C^∞-algebraic geometry with corners

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Dedication

This thesis is dedicated to my Grandmother and in memory of my Gran — for their endless generosity, support and encouragement.

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Abstract

C^∞-schemes are a generalisation of manifolds that have nice properties such as the existence of fibre products. C^∞-schemes have been used as a model for synthetic differential geometry, as in Dubuc [21], Kock [55], and Moerdijk and Reyes [72], and for defining derived differential geometry as in Lurie [62, §4.5], and Spivak [84].

Manifolds with corners are a generalisation of manifolds locally modelled on $[0,\infty)^k \times \mathbb{R}^{n-k}$, and their smooth maps behave well with respect to the corners as in Melrose [68]. In particular, Joyce [47] describes a corner functor from the category of manifolds with corners to the category of 'interior' manifolds with corners with mixed dimension.

C^∞-algebraic geometry with corners is the study of C^∞-rings and C^∞-schemes with corners, which we define in this thesis. We define (local/interior/firm) C^∞-rings with corners, and study categorical properties such as the existence of limits and colimits using various adjoint functors. We describe a spectrum functor from C^∞-rings with corners to local C^∞-ringed spaces with corners, and show this a right adjoint to a global sections functor. We define C^∞-schemes with corners using this spectrum functor.

We show there is a full and faithful embedding of the category of manifolds with corners into the category of firm C^∞-schemes with corners, and that fibre products of firm C^∞-schemes with corners exist. We show that manifolds with corners are affine under geometric conditions. We define (b-)cotangent sheaves of C^∞-schemes with corners and show they correspond to the (b-)cotangent bundles of manifolds with corners of Joyce [47].

We describe the categories of interior local C^∞-ringed spaces with corners and interior firm C^∞-schemes with corners. We construct corner functors for both of these categories, which are right adjoint to the inclusion of these interior spaces/schemes into the non-interior ones. We show that these corner functors correspond to the corner functor for manifolds with corners.

We expect applications of this work in defining derived spaces with corners in derived differential geometry, and we explore the connections of this work to log geometry and the positive log differentiable spaces of Gillam and Molcho [28].

Statement of Originality

I declare that the work contained in this thesis is, to the best of my knowledge, original and my own work, unless indicated otherwise. I declare that the work contained in this thesis has not been submitted towards any other degree or award at this institution or at any other institution.

Chapter 4 is based on joint work with Professor Dominic Joyce.

Kelli L. Francis-Staite
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Contents

Dedication **i**

Acknowledgements **ii**

Abstract **v**

Statement of Originality **vi**

1 Introduction **1**

1.1 Motivation 2

1.1.1 The category of manifolds 2

1.1.2 C^∞-rings and C^∞-schemes 3

1.1.3 Other generalisations of the category of manifolds 5

1.1.4 Derived geometry 6

1.1.5 Manifolds with corners 7

1.1.6 Motivations from symplectic geometry 8

1.2 What is in this thesis 9

1.3 Summary of main results 11

1.3.1 C^∞-rings and C^∞-schemes with corners 11

1.3.2 Finite limits 12

1.3.3 Embedding manifolds (with corners) 13

1.3.4 Corner functors 13

1.4 Future work and applications of C^∞-algebraic geometry with corners 14

2 Background on C^∞-rings and C^∞-schemes **16**

2.1 Two definitions of C^∞-ring 17

2.2 Modules and cotangent modules of C^∞-rings 26

2.3 Sheaves on topological spaces 28

2.4 C^∞-ringed spaces and C^∞-schemes 32
2.4.1 Products of C^∞-schemes 39
2.5 Sheaves of $\mathcal{O}_X$-modules and cotangent modules 43

3 Background on manifolds with (g-)corners 45
3.1 Monoids and the local model 45
3.2 Smooth maps and manifolds with (g-)corners 49
3.3 Boundaries and corners of manifolds with (g-)corners 53
3.4 Tangent bundles and b-tangent bundles 56

4 C^∞-rings with corners 60
4.1 Categorical pre C^∞-rings with corners 60
4.2 Pre C^∞-rings with corners 63
4.3 C^∞-rings with corners 73
4.4 Free C^∞-rings with corners, generators and relations 82
4.5 Special classes of C^∞-rings with corners 88
4.6 Local C^∞-rings with corners, and localisation 93
4.7 Modules, and (b-)cotangent modules 103

5 C^∞-schemes with corners 116
5.1 C^∞-ringed spaces with corners 116
5.1.1 Limits and colimits 120
5.2 Spectrum functor 129
5.3 Semi-complete C^∞-rings with corners 134
5.4 C^∞-schemes with corners 141
5.4.1 Limits and colimits 145
5.5 Relation to manifolds with corners 152
5.6 Sheaves of $\mathcal{O}_X$-modules and cotangent modules 158
5.7 Corner functor for $\mathbf{LC^\infty RS^c}$ 160
5.7.1 Boundary 165
5.8 Corner functor for $\mathbf{C^\infty Sch^c_{fi}}$ 166
5.8.1 Boundary 180
5.9 Log geometry and log schemes 182
5.9.1 Comparison to C^∞-algebraic geometry 186

A Additional Material 193
A.1 Fibre products of manifolds 193

Bibliography **196**

Glossary **203**

Chapter 1

Introduction

Algebraic Geometry was revolutionised in the 1960's when Alexander Grothendieck introduced the concept of a 'scheme' and encouraged the use of category theory to study these objects. Schemes generalise 'varieties', which are the solution sets of polynomial equations. While both concepts (locally) correspond to an algebraic object called a 'commutative ring', schemes allow more general commutative rings. This means they hold more algebraic information about the polynomials and more specific information about the functions between these solution sets. Schemes reflected Grothendieck's sentiment that we should care less about the objects studied and more about the functions between the objects.

Differential Geometry studies 'nice' solutions to differential equations and the geometry of these solutions, which are the spaces known as manifolds. While schemes generalise varieties using commutative rings, in a similar way manifolds can be generalised by C^∞-schemes using C^∞-rings. This is known as C^∞-algebraic geometry, and was originally suggested by William Lawvere in the late 1960's.

Recently, both Algebraic Geometry and Differential Geometry have been further generalised in Derived Geometry, which is based on the notions of schemes and C^∞-schemes. One of the motivations for Derived Geometry is to study the parameter spaces of solutions to equations known as moduli spaces. Moduli spaces appear prolifically in all areas of Geometry, and in Mathematics more generally. In many cases, these moduli spaces are well behaved and we can deduce many facts about possible solutions from their geometry, topology and algebra. However, poorly behaved moduli spaces are also of importance, and one of the aims of Derived Geometry is to understand these more complicated moduli spaces.

Poor behaviour of moduli spaces includes the appearance of boundaries and corners in their geometry, particularly when considering the process of compactification. To study these moduli spaces in Differential Geometry requires understanding manifolds with boundary and corners, and suggests generalising to their corresponding C^∞-rings and C^∞-schemes *with corners*.

This thesis defines these new concepts of C^∞-rings and C^∞-schemes with corners and studies their properties. We call this the study of C^∞-algebraic geometry with corners, and we aim to provide the foundational material necessary to describe moduli spaces with boundary and corners in Derived Geometry.

We now make all of this more precise. We first introduce and motivate the key concepts, then describe the main results and layout of thesis, and finally describe future work and potential applications of C^∞-algebraic geometry with corners.

1.1 Motivation

We start by motivating why we should generalise manifolds using C^∞-algebraic geometry, then consider manifolds with corners.

1.1.1 The category of manifolds

The category of smooth manifolds with smooth morphisms does not have particularly nice properties. Firstly, the space of morphisms between two manifolds is not a manifold, as it is an infinite dimensional space, however it has many similar properties to a manifold. Secondly, fibre products of manifolds do not always exist. Let us be more precise about fibre products.

Take the smooth morphisms $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ and $g(y) = y^2$ for each $x, y \in \mathbb{R}$. The fibre product of the diagram (1.1.1)

$$\begin{array}{ccccc} \mathbb{R} & & & & \mathbb{R} \\ & \searrow^{f} & & \swarrow^{g} & \\ & & \mathbb{R} & & \end{array} \tag{1.1.1}$$

if it exists, is a manifold X with morphisms $p_1, p_2 : X \to \mathbb{R}$ such that $f \circ p_1 = g \circ p_2$. It satisfies a universal property, that is, if any other space X' comes equipped with morphisms $p_1', p_2' : X' \to \mathbb{R}$ with $f \circ p_1' = g \circ p_2'$, then there is a unique map $X' \to X$ that commutes with all the other morphisms. Intuitively, the universal property makes X into the smallest manifold that has the right morphisms p_1, p_2.

Two pieces of information allow the calculation of fibre products of manifolds, with further details in Appendix A.1. The first is that if the fibre product of manifolds exists, its underlying set is equal to the fibre product of sets, which always exists and is well known. Explicitly, for sets A, B, C with set maps $\alpha : A \to C$, and $\beta : B \to C$, then the fibre product is the following set

$$A \times_C B = \{(a, b) \in A \times B | \alpha(a) = \beta(b)\}.$$

This is a subset of the usual cartesian product of sets $A \times B$. Then the underlying set of X is the set

$$X = \{(x, y) \in \mathbb{R}^2 | x^2 = y^2\},$$

depicted in Figure 1.1.1.

The second important fact (Lemma A.1.3) tells us that in this case the topology of X must be the topology from $\mathbb{R}^2$, so that X must be a submanifold of $\mathbb{R}^2$.

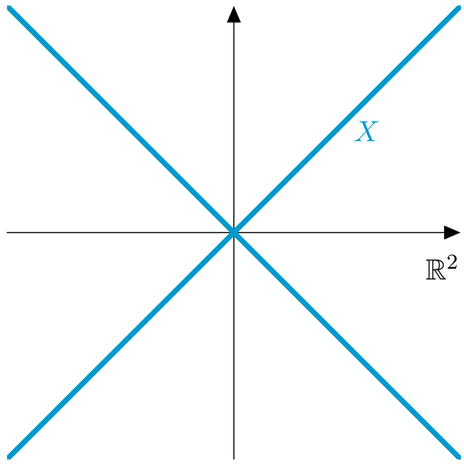


Figure 1.1.1: The fibre product $X \subset \mathbb{R}^2$ as a set.

Yet X can be shown to not be a submanifold of $\mathbb{R}^2$ (as we explain in Example A.1.4), so the fibre product X cannot exist in the category of manifolds.

However, X is not a particularly badly behaved space. For example, there are morphisms between it and manifolds that behave like smooth morphisms. It is also a simple example of an algebraic variety, which can be studied by ordinary algebraic geometry. This motivates considering generalisations of the category of manifolds to include such spaces. C^∞-algebraic geometry is a way of doing this that considers generalising the $\mathbb{R}$-algebra of smooth maps from a manifold to $\mathbb{R}$.

1.1.2 C^∞-rings and C^∞-schemes

For an $\mathbb{R}$-algebra $(R, +, *)$ we have the following maps: $+ : R \times R \to R$ the addition map; $- : R \to R$ the additive inverse map; $* : R \times R \to R$ the multiplication map; and for any scalar $\lambda \in \mathbb{R}$ the scalar multiplication maps $\lambda : R \to R, r \mapsto \lambda r$. We also have two objects $0, 1$, which can be written as maps $R^0 \to R$. These maps obey certain identities, and they imply that all real polynomials $p : \mathbb{R}^n \to \mathbb{R}$ give operations $R^n \to R$.

For a smooth manifold X, the set of smooth functions to the real numbers, $C^\infty(X)$, has a natural $\mathbb{R}$-algebra structure as well as a richer structure: For each smooth map $f : \mathbb{R}^n \to \mathbb{R}$ we can define an operation $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by $\Phi_f(g_1, \ldots, g_n) = f(g_1, \ldots, g_n)$. This motivates our definition of C^∞-ring, as a set $\mathfrak{C}$ such that for all smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ we have an operation $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$. These operations Φ_f behave in a reasonable way with respect to compositions of functions and coordinate projections.

Indeed, all C^∞-rings are commutative $\mathbb{R}$-algebras. Examples of C^∞-rings include the ring of k-th differentiable functions $C^k(X)$ for a manifold X and for all $k = 0, 1, \ldots, \infty$, quotients of C^∞-rings by ideals, and Weil algebras. This category can be thought of as an algebraic way to generalise manifolds.

C^∞-rings were first mentioned in a lecture series by W. Lawvere [59] in 1967, although examples existed before this. C^∞-schemes are analogous to ordinary schemes in Algebraic Geometry, as they are locally ringed spaces that are locally isomorphic to spectra of C^∞-rings. Unlike ordinary schemes, the spectrum functor used for C^∞-rings has topological space constructed from only maximal ideals with residue field $\mathbb{R}$. There is another approach, as in Moerdijk, van Quê and Reyes [70], for defining a spectrum functor that uses a different definition of local, however we do not consider this here. In each case, the spectrum functor is a right adjoint to the global sections functor.

For a smooth manifold X, the spectrum functor applied to the C^∞-ring $C^\infty(X)$ returns the C^∞-scheme with topological space X and the sheaf of smooth functions on X. This gives a full and faithful embedding of the category of manifolds into the category of C^∞-schemes, as in Moerdijk and Reyes [72, Th. I.2.8]. Importantly, the category of C^∞-schemes addresses several shortcomings of the category of smooth manifolds, for instance while the space of smooth maps between two manifolds is not a manifold, nor are arbitrary fibre products of manifolds, both of these are C^∞-schemes. In fact all finite limits exist in the category of C^∞-schemes. In this sense, the category of C^∞-schemes can be considered as geometric way to generalise the category of manifolds so that the resulting category has better categorical properties.

This embedding of the category of manifolds motivated studying C^∞-rings and C^∞-schemes as a model for synthetic differential geometry, which aims to understand differential geometry by using 'infinitesimals' to replace the 'ϵ/δ' limit approach. Work along these lines has been carried out in Moerdijk and Reyes [71–73], Moerdijk, van Quê and

Reyes [70], Kock [55], and Dubuc [19–21].

The study of C^∞-rings and C^∞-schemes has been called C^∞-algebraic geometry. Recent motivation to study C^∞-algebraic geometry is to develop a version of derived geometry for Differential Geometry, as originally suggested in Lurie [62, §4.5], and developed by Spivak [84]. This has led to further studies in derived geometry by Borisov [8], Borisov and Noel [10], and the 'd-manifolds'/'d-orbifolds' of Joyce [41], and further refinement of C^∞-algebraic geometry as in Joyce [40] and in Borisov [9]. Note that a d-manifold is essentially a C^∞-scheme that is isomorphic to the fibre product of manifolds, with an extra sheaf structure. This motivates using a category that contains manifolds and their fibre products.

1.1.3 Other generalisations of the category of manifolds

C^∞-schemes can be viewed as starting with the maps $C^\infty(X)$ and asking how can this structure be generalised. This is an example of the 'maps out' generalisation of manifolds: we generalise smooth maps out of the space X to $\mathbb{R}$. There are several other 'maps out' approaches, such as those defined in Sikorski [81], and several papers by Spallek starting with [83]. Many of the 'maps out' approaches are also summarised in great detail by Buchner et al. [7].

One of the approaches by Spallek has been further studied in the book Navarro González and Sancho de Salas [76]. In this book, it is known as the category of C^∞-differentiable spaces, and this category also has all finite limits. C^∞-differentiable spaces are equivalent to a subcategory of C^∞-schemes, specifically to C^∞-schemes that are locally isomorphic to the spectrum of certain quotients of $C^\infty(\mathbb{R}^n)$ known as differentiable algebras. Then the category of manifolds also embeds fully and faithfully into affine C^∞-differentiable spaces, and this embeds fully and faithfully into the category of C^∞-schemes. The spaces defined in Sikorski [81] are a nice subcategory of C^∞-differentiable spaces, and these have been expanded to a sheaf-theoretic version in Mostow [75], who also compares these notions in more detail.

Reversing the viewpoint, there have been several 'maps in' approaches that generalise the idea of smooth maps from a (subset of) a Euclidean space to the space X. These approaches include the Diffeological Spaces of Souriau [82] described further in Iglesias-Zemmour [36], and the various Chen spaces from Chen [12–15]. These notions work particularly well for considering infinite dimensional spaces (as the morphisms $\mathbb{R}^n \to X$ capture information of finite dimensional subspaces), and to describe quotient spaces.

In each of these 'maps in' approaches, one begins by taking a set (or topological space) and a collection of maps out of the space (often called *plots*) that satisfy certain conditions, such as allowing composition with the usual smooth morphisms and requiring that if a map is a plot locally, then it is a global plot. Stacey [85] compares these various different notions, and their relations to Sikorski's 'maps out' approach. However, while each of these notions generalises smooth manifolds in ways to allow fibre products, they do not do this by considering spectra of rings in ways similar to Algebraic Geometry, and this approach is not well suited for derived geometry.

1.1.4 Derived geometry

We are motivated to develop the theory of C^∞-algebraic geometry with corners so it can be used in derived differential geometry as in Joyce [41]. Let us explain the origins of derived geometry.

Derived geometry was initially conceptualised for algebraic geometry. The motivation arose from trying to define invariants from moduli spaces that were very singular, as in Kontsevich [56]. To say something is singular, usually one means either it has quotient singularities or it has intersection singularities. On the level of spaces, schemes can handle intersection singularities well and stacks can handle quotient singularities well, but cohomology theories do not necessarily behave well without additional assumptions. Here, the usual notion of cotangent bundle is not sufficient to capture the singular nature of the space, instead cotangent complexes are more appropriate.

Bertand Toën, Gabriele Vezzossi and Jacob Lurie developed many of the initial ideas of Derived Algebraic Geometry, and the survey paper Toën [86] details the extensive applications and further developments of this work in the wider mathematical community. A more recent survey paper by Anel [4] also describes the ideas in derived geometry to motivate its use. In Derived Algebraic Geometry, the cotangent complexes live naturally and hold the information required about the singular nature of a space.

Lurie in [62] first described how to apply many of the ideas of Derived Algebraic Geometry to differential geometry. Much of the foundational work was carried out by Lurie's student David Spivak in his thesis [84]. Further work has been undertaken by Borisov [8], and Borisov and Noel [10], although their derived objects formed an ∞-category. The derived differential geometry of Joyce [41] involves only a 2-category of derived spaces. All of these approaches are built from C^∞-rings, C^∞-schemes and C^∞-stacks.

The motivation behind the 'd-manifolds' of [41] is also related to defining invariants of certain moduli spaces. This results in requiring additional structure on the moduli spaces, which may be spaces with corners. A manifold with corners is one such space with corners. The thesis involves building a model of C^∞-rings and C^∞-schemes with corners that describes manifolds with corners, not just manifolds. One can then define C^∞-stacks with corners and derived spaces with corners to capture the structure of these moduli spaces with corners.

1.1.5 Manifolds with corners

The definition of manifold with corners involves generalising the local model of a manifold from $\mathbb{R}^n$ to $\mathbb{R}^n_k = [0,\infty)^k \times \mathbb{R}^{n-k}$, and generalising the smooth maps between the local models, for which there are several different approaches in the literature. We will use the notion of smooth maps of manifolds with corners that are called 'b-maps' in Melrose [68]. These are also used in more recent work by Joyce as in [47]. These b-maps respect the boundary and corners of a manifold with corners. This allows for a definition of a corner functor, that takes manifolds to their 'space of corners', which is a manifold with corners of mixed dimension.

Manifolds with corners have been studied in a variety of contexts, beginning with Cerf [11] and Douady [18] in 1961, as natural ways to extend the notion of a manifold with boundary. Their work was motivated by understanding questions from differential topology, for example, to understand homotopy types of diffeomorphism groups of spheres and other compact manifolds of dimension 3, and they gave many foundational results.

There have been a variety of applications from this work on manifolds with corners in differential topology, including those of Jänich [37]. Jänich considered the classification of manifolds with an $O(n)$ action (called $O(n)$-manifolds) by decomposing into certain 'parts' that are often manifolds with corners. Previously, if such a manifold with corners was obtained, the corners were often smoothed in some way, eliminating the need to study manifolds with corners in general. However, Jänich explains this approach is not helpful for this decomposition, and uses the manifold with corners results of Cerf and Douady to give a 'classification by parts' of certain $O(n)$-manifolds.

Other applications in differential topology include defining the cobordism category of a manifold with corners as in Laures [58], and to define 'extended topological quantum field theories', which are functors between cobordism categories and categories of vector spaces as in Kerler [54].

Manifolds with corners arise naturally in many contexts. They can arise directly such as when considering solutions to the partial differential equation that governs the motion of a square drum when struck. They can also arise indirectly. For example, many results work well for compact manifolds and such results may need to be extended to non-compact manifolds by compactifying them. Upon compactifying, the manifold may become a manifold with corners, so the results need to be generalised for manifolds with corners. One example is from Monthubert and Nistor [74] who recently extended results from index theory to non-compact manifolds using manifolds with corners.

For many such applications in analysis, fundamental theorems on the geometry and analysis of manifolds and manifolds with boundary were extended to manifolds with corners, as in Melrose [68]. There have also been generalisations of manifolds with corners along these lines, including the manifolds with analytic corners of Joyce [48].

Another generalisation of manifolds with corners is manifolds with g-corners, as in Joyce [47]. These allow a more general local model and we will show that many of our results on manifolds with corners extend to these manifolds with g-corners.

1.1.6 Motivations from symplectic geometry

Some of the specific invariants that have motived derived differential geometry have arisen in symplectic geometry, as in Joyce [41].

In symplectic geometry, the objects of interest are symplectic manifolds, and classifying these spaces involves understanding how maps, called J-holomorphic curves, into the manifold behave. J-holomorphic curves, also known as pseudo-holomorphic curves, are curves from a Riemann surface (often the Riemann sphere) to the symplectic manifold that commute with the complex structure from the Riemann surface and an almost complex structure (called J) on the symplectic manifold.

Recent research in symplectic geometry concerns defining invariants (e.g. numbers, cohomology classes, categories) on a symplectic manifold using J-holomorphic curves. Specifically, it aims to define invariants (akin to Gromov-Witten invariants) by 'counting' the moduli spaces of J-holomorphic curves arising from a symplectic manifold.

When the J-holomorphic curves are generic, they create families called moduli spaces, $M(J, A)$, that are parameterised by J and the integer homology classes A of the manifold that this curve represents. In the nice cases, each family is in fact a finite dimensional manifold and, while not necessarily compact, there are ways to define invariants such as the Gromov-Witten invariants as described in McDuff and Salamon [65].

However, to define invariants on these moduli spaces of J-holomorphic curves in general (for example for symplectic manifolds that are not weakly monotone), more structure on the moduli spaces is needed. There are several proposed options for this structure: Kuranishi spaces, polyfolds, and derived spaces. Kuranishi spaces were first defined in

Fukaya and Ono, [24], and expanded upon in Fukaya et al. [25]. While they have made a lot of progress on this, their definition of Kuranishi space has issues, such as not having a nice notion of morphism and relying on many arbitrary choices.

Polyfolds are an alternative theory to Kuranishi spaces. They were first defined by Hofer, and developed in a series of papers by Hofer, Wysocki and Zehnder [35]. They were proposed to solve several issues with Kuranishi spaces. While there has been much work on foundations of this area, there is still progress to be made on the applications of defining invariants.

Joyce has proposed 'd-orbifolds with corners' as the model for the moduli spaces of J-holomorphic curves. This model first uses C^∞-rings and C^∞-schemes to describe C^∞-stacks, as in Joyce [40]. It then considers C^∞-stacks that come from fibre products of orbifolds, and adds an extra sheaf to become a d-orbifold. Then d-orbifolds form a 2-category with nicely behaved morphisms. There is a provisional notion of corners structure on a d-orbifold, which adds another sheaf to the d-orbifolds. Joyce [43] shows d-orbifolds with such corners structure are equivalent to a version of Kuranishi spaces as a 2-category, and that these $M(J, A)$ indeed have such a structure. However, this definition of 'd-orbifold with corners' is provisional, as it currently has problems with identifying the correct corners structure.

This thesis is motivated by ideas to refine the definition of d-orbifolds with corners (and other derived spaces with corners). Instead of adding a sheaf at the end of the construction that defines the corners, one should start with a C^∞-schemes with corners structure (or C^∞-stack with corners structure). This should make the d-orbifolds with corners easier to define for each $M(J, A)$, and also describe properties between the C^∞-schemes and the corners precisely. It is intended by Joyce that final version of a d-orbifold with corners will use the C^∞-schemes with corners defined in this thesis.

1.2 What is in this thesis

This thesis defines C^∞-rings with corners and C^∞-schemes with corners. It explores several properties of both ideas. It shows, under certain conditions, fibre products of C^∞-schemes with corners exist. It describes how the category of manifolds with corners can be fully and faithfully embedded into this category. It also describes a corner functor, which returns the space of corners associated to certain C^∞-schemes with corners. Our C^∞-schemes with corners are related to log geometry, in the sense of 'positive log differentiable spaces' described in Gillam and Molcho [28], which extend the notion of C^∞-differentiable space.

In Chapter 2, we recall background on C^∞-rings and C^∞-schemes; this section is mostly a summary of background material found in Dubuc [21], Joyce [40, §2–§5] and Moerdijk and Reyes [72]. We recall the two definitions of C^∞-rings, and that the category of C^∞-rings is the category of algebras over an algebraic theory (in the sense of Adámek, Rosický and Vitale [3]) so it has all small limits, directed colimits, and small colimits. We recall the definition of local C^∞-rings and discuss their limits and colimits. We recall the definition of C^∞-scheme, and we describe a subcategory of C^∞-rings called complete C^∞-rings for which there is an equivalence of categories with the category of affine C^∞-schemes. We use this to show that finite limits of C^∞-schemes exist. Section 2.4.1 is new, where we discuss infinite products of (affine) C^∞-schemes.

Chapter 3 describes background on manifolds with corners, as in Joyce [39, 47], and Melrose [68], and recalls important facts on monoids to describe manifolds with g-corners. It defines smooth maps of manifolds with (g-)corners, the boundary and corners of a manifold with (g-)corners, and the corner functor. It also describes their (co)tangent bundles and, briefly, how their fibre products behave.

The content of Chapter 4 is mostly new and is joint work with Dominic Joyce. We describe two notions of pre C^∞-ring with corners, one as a functor from Euclidean spaces with corners to sets, and one as a pair $(\mathfrak{C}, \mathfrak{C}_{\rm ex})$ where $\mathfrak{C}$ is a C^∞-ring and $\mathfrak{C}_{\rm ex}$ is a monoid, such that the pair behaves well under smooth maps of manifolds with corners. These notions were first considered in the masters thesis by Kalashnikov [51]. Similar to C^∞-rings, pre C^∞-rings with corners are also algebras over an algebraic theory and have all small limits, directed colimits, and small colimits. We add an additional condition to define C^∞-rings with corners, and show that limits, directed colimits and small colimits exist in this category too. We describe free C^∞-rings with corners, and how to add relations, and then give a notion of local C^∞-rings with corners and localisations, which we use to define C^∞-schemes with corners in Chapter 5. We describe many functors and their adjoints to study limits and colimits of these categories. We also describe modules and (b-)cotangent modules of C^∞-rings with corners, and prove that they are isomorphic to the global sections of the (b-)cotangent bundles of both manifolds with corners and manifolds with g-corners locally and, under certain conditions, globally.

Chapter 5 is new work and comprises of just under half the material in this thesis. It introduces C^∞-ringed spaces with corners and shows small colimits and small limits exist in this category. We construct a spectrum functor that is right adjoint to a global sections functor. We define C^∞-schemes with corners and show that manifolds with (g-) corners embed fully and faithfully into this category. We originally aimed to show that all finite limits of C^∞-schemes with corners exist, however, there are many interesting differences between the category of C^∞-schemes and C^∞-schemes with corners that created difficulties for this. Instead, we show that finite limits exist under a certain finitely generated assumption (which we call *firm*), where manifolds with (g-)corners considered as C^∞-schemes with corners satisfy this assumption. We use the category of semi-complete C^∞-rings with corners to do this, and we study this category for this purpose.

In Chapter 5 we also define the subcategory of interior C^∞-schemes with corners and describe how all our categories relate with functors and their adjoints. We show that there is a corner functor for firm C^∞-rings with corners that is right adjoint to the inclusion of interior firm C^∞-schemes with corners into firm C^∞-schemes with corners. Similarly, we show there is a corner functor between interior and non-interior local C^∞-ringed spaces with corners, and we explain how these two corner functors relate. We describe the boundary and corners of a C^∞-scheme with corners, and match this with the definitions of boundary and corners of a manifold with (g-)corners. Chapter 5 also surveys log geometry, log schemes, and positive log differentiable spaces, and explains how our C^∞-schemes with corners relate to these.

1.3 Summary of main results

1.3.1 C^∞-rings and C^∞-schemes with corners

The new work of Chapter 4 is joint work with Dominic Joyce. We define pre C^∞-rings with corners and categorical pre C^∞-rings with corners, which originally appeared in Kalashnikov [51]. We show these are equivalent, so pre C^∞-rings with corners can be identified as the category of algebras over an algebraic theory. This gives results on existence of small limits and colimits. We describe forgetful functors between pre C^∞-rings with corners and the category of C^∞-rings, and describe adjoints to this. We add an extra condition to define C^∞-rings with corners, and give an adjoint functor from pre C^∞-rings with corners to describe how their limits and colimits relate. We also define subcategories of C^∞-rings with corners (interior, local, finitely generated, free, firm), and explore whether these categories also have colimits and limits using adjoint functors.

We define localisations of C^∞-rings with corners, and explicitly describe localising at an '$\mathbb{R}$-point'. This is important for defining a spectrum functor. We then define modules over C^∞-rings with corners and give notions of cotangent and b-cotangent modules.

The new work of Chapter 5 is in defining C^∞-schemes with corners and their properties. First we describe a suitable category of local C^∞-ringed spaces with corners, and then a spectrum functor for both C^∞-rings with corners and interior C^∞-rings with corners. We show each spectrum functor is right adjoint to a global sections functor. The aim was to show that finite limits in the category of C^∞-schemes existed, but this was more complicated than originally thought.

1.3.2 Finite limits

For an ordinary ring R, then $\Gamma \circ \mathrm{Spec}(R) \cong R$ where Spec is the spectrum functor in ordinary algebraic geometry, and Γ is the global sections functor. Here Spec is right adjoint to Γ considered as functors between ordinary rings and ordinary local ringed spaces with corners. Then these functors give an equivalence of categories between the (opposite) category of ordinary rings and ordinary affine schemes. As finite colimits exist in the category of ordinary rings, then finite limits exist in the category of ordinary affine schemes. One can then show finite limits of ordinary schemes exist, by either glueing together the finite limits of affine neighbourhoods, or describing the finite limits of local ringed spaces with corners and showing these are locally isomorphic to the finite limits of affine neighbourhoods.

For C^∞-ring $\mathfrak{C}$, then $\Gamma \circ \mathrm{Spec}\,\mathfrak{C} \ncong \mathfrak{C}$ in general, where we are now using the spectrum functor for C^∞-rings. However, Spec is still a right adjoint to Γ considered as functors between the (opposite) category of C^∞-rings and local C^∞-ringed spaces with corners, and there is a canonical isomorphism $\mathrm{Spec} \circ \Gamma \circ \mathrm{Spec}\,\mathfrak{C} \cong \mathrm{Spec}\,\mathfrak{C}$. Using this isomorphism, we can define 'complete' C^∞-rings to be C^∞-rings such that $\Gamma \circ \mathrm{Spec}\,\mathfrak{C} \cong \mathfrak{C}$, and show there is an equivalence of categories between complete C^∞-rings and affine C^∞-schemes as in Joyce [40]. As complete C^∞-rings have all finite colimits, then affine C^∞-schemes have all finite limits. Constructing limits of C^∞-schemes in the category of local C^∞-ringed spaces and showing they are locally isomorphic to finite limits of affine C^∞-schemes implies that the category of C^∞-schemes has all finite limits.

For a C^∞-ring with corners $(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$, not only is $\Gamma^{\mathrm{c}} \circ \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}}) \ncong (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$, but we also have $\mathrm{Spec}^{\mathrm{c}} \circ \Gamma^{\mathrm{c}} \circ \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}}) \ncong \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ in general. Here $\mathrm{Spec}^{\mathrm{c}}$ and Γ^{c} are the spectrum and global section functors for C^∞-rings with corners. We can still show that $\mathrm{Spec}^{\mathrm{c}}$ is right adjoint to Γ^{c}, however because $\mathrm{Spec}^{\mathrm{c}} \circ \Gamma^{\mathrm{c}} \circ \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}}) \ncong \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ in general we do not expect an equivalence of categories between a (sub)category of C^∞-rings with corners and affine C^∞-schemes with corners.

Instead we use the category of semi-complete C^∞-rings with corners, and start by showing that the category of local C^∞-ringed spaces with corners has all finite limits. Then we show that when a finitely generated condition on $\mathfrak{C}_{\mathrm{ex}}$ holds, finite limits of

C^∞-schemes with corners exist and are equal to finite limits in the category of local C^∞-ringed spaces with corners using these semi-complete C^∞-rings with corners. This finitely generated condition we call *firm* and manifolds with (g-)corners considered as C^∞-schemes with corners satisfy this condition. We also describe a similar result for interior C^∞-ringed spaces/schemes with corners.

1.3.3 Embedding manifolds (with corners)

As mentioned in the background of §2, the category of manifolds embeds fully and faithfully into the category of C^∞-schemes, in fact into the category of affine C^∞-schemes. Transverse fibre products of manifolds exist in the category of manifolds and respect this embedding. There is a cotangent module for each C^∞-ring and cotangent bundle for each C^∞-scheme that correspond with the cotangent module and bundle of a manifold.

In this thesis, we show the category of manifolds with corners embeds fully and faithfully into the category of C^∞-schemes with corners, but the image is only affine when the manifolds with corners have *faces*, which is a nice geometric property. This geometric property means 'local behaviour comes from global behaviour', which we explain further in Theorem 5.5.2. Manifolds with g-corners also embed fully and faithfully into the category of C^∞-schemes with corners, however an equivalent geometric property to faces does not imply that local behaviour comes from global behaviour, and the image is not affine in general.

This issue extends to cotangent modules and bundles, where there is also another version of this, the b-cotangent module and b-cotangent bundle. These b-cotangent modules and bundles behave better with respect to the smooth maps of manifolds with corners. We show the cotangent module and b-cotangent module are isomorphic to the global sections of the cotangent and b-cotangent bundles on coordinate charts of manifolds with corners and manifolds with g-corners. If we consider manifolds with faces (with finitely many boundary components), then this is true globally not just on coordinate charts, but this does not apply for manifolds with g-corners. However, the cotangent sheaf and b-cotangent sheaf do match the cotangent bundles and b-cotangent bundles of manifolds with (g-)corners globally.

1.3.4 Corner functors

Manifolds with (g-)corners have a notion of corner functor as in Joyce [47], which takes a manifold with (g-)corners to a manifold with corners of mixed dimension with interior maps, and which behaves well when using the smooth maps (called b-maps) of Melrose

[66–68]. Manifolds with (g-)corners also have a boundary and k-corners as defined in §3.3.

We generalise this corners functor in §5.7 and §5.8. We show that there is a corner functor $C^{\rm loc}$ for local C^∞-ringed spaces with corners, which is right adjoint to the inclusion of interior local C^∞-ringed spaces with corners into the category of C^∞-ringed spaces with corners. This means the inclusion preserves colimits and the corner functor preserves limits. The corner functors are related to a description of boundary from Gillam and Molcho [28] for positive log differentiable spaces.

We use a different definition of corner functor C for firm C^∞-schemes with corners, and show this is right adjoint to the inclusion of interior firm C^∞-schemes with corners into firm C^∞-schemes with corners. We show C is equivalent to the restriction of $C^{\rm loc}$ to firm C^∞-schemes with corners. To define C, we could have just restricted $C^{\rm loc}$ to firm C^∞-schemes with corners and showed that its image lies in interior firm C^∞-schemes with corners, however with our definition of C the corners of the schemes can understood and studied without needing to consider ringed spaces. We suspect this may be useful from a derived geometry perspective.

We show that $C^{\rm loc}$ applied to an arbitrary C^∞-scheme with corners is not always a C^∞-scheme with corners, so we do not expect to be able to extend the notion of corners to C^∞-schemes with corners that are not firm. We define the boundary and k-corners of firm C^∞-schemes with corners and local C^∞-ringed spaces with corners, then describe how they match with the boundary and k-corners of manifolds with (g-)corners and how they relate to the boundary defined in [28]. As a corollary we show the corners functors of manifolds with (g-)corners are also right adjoints, and satisfy a universal property.

While we were motivated to study finite limits/fibre products from derived geometry, the corner functor for firm C^∞-schemes with corners is constructed from colimits of C^∞-schemes with corners and motivates studying how colimits behave too. We have done this following colimit results from ordinary (locally) ringed spaces from Demazure and Gabriel [17, Prop. I.1.1.6], and then describing colimits for C^∞-schemes with corners.

1.4 Future work and applications of C^∞-algebraic geometry with corners

There are a few loose ends and potential extensions of C^∞-algebraic geometry. For example, in §4.5 we define various subcategories of C^∞-rings with corners (e.g. toric, integral, saturated), and we expect their corresponding C^∞-schemes with corners to behave better than arbitrary C^∞-schemes with corners, and to have nice results about the corners and boundary.

We would also like to prove that transverse fibre products of manifolds with (g-) corners respect the embedding into C^∞-schemes with corners. In Remark 5.5.5 we suggest appropriate notions of transverse for manifolds with (g-)corners to do this. Some tentative calculation suggests restricting to the category of toric C^∞-rings/schemes with corners may be required here.

Remark 5.4.9 discusses a left adjoint to a certain functor that would describe how limits of C^∞-schemes with corners behave and relate to C^∞-schemes. There is an issue with showing the existence of this adjoint with the current method we have, and further insight on this would be appreciated.

Proposition 5.4.7 characterises interior firm C^∞-schemes with corners as firm C^∞-schemes with corners that are interior C^∞-ringed spaces with corners. It would be interesting to see whether all C^∞-schemes with corners that are interior C^∞-ringed spaces with corners are interior C^∞-schemes with corners, as we mention in Remark 5.4.6.

In Proposition 5.4.10 we show fibre products of C^∞-schemes with corners exist under certain conditions. In Remark 5.4.11 we suggest a counterexample to the existence of fibre products in general, which would be interesting to verify.

Originally C^∞-rings and C^∞-schemes were studied as a model for synthetic differential geometry, and our (firm) C^∞-rings with corners and C^∞-schemes with corners could be investigated as a model for synthetic differential geometry with corners.

The corner functor could possibly motivate a corner functor for log geometry, and some of our ideas of boundary and corners could be translated over to this field.

We should be able to define and study C^∞-stacks with corners and C^∞-orbifolds with corners, and then consider derived spaces with corners. We expect that only firm C^∞-schemes/stacks with corners will be necessary, which will mean fibre products exist and there is a possibility of a corner functor for these derived spaces.

Along these lines, we expect a relationship between Kuranishi spaces and C^∞-schemes with corners. Joyce [43] describes a modification of the Kuranishi spaces (with corners) of Fukaya and Ono [24], which has nice morphisms, and shows that there is an equivalence of 2-categories between these modified Kuranishi spaces (with corners) and d-orbifolds (with corners). The original notion of d-orbifold with corners in Joyce [43] was considered without the definition of C^∞-scheme with corners and Joyce is intending to refine this notion using the work in this thesis, so that there are corner functors for these categories. There is a truncation functor from d-orbifolds to C^∞-schemes, and we expect that there will be a truncation functor from d-orbifolds with corners to C^∞-schemes with corners.

Chapter 2

Background on C^∞-rings and C^∞-schemes

We begin with background material and results on C^∞-rings and C^∞-schemes, which we will later generalise to C^∞-rings with corners and C^∞-schemes with corners. References for this section include Dubuc [20, 21], Moerdijk and Reyes [72], and Kock [55], which all have a view towards synthetic differential geometry, and Adámek, Rosický and Vitale [3] who consider algebraic theories and their algebras, which generalise C^∞-rings from a categorical perspective.

In this chapter, we follow closely the work of Joyce [40, §2–§5], particularly in notation. First, we remark on the notation used from category theory.

Remark 2.0.1. We do not define basic notions of a category nor constructions such as functors, adjoints, limits, colimits, fibre products etc. which can be found in standard texts such as Mac Lane [63], Leinster [61] and Awodey [5]. However, we write the following for notational purposes.

Limits of a diagram in a category, where they exist, are an object in the category with a universal property, such that it has morphisms from the limit into each element of the diagram. Colimits are similar with morphisms to the colimit from each element of the diagram. When we say *small* limit or colimit, we mean a diagram whose collection of objects and morphisms form sets. When we say *finite* limit or colimit we mean the collection of objects in the diagram is finite.

(Co)products are (co)limits over a diagram that has no morphisms between each element of the diagram. If the category has a final/terminal (or initial) object, then (co)products are the same as (co)limits over the same diagram with added morphisms to this final object (from this initial object). When we say *fibre product*, we mean a limit over the diagram of the form $A \to B \leftarrow C$, which is a finite limit. If B is the terminal object, then the fibre product is just the product of A and C. All finite limits exist if and only if there is a terminal object and all fibre products exist in this category, as each finite limit is an iterated number of fibre products over the terminal object. When we say *pushout* we mean fibre coproduct, that is a colimit over the diagram of the form $A \leftarrow B \to C$, and there are similar observations about existence with initial objects.

If a functor is a right adjoint, it preserves limits, and its corresponding left adjoint preserves colimits. Adjoints are defined in several equivalent ways, using unit and counits, using natural transformations, using initial and final objects, and we will make use of all of them. These different definitions can be found for example in Leinster [61, Ch. 2].

2.1 Two definitions of C^∞-ring

Here we recall two different notions of C^∞-rings. This section follows results of Dubuc [21], Joyce [40], and Moerdijk and Reyes [72]. Proposition 2.1.11 expands on details suggested in [21, Prop. 5], but other than this, there is no new material and we keep notation similar to [40].

We first define C^∞-rings as functors using the category of Euclidean spaces, as in Joyce [40].

Definition 2.1.1. Let **Euc** be the category of Euclidean spaces with objects $\mathbb{R}^n$, for non-negative n, and morphisms all smooth maps. Let **Man** be the category of manifolds with smooth morphisms. Let **Sets** be the category of sets with set maps. The notions of finite products in **Euc** and **Sets** are well defined, where $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ is the product of $\mathbb{R}^n$ and $\mathbb{R}^m$, and $A \times B$ is the product of sets A, B.

A product-preserving functor $F : \mathbf{Euc} \to \mathbf{Sets}$ is called a *categorical C^∞-ring*. We require that F preserves the empty product, so it maps $\mathbb{R}^0$ in **Euc** to the point $*$, the final object in **Sets**.

A *morphism* $\eta : F \to G$ between categorical C^∞-rings $F, G : \mathbf{Euc} \to \mathbf{Sets}$ is a natural transformation $\eta : F \Rightarrow G$. These will automatically preserve products. We use the notation $\mathbf{CC^\infty Rings}$ for the category of categorical C^∞-rings and these morphisms. C^∞-rings in this sense are an examples of *algebras* over the *algebraic theory* **Euc** in the sense of Adámek, Rosický and Vitale [3], and many categorical properties of C^∞-rings follow from [3].

Here is an alternative definition of C^∞-rings as in classical algebra:

Definition 2.1.2. A C^∞-*ring* is a set $\mathfrak{C}$ that is equipped with operations

$$\Phi_f : \mathfrak{C}^n = \overbrace{\mathfrak{C} \times \cdots \times \mathfrak{C}}^{n \text{ copies}} \longrightarrow \mathfrak{C}$$

for all non-negative integers n and all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$. We use the convention that when $n = 0$, then $\mathfrak{C}^0$ is the single point $\{\emptyset\}$. We require that these operations satisfy the following composition and projection relations. For the composition relations, take non-negative integers m, n, and smooth functions $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$. Let h be the composition

$$h(x_1, \dots, x_n) = g\big(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)\big),$$

for all $(x_1, \dots, x_n) \in \mathbb{R}^n$. For any $(c_1, \dots, c_n) \in \mathfrak{C}^n$ we require

$$\Phi_h(c_1, \dots, c_n) = \Phi_g\big(\Phi_{f_1}(c_1, \dots, c_n), \dots, \Phi_{f_m}(c_1, \dots, c_n)\big).$$

For the projection relations, let $\pi_j : \mathbb{R}^n \to \mathbb{R}$, $\pi_j : (x_1, \dots, x_n) \mapsto x_j$ be the j-th projection map for each $1 \leqslant j \leqslant n$, then we require $\Phi_{\pi_j}(c_1, \dots, c_n) = c_j$ for all $(c_1, \dots, c_n) \in \mathfrak{C}^n$.

We call each Φ_f a C^∞-*operation*. Usually we refer to $\mathfrak{C}$ as the C^∞-ring, and leave the C^∞-operations implicit.

A *morphism* between C^∞-rings $\phi : \mathfrak{C} \to \mathfrak{D}$ is a map of sets $\phi : \mathfrak{C} \to \mathfrak{D}$ such that for all smooth $f : \mathbb{R}^n \to \mathbb{R}$ and $c_1, \dots, c_n \in \mathfrak{C}$ then $\Psi_f\big(\phi(c_1), \dots, \phi(c_n)\big) = \phi \circ \Phi_f(c_1, \dots, c_n)$, where Φ_f and Ψ_f are the C^∞-operations for $\mathfrak{C}$ and $\mathfrak{D}$ respectively. We will write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

There is a forgetful functor $\Pi : \mathbf{C^\infty Rings} \to \mathbf{Sets}$ mapping a C^∞-ring $\mathfrak{C}$ to its underlying set $\mathfrak{C}$, forgetting the C^∞-operations.

Each C^∞-ring $\mathfrak{C}$ has the structure of a *commutative* $\mathbb{R}$-*algebra*. Here, let $f : \mathbb{R}^2 \to \mathbb{R}$ is $f(x, y) = x + y$ be the smooth addition map, then addition '+' on $\mathfrak{C}$ can be defined by $c + d = \Phi_f(c, d)$ for $c, d \in \mathfrak{C}$. Similarly, the smooth multiplication map $g : \mathbb{R}^2 \to \mathbb{R}$ is $g(x, y) = xy$ gives multiplication '$\cdot$' on $\mathfrak{C}$ by $c \cdot d = \Phi_g(c, d)$. For each $\lambda \in \mathbb{R}$ and scalar multiplication map $\lambda' : \mathbb{R} \to \mathbb{R}$ is $\lambda'(x) = \lambda x$, we define scalar multiplication by $\lambda c = \Phi_{\lambda'}(c)$. Let $0' : \mathbb{R}^0 \to \mathbb{R}$ be the zero map, then we can show that $0 = \Phi_{0'}(\emptyset)$ gives a zero element for $\mathfrak{C}$, and $1 = \Phi_{1'}(\emptyset)$, for the unit map $1' : \emptyset \mapsto 1$, gives an identity element for $\mathfrak{C}$. The projection and composition relations show this gives $\mathfrak{C}$ the structure of a commutative $\mathbb{R}$-algebra.

Remark 2.1.3. There is an equivalence of categories $\mathbf{CC^\infty Rings} \cong \mathbf{C^\infty Rings}$. Here, $F \in \mathbf{CC^\infty Rings}$ is identified with a $\mathfrak{C} \in \mathbf{C^\infty Rings}$ such that $F(\mathbb{R}) = \mathfrak{C}$, and for any smooth $f : \mathbb{R}^n \to \mathbb{R}$, then $F(f)$ is identified with Φ_f.

The following example of smooth functions on a manifold motivates our definitions.

Example 2.1.4. Let X be a smooth manifold. Let $C^\infty(X)$ be the set of smooth functions $c : X \to \mathbb{R}$. For non-negative integers n and smooth $f : \mathbb{R}^n \to \mathbb{R}$, define C^∞-operations $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by composition

$$\big(\Phi_f(c_1, \dots, c_n)\big)(x) = f\big(c_1(x), \dots, c_n(x)\big), \tag{2.1.1}$$

for all $c_1, \dots, c_n \in C^\infty(X)$ and $x \in X$. The composition and projection relations follow directly from the definition of Φ_f, so that $C^\infty(X)$ forms a C^∞-ring. If we consider the $\mathbb{R}$-algebra structure of $C^\infty(X)$ as a C^∞-ring, this is the canonical $\mathbb{R}$-algebra structure on $C^\infty(X)$. If $f : X \to Y$ is a smooth map of manifolds, then $f^* : C^\infty(Y) \to C^\infty(X)$ mapping $c \mapsto c \circ f$ is a morphism of C^∞-rings.

Define a functor $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}} : \mathbf{Man} \to \mathbf{C^\infty Rings^{op}}$ to map $X \mapsto C^\infty(X)$ on objects and $f \mapsto f^*$ on morphisms.

Moerdijk and Reyes show that $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}} : \mathbf{Man} \to \mathbf{C^\infty Rings^{op}}$ is a full and faithful functor [72, Th. I.2.8], and takes transverse fibre products in **Man** to fibre products in $\mathbf{C^\infty Rings^{op}}$.

There are many more C^∞-rings than those that come from manifolds. For example, given any k-differentiable manifold X of $\dim X > 0$, then the set $C^j(X)$ of j-differentiable maps $f : X \to \mathbb{R}$ is a C^∞-ring with operations Φ_f defined as in (2.1.1), and each of these C^∞-rings is different for each integer $0 \geqslant j \geqslant k$.

Example 2.1.5. Consider $X = *$ the point, so $\dim X = 0$, then $C^\infty(*) = \mathbb{R} = C^0(X)$ and Example 2.1.4 shows the C^∞-operations $\Phi_f : \mathbb{R}^n \to \mathbb{R}$ given by $\Phi_f(x_1, \dots, x_n) = f(x_1, \dots, x_n)$ make $\mathbb{R}$ into a C^∞-ring. This is the initial object in $\mathbf{C^\infty Rings}$, and the simplest nonzero example of a C^∞-ring. The zero C^∞-ring is the set $\{0\}$ where all C^∞-operations $\Phi_f : \{0\} \to \{0\}$ send $0 \mapsto 0$, and this is the final object in $\mathbf{C^\infty Rings}$.

By Moerdijk and Reyes [72, p. 21–22] and Adámek et al. [3, Prop. 1.21, Prop. 2.5 & Th. 4.5] we have:

Proposition 2.1.6. *The category* $\mathbf{C^\infty Rings}$ *of* C^∞*-rings has all small limits and all small colimits. The forgetful functor* $\Pi : \mathbf{C^\infty Rings} \to \mathbf{Sets}$ *preserves limits and directed colimits, and can be used to compute such (co)limits, however it does not preserve general colimits such as pushouts.*

This proposition is important for several reasons, including that a C^∞-scheme is defined in terms of sheaves of C^∞-rings, which require (small) limits to exist. Also, for these sheaves to be well behaved, a notion of stalk (which uses directed colimit) and a way to sheafify (which uses small limits and colimits) is needed. We are also particularly interested in fibre products, that is, finite limits of C^∞-schemes, which require pushouts to exist for C^∞-rings.

For the pushout of morphisms $\phi : \mathfrak{C} \to \mathfrak{D}$, $\psi : \mathfrak{C} \to \mathfrak{E}$ in $\mathbf{C^\infty Rings}$, we write $\mathfrak{D} \amalg_{\phi,\mathfrak{C},\psi} \mathfrak{E}$ or $\mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E}$. In the special case $\mathfrak{C} = \mathbb{R}$ the coproduct $\mathfrak{D} \amalg_{\mathbb{R}} \mathfrak{E}$ will be written as $\mathfrak{D} \otimes_\infty \mathfrak{E}$. Recall that coproduct of $\mathbb{R}$-algebras A, B is the tensor product $A \otimes B$, however $\mathfrak{D} \otimes_\infty \mathfrak{E}$ is usually different from their tensor product $\mathfrak{D} \otimes \mathfrak{E}$. For example, for non-negative integers m, n, then $C^\infty(\mathbb{R}^m) \otimes_\infty C^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^{m+n})$ as in [72, p. 22], which contains the tensor product $C^\infty(\mathbb{R}^m) \otimes C^\infty(\mathbb{R}^n)$ but is larger than this, as it includes elements such as $\exp(xy)$.

Definition 2.1.7. An *ideal* I in $\mathfrak{C}$ is an ideal in $\mathfrak{C}$ when $\mathfrak{C}$ is considered as a commutative $\mathbb{R}$-algebra. We do not require it to be closed under all C^∞-operations, as if we did and we consider the smooth function $\exp : \mathbb{R} \to \mathbb{R}$, then $\Phi_{\exp}(0) = 1$, and the ideal would have to be the entire set $\mathfrak{C}$.

We can make the $\mathbb{R}$-algebra quotient $\mathfrak{C}/I$ into a C^∞-ring using Hadamard's Lemma. That is, if $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, define $\Phi_f^I : (\mathfrak{C}/I)^n \to \mathfrak{C}/I$ by

$$\big(\Phi_f^I(c_1 + I, \dots, c_n + I)\big)(x) = \Phi_f\big(c_1(x), \dots, c_n(x)\big) + I.$$

Then Hadamard's Lemma says for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$, there exists $g_i : \mathbb{R}^{2n} \to \mathbb{R}$ for $i = 1, \dots, n$, such that

$$f(x_1, \dots, x_2) - f(y_1, \dots, y_n) = \sum_{i=1}^n (x_i - y_i) g_i(x_1, \dots, x_n, y_1, \dots, y_n).$$

If $d_1, \dots, d_n$ are alternative choices for $c_1, \dots, c_n$, then $c_i - d_i \in I$ for each $i = 1, \dots, n$ and

$$\Phi_f(c_1, \dots, c_n) - \Phi_f(d_1, \dots, d_n) = \sum_{i=1}^n (c_i - d_i) \Phi_f(c_1, \dots, c_n, d_1, \dots, d_n) \in I,$$

so Φ_f^I is independent of the choice of representatives $c_1, \dots, c_n$ in $\mathfrak{C}$ and is well defined.

We can consider the ideal of a C^∞-ring $\mathfrak{C}$ generated by a collection of elements $c_a \in \mathfrak{C}$ with $a \in A$, in the sense of commutative $\mathbb{R}$-algebras. We denote this $(c_a : a \in A)$, so that

$$(c_a : a \in A) = \left\{ \textstyle\sum_{i=1}^n c_{a_i} \cdot d_i : n \geqslant 0,\ a_1, \dots, a_n \in A,\ d_1, \dots, d_n \in \mathfrak{C} \right\}.$$

Definition 2.1.8. Let $\mathfrak{C}$ be a C^∞-ring such that there are a finite number of elements $c_1, \dots, c_n$ in $\mathfrak{C}$ that generate $\mathfrak{C}$ under the C^∞-operations, then $\mathfrak{C}$ is called a *finitely generated* C^∞-ring. Note that then every element of $c \in \mathfrak{C}$ can be written as $\Phi_f(c_1, \dots, c_n)$ for some $c_i \in \mathfrak{C}$. Then $C^\infty(\mathbb{R})$ is finitely generated as a C^∞-ring but not as an $\mathbb{R}$-algebra, so this condition is much weaker than being a finitely generated $\mathbb{R}$-algebra.

In fact, $C^\infty(\mathbb{R}^n)$ is the free C^∞-ring with n generators, as in Kock [55, Prop. III.5.1]. As in Joyce [40], if $\mathfrak{C}$ is finitely generated, then $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ where I is the kernel of the map $\phi : C^\infty(\mathbb{R}^n) \to \mathfrak{C}$, $\phi(f) = \Phi_f(c_1, \ldots, c_n)$.

An ideal I in a C^∞-ring $\mathfrak{C}$ is called *finitely generated* if $I = (c_a : a \in A)$ for A a finite set. A C^∞-ring $\mathfrak{C}$ is called *finitely presented* if there is a finitely generated ideal I in $C^\infty(\mathbb{R}^n)$ such that $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for some $n \geqslant 0$. Note that $C^\infty(\mathbb{R}^n)$ is not noetherian, so ideals in a finitely generated C^∞-ring may not be finitely generated themselves. This implies finitely presented C^∞-rings are a subcategory of finitely generated C^∞-rings, in contrast to ordinary algebraic geometry where they are equal.

Definition 2.1.9. Recall that a local $\mathbb{R}$-algebra, R, is an $\mathbb{R}$-algebra with a unique maximal ideal $\mathfrak{m}$. The residue field of R is the field isomorphic to $R/\mathfrak{m}$. A C^∞-ring $\mathfrak{C}$ is called *local* if, regarded as an $\mathbb{R}$-algebra, $\mathfrak{C}$ is a local $\mathbb{R}$-algebra with residue field $\mathbb{R}$. The quotient morphism gives a (necessarily unique) morphism of C^∞-rings $\pi : \mathfrak{C} \to \mathbb{R}$ with the property that $c \in \mathfrak{C}$ is invertible if and only if $\pi(c) \neq 0$. Equivalently, if such a morphism $\pi : \mathfrak{C} \to \mathbb{R}$ exists with this property, then $\mathfrak{C}$ is local with maximal ideal $\mathfrak{m}_{\mathfrak{C}} \cong \mathrm{Ker}(\pi)$.

Usually morphisms of local rings are required to send maximal ideals into maximal ideals. However, if $\phi : \mathfrak{C} \to \mathfrak{D}$ is any morphism of local C^∞-rings, then because the residue fields in both cases are $\mathbb{R}$, then $\phi^{-1}(\mathfrak{m}_{\mathfrak{D}}) = \mathfrak{m}_{\mathfrak{C}}$, so there is no difference between local morphisms and morphisms for C^∞-rings. This also shows that morphisms of local C^∞-rings commute with the morphisms $\pi : \mathfrak{C} \to \mathbb{R}$.

Remark 2.1.10. We use the term 'local C^∞-ring' following Dubuc [21, Def. 4] and Joyce [40]. They are known by different names in other references, such as *Archimedean local C^∞-rings* in [70, §3], *C^∞-local rings* in Dubuc [20, Def. 2.13], and *pointed local C^∞-rings* in [72, §I.3]. Moerdijk and Reyes [70–72] use 'local C^∞-ring' to mean a C^∞-ring which is a local $\mathbb{R}$-algebra, and require no restriction on its residue field.

Proposition 2.1.11. *The category of local C^∞-rings has all small colimits and small limits. Small colimits commute with small colimits in* $\mathbf{C^\infty Rings}$*, and there is a right adjoint to the inclusion of local C^∞-rings into C^∞-rings. Small limits commute with small limits in* $\mathbf{C^\infty Rings}$ *only in certain cases, outlined in the proof below, so there is no left adjoint.*

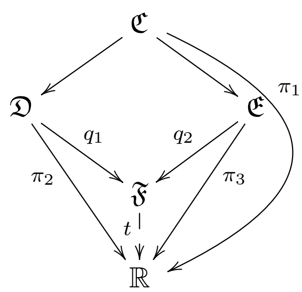
It is already known in the literature that finite colimits (for example, pushouts) of local C^∞-rings exist, for instance in Moerdijk and Reyes [72, §I.3], although their proof is different to the following proof. This proof for colimits expands on the proof of Dubuc [21, Prop. 5] for finite colimits.

Proof. We first consider pushouts of local C^∞-rings.

Let $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ be local C^∞-rings with morphisms $\mathfrak{C} \to \mathfrak{D}$ and $\mathfrak{C} \to \mathfrak{E}$. Let $\mathfrak{F}$ be their pushout in C^∞-rings, with maps $q_1 : \mathfrak{D} \to \mathfrak{F}$ and $q_2 : \mathfrak{E} \to \mathfrak{F}$. By definition of pushout in C^∞-rings, we can show that $\mathfrak{F}$ consists of elements of the form

$$\Phi_g(q_1(d_1), \ldots, q_1(d_m), q_2(e_1), \ldots, q_2(e_n))$$

for smooth $g : \mathbb{R}^{m+n} \to \mathbb{R}$, with $d_1, \ldots, d_m \in \mathfrak{C}$ and $e_1, \ldots, e_n \in \mathfrak{D}$. As $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ are local, there exists unique morphisms $\pi_1 : \mathfrak{C} \to \mathbb{R}, \pi_2 : \mathfrak{D} \to \mathbb{R}, \pi_3 : \mathfrak{E} \to \mathbb{R}$, which define their maximal ideals, and such that the diagram below commutes.



(2.1.2)

As $\mathfrak{F}$ is the pushout, there must be a unique morphism $t : \mathfrak{F} \to \mathbb{R}$ that makes the diagram commute. Take $f \in \mathfrak{F}$ such that $t(f) \neq 0 \in \mathbb{R}$. We need to show f has an inverse in $\mathfrak{F}$, so that t makes $\mathfrak{F}$ into a local C^∞-ring.

As $f \in \mathfrak{F}$, then $f = \Phi_g(\underline{p})$ for $\underline{p} = (\underline{p_1}, \underline{p_2})$ with $\underline{p_1} = (q_1(d_1), \ldots, q_1(d_n))$ and $\underline{p_2} = (q_2(e_1), \ldots, q_2(e_m))$ for some smooth $g : \mathbb{R}^{n+m} \to \mathbb{R}$, $c_1, \ldots, c_n \in \mathfrak{C}$ and $d_1, \ldots, d_m \in \mathfrak{D}$. Then

$$t(f) = g(\pi_1(d_1), \ldots, \pi_1(d_n), \pi_2(e_1), \ldots, \pi_2(e_m)) \neq 0.$$

As g is non-zero at this point, then it must be non-zero in a neighbourhood of this point, and there must be a function $h \neq 0, h : \mathbb{R}^{n+m} \to \mathbb{R}$ such that $gh = 1$ in an open neighbourhood V of the point. We will show that $\Phi_h(\underline{p})$ is the inverse of f.

There are open sets $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^m$ such that $U_1 \times U_2 \subset V$, and functions $h_1 : \mathbb{R}^n \to \mathbb{R}, h_2 : \mathbb{R}^m \to \mathbb{R}$ such that h_1, h_2 are zero outside of U_1 and U_2 respectively, and are equal to 1 in open balls about $t(\underline{p_1}), t(\underline{p_2})$ contained in U_1 and U_2 respectively. Hence $(gh-1)h_1h_2 = 0$ on $\mathbb{R}^{n+m}$, which implies that

$$0 = \Phi_{(gh-1)h_1h_2}(\underline{p}) = \Phi_{(gh-1)}(\underline{p})\Phi_{h_1}(\underline{p_1})\Phi_{h_2}(\underline{p_2}).$$

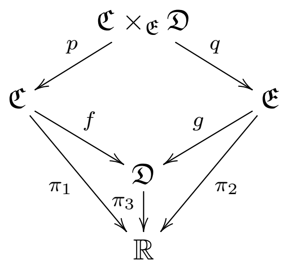
As $\mathfrak{D}$ and $\mathfrak{E}$ are local, and h_1 and h_2 are non-zero at these points, which lie in U_1 and U_2, then $\Phi_{h_1}(d_1, \ldots, d_n)$ is invertible in $\mathfrak{D}$, and $q_1(\Phi_{h_1}(d_1, \ldots, d_n)) = \Phi_{h_1}(\underline{p_1})$ is invertible in $\mathfrak{F}$, and similarly for h_2. So we must have $0 = \Phi_{(gh-1)}(\underline{p})$, which implies that $\Phi_g(\underline{p})\Phi_h(\underline{p}) = f\Phi_h(\underline{p}) = 1$, and $\mathfrak{F}$ is a local C^∞-ring.

We have shown the category of local C^∞-rings is closed under pushouts, and the pushouts are exactly those in $\mathbf{C^\infty Rings}$. As $\mathbb{R}$ is the initial object in local C^∞-rings, then all finite colimits can be written as a combination of (iterated) pushouts, which shows that the category of local C^∞-rings has all finite colimits.

To extend this to small colimits, consider that by Proposition 2.1.6, all small colimits exist for C^∞-rings. Again, we can show each element in the colimit is generated from finitely many elements from the C^∞-rings in the diagram, and if all C^∞-rings in the diagram are local, then there must be a unique morphism from the colimit to $\mathbb{R}$. The same method can then be applied to show that this element is invertible if and only if its image in $\mathbb{R}$ is non-zero. Hence the small colimit of local C^∞-rings exists and commutes with small colimits in the category of C^∞-rings.

One can then construct the right adjoint F to the inclusion of local C^∞-rings into C^∞-rings by taking $F(\mathfrak{C})$ of a C^∞-ring $\mathfrak{C}$ to be the colimit of all the local C^∞-rings $\mathfrak{D}$ that have morphisms $\mathfrak{D} \to \mathfrak{C}$. For a morphism $\phi : \mathfrak{C}_1 \to \mathfrak{C}_2 \in \mathbf{C^\infty Rings}$ then $F(\phi)$ is constructed using the universal property of colimits. The unit is the identity natural transformation, and the counit is the unique morphism from the colimit to $\mathfrak{C}$, and it is straightforward to see that they form an adjoint pair.

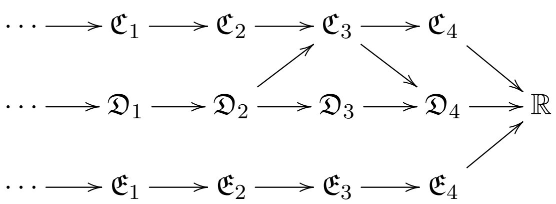
To consider limits of local C^∞-rings let us consider two cases. If we take a fibre product diagram $f : \mathfrak{C} \to \mathfrak{E} \leftarrow \mathfrak{D} : g$ of local C^∞-rings with corners, then the limit in C^∞-rings is $\mathfrak{C} \times_{\mathfrak{E}} \mathfrak{D}$ as in the diagram below.



As $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ are local they each have unique morphisms to $\mathbb{R}$, $\pi_1 : \mathfrak{C} \to \mathbb{R}$, $\pi_2 : \mathfrak{D} \to \mathbb{R}$, $\pi_3 : \mathfrak{E} \to \mathbb{R}$ and, by uniqueness, these morphisms must commute with the diagram $\mathfrak{C} \to \mathfrak{E} \leftarrow \mathfrak{D}$. This gives a morphism $\pi : \mathfrak{C} \times_{\mathfrak{E}} \mathfrak{D} \to \mathbb{R}$. We need only check that for $c \in \mathfrak{C} \times_{\mathfrak{E}} \mathfrak{D}$, then $\pi(c) \neq 0$ if and only if c is invertible. Say $c \in \mathfrak{C} \times_{\mathfrak{E}} \mathfrak{D}$ has $\pi(c) = 0$. Then $\pi_1 \circ p(c) = \pi_2 \circ q(c) = 0$, so $p(c)$ and $q(c)$ are not invertible, so $c = (p(c), q(c))$ cannot be invertible either. So $\mathfrak{C} \times_{\mathfrak{E}} \mathfrak{D}$ is local.

Consider now a pair of local C^∞-rings $\mathfrak{C}$,$\mathfrak{D}$ with no morphisms between them. Their product in C^∞-rings is $\mathfrak{C} \times \mathfrak{D}$, which is not local as it has two distinct $\mathbb{R}$-points. However, if one instead takes the fibre product over their morphisms to $\mathbb{R}$, that is over the diagram $\mathfrak{C} \to \mathbb{R} \leftarrow \mathfrak{D}$, then the fibre product $\mathfrak{C} \times_{\mathbb{R}} \mathfrak{D}$ in C^∞-rings is local. It then follows that this fibre product is actually the product in local C^∞-rings: if any other local C^∞-ring $\mathfrak{E}$ maps into both $\mathfrak{C}$ and $\mathfrak{D}$, then it must commute with their morphisms to $\mathbb{R}$ by uniqueness of its own morphism to $\mathbb{R}$.

Using this it is not hard to show that all small limits of local C^∞-rings exist and are equal to their limits as C^∞-rings taken over diagrams that include the morphisms to $\mathbb{R}$ as a vertex. That is, diagrams like the one below. (One might want to call this a 'directed' or 'inverse' limit but the diagram is in the opposite direction to the usual inverse/directed limit diagrams in the literature.)



(2.1.3)

This implies that all small limits of local C^∞-rings exist, but they are equal to their limits taken in $\mathbf{C^\infty Rings}$ only when the diagrams are already in the form of (2.1.3). □

Remark 2.1.12. The right adjoint to the inclusion of local C^∞-rings into C^∞-rings also follows abstractly from checking that the inclusion satisfies Freyd's Adjoint Functor Theorem (see Awodey [5, Th. 9.28]), or applying a special case of this. One such special case is Riehl [79, Th. 4.6.17(a)] provided one recognises $\mathbf{C^\infty Rings}$ as a *locally (finitely) presentable* category as in Adámek and Rosický [2]. Another special case involves recognising $\mathbf{C^\infty Rings}$ as a *total* category and applying Wood [87, Th. 1].

One can check that this right adjoint applied to a local C^∞-ring returns the local C^∞-ring, and also that applied to $C^\infty(\mathbb{R}^n)$ it returns $\mathbb{R}$. However we have not found a constructive formula for it in general.

Localisations of rings are important in ordinary algebraic geometry, for instance, restricting a scheme to a (nice) open set involves localising the ring, also stalks of schemes are isomorphic to localisations of rings. Localisations of C^∞-rings have been studied in [20, 21, 70, 71], [72, p. 23] and [40].

Definition 2.1.13. A *localisation* $\mathfrak{C}[s^{-1} : s \in S] = \mathfrak{D}$ of a C^∞-ring $\mathfrak{C}$ at a subset $S \subset \mathfrak{C}$ is a C^∞-ring $\mathfrak{D}$ and a morphism $\pi : \mathfrak{C} \to \mathfrak{D}$ such that $\pi(s)$ is invertible in $\mathfrak{D}$ for all $s \in S$.

We call $\pi : \mathfrak{C} \to \mathfrak{D}$ the localisation morphism for $\mathfrak{D}$. This has the universal property that for any morphism of C^∞-rings $\phi : \mathfrak{C} \to \mathfrak{E}$ such that $\phi(s)$ is invertible in $\mathfrak{E}$ for all $s \in S$, then there is a unique morphism $\psi : \mathfrak{D} \to \mathfrak{E}$ with $\phi = \psi \circ \pi$.

Adding an extra generator s^{-1} and extra relation $s \cdot s^{-1} - 1 = 0$ for each $s \in S$ to $\mathfrak{C}$, then it can be shown that localisations $\mathfrak{C}[s^{-1} : s \in S]$ always exist and are unique up to unique isomorphism. When $S = \{c\}$ then $\mathfrak{C}[c^{-1}] \cong \mathfrak{C} \otimes_\infty C^\infty(\mathbb{R})/I$, where I is the ideal generated by $\iota_1(c) \cdot \iota_2(x) - 1$, x is the generator of $C^\infty(\mathbb{R})$, and ι_1, ι_2 are the coproduct morphisms $\iota_1 : \mathfrak{C} \to \mathfrak{C} \otimes_\infty C^\infty(\mathbb{R})$, $\iota_2 : C^\infty(\mathbb{R}) \to \mathfrak{C} \otimes_\infty C^\infty(\mathbb{R})$.

An example of this is that if $f \in C^\infty(\mathbb{R}^n)$ is a smooth function, and $U = f^{-1}(\mathbb{R}^n \setminus 0)$, then partitions of unity show that $C^\infty(U) \cong C^\infty(\mathbb{R}^n)[f^{-1}]$ as in [72, Prop. I.1.6].

The following definition is crucial for defining C^∞-schemes.

Definition 2.1.14. A C^∞-ring morphism $x : \mathfrak{C} \to \mathbb{R}$, where $\mathbb{R}$ is regarded as a C^∞-ring as in Example 2.1.5, is called an $\mathbb{R}$-*point*. Note that a map $x : \mathfrak{C} \to \mathbb{R}$ is a morphism of C^∞-rings whenever it is a morphism of the underlying $\mathbb{R}$-algebras, as in [72, Prop. I.3.6]. We define $\mathfrak{C}_x$ as the localisation $\mathfrak{C}_x = \mathfrak{C}[s^{-1} : s \in \mathfrak{C},\, x(s) \neq 0]$, and denote the projection morphism $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$. Importantly, [71, Lem. 1.1] shows $\mathfrak{C}_x$ is a local C^∞-ring.

There is a one to one correspondence between the $\mathbb{R}$-points of $C^\infty(\mathbb{R}^n)$ and evaluation at points $x \in \mathbb{R}^n$. This also true for $C^\infty(X)$ for any smooth manifold X, which is a consequence of [72, Cor. I.3.7].

We can describe $\mathfrak{C}_x$ explicitly as in Joyce [40, Prop. 2.14].

Proposition 2.1.15. *Let $x : \mathfrak{C} \to \mathbb{R}$ be an $\mathbb{R}$-point of a C^∞-ring $\mathfrak{C}$, and consider the projection morphism $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$. Then $\mathfrak{C}_x \cong \mathfrak{C}/\operatorname{Ker}\pi_x$. This kernel is* $\operatorname{Ker}\pi_x = I$ *where*

$$I = \bigl\{c \in \mathfrak{C} : \textit{there exists } d \in \mathfrak{C} \textit{ with } x(d) \neq 0 \textit{ in } \mathbb{R} \textit{ and } c \cdot d = 0 \textit{ in } \mathfrak{C}\bigr\}. \tag{2.1.4}$$

While this localisation morphism $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$ is surjective, general localisations of C^∞-rings do not have surjective localisation morphisms.

Example 2.1.16. Let $C^\infty_p(\mathbb{R}^n)$ to be the set of germs of smooth functions $c : \mathbb{R}^n \to \mathbb{R}$ at $p \in \mathbb{R}^n$ for $n \geqslant 0$ and $p \in \mathbb{R}^n$. We can give $C^\infty_p(\mathbb{R}^n)$ a C^∞-ring structure by using (2.1.1) on germs of functions. There are many equivalent ways to consider the set germs: as a quotient of $C^\infty(\mathbb{R}^n)$ by an ideal, as a localisation, and as an equivalence class of pairs as in Joyce [40, Ex. 2.15]. As set of germs $[(c, U)]$ for $c \in C^\infty(U)$ for some $U \subseteq X$ with $p \in U$, there is a unique maximal ideal $\mathfrak{m}_p = \bigl\{[(c, U)] \in C^\infty_p(\mathbb{R}^n) : c(x) = 0\bigr\}$ and $C^\infty_p(\mathbb{R}^n)/\mathfrak{m}_p \cong \mathbb{R}$. Then $C^\infty_p(\mathbb{R}^n)$ is a local C^∞-ring.

2.2 Modules and cotangent modules of C^∞-rings

The following is a summary of the theory of modules and cotangent modules for C^∞-rings as defined in Joyce [40, §5], with reference to Fermat Theories in Dubuc and Kock [23].

Definition 2.2.1. A *module* M over a C^∞-ring $\mathfrak{C}$ is a module over $\mathfrak{C}$ as a commutative $\mathbb{R}$-algebra, and morphisms of $\mathfrak{C}$-modules are the usual morphisms of $\mathbb{R}$-algebra modules. Denote $\mu_M : \mathfrak{C} \times M \to M$ the multiplication map, and write $\mu_M(c, m) = c \cdot m$ for $c \in \mathfrak{C}$ and $m \in M$. The category $\mathfrak{C}$-mod of $\mathfrak{C}$-modules is an abelian category.

If a $\mathfrak{C}$-module M fits into an exact sequence $\mathfrak{C} \otimes \mathbb{R}^n \to M \to 0$ in $\mathfrak{C}$-mod then it is *finitely generated*; if it further fits into an exact sequence $\mathfrak{C} \otimes \mathbb{R}^m \to \mathfrak{C} \otimes \mathbb{R}^n \to M \to 0$ it is *finitely presented*. This second condition is not automatic from the first as C^∞-rings are not generally noetherian.

For a morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ of C^∞-rings and $M \in \mathfrak{C}$-mod then we have $\phi_*(M) = M \otimes_{\mathfrak{C}} \mathfrak{D} \in \mathfrak{D}$-mod, which gives a functor $\phi_* : \mathfrak{C}\text{-mod} \to \mathfrak{D}\text{-mod}$. For $N \in \mathfrak{D}$-mod there is a corresponding $\mathfrak{C}$-module $\phi^*(N) = N$ where the $\mathfrak{C}$-action is defined by $\mu_{\phi^*(N)}(c, n) = \mu_N(\phi(c), n)$. This also defines a functor $\phi^* : \mathfrak{D}\text{-mod} \to \mathfrak{C}\text{-mod}$. Here ϕ_* respects the finitely generated and finitely presented properties, however ϕ^* does not.

Example 2.2.2. Let $\Gamma^\infty(E)$ be the collection of smooth sections e of a vector bundle $E \to X$ of a manifold X, so $\Gamma^\infty(E)$ is a vector space and a module over $C^\infty(X)$. If $\lambda : E \to F$ is a morphism of vector bundles over X, then there is a morphism of $C^\infty(X)$-modules $\lambda_* : \Gamma^\infty(E) \to \Gamma^\infty(F)$ where $\lambda_* : e \mapsto \lambda \circ e$.

For each smooth map of manifolds $f : X \to Y$ there is a morphism of C^∞-rings $f^* : C^\infty(Y) \to C^\infty(X)$. Each vector bundle $E \to Y$ gives a vector bundle $f^*(E) \to X$. Using $(f^*)_* : C^\infty(Y)\text{-mod} \to C^\infty(X)\text{-mod}$ from Definition 2.2.1, then $(f^*)_*\big(\Gamma^\infty(E)\big) = \Gamma^\infty(E) \otimes_{C^\infty(Y)} C^\infty(X)$ is isomorphic to $\Gamma^\infty\big(f^*(E)\big)$ in $C^\infty(X)$-mod.

Remark 2.2.3. Let $E \to X$ be a vector bundle over a manifold X. Then $\Gamma^\infty(E)$ is finitely presented. In fact, any manifold has a finite atlas of (disconnected) charts, see for example Greub, Halperin and Valstone, [31, p. 20–21]. Using bump functions, one can extend local bases for the sections of the vector bundle restricted to these charts to global spanning sections $e_1, \dots, e_n \in \Gamma^\infty(E)$ for $n \gg 0$. This gives a surjective morphism $\psi : X \times \mathbb{R}^n \to E$ of vector bundles. The kernel is also a vector bundle F.

For any other surjective morphism $\phi : X \times \mathbb{R}^m \to F$, we then have the following exact sequence of vector bundles

$$X \times \mathbb{R}^m \xrightarrow{\phi} X \times \mathbb{R}^n \xrightarrow{\psi} E \longrightarrow 0.$$

Taking sections, we have an exact sequence of $C^\infty(X)$-modules

$$C^\infty(X) \otimes_{\mathbb{R}} \mathbb{R}^m \xrightarrow{\phi_*} C^\infty(X) \otimes_{\mathbb{R}} \mathbb{R}^n \xrightarrow{\psi_*} \Gamma^\infty(E) \longrightarrow 0.$$

This means $\Gamma^\infty(E)$ is finitely presented as a $C^\infty(X)$-module.

The definition of $\mathfrak{C}$-module only used the commutative $\mathbb{R}$-algebra structure of $\mathfrak{C}$, however the *cotangent module* $\Omega_{\mathfrak{C}}$ of $\mathfrak{C}$ does use the C^∞-ring structure. It is related to the module of *Kähler differentials* (or *module of relative differential forms*) as in Hartshorne [33, p. 172]. In their language, there is a morphism of modules from the module of Kähler differentials of $\mathfrak{C}$ over $\mathbb{R}$ to the module $\Omega_{\mathfrak{C}}$ that is surjective but not in general injective, with further discussion on this available on the nLab page [77].

Cotangent modules are an example of a construction defined in Dubuc and Kock [23] for *Fermat Theories*, which are types of algebraic theories that have derivatives, and so this construction can be applied to C^∞-rings.

Definition 2.2.4. Take a C^∞-ring $\mathfrak{C}$ and $M \in \mathfrak{C}$-mod, then a C^∞-*derivation* is a map $\mathrm{d} : \mathfrak{C} \to M$ that is $\mathbb{R}$-linear and satisfies the following: for any smooth $f : \mathbb{R}^n \to \mathbb{R}$ and elements $c_1, \ldots, c_n \in \mathfrak{C}$, then

$$\mathrm{d}\Phi_f(c_1, \ldots, c_n) = \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}c_i. \tag{2.2.1}$$

The pair (M, d) is called a *cotangent module* for $\mathfrak{C}$ if it is universal in the sense that for any $M' \in \mathfrak{C}$-mod with C^∞-derivation $\mathrm{d}' : \mathfrak{C} \to M'$, there exists a unique morphism of $\mathfrak{C}$-modules $\lambda : M \to M'$ with $\mathrm{d}' = \lambda \circ \mathrm{d}$. Then a cotangent module is unique up to unique isomorphism. We can explicitly construct a cotangent module for $\mathfrak{C}$ by considering the free $\mathfrak{C}$-module over the symbols $\mathrm{d}c$ and quotienting by relations $\mathrm{d}\Phi_f(c_1, \ldots, c_n) - \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}c_i$ where we have smooth $f : \mathbb{R}^n \to \mathbb{R}$ and elements $c_1, \ldots, c_n \in \mathfrak{C}$. We call this construction 'the' cotangent module of $\mathfrak{C}$ and write it as $\mathrm{d}_{\mathfrak{C}} : \mathfrak{C} \to \Omega_{\mathfrak{C}}$

If we have a morphism of C^∞-rings $\mathfrak{C} \to \mathfrak{D}$ then $\Omega_{\mathfrak{D}} = \phi^*(\Omega_{\mathfrak{D}})$ can be considered as a $\mathfrak{D}$-module with C^∞-derivation $\mathrm{d}_{\mathfrak{D}} \circ \phi : \mathfrak{C} \to \Omega_{\mathfrak{D}}$. The universal property of $\Omega_{\mathfrak{C}}$, gives a unique morphism $\Omega_\phi : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{D}}$ of $\mathfrak{C}$-modules such that $\mathrm{d}_{\mathfrak{D}} \circ \phi = \Omega_\phi \circ \mathrm{d}_{\mathfrak{C}}$. From this we have a morphism of $\mathfrak{D}$-modules $(\Omega_\phi)_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{D} \to \Omega_{\mathfrak{D}}$. If we have two morphisms of C^∞-rings $\phi : \mathfrak{C} \to \mathfrak{D}$, $\psi : \mathfrak{D} \to \mathfrak{E}$ then uniqueness means $\Omega_{\psi \circ \phi} = \Omega_\psi \circ \Omega_\phi : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{E}}$.

Example 2.2.5. As in Example 2.2.2 if X is a manifold, then its cotangent bundle T^*X is a vector bundle over X and its global sections $\Gamma^\infty(T^*X)$ are a $C^\infty(X)$-module, with C^∞-derivation $\mathrm{d} : C^\infty(X) \to \Gamma^\infty(T^*X)$, $\mathrm{d} : c \mapsto \mathrm{d}c$ the usual exterior derivative and equation (2.2.1) following from the chain rule.

As in Remark 2.2.3, $\Gamma^\infty(T^*X)$ is a finitely presented module. One can then show that $(\Gamma^\infty(T^*X), \mathrm{d})$ has the universal property in Definition 2.2.4, and so form a cotangent module for $C^\infty(X)$. Joyce [40, Ex. 5.4] states this result without proof for manifolds and, while this result seems known in the literature, we cannot find a proof. We extend this result to manifolds with corners and describe the proof precisely in Proposition 4.7.5 where we consider cotangent modules for C^∞-rings with corners.

If we have a smooth map of manifolds $f : X \to Y$, then $f^*(T^*Y), T^*X$ are vector bundles over X, and the derivative $\mathrm{d}f : f^*(T^*Y) \to T^*X$ is a vector bundle morphism. This induces a morphism of $C^\infty(X)$-modules $(\mathrm{d}f)_* : \Gamma^\infty(f^*(T^*Y)) \to \Gamma^\infty(T^*X)$, which is identified with $(\Omega_{f^*})_*$ from Definition 2.2.4 using that $\Gamma^\infty(f^*(T^*Y)) \cong \Gamma^\infty(T^*Y) \otimes_{C^\infty(Y)} C^\infty(X)$.

This example shows that Definition 2.2.4 abstracts the notion of sections of a cotangent bundle of a manifold to a concept that is well defined for any C^∞-ring.

2.3 Sheaves on topological spaces

Here we consider the definitions of presheaves and sheaves. The standard definition of a presheaf on a category C valued in a category $\mathcal{A}$ is a functor $\mathcal{E} : C^{\mathbf{op}} \to \mathcal{A}$, where $C^{\mathbf{op}}$ is the opposite category of C (as in Kashiwara and Schapira [52, §17]).

For a topological space X, let $\mathbf{Open}(X)$ be the category of open subsets of X with inclusion morphisms. In this thesis, we need only consider presheaves and sheaves $\mathcal{E} : C^{\mathbf{op}} \to \mathcal{A}$ where $C = \mathbf{Open}(X)$ for some topological space X, and $\mathcal{A}$ is some 'nice' category, such as the category of abelian groups, rings, C^∞-rings, monoids etc. We will call these (pre)sheaves of sets, groups, rings, C^∞-rings, monoids etc. over X.

By 'nice' we mean categories that are *complete*, that is, having all (small) limits, and, for this thesis, we will only consider categories $\mathcal{A}$ whose objects are sets with some extra structure, so that there is a faithful functor from these categories to the category of sets that takes each object to their underlying set. Abelian groups, rings, C^∞-rings, monoids etc. are are all algebras over algebraic theories, so they satisfy this and their faithful functor to sets respects limits and colimits, which is important in the definition of sheaf. In §4 we define interior C^∞-rings with corners, and while these are constructed from algebraic theories and are 'nice' in the sense above, they are not algebras over algebraic theories themselves and the functor(s) from interior C^∞-rings with corners to their underlying set(s) respect colimits but not limits, so their sheaves behave differently.

The above discussion gives the following definition, following Godement [30] and Mac

Lane and Moerdijk [64].

Definition 2.3.1. A *presheaf* $\mathcal{E}$ on a topological space X valued in $\mathcal{A}$ is a functor $\mathcal{E} : \mathbf{Open}(X)^{\mathbf{op}} \to \mathcal{A}$. This equivalently means that $\mathcal{E}(U) \in \mathcal{A}$ for every open set $U \subseteq X$, and there is a morphism $\rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ in $\mathcal{A}$ called the *restriction map* for every inclusion $V \subseteq U \subseteq X$ of open sets, satisfying the conditions that

(i) $\rho_{UU} = \mathrm{id}_{\mathcal{E}(U)} : \mathcal{E}(U) \to \mathcal{E}(U)$ for all open $U \subseteq X$; and

(ii) $\rho_{UW} = \rho_{VW} \circ \rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(W)$ for all open $W \subseteq V \subseteq U \subseteq X$.

A presheaf $\mathcal{E} : \mathbf{Open}(X)^{\mathbf{op}} \to \mathcal{A}$ is called a *sheaf* if for all open covers $\{U_i\}_{i\in I}$ of $U \in \mathbf{Open}(X)$, then

$$\mathcal{E}(U) \to \prod_{i\in I} \mathcal{E}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{E}(U_i \cap U_j)$$

forms an equaliser diagram in $\mathcal{A}$. This implies

(iii) $\mathcal{E}(\emptyset) = 0$ where 0 is the final object in $\mathcal{A}$.

If there is a faithful functor $F : \mathcal{A} \to \mathbf{Sets}$ taking an object of $\mathcal{A}$ to its underlying set that preserves limits, then a presheaf $\mathcal{E}$ valued in $\mathcal{A}$ on X is sheaf if it equivalently satisfies the following

(iv) (Uniqueness) If $U \subseteq X$ is open, $\{V_i : i \in I\}$ is an open cover of U, and $s, t \in F(\mathcal{E}(U))$ with $F(\rho_{UV_i})(s) = F(\rho_{UV_i})(t)$ in $F(\mathcal{E}(V_i))$ for all $i \in I$, then $s = t$ in $F(\mathcal{E}(U))$; and

(v) (Glueing) If $U \subseteq X$ is open, $\{V_i : i \in I\}$ is an open cover of U, and we are given elements $s_i \in F(\mathcal{E}(V_i))$ for all $i \in I$ such that $F(\rho_{V_i(V_i\cap V_j)})(s_i) = F(\rho_{V_j(V_i\cap V_j)})(s_j)$ in $F(\mathcal{E}(V_i \cap V_j))$ for all $i, j \in I$, then there exists $s \in F(\mathcal{E}(U))$ with $F(\rho_{UV_i})(s) = s_i$ for all $i \in I$.

Note that (iv) implies (iii) using the empty cover of the empty set. If $s \in F(\mathcal{E}(U))$ and open $V \subset U$, we write $s|_V = F(\rho_{UV})(s)$.

If $\mathcal{E}, \mathcal{F}$ are presheaves or sheaves valued in $\mathcal{A}$ on X, then a *morphism* $\phi : \mathcal{E} \to \mathcal{F}$ is a natural transformation of functors $\mathcal{E} \Rightarrow \mathcal{F}$. That is, for each open $U \subseteq X$, it gives a morphism in $\mathcal{A}$ $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ such that the following diagram commutes for all open $V \subseteq U \subseteq X$

$$\begin{array}{ccc} \mathcal{E}(U) & \xrightarrow{\phi(U)} & \mathcal{F}(U) \\ \downarrow{\rho_{UV}} & & \downarrow{\rho'_{UV}} \\ \mathcal{E}(V) & \xrightarrow{\phi(V)} & \mathcal{F}(V), \end{array}$$

where ρ_{UV} is the restriction map for $\mathcal{E}$, and ρ'_{UV} the restriction map for $\mathcal{F}$. We define $\mathrm{Sh}(X, \mathcal{A})$ as the category of sheaves on a topological space X valued in $\mathcal{A}$.

This second equivalent definition applies for sheaves of C^∞-rings and C^∞-rings with corners. The functor from interior C^∞-rings with corners to sets $(\mathfrak{C}, \mathfrak{C}_{\rm in} \amalg \{0\}) \mapsto \mathfrak{C}_{\rm in} \amalg \{0\}$ does not respect limits (even though the functor $\Pi_{\rm in} : (\mathfrak{C}, \mathfrak{C}_{\rm in} \amalg \{0\}) \mapsto \mathfrak{C}_{\rm in}$ does respect limits as in Theorem 4.3.7), so only the definition of sheaves in terms of equalisers makes sense for this category.

It is often required in ordinary Algebraic Geometry for presheaves to satisfy (iii) as in Hartshorne [33, §II.1]. However, this would imply sheaves of interior C^∞-rings with corners are not presheaves of C^∞-rings with corners and create additional difficulties, so we do not require this. We discuss this further in §5.1 and in Remark 5.1.5.

We now assume that $\mathcal{A}$ is also cocomplete, that is, it has small colimits and equalisers. Abelian groups, rings, C^∞-rings, monoids etc. all satisfy this, as will (interior) C^∞-rings with corners.

