or* H *stabilises a vertex or an edge.*

Finally we get a classification similar to the one in Neumann's Theorem for the subgroups of small index in $\mathrm{Aut}(T_{\aleph_0})$.

Theorem 4.3 *Let* $G := \mathrm{Aut}(T_{\aleph_0})$ *and* $H \leq G$ *with* $|G : H| < 2^{\aleph_0}$. *Then* $H = G$ *or* $H = G_0$ *or there is a finite subtree* Φ *of* $T_{\aleph_0}$ *such that*

$$G_{(\Phi)} \leq H \leq G_{\{\Phi\}}.$$

In Section 4.3 we will prove Theorem 4.2 and Theorem 4.3 will be proved in Section 4.4. In the final chapter of this part we will point out how we can use our results about subgroups of small index in the automorphism groups of regular trees to gain information about the structure of the automorphism groups: we will give short proofs of two theorems proved by D.V. Znoiko in [24] based on Theorem 4.2 and prove that the group G_0 defined above is simple in the case of a regular tree of finite valency. The simplicity of G_0 in these cases is a part of a much more general result proved by Tits in [18]. Finally we make few remarks about how these results could possibly be generalised and extended.

Chapter 2

Preliminaries

In this chapter we go first over some basic definitions. The notation we use is more or less standard (at least according to some standard). In the second section we go over some easily proved results that we will need in the this part of the thesis.

2.1 Definitions

All graphs will be assumed to be undirected. We can therefore realize a graph Γ as a pair $(V\Gamma, E\Gamma)$, where V is a set and $E\Gamma$ is a set of two element subsets of $V\Gamma$. We call elements of $V\Gamma$ vertices and elements of $E\Gamma$ edges.

A tree is a connected graph without any cycles. Let $T = (VT, ET)$ be a tree. If $\alpha, \beta \in VT$ then there is a unique shortest path between α and β. We say that a vertex γ is between α and β if γ lies on that shortest path between α and β, in which case we write $B(\gamma; \alpha, \beta)$. (Note that we say that both α and β are between α and β.)

Definition 2.1 *If $\alpha, \beta \in VT$ then we set*

$$d(\alpha, \beta) := |\{\gamma \mid B(\gamma; \alpha, \beta)\}| - 1$$

and call $d(\alpha, \beta)$ the distance between α and β.

For $A, B \subseteq VT$ we define $d(A, B) := min\{d(\alpha, \beta) \mid \alpha \in A, \beta \in B\}$. If $d(\alpha, \beta) = 1$ then we say that α and β are adjacent.

Definition 2.2 *If $\alpha \in VT$ let*

$$C_i(\alpha) := \{\beta \in VT \mid d(\alpha, \beta) = i\},$$

for $i \in \mathbf{N}$.

The number $|C_1(\alpha)|$ is called the valency of α. That is, the valency of a vertex is the number of adjacent vertices.

It will be convenient in some cases to think of our trees as rooted trees; that is, trees with a special vertex which we call the root of the tree. If T is a tree then we let T^α denote the rooted tree with root α. We define the set of descendants of a vertex γ in T^α by

$$\text{desc}(\gamma) := \{\beta \in VT^\alpha \mid B(\gamma; \alpha, \beta)\}.$$

If $\Gamma \subseteq T^\alpha$ then we define

$$\text{desc}(\Gamma) := \cup_{\gamma \in \Gamma} \text{desc}(\gamma).$$

Definition 2.3 *A tree T is said to be regular if every two vertices of T have the same valency. If n is any cardinal number we let T_n denote the regular tree in which every vertex has valency n.*

The first one T_0 is just a single vertex with no edges, and T_1 consists of two vertices with an edge between them. One sees that T_2 is isomorphic to $\mathbf{Z}$, when $\mathbf{Z}$ is viewed as a graph with edges joining consecutive integers. For $n \geq 3$ the situation is different and more complicated.

Definition 2.4 *Let T be a tree. A subgraph of T that is isomorphic to $\mathbf{Z}$ is called a line. A subgraph isomorphic to $\mathbf{N}$ is called a half-line.*

Let $\mathcal{H}T$ be the set of half-lines in T. We say that $L_1, L_2 \in \mathcal{H}T$ are in the same end if $L_1 \cap L_2$ is a half-line. One easily checks that this defines an equivalence relation on $\mathcal{H}T$. The equivalence classes will be called ends. We let $\mathcal{E}T$ denote the set of ends of T. Note that if $\alpha \in VT$ then there is one and only one half-line belonging to any given end starting at α.

As pointed out above, the trees T_0, T_1 and T_2 are easily described, and so too are their automorphism groups; hence the emphasis will be on $\mathrm{Aut}(T_n)$ for $n \geq 3$.

Now let $G := \mathrm{Aut}(T_n)$ for some cardinal number n. The condition that all vertices have the same valency is exactly what we need to make G transitive. This also ensures that $C_i(\alpha)$ is an orbit of G_α. Furthermore, we can see that G is transitive on the set of lines and half-lines in T_n. If $L_1, L_2 \in \mathcal{H}T_n$ are in the same end then L_1g and L_2g also belong to the same end. So G has a well defined action on $\mathcal{E}T_n$. This action of G is 3-fold transitive; as one can see by imagining that two ends are fixed, which amounts to stabilising a line, but inside the stabiliser of a line we still have freedom to move a third end wherever we like.

The action of G on VT_n is not primitive. We define a binary relation ρ on VT_n by

$$\alpha \, \rho \, \beta :\Leftrightarrow d(\alpha, \beta) \equiv 0 \pmod 2.$$

It is easy to see that ρ is an equivalence relation on VT_n. If $\alpha \in VT$ then $\rho(\alpha)$ is the set of vertices that are at even distance from α, thus ρ has two equivalence classes. Distances in T_n are preserved by G, so ρ is preserved by G. Thus G is not primitive. Define

$$G_0 := \{g \in G \mid \alpha \, \rho \, \alpha g \text{ for all } \alpha \in VT_n\}.$$

It will be understood that if G is defined to be $\mathrm{Aut}(T_n)$ then G_0 denotes this subgroup of G. Note that for $\alpha \in VT_n$, $G_\alpha \leq G_0$.

2.2 Warming up

The aim of this section is to state some basic results we will need later on, and also to introduce the reader to the type of arguments that will be used to prove our main results.

Proposition 2.1 *The group G_0 is normal in G and $|G : G_0| = 2$. The action of G on one of the equivalence classes of ρ is primitive.*

Proof The first statement is trivial. Let R be an equivalence class of ρ. Assume $\Gamma \subseteq R$ is a non-trivial block of imprimitivity. Since Γ is non-trivial we can find two distinct vertices α, β in Γ. Because $G_\alpha \leq G_0$ we have $C_{d(\alpha,\beta)}(\alpha) \subseteq \Gamma$. Therefore Γ contains two vertices at distance 2 from each other, one can thus assume that $d(\alpha, \beta) = 2$ and $C_2(\alpha) \subseteq \Gamma$. Now assume that $C_{2i}(\alpha) \subseteq \Gamma$ for $i \leq k$ with $k \in \mathbb{N}$. If $\beta \in C_{2k}(\alpha)$ then $C_2(\beta) \cap \Gamma \neq \emptyset$ so $C_2(\beta) \subseteq \Gamma$ (because $G_\beta \leq G_0$). Hence $C_{2(k+1)}(\alpha) \cap \Gamma \neq \emptyset$, so $C_{2(k+1)}(\alpha) \subseteq \Gamma$. We have now proved that $\Gamma = R$ and the conclusion follows. □

We have in fact shown that ρ is the only nontrivial proper equivalence relation that G preserves, and thus that if $\alpha \in VT_n$ then G_0 is the only proper subgroup of G that contains G_α. By the primitivity of G_0 we see that G_α is a maximal subgroup of G_0, and G_0 is generated by the stabilisers of vertices.

To end this section we state for completeness those of the results of Tits [18] that will be used in this part. These results tell us that automorphisms, and to a lesser extent groups of automorphisms of trees, can be classified in a geometrical way. The resulting classification corresponds roughly to the usual classification into rotations, reflections and translations.

Proposition 2.2 ([18], Proposition 3.2) *Every automorphism g of a tree T satisfies one and only one of the following conditions:*

(i) *g fixes a vertex of T; (rotation)*

(ii) *g transposes two adjacent vertices; (reflection)*

(iii) *there exists a line L such that L is invariant under g and g induces on L a non-trivial translation. (translation)*

Let $d = \inf_{g \in G} d(\alpha, \alpha g)$. In the case of a translation the line L is unique and g induces on L a translation of degree d, and $d(\alpha, \alpha g) = d$ if and only if $\alpha \in L$.

The ideas used in Tits's proof will be implicit in many of the arguments later on in this thesis; it is hoped that the inclusion of his proof here will help the reader to understand the more involved arguments that follow. But, the main reason for the inclusion of this proof is that I like it. The first step in the proof of the proposition is the following lemma.

Lemma 2.1 *Let* $g \in \mathrm{Aut}(T)$ *and let* α *and* β *be two adjacent points of* T*. Suppose* $B(\beta;\alpha,\alpha g)$ *and* $B(\alpha g;\alpha,\beta g)$*. Then there exists a line* L *such that* g *induces a non-trivial translation on* L *.*

Proof For two vertices α_1 and α_2 in T we define

$$[\alpha_1,\alpha_2] := \{\gamma \in VT \mid B(\gamma;\alpha_1,\alpha_2\}.$$

It follows directly from the hypothesis that

$$[\alpha,\alpha g] \cap [\alpha g,\alpha g^2] = \{\alpha g\}.$$

Then we have

$$[\alpha,\alpha g]g^i \cap [\alpha,\alpha g]g^{i+1} = \{\alpha g^{i+1}\}.$$

The set

$$\cup_{i \in \mathbf{Z}}[\alpha,\alpha g]g^i$$

is then the required line. It is trivial that it is invariant under g and that g induces on L a non-trivial translation.□

Proof of Proposition 2.2 Now back to the proof of the proposition. Let $\alpha \in T$ be such that $d(\alpha,\alpha g) = d$. If $d = 0$ then α is fixed by g. If $d \neq 0$

then let β be a point adjacent to α such that $B(\beta; \alpha, \alpha g)$. If $B(\beta g; \alpha, \alpha g)$ then we must have $\alpha g = \beta$ and $\beta g = \alpha$ [by the choice of α], if this is the case then g transposes two adjacent points. We are then left to consider the case where $B(\alpha g; \alpha, \beta g)$, by Lemma 2.1 there is a line L along which g induces a non-trivial translation. The first part of the proposition is thus proved since it is obvious that the three conditions are mutually exclusive. Suppose now that g is a translation; that is, satisfies the third condition above. Let π_L denote the projection of T down to L; that is, if $\alpha \in T$ then $\pi_L(\alpha)$ is the point on L closest to α. Clearly $[\pi_L(\alpha), \pi_L(\alpha g)] \subseteq [\alpha, \alpha g]$ so $d(\alpha, \alpha g) \geq d$ and we have equality if and only if $\alpha \in L$. □

Before we state the classification of groups of automorphisms it is necessary to make the following definition.

Definition 2.5 *Let T be a tree and ϵ an end of T. We say that $G \leq \mathrm{Aut}(T)$ centralises the end ϵ if every element of G fixes some half-line in ϵ.*

Proposition 2.3 ([18], Proposition 3.4) *Let T be a tree and $G \leq \mathrm{Aut}(T)$. Suppose that G contains no translations. Then G fixes a vertex or an edge or centralises an end.*

Chapter 3

Infinite wreath products

Let $G := \mathrm{Aut}(T_n)$ and $\alpha \in VT_n$. In this chapter we will describe the structure of G_α the stabiliser of a vertex in T_n. We begin by looking at wreath products and the orbits of G_α and determine the action of G_α on those. These observations will indicate how we should describe G_α. This is done in the first section, in the second section we will look at normal subgroups in these infinite wreath products.

3.1 Wreath products and the stabiliser of a vertex

Let A and B be permutation groups, acting faithfully on sets Δ and Γ, respectively. We define $A\mathrm{Wr}_\Gamma B$ to be a permutation group that acts on $\Delta \times \Gamma$. The set of elements of $A\mathrm{Wr}_\Gamma B$ is the set $A^\Gamma \times B$ and the action of $(f,g) \in A^\Gamma \times B$ on $(\delta,\gamma) \in \Delta \times \Gamma$ is given by

$$(\delta,\gamma)(f,g) = (\delta f(\gamma), \gamma g).$$

From this it follows that

$$(f,g)(f',g') = (f'',gg')$$

where $f''(\gamma) = f(\gamma)f'(\gamma g)$. This makes $A\mathrm{Wr}_\Gamma B$ into a semidirect product of A^Γ with B, conjugation of $f \in A^\Gamma$ by $g \in B$ being given by $g^{-1}fg = f'$ where $f'(\gamma) = f(\gamma g^{-1})$. The group B is called the top group of the wreath product and A^Γ is called the base group. By identifying B and the top group we often write, $h = fg$ for $h = (f,g)$ with f in the base group and g in the top group.

We can also describe an element of an iterated wreath product in a similar way. Note also that the wreath product construction on permutation groups is associative.

It is now easily seen that G_α acts on $C_1(\alpha)$ like S_n, the symmetric group on n letters, and that G_α acts on $C_i(\alpha)$ like

$$S_{n-1}\,\mathrm{Wr}\ldots\mathrm{Wr}\,S_{n-1}\,\mathrm{Wr}\,S_n.$$

This is all very well, but it does not fully describe G_α. In order to do so we must extend the notion of a wreath product to infinitely many groups.

The main ideas behind our definition of the wreath product of a family of permutation groups indexed by $\mathbf{N}$ can be found in [15]. Let $\{H_i\}_{i\in\mathbf{N}}$ be a family of groups with H_i acting faithfully on some set Δ_i. Define $\Gamma_1 := \Delta_1$ and then for $k > 1$,

$$\Gamma_k := \Delta_k \times \Gamma_{k-1} = \Delta_k \times \Delta_{k-1} \times \ldots \times \Delta_1.$$

Set $W_1 := H_1$ and define inductively for $k > 1$

$$W_k := H_k\mathrm{Wr}_{\Gamma_{k-1}}W_{k-1} = H_k\mathrm{Wr}_{\Gamma_{k-1}}H_{k-1}\mathrm{Wr}_{\Gamma_{k-2}}\ldots\mathrm{Wr}_{\Gamma_1}H_1.$$

An element g of W_k can be expressed as a sequence $(g_k, g_{k-1}, \ldots, g_1)$ with $g_1 \in H_1$ and $g_i \in H_i^{\Gamma_{i-1}}$ for $2 \leq i \leq k$. Now define $\pi_k : W_k \rightarrow W_{k-1},\ k \geq 2$ by

$$(g_k, g_{k-1}, \ldots, g_1) \mapsto (g_{k-1}, \ldots, g_1).$$

The groups $\{W_k\}_{k \in \mathbf{N}}$ together with the homomorphisms π_k form an inverse system. It is well known that inverse limits always exist in the category of groups ([9], pp. 51-52); the following definition can thus be made in full safety

$$W := \varprojlim W_k.$$

It is also well known that inverse limits of groups can be realized as subgroups of the full cartesian product $\prod W_k$. Following that construction, an element $g \in W$ is expressible as a sequence $(g_k)_{k \in \mathbf{N}} \in \prod W_k$ with $g_{k+1}\pi_{k+1} = g_k$. If we look a little bit closer at the definitions we see that this expression can be simplified, because $g_k \in W_k$, so $g_k = g_k' g_{k-1}$ with $g_k' \in H_k^{\Gamma_{k-1}}$. When we write $g = (\ldots, g_k, g_{k-1}, \ldots, g_1)$ we will be using this latter way of expressing elements of W; that is, g_k will be taken to be an element of $H_k^{\Gamma_{k-1}}$.

We want to define an action of W on the set $\Gamma := \prod \Delta_i$. Elements of Γ will be written as left-infinite sequences. The action of W is defined as the 'limit' of the actions of the W_k. More precisely let $g = (\ldots, g_k, g_{k-1}, \ldots, g_1) \in W$ and $\gamma = (\ldots, \gamma_k, \gamma_{k-1}, \ldots, \gamma_1) \in \Gamma$. Then

$$(\ldots, \gamma_k, \gamma_{k-1}, \ldots, \gamma_1)g = (\ldots, \mu_k, \mu_{k-1}, \ldots, \mu_1),$$

where for all $k \in \mathbf{N}$ we have

$$(\mu_k, \mu_{k-1}, \ldots, \mu_1) = (\gamma_k, \gamma_{k-1}, \ldots, \gamma_1)(g_k, g_{k-1}, \ldots, g_1).$$

This definition makes perfect sense because $(g_k, g_{k-1}, \ldots, g_1) \in W_k$.