Definition 2.3.2. For $\mathcal{E}$ a presheaf valued in $\mathcal{A}$ on a topological space X, then we can define the *stalk* $\mathcal{E}$ at a point $x \in X$ to be the direct limit of the $\mathcal{E}(U)$ in $\mathcal{A}$ for all $U \subseteq X$ with $x \in U$, using the restriction maps ρ_{UV}.

If there is a faithful functor $F : \mathcal{A} \to \textbf{Sets}$ taking an object of $\mathcal{A}$ to its underlying set that preserves colimits, then explicitly it can be written as a set of equivalence classes of sections $s \in F(\mathcal{E}(U))$ for any open U which contains x, where the equivalence relation is such that $s_1 \sim s_2$ for $s_1 \in F(\mathcal{E}(U))$ and $s_2 \in F(\mathcal{E}(V))$ with $x \in U, V$ if there is an open set $W \subset V \cap U$ with $x \in W$ and $s_1|_W = s_2|_W \in F(\mathcal{E}(W))$.

The stalk is also an element of $\mathcal{A}$ and the restriction morphisms give rise to morphisms $\rho_{U,x} : \mathcal{E}(U) \to \mathcal{E}_x$. A morphism $\phi : \mathcal{E} \to \mathcal{F}$ induces morphisms $\phi_x : \mathcal{E}_x \to \mathcal{F}_x$ for all $x \in X$; this is an isomorphism if and only if ϕ_x is an isomorphism for all $x \in X$.

Definition 2.3.3. There is a *sheafification* functor which takes the category of presheaves over a topological space X valued in $\mathcal{A}$ and their natural transformations to $\mathrm{Sh}(X, \mathcal{A})$. This is defined as a left adjoint to the inclusion of $\mathrm{Sh}(X, \mathcal{A})$ into the category of presheaves over X. We say the image of a presheaf $\mathcal{E}$ over X is the sheaf $\hat{\mathcal{E}}$, and the adjoint property gives a morphism $\pi : \mathcal{E} \to \hat{\mathcal{E}}$ and a universal property: whenever we have a morphism $\phi : \mathcal{E} \to \mathcal{F}$ of presheaves of abelian groups on X and $\mathcal{F}$ is a sheaf, then there is a unique morphism $\hat{\phi} : \hat{\mathcal{E}} \to \mathcal{F}$ with $\phi = \hat{\phi} \circ \pi$. This implies sheafification is unique up to canonical isomorphism.

The sheafification always exists for our categories $\mathcal{A}$, and there is an isomorphism of stalks $\mathcal{E}_x \cong \hat{\mathcal{E}}_x$. If there is a faithful functor $F : \mathcal{A} \to \textbf{Sets}$ taking an object of $\mathcal{A}$ to its underlying set that preserves colimits and limits, it can be constructed (as in [33, Prop. II.1.2]) by defining $\hat{\mathcal{E}}(U)$ as the subset of all functions $t : U \to \amalg_{x\in U}\mathcal{E}_x$ such that for all $x \in U$, then $t(x) = F(\rho_{V,x})(s) \in \mathcal{E}_x$ for some $s \in F(\mathcal{E}(V))$ for open $V \subset U$, $x \in V$.

If $f : X \to Y$ is a continuous map of topological spaces, we can consider *pushforwards* and *pullbacks* of sheaves by f. We will use both of these definitions when defining C^∞-schemes (with corners).

Definition 2.3.4. If $f : X \to Y$ is a continuous map of topological spaces, and $\mathcal{E}$ is a sheaf valued in $\mathcal{A}$ on X, then the *direct image* (or *pushforward*) sheaf $f_*(\mathcal{E})$ on Y is defined by $\big(f_*(\mathcal{E})\big)(U) = \mathcal{E}\big(f^{-1}(U)\big)$ for all open $U \subseteq V$. Here, we have restriction maps $\rho'_{UV} = \rho_{f^{-1}(U)f^{-1}(V)} : \big(f_*(\mathcal{E})\big)(U) \to \big(f_*(\mathcal{E})\big)(V)$ for all open $V \subseteq U \subseteq Y$ so that $f_*(\mathcal{E})$ is a sheaf valued in $\mathcal{A}$ on Y.

For a morphism $\phi : \mathcal{E} \to \mathcal{F}$ in $\mathrm{Sh}(X, \mathcal{A})$ we can define $f_*(\phi) : f_*(\mathcal{E}) \to f_*(\mathcal{F})$ by $\big(f_*(\phi)\big)(U) = \phi\big(f^{-1}(U)\big)$ for all open $U \subseteq Y$. This gives a morphism $f_*(\phi)$ in $\mathrm{Sh}(Y, \mathcal{A})$, and a functor $f_* : \mathrm{Sh}(X, \mathcal{A}) \to \mathrm{Sh}(Y, \mathcal{A})$. For two continuous maps of topological spaces, $f : X \to Y$, $g : Y \to Z$, then $(g \circ f)_* = g_* \circ f_*$.

Definition 2.3.5. For a continuous map $f : X \to Y$ topological spaces and a sheaf $\mathcal{E}$ valued in $\mathcal{A}$ on Y, then we define the *pullback* (*inverse image*) of $\mathcal{E}$ under f to be the sheafification of the presheaf $U \mapsto \lim_{A \supseteq f(U)} \mathcal{E}(A)$ for open $U \subseteq X$, where the direct limit is taken over all open $A \subseteq Y$ containing $f(U)$, using the restriction maps ρ_{AB} in $\mathcal{E}$. We write this sheaf as $f^{-1}(\mathcal{E})$.

If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism in $\mathrm{Sh}(Y, \mathcal{A})$, then there is a pullback morphism $f^{-1}(\phi) : f^{-1}(\mathcal{E}) \to f^{-1}(\mathcal{F})$.

Remark 2.3.6. (a) Pullbacks written $f^{-1}(\mathcal{E})$ as in Definition 2.3.5 are used for sheaves of abelian groups or C^∞-rings, however there are different notions $\underline{f}^*(\mathcal{E})$ or $f^*(\mathcal{E})$ for pullbacks of sheaves of $\mathcal{O}_Y$-modules $\mathcal{E}$ that are more involved and discussed in §2.5.

(b) For a continuous map $f : X \to Y$ of topological spaces we have functors $f_* : \mathrm{Sh}(X, \mathcal{A}) \to \mathrm{Sh}(Y, \mathcal{A})$, and $f^{-1} : \mathrm{Sh}(Y, \mathcal{A}) \to \mathrm{Sh}(X, \mathcal{A})$. Hartshorne [33, Ex. II.1.18] gives a natural bijection

$$\mathrm{Hom}_X\big(f^{-1}(\mathcal{E}), \mathcal{F}\big) \cong \mathrm{Hom}_Y\big(\mathcal{E}, f_*(\mathcal{F})\big) \tag{2.3.1}$$

for all $\mathcal{E} \in \mathrm{Sh}(Y, \mathcal{A})$ and $\mathcal{F} \in \mathrm{Sh}(X, \mathcal{A})$, so that f_* is right adjoint to f^{-1}. This will be important in many proofs we consider.

2.4 C^∞-ringed spaces and C^∞-schemes

We now define (local) C^∞-ringed spaces and C^∞-schemes and consider their properties. New material includes the proof of Lemma 2.4.6, a discussion of limits and colimits in Remark 2.4.16, and §2.4.1, which considers products of C^∞-schemes.

Definition 2.4.1. A *C^∞-ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf $\mathcal{O}_X$ of C^∞-rings on X.

A *morphism* $\underline{f} = (f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of C^∞ ringed spaces consists of a continuous map $f : X \to Y$ and a morphism $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves of C^∞-rings on X, for $f^{-1}(\mathcal{O}_Y)$ the inverse image sheaf as in Definition 2.3.5. From (2.3.1), we know f_* is right adjoint to f^{-1}, so there is a natural bijection

$$\mathrm{Hom}_X\big(f^{-1}(\mathcal{O}_Y), \mathcal{O}_X\big) \cong \mathrm{Hom}_Y\big(\mathcal{O}_Y, f_*(\mathcal{O}_X)\big). \tag{2.4.1}$$

We will write $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ for the morphism of sheaves of C^∞-rings on Y corresponding to the morphism of sheaves of C^∞-rings on X $f^\sharp$ under (2.4.1), so that

$$f^\sharp : f^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X \quad \leftrightsquigarrow \quad f_\sharp : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X). \tag{2.4.2}$$

Given two C^∞-ringed space morphisms $\underline{f} : \underline{X} \to \underline{Y}$ and $\underline{g} : \underline{Y} \to \underline{Z}$ we can compose them to form

$$\underline{g} \circ \underline{f} = \big(g \circ f, (g \circ f)^\sharp\big) = \big(g \circ f, f^\sharp \circ f^{-1}(g^\sharp)\big).$$

If we consider $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$, then the composition is

$$(g \circ f)_\sharp = g_*(f_\sharp) \circ g_\sharp : \mathcal{O}_Z \longrightarrow (g \circ f)_*(\mathcal{O}_X) = g_* \circ f_*(\mathcal{O}_X).$$

We call $\underline{X} = (X, \mathcal{O}_X)$ a *local C^∞-ringed space* if it is C^∞-ringed space for which the stalks $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at x are local C^∞-rings for all $x \in X$. As in Definition 2.1.9, since morphisms of local C^∞-rings are automatically local morphisms, morphisms of local C^∞-ringed spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ are just morphisms of C^∞-ringed spaces without any additional locality condition. Local C^∞-ringed spaces are called *Archimedean C^∞-spaces* in Moerdijk, van Quê and Reyes [70, §3].

We will follow the notation of Joyce [40] and write $\mathbf{C^\infty RS}$ for the category of C^∞-ringed spaces, and $\mathbf{LC^\infty RS}$ for the full subcategory of local C^∞-ringed spaces. We write underlined upper case letters such as $\underline{X}, \underline{Y}, \underline{Z}, \ldots$ to represent C^∞-ringed spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y), (Z, \mathcal{O}_Z), \ldots$, and underlined lower case letters $\underline{f}, \underline{g}, \ldots$ to represent morphisms of C^∞-ringed spaces $(f, f^\sharp), (g, g^\sharp), \ldots$. When we write '$x \in \underline{X}$' we mean that $\underline{X} = (X, \mathcal{O}_X)$ and $x \in X$. If we write '*$\underline{U}$ is open in $\underline{X}$*' we will mean that $\underline{U} = (U, \mathcal{O}_U)$ and $\underline{X} = (X, \mathcal{O}_X)$ with $U \subseteq X$ an open set and $\mathcal{O}_U = \mathcal{O}_X|_U$.

Example 2.4.2. For a manifold X, we have a C^∞-ringed space $\underline{X} = (X, \mathcal{O}_X)$ with topological space X and its sheaf of smooth functions $\mathcal{O}_X(U) = C^\infty(U)$ for each open subset $U \subseteq X$, with $C^\infty(U)$ defined in Example 2.1.4. If $V \subseteq U \subseteq X$ then the restriction morphisms $\rho_{UV} : C^\infty(U) \to C^\infty(V)$ are the usual restriction of a function to an open subset $\rho_{UV} : c \mapsto c|_V$.

Partitions of unity allow us to verify that $\mathcal{O}_X$ is a sheaf of C^∞-rings on X (not just a presheaf), so $\underline{X} = (X, \mathcal{O}_X)$ is a C^∞-ringed space. As the stalks $\mathcal{O}_{X,x}$ at $x \in X$ are local C^∞-rings, isomorphic to the ring of germs as in Example 2.1.16, then $\underline{X}$ is a local C^∞-ringed space.

For a smooth map of manifolds $f : X \to Y$ with corresponding local C^∞-ringed spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ as above we define $f_\sharp(U) : \mathcal{O}_Y(U) = C^\infty(U) \to \mathcal{O}_X(f^{-1}(U)) = C^\infty(f^{-1}(U))$ for each open $U \subseteq Y$ by $f_\sharp(U) : c \mapsto c \circ f$ for all $c \in C^\infty(U)$. This gives a morphism $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ of sheaves of C^∞-rings on Y. Then $\underline{f} = (f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of (local) C^∞-ringed spaces with $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ corresponding to $f_\sharp$ under (2.4.2)

Definition 2.4.3. Let $\mathfrak{C}$ be a C^∞-ring, and write $X_{\mathfrak{C}}$ for the set of all $\mathbb{R}$-points x of $\mathfrak{C}$, as in Definition 2.1.13. Write $\mathcal{T}_{\mathfrak{C}}$ for the topology on $X_{\mathfrak{C}}$ that has basis of open sets $U_c = \big\{x \in X_{\mathfrak{C}} : x(c) \neq 0\big\}$ for all $c \in \mathfrak{C}$. For each $c \in \mathfrak{C}$ define a map $c_* : X_{\mathfrak{C}} \to \mathbb{R}$ such that $c_* : x \mapsto x(c)$.

For a morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ of C^∞-rings, we can define $f_\phi : X_{\mathfrak{D}} \to X_{\mathfrak{C}}$ by $f_\phi(x) = x \circ \phi$, which is continuous.

From Joyce [40, Lem. 4.15], this definition implies $\mathcal{T}_{\mathfrak{C}}$ is the weakest topology on $X_{\mathfrak{C}}$ such that the $c_* : X_{\mathfrak{C}} \to \mathbb{R}$ are continuous for all $c \in \mathfrak{C}$, and it implies that $(X_{\mathfrak{C}}, \mathcal{T}_{\mathfrak{C}})$ is a regular, Hausdorff topological space.

Definition 2.4.4. For a C^∞-ring $\mathfrak{C}$, we define the *spectrum* of $\mathfrak{C}$, and write it as $\operatorname{Spec}\mathfrak{C}$. Here, $\operatorname{Spec}\mathfrak{C}$ is a C^∞-ringed space $(X, \mathcal{O}_X)$, with X the topological space $X_{\mathfrak{C}}$ from Definition 2.4.3. For open $U \subseteq X$, then $\mathcal{O}_X(U)$ is the set of functions $s : U \to \coprod_{x\in U} \mathfrak{C}_x$, where we write s_x for the image of x under s, such that around each point $x \in U$ there is an open subset $x \in W \subseteq U$ and element $c \in \mathfrak{C}$ with $s_x = \pi_x(c) \in \mathfrak{C}_x$ for all $x \in W$. This is a C^∞-ring with the operations Φ_f on $\mathcal{O}_X(U)$ defined using the operations Φ_f on $\mathfrak{C}_x$.

For $s \in \mathcal{O}_X(U)$, the restriction map of functions $s \mapsto s|_V$ for open $V \subseteq U \subseteq X$ is a morphism of C^∞-rings, giving the restriction map $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$. The stalk $\mathcal{O}_{X,x}$ at $x \in X$ is isomorphic to $\mathfrak{C}_x$ by Joyce [40, Lem. 4.18], which is a local C^∞-ring. Hence $(X, \mathcal{O}_X)$ is a local C^∞-ringed space.

For a morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ of C^∞-rings, then we have an induced morphism of local C^∞-rings, $\phi_x : \mathfrak{C}_{f_\phi(x)} \to \mathfrak{D}_x$. If we let $(X, \mathcal{O}_X) = \operatorname{Spec}\mathfrak{C}$, $(Y, \mathcal{O}_Y) = \operatorname{Spec}\mathfrak{D}$, then for open $U \subseteq X$ define $(f_\phi)_\sharp(U) : \mathcal{O}_X(U) \to \mathcal{O}_Y(f_\phi^{-1}(U))$ by $(f_\phi)_\sharp(U)s : x \mapsto \phi_x(s_{f_\phi(x)})$. This gives a morphism $(f_\phi)_\sharp : \mathcal{O}_X \to (f_\phi)_*(\mathcal{O}_Y)$ of sheaves of C^∞-rings on X. Then $\underline{f}_\phi = (f_\phi, f_\phi^\sharp) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of local C^∞-ringed spaces, where $f_\phi^\sharp$ corresponds to $(f_\phi)_\sharp$ under (2.4.2). Then Spec is a functor $\mathbf{C^\infty Rings^{op}} \to \mathbf{LC^\infty RS}$, called the *spectrum functor*, where $\operatorname{Spec}\phi : \operatorname{Spec}\mathfrak{D} \to \operatorname{Spec}\mathfrak{C}$ is defined by $\operatorname{Spec}\phi = \underline{f}_\phi$.

Example 2.4.5. For a manifold X then $\operatorname{Spec} C^\infty(X)$ is isomorphic to the local C^∞-ringed space $\underline{X}$ constructed in Example 2.4.2.

The following lemma will be important for considering C^∞-schemes with corners. As the lemma is stated without proof in [40, Lem 4.28], we include a proof here.

Lemma 2.4.6. *Take element $c \in \mathfrak{C}$ in a C^∞-ring $\mathfrak{C}$ and let $\underline{X} = \operatorname{Spec}\mathfrak{C} = (X, \mathcal{O}_X)$. If we consider $U_c = \{x \in X : x(c) \neq 0\}$ as in Definition 2.4.3, then $U_c \subseteq X$ is open and $\underline{X}|_{U_c} = (U_c, \mathcal{O}_X|_{U_c}) \cong \operatorname{Spec}\mathfrak{C}[c^{-1}]$.*

Proof. Let $\phi : \mathfrak{C} \to \mathfrak{C}[c^{-1}]$ be the localisation morphism, and write $\operatorname{Spec}(\mathfrak{C}[c^{-1}]) = (Y, \mathcal{O}_Y) = \underline{Y}$. Consider the morphism $\operatorname{Spec}(\phi) = (\phi^*, \phi^\sharp) : \underline{Y} \to \underline{X}$ and the restriction $\rho_{X,U_c} : \mathcal{O}_X \to \mathcal{O}_X|_{U_c}$. We will show that $\phi^* : Y \to X$ is an isomorphism onto its image, U_c, and that $\phi_\sharp : \mathcal{O}_X \to \phi_*(\mathcal{O}_Y)$ is an isomorphism upon restriction to U_c, so that $\operatorname{Spec}(\phi)$ is an isomorphism onto its image $\underline{X}|_{U_c}$.

Firstly, as $\phi : \mathfrak{C} \to \mathfrak{C}[c^{-1}]$ is a C^∞-ring morphism, any $\mathbb{R}$-point, $\hat{x}$ of $\mathfrak{C}[c^{-1}]$ corresponds to a unique $\mathbb{R}$-point, x, of $\mathfrak{C}$. If $x \in X \backslash U_c$ is an $\mathbb{R}$-point of $\mathfrak{C}$, then $x \notin U_c$, and the definition of U_c means $x(c) = 0$. However, in $\mathfrak{C}[c^{-1}]$, c is invertible, so x cannot give a corresponding an $\mathbb{R}$-point of $\mathfrak{C}[c^{-1}]$.

For any $x \in U_c$, $x(c) \neq 0$, then the universal property of $\mathfrak{C}[c^{-1}]$ implies there is a unique corresponding $\mathbb{R}$-point $\hat{x}$ of $\mathfrak{C}[c^{-1}]$. Hence Y is isomorphic as a set to U_c. Using the definition of ϕ^*, then this correspondence is the continuous map $\phi^* : Y \to X$ with image U_c. That is, any open set $U \subset X$, gives an open set $U \cap U_c$ in Y.

To see it is a homeomorphism, consider a base element of the topology $U_d = \{y \in Y : y(d) \neq 0\} \subset Y \cong U_c$, for some $d \in \mathfrak{C}[c^{-1}]$. We show that around any point $\tilde{y} \in U_d$, there is an open set U_k, with $k \in \phi(\mathfrak{C})$, such that $\tilde{y} \in U_k \subset U_d$, so the topology is the subset topology. Now d must be of the form $\Phi_f(a, c^{-1})$ for some $a \in \mathfrak{C}$ and some smooth $f : \mathbb{R}^2 \to \mathbb{R}$. Take $\tilde{y} \in U_d$, then $\tilde{y}(d) = f(\tilde{y}(a), \tilde{y}(c^{-1}))$, with $\tilde{y}(c^{-1}) = \tilde{y}(c)^{-1} \neq 0$. Assume $r = \tilde{y}(c) > 0$ without loss of generality. Then consider that $\{y \in Y : y(c) \in (\frac{r}{2}, \frac{3r}{2})\} = U_{\Phi_g(c)}$ is open in Y, where $g : \mathbb{R} \to \mathbb{R}$ is a smooth bump function with support $(\frac{r}{2}, \frac{3r}{2})$, and $U_{\Phi_g(c)}$ contains y. Let $h : \mathbb{R} \to \mathbb{R}$ be any smooth function with $h(t) = \frac{1}{t}$ for $t \in (\frac{r}{2}, \frac{3r}{2})$ and positive elsewhere. Then for all $y \in U_d \cap U_{\Phi_g(c)}$, we have

$$0 \neq y(d) = f(y(a), y(c^{-1})) = f(y(a), h(y(c))) = y(\Phi_{f(\cdot,h(\cdot))}(a, c)) = y(b)$$

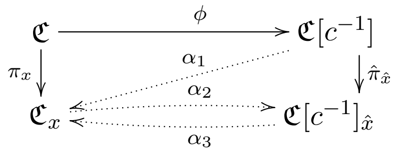
for $b = \Phi_{f(\cdot,h(\cdot))}(a, c) \in \phi(\mathfrak{C})$. So $U_b \supset U_d \cap U_{\Phi_g(c)}$ and it follows that

$$\tilde{y} \in U_{b\Phi_g(c)} = U_b \cap U_{\Phi_g(c)} = U_d \cap U_{\Phi_g(c)} \subset U_d$$

is open in U_c, with $k = b\Phi_g(c) \in \phi(\mathfrak{C})$, as required.

To show the map of sheaves $\phi_\sharp : \mathcal{O}_X \to \phi_*(\mathcal{O}_Y)$ is an isomorphism upon restriction to U_c, we show it is an isomorphism on stalks, that is $\mathfrak{C}_x$ and $\mathfrak{C}[c^{-1}]_{\hat{x}}$ are isomorphic for each $x \in U_c$ and corresponding $\hat{x} \in Y$.

For $x \in U_c$, then the image of c under the map $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$ is invertible. By the universal property of $\mathfrak{C}[c^{-1}]$, there is a unique map $\alpha_1 : \mathfrak{C}[c^{-1}] \to \mathfrak{C}_x$. The universal property of $\mathfrak{C}_x$ then gives a unique morphism $\alpha_2 : \mathfrak{C}_x \to \mathfrak{C}[c^{-1}]_{\hat{x}}$, and the universal property of $\mathfrak{C}[c^{-1}]_{\hat{x}}$ then gives a unique morphism $\alpha_3 : \mathfrak{C}[c^{-1}]_{\hat{x}} \to \mathfrak{C}_x$. As the following diagram commutes, these two maps must be inverses, and the localisations must be isomorphic.



By definition, α_3 is the stalk map $\phi_x^\sharp$, which implies $\mathrm{Spec}(\phi) : \underline{Y} \to \underline{X}$ is an isomorphism onto its image $\underline{X}|_{U_c}$ as required. $\square$

Definition 2.4.7. There is a *global sections functor* $\Gamma : \mathbf{LC^\infty RS} \to \mathbf{C^\infty Rings^{op}}$, which takes $(X, \mathcal{O}_X)$ to $\mathcal{O}_X(X)$ and morphisms $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to $\Gamma : (f, f^\sharp) \mapsto f_\sharp(Y)$, for $f_\sharp$ relating $f^\sharp$ as in (2.4.2).

For each C^∞-ring $\mathfrak{C}$ we can define a morphism $\Xi_\mathfrak{C} : \mathfrak{C} \to \Gamma \circ \mathrm{Spec}\,\mathfrak{C}$. Here, for $c \in \mathfrak{C}$ then $\Xi_\mathfrak{C}(c) : X_\mathfrak{C} \to \coprod_{x \in X_\mathfrak{C}} \mathfrak{C}_x$ is defined by $\Xi_\mathfrak{C}(c)_x = \pi_x(c) \in \mathfrak{C}_x$, so $\Xi_\mathfrak{C}(c) \in \mathcal{O}_{X_\mathfrak{C}}(X_\mathfrak{C}) = \Gamma \circ \mathrm{Spec}\,\mathfrak{C}$. This $\Xi_\mathfrak{C}$ is a C^∞-ring morphism as it it composed of C^∞-ring morphism $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$ and the C^∞-operations on $\mathcal{O}_{X_\mathfrak{C}}(X_\mathfrak{C})$ are defined pointwise in the $\mathfrak{C}_x$. In fact, it defines a natural transformation $\Xi : \mathrm{id}_{\mathbf{C^\infty Rings}} \Rightarrow \Gamma \circ \mathrm{Spec}$ of functors $\mathrm{id}_{\mathbf{C^\infty Rings}}, \Gamma \circ \mathrm{Spec} : \mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings}$.

Theorem 2.4.8. *The functor* Spec : $\mathbf{C^\infty Rings^{op}} \to \mathbf{LC^\infty RS}$ *is* ***right adjoint*** *to* $\Gamma : \mathbf{LC^\infty RS} \to \mathbf{C^\infty Rings^{op}}$. *Here,* Ξ *is the unit of the adjunction between* Γ *and* Spec. *This implies* Spec *preserves limits as in [21, p. 687]. Hence if we have* C^∞*-ring morphisms*

$\phi : \mathfrak{F} \to \mathfrak{D}$, $\psi : \mathfrak{F} \to \mathfrak{E}$ *in* $\mathbf{C^\infty Rings}$ *then their pushout* $\mathfrak{C} = \mathfrak{D} \amalg_{\mathfrak{F}} \mathfrak{E}$ *has image that is isomorphic to the fibre product* $\operatorname{Spec}\mathfrak{C} \cong \operatorname{Spec}\mathfrak{D} \times_{\operatorname{Spec}\mathfrak{F}} \operatorname{Spec}\mathfrak{E}$.

We extend this theorem to C^∞-schemes with corners in §5.1.

Remark 2.4.9. The definition of spectrum functor follows Dubuc [21] and Joyce [40], and it is called the *Archimedean spectrum* in Moerdijk et al. [70, §3]. They also show it is a right adjoint to the global sections functor as above.

In [70, §1] they consider another definition of spectrum $\operatorname{Spec}\mathfrak{C}$ which uses 'C^∞-radical prime ideals' not $\mathbb{R}$-points. This means they use a different, less restrictive, definition of local C^∞-ring. This is not an equivalent definition to our definition, as the image of the functor is not contained in $\mathbf{LC^\infty RS}$, but in a larger subcategory of $\mathbf{C^\infty RS}$ which they call *C^∞-spaces*. However, in [70, §3] they show there is a right adjoint to the inclusion of $\mathbf{LC^\infty RS}$ into the category of C^∞-spaces. Their definition of spectrum composed with this right adjoint gives a right adjoint to our global sections functor. As right adjoints are naturally isomorphic, then this composition is naturally isomorphic to our spectrum functor.

Definition 2.4.10. Elements $\underline{X} \in \mathbf{LC^\infty RS}$ that are isomorphic to $\operatorname{Spec}\mathfrak{C}$ for some $\mathfrak{C} \in \mathbf{C^\infty Rings}$ are called *affine C^∞-schemes*. Elements $\underline{X} \in \mathbf{LC^\infty RS}$ that are locally isomorphic to $\operatorname{Spec}\mathfrak{C}$ for some $\mathfrak{C} \in \mathbf{C^\infty Rings}$ (depending upon the open sets) are called *C^∞-schemes*. We define $\mathbf{C^\infty Sch}$ and $\mathbf{AC^\infty Sch}$ to be the full subcategories of C^∞-schemes and affine C^∞-schemes in $\mathbf{LC^\infty RS}$ respectively.

Remark 2.4.11. Unlike ordinary algebraic geometry, affine C^∞-schemes are very general objects. All manifolds are affine, and all their fibre products are affine, suggesting that the use of C^∞-schemes in Derived Differential Geometry can (and usually is) confined to only affine (finitely presentable) C^∞-schemes. Contrary to this, we will need non-affine C^∞-schemes with corners in §5, which will require understanding non-affine C^∞-schemes.

More generally, all second countable, metrizable C^∞-schemes are affine, and it is necessary that affine C^∞-schemes are Hausdorff and regular. Joyce [40, Th. 4.41] shows that a local C^∞-ringed space that is Hausdorff, Lindelöf and has *smoothly generated topology* is an affine C^∞-scheme. Here, $\underline{X} \in \mathbf{LC^\infty RS}$ has smoothly generated topology if the sets $U_c = \{x \in X : \pi \circ \pi_x(c) \neq 0 \in \mathbb{R}\}$ where $c \in \mathcal{O}_X(X)$ form a basis for the topology of $\underline{X}$.

As in Joyce [40, Prop. 4.34] we have the following crucial isomorphism that allows us to define complete C^∞-rings.

Proposition 2.4.12. *For each C^∞-ring* $\mathfrak{C}$, $\operatorname{Spec}\Xi_{\mathfrak{C}} : \operatorname{Spec}\circ\Gamma\circ\operatorname{Spec}\mathfrak{C} \to \operatorname{Spec}\mathfrak{C}$ *is an isomorphism in* $\mathbf{LC^\infty RS}$.

Definition 2.4.13. A C^∞-ring $\mathfrak{C}$ is called *complete* if $\Xi_{\mathfrak{C}} : \mathfrak{C} \to \Gamma \circ \operatorname{Spec} \mathfrak{C}$ is an isomorphism. We define $\mathbf{C^\infty Rings^{co}}$ to be the full subcategory in $\mathbf{C^\infty Rings}$ of complete C^∞-rings.

Using Proposition 2.4.12 we see that complete C^∞-rings are isomorphic to the image of the functor $\Gamma \circ \operatorname{Spec} : \mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings}$, which gives a left adjoint to the inclusion of $\mathbf{C^\infty Rings^{co}}$ into $\mathbf{C^\infty Rings}$. Let this left adjoint be the functor $R^{\mathrm{co}}_{\mathrm{all}} : \mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings^{co}}$.

An example of a C^∞-ring that is not complete is the quotient of $C^\infty(\mathbb{R}^n)$ by the ideal of compactly supported functions, and $R^{\mathrm{co}}_{\mathrm{all}}$ applied to this quotient returns the zero C^∞-ring.

In ordinary algebraic geometry, we have a contravariant equivalence of categories between ordinary affine schemes and commutative rings. This is used to show all (finite) limits of ordinary schemes exist. However, this is not true in the case of affine C^∞-schemes; in some sense, there are more C^∞-rings than affine C^∞-schemes, and Spec is not full nor faithful on $\mathbf{C^\infty Rings^{op}}$. The next theorem tells us that the category of complete C^∞-rings gives an equivalence of categories instead.

The following is a summary of results from [40, Prop. 4.11,Th. 4.25].

Theorem 2.4.14. **(a)** $\operatorname{Spec}|_{(\mathbf{C^\infty Rings^{co}})^{\mathbf{op}}} : (\mathbf{C^\infty Rings^{co}})^{\mathbf{op}} \to \mathbf{LC^\infty RS}$ *is full and faithful, and an equivalence of categories* $\operatorname{Spec}|_{\dots} : (\mathbf{C^\infty Rings^{co}})^{\mathbf{op}} \to \mathbf{AC^\infty Sch}$.

(b) *Let $\underline{X}$ be an affine C^∞-scheme. Then* $\underline{X} \cong \operatorname{Spec} \mathcal{O}_X(X)$, *where $\mathcal{O}_X(X)$ is a complete C^∞-ring.*

(c) *The functor* $R^{\mathrm{co}}_{\mathrm{all}} : \mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings^{co}}$ *is left adjoint to the inclusion functor* $\mathrm{inc} : \mathbf{C^\infty Rings^{co}} \hookrightarrow \mathbf{C^\infty Rings}$. *That is,* $R^{\mathrm{co}}_{\mathrm{all}}$ *is a* ***reflection functor****.*

(d) *All small colimits exist in* $\mathbf{C^\infty Rings^{co}}$, *although they may not coincide with the corresponding small colimits in* $\mathbf{C^\infty Rings}$.

(e) $\operatorname{Spec}|_{(\mathbf{C^\infty Rings^{co}})^{\mathbf{op}}} = \operatorname{Spec} \circ \mathrm{inc} : (\mathbf{C^\infty Rings^{co}})^{\mathbf{op}} \to \mathbf{LC^\infty RS}$ *is right adjoint to* $R^{\mathrm{co}}_{\mathrm{all}} \circ \Gamma : \mathbf{LC^\infty RS} \to (\mathbf{C^\infty Rings^{co}})^{\mathbf{op}}$. *Thus* $\operatorname{Spec}|_{\dots}$ *takes limits in* $(\mathbf{C^\infty Rings^{co}})^{\mathbf{op}}$ *(equivalently, colimits in* $\mathbf{C^\infty Rings^{co}}$*) to limits in* $\mathbf{LC^\infty RS}$.

Using (a), that small limits exist in the category of $\mathbf{C^\infty Rings}$, and that $\Gamma : \mathbf{LC^\infty RS} \to \mathbf{C^\infty Rings}$ is a left adjoint with image in $(\mathbf{C^\infty Rings^{co}})^{\mathbf{op}}$ when restricted to $\mathbf{AC^\infty Sch}$, then small limits of $\mathbf{C^\infty Rings^{co}}$ exist and coincide with small limits in $\mathbf{C^\infty Rings}$. As we have an equivalence of categories, $(\mathbf{C^\infty Rings^{co}})^{\mathbf{op}} \to \mathbf{AC^\infty Sch}$, then $\mathbf{AC^\infty Sch}$ also has all small colimits and small limits. As Spec is a right adjoint, then limits in $\mathbf{AC^\infty Sch}$ coincide with limits in $\mathbf{C^\infty Sch}$ and $\mathbf{LC^\infty RS}$, however it is not necessarily true that colimits in $\mathbf{AC^\infty Sch}$ coincide with colimits in $\mathbf{C^\infty Sch}$ and $\mathbf{LC^\infty RS}$.

We have the following theorem on limits in $\mathbf{C^\infty RS}$ and subcategories.

Theorem 2.4.15. (i) *All finite limits exist in the category* $\mathbf{C^\infty RS}$.

(ii) *The full subcategory* $\mathbf{LC^\infty RS}$ *is closed under finite limits in* $\mathbf{C^\infty RS}$.

(iii) *The full subcategories* $\mathbf{AC^\infty Sch}$ *and* $\mathbf{C^\infty Sch}$ *of* $\mathbf{LC^\infty RS}$ *are closed under all finite limits in* $\mathbf{LC^\infty RS}$. *Hence fibre products and all finite limits exist in these categories.*

Remark 2.4.16. In the previous Theorem, for (i) and (ii) refer to Joyce [40, Prop. 4.11, Th. 4.25], and for (ii) refer to Dubuc [21, Prop. 7]. Note however that the proof of (iii) relies on the existence of complete C^∞-rings; this part of the theorem is mentioned in [40] however the proof is not detailed.

One can prove this using the following: Theorem 2.4.14(a) shows all limits are exactly those from colimits of complete C^∞-rings, then Theorem 2.4.14(d) implies all small colimits exist, and therefore all small limits exist in $\mathbf{AC^\infty Sch}$. One then shows they coincide with small limits in $\mathbf{LC^\infty RS}$. Finite limits of $\mathbf{C^\infty Sch}$ in $\mathbf{LC^\infty RS}$ are then shown to be locally affine, giving the result. Alternatively, following the standard proof in ordinary algebraic geometry, see Hartshorne [33, Th. 3.3]), one could show that glueing C^∞-schemes together on affine neighbourhoods is a scheme, then glue the fibre products of the local affine neighbourhoods, which gives the same result.

In Section 5, we have similar results for local C^∞-ringed spaces with corners and C^∞-schemes with corners, where we describe the proofs in more detail. In fact, all small limits exist in $\mathbf{C^\infty RS}$ and $\mathbf{LC^\infty RS}$, we show this is true in §5.1.1 with our more general results on C^∞-ringed spaces with corners.

Demazure and Gabriel, [17, I §1 1.6] construct small colimits in the categories of ordinary ringed and locally ringed spaces. This construction also applies for (local) C^∞-ringed spaces; the underlying topological space is the colimit of the underlying topological spaces and the sheaf is essentially the limit of the sheaves. For locally ringed spaces, the main issue is to show the stalks are local, where in the C^∞-ring case, this follows from Proposition 2.1.11. We describe this proof explicitly when we consider C^∞-ringed spaces with corners in §5.1.1.

Finally, we consider the embedding of manifolds into C^∞-schemes and whether fibre products respect this embedding. In the following theorem we summarise results found in Dubuc [21, Th. 16], Moerdijk and Reyes [71, § II. Prop. 1.2], and Joyce [40, Cor. 4.27].

Theorem 2.4.17. *There is a full and faithful functor* $F_{\mathbf{Man}}^{\mathbf{AC^\infty Sch}} : \mathbf{Man} \to \mathbf{AC^\infty Sch}$ *that takes a manifold* X *to the affine* C^∞*-scheme* $\underline{X} = (X, \mathcal{O}_X)$, *where* $\mathcal{O}_X(U) = C^\infty(U)$ *is* *the usual smooth sections on U. Here $(X, \mathcal{O}_X) \cong \mathrm{Spec}(C^\infty(X))$ and hence $\underline{X}$ is affine. The functor $F_{\mathbf{Man}}^{\mathbf{AC^\infty Sch}}$ sends transverse fibre products of manifolds to fibre products of C^∞-schemes with corners.*

2.4.1 Products of C^∞-schemes

This section is new and we discuss products of C^∞-schemes.

Remark 2.4.18. When considering infinite products of schemes we will need to consider whether or not *measurable cardinals* exist. We will not define measurable cardinals, but refer the reader to Jech [38, Ch. 10] and Gillman and Jerison [29, Ch. 12] for further details. Measurable cardinals are a type of large cardinal number, that is, a cardinal number that cannot be accessed by the standard arithmetic operations using $\aleph_0$. This means that $\aleph_0, \aleph_1, \aleph_2 \ldots, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots, 2^{\aleph_1}, 2^{2^{\aleph_0}}, \ldots$ and many other cardinal numbers are not measurable cardinals, (cf. [29, p. 161–166]). We say such non-measurable cardinals are *less than any measurable cardinal.*

The existence of measurable cardinals is an axiom of set theory that is independent from the usual ZFC axioms (Zermelo–Fraenkel set axioms and the Axiom of Choice), as in Jech [38, p. 33, 77]. None of our examples or intended applications would need to assume measurable cardinals exist, however the definitions of C^∞-rings and C^∞-rings with corners do not preclude this assumption, so we note carefully where this assumption impacts the theory.

The author would like to acknowledge assistance from Professor George Bergman for details in the proof of the following lemma.

Lemma 2.4.19. *Let I be a set and define $\mathfrak{C} = \prod_{i \in I} \mathbb{R}$. Then any $\mathbb{R}$-point $x : \mathfrak{C} \to \mathbb{R}$ factors through precisely one of the $\mathbb{R}$ in the product provided the cardinality of I is less than any measurable cardinal.*

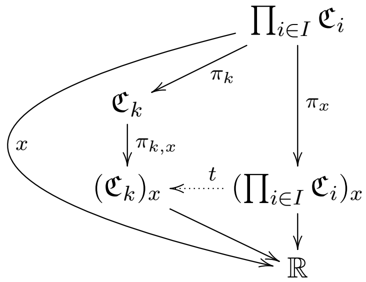
Proof. This follows by considering Bergman and Nahlus [6, Th. 9], which implies that whenever the cardinality of I is less than any measurable cardinal, then $x : \mathfrak{C} \to \mathbb{R}$ factors as a $\mathbb{R}$-algebra morphism through a product of finitely many of the $\mathbb{R}$ in $\mathfrak{C}$, say $\mathbb{R}_{i_1}, \ldots, \mathbb{R}_{i_n}$. This means that there is an element $\hat{c} = (c_i)_{i \in I} \in \mathfrak{C}$, with $c_i = 1$ for $i = i_k$ and zero otherwise, such that $x(\hat{c}) = 1$.

Letting $\hat{d}_{i_k} = (d_i)_{i \in I}$, with $d_i = \delta_i^{i_k}$, then $\sum_{i=1}^k \hat{d}_{i_k} = \hat{c}$, and $x(\hat{c}) = 1 = \sum_{i=1}^k x(\hat{d}_{i_k})$. However, $\hat{d}_{i_{k_1}} \cdot \hat{d}_{i_{k_2}} = 0$, so $x(\hat{d}_{i_{k_1}}) \cdot x(\hat{d}_{i_{k_2}}) = x(\hat{d}_{i_{k_1}} \cdot \hat{d}_{i_{k_2}}) = 0$, so precisely one d_{i_k} is such that $x(d_{i_k}) = 1$ with all others evaluating to 0. Then x must factor through only this $\mathbb{R}_{i_k}$. □

Remark 2.4.20. One can deduce Lemma 2.4.19 using the language of *ultrafilters* (in the sense of Comfort and Negrepontis [16]), which forms part of the reasoning behind [6, Th. 9]. First one shows that any such $x : \mathfrak{C} \to \mathbb{R}$ gives an ultrafilter on I, and that factoring through precisely one of the $\mathbb{R}$ in the product requires it to be a *principal/fixed ultrafilter*. Showing that there are no ultrafilters that do not factor through one of the $\mathbb{R}$ in the product requires showing there are no *non-principal/free ultrafilters* to come from such an x. In Gillman and Jerison [29], this is a property called *realcompact* and they show in [29, p. 163] that such an ultrafilter I only has this property if and only if it has cardinality less that any measurable cardinal, giving the result.

Lemma 2.4.21. *Let I be a set and $\mathfrak{C} = \prod_{i\in I} \mathfrak{C}_i$ be a C^∞-ring that is the product of the C^∞-rings $\mathfrak{C}_i$. If $x : \mathfrak{C} \to \mathbb{R}$ factors through $\mathfrak{C}_k$ then there is a canonical isomorphism $(\mathfrak{C}_k)_x \cong \mathfrak{C}_x$.*

Proof. Consider the following commutative diagram.



(2.4.3)

Here π_k is the projection onto the k-th factor and $\pi_x, \pi_{k,x}$ are the localisation projections, which are surjective. Note that the dotted arrow t exists by the universal property of localisation of $\mathfrak{C}$.

The map $t : (\prod_{i\in I} \mathfrak{C}_i)_x \to (\mathfrak{C}_k)_x$ sends $\pi_x((c_i)_{i\in I}) \in (\prod_{i\in I} \mathfrak{C}_i)_x$ to $\pi_{k,x} \circ \pi_x^{\mathrm{in}}((c_i)_{i\in I}) = \pi_{k,x}(c_k) \in (\mathfrak{C}_k)_x$. This implies t is surjective. To show it is injective, say $t(\pi_x((c_i)_{i\in I})) = t(\pi_x((d_i)_{i\in I})) \in (\mathfrak{C}_k)_x$, then $\pi_{k,x}(c_k) = \pi_{k,x}(d_k)$, so by Proposition 2.1.15 there exists $a \in \mathfrak{C}_k$ with $x(a) \neq 0$ such that $a \cdot (c_k - d_k) = 0$. Then define $(a_i)_{i\in I} \in \prod_{i\in I} \mathfrak{C}_i$ by $a_k = a$ and $a_i = 0$ for $i \neq k$. Then $(a_i)_{i\in I} \cdot ((c_i)_{i\in I} - (d_i)_{i\in I})$ and $x((a_i)_{i\in I}) \neq 0$, which implies $\pi_x((c_i)_{i\in I}) = \pi_x((d_i)_{i\in I}) \in (\prod_{i\in I} \mathfrak{C}_i)_x$, so t is injective and must be an isomorphism. □

Proposition 2.4.22. *If I is a set with cardinality less than any measurable cardinal and if $\{\mathfrak{C}_i\}_{i\in I}$ is a collection of C^∞-rings, then* $\mathrm{Spec}(\prod_{i\in I} \mathfrak{C}_i) \cong \coprod_{i\in I} \mathrm{Spec}(\mathfrak{C}_i)$*. That is, coproducts in C^∞-schemes and affine C^∞-schemes are the same.*

Proof. Let $\underline{X} = \mathrm{Spec}(\prod_{i\in I} \mathfrak{C}_i)$ and $\underline{Y} = \coprod_{i\in I} \mathrm{Spec}(\mathfrak{C}_i)$. Firstly note that any $\mathbb{R}$-point $y_k : \mathfrak{C}_k \to \mathbb{R}$, $k \in I$, gives a unique $\mathbb{R}$-point $x : \prod_{i\in I} \mathfrak{C}_i \to \mathbb{R}$ by $x((c_i)_{i\in I}) \mapsto y_k(c_k)$, so there is an inclusion of sets $Y \hookrightarrow X$.

Take $x \in X$, so that $x : \prod_{i \in I} \mathfrak{C}_i \to \mathbb{R}$ is an $\mathbb{R}$-point. We have an inclusion morphism of C^∞-rings $i : \prod_{i \in I} \mathbb{R}_i \to \prod_{i \in I} \mathfrak{C}_i$, so that the composition $x \circ i : \prod_{i \in I} \mathbb{R}_i \to \mathbb{R}$ is an $\mathbb{R}$-point of $\prod_{i \in I} \mathbb{R}$. By Lemma 2.4.19, this $\mathbb{R}$-point factors through one of the $\mathbb{R}$ in the product; denote this $\mathbb{R}_k$. Then the element $\hat{d}_k = (d_i)_{i \in I} \in \prod_{i \in I} \mathbb{R}$ with $d_i = \delta_i^k$ maps to a similarly described element in $\prod_{i \in I} \mathfrak{C}_i$ and then maps to 1 under x. This gives a morphism from $\mathfrak{C}_k \to \mathbb{R}$ that commutes with the projection from $\prod_{i \in I} \mathfrak{C}_i \to \mathfrak{C}_k$. However, $\hat{d}_{k'} = (d_i)_{i \in I} \in \prod_{i \in I} \mathbb{R}$ with $d_i = \delta_i^{k'}$ and $k' \neq k$ will map to a similarly described element in $\prod_{i \in I} \mathfrak{C}_i$ and then to 0 under x, so that x factors through $\mathfrak{C}_k$ uniquely. Hence $Y = X$ as sets.

Take a basic open set $U_c = \{x : \prod_{i \in I} \mathfrak{C}_i \to \mathbb{R} : x(c) \neq 0\} \subset X$ for some $c = (c_i)_{i \in I} \in \prod_{i \in I} \mathfrak{C}_i$. Then each $x \in U_c$ corresponds to an $\mathbb{R}$-point $y_k : \mathfrak{C}_k \to \mathbb{R}$ for some $k \in I$, and $x(c) \neq 0$ if and only if $y_k(c_k) \neq 0$. Then each U_c is in one-to-one correspondence with disjoint unions of basic opens $\coprod_{i \in I} U^k_{c_k} \subset Y$, where $U^k_{c_k} = \{y_k : \mathfrak{C}_k \to \mathbb{R} : y_k(c_k) \neq 0\}$. As such disjoint unions of basic opens form a basis for the topology of Y then the topologies are the same, and $X = Y$ as topological spaces.

Taking Spec of the projections $\pi_k : \prod_{i \in I} \mathfrak{C}_i \to \mathfrak{C}_k$ gives corresponding morphisms $\mathrm{Spec}(\pi_k) : \mathrm{Spec}(\mathfrak{C}_k) \to \mathrm{Spec}(\prod_{i \in I} \mathfrak{C}_i)$, which we can amalgamate to a morphism

$$\underline{f} = (f, f^\sharp) : \coprod_{i \in I} \mathrm{Spec}(\mathfrak{C}_i) \to \mathrm{Spec}(\prod_{i \in I} \mathfrak{C}_i)$$

using the universal property of a coproduct. We have shown that f is an isomorphism of topological spaces. Take $y_k \in Y$ and $x = f(y_k)$, then $f^\sharp_{y_k} : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y_k}$ is the isomorphism t in Lemma 2.4.21, so $\underline{f}$ is an isomorphism. □

Remark 2.4.23. Proposition 2.4.22 is unlike the case of ordinary algebraic geometry, which we discuss here. In ordinary algebraic geometry, if I is a set and A_i is a collection of rings, then the projections $\prod_{i \in I} A_i \to A_j$ and the universal property of coproducts give a canonical morphism $\underline{\phi} : \coprod_{i \in I} \mathrm{Spec}(A_i) \to \mathrm{Spec}(\prod_{i \in I} A_i)$. This is an isomorphism if I is finite. Let $X = \mathrm{Spec}(\prod_{i \in I} A_i)$, $X_i = \mathrm{Spec}(A_i)$ so $\underline{\phi} = (\phi, \phi^\sharp) : \coprod_{i \in I} X_i \to X$.

If I is infinite (and enough of the A_i are non-zero) then ϕ is not an isomorphism, as then the topological space of the target is larger than the topological space of the domain. To see this, note first that the inclusion of each X_i in X is both open and closed. Here, the prime ideal $p_i \subset A_i$ corresponds to a point $p_i \in \coprod_{j \in I} X_j$, and under ϕ p_i is sent to prime ideal $\hat{p}_i = (a_j)_{j \in I} \in X$ with $a_j = A_j$ for $i \neq j$ and $a_i = p_i$. This means ϕ is a bijection onto its image.

Take ideals in $\prod_{i \in I} A_i$ defined as $b_j = (b_{i,j})_{i \in I}$ with $b_{j,j} = A_j$ and $b_{i,j} = 0$ the zero ideal of A_j whenever $i \neq j$. Let $V(b_j)$ be the basic closed set in $\prod_{i \in I} A_i$ corresponding to

b_j (defined in Hartshorne [33, p. 70] as the set of all prime ideals that contain b_j), then its complement is equal to the image of X_i in X. Hence each image of X_i is open. Also an ideal $q_i \subset A_i$ gives an open set $X_i \setminus V(q_i)$, and then

$$\phi(X_i \setminus V(q_i)) = \phi(X_i) \setminus V(\hat{q}_i)$$

where $\hat{q}_i = (c_j)_{j \in I} \in X$ with $c_j = A_j$ for $i \neq j$ and $c_i = q_i$. This implies ϕ is a homeomorphism onto its image.

Now take ideals $d_j = (d_{i,j})_{i \in I} \subset \prod_{i \in I} A_i$ with $d_{j,j} = 0$ the zero ideal of A_i and $d_{i,j} = A_i$ for all $i \neq j$. These define closed sets $V(d_j)$ that are equal to the image of each X_i in X, so this image is also closed.

However, the direct sum $\sum_{i \in I} A_i$ (which contains all finite linear combinations of elements of the A_i's) is an ideal of $\prod_{i \in I} A_i$. It is equal to $\prod_{i \in I} A_i$ when I is finite. However, when I is infinite it is not equal to $\prod_{i \in I} A_i$, so must be contained in a maximal (prime) ideal that is not of the form $\hat{p}_i$ for some p_i. This prime ideal is not in the image of ϕ, and hence ϕ is not an isomorphism.

We can examine the topology of X and $\coprod_{i \in I} X_i$ when I is infinite in more detail. We see that

$$\phi(\coprod_{i \in I} X_i) = \cup_{i \in I} \phi(X_i) = \cup_{j \in I}(X \setminus V(b_j)) = X \setminus (\cap_{j \in I} V(b_j)) = X \setminus V(\sum_{j \in J} b_j) \subset X,$$

where $\sum_{j \in J} b_j$ is the direct sum of ideals. As $V(\sum_{j \in J} b_j)$ is the complement of $\phi(\coprod_{i \in I} X_i)$, then all prime ideals of $\prod_{i \in I} A_i$ either are equal to $\hat{p}_i$ for some $p_i \subset X_I$ and are in the image of ϕ, or they contain the direct sum and are not in the image of ϕ. Then $V(\sum_{j \in J} b_j)$ is closed in X and not open, and $\phi(\coprod_{i \in I} X_i)$ is open in X but not closed.

This is important because while X is an affine scheme, $\coprod_{i \in I} X_i$ is not affine for I infinite. If it were affine, it would be quasi-compact, that is every open cover would have a finite subcover. [This follows as, say $\{Y \setminus V(d_j)\}_{j \in J}$ is an open cover of an affine scheme $Y = \text{Spec}(B)$ for some ring B, then

$$Y = \cup_{j \in J}(Y \setminus V(d_j)) = Y \setminus \cup_{j \in J} V(d_j) = Y \setminus V(\sum_{j \in J} d_j)$$

implies the direct sum $\sum_{j \in J} d_j = B$, so there is a finite linear combination of the d_j that equals $1 \in B$. The elements in this finite linear combination correspond to the subcover. For more on quasi-compact, see Hartshorne [33, p. 80, 2.13(b)].] However, $\{X_i\}_{i \in I}$ is an open cover of $\coprod_{i \in I} X_i$ and it has no finite subcover, so $\coprod_{i \in I} X_i$ is not affine.

We can also show that any open set of X that contains $V(\sum_{j \in J} b_j)$ must contain all but finitely many of the $\phi(X_i)$. Here an open set is of the form $X \setminus V(K)$ for some ideal

$K \subset \prod_{i\in I} A_i$. If $V(\sum_{j\in J} b_j) \subset X \setminus V(K)$, then

$$\emptyset = V(K) \cap V(\sum_{j\in J} b_j) = V(K + \sum_{j\in J} b_j),$$

so $1 \in K + \sum_{j\in J} b_j = \prod_{i\in I} A_i$. Then K must contain an element that is equal to 1 in all but finitely many of the entries, and the result follows.

We conclude that X is affine, it has (open and closed) affine subschemes $\phi(X_i) \cong X_i$, and has the (non-affine when I is infinite) open subscheme $\phi(\coprod_{i\in I} X_i)$. Hence, the image of $\coprod_{i\in I} X_i$ under ϕ when I is infinite is open but does not cover X, so ϕ is not an isomorphism when I is infinite.

2.5 Sheaves of $\mathcal{O}_X$-modules and cotangent modules

This section follows Joyce [40, §5.3], recalling their definition of cotangent sheaves.

Our definition of sheaf of $\mathcal{O}_X$-module is the usual definition of sheaf of modules on a ringed space as in Hartshorne [33, §II.5] and Grothendieck [32, §0.4.1], using the underlying $\mathbb{R}$-algebra structure on our (sheaves of) C^∞-rings.

Definition 2.5.1. For an element $\underline{X} \in \mathbf{C^\infty RS}$ we define the category $\mathcal{O}_X$-mod. The objects are *sheaves of* $\mathcal{O}_X$*-modules* (or simply $\mathcal{O}_X$*-modules*) $\mathcal{E}$ on X. Here, $\mathcal{E}$ is a functor on open sets $U \subseteq X$ such that $\mathcal{E} : U \mapsto \mathcal{E}(U) \in \mathcal{O}_X(U)$-mod is a sheaf as in Definition 2.3.1. This means we have linear restriction maps $\mathcal{E}_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ for each inclusion of open sets $V \subseteq U \subseteq X$, such that the following commutes

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{E}(U) & \longrightarrow & \mathcal{E}(U) \\ \downarrow{\scriptstyle \rho_{UV}\times\mathcal{E}_{UV}} & & {\scriptstyle \mathcal{E}_{UV}}\downarrow \\ \mathcal{O}_X(V) \times \mathcal{E}(V) & \longrightarrow & \mathcal{E}(V), \end{array} \tag{2.5.1}$$

where the horizontal arrows are module multiplication.

Morphisms in $\mathcal{O}_X$-mod are natural transformations $\phi : \mathcal{E} \to \mathcal{F}$. An $\mathcal{O}_X$-module $\mathcal{E}$ is called a *vector bundle of rank* n if it is locally free, that is, around every point there is an open set $U \subseteq X$ with $\mathcal{E}|_U \cong \mathcal{O}_X|_U \otimes_{\mathbb{R}} \mathbb{R}^n$.

Definition 2.5.2. We define the *pullback* $\underline{f}^*(\mathcal{E})$ of a sheaf of modules $\mathcal{E}$ on $\underline{Y}$ by a morphism $\underline{f} = (f, f^\sharp) : \underline{X} \to \underline{Y}$ of C^∞-ringed spaces as $\underline{f}^*(\mathcal{E}) = f^{-1}(\mathcal{E}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$. Here $f^{-1}(\mathcal{E})$ is as in Definition 2.3.5, so that $\underline{f}^*(\mathcal{E})$ is a sheaf of modules on $\underline{X}$. Morphisms of $\mathcal{O}_Y$-modules $\phi : \mathcal{E} \to \mathcal{F}$ give morphisms of $\mathcal{O}_X$-modules $\underline{f}^*(\phi) = f^{-1}(\phi) \otimes \mathrm{id}_{\mathcal{O}_X} : \underline{f}^*(\mathcal{E}) \to \underline{f}^*(\mathcal{F})$.

Definition 2.5.3. We define the cotangent sheaf $\mathcal{P}T^*\underline{X}$ of a C^∞-ringed space $\underline{X} = (X, \mathcal{O}_X)$ as follows. To each open $U \subseteq X$ we define a presheaf by taking the cotangent module $\Omega_{\mathcal{O}_X(U)}$ of Definition 2.2.4, regarded as a module over the C^∞-ring $\mathcal{O}_X(U)$. Here, for open sets $V \subseteq U \subseteq X$ we have restriction morphisms $\Omega_{\rho_{UV}} : \Omega_{\mathcal{O}_X(U)} \to \Omega_{\mathcal{O}_X(V)}$ associated to the morphisms of C^∞-rings $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ so that the following commutes:

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \Omega_{\mathcal{O}_X(U)} & \xrightarrow[\mu_{\mathcal{O}_X(U)}]{} & \Omega_{\mathcal{O}_X(U)} \\ \big\downarrow{\scriptstyle \rho_{UV} \times \Omega_{\rho_{UV}}} & & {\scriptstyle \Omega_{\rho_{UV}}}\big\downarrow \\ \mathcal{O}_X(V) \times \Omega_{\mathcal{O}_X(V)} & \xrightarrow{\mu_{\mathcal{O}_X(V)}} & \Omega_{\mathcal{O}_X(V)}. \end{array}$$

Definition 2.2.4 implies $\Omega_{\psi \circ \phi} = \Omega_\psi \circ \Omega_\phi$ so that this is a well defined presheaf of $\mathcal{O}_X$-modules on $\underline{X}$. The *cotangent sheaf* $T^*\underline{X}$ *of* X is the sheafification of $\mathcal{P}T^*\underline{X}$.

The universal property of sheafification shows that for open $U \subseteq X$ we have an isomorphism of $\mathcal{O}_X|_U$-modules

$$T^*(U, \mathcal{O}_X|_U) \cong T^*\underline{X}|_U.$$

For a morphism $\underline{f} : \underline{X} \to \underline{Y} \in \mathbf{C^\infty RS}$ then $\underline{f}^*(T^*\underline{Y}) = f^{-1}(T^*\underline{Y}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$. The universal properties of sheafification show that $\underline{f}^*(T^*\underline{Y})$ is the sheafification of the presheaf $\mathcal{P}(\underline{f}^*(T^*\underline{Y}))$ where

$$U \longmapsto \mathcal{P}(\underline{f}^*(T^*\underline{Y}))(U) = \lim_{V \supseteq f(U)} \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U).$$

This gives a morphism of presheaves $\mathcal{P}\Omega_{\underline{f}} : \mathcal{P}(\underline{f}^*(T^*\underline{Y})) \to \mathcal{P}T^*\underline{X}$ on X where

$$(\mathcal{P}\Omega_{\underline{f}})(U) = \lim_{V \supseteq f(U)} (\Omega_{\rho_{f^{-1}(V)\,U} \circ f_\sharp(V)})_*.$$

Here, we have morphisms $f_\sharp(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ from $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ corresponding to $f^\sharp$ in $\underline{f}$ as in (2.4.2), and $\rho_{f^{-1}(V)\,U} : \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_X(U)$ in $\mathcal{O}_X$ so that $(\Omega_{\rho_{f^{-1}(V)\,U} \circ f_\sharp(V)})_* : \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U) \to \Omega_{\mathcal{O}_X(U)} = (\mathcal{P}T^*\underline{X})(U)$ is constructed as in Definition 2.2.4. Then write $\Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \to T^*\underline{X}$ for the induced morphism of the associated sheaves. This corresponds to the morphism $\mathrm{d}f : f^*(T^*Y) \to T^*X$ of vector bundles over manifold X and smooth map of manifolds $f : X \to Y$ as in Example 2.2.2.

Chapter 3

Background on manifolds with (g-)corners

We now give some background material on manifolds with (g-)corners. While manifolds with corners were originally studied in Cerf [11] and Douady [18] to generalise manifolds with boundary, we will focus on the more recent work of Melrose [66–68] and Joyce [39,47], who are of particular interest for their descriptions of smooth maps and the corner functor. In particular, Joyce [47] studied 'manifolds with generalised corners', or 'manifolds with g-corners' for short, a generalisation of manifolds with corners. Here we will present the fundamental definitions of manifold with corners and manifolds with g-corners, and explain how they relate to each other.

3.1 Monoids and the local model

While manifolds are locally modelled on $\mathbb{R}^n$, manifolds with corners are more generally modelled on $\mathbb{R}^n_k = [0,\infty)^k \times \mathbb{R}^{n-k}$. Manifolds with g-corners are more general still, where for any weakly toric commutative monoid P, we have a corresponding local model X_P, and if $P = \mathbb{N}^k \times \mathbb{Z}^{n-k}$ then $X_P \cong \mathbb{R}^n_k$.

To make this precise, we start by recalling facts about monoids in the style of log geometry. References for monoids include Ogus [78, §I], Gillam [26, §1–§2], and Gillam and Molcho [28, §1]. The only thing new here is the definition of firm monoid.

Definition 3.1.1. A (commutative) monoid is a set P equipped with an associative commutative binary operation $+ : P \times P \to P$ that has an identity element 0. All monoids in this thesis will be commutative. A morphism of monoids $P \to Q$ is a morphism of sets that respects the binary operation and sends identity to identity. Write **Mon** for the category of monoids.

For any $n \in \mathbb{N} = \{0, 1, \dots\}$ and $p \in P$ we will write $np = n \cdot p = \overbrace{p + \cdots + p}^{n \text{ copies}}$, and require $0 \cdot p = 0$.

A *submonoid* Q of a monoid P is a subset that is closed under the binary operation and contains the identity element. We can form the *quotient monoid* P/Q which is the set of all $\sim$-equivalence classes $[p]$ of $p \in P$ such that $p \sim p'$ if there are $q, q' \in Q$ with $p + q = p' + q' \in P$. It has an induced monoid structure from the monoid P. There is a morphism $\pi : P \to P/Q$. This quotient satisfies the following universal property: it is a monoid P/Q with a morphism of monoids $\pi : P \to P/Q$ such that $\pi(Q) = \{0\}$ and if $\mu : P \to R$ is a monoid morphism with $\mu(Q) = \{0\}$ then $\mu = \nu \circ \pi$ for a unique morphism $\nu : P/Q \to R$.

A *unit* in P is an element $p \in P$ that has a (necessarily unique) inverse under the binary operation, p', so that $p' + p = 0$. Write $P^\times$ as the set of all units of P. It is a submonoid of P, in fact it is an abelian group. A monoid is an abelian group if and only if it is equal to its set of units.

An *ideal* I in a monoid P is a proper subset $I \subset P$ such that if $p \in P$ and $i \in I$ then $ip \in I$, so it is necessarily closed under P's binary operation. It must not contain any units. An ideal I is called *prime* if whenever $a + b \in I$ for $a, b \in P$ then either a or b is in P. We say the complement $P \setminus I$ of a prime ideal I is a *face* which is automatically a submonoid of P. If we have elements $p_j \in P$ for j in some indexing set J then we can consider the *ideal generated* by the p_j, which we write as $\langle p_j \rangle_{j \in J}$. It consists of all elements in P of the form $a + p_j$ for any $a \in P$ and any $j \in J$. Note that if any of the p_j are units then the 'ideal' generated by these p_j is a misnomer, as $\langle p_j \rangle_{j \in J}$ is not an ideal and instead equal to P. We do not usually consider the empty set to be an ideal.

For any monoid P there is an associated abelian group P^{gp} and morphism $\pi^{\mathrm{gp}} : P \to P^{\mathrm{gp}}$. This has the universal property that any morphism from P to an abelian group factors through π^{gp}, so P^{gp} is unique up to canonical isomorphism. It can be shown to be isomorphic to the quotient monoid $(P \times P)/\Delta_P$, where $\Delta_P = \{(p, p) : p \in P\}$ is the diagonal submonoid of $P \times P$, and $\pi^{\mathrm{gp}} : p \mapsto [p, 0]$.

For a monoid P we have the following properties:

(i) If there is a surjective morphism $\mathbb{N}^k \to P$ for some $k \geqslant 0$, we call P *finitely generated.* This morphism can be uniquely written as $(n_1, \dots, n_k) \mapsto n_1 p_1 + \cdots n_k p_k$ for some $p_1, \dots, p_k \in P$ which we call the *generators* of P. This implies P^{gp} is finitely generated. If there is an isomorphism $P \cong \mathbb{N}^k$ then P is called *free.*

(ii) If the group of units $P^\times$ is only the identity element we call P *sharp*. Any monoid has an associated *sharpening* $P^\sharp$ which is the sharp quotient monoid $P/P^\times$ with surjection $\pi^\sharp : P \to P^\sharp$. If the sharpening of P is finitely generated we call P *firm*, so finitely generated P are firm.

(iii) If $\pi^{\mathrm{gp}} : P \to P^{\mathrm{gp}}$ is injective we call P *integral* or *cancellative*. This occurs if and only if $p + p' = p + p''$ implies $p' = p''$ for all $p, p', p'' \in P$. Then P is isomorphic to its image under π^{gp}, so we can consider it a subset of P^{gp}.

(iv) If P is integral and whenever $p \in P^{\mathrm{gp}}$ with $np \in P \subset P^{\mathrm{gp}}$ for some $n \geqslant 1$ implies $p \in P$ then we call P *saturated*.

(v) If P^{gp} is a torsion free group, then we call P *torsion free*. That is, if there is $n \geqslant 0$ and $p \in P^{\mathrm{gp}}$ such that $np = 0$ then $p = 0$.

(vi) If P is finitely generated, integral, saturated and torsion free then it is called *weakly toric*, so weakly toric implies firm. It has *rank* $\operatorname{rank} P = \dim_{\mathbb{R}}(P \otimes_{\mathbb{N}} \mathbb{R})$. For a weakly toric P then there is an isomorphism $P^\times \cong \mathbb{Z}^l$ and $P^\sharp$ is a toric monoid (defined below). The exact sequence $0 \to P^\times \to P \to P^\sharp \to 0$ splits, so that $P \cong P^\sharp \times \mathbb{Z}^l$. Then the rank of P is equal to $\operatorname{rank} P = \operatorname{rank} P^{\mathrm{gp}} = \operatorname{rank} P^\sharp + l$.

(vii) If P is a weakly toric monoid and is also sharp we call P *toric* (note that saturated and sharp together imply torsion free.) For a toric monoid P its associated group P^{gp} is a finitely generated, torsion-free abelian group, so $P^{\mathrm{gp}} \cong \mathbb{Z}^k$ for $k \geqslant 0$. Then the rank of P is $\operatorname{rank} P = k$.

These definitions are not standard in the literature. For example, Ogus [78, p. 13], refer to our weakly toric monoids as *toric monoids*, and to our toric monoids as *sharp toric monoids*.

We will write $\mathbf{Mon^{fg}}, \mathbf{Mon^{wt}}, \mathbf{Mon^{to}}$ for the full subcategories of $\mathbf{Mon}$ that contain finitely generated, weakly toric, and toric monoids, respectively, so that $\mathbf{Mon^{to}} \subset \mathbf{Mon^{wt}} \subset \mathbf{Mon^{fg}} \subset \mathbf{Mon}$.

In many examples, we may use multiplication $\cdot$ instead of addition as the binary operation with identity element 1. Such a monoid P may have an element $0 \in P$ such that $0p = 0$ for all $p \in P$ which we call a zero element of P. Important examples for this thesis are the following:

Example 3.1.2. **(a)** The most basic toric monoid is $\mathbb{N}^k$ under addition for $k = 0, 1, \ldots$, with $(\mathbb{N}^k)^{\mathrm{gp}} \cong \mathbb{Z}^k$.

(b) $\mathbb{Z}^k$ under addition is weakly toric, but not toric as it is not sharp with $(\mathbb{Z}^k)^\times = \mathbb{Z}^k \neq 0$. An example of generators are $(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,1),(-1,-1,\ldots,-1)$.

(c) $\big([0,\infty),\cdot,1\big)$ under multiplication is a monoid that is not finitely generated. It has identity 1 and zero element 0. We have $[0,\infty)^{\rm gp} = \{0\}$, so $[0,\infty)$ is not integral, and $[0,\infty)^\times = (0,\infty)$, so $[0,\infty)$ is not sharp. However, the sharpening is isomorphic to $\{0\}$ and so it is firm. Similarly, $\big(\mathbb{R},\cdot,1\big)$ is not finitely generated, not integral and not sharp, but it is firm and has zero element 0.

Definition 3.1.3. Let P be a weakly toric monoid (considered under addition). We define $X_P = \mathrm{Hom}(P,[0,\infty))$ as in Joyce [47, §3.2] to be the set of monoid morphisms $x : P \to [0,\infty)$ where the target is considered as a monoid under multiplication as in Example 3.1.2(c). The *interior* of X_P is defined to be $X_P^\circ = \mathrm{Hom}(P,(0,\infty))$ where $(0,\infty)$ is a submonoid of $[0,\infty)$, so that $X_P^\circ \subset X_P$.

For $p \in P$ there is a corresponding function $\lambda_p : X_P \to [0,\infty)$ such that $\lambda_p(x) = x(p)$. For any $p,q \in P$ then $\lambda_{p+q} = \lambda_p \cdot \lambda_q$ and $\lambda_0 = 1$. Then we can define a topology on X_P to be the weakest topology such that each λ_p is continuous. Then X_P is locally compact, Hausdorff and X_P° is an open subset of X_P. The *interior* U° of an open set $U \subset X_P$ is defined to be $U \cap X_P^\circ$.

As P is weakly toric, then we can take a presentation for P with generators $p_1,\ldots,p_m$ and relations

$$a_1^j p_1 + \cdots + a_m^j p_m = b_1^j p_1 + \cdots + b_m^j p_m \quad \text{in } P \text{ for } j = 1,\ldots,k,$$

for $a_i^j, b_i^j \in \mathbb{N}$, $i = 1,\ldots,m$ and $j = 1,\ldots,k$. Then we have a continuous function $\lambda_{p_1} \times \cdots \times \lambda_{p_m} : X_P \to [0,\infty)^m$ that is a homeomorphism onto its image

$$X_P' = \big\{(x_1,\ldots,x_m) \in [0,\infty)^m : x_1^{a_1^j} \cdots x_m^{a_m^j} = x_1^{b_1^j} \cdots x_m^{b_m^j},\ j = 1,\ldots,k\big\},$$

which is closed subset of $[0,\infty)^m$.

Example 3.1.4. If $P = \mathbb{N}^k \times \mathbb{Z}^{m-k}$ then P is weakly toric. We can take generators $p_1 = (1,0,\ldots,0), p_2 = (0,1,0,\ldots,0),\ldots,p_m = (0,\ldots,0,1), p_{m+1} = (0,\ldots,0,-1,\ldots,-1)$ with p_{m+1} having -1 in the $k+1$ to $m+1$ entries, so that the only relation on P is $p_{k+1} + \cdots + p_{m+1} = 0$. Then X_P is homeomorphic to

$$X_P' = \big\{(x_1,\ldots,x_{m+1}) \in [0,\infty)^{m+1} : x_{k+1} \cdots x_{m+1} = 1\big\}.$$

This means that for $(x_1,\ldots,x_{m+1}) \in X_P'$ we have $x_{k+1},\ldots,x_{m+1}$ positive with $x_{m+1}^{-1} = x_{k+1} \cdots x_m$. So there is a homeomorphism from X_P to $\mathbb{R}_k^m$ where $(x_1,\ldots,x_{m+1}) \mapsto (x_1,\ldots,x_k,\log(x_{k+1}),\ldots,\log(x_m))$.

3.2 Smooth maps and manifolds with (g-)corners

We start by defining smooth maps with target and domain (open subsets of) $\mathbb{R}^n_k$, then extend this to define smooth maps with target and domain (open subsets of) X_P. We will use the notion of smooth map from Melrose [66–68] who calls them b-maps.

Definition 3.2.1. Let $f : U \to \mathbb{R}$ be a continuous map for open $U \subseteq \mathbb{R}^n_k$. We say that f is *smooth* if all derivatives $\frac{\partial^{a_1+\cdots+a_m}}{\partial u_1^{a_1}\cdots\partial u_m^{a_m}} f(u_1,\ldots,u_m) : U \to \mathbb{R}$ exist and are continuous for all $a_1,\ldots,a_m \geqslant 0$, including one-sided derivatives where $u_i = 0$ for $i = 1,\ldots,k$.

Let $f : U \to [0,\infty)$ be a continuous map. We say that f is *weakly smooth* if if all derivatives $\frac{\partial^{a_1+\cdots+a_m}}{\partial u_1^{a_1}\cdots\partial u_m^{a_m}} f(u_1,\ldots,u_m) : U \to \mathbb{R}$ exist and are continuous for all $a_1,\ldots,a_m \geqslant 0$, including one-sided derivatives where $u_i = 0$ for $i = 1,\ldots,k$. We say f is *smooth* if it is weakly smooth and it is either identically 0, or for every point in U there is an open neighbourhood in U containing this point such that f is of the form $f(u_1,\ldots,u_n) = u_1^{a_1}\cdots u_k^{a_k} F(u_1,\ldots,u_n)$ for a weakly smooth positive function $F : U \to (0,\infty)$ and non-negative integers $a_1,\ldots,a_k$. In the latter case, f is called *interior*.

Let $\pi_i : \mathbb{R}^n_k \to [0,\infty)$ be the projection onto the i-th factor for $i = 1,\ldots,k$ and let $\pi_i : \mathbb{R}^n_k \to \mathbb{R}$ be the projection onto the i-th factor for $i = k+1,\ldots,n$. For open sets $U \subseteq \mathbb{R}^m_l$ and $V \subseteq \mathbb{R}^n_k$ we say a continuous function $f : U \to V$ is *weakly smooth* if $\pi_i \circ f$ is weakly smooth for $i = 1,\ldots,k$ and smooth for $i = k+1,\ldots,n$. We say f is *smooth* if it is weakly smooth and $\pi_i \circ f$ is smooth for $i = 1,\ldots,k$. We say f is *interior* if it is smooth and no composition $\pi_i \circ f$ is zero for $i = 1,\ldots,k$, which implies $f(U^\circ) \subset V^\circ$. Then we say f is a *diffeomorphism* if it is a smooth bijection with smooth inverse, which requires $n = m, k = l$.

Let P be a weakly toric monoid and X_P its corresponding topological space as in Definition 3.1.3. Let P have generators $p_1,\ldots,p_m$ and take the homeomorphism $\lambda_{p_1}\times\cdots\times\lambda_{p_m} : X_P \to X'_P \subseteq [0,\infty)^m$. Let U be an open subset of X_P, and $U' = \lambda_{p_1}\times\cdots\times\lambda_{p_m}(U) \subset X'_P$. Then we say a continuous function $f : U \to \mathbb{R}$ or $f : U \to [0,\infty)$ is *smooth* if there exists an open neighbourhood W' of U' in $[0,\infty)^m$ and a smooth function $g : W' \to \mathbb{R}$ or $g : W' \to [0,\infty)$ that is smooth in the sense above, such that $f = g \circ \lambda_{p_1}\times\cdots\times\lambda_{p_m}$. This can be shown to be independent of the choice of generators of P, and if $P = \mathbb{N}^k \times \mathbb{Z}^{n-k}$ it matches with the definition above.

If Q is another weakly toric monoid and we consider open $V \subseteq X_Q$ and continuous function $f : U \to V$. Then f is *smooth* if $\lambda_q \circ f : U \to [0,\infty)$ is smooth for all $q \in Q$ in the sense above. We call f *interior* if $f(U^\circ) \subseteq V^\circ$, and a *diffeomorphism* if it bijective with smooth inverse. Again if $P = \mathbb{N}^k \times \mathbb{Z}^{n-k}$ this matches with the definitions above.

We now define charts and atlases to give the definitions of a manifold with corners and a manifold with g-corners, as in Joyce [47, §3].

Definition 3.2.2. For a Hausdorff, second countable topological space X we define an *g-chart* on X to be a triple (P, U, ϕ). Here, P is a weakly toric monoid, so it is a submonoid of $P^{\mathrm{gp}} \cong \mathbb{Z}^n$ for some $n > 0$. We require $U \subset X_P$ to be open and $\phi : U \to X$ to be a homeomorphic onto its image in X. If $\operatorname{rank} P = m$ we call (P, U, ϕ) *m-dimensional.*

If (P, U, ϕ) and (Q, V, ψ) are m-dimensional g-charts on X then they are called *compatible* if we have a diffeomorphism $\psi^{-1} \circ \phi : \phi^{-1}\big(\phi(U) \cap \psi(V)\big) \to \psi^{-1}\big(\phi(U) \cap \psi(V)\big)$ between open subsets of X_P and X_Q. A *g-atlas* on X is a family of pairwise compatible charts with the same dimension where the union of the images of the ϕ in each chart cover X. A *maximal* g-atlas on X is a g-atlas that is not properly contained in any other g-atlas; each g-atlas is contained in a unique maximal g-atlas, which contains all g-charts compatible with each chart in the g-atlas.

We call X a *manifold with g-corners* if it can be equipped with a maximal g-atlas. We say X has *dimension* m if the g-atlas has dimension m. We say that X is a manifold with corners if each g-chart (P, U, ϕ) of X then $P \cong \mathbb{N}^k \times \mathbb{Z}^{m-k}$ for some $k = 1, \dots, m$. In this case, the data of the g-chart (P, U, ϕ) is equivalent to the data of a *chart* (ϕ, U) for open $U \subseteq \mathbb{R}^n_k$ and $\phi : U \to X$ a homeomorphism onto its image, as in [47, §2]. If $k = 0$ for each g-chart, then X is a (smooth) manifold in the usual sense.

If X, Y are two manifolds with g-corners then a continuous map $f : X \to Y$ is *smooth* (or *interior*) if for all g-charts $(P, U, \phi), (Q, V, \psi)$ then

$$\psi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}(\psi(V)) \longrightarrow V$$

is smooth (or interior) between open sets of X_P and X_Q. Then we let $\mathbf{Man^{gc}}$ be the category of manifolds with g-corners and their smooth maps, and $\mathbf{Man^{gc}_{in}}$ the non-full subcategory of manifolds with g-corners and interior maps. We also have $\mathbf{Man^{c}}$ the full subcategory of $\mathbf{Man^{gc}}$ of manifolds with corners, and $\mathbf{Man^{c}_{in}}$ the full subcategory of $\mathbf{Man^{gc}_{in}}$ of manifolds with corners with interior maps.

We will also like to consider Hausdorff, second countable topological spaces that consist of disjoint unions $\coprod_{m=0}^{\infty} X_m$, where X_m is a manifold with g-corners of dimension m. We write $\check{\mathbf{M}}\mathbf{an^{gc}}$ for the category with these objects (which we call of *manifolds with g-corners with mixed dimension*), and morphisms that are continuous morphisms and restrict to smooth morphisms of manifolds with g-corners on each m-dimensional pieces. That is, continuous $f : \coprod_{m=0}^{\infty} X_m \to \coprod_{n=0}^{\infty} Y_n$, such that $f|_{X_m \cap f^{-1}(Y_n)} : X_m \cap f^{-1}(Y_n) \to Y_n$ is smooth for all $m, n \geqslant 0$.

We write $\check{\mathbf{Man}}^{\mathbf{gc}}_{\mathbf{in}} \subset \check{\mathbf{Man}}^{\mathbf{gc}}$ for the subcategory with the same objects, and morphisms f with $f|_{X_m \cap f^{-1}(Y_n)}$ interior for all $m, n \geqslant 0$. We write $\check{\mathbf{Man}}^{\mathbf{c}}$ for the full subcategory of $\check{\mathbf{Man}}^{\mathbf{gc}}$ that consists of disjoint unions of manifolds with corners, and $\check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}}$ for the full subcategory of $\check{\mathbf{Man}}^{\mathbf{gc}}_{\mathbf{in}}$ with objects in $\check{\mathbf{Man}}^{\mathbf{c}}$. We call objects of $\check{\mathbf{Man}}^{\mathbf{c}}$ and $\check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}}$ *manifolds with corners with mixed dimension.*

For a manifold with (g-)corners (of possibly mixed dimension) we will write $C^\infty(X)$ to be the set of smooth functions from X to $\mathbb{R}$, which is a C^∞-ring and an $\mathbb{R}$-algebra (in a similar way to Example 2.1.4). We will write $\mathrm{Ex}(X)$ to be the set smooth functions from X to $[0,\infty)$ and $\mathrm{In}(X)$ to be the set of interior functions from X to $[0,\infty)$, which are both monoids, as in Definition 4.1.1 and discussed further in §4. Similarly to Example 2.1.16, we will also consider the germs of these functions at a point $x \in X$, and write these sets as $C^\infty_x(X)$, $\mathrm{Ex}_x(X)$, $\mathrm{In}_x(X)$.

Remark 3.2.3. We know that a weakly toric monoid P is isomorphic to $P^\sharp \times \mathbb{Z}^l$, where $P^\sharp$ is toric and l is a non-negative integer. This implies $X_P \cong X_{P^\sharp} \times X_{\mathbb{Z}^l} \cong X_{P^\sharp} \times \mathbb{R}^l$. This means that manifolds with g-corners have local models $X_Q \times \mathbb{R}^l$ for toric monoids Q and $l \geqslant 0$, where $X_{\mathbb{N}^k} \cong [0,\infty)^k$.

Each toric monoid Q has a natural point $\delta_0 \in X_Q$ called the *vertex* of X_Q, which acts by taking $0 \in Q$ to $1 \in [0,\infty)$ and all non-zero $q \in Q$ to zero. Given a manifold with g-corners X and a point $x \in X$, there is a neighbourhood of x and a toric monoid Q such that this neighbourhood is modelled on $X_Q \times \mathbb{R}^l$ near $(\delta_0, 0) \in X_Q \times \mathbb{R}^l$, where $\operatorname{rank} Q + l = \dim X$.

From [47, Ex. 3.23] we have the simplest example of a manifold with g-corners that is not a manifold with corners.

Example 3.2.4. Let P be the weakly toric monoid of rank 3 with

$$P = \{(a,b,c) \in \mathbb{Z}^3 : a \geqslant 0,\ b \geqslant 0,\ a+b \geqslant c \geqslant 0\}.$$

This has generators $p_1 = (1,0,0)$, $p_2 = (0,1,1)$, $p_3 = (0,1,0)$, and $p_4 = (1,0,1)$ and one relation $p_1 + p_2 = p_3 + p_4$. The local model it induces is

$$X_P \cong X'_P = \{(x_1,x_2,x_3,x_4) \in [0,\infty)^4 : x_1x_2 = x_3x_4\}. \tag{3.2.1}$$

Figure 3.2.1 is a three dimensional sketch of X'_P as a square-based 3-dimensional infinite pyramid. From Remark 3.2.3, we can see X'_P has a vertex $(0,0,0,0)$ corresponding to $\delta_0 \in X_P$. It also has one-dimensional edges consisting of points $(x_1,0,0,0), (0,x_2,0,0)$, $(0,0,x_3,0), (0,0,0,x_4)$, and 2-dimensional faces of consisting of the points $(x_1,0,x_3,0)$, $(x_1,0,0,x_4), (0,x_2,x_3,0), (0,x_2,0,x_4)$. Its interior $X'^{\circ}_P \cong \mathbb{R}^3$ consists of points (x_1,x_2,x_3,x_4)

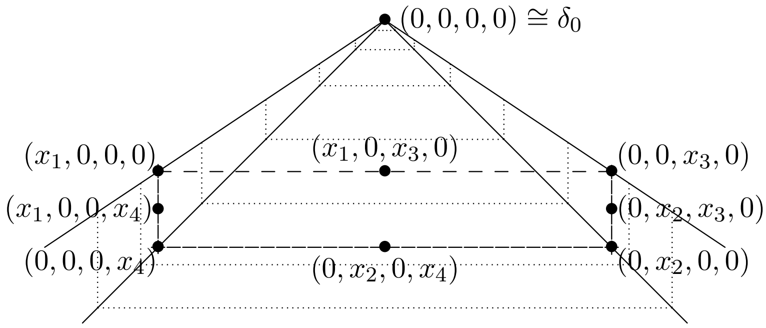


Figure 3.2.1: 3-manifold with g-corners $X'_P \cong X_P$ in (3.2.1)

with $x_1, \ldots, x_4$ non-zero and $x_1 x_2 = x_3 x_4$. Without the vertex, $X_P \setminus \{\delta_0\}$ is a 3-manifold with corners, however X_P is not a manifold with corners near its vertex, as it is not isomorphic to $\mathbb{R}^n_k$ around δ_0.

Remark 3.2.5. References on manifolds with corners include Cerf [11], Douady [18], Gillam and Molcho [28, §6.7], Melrose [66–68], and Joyce [39, 47]. Most authors define manifolds with corners to have local model $\mathbb{R}^n_k$ as we do. In addition, Melrose [66–68] and authors who follow him restrict to *manifolds with faces* as in our Definition 3.3.3 below, which we will not do, although we will make use of this notion.

There is no consensus on the definition of smooth maps/morphisms of manifolds with corners in the literature, for example:

(i) Melrose [68, §1.12], [66, §1], first defined our notion of (interior) smooth maps and calls them *(interior) b-maps.*

(ii) In Joyce [39], the author required smooth maps to be 'strongly smooth maps' (which were just called 'smooth maps' in [39]), where for example if $f : \mathbb{R}^n_k \to [0, \infty)$ is strongly smooth, then it is either identically zero or it is smooth in our sense but with $f(u_1, \ldots, u_n) = u_i^a F(u_1, \ldots, u_n)$ for some $i = 1, \ldots, k$ and $a \in \{0, 1\}$, and weakly smooth $F : \mathbb{R}^n_k \to (0, \infty)$. .

(iii) Our interior maps coincide with Gillam and Molcho's *morphisms of manifolds with corners* [28, §6.7].

(iv) Cerf [11, §I.1.2] and many other authors define smooth maps of manifolds with corners to be only weakly smooth maps in our notation.

Manifolds with g-corners were defined in [47] and we follow this presentation. They have also been studied in Kottke [57], which considers their blow ups, and Joyce [49, §2.4] for their applications for moduli spaces in Symplectic Geometry.

3.3 Boundaries and corners of manifolds with (g-)corners

The material of this section broadly follows Joyce [39], [47, §2.2 & §3.4]. In the following definition we will consider the empty set and the whole set to be prime ideals of a monoid, which we will not need to do later.

Definition 3.3.1. Take a weakly toric monoid P and corresponding topological space X_P. For each $x \in X_P$ define the *support* of x, $\operatorname{supp}_{X_P} x = \{p \in P : x(p) \neq 0\}$. Then $\operatorname{supp}_{X_P} x$ is a face of P. Define the *depth* of x to be $\operatorname{depth}_{X_P} x = \operatorname{rank} P - \operatorname{rank} \operatorname{supp}_{X_P} x$, which is an integer valued in $0, \ldots, \operatorname{rank} P$. Define the *depth l stratrum* of X_P to be

$$S^l(X_P) = \big\{x \in X_P : \operatorname{depth}_{X_P} x = l\big\}.$$

The interior of X_P is $S^0(X)$.

For any face $F \subset P$ there is an inclusion $i_F^P : X_F \to X_P$ where $y \in X_F$ maps to the $x \in X_P$ such that $x(p) = y(p)$ if $p \in F$ and $x(p) = 0$ otherwise. Then $i_F^P(X_F^\circ) = \{x \in X_P : \operatorname{supp}_{X_P} x = F\}$ and this is isomorphic to $\mathbb{R}^{\operatorname{rank} F}$, a manifold without boundary. Then we see that

$$S^l(X_P) = \coprod_F i_F^P(X_F^\circ),$$

where this disjoint union is over the faces F of P that have $\operatorname{rank}(F) + l = \operatorname{rank}(P)$. Also, $S^l(X_P)$ is a smooth manifold without boundary, and we see

$$\coprod_{l=0}^{\operatorname{rank} P} S^l(X_P) = X_P$$

stratifies X_P into manifolds without boundary.

We can restrict this stratification to open sets, $U \subset X_P$, so that $S^l(U) = U \cap S^l(X_P)$ are the depth l elements of U, and

$$U = \coprod_{l=0}^{\operatorname{rank} P} S^l(U).$$

This is shown to be invariant under diffeomorphism in [47, §3.4].

For a manifold with g-corners, X, then for each $x \in X$ we can choose a chart (P, U, ϕ) with $\phi(u) = x$ for some $u \in U$ and let the depth of x be the depth of u. As the depth is invariant under diffeomorphisms, this is independent of the chart. Then we can similarly define the stratification

$$S^l(X) = \big\{x \in X : \operatorname{depth}_X x = l\big\}$$

for $l = 0, 1, \ldots, \dim X$. The disjoint union of the stratum is equal to X, and each stratum is a manifold without boundary of codimension l.

For $x \in \mathbb{R}^n_k$ the depth of x simplifies to the number of zeros of its coordinates. Then for a manifold with corners X the depth of an element $x \in X$ is the depth of its image $\phi(x) \in U \subseteq \mathbb{R}^n_k$ in a coordinate chart (U, ϕ).

Definition 3.3.2. For a manifold with g-corners X of dimension n, take $x \in X$, and $k = 0, 1, \ldots, n$. Then we define a *local k-corner component γ of X at x* to be a local choice of connected component of $S^k(X)$ near x. More precisely, for any open neighbourhood V of x in X that is small enough, then γ is a choice of connected component W of $V \cap S^k(X)$. So x is in the closure of W in X and if we chose another open neighbourhood V' and connected component W' of $V' \cap S^k(X)$ then $x \in \overline{W \cap W'}$. The local 1-corner components are called *local boundary components* of X.

We define the *boundary* and *k-corners* of X to be the sets

$$\partial X = \big\{(x, \beta) : x \in X,\ \beta \text{ is a local boundary component of } X \text{ at } x\big\},$$
$$C_k(X) = \big\{(x, \gamma) : x \in X,\ \gamma \text{ is a local } k\text{-corner component of } X \text{ at } x\big\},$$

for $k = 0, 1, \ldots, n$. This implies $\partial X = C_1(X)$ and $C_0(X) = X$.

One can define charts on $C_k(X)$ so that each are a manifold with g-corners of codimension k. If X is a manifold with corners then $C_k(X)$ is a manifold with corners for each $k = 0, \ldots, n$.

The *corners* of X is the manifold with (g-)corners with mixed dimension

$$C(X) = \coprod_{k=0}^{\dim X} C_k(X).$$

There are canonical smooth maps $i_X : C_k(X) \to X$ where $(x, \beta) \mapsto x$, which are not interior and may also not be injective.

The property that i_X is injective on connected components of ∂X is of particular importance for manifolds with corners, although it is not considered for manifolds with g-corners as discussed in Remark 4.7.10.

Definition 3.3.3. A manifold with corners X is called a *manifold with faces* if for each connected component F of ∂X, the map $i_X|_F : F \to X$ is injective. Then the *faces* of X are the connected components of ∂X, which can be regarded as subsets of X. In his work on analysis on manifolds with corners, Melrose [66–68] restricts to manifolds with faces, as some things can then be done globally.

Here is an example of a manifold with corners that does not have faces that is from [47, Ex. 2.8].

Example 3.3.4. We define the *teardrop* as the subset $T = \{(x, y) \in \mathbb{R}^2 : x \geqslant 0, y^2 \leqslant x^2 - x^4\}$. As shown in Figure 3.3.1, T is a manifold with corners of dimension 2. The teardrop is not a manifold with faces as the boundary is diffeomorphic to $[0, 1]$, and so is connected, while the map $i_T : \partial T \to T$ is not injective.

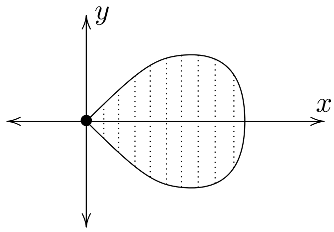


Figure 3.3.1: The teardrop T

We now show that the corners of X is in fact a functor in a specific sense by considering how it acts on morphisms. The following is from [47, Lem. 3.33].

Lemma 3.3.5. *A smooth map of manifolds with g-corners* $f : X \to Y$ ***is compatible with the depth stratifications*** $X = \coprod_{k \geqslant 0} S^k(X)$, $Y = \coprod_{l \geqslant 0} S^l(Y)$ *in Definition* 3.3.1. *That is, if we take a connected subset* $\emptyset \neq W \subseteq S^k(X)$ *for some* $k \geqslant 0$, *then* $f(X)$ *is contained in some* $S^l(Y)$ *for a unique* $l \geqslant 0$.

In general, a smooth $f : X \to Y$ does not induce a map $\partial f : \partial X \to \partial Y$ nor a map $C_k(f) : C_k(Y) \to C_k(Y)$. This means that boundaries and k-corners do not give functors on $\mathbf{Man^c}$ by themselves. If we allow mixed dimension and the full corners $C(X) = \coprod_{k \geqslant 0} C_k(X)$ from Definition 3.3.2, we can define a functor as in Joyce [47, Def. 3.34].

Definition 3.3.6. Take a smooth map $f : X \to Y$ of manifolds with corners, and let γ be a local k-corner component at $x \in X$. For a small enough neighbourhood V of x in X, then γ gives a connected component W of $V \cap S^k(X)$ with x in the closure of W. Then Lemma 3.3.5 says there is an $l \geqslant 0$ such that $f(W) \subseteq S^l(Y)$. Now $f(W)$ is connected as f is continuous, and $f(x) \in \overline{f(W)}$. This gives a unique l-corner component $f_*(\gamma)$ of Y at $f(x)$, such that for a small enough neighbourhood $\tilde{V}$ of $f(x)$ in Y, then $f_*(\gamma)$ has the corresponding connected component $\tilde{W}$ of $\tilde{V} \cap S^l(Y)$ where $\tilde{W} \cap f(W) \neq \emptyset$. This $f_*(\gamma)$ is well defined as it is independent of the choice of sufficiently small $V, \tilde{V}$.

We define a map $C(f) : C(X) \to C(Y)$ by $C(f) : (x, \gamma) \mapsto (f(x), f_*(\gamma))$, which is smooth by [47, Def. 2.10,Def. 3.34]. Then $C(f)$ is a morphism in $\check{\mathbf{Man}}^{\mathbf{c}}$ and this defines the functor $C : \mathbf{Man^{gc}} \to \check{\mathbf{M}}\mathbf{an^{gc}}$, and restriction to manifolds with corners defines a functor $C : \mathbf{Man^{c}} \to \check{\mathbf{M}}\mathbf{an^{c}}$, both of which we call the *corner functor.*

By [47, Prop. 2.11, Prop. 3.36] $C(f)$ is interior for each smooth map of manifolds with (g-)corners $f : X \to Y$. We can also consider taking the domain category to be $\check{\mathbf{M}}\mathbf{an^{c}}$ and $\check{\mathbf{M}}\mathbf{an^{gc}}$ so that the corner functors can be extended to the functors $C : \check{\mathbf{M}}\mathbf{an^{c}} \to \check{\mathbf{M}}\mathbf{an^{c}_{in}}$ and $C : \check{\mathbf{M}}\mathbf{an^{gc}} \to \check{\mathbf{M}}\mathbf{an^{gc}_{in}}$. We will show these are right adjoint to the inclusions $\check{\mathbf{M}}\mathbf{an^{c}_{in}} \to \check{\mathbf{M}}\mathbf{an^{c}}$ and $\check{\mathbf{M}}\mathbf{an^{gc}_{in}} \to \check{\mathbf{M}}\mathbf{an^{gc}}$ in §5.8.

3.4 Tangent bundles and b-tangent bundles

We now discuss tangent bundles of manifolds with (g-)corners. For a manifold with corners there are two relevant constructions: the (*ordinary*) *tangent bundle* which is a standard generalisation of the tangent space of a manifold with corners, and the *b-tangent bundle* defined in Melrose [67, §2.2], [68, §I.10], [66, §2]. Manifolds with g-corners do not have well behaved (ordinary) tangent bundles in that the dimension of the tangent spaces is not locally constant, however their b-tangent bundles are well behaved. The duals of the tangent bundle and b-tangent bundle give the cotangent and b-cotangent bundles respectively. We follow the presentation of Joyce [47, §2.3, §3.5].

Definition 3.4.1. For a manifold with corners X of dimension m we define the *tangent bundle* $\pi : TX \to X$ of X. This is a natural vector bundle on X that is unique up to canonical isomorphism. There are many equivalent ways to characterise TX, let us first consider how to do this using coordinate charts.

For a chart (U, ϕ) on X, with $U \subseteq \mathbb{R}^m_k$ open, then $TX|_{\phi(U)}$ is the trivial bundle. If $(u_1, \ldots, u_m)$ are coordinates on U then $TX|_{\phi(U)}$ has a basis of sections $\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}$. Then TX has a corresponding chart $(TU, T\phi)$ where $TU = U \times \mathbb{R}^m \subseteq \mathbb{R}^{2m}_k$. A point $(u_1, \ldots, u_m, q_1, \ldots, q_m) \in TU$ can be represented by the vector $q_1 \frac{\partial}{\partial u_1} + \cdots + q_m \frac{\partial}{\partial u_m}$ over $(u_1, \ldots, u_m) \in U$ or $\phi(u_1, \ldots, u_m) \in X$.

Where two charts (U, ϕ) to $(\tilde{U}, \tilde{\phi})$ of X overlap, we can consider their change of coordinates $(u_1, \ldots, u_m) \rightsquigarrow (\tilde{u}_1, \ldots, \tilde{u}_m)$, which gives a corresponding change from $(TU, T\phi)$ to $(T\tilde{U}, T\tilde{\phi})$ where $(u_1, \ldots, u_m, q_1, \ldots, q_m) \rightsquigarrow (\tilde{u}_1, \ldots, \tilde{u}_m, \tilde{q}_1, \ldots, \tilde{q}_m)$. This is given by $\frac{\partial}{\partial u_i} = \sum_{j=1}^m \frac{\partial \tilde{u}_j}{\partial u_i}(u_1, \ldots, u_m) \cdot \frac{\partial}{\partial \tilde{u}_j}$, which implies $\tilde{q}_j = \sum_{i=1}^m \frac{\partial \tilde{u}_j}{\partial u_i}(u_1, \ldots, u_m) q_i$.

We can also define TX intrinsically using the elements of $\Gamma^\infty(TX)$ which are called *vector fields.* For each $x \in X$ the fibre of TX at x is denoted T_xX and there is a canonical isomorphism

$$T_xX \cong \big\{\text{linear maps } v : C^\infty(X) \to \mathbb{R} : v(fg) = v(f)g(x) + f(x)v(g) \text{ for all } f, g \in C^\infty(X)\big\}.$$

There is also a canonical isomorphism of $C^\infty(X)$-modules

$$\Gamma^\infty(TX) \cong \big\{\text{linear } v : C^\infty(X) \to C^\infty(X) : v(fg) = v(f)\cdot g + f\cdot v(g) \text{ for all } f,g \in C^\infty(X)\big\}.$$

For a smooth map of manifolds with corners $f : X \to Y$ there is a corresponding smooth map $Tf : TX \to TY$ (as defined in [47, Def. 2.14]), which commutes around the following diagram

$$\begin{array}{ccc} TX & \xrightarrow{\quad Tf \quad} & TY \\ \big\downarrow \pi & & \pi \big\downarrow \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

For another smooth map of manifolds with corners $g : Y \to Z$ we have $T(g \circ f) = Tg \circ Tf : TX \to TZ$, and also $T(\mathrm{id}_X) = \mathrm{id}_{TX} : TX \to TX$. This means that we have a *tangent functor* $X \mapsto TX$, $f \mapsto Tf$, $T : \mathbf{Man^c} \to \mathbf{Man^c}$, which restricts to $T : \mathbf{Man^c_{in}} \to \mathbf{Man^c_{in}}$. Here Tf is also a vector bundle morphism $\mathrm{d}f : TX \to f^*(TY)$ on X.

The dual vector bundle T^*X of TX is called the *cotangent bundle*. This is not functorial, but for smooth maps of manifolds with corners $f : X \to Y$ there are vector bundle morphisms $(\mathrm{d}f)^* : f^*(T^*Y) \to T^*X$ on X

We now define b-(co)tangent bundles for manifolds with corners and manifolds with g-corners.

Definition 3.4.2. For a manifold with g-corners X of dimension m we define the *b-tangent bundle* $\pi : {}^bTX \to X$ of X. This is a natural vector bundle on X that is unique up to canonical isomorphism. For a manifold with corners, there is a natural map $I_X : {}^bTX \to TX$, that is an isomorphism over the interior X°, but is not an isomorphism over the boundary strata $S^k(X)$ for $k \geqslant 1$. We consider three ways to characterise ${}^bTX$ for a manifold with corners, one of which gives a nice characterisation for a manifold with g-corners. We start by considering charts (U, ϕ) on a manifold with corners, X, with $U \subseteq \mathbb{R}^m_k$ open.

If $(u_1, \ldots, u_m)$ the coordinates on U then over $\phi(U)$, ${}^bTX|_{\phi(U)}$ has basis of sections $u_1\frac{\partial}{\partial u_1}, \ldots, u_k\frac{\partial}{\partial u_k}, \frac{\partial}{\partial u_{k+1}}, \ldots, \frac{\partial}{\partial u_m}$. Then ${}^bTX$ has a corresponding chart $({}^bTU, {}^bT\phi)$, where ${}^bTU = U \times \mathbb{R}^m \subseteq \mathbb{R}^{2m}_k$. Then a point $(u_1, \ldots, u_m, s_1, \ldots, s_m) \in {}^bTU$ can be represented by the vector

$$s_1u_1\frac{\partial}{\partial u_1} + \cdots + s_ku_k\frac{\partial}{\partial u_k} + s_{k+1}\frac{\partial}{\partial u_{k+1}} + \cdots + s_m\frac{\partial}{\partial u_m}$$

over $(u_1, \ldots, u_m)$ in U or $\phi(u_1, \ldots, u_m)$ in X.

Where two charts (U, ϕ) and $(\tilde{U}, \tilde{\phi})$ overlap we can consider their change of coordinates $(u_1, \ldots, u_m) \rightsquigarrow (\tilde{u}_1, \ldots, \tilde{u}_m)$, which give a change of coordinates $({}^bTU, {}^bT\phi)$ to $({}^bT\tilde{U}, {}^bT\tilde{\phi})$ where $(u_1, \ldots, u_m, s_1, \ldots, s_m) \rightsquigarrow (\tilde{u}_1, \ldots, \tilde{u}_m, \tilde{s}_1, \ldots, \tilde{s}_m)$ with

$$\tilde{s}_j = \begin{cases} \sum_{i=1}^k \tilde{u}_j^{-1} u_i \frac{\partial \tilde{u}_j}{\partial u_i} s_i + \sum_{i=k+1}^m \tilde{u}_j^{-1} \frac{\partial \tilde{u}_j}{\partial u_i} s_i, & j \leqslant k, \\ \sum_{i=1}^k u_i \frac{\partial \tilde{u}_j}{\partial u_i} s_i + \sum_{i=k+1}^m \frac{\partial \tilde{u}_j}{\partial u_i} s_i, & j > k. \end{cases}$$

In coordinate charts $({}^bTU, {}^bT\phi)$, $(TU, T\phi)$, $I_X : {}^bTX \to TX$ acts by

$$(u_1, \ldots, u_m, s_1, \ldots, s_m) \longmapsto (u_1, \ldots, u_m, u_1 s_1, \ldots, u_k s_k, s_{k+1}, \ldots, s_m).$$

A more intrinsic definition uses elements of $\Gamma^\infty({}^bTX)$, which are called *b-vector fields*. Then there is a canonical isomorphism of $C^\infty(X)$-modules

$$\Gamma^\infty({}^bTX) \cong \big\{ v \in \Gamma^\infty(TX) : v|_{S^k(X)} \text{ is tangent to } S^k(X) \text{ for all } k \big\}. \tag{3.4.1}$$

This gives an inclusion of $\Gamma^\infty({}^bTX)$ into $\Gamma^\infty(TX)$, which corresponds to the morphism $I_X : {}^bTX \to TX$.

Finally, in terms of germs, there is a canonical isomorphism

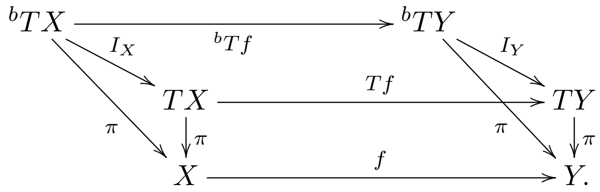
$$\begin{aligned} {}^bT_xX \cong \{(v, v') : \; & v : C_x^\infty(X) \to \mathbb{R} \text{ is a linear map }, \\ & v' : \mathrm{Ex}_x(X) \to \mathbb{R} \text{ is a morphism of monoids}, \\ & v([a][b]) = v([a])\,\mathrm{ev}(b) + v([b])\,\mathrm{ev}(a), \text{ for all } [a],[b] \in C_x^\infty(X), \\ & v' \circ \exp([a]) = v([a]), \text{ for all } [a] \in C_x^\infty(X), \\ & v \circ \mathrm{inc}([b]) = \mathrm{ev}([b]) v'([b]), \text{ for all } [b] \in C_x^\infty(X)\} \end{aligned}$$

Here $\mathrm{ev}([a]) = a(x)$ is evaluation at the point $x \in X$ and $\mathrm{inc} : \mathrm{Ex}_x(X) \to C_x^\infty(X)$ is the natural inclusion. If the manifold with corners has faces then this can be extended to a global definition but otherwise it cannot be. Then $I_X : {}^bTX \to TX$ for a manifold with corners acts by $(v, v') \mapsto v$.

This last definition in terms of germs is also well behaved for a manifold with g-corners, defining ${}^bTX$ for a manifold with g-corners as in [47, §3.5]. For the model spaces X_P, we have ${}^bTX_P \cong X_P \times \mathrm{Hom}_{\mathbf{Mon}}(P, \mathbb{R})$. Also, for a manifold with g-corners X and a point $x \in X$ then Remark 3.2.3 tells us that X near x is locally modelled on $X_Q \times \mathbb{R}^l$ near $(\delta_0, 0)$ for a toric monoid Q, which gives an isomorphism

$${}^bT_xX \cong \mathrm{Hom}_{\mathbf{Mon}}(Q, \mathbb{R}) \times \mathbb{R}^l.$$

For a smooth morphism of manifolds with (g-)corners that is interior there is a corresponding interior map ${}^bTf : {}^bTX \to {}^bTY$ as in [66, §2] that makes the following commute:



In terms of germs, ${}^bTf : (x, v, v') \mapsto (f(x), v \circ f, v' \circ f)$. This gives functors ${}^bT : \mathbf{Man^c_{in}} \to \mathbf{Man^c_{in}}$, ${}^bT : \mathbf{Man^{gc}_{in}} \to \mathbf{Man^{gc}_{in}}$ called the tangent functor where $X \mapsto {}^bTX$, $f \mapsto {}^bTf$. The maps $I_X : {}^bTX \to TX$ correspond to a natural transformations $I : {}^bT \to T$ of functors on $\mathbf{Man^c_{in}}$ and $\mathbf{Man^{gc}_{in}}$. The map ${}^bTf$ is also a vector bundle morphism ${}^b\mathrm{d}f : {}^bTX \to f^*({}^bTY)$ on X.

The dual vector bundle of ${}^bTX$ is called the *b-cotangent bundle* ${}^bT^*X$ of X. This is not functorial, although for an interior map $f : X \to Y$ of manifolds with (g-)corners we have a vector bundle morphism $({}^b\mathrm{d}f)^* : f^*({}^bT^*Y) \to {}^bT^*X$.

One reason for considering manifolds with g-corners is that they are specially well behaved under fibre products, as the next result from [47, §4.3] shows:

Theorem 3.4.3. *Let $g : X \to Z$ and $h : Y \to Z$ be interior maps of manifolds with g-corners. Call g, h* ***b-transverse*** *if ${}^bT_xg \oplus {}^bT_yh : {}^bT_xX \oplus {}^bT_yY \to {}^bT_zZ$ is a surjective linear map for all $x \in X$ and $y \in Y$ with $g(x) = h(y) = z \in Z$.*

If g, h are b-transverse then the fibre product $X \times_{g,Z,h} Y$ exists in $\mathbf{Man^{gc}_{in}}$.

The analogue is false for manifolds with corners, unless we impose complicated extra restrictions on g, h over $\partial^j X, \partial^k Y, \partial^l Z$ so that the maps are what is called *sb-transverse* in Joyce [49]. Note that b-transverse fibre products of manifolds with corners in $\mathbf{Man^{gc}_{in}}$ can be manifolds with g-corners, not corners. For example, from (3.2.1) we see that X_P in Example 3.2.4 may be written as $[0,\infty)^2 \times_{g,[0,\infty),h} [0,\infty)^2$, where $g, h : [0,\infty)^2 \to [0,\infty)$ given by $g(x, y) = h(x, y) = xy$ are b-transverse. So manifolds with g-corners can be seen as a type of completion of the category of manifolds with corners under fibre products of b-transverse maps. We discuss fibre products of manifolds with (g-)corners further in Remark 5.5.5.

Chapter 4

C^∞-rings with corners

We now develop a theory of C^∞*-rings with corners*, a generalisation of C^∞-rings in which manifolds are replaced by manifolds with corners, as in §3. This chapter is based on joint work with Professor Dominic Joyce.

Some of these ideas were introduced in the MSc thesis of Kalashnikov [51], who studied the category of (categorical) pre C^∞-ring with corners $\mathbf{PC^\infty Rings^c_{in}}$($\mathbf{CPC^\infty Rings^c_{in}}$) as in §4.1 and §4.2, although they did not use the pre-fix 'pre'. Apart from this, the author knows of no previous work on this subject. However, the theory is related to 'log geometry' in algebraic geometry, as discussed in §5.9, and our C^∞-rings with corners could in some sense be regarded as 'log C^∞-rings'.

4.1 Categorical pre C^∞-rings with corners

There is a natural generalisation of C^∞-rings to the corners case. We will call these 'pre C^∞-rings with corners', as we will reserve the name 'C^∞-rings with corners' for pre C^∞-rings with corners satisfying an additional condition given in §4.3. The name comes from an analogy with 'pre log rings' and 'log rings' in log geometry, as in §5.9.

Definition 4.1.1. Let X be a manifold with corners (or with g-corners). Smooth maps $g : X \to [0,\infty)$ will be called *exterior maps*, to contrast them with interior maps. We write $\mathrm{In}(X)$ for the set of interior maps $g : X \to [0,\infty)$, and $\mathrm{Ex}(X)$ for the set of exterior maps $g : X \to [0,\infty)$. Thus, we have three sets:

(a) $C^\infty(X)$ of smooth maps $f : X \to \mathbb{R}$;

(b) $\mathrm{In}(X)$ of interior maps $g : X \to [0,\infty)$; and

(c) $\mathrm{Ex}(X)$ of exterior (smooth) maps $g : X \to [0,\infty)$, with $\mathrm{In}(X) \subseteq \mathrm{Ex}(X)$.

Much of this chapter concerns properties of these three sets.

In §2.1 we gave two equivalent definitions for C^∞-rings. Similarly, we will give two equivalent definitions for pre C^∞-rings with corners. We start with the analogue of Definition 2.1.1.

Definition 4.1.2. Let $\mathbf{Euc} \subset \mathbf{Man}$ and $\mathbf{CC^\infty Rings}$ be as in Definition 2.1.1. Write $\mathbf{Euc^c}, \mathbf{Euc^c_{in}}$ for the full subcategories of $\mathbf{Man^c}, \mathbf{Man^c_{in}}$ with objects $\mathbb{R}^m \times [0,\infty)^n$ for all $m, n \geqslant 0$. Note that products are defined in $\mathbf{Euc^c}, \mathbf{Euc^c_{in}}$ with

$$\big(\mathbb{R}^k \times [0,\infty)^l\big) \times \big(\mathbb{R}^m \times [0,\infty)^n\big) = \mathbb{R}^{k+m} \times [0,\infty)^{l+n}, \tag{4.1.1}$$

where if the coordinates on $\mathbb{R}^k \times [0,\infty)^l$ are $(w_1,\ldots,w_k,x_1,\ldots,x_l)$ and on $\mathbb{R}^m \times [0,\infty)^n$ are $(y_1,\ldots,y_m,z_1,\ldots,z_n)$, then the coordinates on $\mathbb{R}^{k+m} \times [0,\infty)^{l+n}$ are

$$(w_1,\ldots,w_k,y_1,\ldots,y_m,x_1,\ldots,x_l,z_1,\ldots,z_n).$$

We have inclusions

$$\mathbf{Euc} \subset \mathbf{Euc^c_{in}} \subset \mathbf{Euc^c}. \tag{4.1.2}$$

Define a *categorical pre C^∞-ring with corners* as in Kalashnikov [51, Def. 4.17] to be a product-preserving functor $F : \mathbf{Euc^c} \to \mathbf{Sets}$. Here F should also preserve the empty product, i.e. it maps $\mathbb{R}^0 \times [0,\infty)^0 = \{\emptyset\}$ in $\mathbf{Euc^c}$ to the terminal object in $\mathbf{Sets}$, the point $*$.

If $F, G : \mathbf{Euc^c} \to \mathbf{Sets}$ are categorical pre C^∞-rings with corners, a *morphism* $\eta : F \to G$ is a natural transformation $\eta : F \Rightarrow G$. Such natural transformations are automatically product-preserving. We write $\mathbf{CPC^\infty Rings^c}$ for the category of categorical pre C^∞-ring with corners.

Define a *categorical interior pre C^∞-ring with corners* to be a product-preserving functor $F : \mathbf{Euc^c_{in}} \to \mathbf{Sets}$. These form a category $\mathbf{CPC^\infty Rings^c_{in}}$, with morphisms natural transformations.

Define functors $\Pi^{C^\infty}_{\rm cor}, \Pi^{\rm int}_{\rm cor}, \Pi^{C^\infty}_{\rm int}$ in a commutative triangle

$$\begin{array}{ccc} \mathbf{CPC^\infty Rings^c} & \xrightarrow{\qquad\Pi^{C^\infty}_{\rm cor}\qquad} & \mathbf{CC^\infty Rings} \\ & {\scriptstyle \Pi^{\rm int}_{\rm cor}} \searrow \quad \nearrow {\scriptstyle \Pi^{C^\infty}_{\rm int}} & \\ & \mathbf{CPC^\infty Rings^c_{in}} & \end{array} \tag{4.1.3}$$

by restriction to subcategories in (4.1.2), so that for example $\Pi^{\rm int}_{\rm cor}$ maps $F : \mathbf{Euc^c} \to \mathbf{Sets}$ to $F|_{\mathbf{Euc^c_{in}}} : \mathbf{Euc^c_{in}} \to \mathbf{Sets}$.

Here is the motivating example.

Example 4.1.3. **(a)** Let X be a manifold with corners. Define a categorical pre C^∞-ring with corners $F : \mathbf{Euc^c} \to \mathbf{Sets}$ by $F = \mathrm{Hom}_{\mathbf{Man^c}}(X, -)$. That is, for objects $\mathbb{R}^m \times [0,\infty)^n$ in $\mathbf{Euc^c} \subset \mathbf{Man^c}$ we have

$$F\bigl(\mathbb{R}^m \times [0,\infty)^n\bigr) = \mathrm{Hom}_{\mathbf{Man^c}}\bigl(X, \mathbb{R}^m \times [0,\infty)^n\bigr),$$

and for morphisms $g : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}^{m'} \times [0,\infty)^{n'}$ in $\mathbf{Euc^c}$ we have

$$F(g) = g\circ : \mathrm{Hom}_{\mathbf{Man^c}}\bigl(X, \mathbb{R}^m \times [0,\infty)^n\bigr) \longrightarrow \mathrm{Hom}_{\mathbf{Man^c}}\bigl(X, \mathbb{R}^{m'} \times [0,\infty)^{n'}\bigr)$$

mapping $F(g) : h \mapsto g\circ h$. Let $f : X \to Y$ be a smooth map of manifolds with corners, and $F, G : \mathbf{Euc^c} \to \mathbf{Sets}$ the functors corresponding to X, Y. Define a natural transformation $\eta : G \Rightarrow F$ by

$$\eta\bigl(\mathbb{R}^m \times [0,\infty)^n\bigr) = \circ f : \mathrm{Hom}\bigl(Y, \mathbb{R}^m \times [0,\infty)^n\bigr) \longrightarrow \mathrm{Hom}\bigl(X, \mathbb{R}^m \times [0,\infty)^n\bigr)$$

mapping $\eta : h \mapsto h \circ f$.

Define a functor $F_{\mathbf{Man^c}}^{\mathbf{CPC^\infty Rings^c}} : \mathbf{Man^c} \to (\mathbf{CPC^\infty Rings^c})^{\mathbf{op}}$ to map $X \mapsto F$ on objects, and $f \mapsto \eta$ on morphisms, for X, Y, F, G, f, η as above.

All of this also works if X, Y are manifolds with g-corners, as in Definition 3.2.2, giving a functor $F_{\mathbf{Man^{gc}}}^{\mathbf{CPC^\infty Rings^c}} : \mathbf{Man^{gc}} \to (\mathbf{CPC^\infty Rings^c})^{\mathbf{op}}$.

(b) Similarly, if X is a manifold with corners, define a categorical interior pre C^∞-ring with corners $F : \mathbf{Euc^c_{in}} \to \mathbf{Sets}$ by $F = \mathrm{Hom}_{\mathbf{Man^c_{in}}}(X, -)$. This gives functors $F_{\mathbf{Man^c_{in}}}^{\mathbf{CPC^\infty Rings^c_{in}}} : \mathbf{Man^c_{in}} \to (\mathbf{CPC^\infty Rings^c_{in}})^{\mathbf{op}}$ and $F_{\mathbf{Man^{gc}_{in}}}^{\mathbf{CPC^\infty Rings^c_{in}}} : \mathbf{Man^{gc}_{in}} \to (\mathbf{CPC^\infty Rings^c_{in}})^{\mathbf{op}}$.

In the language of Algebraic Theories, as in Adámek, Rosický and Vitale [3], $\mathbf{Euc}, \mathbf{Euc^c}$, $\mathbf{Euc^c_{in}}$ are examples of *algebraic theories* (that is, small categories with finite products), and $\mathbf{CC^\infty Rings}, \mathbf{CPC^\infty Rings^c}, \mathbf{CPC^\infty Rings^c_{in}}$ are the corresponding *categories of algebras*. Also the inclusions of subcategories (4.1.2) are *morphisms of algebraic theories*, and the functors (4.1.3) the corresponding morphisms. So, as for Proposition 2.1.6, Adámek et al. [3, Prop.s 1.21, 2.5, 9.3 & Th. 4.5] give important results on their categorical properties:

Theorem 4.1.4. **(a)** *The categories* $\mathbf{CC^\infty Rings}, \mathbf{CPC^\infty Rings^c}, \mathbf{CPC^\infty Rings^c_{in}}$, *have all small limits and directed colimits and they may be computed objectwise in* $\mathbf{Euc}, \mathbf{Euc^c}$, $\mathbf{Euc^c_{in}}$ *by taking the corresponding small limits/directed colimits in* $\mathbf{Sets}$.

(b) *All small colimits exist in* $\mathbf{CC^\infty Rings}, \mathbf{CPC^\infty Rings^c}, \mathbf{CPC^\infty Rings^c_{in}}$, *though in general they are not computed objectwise in* $\mathbf{Euc}, \mathbf{Euc^c}, \mathbf{Euc^c_{in}}$ *by taking colimits in* $\mathbf{Sets}$.

(c) *There are functors* $I_{C^\infty}^{\mathrm{cor}}, I_{C^\infty}^{\mathrm{int}}, I_{\mathrm{int}}^{\mathrm{cor}}$ *which are left adjoints of* $\Pi_{\mathrm{cor}}^{C^\infty}, \Pi_{\mathrm{int}}^{C^\infty}, \Pi_{\mathrm{cor}}^{\mathrm{int}}$ *in* (4.1.3). *As* $\Pi_{\mathrm{cor}}^{C^\infty}, \Pi_{\mathrm{int}}^{C^\infty}, \Pi_{\mathrm{cor}}^{\mathrm{int}}$ *are right adjoints, they preserve limits. Since* $I_{C^\infty}^{\mathrm{cor}}, I_{C^\infty}^{\mathrm{int}}, I_{\mathrm{int}}^{\mathrm{cor}}$ *are left adjoints, they preserve colimits.*

In (a), Adámek et al. prove that all sifted colimits exist in **CC$^\infty$Rings**, … and may be computed objectwise in **Euc**, … by taking sifted colimits in **Sets**. Here *sifted colimits* [3, §2] are a class of limits in categories which include filtered colimits and directed colimits. We only mainly interested in directed colimits.

4.2 Pre C^∞-rings with corners

We now give the analogue of §4.1 using an alternative definition of pre C^∞-ring with corners similar to Definition 2.1.2, as in Kalashnikov [51, Def. 4.18].

Definition 4.2.1. A *pre C^∞-ring with corners* $\mathfrak{C}$ assigns the data:

(a) Two sets $\mathfrak{C}$ and $\mathfrak{C}_{\rm ex}$.

(b) Operations $\Phi_f : \mathfrak{C}^m \times \mathfrak{C}_{\rm ex}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$.

(c) Operations $\Psi_g : \mathfrak{C}^m \times \mathfrak{C}_{\rm ex}^n \to \mathfrak{C}_{\rm ex}$ for all exterior $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$.

Here we allow one or both of m, n to be zero, and consider S^0 to be the single point $\{\emptyset\}$ for any set S. These operations must satisfy the following relations:

(i) Suppose $k, l, m, n \geqslant 0$, and $e_i : \mathbb{R}^k \times [0,\infty)^l \to \mathbb{R}$ is smooth for $i = 1, \ldots, m$, and $f_j : \mathbb{R}^k \times [0,\infty)^l \to [0,\infty)$ is exterior for $j = 1, \ldots, n$, and $g : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ is smooth. Define smooth $h : \mathbb{R}^k \times [0,\infty)^l \to \mathbb{R}$ by

$$\begin{aligned} h(x_1, \ldots, x_k, y_1, \ldots, y_l) = g\big(&e_1(x_1, \ldots, y_l), \ldots, e_m(x_1, \ldots, y_l), \\ &f_1(x_1, \ldots, y_l), \ldots, f_n(x_1, \ldots, y_l)\big). \end{aligned} \tag{4.2.1}$$

Then for all $(c_1, \ldots, c_k, c'_1, \ldots, c'_l) \in \mathfrak{C}^k \times \mathfrak{C}_{\rm ex}^l$ we have

$$\begin{aligned} \Phi_h(c_1, \ldots, c_k, c'_1, \ldots, c'_l) = \Phi_g\big(&\Phi_{e_1}(c_1, \ldots, c'_l), \ldots, \Phi_{e_m}(c_1, \ldots, c'_l), \\ &\Psi_{f_1}(c_1, \ldots, c'_l), \ldots, \Psi_{f_n}(c_1, \ldots, c'_l)\big). \end{aligned}$$

(ii) Suppose $k, l, m, n \geqslant 0$, and $e_i : \mathbb{R}^k \times [0,\infty)^l \to \mathbb{R}$ is smooth for $i = 1, \ldots, m$, and $f_j : \mathbb{R}^k \times [0,\infty)^l \to [0,\infty)$ is exterior for $j = 1, \ldots, n$, and $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ is exterior. Define exterior $h : \mathbb{R}^k \times [0,\infty)^l \to [0,\infty)$ by (4.2.1). Then for all $(c_1, \ldots, c_k, c'_1, \ldots, c'_l) \in \mathfrak{C}^k \times \mathfrak{C}_{\rm ex}^l$ we have

$$\begin{aligned} \Psi_h(c_1, \ldots, c_k, c'_1, \ldots, c'_l) = \Psi_g\big(&\Phi_{e_1}(c_1, \ldots, c'_l), \ldots, \Phi_{e_m}(c_1, \ldots, c'_l), \\ &\Psi_{f_1}(c_1, \ldots, c'_l), \ldots, \Psi_{f_n}(c_1, \ldots, c'_l)\big). \end{aligned}$$

(iii) Write $\pi_i : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ for projection to the i^{th} coordinate of $\mathbb{R}^m$ for $i = 1,\ldots,m$, and $\pi'_j : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ for projection to the j^{th} coordinate of $[0,\infty)^n$ for $j = 1,\ldots,n$. Then for all $(c_1,\ldots,c_m,c'_1,\ldots,c'_n)$ in $\mathfrak{C}^m \times \mathfrak{C}_{\text{ex}}^n$ and all $i = 1,\ldots,m$, $j = 1,\ldots,n$ we have

$$\Phi_{\pi_i}(c_1,\ldots,c_m,c'_1,\ldots,c'_n) = c_i, \;\; \Psi_{\pi'_j}(c_1,\ldots,c_m,c'_1,\ldots,c'_n) = c'_j.$$

We will refer to the operations Φ_f, Ψ_g as the C^∞*-operations*, and we often write a pre C^∞-ring with corners as a pair $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\text{ex}})$, leaving the C^∞ operations implicit. If $c \in \mathfrak{C}$ and $c' \in \mathfrak{C}_{\text{ex}}$, by a slight abuse of notation we will write $\boldsymbol{c} = (c, c') \in \boldsymbol{\mathfrak{C}}$, which will be useful for §5.

Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\text{ex}})$ and $\boldsymbol{\mathfrak{D}} = (\mathfrak{D}, \mathfrak{D}_{\text{ex}})$ be pre C^∞-rings with corners. A *morphism* $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ is a pair $\boldsymbol{\phi} = (\phi, \phi_{\text{ex}})$ of maps $\phi : \mathfrak{C} \to \mathfrak{D}$ and $\phi_{\text{ex}} : \mathfrak{C}_{\text{ex}} \to \mathfrak{D}_{\text{ex}}$, which commute with all the operations Φ_f, Ψ_g on $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}$. That is, whenever $f : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ is smooth, $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ is exterior and $(c_1,\ldots,c_m,c'_1,\ldots,c'_n) \in \mathfrak{C}^m \times \mathfrak{C}_{\text{ex}}^n$ we have

$$\begin{aligned}
\phi \circ \Phi_f(c_1,\ldots,c_m,c'_1,\ldots,c'_n) &= \Phi_f\big(\phi(c_1),\ldots,\phi(c_m),\phi_{\text{ex}}(c'_1),\ldots,\phi_{\text{ex}}(c'_n)\big),\\
\phi_{\text{ex}} \circ \Psi_g(c_1,\ldots,c_m,c'_1,\ldots,c'_n) &= \Psi_g\big(\phi(c_1),\ldots,\phi(c_m),\phi_{\text{ex}}(c'_1),\ldots,\phi_{\text{ex}}(c'_n)\big).
\end{aligned}$$

Morphisms compose in the obvious way. Write $\mathbf{PC^\infty Rings^c}$ for the category of pre C^∞-rings with corners.

Define functors $\Pi_{\text{sm}}, \Pi_{\text{ex}} : \mathbf{PC^\infty Rings^c} \to \mathbf{Sets}$ by $\Pi_{\text{sm}} : \boldsymbol{\mathfrak{C}} \mapsto \mathfrak{C}$, $\Pi_{\text{ex}} : \boldsymbol{\mathfrak{C}} \mapsto \mathfrak{C}_{\text{ex}}$ on objects, and $\Pi_{\text{sm}} : \boldsymbol{\phi} \mapsto \phi$, $\Pi_{\text{ex}} : \boldsymbol{\phi} \mapsto \phi_{\text{ex}}$ on morphisms, where 'sm', 'ex' are short for 'smooth' and 'exterior'.

As for $\Pi_{\text{cor}}^{C^\infty}$ in (4.1.3), there is a natural functor $\Pi_{\text{cor}}^{C^\infty} : \mathbf{PC^\infty Rings^c} \to \mathbf{C^\infty Rings}$ acting on objects by $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\text{ex}}) \mapsto \mathfrak{C}$, where $\mathfrak{C}$ has the C^∞-operations $\Phi_f : \mathfrak{C}^m \to \mathfrak{C}$ from smooth $f : \mathbb{R}^m \to \mathbb{R}$ in (b) above with $n = 0$, and on morphisms by $\boldsymbol{\phi} = (\phi, \phi_{\text{ex}}) \mapsto \phi$.

Here is our motivating example.

Example 4.2.2. Let X be a manifold with corners. Define a pre C^∞-ring with corners $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\text{ex}})$ by $\mathfrak{C} = C^\infty(X)$ and $\mathfrak{C}_{\text{ex}} = \text{Ex}(X)$, as sets. If $f : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ is smooth, define the operation $\Phi_f : \mathfrak{C}^m \times \mathfrak{C}_{\text{ex}}^n \to \mathfrak{C}$ by

$$\Phi_f(c_1,\ldots,c_m,c'_1,\ldots,c'_n) : x \longmapsto f\big(c_1(x),\ldots,c_m(x),c'_1(x),\ldots,c'_n(x)\big). \tag{4.2.2}$$

If $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ is exterior, define $\Psi_g : \mathfrak{C}^m \times \mathfrak{C}_{\text{ex}}^n \to \mathfrak{C}_{\text{ex}}$ by

$$\Psi_g(c_1,\ldots,c_m,c'_1,\ldots,c'_n) : x \longmapsto g\big(c_1(x),\ldots,c_m(x),c'_1(x),\ldots,c'_n(x)\big). \tag{4.2.3}$$

This makes $\mathfrak{C}$ into a pre C^∞-ring with corners.

Suppose $f : X \to Y$ is a smooth map of manifolds with corners, and let $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}$ be the pre C^∞-rings with corners corresponding to X, Y. Write $\boldsymbol{\phi} = (\phi, \phi_{\rm ex})$, where $\phi : \mathfrak{D} \to \mathfrak{C}$ maps $\phi(d) = d \circ f$ and $\phi_{\rm ex} : \mathfrak{D}_{\rm ex} \to \mathfrak{C}_{\rm ex}$ maps $\phi(d') = d' \circ f$. Then $\boldsymbol{\phi} : \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{C}}$ is a morphism of pre C^∞-rings with corners.

Define a functor $F_{\mathbf{Man^c}}^{\mathbf{PC^\infty Rings^c}} : \mathbf{Man^c} \to (\mathbf{PC^\infty Rings^c})^{\mathbf{op}}$ to map $X \mapsto \boldsymbol{\mathfrak{C}}$ on objects, and $f \mapsto \boldsymbol{\phi}$ on morphisms, for $X, Y, \boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}, f, \boldsymbol{\phi}$ as above.

We will also write $\boldsymbol{C}^\infty(X) = F_{\mathbf{Man^c}}^{\mathbf{PC^\infty Rings^c}}(X)$, and write $\boldsymbol{f}^* : \boldsymbol{C}^\infty(Y) \to \boldsymbol{C}^\infty(X)$ for $F_{\mathbf{Man^c}}^{\mathbf{PC^\infty Rings^c}}(f) : F_{\mathbf{Man^c}}^{\mathbf{PC^\infty Rings^c}}(Y) \to F_{\mathbf{Man^c}}^{\mathbf{PC^\infty Rings^c}}(X)$.

All of this also works if X, Y are manifolds with g-corners, as in Definition 3.2.2, giving a functor $F_{\mathbf{Man^{gc}}}^{\mathbf{PC^\infty Rings^c}} : \mathbf{Man^{gc}} \to (\mathbf{PC^\infty Rings^c})^{\mathbf{op}}$.

Remark 4.2.3. An important difference between ordinary manifolds, and manifolds with corners, is that if X is a manifold with corners, $g_i : X \to [0, \infty)$ for $i \in I$ are exterior (or interior) maps, and $\{\eta_i : i \in I\}$ is a smooth partition of unity on X, then $\sum_{i \in I} \eta_i g_i : X \to [0, \infty)$ is generally not exterior (or interior).

This means that the geometry of manifolds with corners is more global. If $g : X \to [0, \infty)$ is an exterior map, then there is a locally constant map $n_g : \partial X \to \{0, 1, 2, \ldots, \infty\}$ such that g vanishes to order n_g along ∂X locally, and if x', x'' lie in the same connected component of ∂X then $n_g(x') = n_g(x'')$ even if $i_X(x'), i_X(x'')$ are far away in X.

Recall the notion of 'manifold with faces' in Definition 3.3.3. If X is a manifold with corners, but not a manifold with faces, then there are not enough exterior maps $g : X \to [0, \infty)$ to properly describe the local geometry of X. That is, X near some x is locally modelled on $\mathbb{R}^{n-k} \times [0, \infty)^k$ near 0 with coordinates $(x_1, \ldots, x_{n-k}, y_1, \ldots, y_k)$, but there do not exist exterior $g_i : X \to [0, \infty)$ locally modelled on $y_i : \mathbb{R}^{n-k} \times [0, \infty)^k \to [0, \infty)$ for all $i = 1, \ldots, k$. For example, the teardrop T in Example 3.3.4 is locally modelled near $(0, 0)$ on $[0, \infty)^2$, but we can only find exterior $g : T \to [0, \infty)$ locally modelled on $y_1^a y_2^b : [0, \infty)^2 \to [0, \infty)$ when $a = b$, as the multiplicities n_g on the y_1- and y_2-axes must be the same.

In the notation of §5, a manifold with corners X corresponds to a natural C^∞-scheme with corners $\boldsymbol{X}$, but $\boldsymbol{X}$ is affine with $\boldsymbol{X} \cong \operatorname{Spec^c} \boldsymbol{C}^\infty(X)$ only if X is a manifold with faces. If X does not have faces then $\boldsymbol{X} \ncong \operatorname{Spec^c} \boldsymbol{C}^\infty(X)$.

In fact $F_{\mathbf{Man^c}}^{\mathbf{PC^\infty Rings^c}} : \mathbf{Man^c} \to (\mathbf{PC^\infty Rings^c})^{\mathbf{op}}$ above is full and faithful. Despite this, if X does not have faces then we regard $\boldsymbol{C}^\infty(X)$ as somehow 'wrong' (e.g. its local C^∞-rings with corners in §4.6 and b-cotangent modules in §4.7 do not behave as expected), and we will not make much use of it.

This also means that results on C^∞-schemes relying on partitions of unity should not be expected to extend to C^∞-schemes with corners.

Similar to Remark 2.1.3, we have the following equivalence.

Proposition 4.2.4. *There is a natural equivalence of categories between* $\mathbf{CPC^\infty Rings^c}$ *and* $\mathbf{PC^\infty Rings^c}$, *which identifies* $F : \mathbf{Euc^c} \to \mathbf{Sets}$ *in* $\mathbf{CPC^\infty Rings^c}$ *with* $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ *in* $\mathbf{PC^\infty Rings^c}$ *such that* $F\bigl(\mathbb{R}^m \times [0,\infty)^n\bigr) = \mathfrak{C}^m \times \mathfrak{C}_{\rm ex}^n$ *for* $m, n \geqslant 0$.

Under this equivalence, for a smooth function $f : \mathbb{R}^n_k \to \mathbb{R}$, we identify $F(f)$ with Φ_f, and for an exterior function $g : \mathbb{R}^n_k \to [0,\infty)$, we identify $F(g)$ with Ψ_g. The proof of this proposition then follows from F being a product preserving functor, and the definition of C^∞-ring with corners, similar to the discussion in Joyce [40, p. 7].

The operations Φ_f, Ψ_g on a pre C^∞-ring with corners $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ comprise a huge amount of data. It is often helpful to work with a small subset of this structure. The next definition explains this small subset. We use the theory of monoids from §3.1.

Definition 4.2.5. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a pre C^∞-ring with corners. Then $\mathfrak{C} = \Pi^{C^\infty}_{\rm cor}(\boldsymbol{\mathfrak{C}})$ is a C^∞-ring, and thus a commutative $\mathbb{R}$-algebra. The $\mathbb{R}$-algebra structure makes $\mathfrak{C}$ into a monoid in two ways: under multiplication '$\cdot$' with identity 1, and under addition '+' with identity 0.

Define $g : [0,\infty)^2 \to [0,\infty)$ by $g(x,y) = xy$. Then g induces $\Psi_g : \mathfrak{C}_{\rm ex} \times \mathfrak{C}_{\rm ex} \to \mathfrak{C}_{\rm ex}$. Define multiplication $\cdot : \mathfrak{C}_{\rm ex} \times \mathfrak{C}_{\rm ex} \to \mathfrak{C}_{\rm ex}$ by $c' \cdot c'' = \Psi_g(c', c'')$. The map $1 : \mathbb{R}^0 \to [0,\infty)$ gives an operation $\Psi_1 : \{\emptyset\} \to \mathfrak{C}_{\rm ex}$. The *identity* in $\mathfrak{C}_{\rm ex}$ is $1_{\mathfrak{C}_{\rm ex}} = \Psi_1(\emptyset)$. This makes $(\mathfrak{C}_{\rm ex}, \cdot, 1_{\mathfrak{C}_{\rm ex}})$ into a monoid.

The functor $\Pi_{\rm ex} : \mathbf{PC^\infty Rings^c} \to \mathbf{Sets}$ in Definition 4.2.1 extends to a functor $\bar\Pi_{\rm ex} : \mathbf{PC^\infty Rings^c} \to \mathbf{Mon}$ mapping $\Pi_{\rm ex} : \boldsymbol{\mathfrak{C}} \mapsto \mathfrak{C}_{\rm ex}$ and $\Pi_{\rm ex} : \boldsymbol{\phi} \mapsto \phi_{\rm ex}$, where $\mathfrak{C}_{\rm ex}$ is now regarded as a monoid.

The map $0 : \mathbb{R}^0 \to [0,\infty)$ gives an operation $\Psi_0 : \{\emptyset\} \to \mathfrak{C}_{\rm ex}$. Thus we have a distinguished element $0_{\mathfrak{C}_{\rm ex}} = \Psi_0(\emptyset)$ in $\mathfrak{C}_{\rm ex}$, which is not the monoid identity element. It is uniquely characterised by the property that $c' \cdot 0_{\mathfrak{C}_{\rm ex}} = 0_{\mathfrak{C}_{\rm ex}}$ for all $c' \in \mathfrak{C}_{\rm ex}$.

Write $i : [0,\infty) \hookrightarrow \mathbb{R}$ for the inclusion. Then we have a map $\Phi_i : \mathfrak{C}_{\rm ex} \to \mathfrak{C}$. This is a monoid morphism, for $\mathfrak{C}$ a monoid under multiplication.

The exterior map $\exp : \mathbb{R} \to [0,\infty)$ induces $\Psi_{\exp} : \mathfrak{C} \to \mathfrak{C}_{\rm ex}$. It is a monoid morphism, for $\mathfrak{C}$ a monoid under addition.

The smooth map $\exp : \mathbb{R} \to \mathbb{R}$ induces $\Phi_{\exp} : \mathfrak{C} \to \mathfrak{C}$, with $\Phi_{\exp} = \Phi_i \circ \Psi_{\exp}$.

To summarise, the following data in $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ are particularly important:

(a) $\mathfrak{C}$ is a commutative $\mathbb{R}$-algebra.

(b) $\mathfrak{C}_{\rm ex}$ is a monoid. It has a special $0_{\mathfrak{C}_{\rm ex}} \in \mathfrak{C}_{\rm ex}$ with $c' \cdot 0_{\mathfrak{C}_{\rm ex}} = 0_{\mathfrak{C}_{\rm ex}}$, all $c' \in \mathfrak{C}_{\rm ex}$.

(c) $\Phi_i : \mathfrak{C}_{\rm ex} \to \mathfrak{C}$ is a monoid morphism, for $\mathfrak{C}$ a monoid under multiplication.

(d) $\Psi_{\exp} : \mathfrak{C} \to \mathfrak{C}_{\rm ex}$ is a monoid morphism, for $\mathfrak{C}$ a monoid under addition.

(e) $\Phi_{\exp} = \Phi_i \circ \Psi_{\exp} : \mathfrak{C} \to \mathfrak{C}$.

Many of our definitions will use only the structures (a)–(e). When we write $\Phi_i, \Psi_{\exp}, \Phi_{\exp}$ without further explanation, we mean those in (c)–(e).

The monoid morphism $\Phi_i : \mathfrak{C}_{\rm ex} \to \mathfrak{C}$ makes $\mathfrak{C}$ into a *prelog ring*, in the sense of log geometry discussed in §5.9. We will define C^∞-rings with corners in §4.3 to be pre C^∞-rings with corners satisfying an additional condition similar to requiring $\Phi_i : \mathfrak{C}_{\rm ex} \to \mathfrak{C}$ to be a *log ring*.

Now we could define 'interior pre C^∞-rings with corners' following Definition 4.2.1, but replacing exterior maps by interior maps throughout. Instead we will do something more complicated. The functor $I_{\rm int}^{\rm cor} : \mathbf{CPC^\infty Rings^c_{in}} \to \mathbf{CPC^\infty Rings^c}$ in Theorem 4.1.4(c) is faithful, and thus an equivalence from $\mathbf{CPC^\infty Rings^c_{in}}$ to a subcategory of $\mathbf{CPC^\infty Rings^c}$, the essential image of $I_{\rm int}^{\rm cor}$.

We will define $\mathbf{PC^\infty Rings^c_{in}} \subset \mathbf{PC^\infty Rings^c}$ to be the subcategory corresponding to the essential image of $I_{\rm int}^{\rm cor}$ under the equivalence $\mathbf{CPC^\infty Rings^c} \cong \mathbf{PC^\infty Rings^c}$ from Proposition 4.2.4. That is, we will define interior pre C^∞-rings with corners as special examples of pre C^∞-rings with corners, and interior morphisms as special morphisms between (interior) pre C^∞-rings with corners.

The advantage of this is that rather than having two separate theories, we will be able to work with both interior and non-interior (pre) C^∞-rings with corners, and both interior and non-interior morphisms, all in a single theory.

Definition 4.2.6. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a pre C^∞-ring with corners. Then $\mathfrak{C}_{\rm ex}$ is a monoid and $0_{\mathfrak{C}_{\rm ex}} \in \mathfrak{C}_{\rm ex}$ with $c' \cdot 0_{\mathfrak{C}_{\rm ex}} = 0_{\mathfrak{C}_{\rm ex}}$ for all $c' \in \mathfrak{C}_{\rm ex}$.

We call $\boldsymbol{\mathfrak{C}}$ an *interior* pre C^∞-ring with corners if $0_{\mathfrak{C}_{\rm ex}} \neq 1_{\mathfrak{C}_{\rm ex}}$, and there do not exist $c', c'' \in \mathfrak{C}_{\rm ex}$ with $c' \neq 0_{\mathfrak{C}_{\rm ex}} \neq c''$ and $c' \cdot c'' = 0_{\mathfrak{C}_{\rm ex}}$. That is, $\mathfrak{C}_{\rm ex}$ should have no zero divisors. Write $\mathfrak{C}_{\rm in} = \mathfrak{C}_{\rm ex} \setminus \{0_{\mathfrak{C}_{\rm ex}}\}$. Then $\mathfrak{C}_{\rm ex} = \mathfrak{C}_{\rm in} \amalg \{0_{\mathfrak{C}_{\rm ex}}\}$, where $\amalg$ is the disjoint union. Since $\mathfrak{C}_{\rm ex}$ has no zero divisors, $\mathfrak{C}_{\rm in}$ is closed under multiplication, and $1_{\mathfrak{C}_{\rm ex}} \in \mathfrak{C}_{\rm in}$ as $0_{\mathfrak{C}_{\rm ex}} \neq 1_{\mathfrak{C}_{\rm ex}}$. Thus $\mathfrak{C}_{\rm in}$ is a submonoid of $\mathfrak{C}_{\rm ex}$. We write $1_{\mathfrak{C}_{\rm in}} = 1_{\mathfrak{C}_{\rm ex}}$. This implies $(\mathfrak{C}, \mathfrak{C}_{\rm ex})$ is in the essential image of $I_{\rm int}^{\rm cor}$ considered as a subcategory of $\mathbf{PC^\infty Rings^c}$.

Let $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}$ be interior pre C^∞-rings with corners, and $\boldsymbol{\phi} = (\phi, \phi_{\rm ex}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ be a morphism in $\mathbf{PC^\infty Rings^c}$. We call $\boldsymbol{\phi}$ *interior* if $\phi_{\rm ex}(\mathfrak{C}_{\rm in}) \subseteq \mathfrak{D}_{\rm in}$. Then we write

$\phi_{\rm in} = \phi_{\rm ex}|_{\mathfrak{C}_{\rm in}} : \mathfrak{C}_{\rm in} \to \mathfrak{D}_{\rm in}$. Interior morphisms are closed under composition and include the identity morphisms.

Write $\mathbf{PC^\infty Rings^c_{in}}$ for the (non-full) subcategory of $\mathbf{PC^\infty Rings^c}$ with objects interior pre C^∞-rings with corners, and morphisms interior morphisms.

As in §4.1, define functors $\Pi_{\rm sm}, \Pi_{\rm in} : \mathbf{PC^\infty Rings^c_{in}} \to \mathbf{Sets}$ by $\Pi_{\rm sm} : \boldsymbol{\mathfrak{C}} \mapsto \mathfrak{C}$, $\Pi_{\rm in} : \boldsymbol{\mathfrak{C}} \mapsto \mathfrak{C}_{\rm in}$ on objects, and $\Pi_{\rm sm} : \boldsymbol{\phi} \mapsto \phi$, $\Pi_{\rm in} : \boldsymbol{\phi} \mapsto \phi_{\rm in}$ on morphisms, where 'sm', 'in' are short for 'smooth' and 'interior'. Also define $\bar\Pi_{\rm in} : \mathbf{PC^\infty Rings^c_{in}} \to \mathbf{Mon}$ to map $\bar\Pi_{\rm in} : \boldsymbol{\mathfrak{C}} \mapsto \mathfrak{C}_{\rm in}$, where $\mathfrak{C}_{\rm in}$ is regarded as a monoid.

Define $\Pi^{C^\infty}_{\rm int} : \mathbf{PC^\infty Rings^c_{in}} \to \mathbf{C^\infty Rings}$ to be the restriction of the functor $\Pi^{C^\infty}_{\rm cor} : \mathbf{PC^\infty Rings^c} \to \mathbf{C^\infty Rings}$ in Definition 4.2.1 to $\mathbf{PC^\infty Rings^c_{in}}$.

Example 4.2.7. Let X be a manifold with corners. Define an interior pre C^∞-ring with corners $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ by $\mathfrak{C} = C^\infty(X)$ and $\mathfrak{C}_{\rm ex} = {\rm In}(X) \amalg \{0\}$ the disjoint union, as sets, where 0 is the zero function $X \to [0,\infty)$.

If $f : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ is smooth and $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ is exterior, define $\Phi_f : \mathfrak{C}^m \times \mathfrak{C}^n_{\rm ex} \to \mathfrak{C}$ and $\Psi_g : \mathfrak{C}^m \times \mathfrak{C}^n_{\rm ex} \to \mathfrak{C}_{\rm ex}$ by (4.2.2)–(4.2.3). Here to check that Ψ_g does map to $\mathfrak{C}_{\rm ex} \subseteq {\rm Ex}(X)$, we verify that $\Psi_g(c_1, \cdots, c_m, c'_1, \ldots, c'_n) : X \to [0,\infty)$ is either interior or zero, since $c'_1, \ldots, c'_n$ are either interior or zero.

Suppose $f : X \to Y$ is an interior map of manifolds with corners, and let $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}$ be the interior pre C^∞-rings with corners corresponding to X, Y. Write $\boldsymbol{\phi} = (\phi, \phi_{\rm ex})$, where $\phi : \mathfrak{D} \to \mathfrak{C}$ maps $\phi(d) = d \circ f$ and $\phi_{\rm ex} : \mathfrak{D}_{\rm ex} \to \mathfrak{C}_{\rm ex}$ maps $\phi(d') = d' \circ f$. Then $\boldsymbol{\phi} : \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{C}}$ is a morphism in $\mathbf{PC^\infty Rings^c_{in}}$.

Define a functor $F^{\mathbf{PC^\infty Rings^c_{in}}}_{\mathbf{Man^c_{in}}} : \mathbf{Man^c_{in}} \to (\mathbf{PC^\infty Rings^c_{in}})^{\mathbf{op}}$ to map $X \mapsto \boldsymbol{\mathfrak{C}}$ on objects, and $f \mapsto \boldsymbol{\phi}$ on morphisms, for $X, Y, \boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}, f, \boldsymbol{\phi}$ as above.

If X is a manifold with corners, then $F^{\mathbf{PC^\infty Rings^c}}_{\mathbf{Man^c}}(X)$ from Example 4.2.2 agrees with $F^{\mathbf{PC^\infty Rings^c_{in}}}_{\mathbf{Man^c_{in}}}(X)$ if and only if X is *nonempty and connected*, since then ${\rm Ex}(X)$ is the disjoint union ${\rm In}(X) \amalg \{0\}$.

We will also write $\boldsymbol{C}^\infty_{\rm in}(X) = F^{\mathbf{PC^\infty Rings^c_{in}}}_{\mathbf{Man^c_{in}}}(X)$, and write $\boldsymbol{f}^*_{\rm in} : \boldsymbol{C}^\infty_{\rm in}(Y) \to \boldsymbol{C}^\infty_{\rm in}(X)$ for $F^{\mathbf{PC^\infty Rings^c_{in}}}_{\mathbf{Man^c_{in}}}(f) : F^{\mathbf{PC^\infty Rings^c_{in}}}_{\mathbf{Man^c_{in}}}(Y) \to F^{\mathbf{PC^\infty Rings^c_{in}}}_{\mathbf{Man^c_{in}}}(X)$. Thus $\boldsymbol{C}^\infty_{\rm in}(X) = \boldsymbol{C}^\infty(X)$ if and only if X is nonempty and connected.

This example also works if X, Y are manifolds with g-corners, as in Definition 3.2.2, giving a functor $F^{\mathbf{PC^\infty Rings^c_{in}}}_{\mathbf{Man^{gc}_{in}}} : \mathbf{Man^{gc}_{in}} \to (\mathbf{PC^\infty Rings^c_{in}})^{\mathbf{op}}$.

Lemma 4.2.8. *Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be an interior pre C^∞-ring with corners. Then $\mathfrak{C}^\times_{\rm ex} \subseteq \mathfrak{C}_{\rm in}$. If $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ is interior, then $\Psi_g : \mathfrak{C}^m \times \mathfrak{C}^n_{\rm ex} \to \mathfrak{C}_{\rm ex}$ maps $\mathfrak{C}^m \times \mathfrak{C}^n_{\rm in} \to \mathfrak{C}_{\rm in}$.*

Proof. As $0_{\mathfrak{C}_{\rm ex}}$ is not invertible, then $0_{\mathfrak{C}_{\rm ex}} \notin \mathfrak{C}_{\rm ex}^\times$, and $\mathfrak{C}_{\rm ex}^\times \subseteq \mathfrak{C}_{\rm ex} \setminus \{0_{\mathfrak{C}_{\rm ex}}\} = \mathfrak{C}_{\rm in}$. As g is interior we may write

$$g(x_1,\ldots,x_m,y_1,\ldots,y_n) = y_1^{a_1}\cdots y_n^{a_n}\cdot \exp\circ h(x_1,\ldots,x_m,y_1,\ldots,y_n), \tag{4.2.4}$$

for $a_1,\ldots,a_n \in \mathbb{N}$ and $h : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ smooth. Then for $c_1,\ldots,c_m \in \mathfrak{C}$ and $c'_1,\ldots,c'_n \in \mathfrak{C}_{\rm in}$ we have

$$\Psi_g\bigl(c_1,\ldots,c_m,c'_1,\ldots,c'_n\bigr) = c_1'^{a_1}\cdots c_n'^{a_n}\cdot \Psi_{\exp}\bigl[\Phi_h\bigl(c_1,\ldots,c_m,c'_1,\ldots,c'_n\bigr)\bigr].$$

Here $c_1'^{a_1}\cdots c_n'^{a_n} \in \mathfrak{C}_{\rm in}$ as $\mathfrak{C}_{\rm in}$ is a submonoid of $\mathfrak{C}_{\rm ex}$, and $\Psi_{\exp}[\cdots] \in \mathfrak{C}_{\rm in}$ as $\Psi_{\exp}$ maps to $\mathfrak{C}_{\rm ex}^\times \subseteq \mathfrak{C}_{\rm in} \subseteq \mathfrak{C}_{\rm ex}$. Thus $\Psi_g\bigl(c_1,\ldots,c_m,c'_1,\ldots,c'_n\bigr) \in \mathfrak{C}_{\rm in}$. □

Here is the analogue of Remark 2.1.3 and Proposition 4.2.4.

Proposition 4.2.9. *There is an equivalence* $\mathbf{CPC^\infty Rings^c_{in}} \cong \mathbf{PC^\infty Rings^c_{in}}$, *which identifies* $F : \mathbf{Euc^c_{in}} \to \mathbf{Sets}$ *in* $\mathbf{CPC^\infty Rings^c_{in}}$ *with* $\boldsymbol{\mathfrak{C}} = (\mathfrak{C},\mathfrak{C}_{\rm ex})$ *in* $\mathbf{PC^\infty Rings^c_{in}}$ *such that* $F\bigl(\mathbb{R}^m \times [0,\infty)^n\bigr) = \mathfrak{C}^m \times \mathfrak{C}_{\rm in}^n$ *for* $m,n \geqslant 0$.

Proof. Let $F : \mathbf{Euc^c_{in}} \to \mathbf{Sets}$ be a categorical interior pre C^∞-ring with corners. Define sets $\mathfrak{C} = F(\mathbb{R})$, $\mathfrak{C}_{\rm in} = F([0,\infty))$, and $\mathfrak{C}_{\rm ex} = \mathfrak{C}_{\rm in} \amalg \{0_{\mathfrak{C}_{\rm ex}}\}$, where $\amalg$ is the disjoint union. Then $F\bigl(\mathbb{R}^m \times [0,\infty)^n\bigr) = \mathfrak{C}^m \times \mathfrak{C}_{\rm in}^n$, as F is product-preserving. Let $f : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ and $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ be smooth. We must define maps $\Phi_f : \mathfrak{C}^m \times \mathfrak{C}_{\rm ex}^n \to \mathfrak{C}$ and $\Psi_g : \mathfrak{C}^m \times \mathfrak{C}_{\rm ex}^n \to \mathfrak{C}_{\rm ex}$.

Let $c_1,\ldots,c_m \in \mathfrak{C}$ and $c'_1,\ldots,c'_n \in \mathfrak{C}_{\rm ex}$. Then some of $c'_1,\ldots,c'_n$ lie in $\mathfrak{C}_{\rm in}$ and the rest in $\{0_{\mathfrak{C}_{\rm ex}}\}$. For simplicity suppose that $c'_1,\ldots,c'_k \in \mathfrak{C}_{\rm in}$ and $c'_{k+1} = \cdots = c'_n = 0_{\mathfrak{C}_{\rm ex}}$ for $0 \leqslant k \leqslant n$. Define smooth $d : \mathbb{R}^m \times [0,\infty)^k \to \mathbb{R}$, $e : \mathbb{R}^m \times [0,\infty)^k \to [0,\infty)$ by

$$\begin{aligned} d(x_1,\ldots,x_m,y_1,\ldots,y_k) &= f(x_1,\ldots,x_m,y_1,\ldots,y_k,0,\ldots,0),\\ e(x_1,\ldots,x_m,y_1,\ldots,y_k) &= g(x_1,\ldots,x_m,y_1,\ldots,y_k,0,\ldots,0). \end{aligned} \tag{4.2.5}$$

Then $F(d)$ maps $\mathfrak{C}^m \times \mathfrak{C}_{\rm in}^k \to \mathfrak{C}$. Set

$$\Phi_f(c_1,\ldots,c_m,c'_1,\ldots,c'_k,0_{\mathfrak{C}_{\rm ex}},\ldots,0_{\mathfrak{C}_{\rm ex}}) = F(d)(c_1,\ldots,c_m,c'_1,\ldots,c'_k).$$

Either $e : \mathbb{R}^m \times [0,\infty)^k \to [0,\infty)$ is interior, or $e = 0$. If e is interior define

$$\Psi_g(c_1,\ldots,c_m,c'_1,\ldots,c'_k,0_{\mathfrak{C}_{\rm ex}},\ldots,0_{\mathfrak{C}_{\rm ex}}) = F(e)(c_1,\ldots,c_m,c'_1,\ldots,c'_k).$$

If $e = 0$ set $\Psi_g(c_1,\ldots,c_m,c'_1,\ldots,c'_k,0_{\mathfrak{C}_{\rm ex}},\ldots,0_{\mathfrak{C}_{\rm ex}}) = 0_{\mathfrak{C}_{\rm ex}}$. This defines Φ_f, Ψ_g, and makes $\boldsymbol{\mathfrak{C}} = (\mathfrak{C},\mathfrak{C}_{\rm ex})$ into an interior pre C^∞-ring with corners.

Conversely, let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be an interior pre C^∞-ring with corners. Then $\mathfrak{C}_{\rm ex} = \mathfrak{C}_{\rm in} \amalg \{0\}$. As in the proof of Proposition 4.2.4 we define a product-preserving functor $F : \mathbf{Euc^c_{in}} \to \mathbf{Sets}$ with $F\big(\mathbb{R}^m \times [0,\infty)^n\big) = \mathfrak{C}^m \times \mathfrak{C}_{\rm in}^n$, using the fact from Lemma 4.2.8 that $\Psi_g : \mathfrak{C}^m \times \mathfrak{C}_{\rm ex}^n \to \mathfrak{C}_{\rm ex}$ maps $\mathfrak{C}^m \times \mathfrak{C}_{\rm in}^n \to \mathfrak{C}_{\rm in}$ for g interior. The rest of the proof follows that of Proposition 4.2.4. □

We define analogues of the functors $I^{\rm int}_{C^\infty}, I^{\rm cor}_{C^\infty}, \Pi^{\rm int}_{\rm cor}$ in Theorem 4.1.4.

Definition 4.2.10. Let $\mathfrak{C}$ be a C^∞-ring, and write $\Phi'_e : \mathfrak{C}^n \to \mathfrak{C}$ for the C^∞-ring operations on $\mathfrak{C}$ for $e : \mathbb{R}^n \to \mathbb{R}$ smooth. Set $\mathfrak{C}_{\rm ex} = \mathfrak{C} \amalg \{0_{\mathfrak{C}_{\rm ex}}\}$, where $\amalg$ is the disjoint union.

Let $f : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ be smooth. We must define $\Phi_f : \mathfrak{C}^m \times \mathfrak{C}^n_{\rm ex} \to \mathfrak{C}$. Let $c_1, \ldots, c_m \in \mathfrak{C}$ and $c'_1, \ldots, c'_n \in \mathfrak{C}_{\rm ex}$. Then some of $c'_1, \ldots, c'_n$ lie in $\mathfrak{C}_{\rm in} = \mathfrak{C}$ and the rest in $\{0_{\mathfrak{C}_{\rm ex}}\}$. For simplicity suppose that $c'_1, \ldots, c'_k \in \mathfrak{C}$ and $c'_{k+1} = \cdots = c'_n = 0_{\mathfrak{C}_{\rm ex}}$ for $0 \leqslant k \leqslant n$. Define smooth $d : \mathbb{R}^{m+k} \to \mathbb{R}$ by

$$d(x_1, \ldots, x_m, y_1, \ldots, y_k) = f(x_1, \ldots, x_m, \exp y_1, \ldots, \exp y_k, 0, \ldots, 0),$$

and set $\Phi_f(c_1, \ldots, c_m, c'_1, \ldots, c'_n) = \Phi'_d(c_1, \ldots, c_m, c'_1, \ldots, c'_k)$. This defines Φ_f.

Suppose $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ is exterior. Let $c_1, \ldots, c_m \in \mathfrak{C}$ and $c'_1, \ldots, c'_n \in \mathfrak{C}_{\rm ex}$, and again for simplicity suppose that $c'_1, \ldots, c'_k \in \mathfrak{C}$ and $c'_{k+1} = \cdots = c'_n = 0_{\mathfrak{C}_{\rm ex}}$. Define exterior $h : \mathbb{R}^m \times [0,\infty)^k \to [0,\infty)$ by

$$h(x_1, \ldots, x_m, y_1, \ldots, y_k) = g(x_1, \ldots, x_m, y_1, \ldots, y_k, 0, \ldots, 0).$$

Then either $h = 0$ or h is interior. If $h = 0$ set $\Psi_g(c_1, \ldots, c_m, c'_1, \ldots, c'_n) = 0_{\mathfrak{C}_{\rm ex}}$. If h is interior it maps $\mathbb{R}^m \times (0,\infty)^k \to (0,\infty)$, so we define $e : \mathbb{R}^{m+k} \to \mathbb{R}$ by

$$e(x_1, \ldots, x_m, y_1, \ldots, y_k) = \log\big(h(x_1, \ldots, x_m, \exp y_1, \ldots, \exp y_k)\big).$$

Set $\Psi_g(c_1, \ldots, c_m, c'_1, \ldots, c'_n) = \Phi'_e(c_1, \ldots, c_m, c'_1, \ldots, c'_k)$. This defines Ψ_g.

These Φ_f, Ψ_g make $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ into a pre C^∞-ring with corners, which has $\Pi^{C^\infty}_{\rm cor}(\boldsymbol{\mathfrak{C}}) = \mathfrak{C}$. Also $\mathfrak{C}_{\rm ex}$ has no zero divisors, so $\boldsymbol{\mathfrak{C}}$ is interior.

Define a functor $I^{\rm int}_{C^\infty} : \mathbf{C^\infty Rings} \to \mathbf{PC^\infty Rings^c_{in}}$ to map $\mathfrak{C} \mapsto \boldsymbol{\mathfrak{C}}$ on objects, for $\boldsymbol{\mathfrak{C}}$ as above, and to map morphisms $\phi : \mathfrak{C} \to \mathfrak{D}$ in $\mathbf{C^\infty Rings}$ to $\boldsymbol{\phi} = (\phi, \phi_{\rm ex}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ in $\mathbf{PC^\infty Rings^c_{in}}$, where $\phi_{\rm ex} : \mathfrak{C} \amalg \{0_{\mathfrak{C}_{\rm ex}}\} \to \mathfrak{D} \amalg \{0_{\mathfrak{D}_{\rm ex}}\}$ maps $\phi_{\rm ex}|_{\mathfrak{C}} = \phi$ and $\phi_{\rm ex}(0_{\mathfrak{C}_{\rm ex}}) = 0_{\mathfrak{D}_{\rm ex}}$. Define $I^{\rm cor}_{C^\infty} : \mathbf{C^\infty Rings} \to \mathbf{PC^\infty Rings^c}$ by $I^{\rm cor}_{C^\infty} = I^{\rm int}_{C^\infty}$. Both $I^{\rm int}_{C^\infty}, I^{\rm cor}_{C^\infty}$ are full and faithful.

Suppose now that $\mathfrak{C}$ is a C^∞-ring, and $\boldsymbol{\mathfrak{D}}$ an interior pre C^∞-ring with corners, and set $\boldsymbol{\mathfrak{C}} = I^{\rm int}_{C^\infty}(\mathfrak{C})$. Then we can define a 1-1 correspondence

$$\begin{aligned}\mathrm{Hom}_{\mathbf{C^\infty Rings}}\bigl(\mathfrak{C},\Pi^{C^\infty}_{\rm int}(\boldsymbol{\mathfrak{D}})\bigr) &= \mathrm{Hom}_{\mathbf{C^\infty Rings}}\bigl(\mathfrak{C},\mathfrak{D}\bigr) \cong \mathrm{Hom}_{\mathbf{PC^\infty Rings^c_{in}}}(\boldsymbol{\mathfrak{C}},\boldsymbol{\mathfrak{D}})\\ &= \mathrm{Hom}_{\mathbf{PC^\infty Rings^c_{in}}}\bigl(I^{\rm int}_{C^\infty}(\mathfrak{C}),\boldsymbol{\mathfrak{D}}\bigr),\end{aligned}$$

identifying $\phi:\mathfrak{C}\to\mathfrak{D}$ with $\hat{\boldsymbol\phi}=(\phi,\phi_{\rm ex}):\boldsymbol{\mathfrak{C}}\to\boldsymbol{\mathfrak{D}}$, where $\phi_{\rm ex}:\mathfrak{C}\amalg\{0_{\mathfrak{C}_{\rm ex}}\}\to\mathfrak{D}_{\rm ex}$ is given by $\phi_{\rm ex}|_{\mathfrak{C}}=\Psi_{\rm exp}\circ\phi$ and $\phi_{\rm ex}(0_{\mathfrak{C}_{\rm ex}})=0_{\mathfrak{D}_{\rm ex}}$. This is functorial in $\mathfrak{C},\boldsymbol{\mathfrak{D}}$, and so shows that $I^{\rm int}_{C^\infty}$ is left adjoint to $\Pi^{C^\infty}_{\rm int}$. The same proof shows that $I^{\rm cor}_{C^\infty}$ is left adjoint to $\Pi^{C^\infty}_{\rm cor}$. There is also a right adjoint to $\Pi^{C^\infty}_{\rm cor}$, which we define for C^∞-rings with corners in Theorem 4.3.9.

Definition 4.2.11. We define a functor $\Pi^{\rm int}_{\rm cor}:\mathbf{PC^\infty Rings^c}\hookrightarrow\mathbf{PC^\infty Rings^c_{in}}$ right adjoint to $\mathrm{inc}:\mathbf{PC^\infty Rings^c_{in}}\hookrightarrow\mathbf{PC^\infty Rings^c}$. Let $\boldsymbol{\mathfrak{C}}=(\mathfrak{C},\mathfrak{C}_{\rm ex})$ be a pre C^∞-ring with corners. We will define an interior pre C^∞-ring with corners $\tilde{\boldsymbol{\mathfrak{C}}}=(\mathfrak{C},\tilde{\mathfrak{C}}_{\rm ex})$ where $\tilde{\mathfrak{C}}_{\rm ex}=\mathfrak{C}_{\rm ex}\amalg\{0_{\tilde{\mathfrak{C}}_{\rm ex}}\}$, and set $\Pi^{\rm int}_{\rm cor}(\boldsymbol{\mathfrak{C}})=\tilde{\boldsymbol{\mathfrak{C}}}$. Here $\mathfrak{C}_{\rm ex}$ already contains a zero element $0_{\mathfrak{C}_{\rm ex}}$, but we are adding an extra $0_{\tilde{\mathfrak{C}}_{\rm ex}}$ with $0_{\mathfrak{C}_{\rm ex}}\neq 0_{\tilde{\mathfrak{C}}_{\rm ex}}$.

Let $f:\mathbb{R}^m\times[0,\infty)^n\to\mathbb{R}$ and $g:\mathbb{R}^m\times[0,\infty)^n\to[0,\infty)$ be smooth, and write Φ_f,Ψ_g for the operations in $\boldsymbol{\mathfrak{C}}$. We must define maps $\tilde\Phi_f:\mathfrak{C}^m\times\tilde{\mathfrak{C}}^n_{\rm ex}\to\mathfrak{C}$ and $\tilde\Psi_g:\mathfrak{C}^m\times\tilde{\mathfrak{C}}^n_{\rm ex}\to\tilde{\mathfrak{C}}_{\rm ex}$. Let $c_1,\ldots,c_m\in\mathfrak{C}$ and $c'_1,\ldots,c'_n\in\tilde{\mathfrak{C}}_{\rm ex}$. Then some of $c'_1,\ldots,c'_n$ lie in $\mathfrak{C}_{\rm ex}$ and the rest in $\{0_{\tilde{\mathfrak{C}}_{\rm ex}}\}$. For simplicity suppose that $c'_1,\ldots,c'_k\in\mathfrak{C}_{\rm ex}$ and $c'_{k+1}=\cdots=c'_n=0_{\tilde{\mathfrak{C}}_{\rm ex}}$ for $0\leqslant k\leqslant n$. Define smooth $d:\mathbb{R}^m\times[0,\infty)^k\to\mathbb{R}$, $e:\mathbb{R}^m\times[0,\infty)^k\to[0,\infty)$ by (4.2.5). Set

$$\tilde\Phi_f(c_1,\ldots,c_m,c'_1,\ldots,c'_k,0_{\tilde{\mathfrak{C}}_{\rm ex}},\ldots,0_{\tilde{\mathfrak{C}}_{\rm ex}})=\Phi_d(c_1,\ldots,c_m,c'_1,\ldots,c'_k).$$

Either $e:\mathbb{R}^m\times[0,\infty)^k\to[0,\infty)$ is interior, or $e=0$. If e is interior define

$$\tilde\Psi_g(c_1,\ldots,c_m,c'_1,\ldots,c'_k,0_{\tilde{\mathfrak{C}}_{\rm ex}},\ldots,0_{\tilde{\mathfrak{C}}_{\rm ex}})=\Psi_e(c_1,\ldots,c_m,c'_1,\ldots,c'_k).$$

If $e=0$ define $\tilde\Psi_g(c_1,\ldots,c_m,c'_1,\ldots,c'_k,0_{\tilde{\mathfrak{C}}_{\rm ex}},\ldots,0_{\tilde{\mathfrak{C}}_{\rm ex}})=0_{\tilde{\mathfrak{C}}_{\rm ex}}$. This defines the maps $\tilde\Phi_f,\tilde\Psi_g$. This makes $\tilde{\boldsymbol{\mathfrak{C}}}=(\mathfrak{C},\tilde{\mathfrak{C}}_{\rm ex})$ into an interior pre C^∞-ring with corners.

Now let $\boldsymbol\phi:\boldsymbol{\mathfrak{C}}\to\boldsymbol{\mathfrak{D}}$ be a morphism in $\mathbf{PC^\infty Rings^c}$, and define $\tilde{\boldsymbol{\mathfrak{C}}},\tilde{\boldsymbol{\mathfrak{D}}}$ as above. Define $\tilde\phi_{\rm ex}:\tilde{\mathfrak{C}}_{\rm ex}\to\tilde{\mathfrak{D}}_{\rm ex}$ by $\tilde\phi_{\rm ex}|_{\mathfrak{C}_{\rm ex}}=\phi_{\rm ex}$ and $\tilde\phi_{\rm ex}(0_{\tilde{\mathfrak{C}}_{\rm ex}})=0_{\tilde{\mathfrak{D}}_{\rm ex}}$. Then $\tilde{\boldsymbol\phi}=(\phi,\tilde\phi_{\rm ex}):\tilde{\boldsymbol{\mathfrak{C}}}\to\tilde{\boldsymbol{\mathfrak{D}}}$ is a morphism in $\mathbf{PC^\infty Rings^c_{in}}$.

Define a functor $\Pi^{\rm int}_{\rm cor}:\mathbf{PC^\infty Rings^c}\hookrightarrow\mathbf{PC^\infty Rings^c_{in}}$ by $\Pi^{\rm int}_{\rm cor}:\boldsymbol{\mathfrak{C}}\mapsto\tilde{\boldsymbol{\mathfrak{C}}}$ on objects and $\Pi^{\rm int}_{\rm cor}:\boldsymbol\phi\mapsto\tilde{\boldsymbol\phi}$ on morphisms, for $\boldsymbol{\mathfrak{C}},\boldsymbol{\mathfrak{D}},\tilde{\boldsymbol{\mathfrak{C}}},\tilde{\boldsymbol{\mathfrak{D}}},\boldsymbol\phi,\tilde{\boldsymbol\phi}$ as above.

Suppose now that $\boldsymbol{\mathfrak{C}},\boldsymbol{\mathfrak{D}}$ are pre C^∞-rings with corners with $\boldsymbol{\mathfrak{C}}$ interior. Then we can define a 1-1 correspondence

$$\begin{aligned}\mathrm{Hom}_{\mathbf{PC^\infty Rings^c}}\bigl(\mathrm{inc}(\mathfrak{C}),\mathfrak{D}\bigr) &= \mathrm{Hom}_{\mathbf{PC^\infty Rings^c}}(\mathfrak{C},\mathfrak{D})\\ &\cong \mathrm{Hom}_{\mathbf{PC^\infty Rings^c_{in}}}\bigl(\mathfrak{C},\Pi^{\mathrm{int}}_{\mathrm{cor}}(\mathfrak{D})\bigr),\end{aligned}$$

identifying $\boldsymbol{\phi}:\mathrm{inc}(\mathfrak{C})\to\mathfrak{D}$ with $\hat{\boldsymbol{\phi}}:\mathfrak{C}\to\Pi^{\mathrm{int}}_{\mathrm{cor}}(\mathfrak{D})$, where $\boldsymbol{\phi}=(\phi,\phi_{\mathrm{ex}})$ and $\hat{\boldsymbol{\phi}}=(\phi,\hat{\phi}_{\mathrm{ex}})$ with $\hat{\phi}_{\mathrm{ex}}|_{\mathfrak{C}_{\mathrm{in}}}=\phi_{\mathrm{ex}}|_{\mathfrak{C}_{\mathrm{in}}}$ and $\hat{\phi}_{\mathrm{ex}}(0_{\mathfrak{C}_{\mathrm{ex}}})=0_{\tilde{\mathfrak{D}}_{\mathrm{ex}}}$. This is functorial in $\mathfrak{C},\mathfrak{D}$, and so shows that $\Pi^{\mathrm{int}}_{\mathrm{cor}}$ is right adjoint to $\mathrm{inc}:\mathbf{PC^\infty Rings^c_{in}}\hookrightarrow\mathbf{PC^\infty Rings^c}$.

We have now given the analogue of all of §4.1 in terms of our new definitions. Remark 2.1.3, and Propositions 4.2.4 and 4.2.9 give equivalences

$$\begin{gathered}\mathbf{CC^\infty Rings}\cong\mathbf{C^\infty Rings},\quad \mathbf{CPC^\infty Rings^c}\cong\mathbf{PC^\infty Rings^c},\\ \mathbf{CPC^\infty Rings^c_{in}}\cong\mathbf{PC^\infty Rings^c_{in}}.\end{gathered}\tag{4.2.6}$$

These identify $\Pi^{C^\infty}_{\mathrm{cor}},\Pi^{C^\infty}_{\mathrm{int}},\Pi^{\mathrm{int}}_{\mathrm{cor}}$ in (4.1.3) with $\Pi^{C^\infty}_{\mathrm{cor}},\Pi^{C^\infty}_{\mathrm{int}},\Pi^{\mathrm{int}}_{\mathrm{cor}}$ in Definitions 4.2.6 and 4.2.11. Theorem 4.1.4(c) gives left adjoints $I^{\mathrm{cor}}_{C^\infty},I^{\mathrm{int}}_{C^\infty},I^{\mathrm{cor}}_{\mathrm{int}}$ for $\Pi^{C^\infty}_{\mathrm{cor}},\Pi^{C^\infty}_{\mathrm{int}},\Pi^{\mathrm{int}}_{\mathrm{cor}}$ in (4.1.3). Definitions 4.2.10 and 4.2.11 give left adjoints $I^{\mathrm{cor}}_{C^\infty},I^{\mathrm{int}}_{C^\infty}$ and $\mathrm{inc}:\mathbf{PC^\infty Rings^c_{in}}\hookrightarrow\mathbf{PC^\infty Rings^c}$ for $\Pi^{C^\infty}_{\mathrm{cor}},\Pi^{C^\infty}_{\mathrm{int}},\Pi^{\mathrm{int}}_{\mathrm{cor}}$ above. Therefore (4.2.6) identifies $I^{\mathrm{cor}}_{C^\infty},I^{\mathrm{int}}_{C^\infty},I^{\mathrm{cor}}_{\mathrm{int}}$ in §4.1 with $I^{\mathrm{cor}}_{C^\infty},I^{\mathrm{int}}_{C^\infty},\mathrm{inc}$ above.

Thus from Theorem 4.1.4 we deduce:

Theorem 4.2.12. **(a)** *In the categories* $\mathbf{PC^\infty Rings^c},\mathbf{PC^\infty Rings^c_{in}}$ *of (interior) pre* C^∞*-rings with corners, all small limits and all directed colimits exist. The functors* $\Pi_{\mathrm{sm}},\Pi_{\mathrm{ex}}:\mathbf{PC^\infty Rings^c}\to\mathbf{Sets}$, $\bar{\Pi}_{\mathrm{ex}}:\mathbf{PC^\infty Rings^c}\to\mathbf{Mon}$, $\Pi_{\mathrm{sm}},\Pi_{\mathrm{in}}:\mathbf{PC^\infty Rings^c_{in}}\to\mathbf{Sets}$ *and* $\bar{\Pi}_{\mathrm{in}}:\mathbf{PC^\infty Rings^c_{in}}\to\mathbf{Mon}$ *preserve limits and directed colimits, so these may be used to compute such (co)limits.*

(b) *All small colimits exist in* $\mathbf{PC^\infty Rings^c},\mathbf{PC^\infty Rings^c_{in}}$*, though in general they are not preserved by* $\Pi_{\mathrm{sm}},\Pi_{\mathrm{ex}},\bar{\Pi}_{\mathrm{ex}}$ *and* $\Pi_{\mathrm{sm}},\Pi_{\mathrm{in}},\bar{\Pi}_{\mathrm{in}}$.

(c) *The functors* $\Pi^{C^\infty}_{\mathrm{cor}},\Pi^{C^\infty}_{\mathrm{int}},\Pi^{\mathrm{int}}_{\mathrm{cor}}$ *described above are right adjoint to* $I^{\mathrm{cor}}_{C^\infty},I^{\mathrm{int}}_{C^\infty}$ *and* $\mathrm{inc}:\mathbf{PC^\infty Rings^c_{in}}\hookrightarrow\mathbf{PC^\infty Rings^c}$. *Since* $\Pi^{C^\infty}_{\mathrm{cor}},\Pi^{C^\infty}_{\mathrm{int}},\Pi^{\mathrm{int}}_{\mathrm{cor}}$ *are right adjoints, they preserve limits. Since* $I^{\mathrm{cor}}_{C^\infty},I^{\mathrm{int}}_{C^\infty}$ *and* $\mathrm{inc}:\mathbf{PC^\infty Rings^c_{in}}\hookrightarrow\mathbf{PC^\infty Rings^c}$ *are left adjoints, they preserve colimits.*

Remark 4.2.13. Kalashnikov [51, §4.6] also showed small colimits of pre C^∞-rings with corners exist using an argument similar to Moerdijk and Reyes proof that small colimits of C^∞-rings exist as in [72, p. 21–23].

Example 4.2.14. The inclusion $\mathrm{inc} : \mathbf{PC^\infty Rings^c_{in}} \hookrightarrow \mathbf{PC^\infty Rings^c}$ in general does not preserve limits, and therefore cannot have a left adjoint. Let $\mathcal{J}$ be a small category and $\boldsymbol{A} : \mathcal{J} \to \mathbf{PC^\infty Rings^c_{in}}$ be a functor. Then by Theorem 4.2.12(a), limits $\varprojlim_{j\in\mathcal{J}} \boldsymbol{A}(j)$ exist in both $\mathbf{PC^\infty Rings^c}$ and $\mathbf{PC^\infty Rings^c_{in}}$. But these limits may not be the same, and the limit in $\mathbf{PC^\infty Rings^c}$ may not be an object in $\mathbf{PC^\infty Rings^c_{in}}$.

We illustrate this for products in $\mathbf{PC^\infty Rings^c}$ and $\mathbf{PC^\infty Rings^c_{in}}$. Suppose $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}$ are interior pre C^∞-rings with corners, and write $\boldsymbol{\mathfrak{E}} = \boldsymbol{\mathfrak{C}} \times \boldsymbol{\mathfrak{D}}$ and $\boldsymbol{\mathfrak{F}} = \boldsymbol{\mathfrak{C}} \times_{\mathrm{in}} \boldsymbol{\mathfrak{D}}$ for the products in $\mathbf{PC^\infty Rings^c}$ and $\mathbf{PC^\infty Rings^c_{in}}$. Then Theorem 4.2.12(a) implies that $\mathfrak{E} = \mathfrak{C} \times \mathfrak{D}$, $\mathfrak{E}_{\mathrm{ex}} = \mathfrak{C}_{\mathrm{ex}} \times \mathfrak{D}_{\mathrm{ex}}$, $\mathfrak{F} = \mathfrak{C} \times \mathfrak{D}$, $\mathfrak{F}_{\mathrm{in}} = \mathfrak{C}_{\mathrm{in}} \times \mathfrak{D}_{\mathrm{in}}$. Since $\mathfrak{C}_{\mathrm{ex}} = \mathfrak{C}_{\mathrm{in}} \amalg \{0_{\mathfrak{C}_{\mathrm{ex}}}\}$, etc., where $\amalg$ is the disjoint union, this gives

$$
\begin{aligned}
\mathfrak{E}_{\mathrm{ex}} &= (\mathfrak{C}_{\mathrm{in}} \times \mathfrak{D}_{\mathrm{in}}) \amalg (\mathfrak{C}_{\mathrm{in}} \times \{0_{\mathfrak{D}_{\mathrm{ex}}}\}) \amalg (\{0_{\mathfrak{C}_{\mathrm{ex}}}\} \times \mathfrak{D}_{\mathrm{in}}) \amalg (\{0_{\mathfrak{C}_{\mathrm{ex}}}\} \times \{0_{\mathfrak{D}_{\mathrm{ex}}}\}),\\
\mathfrak{F}_{\mathrm{ex}} &= (\mathfrak{C}_{\mathrm{in}} \times \mathfrak{D}_{\mathrm{in}}) \amalg (\{0_{\mathfrak{C}_{\mathrm{ex}}}\} \times \{0_{\mathfrak{D}_{\mathrm{ex}}}\}).
\end{aligned}
$$

Thus $\boldsymbol{\mathfrak{E}} \not\cong \boldsymbol{\mathfrak{F}}$. Moreover in $\mathfrak{E}_{\mathrm{ex}}$ we have $(1_{\mathfrak{C}_{\mathrm{ex}}}, 0_{\mathfrak{D}_{\mathrm{ex}}}) \cdot (0_{\mathfrak{C}_{\mathrm{ex}}}, 1_{\mathfrak{D}_{\mathrm{ex}}}) = (0_{\mathfrak{C}_{\mathrm{ex}}}, 0_{\mathfrak{D}_{\mathrm{ex}}})$, so $\mathfrak{E}_{\mathrm{ex}}$ has zero divisors, and $\boldsymbol{\mathfrak{E}}$ is not an object in $\mathbf{PC^\infty Rings^c_{in}}$.