But there is another natural permutation action of W. Let $VT := \{\alpha\} \cup (\bigcup_{i \in \mathbf{N}} \Gamma_i)$, where α is not in any of the sets Γ_i. We define an action of W on VT by specifying that

$$\alpha g := \alpha \text{ for all } g \in W,$$

and if $\gamma \in VT$ is an element of Γ_i and $g = (\ldots, g_k, g_{k-1}, \ldots, g_1) \in W$ then

$$\gamma g := \gamma(g_i, \ldots, g_1).$$

We can easily define ET such that $T = (VT, ET)$ becomes a tree: we say that $\beta, \beta' \in VT$ are adjacent if and only if

(i) $\beta = \alpha$ and $\beta' \in \Gamma_1$ or vice versa;

(ii) $\beta = (\beta_i, \ldots, \beta_1) \in \Gamma_i$ and $\beta' = (\beta'_{i+1}, \beta_i, \ldots, \beta_1) \in \Gamma_{i+1}$ or vice versa.

One easily checks that T with this relation is a tree and that W acts on T as a group of automorphisms, more precisely as a group of automorphisms of the rooted tree T^α. Let $H_1 := S_n$ and $H_i := S_{n-1}$ for $i \geq 2$, with the usual permutation representation. Then the tree constructed above is the regular tree of valency n and W acts on it like the stabiliser of α in $\mathrm{Aut}(T_n)$. In Section 2.1 we pointed out that for any given vertex in a tree there is exactly one half-line starting at that vertex in any given end. The set Γ defined above is thus a set of representatives for the ends of T and the action of W on Γ

described above is the action induced by W on the ends of T. We say that the group W is the wreath product of the family $\{H_k\}$ and write $W := \mathrm{Wr}\ H_k$. This definition depends upon the specific permutation representations of the groups H_k but it will always be clear from the context which representation we have in mind. In what follows we will only be looking at W with the permutation representation on the rooted tree T^α.

3.2 Normal subgroups of infinite wreath products

We will continue to use the same notation as in the last section. For $k \in \mathrm{N}$ define

$$U_k := \mathrm{Wr}_{i>k} H_i.$$

From the definition in the last section it follows that

$$W \cong U_k \mathrm{Wr}_{\Gamma_k} W_k.$$

We define K_k as the base group of this wreath product; that is, $K_k = U_k^{\Gamma_k}$. Note that K_k is normal in W.

Our next object is to gain some information on the normal subgroups of W and to place some constraints on those.

Proposition 3.1 *Suppose the sets Δ_i are all finite and the groups H_i all transitive. If N is a non-trivial normal subgroup of W, then there is $n \in \mathrm{N}$ such that $K_n' \leq N$.*

Proof Find $n \in \mathrm{N}$ such that $N \leq K_{n-1}$ and $N \not\leq K_n$. Thus N contains an element h that fixes $C_{n-1}(\alpha)$ but not $C_n(\alpha)$. If we express h as an element of $U_n \mathrm{Wr}_{\Gamma_n} W_n$ then $h = gf$ with $f = (f', 1, \ldots, 1) \in W_n$. Now $f' \in H_n^{\Gamma_{n-1}}$ and f' is acting on $\Delta_n^{\Gamma_{n-1}}$ which by assumption is finite. Therefore f' has finite order, say m. If p is a prime and p divides m then, by replacing f' with $(f')^{m/p}$ (more precisely h by $h^{m/p}$), one can assume that f' has order p.

Let $W_0 := \langle K_n, h \rangle$. The next objective is to find a more concrete way of expressing W_0; having done so we can get the result by brutal calculations. The first thing we know is that f' acts on $C_n(\alpha)$ like a product of disjoint p-cycles. Let d_1 be the number of those p-cycles and let $d_2 := |C_n(\alpha)| - pd_1$; that is, d_2 is the number of vertices in $C_n(\alpha)$ that are fixed by f'. Now W_0 acts on the descendants of these vertices like K_n does; that is, like $U_n^{d_2}$. Let Θ_1 contain a representative from each p-cycle of f' and let $\Theta_i := \Theta_1 (f')^{i-1}$ for $1 \leq i \leq p$. The sets Θ_i are permuted cyclically by f'. The order of Θ_1 is d_1. Now we see that W_0 acts on the descendants of $\mathrm{supp}(f')$ (where $\mathrm{supp}(f)$ is the set of elements that are not fixed by f) like

$$U_n^{d_1} \mathrm{Wr}_{\{\Theta_1, \ldots, \Theta_p\}} \langle f' \rangle.$$

Noting that W_0 acts independently on the descendants of $\mathrm{fix}(f')$ (the set of elements fixed by f') and on the descendants of $\mathrm{supp}(f')$, we conclude that

$$W_0 \cong (U_n^{d_1} \mathrm{Wr}_{\{\Theta_1, \ldots, \Theta_p\}} \langle f' \rangle) \times U_n^{d_2}.$$

Of course the element h, which we took originally from N, is in W_0. Using the above result write h as $((h_1, \ldots, h_p) f', h_0)$ with $(h_1, \ldots, h_p) \in (U_n^{d_1})^p$ and $h_0 \in U_n^{d_2}$. For $y \in U_n^{d_1}$, let $y_0 := ((y, 1, \ldots, 1), 1) \in W_0$. Because N is normal we have $y_0 h y_0^{-1} h^{-1} \in N$. But

$$y_0 h y_0^{-1} h^{-1} = y_0((h_1, \ldots, h_p) f', h_0) y_0^{-1} (f'^{-1}(h_1^{-1}, \ldots, h_p^{-1}), h_0^{-1})$$

$$\begin{aligned}
&= ((y,1,\ldots,1)(h_1,\ldots,h_p)f'(y^{-1},1,\ldots,1)f'^{-1}(h_1^{-1},\ldots,h_p^{-1}),1)\\
&= ((yh_1,h_2,\ldots,h_p)(1,y^{-1},1,\ldots,1)(h_1^{-1},\ldots,h_p^{-1}),1)\\
&= ((y,h_2y^{-1}h_2^{-1},1,\ldots,1),1).
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
&((y,h_2y^{-1}h_2^{-1},1,\ldots,1),1) =\\
&\qquad ((1,h_2,1,\ldots,1),1)((y,y^{-1},1,\ldots,1),1)((1,h_2^{-1},1,\ldots,1),1).
\end{aligned}$$

Hence, since N is normal, we get that for all $y \in U_n^{d_1}$ the element

$$\bar{y} := ((y,y^{-1},1,\ldots,1),1)$$

of W_0 is in N. (This fact will be used later on in the proof of Theorem 4.2.) These elements generate U_n', but then we get that $(U_n^{d_1})' \leq N$ and, by normality, that $K_n' \leq N$. $\Box$

Chapter 4

Subgroups of small index

4.1 Theorem 4.1

Theorem 4.1 *Let* $G := \mathrm{Aut}(T_n)$ *with* $3 \leq n < \aleph_0$ *and* $\alpha \in VT_n$. *Then* G_α *contains* $2^{2^{\aleph_0}}$ *subgroups of index less than* $2^{\aleph_0}$, *indeed, of index 2.*

Proof Let

$$\phi_i : G_\alpha \to C_2;\ g \mapsto \mathrm{sign}(g|C_i(\alpha)).$$

That is, ϕ_i is the sign of the action of g on $C_i(\alpha)$. Note that by assumption $C_i(\alpha)$ is finite. Define

$$\phi : G_\alpha \to \prod_{i=1}^{\infty} C_2;\ g \mapsto (g\phi_i)_{i=1}^{\infty}.$$

Obviously ϕ is a homomorphism. But $\prod C_2$ contains $2^{2^{\aleph_0}}$ subgroups of index 2, so in order to prove the theorem it suffices to show that ϕ is onto.

Now let $(a_i)_{i=1}^{\infty}$ be a sequence in $\prod C_2$. Define a sequence $(g_i)_{i=1}^{\infty}$ in G_α by induction such that

(i) $g_1\phi_1 = a_1$, and

(ii) assume that g_i has already been defined. If $g_i\phi_{i+1} = a_{i+1}$ then let $g_{i+1} = g_i$; if not, find $h \in G_{(C_i(\alpha))}$ such that $(g_ih)\phi_{i+1} = a_{i+1}$ and set $g_{i+1} := g_ih$.

This can clearly be done because G acts on $C_1(\alpha)$ like S_n and the action of $G_{(C_i(\alpha))}$ on $C_{i+1}(\alpha)$ is like the natural action of $\prod S_{n-1}$. Let g be the limit of the sequence $(g_i)_{i=1}^{\infty}$; that is, if $\beta \in VT_n$ and $i = d(\alpha, \beta)$ then $\beta g = \beta g_i$. The construction of the sequence $(g_i)_{i=1}^{\infty}$ ensures that g is an automorphism of T_n, since if $i < j$ then g_j acts like g_i on $C_i(\alpha)$. Now $g\phi = (a_i)_{i=1}^{\infty}$. We have now shown that Φ is onto and thus proved the theorem. □

Corollary 4.1 *It is not possible to characterize the subgroups of small index in* $\mathrm{Aut}(T_n)$ *for n finite in the same way as in Neumann's Theorem.*

Proof The characterization in Neumann's Theorem implies that there are only $\aleph_0$ subgroups of small index. □

One can also note that a classification like in Neumann's Theorem implies that a proper subgroup of small index has index precisely $\aleph_0$. In the case of a stabiliser of a vertex in a regular tree of finite valency we have that if $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$ then there are $2^{2^{\aleph_0}}$ subgroups of index κ.

4.2 Subgroups of small index in infinite wreath products

In the last section we saw that in general we can not expect to get the usual classification of subgroups of small index, yet we are able to prove a result that gives us an idea of what the subgroups of small index look like.

Proposition 4.1 *Suppose the sets* Δ_i *are all finite and the groups* H_i *all transitive. Let* $H \leq W$ *and* $|W : H| < 2^{\aleph_0}$. *Then there is* $n \in \mathbb{N}$ *such that* $K_n' \leq H$.

Proof The proof is an application of the argument with cartesian products used first in [3]. First we must go through some technicalities. We think of W as acting on a tree T, as described in Section 3.1.

Step 1: There is a sequence $i_0 < i_1 < i_2 < \ldots$ of integers such that

(i) all H-orbits in $C_{i_0}(\alpha)$ have length ≥ 2;

(ii) if Δ is an H-orbit in $C_{i_j}(\alpha)$ then the H-orbits in $C_{i_{j+1}}(\alpha) \cap \mathrm{desc}(\Delta)$ are all longer than Δ.

Assume that we cannot find i_0 as in (i). This implies that H fixes at least one vertex in $C_i(\alpha)$ for all i. The vertices fixed by H form a subtree Λ of T and the assumption is that it is infinite. Since all the sets Δ_i are finite and the tree T is locally finite then by König's Lemma we know that Λ contains an infinite path L. So H fixes an infinite path in T, but that is impossible since all infinite paths have $2^{\aleph_0}$ translates under W, so

$$2^{\aleph_0} = |W : W_{(L)}| \leq |W : H|.$$

This contradicts the assumption that $|W : H| < 2^{\aleph_0}$. Thus we can find i_0 as in (i).

Assume that we have found $i_0, i_1, \ldots, i_k$ such that (i) and (ii) are satisfied. Suppose that Δ is an H-orbit in $C_{i_k}(\alpha)$ and that for all $m > i_k$ we have that

$C_m(\alpha) \cap \mathrm{desc}(\Delta)$ contains an H-orbit of length $|\Delta|$. (Note that an H-orbit in $\mathrm{desc}(\Delta)$ must have length at least $|\Delta|$.) Let $\Delta_0 := \Delta$ and choose for $j \geq 1$ an H-orbit Δ_j in $C_{i_k+j}(\alpha) \cap \mathrm{desc}(\Delta_{j-1})$ such that $|\Delta_j| = |\Delta|$ and $\mathrm{desc}(\Delta_j)$ contains infinitely many H-orbits of length $|\Delta|$. Then $\Phi := \bigcup_{j=0}^{\infty} \Delta_j$ is a union of $|\Delta|$ half-lines and Φ is H-invariant. But Φ has $2^{\aleph_0}$ translates under W, and by the same argument as above we arrive at a contradiction.

Thus our assumption that we can find an H-orbit of length $|\Delta|$ in $C_m(\alpha) \cap \mathrm{desc}(\Delta)$ for all $m > i_k$ must be wrong. There are only finitely many H-orbits in $C_{i_k}(\alpha)$ so we can choose i_{k+1} large enough so that the induction hypothesis is satisfied.

For the next step assume that $i_0 < i_1 < i_2 \dots$ is a sequence like the one described above.

Step 2: It is possible to find a sequence $\{\beta_j\}_{j=1}^{\infty}$ of vertices in $\cup C_{i_k}(\alpha)$ such that

(i) $\{\beta_j\}_{j=1}^{\infty}$ contains exactly one element from each H-orbit in $C_{i_k}(\alpha)$ for all k;

(ii) if $i \neq j$ then $\beta_i \notin \mathrm{desc}(\beta_j)$.

We will be using induction to prove that we can find such a sequence. The only difficulty is to formulate the induction hypothesis: assume that we have found sequences $\{\beta_j\}_{j=1}^{k}$ and $\{\gamma_j\}_{j=1}^{k}$ in $\cup C_{i_k}(\alpha)$ both containing exactly one element from each H-orbit in $C_{i_0}(\alpha), \dots, C_{i_m}(\alpha)$; and that $\beta_i \notin \mathrm{desc}(\beta_j)$ if $i \neq j$; and that $\gamma_i \notin \mathrm{desc}(\beta_j)$ for all $i, j \in \{1, \dots, k\}$. Then the induction hypothesis is that we can add representatives from the H-orbits in $C_{i_{m+1}}(\alpha)$ to the sequences and still satisfy the above conditions.

The condition that all H-orbits in $C_{i_0}(\alpha)$ have length at least 2 ensures that we can choose two such sequences of representatives from $C_{i_0}(\alpha)$.

For the induction step we need to show that if Δ is an H-orbit in $C_{i_{m+1}}(\alpha)$ then we can find elements β and γ which we can add to the sequences $\{\beta_j\}_{j=1}^{k}$ and $\{\gamma_j\}_{j=1}^{k}$, respectively, and with the new sequences still satisfying the conditions in the induction hypothesis.

Let Δ' be the H-orbit in $C_{i_m}(\alpha)$ with $\Delta \subseteq \mathrm{desc}(\Delta')$ and let γ' be the representative of Δ' in $\{\gamma_j\}_{j=1}^{k}$. But the sequence $i_0 < i_1 < \ldots$ was defined such that $|\Delta| > |\Delta'|$, and so $|\Delta \cap \mathrm{desc}(\gamma')| \geq 2$. It is clear from the above that any two distinct memebers of $\Delta \cap \mathrm{desc}(\gamma')$ will do.

The sequence $\{\beta_j\}_{j=1}^{\infty}$ which we get by this method is exactly what we want.

Step 3: The third and last step is finally to apply the argument used in [3] to prove the theorem.

Let $\Psi := T \setminus \cup_{i=1}^{\infty} \mathrm{desc}(\beta_i)$. The conditions on the sequence $\{\beta_i\}_{i=1}^{\infty}$ ensure that the sets $\mathrm{desc}(\beta_i)$ are pairwise disjoint. Let $X_i := W_{(T \setminus \mathrm{desc}(\beta_i))}^{\mathrm{desc}(\beta_i)}$, thought of as a permutation group acting on $\mathrm{desc}(\beta_i)$. One sees that $W_{(\Psi)}$ acts on $\cup_{i=1}^{\infty} \mathrm{desc}(\beta_i)$ like $\prod X_i$. Define $F_i := (H_{(\Psi)})_{\{\mathrm{desc}(\beta_i)\}}^{\mathrm{desc}(\beta_i)}$. Then $H_{(\Psi)} \leq \prod F_i$ and by assumption $|W_{(\Psi)} : H_{(\Psi)}| < 2^{\aleph_0}$. Thus

$$2^{\aleph_0} > |W_{(\Psi)} : H_{(\Psi)}| \geq |\prod X_i : \prod F_i| = \prod |X_i : F_i|.$$

So $F_i = X_i$ for almost all i. Hence $H_{(T\backslash\mathrm{desc}(\beta_i))} \trianglelefteq W_{(T\backslash\mathrm{desc}(\beta_i))}$ for almost all i, and we can therefore find an integer m such that if $\beta_i \in C_{i_m}(\alpha)$ then $H_{(T\backslash\mathrm{desc}(\beta_i))} \trianglelefteq W_{(T\backslash\mathrm{desc}(\beta_i))}$. But $W_{(T\backslash\mathrm{desc}(\beta_i))} = U_{d(\alpha,\beta_i)}$; that is, $W_{(T\backslash\mathrm{desc}(\beta_i))}$ can be expressed as an infinite wreath product. The last proposition thus applies, so that if $\beta_j \in C_{i_m}(\alpha)$ then we can find a natural number l_j such that $U'_{i_m+l_j} \leq H_{(T\backslash\mathrm{desc}(\beta_j))}$. But the sequence $\{\beta_i\}_{i=1}^{\infty}$ contains a representative from each H-orbit in $C_{i_m}(\alpha)$. We find l_j as described above for each representative β_j of an H-orbit in $C_{i_m}(\alpha)$, and let $L := \max\{l_j \mid \beta_j \in C_{i_m}(\alpha)\}$. Then $K_{L'+i_m} \leq H$. $\square$

If we could calculate the derived group for such a wreath product then we would have a good idea of what the normal subgroups and the subgroups of small index look like. In [15] it is shown that W is perfect if $H_i = A_n$ for all i and $n \geq 5$. But even in the case where $H_i = S_n$ it does not seem to be obvious how one should determine the derived group. It seems likely the answer is that $g = (\ldots, g_i, \ldots, g_1) \in W$ with $g_i = (g_{i\,0}, \ldots, g_{i\,n_i}) \in H_i^{n_i}$ is in W' if and only if $\prod_{j=1}^{n_i} g_{i\,j}$ is in H'_i for all i.

We have seen that in spite of the negative result in the last section we can nevertheless get a fairly accurate description of the subgroups of small index in infinite wreath products of finite groups. Now we will look briefly at the case of infinite groups. Our aim is to prove a lemma about the stabiliser of a vertex in $\mathrm{Aut}(T_{\aleph_0})$, a result which will be essential for the proofs of Theorems 4.2 and 4.3. The first step is the following lemma, which is just a slight amendment of Lemma 5.1 in [14].

Lemma 4.1 *Let B be a group acting faithfully on an infinite set Γ and A a group acting faithfully on some set Δ. If $b \in B$ is a product of disjoint infinite cycles and* $\mathrm{supp}(b) = \Gamma$, *then the normal closure of b in $A\mathrm{Wr}_{\Gamma}B$ contains the base group.*

Proof The proof in [14] works equally well for this extended version. □

Lemma 4.2 *Let* $G := \mathrm{Aut}(T_{\aleph_0})$ *and let $\alpha \in VT_{\aleph_0}$. If $N \trianglelefteq G_\alpha$ and the index of N in G_α is less than $2^{\aleph_0}$ then $N = G_\alpha$.*

Proof We have from the last section that $G_\alpha \cong G_\alpha \mathrm{Wr}\,\mathrm{Sym}(\aleph_0)$. The projection of N on the top group is normal and has index less than $2^{\aleph_0}$. According to an old theorem of Schreier and Ulam (cf. [20], §2) the only proper non-trivial normal subgroups of $\mathrm{Sym}(\aleph_0)$ are the finitary group and the alternating finitary group. But these two both have index $2^{\aleph_0}$ in $\mathrm{Sym}(\aleph_0)$. Thus N projects onto the top group. We can therefore find an element h in N where $h = gb$ with g in the base group and b in $\mathrm{Sym}(\aleph_0)$ and b satisfies the hypothesis in the last lemma, so the result follows. □

4.3 Theorem 4.2

Theorem 4.2 *Let $3 \leq n \leq \aleph_0$ and* $G := \mathrm{Aut}(T_n)$. *If H is a subgroup of G with index less than $2^{\aleph_0}$ then $H = G$ or $H = G_0$ or H stabilises a vertex or an edge.*

Before we prove this theorem we must first prove the following lemma:

Lemma 4.3 *Let* $3 \leq n \leq \aleph_0$ *and* $H \leq \mathrm{Aut}(T_n)$. *Suppose* H *contains a translation* g *along a line* L *and* H *does not stabilise any end or line. Then there is for any* $n \in \mathbf{N}$ *an element* $h \in H$ *such that* $d(L, Lh) \geq n$.

Proof of Lemma 4.3 Find an element $h_0 \in H$ such that $Lh_0 \neq L$ and set $g_0 := h_0^{-1} g h_0$. Now we have to consider three cases.

(A) $Lh_0 \cap L = \emptyset$

Then we also have that $Lh_0 \cap Lh_0 g = \emptyset$. By choosing $i \in \mathbf{Z}$ suitably we see that $d((Lh_0 g)g_0^i, L)$ can be made as large as desired.

(B) $Lh_0 \cap L$ is finite.

Then we can find $i \in \mathbf{Z}$ such that $Lh_0 g^i h_0^{-1} \cap L = \emptyset$ and then apply the method in case (A).

(C) $Lh_0 \cap L$ is infinite.

Then $Lh_0 \cap L$ is a half-line. Let ϵ_1 and ϵ_2 be the ends of L. Suppose $Lh_0 \cap L$ is in the end ϵ_1. Now we have two cases; either $\epsilon_1 h_0 = \epsilon_1$ or $\epsilon_2 h_0 = \epsilon_1$. Assume that $\epsilon_1 h_0 = \epsilon_1$. Find $h_0' \in H$ such that $\epsilon_1 h_0' \neq \epsilon_1$. If Lh_0' comes under cases (A) or (B) then we are done. So assume $Lh_0' \cap L$ is infinite. If it is in the end ϵ_2 then we can use g to separate Lh_0 and Lh_0' as above and then use the method of (A). So assume $Lh_0 \cap L$ and $Lh_0' \cap L$ are in the same end. We may suppose that Lh_0 is not equal to $Lh_0'^{-1}$. Then looking at $Lh_0 h_0' \cap L$ which is not in the end ϵ_1, we can apply the methods above. The other cases can be tackled similarly. □

Proof of Theorem 4.2 First we show that H can not fix an end or a line. Let ϵ be an end of T_n. As was pointed out in Section 2.1 there are $2^{\aleph_0}$ ends of T_n and G acts transitively on them. Thus we have $|G : G_\epsilon| = 2^{\aleph_0}$ and hence it is impossible that $H \leq G_\epsilon$. Exactly the same argument shows that H can not stabilise a line.

Proposition 2.3 tells us that if H does not contain any translations then H fixes a vertex or an edge or an end. Therefore let us assume that H does not fix a vertex or an edge. We have already shown that H cannot fix an end so we can assume that H contains a translation along some line L. The aim is to prove that $G_0 \leq H$.

Now let π_L be the projection of T_n on L. More precisely, define $\pi_L(\beta)$ to be the vertex on L that is closest to β. For $\alpha \in L$ define $B_\alpha := \pi_L^{-1}(\alpha)$. Let $G_{(L)}^{B_\alpha}$, $H_{(L)}^{B_\alpha}$ be the permutation groups induced upon B_α by $G_{(L)}$ and $H_{(L)}$ respectively. Note that $G_{(T_n \backslash B_\alpha)} = G_{(L)}^{B_\alpha}$. Now $G_{(L)} = \prod_{\alpha \in L} G_{(L)}^{B_\alpha}$ and $H_{(L)} \leq \prod_{\alpha \in L} H_{(L)}^{B_\alpha}$ so

$$2^{\aleph_0} > |G_{(L)} : H_{(L)}| \geq |\prod_{\alpha \in L} G_{(L)}^{B_\alpha} : \prod_{\alpha \in L} H_{(L)}^{B_\alpha}| = \prod_{\alpha \in L} |G_{(L)}^{B_\alpha} : H_{(L)}^{B_\alpha}|.$$

Thus $H_{(L)}^{B_\alpha} = G_{(L)}^{B_\alpha}$ for almost all $\alpha \in L$, indeed, because of our translation along L we have that $H_{(L)}^{B_\alpha} = G_{(L)}^{B_\alpha}$ for all $\alpha \in L$. It follows that $H_{(T_n \backslash B_\alpha)}$ is normal in $G_{(T_n \backslash B_\alpha)}$.

Now the proof has to be split up into two cases according to whether n is finite or not. In both cases the plan is to prove that $G_\alpha \leq H$. Earlier it was proved that G_0 is primitive so G_α is a maximal subgroup of G_0, but by assumption we have that $G_\alpha \neq H$ and, as remarked in Section 2.2, then $G_0 \leq H$.

First the case where $n = \aleph_0$. By Lemma 4.2 we have that $H_{(T_n \setminus B_\alpha)} = G_{(T_n \setminus B_\alpha)}$ (because $G_{(T_n \setminus B_\alpha)} \cong G_\alpha \mathrm{Wr}\,\mathrm{Sym}(\aleph_0)$ and $H_{(T_n \setminus B_\alpha)} \trianglelefteq G_{(T_n \setminus B_\alpha)}$ and $|G_{(T_n \setminus B_\alpha)} : H_{(T_n \setminus B_\alpha)}| < 2^{\aleph_0}$.) Now find two lines L and L' such that H contains translations along both of them and $d(L, L') \geq 2$, here we use the previous lemma. Let α be a vertex on the path between them but not on either L or L'. We set $\gamma := \pi_L(\alpha)$ and $\gamma' := \pi_{L'}(\alpha)$. Let β, β' be vertices adjacent to α such that $B(\beta; \alpha, \gamma)$ and $B(\beta'; \alpha, \gamma')$. Let $B_\gamma := \pi_L^{-1}(\gamma)$ and $B_{\gamma'} := \pi_{L'}^{-1}(\gamma')$. Let $A := \langle H_{(T_n \setminus B_\gamma)}, H_{(T_n \setminus B_{\gamma'})} \rangle_\alpha$: we want to show that $G_\alpha \leq A$.

Step 1: The group A acts as S_n on $C_1(\alpha)$.

The group $H_{(T_n \setminus B_\gamma)\alpha}$ acts on $C_1(\alpha)$ like the stabiliser in S_n of β and the group $H_{(T_n \setminus B_{\gamma'})\alpha}$ acts on $C_1(\alpha)$ like the stabiliser in S_n of β'. Thus A acts as S_n on $C_1(\alpha)$.

Step 2: $G_{(C_1(\alpha))} \leq A$.

Let $g \in G_{(C_1(\alpha))}$. We can find an element $h \in H_{(T_n \setminus B_\gamma)}$ such that h fixes the set $\Gamma := \{\delta \in VT_n \mid B(\beta; \alpha, \delta)\}$ and agrees with g on $T_n \setminus \Gamma$. Then $h' := gh^{-1}$ fixes $T_n \setminus \Gamma$, which includes $T_n \setminus B_{\gamma'}$, so $h' \in H_{(T_n \setminus B_{\gamma'})}$. Hence $g = h'h \in A$. We have now shown that $G_\alpha \leq A$. As said before this implies that $G_0 \leq H$ so we must have $H = G_0$ or $H = G$, which is what we wanted to prove.

Now we deal with the case where $3 \leq n < \aleph_0$. Because we do not have Lemma 4.2 it is more difficult.

First let us look at $G_{(T_n\backslash B_\gamma)}$ with $\gamma \in L$. Clearly

$$G_{(T_n\backslash B_\gamma)} = \ldots \mathrm{Wr} S_{n-1} \mathrm{Wr} S_{n-1} \mathrm{Wr} \ldots \mathrm{Wr} S_{n-2},$$

and $H_{(T_n\backslash B_\gamma)} \trianglelefteq G_{(T_n\backslash B_\gamma)}$. Let U_k be defined in the same way as in Section 3.2 and let K_k be the k-th base group of $G_{(T_n\backslash B_\gamma)}$. Then $K_k = \prod_{\beta\in C_k(\gamma)} U_k$. In the proof of Proposition 3.1 we found an integer l such that $C_l(\gamma)$ contained two disjoint sets Θ_1 and Θ_2 such that for any $y \in U_l$ the element $\bar{y} \in K_n$ defined by

$$\bar{y}(\beta) = \begin{cases} y & \text{if } \beta \in \Theta_1, \\ y^{-1} & \text{if } \beta \in \Theta_2, \\ 1 & \text{otherwise} \end{cases}$$

is also in $H_{(T_n\backslash B_\gamma)}$. It is also clear that if $\beta_0 \in C_l(\gamma)$ then we can arrange things so that β_0 is in Θ_1. (Here we are using the normality of $H_{(T_n\backslash B_\gamma)}$ in $G_{(T_n\backslash B_\gamma)}$.)

Now let us find $l \in \mathbb{N}$ such that for all $\gamma \in L$ we can find sets Θ_1 and Θ_2 as above in $C_l(\gamma)$. This can be done because we have a translation along L in H. Now appeal to Lemma 4.3 to find $h \in H$ such that $d(L, Lh) \geq 2l + 1$, and set $L' := Lh$. Because of the way L' is defined we see that l has the same property relative to L' as to L. Let α be a vertex on the path between L and L' such that $d(\alpha, L) > l$ and $d(\alpha, L') > l$. Define $\beta, \beta', \gamma, \gamma', B_\gamma$ and $B_{\gamma'}$ in the same way as in the case where $n = \aleph_0$. Furthermore, define

$$D_\beta := \{\delta \in VT_n \mid B(\alpha; \beta, \delta)\}$$

and

$$D_{\beta'} := \{\delta \in VT_n \mid B(\alpha; \beta', \delta)\}.$$

Set $d := d(\gamma, \beta)$ and $d' := d(\gamma', \beta')$. Then we identify U_d with $G_{(T_{\aleph_0}\setminus D_\beta)}$ and $U_{d'}$ with $G_{(T_n \setminus D_{\beta'})}$. Let Θ_1 and Θ_2 be subsets of $C_d(\gamma)$ having the property that if $y \in U_d$ then $\bar{y}$, as defined above, is in $H_{(T_n\setminus B_\gamma)}$; and similarly find Θ_1' and Θ_2', subsets of $C_{d'}(\gamma')$, having the property that if $x \in U_{d'}$ then $\bar{x}$, as defined above, is in $H_{(T_n \setminus B_{\gamma'})}$ (possible since $d \geq l$). Now we want to prove that $G_\alpha \leq \langle H_{(T_n\setminus B_\gamma)}, H_{(T_n\setminus B_{\gamma'})}\rangle_\alpha =: A$. The steps in the proof of this are the same as in the case when n is infinite.

Step 1: The group A acts as S_n on $C_1(\alpha)$.

This is clear, because $(U_d)_\alpha$ acts like $(S_n)_\beta$ on $C_1(\alpha)$; similarly $(U_{d'})_\alpha$ acts like $(S_n)_{\beta'}$ on $C_1(\alpha)$. Thus A acts as S_n on $C_1(\alpha)$.

Step 2: $A_{(C_1(\alpha))} = G_{(C_1(\alpha))}$.

Let $g \in G_{(C_1(\alpha))}$. Find an element $y_1 \in U_d$ that fixes $C_1(\alpha)$ and likewise an element $x_1 \in U_{d'}$ that fixes $C_1(\alpha)$. We want to define two sequences $(y_i)_{i=1}^\infty$ in U_d and $(x_i)_{i=1}^\infty$ in $U_{d'}$ such that

(i) $y_j | C_{i+1}(\beta) = y_i | C_{i+1}(\beta)$ and $x_j | C_{i+1}(\beta') = x_i | C_{i+1}(\beta')$ for all $j \geq i$;

(ii) $\bar{y}_i \bar{x}_i | C_i(\alpha) = g | C_i(\alpha)$.

Assume that we have defined y_i and x_i. We then let y_{i+1} agree with y_i on $C_{i+1}(\beta)$ and we y_{i+1} agree with $g\bar{x}_i^{-1}$ on $C_{i+1}(\alpha) \cap C_{i+2}(\beta)$. These are the only demands we make of y_{i+1}. Now we define x_{i+1}. First let x_{i+1} agree with x_i on $C_{i+1}(\beta')$. On $C_{i+1}(\alpha) \setminus C_{i+2}(\beta)$ we let x_{i+1} agree with $\bar{y}_{i+1}^{-1} g$ and then act like $\bar{x}_i$ on the rest of $C_{i+1}(\alpha)$. It may then do what it likes to all other vertices. Condition (i) above is then automatically satisfied and we are left to check condition (ii). The element y_{i+1} was defined such that on $C_{i+1}(\alpha) \cap C_{i+2}(\beta)$ we have $\bar{y}_{i+1}\bar{x}_i = g$. But we defined x_{i+1} to act like $\bar{x}_i$ on the set $C_{i+1}(\alpha) \cap C_{i+2}(\beta) \cap C_{i+2}(\beta')$ so that $\bar{y}_{i+1}\bar{x}_{i+1} = g$ on that set. But $C_{i+1}(\alpha) \cap C_{i+2}(\beta) \setminus C_{i+2}(\beta') \subseteq C_{i+1}(\beta')$ and there we have $\bar{x}_{i+1}|C_{i+1}(\beta') = \bar{x}_i|C_{i+1}(\beta')$. So

$$\bar{y}_{i+1}\bar{x}_{i+1}|(C_{i+1}(\alpha) \cap C_{i+2}(\beta)) = g|(C_{i+1}(\alpha) \cap C_{i+2}(\beta)).$$

On $C_{i+1}(\alpha) \setminus C_{i+2}(\beta)$ we defined x_{i+1} in such a way that the condition is obvious. We have now proved that condition (ii) is satisfied. Let y be the limit of $(y_i)_{i=1}^{\infty}$ and x the limit of $(x_i)_{i=1}^{\infty}$. One way to think of this is to say that $\delta y := \delta y_i$ if $d(\delta, \beta) = i$; then condition (i) above ensures that y is well defined. Define x in the same way. The groups U_d and $U_{d'}$ are both closed so $y \in U_d$ and $x \in U_{d'}$. Then $\bar{y}\bar{x} = g$ and both $\bar{y}$, $\bar{x}$ are in A so g is also in A

Now we have proved that $G_\alpha \leq H$ and the rest follows as indicated above.
□

4.4 Theorem 4.3

For the regular tree of countable infinite valency the situation turns out to be quite different. We are able to prove a characterisation of the subgroups of small index like the one in Neumann's Theorem.