Another way to see this is that the final object in the category of pre C^∞-rings with corners is $(\{0\}, \{0\})$, however the final object in the category of interior pre C^∞-rings with corners is $(\{0\}, \{0, 1\})$. Taking the product of two objects in interior pre C^∞-rings with corners is the same as taking the fibre product (in both $\mathbf{PC^\infty Rings^c_{in}}$ and $\mathbf{PC^\infty Rings^c}$) over these two objects and their unique morphisms to $(\{0\}, \{0, 1\})$. This is different to taking their product in pre C^∞-rings with corners, which is a fibre product over these two objects and their unique morphisms to $(\{0\}, \{0\})$ in $\mathbf{PC^\infty Rings^c}$ only.

In contrast, the initial object in both categories is $(\mathbb{R}, [0, \infty))$, and their coproducts are the same.

4.3 C^∞-rings with corners

Here are some properties of $\Phi_i, \Phi_{\exp}$ in Definition 4.2.5(c),(e).

Proposition 4.3.1. (a) *Let $\mathfrak{C}$ be a C^∞-ring. Then the C^∞-ring operation $\Phi_{\exp} : \mathfrak{C} \to \mathfrak{C}$ induced by* $\exp : \mathbb{R} \to \mathbb{R}$ *is injective.*

(b) *Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ be a pre C^∞-ring with corners, and suppose c' lies in the group $\mathfrak{C}_{\mathrm{ex}}^\times$ of invertible elements in the monoid $\mathfrak{C}_{\mathrm{ex}}$. Then there exists a unique $c \in \mathfrak{C}$ such that $\Phi_{\exp}(c) = \Phi_i(c')$ in $\mathfrak{C}$, for $i : [0, \infty) \hookrightarrow \mathbb{R}$ the inclusion.*

Proof. For (a), let $a \in \mathfrak{C}$ with $b = \Phi_{\exp}(a) \in \mathfrak{C}$. Then $\Phi_{\exp}(-a)$ is the inverse b^{-1} of b. The map $t \mapsto \exp(t) - \exp(-t)$ is a diffeomorphism $\mathbb{R} \to \mathbb{R}$. Let $e : \mathbb{R} \to \mathbb{R}$ be its inverse.

Define smooth $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = e(x - y)$. Then $f(\exp t, \exp(-t)) = t$. Hence in the C^∞-ring $\mathfrak{C}$ we have

$$\Phi_f(b, b^{-1}) = \Phi_f(\Phi_{\exp}(a), \Phi_{\exp\circ -}(a)) = \Phi_{f\circ(\exp,\exp\circ -)}(a) = \Phi_{\mathrm{id}}(a) = a.$$

But b determines b^{-1} uniquely, so $\Phi_f(b, b^{-1}) = a$ implies that $b = \Phi_{\exp}(a)$ determines a uniquely, and $\Phi_{\exp} : \mathfrak{C} \to \mathfrak{C}$ is injective.

For (b), as $c' \in \mathfrak{C}_{\mathrm{ex}}^\times$ we have a unique inverse $c'^{-1} \in \mathfrak{C}_{\mathrm{ex}}^\times$. Define smooth $g : [0, \infty)^2 \to \mathbb{R}$ by $g(x, y) = e(x - y)$, for $e : \mathbb{R} \to \mathbb{R}$ as above. Observe that if $(x, y) \in [0, \infty)^2$ with $xy = 1$ then $x = \exp t$, $y = \exp(-t)$ for $t = \log x$, and so

$$\exp\circ g(x, y) = \exp\circ g(\exp t, \exp(-t)) = \exp\circ e(\exp t - \exp(-t)) = \exp(t) = x.$$

Therefore there is a unique smooth function $h : [0, \infty)^2 \to \mathbb{R}$ with

$$\exp\circ g(x, y) - x = h(x, y)(xy - 1). \tag{4.3.1}$$

We have operations $\Phi_g, \Phi_h : \mathfrak{C}_{\mathrm{ex}}^2 \to \mathfrak{C}$. Define $c = \Phi_g(c', c'^{-1})$. Then

$$\begin{aligned}\Phi_{\exp}(c) - \Phi_i(c') &= \Phi_{\exp\circ g(x,y)-x}(c', c'^{-1}) = \Phi_{h(x,y)(xy-1)}(c', c'^{-1}) \\ &= \Phi_h(c', c'^{-1}) \cdot (\Phi_i(c' \cdot c'^{-1}) - 1_{\mathfrak{C}}) = 0,\end{aligned}$$

using Definition 4.1.2(i) in the first and third steps, and (4.3.1) in the second. Hence $\Phi_{\exp}(c) = \Phi_i(c')$. Uniqueness of c follows from part (a). □

We can now define C^∞-rings with corners.

Definition 4.3.2. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ be a pre C^∞-ring with corners. We call $\boldsymbol{\mathfrak{C}}$ a *C^∞-ring with corners* if any (hence all) of the following hold:

(i) $\Phi_i|_{\mathfrak{C}_{\mathrm{ex}}^\times} : \mathfrak{C}_{\mathrm{ex}}^\times \to \mathfrak{C}$ is injective.

(ii) $\Psi_{\exp} : \mathfrak{C} \to \mathfrak{C}_{\mathrm{ex}}^\times$ is surjective.

(iii) $\Psi_{\exp} : \mathfrak{C} \to \mathfrak{C}_{\mathrm{ex}}^\times$ is a bijection.

Here Proposition 4.3.1(b) implies that (i),(ii) are equivalent, and Definition 4.2.5(e) and Proposition 4.3.1(a) imply that $\Psi_{\exp} : \mathfrak{C} \to \mathfrak{C}_{\mathrm{ex}}^\times$ is injective, so (ii),(iii) are equivalent, and therefore (i)–(iii) are equivalent. Write $\mathbf{C^\infty Rings^c}$ for the full subcategory of C^∞-rings with corners in $\mathbf{PC^\infty Rings^c}$.

We call a C^∞-ring with corners $\boldsymbol{\mathfrak{C}}$ *interior* if it is an interior pre C^∞-ring with corners, as in §4.2. Write $\mathbf{C^\infty Rings^c_{in}} \subset \mathbf{C^\infty Rings^c}$ for the subcategory of interior C^∞-rings with corners, and interior morphisms between them.

Remark 4.3.3. We can interpret the condition that $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a C^∞-ring with corners as follows. Imagine there is some 'space with corners' X, such that $\mathfrak{C} = \{$smooth maps $c : X \to \mathbb{R}\}$, and $\mathfrak{C}_{\rm ex} = \{$exterior maps $c' : X \to [0,\infty)\}$. If $c' \in \mathfrak{C}_{\rm ex}$ is invertible (that is, $c' \in \mathfrak{C}_{\rm ex}^\times$) then c' should map $X \to (0,\infty)$, and we require that there should exist smooth $c = \log c' : X \to \mathbb{R}$ in $\mathfrak{C}$ with $c' = \exp c$.

Example 4.3.4. The functors $F_{\mathbf{Man^c}}^{\mathbf{PC^\infty Rings^c}}, F_{\mathbf{Man^{gc}}}^{\mathbf{PC^\infty Rings^c}}$ in Example 4.2.2 map to the functors $(\mathbf{C^\infty Rings^c})^{\mathbf{op}} \subset (\mathbf{PC^\infty Rings^c})^{\mathbf{op}}$, and so we will write them as $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Rings^c}}$, $F_{\mathbf{Man^{gc}}}^{\mathbf{C^\infty Rings^c}}$. To see this, note that if X is a manifold with corners and $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex}) = \bigl(C^\infty(X), {\rm Ex}(X)\bigr)$ as in Example 4.2.2, and $c' \in \mathfrak{C}_{\rm ex}^\times$, then $c' : X \to [0,\infty)$ is smooth and invertible, so c' actually maps $X \to (0,\infty)$. Thus $c = \log c' : X \to \mathbb{R}$ is smooth and lies in $\mathfrak{C} = C^\infty(X)$, with $\Psi_{\exp}(c) = c'$. Hence $\boldsymbol{\mathfrak{C}}$ is a C^∞-ring with corners by Definition 4.3.2(ii). Similarly, $F_{\mathbf{Man^c_{in}}}^{\mathbf{PC^\infty Rings^c_{in}}}$, $F_{\mathbf{Man^{gc}_{in}}}^{\mathbf{PC^\infty Rings^c_{in}}}$ in Example 4.2.7 map to the functors $(\mathbf{C^\infty Rings^c_{in}})^{\mathbf{op}} \subset (\mathbf{PC^\infty Rings^c_{in}})^{\mathbf{op}}$, and so we will write them as $F_{\mathbf{Man^c_{in}}}^{\mathbf{C^\infty Rings^c_{in}}}$ and $F_{\mathbf{Man^{gc}_{in}}}^{\mathbf{C^\infty Rings^c_{in}}}$.

Proposition 4.3.5. *The inclusion functor* inc : $\mathbf{C^\infty Rings^c} \hookrightarrow \mathbf{PC^\infty Rings^c}$ *has a left adjoint* $\Pi_{{\rm pre}\,C^\infty}^{C^\infty} : \mathbf{PC^\infty Rings^c} \to \mathbf{C^\infty Rings^c}$ *and a right adjoint* $\Pi_{{\rm pre}\,C^\infty}^{\tilde{C^\infty}}$. *Their restrictions to* $\mathbf{PC^\infty Rings^c_{in}}$ *are left and right adjoints respectively for* inc : $\mathbf{C^\infty Rings^c_{in}} \hookrightarrow \mathbf{PC^\infty Rings^c_{in}}$, *so the inclusion respects limits and colimits.*

Proof. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a pre C^∞-ring with corners. We will define a C^∞-ring with corners $\hat{\boldsymbol{\mathfrak{C}}} = (\mathfrak{C}, \hat{\mathfrak{C}}_{\rm ex})$. As a set, define $\hat{\mathfrak{C}}_{\rm ex} = \mathfrak{C}_{\rm ex}/{\sim}$, where $\sim$ is the equivalence relation on $\mathfrak{C}_{\rm ex}$ given by $c' \sim c''$ if there exists $c''' \in \mathfrak{C}_{\rm ex}^\times$ with $\Phi_i(c''') = 1$ and $c' = c'' \cdot c'''$. That is, $\hat{\mathfrak{C}}_{\rm ex}$ is the quotient of $\mathfrak{C}_{\rm ex}$ by the group ${\rm Ker}\bigl(\Phi_i|_{\mathfrak{C}_{\rm ex}^\times}\bigr) \subseteq \mathfrak{C}_{\rm ex}^\times$. There is a natural surjective projection $\hat{\pi} : \mathfrak{C}_{\rm ex} \to \hat{\mathfrak{C}}_{\rm ex}$.

If $f : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ is smooth and $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ is exterior, then we can show there exist unique maps $\hat{\Phi}_f : \mathfrak{C}^m \times \hat{\mathfrak{C}}_{\rm ex}^n \to \mathfrak{C}$ and $\hat{\Psi}_g : \mathfrak{C}^m \times \hat{\mathfrak{C}}_{\rm ex}^n \to \hat{\mathfrak{C}}_{\rm ex}$ making the following diagrams commute:

$$\begin{array}{ccc} \mathfrak{C}^m \times \mathfrak{C}_{\rm ex}^n & \xrightarrow{\Phi_f} & \mathfrak{C} \\ \downarrow {\rm id}^m \times \hat{\pi}^n & & {\rm id} \downarrow \\ \mathfrak{C}^m \times \hat{\mathfrak{C}}_{\rm ex}^n & \xrightarrow{\hat{\Phi}_f} & \mathfrak{C}, \end{array} \qquad \begin{array}{ccc} \mathfrak{C}^m \times \mathfrak{C}_{\rm ex}^n & \xrightarrow{\Psi_g} & \mathfrak{C}_{\rm ex} \\ \downarrow {\rm id}^m \times \hat{\pi}^n & & \hat{\pi} \downarrow \\ \mathfrak{C}^m \times \hat{\mathfrak{C}}_{\rm ex}^n & \xrightarrow{\hat{\Psi}_g} & \hat{\mathfrak{C}}_{\rm ex}, \end{array} \tag{4.3.2}$$

and these make $\hat{\boldsymbol{\mathfrak{C}}}$ into a pre C^∞-ring with corners. Also, $\hat{\boldsymbol{\mathfrak{C}}}$ satisfies Definition 4.3.2(i). Therefore $\hat{\boldsymbol{\mathfrak{C}}}$ is a C^∞-ring with corners.

Suppose $\boldsymbol{\phi} = (\phi, \phi_{\rm ex}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ is a morphism of pre C^∞-rings with corners, and define $\hat{\boldsymbol{\mathfrak{C}}}, \hat{\boldsymbol{\mathfrak{D}}}$ as above. Then by a similar argument to (4.3.2), we find that there is a unique map

$\hat{\phi}_{\rm ex}$ such that the following commutes:

$$\begin{array}{ccc} \mathfrak{C}_{\rm ex} & \xrightarrow{\phi_{\rm ex}} & \mathfrak{D}_{\rm ex} \\ \downarrow \hat{\pi} & & \hat{\pi} \downarrow \\ \hat{\mathfrak{C}}_{\rm ex} & \xrightarrow{\hat{\phi}_{\rm ex}} & \hat{\mathfrak{D}}_{\rm ex}, \end{array}$$

and then $\hat{\boldsymbol{\phi}} = (\phi, \hat{\phi}_{\rm ex}) : \hat{\boldsymbol{\mathfrak{C}}} \to \hat{\boldsymbol{\mathfrak{D}}}$ is a morphism of C^∞-rings with corners. Define a functor $\Pi^{C^\infty}_{\mathrm{pre}\,C^\infty} : \mathbf{PC^\infty Rings^c} \to \mathbf{C^\infty Rings^c}$ to map $\boldsymbol{\mathfrak{C}} \mapsto \hat{\boldsymbol{\mathfrak{C}}}$ on objects and $\boldsymbol{\phi} \mapsto \hat{\boldsymbol{\phi}}$ on morphisms.

Now let $\boldsymbol{\mathfrak{C}}$ be a pre C^∞-ring with corners, $\boldsymbol{\mathfrak{D}}$ a C^∞-ring with corners, and $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ a morphism. Then we have a morphism $\hat{\boldsymbol{\phi}} : \hat{\boldsymbol{\mathfrak{C}}} \to \boldsymbol{\mathfrak{D}}$, as $\hat{\boldsymbol{\mathfrak{D}}} = \boldsymbol{\mathfrak{D}}$, and $\boldsymbol{\phi} \leftrightarrow \hat{\boldsymbol{\phi}}$ gives a 1-1 correspondence

$$\begin{aligned} \mathrm{Hom}_{\mathbf{PC^\infty Rings^c}}\big(\boldsymbol{\mathfrak{C}}, \mathrm{inc}(\boldsymbol{\mathfrak{D}})\big) &= \mathrm{Hom}_{\mathbf{PC^\infty Rings^c}}(\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}) \\ &\cong \mathrm{Hom}_{\mathbf{C^\infty Rings^c}}\big(\Pi^{C^\infty}_{\mathrm{pre}\,C^\infty}(\boldsymbol{\mathfrak{C}}), \boldsymbol{\mathfrak{D}}\big), \end{aligned} \tag{4.3.3}$$

which is functorial in $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}$. Hence $\Pi^{C^\infty}_{\mathrm{pre}\,C^\infty}$ is left adjoint to inc.

If $\boldsymbol{\mathfrak{C}}$ above is interior then $\hat{\boldsymbol{\mathfrak{C}}}$ is interior, and if $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ is interior then $\hat{\boldsymbol{\phi}}$ is interior, so $\Pi^{C^\infty}_{\mathrm{pre}\,C^\infty}$ restricts to $\Pi^{C^\infty}_{\mathrm{pre}\,C^\infty} : \mathbf{PC^\infty Rings^c_{in}} \to \mathbf{C^\infty Rings^c_{in}}$. The 1-1 correspondence (4.3.3) restricts to a 1-1 correspondence on interior morphisms. Hence $\Pi^{C^\infty}_{\mathrm{pre}\,C^\infty}$ is left adjoint to $\mathrm{inc} : \mathbf{C^\infty Rings^c_{in}} \hookrightarrow \mathbf{PC^\infty Rings^c_{in}}$.

For the right adjoint, take $(\mathfrak{C}, \mathfrak{C}_{\rm ex}) = \boldsymbol{\mathfrak{C}}$ a pre C^∞-ring with corners and consider the C^∞-ring with corners $(\mathfrak{C}, \tilde{\mathfrak{C}}_{\rm ex}) = \tilde{\boldsymbol{\mathfrak{C}}}$ where $\tilde{\mathfrak{C}}_{\rm ex}$ is the submonoid of $\mathfrak{C}_{\rm ex}$ generated by $\Psi_{\exp}(\mathfrak{C})$ and $\mathfrak{C}_{\rm ex} \setminus \mathfrak{C}_{\rm ex}^\times$. We see that $\tilde{\mathfrak{C}}_{\rm ex} = \Psi_{\exp}(\mathfrak{C}) \cup (\mathfrak{C}_{\rm ex} \setminus \mathfrak{C}_{\rm ex}^\times) \subset \mathfrak{C}_{\rm ex}$. To make this a C^∞-ring with corners, we take the C^∞-operations on $\tilde{\boldsymbol{\mathfrak{C}}}$ corresponding to smooth maps $f : \mathbb{R}^n_k \to \mathbb{R}$ to be the restrictions of these C^∞-operations on $\boldsymbol{\mathfrak{C}}$ to $\tilde{\boldsymbol{\mathfrak{C}}}$.

Non-zero smooth maps $f : \mathbb{R}^n_k \to [0, \infty)$ are of the form

$$f(x_1, \ldots, x_n) = x_1^{a_1} \cdots x_k^{a_k} \exp(g(x_1, \ldots, x_n))$$

for a smooth function $g : \mathbb{R}^n_k \to \mathbb{R}$. Then we see that

$$\Psi_f(c_1', \ldots, c_k', c_{k+1}, \ldots, c_n) = c_1'^{\,a_1} \cdots c_k'^{\,a_k} \Psi_{\exp}(g(c_1', \ldots, c_k', c_{k+1}, \ldots, c_n)))$$

is indeed an element of $\Psi_{\exp}(\mathfrak{C}) \cup (\mathfrak{C}_{\rm ex} \setminus \mathfrak{C}_{\rm ex}^\times)$. This tells us that the images of the C^∞-operations corresponding to these f are already in $\tilde{\mathfrak{C}}_{\rm ex}$, so that $\tilde{\boldsymbol{\mathfrak{C}}} = \Pi^{\tilde{C^\infty}}_{\mathrm{pre}\,C^\infty}(\boldsymbol{\mathfrak{C}})$ is a well defined sub pre C^∞-ring with corners of $\boldsymbol{\mathfrak{C}}$. As $\Psi_{\exp} : \mathfrak{C} \to \tilde{\mathfrak{C}}_{\rm ex}^\times$ is now surjective, then $\tilde{\boldsymbol{\mathfrak{C}}}$ is a C^∞-ring with corners. (We could in fact define $\tilde{\boldsymbol{\mathfrak{C}}}$ to be the largest sub pre C^∞-ring with corners of $\boldsymbol{\mathfrak{C}}$ such that $\Psi_{\exp} : \mathfrak{C} \to \tilde{\mathfrak{C}}_{\rm ex}^\times$ is surjective, so that $\tilde{\boldsymbol{\mathfrak{C}}}$ is a C^∞-ring with corners.)

If we have a pre C^∞-rings with corners morphism $\boldsymbol{\phi} : \mathfrak{C} \to \mathfrak{D}$ then we can consider the restriction $\boldsymbol{\phi}|_{\tilde{\mathfrak{C}}}$. On the C^∞-ring this is the identity, on the monoid we see that elements of $\Psi_{\exp}(\mathfrak{C})$ have image in $\Psi_{\exp}(\mathfrak{D})$ as $\phi_{\rm ex}$ commutes with the C^∞-operations, however there may be non-invertible elements c' that map into $\mathfrak{D}^\times \setminus \Psi_{\exp}(\mathfrak{D})$. We define $\tilde{\boldsymbol{\phi}} = \Pi^{\tilde{C^\infty}}_{\mathrm{pre}\, C^\infty}(\boldsymbol{\phi}) : \tilde{\mathfrak{C}} \to \tilde{\mathfrak{D}}$ to be the restriction of $\boldsymbol{\phi}$ to $\tilde{\mathfrak{C}}$ except for elements c' that map into $\mathfrak{D}^\times \setminus \Psi_{\exp}(\mathfrak{D})$. These elements we define to map to their corresponding unique $d' \in \Psi_{\exp}(\mathfrak{D})$ from Proposition 4.3.1(b) such that $\Phi_i(\phi_{\rm ex}(c')) = \Phi_i(d')$. It follows that $\tilde{\boldsymbol{\phi}}$ respects the C^∞-operations and is a morphism of C^∞-rings with corners.

To check that $\Pi^{\tilde{C^\infty}}_{\mathrm{pre}\, C^\infty}$ is right adjoint to $\mathrm{inc} : \mathbf{C^\infty Rings^c} \hookrightarrow \mathbf{PC^\infty Rings^c}$, we see that the unit is the identity natural transformation and the counit is the inclusion morphism. The compositions $\mathrm{inc} \Rightarrow \mathrm{inc}\, \Pi^{\tilde{C^\infty}}_{\mathrm{pre}\, C^\infty}\, \mathrm{inc} \Rightarrow \mathrm{inc}$ and $\Pi^{\tilde{C^\infty}}_{\mathrm{pre}\, C^\infty} \Rightarrow \Pi^{\tilde{C^\infty}}_{\mathrm{pre}\, C^\infty}\, \mathrm{inc}\, \Pi^{\tilde{C^\infty}}_{\mathrm{pre}\, C^\infty} \Rightarrow \Pi^{\tilde{C^\infty}}_{\mathrm{pre}\, C^\infty}$ are the identity morphisms as $\Pi^{\tilde{C^\infty}}_{\mathrm{pre}\, C^\infty}$ takes a C^∞-ring with corners to itself.

If $\mathfrak{C}$ above is interior then $\hat{\mathfrak{C}}$ and $\tilde{\mathfrak{C}}$ are interior, and if $\boldsymbol{\phi} : \mathfrak{C} \to \mathfrak{D}$ is interior then $\hat{\boldsymbol{\phi}}$ and $\tilde{\boldsymbol{\phi}}$ are interior, so $\Pi^{C^\infty}_{\mathrm{pre}\, C^\infty}$ and $\Pi^{\tilde{C^\infty}}_{\mathrm{pre}\, C^\infty}$ restrict to functors $\mathbf{PC^\infty Rings^c_{in}} \to \mathbf{C^\infty Rings^c_{in}}$, and considering the proofs above we see that they are left and right adjoint respectively to $\mathrm{inc} : \mathbf{C^\infty Rings^c_{in}} \hookrightarrow \mathbf{PC^\infty Rings^c_{in}}$. □

Remark 4.3.6. Note that the $\hat{\mathfrak{C}}_{\rm ex}$ defined above is actually the pushout of monoids $\Phi_{\exp}(\mathfrak{C}) \amalg_{\Phi_i^{-1}(\Phi_{\exp}(\mathfrak{C}))} \mathfrak{C}_{\rm ex}$, and we discuss this further in §5.9.1.

The categories $\mathbf{C^\infty Rings^c}, \mathbf{C^\infty Rings^c_{in}}$ behave well under (co)limits.

Theorem 4.3.7. (a) $\mathbf{C^\infty Rings^c}, \mathbf{C^\infty Rings^c_{in}}$ *are closed under limits and colimits in* $\mathbf{PC^\infty Rings^c}, \mathbf{PC^\infty Rings^c_{in}}$*, respectively. Thus, all small limits and small colimits exist in* $\mathbf{C^\infty Rings^c}, \mathbf{C^\infty Rings^c_{in}}$*. The functors*

$$\Pi_{\rm sm}, \Pi_{\rm ex} : \mathbf{C^\infty Rings^c} \longrightarrow \mathbf{Sets}, \qquad \bar{\Pi}_{\rm ex} : \mathbf{C^\infty Rings^c} \longrightarrow \mathbf{Mon},$$
$$\Pi_{\rm sm}, \Pi_{\rm in} : \mathbf{C^\infty Rings^c_{in}} \longrightarrow \mathbf{Sets}, \qquad \bar{\Pi}_{\rm in} : \mathbf{C^\infty Rings^c_{in}} \longrightarrow \mathbf{Mon},$$

preserve limits and directed colimits, and may be used to compute such (co)limits. The inclusion $\mathrm{inc} : \mathbf{C^\infty Rings^c_{in}} \hookrightarrow \mathbf{C^\infty Rings^c}$ *has a right adjoint* $(\mathfrak{C}, \mathfrak{C}_{\rm ex}) \mapsto (\mathfrak{C}, \mathfrak{C}_{\rm ex} \amalg \{0_{\rm ex}\})$*, and hence preserves colimits; it does not preserve limits, hence it does not have a left adjoint.*

Proof. This proof follows from applying Proposition 4.3.5 to Theorem 4.2.12. The same proof from Example 4.2.14 shows $\mathrm{inc} : \mathbf{C^\infty Rings^c_{in}} \hookrightarrow \mathbf{C^\infty Rings^c}$ does not preserve limits, hence there is no left adjoint. □

Remark 4.3.8. Theorem 4.3.7 is essential for sheaves of (interior) C^∞-rings with corners to be well behaved. In particular, to construct the sheafification $\boldsymbol{\mathcal{O}}_X$ of a presheaf of C^∞-rings with corners $\mathcal{P}\boldsymbol{\mathcal{O}}_X$ on X we need to take small limits in $\mathbf{C^\infty Rings^c}$, and to define the stalk $\boldsymbol{\mathcal{O}}_{X,x}$ of $\boldsymbol{\mathcal{O}}_X$ at $x \in X$ we need to take a directed colimit in $\mathbf{C^\infty Rings^c}$.

Theorem 4.3.9. *The forgetful functor* $G : \mathbf{C^\infty Rings^c} \to \mathbf{C^\infty Rings}$ *has a left adjoint* $F_{\exp} : \mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings^c}$. *Hence* G *preserves limits.* G *also has a right adjoint,* $F_{\geqslant 0} : \mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings^c}$, *hence* G *preserves colimits.*

Proof. We construct $F_{\exp}$ on objects by $\mathfrak{C} \mapsto (\mathfrak{C}, \Phi_{\exp}(\mathfrak{C}) \amalg \{0\}) = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$, where $\amalg$ is the disjoint union. Here $\mathfrak{C}_{\rm ex}$ is a subset of $\mathfrak{C}$ and $\Phi_i : \mathfrak{C}_{\rm ex} \to \mathfrak{C}$ is the inclusion morphism, and consequently injective. The C^∞-operations are defined as follows. For elements $c_1', \ldots, c_k' \in \mathfrak{C}_{\rm ex}, c_{k+1}, \ldots, c_n \in \mathfrak{C}$ and for a smooth function $f : \mathbb{R}^n_k \to \mathbb{R}$ we have

$$\Phi_f(c_1', \ldots, c_k', c_{k+1}, \ldots, c_n) = \Phi_f(\Phi_{\exp}(c_1), \ldots, \Phi_{\exp}(c_k), c_{k+1}, \ldots, c_n).$$

For a smooth function $g : \mathbb{R}^n_k \to [0, \infty)$, if $g = 0$, then $\Psi_g(c_1', \ldots, c_k', c_{k+1}, \ldots, c_n) = 0 \in \mathfrak{C}_{\rm ex}$.

Otherwise, we have $g(x_1, \ldots, x_n) = x_1^{a_1} \cdots x_k^{a_k} F(x_1, \ldots, x_2)$ for a positive smooth function $F : \mathbb{R}^n_k \to (0, \infty)$. Then F can be extended to a smooth positive function $\hat{F} : \mathbb{R}^n \to \mathbb{R}$ such that $\hat{F}|_{\mathbb{R}^n_k} = i \circ F$. Then we define

$$\Psi_g(c_1', \ldots, c_k', c_{k+1}, \ldots, c_n) = (c_1')^{a_1} \cdots (c_k')^{a_k} \Phi_{\hat{F}}(c_1', \ldots, c_k', c_{k+1}, \ldots, c_n),$$

which is independent of the choice of extension of F.

For a morphism $\phi : \mathfrak{C} \to \mathfrak{D}$, then we define $F_{\exp}(\phi) : F_{\exp}(\mathfrak{C}) \to F_{\exp}(\mathfrak{D})$, such that $F_{\exp}(\phi)(\Phi_{\exp}(c)) = \Phi_{\exp}(F_{\exp}(\phi(c)))$ for all $c \in \mathfrak{C}$ and $F_{\exp}(\phi)(0) = 0$. This is well defined as ϕ respects the C^∞-operations.

Then $F_{\exp}$ is a left adjoint to G, where the unit of the adjunction $\eta : \mathrm{id} \Rightarrow GF_{\exp}$ is the identity natural transformation, as $GF_{\exp}$ is the identity functor $GF_{\exp} : \mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings}$. The counit is $\epsilon : F_{\exp}G \Rightarrow \mathrm{id}$, where for any $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex}) \in \mathbf{C^\infty Rings^c}$, then $\epsilon_{\boldsymbol{\mathfrak{C}}} : (\mathfrak{C}, \Phi_{\exp}(\mathfrak{C}) \amalg \{0\}) \to \boldsymbol{\mathfrak{C}}$ is the identity on the C^∞-ring, and it is the injective map $\Phi_{\exp}(\mathfrak{C}) \amalg \{0\} \ni \Phi_{\exp}(c) \mapsto \Psi_{\exp}(c) \in \mathfrak{C}_{\rm ex}$, $0 \mapsto 0$ on the monoid. Proposition 4.3.1 implies this is well defined and forms a natural transformation.

The compositions $F_{\exp} \Rightarrow F_{\exp}GF_{\exp} \Rightarrow F_{\exp}$ and $G \Rightarrow GF_{\exp}G \Rightarrow G$ are the identity natural transformations, and this gives the required adjunction. This also implies that

$$\mathrm{Hom}_{\mathbf{C^\infty Rings^c}}(F_{\exp}(\mathfrak{C}), \boldsymbol{\mathfrak{D}}) \cong \mathrm{Hom}_{\mathbf{C^\infty Rings}}(\mathfrak{C}, G(\boldsymbol{\mathfrak{D}})),$$

for a C^∞-ring $\mathfrak{C}$ and a C^∞-ring with corners $\boldsymbol{\mathfrak{D}} = (\mathfrak{D}, \mathfrak{D}_{\rm ex})$. That is,

$$\mathrm{Hom}_{\mathbf{C^\infty Rings^c}}((\mathfrak{C}, \Phi_{\exp}(\mathfrak{C}) \amalg \{0\}), (\mathfrak{D}, \mathfrak{D}_{\rm ex})) \cong \mathrm{Hom}_{\mathbf{C^\infty Rings}}(\mathfrak{C}, \mathfrak{D}).$$

This adjunction is a different but equivalent construction to the left adjoint constructed in Definition 4.2.10.

For the second part of the theorem, we begin by constructing $F_{\geqslant 0}$ on objects. Here $\mathfrak{C} \mapsto (\mathfrak{C}, \mathfrak{C}_{\geqslant 0})$ where $\mathfrak{C}_{\geqslant 0}$ is the subset of elements c of $\mathfrak{C}$ that satisfy the following condition: for all smooth $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_{[0,\infty)\times\mathbb{R}^{n-1}} = 0$ and for $d_1, \ldots, d_{n-1} \in \mathfrak{C}$, then $\Phi_f(c, d_1, \ldots, d_{n-1}) = 0 \in \mathfrak{C}$. Note that $\mathfrak{C}_{\geqslant 0}$ is non-empty, as it contains $\Phi_{\exp}(\mathfrak{C}) \amalg \{0\}$. Also, any C^∞-ring with corners $(\mathfrak{C}, \mathfrak{C}_{\rm ex})$ has $\Phi_i(\mathfrak{C}_{\rm ex}) \subseteq \mathfrak{C}_{\geqslant 0}$.

We need to show that $(\mathfrak{C}, \mathfrak{C}_{\geqslant 0})$ is a C^∞-ring with corners. Take a smooth function $f : \mathbb{R}^n_k \to [0, \infty)$. Then there is a (non-unique) smooth extension $g : \mathbb{R}^n \to \mathbb{R}$ such that $g|_{\mathbb{R}^n_k} = i \circ f$ for $i : [0, \infty) \to \mathbb{R}$ the inclusion function. We define

$$\Psi_f(c'_1, \ldots, c'_k, c_{k+1}, \ldots, c_n) = \Phi_g(c'_1, \ldots, c'_k, c_{k+1}, \ldots, c_n)$$

for $c'_i \in \mathfrak{C}_{\geqslant 0}$ and $c_i \in \mathfrak{C}$. Then $\Phi_i : \mathfrak{C}_{\geqslant 0} \to \mathfrak{C}$ is the inclusion morphism.

To show this is well defined, say h is another such extension, then $g - h|_{\mathbb{R}^n_k} = 0$. We want to show that

$$\Phi_{g-h}(c'_1, \ldots, c'_k, c_{k+1}, \ldots, c_n) = 0$$

for $c'_i \in \mathfrak{C}_{\geqslant 0}$ and $c_i \in \mathfrak{C}$. As $g-h$ satisfies the hypothesis of Lemma 4.3.10, we can assume $g - h = f_1 + \cdots + f_k$ with $f_i|_{\mathbb{R}^{i-1}\times[0,\infty)\times\mathbb{R}^{n-i-1}} = 0$. Then

$$\begin{aligned}\Phi_{g-h}(c'_1, \ldots, c'_k, c_{k+1}, \ldots, c_n) &= \Phi_{f_1+\cdots+f_k}(c'_1, \ldots, c'_k, c_{k+1}, \ldots, c_n)\\ &= \Phi_{f_1}(c'_1, \ldots, c'_k, c_{k+1}, \ldots, c_n) + \cdots\\ &\quad + \Phi_{f_k}(c'_1, \ldots, c'_k, c_{k+1}, \ldots, c_n)\\ &= 0 + \cdots + 0\end{aligned}$$

as c'_i are in $\mathfrak{C}_{\geqslant 0}$. Therefore Ψ_f is a well defined C^∞-operation. A similar approach shows smooth functions $f : \mathbb{R}^n_k \to \mathbb{R}$ give well defined C^∞-operations. Hence $(\mathfrak{C}, \mathfrak{C}_{\geqslant 0})$ is a C^∞-ring with corners.

On morphisms, $F_{\geqslant 0}$ sends $\phi : \mathfrak{C} \to \mathfrak{D}$ to $(\phi, \phi_{\rm ex}) : (\mathfrak{C}, \mathfrak{C}_{\geqslant 0}) \to (\mathfrak{D}, \mathfrak{D}_{\geqslant 0})$ where $\phi_{\rm ex} = \phi|_{\mathfrak{C}_{\geqslant 0}}$. As ϕ respects the C^∞-operations, then the image of $\mathfrak{C}_{\geqslant 0}$ under ϕ is contained in $\mathfrak{D}_{\geqslant 0}$, so $\phi_{\rm ex}$ is well defined.

We describe the unit and counit of the adjunction to show that $F_{\geqslant 0}$ is right adjoint to G. The counit is the identity natural transformation. The unit is the identity on the C^∞-ring and it is Φ_i on the monoid. It then follows that the compositions $F_{\geqslant 0} \Rightarrow F_{\geqslant 0} G F_{\geqslant 0} \Rightarrow F_{\geqslant 0}$ and $G \Rightarrow G F_{\geqslant 0} G \Rightarrow G$ are the identity compositions, so $F_{\geqslant 0}$ is a right adjoint to G. This also implies that

$$\mathrm{Hom}_{\mathbf{C^\infty Rings^c}}(\mathfrak{C}, F_{\geqslant 0}(\mathfrak{D})) \cong \mathrm{Hom}_{\mathbf{C^\infty Rings}}(G(\mathfrak{C}), \mathfrak{D}),$$

for a C^∞-ring with corners $\mathfrak{C}$ and a C^∞-ring $\mathfrak{D}$. □

Lemma 4.3.10. *If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth such that $f|_{\mathbb{R}^n_k} = 0$, then there are smooth $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, k$ such that $f_i|_{\mathbb{R}^{i-1} \times [0,\infty) \times \mathbb{R}^{n-i-1}} = 0$ for $i = 1, \ldots, k$, and $f = f_1 + \ldots + f_k$.*

Proof. Let f be as in the statement of the lemma. Consider the following open subset $U = S^{k-1} \setminus \{(x_1, \ldots, x_k) : x_i \geqslant 0\}$ of the dimension $k-1$ unit sphere $S^{k-1} \subset \mathbb{R}^k$. Take an open cover $U_1, \ldots, U_k$ of U such that $U_i = \{(x_1, \ldots, x_k) \in U | x_i < 0\}$. Take a partition of unity $\rho_i : U \to [0, 1]$ for $i = 1, \ldots, k$, subordinate to $\{U_1, \ldots, U_n\}$, with ρ_i having support on U_i and $\sum_{i=1}^k \rho_i = 1$.

Define the f_i as follows

$$f_i = \begin{cases} f(x_1, \ldots, x_n)\rho_i \left(\frac{(x_1, \ldots, x_k)}{|(x_1, \ldots, x_k)|} \right), & \text{if } x_i < 0 \text{ for some } i = 1, \ldots, k, \\ 0, & \text{otherwise,} \end{cases}$$

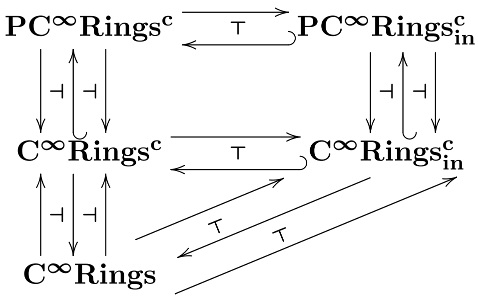
where $|(x_1, \ldots, x_k)|$ is the length of the vector $(x_1, \ldots, x_k) \in \mathbb{R}^k$. These f_i are smooth where the ρ_i are defined. The ρ_i are not defined in the first quadrant of $\mathbb{R}^k$, where all $x_i \geqslant 0$, however approaching the boundary of this quadrant, the ρ_i are all constant. As $f|_{\mathbb{R}^n_k} = 0$, then all derivatives of f are zero in this quadrant, so the f_i are smooth on $\mathbb{R}^n$ and identically zero on $\mathbb{R}^n_k$. In addition, $f_i|_{\mathbb{R}^{i-1} \times [0,\infty) \times \mathbb{R}^{n-i-1}} = 0$, as the ρ_i are zero outside of U_i. Finally, as $\sum \rho_i = 1$ and $f|_{\mathbb{R}^n_k} = 0$, then $f = f_1 + \ldots + f_k$ as required. □

Remark 4.3.11. The definition of the left adjoint F_{exp} in Theorem 4.3.9 shows its image actually lies in $\mathbf{C^\infty Rings^c_{in}}$, and the first part of this theorem is then also true for interior C^∞-rings with corners. Also, $F_{\geqslant 0}$ composed with the right adjoint $\mathbf{C^\infty Rings^c} \to \mathbf{C^\infty Rings^c_{in}}$ defined in Theorem 4.3.7(b) gives a right adjoint to $\mathbf{C^\infty Rings^c_{in}} \to \mathbf{C^\infty Rings}$, and so the second part of Theorem 4.3.9 is also true for interior C^∞-rings with corners.

In Section 4.6, we define local C^∞-rings with corners, and, in this case, the functors F_{exp} and $F_{\geqslant 0}$ restricted to local C^∞-rings are also left and right adjoints respectively to G restricted to local C^∞-rings with corners. This implies G preserves limits and colimits of local C^∞-rings with corners.

Relating to log geometry, discussed in §5.9, if $\mathfrak{C}$ is a C^∞-ring, we could say that $F_{\exp}(\mathfrak{C})$ is the *trivial corners structure* on $\mathfrak{C}$, as it is the initial object in the category of C^∞-rings with corners that have C^∞-ring $\mathfrak{C}$. Also, $F_{\geqslant 0}(\mathfrak{C})$ is the final object in this category.

We can summarise the adjoints from Definition 4.2.10, Definition 4.2.11, Theorem 4.2.12, Proposition 4.3.5, Theorem 4.3.7 and Theorem 4.3.9 in the following diagram.



The following lemma considers how localisations of C^∞-rings (with corners) behave with respect to $F_{\geqslant 0}$, and we use this in Remark 5.4.9. It also gives some intuition into what elements of a C^∞-ring $\mathfrak{C}$ are in $\mathfrak{C}_{\geqslant 0}$. For example, it implies that $c^2 \in \mathfrak{C}_{\geqslant 0}$ for all $c \in \mathfrak{C}$.

Lemma 4.3.12. *Let $\mathfrak{C}$ be a C^∞-ring. If $g : \mathbb{R}^n \to \mathbb{R}$ is smooth such that $g(\mathbb{R}^n) \subseteq [0, \infty)$, then $\Phi_g(c_1, \ldots, c_n) \in \mathfrak{C}_{\geqslant 0}$ for all $c_i \in \mathfrak{C}$.*

Proof. Let $\mathfrak{C}$ be a C^∞-ring and $g : \mathbb{R}^n \to \mathbb{R}$ smooth such that $g(\mathbb{R}^n) \subseteq [0, \infty)$. Take any $c_1, \ldots, c_n \in \mathfrak{C}$. Take smooth $f : \mathbb{R}^{m+1} \to \mathbb{R}$ such that $f|_{[0,\infty)\times\mathbb{R}^m} = 0$. We need to show that for any $d_1, \ldots, d_m \in \mathfrak{C}$ then we have that $\Phi_f(\Phi_g(c_1, \ldots, c_n), d_1, \ldots, d_m) = 0$.

However

$$\Phi_f(\Phi_g(c_1, \ldots, c_n), d_1, \ldots, d_m) = \Phi_h(c_1, \ldots, c_n, d_1, \ldots, d_m)$$

where $h(x_1, \ldots, x_n, y_1, \ldots, y_m) = f(g(x_1, \ldots, x_n), y_1, \ldots, y_m) = 0$ from the definition of f and g. Hence $\Phi_f(\Phi_g(c_1, \ldots, c_n), d_1, \ldots, d_m) = 0$ as required. $\square$

Definition 4.3.13. A morphism $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ in $\mathbf{C^\infty Rings^c}$ or $\mathbf{PC^\infty Rings^c}$ is called *injective*, or *surjective*, if the maps of sets $\phi : \mathfrak{C} \to \mathfrak{D}$ and $\phi_{\rm ex} : \mathfrak{C}_{\rm ex} \to \mathfrak{D}_{\rm ex}$ are injective, or surjective, respectively.

Let $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ be a morphism, and define $\mathfrak{E} = \phi(\mathfrak{C}) \subseteq \mathfrak{D}$ and $\mathfrak{E}_{\rm ex} = \phi_{\rm ex}(\mathfrak{C}_{\rm ex}) \subseteq \mathfrak{D}_{\rm ex}$. If $f : \mathbb{R}^m \times [0, \infty)^n \to \mathbb{R}$ is smooth and $g : \mathbb{R}^m \times [0, \infty)^n \to [0, \infty)$ is exterior, then since $\phi, \phi_{\rm ex}$ commute with operations Φ_f, Ψ_g, we see that $\Phi_f : \mathfrak{D}^m \times \mathfrak{D}_{\rm ex}^m \to \mathfrak{D}$ maps $\mathfrak{E}^m \times \mathfrak{E}_{\rm ex}^m \to \mathfrak{E}$, and $\Psi_g : \mathfrak{D}^m \times \mathfrak{D}_{\rm ex}^m \to \mathfrak{D}_{\rm ex}$ maps $\mathfrak{E}^m \times \mathfrak{E}_{\rm ex}^m \to \mathfrak{E}_{\rm ex}$. Define $\Phi'_f = \Phi_f|_{\cdots} : \mathfrak{E}^m \times \mathfrak{E}_{\rm ex}^m \to \mathfrak{E}$ and

$\Psi'_g = \Psi_g|_{\cdots} : \mathfrak{E}^m \times \mathfrak{E}_{\rm ex}^m \to \mathfrak{E}_{\rm ex}$. Then these operations Φ'_f, Ψ'_g for all f, g make $\boldsymbol{\mathfrak{E}} = (\mathfrak{E}, \mathfrak{E}_{\rm ex})$ into a pre C^∞-ring with corners, and a C^∞-ring with corners if $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}$ are. We call $\boldsymbol{\mathfrak{E}}$ the *image of* $\boldsymbol{\phi}$, and write it $\operatorname{Im}\boldsymbol{\phi}$.

Write $\pi = \phi : \mathfrak{C} \to \mathfrak{E}$ and $\pi_{\rm ex} = \phi_{\rm ex} : \mathfrak{C}_{\rm ex} \to \mathfrak{E}_{\rm ex}$, and let $\imath : \mathfrak{E} \hookrightarrow \mathfrak{D}$, $\imath_{\rm ex} : \mathfrak{E}_{\rm ex} \hookrightarrow \mathfrak{D}_{\rm ex}$ be the inclusions. Then $\boldsymbol{\pi} = (\pi, \pi_{\rm ex}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{E}}$ and $\boldsymbol{\imath} = (\imath, \imath_{\rm ex}) : \boldsymbol{\mathfrak{E}} \to \boldsymbol{\mathfrak{D}}$ are morphisms of (pre) C^∞-rings with corners, with $\boldsymbol{\pi}$ surjective and $\boldsymbol{\imath}$ injective. This shows that every morphism $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ in $\mathbf{C^\infty Rings^c}$ or $\mathbf{PC^\infty Rings^c}$ fits into a commutative triangle

$$\begin{array}{ccc} \boldsymbol{\mathfrak{C}} & \xrightarrow{\quad\boldsymbol{\phi}\quad} & \boldsymbol{\mathfrak{D}}, \\ & {\scriptstyle\boldsymbol{\pi}}\searrow \quad \nearrow{\scriptstyle\boldsymbol{\imath}} & \\ & \operatorname{Im}\boldsymbol{\phi} & \end{array}$$

with $\boldsymbol{\pi}$ surjective and $\boldsymbol{\imath}$ injective, and this characterises $\operatorname{Im}\boldsymbol{\phi}, \boldsymbol{\pi}, \boldsymbol{\imath}$ uniquely up to canonical isomorphism.

If $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}, \boldsymbol{\phi}$ are interior, then $\operatorname{Im}\boldsymbol{\phi}, \boldsymbol{\pi}, \boldsymbol{\imath}$ are also interior.

If $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ is an injective morphism in $\mathbf{C^\infty Rings^c}$ or $\mathbf{PC^\infty Rings^c}$ with $\boldsymbol{\mathfrak{D}}$ interior, then $\boldsymbol{\mathfrak{C}}$ is also interior, since $\phi_{\rm ex} : \mathfrak{C}_{\rm ex} \to \mathfrak{D}_{\rm ex}$ maps zero divisors to zero divisors as it is injective, but there are no zero divisors in $\mathfrak{D}_{\rm ex}$.

4.4 Free C^∞-rings with corners, generators and relations

We define free C^∞-rings with corners.

Definition 4.4.1. We defined categorical pre C^∞-rings with corners as product-preserving functors $F : \mathbf{Euc^c} \to \mathbf{Sets}$, so that we have a full embedding of $\mathbf{C^\infty Rings^c} \subset \mathbf{PC^\infty Rings^c}$ in the functor category $\mathrm{Fun}(\mathbf{Euc^c}, \mathbf{Sets})$. The Yoneda embedding $Y : (\mathbf{Euc^c})^{\mathbf{op}} \hookrightarrow \mathrm{Fun}(\mathbf{Euc^c}, \mathbf{Sets})$ maps to $\mathbf{C^\infty Rings^c} \subset \mathrm{Fun}(\mathbf{Euc^c}, \mathbf{Sets})$, which gives a full embedding $(\mathbf{Euc^c})^{\mathbf{op}} \hookrightarrow \mathbf{C^\infty Rings^c}$. Explicitly, this embedding maps $\mathbb{R}^m \times [0,\infty)^n$ in $\mathbf{Euc^c}$ to the C^∞-ring with corners $\boldsymbol{\mathfrak{F}}^{m,n} := \boldsymbol{C^\infty}\big(\mathbb{R}^m \times [0,\infty)^n\big)$ from Example 4.2.2 with $X = \mathbb{R}^m \times [0,\infty)^n$.

These C^∞-rings with corners $\boldsymbol{\mathfrak{F}}^{m,n}$ corresponding to objects $\mathbb{R}^m \times [0,\infty)^n$ in $\mathbf{Euc^c}$ are important from the point of view of Algebraic Theories [3], where they are called *finitely generated free algebras* [3, Rem. 14.12], and $\boldsymbol{\mathfrak{F}}^{m,n}$ is the *free C^∞-ring with corners with (m,n) generators* in this 2-sorted case. Free C^∞-rings with corners were considered in Kalashnikov [51, Lem. 4.21].

It has the universal property that if $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ is any (pre) C^∞-ring with corners and $c_1, \ldots, c_m \in \mathfrak{C}$, $c'_1, \ldots, c'_n \in \mathfrak{C}_{\rm ex}$, then there is a unique morphism $\boldsymbol{\phi} = (\phi, \phi_{\rm ex}) :$

$\mathfrak{F}^{m,n} \to \mathfrak{C}$ such that $\phi(x_i) = c_i$ for $i = 1,\dots,m$ and $\phi_{\rm ex}(y_j) = c'_j$ for $j = 1,\dots,n$, where $(x_1,\dots,x_m,y_1,\dots,y_n)$ are the coordinates on $\mathbb{R}^m \times [0,\infty)^n$.

Explicitly, if $f \in \mathfrak{F}^{m,n}$, so that $f : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ is smooth, we define $\phi(f) = \Phi_f(c_1,\dots,c_m,c'_1,\dots,c'_n) \in \mathfrak{C}$, and if $g \in \mathfrak{F}^{m,n}_{\rm ex}$, so that $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ is exterior, we set $\phi_{\rm ex}(g) = \Psi_g(c_1,\dots,c_m,c'_1,\dots,c'_n) \in \mathfrak{C}_{\rm ex}$.

More generally, if $A, A_{\rm ex}$ are sets then by [3, Rem. 14.12] we can define the *free C^∞-ring with corners $\boldsymbol{\mathfrak{F}}^{A,A_{\rm ex}} = (\mathfrak{F}^{A,A_{\rm ex}}, \mathfrak{F}^{A,A_{\rm ex}}_{\rm ex})$ generated by* $(A, A_{\rm ex})$. We may think of $\boldsymbol{\mathfrak{F}}^{A,A_{\rm ex}}$ as $\boldsymbol{C}^\infty\bigl(\mathbb{R}^A \times [0,\infty)^{A_{\rm ex}}\bigr)$, where $\mathbb{R}^A = \bigl\{(x_a)_{a\in A} : x_a \in \mathbb{R}\bigr\}$ and $[0,\infty)^{A_{\rm ex}} = \bigl\{(y_{a'})_{a'\in A_{\rm ex}} : y_{a'} \in [0,\infty)\bigr\}$. Explicitly, we define $\mathfrak{F}^{A,A_{\rm ex}}$ to be the set of maps $c : \mathbb{R}^A \times [0,\infty)^{A_{\rm ex}} \to \mathbb{R}$ which depend smoothly on only finitely many variables $x_a, y_{a'}$, and $\mathfrak{F}^{A,A_{\rm ex}}_{\rm ex}$ to be the set of maps $c' : \mathbb{R}^A \times [0,\infty)^{A_{\rm ex}} \to [0,\infty)$ which depend smoothly on only finitely many variables $x_a, y_{a'}$, and operations Φ_f, Ψ_g are defined as in (4.2.2)–(4.2.3). Regarding $x_a : \mathbb{R}^A \to \mathbb{R}$ and $y_{a'} : [0,\infty)^{A_{\rm ex}} \to [0,\infty)$ as functions for $a \in A$, $a' \in A_{\rm ex}$, we have $x_a \in \mathfrak{F}^{A,A_{\rm ex}}$ and $y_{a'} \in \mathfrak{F}^{A,A_{\rm ex}}_{\rm ex}$, and we call $x_a, y_{a'}$ the *generators* of $\boldsymbol{\mathfrak{F}}^{A,A_{\rm ex}}$.

Then $\boldsymbol{\mathfrak{F}}^{A,A_{\rm ex}}$ has the property that if $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ is any (pre) C^∞-ring with corners then a choice of maps $\alpha : A \to \mathfrak{C}$ and $\alpha_{\rm ex} : A_{\rm ex} \to \mathfrak{C}_{\rm ex}$ uniquely determine a morphism $\boldsymbol{\phi} : \boldsymbol{\mathfrak{F}}^{A,A_{\rm ex}} \to \boldsymbol{\mathfrak{C}}$ with $\phi(x_a) = \alpha(a)$ for $a \in A$ and $\phi_{\rm ex}(y_{a'}) = \alpha_{\rm ex}(a')$ for $a' \in A_{\rm ex}$. We have $\mathfrak{F}^{A,A_{\rm ex}} = \mathfrak{F}^{m,n}$ when $A = \{1,\dots,m\}$ and $A_{\rm ex} = \{1,\dots,n\}$.

The analogue of all this also holds in $\mathbf{C^\infty Rings^c_{in}} \subset \mathbf{PC^\infty Rings^c_{in}}$, with the same objects $\boldsymbol{\mathfrak{F}}^{m,n}$ and $\boldsymbol{\mathfrak{F}}^{A,A_{\rm in}}$, which are interior C^∞-rings with corners, and the difference that (interior) morphisms $\boldsymbol{\mathfrak{F}}^{m,n} \to \boldsymbol{\mathfrak{C}}$ in $\mathbf{PC^\infty Rings^c_{in}}$ are uniquely determined by elements $c_1,\dots,c_m \in \mathfrak{C}$ and $c'_1,\dots,c'_n \in \mathfrak{C}_{\rm in}$ (rather than $c'_1,\dots,c'_n \in \mathfrak{C}_{\rm ex}$), and similarly for $\boldsymbol{\mathfrak{F}}^{A,A_{\rm in}}$ with $\alpha_{\rm in} : A_{\rm in} \to \mathfrak{C}_{\rm in}$.

As in Adámek et al. [3, §5, §11, §14], see in particular [3, Prop.s 11.26, 11.28, 11.30, Cor. 11.33, & Rem. 14.14], every object in $\mathbf{PC^\infty Rings^c}, \mathbf{PC^\infty Rings^c_{in}}$ can be built out of free C^∞-rings with corners, in a certain sense.

Proposition 4.4.2. (a) *Every object $\boldsymbol{\mathfrak{C}}$ in $\mathbf{PC^\infty Rings^c}$ admits a surjective morphism $\boldsymbol{\phi} : \boldsymbol{\mathfrak{F}}^{A,A_{\rm ex}} \to \boldsymbol{\mathfrak{C}}$ from some free C^∞-ring with corners $\boldsymbol{\mathfrak{F}}^{A,A_{\rm ex}}$. We call $\boldsymbol{\mathfrak{C}}$* ***finitely generated*** *if this holds with $A, A_{\rm ex}$ finite sets.*

(b) *Every object $\boldsymbol{\mathfrak{C}}$ in $\mathbf{PC^\infty Rings^c}$ fits into a* ***coequaliser diagram***

$$\boldsymbol{\mathfrak{F}}^{B,B_{\rm ex}} \underset{\boldsymbol{\beta}}{\overset{\boldsymbol{\alpha}}{\rightrightarrows}} \boldsymbol{\mathfrak{F}}^{A,A_{\rm ex}} \xrightarrow{\ \boldsymbol{\phi}\ } \boldsymbol{\mathfrak{C}}, \tag{4.4.1}$$

that is, $\boldsymbol{\mathfrak{C}}$ is the colimit of the diagram $\boldsymbol{\mathfrak{F}}^{B,B_{\rm ex}} \rightrightarrows \boldsymbol{\mathfrak{F}}^{A,A_{\rm ex}}$ in $\mathbf{PC^\infty Rings^c}$, where $\boldsymbol{\phi} : \boldsymbol{\mathfrak{F}}^{A,A_{\rm ex}} \to \boldsymbol{\mathfrak{C}}$ is automatically surjective. We call $\boldsymbol{\mathfrak{C}}$ ***finitely presented*** *if this holds with $A, A_{\rm ex}, B, B_{\rm ex}$ finite sets.*

The analogues of (a) and (b) also hold in $\mathbf{PC^\infty Rings^c_{in}}$.

In (b), if $\mathfrak{C}\in\mathbf{C^\infty Rings^c}$ then (4.4.1) is a coequaliser diagram in $\mathbf{C^\infty Rings^c}$, since $\Pi^{C^\infty}_{\rm cor}$ preserves colimits, and similarly for $\mathfrak{C}\in\mathbf{C^\infty Rings^c_{in}}$.

For C^∞-rings in §2, we often wrote a C^∞-ring $\mathfrak{C}$ as a quotient $\mathfrak{C}=C^\infty(\mathbb{R}^m)/I$ for $I\subseteq C^\infty(\mathbb{R}^m)$ an ideal. Suppose I is finitely generated by $f_1,\ldots,f_k\in C^\infty(\mathbb{R}^m)$. Then we have a coequaliser diagram in $\mathbf{C^\infty Rings}$

$$C^\infty(\mathbb{R}^k)\underset{0^*}{\overset{(f_1,\ldots,f_k)^*}{\rightrightarrows}}C^\infty(\mathbb{R}^m)\longrightarrow\mathfrak{C},$$

an analogue of (4.4.1). That is, $\mathfrak{C}$ is the C^∞-ring we get by imposing the relations $f_1=0,\ldots,f_k=0$ in $C^\infty(\mathbb{R}^m)$. This is the general finitely presented C^∞-ring.

For C^∞-rings with corners, things are more complicated in two ways. Firstly, we have *two kinds of generators*, and *two kinds of relations*, corresponding to the two generating objects $\mathbb{R},[0,\infty)$ of $\mathbf{Euc^c}$. And secondly, we must now write the $[0,\infty)$-type relations in the form $g_j=h_j$, rather than in the form $f_j=0$.

As in (4.4.1) the general finitely presented C^∞-ring with corners $\mathfrak{C}$ fits into a coequaliser diagram in $\mathbf{C^\infty Rings^c}$

$$\boldsymbol{C^\infty}\bigl(\mathbb{R}^k\times[0,\infty)^l\bigr)\underset{(f_1,\ldots,f_k,h_1,\ldots,h_l)^*}{\overset{(e_1,\ldots,e_k,g_1,\ldots,g_l)^*}{\rightrightarrows}}\boldsymbol{C^\infty}\bigl(\mathbb{R}^m\times[0,\infty)^n\bigr)\longrightarrow\mathfrak{C},$$

where $e_1,\ldots,e_k,f_1,\ldots,f_k$ lie in $C^\infty\bigl(\mathbb{R}^m\times[0,\infty)^n\bigr)$ and $g_1,\ldots,g_l,h_1,\ldots,h_l$ in $\mathrm{Ex}\bigl(\mathbb{R}^m\times[0,\infty)^n\bigr)$. That is, $\mathfrak{C}$ has m generators $x_1\ldots,x_m$ of type $\mathbb{R}$ and n generators $y_1,\ldots,y_n$ of type $[0,\infty)$, where $(x_1,\ldots,x_m,y_1,\ldots,y_n)$ are the coordinates on $\mathbb{R}^m\times[0,\infty)^n$, and these generators satisfy k relations $e_j=f_j$ in $\mathfrak{C}$ of type $\mathbb{R}$, and l relations $g_j=h_j$ in $\mathfrak{C}_{\rm ex}$ of type $[0,\infty)$.

By replacing e_i,f_i by $e_i-f_i,0$ we may suppose that $f_1=\cdots=f_k=0$, and so write the $\mathbb{R}$ type relations as $e_1=0,\ldots,e_k=0$ in $\mathfrak{C}$, as for ideals in C^∞-rings. However, for the $[0,\infty)$-type relations $g_j=h_j$ in $\mathfrak{C}_{\rm ex}$, we are *not* able to replace g_j,h_j by $g_j-h_j,0$, since g_j-h_j does not make sense in the monoid $\mathrm{Ex}\bigl(\mathbb{R}^m\times[0,\infty)^n\bigr)$. Thus $[0,\infty)$ type relations must be written as $g_j=h_j$.

We can also modify a given C^∞-ring with corners $\mathfrak{C}$ by adding extra generators and relations. We will use the following notation for this:

Definition 4.4.3. Let $\mathfrak{C}$ be a C^∞-ring with corners, and $A,A_{\rm ex}$ be sets. We will write $\mathfrak{C}(x_a:a\in A)[y_{a'}:a'\in A_{\rm ex}]$ for the C^∞-ring with corners obtained by adding extra generators x_a for $a \in A$ of type $\mathbb{R}$ and $y_{a'}$ for $a' \in A_{\rm ex}$ of type $[0,\infty)$ to $\mathfrak{C}$. That is, by definition

$$\mathfrak{C}(x_a : a \in A)[y_{a'} : a' \in A_{\rm ex}] := \mathfrak{C} \otimes_\infty \mathfrak{F}^{A,A_{\rm ex}}, \tag{4.4.2}$$

where $\mathfrak{F}^{A,A_{\rm ex}}$ is the free C^∞-ring with corners from Definition 4.4.1, with generators x_a for $a \in A$ of type $\mathbb{R}$ and $y_{a'}$ for $a' \in A_{\rm ex}$ of type $[0,\infty)$, and $\otimes_\infty$ is the coproduct in $\mathbf{C^\infty Rings^c}$. As coproducts are a type of colimit, Theorem 4.3.7(b) implies that $\mathfrak{C}(x_a : a \in A)[y_{a'} : a' \in A_{\rm ex}]$ is well defined. Since $\mathfrak{F}^{A,A_{\rm ex}}$ is interior, if $\mathfrak{C}$ is interior then (4.4.2) is a coproduct in both $\mathbf{C^\infty Rings^c}$ and $\mathbf{C^\infty Rings^c_{in}}$, so $\mathfrak{C}(x_a : a \in A)[y_{a'} : a' \in A_{\rm ex}]$ is interior by Theorem 4.3.7(b).

By properties of coproducts and free C^∞-rings with corners, morphisms $\boldsymbol{\phi} : \mathfrak{C}(x_a : a \in A)[y_{a'} : a' \in A_{\rm ex}] \to \mathfrak{D}$ in $\mathbf{C^\infty Rings^c}$ are uniquely determined by a morphism $\boldsymbol{\psi} : \mathfrak{C} \to \mathfrak{D}$ and maps $\alpha : A \to \mathfrak{D}$, $\alpha_{\rm ex} : A_{\rm ex} \to \mathfrak{D}_{\rm ex}$. If $\mathfrak{C}, \mathfrak{D}$ are interior and $\alpha_{\rm ex}(A_{\rm ex}) \subseteq \mathfrak{D}_{\rm in}$ then $\boldsymbol{\phi}$ is interior.

Next suppose $B, B_{\rm ex}$ are sets and $f_b \in \mathfrak{C}$ for $b \in B$, $g_{b'}, h_{b'} \in \mathfrak{C}_{\rm ex}$ for $b' \in B_{\rm ex}$. We will write $\mathfrak{C}/(f_b = 0 : b \in B)[g_{b'} = h_{b'} : b' \in B_{\rm ex}]$ for the C^∞-ring with corners obtained by imposing relations $f_b = 0$, $b \in B$ in $\mathfrak{C}$ of type $\mathbb{R}$, and $g_{b'} = h_{b'}$, $b' \in B_{\rm ex}$ in $\mathfrak{C}_{\rm ex}$ of type $[0,\infty)$. That is, we have a coequaliser diagram

$$\mathfrak{F}^{B,B_{\rm ex}} \underset{\boldsymbol{\beta}}{\overset{\boldsymbol{\alpha}}{\rightrightarrows}} \mathfrak{C} \xrightarrow{\ \boldsymbol{\pi}\ } \mathfrak{C}/(f_b = 0 : b \in B)[g_{b'} = h_{b'} : b' \in B_{\rm ex}], \tag{4.4.3}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are determined uniquely by $\alpha(x_b) = f_b$, $\alpha_{\rm ex}(y_{b'}) = g_{b'}$, $\beta(x_b) = 0$, $\beta_{\rm ex}(y_{b'}) = h_{b'}$ for all $b \in B$ and $b' \in B_{\rm ex}$. As coequalisers are a type of colimit, Theorem 4.3.7(b) shows that $\mathfrak{C}/(f_b = 0 : b \in B)[g_{b'} = h_{b'} : b' \in B_{\rm ex}]$ is well defined. If $\mathfrak{C}$ is interior and $g_{b'}, h_{b'} \in \mathfrak{C}_{\rm in}$ for all $b' \in B_{\rm ex}$ (that is, $g_{b'}, h_{b'} \neq 0_{\mathfrak{C}_{\rm ex}}$) then (4.4.2) is also a coequaliser in $\mathbf{C^\infty Rings^c_{in}}$, so Theorem 4.3.7(b) implies that $\mathfrak{C}/(f_b = 0 : b \in B)[g_{b'} = h_{b'} : b' \in B_{\rm ex}]$ and $\boldsymbol{\pi}$ are interior.

Note that round brackets $(\cdots)$ denote generators or relations of type $\mathbb{R}$, and square brackets $[\cdots]$ generators or relations of type $[0,\infty)$. If we add generators or relations of only one type, we use only these brackets.

We construct two explicit examples of quotients in C^∞-rings with corners, which we will use in §5.

Example 4.4.4. **(a)** Say we wish to quotient a C^∞-ring with corners, $(\mathfrak{C}, \mathfrak{C}_{\rm ex})$ by an ideal I in $\mathfrak{C}$. Quotienting the C^∞-ring by the ideal will result in additional relations on the monoid. While this quotient, $(\mathfrak{D}, \mathfrak{D}_{\rm ex})$ is the coequaliser of a diagram such as (4.4.3), it is also equivalent to the following construction:

(*) The quotient is a C^∞-ring with corners $(\mathfrak{D}, \mathfrak{D}_{\rm ex})$ with a morphism $\boldsymbol{\pi} = (\pi, \pi_{\rm ex}) : (\mathfrak{C}, \mathfrak{C}_{\rm ex}) \to (\mathfrak{D}, \mathfrak{D}_{\rm ex})$ such that I is contained in the kernel of π, and is universal with respect to this property. That is, if $(\mathfrak{E}, \mathfrak{E}_{\rm ex})$ is another C^∞-ring with corners with morphism $\boldsymbol{\pi'} = (\pi', \pi'_{\rm ex}) : (\mathfrak{C}, \mathfrak{C}_{\rm ex}) \to (\mathfrak{E}, \mathfrak{E}_{\rm ex})$ with I contained in the kernel of π', then there is a unique morphism $\boldsymbol{p} : (\mathfrak{D}, \mathfrak{D}_{\rm ex}) \to (\mathfrak{E}, \mathfrak{E}_{\rm ex})$ such that $\boldsymbol{p} \circ \boldsymbol{\pi} = \boldsymbol{\pi'}$.

As a coequaliser is a colimit, by Theorem 4.3.9 we have $\mathfrak{D} \cong \mathfrak{C}/I$, the quotient in C^∞-rings. For the monoid, we require that smooth $f : \mathbb{R} \to [0, \infty)$ give well defined operations $\Psi_f : \mathfrak{D} \to \mathfrak{D}_{\rm ex}$. This means we require that if $a - b \in I$, then $\Psi_f(a) \sim \Psi_f(b) \in \mathfrak{C}_{\rm ex}$, and this needs to generate a monoid equivalence relation on $\mathfrak{C}_{\rm ex}$, so that a quotient by this relation is well defined. If $f : \mathbb{R} \to [0, \infty)$ is identically zero, this follows. If $f : \mathbb{R} \to [0, \infty)$ is non-zero and smooth this means that f is positive, and hence that $f = \exp \circ \log \circ f$ is well defined. By Hadamards lemma, if $a - b \in I$, then $\Phi_g(a) - \Phi_g(b) \in I$, for all $g : \mathbb{R} \to \mathbb{R}$ smooth, and therefore $\Psi_{\log \circ f}(a) - \Psi_{\log \circ f}(b) \in I$. Hence in $\mathfrak{C}_{\rm ex}$ we only require that if $a - b \in I$, then $\Psi_{\exp}(a) \sim \Psi_{\exp}(b) \in \mathfrak{C}_{\rm ex}$. The monoid equivalence relation that this generates is equivalent to $c'_1 \sim_I c'_2 \in \mathfrak{C}_{\rm ex}$ if there exists $d \in I$ such that $c'_1 = \Psi_{\exp}(d) \cdot c'_2$.

We claim that $(\mathfrak{C}/I, \mathfrak{C}_{\rm ex}/{\sim_I})$ is indeed the required C^∞-ring with corners. If $f : [0, \infty) \to \mathbb{R}$ is smooth, and $c'_1 \sim_I c'_2 \in \mathfrak{C}_{\rm ex}$, then $\Phi_f(c'_1) = \Phi_f(\Psi_{\exp}(d)c'_2) \in \mathfrak{C}$ for some $d \in I$. Then applying Hadamard's lemma twice, we have

$$\begin{aligned}\Phi_f(c'_1) - \Phi_f(c'_2) =& \Phi_{(x-y)g(x,y)}(\Psi_{\exp}(d)c'_2, c'_2)\\ &= \Phi_i(c'_2)(\Phi_{\exp}(d) - 1)\Phi_g(\Psi_{\exp}(d)c'_2, c'_2)\\ &= \Phi_i(c'_2)(d - 0)\Phi_{h(x,y)}(d, 0)\Phi_g(\Psi_{\exp}(d)c'_2, c'_2),\end{aligned}$$

for smooth maps $g, h : \mathbb{R}^2 \to \mathbb{R}$. As $d \in I$, then $\Phi_f(c'_1) - \Phi_f(c'_2) \in I$, and Φ_f is well defined. A similar proof shows all the C^∞-operations are well defined, and so $(\mathfrak{C}/I, \mathfrak{C}_{\rm ex}/{\sim_I})$ is a pre C^∞-ring with corners.

We must show that $\Psi_{\exp} : \mathfrak{C}/I \to \mathfrak{C}_{\rm ex}/{\sim_I}$ has image equal to (not just contained in) the invertible elements $(\mathfrak{C}_{\rm ex}/{\sim_I})^\times$. Say $[c'_1] \in (\mathfrak{C}_{\rm ex}/{\sim_I})^\times$, then there is $[c'_2] \in (\mathfrak{C}_{\rm ex}/{\sim_I})^\times$ such that $[c'_1][c'_2] = [c'd'] = [1]$. So there is $d \in I$ such that $c'_1 c'_2 = \Psi_{\exp}(d)$. However, $\Psi_{\exp}(d)$ is invertible, so each of c'_1, c'_2 must be invertible in $\mathfrak{C}_{\rm ex}$, and using that $\Psi_{\exp}$ is surjective onto invertible elements in $\mathfrak{C}_{\rm ex}^\times$ gives the result. Thus $(\mathfrak{C}/I, \mathfrak{C}_{\rm ex}/{\sim_I})$ is a C^∞-ring with corners. The quotient morphisms $\mathfrak{C} \to \mathfrak{C}/I$ and $\mathfrak{C}_{\rm ex} \to \mathfrak{C}_{\rm ex}/{\sim_I}$ give the required map $\boldsymbol{\pi}$.

To show that this satisfies the required universal property of either (4.4.3) or **(*)**, take $(\mathfrak{E}, \mathfrak{E}_{\rm ex})$ another C^∞-ring with corners with morphism $\boldsymbol{\pi'} = (\pi', \pi'_{\rm ex}) : (\mathfrak{C}, \mathfrak{C}_{\rm ex}) \to (\mathfrak{E}, \mathfrak{E}_{\rm ex})$. Then the unique morphism $\boldsymbol{p} : (\mathfrak{D}, \mathfrak{D}_{\rm ex}) \to (\mathfrak{E}, \mathfrak{E}_{\rm ex})$ is defined by $\boldsymbol{p}([c], [c']) = [\pi'(c), \pi'(c')]$.

The requirement of $(\mathfrak{C}, \mathfrak{C}_{\rm ex})$ to factor through each diagram shows that this morphism is well defined and unique, giving the result.

(b) Say we wish to quotient a C^∞-ring with corners, $(\mathfrak{C}, \mathfrak{C}_{\rm ex})$ by an ideal P in the monoid $\mathfrak{C}_{\rm ex}$. By this we mean quotient $\mathfrak{C}_{\rm ex}$ by the equivalence relation $c_1' \sim c_2'$ if $c_1' = c_2'$ or $c_1', c_2' \in P$. This is known as a Rees quotient of semigroups, see Rees [80, p. 389]. Quotienting the monoid by this ideal will result in additional relations on both the monoid and the C^∞-ring, which we will now show. While this quotient, $(\mathfrak{D}, \mathfrak{D}_{\rm ex})$ is the coequaliser of a diagram such as (4.4.3), it is also equivalent to the following construction:

(**) The quotient is a C^∞-ring with corners $(\mathfrak{D}, \mathfrak{D}_{\rm ex})$ with a morphism $\boldsymbol{\pi} = (\pi, \pi_{\rm ex}) : (\mathfrak{C}, \mathfrak{C}_{\rm ex}) \to (\mathfrak{D}, \mathfrak{D}_{\rm ex})$ such that P is contained in the kernel of $\pi_{\rm ex}$, and is universal with respect to this property. That is, if $(\mathfrak{E}, \mathfrak{E}_{\rm ex})$ is another C^∞-ring with corners with morphism $\boldsymbol{\pi'} = (\pi', \pi'_{\rm ex}) : (\mathfrak{C}, \mathfrak{C}_{\rm ex}) \to (\mathfrak{E}, \mathfrak{E}_{\rm ex})$ with P contained in the kernel of $\pi'_{\rm ex}$, then there is a unique morphism $\boldsymbol{p} : (\mathfrak{D}, \mathfrak{D}_{\rm ex}) \to (\mathfrak{E}, \mathfrak{E}_{\rm ex})$ such that $\boldsymbol{p} \circ \boldsymbol{\pi} = \boldsymbol{\pi'}$.

Similar to part (a), we being with quotienting $\mathfrak{C}_{\rm ex}$ by P, and then require that the C^∞-operations are well defined. As all smooth $f : [0, \infty) \to \mathbb{R}$ are equal to $\hat{f} \circ i : [0, \infty) \to \mathbb{R}$ for a smooth function $\hat{f} : \mathbb{R} \to \mathbb{R}$, we need only require that if $c_1' \sim c_2'$, then $\Phi_i(c_1') \sim \Phi_i(c_2')$. This generates a C^∞-ring equivalence relation on the C^∞-ring $\mathfrak{C}$; such a C^∞-ring equivalence relation is the same data as giving an ideal $I \subset \mathfrak{C}$ such that $c_1 \sim c_2 \in \mathfrak{C}$ whenever $c_1 - c_2 \in I$. Here, this equivalence relation will be given by the ideal $\langle \Phi_i(P) \rangle$, that is, the ideal generated by the image of P under Φ_i. Quotienting $\mathfrak{C}$ by this ideal generates a further condition on the monoid $\mathfrak{C}_{\rm ex}$, as in part (a), that is $c_1' \sim c_2'$ if there is $d \in \langle \Phi_i(P) \rangle$ such that $c_1' = \Psi_{\exp}(d) c_2'$.

The claim then is that we may take $(\mathfrak{D}, \mathfrak{D}_{\rm ex})$ equal to $(\mathfrak{C}/\langle \Phi_i(P) \rangle, \mathfrak{C}_{\rm ex}/\sim_P)$ where $c_1' \sim_P c_2'$ if either $c_1', c_2' \in P$ or there is $d \in \langle \Phi_i(P) \rangle$ such that $c_1' = \Psi_{\exp}(d)(c_2')$. Similar applications of Hadamard's lemma as in (a) show that $(\mathfrak{D}, \mathfrak{D}_{\rm ex})$ is a C^∞-ring with corners, and similar discussions show that this is isomorphic to the quotient. We will use the notation

$$(\mathfrak{D}, \mathfrak{D}_{\rm ex}) = (\mathfrak{C}, \mathfrak{C}_{\rm ex})/\sim_P = (\mathfrak{C}/\langle \Phi_i(P) \rangle, \mathfrak{C}_{\rm ex}/\sim_P) = (\mathfrak{C}/\sim_P, \mathfrak{C}_{\rm ex}/\sim_P)$$

to refer to this quotient in §5.

Remark 4.4.5. Say $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ is a morphism of C^∞-rings with corners and we quotient $\boldsymbol{\mathfrak{C}}$ by relations. If we quotient $\boldsymbol{\mathfrak{D}}$ by relations that include the image of all relations of $\boldsymbol{\mathfrak{C}}$ under $\boldsymbol{\phi}$ then there is a unique map from one quotient to the other that commutes with $\boldsymbol{\phi}$ and the projections to the quotients. For example, if we can take a prime ideal $P \subset \mathfrak{C}_{\rm ex}$, such that $\phi(P) \subseteq (Q)$ for some prime ideal $Q \subset \mathfrak{D}_{\rm ex}$, then the universal property of quotients tells us there is a unique morphism $\boldsymbol{\phi}_P : \mathfrak{C}/\sim_P \to \mathfrak{D}/\sim_Q$. We will use this in §5.

Kalashnikov [51, §4.4] considered taking a pre C^∞-rings with corners and considering what conditions on an ideal of $\mathfrak{C}$, and a submonoid or monoidal equivalence relation in $\mathfrak{C}_{\rm ex}$, allow a quotient to be defined without needing to add additional relations. Such pairs of ideals and monoidal equivalences/submonoids they denoted *corner equivalences*. They then considered similar results to Example 4.4.4 in [51, §4.6].

4.5 Special classes of C^∞-rings with corners

We use the theory of monoids from §3.1 to give special classes of C^∞-rings with corners. The first definition is important for properties of C^∞-schemes with corners.

Definition 4.5.1. We call a C^∞-ring with corners, $(\mathfrak{C}, \mathfrak{C}_{\rm ex})$, *firm* if the sharpening $\mathfrak{C}^\sharp_{\rm ex}$ is a finitely generated monoid. We denote by $\mathbf{C^\infty Rings^c_{fi}}$ the full subcategory of $\mathbf{C^\infty Rings^c}$ consisting of firm C^∞-rings with corners.

Note that if $(\mathfrak{C}, \mathfrak{C}_{\rm ex})$ is a firm C^∞-ring with corners, then there are $c_i \in \mathfrak{C}_{\rm ex}$, for $i = 1, \ldots, n$ such that the images of c_i' under the quotient $\mathfrak{C}_{\rm ex} \to \mathfrak{C}^\sharp_{\rm ex}$ generate $\mathfrak{C}^\sharp_{\rm ex}$. This implies each element in $\mathfrak{C}_{\rm ex}$ can be written as $\Psi_{\exp}(c) c_1^{a_1} \cdots c_n^{a_n}$ for some smooth $f : \mathbb{R}^{k+1}_k \to [0, \infty)$, $c \in \mathfrak{C}$, and non-negative integers a_i.

Proposition 4.5.2. $\mathbf{C^\infty Rings^c_{fi}}$ *is closed under finite colimits.*

Proof. $(\mathbb{R}, [0, \infty))$ is an initial object in this category, where the sharpening of $[0, \infty)$ is generated by 0. As our category has an initial object then all finite colimits are composed of a finite number of (iterated) pushouts, hence we need only to show the category is closed under pushouts. We show that the pushout of elements in $\mathbf{C^\infty Rings^c_{fi}}$ in the category of C^∞-rings with corners is an element of $\mathbf{C^\infty Rings^c_{fi}}$ and is therefore the pushout in this full subcategory.

Take $\mathfrak{C}, \mathfrak{D}, \mathfrak{E} \in \mathbf{C^\infty Rings^c_{fi}}$ with morphisms $\mathfrak{C} \to \mathfrak{D}$ and $\mathfrak{C} \to \mathfrak{E}$, and consider the pushout $\mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E}$, with its morphisms $\boldsymbol{\phi} : \mathfrak{D} \to \mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E}$ and $\psi : \mathfrak{E} \to \mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E}$. Then every element of $(\mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E})_{\rm ex}$ is of the form

$$\Psi_f(\phi(d_1), \ldots, \phi(d_n), \psi(e_1), \ldots, \psi(e_m), \phi_{\rm ex}(d_1'), \ldots, \phi_{\rm ex}(d_k'), \psi_{\rm ex}(e_1'), \ldots, \psi_{\rm ex}(e_l')),$$

for $f : \mathbb{R}^{m+n}_{k+l} \to [0, \infty)$, where $d_i \in \mathfrak{D}, d_i' \in \mathfrak{D}_{\rm ex}, e_i \in \mathfrak{E}, e_i' \in \mathfrak{E}_{\rm ex}$, and d_i' are generators of the sharpening of $\mathfrak{D}_{\rm ex}$, and e_i' generate the sharpening of $\mathfrak{E}_{\rm ex}$. We may write

$$f(x_1, \ldots, x_{m+n}, y_1, \ldots, y_{k+l}) = y_1^{a_1} \ldots y_{k+l}^{a_{k+l}} F(x_1, \ldots, x_{m+n}, y_1, \ldots, y_{k+l}),$$

for $F:[0,\infty)^{n+m}\to(0,\infty)$ smooth.

In the sharpening of $(\mathfrak{D}\amalg_{\mathfrak{C}}\mathfrak{E})_{\rm ex}$, the above element corresponds to

$$\phi_{\rm ex}(d_1')^{a_1}\dots\phi_{\rm ex}(d_k')^{a_k}\psi_{\rm ex}(e_1')^{a_{k+1}}\dots\psi_{\rm ex}(e_l')^{a_{k+l}}.$$

Hence every element $(\mathfrak{D}\amalg_{\mathfrak{C}}\mathfrak{E})^\sharp_{\rm ex}$ is the generated by the images of the generators of $\mathfrak{D}^\sharp_{\rm ex}$ and $\mathfrak{E}^\sharp_{\rm ex}$, and therefore $(\mathfrak{D}\amalg_{\mathfrak{C}}\mathfrak{E})^\sharp_{\rm ex}$ is finitely generated. Thus $\mathbf{C^\infty Rings^c_{fi}}$ is closed under pushouts. □

Remark 4.5.3. A finitely generated C^∞-ring with corners is always firm, however the reverse is not true. For example, take an infinitely generated C^∞-ring, $\mathfrak{C}$, and apply the functor $F_{\rm exp}$ from Theorem 4.3.9, to get a C^∞-ring with corners $(\mathfrak{C},\Phi_{\rm exp}(\mathfrak{C})\amalg\{0\})$, where $\amalg$ is the disjoint union. The sharpening of $\Phi_{\rm exp}(\mathfrak{C})\amalg\{0\}$ is $\{0\}$ and therefore it is a firm C^∞-ring with corners, but it is not finitely generated. In other words, firm C^∞-rings with corners may have infinitely many generators of their C^∞-ring, but finitely many generators of the non-invertible elements of their monoid.

The difference between firm C^∞-rings with corners and interior C^∞-rings with corners is that the former has finitely generated sharpening, whereas the latter has sharpening with no zero-divisors.

We now use §3.1 to define some important classes of interior C^∞-rings with corners:

Definition 4.5.4. Suppose $\boldsymbol{\mathfrak{C}}=(\mathfrak{C},\mathfrak{C}_{\rm ex})$ is an interior C^∞-ring with corners, and let $\mathfrak{C}_{\rm in}\subseteq\mathfrak{C}_{\rm ex}$ be the submonoid of §4.2. Then:

(i) We call $\boldsymbol{\mathfrak{C}}$ *integral* if $\mathfrak{C}_{\rm in}$ is an integral monoid.

(ii) We call $\boldsymbol{\mathfrak{C}}$ *torsion-free* if it is integral, and $\mathfrak{C}_{\rm in}$ is a torsion-free monoid.

(iii) We call $\boldsymbol{\mathfrak{C}}$ *saturated* if it is integral, and $\mathfrak{C}_{\rm in}$ is a saturated monoid. Note that $\mathfrak{C}_{\rm in}^\times\cong\mathfrak{C}$ as abelian groups since $\boldsymbol{\mathfrak{C}}$ is a C^∞-ring with corners, so $\mathfrak{C}_{\rm in}^\times$ is torsion-free. Therefore $\boldsymbol{\mathfrak{C}}$ saturated implies that $\boldsymbol{\mathfrak{C}}$ is torsion-free.

(iv) We call $\boldsymbol{\mathfrak{C}}$ *toric* if it is saturated, and the sharpening $\mathfrak{C}_{\rm in}^\sharp=\mathfrak{C}_{\rm in}/\mathfrak{C}_{\rm in}^\times$ is a toric monoid. This implies $\boldsymbol{\mathfrak{C}}$ is firm. Here $\boldsymbol{\mathfrak{C}}$ saturated implies $\mathfrak{C}_{\rm in}^\sharp$ is integral, torsion-free, and sharp. Thus $\mathfrak{C}_{\rm in}^\sharp$ is toric if and only if it is saturated and finitely generated.

(v) We call $\boldsymbol{\mathfrak{C}}$ *simplicial* if it is saturated, and $\mathfrak{C}_{\rm in}^\sharp\cong\mathbb{N}^k$ for some $k\in\mathbb{N}$. Simplicial implies toric which implies firm.

We will write

$$\mathbf{C^\infty Rings^c_\Delta}\subset\mathbf{C^\infty Rings^c_{to}}\subset\mathbf{C^\infty Rings^c_{sa}}\subset\mathbf{C^\infty Rings^c_{tf}}\subset\mathbf{C^\infty Rings^c_{\mathbb{Z}}}\subset\mathbf{C^\infty Rings^c_{in}}$$

for the full subcategories of simplicial, toric, saturated, torsion-free, and integral objects in $\mathbf{C^\infty Rings^c_{in}}$, respectively.

Example 4.5.5. Let X be a manifold with corners, and $\boldsymbol{C}^\infty_{\rm in}(X)$ be the interior C^∞-ring with corners from Example 4.2.7. Let S be the set of connected components of ∂X. For each $F \in S$, we choose an interior map $c_F : X \to [0,\infty)$ which vanishes to order 1 on F, and to order zero on $\partial X \setminus F$, such that $c_F = 1$ outside a small neighbourhood U_F of $i_X(F)$ in X, where we choose $\{U_F : F \in S\}$ to be locally finite in X. Then every interior map $g : X \to [0,\infty)$ may be written uniquely as $g = \exp(f) \cdot \prod_{F\in S} c_F^{a_F}$, for $f \in C^\infty(X)$ and $a_F \in \mathbb{N}$, $F \in S$.

Hence as monoids we have $\mathrm{In}(X) \cong C^\infty(X) \times \mathbb{N}^S$. Therefore $\boldsymbol{C}^\infty_{\rm in}(X)$ is integral, torsion-free, and saturated, and it is simplicial and toric if and only if ∂X has finitely many connected components. A more complicated proof shows that if X is a manifold with g-corners then $\boldsymbol{C}^\infty_{\rm in}(X)$ is integral, torsion-free, and saturated, and it is toric if ∂X has finitely many connected components.

The next proposition is proved as for Theorem 4.3.7(a), noting that writing $\mathbf{Mon_{sa}} \subset \mathbf{Mon_{tf}} \subset \mathbf{Mon}_{\mathbb{Z}}$ for the full subcategories in $\mathbf{Mon}$ of saturated, torsion free, integral monoids, and torsion free, integral monoids, and integral monoids, respectively, then $\mathbf{Mon_{sa}}, \mathbf{Mon_{tf}}, \mathbf{Mon}_{\mathbb{Z}}$ are closed under limits and directed colimits in $\mathbf{Mon}$.

Proposition 4.5.6. *$\mathbf{C^\infty Rings^c_{sa}}, \mathbf{C^\infty Rings^c_{tf}}$ and $\mathbf{C^\infty Rings^c_{\mathbb{Z}}}$ are closed under limits and under directed colimits in $\mathbf{C^\infty Rings^c_{in}}$. Thus, all small limits and directed colimits exist in $\mathbf{C^\infty Rings^c_{sa}}, \mathbf{C^\infty Rings^c_{tf}}, \mathbf{C^\infty Rings^c_{\mathbb{Z}}}$.*

Recall that if $\mathcal{D}$ is a category and $\mathcal{C} \subset \mathcal{D}$ a full subcategory, then $\mathcal{C}$ is a *reflective subcategory* if the inclusion $\mathrm{inc} : \mathcal{C} \hookrightarrow \mathcal{D}$ has a left adjoint $\Pi : \mathcal{D} \to \mathcal{C}$, which is called a *reflection functor*. Proposition 4.3.5 shows that $\mathbf{C^\infty Rings^c} \subset \mathbf{PC^\infty Rings^c}$ and $\mathbf{C^\infty Rings^c_{in}} \subset \mathbf{PC^\infty Rings^c_{in}}$ are reflective subcategories. We will show that $\mathbf{C^\infty Rings^c_{sa}}, \mathbf{C^\infty Rings^c_{tf}}, \mathbf{C^\infty Rings^c_{\mathbb{Z}}} \subset \mathbf{C^\infty Rings^c_{in}}$ are too.

Theorem 4.5.7. *There are reflection functors $\Pi_{\rm in}^{\mathbb{Z}}, \Pi_{\mathbb{Z}}^{\rm tf}, \Pi_{\rm tf}^{\rm sa}, \Pi_{\rm in}^{\rm sa}$ in a diagram*

$$\mathbf{C^\infty Rings^c_{sa}} \underset{\mathrm{inc}}{\overset{\Pi_{\rm tf}^{\rm sa}}{\rightleftarrows}} \mathbf{C^\infty Rings^c_{tf}} \underset{\mathrm{inc}}{\overset{\Pi_{\mathbb{Z}}^{\rm tf}}{\rightleftarrows}} \mathbf{C^\infty Rings^c_{\mathbb{Z}}} \underset{\mathrm{inc}}{\overset{\Pi_{\rm in}^{\mathbb{Z}}}{\rightleftarrows}} \mathbf{C^\infty Rings^c_{in}}, \quad (\text{with } \Pi_{\rm in}^{\rm sa} : \mathbf{C^\infty Rings^c_{in}} \to \mathbf{C^\infty Rings^c_{sa}}) \tag{4.5.1}$$

such that each of $\Pi_{\rm in}^{\mathbb{Z}}, \Pi_{\mathbb{Z}}^{\rm tf}, \Pi_{\rm tf}^{\rm sa}, \Pi_{\rm in}^{\rm sa}$ is left adjoint to the corresponding inclusion functor inc.