Theorem 4.3 *Let* $G := \mathrm{Aut}(T_{\aleph_0})$. *If* $H \leq G$ *has index less than* $2^{\aleph_0}$ *then* $H = G$ *or* $H = G_0$ *or there is a finite subtree* Φ *of* $T_{\aleph_0}$ *such that* $G_{(\Phi)} \leq H \leq G_{\{\Phi\}}$.

For the rest of this section let T denote the regular tree of countably infinite valency $T_{\aleph_0}$. The first step in the prove of the theorem is to establish the following lemma. In the proof of the lemma, as well as in the proof of the theorem, we fix a vertex α in T and view T as a rooted tree with root α.

Lemma 4.4 *Let $H \leq G_\alpha$ and $|G_\alpha : H| < 2^{\aleph_0}$. If H acts like* $\mathrm{Sym}(C_1(\alpha))$ *on $C_1(\alpha)$ then $H = G_\alpha$.*

Proof Write $C_1(\alpha)$ as a disjoint union of $\aleph_0$ moieties, $\{\Gamma_i\}_{i\in\mathbf{N}}$. (A moiety is a subset which has the same cardinality as its complement.) Let $G_i := G_{(T\setminus\mathrm{desc}(\Gamma_i))}$; that is, G_i is the group of permutations that fix everything except $\mathrm{desc}(\Gamma_i)$ pointwise. Similarly define $H_i := H_{(T\setminus\mathrm{desc}(\Gamma_i))}$. Let P_i be the 'projection' of $H_{\{\mathrm{desc}(\Gamma_i)\}}$ on G_i; that is, P_i is the permutation group induced upon $\mathrm{desc}(\Gamma_i)$ by $H_{\{\mathrm{desc}(\Gamma_i)\}}$. Then $P_i \leq G_i$ and $\prod_{i\in\mathbf{N}} G_i$ is precisely the subgroup of G_α that stabilises the sets $\{\Gamma_i\}_{i\in\mathbf{N}}$. The corresponding subgroup of H is contained in $\prod_{i\in\mathbf{N}} P_i$. We can therefore use the same argument with cartesian products as we have used before. That is

$$2^{\aleph_0} > |\prod_{i\in\mathbf{N}} G_i : \prod_{i\in\mathbf{N}} P_i| = \prod_{i\in\mathbf{N}} |G_i : P_i|,$$

which shows that $P_i = G_i$ for almost all i. Choose i so that $P_i = G_i$. Then $H_i \trianglelefteq G_i$ and, by application of Lemma 4.2, we get that $H_i = G_i$. Let $h \in H$ interchange Γ_i and $C_1(\alpha) \setminus \Gamma_i$. Then

$$\begin{aligned} G_{(C_1(\alpha))} &\leq G_i \times G_{(\mathrm{desc}(\Gamma_i))} \\ &= G_i \times h^{-1} G_i h \\ &= H_i \times h^{-1} H_i h \leq H. \end{aligned}$$

We have been assuming that H acted like $\mathrm{Sym}(C_1(\alpha))$ on $C_1(\alpha)$; since we also have that $G_{(C_1(\alpha))} \leq H$ we are able to conclude that $H = G_\alpha$. □

Proof of Theorem 4.3 Let us assume that $H \neq G$ and $H \neq G_0$. From Theorem 4.2 we know that H must then either fix a vertex or stabilise an edge. Assume that H fixes a vertex α. We want to show that there is either a finite subtree Φ such that $G_{(\Phi)} \leq H \leq G_{\{\Phi\}}$ or there is a strictly increasing chain $\Phi_0 \subset \Phi_1 \subset \Phi_2 \ldots$ of finite subtrees with the following properties for all $i \in \mathbf{N}$:

(i) $\Phi_i \cap C_{i+1}(\alpha) = \emptyset$ and $\Phi_{i+1} \setminus \Phi_i \subseteq C_{i+1}(\alpha)$;

(ii) Φ_i is invariant under H;

(iii) let $G_i := G_{(\Phi_i \cup \mathrm{desc}(\Phi_i \cap C_i(\alpha)))}$. We can describe G_i as the group of those permutations that fix the set Φ_i and the descendants of those vertices of Φ_i that are in distance i from α. Similarly let $H_i := H_{(\Phi_i \cup \mathrm{desc}(\Phi_i \cap C_i(\alpha)))}$. Then the condition is that $H_i = G_i$.

We will show that the existence of such a chain contradicts the hypothesis that $|G_\alpha : H| < 2^{\aleph_0}$.

We start naturally enough by setting $\Phi_0 := \{\alpha\}$. The plan is now to define the rest of the chain using induction. We start with a detailed account of how Φ_1 is defined. The argument we use to define Φ_1 wil be refered to when it comes to the general induction step. We saw in Section 3.1 that G_α can be expressed as a wreath product. The top group, which we call E, acts like $\mathrm{Sym}(C_1(\alpha))$ on $C_1(\alpha)$ and is indeed isomorphic to $\mathrm{Sym}(C_1(\alpha))$.

Furthermore, E has the property that if $\beta \in C_1(\alpha)$, g an element of E that fixes β then g fixes desc(β). But $|E : H \cap E| < 2^{\aleph_0}$. Hence it follows from Neumanns's Theorem that there is a finite set $\Delta \in C_1(\alpha)$ such that

$$\mathrm{Sym}(C_1(\alpha))_{(\Delta)} \leq (H \cap E)^{C_1(\alpha)} \leq \mathrm{Sym}(C_1(\alpha))_{\{\Delta\}}.$$

If $\Delta = \emptyset$ then H acts like $\mathrm{Sym}(C_1(\alpha))$ on the set $C_1(\alpha)$, and one can apply Lemma 4.4 to get that $H = G_\alpha$. So we may assume that $\Delta \neq \emptyset$. If $g \in (H \cap E)_{(\Delta)}$ then g fixes desc(Δ) pointwise. So $H_{(\mathrm{desc}(\Delta))}$ acts like $\mathrm{Sym}(C_1(\alpha) \setminus \Delta)$ on $C_1(\alpha) \setminus \Delta$. Now $T \setminus \mathrm{desc}(\Delta)$ is isomorphic to T and $G_{(\mathrm{desc}(\Delta))} \cong G_\alpha$ so by applying Lemma 4.4 we get that $H_{(\mathrm{desc}(\Delta))} = G_{(\mathrm{desc}(\Delta))}$. Let Δ_0 be a minimal finite subset of $C_1(\alpha)$ such that $H_{(\mathrm{desc}(\Delta_0))} = G_{(\mathrm{desc}(\Delta_0))}$ and let $h \in H$. Then

$$\begin{aligned} H &\geq \langle H_{(\mathrm{desc}(\Delta_0))}, h^{-1} H_{(\mathrm{desc}(\Delta_0))} h \rangle \\ &= \langle G_{(\mathrm{desc}(\Delta_0))}, h^{-1} G_{(\mathrm{desc}(\Delta_0))} h \rangle \\ &= \langle G_{(\mathrm{desc}(\Delta_0))}, G_{(\mathrm{desc}(\Delta_0 h))} \rangle \\ &= G_{(\mathrm{desc}(\Delta_0 \cap \Delta_0 h))}. \end{aligned}$$

By minimality of Δ_0 we have that Δ_0 is invariant under H. This also shows that Δ_0 is unique, subject to these conditions. Now set $\Phi_1 := \Phi_0 \cup \Delta_0$. Conditions (i),(ii) and (iii) are readily seen to be satisfied.

Now the induction step. Assume that Φ_N has already been found. Let $\phi \in \Phi_N \cap C_N(\alpha)$. The group $H_{(T \setminus \mathrm{desc}(\phi))}$ has index less than $2^{\aleph_0}$ in $G_{(T \setminus \mathrm{desc}(\phi))}$ and desc$(\phi) \cong T$. We can then use the same argument as above to find a finite set Δ_ϕ in $C_1(\phi)$ such that $H_{(T \setminus \mathrm{desc}(\phi))(\Delta_\phi)} = G_{(T \setminus \mathrm{desc}(\phi))(\Delta_\phi)}$ and Δ_ϕ is invariant under $H_{(T\setminus \mathrm{desc}(\phi))}$. Let

$$\Psi := \cup_{\phi \in \Phi_N \cap C_N(\alpha)} \Delta_\phi.$$

For the time being assume that Ψ is non-empty. Set

$$\Phi_{N+1} := \Phi_N \cup \Psi.$$

Each Δ_ϕ is finite and $\Phi_N \cap C_N(\alpha)$ is also finite so Ψ is finite. It is also clear that Φ_{N+1} is a subtree of T and that condition (i) is satisfied.

We want to show that Φ_{N+1} is invariant under $h \in H$. By the induction hypothesis we know that Φ_N is invariant under h, so if $\phi \in \Phi_N \cap C_N(\alpha)$ then ϕh is also in $\Phi_N \cap C_N(\alpha)$. But then $H_{(T\setminus \mathrm{desc}(\phi))}$ is isomorphic, as a permutation group to $H_{(T\setminus \mathrm{desc}(\phi h))}$. Then, because of the way Δ_ϕ was defined and because it is unique, we have that $\Delta_\phi h = \Delta_{\phi h}$. Hence Φ_{N+1} is invariant under H.

Now we are left to show that (iii) holds. Let $g \in G_{N+1}$. We want to show that $g \in H$. By the induction hypothesis we can find $h \in H_N$ agreeing with g on $T \setminus (\Phi_N \cup \mathrm{desc}(\Phi_N \cap C_N(\alpha)))$. For each $\phi \in \Phi_N \cup C_N(\alpha)$ we can find an element $h_\phi \in H_{(T\setminus \mathrm{desc}(\phi))}$ that agrees with g on $\mathrm{desc}(\phi)$. Then

$$g = h(\prod h_\phi) \in H_{N+1}.$$

So condition (iii) is also satisfied.

This was all done assuming that the set Ψ defined above was non-empty. But if Ψ is empty then we have nothing more to do: because the same argument as used to establish condition (iii) shows that we have $G_{(\Phi_N)} \leq H$, and because Φ_N satisfies (ii) we also have $H \leq G_{\{\Phi_N\}}$ – exactly what we want. Now we need to show that the existence of the chain $\{\Phi_i\}_{i\in \mathbf{N}\cup\{0\}}$ leads to a contradiction.

Assume the chain $\{\Phi_i\}_{i\in \mathbf{N}\cup\{0\}}$ satisfies (i),(ii) and (iii). Let

$$\Phi := \cup_{i\in \mathbf{N}\cup\{0\}}\Phi_i.$$

Then Φ is a subtree of T and Φ is infinite. But for all i we have that $\Phi \cap C_i(\alpha)$ is finite. It follows also from the definition of the trees Φ_i that Φ is invariant under H. But the orbit of Φ under G has cardinality $2^{\aleph_0}$. This can be seen by imagining that for each vertex in T we split its children up into two moieties which we colour red and blue respectively. It is clear that if $(c_i)_{i=1}^{\infty}$ is a sequence in the colours red and blue then we can find $g \in G_\alpha$ such that $\Phi g \cap C_i(\alpha)$ has the colour c_i. There are $2^{\aleph_0}$ such sequences so the orbit of Φ under G_α has length $2^{\aleph_0}$. But that is a contradiction because H stabilises Φ yet H has index less than $2^{\aleph_0}$ in G_α.

This proves the theorem in the case where H fixes a vertex. In the other case H does not fix a vertex but stabilises an edge $\{\alpha, \beta\}$. We can then apply the above argument to $H_{\alpha\beta}$ and the result follows immediately. □

Chapter 5

Applications and remarks

In this chapter we show how one can use our earlier results to prove the results of Znoiko in [24], and also show how we can deduce the simplicity of the group G_0 in the case of a regular tree of finite valency, a result which is a special case of a much more general result of Tits in [18]. First we look at the two theorems that Znoiko proves in [24].

5.1 Two theorems of Znoiko

Theorem 5.1 *For* $3 \leq n \leq \aleph_0$ *the group* $\mathrm{Aut}(T_n)$ *is complete.*

Theorem 5.2 *If* $0 \leq n < m \leq \aleph_0$ *then* $\mathrm{Aut}(T_n) \ncong \mathrm{Aut}(T_m)$.

In order to prove Theorem 5.1 we must show that $G := \mathrm{Aut}(T_n)$ is centreless and every automorphism of G is an inner automorphism.

Lemma 5.1 *G is centreless.*

Proof of Lemma 5.1 Let $g \in G$ and $g \neq 1$. Then there is $\alpha \in VT_n$ such that $\alpha g \neq \alpha$. Find $h \in G_\alpha$ such that $\alpha g h \neq \alpha g = \alpha h g$. Then we have that $hg \neq gh$. □

Proof of Theorem 5.1 Let ϕ be an automorphism of G. Then ϕ maps G_0 to G_0; the stabiliser of a vertex is a maximal subgroup in G_0, and the stabilisers of vertices are the only maximal subgroups of G_0 that have index less than $2^{\aleph_0}$ in G so ϕ must map a stabiliser of a vertex bijectively to a stabiliser of a vertex. A similiar argument shows that ϕ must map a stabiliser of an edge to a stabiliser of an edge. Define

$$\bar{\phi} : T_n \to T_n;\ \alpha \mapsto \bar{\alpha} \text{ if } G_\alpha\phi = G_{\bar{\alpha}}.$$

We want to show that $\bar{\phi}$ is an automorphism of T_n. Suppose that α and β are two adjacent vertices in T_n. The stabiliser of the edge $\{\alpha, \beta\}$ is mapped by ϕ to the stabiliser of some edge $\{\gamma, \delta\}$. Now we have

$$G_{\alpha\beta}\phi = (G_\alpha \cap G_\beta)\phi = G_\alpha\phi \cap G_\beta\phi = G_{\alpha\bar{\phi}} \cap G_{\beta\bar{\phi}} = G_{\alpha\bar{\phi}\beta\bar{\phi}}.$$

But we also have that $G_{\alpha\beta}\phi \leq G_{\{\gamma,\delta\}}$ and, as we saw above, $G_{\alpha\beta}\phi$ fixes two vertices so $G_{\alpha\beta}\phi \leq G_{\gamma\delta}$. Also, by the same argument, we see that $G_{\gamma\delta}\phi^{-1} \leq G_{\alpha\beta}$ so $G_{\alpha\beta}\phi = G_{\gamma\delta}$. Thus $\{\alpha, \beta\}\bar{\phi} = \{\gamma, \delta\}$. We have now proved that $\bar{\phi}$ is an automorphism of T_n.

Let $g \in G$ and $\alpha \in VT_n$. We know that $g^{-1}G_\alpha g = G_{\alpha g}$ and that $G_{\alpha g}\phi = G_{(\alpha g)\bar{\phi}}$, but

$$(g^{-1}G_\alpha g)\phi = (g^{-1}\phi)(G_\alpha\phi)(g\phi) = G_{(\alpha\bar{\phi})g\phi}.$$

Hence $g\phi = \bar{\phi}^{-1}g\bar{\phi}$. □

Proof of Theorem 5.2 We can assume that $n \geq 3$ since the other cases are trivial. The argument used in the proof of the last theorem shows that an isomorphism between $\mathrm{Aut}(T_n)$ and $\mathrm{Aut}(T_m)$ would induce an isomorphism between T_n and T_m. That is absolutely impossible. □

5.2 A theorem of Tits

Theorem 5.3 *Let* $G := \mathrm{Aut}(T_n)$ *with* $3 \leq n < \aleph_0$. *Then the group* G_0 *is simple.*

Proof Let $N \trianglelefteq G_0$ and assume that $N \neq 1$. Let $\alpha \in VT_n$. Then $N_\alpha \trianglelefteq G_\alpha$. It is clear that N does not act regularly on VT_n. By Proposition 3.1 we can then find an integer m such that $K'_m \leq N$. (Here we are using the same notation as in Section 3.2.) In Section 2.2 we showed that G_0 acts primitively on the set Γ of vertices in even distance from α. It is well known that a non-trivial normal subgroup of a primitive group is transitive so N acts transitively on Γ. Therefore for all vertices γ in Γ we have that $N_\gamma \cong N_\alpha$. Find a vertex α' in Γ such that $d(\alpha, \alpha') > 2m$. Let β be a vertex between α and α' and in distance greater than m from both α and α'. Now the same argument as in the last part of the proof of Theorem 4.2 can be applied to show that $G_\beta \leq N$. It then follows that $N = G_0$. □

Tits proves a much more general result, in particular his proof also works in the case of the regular tree of countably infinite valency.