Proof. Let $\boldsymbol{\mathfrak{C}}$ be an object in $\mathbf{C^\infty Rings^c_{in}}$. We will construct an object $\boldsymbol{\mathfrak{D}} = \Pi^{\mathbb{Z}}_{\rm in}(\boldsymbol{\mathfrak{C}})$ in $\mathbf{C^\infty Rings^c_{\mathbb{Z}}}$ and a projection $\boldsymbol{\pi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$, with the property that if $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{E}}$ is a morphism in $\mathbf{C^\infty Rings^c_{in}}$ with $\boldsymbol{\mathfrak{E}} \in \mathbf{C^\infty Rings^c_{\mathbb{Z}}}$ then $\boldsymbol{\phi} = \boldsymbol{\psi} \circ \boldsymbol{\pi}$ for a unique morphism $\boldsymbol{\psi} : \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{E}}$. Consider the diagram:

$$\begin{array}{ccccccccc}
\boldsymbol{\mathfrak{C}} = \boldsymbol{\mathfrak{C}}^0 & \xrightarrow{\boldsymbol{\alpha}^0} & \boldsymbol{\mathfrak{C}}^1 & \xrightarrow{\boldsymbol{\alpha}^1} & \boldsymbol{\mathfrak{C}}^2 & \xrightarrow{\boldsymbol{\alpha}^2} & \cdots & \longrightarrow & \boldsymbol{\mathfrak{D}} \\
&&&&&&&& \Big\downarrow{\scriptstyle \boldsymbol{\psi}} \\
&&&&&&&& \boldsymbol{\mathfrak{E}}.
\end{array}
\qquad \boldsymbol{\pi}=\boldsymbol{\pi}^0,\ \boldsymbol{\pi}^1 : \boldsymbol{\mathfrak{C}}^n \to \boldsymbol{\mathfrak{D}}, \quad \boldsymbol{\phi}=\boldsymbol{\phi}^0,\ \boldsymbol{\phi}^1,\ \boldsymbol{\phi}^2 : \boldsymbol{\mathfrak{C}}^n \to \boldsymbol{\mathfrak{E}} \tag{4.5.2}$$

Define $\boldsymbol{\mathfrak{C}}^0 = \boldsymbol{\mathfrak{C}}$ and $\boldsymbol{\phi}^0 = \boldsymbol{\phi}$. By induction on $n = 0, 1, \ldots,$ if $\boldsymbol{\mathfrak{C}}^n, \boldsymbol{\phi}^n$ are defined, define an object $\boldsymbol{\mathfrak{C}}^{n+1} \in \mathbf{C^\infty Rings^c_{in}}$ and morphisms $\boldsymbol{\alpha}^n : \boldsymbol{\mathfrak{C}}^n \to \boldsymbol{\mathfrak{C}}^{n+1}$, $\boldsymbol{\phi}^{n+1} : \boldsymbol{\mathfrak{C}}^{n+1} \to \boldsymbol{\mathfrak{E}}$ as follows. We have a monoid $\mathfrak{C}^n_{\rm in}$, which as in §3.1 has an abelian group $(\mathfrak{C}^n_{\rm in})^{\rm gp}$ with projection $\pi^{\rm gp} : \mathfrak{C}^n_{\rm in} \to (\mathfrak{C}^n_{\rm in})^{\rm gp}$, where $\boldsymbol{\mathfrak{C}}^n_{\rm in}, \boldsymbol{\mathfrak{C}}^n$ are integral if $\pi^{\rm gp}$ is injective. Using the notation of Definition 4.4.3, we define

$$\boldsymbol{\mathfrak{C}}^{n+1} = \boldsymbol{\mathfrak{C}}^n \big/ \big[c' = c'' \text{ if } c', c'' \in \mathfrak{C}^n_{\rm in} \text{ with } \pi^{\rm gp}(c') = \pi^{\rm gp}(c'')\big]. \tag{4.5.3}$$

Write $\boldsymbol{\alpha}^n : \boldsymbol{\mathfrak{C}}^n \to \boldsymbol{\mathfrak{C}}^{n+1}$ for the natural surjective projection. Then $\boldsymbol{\mathfrak{C}}^{n+1}, \boldsymbol{\alpha}^n$ are both interior, since the relations $c' = c''$ in (4.5.3) are all interior.

We have a morphism $\boldsymbol{\phi}^n : \boldsymbol{\mathfrak{C}}^n \to \boldsymbol{\mathfrak{E}}$ with $\boldsymbol{\mathfrak{E}}$ integral, so by considering the diagram with bottom morphism injective

$$\begin{array}{ccc}
\mathfrak{C}^n_{\rm in} & \xrightarrow{\pi^{\rm gp}} & (\mathfrak{C}^n_{\rm in})^{\rm gp} \\
\Big\downarrow{\scriptstyle \phi^n_{\rm in}} & & \Big\downarrow{\scriptstyle (\phi^n_{\rm in})^{\rm gp}} \\
\mathfrak{E}_{\rm in} & \xhookrightarrow{\pi^{\rm gp}} & (\mathfrak{E}_{\rm in})^{\rm gp},
\end{array}$$

we see that if $c', c'' \in \mathfrak{C}^n_{\rm in}$ with $\pi^{\rm gp}(c') = \pi^{\rm gp}(c'')$ then $\phi^n_{\rm in}(c') = \phi^n_{\rm in}(c'')$. Thus by the universal property of (4.5.3), there is a unique morphism $\boldsymbol{\phi}^{n+1} : \boldsymbol{\mathfrak{C}}^{n+1} \to \boldsymbol{\mathfrak{E}}$ with $\boldsymbol{\phi}^n = \boldsymbol{\phi}^{n+1} \circ \boldsymbol{\alpha}^n$. This completes the inductive step, so we have defined $\boldsymbol{\mathfrak{C}}^n, \boldsymbol{\alpha}^n, \boldsymbol{\phi}^n$ for all $n = 0, 1, \ldots,$ where $\boldsymbol{\mathfrak{C}}^n, \boldsymbol{\alpha}^n$ are independent of $\boldsymbol{\mathfrak{E}}, \boldsymbol{\phi}$.

Now define $\boldsymbol{\mathfrak{D}}$ to be the directed colimit $\boldsymbol{\mathfrak{D}} = \varinjlim_{n=0}^{\infty} \boldsymbol{\mathfrak{C}}^n$ in $\mathbf{C^\infty Rings^c_{in}}$, using the morphisms $\boldsymbol{\alpha}^n : \boldsymbol{\mathfrak{C}}^n \to \boldsymbol{\mathfrak{C}}^{n+1}$. This exists by Theorem 4.3.7(a), and commutes with $\bar{\Pi}_{\rm in} : \mathbf{C^\infty Rings^c_{in}} \to \mathbf{Mon}$. It has a natural projection $\boldsymbol{\pi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$, and also projections $\boldsymbol{\pi}^n : \boldsymbol{\mathfrak{C}}^n \to \boldsymbol{\mathfrak{D}}$ for all n. By the universal property of colimits, there is a unique morphism $\boldsymbol{\psi}$ in $\mathbf{C^\infty Rings^c_{in}}$ making (4.5.2) commute.

The purpose of the quotient (4.5.3) is to modify $\boldsymbol{\mathfrak{C}}^n$ to make it integral, since if $\boldsymbol{\mathfrak{C}}^n$ were integral then $\pi^{\rm gp}(c') = \pi^{\rm gp}(c'')$ implies $c' = c''$. It is not obvious that $\boldsymbol{\mathfrak{C}}^{n+1}$ in (4.5.3) is integral, as the quotient modifies $(\mathfrak{C}^n_{\rm in})^{\rm gp}$. However, the direct limit $\boldsymbol{\mathfrak{D}}$ is integral. To see this, suppose $d', d'' \in \mathfrak{D}_{\rm in}$ with $\pi^{\rm gp}(d') = \pi^{\rm gp}(d'')$ in $(\mathfrak{D}_{\rm in})^{\rm gp}$. Since $\mathfrak{D}_{\rm in} = \varinjlim_{m=0}^{\infty} \mathfrak{C}_{\rm in}^m$ in **Mon**, for $m \gg 0$ we may write $d' = \pi_{\rm in}^m(c')$, $d'' = \pi_{\rm in}^m(c'')$ for $c', c'' \in \mathfrak{C}_{\rm in}^m$. As $(\mathfrak{D}_{\rm in})^{\rm gp} = \varinjlim_{n=0}^{\infty} (\mathfrak{C}_{\rm in}^n)^{\rm gp}$ and $\pi^{\rm gp}(d') = \pi^{\rm gp}(d'')$, for some $n \gg m$ we have

$$\pi^{\rm gp} \circ \alpha_{\rm in}^{n-1} \circ \cdots \circ \alpha_{\rm in}^m(c') = \pi^{\rm gp} \circ \alpha_{\rm in}^{n-1} \circ \cdots \circ \alpha_{\rm in}^m(c'') \quad \text{in } (\mathfrak{C}_{\rm in}^n)^{\rm gp}.$$

But then (4.5.3) implies that $\alpha_{\rm in}^n \circ \cdots \circ \alpha_{\rm in}^n(c') = \alpha_{\rm in}^n \circ \cdots \circ \alpha_{\rm in}^m(c'')$ in $\mathfrak{C}_{\rm in}^{n+1}$, so $d' = d''$. Therefore $\pi^{\rm gp} : \mathfrak{D}_{\rm in} \to (\mathfrak{D}_{\rm in})^{\rm gp}$ is injective, and $\boldsymbol{\mathfrak{D}}$ is integral.

Set $\Pi_{\rm in}^{\mathbb{Z}}(\boldsymbol{\mathfrak{C}}) = \boldsymbol{\mathfrak{D}}$. If $\boldsymbol{\xi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{C}}'$ is a morphism in $\mathbf{C^\infty Rings^c_{in}}$, by taking $\boldsymbol{\mathfrak{E}} = \Pi_{\rm in}^{\mathbb{Z}}(\boldsymbol{\mathfrak{C}}')$ and $\boldsymbol{\phi} = \boldsymbol{\pi}' \circ \boldsymbol{\xi}$ in (4.5.2) we see that there is a unique morphism $\Pi_{\rm in}^{\mathbb{Z}}(\boldsymbol{\xi})$ in $\mathbf{C^\infty Rings^c_{\mathbb{Z}}}$ making the following commute:

$$\begin{array}{ccc} \boldsymbol{\mathfrak{C}} & \xrightarrow{\quad \boldsymbol{\pi} \quad} & \Pi_{\rm in}^{\mathbb{Z}}(\boldsymbol{\mathfrak{C}}) \\ \downarrow{\scriptstyle \boldsymbol{\xi}} & & {\scriptstyle \Pi_{\rm in}^{\mathbb{Z}}(\boldsymbol{\xi})}\downarrow \\ \boldsymbol{\mathfrak{C}}' & \xrightarrow{\quad \boldsymbol{\pi}' \quad} & \Pi_{\rm in}^{\mathbb{Z}}(\boldsymbol{\mathfrak{C}}'). \end{array}$$

This defines the functor $\Pi_{\rm in}^{\mathbb{Z}}$. For any $\boldsymbol{\mathfrak{E}} \in \mathbf{C^\infty Rings^c_{\mathbb{Z}}}$, the correspondence between $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ in (4.5.2) implies that we have a natural bijection

$$\mathrm{Hom}_{\mathbf{C^\infty Rings^c_{in}}}\bigl(\boldsymbol{\mathfrak{C}}, \mathrm{inc}(\boldsymbol{\mathfrak{E}})\bigr) \cong \mathrm{Hom}_{\mathbf{C^\infty Rings^c_{\mathbb{Z}}}}\bigl(\Pi_{\rm in}^{\mathbb{Z}}(\boldsymbol{\mathfrak{C}}), \boldsymbol{\mathfrak{E}}\bigr).$$

This is functorial in $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{E}}$, and so $\Pi_{\rm in}^{\mathbb{Z}}$ is left adjoint to $\mathrm{inc} : \mathbf{C^\infty Rings^c_{\mathbb{Z}}} \hookrightarrow \mathbf{C^\infty Rings^c_{in}}$, as we have to prove.

The constructions of $\Pi_{\mathbb{Z}}^{\rm tf}, \Pi_{\rm tf}^{\rm sa}$ are very similar. For $\Pi_{\mathbb{Z}}^{\rm tf}$, if $\boldsymbol{\mathfrak{C}}$ is an object in $\mathbf{C^\infty Rings^c_{\mathbb{Z}}}$, the analogue of (4.5.3) is

$$\boldsymbol{\mathfrak{C}}^{n+1} = \boldsymbol{\mathfrak{C}}^n \big/ \bigl[c' = c'' \text{ if } c', c'' \in \mathfrak{C}_{\rm in}^n \text{ with } \pi^{\rm tf}(c') = \pi^{\rm tf}(c'')\bigr],$$

where $\pi^{\rm tf} : \mathfrak{C}_{\rm in}^n \to (\mathfrak{C}_{\rm in}^n)^{\rm gp}/\text{torsion}$ is the natural projection. For $\Pi_{\rm tf}^{\rm sa}$, if $\boldsymbol{\mathfrak{C}}$ is an object in $\mathbf{C^\infty Rings^c_{tf}}$, the analogue of (4.5.3) is

$$\begin{aligned} \boldsymbol{\mathfrak{C}}^{n+1} = \boldsymbol{\mathfrak{C}}^n \big/ \bigl(s_{c'} : c' \in \mathfrak{C}_{\rm in}^n \subseteq (\mathfrak{C}_{\rm in}^n)^{\rm gp} & \text{ and there exists } c'' \in (\mathfrak{C}_{\rm in}^n)^{\rm gp} \setminus \mathfrak{C}_{\rm in}^n \\ & \text{with } c' = n_{c'} \cdot c'',\ n_c' = 2, 3, \ldots\bigr)\bigl[n_{c'} \cdot s_{c'} = c', \text{ all } c', n_{c'}, s_{c'}\bigr]. \end{aligned}$$

Finally we set $\Pi_{\rm in}^{\rm sa} = \Pi_{\rm tf}^{\rm sa} \circ \Pi_{\mathbb{Z}}^{\rm tf} \circ \Pi_{\rm in}^{\mathbb{Z}}$. This completes the proof. □

Remark 4.5.8. One can prove Theorem 4.5.7 without constructing $\Pi_{\rm in}^{\mathbb{Z}}, \Pi_{\mathbb{Z}}^{\rm tf}, \Pi_{\rm tf}^{\rm sa}, \Pi_{\rm in}^{\rm sa}$ explicitly, using Freyd's *Adjoint Functor Theorem*, as in Mac Lane [63, Th. V.6.2]. This says that given a functor $F : \mathcal{C} \to \mathcal{D}$, if (i) $\mathcal{C}$ has all small limits, (ii) F preserves small limits, and (iii) the 'solution set condition' holds, then F has a left adjoint $G : \mathcal{D} \to \mathcal{C}$. For the functors inc in (4.4.2), conditions (i),(ii) follow from Proposition 4.5.6.

Condition (iii) is set-theoretic. For $\Pi_{\rm in}^{\mathbb{Z}}$ one can it check by considering the set of surjective morphisms $\boldsymbol{\phi}:\boldsymbol{\mathfrak{C}}\to\boldsymbol{\mathfrak{D}}$ in $\mathbf{C^\infty Rings^c_{in}}$ with $\boldsymbol{\mathfrak{D}}\in\mathbf{C^\infty Rings^c_{\mathbb{Z}}}$ and $\mathfrak{D},\mathfrak{D}_{\rm ex}$ the quotients of $\mathfrak{C},\mathfrak{C}_{\rm ex}$ by equivalence relations, and similarly for $\Pi_{\mathbb{Z}}^{\rm tf}$. As in Adámek and Rosický [2, §6.D], if we assume the large-cardinal axiom in Set Theory known as 'Vopěnka's Principle', then the Adjoint Functor Theorem holds in this case without (iii), so Proposition 4.5.6 implies Theorem 4.5.7.

As for Theorem 4.3.7(b), we deduce:

Corollary 4.5.9. *Small colimits exist in* $\mathbf{C^\infty Rings^c_{sa}}$,$\mathbf{C^\infty Rings^c_{tf}}$,$\mathbf{C^\infty Rings^c_{\mathbb{Z}}}$.

In Definition 4.4.3 we explained how to modify a C^∞-ring with corners $\boldsymbol{\mathfrak{C}}$ by adding generators $\boldsymbol{\mathfrak{C}}(x_a : a\in A)[y_{a'} : a'\in A_{\rm ex}]$ and imposing relations $\boldsymbol{\mathfrak{C}}/(f_b=0 : b\in B)[g_{b'}=h_{b'} : b'\in B_{\rm ex}]$. This is just notation for certain small colimits in $\mathbf{C^\infty Rings^c}$ or $\mathbf{C^\infty Rings^c_{in}}$. Corollary 4.5.9 implies that we can also add generators and relations in $\mathbf{C^\infty Rings^c_{sa}}, \mathbf{C^\infty Rings^c_{tf}}, \mathbf{C^\infty Rings^c_{\mathbb{Z}}}$, provided the relations $g_{b'}=h_{b'}$ are interior, that is, $g_{b'},h_{b'}\neq 0_{\mathfrak{C}_{\rm ex}}$.

Remark 4.5.10. We do not expect to have arbitary limits or colimits in $\mathbf{C^\infty Rings^c_{\Delta}}$, $\mathbf{C^\infty Rings^c_{to}}$ or $\mathbf{C^\infty Rings^c_{fi}}$, as the finitely generated conditions could not be expected to hold for infinite limits or colimits. However, finite limits and finite colimits in $\mathbf{C^\infty Rings^c_{to}}$ or $\mathbf{C^\infty Rings^c_{fi}}$ do exist. We can show finite colimits exist in $\mathbf{C^\infty Rings^c_{to}}$ by using a similar proof to showing they exist in $\mathbf{C^\infty Rings^c_{fi}}$ as in Proposition 4.5.2. Finite limits can be shown to exist by considering the finite limit in $\mathbf{C^\infty Rings^c}$ using Theorem 4.3.7(a), and checking it is finitely generated, and then applying Proposition 4.5.6.

Finite products and coproducts exist in $\mathbf{C^\infty Rings^c_{\Delta}}$, however fibre products and fibre coproducts do not exist in general. For example, the fibre product of $\mathbb{N}^2\to\mathbb{N}$, $(a,b)\mapsto a+b$, and $\mathbb{N}^2\to\mathbb{N}$, $(c,d)\mapsto c+d$ is the set $\{(a,b,c,d)\in\mathbb{N}^4 | a+b=c+d\}$, which has generators $(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1)$ and relations between them. This is not isomorphic to $\mathbb{N}^k$ for any non-negative integer k. This example relates to elements in $\mathbf{C^\infty Rings^c_{\Delta}}$ where $\mathbb{N}^2$ correspond to the sharpening of the monoid part of $\boldsymbol{C}^\infty([0,\infty))$, and the $\mathbb{N}$ to the sharpening of the monoid part of $\boldsymbol{C}^\infty([0,\infty))$, and it is related to Example 3.2.4.

4.6 Local C^∞-rings with corners, and localisation

Here is the corners analogue of local C^∞-rings. To understand the first definition, recall from Definition 2.1.9 that a C^∞-ring $\mathfrak{C}$ is local if and only if there exists a (necessarily unique) $\mathbb{R}$-algebra morphism $\pi:\mathfrak{C}\to\mathbb{R}$ such that $c\in\mathfrak{C}$ is invertible if and only if $\pi(c)\neq 0$.

Definition 4.6.1. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a C^∞-ring with corners. We say that $\boldsymbol{\mathfrak{C}}$ is *local* if there exists a C^∞-ring morphism (or equivalently, an $\mathbb{R}$-algebra morphism) $\pi : \mathfrak{C} \to \mathbb{R}$ such that each $c \in \mathfrak{C}$ is invertible in $\mathfrak{C}$ if and only if $\pi(c) \neq 0$ in $\mathbb{R}$, and each $c' \in \mathfrak{C}_{\rm ex}$ is invertible in $\mathfrak{C}_{\rm ex}$ if and only if $\pi \circ \Phi_i(c') \neq 0$ in $\mathbb{R}$, where $\Phi_i : \mathfrak{C}_{\rm ex} \to \mathfrak{C}$ is induced by the inclusion $i : [0, \infty) \hookrightarrow \mathbb{R}$. Note that if $\boldsymbol{\mathfrak{C}}$ is local then $\pi : \mathfrak{C} \to \mathbb{R}$ is determined uniquely by $\operatorname{Ker} \pi = \{c \in \mathfrak{C} : c \text{ is not invertible}\}$, and $\mathfrak{C}$ is a local C^∞-ring.

Alternatively, we could say $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ is a local C^∞-ring with corners, if $\mathfrak{C}$ is local, and $c' \in \mathfrak{C}_{\rm ex}$ is invertible if and only if $\Phi_i(c') \in \mathfrak{C}$ is invertible.

Remark 4.6.2. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ and $\boldsymbol{\mathfrak{D}} = (\mathfrak{D}, \mathfrak{D}_{\rm ex})$ be local C^∞-rings with corners and $\boldsymbol{\phi} = (\phi, \phi_{\rm ex}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ be a morphism of C^∞-rings with corners. As in Definition 2.1.9, $\phi : \mathfrak{C} \to \mathfrak{D}$ is already a local morphism, which is equivalent to requiring that $c \in \mathfrak{C}$ is invertible if and only if $\phi(c)$ is invertible in $\mathfrak{D}$. The definition of local C^∞-ring with corners then ensures that $c' \in \mathfrak{C}_{\rm ex}$ is invertible if and only if $\phi_{\rm ex}(c')$ is invertible in $\mathfrak{D}_{\rm ex}$. Thus we do not define local morphisms of local C^∞-rings with corners, as morphisms already respect locality conditions.

Proposition 4.6.3. *The category of (interior) local C^∞-rings with corners has all small colimits, and they commute with colimits in* $\mathbf{C^\infty Rings^c}$*, and there is a right adjoint functor to the inclusion of local C^∞-rings with corners into non-local ones. The category of (interior) local C^∞-rings with corners has all small limits, and they commute with limits in (interior)* $\mathbf{C^\infty Rings^c}$ *in certain cases. The forgetful functor from (interior) local C^∞-rings with corners to local C^∞-rings, $(\mathfrak{C}, \mathfrak{C}_{\rm ex}) \mapsto \mathfrak{C}$, has both a left and right adjoint, so it respects colimits and limits.*

This proof is similar to that of Proposition 2.1.11, and uses Theorem 4.3.9.

Proof. As in Proposition 2.1.11, we first consider pushouts of C^∞-rings with corners.

Take C^∞-rings with corners $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}, \boldsymbol{\mathfrak{E}}$ and morphisms $\boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ and $\boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{E}}$, and let $\boldsymbol{\mathfrak{F}} = (\mathfrak{F}, \mathfrak{F}_{\rm ex})$ be the pushout in C^∞-rings with corners, with morphisms $\boldsymbol{p} = (p, p_{\rm ex}) : \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{F}}$ and $\boldsymbol{q} = (q, q_{\rm ex}) : \boldsymbol{\mathfrak{E}} \to \boldsymbol{\mathfrak{F}}$. From Theorem 4.3.9, $\mathfrak{F}$ is the pushout of the C^∞-rings $\mathfrak{C}, \mathfrak{D}$ and $\mathfrak{E}$, and it is local by Proposition 2.1.11.

Elements of $\mathfrak{F}_{\rm ex}$ are generated by elements of the form $\Psi_f(\underline{p})$ for some smooth $f : \mathbb{R}^{m+n}_{j+k} \to [0, \infty)$, with

$$\underline{p} = (p(d_1), \ldots, p(d_m), q(e_1), \ldots, q(e_n), p_{\rm ex}(d'_1), \ldots, p_{\rm ex}(d'_j), q_{\rm ex}(e'_1), \ldots, q_{\rm ex}(e'_k)),$$

where $d_1, \ldots, d_m \in \mathfrak{D}$, $e_1, \ldots, e_m \in \mathfrak{E}$, $d'_1, \ldots, d'_m \in \mathfrak{D}_{\rm ex}$, $e'_1, \ldots, e'_m \in \mathfrak{E}_{\rm ex}$. Therefore, all elements of $\mathfrak{F}_{\rm ex}$ are of this form.

Take $w' = \Psi_f(\underline{p}) \in \mathfrak{F}_{\rm ex}$ such that $\Phi_i(w')$ is invertible in $\mathfrak{F}$. We need to show that w' is invertible in $\mathfrak{F}_{\rm ex}$. We may write $f = h_1 h_2 F$ for smooth $h_1 : [0,\infty)^j \to [0,\infty)$, $h_2 : [0,\infty)^k \to [0,\infty)$, $F : \mathbb{R}^{m+n}_{j+k} \to (0,\infty)$, where $h_1(x'_1,\ldots,x'_j) = x'^{a_1}_1 \ldots x'^{a_j}_j$ and $h_2(y'_1,\ldots,y'_k) = y'^{b_1}_1 \ldots y'^{b_k}_k$. Then

$$w' = \Psi_{h_1}(p_{\rm ex}(d'_1),\ldots,p_{\rm ex}(d'_j))\Psi_{h_2}(q_{\rm ex}(e'_1),\ldots,q_{\rm ex}(e'_k))\Psi_F(\underline{p}).$$

As F is positive, it has an inverse G, and then $\Psi_F(\underline{p})$ is invertible with inverse $\Psi_G(\underline{p})$. As $\Phi_i(w')$ is invertible, the both $\Phi_{i\circ h_1}(p_{\rm ex}(d'_1),\ldots,p_{\rm ex}(d'_j))$ and $\Phi_{i\circ h_2}(q_{\rm ex}(e'_1),\ldots,q_{\rm ex}(e'_k))$ must be invertible.

Now $\Phi_{i\circ h_1}(p_{\rm ex}(d'_1),\ldots,p_{\rm ex}(d'_j)) = p(\Phi_{i\circ h_1}(d'_1,\ldots,d'_j))$. As p is a map of local C^∞-rings, then $\Phi_{i\circ h_1}(d'_1,\ldots,d'_j)$ is invertible in $\mathfrak{D}$. As $\boldsymbol{\mathfrak{D}}$ is a local C^∞-ring with corners, this holds if and only if $\Psi_{h_1}(d'_1,\ldots,d'_n)$ is invertible in $\mathfrak{D}_{\rm ex}$. Applying p, then $\Psi_{h_1}(p_{\rm ex}(d'_1),\ldots,p_{\rm ex}(d'_j))$ must be invertible. The same argument applied to $\Phi_{i\circ h_2}(q_{\rm ex}(e'_1),\ldots,q_{\rm ex}(e'_k))$ shows that $\Psi_{h_2}(q_{\rm ex}(e'_1),\ldots,q_{\rm ex}(e'_k))$ is invertible. Therefore w' is a product of three invertible elements and is invertible itself.

Hence w' is invertible in $\mathfrak{F}_{\rm ex}$, and local C^∞-rings with corners are closed under pushouts in $\mathbf{C^\infty Rings^c}$. As pushouts in $\mathbf{C^\infty Rings^c_{in}}$ coincide with pushouts in $\mathbf{C^\infty Rings^c}$, using Theorem 4.3.7(b), then this is also true for interior local C^∞-rings with corners.

As the initial object $(\mathbb{R},[0,\infty))$ is local (and interior), then the category of (interior) local C^∞-rings with corners has all finite colimits, and the construction shows they commute with colimits taken in $\mathbf{C^\infty Rings^c}$.

For a small colimit of local C^∞-rings with corners, again take the colimit in C^∞-rings with corners. The C^∞-ring part is local by Theorem 4.3.9 and Proposition 2.1.11. We can observe that every element in the monoid is again generated by finitely many elements from the monoids in the diagram, and that there must be a unique morphism from the colimit to $\mathbb{R}$. Applying the same proof above shows that elements in the colimit are invertible if and only if their image in $\mathbb{R}$ is non-zero, showing that the colimit is local, as required. If the C^∞-rings with corners in the diagram are also interior, Theorem 4.3.7(b) shows that the colimit is also interior.

As in Proposition 2.1.11, one can construct a right adjoint F to the inclusion of local C^∞-rings with corners into C^∞-rings with corners by taking $F(\boldsymbol{\mathfrak{C}})$ to be the colimit of all local C^∞-rings with corners that have morphisms into $\boldsymbol{\mathfrak{C}}$.

If one takes a diagram of local C^∞-rings with corners in the form of (2.1.3), then the limit in C^∞-rings with corners exists and it is local by the same reasoning as Proposition 2.1.11, so this is the limit in local C^∞-rings with corners. Theorem 4.3.9 shows its underlying C^∞-ring is the limit of the underlying local C^∞-rings. If the diagram is not of the form of (2.1.3) we can add all the morphisms to $\mathbb{R}$ to form a vertex so that it is of the form of (2.1.3) and then take the limit in C^∞-rings with corners. This will be the limit of the original diagram in the category of local C^∞-rings with corners, so all small limits exist in the category of local C^∞-rings with corners, and they are equal to their limits in C^∞-rings with corners when their diagrams are of the form of (2.1.3).

If we take a diagram of interior local C^∞-rings with corners, the same reasoning above shows that the limit exists, and it commutes with limits in interior C^∞-rings with corners only when the diagram has the form of (2.1.3). For example, the product of interior local C^∞-rings with corners $(\mathfrak{C}, \mathfrak{C}_{\rm ex}), (\mathfrak{D}, \mathfrak{D}_{\rm ex})$ will be $(\mathfrak{C} \times_{\mathbb{R}} \mathfrak{D}, \mathfrak{C}_{\rm in} \times_{[0,\infty)} \mathfrak{D}_{\rm in} \amalg \{0\})$. Note that limits of interior C^∞-rings with corners do not in general commute with limits of C^∞-rings with corners due to Theorem 4.3.7(b), so it is also not true that limits of interior local C^∞-rings with corners commute with limits of local C^∞-rings with corners.

From Theorem 4.3.9, the forgetful functor from (interior) local C^∞-rings with corners to local C^∞-rings preserves both limits and colimits; then as in Theorem 4.3.9 the left adjoint and right adjoints constructed in Theorem 4.3.9 are left and right adjoints respectively when restricted to local C^∞-rings. This completes the proof. □

Definition 4.6.4. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a C^∞-ring with corners, and $A \subset \mathfrak{C}$, $A_{\rm ex} \subset \mathfrak{C}_{\rm ex}$ be subsets. A *localisation* $\boldsymbol{\mathfrak{C}}(a^{-1} : a \in A)[a'^{-1} : a' \in A_{\rm ex}]$ of $\boldsymbol{\mathfrak{C}}$ at $(A, A_{\rm ex})$ is a C^∞-ring with corners $\boldsymbol{\mathfrak{D}} = \boldsymbol{\mathfrak{C}}(a^{-1} : a \in A)[a'^{-1} : a' \in A_{\rm ex}]$ and a morphism $\boldsymbol{\pi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ such that $\pi(a)$ is invertible in $\mathfrak{D}$ for all $a \in A$ and $\pi_{\rm ex}(a')$ is invertible in $\mathfrak{D}_{\rm ex}$ for all $a' \in A_{\rm ex}$, with the universal property that if $\boldsymbol{\mathfrak{E}} = (\mathfrak{E}, \mathfrak{E}_{\rm ex})$ is a C^∞-ring with corners and $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{E}}$ a morphism with $\phi(a)$ invertible in $\mathfrak{E}$ for all $a \in A$ and $\phi_{\rm ex}(a')$ invertible in $\mathfrak{E}_{\rm ex}$ for all $a' \in A_{\rm ex}$, then there is a unique morphism $\boldsymbol{\psi} : \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{E}}$ with $\boldsymbol{\phi} = \boldsymbol{\psi} \circ \boldsymbol{\pi}$.

A localisation $\boldsymbol{\mathfrak{C}}(a^{-1} : a \in A)[a'^{-1} : a' \in A_{\rm ex}]$ always exists (as proved in Kalashnikov [51, §4.7] for localisations of pre C^∞-rings with corners), and is unique up to unique isomorphism. In the notation of Definition 4.4.3 we may write

$$\begin{aligned}&\boldsymbol{\mathfrak{C}}(a^{-1} : a \in A)[a'^{-1} : a' \in A_{\rm ex}] = \\ &\big(\boldsymbol{\mathfrak{C}}(x_a : a \in A)[y_{a'} : a' \in A_{\rm ex}]\big)\big/\big(a \cdot x_a = 1 : a \in A\big)\big[a' \cdot y_{a'} = 1 : a' \in A_{\rm ex}\big].\end{aligned}$$

That is, we add an extra generator x_a of type $\mathbb{R}$ and an extra relation $a \cdot x_a = 1$ of type $\mathbb{R}$ for each $a \in A$, so that $x_a = a^{-1}$, and similarly for each $a' \in A_{\rm ex}$.

If $\boldsymbol{\mathfrak{C}}$ is interior and $A_{\rm ex} \subseteq \mathfrak{C}_{\rm in}$ then $\boldsymbol{\mathfrak{C}}(a^{-1} : a \in A)[a'^{-1} : a' \in A_{\rm ex}]$ makes sense and exists in $\mathbf{C^\infty Rings^c_{in}}$ as well as in $\mathbf{C^\infty Rings^c}$, and Theorem 4.3.7(b) implies that the two localisations are the same.

The following lemma will be important in the theory of C^∞-schemes with corners.

Lemma 4.6.5. *Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a C^∞-ring with corners and take $c \in \mathfrak{C}$. Let $\boldsymbol{\mathfrak{C}}(c^{-1}) = (\mathfrak{D}, \mathfrak{D}_{\rm ex})$ be the localisation, then $\mathfrak{D} \cong \mathfrak{C}[c^{-1}]$, the localisation of the C^∞-ring.*

Proof. As localisation is a colimit, this follows directly from Theorem 4.3.9. Explicitly, let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$, c, and $\boldsymbol{\mathfrak{D}} = (\mathfrak{D}, \mathfrak{D}_{\rm ex})$ be as in the statement. Then $\boldsymbol{\mathfrak{D}} = \boldsymbol{\mathfrak{C}}(x_c)/(c \cdot x_c = 1) = (\boldsymbol{\mathfrak{C}} \otimes_\infty \boldsymbol{C^\infty}(\mathbb{R}))/(c \cdot x_c = 1)$, where x_c is the generator of $C^\infty(\mathbb{R})$. Theorem 4.3.9 implies the underlying C^∞-ring of $\boldsymbol{\mathfrak{C}} \otimes_\infty \boldsymbol{C^\infty}(\mathbb{R})$ is equal to $\mathfrak{C} \otimes_\infty C^\infty(\mathbb{R})$. Example 4.4.4(a) shows that the quotient $(\boldsymbol{\mathfrak{C}} \otimes_\infty \boldsymbol{C^\infty}(\mathbb{R}))/(c \cdot x_c = 1)$ must have underlying C^∞-ring, $\mathfrak{C} \otimes_\infty C^\infty(\mathbb{R})/(c \cdot x_c = 1) = \mathfrak{C}[c^{-1}] = \mathfrak{D}$. In fact, we can conclude that $\boldsymbol{\mathfrak{D}} = (\mathfrak{D}, \mathfrak{D}_{\rm ex}) = (\mathfrak{C}[c^{-1}], (\boldsymbol{\mathfrak{C}} \otimes_\infty \boldsymbol{C^\infty}(\mathbb{R}))_{\rm ex}/\sim_I)$ where I is the ideal $(c \cdot x_c = 1)$, using the notation of Example 4.4.4(a). □

Lemma 4.6.6. *If $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ is a (pre) C^∞-ring with corners and $x : \mathfrak{C} \to \mathbb{R}$ a (pre) C^∞-ring morphism, then we have a morphism of (pre) C^∞-rings with corners $(x, x_{\rm ex}) : (\mathfrak{C}, \mathfrak{C}_{\rm ex}) \to (\mathbb{R}, [0, \infty))$, where $x_{\rm ex}(c') = x \circ \Phi_i(c')$ for $c' \in \mathfrak{C}_{\rm ex}$.*

Proof. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ and $x : \mathfrak{C} \to \mathbb{R}$ be as in the statement. Take $c' \in \mathfrak{C}_{\rm ex}$. To show $x_{\rm ex} = x \circ \Phi_i : \mathfrak{C}_{\rm ex} \to [0, \infty)$ is well defined, assume for a contradiction that $x \circ \Phi_i(c') = \epsilon < 0 \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that f is the identity on $[0, \infty)$ and it is zero on $(-\infty, \epsilon/2)$. Then $f \circ i = i$ for $i : [0, \infty) \to \mathbb{R}$ the inclusion. So we have $0 > \epsilon = x \circ \Phi_i(c') = x \circ \Phi_f \circ \Phi_i(c') = f(x \circ \Phi_i(c')) = f(\epsilon) = 0$, and $x_{\rm ex} = x \circ \Phi_i : \mathfrak{C}_{\rm ex} \to [0, \infty)$ is well defined.

For $(x, x_{\rm ex})$ to be a morphism of (pre) C^∞-rings with corners, it must respect the C^∞-operations. For example, let $f : [0, \infty) \to [0, \infty)$ be smooth, then there is a smooth function $g : \mathbb{R} \to \mathbb{R}$ that extends f, so that $g \circ i = i \circ f$. Then $x_{\rm ex}(\Psi_f(c')) = x \circ \Phi_i(\Psi_f(c')) = \Phi_g(x \circ \Phi_i(c')) = \Phi_g(x_{\rm ex}(c'))$ as required. A similar proof holds for the other C^∞-operations. □

Definition 4.6.7. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a C^∞-ring with corners. An $\mathbb{R}$-*point* x of $\boldsymbol{\mathfrak{C}}$ is a C^∞-ring morphism (or equivalently, an $\mathbb{R}$-algebra morphism) $x : \mathfrak{C} \to \mathbb{R}$. Define $\boldsymbol{\mathfrak{C}}_x$ to be the localisation

$$\boldsymbol{\mathfrak{C}}_x = \boldsymbol{\mathfrak{C}}\big(c^{-1} : c \in \mathfrak{C},\ x(c) \neq 0\big)\big[c'^{-1} : c' \in \mathfrak{C}_{\rm ex},\ x \circ \Phi_i(c') \neq 0\big], \tag{4.6.1}$$

with projection $\boldsymbol{\pi}_x : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{C}}_x$. Lemma 4.6.6 shows $x \circ \Phi_i(c') \geqslant 0$ so the localisation is well defined.

If $\boldsymbol{\mathfrak{C}}$ is interior then $\boldsymbol{\mathfrak{C}}_x$ is interior by Definition 4.6.4. Theorem 4.6.8 shows $\boldsymbol{\mathfrak{C}}_x$ is local. Part (c) of Theorem 4.6.8 is the analogue of Proposition 2.1.15. The point of the proof is to give an alternative construction of $\boldsymbol{\mathfrak{C}}_x$ from $\boldsymbol{\mathfrak{C}}$ by imposing relations, but adding no new generators.

Theorem 4.6.8. *Let* $\boldsymbol{\pi}_x : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{C}}_x$ *be as in Definition* 4.6.7. *Then:*

(a) $\boldsymbol{\mathfrak{C}}_x$ *is a local* C^∞*-ring with corners.*

(b) $\boldsymbol{\mathfrak{C}}_x = (\mathfrak{C}_x, \mathfrak{C}_{x,\mathrm{ex}})$ *and* $\boldsymbol{\pi}_x = (\pi_x, \pi_{x,\mathrm{ex}})$, *where* $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$ *is the local* C^∞*-ring associated to* $x : \mathfrak{C} \to \mathbb{R}$ *in Definition* 2.1.13.

(c) $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$ *and* $\pi_{x,\mathrm{ex}} : \mathfrak{C}_{\mathrm{ex}} \to \mathfrak{C}_{x,\mathrm{ex}}$ *are surjective.*

Proof. Proposition 2.1.15 says that the local C^∞-ring $\mathfrak{C}_x$ is $\mathfrak{C}/I$, for $I \subset \mathfrak{C}$ the ideal defined in (2.1.4), with $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$ the projection $\mathfrak{C} \to \mathfrak{C}/I$. Define

$$\mathfrak{D} = \mathfrak{C}_x = \mathfrak{C}/I \quad \text{and} \quad \mathfrak{D}_{\mathrm{ex}} = \mathfrak{C}_{\mathrm{ex}}/\sim,$$

where $\sim$ is the equivalence relation on $\mathfrak{C}_{\mathrm{ex}}$ given by $c' \sim c''$ if there exists $i \in I$ with $c'' = \Psi_{\exp}(i) \cdot c'$, and $\phi : \mathfrak{C} \to \mathfrak{D} = \mathfrak{C}/I$, $\phi_{\mathrm{ex}} : \mathfrak{C}_{\mathrm{ex}} \to \mathfrak{D}_{\mathrm{ex}} = \mathfrak{C}_{\mathrm{ex}}/\sim$ to be the natural surjective projections. Let $f : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ be smooth and $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ be exterior, and write Φ_f, Ψ_g for the operations in $\boldsymbol{\mathfrak{C}}$. Then as for (4.3.2), we can show there exist unique maps Φ'_f, Ψ'_g making the following diagrams commute:

$$\begin{array}{ccccccc} \mathfrak{C}^m \times \mathfrak{C}_{\mathrm{ex}}^n & \xrightarrow{\Phi_f} & \mathfrak{C} & & \mathfrak{C}^m \times \mathfrak{C}_{\mathrm{ex}}^n & \xrightarrow{\Psi_g} & \mathfrak{C}_{\mathrm{ex}} \\ \downarrow{\scriptstyle \phi^m \times \phi_{\mathrm{ex}}^n} & & {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi^m \times \phi_{\mathrm{ex}}^n} & & {\scriptstyle \phi_{\mathrm{ex}}}\downarrow \\ \mathfrak{D}^m \times \mathfrak{D}_{\mathrm{ex}}^n & \xrightarrow{\Phi'_f} & \mathfrak{D}, & & \mathfrak{D}^m \times \mathfrak{D}_{\mathrm{ex}}^n & \xrightarrow{\Psi'_g} & \mathfrak{D}_{\mathrm{ex}}, \end{array} \tag{4.6.2}$$

and these Φ'_f, Ψ'_g make $\boldsymbol{\mathfrak{D}} = (\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ into a C^∞-ring with corners, and $\boldsymbol{\phi} = (\phi, \phi_{\mathrm{ex}}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ into a surjective morphism.

Suppose that $\boldsymbol{\mathfrak{F}}$ is a C^∞-ring with corners and $\boldsymbol{\chi} = (\chi, \chi_{\mathrm{ex}}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{F}}$ a morphism such that $\chi(c)$ is invertible in $\mathfrak{F}$ for all $c \in \mathfrak{C}$ with $x(c) \neq 0$. The definition $\mathfrak{D} = \mathfrak{C}_x = \mathfrak{C}[c^{-1} : c \in \mathfrak{C},\ x(c) \neq 0]$ in $\mathbf{C^\infty Rings}$ in Definition 2.1.13 implies that $\chi : \mathfrak{C} \to \mathfrak{F}$ factorises uniquely as $\chi = \xi \circ \phi$ for $\xi : \mathfrak{D} \to \mathfrak{F}$ a morphism in $\mathbf{C^\infty Rings}$. Hence $\chi(i) = 0$ in $\mathfrak{F}$ for all $i \in I$, so $\chi_{\mathrm{ex}}(\Psi_{\exp}(i)) = 1_{\mathfrak{F}_{\mathrm{ex}}}$ for all $i \in I$. Thus if $c', c'' \in \mathfrak{C}_{\mathrm{ex}}$ with $c'' = \Psi_{\exp}(i) \cdot c'$ for $i \in I$ then $\chi_{\mathrm{ex}}(c') = \chi_{\mathrm{ex}}(c'')$. Hence χ_{ex} factorises uniquely as $\chi_{\mathrm{ex}} = \xi_{\mathrm{ex}} \circ \phi_{\mathrm{ex}}$ for $\xi_{\mathrm{ex}} : \mathfrak{D}_{\mathrm{ex}} \to \mathfrak{F}_{\mathrm{ex}}$.

As $\boldsymbol{\chi}, \boldsymbol{\phi}$ are morphisms in $\mathbf{C^\infty Rings^c}$ with $\boldsymbol{\phi}$ surjective we see that $\boldsymbol{\xi} = (\xi, \xi_{\mathrm{ex}}) : \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{F}}$ is a morphism. Therefore $\boldsymbol{\chi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{F}}$ factorises uniquely as $\boldsymbol{\chi} = \boldsymbol{\xi} \circ \boldsymbol{\phi}$. Also $\phi(c)$ is invertible in $\mathfrak{D} = \mathfrak{C}_x$ for all $c \in \mathfrak{C}$ with $x(c) \neq 0$, by definition of $\mathfrak{C}_x$ in Definition 2.1.13. Therefore we have a canonical isomorphism

$$\boldsymbol{\mathfrak{D}} \cong \boldsymbol{\mathfrak{C}}\big(c^{-1} : c \in \mathfrak{C},\ x(c) \neq 0\big)$$

identifying $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ with the projection $\boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{C}}\big(c^{-1} : c \in \mathfrak{C},\ x(c) \neq 0\big)$. Note that $x : \mathfrak{C} \to \mathbb{R}$ factorises as $x = \tilde{x} \circ \phi$ for a unique morphism $\tilde{x} : \mathfrak{D} \to \mathbb{R}$.

Next define

$$\mathfrak{E} = \mathfrak{D} = \mathfrak{C}_x \quad \text{and} \quad \mathfrak{E}_{\rm ex} = \mathfrak{D}_{\rm ex}/\approx,$$

where $\approx$ is the monoidal equivalence relation on $\mathfrak{D}_{\rm ex}$ generated by the conditions that $d' \approx d''$ whenever $d', d'' \in \mathfrak{D}_{\rm ex}$ with $\Phi'_i(d') = \Phi'_i(d'')$ in $\mathfrak{D}$ and $\tilde{x} \circ \Phi'_i(d') \neq 0$. Write $\psi = \mathrm{id} : \mathfrak{D} \to \mathfrak{E}$, and let $\psi_{\rm ex} : \mathfrak{D}_{\rm ex} \to \mathfrak{E}_{\rm ex}$ be the natural surjective projection. Suppose $f : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ is smooth and $g : \mathbb{R}^m \times [0,\infty)^n \to [0,\infty)$ is exterior. Then as for (4.6.2), we claim there are unique maps Φ''_f, Ψ''_g making the following diagrams commute:

$$\begin{array}{ccc} \mathfrak{D}^m \times \mathfrak{D}^n_{\rm ex} & \xrightarrow{\Phi'_f} & \mathfrak{D} \\ \downarrow{\scriptstyle \psi^m \times \psi^n_{\rm ex}} & & \downarrow{\scriptstyle \psi} \\ \mathfrak{E}^m \times \mathfrak{E}^n_{\rm ex} & \xrightarrow{\Phi''_f} & \mathfrak{E}, \end{array} \qquad \begin{array}{ccc} \mathfrak{D}^m \times \mathfrak{D}^n_{\rm ex} & \xrightarrow{\Psi'_g} & \mathfrak{D}_{\rm ex} \\ \downarrow{\scriptstyle \psi^m \times \psi^n_{\rm ex}} & & \downarrow{\scriptstyle \psi_{\rm ex}} \\ \mathfrak{E}^m \times \mathfrak{E}^n_{\rm ex} & \xrightarrow{\Psi''_g} & \mathfrak{E}_{\rm ex}. \end{array} \tag{4.6.3}$$

To see that Φ''_f in (4.6.3) is well defined, note that as $\Phi'_i : \mathfrak{D}_{\rm ex} \to \mathfrak{D}$ is a monoid morphism and $\approx$ is a monoidal equivalence relation generated by $d' \approx d''$ when $\Phi'_i(d') = \Phi'_i(d'')$, we have a factorisation $\Phi'_i = \tilde{\Phi}'_i \circ \psi_{\rm ex}$. We may extend f to smooth $\tilde{f} : \mathbb{R}^{m+n} \to \mathbb{R}$, and then Φ'_f factorises as

$$\begin{array}{ccc} \mathfrak{D}^m \times \mathfrak{D}^n_{\rm ex} & \xrightarrow{\qquad\Phi'_f\qquad} & \mathfrak{D}. \\ & {\scriptstyle \mathrm{id}^m_{\mathfrak{D}} \times (\Phi'_i)^n} \searrow \quad \mathfrak{D}^{m+n} \quad \nearrow {\scriptstyle \Phi'_{\tilde{f}}} & \end{array}$$

Using $\Phi'_i = \tilde{\Phi}'_i \circ \psi_{\rm ex}$, we see that Φ''_f in (4.6.3) exists and is unique.

For Ψ''_g, if $g = 0$ then $\Psi''_g = 0_{\mathfrak{E}_{\rm ex}} = [0_{\mathfrak{D}_{\rm ex}}]$ in (4.6.3). Otherwise we may write g using $a_1, \ldots, a_n$ and $h : \mathbb{R}^m \times [0,\infty)^n \to \mathbb{R}$ as in (4.2.4), and then

$$\Psi_{g'}(d_1, \ldots, d_m, d'_1, \ldots, d'_n) = (d'_1)^{a_1} \cdots (d'_n)^{a_n} \cdot \Psi'_{\exp}\big[\Phi'_h(d_1, \ldots, d_m, d'_1, \ldots, d'_n)\big].$$

Since $\psi_{\rm ex} : \mathfrak{D}_{\rm ex} \to \mathfrak{E}_{\rm ex}$ is a monoid morphism, as it is a quotient by a monoidal equivalence relation, we see from this and the previous argument applied to $\Phi_h(d_1, \ldots, d_m, d'_1, \ldots, d'_n)$ that Ψ''_g in (4.6.3) exists and is unique. These Φ''_f, Ψ''_g make $\boldsymbol{\mathfrak{E}} = (\mathfrak{E}, \mathfrak{E}_{\rm ex})$ into a C^∞-ring with corners, and $\boldsymbol{\psi} = (\psi, \psi_{\rm ex}) : \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{E}}$ into a surjective morphism.

We will show that there is a canonical isomorphism $\boldsymbol{\mathfrak{E}} \cong \boldsymbol{\mathfrak{C}}_x$ which identifies $\boldsymbol{\psi} \circ \boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{E}}$ with $\boldsymbol{\pi}_x : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{C}}_x$. Firstly, suppose $c \in \mathfrak{C}$ with $x(c) \neq 0$. Then $\psi \circ \phi(c)$ is invertible in $\mathfrak{E} = \mathfrak{C}_x$ by definition of $\mathfrak{C}_x$ in Definition 2.1.13. Secondly, suppose $c' \in \mathfrak{C}_{\rm ex}$ with $x \circ \Phi_i(c') \neq 0$. Set $d' = \phi_{\rm ex}(c')$. Then $\Phi'_i(d') = \phi \circ \Phi_i(c')$ is invertible in $\mathfrak{D}$. Now in the proof of Proposition 4.3.1(b), we do not actually need c' to be invertible in $\mathfrak{C}_{\rm ex}$, it is enough that $\Phi_i(c')$ is invertible in $\mathfrak{C}$. Thus this proof shows that there exists a unique $d \in \mathfrak{D}$ with $\Phi'_i(d') = \Phi'_{\exp}(d) = \Phi'_i \circ \Psi'_{\exp}(d)$. But then $d' \approx \Psi'_{\exp}(d)$, so $\psi_{\rm ex}(d') = \psi_{\rm ex} \circ \Psi'_{\exp}(d) = \Psi''_{\exp}(\psi(d))$. Hence $\psi_{\rm ex} \circ \phi_{\rm ex}(c') = \psi_{\rm ex}(d')$ is invertible in $\mathfrak{E}_{\rm ex}$, with inverse $\Psi''_{\exp}(-\psi(d))$.

Thirdly, suppose that $\boldsymbol{\mathfrak{G}}$ is a C^∞-ring with corners and $\boldsymbol{\zeta} = (\zeta, \zeta_{\rm ex}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{G}}$ a morphism such that $\zeta(c)$ is invertible in $\mathfrak{G}$ for all $c \in \mathfrak{C}$ with $x(c) \neq 0$ and $\zeta_{\rm ex}(c')$ is invertible in $\mathfrak{G}_{\rm ex}$ for all $c' \in \mathfrak{C}_{\rm ex}$ with $x \circ \Phi_i(c') \neq 0$. Then $\boldsymbol{\zeta} = \boldsymbol{\eta} \circ \boldsymbol{\phi}$ for a unique $\boldsymbol{\eta} : \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{G}}$, by the universal property of $\boldsymbol{\mathfrak{D}}$. Since $\phi_{\rm ex} : \mathfrak{C}_{\rm ex} \to \mathfrak{D}_{\rm ex}$ is surjective and $x = \tilde{x} \circ \phi$ we see that $\eta_{\rm ex}(d')$ is invertible in $\mathfrak{G}_{\rm ex}$ for all $d' \in \mathfrak{D}_{\rm ex}$ with $\tilde{x} \circ \Phi'_i(d') \neq 0$.

Let $d', d'' \in \mathfrak{D}_{\rm ex}$ with $\Phi'_i(d') = \Phi'_i(d'')$ in $\mathfrak{D}$ and $\tilde{x} \circ \Phi'_i(d') \neq 0$, so that $d' \approx d''$. Then $\eta_{\rm ex}(d'), \eta_{\rm ex}(d'')$ are invertible in $\mathfrak{G}_{\rm ex}$ with $\Phi_i \circ \eta_{\rm ex}(d') = \Phi_i \circ \eta_{\rm ex}(d'')$ in $\mathfrak{G}$, so Definition 4.3.2(i) for $\boldsymbol{\mathfrak{G}}$ implies that $\eta_{\rm ex}(d') = \eta_{\rm ex}(d'')$. Since $\eta_{\rm ex} : \mathfrak{D}_{\rm ex} \to \mathfrak{G}_{\rm ex}$ is a monoid morphism, and $\approx$ is a monoidal equivalence relation, and $\eta_{\rm ex}(d') = \eta_{\rm ex}(d'')$ for the generating relations $d' \approx d''$, we see that $\eta_{\rm ex}$ factorises via $\mathfrak{D}_{\rm ex}/\approx$. Thus there exists unique $\theta_{\rm ex} : \mathfrak{E}_{\rm ex} \to \mathfrak{F}_{\rm ex}$ with $\eta_{\rm ex} = \theta_{\rm ex} \circ \psi_{\rm ex}$. Set $\theta = \eta : \mathfrak{E} = \mathfrak{D} \to \mathfrak{G}$. Then $\eta = \theta \circ \psi$ as $\psi = {\rm id}_{\mathfrak{D}}$. As $\boldsymbol{\psi}$ is surjective we see that $\boldsymbol{\theta} = (\theta, \theta_{\rm ex}) : \boldsymbol{\mathfrak{E}} \to \boldsymbol{\mathfrak{F}}$ is a morphism in $\mathbf{C^\infty Rings^c}$, with $\boldsymbol{\eta} = \boldsymbol{\theta} \circ \boldsymbol{\psi}$, so that $\boldsymbol{\zeta} = \boldsymbol{\theta} \circ \boldsymbol{\psi} \circ \boldsymbol{\phi}$.

This proves that $\boldsymbol{\psi} \circ \boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{E}}$ satisfies the universal property of $\boldsymbol{\pi}_x : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{C}}_x$ from the localisation (4.6.1), so $\boldsymbol{\mathfrak{E}} \cong \boldsymbol{\mathfrak{C}}_x$ as we claimed. Parts (b),(c) of the theorem are now immediate, as $\boldsymbol{\mathfrak{E}} = \boldsymbol{\mathfrak{C}}_x$ and $\boldsymbol{\phi}, \boldsymbol{\psi}$ are surjective. For (a), observe that $x : \mathfrak{C} \to \mathbb{R}$ factorises as $\pi \circ \pi_x$ for $\pi : \mathfrak{C}_x \to \mathbb{R}$ a morphism. If $\bar{c} \in \mathfrak{C}_x$ with $\pi(\bar{c}) \neq 0$ then as $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$ is surjective by (c) we have $\bar{c} = \pi_x(c)$ with $x(c) \neq 0$, so $\bar{c} = \pi_x(c)$ is invertible in $\mathfrak{C}_x$ by (4.6.1). Similarly, if $\bar{c}' \in \mathfrak{C}_{x,{\rm ex}}$ with $\pi \circ \Phi_i(\bar{c}') \neq 0$ then as $\pi_{x,{\rm ex}} : \mathfrak{C}_{\rm ex} \to \mathfrak{C}_{x,{\rm ex}}$ is surjective we find that $\bar{c}'$ is invertible in $\mathfrak{C}_{x,{\rm ex}}$. Hence $\boldsymbol{\mathfrak{C}}_x$ is a local C^∞-ring with corners. $\square$

Note that the first equivalence relation in this proof is an example of the quotient from Example 4.4.4(a), which enforces the correct invertibility condition ($c \in \mathfrak{C}_x$ is invertible if and only if $x(c) \neq 0$) on the C^∞-ring. The second equivalence relation enforces the correct invertibility condition ($c' \in \mathfrak{C}_{x,{\rm ex}}$ is invertible if and only if $x \circ \Phi_i(c') \neq 0$) on the monoid. We can characterise the equivalence relations that define $\mathfrak{C}_{x,{\rm ex}} = \mathfrak{E}_{\rm ex}$ in the above proof using the following lemma.

Lemma 4.6.9. *Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a C^∞-ring with corners and $x : \mathfrak{C} \to \mathbb{R}$ an $\mathbb{R}$-point of $\mathfrak{C}$. Let $\boldsymbol{\pi}_x : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{C}}_x$ be as in Definition* 4.6.7*, and I the ideal defined in* (2.1.4)*. For any $c'_1, c'_2 \in \mathfrak{C}_{\rm ex}$, then $\pi_{x,{\rm ex}}(c'_1) = \pi_{x,{\rm ex}}(c'_2)$ if and only if there are elements $a', b' \in \mathfrak{C}_{\rm ex}$ such that $\Phi_i(a') - \Phi_i(b') \in I$, $x \circ \Phi_i(a') \neq 0$ and $a'c'_1 = b'c'_2$. Hence $\boldsymbol{\mathfrak{C}}_x = (\mathfrak{C}/I, \mathfrak{C}_{\rm ex}/{\sim})$ where $c'_1 \sim c'_2 \in \mathfrak{C}_{\rm ex}$ if and only if there are elements $a', b' \in \mathfrak{C}_{\rm ex}$ such that $\Phi_i(a') - \Phi_i(b') \in I$, $x \circ \Phi_i(a') \neq 0$ and $a'c'_1 = b'c'_2$.*

Proof. We first show that if $\pi_{x,\mathrm{ex}}(c_1') = \pi_{x,\mathrm{ex}}(c_2')$, then there are a', b' satisfying the conditions.

In Theorem 4.6.8, we constructed C^∞-rings with corners $\boldsymbol{\mathfrak{D}} = (\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ and $\boldsymbol{\mathfrak{E}} = (\mathfrak{E}, \mathfrak{E}_{\mathrm{ex}})$ and surjective morphisms $\boldsymbol{\phi} = (\phi, \phi_{\mathrm{ex}}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$, $\boldsymbol{\psi} = (\psi, \psi_{\mathrm{ex}}) : \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{E}}$, where

$$\mathfrak{E} = \mathfrak{D} = \mathfrak{C}_x = \mathfrak{C}/I, \quad \mathfrak{D}_{\mathrm{ex}} = \mathfrak{C}_{\mathrm{ex}}/\sim \quad \text{and} \quad \mathfrak{E}_{\mathrm{ex}} = \mathfrak{D}_{\mathrm{ex}}/\approx,$$

and $I \subset \mathfrak{C}$ is the ideal defined in (2.1.4), and $\sim, \approx$ are explicit equivalence relations. Then we showed that there is a unique isomorphism $\boldsymbol{\mathfrak{E}} \cong \boldsymbol{\mathfrak{C}}_x$ identifying $\boldsymbol{\psi} \circ \boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{E}}$ with $\boldsymbol{\pi}_x : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{C}}_x$. As $\pi_{x,\mathrm{ex}}(c_1') = \pi_{x,\mathrm{ex}}(c_2')$ we have $\psi_{\mathrm{ex}} \circ \phi_{\mathrm{ex}}(c_1') = \psi_{\mathrm{ex}} \circ \phi_{\mathrm{ex}}(c_2')$ in $\mathfrak{E}_{\mathrm{ex}}$. Thus $\phi_{\mathrm{ex}}(c_1') \approx \phi_{\mathrm{ex}}(c_2')$.

By definition $\approx$ is the monoidal equivalence relation on $\mathfrak{D}_{\mathrm{ex}}$ generated by the condition that $d' \approx d''$ whenever $d', d'' \in \mathfrak{D}_{\mathrm{ex}}$ with $\Phi_i'(d') = \Phi_i'(d'')$ in $\mathfrak{D}$ and $\tilde{x} \circ \Phi_i'(d') \neq 0$, where $\Phi_i' : \mathfrak{D}_{\mathrm{ex}} \to \mathfrak{D}$ is the C^∞-ring operation from the inclusion $i : [0, \infty) \hookrightarrow \mathbb{R}$, and $x : \mathfrak{C} \to \mathbb{R}$ factorises as $x = \tilde{x} \circ \phi$ for a unique morphism $\tilde{x} : \mathfrak{D} \to \mathbb{R}$. Hence $\phi_{\mathrm{ex}}(c_1') \approx \phi_{\mathrm{ex}}(c_2')$ means that there is a finite sequence $\phi_{\mathrm{ex}}(c_1') = d_0', d_1', d_2', \ldots, d_{n-1}', d_n' = \phi_{\mathrm{ex}}(c_2')$ in $\mathfrak{D}_{\mathrm{ex}}$, and elements $e_i', f_i', g_i' \in \mathfrak{D}_{\mathrm{ex}}$ such that $\Phi_i'(e_i') = \Phi_i'(f_i')$, $\tilde{x} \circ \Phi_i'(e_i') \neq 0$, and $d_{i-1}' = e_i' \cdot g_i'$, $d_i' = f_i' \cdot g_i'$ in $\mathfrak{D}_{\mathrm{ex}}$ for $i = 1, \ldots, n$.

As $\phi_{\mathrm{ex}} : \mathfrak{C}_{\mathrm{ex}} \to \mathfrak{D}_{\mathrm{ex}}$ is surjective we can choose $e_i, f_i, g_i \in \mathfrak{C}_{\mathrm{ex}}$ with $e_i' = \phi_{\mathrm{ex}}(e_i)$, $f_i' = \phi_{\mathrm{ex}}(f_i)$, $g_i' = \phi_{\mathrm{ex}}(g_i)$ for $i = 1, \ldots, n$. Then the conditions become

$$\begin{gathered} \Phi_i(e_i) - \Phi_i(f_i) \in I, \quad x \circ \Phi_i(e_i) \neq 0 \in \mathbb{R}, \quad i = 1, \ldots, n, \\ c_1' \sim e_1 \cdot g_1, \quad e_{i+1} \cdot g_{i+1} \sim f_i \cdot g_i, \quad i = 1, \ldots, n-1, \quad c_2' \sim f_n \cdot g_n, \end{gathered} \tag{4.6.4}$$

since equality in $\mathfrak{D}_{\mathrm{ex}}$ lifts to $\sim$-equivalence in $\mathfrak{C}_{\mathrm{ex}}$. By definition of $\sim$, this means that there exist elements $h_0, h_1, \ldots, h_n$ in the ideal $I \subset \mathfrak{C}$ in (2.1.4) such that

$$\begin{gathered} c_1' = \Psi_{\exp}(h_0) \cdot e_1 \cdot g_1, \quad c_2' = \Psi_{\exp}(h_n) \cdot f_n \cdot g_n, \\ \text{and} \quad e_{i+1} \cdot g_{i+1} = \Psi_{\exp}(h_i) \cdot f_i \cdot g_i, \quad i = 1, \ldots, n-1. \end{gathered} \tag{4.6.5}$$

In fact, for any element $h \in I$, then the conditions $\Phi_i(e_i) - \Phi_i(f_i) \in I$ and $x \circ \Phi_i(e_i) \neq 0 \in \mathbb{R}$ hold if and only if $\Phi_i(\Psi_{\exp}(h)e_i) - \Phi_i(f_i) \in I$ and $x \circ \Phi_i(\Psi_{\exp}(h)e_i) \neq 0 \in \mathbb{R}$ hold. So we can remove the h_i in (4.6.5). We have that $\pi_{x,\mathrm{ex}}(c_1') = \pi_{x,\mathrm{ex}}(c_2')$ if and only if there are elements $e_i, f_i, g_i \in \mathfrak{C}_{\mathrm{ex}}$ such that

$$\begin{gathered} c_1' = e_1 \cdot g_1, \quad c_2' = f_n \cdot g_n, \\ e_{i+1} \cdot g_{i+1} = f_i \cdot g_i, \quad i = 1, \ldots, n-1, \\ \text{and} \quad x \circ \Phi_i(e_i) \neq 0 \in \mathbb{R}, \quad i = 1, \ldots, n. \end{gathered} \tag{4.6.6}$$

We define $a' = f_1 \cdot f_2 \cdot \ldots \cdot f_n$ and $b' = e_1 \cdot e_2 \cdot \ldots \cdot e_n$. Then using (4.6.6), we see that $a'c'_1 = b'c'_2$, $\Phi_i(a') - \Phi_i(b') \in I$ and $x \circ \Phi_i(a') \neq 0$ as required.

For the reverse argument, say we have $a', b' \in \mathfrak{C}_{\rm ex}$ with $\Phi_i(a') - \Phi_i(b') \in I$ and $x \circ \Phi_i(a') \neq 0$. Let $n = 1$, $e_1 = a'$, $g_1 = 0$ and $f_1 = b'$ in (4.6.5) then we see that $\pi_{x,\rm ex}(a') = \pi_{x,\rm ex}(b')$. As $x \circ \Phi_i(a') \neq 0$, then $\pi_{x,\rm ex}(a')$ is invertible in $\mathfrak{C}_{\rm ex}$. If we also have that $a'c'_1 = b'c'_2$, then as $\pi_{x,\rm ex}$ is a morphism, $\pi_{x,\rm ex}(c'_1) = \pi_{x,\rm ex}(c'_2)$ and the result follows. □

This lemma is important as it characterises localising at a point in the monoid as a global condition, that is we have a global equality $a'c'_1 = b'c'_2$. In the C^∞-ring, this condition can be made local as a', b' can be described using bump functions, however bump functions do not exist in the monoid in general. This creates interesting issues for C^∞-schemes with corners, discussed in §5.3.

Remark 4.6.10. Let $(\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a C^∞-ring with $a', b' \in \mathfrak{C}_{\rm ex}$, and, such that for an $\mathbb{R}$-point $x : \mathfrak{C} \to \mathbb{R}$, we have $x \circ \Phi_i(a') \neq 0$ and $\Phi_i(a') - \Phi_i(b') \in I$. Then if we take $c' = a'$ and $d' = b'$ in Lemma 4.6.9 we have $\pi_{x,\rm ex}(a') = \pi_{x,\rm ex}(b')$. That is, invertible elements in $\mathfrak{C}_{\rm ex}$ are equal in the stalk $\mathfrak{C}_{x,\rm ex}$ whenever their images under Φ_i are equal in $\mathfrak{C}_x$. This is also a consequence of the definition of C^∞-rings with corners, as Φ_i must be injective on invertible elements of $\mathfrak{C}_{\rm ex}$, and $\mathfrak{C}_{x,\rm ex}$.

Also, say $c' \in \mathfrak{C}_{\rm ex}$ such that $\pi_{x,\rm ex}(c') = 0$. Then Lemma 4.6.9 implies there must be an $a' \in \mathfrak{C}_{\rm ex}$ such that $x \circ \Phi_i(a') \neq 0$, with $a'c' = 0$. As $x \circ \Phi_i(a') \neq 0$ then $a' \neq 0$. If $c' \neq 0$, then a' and c' must be a zero divisors.

Example 4.6.11. Let X be a manifold with corners (or with g-corners), and $x \in X$. Define a C^∞-ring with corners $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ such that $\mathfrak{C}$ is the set of germs at x of smooth functions $c : X \to \mathbb{R}$, and $\mathfrak{C}_{\rm ex}$ is the set of germs at x of exterior functions $c' : X \to [0, \infty)$.

That is, elements of $\mathfrak{C}$ are $\sim$-equivalence classes $[U, c]$ of pairs (U, c), where U is an open neighbourhood of x in X and $c : U \to \mathbb{R}$ is smooth, and $(U, c) \sim (\tilde{U}, \tilde{c})$ if there exists an open neighbourhood $\hat{U}$ of x in $U \cap \tilde{U}$ with $c|_{\hat{U}} = \tilde{c}|_{\hat{U}}$. Similarly, elements of $\mathfrak{C}_{\rm ex}$ are equivalence classes $[U, c']$, where U is an open neighbourhood of x in X and $c' : U \to [0, \infty)$ is exterior. The C^∞-ring operations Φ_f, Ψ_g are defined as in (4.2.2)–(4.2.3), but for germs.

As the set of germs only depends on the local behaviour, then the set of germs at x of exterior functions is equal to the set of germs at x of interior functions and the zero function. In particular, for an exterior function defined locally on an open set U of x, we can shrink the open set around x until the function is either interior or zero in that set. This implies that $\boldsymbol{\mathfrak{C}}$ is an interior C^∞-ring with corners.

There is a morphism $\pi : \mathfrak{C} \to \mathbb{R}$ mapping $\pi : [U, c] \mapsto c(x)$. If $\pi([U, c]) \neq 0$ then $[U, c]$ is invertible in $\mathfrak{C}$, and if $\pi \circ \Phi_i([U, c']) = c'(x) \neq 0$ then $[U, c']$ is invertible in $\mathfrak{C}_{\rm ex}$. Hence $\boldsymbol{\mathfrak{C}}$ is a local C^∞-ring with corners. Write $\boldsymbol{C}^\infty_x(X) = \boldsymbol{\mathfrak{C}}$. Then $\boldsymbol{C}^\infty_x(X)$ depends only on an arbitrarily small neighbourhood of x in X, so if X has corners then $\boldsymbol{C}^\infty_x(X) \cong \boldsymbol{C}^\infty_0(\mathbb{R}^n_k)$ for $n = \dim X$ and $0 \leqslant k \leqslant n$.

In Remark 4.2.3 we noted that the C^∞-ring with corners $\boldsymbol{C}^\infty(X)$ in Example 4.2.2 captures the geometric structure of X more faithfully if X is a manifold with faces. If X has faces, then a smooth, exterior or interior function that is defined locally around a point can be extended to a smooth, exterior or interior function defined globally. In particular, for an exterior function defined locally on an open set U, we can shrink the open set until the function is either interior or zero, and then extend it to an interior function or the zero function respectively.

We conclude that if X has faces then

$$\boldsymbol{C}^\infty_x(X) \cong \big(\boldsymbol{C}^\infty(X)\big)_{x_*} \cong \big(\boldsymbol{C}^\infty_{\rm in}(X)\big)_{x_*} \text{ for all } x \in X.$$

As in Definition 4.6.7, here $(\boldsymbol{C}^\infty(X))_{x_*}$ is the localisation of $\boldsymbol{C}^\infty(X)$ at $x_* : C^\infty(X) \to \mathbb{R}$ where for $f \in C^\infty(X)$ then $x^*(f) = f(x)$, and $(\boldsymbol{C}^\infty_{\rm in}(X))$ is the localisation at x_* of $\boldsymbol{C}^\infty_{\rm in}(X)$, the interior C^∞-ring with corners of Example 4.2.7.

However, if X does not have faces then there is a point $x \in X$ such that $(\boldsymbol{C}^\infty(X))_{x_*} \subsetneq \boldsymbol{C}^\infty_x(X)$. That is, there are elements of the exterior germs in $\boldsymbol{C}^\infty_x(X)$ that correspond to exterior maps defined locally, but cannot be extended to global exterior maps. Again recall the example of the teardrop T in Example 3.3.4. As in Remark 4.2.3, we noted that exterior maps locally modelled near $(0, 0)$ on $y_1^a y_2^b : [0, \infty)^2 \to [0, \infty)$ when $a \neq b$ cannot be extended to the entire teardrop. In this case, $(\boldsymbol{C}^\infty(T))_{x_*} \subsetneq \boldsymbol{C}^\infty_x(T)$ where $x_* : C^\infty(T) \to \mathbb{R}$ is the C^∞-ring morphism which evaluates the function at $x = (0, 0)$.

4.7 Modules, and (b-)cotangent modules

In §2.2 we discussed modules over C^∞-rings. Here is the corners analogue:

Definition 4.7.1. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a C^∞-ring with corners. A *module M over* $\boldsymbol{\mathfrak{C}}$, or $\boldsymbol{\mathfrak{C}}$-*module*, is a module over $\mathfrak{C}$ regarded as a commutative $\mathbb{R}$-algebra as in Definition 2.2.1, and morphisms of $\boldsymbol{\mathfrak{C}}$-modules are morphisms of $\mathbb{R}$-algebra modules. Then $\boldsymbol{\mathfrak{C}}$-modules form an abelian category, which we write as $\boldsymbol{\mathfrak{C}}$-mod.

The basic theory of §2.2 extends trivially to the corners case. So if $\boldsymbol{\phi} = (\phi, \phi_{\rm ex}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ is a morphism in $\mathbf{C^\infty Rings^c}$ then using $\phi : \mathfrak{C} \to \mathfrak{D}$ we get functors $\boldsymbol{\phi}_* : \boldsymbol{\mathfrak{C}}$-mod $\to \boldsymbol{\mathfrak{D}}$-mod mapping $M \mapsto M \otimes_{\mathfrak{C}} \mathfrak{D}$, and $\boldsymbol{\phi}^* : \boldsymbol{\mathfrak{D}}$-mod $\to \boldsymbol{\mathfrak{C}}$-mod mapping $N \mapsto N$.

One might expect that modules over $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ should also include some kind of monoid module over $\mathfrak{C}_{\rm ex}$, but we do not do this.

Example 4.7.2. Let X be a manifold with corners (or g-corners) and $E \to X$ be a vector bundle, and write $\Gamma^\infty(E)$ for the vector space of smooth sections e of E. This is a module over $C^\infty(X)$, and hence over both the C^∞-ring with corners $\boldsymbol{C^\infty}(X)$ from Example 4.2.2, and also over the interior C^∞-ring with corners $\boldsymbol{C^\infty_{\rm in}}(X)$ from Example 4.2.7.

Section 2.2 studied cotangent modules of C^∞-rings, the analogues of (co)tangent bundles of manifolds. As in §3.4 manifolds with corners X have (co)tangent bundles TX, T^*X which are functorial over smooth maps, and b-(co)tangent bundles ${}^bTX, {}^bT^*X$, which are functorial only over interior maps. In a similar way, for a C^∞-ring with corners $\boldsymbol{\mathfrak{C}}$ we will define the *cotangent module* $\Omega_{\boldsymbol{\mathfrak{C}}}$, and if $\boldsymbol{\mathfrak{C}}$ is interior we will also define the *b-cotangent module* ${}^b\Omega_{\boldsymbol{\mathfrak{C}}}$.

Definition 4.7.3. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a C^∞-ring with corners. Define the *cotangent module* $\Omega_{\boldsymbol{\mathfrak{C}}}$ of $\boldsymbol{\mathfrak{C}}$ to be the cotangent module $\Omega_{\mathfrak{C}}$ of §2.2, regarded as a $\boldsymbol{\mathfrak{C}}$-module. If $\boldsymbol{\phi} = (\phi, \phi_{\rm ex}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ is a morphism in $\mathbf{C^\infty Rings^c}$ then from Ω_ϕ in §2.2 we get functorial morphisms $\Omega_{\boldsymbol{\phi}} : \Omega_{\boldsymbol{\mathfrak{C}}} \to \Omega_{\boldsymbol{\mathfrak{D}}} = \boldsymbol{\phi}^*(\Omega_{\boldsymbol{\mathfrak{D}}})$ in $\boldsymbol{\mathfrak{C}}$-mod and $(\Omega_\phi)_* : \boldsymbol{\phi}_*(\Omega_{\boldsymbol{\mathfrak{C}}}) = \Omega_{\boldsymbol{\mathfrak{C}}} \otimes_{\mathfrak{C}} \mathfrak{D} \to \Omega_{\boldsymbol{\mathfrak{D}}}$ in $\boldsymbol{\mathfrak{D}}$-mod.

Example 4.7.4. Let X be a manifold with corners. Then $\Gamma^\infty(T^*X)$ is a module over the $\mathbb{R}$-algebra $C^\infty(X)$, and so over the C^∞-rings with corners $\boldsymbol{C^\infty}(X)$ and $\boldsymbol{C^\infty_{\rm in}}(X)$ from Examples 4.2.2 and 4.2.7. The exterior derivative $\mathrm{d} : C^\infty(X) \to \Gamma^\infty(T^*X)$ is a C^∞-derivation, and there is a unique morphism $\lambda : \Omega_{\boldsymbol{C^\infty}(X)} \to \Gamma^\infty(T^*X)$ such that $\mathrm{d} = \lambda \circ \mathrm{d}$.

Proposition 4.7.5. *Let X be a manifold with corners. Then $\Gamma^\infty(T^*X)$ is the cotangent module of $C^\infty(X)$. That is, λ from Example 4.7.4 is an isomorphism.*

Proof. To show that $\Gamma^\infty(T^*X)$ is the cotangent module of both $\boldsymbol{C^\infty}(X)$ and $\boldsymbol{C^\infty_{\rm in}}(X)$, we will first show that λ is surjective by exhibiting a finite set of globally generating sections of $\Gamma^\infty(T^*X)$ of the form $\mathrm{d}c$ for some $c \in C^\infty(X)$. By Melrose [68, Prop, 1.14.1], any manifold with corners can be embedded into a manifold without boundary. As every manifold without boundary admits a finite atlas (with charts having possibly disconnected, uncontractable open sets, see for example Greub et al. [31, p. 20–21]), we can take coordinate functions $x_1^i, \ldots, x_n^i$ for each coordinate patch U_i, $i = 1, \ldots, k$. On U_i, the $\mathrm{d}x_1^i, \ldots, \mathrm{d}x_n^i$ span the cotangent bundle restricted to this neighbourhood.

By Melrose [68, Lem. 1.6.1], we can take partitions of unity for manifolds with corners, so we can take a partition of unity $\{\rho_i\}_{i=1,\ldots,k}$ subordinate to the open cover $\{U_i\}_{i=1,\ldots,k}$.

Rescaling the ρ_i, we can assume that $\sum_{i=1}^{k} \rho_j^2 = 1$. Then we can extend the coordinate functions to the entire manifold by taking $\hat{x}_j^i = \rho_i x_j^i \in C^\infty(X)$ for each $i = 1, \ldots, k$, $j = 1, \ldots, n$. Using the local relation

$$\rho_i \mathrm{d}(\rho_i x_j^i) = \rho_i^2 \mathrm{d}x_j^i + \rho_i x_j^i \mathrm{d}\rho_i$$

we deduce that

$$\rho_i^2 \mathrm{d}x_j^i = \rho_i \mathrm{d}(\rho_i x_j^i) - \rho_i x_j^i \mathrm{d}\rho_i = \rho_i \mathrm{d}\hat{x}_j^i - \hat{x}_j^i \mathrm{d}\rho_i, \tag{4.7.1}$$

where the right hand side is defined globally. We can then show that the collection of global sections $\{\mathrm{d}\hat{x}_j^i, \mathrm{d}\rho_i\}$ span the cotangent bundle at each point, and therefore they span the cotangent bundle. That is, if we have a one form $\phi \in \Gamma^\infty(T^*X)$ such that $\phi|_{U_i} = \sum_{j=1}^{n} f_j^i \mathrm{d}x_j^i$ for $f_j^i \in C^\infty(X)$, then

$$\phi = \sum_{i=1}^{k} \sum_{j=1}^{n} \rho_i^2 f_j^i \mathrm{d}x_j^i = \sum_{i=1}^{k} \sum_{j=1}^{n} f_j^i (\rho_i \mathrm{d}\hat{x}_j^i - \hat{x}_j^i \mathrm{d}\rho_i).$$

Hence $\Gamma^\infty(T^*X)$ is globally generated by sections of the form $\mathrm{d}c$ for some $c \in C^\infty(X)$, and so λ is surjective.

To show λ is injective, we first proceed by making a series of embeddings. From Melrose [68, Prop. 1.14.1] we can embed X into a manifold without boundary U, and we can use the Whitney Embedding Theorem to embed this first embedding into $\mathbb{R}^N$ for some $N >> 0$. We will then use, as in Lee [60, Th. 6.24, Prop. 6.25], that there is a tubular open neighbourhood $V \subset \mathbb{R}^N$ of U and a smooth submersion $r : V \to U$ that is a retraction.

Now, take elements $a_i, b_i \in C^\infty(X)$, $i = 1, \ldots, n$ such that $\sum_i a_i \mathrm{d}b_i \in \Omega_{\boldsymbol{C}^\infty(X)}$ and say that their image in $\Gamma^\infty(T^*X)$ under λ is 0, that is $\sum_i a_i \mathrm{d}b_i = 0 \in \Gamma^\infty(T^*X)$. As X can be embedded as a submanifold with corners of U, then these functions a_i, b_i can be extended to functions on U using Seeley's Extension theorem and Borel's Lemma. While $\sum_i a_i \mathrm{d}b_i = 0$ on X it is not clear that this is true on $U \setminus X$.

As U is a manifold, then, as in the previous part of the proof, there is a finite atlas $\{U_k\}_{k=1,\ldots,m}$ of U and we can take coordinate functions $x_1^k, \ldots, x_p^k$ for each coordinate patch U_k, $k = 1, \ldots, m$. On U_k, the $\mathrm{d}x_1^k, \ldots, \mathrm{d}x_n^k$ span the cotangent bundle restricted to this neighbourhood. Take a partition of unity $\{\rho_k\}_{k=1,\ldots,m}$ subordinate to the open cover $\{U_i\}_{i=1,\ldots,k}$ and rescale the ρ_i so that $\sum_{k=1}^{m} \rho_k^2 = 1$. Then the a_i and b_i are functions of these x_j^k locally, and we can use eq. (4.7.1) to write

$$\begin{aligned}\sum_i a_i \mathrm{d}b_i = \sum_k \rho_k^2 \sum_i a_i \mathrm{d}b_i &= \sum_{i,j,k} a_i|_{U_k} \frac{\partial b_i|_{U_k}}{\partial x_j^k} \rho_k^2 \mathrm{d}x_j^k \\ &= \sum_{i,j,k} a_i|_{U_k} \frac{\partial b_i|_{U_k}}{\partial x_j^k} (\rho_k \mathrm{d}\hat{x}_j^k - \hat{x}_j^k \mathrm{d}\rho_k) \\ &= \sum_{i,j,k} (f_{i,j,k} \mathrm{d}\hat{x}_j^k - g_{i,j,k} \mathrm{d}\rho_k), \end{aligned} \tag{4.7.2}$$

where $f_{i,j,k} : U \to \mathbb{R}$, $g_{i,j,k} : U \to \mathbb{R}$ are smooth functions with $f_{i,j,k} = a_i|_{U_k} \frac{\partial b_i|_{U_k}}{\partial x_j^k} \rho_k$ and $g_{i,j,k} = a_i|_{U_k} \frac{\partial b_i|_{U_k}}{\partial x_j^k} \hat{x}_j^k$. These are both defined on all of U and are zero outside of U_k, and for $x \in X \cap U_k$, we have $0 = \sum_i a_i \mathrm{d}b_i = \sum_{i,j,k} a_i|_{U_k} \frac{\partial b_i|_{U_k}}{\partial x_j^k} dx_j^k$. As $\mathrm{d}x_j^k$ are a basis of $\Gamma^\infty(T^*U_k)$ then $\sum_i a_i|_{U_k} \frac{\partial b_i|_{U_k}}{\partial x_j^k} = 0$ for $x \in X \cap U_k$, which implies that $f_{i,j,k}, g_{i,j,k}$ are also zero on $X \cap U_k$. In particular, $f_{i,j,k}, g_{i,j,k}$ are zero on all of X. We have that

$$\sum_i a_i \mathrm{d}b_i - \sum_{i,j,k} (f_{i,j,k} \mathrm{d}\hat{x}_j^k - g_{i,j,k} \mathrm{d}\rho_k) = 0$$

on U. Relabel and define $\hat{a}_j$, $\hat{b}_j$ so that on U

$$\sum_i a_i \mathrm{d}b_i + \sum_j \hat{a}_j \mathrm{d}\hat{b}_j = 0,$$

where the $\hat{b}_j$ correspond to the $\hat{x}_j^k, \rho_k$, and the $\hat{a}_j$ correspond to the $f_{i,j,k}, g_{i,j,k}$.

We can now pull back $\sum_i a_i \mathrm{d}b_i + \sum_j \hat{a}_j \hat{b}_j = 0 \in \Gamma^\infty(T^*U)$ to V using r so that we have $0 = \sum_i (a_i \circ r) \mathrm{d}(b_i \circ r) + \sum_j (\hat{a}_j \circ r) \mathrm{d}(\hat{b}_j \circ r) \in \Gamma^\infty(T^*V)$. Take coordinate functions $x_1, \ldots, x_N$ on $\mathbb{R}^N$, then $a_i \circ r, b_i \circ r, \hat{a}_j \circ r, \hat{b}_j \circ r$ are functions of the x_i restricted to V.

Take a smooth bump function $\rho : \mathbb{R}^N \to [0,1]$ that is 1 near U and (the closure of) the support is contained in V. Then define $f_i : \mathbb{R}^N \to \mathbb{R}$ by $f_i = \rho \cdot a_i \circ r$ for each $i = 1, \ldots, n$, and $\hat{f}_j : \mathbb{R}^N \to \mathbb{R}$ by $\hat{f}_j = \rho \cdot \hat{a}_i \circ r$ for each $j = 1, \ldots, N$. So we have $f_i|_U = a_i \circ r|_U = a_i$, $\hat{f}_i|_U = \hat{a}_i \circ r|_U = \hat{a}_i$, and we still have

$$0 = \sum_i (f_i|_V) \mathrm{d}(b_i \circ r) + \sum_j (\hat{f}_j)|_V \mathrm{d}(\hat{b}_j \circ r) \in \Gamma^\infty(T^*V).$$

Then take smooth bump functions $\rho_i, \hat{\rho}_j : \mathbb{R}^N \to [0,1]$ that are 1 in a neighbourhood of the closed support of $f_i, \hat{f}_j$ respectively, and have closed support inside U, and define $g_i = \rho_i \cdot b_i \circ r : \mathbb{R}^N \to \mathbb{R}$ and $\hat{g}_i = \hat{\rho}_j \cdot \hat{b}_i \circ r$. So $g_i|_U = b_i \circ \pi$, $\hat{g}_i|_U = \hat{b}_i \circ \pi$, and $\sum_i f_i \mathrm{d}g_i + \sum_j \hat{f}_i \mathrm{d}\hat{g}_i = 0$, as this is zero outside of V by definition of $g_i, \hat{g}_j$ and zero inside

V by definition of $f_i, \hat{f}_j$ and $g_i, \hat{g}_j$. This implies that

$$0 = \sum_i f_i \mathrm{d}g_i + \sum_j \hat{f}_i \mathrm{d}\hat{g}_i = \sum_{i,j} f_i \frac{\partial g_i}{\partial x_j} \mathrm{d}x_j + \sum_{j,k} \hat{f}_j \frac{\partial \hat{g}_j}{\partial x_k} \mathrm{d}x_k \in \Gamma^\infty(T^*\mathbb{R}^N),$$

with the sums over $i = 1, \ldots, n, j = 1, \ldots, N, k = 1, \ldots, N$. As $\mathrm{d}x_1, \ldots, \mathrm{d}x_N$ are a basis for $\Gamma^\infty(T^*\mathbb{R}^N)$, then this implies $\sum_i f_i \frac{\partial g_i}{\partial x_k} + \sum_j \hat{f}_j \frac{\partial \hat{g}_j}{\partial x_k} = 0$ for each $k = 1, \ldots, N$.

Now, in $\Omega_{\boldsymbol{C}^\infty(X)}$ we have that

$$\begin{aligned}\sum_i a_i \mathrm{d}b_i + \sum_j \hat{a}_j \mathrm{d}\hat{d}_j =& \sum_i \Phi_{f_i}(x_1|_X, \ldots, x_N|_X) \mathrm{d}\Phi_{g_i}(x_1|_X, \ldots, x_N|_X) \\ &+ \sum_j \Phi_{\hat{f}_i}(x_1|_X, \ldots, x_N|_X) \mathrm{d}\Phi_{\hat{g}_i}(x_1|_X, \ldots, x_N|_X) \\ =& \sum_{i,j} \Phi_{f_i}(x_1|_X, \ldots, x_N|_X) \Phi_{\frac{\partial g_i}{\partial x_j}}(x_1|_X, \ldots, x_N|_X) \mathrm{d}x_j|_X \\ &+ \sum_{j,k} \Phi_{\hat{f}_j}((x_1|_X, \ldots, x_N|_X) \Phi_{\frac{\partial \hat{g}_j}{\partial x_k}} \mathrm{d}x_k|_X \\ =& \sum_k \Phi_{\sum_i f_i \frac{\partial g_i}{\partial x_k} + \sum_j \hat{f}_j \frac{\partial \hat{g}_j}{\partial x_k}}(x_1|_X, \ldots, x_N|_X) \mathrm{d}x_k|_X \\ =& \sum_k \Phi_0(x_1|_X, \ldots, x_N|_X) \mathrm{d}x_k|_X = 0 \in \Omega_{\boldsymbol{C}^\infty(X)}.\end{aligned}$$

However, $\hat{a}_j|_X : X \to \mathbb{R}$ are all identically zero, so that $\sum_j \hat{a}_j \mathrm{d}\hat{b}_j = 0 \in \Omega_{\boldsymbol{C}^\infty(X)}$. Hence we have that $\sum_i a_i \mathrm{d}b_i = 0 \in \Omega_{\boldsymbol{C}^\infty(X)}$ and λ is injective, so $\Gamma^\infty(T^*X)$ is the cotangent module of both $\boldsymbol{C}^\infty(X)$ and $\boldsymbol{C}^\infty_{\mathrm{in}}(X)$, extending Joyce [40, Ex. 5.4]. Note that if X is a manifold without boundary, we can skip the first part of the proof and just embed X in $\mathbb{R}^N$ with a tubular neighbourhood and use the retract as above. □

Definition 4.7.6. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ be an interior C^∞-ring with corners, so that $\mathfrak{C}_{\mathrm{ex}} = \mathfrak{C}_{\mathrm{in}} \amalg \{0_{\mathfrak{C}_{\mathrm{ex}}}\}$ with $\mathfrak{C}_{\mathrm{in}}$ a monoid. Let M be a $\mathfrak{C}$-module. A *b-derivation* is a monoid morphism $\mathrm{d}_{\mathrm{in}} : \mathfrak{C}_{\mathrm{in}} \to M$, where M is a monoid over addition, such that $\mathrm{d} = \mathrm{d}_{\mathrm{in}} \circ \Psi_{\exp} : \mathfrak{C} \to M$ is a C^∞-derivation in the sense of Definition 2.2.4 and we require that $\mathrm{d}_{\mathrm{in}} \circ \Psi_{\exp}(\Phi_i(c')) = \Phi_i(c')\mathrm{d}_{\mathrm{in}}c'$ for all $c' \in \mathfrak{C}_{\mathrm{in}}$.

We call such a pair $(M, \mathrm{d}_{\mathrm{in}})$ a *b-cotangent module* for $\boldsymbol{\mathfrak{C}}$ if it has the universal property that for any b-derivation $\mathrm{d}'_{\mathrm{in}} : \mathfrak{C}_{\mathrm{in}} \to M'$, there exists a unique morphism of $\mathfrak{C}$-modules $\lambda : M \to M'$ with $\mathrm{d}'_{\mathrm{in}} = \lambda \circ \mathrm{d}_{\mathrm{in}}$.

There is a natural construction for a b-cotangent module: we take M to be the quotient of the free $\mathfrak{C}$-module with basis of symbols $\mathrm{d}_{\mathrm{in}}c'$ for $c' \in \mathfrak{C}_{\mathrm{in}}$ by the $\mathfrak{C}$-submodule spanned by all expressions of the form

(i) $\mathrm{d}_{\mathrm{in}}(c' \cdot c'') - \mathrm{d}_{\mathrm{in}}c' - \mathrm{d}_{\mathrm{in}}c''$ for all $c', c'' \in \mathfrak{C}_{\mathrm{in}}$, and

(ii) $\mathrm{d}_{\mathrm{in}} \circ \Psi_{\exp\circ f}(c_1, \ldots, c_n) - \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}_{\mathrm{in}} \circ \Psi_{\exp}(c_i)$ for all $f : \mathbb{R}^n \to \mathbb{R}$ smooth and $c_1, \ldots, c_n \in \mathfrak{C}$.

(iii) $\mathrm{d}_{\mathrm{in}} \circ \Psi_{\exp}(\Phi_i(c')) - \Phi_i(c')\mathrm{d}_{\mathrm{in}}c'$ for all $c' \in \mathfrak{C}_{\mathrm{in}}$.

Here (i) makes $\mathrm{d}_{\mathrm{in}} : \mathfrak{C}_{\mathrm{in}} \to M$ a monoid morphism, and (ii) makes $\mathrm{d}_{\mathrm{in}} \circ \Psi_{\exp} : \mathfrak{C} \to M$ a C^∞-derivation. Thus b-cotangent modules exist, and are unique up to unique isomorphism. When we speak of 'the' b-cotangent module, we mean that constructed above, and we write it as $\mathrm{d}_{\boldsymbol{\mathfrak{C}},\mathrm{in}} : \mathfrak{C}_{\mathrm{in}} \to {}^b\Omega_{\boldsymbol{\mathfrak{C}}}$.

Since $\mathrm{d}_{\boldsymbol{\mathfrak{C}},\mathrm{in}} \circ \Psi_{\exp} : \mathfrak{C} \to {}^b\Omega_{\boldsymbol{\mathfrak{C}}}$ is a C^∞-derivation, the universal property of $\Omega_{\mathfrak{C}} = \Omega_{\boldsymbol{\mathfrak{C}}}$ in §2.2 implies that there is a unique $\boldsymbol{\mathfrak{C}}$-module morphism $I_{\boldsymbol{\mathfrak{C}}} : \Omega_{\boldsymbol{\mathfrak{C}}} \to {}^b\Omega_{\boldsymbol{\mathfrak{C}}}$ with $\mathrm{d}_{\boldsymbol{\mathfrak{C}},\mathrm{in}} \circ \Psi_{\exp} = I_{\boldsymbol{\mathfrak{C}}} \circ \mathrm{d}_{\mathfrak{C}} : \mathfrak{C} \to {}^b\Omega_{\boldsymbol{\mathfrak{C}}}$.

Let $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ be a morphism in $\mathbf{C^\infty Rings^c_{in}}$. Then we have a monoid morphism $\phi_{\mathrm{in}} : \mathfrak{C}_{\mathrm{in}} \to \mathfrak{D}_{\mathrm{in}}$. Regarding ${}^b\Omega_{\boldsymbol{\mathfrak{D}}}$ as a $\mathfrak{C}$-module using $\phi : \mathfrak{C} \to \mathfrak{D}$, then $\mathrm{d}_{\boldsymbol{\mathfrak{D}},\mathrm{in}} \circ \phi_{\mathrm{in}} : \mathfrak{C}_{\mathrm{in}} \to {}^b\Omega_{\boldsymbol{\mathfrak{D}}}$ becomes a b-derivation. Thus by the universal property of ${}^b\Omega_{\boldsymbol{\mathfrak{C}}}$, there exists a unique $\boldsymbol{\mathfrak{C}}$-module morphism ${}^b\Omega_{\boldsymbol{\phi}} : {}^b\Omega_{\boldsymbol{\mathfrak{C}}} \to {}^b\Omega_{\boldsymbol{\mathfrak{D}}}$ with $\mathrm{d}_{\boldsymbol{\mathfrak{D}},\mathrm{in}} \circ \phi_{\mathrm{in}} = {}^b\Omega_{\boldsymbol{\phi}} \circ \mathrm{d}_{\boldsymbol{\mathfrak{C}},\mathrm{in}}$. This then induces a morphism of $\boldsymbol{\mathfrak{D}}$-modules $({}^b\Omega_{\boldsymbol{\phi}})_* : \boldsymbol{\phi}_*({}^b\Omega_{\boldsymbol{\mathfrak{C}}}) = {}^b\Omega_{\boldsymbol{\mathfrak{C}}} \otimes_{\mathfrak{C}} \mathfrak{D} \to {}^b\Omega_{\boldsymbol{\mathfrak{D}}}$. If $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$, $\boldsymbol{\psi} : \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{E}}$ are morphisms in $\mathbf{C^\infty Rings^c_{in}}$ then ${}^b\Omega_{\boldsymbol{\psi}\circ\boldsymbol{\phi}} = {}^b\Omega_{\boldsymbol{\psi}} \circ {}^b\Omega_{\boldsymbol{\phi}} : {}^b\Omega_{\boldsymbol{\mathfrak{C}}} \to {}^b\Omega_{\boldsymbol{\mathfrak{E}}}$.

Remark 4.7.7. In Definition 4.7.6 we could have omitted the condition that $\boldsymbol{\mathfrak{C}}$ be interior, and considered monoid morphisms $\mathrm{d}_{\mathrm{ex}} : \mathfrak{C}_{\mathrm{ex}} \to M$ such that $\mathrm{d} = \mathrm{d}_{\mathrm{ex}} \circ \Psi_{\exp} : \mathfrak{C} \to M$ is a C^∞-derivation and $\mathrm{d}_{\mathrm{ex}} \circ \Psi_{\exp}(\Phi_i(c')) = \Phi_i(c')\mathrm{d}_{\mathrm{ex}}c'$ for all $c' \in \mathfrak{C}_{\mathrm{ex}}$. However, since $c' \cdot 0_{\mathfrak{C}_{\mathrm{ex}}} = 0_{\mathfrak{C}_{\mathrm{ex}}}$ for all $c' \in \mathfrak{C}_{\mathrm{ex}}$ we would have $\mathrm{d}_{\mathrm{ex}}c' + \mathrm{d}_{\mathrm{ex}}0_{\mathfrak{C}_{\mathrm{ex}}} = \mathrm{d}_{\mathrm{ex}}0_{\mathfrak{C}_{\mathrm{ex}}}$ in M, so $\mathrm{d}_{\mathrm{ex}}c' = 0$, and this modified definition would give ${}^b\Omega_{\boldsymbol{\mathfrak{C}}} = 0$ for any $\boldsymbol{\mathfrak{C}}$. To get a nontrivial definition we took $\boldsymbol{\mathfrak{C}}$ to be interior, and defined d_{in} only on $\mathfrak{C}_{\mathrm{in}} = \mathfrak{C}_{\mathrm{ex}} \setminus \{0_{\mathfrak{C}_{\mathrm{ex}}}\}$.

If $\boldsymbol{\mathfrak{C}}$ lies in the image of the functor $\Pi^{\mathrm{int}}_{\mathrm{cor}} : \mathbf{C^\infty Rings^c} \hookrightarrow \mathbf{C^\infty Rings^c_{in}}$ of Definition 4.2.11 then ${}^b\Omega_{\boldsymbol{\mathfrak{C}}} = 0$, since $\mathfrak{C}_{\mathrm{in}}$ then contains a zero element $0_{\mathfrak{C}_{\mathrm{in}}}$ with $c' \cdot 0_{\mathfrak{C}_{\mathrm{in}}} = 0_{\mathfrak{C}_{\mathrm{in}}}$ for all $c' \in \mathfrak{C}_{\mathrm{in}}$.

If $\mathrm{d}_{\mathrm{in}} : \mathfrak{C}_{\mathrm{in}} \to M$ is a b-derivation then it is a morphism from a monoid to an abelian group, and so factors through $\pi^{\mathrm{gp}} : \mathfrak{C}_{\mathrm{in}} \to (\mathfrak{C}_{\mathrm{in}})^{\mathrm{gp}}$. This suggests that b-cotangent modules may be most interesting for *integral* C^∞-rings with corners, as in §4.5, for which $\pi^{\mathrm{gp}} : \mathfrak{C}_{\mathrm{in}} \to (\mathfrak{C}_{\mathrm{in}})^{\mathrm{gp}}$ is injective.

Example 4.7.8. Let X be a manifold with corners (or g-corners), with b-cotangent bundle ${}^bT^*X$ as in §3.4. Example 4.2.7 defines a C^∞-ring with corners $\boldsymbol{C}^\infty_{\mathrm{in}}(X) = \boldsymbol{\mathfrak{C}}$ with $\mathfrak{C}_{\mathrm{in}} = \mathrm{In}(X)$, the monoid of interior maps $c' : X \to [0, \infty)$. We have a $C^\infty(X)$-module $\Gamma^\infty({}^bT^*X)$. Define $\mathrm{d}_{\mathrm{in}} : \mathrm{In}(X) \to \Gamma^\infty({}^bT^*X)$ by

$$\mathrm{d}_{\mathrm{in}}(c') = c'^{-1} \cdot {}^b\mathrm{d}c' = {}^b\mathrm{d}(\log c'), \qquad (4.7.3)$$

where ${}^b\mathrm{d} = I_X^* \circ \mathrm{d}$ is the composition of the exterior derivative $\mathrm{d} : C^\infty(X) \to \Gamma^\infty(T^*X)$ with the projection $I_X^* : T^*X \to {}^bT^*X$. Here (4.7.3) makes sense on the interior X° where $c' > 0$, but has a unique smooth extension over $X \setminus X^\circ$.

We can now show that $\mathrm{d_{in}} : \mathrm{In}(X) \to \Gamma^\infty({}^bT^*X)$ is a b-derivation in the sense of Definition 4.7.6, so there is a unique morphism $\lambda : {}^b\Omega_{\boldsymbol{C}^\infty(X)} \to \Gamma^\infty({}^bT^*X)$ such that $\mathrm{d_{in}} = \lambda \circ \mathrm{d_{in}}$.

Proposition 4.7.9. *If X is a manifold with faces with finitely many boundary components then $\Gamma^\infty({}^bT^*X)$ is the b-cotangent module of $\boldsymbol{C}^\infty_{\mathrm{in}}(X)$. That is, λ from Example 4.7.8 is an isomorphism.*

Proof. Say X has dimension n and let $\mathfrak{C} = (C^\infty(X), \mathrm{In}(X) \amalg \{0\})$. Each element of ${}^b\Omega_{\mathfrak{C}}$ is a linear combination of elements of the form $\mathrm{d_{in}}c'$ for some $c' \in \mathrm{In}(X)$. On the other hand, $\Gamma^\infty({}^bT^*X)$ contains elements that are locally spanned by $\mathrm{d_{in}}c'$ for some $c' \in \mathrm{In}(X)$ restricted to the local neighbourhood. As in Example 4.7.8, the universal property of ${}^b\Omega_{\mathfrak{C}}$ gives a morphism $\lambda : {}^b\Omega_{\mathfrak{C}} \to \Gamma^\infty({}^bT^*X)$. We break this proof into two parts: showing λ is surjective and injective.

To show λ is surjective, we need to show that there is a global spanning set for $\Gamma^\infty({}^bT^*X)$ of elements of the form $\mathrm{d_{in}}c'$ for $c' \in \mathrm{In}(X)$. We will not need to use that X has finitely many boundary components for this part of the proof.

Firstly, using paracompactness of X, it can be shown that there are a countable number of boundary components. We label these boundary components X_i, $i \in \mathbb{N}$, so that $\partial X = \cup X_i$. Take elements $f_1, \ldots, f_n \in \mathrm{In}(X)$ such that in a neighbourhood of boundary component X_i (away from any $\geqslant$ 2-corners), then $f_j = x_i^{a_{i,j}} F$ for a smooth positive function F. Here x_i is the coordinate corresponding to boundary component X_i. As X has faces, we can prescribe the values of $a_{i,j} \in \mathbb{N}$ independently, as we do below.

Near a k-corner, where boundaries $X_{i_1}, \ldots, X_{i_k}$ meet, $f_j = x_{i_1}^{a_{i_1,j}} \ldots x_{i_k}^{a_{i_k,j}} F'$ for a positive smooth function F'. Now,

$$\mathrm{d_{in}}(f_j) = a_{i_1,j} \frac{1}{x_{i_1}} \mathrm{d}x_{i_1} + \ldots + a_{i_k,j} \frac{1}{x_{i_k}} \mathrm{d}x_{i_k} + \frac{1}{F} \sum_{t=1}^{n} \frac{\partial F}{\partial x_{i_t}} \mathrm{d}x_{i_t}$$

near this k-corner.

From the proof of Proposition 4.7.5, we have elements $\hat{x}_l^r = \rho_r x_l^r \in C^\infty(X)$, where $\rho_r \in C^\infty(X)$ is a partition of unity for an open cover $\{U_r\}_{r=1,\ldots,n}$. We see that $\mathrm{d}\hat{x}_l^r = \mathrm{d_{in}}(\exp \circ \hat{x}_l^r) = {}^b\mathrm{d}\hat{x}_l^r$ and $\mathrm{d}\rho_r = \mathrm{d_{in}}(\exp \circ \rho_r) = {}^b\mathrm{d}\rho_r$ in $\Gamma^\infty({}^bT^*X)$. We will show that we can pick the set $\{(a_{i_1,j}, \ldots, a_{i_k,j}) : j = 1, \ldots, n\}$ so that

$$\{\mathrm{d}\hat{x}_l^r, \mathrm{d_{in}}f_j, \mathrm{d}\rho_r : j = 1, \ldots, n, r = 1, \ldots, n+1, l = 1, \ldots, n\}$$

spans $\Gamma^\infty({}^bT^*X)$. To do this, we first ensure that the set $\{(a_{i_1,j},\dots,a_{i_k,j}) : j = 1,\dots,n\}$ is linearly independent over $\mathbb{R}$ for all distinct selections of $i_1,\dots,i_k \in \mathbb{N}$ for all $k = 1,\dots,n$. We assume, without loss of generality, that $k = n$.

Let $a_{i,j} = (i+1)^{j-1}$. Then linear dependence is equivalent to showing that for choice of distinct $i_1,\dots,i_n \in \mathbb{N}$, there are non-zero $b_0,\dots,b_{n-1}$ such that $b_0 + b_1(i_t+1) + b_2(i_t+1)^2 + \dots + b_{n-1}(i_t+1)^{n-1} = 0$ for each $t = 1,\dots,n$. However, this requires n distinct roots to a non-zero degree $n-1$ polynomial, giving a contradiction. Hence the coefficients are linearly independent.

Then we have $f_1,\dots,f_n$, such that $\mathrm{d_{in}}(f_1),\dots,\mathrm{d_{in}}(f_n)$, locally span the b-cotangent bundle's 'corner elements' at each k-corner.

That is, if we take a point $x \in X$, and a coordinate neighbourhood U_x of x, such that $U_x \cong \mathbb{R}^n_k$, then an element $s \in \Gamma^\infty({}^bT^*X)$ is locally of the form $s_1\frac{1}{x_1}\mathrm{d}x_1 + \dots, s_k\frac{1}{x_k}\mathrm{d}x_k + s_{k+1}\mathrm{d}x_{k+1} + \dots + s_n\mathrm{d}x_n$. Here $s_i \in C^\infty(U_x)$ and $x_1,\dots,x_n \in C^\infty(U_x)$ are coordinate charts on U_x. From the proof of Proposition 4.7.5, we can write $s_{k+1}\mathrm{d}x_{k+1} + \dots + s_n\mathrm{d}x_n$ as a linear combination of the $\mathrm{d}\hat{x}^r_l$'s and $\mathrm{d}\rho_r$'s, with coefficients $v^l_{x,r} \in C^\infty(U_x)$ and $w^r_x \in C^\infty(U_x)$ respectively. From the definition of the f'_js, we can write $s_1\frac{1}{x_1}\mathrm{d}x_1 + \dots, s_k\frac{1}{x_k}\mathrm{d}x_k$ as a linear combination of $\mathrm{d_{in}}f_j$'s, with coefficients $u^x_j \in C^\infty(U_x)$.

Take a cover $\{U_x\}$ of X by coordinate patches, and a partition of unity $\{\phi_x\}$ subordinate to this cover. Then we can glue the coefficients $v^l_{x,r}, w^r_x, u^x_j \in C^\infty(X)$ together to define $v^l_r = \sum_x v^l_{x,r}\phi_x$, $w^r = \sum_x w^r_x\phi_x$ and $u_j = \sum_x u^x_j\phi_x$. Then s is a global linear combination of the $\mathrm{d}\hat{x}^r_l$'s, $\mathrm{d}\rho_r$'s and $\mathrm{d_{in}}f_j$'s with coefficients v^l_r, w^r and u_j respectively.

Hence, any element in $\Gamma^\infty({}^bT^*X)$ is generated by elements of the form $\mathrm{d_{in}}(c')$ with $c' \in \mathrm{In}(X)$, and so $\lambda : {}^b\Omega_{\mathfrak{C}} \to \Gamma^\infty({}^bT^*X)$ is surjective when X has faces.

To show that λ is injective, we follow a similar method to the proof of Proposition 4.7.5. Firstly, take $X = \mathbb{R}^n_k$ and assume we have $a_i \in C^\infty(X)$ and $b_i \in \mathrm{In}(X)$ such that $\sum_i a_i\mathrm{d_{in}}b_i \in {}^b\Omega_{\mathfrak{C}}$ but with image under λ such that $\sum_i a_i\mathrm{d_{in}}b_i = 0 \in \Gamma^\infty({}^bT^*X)$. Write $b_i = x_1^{c^i_1}\dots x_k^{c^i_k}\exp(f_i)$ for some smooth function $f_i : \mathbb{R}^n_k \to \mathbb{R}$ and non-negative integers

c^i_j. We have $\mathrm{d_{in}} = {}^b\mathrm{d} \circ \log$ in $\Gamma^\infty({}^bT^*X)$, and using this we can write

$$\begin{aligned}
0 = \sum_i a_i \mathrm{d_{in}} b_i &= \sum_i a_i \left(\sum_{j=1}^k \frac{c^i_j}{x_j} {}^b\mathrm{d}x_j + \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} {}^b\mathrm{d}x_j \right) \\
&= \sum_i a_i \left(\sum_{j=1}^k (\frac{c^i_j}{x_j} + \frac{\partial f_i}{\partial x_j}) {}^b\mathrm{d}x_j + \sum_{j=k+1}^n \frac{\partial f_i}{\partial x_j} {}^b\mathrm{d}x_j \right) \\
&= \sum_i a_i \left(\sum_{j=1}^k (c^i_j + x_j \frac{\partial f_i}{\partial x_j}) \mathrm{d_{in}} x_j + \sum_{j=k+1}^n \frac{\partial f_i}{\partial x_j} \mathrm{d_{in}} \Psi_{\exp}(x_j) \right).
\end{aligned}$$

Here the $\frac{1}{x_j}$ make sense on the interior of $\mathbb{R}^n_k$ and have unique smooth extension over the boundary, and we use that ${}^b\mathrm{d}x_j = x_j \mathrm{d_{in}} x_j$. As the $\mathrm{d_{in}} x_j$ and $\mathrm{d_{in}} \Psi_{\exp}(x_j)$ are a basis for $\Gamma^\infty({}^bT^*X)$ then this implies $\sum_i a_i(c^i_j + x_j \frac{\partial f_i}{\partial x_j}) = 0$ for each $j = 1, \ldots, k$ and $\sum_i a_i \frac{\partial f_i}{\partial x_j} = 0$ for each $j = k+1, \ldots, n$.

We note that each x_j for $j = 1, \ldots, k$ are coordinate projections, so $x_j : \mathbb{R}^n_k \to [0, \infty) \in \mathrm{In}(X)$ and we have $b_i = x_1^{c^i_1} \ldots x_k^{c^i_k} \Psi_{\exp \circ f_i}(\Phi_i(x_1), \ldots, \Phi_i(x_k), x_{k+1}, \ldots, x_n)$. In ${}^b\Omega_{\mathfrak{C}}$ we use (i),(ii),(iii) from Definition 4.7.6 to write

$$\begin{aligned}
\sum_i a_i \mathrm{d_{in}} b_i = \sum_i a_i \Big(&\sum_{j=1}^k c^i_j \mathrm{d_{in}} x_j + \sum_{j=1}^k \Phi_i(x_j) \Phi_{\frac{\partial f_i}{\partial x_j}} (\Phi_i(x_1), \ldots, \Phi_i(x_k), x_{k+1}, \ldots, x_n) \mathrm{d_{in}} x_j \\
&+ \sum_{j=k+1}^n \Phi_{\frac{\partial f_i}{\partial x_j}} (\Phi_i(x_1), \ldots, \Phi_i(x_k), x_{k+1}, \ldots, x_n) \mathrm{d_{in}} \Psi_{\exp}(x_j) \Big) \\
= \sum_{j=1}^k &(\Phi_{\sum_i a_i (c^i_j + x_j \frac{\partial f_i}{\partial x_j})} (\Phi_i(x_1), \ldots, \Phi_i(x_k), x_{k+1}, \ldots, x_n)) \mathrm{d_{in}} x_j \\
&+ \sum_{j=k+1}^n \Phi_{\sum_i a_i (\frac{\partial f_i}{\partial x_j})} (\Phi_i(x_1), \ldots, \Phi_i(x_k), x_{k+1}, \ldots, x_n) \mathrm{d_{in}} \Psi_{\exp}(x_j) \\
= \sum_{j=1}^k &(\Phi_0(\Phi_i(x_1), \ldots, \Phi_i(x_k), x_{k+1}, \ldots, x_n)) \mathrm{d_{in}} x_j \\
&+ \sum_{j=k+1}^n \Phi_0(\Phi_i(x_1), \ldots, \Phi_i(x_k), x_{k+1}, \ldots, x_n) \mathrm{d_{in}} \Psi_{\exp}(x_j) = 0.
\end{aligned}$$

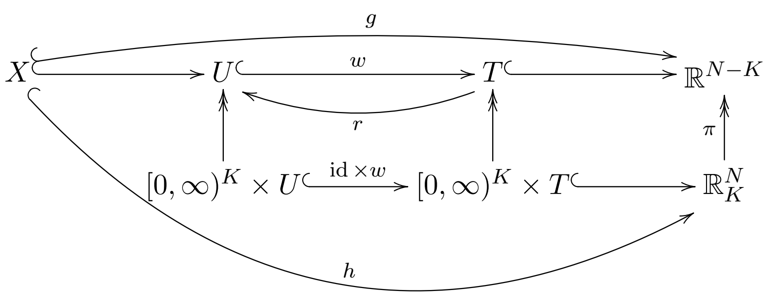
So λ is injective for $X = \mathbb{R}^n_k$.

We now show λ is injective more generally. Let X be a manifold with corners with faces with finitely many boundary components $\{(\partial X)_i\}_{i=1,\ldots,K}$. We will show X can be embedded in $\mathbb{R}^N_K$ for some $N > K$ and use this embedding to show λ is injective for this X. In this 'embedding' the boundary and corners of X are embedded into the boundary and corners of $\mathbb{R}^N_K$, so that each boundary component of X is in a different boundary component of $\mathbb{R}^N_K$. (This means X is embedded as a 'p-submanifold' of $\mathbb{R}^N_K$ in the language of Melrose [68, Def. 1.7.4].)

To do this, we see that by Melrose [68, Lem. 1.8.1] there are smooth functions $\eta_i : X \to [0,\infty)$ for $i = 1,\dots,K$, such that if we take a coordinate chart V of X that intersects boundary component $i_X((\partial X)_i)$ and let x_i be the coordinate of V that is zero along $i_X((\partial X)_i)$, then we have $\eta_i(x) = x_i H_i(x)$ for some positive function H_i. In other coordinate charts, η_i is strictly positive.

By Melrose [68, Prop. 1.14.1] we can embed X into a manifold without boundary U, and the Whitney embedding theorem says we can embed U into $\mathbb{R}^M$ via $w : U \to \mathbb{R}^M$ for some large M. Let $g : X \to U \to \mathbb{R}^M$ be the composition of these embeddings. Then define $h : X \to \mathbb{R}^N_K$ by $h(x) = (\eta_1(x),\dots,\eta_K(x), g(x))$ so that $N = M + K$. As g is an embedding, then h is an embedding, and we have each different boundary component $(\partial X)_i$ embedding into a different boundary component of $\mathbb{R}^N_K$, so X embeds as a p-submanifold of $\mathbb{R}^N_K$. In fact, this is an even stronger form of embedding where locally the target is of the form $X \times \mathbb{R}^M$ for some large M, so that the b-tangent and b-cotangent spaces also respect this decomposition. More details on this type of embedding can be found in Joyce [47, Def. 4.8].

Lee [60, Th. 6.24, Prop. 6.25] tells us we can take a tubular neighbourhood T of $w(U)$ in $\mathbb{R}^{N-K}$, and this has a smooth retract $r : T \to w(U)$. Then our embedding of $h : X \to \mathbb{R}^N_K$ breaks into a series of embeddings along the bottom line of the commutative diagram (4.7.4). Note that the upward arrows in the diagram are the canonical projections, and that r is really $w^{-1}|_{w(U)} \circ r$.



(4.7.4)

Now assume we have $a_i \in C^\infty(X)$ and $b_i \in \mathrm{In}(X)$ for $i = 1,\dots,m$, such that $\sum_i a_i \mathrm{d}_{\mathrm{in}} b_i \in {}^b\Omega_{\mathfrak{C}}$ but with image under λ such that $\sum_i a_i \mathrm{d}_{\mathrm{in}} b_i = 0 \in \Gamma^\infty({}^bT^*X)$. For a coordinate patch $V \cong \mathbb{R}^n_k$ in X with coordinates $x_1,\dots,x_n$, then $b_i(x_1,\dots,x_n) = x_1^{c_1^i} \cdots x_k^{c_k^i} F_i(x_1,\dots,x_n)$ for a positive smooth function F_i and non-negative integers c_i^j. Then for (global) coordinates $y_1,\dots,y_N$ of $\mathbb{R}^N_K$, we have $b_i \circ g^{-1}(y_1,\dots,y_N) \in [0,\infty)$ for $(y_1,\dots,y_N) \in g(X) \subset \mathbb{R}^n_k$. However, we know that $\eta_i(x) = x_i H_i(x)$ for positive smooth function H_i and x_i the coordinate on $[0,\infty)$, so $x_i = \eta_i(x)/H_i(x)$, so if

$g(x_1,\ldots,x_N)=(y_1,\ldots,y_N)$ then $x_i=y_i/H_i(x)$ and

$$\begin{aligned} b_i\circ g^{-1}(y_1,\ldots,y_N) &= x_1^{c_1^i}\cdots x_k^{c_k^i}F_i(x_1,\ldots,x_n)\\ &= \frac{\eta_1(x)}{H_1(x)}^{c_1^i}\cdots\frac{\eta_k(x)}{H_k(x)}^{c_k^i}F_i\circ g^{-1}(y_1,\ldots,y_N)\\ &= y_1^{c_1^j}\cdots y_k^{c_k^i}G_i(y_1,\ldots,y_N), \end{aligned}$$

for a positive smooth function $G_i : g(V)\to[0,\infty)$. Note that G_i is only defined locally on the image of the coordinate patch V. Also, note that on the k-th boundary component, for each $i=1,\ldots,m$ and $j=1,\ldots,K$ the c_j^i are the same integer in any coordinate patch V, so we can relabel the G_i to write

$$b_i\circ g^{-1}(y_1,\ldots,y_N)=y_1^{c_1^j}\cdots y_K^{c_K^i}G_i(y_1,\ldots,y_N)$$

with the same c_j^i in each coordinate patch V of X. We can then take a partition of unity on the coordinate patches of $g(X)\subset\mathbb{R}^N_K$ to glue the G_i so that we can globally write

$$b_i\circ g^{-1}(y_1,\ldots,y_N)=y_1^{c_1^j}\cdots y_K^{c_K^i}G_i(y_1,\ldots,y_N)$$

for a positive smooth function $G_i : g(X)\to(0,\infty)$ for each $i=1,\ldots,m$. For $j=1,\ldots,K$ we have

$$y_j=\eta_j\circ g^{-1}\circ\pi(y_1,\ldots,y_N)=\eta_j(g^{-1}(y_{K+1},\ldots,y_N))$$

then each y_j for $j=1,\ldots,K$ is dependent upon the other y_j for $j=K+1,\ldots,N$ so we can consider G_i as a function of y_j for $j=K+1,\ldots,N$ only.

Similarly we can also see that $a_i\circ g^{-1} : g(X)\to\mathbb{R}$ is smooth and we can consider it as a function of the $y_K,\ldots,y_N$ only, and we have that $\sum_i a_i\circ g^{-1}d_{\mathrm{in}}(b_i\circ g^{-1})=0$ in $\Gamma^\infty({}^bT^*g(X))$. Relabel $a_i\circ g^{-1}$ as a_i and $b_i\circ g^{-1}$ as b_i.

As G_i and a_i are functions of the $y_{K+1},\ldots,y_N$ only, then we can use Seeley's Extension Theorem and Borel's Lemma to extend them to functions on $w(U)$, so that composing with restriction implies they are functions on $[0,\infty)^k\times w(U)$.

Then $\sum_i a_i\mathrm{d}_{\mathrm{in}}b_i=\sum_{i,j}a_ic_j^i\mathrm{d}_{\mathrm{in}}y_j+\sum_i a_i\mathrm{d}_{\mathrm{in}}G_i\in\Gamma^\infty({}^bT^*[0,\infty)^k\times w(U))$. As in the proof of Proposition 4.7.5, we have a finite atlas $\{U_k\}_{k=1,\ldots,p}$ of $w(U)$ and coordinate functions $x_1^k,\ldots,x_n^k$ for each coordinate patch U_k, $k=1,\ldots,p$. We take $\{\rho_k\}_{k=1,\ldots,p}$ a partition of unity subordinate to this open cover that have been rescaled so that $\sum_{k=1}^m\rho_k^2=1$. On U_k, the $\mathrm{d}_{\mathrm{in}}\circ\exp(x_1^k),\ldots,\mathrm{d}_{\mathrm{in}}\circ\exp(x_n^k)$ span the cotangent bundle of U_k. We see we have an analogous relation to (4.7.1)

$$\rho_k^2\mathrm{d}_{\mathrm{in}}\circ\exp(x_j^k)=\rho_k\mathrm{d}_{\mathrm{in}}\circ\exp(\rho_kx_j^k)-\rho_kx_k^i\mathrm{d}_{\mathrm{in}}\circ\exp\rho_k$$

so that if $\hat{x}_j^k = \rho_k x_j^k$ the collection $\{\mathrm{d_{in}} \circ \exp \hat{x}_j^k, \mathrm{d_{in}} \circ \exp \rho_k\}_{j=1,\dots,n,k=1,\dots,p}$ span the cotangent bundle of $w(U)$. We can write, as in eq. (4.7.2),

$$\begin{aligned}\sum_i a_i \mathrm{d_{in}} b_i &= \sum_{i,j} a_i c_j^i \mathrm{d_{in}} y_j + \sum_i a_i \mathrm{d_{in}} G_i \in \Gamma^\infty({}^bT^*[0,\infty)^k \times w(U)) \\ &= \sum_{i,j} a_i c_j^i \mathrm{d_{in}} y_j + \sum_{i,j,k} (f_{i,j,k} \mathrm{d_{in}} \circ \exp \hat{x}_j^k - g_{i,j,k} \mathrm{d_{in}} \circ \exp \rho_k).\end{aligned}$$

In the first sum, j sums from 1 to K. In the second sum, j sums from 1 to n. Here, $f_{i,j,k} : U \to \mathbb{R}$, $g_{i,j,k} : U \to \mathbb{R}$ are smooth functions with $f_{i,j,k} = a_i|_{U_k} \frac{\partial \log(G_i)|_{U_k}}{\partial x_j^k} \rho_k$ and $g_{i,j,k} = a_i|_{U_k} \frac{\log(G_i)|_{U_k}}{\partial x_j^k} \hat{y}_j^k$. These are both defined on all of U and are zero outside of U_k, and in $g(X) \cap U_k$, we have

$$0 = \sum_i a_i \mathrm{d} b_i = \sum_{i,j} a_i c_j^i \mathrm{d_{in}} y_j + \sum_{i,j,k} a_i|_{U_k} \frac{\partial \log(G_i)|_{U_k}}{\partial y_j^k} \mathrm{d_{in}} \circ \exp x_j^k.$$

As dx_j^k are a basis of $\Gamma^\infty(T^*U_k)$ then $\sum_i a_i|_{U_k} \frac{\partial \log(G_i)|_{U_k}}{\partial y_j^k} = 0$ in $g(X) \cap U_k$, which implies that $f_{i,j,k}, g_{i,j,k}$ are also zero in $g(X) \cap U_k$. In particular, $f_{i,j,k}, g_{i,j,k}$ are zero on all of $g(X)$.

Hence, in $\Gamma^\infty({}^bT^*([0,\infty)^K \times g(U)))$ have that

$$\sum_i a_i \mathrm{d_{in}} b_i - \sum_{i,j} a_i c_j^i \mathrm{d_{in}} y_j + \sum_{i,j,k} (f_{i,j,k} \mathrm{d_{in}} \circ \exp \hat{x}_j^k - g_{i,j,k} \mathrm{d_{in}} \circ \exp \rho_k) = 0,$$

Relabel so that

$$\sum_i a_i \mathrm{d_{in}} b_i + \sum_j \hat{a}_j \mathrm{d_{in}} \circ \exp(\hat{b}_j) = 0 \in \Gamma^\infty({}^bT^*([0,\infty)^K \times U))$$

and the $\hat{a}_j$ correspond to the $f_{i,j,k}, g_{i,j,k}$, and the $\hat{b}_j$ correspond to the $\hat{x}_j^k$ and ρ_k which are all smooth functions on $g(U)$ only, and composition with restriction implying they are smooth functions on $[0,\infty)^k \times g(U)$.

Consider now using the retraction $r : T \to U$ to pull back the $a_i, b_i, \hat{a}_j, \hat{b}_j$ as we did in Proposition 4.7.5, so that

$$\sum_i (a_i \circ r) \mathrm{d_{in}} (b_i \circ r) + \sum_j (\hat{a}_j \circ r) \mathrm{d_{in}} \circ \exp(\hat{b}_j \circ r) = 0 \in \Gamma^\infty({}^bT^*([0,\infty)^K \times T)).$$

Note that here that by $b_i \circ r$ we really mean composing with $\mathrm{id} \times r : [0,\infty)^K \times T \to [0,\infty)^K \times g(U)$. The rest of the proof follows using the method in Proposition 4.7.5 and the relations we used to show λ is injective for $X = \mathbb{R}_k^n$.

Hence $\Gamma^\infty({}^bT^*X)$ is the cotangent module of $\boldsymbol{C}_{\mathrm{in}}^\infty(X)$ when X has faces and finitely many boundary components.

Note that if X does not have faces, then there are not enough global elements to generate $\Gamma^\infty({}^bT^*X)$ (as there will be restrictions on the powers $a_{i,j}$), and if it does not have finitely many boundary components we cannot use the embeddings in (4.7.4). $\square$

Remark 4.7.10. Note that a compact manifold with faces will have finitely many boundary components and satisfy Proposition 4.7.9.

Let X be a manifold with g-corners. Say that for each connected component F of ∂X, the map $i_X|_F : F \to X$ is injective. Here, $i_X : (x, \beta) \mapsto x$, where $x \in X$ and β is a local boundary component of X at x. If X is also a manifold with corners, then it would be a manifold with faces and $\mathrm{Ex}_x(X) \cong \mathrm{Ex}(X)_{x*}$ for all $x \in X$. However, for general manifolds with g-corners satisfying this condition, it is not necessarily true that $\mathrm{Ex}_x(X) \cong \mathrm{Ex}(X)_{x*}$, as we show in Example 5.5.4.

This means that there are conditions on the powers $a_{i,j}$ for global sections and hence there are not enough global sections of the form $\mathrm{d_{in}}(f)$ for $f \in \mathrm{In}(X)$ to show that $\mathrm{d_{in}} : \mathrm{In}(X) \to \Gamma^\infty({}^bT^*X)$ is the b-cotangent module. This suggests defining a manifold with g-corners *with faces* to satisfy $\mathrm{Ex}_x(X) \cong \mathrm{Ex}(X)_{x*}$ for all $x \in X$, a stronger condition than injectivity of $i_X|_F : F \to X$.

However, for X_P, the local model for a manifold with g-corners, we will be able to show both surjectivity and injectivity in an analogous proof to Proposition 4.7.9, so that the b-cotangent modules of charts of a manifold with g-corners are isomorphic to the sections of ${}^bT^*X$ over g-charts.

Example 4.7.11. Suppose $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ is a C^∞-ring with corners such that $\mathfrak{C}_{\mathrm{ex}} = \mathfrak{C}_{\mathrm{ex}}^\times \amalg \{0_{\mathfrak{C}_{\mathrm{ex}}}\}$, where $\amalg$ is the disjoint union. Then $\boldsymbol{\mathfrak{C}}$ is interior, with $\mathfrak{C}_{\mathrm{in}} = \mathfrak{C}_{\mathrm{ex}}^\times$, and lies in the essential image of $I_{C^\infty}^{\mathrm{int}} : \mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings^c_{in}}$ in Definition 4.2.10, so $\boldsymbol{\mathfrak{C}}$ holds only information from a C^∞-ring. An example of this is when $\boldsymbol{\mathfrak{C}} = \boldsymbol{C^\infty}(X)$ for X a connected manifold without boundary.

Then $\Psi_{\exp} : \mathfrak{C} \to \mathfrak{C}_{\mathrm{in}}$ is a bijection, and setting $\mathrm{d_{in}} = \mathrm{d} \circ \Psi_{\exp}$ gives a 1-1 correspondence between C^∞-derivations $\mathrm{d} : \mathfrak{C} \to M$ and b-derivations $\mathrm{d_{in}} : \mathfrak{C}_{\mathrm{in}} \to M$, for $\mathfrak{C}$-modules M. It follows that $I_{\boldsymbol{\mathfrak{C}}} : \Omega_{\boldsymbol{\mathfrak{C}}} \to {}^b\Omega_{\boldsymbol{\mathfrak{C}}}$ is an isomorphism, so the cotangent and b-cotangent modules of $\boldsymbol{\mathfrak{C}}$ coincide.

Chapter 5

C^∞-schemes with corners

We define and study C^∞-schemes with corners, generalising the results of §2. We begin with describing local C^∞-ringed spaces with corners, and a corners version of a spectrum functor.

5.1 C^∞-ringed spaces with corners

Definition 5.1.1. A *C^∞-ringed space with corners* $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ is a topological space X with a sheaf $\boldsymbol{\mathcal{O}}_X$ of C^∞-rings with corners on X. That is, for each open set $U \subseteq X$, then $\boldsymbol{\mathcal{O}}_X(U) = (\mathcal{O}_X(U), \mathcal{O}_X^{\rm ex}(U))$ is a C^∞-ring with corners and $\boldsymbol{\mathcal{O}}_X$ satisfies the sheaf axioms in §2.3. With a slight abuse of notation, we will write elements $s \in \mathcal{O}_X(U), s' \in \mathcal{O}_X^{\rm ex}(U)$ as $\boldsymbol{s} = (s, s') \in \boldsymbol{\mathcal{O}}_X(U)$.

A *morphism* $\boldsymbol{f} = (f, \boldsymbol{f}^\sharp) : (X, \boldsymbol{\mathcal{O}}_X) \to (Y, \boldsymbol{\mathcal{O}}_Y)$ of C^∞-ringed spaces with corners is a continuous map $f : X \to Y$ and a morphism $\boldsymbol{f}^\sharp = (f^\sharp, f^\sharp_{\rm ex}) : f^{-1}(\boldsymbol{\mathcal{O}}_Y) \to \boldsymbol{\mathcal{O}}_X$ of sheaves of C^∞-rings with corners on X, for $f^{-1}(\boldsymbol{\mathcal{O}}_Y) = (f^{-1}(\mathcal{O}_Y), f^{-1}(\mathcal{O}_Y^{\rm ex}))$ as in Definition 2.3.5. Note that $\boldsymbol{f}^\sharp$ is adjoint to a morphism $\boldsymbol{f}_\sharp = (f_\sharp, f_\sharp^{\rm ex}) : \boldsymbol{\mathcal{O}}_Y \to f_*(\boldsymbol{\mathcal{O}}_X)$ on $\boldsymbol{Y}$ as in (2.4.2).

A *local C^∞-ringed space with corners* $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ is a C^∞-ringed space for which the stalks $\boldsymbol{\mathcal{O}}_{X,x} = (\mathcal{O}_{X,x}, \mathcal{O}_{X,x}^{\rm ex})$ of $\boldsymbol{\mathcal{O}}_X$ at x are local C^∞-rings with corners for all $x \in X$. As in Remark 4.6.2, we define morphisms of local C^∞-ringed spaces with corners $(X, \boldsymbol{\mathcal{O}}_X), (Y, \boldsymbol{\mathcal{O}}_Y)$ to be morphisms of C^∞-ringed spaces with corners, without any additional locality condition.

Write $\mathbf{C^\infty RS^c}$ for the category of C^∞-ringed spaces with corners, and $\mathbf{LC^\infty RS^c}$ for the full subcategory of local C^∞-ringed spaces with corners.

For brevity, we will use the notation that bold upper case letters $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \ldots$ represent C^∞-ringed spaces with corners $(X, \boldsymbol{\mathcal{O}}_X)$, $(Y, \boldsymbol{\mathcal{O}}_Y)$, $(Z, \boldsymbol{\mathcal{O}}_Z)$, $\ldots$, and bold lower case letters

$\boldsymbol{f}, \boldsymbol{g}, \ldots$ represent morphisms of C^∞-ringed spaces with corners $(f, \boldsymbol{f}^\sharp), (g, \boldsymbol{g}^\sharp), \ldots$. When we write '$x \in \boldsymbol{X}$' we mean that $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ and $x \in X$. When we write '$\boldsymbol{U}$ *is open in* $\boldsymbol{X}$' we mean that $\boldsymbol{U} = (U, \boldsymbol{\mathcal{O}}_U)$ and $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ with $U \subseteq X$ an open set and $\boldsymbol{\mathcal{O}}_U = \boldsymbol{\mathcal{O}}_X|_U$.

Let $\boldsymbol{X} = (X, \mathcal{O}_X, \mathcal{O}_X^{\rm ex}) \in \mathbf{LC^\infty RS^c}$, and let U be open in X. Take elements $s \in \mathcal{O}_X(U)$ and $s' \in \mathcal{O}_X^{\rm ex}(U)$. Then s and s' induce functions $s : U \to \mathbb{R}$, $s' : U \to [0, \infty)$, that at each $x \in U$ are the compositions

$$\mathcal{O}_X(U) \xrightarrow{\rho_{X,x}} \mathcal{O}_{X,x} \xrightarrow{\pi_x} \mathbb{R}, \text{ and } \mathcal{O}_X^{\rm ex}(U) \xrightarrow{\rho_{X,x}^{\rm ex}} \mathcal{O}_{X,x}^{\rm ex} \xrightarrow{\pi_x^{\rm ex}} [0, \infty).$$

Here, $\rho_{X,x}, \rho_{X,x}^{\rm ex}$ are the restriction morphism to the stalks, and $\pi_x, \pi_x^{\rm ex}$ are the unique morphisms that exist as $\mathcal{O}_{X,x}$ is local for each $x \in X$, as in Definition 4.6.1 and Lemma 4.6.6. We denote $s(x)$ and $s'(x)$ the values of $s : U \to \mathbb{R}$ and $s' : U \to [0, \infty)$ respectively at the point $x \in U$. We denote $s_x \in \mathcal{O}_{X,x}$ and $s'_x \in \mathcal{O}_{X,x}^{\rm ex}$ the values of s and s' under the restriction morphisms to the stalks $\rho_{X,x}$ and $\rho_{X,x}^{\rm ex}$ respectively.

Lemma 5.1.2. *The functions $s : U \to \mathbb{R}$ and $s' : U \to [0, \infty)$ are continuous.*

Proof. Let $\boldsymbol{X}$, U, s and s' be as in the statement. Assume for a contradiction that $s : U \to \mathbb{R}$ is not continuous. So there is an open set $V \subset \mathbb{R}$ such that $s^{-1}(V) \subset U$ is not open. Hence there is a $u \in s^{-1}(V)$ such that for every open set $U' \subset U$ with $u \in U'$, then there is a $u' \in U'$ with $s(u') \notin V$. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth bump function with support on V, and let $s_1 = \Phi_\rho(s)$ so that $s_1(u) = \rho(s(u)) \neq 0$.

As $\mathcal{O}_{X,u}$ is local and $s_1(u) = \rho(s(u)) \neq 0$, then s_1 is invertible in $\mathcal{O}_{X,u}$. So there is an open set $W \subset X$ with $u \in W$ and $t \in \mathcal{O}_X(W)$ such that $ts_1|_{W'} = 1 \in \mathcal{O}_X(W')$ for an open set $W' \subset W \cap U$ with $u \in W'$. However, as s is not continuous, there is a $u' \in W'$ such that $s(u') \notin V$, so $s_1(u') = \rho(s(u')) = 0$. Then $1 = t(u)s_1(u) = t(u')s_1(u') = t(u')0 = 0 \in \mathcal{O}_X(W')$. However, as $\boldsymbol{X}$ has local stalks, then $0 \neq 1 \in \mathcal{O}_X(W')$, which gives the required contradiction.

For s', note that $s' : U \to [0, \infty)$ is continuous if and only if $\Phi_i(s') : U \to \mathbb{R}$ is continuous, as $s'^{-1}([0, a)) = \Phi_i(s')^{-1}((-\infty, a))$ for any element $a > 0 \in \mathbb{R}$. Then $\Phi_i(s') : U \to \mathbb{R}$ is continuous by the discussion above, so $s' : U \to [0, \infty)$ is continuous. □

Definition 5.1.3. Let $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ be a C^∞-ringed space with corners. We call $\boldsymbol{X}$ an *interior* C^∞-ringed space with corners if one of the following equivalent conditions hold:

(a) For all open $U \subset X$ and each $s' \in \mathcal{O}_X^{\rm ex}(U)$, then $U_{s'} = \{x \in U : s'_x \neq 0 \in \mathcal{O}_{X,x}^{\rm ex}\}$, which is always closed in U, is open in U, and the stalks $\boldsymbol{\mathcal{O}}_{X,x}$ are interior C^∞-rings with corners.

(b) For all open $U \subset X$ and each $s' \in \mathcal{O}_X^{\text{ex}}(U)$, then $U \setminus U_{s'} = \hat{U}_{s'} = \{x \in U : s'_x = 0 \in \mathcal{O}_{X,x}^{\text{ex}}\}$, which is always open in U, is closed in U, and the stalks $\boldsymbol{\mathcal{O}}_{X,x}$ are interior C^∞-rings with corners.

(c) $\mathcal{O}_X^{\text{ex}}$ is the sheafification of a presheaf of the form $\mathcal{O}_X^{\text{in}} \amalg \{0\}$, where $\mathcal{O}_X^{\text{in}}$ is a sheaf of monoids, such that $(\mathcal{O}_X(U), \mathcal{O}_X^{\text{in}}(U) \amalg \{0\})$ is an interior C^∞-ring with corners for each open $U \subset X$.

In each case, we can define a sheaf of monoids $\mathcal{O}_X^{\text{in}}$, such that $\mathcal{O}_X^{\text{in}}(U) = \{s' \in \mathcal{O}_X^{\text{ex}}(U) | s'_x \neq 0 \in \mathcal{O}_{X,x}^{\text{ex}}$ for all $x \in U\}$.

We call $\boldsymbol{X}$ an *interior* local C^∞-ringed space with corners if $\boldsymbol{X}$ is a local C^∞-ringed space with corners that is also an interior C^∞-ringed space with corners.

If $\boldsymbol{X}, \boldsymbol{Y}$ are interior (local) C^∞-ringed spaces with corners, a morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is called *interior* if the induced maps on stalks $\boldsymbol{f}_x^\sharp : \boldsymbol{\mathcal{O}}_{Y,f(x)} \to \boldsymbol{\mathcal{O}}_{X,x}$ are interior morphisms of interior C^∞-rings with corners for all $x \in X$. This gives a morphism of sheaves $f^{-1}(\mathcal{O}_Y^{\text{in}}) \to \mathcal{O}_X^{\text{in}}$. Write $\mathbf{C^\infty RS^c_{in}} \subset \mathbf{C^\infty RS^c}$ (and $\mathbf{LC^\infty RS^c_{in}} \subset \mathbf{LC^\infty RS^c}$) for the non-full subcategories of interior (local) C^∞-ringed spaces with corners and interior morphisms.

Lemma 5.1.4. *(a)-(c) in Definition 5.1.3 are equivalent.*

Proof. (a) and (b) are equivalent by definition. The set $\hat{U}_{s'}$ is open, as the requirement that an element is zero in the stalk is a local requirement. That is, $s'_x = 0$ if and only if $s'|_V = 0 \in \mathcal{O}_X^{\text{ex}}(V)$ for some $V \subseteq U$.

Suppose (a) and (b) hold, then we will show they imply (c). Define $\mathcal{O}_X^{\text{in}}(U) = \{s' \in \mathcal{O}_X^{\text{ex}}(U) | s'_x \neq 0 \in \mathcal{O}_{X,x}^{\text{ex}}$ for all $x \in U\}$. If $s'_1, s'_2 \in \mathcal{O}_X^{\text{in}}(U)$ then $s'_{1,x}, s'_{2,x} \neq 0$, and as the stalks are interior, then $s'_{1,x} \cdot s'_{2,x} \neq 0 \in \mathcal{O}_{X,x}^{\text{ex}}$. So $s'_1 \cdot s'_2 \in \mathcal{O}_X^{\text{in}}(U)$, and $\mathcal{O}_X^{\text{in}}$ is a monoid. Then $(\mathcal{O}_X(U), \mathcal{O}_X^{\text{in}}(U) \amalg \{0\})$ is a pre C^∞-ring with corners, where the C^∞-operations come from restriction from $\boldsymbol{\mathcal{O}}_X(U)$. As the invertible elements of the monoid and the C^∞-rings of $(\mathcal{O}_X(U), \mathcal{O}_X^{\text{in}}(U) \amalg \{0\})$ are the same as those from $\boldsymbol{\mathcal{O}}_X(U)$, this is a C^∞-ring with corners. Let $\hat{\mathcal{O}_X^{\text{ex}}}$ be the sheafification of $\mathcal{O}_X^{\text{in}} \amalg \{0\}$, which is a subsheaf of $\mathcal{O}_X^{\text{ex}}$. Note that $\mathcal{O}_X^{\text{in}}(U) \amalg \{0\}$ already satisfies uniqueness, so the sheafification process means $\hat{\mathcal{O}_X^{\text{ex}}}$ now satisfies glueing. Then $(\mathcal{O}_X, \hat{\mathcal{O}_X^{\text{ex}}})$ is a sheaf of C^∞-rings with corners.

There is a morphism $(\text{id}, \text{id}^\sharp, \iota_{\text{ex}}^\sharp) : (X, \mathcal{O}_X, \mathcal{O}_X^{\text{ex}}) \to (X, \mathcal{O}_X, \hat{\mathcal{O}_X^{\text{ex}}})$. This is the identity on the topological spaces and the sheaves of C^∞-rings. On the sheaves of monoids, we have an inclusion $\iota_{\text{ex}}^\sharp(U) : \hat{\mathcal{O}_X^{\text{ex}}}(U) \to \mathcal{O}_X^{\text{ex}}(U)$. On stalks, any non-zero element of $\mathcal{O}_{X,x}^{\text{ex}}$ is an equivalence class represented by a section $s' \in \mathcal{O}_X^{\text{ex}}(U)$. As (a) is true, we can choose U so that $s'_x \neq 0$ for all $x \in U$. Then $s' \in \mathcal{O}_X^{\text{in}}(U)$, so there is an element $s'' \in \hat{\mathcal{O}_X^{\text{ex}}}(U)$

that maps to s' under $\iota_{\rm ex}^\sharp(U)$. Then $s''_x \mapsto s'_x$, and, as $0 \mapsto 0$, then $\iota_{\rm ex}^\sharp$ is surjective on stalks. As $\hat{\mathcal{O}_X^{\rm ex}}$ is a subsheaf of $\mathcal{O}_X^{\rm ex}$, then $\iota_{\rm ex}^\sharp$ is injective on stalks. Then $({\rm id}, {\rm id}^\sharp, \iota_{\rm ex}^\sharp)$ is an isomorphism.

Suppose (c) holds, then we will show it implies (a), firstly if $s' \in \mathcal{O}_X^{\rm ex}(U)$, where $\mathcal{O}_X^{\rm ex}$ is the sheafification of $\mathcal{O}_X^{\rm in} \amalg \{0\}$, then if $s'_x \neq 0 \in \mathcal{O}_{X,x}^{\rm ex}$, then there is an open set V and an element $s'' \in \mathcal{O}_X^{\rm in}(V)$ that represents s' on $x \in V \subset U$, and therefore $s'_{x'} \neq 0 \in \mathcal{O}_{X,x'}^{\rm ex}$ for all $x' \in V$, and the $U_{s'}$ defined in (a) is open.

Now if $s'_1, s'_2 \in \mathcal{O}_X^{\rm ex}(U)$, and $s'_{1,x}, s'_{2,x} \neq 0 \in \mathcal{O}_{X,x}^{\rm ex}$, then there is an open set V and elements $s''_1, s''_2 \in \mathcal{O}_X^{\rm in}(V)$ that represent these s'_1, s'_2 upon restriction to $\mathcal{O}_X^{\rm ex}(V)$. Then $s''_{1,x} \cdot s''_{2,x} \in \mathcal{O}_{X,x}^{\rm in} \not\ni 0$, so the stalk $\boldsymbol{\mathcal{O}}_{X,x} = (\mathcal{O}_{X,x}, \mathcal{O}_{X,x}^{\rm in} \amalg \{0\})$ is interior. □

Remark 5.1.5. Note that, for an element $\boldsymbol{X} = (X, \mathcal{O}_X, \mathcal{O}_X^{\rm ex}) \in \mathbf{LC^\infty RS^c_{in}}$, then $\boldsymbol{\mathcal{O}}_X$ is not a sheaf of interior C^∞-rings with corners even if X is connected. For example, $\boldsymbol{\mathcal{O}}_X(\emptyset) = (\{0\}, \{0\})$ where $(\{0\}, \{0\})$ is the final object in C^∞-rings with corners, and it is not equal to $(\{0\}, \{0, 1\})$, the final object in interior C^∞-rings with corners. This is important for considering colimits and limits in both $\mathbf{LC^\infty RS^c}$ and $\mathbf{LC^\infty RS^c_{in}}$, and for defining the corner functors of §5.8.

In fact something more subtle occurs here: $(\mathcal{O}_X, \mathcal{O}_X^{\rm in} \amalg \{0\})$ is a sheaf of interior C^∞-rings with corners in the sense of sheaves valued in arbitrary categories, not those valued in the category of sets (as discussed in §2.3). The issue is that products (limits) of interior C^∞-rings with corners are not products of their underlying sets (as in Theorem 4.3.7(b) and Example 4.2.14). This means the glueing condition (stated in Definition 2.3.1 for abelian groups), which is formed by considering equalisers of products in Sets (or Abelian groups), is different from considering such equalisers of products in interior C^∞-rings with corners. However, $(\mathcal{O}_X, \mathcal{O}_X^{\rm in} \amalg \{0\})$ is a presheaf of the underlying sets, so we can sheafify to form $\boldsymbol{\mathcal{O}}_X$, which is a sheaf of sets as well as a sheaf of C^∞-rings with corners. Conditions (a)-(c) characterise all sheaves of C^∞-rings with corners that come from sheafifications of sheaves of interior C^∞-rings with corners.

Notably, conditions (a)-(c) are stronger than just requiring that the stalks are interior, as we show in the following example. This is important as we would like sections of our sheaves to identify the boundary and corners of the underlying spaces, so that the corner functor studied in §5.8 is well behaved.

In particular, the boundary and corners of elements of $\mathbf{LC^\infty RS^c}, \mathbf{LC^\infty RS^c_{in}}$, should result from points in the space where sections change from invertible to a non-invertible in the stalks at these points. However, in the following example, adding a 'bump function' that changes from zero to non-zero to invertible in the interior of the topological space suggests the topological space should have boundary in the interior. Imposing (a)-(c) removes cases such as this for interior C^∞-ringed spaces with corners. Note also that when we define interior C^∞-schemes with corners as spectra of interior C^∞-rings with corners, then (a)-(c) are already satisfied.

Example 5.1.6. Let $X = \mathbb{R}$, and take K any non-empty closed subset of $\mathbb{R}$. Define a sheaf of C^∞-rings with corners on X by $\mathcal{O}_X(U) = C^\infty(U)$ and $\mathcal{O}_X^{\rm ex}(U) = \text{In}(U) \amalg \{\text{constant functions } U \cap K \to \{0, 0'\}\}$ where $0, 0'$ act as zeros with $0 \cdot 0' = 0$. Any constant function $f : U \cap K \to \{0', 0\}$ is the zero function under Φ_i.

If $U \cap K$ is empty, then $\mathcal{O}_X^{\rm ex} \cong \text{In}(U) \amalg \{0\} = \text{Ex}(U)$. At points $x \in K$, we have $\boldsymbol{\mathcal{O}}_{X,x} \cong (C_x^\infty(X), \text{Ex}_x(X) \amalg \{0'\})$, and at points $x \in X \setminus K$ we have $\boldsymbol{\mathcal{O}}_{X,x} \cong (C_x^\infty(X), \text{Ex}_x(X))$. These stalks are local C^∞-rings with corners where the localisation morphism is evaluation at x and $0'$ evaluates to $0 \in \mathbb{R}$. There are no zero divisors so each stalk is interior. This gives an interior local C^∞-ringed space with corners.

Let $s' \in \mathcal{O}_X^{\rm ex}(X)$ such that $s' = 0$ for $x \in X \setminus K$ and $s' = 0'$ for $x \in K$. Then the set $\{x \in X | s'_x \neq 0 \in \mathcal{O}_{X,x}^{\rm ex}\} = K$ is not open in X, so this is not an interior C^∞-ringed space with corners.

In particular, if we consider the corner functor $C^{\rm loc}$ for $\mathbf{LC^\infty RS^c}$ as defined in §5.7, and apply it to $(X, \boldsymbol{\mathcal{O}}_X)$, it will have underlying set $\mathbb{R} \amalg K$, so this has extra boundary over the set K.

5.1.1 Limits and colimits

Proposition 5.1.7. *The categories* $\mathbf{C^\infty RS^c}$, $\mathbf{LC^\infty RS^c}$, $\mathbf{C^\infty RS^c_{in}}$ *and* $\mathbf{LC^\infty RS^c_{in}}$ *have all small colimits.*

Proof. This follows from the construction of small colimits of ordinary ringed and local ringed spaces in Demazure and Gabriel, [17, I §1 1.6]. As in Remark 2.4.16, the hard part of the proof for $\mathbf{LC^\infty RS^c}$ and $\mathbf{LC^\infty RS^c_{in}}$ involves showing that fibre products of local C^∞-rings with corners are local, which follows from Proposition 4.6.3.

One also needs to check that small colimits in $\mathbf{C^\infty RS^c_{in}}$ and $\mathbf{LC^\infty RS^c_{in}}$ constructed in the same way are interior. We explain the details for the pushout $\boldsymbol{W} = (W, \mathcal{O}_W, \mathcal{O}_W^{\rm ex})$ of $\boldsymbol{f} = (f, f^\sharp, f^\sharp_{\rm ex}) : \boldsymbol{X} = (X, \mathcal{O}_X, \mathcal{O}_X^{\rm ex}) \to (Y, \mathcal{O}_Y, \mathcal{O}_Y^{\rm ex}) = \boldsymbol{Y}$ and $\boldsymbol{g} = (g, g^\sharp, g^\sharp_{\rm ex}) : \boldsymbol{X} \to (Z, \mathcal{O}_Z, \mathcal{O}_Z^{\rm ex}) = \boldsymbol{Z}$ as elements of $\mathbf{LC^\infty RS^c_{in}}$. We will construct morphisms $\boldsymbol{p} = (p, p^\sharp, p^\sharp_{\rm ex}) : \boldsymbol{Y} \to \boldsymbol{W}$ and $\boldsymbol{q} = (q, q^\sharp, q^\sharp_{\rm ex}) : \boldsymbol{Z} \to \boldsymbol{W}$ for the morphisms from the definition of the pushout, as in the diagram below.

$$
\begin{array}{ccc}
 & \boldsymbol{X} & \\
\boldsymbol{f} \swarrow & & \searrow \boldsymbol{g} \\
\boldsymbol{Y} & & \boldsymbol{Z} \\
\boldsymbol{p} \searrow & & \swarrow \boldsymbol{q} \\
 & \boldsymbol{W} &
\end{array} \tag{5.1.1}
$$

Here the topological space $W \cong Y \amalg_X Z$ is the pushout of the topological spaces X, Y, Z, that is, it is the disjoint union of Y and Z quotiented by the equivalence relation generated by $Y \ni y \sim z \in Z$ if there is $x \in X$ such that $f(x) = y$ and $g(x) = z$. Then $p : Y \to W$ and $q : Z \to W$ are the morphisms from the pushout of topological spaces.

If we take an open set $U \subset W \cong Y \amalg_X Z$, then $U \cong U_1 \amalg_{f^{-1}(U_1) \cap g^{-1}(U_2)} U_2$ for some $p^{-1}(U) = U_1 \subset Y, q^{-1}(U) = U_2 \subset Z$. For the sheaf of C^∞-rings with corners $\boldsymbol{\mathcal{O}}_W$, we have $\boldsymbol{\mathcal{O}}_W(U) = \boldsymbol{\mathcal{O}}_Y(U_1) \times_{\boldsymbol{\mathcal{O}}_X(f^{-1}(U_1) \cap g^{-1}(U_2))} \boldsymbol{\mathcal{O}}_Z(U_2))$, the fibre product of C^∞-rings with corners. Then $(p_\sharp, p_\sharp^{\rm ex})(U) : \boldsymbol{\mathcal{O}}_W(U) \to \boldsymbol{\mathcal{O}}_Y(U_1)$ and $(q_\sharp, q_\sharp^{\rm ex})(U) : \boldsymbol{\mathcal{O}}_W(U) \to \boldsymbol{\mathcal{O}}_Z(U_2)$ are the canonical maps coming from the fibre product. Note that any $s' \in \mathcal{O}_W^{\rm ex}(U)$ is represented by $(s_1', s_2') \in \mathcal{O}_Y^{\rm ex}(U_1) \times \mathcal{O}_Z^{\rm ex}(U_2)$.

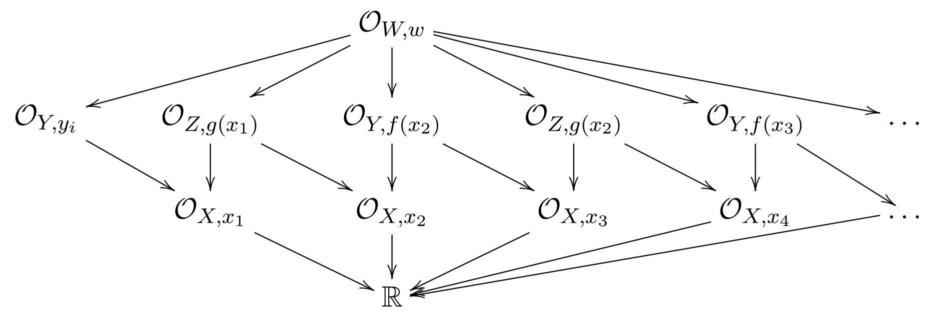
We will show $\boldsymbol{W}$ is interior and local. Take an open set $U \subset W$ and $s' \in \mathcal{O}_W^{\rm ex}(U)$, and say at $w \in U$ we have $s_w' \neq 0 \in \mathcal{O}_{W,w}$. Then s' is represented by $(s_1', s_2') \in \mathcal{O}_Y(U_1) \times \mathcal{O}_Z(U_2)$, and we must have $(s_1')_y \in \mathcal{O}_{Y,y}$ non-zero and $(s_2')_z \in \mathcal{O}_{Z,z}$ non-zero, for any $y \in p^{-1}(w)$ and $z \in q^{-1}(w)$. As $\boldsymbol{Y}, \boldsymbol{Z}$ are interior, then s_1' and s_2' must be non-zero locally, so s' must be non-zero locally, and the set $\{w \in W | s_w' \neq 0 \in \mathcal{O}_{W,w}^{\rm ex}\}$ is open. We now need to show that the stalks are interior and local.

Say $p^{-1}(w) = y$ is non-empty but $q^{-1}(w)$ is empty, and such that y is not on the boundary $f(X) \cap U_1 \subset Y$. Then the stalk $\boldsymbol{\mathcal{O}}_{W,w}$ is isomorphic to $\boldsymbol{\mathcal{O}}_{Y,y}$, which is both local and interior. Similarly, if $q^{-1}(w) = z$ is non-empty but $p^{-1}(w)$ is empty, and z is not on the boundary of $g(X) \cap U_2 \subset Z$, the stalk is isomorphic to $\boldsymbol{\mathcal{O}}_{Z,z}$, which is both local and interior.

Say $p^{-1}(w) = y$ and $q^{-1}(w) = z$ are both non-empty. Pick any $x \in f^{-1}(y) \cup g^{-1}(z)$. As $\boldsymbol{X}$ is local, there is a unique morphism $\mathcal{O}_{X,x} \to \mathbb{R}$, and the morphisms of sheaves give the composition $\mathcal{O}_{W,w} \to \mathcal{O}_{X,x} \to \mathbb{R}$ factoring through either $\mathcal{O}_{Y,y}$ or $\mathcal{O}_{Z,z}$. There may be more than one x in $f^{-1}(w) \cup g^{-1}(w)$, therefore more than one morphism $\mathcal{O}_{W,w} \to \mathcal{O}_{X,x} \to \mathbb{R}$, however as $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,z}$ are local, any morphism to $\mathbb{R}$ with the local property is necessarily unique, so these morphisms are identical. An element $s_w' \in \mathcal{O}_{W,w}^{\rm ex}$ is represented by an element $s' \in \mathcal{O}_W^{\rm ex}(U)$, which is represented by $(s_1', s_2') \in \mathcal{O}_Y(U_1) \times \mathcal{O}_Z(U_2)$ such that $f^\sharp(s_1) = g^\sharp(s_2) \in \mathcal{O}_X(f^{-1}(U_1) \cap g^{-1}(U_2))$. Then s_w' is invertible if and only if s' is invertible locally, which is if and only if s_1' and s_2' are invertible locally, which is if and only if the image of s'_w under this unique morphism is non-zero, which makes $\boldsymbol{\mathcal{O}}_{W,w}$ into a local C^∞-ring with corners.

Additionally, if $s' \in \mathcal{O}^{\rm ex}_{W,w}$ is a non-zero zero divisor, then it must be represented by zero-divisor $(s'_1, s'_2) \in \mathcal{O}^{\rm ex}_{Y,y} \times \mathcal{O}_{Z,z}$, and requiring that it factor through to $\mathcal{O}_{X,x}$ means that both s'_1, s'_2 are non-zero. However, as $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,z}$ are interior, the only zero divisors in their product have one zero entry (as in Example 4.2.14), so s' cannot be a zero divisor. Hence $\mathcal{O}_{W,w}$ is interior.

Here we have assumed there is only one element in both $p^{-1}(w)$ and $q^{-1}(w)$, however it is possible there is more than one element. In this case, we have several maps $\mathcal{O}_{W,w} \to \mathcal{O}_{Y,y_j} \to \mathcal{O}_{X,x_i} \to \mathbb{R}$ and $\mathcal{O}_{W,w} \to \mathcal{O}_{Z,z_k} \to \mathcal{O}_{X,x_i} \to \mathbb{R}$ for elements x_i, y_k, z_k. Each pair (y_j, z_k) are related by a finite number of relations such as $f(x_1) = y_i, g(x_1) = g(x_2), f(x_2) = f(x_3), g(x_3) = g(x_4), \ldots, g(x_n) = z_j$, and we get a diagram of maps as below.



However, the top rectangles commute by definition of the pushout, and the lower rectangles commute by definition of local morphisms, so all these compositions are the same, and we can again show these maps make $\mathcal{O}_{W,w}$ into a local interior C^∞-ring with corners.

Now consider $p^{-1}(w) = y$ but on the boundary of $f(X) \cap U_1 \subset Y$. There is still a morphism from $\mathcal{O}_{W,w} \to \mathcal{O}_{Y,y} \to \mathbb{R}$, but we do not necessarily know that $\mathcal{O}_{W,w} \cong \mathcal{O}_{Y,y}$. However, this morphism is well defined, and $s_w \in \mathcal{O}_{W,w}$ is sent to $s_{1,w} \in \mathcal{O}_{Y,y}$ which is non-zero under the map to $\mathbb{R}$ if and only if $s_{1,w}$ is invertible and hence if and only if s_w is invertible, so $\mathcal{O}_{W,w}$ is local. A similar proof shows $\boldsymbol{\mathcal{O}}_{W,w}$ is local, and a similar proof as above shows $\boldsymbol{\mathcal{O}}_{W,w}$ is also interior. Hence $\boldsymbol{W}$ is an interior local C^∞-ringed space with corners.

Note that $\boldsymbol{W}$ satisfies the universal property of the pushout by using the universal properties of the pushout of topological spaces and of the pullbacks of (interior/local) C^∞-rings with corners.

As $\mathbf{C^\infty RS^c}$, $\mathbf{LC^\infty RS^c}$, $\mathbf{C^\infty RS^c_{in}}$ and $\mathbf{LC^\infty RS^c_{in}}$ all have an initial object, $(\emptyset, \{0\}, \{0\})$ the empty set with the zero C^∞-ring with corners sheaf, and we can construct finite colimits as iterated pushouts using the initial objects. This shows that all finite colimits in these categories exist. This result can be extended to show all small colimits exist by showing small products exist. In this case, again the topological space is the coproduct of the topological spaces, the sheaves are the product of the sheaves, local and interior follow as above, and the universal properties follow from the universal properties of coproducts and products. □

Corollary 5.1.8. *The inclusion and forgetful functors in the following diagram respect small colimits.*

$$\begin{array}{ccccc} \mathbf{LC^\infty RS^c_{in}} & \longrightarrow & \mathbf{C^\infty RS^c_{in}} & & \\ \downarrow & & \downarrow & & \\ \mathbf{LC^\infty RS^c} & \longrightarrow & \mathbf{C^\infty RS^c} & & \\ \downarrow & & \downarrow & & \\ \mathbf{LC^\infty RS} & \longrightarrow & \mathbf{C^\infty RS} & \longrightarrow & \mathbf{Top} \end{array}$$

Proposition 5.1.9. *The forgetful functor* $\mathbf{LC^\infty RS^c_{in}} \to \mathbf{LC^\infty RS^c}$ *has a left adjoint, therefore it preserves limits.*

This proof uses the right adjoint to $\mathrm{inc}: \mathbf{C^\infty Rings^c_{in}} \to \mathbf{C^\infty Rings^c}$ defined in Theorem 4.3.7(b). We also show this forgetful functor preserves small limits directly in Theorem 5.1.10.

In contrast to Example 4.2.14, which shows that $\mathrm{inc}: \mathbf{C^\infty Rings^c_{in}} \to \mathbf{C^\infty Rings^c}$ has no left adjoint, in §5.8 we show that $\mathbf{LC^\infty RS^c_{in}} \to \mathbf{LC^\infty RS^c}$ also has a right adjoint, and hence preserves colimits, extending Proposition 5.1.7. While interior local C^∞-ringed spaces with corners have interior stalks, they do not have sheaves of interior C^∞-rings with corners. This means this result does not contradict Example 4.2.14, as, from Proposition 5.1.7, colimits of interior local C^∞-ringed spaces with corners have stalks that are constructed using only certain types of limits of interior C^∞-rings with corners, which are also interior.

Proof. A left adjoint can be constructed as follows. On objects, take $(X, \mathcal{O}_X, \mathcal{O}_X^{\mathrm{ex}}) \in \mathbf{LC^\infty RS^c}$ to $(X, \mathcal{O}_X, \hat{\mathcal{O}}_X^{\mathrm{ex}})$ where $\hat{\mathcal{O}}_X^{\mathrm{ex}}$ is the sheafification of the presheaf $\mathcal{O}_X^{\mathrm{ex}} \amalg \{0_{\mathrm{ex}}\}$, and $\amalg$ is the disjoint union. Here 0_{ex} becomes the new zero object. On connected components $U \subset X$, $\hat{\mathcal{O}}_X^{\mathrm{ex}}(U) \cong \mathcal{O}_X^{\mathrm{ex}}(U) \amalg \{0_{\mathrm{ex}}\}$. This is an interior C^∞-ring with corners, as it is the image of $\boldsymbol{\mathcal{O}}_X(U)$ under the right adjoint to the inclusion $\mathrm{inc}: \mathbf{C^\infty Rings^c_{in}} \to \mathbf{C^\infty Rings^c}$ defined in Theorem 4.3.7(b).

A morphism $(\phi, \phi^\sharp, \phi^\sharp_{\mathrm{ex}}) : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{LC^\infty RS^c}$ is mapped to the morphism $(\phi, \phi^\sharp, \hat{\phi}^\sharp_{\mathrm{ex}})$, where $\hat{\phi}^\sharp_{\mathrm{ex}}$ sends $s' \in \phi^{-1}(\mathcal{O}_X^{\mathrm{ex}})(U)$ to $\phi^\sharp_{\mathrm{ex}}(U)(s') \in \mathcal{O}_Y^{\mathrm{ex}}(U)$, and 0_{ex} to 0_{ex}. This defines

$\hat{\phi}^{\sharp}_{\text{ex}}$ on connected open sets U of X. The glueing property of sheaves then defines this on all of X.

To show this is an adjoint, we construct the unit and counit at elements $\boldsymbol{X} \in \mathbf{LC^{\infty}RS^{c}}$ and $\boldsymbol{Y} \in \mathbf{LC^{\infty}RS^{c}_{in}}$ respectively. They are both the identity on the topological space and the sheaves of C^{∞}-rings. On the monoid scheaves, the unit sends $\hat{\mathcal{O}}^{\text{ex}}_{X}$ to $\mathcal{O}^{\text{ex}}_{X}$ as sheaves on X. On a connected component $U \subset X$, it does this by sending $s' \in \mathcal{O}^{\text{ex}}_{X}(U)$ to s' and 0_{ex} to 0, and the glueing property of sheaves defines this on all of X. The counit is defined by the inclusion of $\mathcal{O}^{\text{ex}}_{Y}$ into $\hat{\mathcal{O}}^{\text{ex}}_{Y}$ as sheaves on Y. Checking that the unit and counit are natural transformations and that they form an adjunction follows immediately from the definitions. □

Theorem 5.1.10. *The categories* $\mathbf{C^{\infty}RS^{c}}$, $\mathbf{LC^{\infty}RS^{c}}$, $\mathbf{C^{\infty}RS^{c}_{in}}$ *and* $\mathbf{LC^{\infty}RS^{c}_{in}}$ *have all small limits. Small limits commute with the inclusion and forgetful functors in the following diagram, where* **Top** *is the category of topological spaces and continuous maps.*

$$\begin{array}{ccccc} \mathbf{LC^{\infty}RS^{c}_{in}} & \longrightarrow & \mathbf{C^{\infty}RS^{c}_{in}} & & \\ \downarrow & & \downarrow & & \\ \mathbf{LC^{\infty}RS^{c}} & \longrightarrow & \mathbf{C^{\infty}RS^{c}} & & \\ \downarrow & & \downarrow & & \\ \mathbf{LC^{\infty}RS^{c}} & \longrightarrow & \mathbf{C^{\infty}RS^{c}} & \longrightarrow & \mathbf{Top} \end{array}$$

The proof is essentially the same as showing ordinary ringed spaces have all small limits, however as this is not well known in the literature (see the discussion in the introduction to Gillam [27]), we include the proof here. Note that limits in ordinary locally ringed spaces are different from their limits as ordinary ringed spaces; this is due the fact that pushouts of local rings are not always local. However pushouts of local C^{∞}-rings (with corners) are local, and so small limits of local C^{∞}-ringed spaces (with corners) coincide with small limits of C^{∞}-ringed spaces (with corners).

Proof. We first show that all fibre products of (local) C^{∞}-ringed spaces with corners exist; as there is also a final object, $(*, (\mathbb{R}, [0, \infty))$ the point with local C^{∞}-ring with corners $(\mathbb{R}, [0, \infty))$, then all finite limits exist. When then explain how to show all small products exist, which shows all small limits exist.

Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be (local) C^{∞}-ringed spaces with corners, and let there be morphisms $(f, \boldsymbol{f}^{\sharp}) = \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$, $(g, \boldsymbol{g}^{\sharp}) = \boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$. We will construct the fibre product $\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Z},\boldsymbol{g}} \boldsymbol{Y} = \boldsymbol{W} = (W, \boldsymbol{\mathcal{O}}_{W})$ in $\mathbf{LC^{\infty}RS^{c}}$. We define $W = X \times_{f,Z,g} Y$ to be the fibre product of the topological spaces of $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$. At each point $(x, y) \in W$, where

$f(x) = g(y) = z$, we define the C^∞-ring with corners $\mathfrak{W}_{x,y} = \boldsymbol{\mathcal{O}}_{X,x} \amalg_{\boldsymbol{f}_x^\sharp, \boldsymbol{\mathcal{O}}_{Z,z}, \boldsymbol{g}_y^\sharp} \boldsymbol{\mathcal{O}}_{Y,y}$ to be the pushout of the stalks, with projections $\boldsymbol{q_{1,(x,y)}} : \boldsymbol{\mathcal{O}}_{X,x} \to \mathfrak{W}_{x,y}$ and $\boldsymbol{q_{2,(x,y)}} : \boldsymbol{\mathcal{O}}_{Y,y} \to \mathfrak{W}_{x,y}$. Note that $\mathfrak{W}_{x,y}$ is a local C^∞-ring with corners if each stalk is also local, by Proposition 4.6.3.

Let U be an open set in W. We define $\boldsymbol{\mathcal{O}}_W(U) = \{\boldsymbol{s} : U \to \coprod_{(x,y)\in U} \mathfrak{W}_{x,y}\}$ such that for all $\boldsymbol{s} \in \boldsymbol{\mathcal{O}}_W(U)$, for all $(\tilde{x}, \tilde{y}) \in U$, there are open sets $V_1 \subset X$, $V_2 \subset Y$, and $V_3 \subset Z$, with $f(V_1) \subset V_3, g(V_2) \subset V_3$, and $(\tilde{x}, \tilde{y}) \in V_1 \times_{V_3} V_2 \subset U$ such that $\boldsymbol{s}_{(x,y)} = \boldsymbol{\pi}_{x,y}(\boldsymbol{w})$ for some $\boldsymbol{w}$ in the pushout $\boldsymbol{\mathcal{O}}_X(V_1) \amalg_{\boldsymbol{\mathcal{O}}_Z(V_3)} \boldsymbol{\mathcal{O}}_Y(V_2)$ for all $(x, y) \in V_1 \times_{V_3} V_2$. Here $\boldsymbol{\pi}_{x,y}$ is the unique map $\boldsymbol{\pi}_{x,y} : \boldsymbol{\mathcal{O}}_X(V_1) \amalg_{\boldsymbol{\mathcal{O}}_Z(V_3)} \boldsymbol{\mathcal{O}}_Y(V_2) \to \boldsymbol{\mathcal{O}}_{X,x} \amalg_{\boldsymbol{\mathcal{O}}_{Z,z}} \boldsymbol{\mathcal{O}}_{Y,y}$, which exists by the universal property of the pushout in the domain.

We give $\boldsymbol{\mathcal{O}}_W(U)$ the structure of a C^∞-ring with corners using the C^∞-ring with corners structure from the stalks $\mathfrak{W}_{x,y}$. For example, for a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, then $\Phi_f(\boldsymbol{s})_{(x,y)} = \boldsymbol{\pi}_{x,y}(\Phi_f(\boldsymbol{w}))$. Then $\boldsymbol{\mathcal{O}}_W$ is a sheaf of (local) C^∞-rings with corners on W, with stalks $\mathfrak{W}_{x,y}$ at points $(x, y) \in W$.

We define maps $(q_1, \boldsymbol{q_1}^\sharp) = \boldsymbol{q_1} : \boldsymbol{W} \to \boldsymbol{X}$ and $(q_2, \boldsymbol{q_2}^\sharp) = \boldsymbol{q_2} : \boldsymbol{W} \to \boldsymbol{Y}$, where $q_1 : W \to X, (x, y) \mapsto x$ and $q_2 : W \to Y, (x, y) \mapsto y$ are the usual projection maps defined in the fibre product of topological spaces. On stalks, we have that $\boldsymbol{q_{1,x}}^\sharp : \boldsymbol{\mathcal{O}}_{X,x} \to \boldsymbol{\mathcal{O}}_{W,(x,y)}$ is the map $\boldsymbol{q_{1,(x,y)}}$, and similarly $\boldsymbol{q_{2,y}}^\sharp = \boldsymbol{q_{2,(x,y)}}$. We need to show that these maps glue to form maps $\boldsymbol{q_1}^\sharp$ and $\boldsymbol{q_2}^\sharp$. We describe this for $\boldsymbol{q_1}_{,\sharp}(U) : \boldsymbol{\mathcal{O}}_X(U) \to \boldsymbol{\mathcal{O}}_W(q_1^{-1}(U))$, where $\boldsymbol{q_1}_{,\sharp}$ corresponds to $\boldsymbol{q_1}^\sharp$ by (2.4.2).

Take $\boldsymbol{s} \in \boldsymbol{\mathcal{O}}_X(U)$, then, by definition of pushout, there is a map $\boldsymbol{q_{1,U}} : \boldsymbol{\mathcal{O}}_X(U) \to \boldsymbol{\mathcal{O}}_X(U) \amalg_{\boldsymbol{\mathcal{O}}_Z(Z)} \boldsymbol{\mathcal{O}}_Y(Y)$. This pushout is not necessarily isomorphic to $\boldsymbol{\mathcal{O}}_W(q_1^{-1}(U))$, however we can define $\boldsymbol{q_1}_{,\sharp}(\boldsymbol{s})_{(x,y)} = \boldsymbol{\pi}_{x,y} \circ \boldsymbol{q_{1,U}}(\boldsymbol{s})$ for all $(x, y) \in q_1^{-1}(U) = U \times_Z Y$, and this is a well defined element of $\boldsymbol{\mathcal{O}}_W(q_1^{-1}(U))$. Using the universal properties of pushouts, this map at the level of stalks is the stalk map $\boldsymbol{q_{1,(x,y)}}$, and that $\boldsymbol{q_1}^\sharp : \boldsymbol{\mathcal{O}}_X \to f_*(\boldsymbol{\mathcal{O}}_W)$ is a well behaved map of sheaves. A similar construction gives $\boldsymbol{q_2}^\sharp$.

Finally, we must show the universal property holds for $\boldsymbol{W}$. This follows from the universal property of the fibre product $X \times_Z Y$. Here, we again glue maps that result from the universal properties from the pushouts of the stalks $\boldsymbol{\mathcal{O}}_{X,x} \amalg_{f_x^\sharp, \boldsymbol{\mathcal{O}}_{Z,z}, g_y^\sharp} \boldsymbol{\mathcal{O}}_{Y,y}$, using the universal properties from $\boldsymbol{\mathcal{O}}_X(V_1) \amalg_{\boldsymbol{\mathcal{O}}_Z(V_3)} \boldsymbol{\mathcal{O}}_Y(V_2)$ for open sets $V_1 \subset X$, $V_2 \subset Y$, and $V_3 \subset Z$, with $f(V_1) \subset V_3, g(V_2) \subset V_3$.

Hence $(W, \boldsymbol{\mathcal{O}}_W) = \boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ is the fibre product and all finite limits exist in the categories $\mathbf{C^\infty RS^c}$ and $\mathbf{LC^\infty RS^c}$. These fibre products commute with the forgetful functor $\mathbf{LC^\infty RS^c} \to \mathbf{C^\infty RS^c}$, as the pushout of local C^∞-rings is local. The construction of $\boldsymbol{W}$ shows that the limits in both $\mathbf{C^\infty RS^c}$ and $\mathbf{LC^\infty RS^c}$ commute with the forgetful functor to topological spaces.

To show the same construction applies for fibre products in the categories of $\mathbf{C^\infty RS^c_{in}}$ and $\mathbf{LC^\infty RS^c_{in}}$, we need to check that the resulting sheaf is interior. From Proposition 4.6.3, we know the stalks are interior. Take an open set $U \subset W$ and $s' \in \mathcal{O}^{\mathrm{ex}}_W(U)$. Say that $s'_{w_1} \neq 0 \in \mathcal{O}^{\mathrm{ex}}_{W,w_1}$ for some $w_1 \in U$, we need to show that s' is non-zero in every stalk in a neighbourhood of w_1. Now, in a neighbourhood $V \subset U$ of w_1, we have $s'_w = \pi^{\mathrm{ex}}_{x,y}(c')$ for all $w \in V$, and some c' in the monoid part of the pushout $\boldsymbol{\mathcal{O}}_X(V_1) \amalg_{\boldsymbol{\mathcal{O}}_Z(V_3)} \boldsymbol{\mathcal{O}}_Y(V_2)$, where $V_1 \times_{V_3} V_2 = V$. Then $s'_{w_1} \neq 0$ implies $\pi^{\mathrm{ex}}_{x,y}(c') \neq 0$ in the monoid part of $\boldsymbol{\mathcal{O}}_{X,x} \amalg_{\boldsymbol{\mathcal{O}}_{Z,z}} \boldsymbol{\mathcal{O}}_{Y,y}$, and that $c' \neq 0$.

This means

$$\begin{aligned} c' = \Psi_h(&q_1(a_1), \ldots, q_1(a_m), q_2(b_1), \ldots, q_2(b_n), \\ &q_1^{\mathrm{ex}}(a'_1), \ldots, q_1^{\mathrm{ex}}(a'_k), q_2^{\mathrm{ex}}(b'_1), \ldots, q_2^{\mathrm{ex}}(b'_l)) \end{aligned}$$

for some smooth $h : \mathbb{R}^{m+n}_{k+l} \to [0, \infty)$ where $a_i \in \mathcal{O}_X(V_1)$, $b_i \in \mathcal{O}_Y(V_2)$, $a'_i \in \mathcal{O}^{\mathrm{ex}}_X(V_1)$ and $b'_i \in \mathcal{O}^{\mathrm{ex}}_Y(V_2)$. Here, $\boldsymbol{q_1} : \boldsymbol{\mathcal{O}}_X(V_1) \to \boldsymbol{\mathcal{O}}_X(V_1) \amalg_{\boldsymbol{\mathcal{O}}_Z(V_3)} \boldsymbol{\mathcal{O}}_Y(V_2)$ and $\boldsymbol{q_2} : \boldsymbol{\mathcal{O}}_Y(V_2) \to \boldsymbol{\mathcal{O}}_X(V_1) \amalg_{\boldsymbol{\mathcal{O}}_Z(V_3)} \boldsymbol{\mathcal{O}}_Y(V_2)$ are the inclusion morphisms coming from the fibre product.

Then

$$h(x_1, \ldots, x_{m+n}, y_1, \ldots, y_{k+l}) = y_1^{t_1} \cdots y_{n+m}^{t_{m+n}} \Psi_F(x_1, \ldots, x_{m+n}, y_1, \ldots, y_{k+l})$$

with $F : \mathbb{R}^{m+n}_{k+l} \to [0, \infty)$ smooth and positive, and t_i non-negative integers. As $\pi^{\mathrm{ex}}_{x,y}$ respects the C^∞-operations, then applying it to c', and given that the stalks are interior, we see that $\pi^{\mathrm{ex}}_{x,y}(c') \neq 0$ if and only for all non-zero t_i then $\pi^{\mathrm{ex}}_{x,y} \circ q_1^{\mathrm{ex}}(a'_i) \neq 0$ and $\pi^{\mathrm{ex}}_{x,y} \circ q_2^{\mathrm{ex}}(b'_i) \neq 0$. This implies these a'_i and b'_i are non-zero in the stalks $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$. As $\boldsymbol{X}$ and $\boldsymbol{Y}$ are interior, then these a'_i and b'_i must be non-zero in open neighbourhoods containing x and y respectively.

As there are finitely many a'_i and b'_i, intersecting these open neighbourhoods, we must have an open neighbourhood $V_x \subset V_1$ of x and an open neighbourhood $V_y \subset V_2$ of y where these a'_i and b'_i are all non-zero in their respective neighbourhoods. Then c' must be non-zero in every stalk in the open neighbourhood $V_x \times_{V_3} V_y \subset W$ of w_1, and so s' must be non-zero in this open neighbourhood, as required. Therefore the fibre product of interior (local) C^∞-ringed spaces with corners is interior, and all fibre products exist in $\mathbf{C^\infty RS^c_{in}}$ and $\mathbf{LC^\infty RS^c_{in}}$.