5.3 Remarks

1. Here we have for mostly only been dealing with regular trees. But, it is easy to see that the same proofs also work if our trees are semi-regular (a tree T is semi-regular if $\mathrm{Aut}(T)$ has at most two orbits on VT). Perhaps it is also possible to extend these ideas such that we get similar results for trees where $\mathrm{Aut}(T)$ has only finitely many orbits on VT.

2. As it is, our definition of infinite wreath products only works if the index set is $\mathbf{N}$. The same basic method can be used in the case where the index set is a rooted tree. By using that method we get a description of the stabiliser of a vertex in an arbitrary tree.

3. The ideas used to prove the simplicity of G_0 in Theorem 5.3 can also be used to prove the simplicity of other similar groups. It is quite likely that one could get a proof of the general version of Tits result by these methods. If such a proof, indeed, exists then I suspect that it might be far more complicated than Tits original proof.

Part II

Ends of graphs

Chapter 6

Introduction

In their book *Groups acting on graphs*, Warren Dicks and M. J. Dunwoody developed powerful techniques for forcing a group to act on a tree. One of their results is a method of building a tree from a graph Γ such that Aut(Γ) acts naturally on the tree. We call a tree constructed in this way from a graph Γ a structure tree of Γ. The aim of this part of the thesis is to show how this technique can be utilised to investigate connected graphs with infinitely many ends.

We start in Chapter 7 by reviewing the the definitions and background material needed. In Section 7.1 we define the concept of an end of a graph and state various general results. The second section of Chapter 7 contains a brief account of the necessary results and terminology we need from [2]. In the third chapter we then state and prove the connections between the ends of Γ and the structure trees of Γ. We will give five applications where we use the above mentioned connections. The first of these applications is to determine the kernel of the action of Aut(Γ) on the set of ends of Γ when Γ is locally finite and transitive. In the second one we investigate the structure of the set of ends, trying to show that the ends of a graph have a strong resemblance to structures investigated by Adeleke and Neumann in [1]. These two are treated in Chapter 9. In the next chapter we get to the third application. We show how our methods are ideal to prove 'fixed point theorems' for groups acting on graphs. The last two chapters then contain solutions to two problems posed by W. Woess in [23]. The first one is to classify graphs with an automorphism group that acts transitively on the ends. The second one is to classify those graphs Γ which have an end ω such that $\mathrm{Aut}(\Gamma)_\omega$, the stabiliser in $\mathrm{Aut}(\Gamma)$ of ω, acts transitively on $V\Gamma$.

Chapter 7

Setting the scene

7.1 Notation and preliminaries

Let Γ denote an undirected connected graph. We use $V\Gamma$ and $E\Gamma$ to denote the set of vertices and edges of Γ, respectively. We will use $d_\Gamma(-,-)$ to denote the distance between two vertices, if there is no danger of confusion we will drop the subscript. The graph Γ is said to be transitive if $\mathrm{Aut}(\Gamma)$ acts transitively on $V\Gamma$. Even if we are only working with undirected graphs it is useful to think of the edges as having directions. Thus an edge $e \in E\Gamma$ is represented by an ordered pair of vertices, that is $e = (o(e), t(e))$. We call $o(e)$ the origin of e and $t(e)$ the terminus of e. Because our graphs are undirected, we have that if $e = (o(e), t(e))$ is in $E\Gamma$ then the edge $e^* := (t(e), o(e))$ is also in $E\Gamma$. If $s \subseteq V\Gamma$ then define

$$\partial s := \{e \in E\Gamma \mid \text{exactly one of } o(e) \text{ and } t(e) \text{ is in } s\}.$$

The set ∂s is called the boundary of s.

Let $\mathcal{B}\Gamma$ denote the boolean ring generated by subsets s of $V\Gamma$ with $|\partial s| < \infty$. Indeed, because the intersection and symmetric difference of two such sets again has finite boundary and because for any $s \subseteq V\Gamma$ we have that $\partial s = \partial s^*$ (where s^* denotes the complement of s), we see that

$$\mathcal{B}\Gamma = \{s \subseteq V\Gamma \mid |\partial s| < \infty\}.$$

Define $\mathcal{B}_n\Gamma$ as the subring of $\mathcal{B}\Gamma$ generated by those $s \subseteq V\Gamma$ with $|\partial s| \leq n$. We say that s is n-thin if $s \notin \mathcal{B}_{n-1}\Gamma$ where $n = |\partial s|$, if $s \in \mathcal{B}\Gamma$ is n-thin for some n then we say that s is thin. Clearly a thin set is connected.

A line in a graph is a sequence of distinct vertices $\{\alpha_i\}_{i \in \mathbf{Z}}$ such that α_i is adjacent to α_{i+1} for all i. A half-line is a sequence of distinct vertices $\{\alpha_i\}_{i \in \mathbf{N}}$ such that α_i is adjacent to α_{i+1} for all i. An end of a graph is defined as an equivalence class of half-lines: we say that two half-lines L_1 and L_2 are in the same end if there is a third half-line L_3 that contains infinitely many vertices of both L_1 and L_2. Equivalently, we could say that L_1 and L_2 are in different ends if and only if there is a finite subset $\Phi \subseteq \Gamma$ such that there are distinct connected components C_1 and C_2 of $\Gamma \setminus \Phi$ such that C_1 contains infinitely many vertices of L_1 and C_2 contains infinitely many vertices of L_2. Yet another equivalent definition is to say that L_1 and L_2 are in the same end if and only if there are infinitely many disjoint paths connecting vertices on L_1 to vertices on L_2. Following [7], we define the 'thickness', denoted by $m_1(\epsilon)$, of an end ϵ as the maximum number of disjoint half-lines it contains. The following theorem summarises various results from [8].

Theorem 7.1 *Let Γ be a connected locally finite graph, and let $g \in \mathrm{Aut}(\Gamma)$. Then either there is a finite subgraph Φ invariant under g or we can find a line L and a natural number n such that g^n induces a non-trivial translation* *on L. In the latter case the ends which are determined by L are invariant under g. Let L^+ be a half-line contained in L such that $L^+g^n \subset L^+$. Let $\mathcal{D}(g)$ denote the end of Γ that contains L^+, and call $\mathcal{D}(g)$ the direction of g. Then the following hold*

(i) *$\mathcal{D}(g) = \mathcal{D}(g^{-1})$ if and only if $m_1(\mathcal{D}(g)) = \infty$.*

(ii) *If $\mathcal{D}(g) \neq \mathcal{D}(g^{-1})$ then $m_1(\mathcal{D}(g)) < \infty$ and*

$$m_1(\mathcal{D}(g)) = m_1(\mathcal{D}(g^{-1})).$$

Definition 7.1 *If $g \in \mathrm{Aut}(\Gamma)$ does not leave any non-empty finite subgraph invariant then we call g a translation. If g satisfies (ii) in the theorem above then we say that g is hyperbolic.*

We will write $\mathcal{E}\Gamma$ for the set of ends of Γ. If $\epsilon \in \mathcal{E}\Gamma$ and there is $g \in \mathrm{Aut}(\Gamma)$ such that $\epsilon = \mathcal{D}(g)$ then we call ϵ a direction of Γ. The set of directions of Γ is denoted by $\mathcal{D}\Gamma$. Furthermore, let $\mathcal{FE}\Gamma$ (respectively, $\mathcal{FD}\Gamma$) be the set of those ends (directions) that have finite thickness.

If $s \in \mathcal{B}\Gamma$ then we say that an end ϵ of Γ belongs to s if s contains one half-line in ϵ. We see that if ϵ belongs to s then $L \cap s$ contains a half-line for every half-line L in ϵ. By slight abuse of notation we will write $\epsilon \in s$ if ϵ belongs to s.

The definition of an end of a graph given above is due to Halin, [6]. Before that the same concept occurs, in a more topological setting, in the writings of Hans Freudenthal and Heinz Hopf. There the ends arise as the points in a certain compactification $\tilde{\Gamma}$ of $V\Gamma$ with the discrete topology. If Γ is locally finite we can use our terminology to describe $\tilde{\Gamma}$. We let $\tilde{\Gamma} := V\Gamma \cup \mathcal{E}\Gamma$. A basis for the topology on $\tilde{\Gamma}$ is given by the set $\mathcal{B}\Gamma$, where we think of $s \in \mathcal{B}\Gamma$ as containing also the ends that belong to s. It is now easy to prove that $\tilde{\Gamma}$ is compact and that $\mathcal{E}\Gamma$ is a closed subset of $\tilde{\Gamma}$. Some of the arguments we will use later on will be in the spirit of this earlier definition and we will also on one occasion use the fact that $\mathcal{E}\Gamma$ is compact in the topology defined above.

We will also have occasion to refer to Theorem 1 in [10]. Here is a version of that theorem which has been slightly rephrased to suit our needs better. A short proof can be found in [17, Lemma 5].

Theorem 7.2 *Assume Γ is an infinite locally finite connected graph and that s is an infinite connected element of $\mathcal{B}\Gamma$ such that S^* is also infinite . If a group G acts transitively on Γ then G contains a hyperbolic element g such that $\mathcal{D}(g) \in s$ and $\mathcal{D}(g^{-1}) \in s^*$.*

Theorem 7.2 remains true if we replace the condition that G acts transitively on Γ with the condition that G has only finitely many orbits on $V\Gamma$.

Let G be a group acting on a set Ω. We can make G into a topological group by taking as a basis of neighbourhoods around the identity the pointwise stabilisers of finite subsets of Ω. Another way to phrase this is to say that we endow Ω with the discrete topology and let G have the topology of pointwise convergence. This topology is called the permutation group topology on G. For the purpose of the present discusion we need only to know that if G is acting on a locally finite graph Γ then the setwise stabiliser in G of a finite subgraph in Γ is compact. For a proof of this and further information about this topology the reader is refered to W. Woess' survey paper [23].

7.2 Structure Trees

In [4] and [2] a very powerful technique for building trees from graphs is developed. The aim of this section is to give a description of this machinery, and to establish the notation needed to make it work.

Let Ω be some non-empty set and let $\mathcal{P}(\Omega)$ denote the power set of Ω. We say that $E \subseteq \mathcal{P}(\Omega)$ is nested if for every $e, f \in E$ we have that one of

$$e \cap f,\ e \cap f^*,\ e^* \cap f,\ e^* \cap f^*$$

is empty, (i.e. $e \subseteq f, e \subseteq f^*, e^* \subseteq f$ or $f \subseteq e$).

If E is nested and for all $\alpha, \beta \in \Omega$ we have that the number of elements of E containing α but not β is finite then we say that E is a tree set. An element $\alpha \in \Omega$ can be treated as a function $E \to \mathbf{Z}_2$, where $\mathbf{Z}_2$ denotes the integers modulo 2. This function, denoted by $\alpha|E$, is defined by

$$\alpha|E(e) := \begin{cases} 1 & \text{if } \alpha \in e \\ 0 & \text{if } \alpha \notin e \end{cases},$$

where $e \in E$. Then E is a tree set if and only if it is nested and for all $\alpha, \beta \in \Omega$ the functions $\alpha|E$ and $\beta|E$ agree almost everywhere. We say that a tree set E is undirected if it has the property that whenever $e \in E$ then $e^* \in E$. Clearly if E is a tree set then the set $E \cup \{e^* \mid e \in E\}$ is also a tree set so every tree set is a subset of an undirected tree set.

Let T be an undirected tree. If $e, f \in ET$ then define $e \leq f$ to hold if and only if there is an edge path

$$e = e_0, \ldots, e_n = f$$

such that $t(e_{i-1}) = o(e_i)$ and $e_{i-1} \neq e_i^*$ for $i = 1, \ldots, n$. One shows easily that this relation is a partial order. It is also clear that * acts on ET as an order reversing involution.

Theorem 7.3 [4, Theorem 2.1] *Suppose E is a partially ordered set with an order reversing involution *. Furthermore, suppose that for all $e, f \in E$ exactly one of*

$$e < f,\ e = f,\ e > f,\ e < f^*,\ e = f^*,\ e > f*$$

holds and that there are only finitely many $g \in E$ such that $e < g < f$. Then there is an undirected tree T such that E is isomorphic (as a partially ordered set with an involution) to ET. Also, ET must satisfy the above conditions for any undirected tree T.

If E satisfies the conditions in the theorem then we use $T(E)$ to denote the associated tree. Thus if E is an undirected tree set then there is an undirected tree $T := T(E)$ such that E can be identified with ET in a natural way. Then inclusion in E corresponds to the relation $\leq$ defined above on ET and taking complements in E corresponds to the operation *, as defined on ET. We also see that if some group G acts on Ω such that E is invariant under G then there is a natural action of G on T.

Definition 7.2 *Let T be a tree. If $e \in ET$ then e points towards $\alpha \in VT$ if $d(\alpha, o(e)) > d(\alpha, t(e))$. If $\alpha \in VT$ then let $\alpha|E$ denote the function from ET to $\mathbf{Z}_2$ that takes the value 1 on those edges that point towards α and the value 0 on those edges that do not point towards α.*

Let E be an undirected tree set. Consider functions $\phi : E \to \mathbf{Z}_2$ that satisfy the following conditions.

(i) There is a point $\alpha \in \Omega$ such that ϕ and $\alpha|E$ agree almost everywhere.

(ii) If $e, f \in E$ such that $e \cap f = \emptyset$ then $\phi(e)\phi(f) = 0$.

Call the set of all such functions V.

Let T be an undirected tree. We construct a new directed tree T^d from T in the following way: the set of vertices of T^d is the union of VT and the set $\{\{e, e^*\} \mid e \in ET\}$, and the set of edges of T^d is the set $\{e^d \mid e \in ET\}$ where $e^d := (\{e, e^*\}, t(e))$. We can identify ET and ET^d in a natural way, in particular the vertices of T^d can be thought of as functions $E \to \mathbf{Z}_2$. If $\alpha \in VT^d$ and $\alpha = \{e, e^*\}$ then $\alpha|E$ agrees on $E \setminus \{e, e^*\}$ with the functions given by the endpoints of e, but takes the value zero on both e and e^*. Informally, T^d is a kind of barycentric division of T.

Theorem 7.4 [2, Theorem II.1.8] *Let $E \subseteq \mathcal{P}(\Omega)$ be an undirected tree set and $T := T(E)$. Suppose V is as above and let $\phi \in V$. Then there is a vertex $\alpha \in VT^d$ such that $\phi = \alpha|E$.*

Let us now again consider general graphs.

Definition 7.3 *If $E \subseteq \mathcal{P}(V\Gamma)$ is an undirected tree set, invariant under* $\mathrm{Aut}(\Gamma)$, *then we call the tree $T(E)$ a structure tree of Γ.*

Theorem 7.5 [2, Theorem II.2.20] *Let Γ be a connected infinite graph and let $G \leq \mathrm{Aut}(\Gamma)$. There is a chain of G-invariant undirected tree sets $E_1 \subseteq$*

$E_2 \subsetneq \ldots$ *in* $\mathcal{P}(V\Gamma)$ *such that all elements in* E_n *are* n*-thin and* E_n *generates* $\mathcal{B}_n\Gamma$ *for all* n.

The condition that E_n generates $\mathcal{B}_n\Gamma$ implies that if there is an $s \in \mathcal{B}_n\Gamma$ separating two vertices (edges, ends) then there is an element $e \in E_n$ that separates them also. This will be very useful in what follows. We will also need to use the following lemma.

Lemma 7.1 [2, Lemma II.2.5] *If* $\alpha, \beta \in V\Gamma$ *then every descending chain in* $\{e \in E_n \mid \alpha \in e\}$ *is finite and every chain in* $\{e \in E_n \mid \alpha \in e, \beta \notin e\}$ *is finite.*

Chapter 8

Ends and structure trees

In this chapter Γ will denote some connected graph. We will be establishing the connections between the ends of a graph and the structure trees of the graph.

Lemma 8.1 *Let $T := T(E)$ be some structure tree (undirected) of Γ and let ϵ be an end of T. Then there is an unique end in Γ that corresponds to ϵ in a natural way.*

Proof Let $L = \{\alpha_0, \alpha_1, \ldots\}$ be some half-line in ϵ and $e_i \in E$ be the edge (α_i, α_{i+1}). Then $e_0 \supset e_1 \supset e_2 \supset \ldots$, so by Lemma 7.1 we have that $\bigcap_{i \geq 0} e_i = \emptyset$. Clearly we can find a half-line L that contains at least one vertex from infinitely many of the sets e_i. Assume L' and L'' are two such half-lines. Find i_0 such that e_{i_0} contains vertices β_0' and β_0'' from L' and L'', respectively. Because e_{i_0} is connected we can find a path P_0 in e_{i_0} connecting β_0' to β_0''. Then we find i_1 such that $P_0 \cap e_{i_1} = \emptyset$. Take vertices β_1' and β_1'' from L' and L'', respectively, that are in e_{i_1}. We let P_1 denote a path in e_{i_1} connecting β_1' and β_1''. Then we go on to find a number i_2 such that $P_1 \cap e_{i_2} = \emptyset$ and so on. The paths $P_0, P_1, \ldots$ form an infinite family of disjoint paths connecting L' to L'', thus L' and L'' are in the same end. Whence all half-lines L such that infinitely many of the sets e_i contain vertices from L are in the same end. This end is then the only end that is contained in e_i for all i. □

A similar argument can be found in [22, §4]. On the other hand it does not follow that every end in Γ corresponds in this way to an end in a structure tree T, but we do have the following lemma.

Lemma 8.2 *Let ϵ be an end of Γ and let $T := T(E)$ be a structure tree of Γ. Then there is either a vertex in T or an end in T that corresponds to ϵ.*

Proof Assume that no end of T corresponds to ϵ, that is to say that there is no infinite strictly descending chain in E such that all the elements in the chain contain ϵ. Define $\epsilon|E : E \to \mathbf{Z}_2$ by

$$\epsilon|E(e) := \begin{cases} 1 & \text{if } \epsilon \in e \\ 0 & \text{if } \epsilon \notin e. \end{cases}$$

First of all it is clear that if $e, f \in E$ and $e \cap f = \emptyset$ then $(\epsilon|E(e))\,(\epsilon|E(f)) = 0$. In order to prove our lemma we have to show that if $\alpha \in V\Gamma$ then $\alpha|E$ and $\epsilon|E$ are almost equal. Let

$$A := \{e \in E \mid \epsilon|E(e) = 1 \text{ and } \alpha|E(e) = 0\}.$$

Because E is nested, we have that for all $e, f \in A$ one of

$$e \leq f,\ f \leq e,\ e^* \leq f,\ f \leq e^*,$$

holds. But, because $\epsilon \in f$ and $\epsilon \notin e^*$, we can exclude the last possibility and the second last one can also be excluded because $\alpha \in e^*$ yet $\alpha \notin f$. Thus A

is a chain. By assumption A cannot contain an infinite strictly descending chain and an infinite strictly ascending chain would contradict Lemma 7.11. Hence A must be finite. The elements of the set

$$\{e \in E \mid \epsilon|E(e) = 0 \text{ and } \alpha|E(e) = 1\},$$

are just the complements of the elements in A. Therefore $\alpha|E$ and $\epsilon|E$ agree almost everywhere. By Theorem 7.4, there is a vertex in T^d that corresponds to $\epsilon|E$. If that vertex was not in VT then there would be a set $e \in E$ such that $\epsilon|E(e) = \epsilon|E(e^*) = 0$; but this is clearly out of the question. □

Now we can ask whether one can recognise those vertices in a structure tree that correspond to ends in Γ. The next lemma contains a complete answer to this question, and the proof also illustrates yet another way of linking ends of graphs with structure trees.

Lemma 8.3 *Let Γ be a locally finite connected graph, and let $T := T(E)$ be a structure tree (undirected) of Γ. Define $\phi : V\Gamma \to VT; \alpha \mapsto \alpha|E$. If $\alpha \in V\Gamma$ has infinite valency or $\phi^{-1}(\alpha)$ is infinite then there is an end ϵ in Γ such that $\epsilon|E = \alpha$. Furthermore, if $\alpha \in VT$ corresponds to an end ϵ then $\phi^{-1}(\alpha)$ is infinite or α has infinite valency.*

Proof Suppose that $\alpha \in VT$ has infinite valency. Let C be a component of $T \setminus \{\alpha\}$. If e is an edge in C pointing away from α then we know that e points towards the elements of $\phi(e)$. Thus every component of $T \setminus \{\alpha\}$ contains some elements of $\mathrm{Im}\phi$.

If $\alpha \in V\Gamma$ has infinite valency then we take an element β_C from $\phi^{-1}(C)$ for each component C of $T \setminus \{\alpha\}$. Let Λ be a minimal spanning tree in Γ for all the vertices β_C. By König's Lemma we can find a half-line L in Λ. Denote by ϵ the end of Γ that L belongs to. If e is a connected element of $\mathcal{B}\Gamma$ which contains all but finitely many of the vertices β_C then e must contain ϵ. This is because δe is finite and thus $e^* \cap \Lambda$ will also be finite. If $e \in E$ points away from α then e contains at most one the vertices β_C and thus cannot contain ϵ. On the other hand, if e points towards α then e must contain ϵ. But now we have that α, as a function on E, is equal to $\epsilon|E$.

If $\phi^{-1}(\alpha)$ is infinite then we let Λ be a spanning tree of $\phi^{-1}(\alpha)$ and the result follows by the same argument.

Assume that $\alpha \in VT$ and $\alpha|E = \epsilon|E$ for some end ϵ of Γ. Suppose that α has finite valency and that $\phi^{-1}(\alpha)$ is finite. Take a half-line L that belongs to the end ϵ. The conditions imply that infinitely many of the vertices in L must be mapped by ϕ to some component C of $T \setminus \{\alpha\}$. If e is the edge in T that has α as is origin and points towards C, then e must contain ϵ but e points away from α. Thus $\epsilon|E(e) = 1$ but $\alpha|E(e) = 0$, which contradicts the assumption that $\alpha|E = \epsilon|E$. □

In the following corollary we collect together the information we can deduce from the above lemmas about connections between actions of elements of $\mathrm{Aut}(\Gamma)$ on Γ and on structure trees of Γ.

Corollary 8.1 *Let Γ be a connected locally finite graph and $T := T(E)$ be some structure tree of Γ.*

(i) *If $g \in \mathrm{Aut}(\Gamma)$ acts like a translation on T then g acts like a translation on Γ and $m_1(\mathcal{D}(g)) < \infty$.*

(ii) *If $g \in \mathrm{Aut}(\Gamma)$ is a translation then either g acts as a translation on T or there is an unique vertex of T fixed by g and that vertex has infinite valency.*

(iii) *If $g \in \mathrm{Aut}(\Gamma)$ is a translation and $m_1(\mathcal{D}(g)) < \infty$ then there is an undirected tree set E_g such that g acts as a translation on $T(E_g)$.*

Proof **(i)** Let L be the line in T that is invariant under g. Suppose that e is an edge in L. Then δe has an infinite orbit under g, and therefore g cannot fix any nonempty finite subgraph. There are two ends of T that are invariant under g, so there are also two ends of Γ invariant under g, which means that $m_1(\mathcal{D}(g)) < \infty$.

(ii) If $\mathcal{D}(g)$ does not appear as an end in T then there is an unique vertex α in T that corresponds to ϵ. Clearly α is fixed by g. And α must have infinite valency, because otherwise some power of g would fix an edge of T which is impossible.

(iii) Assume $\mathcal{D}(g) \neq \mathcal{D}(g^{-1})$. Then there is $s \in \mathcal{B}\Gamma$ such that $\mathcal{D}(g) \in s$ but $\mathcal{D}(g^{-1}) \notin s$. We can then find a tree set E_g (by Theorem 7.5) that contains an element e that separates $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$. Clearly g acts on $T(E_g)$ as a translation. □

Chapter 9

The structure of the ends

In this chapter we look first at the question whether the action of the automorphism group on the ends is faithful. It is very easy to construct examples where this action is not faithful, but in the case of a locally finite connected, transitive graph we manage to get an explicit description of the kernel. This description indicates that in this case at least the action on the ends is very close to being faithful.

In Section 9.2 we try to link the ends of a graph to recent work of Adeleke and Neumann. We prove that the set of ends resembles the structures that Adeleke and Neumann treat but is not quite as nice as their structures. Still these results give another indication of how one might capture the 'treeness' of graphs with infinitely many ends.

9.1 The action on the ends

Let Γ be a connected locally finite, graph. We say that $g \in \mathrm{Aut}(\Gamma)$ is bounded if there is a natural number k such that $d(\alpha, \alpha g) \leq k$ for all $\alpha \in \Gamma$. The bounded automorphisms of Γ form a subgroup $B(\Gamma)$ of $\mathrm{Aut}(\Gamma)$.

Theorem 9.1 *Assume Γ is connected, locally finite and transitive and, furthermore, that Γ has infinitely many ends. Then the kernel K of the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{E}\Gamma$ is the group $B(\Gamma)$.*

Proof First we will show that $B(\Gamma) \leq K$. Take $g \in B(\Gamma)$. Assume there is an end $\epsilon \in \mathcal{E}\Gamma$ such that $\epsilon \neq \epsilon g$. Then we can find a finite subgraph Φ of Γ such that ϵ and ϵg belong to different components of $\Gamma \setminus \Phi$. Let L_1 be a half-line in ϵ. Then $L_1 g$ is in ϵg. We can assume that both $L_1 \cap \Phi$ and $L_1 g \cap \Phi$ are empty. All paths connecting vertices in L_1 to vertices in $L_1 g$ must go through Φ. Because Γ is locally finite we have a contradiction. Note that here we have not made any use of the transitivity of Γ.

Let $e_0 \in \mathcal{B}\Gamma$ be an infinite and co-infinite set such that the set $E := \{e_0\mathrm{Aut}(\Gamma), e_0^*\mathrm{Aut}(\Gamma)\}$ is a tree set. Put $T := T(E)$. Then $\mathrm{Aut}(\Gamma)$ is transitive on the edges of T, and (as pointed out in [4]) this fact together with the transitivity of Γ ensures that T has no leaves (i.e. there are no vertices of valency one). Take $h \in K$. Suppose h does not act trivially on T. Then either h moves an end of T, which (by Lemma 8.1) would mean that h moved an end of Γ, or T has only two ends and h acts like a translation on T. But, by Corollary 8.1 this would imply that h acts like a translation on Γ and a translation fixes at most two ends. Thus K acts trivially on T. For $e \in E$, set

$$\partial_v e := \{\alpha \in V\Gamma \mid \alpha \notin e \text{ but } \alpha \text{ adjacent to a vertex in } e\}.$$

Because Γ is transitive we have that every vertex in Γ is contained in $\partial_v e$ for some $e \in E$. The sets $\partial_v e$ are all invariant under $\mathrm{Aut}(\Gamma)$ and all have the same diameter (the diameter of a set is the supremum of the distances between its elements). Whence, for all $h \in K$ and $\alpha \in V\Gamma$ we have

$$d(\alpha, \alpha h) \leq \operatorname{diam}(\partial_v e_0) < \infty.$$

Thus $K \leq B(\Gamma)$. □

In fact we have proved more: $B(\Gamma)$ is a sub-cartesian product of isomorphic finite groups (given by the action of $B(\Gamma)$ on its orbits). In particular this gives us:

Corollary 9.1 [5, Theorem 5] *The group $B(\Gamma)$ is locally finite.*

Remark Theorem 9.1 remains true if we do not assume that Γ is transitive but instead assume that $\operatorname{Aut}(\Gamma)$ has only finitely many orbits on $V\Gamma$. Choose $\alpha_0 \in \partial_v e_0$. Then there is a number n_0 such that $d(\beta, \alpha_0 \operatorname{Aut}(\Gamma)) \leq n_0$ for all $\beta \in V\Gamma$. Now we see that

$$d(\beta, \beta h) \leq \operatorname{diam}(\partial_v e_0) + 2n_0,$$

for all $\beta \in V\Gamma$ and all $h \in K$.

9.2 D-relations

In the study of infinite permutation groups certain types of relations seem to occur naturally. These types of relations have been axiomatised and investigated by Adeleke and Neumann ([1]). We are interested in what they call D-relationsl; these arise naturally on the set of ends of sufficiently homogeneous 'tree-like' structures. It would take us too far afield to describe precisely what these 'tree-like' structures are. The remarkable thing about these D-relations is that a set carrying a D-relation can be represented as the set of ends of a 'tree-like' structure. A quaternary relation D on a set Ω is a D-relation if it satisfies the following conditions:

D1: if $D(\alpha,\beta;\gamma,\delta)$ then $D(\beta,\alpha;\gamma,\delta)$ and $D(\gamma,\delta;\alpha,\beta)$,

D2: if $D(\alpha,\beta;\gamma,\delta)$ then $\neg D(\alpha,\gamma;\beta,\delta)$,

D3: if $D(\alpha,\beta;\gamma,\delta)$ then $D(\alpha,\beta;\gamma,\epsilon)$ or $D(\alpha,\epsilon;\gamma,\delta)$,

D4: if α,β,γ are all distinct then there is $\delta \in (\Omega \setminus \{\alpha,\beta,\gamma\})$ such that $D(\alpha,\beta;\gamma,\delta)$.

A quaternary relation D defined on the ends of a tree T by $D(\alpha,\beta;\gamma,\delta)$ if and only if the lines having ends α,β and γ,δ respectively do not intersect will automatically satisfy D1, D2, D3. If, for instance, our tree T is transitive then this relation satisfies also D4. If we could define a D-relation on a subset of ends of a graph Γ then the results of Adeleke and Neumann would imply that this subset of ends of Γ could be represented as a set of ends of a tree. This would give us another way to construct a tree from the graph Γ.

Assume that $E \subseteq \mathcal{B}\Gamma$ is an undirected tree set. Define a quaternary relation D_E on $\mathcal{E}\Gamma$ by

$$D_E(\alpha,\beta;\gamma,\delta) \Leftrightarrow \exists e \in E \text{ with } \alpha,\beta \in e \text{ and } \gamma,\delta \in e^*.$$

Proposition 9.1 *The relation D_E defined above satisfies* D1, D2, D3.

Proof

D1: Follows straight from the definition.

D2: Assume that $D(\alpha,\beta;\gamma,\delta)$ holds and we have $e \in E$ such that $\alpha,\beta \in e$ and $\gamma,\delta \in e^*$. Suppose that $D(\alpha,\gamma;\beta,\delta)$ holds and is witnessed by some element $f \in E$. Then we have $\alpha \in e \cap f$, $\beta \in e \cap f^*$, $\gamma \in e^* \cap f$ and $\delta \in e^* \cap f^*$. But that is impossible, because E is a tree set and therefore one of the intersections above must be empty. Hence $\neg D(\alpha,\gamma;\beta,\delta)$.

D3: Assume $D(\alpha,\beta;\gamma,\delta)$ holds and is witnessed by some $e \in E$. Let ϵ be some end of Γ. Clearly, either $\epsilon \in e$ or $\epsilon \in e^*$. If $\epsilon \in e$ then $D(\alpha,\epsilon;\gamma,\delta)$ and if $\epsilon \in e^*$ then $D(\alpha,\beta;\gamma,\epsilon)$ holds. □

If we restrict this relation D_E to those ends of Γ that actually appear as ends of $T(E)$ then we see that D_E agrees with the relation D, as defined above on the ends of a tree.

Now let $\{E_n\}$ be a chain of undirected tree sets as described in Theorem 7.5. Put $E_\omega := \cup E_n$. We then get a tower of relations $D_{E_1} \subseteq D_{E_2} \subseteq \ldots$, with $D_{E_\omega} = \cup D_{E_n}$.

Proposition 9.2 *Assume Γ is locally finite and transitive. Then the restriction of D_{E_ω} to $\mathcal{FD}\Gamma$ is a D-relation.*

Proof The only thing we need to check is condition D4: the others follow straight from the previous proposition. Let $\alpha,\beta,\gamma \in \mathcal{FD}\Gamma$ be distinct. We need to find $\delta \in \mathcal{FD}\Gamma \setminus \{\alpha,\beta,\gamma\}$ such that there is a set $e_0 \in E_\omega$ containing γ and δ but neither α nor β. Because $\gamma \in \mathcal{FD}\Gamma$, we have a translation g such that $\mathcal{D}(g) = \gamma$. Find $e \in E_\omega$ such that $\gamma \in e$ and $\mathcal{D}(g^{-1}) \notin e$. Then there is a natural number j such that $\alpha, \beta \notin eg^j$. Set $e_0 := eg^j$. It is clear that e_0 must contain some end δ_0 of Γ different from γ. Whereupon we find $f \in \mathcal{B}e_0$ such that $\delta_0 \in f$ but $\gamma \notin f$. Now we apply Theorem 7.2 which tells us that there is a hyperbolic element h of $\mathrm{Aut}(\Gamma)$ such that $\mathcal{D}(g) \in f$. Put $\delta := \mathcal{D}(g)$. Because $\delta \in f$ and none of α, β, γ is in f we have that δ is not equal to any of α, β, γ and $D(\alpha, \beta; \gamma, \delta)$. □

Corollary 9.2 *Let* Γ *be locally finite and transitive. If* $\mathrm{Aut}(\Gamma)$ *is transitive on the ends of* Γ *then* D_{E_ω} *is a D-relation.*

Proof Theorem 7.2 above, gives us that $\mathcal{FD}\Gamma$ is not empty. Therefore $\mathcal{E}\Gamma = \mathcal{FD}\Gamma$. □

Remark In the proof of the above proposition the only time we require the transitivity of Γ is when we apply Theorem 7.2. But, as pointed out earlier, we only need to assume that $\mathrm{Aut}(\Gamma)$ has only finitely many orbits on $V\Gamma$ for Theorem 7.2 to hold. The above proposition and the corollary also hold if we replace transitivity by this weaker condition.