We can extend this proof to small products. If $\{\boldsymbol{X}_i\}_{i \in I}$ is a collection of (local) C^∞-ringed spaces with corners, then we can construct the product $\boldsymbol{X}$. Its underlying topological space is the product of the X_i's. Its stalks are the coproduct of the stalks, and its sheaf of C^∞-rings with corners is constructed in the same way as above. Proposition 4.6.3 again says that if the $\boldsymbol{X}_i$ are local, then $\boldsymbol{X}$ is local, and that if the $\boldsymbol{X}_i$ are interior, the stalks of $\boldsymbol{X}$ are interior. To show that $\boldsymbol{X}$ is interior follows as above, by understanding that each element c' in the monoid part of a coproduct is generated by finitely many elements from the monoids in the coproduct. The universal properties follow directly from the universal properties of C^∞-ring with corners coproducts and topological space products. $\square$

Definition 5.1.11. Let $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ be a local C^∞-ringed space with corners. We have that $\boldsymbol{\mathcal{O}}_X = (\mathcal{O}_X, \mathcal{O}_X^{\rm ex})$, where $\mathcal{O}_X$ is a sheaf of C^∞-rings on X. We define a forgetful functor $\tau : \mathbf{LC^\infty RS^c} \to \mathbf{LC^\infty RS}$ by sending objects $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X) \mapsto (X, \mathcal{O}_X)$ and morphisms $\boldsymbol{f} = (f, \boldsymbol{f}^\sharp) = (f, (f^\sharp, f^\sharp_{\rm ex})) \mapsto (f, f^\sharp)$. We define a forgetful functor $\tau_{\rm in}$: $\mathbf{LC^\infty RS^c_{in}} \to \mathbf{LC^\infty RS}$ by $\tau_{\rm in} = \tau|_{\mathbf{LC^\infty RS^c_{in}}}$.

Proposition 5.1.12. *The forgetful functor* $\tau : \mathbf{LC^\infty RS^c} \to \mathbf{LC^\infty RS}$ *has a right adjoint, hence it preserves colimits. It also has a left adjoint, so it preserves limits.*

Proof. We begin by constructing the right adjoint on objects. Take $(X, \mathcal{O}_X) \in \mathbf{LC^\infty RS}$ and construct $\hat{\mathcal{O}}_X^{\rm ex}$ by sheafifying the following presheaf of monoids $\mathcal{P}\hat{\mathcal{O}}_X^{\rm ex}$ where, for an open set $U \subset X$, then $\mathcal{P}\hat{\mathcal{O}}_X^{\rm ex}(U) = \Phi_{\rm exp}(\mathcal{O}_X(U)) \amalg \{0_{\rm ex}\}$, for $\amalg$ the disjoint union. The restriction map on the presheaf is $\rho_{UV}(s') = \Phi_{\rm exp}(\rho_{UV}(s))$ where $s' = \Phi_{\rm exp}(s)$, and zero otherwise. Then $(\mathcal{O}_X, \hat{\mathcal{O}}_X^{\rm ex})$ is a sheaf of C^∞-rings with corners, as on connected components, it is defined using the $F_{\rm exp}$ from Theorem 4.3.9. The right adjoint then sends $(X, \mathcal{O}_X) \in \mathbf{LC^\infty RS}$ to $(X, \mathcal{O}_X, \hat{\mathcal{O}}_X^{\rm ex})$.

A morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y))$ naturally extends to a morphism $(f, f^\sharp, f^\sharp_{\rm ex})$: $(X, \mathcal{O}_X, \hat{\mathcal{O}}_X^{\rm ex}) \to (Y, \mathcal{O}_Y, \hat{\mathcal{O}}_Y^{\rm ex})$. Here, on a connected component $U \subset X$, then $s' \in f^{-1}(\hat{\mathcal{O}}_Y^{\rm ex})(U)$ is either in $f^{-1}(\Phi_{\rm exp}(\mathcal{O}_Y))(U) \cong \Phi_{\rm exp}(f^{-1}(\mathcal{O}_Y))(U)$ or is in $f^{-1}(0_{\rm ex})(U) = 0_{\rm ex}$. If it is in $\Phi_{\rm exp}(f^{-1}(\mathcal{O}_X))(U)$, then $s' = \Phi_{\rm exp}(s) \mapsto \Phi_{\rm exp}(f^\sharp(s)) \in \hat{\mathcal{O}}_X^{\rm ex}(U)$ and otherwise it maps to $0_{\rm ex} \in \hat{\mathcal{O}}_X^{\rm ex}(U)$.

This construction gives the right adjoint to the forgetful functor. The counit is the identity, and the unit at the object $(X, \mathcal{O}_X, \mathcal{O}_X^{\rm ex}) \in \mathbf{LC^\infty RS^c}$ is the identity on the sheaves of C^∞-rings and on the topological spaces. On the sheaves of monoids, for a connected component $U \subset X$, then $s' \in \hat{\mathcal{O}}_Y^{\rm ex}(U)$ is either in $\Phi_{\rm exp}(\mathcal{O}_X(U))$ or it is equal to $0_{\rm ex}$. In the former case, the unit maps s' to $\Psi_{\rm exp}(s') \in \mathcal{O}_X^{\rm ex}(U)$, and in the latter case it maps to $0 \in \mathcal{O}_X^{\rm ex}(U)$.

As the forgetful functor is then a left adjoint, it preserves colimits.

Recalling in the proof of the existence of fibre products from Theorem 5.1.10, fibre products in $\mathbf{LC^\infty RS^c}$ are constructed by glueing the pushout of the stalks using the pushouts of C^∞-rings with corners. By Theorem 4.3.9, the C^∞-ring of these pushouts is the pushout of the underlying C^∞-rings. As the construction for fibre products in $\mathbf{LC^\infty RS}$ is the same as in $\mathbf{LC^\infty RS^c}$ just using C^∞-rings, and all finite limits in a category with a terminal object are composed of a finite number of iterated fibre products, then τ preserves finite limits. However, more generally, we can construct a left adjoint to τ, showing that τ preserves all limits (not just finite ones.)

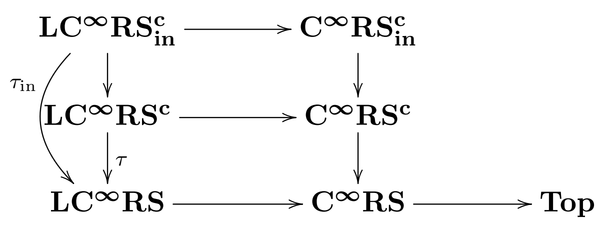
The left adjoint is constructed as follows. On objects, $(X, \mathcal{O}_X) \in \mathbf{LC^\infty RS}$, construct $\mathcal{O}^{\rm ex}_{X,\geqslant 0}$ as the sheaf $\mathcal{O}^{\rm ex}_{X,\geqslant 0}(U) = F_{\geqslant 0}(\mathcal{O}_X(U))$, where $F_{\geqslant 0}$ is defined in Theorem 4.3.9. The definition of $F_{\geqslant 0}$ ensures this is already a sheaf, and that $(X, \mathcal{O}_X, \mathcal{O}^{\rm ex}_{X,\geqslant 0})$ is in $\mathbf{LC^\infty RS^c}$.

On morphisms, $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ naturally extends to a morphism $(f, f^\sharp, f^\sharp_{\rm ex}) : (X, \mathcal{O}_X, \mathcal{O}^{\rm ex}_{X,\geqslant 0}) \to (Y, \mathcal{O}_Y, \mathcal{O}^{\rm ex}_{Y,\geqslant 0})$. Here, for open $U \subset X$, then $s' \in f^{-1}(\mathcal{O}^{\rm ex}_{Y,\geqslant 0})(U)$ maps to $f^\sharp(U)(s') \in \mathcal{O}^{\rm ex}_{X,\geqslant 0}(U)$, which is well defined as $\mathcal{O}^{\rm ex}_{X,\geqslant 0}$ is a subsheaf of $\mathcal{O}_X$, and $f^\sharp$ respects the C^∞-operations.

This functor is a left adjoint to the forgetful functor. The unit is the identity natural transformation. The counit is the identity on the topological space and the sheaves of C^∞-rings. On the sheaves of monoids, the counit is the C^∞-operation Φ_i. This gives a well defined natural transformation, and makes this functor the left adjoint, as required. □

Remark 5.1.13. The forgetful functor $\tau : \mathbf{LC^\infty RS^c} \to \mathbf{LC^\infty RS}$ in Proposition 5.1.12 restricted to $\mathbf{LC^\infty RS^c_{in}}$ has the same right adjoint, and therefore the restriction of this functor, $\tau_{\rm in}$, also preserves colimits in $\mathbf{LC^\infty RS^c_{in}}$. Composing the left adjoint in Proposition 5.1.12 and the left adjoint in Proposition 5.1.9 gives a left adjoint to $\mathbf{LC^\infty RS^c_{in}} \to \mathbf{LC^\infty RS}$, showing that $\tau_{\rm in}$, also preserves limits in $\mathbf{LC^\infty RS^c_{in}}$.

We have that small limits and small colimits commute around the following diagram of forgetful functors and inclusion functors.



5.2 Spectrum functor

Note that as in §5.1 we are using the notation that for a sheaf $\mathcal{O}_X$ of local C^∞-rings and an element $s \in \mathcal{O}_X(U)$ for open $U \subset X$, then s_x is value of s in the stalk $\mathcal{O}_{X,x}$ and $s(x)$ value of s_x under the stalk map $\mathcal{O}_{X,x} \to \mathbb{R}$.

We now define a spectrum functor for C^∞-rings with corners, in a similar way to Definition 2.4.4.

Definition 5.2.1. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a C^∞-ring with corners, and use the notation from Definition 4.6.7. As in Definition 2.4.3, write $X_{\mathfrak{C}}$ for the set of $\mathbb{R}$-points of $\mathfrak{C}$ with topology $\mathcal{T}_{\mathfrak{C}}$. For each open $U \subseteq X_{\mathfrak{C}}$, define $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}(U) = (\mathcal{O}_{X_{\mathfrak{C}}}(U), \mathcal{O}^{\rm ex}_{X_{\mathfrak{C}}}(U))$. Here $\mathcal{O}_{X_{\mathfrak{C}}}(U)$ is the set of functions $s : U \to \coprod_{x\in U} \mathfrak{C}_x$ (where we write s_x for its value at the point $x \in U$) such that $s_x \in \mathfrak{C}_x$ for all $x \in U$, and such that U may be covered by open $W \subseteq U$ for which there exist $c \in \mathfrak{C}$ with $s_x = \pi_x(c)$ in $\mathfrak{C}_x$ for all $x \in W$. Similarly, $\mathcal{O}^{\rm ex}_{X_{\mathfrak{C}}}(U)$ is the set of $s' : U \to \coprod_{x\in U} \mathfrak{C}_{x,{\rm ex}}$ with $s'_x \in \mathfrak{C}_{x,{\rm ex}}$ for all $x \in U$, and such that U may be covered by open $W \subseteq U$ for which there exist $c' \in \mathfrak{C}_{\rm ex}$ with $s'_x = \pi_{x,{\rm ex}}(c')$ in $\mathfrak{C}_{x,{\rm ex}}$ for all $x \in W$.

Define operations Φ_f and Ψ_g on $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}(U)$ pointwise in $x \in U$ using the operations Φ_f and Ψ_g on $\boldsymbol{\mathfrak{C}}_x$. This makes $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}(U)$ into a C^∞-ring with corners. If $V \subseteq U \subseteq X_{\mathfrak{C}}$ are open, the restriction maps $\boldsymbol{\rho}_{UV} = (\rho_{UV}, \rho_{UV,{\rm ex}}) : \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}(U) \to \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}(V)$ mapping $\rho_{UV} : s \mapsto s|_V$ and $\rho_{UV,{\rm ex}} : s' \mapsto s'|_V$ are morphisms of C^∞-rings with corners.

The local nature of the definition implies that $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}} = (\mathcal{O}_{X_{\mathfrak{C}}}, \mathcal{O}^{\rm ex}_{X_{\mathfrak{C}}})$ is a sheaf of C^∞-rings with corners on $X_{\mathfrak{C}}$. In fact, $\mathcal{O}_{X_{\mathfrak{C}}}$ is the sheaf of C^∞-rings in Definition 2.4.4. By Proposition 5.2.3 below, the stalk $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}},x}$ at $x \in X_{\mathfrak{C}}$ is naturally isomorphic to $\boldsymbol{\mathfrak{C}}_x$, which is a local C^∞-ring with corners by Theorem 4.6.8(a). Hence $(X_{\mathfrak{C}}, \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}})$ is a local C^∞-ringed space with corners, which we call the *spectrum* of $\boldsymbol{\mathfrak{C}}$, and write as $\operatorname{Spec}^{\rm c} \boldsymbol{\mathfrak{C}}$.

Now let $\boldsymbol{\phi} = (\phi, \phi_{\rm ex}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ be a morphism of C^∞-rings with corners. As in Definition 2.4.4, define the continuous function $f_\phi : X_{\mathfrak{D}} \to X_{\mathfrak{C}}$ by $f_\phi(x) = x \circ \phi$. For $U \subseteq X_{\mathfrak{C}}$ open define $(\boldsymbol{f}_\phi)_\sharp(U) : \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}(U) \to \boldsymbol{\mathcal{O}}_{X_{\mathfrak{D}}}(f_\phi^{-1}(U))$ by $(\boldsymbol{f}_\phi)_\sharp(U)\boldsymbol{s}_x = \boldsymbol{\phi}_x(\boldsymbol{s}_{f_\phi(x)})$, where $\boldsymbol{\phi}_x : \boldsymbol{\mathfrak{C}}_{f_\phi(x)} \to \boldsymbol{\mathfrak{D}}_x$ is the induced morphism of local C^∞-rings with corners and $\boldsymbol{s} = (s, s') \in \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}(U)$. Then $(\boldsymbol{f}_\phi)_\sharp : \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}} \to (f_\phi)_*(\boldsymbol{\mathcal{O}}_{X_{\mathfrak{D}}})$ is a morphism of sheaves of C^∞-rings with corners on $X_{\mathfrak{C}}$.

Let $\boldsymbol{f}^\sharp_\phi : f_\phi^{-1}(\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}) \to \boldsymbol{\mathcal{O}}_{X_{\mathfrak{D}}}$ be the corresponding morphism of sheaves of C^∞-rings with corners on $X_{\mathfrak{D}}$ under (2.4.2). The stalk map $\boldsymbol{f}^\sharp_{\phi,x} : \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}},f_\phi(x)} \to \boldsymbol{\mathcal{O}}_{X_{\mathfrak{D}},x}$ of $\boldsymbol{f}^\sharp_\phi$ at $x \in X_{\mathfrak{D}}$ is identified with $\boldsymbol{\phi}_x : \boldsymbol{\mathfrak{C}}_{f_\phi(x)} \to \boldsymbol{\mathfrak{D}}_x$ under the isomorphisms $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}},f_\phi(x)} \cong \boldsymbol{\mathfrak{C}}_{f_\phi(x)}$, $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{D}},x} \cong \boldsymbol{\mathfrak{D}}_x$ in Proposition 5.2.3. Then $\boldsymbol{f}_\phi = (f_\phi, \boldsymbol{f}^\sharp_\phi) : (X_{\mathfrak{D}}, \boldsymbol{\mathcal{O}}_{X_{\mathfrak{D}}}) \to (X_{\mathfrak{C}}, \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}})$ is a morphism of local C^∞-ringed spaces with corners. Define $\operatorname{Spec}^{\rm c} \boldsymbol{\phi} : \operatorname{Spec}^{\rm c} \boldsymbol{\mathfrak{D}} \to \operatorname{Spec}^{\rm c} \boldsymbol{\mathfrak{C}}$

by $\mathrm{Spec^c}\,\boldsymbol{\phi} = \boldsymbol{f}_\phi$. Then $\mathrm{Spec^c}$ is a functor $(\mathbf{C^\infty Rings^c})^{\mathbf{op}} \to \mathbf{LC^\infty RS^c}$, the *spectrum functor*.

Remark 5.2.2. Kalashnikov [51, §4.8] defined the *real spectrum* of a pre C^∞-ring with corners using the same topological space as our spectrum. However they required it to have a sheaf of pre C^∞-rings with corners that uses localisations of the pre C^∞-ring with corners on subsets that correspond to open sets. Lemma 5.4.4 and an argument using universal properties will show the stalks are isomorphic in both definitions so that the real spectrum is equivalent to our spectrum on C^∞-rings with corners.

Proposition 5.2.3. *In Definition* 5.2.1, *the stalk* $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}},x}$ *of* $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}$ *at* $x \in X_{\mathfrak{C}}$ *is naturally isomorphic to* $\boldsymbol{\mathfrak{C}}_x$.

Proof. We have $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}},x} = (\mathcal{O}_{X_{\mathfrak{C}},x}, \mathcal{O}^{\mathrm{ex}}_{X_{\mathfrak{C}},x})$ where elements $[U,s] \in \mathcal{O}_{X_{\mathfrak{C}},x}$ and $[U,s'] \in \mathcal{O}^{\mathrm{ex}}_{X_{\mathfrak{C}},x}$ are $\sim$-equivalence classes of pairs (U,s) and (U,s'), where U is an open neighbourhood of x in $X_{\mathfrak{C}}$ and $s \in \mathcal{O}_{X_{\mathfrak{C}}}(U)$, $s' \in \mathcal{O}^{\mathrm{ex}}_{X_{\mathfrak{C}}}(U)$, and $(U,s) \sim (V,t)$, $(U,s') \sim (V,t')$ if there exists open $x \in W \subseteq U \cap V$ with $s|_W = t|_W$ in $\mathcal{O}_{X_{\mathfrak{C}}}(W)$ and $s'|_W = t'|_W$ in $\mathcal{O}^{\mathrm{ex}}_{X_{\mathfrak{C}}}(W)$. Define a morphism of C^∞-rings with corners $\boldsymbol{\Pi} = (\Pi, \Pi_{\mathrm{ex}}) : \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}},x} \to \boldsymbol{\mathfrak{C}}_x$ by $\Pi : [U,s] \mapsto s_x \in \mathfrak{C}_x$ and $\Pi_{\mathrm{ex}} : [U,s'] \mapsto s'_x \in \mathfrak{C}_{x,\mathrm{ex}}$.

Suppose $c_x \in \mathfrak{C}_x$ and $c'_x \in \mathfrak{C}_{x,\mathrm{ex}}$. Then $c_x = \pi_x(c)$ for $c \in \mathfrak{C}_x$ and $c'_x = \pi_{x,\mathrm{ex}}(c')$ for $c' \in \mathfrak{C}_{x,\mathrm{ex}}$ by Theorem 4.6.8(c). Define $s : X_{\mathfrak{C}} \to \coprod_{y \in X_{\mathfrak{C}}} \mathfrak{C}_y$ and $s' : X_{\mathfrak{C}} \to \coprod_{y \in X_{\mathfrak{C}}} \mathfrak{C}_{y,\mathrm{ex}}$ by $s_y = \pi_y(c)$ and $s'_y = \pi_{y,\mathrm{ex}}(c')$. Then $s \in \mathcal{O}_{X_{\mathfrak{C}}}(X_{\mathfrak{C}})$, so that $[X_{\mathfrak{C}}, s] \in \mathcal{O}_{X_{\mathfrak{C}},x}$ with $\Pi([X_{\mathfrak{C}}, s]) = s_x = \pi_x(c) = c_x$, and similarly $s' \in \mathcal{O}^{\mathrm{ex}}_{X_{\mathfrak{C}}}(X_{\mathfrak{C}})$ with $\Pi_{\mathrm{ex}}([X_{\mathfrak{C}}, s']) = c'_x$. Hence $\Pi : \mathcal{O}_{X_{\mathfrak{C}},x} \to \mathfrak{C}_x$ and $\Pi_{\mathrm{ex}} : \mathcal{O}^{\mathrm{ex}}_{X_{\mathfrak{C}},x} \to \mathfrak{C}_{x,\mathrm{ex}}$ are surjective.

Let $[U_1, s_1], [U_2, s_2] \in \mathcal{O}_{X_{\mathfrak{C}},x}$ with $\Pi([U_1, s_1]) = s_{1,x} = s_{2,x} = \Pi([U_2, s_2])$. Then by definition of $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}(U_1), \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}(U_2)$ there exists an open neighbourhood V of x in $U_1 \cap U_2$ and $c_1, c_2 \in \mathfrak{C}$ with $s_{1,v} = \pi_v(c_1)$ and $s_{2,v} = \pi_v(c_2)$ for all $v \in V$. Thus $\pi_x(c_1) = \pi_x(c_2)$ as $s_{1,x} = s_{2,x}$. Hence $c_1 - c_2$ lies in the ideal I in (2.1.4) by Proposition 2.1.15. Thus there exists $d \in \mathfrak{C}$ with $x(d) \neq 0 \in \mathbb{R}$ and $d \cdot (c_1 - c_2) = 0 \in \mathfrak{C}$.

Making V smaller we can suppose that $v(d) \neq 0$ for all $v \in V$, as this is an open condition. Then $\pi_v(c_1) = \pi_v(c_2) \in \mathfrak{C}_v$ for $v \in V$, since $\pi_v(d) \cdot \pi_v(c_1) = \pi_v(d) \cdot \pi_v(c_2)$ as $d \cdot c_1 = d \cdot c_2$ and $\pi_v(d)$ is invertible in $\mathfrak{C}_v$. Thus $s_{1,v} = \pi_v(c_1) = \pi_v(c_2) = s_{2,v}$ for $v \in V$, so $s_1|_V = s_2|_V$, and $[U_1, s_1] = [V, s_1|_V] = [V, s_2|_V] = [U_2, s_2]$. Therefore $\Pi : \mathcal{O}_{X_{\mathfrak{C}},x} \to \mathfrak{C}_x$ is injective, and an isomorphism.

Suppose $[U_1, s'_1], [U_2, s'_2] \in \mathcal{O}^{\mathrm{ex}}_{X_{\mathfrak{C}},x}$ with $\Pi_{\mathrm{ex}}([U_1, s'_1]) = s'_{1,x} = s'_{2,x} = \Pi([U_2, s'_2])$. As above there exist an open neighbourhood V of x in $U_1 \cap U_2$ and $c'_1, c'_2 \in \mathfrak{C}_{\mathrm{ex}}$ with $s'_{1,v} = \pi_{v,\mathrm{ex}}(c'_1)$ and $s'_{2,v} = \pi_{v,\mathrm{ex}}(c'_2)$ for all $v \in V$. At this point we can use Lemma 4.6.9, which says that $\pi_{x,\mathrm{ex}}(x'_2) = \pi_{x,\mathrm{ex}}(c'_1)$ if and only if there are $a, b \in \mathfrak{C}_{\mathrm{ex}}$ such that $\Phi_i(a) - \Phi_i(b) \in I_x$, $x \circ \Phi_i(a) \neq 0$ and $ac_1 = bc_2$, where I_x is the ideal in (2.1.4). The third condition does not depend on x, whereas the first two conditions are open conditions in x, that is, if $\Phi_i(a) - \Phi_i(b) \in I_x$, $x \circ \Phi_i(a) \neq 0$, then there is an open neighbourhood of X such that $\Phi_i(a) - \Phi_i(b) \in I_v$, $v \circ \Phi_i(a) \neq 0$ for all v in that neighbourhood.

Making V above smaller if necessary, we can suppose that these conditions hold in V and thus that $\pi_{v,\mathrm{ex}}(c'_1) = \pi_{v,\mathrm{ex}}(c'_2)$ for all $v \in V$. Hence $s'_{1,v} = s'_{2,v}$ for all $v \in V$, and $s'_1|_V = s'_2|_V$, so that $[U_1, s'_1] = [V, s'_1|_V] = [V, s'_2|_V] = [U_2, s'_2]$. Therefore $\Pi_{\mathrm{ex}} : \mathcal{O}^{\mathrm{ex}}_{X_{\mathfrak{C}},x} \to \mathfrak{C}_{x,\mathrm{ex}}$ is injective, and an isomorphism. So $\mathbf{\Pi} = (\Pi, \Pi_{\mathrm{ex}}) : \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}},x} \to \mathfrak{C}_x$ is an isomorphism, as we have to prove. $\square$

Definition 5.2.4. As $\mathbf{C^\infty Rings^c_{in}}$ is a subcategory of $\mathbf{C^\infty Rings^c}$ we can define the functor $\mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}}$ by restricting $\mathrm{Spec}^{\mathrm{c}}$ to $(\mathbf{C^\infty Rings^c_{in}})^{\mathbf{op}}$. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ be an interior C^∞-ring with corners, and $\boldsymbol{X} = \mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}}(\boldsymbol{\mathfrak{C}}) = (X, \mathcal{O}_X)$. Definition 4.6.4 implies the localisations $\boldsymbol{\mathfrak{C}}_x$ are interior C^∞-rings with corners, and $\boldsymbol{\mathfrak{C}}_x \cong \boldsymbol{\mathcal{O}}_{X,x}$ by Proposition 5.2.3.

If $s' \in \mathcal{O}^{\mathrm{ex}}_X(U)$ such that $s'_x \neq 0 \in \boldsymbol{\mathcal{O}}_{X,x}$ at some $x \in X$, then we know $s'_{x'} = \pi^{\mathrm{ex}}_{x'}(c')$ for some c' for all x' an open set V containing x. As $\boldsymbol{\mathfrak{C}}_x \cong \boldsymbol{\mathcal{O}}_{X,x}$, and $s'_x \neq 0$ then $\pi^{\mathrm{ex}}_{x'}(c') \neq 0 \in \mathfrak{C}_{x,\mathrm{ex}}$ and $c' \neq 0$ in $\mathfrak{C}_{\mathrm{ex}}$. As $\boldsymbol{\mathfrak{C}}$ is interior, then c' must be non-zero in every stalk by Remark 4.6.10, hence s' must be non-zero in U. So $\boldsymbol{X}$ is an interior local C^∞-ringed space with corners as in Definition 5.1.3.

If $\boldsymbol{\phi} : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ is a morphism of interior C^∞-rings with corners, then $\mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}}\, \boldsymbol{\phi} = (f, \boldsymbol{f}^\sharp)$ has stalk map $\boldsymbol{f}^\sharp_x = \boldsymbol{\phi}_x : \boldsymbol{\mathfrak{C}}_{f_\phi(x)} \to \boldsymbol{\mathfrak{D}}_x$. This map fits into the commutative diagram

$$\begin{array}{ccc} \boldsymbol{\mathfrak{C}} & \xrightarrow{\quad\boldsymbol{\phi}\quad} & \boldsymbol{\mathfrak{D}} \\ \downarrow{\scriptstyle \boldsymbol{\pi}_{f_\phi(x)}} & & {\scriptstyle \boldsymbol{\pi}_x}\downarrow \\ \boldsymbol{\mathfrak{C}}_{f_\phi(x)} & \xrightarrow{\quad\boldsymbol{\phi}_x\quad} & \boldsymbol{\mathfrak{D}}_x. \end{array} \tag{5.2.1}$$

As $\boldsymbol{\phi}$ is interior, and the maps $\boldsymbol{\pi}_{f_\phi(x)}, \boldsymbol{\pi}_x$ are interior and surjective, then $\boldsymbol{f}^\sharp_x$ is interior. This implies $\mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}}\, \boldsymbol{\phi}$ is an interior morphism of interior local C^∞-ringed spaces with corners. Hence $\mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}} : (\mathbf{C^\infty Rings^c_{in}})^{\mathbf{op}} \to \mathbf{LC^\infty RS^c_{in}}$ is a well defined functor, which we call the *interior spectrum functor*.

Definition 5.2.5. The *global sections functor* $\Gamma^{\mathrm{c}} : \mathbf{LC^\infty RS^c} \to (\mathbf{C^\infty Rings^c})^{\mathbf{op}}$ takes element $(X, \boldsymbol{\mathcal{O}}_X) \in \mathbf{LC^\infty RS^c}$ to $\boldsymbol{\mathcal{O}}_X(X)$ and takes morphisms $(f, \boldsymbol{f}^\sharp) : (X, \boldsymbol{\mathcal{O}}_X) \to (Y, \boldsymbol{\mathcal{O}}_Y)$ to $\Gamma^{\mathrm{c}} : (f, \boldsymbol{f}^\sharp) \mapsto \boldsymbol{f}_\sharp(Y)$. Here $\boldsymbol{f}_\sharp : \boldsymbol{\mathcal{O}}_Y \to f_*(\boldsymbol{\mathcal{O}}_X)$ corresponds to $\boldsymbol{f}^\sharp$ under (2.4.2).

The composition $\Gamma^{\mathrm{c}} \circ \mathrm{Spec}^{\mathrm{c}}$ is a functor $(\mathbf{C^\infty Rings^c})^{\mathbf{op}} \to (\mathbf{C^\infty Rings^c})^{\mathbf{op}}$, or equivalently a functor $\mathbf{C^\infty Rings^c} \to \mathbf{C^\infty Rings^c}$. For each C^∞-ring with corners $\boldsymbol{\mathfrak{C}}$ and

$\boldsymbol{c} = (c, c') \in \mathfrak{C}$, we define $\boldsymbol{\Xi}_{\mathfrak{C}}(\boldsymbol{c}) = (\Xi(c), \Xi^{\rm ex}(c'))$ where $\Xi(c) : X_{\mathfrak{C}} \to \coprod_{x \in X_{\mathfrak{C}}} \mathfrak{C}_x$ with $\Xi(c) : x \mapsto \pi_x(c)$, and $\Xi^{\rm ex}(c) : X_{\mathfrak{C}} \to \coprod_{x \in X_{\mathfrak{C}}} \mathfrak{C}_{x,\rm ex}$ with $\Xi^{\rm ex}(c) : x \mapsto \pi_{x,\rm ex}(c')$. We will write this as $\boldsymbol{\Xi}_{\mathfrak{C}}(\boldsymbol{c}) : x \mapsto \boldsymbol{\pi}_x(\boldsymbol{c}) \in \mathfrak{C}_x$. Then $\boldsymbol{\Xi}_{\mathfrak{C}}(\boldsymbol{c}) \in \boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}(X_{\mathfrak{C}}) = \Gamma^{\rm c} \circ \operatorname{Spec}^{\rm c} \mathfrak{C}$ by Definition 5.2.1, so $\boldsymbol{\Xi}_{\mathfrak{C}} : \mathfrak{C} \to \Gamma^{\rm c} \circ \operatorname{Spec}^{\rm c} \mathfrak{C}$ is a map. Since $\boldsymbol{\pi}_x : \mathfrak{C} \to \mathfrak{C}_x$ is a morphism of C^∞-rings with corners and the C^∞-ring with corners operations on $\boldsymbol{\mathcal{O}}_{X_{\mathfrak{C}}}(X_{\mathfrak{C}})$ are defined pointwise in the $\mathfrak{C}_x$, this $\boldsymbol{\Xi}_{\mathfrak{C}}$ is a C^∞-ring with corners morphism. It is functorial in $\mathfrak{C}$, so that the $\boldsymbol{\Xi}_{\mathfrak{C}}$ for all $\mathfrak{C}$ define a natural transformation $\boldsymbol{\Xi} : \mathrm{id}_{\mathbf{C^\infty Rings^c}} \Rightarrow \Gamma^{\rm c} \circ \operatorname{Spec}^{\rm c}$ of functors $\mathbf{id}_{\mathbf{C^\infty Rings^c}}, \Gamma^{\rm c} \circ \operatorname{Spec}^{\rm c} : \mathbf{C^\infty Rings^c} \to \mathbf{C^\infty Rings^c}$.

Theorem 5.2.6. *The functor* $\operatorname{Spec}^{\rm c} : (\mathbf{C^\infty Rings^c})^{\mathbf{op}} \to \mathbf{LC^\infty RS^c}$ *is* ***right adjoint*** *to* $\Gamma^{\rm c} : \mathbf{LC^\infty RS^c} \to (\mathbf{C^\infty Rings^c})^{\mathbf{op}}$. *This implies for all* $\mathfrak{C} \in \mathbf{C^\infty Rings^c}$ *and all* $\boldsymbol{X} \in \mathbf{LC^\infty RS^c}$ *there are inverse bijections*

$$\operatorname{Hom}_{\mathbf{C^\infty Rings^c}}(\mathfrak{C}, \Gamma^{\rm c}(\boldsymbol{X})) \underset{R_{\mathfrak{C},\boldsymbol{X}}}{\overset{L_{\mathfrak{C},\boldsymbol{X}}}{\rightleftarrows}} \operatorname{Hom}_{\mathbf{LC^\infty RS^c}}(\boldsymbol{X}, \operatorname{Spec}^{\rm c} \mathfrak{C}). \tag{5.2.2}$$

If we let $\boldsymbol{X} = \operatorname{Spec}^{\rm c} \mathfrak{C}$ *then* $\boldsymbol{\Xi}_{\mathfrak{C}} = R_{\mathfrak{C},\boldsymbol{X}}(\mathbf{id}_{\boldsymbol{X}})$, *and* $\boldsymbol{\Xi}$ *is the unit of the adjunction between* $\Gamma^{\rm c}$ *and* $\operatorname{Spec}^{\rm c}$.

Proof. This proof follows the proof of [40, Th. 4.20]. Take $\boldsymbol{X} \in \mathbf{LC^\infty RS^c}$ and $\mathfrak{C} \in \mathbf{C^\infty Rings^c}$, and let $\boldsymbol{Y} = (Y, \boldsymbol{\mathcal{O}}_Y) = \operatorname{Spec}^{\rm c} \mathfrak{C}$. Define a functor $R_{\mathfrak{C},\boldsymbol{X}}$ in (5.2.2) by taking $R_{\mathfrak{C},\boldsymbol{X}}(\boldsymbol{f}) : \mathfrak{C} \to \Gamma^{\rm c}(\boldsymbol{X})$ to be the composition

$$\mathfrak{C} \xrightarrow{\boldsymbol{\Xi}_{\mathfrak{C}}} \Gamma^{\rm c} \circ \operatorname{Spec}^{\rm c} \mathfrak{C} = \Gamma^{\rm c}(\boldsymbol{Y}) \xrightarrow{\Gamma^{\rm c}(\boldsymbol{f})} \Gamma^{\rm c}(\boldsymbol{X}) \tag{5.2.3}$$

for each morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{LC^\infty RS^c}$. If $\boldsymbol{X} = \operatorname{Spec}^{\rm c} \mathfrak{C}$ then we have $\boldsymbol{\Xi}_{\mathfrak{C}} = R_{\mathfrak{C},\boldsymbol{X}}(\mathbf{id}_{\boldsymbol{X}})$. We see that $R_{\mathfrak{C},\boldsymbol{X}}$ is an extension of the functor $R_{\mathfrak{C},X}$ constructed in [40, Th. 4.20] for the adjunction between Spec and Γ. This will also occur for $L_{\mathfrak{C},\boldsymbol{X}}$.

In fact, if we take a morphism $\boldsymbol{\phi} = (\phi, \phi_{\rm ex}) : \mathfrak{C} \to \Gamma^{\rm c}(\boldsymbol{X})$ in $\mathbf{C^\infty Rings^c}$ then we define $L_{\mathfrak{C},\boldsymbol{X}}(\boldsymbol{\phi}) = \boldsymbol{g} = (g, g^\sharp, g^\sharp_{\rm ex})$ where $(g, g^\sharp) = L_{\mathfrak{C},X}(\phi)$ with $L_{\mathfrak{C},X}$ constructed in [40, Th. 4.20]. Here, g acts by $x \mapsto x_* \circ \phi$ where $x_* : \mathcal{O}_X(X) \to \mathbb{R}$ is the composition of the $\sigma_x : \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ with the unique morphism $\pi : \mathcal{O}_{X,x} \to \mathbb{R}$, as $\boldsymbol{\mathcal{O}}_{X,x}$ is a local C^∞-ring with corners. The morphisms $g^\sharp, g^\sharp_{\rm ex}$ are constructed as $g^\sharp$ is constructed for [40, Th. 4.20], and we explain this explicitly now.

For $x \in X$ and $g(x) = y \in Y$, take the stalk map $\boldsymbol{\sigma}_x = (\sigma_x, \sigma_x^{\rm ex}) : \boldsymbol{\mathcal{O}}_X(X) \to \boldsymbol{\mathcal{O}}_{X,x}$. This gives the following diagram of C^∞-rings with corners

$$\begin{array}{ccccc} \mathfrak{C} & \xrightarrow{\quad\boldsymbol{\phi}\quad} & \Gamma^{\rm c}(\boldsymbol{X}) & & \\ \big\downarrow \boldsymbol{\pi}_y & & \boldsymbol{\sigma}_x \big\downarrow & & \\ \mathfrak{C}_y \cong \boldsymbol{\mathcal{O}}_{Y,y} & \xrightarrow{\boldsymbol{\phi}_x} & \boldsymbol{\mathcal{O}}_{X,x} & \xrightarrow{\quad\pi\quad} & \mathbb{R}. \end{array} \tag{5.2.4}$$

We know $\mathfrak{C}_y \cong \boldsymbol{\mathcal{O}}_{Y,y}$ by Proposition 5.2.3 and $\pi : \mathcal{O}_{X,x} \to \mathbb{R}$ is the unique local morphism. If we have $(c, c') \in \mathfrak{C}$ with $y(c) \neq 0$, and $y \circ \Phi_i(c') \neq 0$ then $\boldsymbol{\sigma}_x \circ \boldsymbol{\phi}(c, c') \in \boldsymbol{\mathcal{O}}_{X,x}$ with $\pi[\sigma_x \circ \phi(c)] \neq 0$ and

$$\pi[\Phi_i \circ \sigma_x^{\mathrm{ex}} \circ \phi_{\mathrm{ex}}(c)] = \pi[\sigma_x \circ \phi \circ \Phi_i(c)] \neq 0.$$

As $\boldsymbol{\mathcal{O}}_{X,x}$ is a local C^∞-ring with corners then $\boldsymbol{\sigma}_x \circ \boldsymbol{\phi}(c, c')$ is invertible in $\boldsymbol{\mathcal{O}}_{X,x}$. The universal property of $\boldsymbol{\pi}_y : \mathfrak{C} \to \mathfrak{C}_y$ gives a unique morphism $\boldsymbol{\phi}_x : \boldsymbol{\mathcal{O}}_{Y,y} \to \boldsymbol{\mathcal{O}}_{X,x}$ that makes (5.2.4) commute.

We define

$$\boldsymbol{g}_\sharp(V) = (g_\sharp(V), g_\sharp^{\mathrm{ex}}(V)) : \boldsymbol{\mathcal{O}}_Y(V) \to g_*(\boldsymbol{\mathcal{O}}_X)(V) = \boldsymbol{\mathcal{O}}_X(U)$$

for each open $V \subseteq Y$ with $U = g^{-1}(V) \subseteq X$ by

$$\boldsymbol{g}_\sharp(V)(\boldsymbol{s})_x \mapsto \boldsymbol{\phi}_x(\boldsymbol{s}_{g(x)})$$

for $\boldsymbol{s} = (s, s') \in \boldsymbol{\mathcal{O}}_Y(V)$ and $x \in U \subseteq X$, which means $g(x) \in V$, $\boldsymbol{s}_{g(x)} = (s_{g(x)}, s'_{g(x)}) \in \boldsymbol{\mathcal{O}}_{Y,g(x)}$, and $\boldsymbol{\phi}_x(\boldsymbol{s}_{g(x)}) \in \boldsymbol{\mathcal{O}}_{X,x}$. We can identify elements, p, of $\boldsymbol{\mathcal{O}}_X(U)$ with maps $\boldsymbol{t} = (t, t')$ where $t : U \to \coprod_{x \in U} \mathcal{O}_{X,x}$ and $t : U \to \coprod_{x \in U} \mathcal{O}_{X,x}^{\mathrm{ex}}$ with $t_x = \sigma_x(p) \in \mathcal{O}_{X,x}$ and $t'_x = \sigma_x^{\mathrm{ex}}(p) \in \mathcal{O}_{X,x}^{\mathrm{ex}}$ for $x \in U$. For $\boldsymbol{s} \in \boldsymbol{\mathcal{O}}_Y(V)$ and $x \in U \subseteq X$, $g(x) = y \in V \subseteq Y$, then Definition 5.2.1 tells us around y there is an open neighbourhood W_y in V and there is $(c, c') \in \mathfrak{C}$ such that

$$\boldsymbol{s}_{y'} = \boldsymbol{\pi}_{y'}(c, c') \in \mathfrak{C}_{y'} \cong \boldsymbol{\mathcal{O}}_{Y,y'}$$

for all $y' \in W_y$. This means

$$\boldsymbol{g}_\sharp(V)\boldsymbol{s}(x') = \boldsymbol{\sigma}_{x'}(\boldsymbol{\phi}(c, c'))$$

for all $x' \in g^{-1}(W_y)$, which is an open neighbourhood of x in U, by (5.2.4). These subsets $g^{-1}(W_y)$ cover U so by Definition 2.3.1(v), $\boldsymbol{g}_\sharp(V)\boldsymbol{s}$ is a section of $\boldsymbol{\mathcal{O}}_X|_U$, and $\boldsymbol{g}_\sharp(V)$ is well defined.

This defines a morphism $\boldsymbol{g}_\sharp : \boldsymbol{\mathcal{O}}_Y \to g_*(\boldsymbol{\mathcal{O}}_X)$ of sheaves of C^∞-rings with corners on Y, and $\boldsymbol{g}^\sharp : g^{-1}(\boldsymbol{\mathcal{O}}_Y) \to \boldsymbol{\mathcal{O}}_X$ is the corresponding morphism of sheaves of C^∞-rings with corners on X under (2.4.2). At a point $x \in X$ such that $g(x) = y \in Y$, then the stalk map is $\boldsymbol{g}_x^\sharp : \boldsymbol{\mathcal{O}}_{Y,y} \to \boldsymbol{\mathcal{O}}_{X,x}$ is equal to $\boldsymbol{\phi}_x$. Then $\boldsymbol{g} = (g, \boldsymbol{g}^\sharp)$ is a morphism in $\mathbf{LC^\infty RS^c}$, and $L_{\mathfrak{C},\boldsymbol{X}}(\boldsymbol{\phi}) = \boldsymbol{g}$.

It now remains to show that these define natural bijections, but this follows very similarly to [40, Th. 4.20]. $\square$

Definition 5.2.7. We define the *interior global sections functor* $\Gamma^{\mathrm{c}}_{\mathrm{in}} : \mathbf{LC^\infty RS^c_{in}} \to (\mathbf{C^\infty Rings^c_{in}})^{\mathbf{op}}$ to act on objects $(X, \boldsymbol{\mathcal{O}}_X)$ by $\Gamma^{\mathrm{c}}_{\mathrm{in}} : (X, \boldsymbol{\mathcal{O}}_X) \mapsto (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ where $\mathfrak{C} = \mathcal{O}_X(X)$ and $\mathfrak{C}_{\mathrm{ex}}$ to be the set containing the zero element of $\mathcal{O}^{\mathrm{ex}}_X(X)$ and the elements of $\mathcal{O}^{\mathrm{ex}}_X(X)$ that are non-zero in every stalk. That is,

$$\mathfrak{C}_{\mathrm{ex}} = \{c' \in \mathcal{O}^{\mathrm{ex}}_X(X) : c' = 0 \in \mathcal{O}^{\mathrm{ex}}_X(X), \text{ or } \sigma^{\mathrm{ex}}_x(c') \neq 0 \in \mathcal{O}^{\mathrm{ex}}_{X,x} \; \forall x \in X\}, \tag{5.2.5}$$

where σ^{ex}_x is the stalk map $\sigma^{\mathrm{ex}}_x : \mathcal{O}^{\mathrm{ex}}_X \to \mathcal{O}^{\mathrm{ex}}_{X,x}$. This is an interior C^∞-ring with corners, where the C^∞-ring with corners structure is given by restriction from $(\mathcal{O}_X(X), \mathcal{O}^{\mathrm{ex}}_X(X))$. We define $\Gamma^{\mathrm{c}}_{\mathrm{in}}$ to act on morphisms $(f, \boldsymbol{f}^\sharp) : (X, \boldsymbol{\mathcal{O}}_X) \to (Y, \boldsymbol{\mathcal{O}}_Y)$ by $\Gamma^{\mathrm{c}}_{\mathrm{in}} : (f, \boldsymbol{f}^\sharp) \mapsto \boldsymbol{f}_\sharp(Y)|_{(\mathfrak{C},\mathfrak{C}_{\mathrm{ex}})}$ for $\boldsymbol{f}_\sharp : \boldsymbol{\mathcal{O}}_Y \to f_*(\boldsymbol{\mathcal{O}}_X)$ corresponding to $\boldsymbol{f}^\sharp$ under (2.4.2).

In a similar way to Definition 5.2.5, for each interior C^∞-ring with corners $\boldsymbol{\mathfrak{C}}$, we can define $\boldsymbol{\Xi}^{\mathrm{in}}_{\boldsymbol{\mathfrak{C}}}(\boldsymbol{c}) = (\Xi^{\mathrm{in}}(c), \Xi^{\mathrm{in}}_{\mathrm{ex}}(c'))$ for each element $\boldsymbol{c} = (c, c') \in \boldsymbol{\mathfrak{C}}$ where $\Xi^{\mathrm{in}}(c) : X_{\mathfrak{C}} \to \coprod_{x \in X_{\mathfrak{C}}} \mathfrak{C}_x$ with $\Xi^{\mathrm{in}}(c) : x \mapsto \pi_x(c)$, and $\Xi^{\mathrm{in}}_{\mathrm{ex}}(c) : X_{\mathfrak{C}} \to \coprod_{x \in X_{\mathfrak{C}}} \mathfrak{C}_{x,\mathrm{ex}}$ with $\Xi^{\mathrm{in}}_{\mathrm{ex}}(c) : x \mapsto \pi_{x,\mathrm{ex}}(c')$. We will write this as $\boldsymbol{\Xi}^{\mathrm{in}}_{\boldsymbol{\mathfrak{C}}}(\boldsymbol{c}) : x \mapsto \boldsymbol{\pi}_x(\boldsymbol{c}) \in \boldsymbol{\mathfrak{C}}_x$. We need to check that $\Xi^{\mathrm{in}}_{\mathrm{ex}}(c')$ is an element of (5.2.5), that is, whether $\sigma^{\mathrm{ex}}_x(\Xi^{\mathrm{in}}_{\mathrm{ex}}(c')) = \pi^{\mathrm{ex}}_x(c') = 0 \in \mathfrak{C}_{x,\mathrm{ex}} \cong \mathcal{O}^{\mathrm{ex}}_{X_{\mathfrak{C}},x}$ for some $c' \neq 0$ and for some $x \in X_{\mathfrak{C}}$. However, as $\boldsymbol{\pi}_x = (\pi_x, \pi_{x,\mathrm{ex}}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{C}}_x$ is interior, then this is immediate and hence $\boldsymbol{\Xi}^{\mathrm{in}}_{\boldsymbol{\mathfrak{C}}}(\mathbf{c})$ is a well defined element of $\Gamma^{\mathrm{c}}_{\mathrm{in}} \circ \mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}}(\boldsymbol{\mathfrak{C}})$.

As in Definition 5.2.5, $\boldsymbol{\Xi}^{\mathrm{in}}_{\boldsymbol{\mathfrak{C}}}$ is a C^∞-ring with corners morphism $\boldsymbol{\Xi}^{\mathrm{in}}_{\boldsymbol{\mathfrak{C}}} : \boldsymbol{\mathfrak{C}} \to \Gamma^{\mathrm{c}}_{\mathrm{in}} \circ \mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}} \boldsymbol{\mathfrak{C}}$, and it is functorial in $\boldsymbol{\mathfrak{C}}$, so that the $\boldsymbol{\Xi}^{\mathrm{in}}_{\boldsymbol{\mathfrak{C}}}$ for all $\boldsymbol{\mathfrak{C}}$ define a natural transformation $\boldsymbol{\Xi}^{\mathrm{in}} : \mathrm{id}_{\mathbf{C^\infty Rings^c_{in}}} \Rightarrow \Gamma^{\mathrm{c}}_{\mathrm{in}} \circ \mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}}$ of functors $\mathbf{id}_{\mathbf{C^\infty Rings^c_{in}}}, \Gamma^{\mathrm{c}}_{\mathrm{in}} \circ \mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}} : \mathbf{C^\infty Rings^c_{in}} \to \mathbf{C^\infty Rings^c_{in}}$.

Theorem 5.2.8. *The functor* $\mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}} : (\mathbf{C^\infty Rings^c_{in}})^{\mathbf{op}} \to \mathbf{LC^\infty RS^c_{in}}$ *is right adjoint to* $\Gamma^{\mathrm{c}}_{\mathrm{in}} : \mathbf{LC^\infty RS^c_{in}} \to (\mathbf{C^\infty Rings^c_{in}})^{\mathbf{op}}$.

Proof. This proof is identical to that of Theorem 5.2.6. We need only check that the definition of $\Gamma^{\mathrm{c}}_{\mathrm{in}}(\boldsymbol{X})$, which may not be equal to $\boldsymbol{\mathcal{O}}_X(X)$, gives well defined maps $\boldsymbol{\sigma}^{\mathrm{in}}_x : \Gamma^{\mathrm{c}}_{\mathrm{in}}(X) \to \boldsymbol{\mathcal{O}}_{X,x}$. As $\Gamma^{\mathrm{c}}_{\mathrm{in}}(X)$ is a subobject of $\boldsymbol{\mathcal{O}}_X(X)$, these maps are the restriction of the stalk maps $\boldsymbol{\sigma}_x : \boldsymbol{\mathcal{O}}_X(X) \to \boldsymbol{\mathcal{O}}_{X,x}$ to $\Gamma^{\mathrm{c}}_{\mathrm{in}}(\boldsymbol{X})$. The definition of $\Gamma^{\mathrm{c}}_{\mathrm{in}}(\boldsymbol{X})$ implies these maps are interior. □

5.3 Semi-complete C^∞-rings with corners

There is an equivalence of categories between complete C^∞-rings and affine C^∞-schemes, and one uses this equivalence to show that fibre products of C^∞-schemes exist. Complete C^∞-rings $\mathfrak{C}$ are such that $\Gamma \circ \mathrm{Spec}\,\mathfrak{C} \cong \mathfrak{C}$, which form a particularly nice category due to canonical isomorphisms $\mathrm{Spec} \circ \Gamma \circ \mathrm{Spec}\,\mathfrak{C} \cong \mathrm{Spec}\,\mathfrak{C}$ for all $\mathfrak{C} \in \mathbf{C^\infty Rings}$ as in Proposition

2.4.12. These isomorphisms imply that $\operatorname{Spec}\circ\Gamma$ is the identity functor on affine C^∞-schemes and other nice results listed in Theorem 2.4.14.

Using this, we see that

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{AC^\infty Sch}}(\operatorname{Spec}\circ\Gamma(X,\mathcal{O}_X),(Y,\mathcal{O}_Y))&\\
&\cong \operatorname{Hom}_{\mathbf{AC^\infty Sch}}(\operatorname{Spec}\circ\Gamma\operatorname{Spec}\circ\Gamma(X,\mathcal{O}_X),(Y,\mathcal{O}_Y))\\
&\cong \operatorname{Hom}_{\mathbf{AC^\infty Sch}}(\operatorname{Spec}\circ\Gamma\operatorname{Spec}\circ\Gamma(X,\mathcal{O}_X),\operatorname{Spec}\circ\Gamma(Y,\mathcal{O}_Y))\\
&\cong \operatorname{Hom}_{(\mathbf{C^\infty Rings^{co}})^{\mathbf{op}}}(\Gamma\circ\operatorname{Spec}\circ\Gamma(X,\mathcal{O}_X),\Gamma(Y,\mathcal{O}_Y))\\
&\cong \operatorname{Hom}_{\mathbf{LC^\infty RS}}((X,\mathcal{O}_X),\operatorname{Spec}\circ\Gamma(Y,\mathcal{O}_Y))\\
&\cong \operatorname{Hom}_{\mathbf{LC^\infty RS}}((X,\mathcal{O}_X),(Y,\mathcal{O}_Y))
\end{aligned}$$

is functorial in both $(X,\mathcal{O}_X)\in\mathbf{LC^\infty RS}$ and $(Y,\mathcal{O}_Y)\in\mathbf{AC^\infty Sch}$. Here the first isomorphism follows from Proposition 2.4.12, the second from Theorem 2.4.14(b), the third from Theorem 2.4.14(a), the fourth from Theorem 2.4.14(e) and the fifth from Theorem 2.4.14(b). These isomorphisms imply that $\operatorname{Spec}\circ\Gamma:\mathbf{LC^\infty RS}\to\mathbf{AC^\infty Sch}$ is left adjoint to the inclusion $\mathbf{AC^\infty Sch}\to\mathbf{LC^\infty RS}$, so this inclusion respects limits. An equivalent result holds in ordinary algebraic geometry.

In this section we show that, unlike C^∞-rings, it is not true that $\operatorname{Spec^c}\circ\Gamma^{\mathrm{c}}\circ\operatorname{Spec^c}\boldsymbol{\mathfrak{C}}\cong\operatorname{Spec^c}\boldsymbol{\mathfrak{C}}$ for all C^∞-rings with corners $\boldsymbol{\mathfrak{C}}$. This shows that C^∞-schemes with corners will not be as well behaved as C^∞-schemes (or ordinary schemes), and that the inclusion of (affine) C^∞-schemes with corners into $\mathbf{LC^\infty RS^c}$ may not respect limits. We then define a category of 'semi-complete' C^∞-rings with corners. In §5.4.1 we will use this category to prove existence of fibre products (and finite limits) of C^∞-schemes with corners under certain conditions.

Remark 5.3.1. For C^∞-rings, we have a canonical isomorphism $\operatorname{Spec}\phi:\operatorname{Spec}\circ\Gamma\circ\operatorname{Spec}\mathfrak{C}\to\operatorname{Spec}\mathfrak{C}\cong(X,\mathcal{O}_X)$ for all $\mathfrak{C}\in\mathbf{C^\infty Rings}$ as in Proposition 2.4.12. This means there is an isomorphism ϕ_x between the stalks of $\operatorname{Spec}\mathfrak{C}$ at $\mathbb{R}$-points $x:\mathfrak{C}\to\mathbb{R}$ and the localisations of the global sections, $\mathcal{O}_X(X)$ of $\operatorname{Spec}\mathfrak{C}$ at $\mathbb{R}$-points $x_*:\mathcal{O}_X(X)\to\mathbb{R}$, with x corresponding to x_* as in the proof of Theorem 5.2.6.

The following example describes a C^∞-ring with corners $\boldsymbol{\mathfrak{C}}=(\mathfrak{C},\mathfrak{C}_{\mathrm{ex}})$ such that the canonical morphism $\operatorname{Spec^c}(\phi,\phi_{\mathrm{ex}}):\operatorname{Spec^c}\circ\Gamma^{\mathrm{c}}\circ\operatorname{Spec^c}\boldsymbol{\mathfrak{C}}\to\operatorname{Spec^c}\boldsymbol{\mathfrak{C}}\cong(X,\mathcal{O}_X,\mathcal{O}_X^{\mathrm{ex}})$ is not an isomorphism. Explicitly, it describes a case where the lower triangle in the following diagram does not commute for an $x\in X$, and hence $\phi_{x,\mathrm{ex}}$ is not an isomorphism. Note that we always have the outer rectangle commuting and the upper triangle commuting, and $\phi_{x,\mathrm{ex}}$ is injective but not necessarily surjective. This example says that if two elements of $\mathcal{O}_X^{\mathrm{ex}}(X)$ agree locally, then while they have the same image in the stalk $\mathcal{O}_{X,x}^{\mathrm{ex}}$ they do not necessarily have the same value in the localisation of $\mathcal{O}_X^{\text{ex}}(X)$ at x_*, so $\phi_{x,\text{ex}}$ is not always surjective.

$$\begin{array}{ccc} \mathfrak{C}_{\text{ex}} & \xrightarrow{\pi_{x,\text{ex}}} & \mathfrak{C}_{x,\text{ex}} \cong \mathcal{O}_{X,x}^{\text{ex}} \\ \downarrow \phi_{\text{ex}} & \overset{\rho_{X,x}^{\text{ex}}}{\nearrow} & \downarrow \phi_{x,\text{ex}} \\ \mathcal{O}_X^{\text{ex}}(X) & \xrightarrow{\hat{\pi}_{x,\text{ex}}} & (\mathcal{O}_X^{\text{ex}}(X))_{x_*} \end{array} \tag{5.3.1}$$

Here $(\phi, \phi_{\text{ex}}) : (\mathfrak{C}, \mathfrak{C}_{\text{ex}}) \to \Gamma^{\text{c}} \circ \text{Spec}^{\text{c}}(\mathfrak{C}, \mathfrak{C}_{\text{ex}}) \cong (\mathcal{O}_X(X), \mathcal{O}_X^{\text{ex}}(X))$ is the canonical morphism taking $c \in \mathfrak{C}$ to the section $s \in \mathcal{O}_X(X)$ where $s_x = \pi_x(c)$ for all $x \in X$, and similarly for $\mathfrak{C}_{\text{ex}}$. Also, $\rho_{X,x}$ is the morphism that takes an element $s \in \mathcal{O}_X(X)$ to its value in the stalk $\mathcal{O}_{X,x}$, and similarly for $\rho_{X,x}^{\text{ex}}$.

One reason this diagram may not commute for the corners case is the following: for two elements, $c', d' \in \mathfrak{C}_{\text{ex}}$, equality in $\mathfrak{C}_{\text{ex},x}$ requires a global equality. That is, there need to be $a', b' \in \mathfrak{C}_{\text{ex}}$ such that $a'c' = b'd' \in \mathfrak{C}_{\text{ex}}$ with a', b' satisfying additional conditions as in Lemma 4.6.9. In the C^∞-ring $\mathfrak{C}$, this equality is only a local equality, as the a' and b' can come from bump functions. However, in the monoid, bump functions do not necessarily exist, meaning this condition is stronger and harder to satisfy.

In the following example, the C^∞-ring with corners $\mathfrak{C}$ is interior, and here $\Gamma^{\text{c}}_{\text{in}}(\mathfrak{C}) = \Gamma^{\text{c}}(\mathfrak{C})$, so the above discussion for C^∞-rings with corners is also true for interior C^∞-rings with corners.

Example 5.3.2. Let $X = \mathbb{R}^2$ and $\mathfrak{C} = (C^\infty(X), \mathfrak{C}_{\text{in}} \amalg \{0_{\text{ex}}\})$. Here $\amalg$ is the disjoint union of sets, and $\mathfrak{C}_{\text{in}}$ is the monoid generated by $\text{Ex}(X)$ and the bump functions c_1, c_2, c_3, c_4, where each $c_j \in C^\infty(X)$ has support in the region A_j (defined in Figure 5.3.1) and zero elsewhere. We have that $c_1 c_j = c_2 c_j = c_1 c_2 = 0 \in \text{Ex}(X)$ for $j = 3, 4$. Note that $\text{Ex}(X) \ni 0 \neq 0_{\text{ex}}$, so $\mathfrak{C}_{\text{in}} \amalg \{0_{\text{ex}}\} = \mathfrak{C}_{\text{ex}}$ has no zero divisors.

We make $\mathfrak{C}$ into a C^∞-ring with corners using composition of functions. That is, for a non-zero smooth function $f : [0, \infty)^n \times \mathbb{R}^m \to [0, \infty)$, we define the C^∞-operation $\Psi_f : \mathfrak{C}_{\text{ex}}^n \times \mathfrak{C}^m \to \mathfrak{C}_{\text{ex}}$ by

$$\Psi_f(d_1, \ldots, d_n, g_1, \ldots, g_m) = f(d_1, \ldots, d_n, g_1, \ldots, g_m) \in \mathfrak{C}_{\text{in}},$$

where $d_i \in \mathfrak{C}_{\text{ex}}$ and $g_j \in \mathfrak{C}$. For the zero function $0 : [0, \infty)^n \times \mathbb{R}^m \to [0, \infty)$, we define

$$\Psi_0(d_1, \ldots, d_n, g_1, \ldots, g_m) = 0_{\text{ex}}.$$

If $f : [0, \infty)^n \times \mathbb{R}^m \to \mathbb{R}$ is smooth, then we define the C^∞-operation $\Phi_f : \mathfrak{C}_{\text{ex}}^n \times \mathfrak{C}^m \to \mathfrak{C}$ by

$$\Phi_f(d_1, \ldots, d_n, g_1, \ldots, g_m) = f(d_1, \ldots, d_n, g_1, \ldots, g_m).$$

As $\mathfrak{C}_{\rm ex}^\times = {\rm Ex}(X)^\times = {\rm In}(X)$ and $(C^\infty(X), {\rm Ex}(X))$ is a C^∞-ring with corners, then $\mathfrak{C}$ is a C^∞-ring with corners. As $\mathfrak{C}_{\rm ex}$ has no zero divisors, then $\mathfrak{C}$ is an interior C^∞-ring with corners.

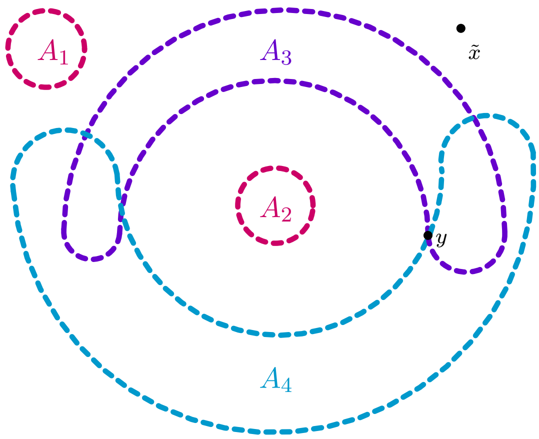


Figure 5.3.1: Region of $X = \mathbb{R}^2$ with open sets A_i, and points $\tilde{x}$ and y

By Proposition 5.2.3, we know $\boldsymbol{\mathcal{O}}_{X,x}$ is isomorphic to $\mathfrak{C}_x$, and we know $(\mathcal{O}_X(X))_x \cong \mathcal{O}_{X,x} \cong \mathfrak{C}_x$ for all $x \in X$. We will show that $(\mathcal{O}_X^{\rm ex}(X))_{\tilde{x}} \ncong \mathfrak{C}_{\tilde{x},{\rm ex}} \cong \mathcal{O}_{X,\tilde{x}}$, where $\tilde{x}$ is a point in $\mathbb{R}^2$ outside of the regions A_1, A_2, A_3, A_4, as in Figure 5.3.1. This will show that ${\rm Spec}^{\rm c} \circ \Gamma^{\rm c} \circ {\rm Spec}^{\rm c}\, \mathfrak{C} \ncong {\rm Spec}^{\rm c}\, \mathfrak{C}$. Note that $\mathbb{R}$-points of $\mathcal{O}_X(X)$ are in 1-1 correspondence with elements of X by Example 2.4.5, so localising at $\tilde{x} \in X$ makes sense.

Firstly, note that in $\mathcal{O}_{X,\tilde{x}}$, the elements c_j under the map $\pi_{\tilde{x},{\rm ex}} : \mathfrak{C}_{\rm ex} \to \mathfrak{C}_{\tilde{x},{\rm ex}}$ are distinct for each $j = 1, 2, 3, 4$, using Lemma 4.6.9 and that if $a \in \mathfrak{C}_{\rm ex}$ such that $\tilde{x} \circ \Phi_i(a) \neq 0$ then $a \in {\rm In}(X)$. We can define the element $s_1' \in \mathcal{O}_X^{\rm ex}(X)$ such that $s_{1,x}' = \pi_{x,{\rm ex}}(c_1)$ for all $x \in X$. Now using Lemma 4.6.9, for any $x \in A_3 \cup A_4$, we have that $\pi_{x,{\rm ex}}(c_1) = \pi_{x,{\rm ex}}(c_2)$. Using this and the locations of A_1, A_2, A_3, A_4 in Figure 5.3.1, then we can also define $s_2' \in \mathcal{O}_X^{\rm ex}(X)$ such that $s_{2,x}' = \pi_{x,{\rm ex}}(c_1)$ for all x outside of $A_3 \cup A_4$, and $s_{2,x}' = \pi_{x,{\rm ex}}(c_2)$ for all x inside of $A_3 \cup A_4$, and $s_{2,x}' = \pi_{x,{\rm ex}}(c_1) = \pi_{x,{\rm ex}}(c_2)$ for all $x \in A_3 \cup A_4$.

The canonical map $(\mathcal{O}_X^{\rm ex}(X))_{\tilde{x}} \to \mathfrak{C}_{\rm ex} \cong \mathcal{O}_{X,\tilde{x}}$ takes $\pi_{\tilde{x},{\rm ex}}(s') \mapsto \pi_{\tilde{x},{\rm ex}}(c')$ for $s' \in \mathcal{O}_X^{\rm ex}(X)$ where $s_{\tilde{x}}' = \pi_{\tilde{x},{\rm ex}}(c')$. This means that s_1' and s_2' have the same image under this map. However, we will show that $\pi_{\tilde{x},{\rm ex}}(s_1') \neq \pi_{\tilde{x},{\rm ex}}(s_2')$ despite $s_{1,\tilde{x}}' = s_{2,\tilde{x}}'$ in an open neighbourhood of $\tilde{x} \in X$.

If $\pi_{\tilde{x},{\rm ex}}(s_1')$ was equal to $\pi_{\tilde{x},{\rm ex}}(s_2')$, then by Lemma 4.6.9, there would be $s_3', s_4' \in \mathcal{O}_X(X)$ with $s_{3,\tilde{x}}' = s_{4,\tilde{x}}' \neq 0$ and $s_3' = s_4'$ in an open neighbourhood of $\tilde{x}$, such that $s_1's_3' = s_2's_4'$. If $s_{3,\tilde{x}}' = s_{4,\tilde{x}}' \neq 0$, then we have that $s_{3,x}' = \pi_{x,{\rm ex}}(e_1)$ with $e_1 \in {\rm In}(X)$, and $s_{4,x}' = \pi_{x,{\rm ex}}(e_2)$

with $e_2 \in \mathrm{In}(X)$, for all x in a neighbourhood of $\tilde{x}$.

As $s'_1 s'_3 = s'_2 s'_4$, then for $x \in A_4$ we must have $s'_4(x) = 0 \in \mathrm{Ex}(X)$. However, the only way for s'_4 to go from $s'_{4,x} = \pi_{x,\mathrm{ex}}(e_2)$, which is invertible at $\tilde{x}$, to 0 in A_4 would mean that $s'_{4,x} = \pi_{x,\mathrm{ex}}(a_3)$ for $x \in A_3$ and $s'_{4,x} = \pi_{x,\mathrm{ex}}(a_4)$ for all $x \in A_4$, where $a_3 = c_3^n h_1$ for some positive integer n and some $h_1 \in \mathrm{In}(x)$, and $a_4 = c_4^m h_2$ for some positive integer m and some $h_2 \in \mathrm{In}(x)$. In particular, this must be true at the point y in Figure 5.3.1, which lies on the intersection of the boundaries of A_3 and A_4. However, Lemma 4.6.9 shows that $\pi_{x,\mathrm{ex}}(c_3) \neq \pi_{x,\mathrm{ex}}(c_4)$ and that $\pi_{x,\mathrm{ex}}(a_3) \neq \pi_{x,\mathrm{ex}}(a_4)$ at this point, as there are no elements of $\mathrm{In}(X)$ that are non-zero at y and zero in $A_3 \setminus A_4$ and $A_4 \setminus A_3$. So this s'_4 cannot exist.

This means $\pi_{\tilde{x},\mathrm{ex}}(s'_1) \neq \pi_{\tilde{x},\mathrm{ex}}(s'_2)$. Therefore, the canonical map $(\mathcal{O}_X^{\mathrm{ex}}(X))_{\tilde{x}} \to \mathfrak{C}_{\mathrm{ex},x} \cong \mathcal{O}_{X,x}$ is not injective. Hence $\mathrm{Spec}^{\mathrm{c}} \circ \Gamma^{\mathrm{c}} \circ \mathrm{Spec}^{\mathrm{c}}\, \boldsymbol{\mathfrak{C}} \not\cong \mathrm{Spec}^{\mathrm{c}}\, \boldsymbol{\mathfrak{C}}$.

If $\underline{X}$ is an affine C^∞-scheme, then Joyce [40, Th. 4.36(a)] tells us that $\mathrm{Spec}(\mathcal{O}_X(X)) \cong \underline{X}$. Also, [40, §4.6] tells us that C^∞-rings that are isomorphic to $\mathcal{O}_X(X)$ for some affine C^∞-scheme $\underline{X}$ are called complete C^∞-rings and there is an equivalence of categories between complete C^∞-rings and affine C^∞-schemes.

However, if $\boldsymbol{X}$ is an affine C^∞-scheme with corners, then Example 5.3.2 shows that $\mathrm{Spec}^{\mathrm{c}}(\boldsymbol{\mathcal{O}}_X(X)) \not\cong \boldsymbol{X}$ in general, as the sheaf of monoids may be different. This means we do not expect there to be a subcategory of C^∞-rings with corners that gives an equivalence of categories to affine C^∞-schemes with corners. However, we use the following lemma to define semi-complete C^∞-rings with corners, which have similar properties to complete C^∞-rings.

Lemma 5.3.3. *Let $(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ be a C^∞-ring with corners and let $\boldsymbol{X} = (X, \mathcal{O}_X, \mathcal{O}_X^{\mathrm{ex}}) = \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$. Then there is a C^∞-ring with corners $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ with $\mathfrak{D} \cong \Gamma \circ \mathrm{Spec}(\mathfrak{C})$ a complete C^∞-ring, such that $\mathrm{Spec}^{\mathrm{c}}(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}}) \cong \boldsymbol{X}$ and the canonical map $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}}) \to \Gamma \circ \mathrm{Spec}^{\mathrm{c}}(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ is an isomorphism on $\mathfrak{D}$, and injective on $\mathfrak{D}_{\mathrm{ex}}$. If $(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ is firm, then $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ is firm. If $(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ is interior, then $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ is interior.*

Note that Example 5.3.2 gives an example where no choice of $\mathfrak{D}_{\mathrm{ex}}$ can make the canonical map $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}}) \to \Gamma \circ \mathrm{Spec}^{\mathrm{c}}(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ surjective on the monoids.

Proof. We define $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ such that $\mathfrak{D} = \Gamma \circ \mathrm{Spec}\, \mathfrak{C} = \mathcal{O}_X(X)$, and let $\mathfrak{D}_{\mathrm{ex}}$ be the submonoid of $\mathcal{O}_X^{\mathrm{ex}}(X)$ generated by the invertible elements $\Psi_{\exp}(\mathfrak{D})$ and the image $\phi_{\mathrm{ex}}(\mathfrak{C}_{\mathrm{ex}})$. One can check that the C^∞-operations from $(\mathcal{O}_X(X), \mathcal{O}_X^{\mathrm{ex}}(X))$ restrict to C^∞-operations on $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$, and make $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ into a C^∞-ring with corners. Let $\boldsymbol{Y} = \mathrm{Spec}^{\mathrm{c}}(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$. If $(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ is firm, then $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ is firm, as the sharpening $\mathfrak{D}_{\mathrm{ex}}^\sharp$ is the image of $\mathfrak{C}_{\mathrm{ex}}^\sharp$ under

$\phi_{\rm ex}$, hence the image of the generators generates $\mathfrak{D}_{\rm ex}^\sharp$. If $(\mathfrak{C},\mathfrak{C}_{\rm ex})$ is interior, then $(\mathfrak{D},\mathfrak{D}_{\rm ex})$ is interior, as elements in both $\Psi_{\rm exp}(\mathfrak{D})$ and $\phi_{\rm ex}(\mathfrak{C}_{\rm ex})$ have no zero divisors.

Now the canonical morphism $(\phi,\phi_{\rm ex}):(\mathfrak{C},\mathfrak{C}_{\rm ex})\to\Gamma\circ\mathrm{Spec}^{\rm c}(\mathfrak{C},\mathfrak{C}_{\rm ex})$ gives a morphism $(\psi,\psi_{\rm ex}):(\mathfrak{C},\mathfrak{C}_{\rm ex})\to(\mathfrak{D},\mathfrak{D}_{\rm ex})$ where $\psi=\phi$, and $\psi_{\rm ex}=\phi_{\rm ex}$ with its image restricted to the submonoid $\mathfrak{D}_{\rm ex}$ of $\mathcal{O}_{X,{\rm ex}}(X)$. As $\mathfrak{D}$ is complete, then $\mathrm{Spec}(\mathfrak{D})\cong\mathrm{Spec}(\mathfrak{C})\cong(X,\mathcal{O}_X)$, and $\mathrm{Spec}^{\rm c}(\psi,\psi_{\rm ex}):\boldsymbol{Y}\cong\mathrm{Spec}^{\rm c}(\mathfrak{D},\mathfrak{D}_{\rm ex})\to\mathrm{Spec}^{\rm c}(\mathfrak{C},\mathfrak{C}_{\rm ex})\cong\boldsymbol{X}$ is an isomorphism on the topological space and the sheaves of C^∞-rings. To show that $\mathrm{Spec}^{\rm c}(\psi,\psi_{\rm ex})$ is an isomorphism on the sheaves of monoids, we show $\psi_{\rm ex}$ induces an isomorphism on the stalks $\mathcal{O}_{X,x}^{\rm ex}\cong\mathfrak{C}_{x,{\rm ex}}$ and $\mathcal{O}_{Y,x}^{\rm ex}\cong\mathfrak{D}_{x,{\rm ex}}$, for all $\mathbb{R}$-points $x\in X$.

The stalk map corresponds to the morphism $\psi_{x,{\rm ex}}:\mathfrak{C}_{x,{\rm ex}}\to\mathfrak{D}_{x,{\rm ex}}$, which is defined by $\psi_{x,{\rm ex}}(\pi_{x,{\rm ex}}(c'))=\hat\pi_{x,{\rm ex}}(\psi_{\rm ex}(c'))$, where $\pi_{x,{\rm ex}}:\mathfrak{C}_{\rm ex}\to\mathfrak{C}_{x,{\rm ex}}$ and $\hat\pi_{x,{\rm ex}}:\mathfrak{D}_{\rm ex}\to\mathfrak{D}_{x,{\rm ex}}$ are the localisation morphisms. Now, as $\mathrm{Spec}^{\rm c}(\psi,\psi_{\rm ex})$ is an isomorphism on the sheaves of C^∞-rings, we know that $\psi_x:\mathfrak{C}_x\to\mathfrak{D}_x$ is an isomorphism, which implies we have an isomorphism $\psi_{x,{\rm ex}}|_{\mathfrak{C}^\times_{x,{\rm ex}}}:\mathfrak{C}^\times_{x,{\rm ex}}\to\mathfrak{D}^\times_{x,{\rm ex}}$ of invertible elements in the monoids. This gives the following commutative diagram of monoids.

$$
\begin{array}{ccccccc}
 & & \mathfrak{C}_{\rm ex} & \xrightarrow{\ \psi_{\rm ex}\ } & \mathfrak{D}_{\rm ex}\subset\mathcal{O}_X^{\rm ex}(X) & & \\
 & & \downarrow{\scriptstyle\pi_{x,{\rm ex}}} & & \downarrow{\scriptstyle\hat\pi_{x,{\rm ex}}} & & \\
\mathfrak{C}^\times_{x,{\rm ex}} & \hookrightarrow & \mathfrak{C}_{x,{\rm ex}} & \xrightarrow{\ \psi_{x,{\rm ex}}\ } & \mathfrak{D}_{x,{\rm ex}} & \hookleftarrow & \mathfrak{D}^\times_{x,{\rm ex}}
\end{array}
\qquad (\mathfrak{C}^\times_{x,{\rm ex}}\xrightarrow{\ \cong\ }\mathfrak{D}^\times_{x,{\rm ex}})
\tag{5.3.2}
$$

To show $\psi_{x,{\rm ex}}$ is injective, first use that $\pi_{x,{\rm ex}}$ is surjective so that for $a'_x,b'_x\in\mathfrak{C}_{x,{\rm ex}}$, then $a'_x=\pi_{x,{\rm ex}}(a')$, $b'_x=\pi_{x,{\rm ex}}(b')\in\mathfrak{C}_{x,{\rm ex}}$ for $a',b'\in\mathfrak{C}_{\rm ex}$. Assume that $\psi_{x,{\rm ex}}(a'_x)=\psi_{x,{\rm ex}}(b'_x)$. Then we have

$$\hat\pi_{x,{\rm ex}}(\psi_{\rm ex}(a'))=\psi_{x,{\rm ex}}(\pi_{x,{\rm ex}}(a'))=\psi_{x,{\rm ex}}(\pi_{x,{\rm ex}}(ab))=\hat\pi_{x,{\rm ex}}(\psi_{\rm ex}(b'))\in\mathfrak{D}_{x,{\rm ex}},$$

so by Lemma 4.6.9, there are $e',f'\in\mathfrak{D}$ such that $e'\psi_{\rm ex}(a')=f'\psi_{\rm ex}(b')\in\mathfrak{D}$, with $\Phi_i(e')-\Phi_i(f')\in I$ and $x\circ\Phi_i(e')\neq 0$. This implies $\hat\pi_{x,{\rm ex}}(e')=\hat\pi_{x,{\rm ex}}(f')\in\mathfrak{D}^\times_{x,{\rm ex}}$, and as $\psi_{x,{\rm ex}}|_{\mathfrak{C}^\times_{x,{\rm ex}}}:\mathfrak{C}^\times_{x,{\rm ex}}\to\mathfrak{D}^\times_{x,{\rm ex}}$ is an isomorphism and $\pi_{x,{\rm ex}}$ is surjective, there must be $e'',f''\in\mathfrak{C}_{\rm ex}$ such that $\psi_{\rm ex}(e'')=e'\in\mathfrak{D}_{\rm ex}$ and $\psi_{\rm ex}(f'')=f'\in\mathfrak{D}_{\rm ex}$, $\pi_{x,{\rm ex}}(e'')=\pi_{x,{\rm ex}}(f'')\in\mathfrak{C}_{x,{\rm ex}}$, and we have $\psi_{\rm ex}(e''a')=\psi_{\rm ex}(f''b')\in\mathfrak{D}_{\rm ex}$.

Recall that the map $\psi_{\rm ex}$ sends $c'\mapsto s'\in\mathfrak{D}_{\rm ex}\subset\mathcal{O}_{X,{\rm ex}}(X)$ where $s'(\hat x)=\pi_{x,{\rm ex}}(c')$ for all $\hat x\in X$. So $\psi_{\rm ex}(e''a')=\psi_{\rm ex}(f''b')$ implies $\pi_{\hat x,{\rm ex}}(e''a')=\pi_{\hat x,{\rm ex}}(f''b')$ for all $\hat x\in X$. At our value of $x\in X$, we have that $\pi_{x,{\rm ex}}(e'')=\pi_{x,{\rm ex}}(f'')\in\mathfrak{C}^\times_{x,{\rm ex}}$, hence we have $a'_x=\pi_{x,{\rm ex}}(a')=\pi_{x,{\rm ex}}(b')=b'_x$, so the map is injective.

To show $\psi_{x,{\rm ex}}$ is surjective, take an element $\hat\pi_{x,{\rm ex}}(d')=d'_x\in\mathfrak{D}_{x,{\rm ex}}$ for $d'\in\mathfrak{D}$. Then $d'=\psi_{\rm ex}(c')\cdot e'$ where $e'\in\Psi_{\rm exp}(\mathfrak{D})$ is invertible, and $c'\in\mathfrak{C}$. Then $d'_x=\hat\pi_{x,{\rm ex}}(\psi_{\rm ex}(c'))\cdot$

$\hat{\pi}_{x,\mathrm{ex}}(e')$. As e' is invertible, then $\hat{\pi}_{x,\mathrm{ex}}(e') \in \mathfrak{D}_{x,\mathrm{ex}}$ is invertible. Then as $\psi_{x,\mathrm{ex}}|_{\mathfrak{C}^{\times}_{x,\mathrm{ex}}}$: $\mathfrak{C}^{\times}_{x,\mathrm{ex}} \to \mathfrak{D}^{\times}_{x,\mathrm{ex}}$ is an isomorphism and $\pi_{x,\mathrm{ex}}$ is surjective, there must be $e'' \in \mathfrak{C}_{\mathrm{ex}}$ such that $\psi_{\mathrm{ex}}(e'') = e'$, with $\psi_{x,\mathrm{ex}}(\pi_{x,\mathrm{ex}}(e'')) = \hat{\pi}_{x,\mathrm{ex}}(e')$. Then $\psi_{x,\mathrm{ex}} \circ \pi_{x,\mathrm{ex}}(e'c') = \hat{\pi}_{x,\mathrm{ex}}(d') = d'_x$ and the map is surjective. Hence, $\psi_{x,\mathrm{ex}}$ is an isomorphism, and $\mathrm{Spec}^{\mathrm{c}}(\psi, \psi_{\mathrm{ex}}) : \boldsymbol{Y} \cong \mathrm{Spec}^{\mathrm{c}}(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}}) \to \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}}) \cong \boldsymbol{X}$ is an isomorphism.

Finally, as $\mathfrak{D}$ is complete, we known that the canonical morphism $(\varphi, \varphi_{\mathrm{ex}}) : (\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}}) \to \Gamma \circ \mathrm{Spec}^{\mathrm{c}}(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ is an isomorphism on the C^∞-rings. On the monoids, say $d'_1, d'_2 \in \mathfrak{D}_{\mathrm{ex}}$ such that $\varphi_{\mathrm{ex}}(d'_1) = \varphi_{\mathrm{ex}}(d'_2)$. Then $\varphi_{\mathrm{ex}}(d'_1), \varphi_{\mathrm{ex}}(d'_2)$ are sections of $\mathcal{O}^{\mathrm{ex}}_Y(Y)$ and their equality implies

$$\pi_{y,\mathrm{ex}}(d'_1) = \varphi_{\mathrm{ex}}(d'_1)_y = \varphi_{\mathrm{ex}}(d'_2)_y = \pi_{y,\mathrm{ex}}(d'_2) \in \mathcal{O}_{Y,y} \cong \mathfrak{D}_{y,\mathrm{ex}}$$

for all $y \in Y$. We have $d'_1, d'_2 \in \mathcal{O}^{\mathrm{ex}}_X(X)$, so say that $d'_{1,x} = \pi_{x,\mathrm{ex}}(c'_1)$ and $d'_{2,x} = \pi_{x,\mathrm{ex}}(c'_2)$ at a point $x \in X$. From the above, we know $\mathfrak{D}_{y,\mathrm{ex}} \cong \mathfrak{C}_{x,\mathrm{ex}}$ where $y \in Y$ corresponds to $x \in X$ via the isomorphism $\mathrm{Spec}^{\mathrm{c}}(\psi, \psi_{\mathrm{ex}})$. This isomorphism implies that $\pi_{y,\mathrm{ex}}(d'_1) = \psi_{x,\mathrm{ex}}(\pi_{x,\mathrm{ex}}(c'_1))$ and $\pi_{y,\mathrm{ex}}(d'_2) = \psi_{x,\mathrm{ex}}(\pi_{x,\mathrm{ex}}(c'_2))$, so we have $\psi_{x,\mathrm{ex}}(\pi_{x,\mathrm{ex}}(c'_1)) = \psi_{x,\mathrm{ex}}(\pi_{x,\mathrm{ex}}(c'_2))$. However, as $\psi_{x,\mathrm{ex}}$ is an isomorphism, we must have $\pi_{x,\mathrm{ex}}(c'_1) = \pi_{x,\mathrm{ex}}(c'_2)$. This means that $d'_{1,x} = d'_{2,x}$ for all $x \in X$. As d'_1, d'_2 are sections of $\mathcal{O}^{\mathrm{ex}}_Y(Y)$, then this is only possible if $d'_1 = d'_2$. Hence φ_{ex} is injective. □

Definition 5.3.4. Let $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ be a C^∞-ring with corners, and let $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}}) = \boldsymbol{\mathfrak{D}}$ be the C^∞-ring with corners constructed in Lemma 5.3.3 corresponding to $\boldsymbol{\mathfrak{C}}$. We call $\boldsymbol{\mathfrak{D}}$ a *semi-complete* C^∞-ring with corners, and the category of semi-complete C^∞-rings with corners, denoted $\mathbf{C^\infty Rings^c_{sc}}$, is a full subcategory of C^∞-rings with corners. The map $\boldsymbol{\mathfrak{C}} \mapsto \boldsymbol{\mathfrak{D}}$ gives a functor $R^{\mathrm{sc}}_{\mathrm{all}} : \mathbf{C^\infty Rings^c} \to \mathbf{C^\infty Rings^c_{sc}}$, which is left adjoint to the inclusion functor $\mathbf{C^\infty Rings^c_{sc}} \to \mathbf{C^\infty Rings^c}$. Here the counit is the identity morphism, and the unit is the morphism $(\psi, \psi_{\mathrm{ex}})$ defined in the proof of Lemma 5.3.3.

Composing the functor $R^{\mathrm{sc}}_{\mathrm{all}}$ with the forgetful functor $\mathbf{C^\infty Rings^c} \to \mathbf{C^\infty Rings}$, $(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}}) \mapsto \mathfrak{C}$ is the same as applying the forgetful functor first and then applying the completion functor $R^{\mathrm{co}}_{\mathrm{all}}$ for C^∞-rings defined in Definition 2.4.13.

If $\boldsymbol{\mathfrak{C}}$ is a C^∞-ring with corners and $\boldsymbol{X} = \mathrm{Spec}^{\mathrm{c}}\,\boldsymbol{\mathfrak{C}}$, $\boldsymbol{Y} = \mathrm{Spec}^{\mathrm{c}} \circ \Gamma^{\mathrm{c}} \circ \mathrm{Spec}^{\mathrm{c}}\,\boldsymbol{\mathfrak{C}}$, then taking $\mathrm{Spec}^{\mathrm{c}}$ of the unit $\boldsymbol{\Xi}_{\boldsymbol{\mathfrak{C}}} : \boldsymbol{\mathfrak{C}} \to \Gamma^{\mathrm{c}} \circ \mathrm{Spec}^{\mathrm{c}}\,\boldsymbol{\mathfrak{C}}$ from Theorem 5.2.6 gives a morphism $\boldsymbol{Y} \to \boldsymbol{X}$, while the counit on $\boldsymbol{X}$ gives a morphism $\boldsymbol{X} \to \boldsymbol{Y}$. The composition $\boldsymbol{X} \to \boldsymbol{Y} \to \boldsymbol{X}$ is the identity, by definition of the counit. Applying the global sections functor Γ^{c} implies that $\Gamma^{\mathrm{c}} \circ \mathrm{Spec}^{\mathrm{c}}\,\boldsymbol{\mathfrak{C}}$ is a semi-complete C^∞-ring with corners for all C^∞-rings with corners $\boldsymbol{\mathfrak{C}}$.

However the composition $\boldsymbol{Y} \to \boldsymbol{X} \to \boldsymbol{Y}$ is not the identity in general, in particular if $\boldsymbol{\mathfrak{C}}$ is the C^∞-ring with corners from Example 5.3.2. Instead, this second composition gives a morphism $\boldsymbol{Y} \to \boldsymbol{Y}$ that is not $\mathrm{Spec^c}\,\boldsymbol{\phi}$ for some morphism $\boldsymbol{\phi} : \Gamma^{\mathrm{c}}(\boldsymbol{X}) \to \Gamma^{\mathrm{c}}(\boldsymbol{X})$. What this means is that, if we restrict $\mathrm{Spec^c}$ to $\mathbf{C^\infty Rings^c_{sc}}$, then $\mathrm{Spec^c}$ is essentially surjective onto the category of affine C^∞-schemes with corners ($\mathbf{AC^\infty Sch^c}$ defined in Definition 5.4.1), and it is faithful, but it is not full. So we do not have an equivalence of categories between $\mathbf{C^\infty Rings^c_{sc}}$ and $\mathbf{AC^\infty Sch^c}$.

Remark 5.3.5. To show the category of C^∞-schemes has all fibre products, we first take the fibre product of the C^∞-schemes in $\mathbf{LC^\infty RS}$ using Theorem 2.4.15. We then use the contravariant equivalence of categories between complete C^∞-rings and affine C^∞-schemes on affine neighbourhoods from Theorem 2.4.14(a). This translates a fibre product of affine C^∞-schemes to a pushout of complete C^∞-rings. As complete C^∞-rings have all pushouts (Theorem 2.4.14(d)), then the fibre product of the affine neighbourhoods of the C^∞-schemes exist, and can be shown to be isomorphic to open neighbourhoods of the fibre product of the C^∞-schemes. This means that the fibre product of the C^∞-schemes exist and are equal to the fibre product taken in $\mathbf{LC^\infty RS}$, which we expect from the discussion at the start of this section.

As there is no such equivalence of categories for affine C^∞-schemes with corners, we cannot use this proof for the corners case. Importantly, we cannot show that morphisms between affine C^∞-schemes with corners $\mathrm{Spec^c}(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}}) \to \mathrm{Spec^c}(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$ give morphisms $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}}) \to (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$, even if the C^∞-rings are semi-complete. However, in §5.4.1, we show that we can do this under certain conditions on $(\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$.

5.4 C^∞-schemes with corners

We define the categories of C^∞-schemes with corners and firm C^∞-schemes with corners. Firm C^∞-schemes with corners have important properties that allow fibre products to exist.

Definition 5.4.1. A local C^∞-ringed space with corners that is isomorphic in $\mathbf{LC^\infty RS^c}$ to $\mathrm{Spec^c}\,\boldsymbol{\mathfrak{C}}$ for some C^∞-ring with corners $\boldsymbol{\mathfrak{C}}$ is called an *affine C^∞-scheme with corners*. We define the category $\mathbf{AC^\infty Sch^c}$ of affine C^∞-schemes with corners to be the full subcategory of affine C^∞-schemes with corners in $\mathbf{LC^\infty RS^c}$. If $X \in \mathbf{AC^\infty Sch^c}$ is isomorphic to $\mathrm{Spec^c}\,\boldsymbol{\mathfrak{C}}$ for a firm C^∞-ring with corners, $\boldsymbol{\mathfrak{C}}$, we call X a *firm* affine C^∞-scheme with corners, and denote $\mathbf{AC^\infty Sch^c_{fi}}$ the full subcategory of $\mathbf{AC^\infty Sch^c}$ of firm affine C^∞-schemes with corners.