Chapter 10

Fixed point theorems

Through out this chapter Γ is a locally finite, connected graph and H will be a subgroup of $\mathrm{Aut}(\Gamma)$. Consider the following properties:

(a) There is a non-empty finite subgraph of Γ that is invariant under H.

(b) There is an end of Γ fixed by H.

(c) There is a pair of ends of Γ that is invariant under H.

The first 'fixed point theorem' was the following.

Proposition 10.1 [18, Proposition 3.4] *Let T be a tree and H a group acting on T. If H contains no translations then either (a) or (b) hold.*

This result has been generalised to locally finite graphs by W. Woess ([21]). The proof given here is perhaps not simpler than Woess original proof, but it illustrates how the results in Chapter 8 can be used to take care of the combinatorial messiness that results from working with general graphs but not just trees.

Proposition 10.2 [21, Proposition 1] *Suppose* $H \leq \mathrm{Aut}(\Gamma)$ *contains no hyperbolic elements. Then (a) or (b) holds and, furthermore, if (b) holds then there is an unique end that* H *fixes.*

Proof Let T be some undirected structure tree of Γ. We know that no element of H acts like a translation on T, so we can apply Proposition 10.1. If H fixes an end of T or has a finite orbit on ET then we are clearly finished. We can therefore assume, without loss of generality, that for all structure trees T of Γ that H fixes precisely one vertex of T. Furthermore, we can also assume that this vertex has infinite valency. Let $E_1 \subset E_2 \subset \ldots$ be a sequence of undirected tree sets like in Theorem 7.5. In each of the trees $T(E_j)$ there is a unique vertex α_j fixed by H, and since this vertex has infinite valency we get by Lemma 8.3 that there is an non-empty H-invariant set I_j of ends of Γ corresponding to α_j. Clearly the sets I_j form a decreasing sequence. Put $I := \cap I_j$. It follows from Theorem 7.5 that either I is empty or it contains precisely one element. As explained in Section 7.1, the set $\mathcal{E}\Gamma$ is a compact topological space. The ends in I_j are precisely those that are contained in every element of E_j that points towards α_j. If $e \in E_j$ then the set of ends contained in e_j is closed in the topology on $\mathcal{E}\Gamma$. Hence all the sets I_j are closed, because they are intersections of closed sets. So by a standard fact about compact sets I is non-empty. □

We also have the following theorem (cf. [13]), which gives a kind of a 'Tits alternative' for groups acting on trees.

Theorem 10.1 *Let T be a tree and $H \leq \mathrm{Aut}(T)$. If H satisfies none of (a), (b), (c) then H contains a free group on two generators.*

Now we are able to use the results in Chapter 8 to prove the following generalisation of this theorem.

Theorem 10.2 *Let Γ be a connected, locally finite graph and $H \leq \mathrm{Aut}(\Gamma)$. If H satisfies none of (a), (b), (c) then H contains a free group on two generators.*

Proof By Proposition 10.2 we can assume that H contains a hyperbolic element g. Let $T := T(E)$ be a structure tree of Γ such that g acts as a translation on T. Because we have this element g acting as a translation on T it is impossible that the action of H on T satisfies (a). If (b) or (c) were to hold for the action of H on T then they would also hold for the action of H on Γ, because the ends of T correspond to ends of Γ. Thus none of (a), (b), (c) holds for the action of H on T. Therefore, by Theorem 10.1, we get that H contains a free group on two generators. □

I have been informed by W. Woess that H. A. Jung also has a proof of the above theorem and that his proof works in more generality, for example for all countable graphs. By using arguments similar to the ones in the proof of the above theorem one is able to deduce the following theorem of W. Woess from the corresponding result for trees. W. Woess, in his paper ([21]), does indeed suggest that this should be possible.

Theorem 10.3 [21, Theorem 1] *Let $H \leq \mathrm{Aut}(\Gamma)$. If H acts amenably on Γ then one of (a), (b), (c) holds.*

First the definition of an amenable group action.

Definition 10.1 *A group G acts amenably on a set Ω if there is a non-negative function μ defined on the power set of Ω such that*

(i) $\mu(\Omega) = 1$,

(ii) *μ is finitely additive,*

(iii) *for every $\Delta \subset \Omega$ and every $g \in G$ we have $\mu(\Delta g) = \mu(\Delta)$.*

We call μ a G-invariant measure on Ω.

Proof of Theorem 10.3 Let μ be a H-invariant measure on $V\Gamma$. Let $T := T(E)$ be some undirected structure tree of Γ. Put

$$\phi : V\Gamma \to VT;\ \alpha \mapsto \alpha|E.$$

Define a function ν on the power set of VT by

$$\nu(\Delta) := \mu(\phi^{-1}(\Delta)),$$

for $\Delta \subset VT$. Clearly ν satisfies all the conditions in the definition of an H-invariant measure. Thus H acts amenably on the tree T. By Proposition 10.2 we can assume that H contains a hyperbolic element h and by Corollary 8.1 we can assume that h acts like a translation on T. Now we see that it suffices to prove the theorem in the case where Γ is a tree.

The following argument is based on [21] and proves the theorem if Γ is a tree (not necessarily locally finite), and thus completes the prove of the general case. We can clearly assume that H contains a translation h. Let C

be a subtree of T such that $|\partial C| = 1$. If neither $\mathcal{D}(h)$ nor $\mathcal{D}(h^{-1})$ is contained in C then we have that the sets Ch^i are all pairwise disjoint, and thus we must have $\mu(C) = 0$. But then we also get that $\mu(C) \neq 0$ if and only if C contains either $\mathcal{D}(h)$ or $\mathcal{D}(h^{-1})$. Now we see that either (b) or (c) must hold.
$\square$

Chapter 11

End-transitive graphs

We say that a graph Γ is end-transitive if $\mathrm{Aut}(\Gamma)$ acts transitively on the ends of Γ. The aim of this chapter is to classify the connected locally finite end-transitive graphs that have infinitely many ends. This is an answer to Problem 2 in [23].

Before we can state our theorem we need some additional notation. Let G be a group acting on some set Ω, and let $\alpha \in \Omega$. The orbits of G_α on Ω are called suborbits. The suborbit βG_α corresponds in a natural way to the orbit $(\alpha, \beta)G$ in $\Omega \times \Omega$. In the case of transitive groups this gives us a 1-1 correspondance between the G_α-orbits, for some fixed $\alpha \in \Omega$, and the G-orbits on $\Omega \times \Omega$. Orbits of G on $\Omega \times \Omega$ are called orbitals. We call a graph Δ a poly-orbital graph of Ω if $V\Delta$ is a union of G-orbits on Ω and $E\Delta$ is a union of orbitals (tacitly we exclude graphs where all the edges are of the form (α, α)).

Let T be a tree. There is a natural bipartition of T. Define $\mathrm{Aut}^+(T)$ as the subgroup of $\mathrm{Aut}(T)$ that stabilises the blocks of this bipartition. We say that T is semi-regular if the vertices in each of the blocks of that bipartition all have the same valency.

Theorem 11.1 *Let* Γ *be a connected locally finite graph with infinitely many ends. Suppose* Γ *is end-transitive. Then there is an* $\mathrm{Aut}(\Gamma)$*-congruence* π *with finite classes such that* Γ/π *is isomorphic to a connected component of a poly-orbital graph arising from the action of* $\mathrm{Aut}^+(T)$ *on* VT*, where* T *is a semi-regular tree, or there is a non-empty finite subset of* $V\Gamma$ *invariant under* $\mathrm{Aut}(\Gamma)$*. In particular, if there is no non-empty* $\mathrm{Aut}(\Gamma)$*-invariant finite set of vertices then* $\mathrm{Aut}(\Gamma)$ *acts with finitely many orbits on* $V\Gamma$.

Let us fix some notation. Set $G := \mathrm{Aut}(\Gamma)$. If $\mathrm{Aut}(\Gamma)$ contains no hyperbolic elements then by Proposition 10.2 then there is a non-empty finite $\mathrm{Aut}(\Gamma)$-invariant subset of $V\Gamma$. We may thus assume that $\mathrm{Aut}(\Gamma)$ contains a hyperbolic element. Let $e_0 \in \mathcal{B}\Gamma$ be a thin element, both infinite and co-infinite, such that $E := \{e_0\mathrm{Aut}(\Gamma), e_0^*\mathrm{Aut}(\Gamma)\}$ is a tree set. Put $T = T(E)$. We can assume that some element of $\mathrm{Aut}(\Gamma)$ acts like a translation on T. We clearly have a 1-1 correspondence between the ends of T and the ends of Γ, and G acts transitively on $\mathcal{E}T$. We have a map

$$\phi : V\Gamma \to VT;\ \alpha \mapsto \alpha|E.$$

The fibres of ϕ give us a G-congruence π on $V\Gamma$.

Lemma 11.1 *The fibres of* ϕ *are finite and* T *is locally finite.*

Proof This is an easy application of Lemma 8.3. Because no vertex in T corresponds to an end of Γ then for all vertices α in T we know that α has finite valency and $\phi^{-1}(\alpha)$ is finite. □

Consider the graph Γ/π. Define $P : \Gamma \to \Gamma/\pi$ to be the natural projection. Clearly there is a 1-1 correspondence between the ends of Γ and the ends of Γ/π. Because G has at most two orbits on the vertices of T we see that G has at most two orbits on the vertices of Γ/π. Because the π-classes are finite that in turns implies that G has only finitely many orbits on $V\Gamma$. One can also note that the set $EP := \{eP \mid e \in E\}$ is a tree set and that $T(EP) = T(E)$. From now one we are only interested in the graph Γ/π and to simplify the exposition we will be replacing Γ with Γ/π and G will be any closed end-transitive subgroup of $\mathrm{Aut}(\Gamma)$. (We cannot be sure that EP is invariant under $\mathrm{Aut}(\Gamma/\pi)$ but the projection of $\mathrm{Aut}(\Gamma)$ on $\mathrm{Aut}(\Gamma/\pi)$ is end-transitive and closed.) The reader will easily convince himself that this makes no difference in what follows. It is thus possible to assume that ϕ is injective. We will be identifying vertices in Γ with their ϕ-images in T.

The next lemma is based on Proposition 1 in [12]. For completeness a full proof of the lemma will be included, this proof follows the proof given by Nebbia in [12].

Lemma 11.2 *The stabiliser of a vertex $\alpha \in V\Gamma$ acts transitively on the ends of Γ.*

From now on most of the action will take place inside T. We will represent the set of ends as the set of half-lines that originate at α. We can define a metric t on $\mathcal{E}T$ by the formula

$$t(L_1, L_2) := 2^{-k},$$

where L_1 and L_2 are two half-lines starting at α and $k = |L_1 \cap L_2| - 1$. One easily sees that $\mathcal{E}T$ with this metric is a complete metric space.

Proof of Lemma 11.2 We first prove that the orbits of G_α on $\mathcal{E}T$ are all closed. Let Δ be an orbit of G_α on $\mathcal{E}T$. Assume $\{\delta_i\}_{i\in\mathbf{N}}$ is a sequence in Δ that converges to an end δ. Now find a sequence $\{g_i\}_{i\in\mathbf{N}}$ such that $\delta_i = \delta_1 g_i$. Because Γ is a locally finite graph we know that G_α is compact in the permutation group topology induced on G_α by the action on Γ. Thus there is a convergent subsequence $\{g_{i_j}\}_{j\in\mathbf{N}}$, which converges to some element g in G_α. But this subsequence must also converge to g in the permutation group topology induced by the action on T. Whence we see that $\delta_1 g = \delta$. Therefore Δ is closed.

Above we showed that $\mathcal{E}T$ is a complete metric space, then automatically it is also a Baire space. Let $\{g_i\}_{i\in\mathbf{N}}$ be a complete set of coset representatives for G_α in G. Suppose Δ is a G_α orbit on $\mathcal{E}T$. Then $\mathcal{E}T = \cup_{i\in\mathbf{N}}\Delta g_i$. By Baire's Theorem at least one of the sets Δg_i (therefore all) must have non-empty interior. Because Δ is a G_α orbit we can conclude that Δ is open. Whence all G_α orbits on $\mathcal{E}T$ are open.

Let $g \in G$ be a translation along some line L in T. Because G acts transitively on the undirected edges of T we can assume that $\alpha \in L$. Say $L = \{\alpha_i\}_{i\in\mathbf{Z}}$ with $\alpha_0 = \alpha$. Let ϵ_1 be the end represented by $\{\alpha_i\}_{i\geq 0}$ and ϵ_2 be the end represented by $\{\alpha_i\}_{i\leq 0}$. For $\beta \in V\Gamma$ define $E(\beta)$ as the set of ends that are represented by half-lines that go through β. These sets form a basis for the topology on $\mathcal{E}T$. Because $\epsilon_1 G_\alpha$ and $\epsilon_2 G_\alpha$ are open we can find a number j such that $E(\alpha_j) \subseteq \epsilon_1 G_\alpha$ and $E(\alpha_{-j}) \subseteq \epsilon_2 G_\alpha$. Now we see by using the translation g that G_α must act transitively on the points that are adjacent to α in T and that G_α must act transitively on $E(\alpha_1)$. Hence G_α acts transitively on $\mathcal{E}T$. $\square$

For simplicity let us assume that G acts transitively on Γ, the case where G has two orbits on Γ is similar. Let α be a vertex in Γ. Let $C_i(\alpha)$ denote the set of vertices in T that are at distance i (in T) from α. The real point of the last lemma is that it implies that G_α acts transitively on $C_i(\alpha)$ for all i. Let Φ be the set of vertices in Γ that are adjacent to α in Γ. Let F be the union of all the orbitals $(\alpha,\beta)G$ and $(\beta,\alpha)G$ where $\beta \in \Phi$. We see now that $E\Gamma = F$ and thus we have proved the theorem.

Remarks 1. Conversely, one can note that the connected components of the poly-orbital graphs one gets from semi-regular trees are all end-transitive.
2. A graph Γ is said to be distance transitive if for all vertices $\alpha, \beta, \gamma, \delta$ with $d(\alpha,\beta) = d(\gamma,\delta)$ we can find $g \in \mathrm{Aut}(\Gamma)$ such that $\alpha g = \gamma$ and $\beta g = \delta$. Clearly a locally finite regular tree is distance transitive. We get another example of a distance transitive graph by taking a locally finite semi-regular tree T and looking at the orbital graph we get by saying that two vertices are adjacent if and only if their distance in T is 2; the connected components of this orbital graph are distance transitive. Macpherson ([11]) has proved that every connected locally finite infinite distance transitive graph is of one of the above types. A simplified version of Macpherson's argument can be found in Chapter II, §3 in [2]. With very little addition the preceding argument will also give a proof of Macpherson's theorem.

Chapter 12

Graphs with a transitive amenable group action and infinitely many ends

Let Γ be a locally finite connected graph with infinitely many ends. Put $G := \mathrm{Aut}(\Gamma)$. We are interested in gaining information about the structure of Γ if we know that there is an end ω such that G_ω (the stabiliser of ω in G) acts transitively on $V\Gamma$. It is shown in [21] that this is equivalent to there being a group with a transitive amenable action on Γ. These graphs have also cropped up in some work of W. Woess on random walks on graphs ([22] and [17]). In [23] W. Woess asks for a classification of such graphs and conjectures that they "look like" trees. More precisely, he conjectures that a graph with these properties is quasi-isometric to a tree. Two graphs Γ_1 and Γ_2 are said to be quasi-isometric if there are maps $\phi : V\Gamma_1 \to V\Gamma_2$ and $\psi : V\Gamma_2 \to V\Gamma_1$ and constants k_1, k_2 such that for all $\alpha_1, \alpha_2 \in V\Gamma_1$ and all $\beta_1, \beta_2 \in V\Gamma_2$ the following hold

(a) $d_{\Gamma_2}(\phi(\alpha_1), \phi(\alpha_2)) \leq k_1 d_{\Gamma_1}(\alpha_1, \alpha_2) + k_2$,

(b) $d_{\Gamma_1}(\psi(\beta_1), \psi(\beta_2)) \leq k_1 d_{\Gamma_2}(\beta_1, \beta_2) + k_2$,

(c) $d_{\Gamma_1}(\psi\phi(\alpha_1), \alpha_1) \leq k_2$,

(d) $d_{\Gamma_2}(\phi\psi(\beta_1), \beta_1) \leq k_2$.

We prove:

Theorem 12.1 *Let Γ be a locally finite connected graph with infinitely many ends. Put $G := \mathrm{Aut}(\Gamma)$. Assume there is an end ω of Γ such that G_ω acts transitively on $V\Gamma$. Then Γ is quasi-isometric to a tree.*

Theorem 12.2 *Let Γ and G be as in the above theorem. Then G acts transitively on $\mathcal{E}\Gamma \setminus \{\omega\}$, where $\mathcal{E}\Gamma$ is the set of ends of Γ. In particular, if ω is not fixed by G then G acts transitively on $\mathcal{E}\Gamma$*

For an example of a graph Γ where the end ω is fixed by G see Example 2 in [17]. Our results imply that all such graphs must arise in a similar way from semi-regular trees. But it is difficult to see how this knowledge would enable us to get a more precise description. In the first section we look more closely at quasi-isometries, in the second section we prove the Theorems 12.1 and 12.2.

12.1 Quasi-isometries

Let Γ_1 and Γ_2 be locally finite connected graphs. Assume that the maps $\phi : V\Gamma_1 \to V\Gamma_2$ and $\psi : V\Gamma_2 \to V\Gamma_1$ satisfy conditions (a) to (d) above for some constants k_1 and k_2.

Proposition 12.1 *There is a natural bijection* $\Phi : \mathcal{E}\Gamma_1 \to \mathcal{E}\Gamma_2$.

To prove this it is convenient to introduce some notation which is taken from [16].

Definition 12.1 *Let* Γ *be a locally finite connected graph. If* Δ *is a finite subset of* $V\Gamma$ *and* ϵ *is an end of* Γ *then let* $C(\Delta, \epsilon)$ *denote the component of* $\Gamma \setminus \Delta$ *that contains* ϵ. *If* $\{\Delta_i\}_{i \geq 1}$ *is a sequence of finite subsets of* $V\Gamma$ *such that*

$$C(\Delta_i, \epsilon) \supseteq \Delta_{i+1} \cup C(\Delta_{i+1}, \epsilon),$$

then we say that $\{\Delta_i\}_{i \geq 1}$ *contracts towards* ϵ.

Clearly every end can be described by sequence of finite sets that contracts towards it. And if we have a sequence that contracts towards an end then that end is uniquely determined.

Proof of Proposition 12.1 Let $L := \{\alpha_1, \alpha_2, \ldots\}$ be some half-line in Γ_1. The set $\phi(L)$ need not be a half-line, but if S is a minimal spanning tree of $\phi(L)$ then S must contain some half-line Λ. Denote by ϵ_L the end of Γ_2 that Λ belongs to. Take a sequence $\{\Delta_i\}_{i \geq 1}$ of finite subsets of $V\Gamma$ that contracts towards ϵ. We clearly have that $C(\Delta_i, \epsilon) \cap \phi(L)$ is infinite for all i. The next step is to show that ϵ_L does not depend on the choice of S and Λ. In order to do so it is enough to prove that $\phi(L) \setminus C(\Delta_i, \epsilon)$ is finite for all i. We know that $S \cap C(\Delta_i, \epsilon)$ is infinite. Both Γ_1 and Γ_2 are locally finite, so, by condition (c) no fiber of ϕ is infinite. Again, because Γ_2 is locally finite, we know that the set of vertices in Γ_2 in distance less than $k_1 + k_2$ from Δ_i is finite.

Thus there is a number j_i such that if $j \geq j_i$ then $d_{\Gamma_2}(\phi(\alpha_j), \Delta_i) > k_1 + k_2$ and $\phi(\alpha_{j_i}) \in \Delta_i$. Now $d_{\Gamma_2}(\phi(\alpha_{j_i}), \phi(\alpha_{j_i+1})) \leq k_1 + k_2$ (by (a)), and thus $\alpha_{j_i+1} \in C(\Delta_i, \epsilon)$. By induction we get that $\phi(\{\alpha_{j_i}, \alpha_{j_i+1}, \ldots\}) \subseteq C(\Delta_i, \epsilon)$.

Suppose that L_1 and L_2 are two half-lines in Γ_1 that belong to the same end. Two half-lines L_1 and L_2 belong to the same end if and only if there is a third half-line L_3 that contains infinitely many vertices from both L_1 and L_2. From the above argument we see that $\epsilon_{L_1} = \epsilon_{L_3} = \epsilon_{L_2}$. Thus ϵ_L depends only on the end that L belongs to but not on the particular half-line L. Now we have a well defined function

$$\Phi : \mathcal{E}\Gamma_1 \to \mathcal{E}\Gamma_2,$$

which maps an end ϵ to the end ϵ_L where L is some half-line in ϵ. One can now define

$$\Psi : \mathcal{E}\Gamma_2 \to \mathcal{E}\Gamma_1$$

in the same way.

By using similar ideas to those above and conditions (c) and (d), we get that $\Psi\Phi = id_{\mathcal{E}\Gamma_1}$ and $\Phi\Psi = id_{\mathcal{E}\Gamma_2}$. □

Let Γ be a locally finite, transitive, connected graph with infinitely many ends. Put $G := \mathrm{Aut}(\Gamma)$. Take an element $e_0 \in \mathcal{B}\Gamma$ that is both infinite and co-infinite and such that $E := \{e_0 G, e_0^* G\}$ is a tree set. Set $T := T(E)$. Define

$$\phi : V\Gamma \to VT;\ \alpha \mapsto \alpha | E.$$

Lemma 12.1 *The following are equivalent:*

(i) *The tree T is locally finite and the fibers of ϕ are all finite;*

(ii) *Γ is quasi-isometric to T.*

Proof First we show that (i) implies (ii). We know that either $\mathrm{Im}\phi = VT$ or $\mathrm{Im}\phi$ is one of the natural bi-partite blocks of T. For each vertex $\alpha \in \mathrm{Im}\phi$ choose a vertex $s(\alpha)$ from $\phi^{-1}(\alpha)$. If $\alpha \in VT$ is in $\mathrm{Im}\phi$ then let $t(\alpha) = \alpha$; otherwise let $t(\alpha)$ be some vertex in T adjacent to α. Define a map $\psi : VT \to V\Gamma$ by

$$\psi(\alpha) := \begin{cases} s(\alpha) & \text{if } \alpha \in \mathrm{Im}\phi, \\ s(t(\alpha)) & \text{if } \alpha \notin \mathrm{Im}\phi. \end{cases}$$

Now we just go through the conditions in the definition of a quasi-isometry one-by-one, showing that for each condition there is a choice of the constants k_1 and k_2 so that the condition is satisfied. Let $\alpha_1, \alpha_2 \in V\Gamma$ and $\beta_1, \beta_2 \in VT$.

(**a**) Because Γ is transitive and locally finite we know that G has only finitely many orbits on adjacent vertices. It follows now from $\phi(\alpha g) = \phi(\alpha)g$ that

$$d_T(\phi(\alpha_1), \phi(\alpha_2)) = d_T(\phi(\alpha_1 g), \phi(\alpha_2 g)),$$

for all $\alpha_1, \alpha_2 \in V\Gamma$ and $g \in G$. Therefore we can make the following definition

$$k_1 := \max\{d_T(\phi(\alpha_1), \phi(\alpha_2)) \mid d(\alpha_1, \alpha_2) = 1\}.$$

Then, by the triangle inequality, we get that for all $\alpha_1, \alpha_2 \in V\Gamma$ the following holds

$$d_T(\phi(\alpha_1), \phi(\alpha_2)) \leq k_1 d_\Gamma(\alpha_1, \alpha_2).$$

(**b**) Let us assume that ϕ is not onto. Define

$$k_1 := \max\{d_\Gamma(\nu, \mu) \mid \nu \in \phi^{-1}(\beta_1), \mu \in \phi^{-1}(\beta_2) \text{ and } d_T(\beta_1, \beta_2) = 2\}.$$

Here we use that T is locally finite and that the fibres of ϕ all have the same finite diameter. If $\beta_1, \beta_2 \in \mathrm{Im}\phi$ then we get

$$d_\Gamma(\psi(\beta_1), \psi(\beta_2)) \leq k_1 d_T(\beta_1, \beta_2).$$

By the way we defined the function t we have

$$d_T(t(\beta_1), t(\beta_2)) \leq d_T(\beta_1, \beta_2) + 2.$$

Whence, in general, we get

$$d_\Gamma(\psi(\beta_1), \psi(\beta_2)) \leq k_1 d_T(\beta_1, \beta_2) + 2k_1.$$

(c) Let $\gamma \in \mathrm{Im}\phi$ and set

$$k_2 := \mathrm{diam}(\phi^{-1}(\gamma)).$$

Now, $\psi\phi(\alpha_1)$ is in the same ϕ-fibre as α. Because Γ is transitive, we then have

$$d_\Gamma(\psi\phi(\alpha_1), \alpha_1) \leq k_2.$$

(d) If $\beta_1 \in \mathrm{Im}\phi$ then $\phi\psi(\beta_1) = \beta_1$. If $\beta_1 \notin \mathrm{Im}\phi$ then $\phi\psi(\beta_1) = \phi\psi(t(\beta_1)) = t(\beta_1)$, and $d_T(\beta_1, t(\beta_1)) = 1$. Thus

$$d_T(\phi\psi(\beta_1), \beta_1) \leq 1.$$

Now we show that (ii) implies (i). If Γ and T are quasi-isometric then there is a natural bijection between $\mathcal{E}\Gamma$ and $\mathcal{E}T$. Therefore every end of Γ appears as an end T. It follows now from Lemma 8.3 that T must be locally finite and the fibers of ϕ are all finite. □

Once again we can note that the condition that Γ is transitive is unnecessarily strong. The reader can convince herself that the above lemma holds if instead we assume that $\mathrm{Aut}(\Gamma)$ has finitely many orbits on $V\Gamma$.

Under the above conditions we get that by Proposition 12.1 there is a natural bijection between the set $\mathcal{E}T$ and the set $\mathcal{E}\Gamma$. The reader can easily check that the injection from $\mathcal{E}T$ into $\mathcal{E}\Gamma$ given in Lemma 8.1 is indeed a bijection.

12.2 The results

For the rest of this chapter we will assume that there is an end ω of Γ such that G_ω acts transitively on $V\Gamma$. By Lemma 8.2, we know that ω appears as an end of T.

Lemma 12.2 *The fibres of ϕ are finite.*

Before going on to prove the lemma, we must first state a result of M. E. Watkins, [19].

Definition 12.2 *If L is a line and $\alpha, \beta \in VL$ then let $[\alpha, \beta]_L$ denote the path (in L) between α and β. If ϵ is an end of L then $[\alpha, \epsilon)_L$ denotes the half-line in L that starts at α and belongs to ϵ. A line L in Γ is called an axis if for all $\alpha, \beta \in VL$ we have $d_\Gamma(\alpha, \beta) = d_L(\alpha, \beta)$.*

Theorem 12.3 [19, Theorem 3.3] *Let Γ be a locally finite connected graph with more than one end. If ϵ_1 and ϵ_2 are two distinct ends of Γ then there is an axis L in Γ such that the ends of L are ϵ_1 and ϵ_2.*

Proof of Lemma 12.2 Let $\alpha \in V\Gamma$. We aim to show that there is a number n_0 such that if $d_\Gamma(\alpha, \beta) \geq n_0$ then there is an element $e \in E$ with $\alpha \in e$ but $\beta \notin e$. Because Γ is locally finite, it suffices to show that the fibres of ϕ are finite. For $e \in E$ define

$$\partial_v e := \{\gamma \in V\Gamma \mid \gamma \notin e \text{ but } \gamma \text{ adjacent to a vertex in } e\}.$$

Since G_ω is transitive we can find $e \in E$ such that $\alpha \in \partial_v e$ and $\omega \in e$. An application of Theorem 7.2 gives us a hyperbolic element $g \in G_\omega$ with $\mathcal{D}(g^{-1}) = \omega$ and $\epsilon := \mathcal{D}(g) \in e^*$. Replacing g with a suitable power of g we can assume that $\partial_v e \subset eg$. By the above theorem of Watkins we can find an axis L in Γ with ends ϵ and ω.

Claim 1 Assume $\alpha \in \partial_v e \cap L$. Then there is a number k_0 such that if $\beta \in L \cap e^*$ and $d_\Gamma(\alpha, \beta) \geq k_0$ then $\alpha \in eg$ and $\beta \in e^* g$.

We can find $\gamma \in \partial_v eg \cap L$ such that $[\gamma, \epsilon)_L \subset e^* g$. If $d_\Gamma(\alpha, \beta) \geq d_\Gamma(\alpha, \gamma)$ then we have that $\alpha \in eg$ but $\beta \in e^* g$.

Claim 2 There is a number l_0 such that if $d_\Gamma(\beta, \partial_v e) \geq l_0$ and $h \in G_\omega$ such that $\beta \in Lh$ then $\epsilon h \in e^*$.

Put

$$l_0 := \frac{1}{2}\text{diam}(\partial_v e) + 2.$$

Take β and h as above. Suppose $\epsilon h \in e$. Then we could find $\gamma_1, \gamma_2 \in \partial_v e^*$ such that $[\gamma_1, \omega)_L \subset e$ and $[\gamma_2, \epsilon)_L \subset e$. Of course

$$d_\Gamma(\gamma_1, \gamma_2) \leq \text{diam}(\partial \delta_v e^*) \leq \text{diam}(\partial_v e) + 2,$$

but, at the same time we have

$$d_\Gamma(\gamma_1,\gamma_2) = d_{Lh}(\gamma_1,\gamma_2) \geq 2d_\Gamma(\beta,\partial_v e).$$

Clearly we can now choose n_0 such that if $d(\alpha,\beta) \geq n_0$ with $\beta \in e^*$ then $d(\beta,\partial_v e) \geq \max\{l_0,k_0\}$. Take $h \in G_\omega$ such that β lies on Lh. Now we can apply Claim 1 with L replaced by Lh and g replaced by $h^{-1}gh$. Note that the value of k_0 is not changed when g is replaced by $h^{-1}gh$. Thus the lemma is now proved. □

Lemma 12.3 *The tree T is locally finite.*

Proof Let $\alpha \in VT$. Define $u(\alpha)$ as the vertex in T adjacent to α such that the half-line originating at α and belonging to ω goes through $u(\alpha)$. If β_1,β_2 are in $\mathrm{Im}\phi$ with $d(\alpha,\beta_1) = d(\alpha,\beta_2)$ and $d(u(\alpha),\beta_1) = d(u(\alpha),\beta_2) = d(\alpha,\beta_1) + 1$ then every element in G_ω moving β_1 to β_2 must fix α. Let $U(\alpha)$ be the set of vertices γ in T such that the α,γ geodesic passes through $u(\alpha)$. Because G_ω acts transitively on $\mathrm{Im}\phi$ we can conclude that $G_{\alpha,\omega}$ acts transitively on the set

$$\{\beta \in T \mid d(\alpha,\beta) = i\} \setminus U(\alpha),$$

for all $i \geq 1$.

The group $G_{\alpha,\omega}$ fixes the half-line starting at α and going into ω. Thus $G_{\alpha,\omega}$ must fix some vertex belonging to $\mathrm{Im}\phi$, and therefore $G_{\alpha,\omega}$ must leave some non-empty finite set of $V\Gamma$ invariant. Hence all orbits on $V\Gamma$, and therefore also on VT, are finite. Now we see that T is locally finite. □

Again the reader is invited to convince herself that here too the condition that Γ is transitive can be replaced by the weaker condition that G acts on $V\Gamma$ with only finitely many orbits.

Proof of Theorem 12.1 Theorem 12.1 follows straight from Lemmas 12.1, 12.2 and 12.3. □

Proof of Theorem 12.2 The argument given in the proof of Lemma 12.3 indeed proves more. The group G is closed in the permutation group topology given by the action of G on VT. Thus we see that G_ω must act transitively on $\mathcal{E}T \setminus \{\omega\}$. The result then follows from Proposition 12.1. □

If the end ω is not fixed by G then the classification given in Theorem 11.1 applies. We can also give the following characterisation in the same spirit as Theorem 11.1.

Theorem 12.4 *Let Γ be a locally finite connected graph with infinitely many ends. Put $G := \mathrm{Aut}(\Gamma)$. Assume there is an end ω of Γ such that G_ω acts transitively on $V\Gamma$. Then there is a G-congruence π with finite classes on $V\Gamma$ such that Γ/π is a poly-orbital graph arising from the action of a transitive group H on a locally finite distance transitive graph Γ_0. We can also find an end ω_0 in Γ_0 such that H_{ω_0} is transitive on $V\Gamma_0$.*

Remarks 1. Theorem 12.4 can also be formulated in terms of group actions on semi-regular trees. The problem of getting a more precise classification thus comes down to classifying those transitive groups acting on a distance transitive graphs and fixing an end. This seems to be a very hard problem.

2. If we instead of insisting that G_ω acts transitively (or with finitely many orbits) on $V\Gamma$ suppose that G_ω acts transitively on $\mathcal{E}\Gamma \setminus \{\omega\}$ then we can use the ideas in Chapter 11 to show that G_ω has at most finitely many orbits on $V\Gamma$.

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