Let $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ be a local C^∞-ringed space with corners. We call $\boldsymbol{X}$ a *(firm) C^∞-scheme with corners* if X can be covered by open sets $U \subseteq X$ such that $(U, \boldsymbol{\mathcal{O}}_X|_U)$ is a (firm) affine C^∞-scheme with corners. We define the category ($\mathbf{C^\infty Sch^c_{fi}}$) $\mathbf{C^\infty Sch^c}$ of (firm) C^∞-schemes with corners to be the full sub-category of (firm) C^∞-schemes with corners in $\mathbf{LC^\infty RS^c}$. Then $\mathbf{C^\infty Sch^c_{fi}}$, $\mathbf{AC^\infty Sch^c_{fi}}$ and $\mathbf{AC^\infty Sch^c}$ are a full subcategories of $\mathbf{C^\infty Sch^c}$, and $\mathbf{AC^\infty Sch^c_{fi}}$ is a full subcategory of $\mathbf{C^\infty Sch^c_{fi}}$.

Remark 5.4.2. Kalashnikov [51, §4.8] defined a different notion of (affine) C^∞-scheme with corners using their 'real spectrum' functor (as in Remark 5.2.3). This definition is more general than our definition as they use spectrums of pre C^∞-rings with corners.

Definition 5.4.3. A local C^∞-ringed space with corners that is isomorphic in $\mathbf{LC^\infty RS^c_{in}}$ to $\mathrm{Spec}^c_{\mathrm{in}}\,\mathfrak{C}$ for some interior C^∞-ring with corners $\mathfrak{C}$ is called an *interior affine C^∞-scheme with corners*. We define the category $\mathbf{AC^\infty Sch^c_{in}}$ of interior affine C^∞-schemes with corners to be the full sub-category of interior affine C^∞-schemes with corners in $\mathbf{LC^\infty RS^c_{in}}$, so $\mathbf{AC^\infty Sch^c_{in}}$ is a non-full subcategory of $\mathbf{AC^\infty Sch^c}$. A *firm* interior affine C^∞-scheme with corners is an interior affine C^∞-scheme with corners that is also a firm affine C^∞-scheme with corners. We denote $\mathbf{AC^\infty Sch^c_{fi,in}}$ the full subcategory of $\mathbf{AC^\infty Sch^c}$ of firm interior affine C^∞-schemes with corners.

We call an object $\boldsymbol{X} \in \mathbf{LC^\infty RS^c_{in}}$ a *(firm) interior C^∞-scheme with corners* if it can be covered by open sets $U \subseteq X$ such that $(U, \boldsymbol{\mathcal{O}}_X|_U)$ is a (firm) interior affine C^∞-scheme with corners. We define the category ($\mathbf{C^\infty Sch^c_{fi,in}}$) $\mathbf{C^\infty Sch^c_{in}}$ of (firm) interior C^∞-schemes with corners to be the full sub-category of (firm) interior C^∞-schemes with corners in $\mathbf{LC^\infty RS^c_{in}}$. This implies that ($\mathbf{AC^\infty Sch^c_{fi,in}}$) $\mathbf{AC^\infty Sch^c_{in}}$ is a full subcategory of ($\mathbf{C^\infty Sch^c_{fi,in}}$) $\mathbf{C^\infty Sch^c_{in}}$.

The following lemma will be useful when considering fibre products of C^∞-schemes with corners in this section. It also tells us that open subsets of (interior/firm) C^∞-schemes with corners are (interior/firm) C^∞-schemes with corners.

Lemma 5.4.4. *Let $\mathfrak{C}$ be a C^∞-ring with corners, and $\boldsymbol{X} = \mathrm{Spec}^c\,\mathfrak{C}$. For any element $c \in \mathfrak{C}$, let U_c be as in Definition 2.4.4. Then $\boldsymbol{X}|_{U_c} \cong \mathrm{Spec}^c(\mathfrak{C}[c^{-1}])$. Note that if $\mathfrak{C}$ is firm, then so is $\mathfrak{C}[c^{-1}]$; if $\mathfrak{C}$ is interior, then $\mathfrak{C}[c^{-1}]$ is also interior.*

Proof. Write $\mathfrak{C}[c^{-1}] = (\mathfrak{D}, \mathfrak{D}_{\mathrm{ex}})$. By Lemma 4.6.5, then $\mathfrak{D} \cong \mathfrak{C}[c^{-1}]$. By Lemma 2.4.6, we need only show there is an isomorphism of stalks $\mathfrak{C}_{x,\mathrm{ex}} \to \mathfrak{D}_{\hat{x},\mathrm{ex}}$. However, using the universal properties of $\mathfrak{C}_x$, $\mathfrak{C}[c^{-1}]$ and $\mathfrak{C}[c^{-1}]_{\hat{x}}$ this follows by the same reasoning as Lemma 2.4.6.

If $\mathfrak{C}$ is firm, then so is $\mathfrak{C}[c^{-1}]$, as the sharpening of the monoid of $\mathfrak{C}[c^{-1}]$ is generated by the image of $\mathfrak{C}^\sharp_{\mathrm{ex}}$ under the morphism $\mathfrak{C} \to \mathfrak{C}[c^{-1}]$. If $\mathfrak{C}$ is interior, then $\mathfrak{C}[c^{-1}]$ is also interior, as otherwise zero-divisors would have to come from zero divisors in $\mathfrak{C}_{\mathrm{ex}}$. ☐

In the following proposition, we find a sufficient condition for affine C^∞-schemes with corners that have interior stalks to be interior affine C^∞-schemes with corners.

Proposition 5.4.5. *Let $(\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be a C^∞-ring with corners and let $(X, \mathcal{O}_X) = \mathrm{Spec^c}(\mathfrak{C}, \mathfrak{C}_{\rm ex})$ be an affine C^∞-scheme with corners such that all stalks are interior C^∞-rings with corners. Then for each $x \in X$, if every zero-divisor $c' \in \mathfrak{C}_{\rm ex}$ has $\pi_{x,\rm ex}(c') \in \mathfrak{C}_{x,\rm ex}$ either invertible or zero, then $(X, \mathcal{O}_X)$ is an interior affine C^∞-scheme with corners.*

Proof. Let $(\mathfrak{C}, \mathfrak{C}_{\rm ex}) = \boldsymbol{\mathfrak{C}}$ be a C^∞-ring with corners such that the conditions in the proposition hold. Let $Z = \{c \in \mathfrak{C}_{\rm ex} : c \neq 0, dc = 0 \text{ for some } d \neq 0, d \in \mathfrak{C}\}$ be the set of all zero divisors in $\mathfrak{C}_{\rm ex}$. Then $Z \cup \{0\}$ is an ideal of $\mathfrak{C}_{\rm ex}$. Consider $\mathfrak{C}'_{\rm ex} = \mathfrak{C}_{\rm ex} \setminus Z$. We claim that $(\mathfrak{C}, \mathfrak{C}'_{\rm ex}) = \boldsymbol{\mathfrak{C}}'$ is a C^∞-ring with corners and $\mathrm{Spec^c}(\mathfrak{C}, \mathfrak{C}'_{\rm ex}) = (X, \mathcal{O}_X)$.

We still have $\Phi_i : \mathfrak{C}'^{\times}_{\rm ex} \to \mathfrak{C}$ injective, as it is a restriction and $Z \cap \mathfrak{C}^{\times}_{\rm ex} = \emptyset$. If we take a function $f : \mathbb{R}^n_k \to [0, \infty)$, we must show that the image of ${\mathfrak{C}'_{\rm ex}}^k \times \mathfrak{C}^{n-k}$ under Ψ_f is contained in $\mathfrak{C}'_{\rm ex}$. Assume f is non-zero, then $f(x_1, \ldots, x_n) = x^{a_1} \cdots x^{a_k} F(x_1, \ldots, x_n)$ where $F : \mathbb{R}^n_k \to (0, \infty)$ and $a_1, \ldots, a_k \in \mathbb{N}$.

Assume for a contradiction that $\Psi_f(c'_1, \ldots, c'_k, c_{k+1}, \ldots, c_n) \in Z$, then

$$c'^{a_1}_1 \cdots c'^{a_k}_k \Psi_F(c'_1, \ldots, c'_k, c_{k+1}, \ldots, c_n) d = 0$$

for some $d \in Z$. As F is non-zero at all points, it is invertible and hence so is element $\Psi_F(c'_1, \ldots, c'_k, c_{k+1}, \ldots, c_n)$. Then $c'^{a_1}_1 \ldots c'^{a_k}_k d = 0$, and at least one a_i must be non-zero. But then the corresponding c'_i is a zero-divisor. Hence, the image of ${\mathfrak{C}'_{\rm ex}}^k \times \mathfrak{C}^{n-k}$ under Ψ_f is contained in $\mathfrak{C}'_{\rm ex}$ and $(\mathfrak{C}, \mathfrak{C}'_{\rm ex})$ is a C^∞-ring with corners.

We now show that the spectra are isomorphic. We have an inclusion $(\mathfrak{C}, \mathfrak{C}'_{\rm ex}) \to (\mathfrak{C}, \mathfrak{C}_{\rm ex})$. Taking the spectrum of this morphism, we have an isomorphism on the topological spaces and the sheaves of C^∞-rings. On the sheaves of monoids, we show the stalks are isomorphic. That is, we show that for any element $z \in Z$, then for each $x \in X$, $\pi_{x',\rm ex}(z) = \pi_{x',\rm ex}(c')$ for some $c' \in \mathfrak{C}'_{\rm ex}$ for all x' in some open neighbourhood around $x \in X$.

Take $z \in Z$, then by assumption, we know $\pi_{x,\rm ex}(z)$ is either invertible or zero. If it is zero, then by Lemma 4.6.9, there is an element $a \in \mathfrak{C}_{\rm ex}$ such that $x \circ \Phi_i(a) \neq 0$ and $az = 0$. Then, by the definition of topology of X, there is an open set $U \subset X$ such that $x' \circ \Phi_i(a) \neq 0$ in this neighbourhood. So again by Lemma 4.6.9, we have that $\pi_{x',\rm ex}(z) = 0$ in this neighbourhood, as required.

If $\pi_{x,\rm ex}(z)$ is invertible, then there is an element c_x in the localisation $\mathfrak{C}_x$, such that $\Psi_{\exp}(c) = \pi_{x,\rm ex}(z)$. As $c \in \mathfrak{C}_x$ and $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$ is surjective, then there is an element $c \in \mathfrak{C}$ such that $\pi_x(c) = c_x$. So we have $\pi_{x,\rm ex}(z) = \Psi_{\exp}(\pi_x(c)) = \pi_{x,\rm ex}(\Psi_{\exp}(c))$ and by Lemma

4.6.9, there are $a, b \in \mathfrak{C}_{\text{ex}}$ such that $\Phi_i(a) - \Phi_i(b) \in I$, $x \circ \Phi_i(a) \neq 0$ and $a\Psi_{\exp}(c) = bz$, where I is as defined in (2.1.4). Now $\Phi_i(a) - \Phi_i(b) \in I$ is a local condition, and so is $x \circ \Phi_i(a) \neq 0$ by the previous paragraph, and $a\Psi_{\exp}(c) = bz$ is a global condition, so we have that $\pi_{x,\text{ex}}(z) = \pi_{x,\text{ex}}(\Psi_{\exp}(c))$ locally as required.

Hence, $\operatorname{Spec^c} \boldsymbol{\mathfrak{C}} = \operatorname{Spec^c} \boldsymbol{\mathfrak{C}}'$ is an interior affine C^∞-scheme with corners. □

Remark 5.4.6. In this proposition, $(\mathfrak{C}, \mathfrak{C}_{\text{ex}})$ may not be firm, however the resulting C^∞-ring with corners $(\mathfrak{C}, \mathfrak{C}'_{\text{ex}})$ may be firm. Example 5.8.8 involves such a case.

This proposition uses that if $\pi_{x,\text{ex}}(c') \in \mathfrak{C}_{x,\text{ex}}$ is either invertible or zero for zero divisors $c' \in \mathfrak{C}_{\text{ex}}$ then $\pi_{x,\text{ex}}(c')$ is equal to $\pi_{x,\text{ex}}(d')$ for some other element $d' \in \mathfrak{C}_{\text{ex}}$ that is not a zero divisor of $\mathfrak{C}_{\text{ex}}$ in every stalk. So it might be possible to generalise this proposition by requiring that for all zero-divisors $c' \in \mathfrak{C}_{\text{ex}}$ and all $\mathbb{R}$-points x, then whenever $\pi_{x,\text{ex}}(c') \in \mathfrak{C}_{x,\text{ex}}$ is not invertible or zero, it must be equal to $\pi_{x,\text{ex}}(d')$ for $d' \in \mathfrak{C}_{\text{ex}}$ such that d' is not a zero-divisor.

It would be interesting to consider whether this is true for all C^∞-rings with corners $\boldsymbol{\mathfrak{C}}$ such that $\operatorname{Spec^c}(\boldsymbol{\mathfrak{C}}) \in \mathbf{LC^\infty RS^c_{in}}$; we show in the following proposition that this is true at least locally for firm C^∞-schemes with corners.

Proposition 5.4.7. *Let $\boldsymbol{X}$ be a firm C^∞-scheme with corners that is an interior C^∞-ringed space with corners. Then $\boldsymbol{X}$ is an interior firm C^∞-scheme with corners.*

Proof. Assume $\boldsymbol{X} \cong \operatorname{Spec^c}(\boldsymbol{\mathfrak{C}})$ is affine with $\boldsymbol{\mathfrak{C}}$ semi-complete and firm. Take a point $x \in X$. We will construct an interior C^∞-ring with corners $\boldsymbol{\mathfrak{D}}$ such that $\boldsymbol{X}|_U \cong \operatorname{Spec^c}(\boldsymbol{\mathfrak{D}})$ for some neighbourhood $U \subset X$ with $x \in U$.

Take generators $c'_1, \ldots, c'_n$ in $\mathfrak{C}_{\text{ex}}$ that generate the sharpening. Assume, without loss of generality, that $\pi_{x,\text{ex}}(c'_k) \neq 0 \in \mathfrak{C}_{x,\text{ex}}$ for $i = 1, \ldots, k$ and $\pi_{x,\text{ex}}(c'_k) = 0 \in \mathfrak{C}_{x,\text{ex}}$ for $i = k+1, \ldots, n$ for some integer $1 \leqslant k \leqslant n$. As $\boldsymbol{X}$ is an interior C^∞-ringed space with corners then the sets

$$X_{c'_i} = \{x \in X : \pi_{x,\text{ex}}(c'_k) \neq 0 \in \mathfrak{C}_{x,\text{ex}}\}$$

and

$$X \setminus X_{c'_i} = \hat{X}_{c'_i} = \{x \in X : \pi_{x,\text{ex}}(c'_k) = 0 \in \mathfrak{C}_{x,\text{ex}}\}$$

are both open and closed in X. If we consider $V = (\cap_{i=1}^k X_{c'_i}) \cap (\cap_{i=k+1}^n \hat{X}_{c'_i})$, then $x \in V$ and V is open. As the topology on X is generated by basic open sets of the form $U_c = \{x \in X : x(c) \neq 0\}$ for some $c \in \mathfrak{C}$, then there is a $c \in \mathfrak{C}$ such that $U_c \subset V$ with $x \in U_c$.

Let $\boldsymbol{\mathfrak{D}}$ be the semi-complete C^∞-ring with corners corresponding to $\boldsymbol{\mathfrak{C}}[c^{-1}]$. Then we know $\boldsymbol{X}|_{U_c} \cong \operatorname{Spec^c}(\boldsymbol{\mathfrak{C}}[c^{-1}]) \cong \operatorname{Spec^c}(\boldsymbol{\mathfrak{D}})$ by Lemma 5.4.4 and Definition 5.3.4. We claim that $\boldsymbol{\mathfrak{D}}$ is an interior C^∞-scheme with corners.

We need to show that there are no zero-divisors in $\mathfrak{D}_{\text{ex}}$. Firstly, note that there is a morphism $\boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ and that under this morphism, the c_i also generate the sharpening of $\mathfrak{D}_{\text{ex}}$ and so $\boldsymbol{\mathfrak{D}}$ is firm. So the candidates for zero-divisors in $\mathfrak{D}_{\text{ex}}$ are these generators. Now if $\pi_{x,\text{ex}}(c_i') = 0 \in \mathfrak{C}_{x,\text{ex}}$ for all $x \in U_c$, we claim that the image of c_i' is zero in $\mathfrak{D}_{\text{ex}}$. This follows as the morphism $\boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ induces a morphism $\boldsymbol{\mathfrak{C}}_x \to \boldsymbol{\mathfrak{D}}_x$, which commutes with the localisation morphisms $\mathfrak{C}_{\text{ex}} \to \mathfrak{C}_{x,\text{ex}}$ and $\mathfrak{D}_{\text{ex}} \to \mathfrak{D}_{x,\text{ex}}$. This means that c_i' is sent to zero in $\mathfrak{D}_{x,\text{ex}}$ whenever it is zero in $\mathfrak{C}_{x,\text{ex}}$. As $\boldsymbol{\mathfrak{D}}$ is semi complete we have that $\mathfrak{D}_{\text{ex}} \subset \mathcal{O}_X(U_c)$, and if $\pi_{x,\text{ex}}(c_i') = 0 \in \mathfrak{D}_{x,\text{ex}}$ for all $x \in U_c$ then it must be the zero section of $\mathcal{O}_X(U_c)$ so it must be zero in $\mathfrak{D}_{\text{ex}}$, as required.

Similarly, if $\pi_{x,\text{ex}}(c_i') \neq 0 \in \mathfrak{C}_{x,\text{ex}}$ for some $x \in U_c$, then the image of c_i' under this morphism is non-zero. So the only candidates for zero-divisors of $\mathfrak{D}_{\text{ex}}$ are the c_i' for $i = 1, \ldots, k$. These are all non-zero in each stalk in $\mathfrak{D}_{\text{ex}}$. If they were zero-divisors, then they would be zero-divisors in the stalks. However, we know $\boldsymbol{X}$ has interior stalks, so these c_i' cannot be zero-divisors. So there must be no zero divisors in $\mathfrak{D}_{\text{ex}}$. Hence $\boldsymbol{\mathfrak{D}}$ is interior and semi-complete, with $\boldsymbol{X}|_{U_c} \cong \operatorname{Spec}^{\text{c}}(\boldsymbol{\mathfrak{D}})$. As we can do this around any point, then $\boldsymbol{X}$ is an interior firm C^∞-scheme with corners.

If $\boldsymbol{X}$ is not affine, we can do this on affine open covers and again show that $\boldsymbol{X}$ is an interior firm C^∞-scheme with corners. □

5.4.1 Limits and colimits

Let us consider how limits and colimits behave in the category of C^∞-schemes with corners. We start by considering adjoints.

Proposition 5.4.8. *There are right adjoints to the forgetful functors* $\mathbf{C^\infty Sch^c} \to \mathbf{C^\infty Sch}$, $\mathbf{AC^\infty Sch^c} \to \mathbf{AC^\infty Sch}$, $\mathbf{C^\infty Sch^c_{in}} \to \mathbf{C^\infty Sch}$ *and* $\mathbf{AC^\infty Sch^c_{in}} \to \mathbf{AC^\infty Sch}$ *hence the forgetful functors preserve colimits.*

This result uses the adjoints constructed in Proposition 5.1.12 and Theorem 4.3.9.

Proof. Let $(X, \mathcal{O}_X) \cong \operatorname{Spec}(\mathfrak{C})$, where $\mathfrak{C} = \mathcal{O}_X(X)$ is complete. We show that the image of $(X, \mathcal{O}_X)$ under the right adjoint $\mathbf{LC^\infty RS} \to \mathbf{LC^\infty RS^c}$ constructed in Proposition 5.1.12 ies in $\mathbf{AC^\infty Sch^c}$, and its image is $(X, \mathcal{O}_X, \hat{\mathcal{O}}_X^{\text{ex}}) \cong \operatorname{Spec}^{\text{c}}(F_{\text{exp}}(\mathfrak{C}))$ where $F_{\text{exp}}(\mathfrak{C}) = (\mathfrak{C}, \Phi_{\text{exp}}(\mathfrak{C}) \amalg \{0_{\text{ex}}\})$ is the left adjoint constructed in Theorem 4.3.9. This means that $\operatorname{Spec}^{\text{c}} \circ F_{\text{exp}} : (\mathbf{C^\infty Rings^{co}})^{\mathbf{op}} \to \mathbf{AC^\infty Sch^c}$ is the right adjoint under the equivalence of categories $\mathbf{AC^\infty Sch} \cong (\mathbf{C^\infty Rings^{co}})^{\mathbf{op}}$ from Theorem 2.4.14(b).

Note that for an $\mathbb{R}$-point $x : \mathfrak{C} \to \mathbb{R}$, then we have $F_{\text{exp}}(\mathfrak{C})_x \cong F_{\text{exp}}(\mathfrak{C}_x)$, as all non-zero elements of the monoid of $F_{\text{exp}}(\mathfrak{C})$ are already invertible. There is a canonical morphism

$(X, \mathcal{O}_X, \hat{\mathcal{O}}_X^{\mathrm{ex}}) \to \mathrm{Spec}^{\mathrm{c}}(F_{\exp}(\mathfrak{C}))$, which is the identity on topological spaces and on C^∞-rings. We will define it on the sheaves of monoids, and show it is an isomorphism.

On the monoid sheaf, the section $s' : U \to \amalg_{x \in U}(\Phi_{\exp}(\mathfrak{C}_x) \amalg \{0_{\mathrm{ex}}\})$ is locally such that $s'(x) = \Phi_{\exp}(\pi_x(c))$ or $s'(x) = 0_{\mathrm{ex}}$. Then locally it is either 0_{ex} or it is $\Phi_{\exp}(s)$ for some section s in the C^∞-ring sheaf. Now if $s'(x) = 0_{\mathrm{ex}}$, it must be zero in the connected component V of U that contains x. Otherwise, $s'(x)$ is non-zero in the connected component, and, using the glueing and uniqueness property of $\mathcal{O}_X$, it is equal to $\Phi_{\exp}(s)$ on V, for a unique element $s \in \mathcal{O}_X(V)$. This means that s' corresponds to a unique element of the sheafification of $\Phi_{\exp}(\mathcal{O}_X) \amalg \{0_{\mathrm{ex}}\}$, which is the definition of $\hat{\mathcal{O}}_X^{\mathrm{ex}}$. So the morphism of sheaves of monoids is well defined. Now if $s' \in \hat{\mathcal{O}}_X^{\mathrm{ex}}$, we can run this argument in reverse to see that s' corresponds to a unique element of the monoid sheaf of $\mathrm{Spec}(F_{\exp}(\mathfrak{C}))$, giving the result.

As $\mathbf{AC^\infty Sch^c_{in}}$ is a subcategory of $\mathbf{AC^\infty Sch^c}$ and $F_{\exp}(\mathfrak{C})$ is interior, then $\mathrm{Spec}^{\mathrm{c}} \circ F_{\exp}$: $\mathbf{C^\infty Rings^{co})^{op}} \to \mathbf{AC^\infty Sch^c_{in}}$ is the right adjoint to $\mathbf{AC^\infty Sch^c_{in}} \to \mathbf{AC^\infty Sch}$.

For the right adjoint to $\mathbf{C^\infty Sch^c} \to \mathbf{C^\infty Sch}$, take $\underline{X} = (X, \mathcal{O}_X) \in \mathbf{C^\infty Sch}$ and let U be an affine open set of X, so that $\underline{X}|_U \cong \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C})$ for $\mathfrak{C}$ complete. Then, by the discussion above, the right adjoint from Proposition 5.1.12 applied to $\underline{X}$ will be locally isomorphic to $\mathrm{Spec}^{\mathrm{c}}(F_{\exp}(\mathfrak{C}))$, and hence in $\mathbf{C^\infty Sch^c}$. Again as $\mathbf{C^\infty Sch^c}$ is a full subcategory of $\mathbf{LC^\infty RS^c}$, then this must be the right adjoint to $\mathbf{C^\infty Sch^c} \to \mathbf{C^\infty Sch}$. The same discussion describes the right adjoint to $\mathbf{C^\infty Sch^c_{in}} \to \mathbf{C^\infty Sch}$. □

Remark 5.4.9. Ideally we would also like left adjoints to the functors $\mathbf{C^\infty Sch^c} \to \mathbf{C^\infty Sch}$, $\mathbf{AC^\infty Sch^c} \to \mathbf{AC^\infty Sch}$, $\mathbf{C^\infty Sch^c_{in}} \to \mathbf{C^\infty Sch}$ and $\mathbf{AC^\infty Sch^c_{in}} \to \mathbf{AC^\infty Sch}$, as this would show that these functors preserve limits. This would be helpful in our understanding of fibre products of C^∞-schemes with corners. A candidate for the left adjoint to $\mathbf{AC^\infty Sch^c} \to \mathbf{AC^\infty Sch}$ is $\mathrm{Spec}(\mathfrak{C}) \mapsto \mathrm{Spec}^{\mathrm{c}}(F_{\geqslant 0}(\mathfrak{C}))$ where $F_{\geqslant 0}(\mathfrak{C})$ is the right adjoint constructed in Theorem 4.3.9. We would like to proceed as follows:

There is a canonical morphism $(X, \mathcal{O}_X, \mathcal{O}^{\mathrm{ex}}_{X\geqslant 0}) \to \mathrm{Spec}(F_{\geqslant 0}(\mathfrak{C}))$. On topological spaces and on the sheaves of C^∞-rings, it is the identity morphism. On the sheaves of monoids, if $s' \in \mathcal{O}^{\mathrm{ex}}_X(U)$ is such that $s'_x = \pi^{\mathrm{ex}}_x(c')$ for all $x \in V \subset U$, and $c' \in \mathfrak{C}_{\geqslant 0}$, then s' comes from an element $s \in \mathcal{O}_X(U)$ with $s_x = \pi_x(c')$ as $c' \in \mathfrak{C}_{\geqslant 0} \subset \mathfrak{C}$. As $c' \in \mathfrak{C}_{\geqslant 0}$, then if $s_1, \ldots, s_n \in \mathcal{O}_X(U)$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is smooth such that $f|_{[0,\infty)\times\mathbb{R}^n} = 0$, then

$$\Phi_f(s, s_1, \ldots, s_n)_x = \Phi_f(s_x, s_{1,x}, \ldots, s_{n,x}) = \Phi_f(\pi_x(c), s_{1,x}), \ldots, s_{n,x}) = 0 \in \mathfrak{C},$$

hence $\Phi_f(s, s_1, \ldots, s_n)$ is locally 0, and the glueing property of sheaves says this is globally 0. Therefore s is an element in $\mathcal{O}^{\mathrm{ex}}_{X\geqslant 0}$, and this gives a well defined morphism of sheaves

$\mathcal{O}_X^{\mathrm{ex}} \to \mathcal{O}_{X\geqslant 0}^{\mathrm{ex}}$.

Considering the stalks of the map, we need to show for any $\mathbb{R}$-point $x : \mathfrak{C} \to \mathbb{R}$, the localisation $(\mathfrak{C}, \mathfrak{C}_{\geqslant 0})_x$ is isomorphic to $F_{\geqslant 0}(\mathfrak{C}_x) = (\mathfrak{C}_x, (\mathfrak{C}_x)_{\geqslant 0})$. The universal property of the localisation shows there exists a unique morphism $(\mathfrak{C}, \mathfrak{C}_{\geqslant 0})_x \to (\mathfrak{C}_x, (\mathfrak{C}_x)_{\geqslant 0})$, which is the stalk map of our morphism, and Theorem 4.6.8 says this is an isomorphism on the C^∞-rings. On the monoids, surjectivity of the localisation $(\pi_x, \pi_x^{\mathrm{ex}}) : (\mathfrak{C}, \mathfrak{C}_{\geqslant 0}) \to (\mathfrak{C}, \mathfrak{C}_{\geqslant 0})_x$ says elements in $(\mathfrak{C}_{\geqslant 0})_x$ can be represented as $\pi_x^{\mathrm{ex}}(c)$ for some $c \in \mathfrak{C}_{\geqslant 0}$. Then, for an element $c \in \mathfrak{C}_{\geqslant 0}$, the monoid morphism sends $(\mathfrak{C}_{\geqslant 0})_x \ni \pi_x^{\mathrm{ex}}(c) \to \pi_x(c) \in (\mathfrak{C}_x)_{\geqslant 0}$.

To show injectivity of this map, take $c, d \in \mathfrak{C}$ such that $\pi_x(c) = \pi_x(d) \in \mathfrak{C}_x$ are in $(\mathfrak{C}_x)_{\geqslant 0}$. By Proposition 2.1.15, $\pi_x(c) = \pi_x(d)$ if and only if there is a $k \in \mathfrak{C}$ such that $k \cdot (c - d) = 0 \in \mathfrak{C}$, with $x(k) \neq 0$. Then $k^2 \in \mathfrak{C}_{\geqslant 0}$ by Lemma 4.3.12, and $k^2 \cdot c = k^2 \cdot d \in \mathfrak{C}_{\geqslant 0}$ with $k^2(x) > 0$. Hence $\pi_x^{\mathrm{ex}}(c) = \pi_x^{\mathrm{ex}}(d) \in (\mathfrak{C}_{\geqslant 0})_x$, and the map is injective.

However, we cannot at present show this map is surjective in general, although it is surjective when $X = \mathbb{R}^n$ for example.

We now consider fibre products of C^∞-schemes with corners.

Proposition 5.4.10. *Let $\boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{Y} \to \boldsymbol{Z}$ be morphisms of C^∞-schemes with corners. If $\boldsymbol{Z}$ is a firm C^∞-scheme with corners, then the fibre product $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ exists in the category of C^∞-schemes with corners, and is equal to the fibre product in the category $\mathbf{LC^\infty RS^c}$.*

Similarly, if $\boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{Y} \to \boldsymbol{Z}$ are morphisms of interior C^∞-schemes with corners and $\boldsymbol{Z}$ is a firm interior C^∞-scheme with corners, then the fibre product $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ exists in the category of interior C^∞-schemes with corners, and is equal to the fibre product in the category $\mathbf{LC^\infty RS^c_{in}}$.

Proof. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be morphisms of C^∞-schemes with corners, $\boldsymbol{Z}$ a firm C^∞-scheme with corners. Take the fibre product $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y} \in \mathbf{LC^\infty RS^c}$, constructed as in Theorem 5.1.10.

Pick a point $(x, y) \in X \times_Z Y$ with $f(x) = g(y) = z \in Z$, and take affine open sets $x \in V_1 \subset X$, $y \in V_2 \subset Y$, $z \in V_3 \subset Z$ with $\boldsymbol{Z}|_{V_3}$ isomorphic to the spectrum of a firm C^∞-ring with corners. If necessary, use Lemma 5.4.4 to shrink V_1 and V_2 to affine open sets so that $f(V_1) \subset V_3 \supset g(V_2)$. Then choosing $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}, \boldsymbol{\mathfrak{E}}$ to be semi-complete C^∞-rings with corners (using Lemma 5.3.3 and Definition 5.3.4), we have $\operatorname{Spec}(\boldsymbol{\mathfrak{C}}) = (V_1, \mathcal{O}_X|_{V_1})$, $\operatorname{Spec} \boldsymbol{\mathfrak{D}} = (V_2, \mathcal{O}_Y|_{V_2})$, and $\operatorname{Spec} \boldsymbol{\mathfrak{E}} = (V_3, \mathcal{O}_Z|_{V_3})$, with $\boldsymbol{\mathfrak{E}}$ firm, and the morphisms $\boldsymbol{\phi}_1 : \boldsymbol{\mathfrak{C}} \to \Gamma \circ \operatorname{Spec}^{\mathrm{c}} \boldsymbol{\mathfrak{C}} \cong \boldsymbol{\mathcal{O}}_X(V_1)$, $\boldsymbol{\phi}_2 : \boldsymbol{\mathfrak{D}} \to \Gamma \circ \operatorname{Spec}^{\mathrm{c}} \boldsymbol{\mathfrak{D}} \cong \boldsymbol{\mathcal{O}}_y(V_2)$, $\boldsymbol{\phi}_3 : \boldsymbol{\mathfrak{E}} \to \Gamma \circ \operatorname{Spec}^{\mathrm{c}} \boldsymbol{\mathfrak{E}} \cong \boldsymbol{\mathcal{O}}_Z(V_3)$ are isomorphisms on their C^∞-rings, and injective on their monoids.

The morphism $\boldsymbol{f}$ gives the morphism of C^∞-rings with corners $\boldsymbol{f}_\sharp(V_3) : \boldsymbol{\mathcal{O}}_Z(V_3) \to \boldsymbol{\mathcal{O}}_X(f^{-1}(V_3))$ which we can compose with the restriction map $\boldsymbol{\rho}_{f^{-1}(V_3),V_1}$ to get a morphism of C^∞-rings with corners $\hat{\boldsymbol{f}} : \boldsymbol{\mathcal{O}}_Z(V_3) \cong \Gamma \circ \mathrm{Spec^c}\,\mathfrak{E} \to \Gamma \circ \mathrm{Spec^c}\,\mathfrak{C} \cong \boldsymbol{\mathcal{O}}_X(V_1)$ and similarly we can define a $\hat{\boldsymbol{g}}$ for $\boldsymbol{g}$. Now, we would like to find morphisms $\mathfrak{E} \to \mathfrak{C}$, $\mathfrak{D} \to \mathfrak{C}$ that have spectra $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{g}}$ respectively. We can do this on the C^∞-rings as ϕ_1, ϕ_2, ϕ_3 are isomorphisms. On the monoid $\mathfrak{E}_{\mathrm{ex}}$, the invertible elements are generated by $\Psi_{\exp}(\mathfrak{E})$. As $\mathfrak{E}$ is firm, every element in $\mathfrak{E}_{\mathrm{ex}}$ is of the form $a'e_1 \dots e_n$ where $a \in \Psi_{\exp}(\mathfrak{E})$ is invertible, and the image of the $e_1, \dots, e_n$ under the quotient morphism $\mathfrak{E}_{\mathrm{ex}} \to \mathfrak{E}^\sharp_{\mathrm{ex}}$ generate the sharpening $\mathfrak{E}^\sharp_{\mathrm{ex}}$.

Now, for each $i = 1, \dots, n$, $\hat{f}_{\mathrm{ex}} \circ \phi_3^{\mathrm{ex}}(e_i) \in \mathcal{O}_X^{\mathrm{ex}}(V_1)$, but it may not 'lift' to $\mathfrak{C}_{\mathrm{ex}}$ (by 'lift' we mean that it has an inverse image under ϕ_1^{ex}). However, locally around x we have that $\hat{f}_{\mathrm{ex}} \circ \phi_3^{\mathrm{ex}}(e_i)_{\hat{x}} = \pi_{\hat{x},\mathrm{ex}}(c')$ for all $\hat{x}$ in some open set $x \in V_1^i \subset V_1$. Taking the intersection $V_1' = \cap_{i=1}^n V_1^j$, we can restrict further to an open set $x \in U_c \subset V_1'$ for some $c \in \mathfrak{C}$, using Lemma 5.4.4. Then each $\hat{f} \circ \phi_3^{\mathrm{ex}}(e_i) \in \mathcal{O}_X^{\mathrm{ex}}(U_c)$ does lift to an element of $(\mathfrak{C}[c^{-1}])_{\mathrm{ex}}$. Then every element $a'e_1 \dots e_n$ lifts in this neighbourhood U_c. This gives a well defined C^∞-ring with corners morphism from $\mathfrak{E}$ to $\mathfrak{C}[c^{-1}]$. Similarly, we have a well defined morphism between $\mathfrak{E}$ and $\mathfrak{D}[d^{-1}]$ for some element of $d \in \mathfrak{D}$. This gives a diagram of C^∞-rings with corners, and we can take the pushout $\mathfrak{C}[c^{-1}] \amalg_{\mathfrak{E}} \mathfrak{D}[d^{-1}]$.

As $\mathrm{Spec^c} : (\mathbf{C^\infty Rings^c})^{\mathbf{op}} \to \mathbf{LC^\infty RS^c}$ is a right adjoint, then it preserves limits, so $\mathrm{Spec^c}(\mathfrak{C}[c^{-1}] \amalg_{\mathfrak{E}} \mathfrak{D}[d^{-1}]) \cong \boldsymbol{X}|_{U_c} \times_{\boldsymbol{Z}|_{V_3}} \boldsymbol{Y}|_{U_d}$. However, the construction of $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ in Theorem 5.1.10 implies $(\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y})|_{U_c \times_{V_3} U_d} \cong \boldsymbol{X}|_{U_c} \times_{\boldsymbol{Z}|_{V_3}} \boldsymbol{Y}|_{U_d}$, and hence $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ is a C^∞-scheme with corners. As $\mathbf{C^\infty Sch^c}$ is a full subcategory of $\mathbf{LC^\infty RS^c}$, then $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ is the fibre product in $\mathbf{C^\infty Sch^c}$.

Similarly, if $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ are morphisms of interior C^∞-schemes with corners, $\boldsymbol{Z}$ a firm interior C^∞-scheme with corners, we can construct the fibre product as above in the category of $\mathbf{LC^\infty RS^c_{in}}$. The same method will show that it is an interior C^∞-scheme with corners; one needs to check that if $\mathfrak{C}$ is interior then $\mathfrak{C}[c^{-1}]$ in $\mathbf{C^\infty Rings^c_{in}}$ is interior and matches with this localisation in $\mathbf{C^\infty Rings^c}$, but this follows from Theorem 4.3.7(b) and Lemma 5.4.4. □

Remark 5.4.11. If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ are morphisms of affine C^∞-schemes with corners where $\boldsymbol{X} \cong \mathrm{Spec^c}\,\mathfrak{C}$, $\boldsymbol{Y} \cong \mathrm{Spec^c}\,\mathfrak{D}$, $\boldsymbol{Z} \cong \mathrm{Spec^c}\,\mathfrak{E}$, and such that $\boldsymbol{f} = \mathrm{Spec^c}(\boldsymbol{\phi} : \mathfrak{E} \to \mathfrak{D})$, $\boldsymbol{g} = \mathrm{Spec^c}(\boldsymbol{\psi} : \mathfrak{E} \to \mathfrak{C})$, then using that $\mathrm{Spec^c}$ is a right adjoint we can show directly that the fibre product in both $\mathbf{LC^\infty RS^c}$ and $\mathbf{AC^\infty Sch^c}$ exists and is isomorphic to $\mathrm{Spec^c}(\mathfrak{C} \amalg_{\mathfrak{E}} \mathfrak{D})$. In Proposition 5.4.10 we see that if $\boldsymbol{Z}$ if firm, then we call locally find such $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ for any morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$.

However, as in Remark 5.3.5 and Definition 5.3.4, there is no equivalence of categories between C^∞-rings with corners and affine C^∞-schemes with corners. This means morphisms of affine C^∞-schemes with corners may not correspond to morphisms of C^∞-rings with corners. So while we can take the fibre product of affine C^∞-schemes with corners in $\mathbf{LC^\infty RS^c}$, it is unclear if this should be the spectrum of a C^∞-ring with corners.

Proposition 5.4.10 suggests fibre products of affine C^∞-schemes with corners where they exist may not even be affine, as it involves shrinking open neighbourhoods to find affine neighbourhoods. Also, if the C^∞-schemes are not firm then the neighbourhoods may need to be shrunk to a set that is not longer open, which suggests that fibre products of arbitrary C^∞-schemes with corners may not exist.

We suspect a counterexample to the existence of arbitrary fibre products of (affine) C^∞-schemes with corners may be constructed using a version of Example 5.3.2. While the C^∞-ring with corners in this example is firm, potentially one could construct a counterexample using some decreasing sequence of open sets and appropriate monoid generators (as in Figure 5.3.1) around a particular point, which would create infinitely many generators in the monoid. This would no longer be firm. Whether this does give a counter-example or not we will not consider this here, as all the examples we would like to consider are firm.

Corollary 5.4.12. *The subcategories* $\mathbf{C^\infty Sch^c_{fi}}$ *and* $\mathbf{C^\infty Sch^c_{fi,in}}$ *are closed under finite limits in* $\mathbf{LC^\infty RS^c}$ *and* $\mathbf{LC^\infty RS^c_{in}}$*. Hence, fibre products and all finite limits exist in these subcategories.*

Proof. The proof of Proposition 5.4.10 shows that the fibre product in $\mathbf{LC^\infty RS^c}$ is a C^∞-scheme with corners and is locally isomorphic to $\mathrm{Spec^c}(\mathfrak{C}[c^{-1}]\amalg_{\mathfrak{E}}\mathfrak{D}[d^{-1}])$. The application of Lemma 5.3.3 allows $\mathfrak{C},\mathfrak{D},\mathfrak{E}$ to be firm, then using Lemma 5.4.4 and Proposition 4.5.2, shows $\mathfrak{C}[c^{-1}]\amalg_{\mathfrak{E}}\mathfrak{D}[d^{-1}]$ is firm, so the fibre product is a firm C^∞-scheme with corners. As $\mathbf{C^\infty Sch^c_{fi}}$ is a full subcategory of $\mathbf{LC^\infty RS^c}$, then this is the fibre product in $\mathbf{C^\infty Sch^c_{fi}}$.

As $\mathbf{C^\infty Sch^c_{fi}}$ has a final object $\mathrm{Spec^c}(\mathbb{R},[0,\infty))$, and all fibre products in a category with a final object are (iterated) fibre products, then $\mathbf{C^\infty Sch^c_{fi}}$ is closed under finite limits in $\mathbf{LC^\infty RS^c}$, and all such fibre products and finite limits exist in $\mathbf{C^\infty Sch^c_{fi}}$. A similar argument holds for $\mathbf{C^\infty Sch^c_{fi,in}}$. ◻

Now we consider coproducts and colimits of C^∞-schemes with corners. We will use $\prod$ for the product of C^∞-rings with corners and $\prod^{\mathrm{in}}$ for products of interior C^∞-rings with corners. We will refer to results in §2.4.1.

Remark 5.4.13. Firstly, some general colimits of C^∞-schemes with corners exist. That is, if we have a collection of C^∞-schemes with corners such that there are isomorphisms between open sets of the schemes, then we can 'glue together' to create another C^∞-scheme with corners.

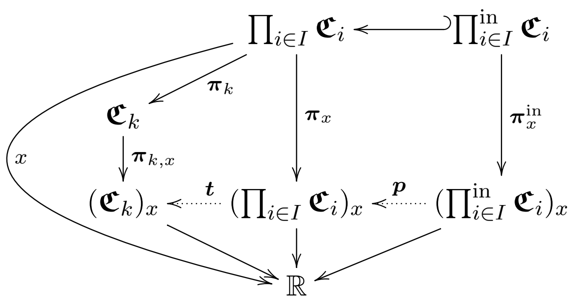
This works as follows: Let $\{\boldsymbol{X}_i\}_{i\in I}$ be a collection of C^∞-schemes with corners and say there are open subsets $\boldsymbol{U}_{i,j}$ for each $i,j \in I$ with the induced C^∞-scheme with corners structure as in Lemma 5.4.4. Say we have isomorphisms of C^∞-schemes with corners $\boldsymbol{\phi}_{i,j} : \boldsymbol{U}_{i,j} \to \boldsymbol{U}_{j,i}$, which are the identity morphism when $i = j$, and such that $\boldsymbol{\phi}_{i,j} = \boldsymbol{\phi}_{j,i}^{-1}$, $\boldsymbol{\phi}_{i,j}(\boldsymbol{U}_{i,j} \cap \boldsymbol{U}_{i,k}) = \boldsymbol{U}_{j,i} \cap \boldsymbol{U}_{j,k}$, and $\boldsymbol{\phi}_{i,j} = \boldsymbol{\phi}_{k,j} \circ \boldsymbol{\phi}_{i,k}$. Then this gives a diagram of C^∞-schemes with corners with objects $\{\boldsymbol{X}_i, \boldsymbol{U}_{i,j}\}_{i,j\in I}$, and inclusion morphisms $\boldsymbol{U}_{i,j} \to \boldsymbol{X}_i$ and isomorphisms $\boldsymbol{\phi}_{i,j} : \boldsymbol{U}_{i,j} \to \boldsymbol{U}_{j,k}$. By Proposition 5.1.7, we can take the colimit of this diagram in local C^∞-ringed spaces with corners. However, the construction of this colimit is such that every point in the colimit has an open set isomorphic to an open subset of one of the $\boldsymbol{X}_i$, so this colimit is a C^∞-scheme with corners.

For ordinary algebraic geometry the proof is the same, and this result appears in Hartshorne [33, Ex. I.2.12]. This tells us, for example, if we take each $\boldsymbol{U}_{i,j}$ to be the empty set, that all coproducts of C^∞-schemes with corners exist and are equal to their products in $\mathbf{LC^\infty RS^c}$.

The following lemma extends Lemma 2.4.21.

Lemma 5.4.14. *Let I be a set and $\boldsymbol{\mathfrak{C}} = \prod_{i\in I}(\mathfrak{C}_i, \mathfrak{C}_{i,\mathrm{ex}})$. If $x : \boldsymbol{\mathfrak{C}} \to \mathbb{R}$ factors through $\boldsymbol{\mathfrak{C}}_k = (\mathfrak{C}_k, \mathfrak{C}_{k,\mathrm{ex}})$ then there is a canonical isomorphism $(\boldsymbol{\mathfrak{C}}_k)_x \cong \boldsymbol{\mathfrak{C}}_x$. If each $\boldsymbol{\mathfrak{C}}_i$ is interior, and $\boldsymbol{\mathfrak{C}}^{\mathrm{in}} = \prod_{i\in I}^{\mathrm{in}}(\mathfrak{C}_i, \mathfrak{C}_{i,\mathrm{ex}})$ then there is a canonical isomorphism $(\boldsymbol{\mathfrak{C}}_k)_x \cong \boldsymbol{\mathfrak{C}}_x \cong \boldsymbol{\mathfrak{C}}_x^{\mathrm{in}}$.*

Proof. Consider the following commutative diagram, where the right hand side exists when all $\boldsymbol{\mathfrak{C}}_i$ are interior.



(5.4.1)

Here $\boldsymbol{\pi}_k$ is the projection onto the k-th factor and $\boldsymbol{\pi}_x, \boldsymbol{\pi}_x^{\mathrm{in}}, \boldsymbol{\pi}_{k,x}$ are the localisation projections, which are surjective. Note that the dotted arrows exist by the universal properties of localisations of $\boldsymbol{\mathfrak{C}}$ and $\boldsymbol{\mathfrak{C}}_{\mathrm{in}}$, and that if each $\boldsymbol{\mathfrak{C}}_i$ is interior, then $\boldsymbol{t} \circ \boldsymbol{p}$ is interior.

On the C^∞-rings, the map $t : (\prod_{i\in I}\mathfrak{C}_i)_x \to (\mathfrak{C}_k)_x$ sends $\pi_x((c_i)_{i\in I}) \in (\prod_{i\in I}\mathfrak{C}_i)_x$ to $\pi_{k,x} \circ \pi_x^{\mathrm{in}}((c_i)_{i\in I}) = \pi_{k,x}(c_k) \in (\mathfrak{C}_k)_x$, and similarly on the monoids. This implies $\boldsymbol{t}$ is surjective. To show it is injective, say $t(\pi_x((c_i)_{i\in I})) = t(\pi_x((d_i)_{i\in I})) \in (\mathfrak{C}_k)_x$, then $\pi_{k,x}(c_k) = \pi_{k,x}(d_k)$, so by Proposition 2.1.15 there exists $a \in \mathfrak{C}_k$ with $x(a) \neq 0$ such that

$a \cdot (c_k - d_k) = 0$. Then define $(a_i)_{i\in I} \in \prod_{i\in I} \mathfrak{C}_i$ by $a_k = a$ and $a_i = 0$ for $i \neq k$. Then $(a_i)_{i\in I} \cdot ((c_i)_{i\in I} - (d_i)_{i\in I})$ and $x((a_i)_{i\in I}) \neq 0$, which implies $\pi_x((c_i)_{i\in I}) = \pi_x((d_i)_{i\in I}) \in (\prod_{i\in I} \mathfrak{C}_i)_x$, so t is injective on the C^∞-rings. On the monoids similar reasoning gives the same result showing $\boldsymbol{t}$ is an isomorphism.

We now consider the interior case. Note that $\boldsymbol{p}$ is the identity on the C^∞-rings. Say each $\mathfrak{C}_i$ is interior so $\boldsymbol{t} \circ \boldsymbol{p} : (\prod^{\text{in}}_{i\in I} \mathfrak{C}_i)_x \to (\mathfrak{C}_k)_x$ is an interior morphism, then $(t_{\text{ex}} \circ p_{\text{ex}})^{-1}(0) = 0 \in (\prod^{\text{in}}_{i\in I} \mathfrak{C}_i)_x$. So say $t_{\text{ex}} \circ p_{\text{ex}}(\pi_{x,\text{ex}}((c'_i)_{i\in I})) = t_{\text{ex}} \circ p_{\text{ex}}(\pi_{x,\text{ex}}((d'_i)_{i\in I})) \in (\mathfrak{C}_k)_{x,\text{ex}}$ where $(c'_i)_{i\in I}, (d'_i)_{i\in I} \in \prod^{\text{in}}_{i\in I} \mathfrak{C}_i$ are non-zero. Then Lemma 4.6.9 says there are $a, b \in \mathfrak{C}_{k,\text{ex}}$ such that $x \circ \Phi_i(a) \neq 0$ and $ac_k = bd_k$, and there is $e \in \mathfrak{C}_k$ such that $e(\Phi_i(a) - \Phi_i(b)) \neq 0$ with $x(e) \neq 0$. Define $(a_i)_{i\in I}$ in the monoid of $\prod^{\text{in}}_{i\in I} \mathfrak{C}_i$ such that $a_k = a$ and $a_i = d_i$ for all $i \neq k$, define $(b_i)_{i\in I}$ in the monoid of $\prod^{\text{in}}_{i\in I} \mathfrak{C}_i$ such that $b_k = b$ and $b_i = c_i$ for all $i \neq k$, and define $(e_i)_{i\in I} \in \prod_{i\in I} \mathfrak{C}_i$ with $e_k = e$ and $e_i = 0$ for all $i \neq k$. Then we have $x \circ \Phi_i((a_i)_{i\in I})) \neq 0$, $(a_i)_{i\in I}(c'_i)_{i\in I} = (b_i)_{i\in I}(d'_i)_{i\in I}$, and $(e_i)_{i\in I}(\Phi_i((a_i)_{i\in I} - \Phi_i((b_i)_{i\in I})) = 0$, which implies that $\pi_{x,\text{ex}}((c'_i)_{i\in I}) = (\pi_{x,\text{ex}}((d'_i)_{i\in I}))$, so that $t_{\text{ex}} \circ p_{\text{ex}}$ is injective. Hence $\boldsymbol{t}$ and $\boldsymbol{t} \circ \boldsymbol{p}$ are isomorphisms. □

The following proposition extends Proposition 2.4.22 to C^∞-schemes with corners.

Proposition 5.4.15. *If I is a set with cardinality less than any measurable cardinal (c.f Remark 2.4.18) and $\{\mathfrak{C}_i\}_{i\in I}$ is a collection of C^∞-rings with corners, then there is a canonical isomorphism* $\text{Spec}^{\text{c}}(\prod_{i\in I} \mathfrak{C}_i) \cong \coprod_{i\in I} \text{Spec}^{\text{c}}(\mathfrak{C}_i)$*. If $\{\mathfrak{C}_i\}_{i\in I}$ is a collection of interior C^∞-rings with corners, then*

$$\text{Spec}^{\text{c}}(\prod_{i\in I}^{\text{in}} \mathfrak{C}_i) \cong \coprod_{i\in I} \text{Spec}^{\text{c}}(\mathfrak{C}_i) \cong \text{Spec}^{\text{c}}(\prod_{i\in I} \mathfrak{C}_i).$$

Proof. As in the proof of Proposition 2.4.22, the projections $\boldsymbol{\pi}_k : \prod_{i\in I} \mathfrak{C}_i \to \mathfrak{C}_k$ give morphisms $\text{Spec}^{\text{c}}(\boldsymbol{\pi}_k) : \text{Spec}^{\text{c}}(\mathfrak{C}_k) \to \text{Spec}^{\text{c}}(\prod^{(\text{in})}_{i\in I} \mathfrak{C}_i)$, which we can amalgamate to a morphism $\boldsymbol{f} = (f, f^\sharp, f^\sharp_{\text{ex}}) : \coprod_{i\in I} \text{Spec}(\mathfrak{C}_i) \to \text{Spec}(\prod^{(\text{in})}_{i\in I} \mathfrak{C}_i)$ using the universal property of a coproduct. Proposition 2.4.22 shows that $\boldsymbol{f}$ is an isomorphism on the topological spaces and the sheaves of C^∞-rings, and Lemma 5.4.14 shows that the stalks of monoids are isomorphic, so that $\boldsymbol{f}$ is an isomorphism of (interior) C^∞-schemes with corners. □

Remark 5.4.16. This proposition tells us that provided the cardinality of I is less than any measurable cardinal, then the coproducts of interior and non-interior affine C^∞-*schemes* with corners are the same despite Theorem 4.3.7(b) telling us the interior and non-interior products of interior C^∞-*rings* with corners are different.

In addition, Joyce [40, Th. 4.41] gives a criterion for a C^∞-scheme to be affine: the sufficient conditions are Hausdorff, and Lindelöf with smoothly generated topology. The coproduct $\coprod_{i\in I}\mathrm{Spec^c}(\mathfrak{C}_i)$ always has smoothly generated topology and is Hausdorff. However, it is not Lindelöf unless I is countable, as otherwise it has an open cover $\{U_i\}_{i\in I}$ such that each $\mathrm{Spec^c}(\mathfrak{C}_i)$ is in exactly one of the U_i and does not intersect any of the others and this open cover has no countable subcover if I has uncountable cardinality. This means that if I has uncountable cardinality but has cardinality less than any measurable cardinal, then the underlying C^∞-scheme of $\coprod_{i\in I}\mathrm{Spec^c}(\mathfrak{C}_i)$ is an example of an C^∞-scheme that does not satisfy the sufficient conditions, yet is affine nonetheless.

Finally, for any cardinality of I, if $\coprod_{i\in I}\mathrm{Spec^c}(\mathfrak{C}_i)$ is affine, then its underlying C^∞-scheme must be isomorphic to the spectrum of its global sections, but its global sections are $\prod_{i\in I}\mathfrak{C}_i$. This means that if I has measurable cardinality, we know that $\coprod_{i\in I}\mathrm{Spec^c}(\mathfrak{C}_i)$ is not affine.

5.5 Relation to manifolds with corners

Here we describe how the category of manifolds with corners can be embedded into the category of C^∞-schemes with corners.

Definition 5.5.1. Define a functor $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}:\mathbf{Man^c}\to\mathbf{C^\infty Sch^c_{fi}}$ that acts on objects $X\in\mathbf{Man^c}$ by $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X)=(X,\boldsymbol{\mathcal{O}}_X)$, where we have $\boldsymbol{\mathcal{O}}_X(U)=\boldsymbol{C}^\infty(U)=(C^\infty(U),\mathrm{Ex}(U))$ for each open subset $U\subseteq X$. That is, $C^\infty(U)$ is the C^∞-ring of smooth maps $c:U\to\mathbb{R}$ and $\mathrm{Ex}(U)$ is the monoid of exterior (smooth) maps $c':U\to[0,\infty)$, as in Definition 4.1.1.

Example 4.3.4 shows that $\boldsymbol{C}^\infty(U)$ is a C^∞-ring with corners for each open U. If $V\subseteq U\subseteq X$ are open we define $\boldsymbol{\rho}_{UV}=(\rho_{UV},\rho_{UV}^{\mathrm{ex}}):\boldsymbol{C}^\infty(U)\to\boldsymbol{C}^\infty(V)$ by $\rho_{UV}:c\mapsto c|_V$ and $\rho^{\mathrm{ex}}_{UV}:c'\mapsto c'|_V$.

One can verify that $\boldsymbol{\mathcal{O}}_X$ is not just a presheaf but a sheaf of C^∞-rings with corners on X, so $\boldsymbol{X}=(X,\boldsymbol{\mathcal{O}}_X)$ is a C^∞-ringed space with corners. We show in Theorem 5.5.2(b) that $\boldsymbol{X}$ is a firm C^∞-scheme with corners, and it is also interior.

Let $f:X\to Y$ be a smooth map of manifolds with corners. Writing $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X)=(X,\boldsymbol{\mathcal{O}}_X)$, and $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(Y)=(Y,\boldsymbol{\mathcal{O}}_Y)$, then for all open $U\subseteq Y$, we define

$$\boldsymbol{f}_\sharp(U):\boldsymbol{\mathcal{O}}_Y(U)=\boldsymbol{C}^\infty(U)\to f_*(\boldsymbol{\mathcal{O}}_x)(U)=\boldsymbol{\mathcal{O}}_X(f^{-1}(U))=\boldsymbol{C}^\infty(f^{-1}(U))$$

by $f_\sharp(U):c\mapsto c\circ f$ for all $c\in C^\infty(U)$ and $f^{\mathrm{ex}}_\sharp(U):c'\mapsto c'\circ f$ for all $c'\in\mathrm{Ex}(U)$. Then $\boldsymbol{f}_\sharp(U)$ is a morphism of C^∞ rings with corners, and $\boldsymbol{f}_\sharp:\boldsymbol{\mathcal{O}}_Y\to f_*(\boldsymbol{\mathcal{O}}_X)$ is a morphism of sheaves of C^∞-rings with corners on Y. Let $\boldsymbol{f}^\sharp:f^{-1}(\boldsymbol{\mathcal{O}}_Y)\to\boldsymbol{\mathcal{O}}_X$ correspond to $\boldsymbol{f}_\sharp$ under (2.4.2). Then $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(f)=\boldsymbol{f}=(f,\boldsymbol{f}^\sharp):(X,\boldsymbol{\mathcal{O}}_X)\to(Y,\boldsymbol{\mathcal{O}}_Y)$ is a morphism of C^∞-schemes with corners.

Define a functor $F_{\mathbf{Man^c_{in}}}^{\mathbf{C^\infty Sch^c_{fi,in}}} : \mathbf{Man^c_{in}} \to \mathbf{C^\infty Sch^c_{fi,in}}$ by restriction of $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}$ to $\mathbf{Man^c_{in}}$.

Theorem 5.5.2. *Let X be a manifold with corners and $\boldsymbol{X} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X)$.*

(a) *If X has faces, then $\boldsymbol{X}$ is an affine C^∞-scheme with corners, and is isomorphic to $\mathrm{Spec^c}(C^\infty(X), \mathrm{Ex}(X))$. It is also an interior affine C^∞-scheme with corners, with $\boldsymbol{X} \cong \mathrm{Spec^c_{in}}(C^\infty(X), \mathrm{In}(X) \amalg \{0\})$. Here $\amalg$ is the disjoint union.*

(b) *In general, $\boldsymbol{X}$ is a firm interior C^∞-scheme with corners.*

(c) *The functors $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}$ and $F_{\mathbf{Man^c_{in}}}^{\mathbf{C^\infty Sch^c_{fi,in}}}$ are fully faithful.*

Proof. For (a), let X be a manifold with corners and write $\boldsymbol{X} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X) = (X, \boldsymbol{\mathcal{O}}_X)$. Note that $\Gamma^c(\boldsymbol{X}) = \boldsymbol{\mathcal{O}}_X(X) = \boldsymbol{C}^\infty(X)$ by the definition of $\boldsymbol{\mathcal{O}}_X$. Consider the functor $L_{\boldsymbol{\mathfrak{C}},\boldsymbol{X}}$ in (5.2.2). In the notation of Theorem 5.2.6, if we let $\boldsymbol{\mathfrak{C}} = \Gamma^c(\boldsymbol{X}) = \boldsymbol{C}^\infty(X)$, then $L_{\boldsymbol{C}^\infty(X),\boldsymbol{X}}$ is a bijection

$$L_{\boldsymbol{C}^\infty(X),\boldsymbol{X}} : \mathrm{Hom}_{\mathbf{C^\infty Rings^c}}(\boldsymbol{C}^\infty(X), \boldsymbol{C}^\infty(X)) \longrightarrow \mathrm{Hom}_{\mathbf{LC^\infty RS^c}}(\boldsymbol{X}, \mathrm{Spec^c}\, \boldsymbol{C}^\infty(X)).$$

Let $\mathrm{id}_{\boldsymbol{C}^\infty(X)}$ be the identity morphism in $\mathrm{Hom}_{\mathbf{C^\infty Rings^c}}(\boldsymbol{C}^\infty(X), \boldsymbol{C}^\infty(X))$ and define $(g, \boldsymbol{g}^\sharp) = \boldsymbol{g} = L_{\boldsymbol{C}^\infty(X),\boldsymbol{X}}(\mathrm{id}_{\boldsymbol{C}^\infty(X)})$. We will show that $\boldsymbol{g} : \boldsymbol{X} \to \mathrm{Spec^c}\, \boldsymbol{C}^\infty(X)$ is an isomorphism in $\mathbf{LC^\infty RS^c}$ when X is a manifold with faces.

Denote $\mathrm{Spec^c}\, \boldsymbol{C}^\infty(X) = \boldsymbol{Y}$, then the continuous map $g : X \to Y$ is defined in the proof of Theorem 5.2.6 by $g(x) = x_* \circ \mathrm{id}_X$ where $\mathrm{id}_X : C^\infty(X) \to C^\infty(X)$ is the identity morphism and x_* is the evaluation map at the point $x \in X$. This is a homeomorphism of topological spaces, as shown in the proof of [40, Th. 4.41].

The map $\boldsymbol{g}^\sharp$ corresponds to $\boldsymbol{g}_\sharp$ by (2.4.2). For each open $V \subset Y$ with $U = g^{-1}(V)$ then $\boldsymbol{g}_\sharp$ is defined by $\boldsymbol{g}_\sharp(V)(\boldsymbol{s})_x \mapsto i(\boldsymbol{s}_{x_*})$, where $\boldsymbol{s} \in \boldsymbol{\mathcal{O}}_Y(V)$, $x \in U \subseteq X$, and i is the inclusion C^∞-ring with corners morphism $i : (\boldsymbol{C}^\infty(X))_{x_*} \to \boldsymbol{C}^\infty_x(X)$ for the localisation $(\boldsymbol{C}^\infty(X))_{x_*}$ and the germs of functions $\boldsymbol{C}^\infty_x(X)$ as defined in Example 4.6.11. On stalks, we have $\boldsymbol{g}^\sharp_x = i : (\boldsymbol{C}^\infty(X))_{x_*} \to \boldsymbol{C}^\infty_x(X)$. In Example 4.6.11, we showed that if X has faces, then $(\boldsymbol{C}^\infty(X))_{x_*} \cong \boldsymbol{C}^\infty_x(X)$ and then i is the identity map, which is indeed an isomorphism of local C^∞-rings with corners. This implies $\boldsymbol{g}^\sharp$ is an isomorphism on stalks, so $\boldsymbol{g}$ is an isomorphism of local C^∞-ringed spaces with corners. Hence $\boldsymbol{X} \cong \mathrm{Spec^c}\, \boldsymbol{C}^\infty(X)$ is an affine C^∞-scheme with corners.

Now let $\boldsymbol{C}^\infty_{\mathrm{in}}(X) = (C^\infty(X), \mathrm{In}(X) \amalg \{0\})$, where $\amalg$ is the disjoint union. To show that if X has faces, then $\boldsymbol{X} \cong \mathrm{Spec^c}\, \boldsymbol{C}^\infty_{\mathrm{in}}(X)$, we follow the same method above. Here we use the bijection

$$L_{\boldsymbol{C}^\infty_{\mathrm{in}}(X),\boldsymbol{X}} : \mathrm{Hom}_{\mathbf{C^\infty Rings^c}}(\boldsymbol{C}^\infty_{\mathrm{in}}(X), \boldsymbol{C}^\infty(X)) \longrightarrow \mathrm{Hom}_{\mathbf{LC^\infty RS^c}}(\boldsymbol{X}, \mathrm{Spec^c}\, \boldsymbol{C}^\infty_{\mathrm{in}}(X))$$

from Theorem 5.2.6, and the result from Example 4.6.11, that if X has faces, then

$$\boldsymbol{C}^\infty_x(X) \cong \big(\boldsymbol{C}^\infty(X)\big)_{x*} \cong \big(\boldsymbol{C}^\infty_{\mathrm{in}}(X)\big)_{x*}.$$

The same reasoning above gives the result.

For (b), for any point $x \in X$, we can find a neighbourhood U_x of x such that U_x is a manifold with faces. For example, U_x can be the coordinate neighbourhood of x, which is diffeomorphic to $\mathbb{R}^n_k$ for some $n \geqslant k$ with $k, n \in \mathbb{N}$. Then if $\mathfrak{C}_{U_x} = (C^\infty(U_x), \mathrm{Ex}(U_x))$, by the argument for part (a), we have $\boldsymbol{X}|_{U_x} \cong \mathrm{Spec}^{\mathrm{c}}\,\mathfrak{C}_{U_x}$ for each $x \in X$. That is, $\boldsymbol{X}$ is a C^∞-scheme with corners.

As $U_x \cong \mathbb{R}^n_k$, it is connected and there are k connected components of the boundary in U_x. From Example 4.5.5, we have that $\mathrm{Ex}(U_x) = \mathrm{In}(U_x) \amalg \{0\} \cong C^\infty(X) \times \mathbb{N}^k \amalg \{0\}$. Hence $\boldsymbol{X}$ is locally isomorphic to the spectrum of a firm interior C^∞-ring with corners, and is therefore a firm interior C^∞-scheme with corners.

For (c), we know that $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Rings^c}}|_{\mathbf{Euc^c}} : \mathbf{Euc^c} \to (\mathbf{C^\infty Rings^c})^{\mathbf{op}}$, as defined in Example 4.3.4, is full and faithful as it is the Yoneda embedding. Note that the composition $\mathrm{Spec}^{\mathrm{c}} \circ F_{\mathbf{Man^c}}^{\mathbf{C^\infty Rings^c}}$ is equal to $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}$ when restricted to manifolds with faces, by the proof of (a).

Let X and Y be manifolds with faces. Let $\boldsymbol{X} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X)$ and $\boldsymbol{Y} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(Y)$. As above, we know $\Gamma^{\mathrm{c}}(\boldsymbol{X}) = \boldsymbol{C}^\infty(X)$ and $\Gamma^{\mathrm{c}}(\boldsymbol{Y}) = \boldsymbol{C}^\infty(Y)$, and (a) above shows that $\mathrm{Spec}^{\mathrm{c}}\,\boldsymbol{C}^\infty(X)$ is naturally isomorphic to $\boldsymbol{X}$, and $\mathrm{Spec}^{\mathrm{c}}\,\boldsymbol{C}^\infty(Y)$ is naturally isomorphic to $\boldsymbol{Y}$. Using these isomorphisms, Theorem 5.2.6 gives a bijection

$$\mathrm{Hom}_{\mathbf{C^\infty Rings^c}}(\boldsymbol{C}^\infty(Y), \boldsymbol{C}^\infty(X)) \cong \mathrm{Hom}_{\mathbf{LC^\infty RS^c}}(\mathrm{Spec}^{\mathrm{c}}\,\boldsymbol{C}^\infty(X), \mathrm{Spec}^{\mathrm{c}}\,\boldsymbol{C}^\infty(Y)).$$

This shows that $\mathrm{Spec}^{\mathrm{c}}$ is full and faithful on the image of $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Rings^c}}$ restricted to manifolds with faces, in particular when restricted to Euclidean spaces with corners, $\mathbf{Euc^c}$. We conclude that $\mathrm{Spec}^{\mathrm{c}} \circ F_{\mathbf{Man^c}}^{\mathbf{C^\infty Rings^c}}|_{\mathbf{Euc^c}} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}|_{\mathbf{Euc^c}}$ is full and faithful.

Now if f, g are smooth maps $f, g : X \to Y$ and $(f, \boldsymbol{f}^\sharp) = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(f) = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(g) = (g, \boldsymbol{g}^\sharp)$, this directly implies $f = g$, so $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}$ is faithful.

To show $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}$ is full, let $\boldsymbol{g} = (g, \boldsymbol{g}^\sharp) \in \mathrm{Hom}_{\mathbf{LC^\infty RS^c}}(\boldsymbol{X}, \boldsymbol{Y})$. We want to show that $g : X \to Y$ is smooth and $\boldsymbol{g} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(g)$. Let $x \in X$ and say $g(x) = y \in Y$. As X and Y are manifolds, there are coordinate neighbourhoods $U_x \subset X$ of x and $V_y \subset Y$ that are isomorphic to $\mathbb{R}^n_k, \mathbb{R}^m_l \subset \mathbf{Euc^c}$ for $n = \dim X, m = \dim Y$.

Shrinking U_x and V_y if necessary, we can consider the restriction $\boldsymbol{g}|_{U_x} : \boldsymbol{X}|_{U_x} \to \boldsymbol{Y}|_{V_y}$. As U_x and V_y are isomorphic to Euclidean spaces with corners, then as $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}|_{\mathbf{Euc^c}}$ is full, then there is a smooth map $h_x : U_x \to V_y$ such that $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}|_{\mathbf{Euc^c}}(h_x) = (h_x, \boldsymbol{h_x}^\sharp) =$

$(g|_{U_x}, \boldsymbol{g}^\sharp|_{U_x}) = \boldsymbol{g}|_{U_x}$. That is, $g|_{U_x} = h_x$. So g is smooth in a neighbourhood of x for all $x \in X$, so g is smooth on X.

Now $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(g)|_{U_x} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(g|_{U_x}) = \boldsymbol{g}|_{U_x}$. As the morphisms $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(g)$ and $\boldsymbol{g}$ agree on the open sets U_x for each $x \in X$, and U_x cover X, then the sheaf property of $\boldsymbol{\mathcal{O}}_X$ and the definition of morphism of sheaves, implies that $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(g)$ and $\boldsymbol{g}$ are equal, as required. □

Remark 5.5.3. Note that if X is a manifold with faces then $\boldsymbol{X} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X)$ is isomorphic to $\mathrm{Spec^c}(C^\infty(X), \mathrm{Ex}(X))$ but $(C^\infty(X), \mathrm{Ex}(X))$ is not firm unless X has finitely many boundary components. This occurs, for example, when X is compact.

We can also consider the functors $F_{\mathbf{Man^{gc}}}^{\mathbf{C^\infty Sch^c_{fi}}} : \mathbf{Man^{gc}} \to \mathbf{C^\infty Sch^c_{fi}}$, $X \mapsto (X, \boldsymbol{\mathcal{O}}_X)$ with $\boldsymbol{\mathcal{O}}_X(U) = (C^\infty(U), \mathrm{Ex}(U))$, and $F_{\mathbf{Man^{gc}_{in}}}^{\mathbf{C^\infty Sch^c_{fi,in}}} : \mathbf{Man^{gc}_{in}} \to \mathbf{C^\infty Sch^c_{fi,in}}$ the restriction of $F_{\mathbf{Man^{gc}}}^{\mathbf{C^\infty Sch^c_{fi}}}$ to $\mathbf{Man^{gc}_{in}}$. We then can consider whether part (a) or (b) hold for $\boldsymbol{X} = F_{\mathbf{Man^{gc}}}^{\mathbf{C^\infty Sch^c_{fi}}}(X)$. As a manifold with g-corners is Hausdorff and second countable, and the topology is smoothly generated, we can apply [40, Th. 4.41] as in the proof, and we find is that (a) is true for all open subsets $U \subset X_P$ for X_P as in Definition 3.1.3, as on these subsets can we guarantee that we have

$$\boldsymbol{C}_x^\infty(X) \cong \big(\boldsymbol{C}^\infty(X)\big)_{x_*}. \tag{5.5.1}$$

This means that part (b) above is then true for all manifolds with g-corners.

We would like to have some geometrical condition (such as 'with faces') that would allow part (a) to be true more generally. That is, a geometric condition that ensures a manifold with g-corners satisfies (5.5.1) at all points $x \in X$, so that it is an affine C^∞-scheme with corners. We could consider the following condition as a candidate: (*) the map $i_X : \partial X \to X$ is injective on connected components of the boundary map, as in Definition 3.3.3. However, we will show below that (*) is not enough to guarantee that part (a) above holds for all of manifolds with g-corners.

Example 5.5.4. Recall Example 3.2.4 of a manifold with g-corners that is not a manifold with corners. We have

$$X'_P = \big\{(x_1, x_2, x_3, x_4) \in [0,\infty)^4 : x_1 x_2 = x_3 x_4\big\}. \tag{5.5.2}$$

Figure 5.5.1 shows a three dimensional representation of this manifold as a square based infinite pyramid such that the 'faces of the pyramid' are actually the 2-corners, where two of the coordinates are zero. Consider compactifying this pyramid by attaching another pyramid to the bottom as in Figure 5.5.2.

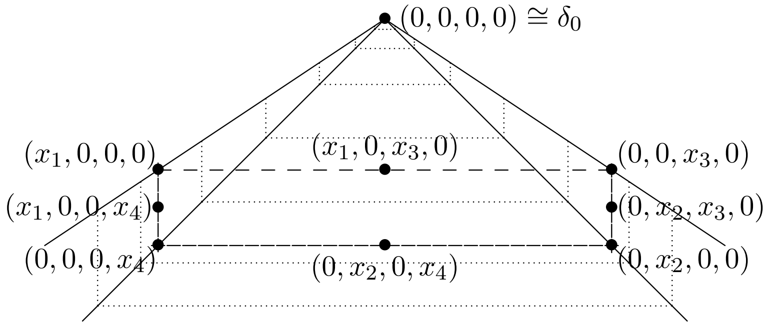


Figure 5.5.1: 3-manifold with g-corners X'_P in (5.5.2)

This is a manifold with g-corners satisfying condition (*) in Remark 5.5.3. Around δ_0 and δ'_0, the manifold is isomorphic to $X_{P'}$, and at all other points, the manifold is a manifold with corners. The dashed lines indicate open lines and are not boundary or corners. The 'faces', F_i, for $i = 1, \ldots, 4$, near δ_0 and δ'_0 are described by the pairs of the variables x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 set equal to zero as in the following table. Here, F_4 would be the 'face' on the back. Note that there is a 'tunnel' where F_2 is the 'face' on the inside of the tunnel and F_1 is the face on the top of the tunnel.

	F_1	F_2	F_3	F_4
δ_0	x_1, x_3	x_2, x_3	x_1, x_4	x_2, x_4
δ'_0	y_2, y_3	y_1, y_3	y_1, y_4	y_2, y_4

Table 5.1: The faces F_i are described by two variables vanishing near δ_0 and δ'_0.

If we take a smooth non-zero function $f : X \to [0, \infty)$ then in an open neighbourhood around δ_0, $f(x_1, x_2, x_3, x_4) = x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} F(x_1, x_2, x_3, x_4)$ such that $a_1 + a_2 = a_3 + a_4$ and F, a smooth function, is greater than zero in this neighbourhood, and around δ'_0, $f(y_1, y_2, y_3, y_4) = y_1^{b_1} y_2^{b_2} y_3^{b_3} y_4^{b_4} F'(y_1, y_2, y_3, y_4)$ such that $b_1 + b_2 = b_3 + b_4$ and smooth F' is greater than zero in this neighbourhood. However, as F_1 is the vanishing of two variables, then a_1 and a_3 must match up with b_2 and b_3. Similarly with all other 'faces'.

Using this, we can deduce that the only smooth functions $f : X \to [0, \infty)$ must have $a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4$, and equivalence classes in $(\mathrm{Ex}(X))_{\delta_0 *}$ are of the form $[(x_1 x_2 x_3 x_4)^n F]$ and $[0]$, for $n = 0, 1, 2, \ldots$ and F a smooth non-zero function defined in a neighbourhood of δ_0.

However, for a local function $f : U \to [0, \infty)$ for U a small neighbourhood around δ'_0, then we may have $a_1 \neq a_2$, and hence there are equivalence classes in $\mathrm{Ex}_{\delta_0}(X)$ of the form $[x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} F]$ for F a smooth non-zero function defined in a neighbourhood of δ_0, and the only requirement on the integers a_i is that they are non-negative and

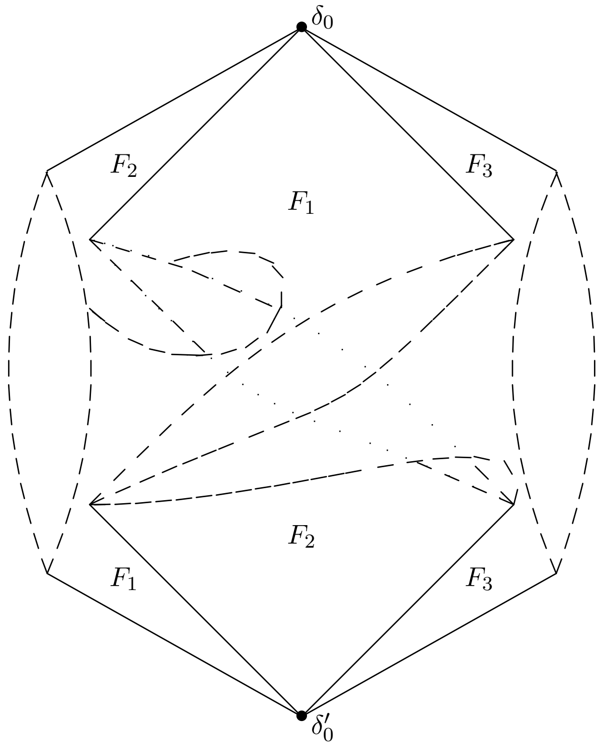


Figure 5.5.2: The 'faces' of the manifold with g-corners, X.

$a_1 + a_2 = a_3 + a_4$. As any C^∞-ring morphism between $(\mathrm{Ex}(X))_{\delta_0 *}$ and $\mathrm{Ex}_{\delta_0}(X)$ must send $[(x_1x_2x_3x_4)^nF] \in (\mathrm{Ex}(X))_{\delta_0 *}$ to $[(x_1x_2x_3x_4)^nF] \in \mathrm{Ex}_{\delta_0}(X)$, then this can never be surjective and hence

$$\boldsymbol{C}^\infty_x(X) \ncong \big(\boldsymbol{C}^\infty(X)\big)_{x_*} \cong \big(\boldsymbol{C}^\infty_{\mathrm{in}}(X)\big)_{x_*},$$

for $x = \delta_0$, and similarly at the point $x = \delta_0'$.

Remark 5.5.5. We would also like to consider whether the functors $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}$ and $F_{\mathbf{Man^{gc}}}^{\mathbf{C^\infty Sch^c_{fi}}}$ respect fibre products, so here is some discussion related to this. As all manifolds with (g-)corners are locally firm, we know that all fibre products of manifolds with (g-)corners exist in $\mathbf{C^\infty Sch^c_{fi}}$ by Corollary 5.4.12.

From Theorem 2.4.17 we know $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}} : X \mapsto \mathrm{Spec}(C^\infty(X))$ preserves transverse fibre products of manifolds without boundary. From Proposition 5.4.10, fibre products in $\mathbf{C^\infty Sch^c_{fi}}$ have C^∞-scheme that is the underlying fibre product of the C^∞-schemes, which implies $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}$ respects transverse fibre products of manifolds without boundary on the topological spaces and the sheaves of C^∞-rings, but we do not know what happens on the sheaves of monoid.

To extend this to manifolds with corners, we also need an appropriate definition of transverse for manifolds with corners. Joyce [47, §4.3] and particularly [49, §2.5] have explored fibre products of manifolds with (g-)corners. Notably, manifolds with g-corners enlarge the category of manifolds with corners under certain types of fibre products, but in both cases, the fibre products may not exist in general, or they may exist but have topological space that is not the fibre product of the topological spaces. They also may not coincide with fibre products in the category of manifolds with (g-)corners and interior maps.

In [49, §2.5.4] two definitions of transverse are given for manifolds with corners, *sb-transverse* and the more restrictive *sc-transverse*. In [49, Th. 2.32, §2.5.4], we see that only sc-transverse guarantees that the topological space of the fibre product is the fibre product of the topological spaces. It is possible that $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}$ respects sc-transverse fibre products.

In [47, §4.3] and [49, §2.5] there are two corresponding definitions of transverse for manifolds with g-corners, *b-transverse* (as in Theorem 3.4.3) and the more restrictive *c-transverse*, and similarly only c-transverse guarantees that the topological space of the fibre product is the fibre product of the topological spaces. It is possible that $F_{\mathbf{Man^{gc}}}^{\mathbf{C^\infty Sch^c_{fi}}}$ respects c-transverse fibre products.

Preliminary investigations of this suggests restricting to toric C^∞-rings/C^∞-schemes with corners may be necessary to prevent torsion occurring in the fibre products.

5.6 Sheaves of $\boldsymbol{\mathcal{O}}_X$-modules and cotangent modules

We define sheaves of $\boldsymbol{\mathcal{O}}_X$-modules on a C^∞-ringed space with corners, as in §2.5.

Definition 5.6.1. Let $(X, \boldsymbol{\mathcal{O}}_X)$ be a C^∞-ringed space with corners. A *sheaf of $\boldsymbol{\mathcal{O}}_X$-modules*, or simply an *$\boldsymbol{\mathcal{O}}_X$-module*, $\mathcal{E}$ on X, is a sheaf of $\mathcal{O}_X$-modules on X, as in Definition 2.5.1. A *morphism of sheaves of $\boldsymbol{\mathcal{O}}_X$-modules* is a morphism of sheaves of $\mathcal{O}_X$-modules. Then $\boldsymbol{\mathcal{O}}_X$-modules form an abelian category, which we write as $\boldsymbol{\mathcal{O}}_X$-mod.

An $\boldsymbol{\mathcal{O}}_X$-module $\mathcal{E}$ is called a *vector bundle of rank n* if we may cover X by open $U \subseteq X$ with $\mathcal{E}|_U \cong \mathcal{O}_X|_U \otimes_{\mathbb{R}} \mathbb{R}^n$.

Pullback sheaves are defined analogously to Definition 2.5.2.

Definition 5.6.2. Let $\boldsymbol{f} = (f, f^\sharp, f^\sharp_{\rm ex}) : (X, \mathcal{O}_X, \mathcal{O}_X^{\rm ex}) \to (Y, \mathcal{O}_Y, \mathcal{O}_Y^{\rm ex})$ be a morphism of C^∞-ringed spaces with corners, and $\mathcal{E}$ be an $\boldsymbol{\mathcal{O}}_Y$-module. Define the *pullback* $\boldsymbol{f}^*(\mathcal{E})$ by $\boldsymbol{f}^*(\mathcal{E}) = \underline{f}^*(\mathcal{E})$, the pullback in Definition 2.5.2 for $\underline{f} = (f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism of $\boldsymbol{\mathcal{O}}_Y$-modules we have a morphism of $\boldsymbol{\mathcal{O}}_X$-modules $\boldsymbol{f}^*(\phi) = \underline{f}^*(\phi) = f^{-1}(\phi) \otimes \mathrm{id}_{\mathcal{O}_X} : \boldsymbol{f}^*(\mathcal{E}) \to \boldsymbol{f}^*(\mathcal{F})$.

Definition 5.6.3. Let $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ be a C^∞-ringed space with corners, and $\underline{X} = (X, \mathcal{O}_X)$ the underlying C^∞-ringed space. Define the *cotangent sheaf* $T^*\boldsymbol{X}$ *of* $\boldsymbol{X}$ to be the cotangent sheaf $T^*\underline{X}$ of $\underline{X}$, as defined in Definition 2.5.3.

If $U \subseteq X$ is open then we have an equality of sheaves of $\boldsymbol{\mathcal{O}}_X|_U$-modules

$$T^*(U, \boldsymbol{\mathcal{O}}_X|_U) = T^*\boldsymbol{X}|_U.$$

Let $\boldsymbol{f} = (f, f^\sharp, f^\sharp_{\rm ex}) : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism of C^∞-ringed spaces. Define $\Omega_{\boldsymbol{f}} : \boldsymbol{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ to be a morphism of the cotangent sheaves by $\Omega_{\boldsymbol{f}} = \Omega_{\underline{f}}$, where $\underline{f} = (f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

Definition 5.6.4. Let $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ be an interior local C^∞-ringed space with corners. For each open $U \subset X$, let $\mathrm{d}_{U,\mathrm{in}} : \mathcal{O}^{\rm in}_X(U) \to {}^b\Omega_{\mathfrak{C}_U}$ be the b-cotangent module associated to the interior C^∞-ring with corners $\mathfrak{C}_U = (\mathcal{O}_X(U), \mathcal{O}^{\rm in}_X(U) \amalg \{0\})$, where $\amalg$ is the disjoint union. Here $\mathcal{O}^{\rm in}_X(U)$ is the set of all elements of $\mathcal{O}^{\rm ex}_X(U)$ that are non-zero in each stalk $\mathcal{O}^{\rm ex}_{X,x}$ for all $x \in U$. Note that $\mathcal{O}^{\rm in}_X$ is a sheaf of monoids on X.

For each open $U \subset X$, the b-cotangent modules ${}^b\Omega_{\mathfrak{C}_U}$ define a presheaf $\boldsymbol{\mathcal{P}}^bT^*\boldsymbol{X}$ of $\boldsymbol{\mathcal{O}}_X$-modules, with restriction map ${}^b\Omega_{\rho_{UV}} : {}^b\Omega_{\mathfrak{C}_U} \to {}^b\Omega_{\mathfrak{C}_V}$ defined as the unique map from Definition 4.7.6. This exists by the universal property of ${}^b\Omega_U$ as the b-cotangent module associated to $(\mathcal{O}_X(U), \mathcal{O}^{\rm in}_X(U) \amalg \{0\})$, and makes the diagram in (2.5.1) commute for this setup. Denote the sheafification of this presheaf the *b-cotangent sheaf* ${}^bT^*\boldsymbol{X}$ of $\boldsymbol{X}$.

The definition of sheafification implies that, for each open set U, there is a canonical morphism ${}^b\Omega_{\mathfrak{C}_U} \to {}^bT^*\boldsymbol{X}(U)$, and we have an equality of sheaves of $\boldsymbol{\mathcal{O}}_X|_U$-modules

$${}^bT^*(U, \boldsymbol{\mathcal{O}}_X|_U) = {}^bT^*\boldsymbol{X}|_U.$$

Also, for each $x \in X$, the stalk ${}^bT^*\boldsymbol{X}|_x \cong {}^b\Omega_{\boldsymbol{\mathcal{O}}_{X,x}}$, where ${}^b\Omega_{\boldsymbol{\mathcal{O}}_{X,x}}$ is the b-cotangent module of the interior C^∞-ring with corners $\boldsymbol{\mathcal{O}}_{X,x}$.

For a morphism $\boldsymbol{f} = (f, f^\sharp, f^\sharp_{\rm ex}) : \boldsymbol{X} \to \boldsymbol{Y}$ of interior local C^∞-ringed spaces, then we define the morphism of b-cotangent sheaves ${}^b\Omega_{\boldsymbol{f}} : \boldsymbol{f}^*({}^bT^*\boldsymbol{Y}) \to {}^bT^*\boldsymbol{X}$ by firstly noting that $\boldsymbol{f}^*({}^bT^*\boldsymbol{Y})$ is the sheafification of the presheaf $\boldsymbol{\mathcal{P}}(\boldsymbol{f}^*({}^bT^*\boldsymbol{Y}))$ acting by

$$U \longmapsto \boldsymbol{\mathcal{P}}(\boldsymbol{f}^*({}^bT^*\boldsymbol{Y}))(U) = \lim\nolimits_{V \supseteq f(U)} {}^b\Omega_{\boldsymbol{\mathcal{O}}_Y(V)} \otimes_{\boldsymbol{\mathcal{O}}_Y(V)} \boldsymbol{\mathcal{O}}_X(U),$$

as in Definition 2.5.3. Then, following Definition 2.5.3, define a morphism of presheaves $\boldsymbol{\mathcal{P}}^b\Omega_{\boldsymbol{f}} : \boldsymbol{\mathcal{P}}(\boldsymbol{f}^*({}^bT^*\boldsymbol{Y})) \to \boldsymbol{\mathcal{P}}T^*\boldsymbol{X}$ on X by

$$(\boldsymbol{\mathcal{P}}^b\Omega_{\boldsymbol{f}})(U) = \lim\nolimits_{V \supseteq f(U)} ({}^b\Omega_{\boldsymbol{\rho}_{f^{-1}(V)\,U} \circ \boldsymbol{f}_\sharp(V)})_*,$$

where $({}^b\Omega_{\boldsymbol{\rho}_{f^{-1}(V)U}\circ\boldsymbol{f}_\sharp(V)})_* : {}^b\Omega_{\boldsymbol{\mathcal{O}}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U) \to {}^b\Omega_{\boldsymbol{\mathcal{O}}_X(U)} = (\boldsymbol{\mathcal{P}}{}^bT^*\boldsymbol{X})(U)$ is constructed as in Definition 2.2.4 from the C^∞-ring with corners morphisms $\boldsymbol{f}_\sharp(V) : \boldsymbol{\mathcal{O}}_Y(V) \to \boldsymbol{\mathcal{O}}_X(f^{-1}(V))$ from $\boldsymbol{f}_\sharp : \boldsymbol{\mathcal{O}}_Y \to f_*(\boldsymbol{\mathcal{O}}_X)$ corresponding to $\boldsymbol{f}^\sharp$ in $\boldsymbol{f}$ as in (2.4.2), and $\boldsymbol{\rho}_{f^{-1}(V)U} : \boldsymbol{\mathcal{O}}_X(f^{-1}(V)) \to \boldsymbol{\mathcal{O}}_X(U)$ in $\boldsymbol{\mathcal{O}}_X$. Define ${}^b\Omega_{\boldsymbol{f}} : \boldsymbol{f}^*({}^bT^*\boldsymbol{Y}) \to {}^bT^*\boldsymbol{X}$ to be the induced morphism of the associated sheaves.

Example 5.6.5. Let X be a manifold with (g-)corners and $\boldsymbol{X}$ the associated C^∞-scheme with corners. For each open set $U \subset X$ we can show ${}^bT^*\boldsymbol{X}(U) \cong \Gamma^\infty({}^bT^*X, U)$ and $T^*\boldsymbol{X}(U) \cong \Gamma^\infty(T^*X, U)$. That is, as sheaves, ${}^bT^*\boldsymbol{X} \cong \Gamma^\infty({}^bT^*X, -)$ and $T^*\boldsymbol{X} \cong \Gamma^\infty(T^*X, -)$. To do this, consider that for every open set $U \subset X$, there is a unique map $\psi_U : \boldsymbol{\mathcal{P}}{}^bT^*\boldsymbol{X}(U) \to \Gamma^\infty({}^bT^*X, U)$ using the universal property of the b-cotangent module as in Example 4.7.8.

If we shrink U to a coordinate patch, then U is a manifold with (g-)corners (with faces) such that $\boldsymbol{C}^\infty_x(U) = (\boldsymbol{C^\infty}(U))_{x_*}$ for each $x \in U$, and corresponding $\mathbb{R}$-point $x_* : C^\infty(U) \to \mathbb{R}$. Hence, by Proposition 4.7.9 (and Remark 4.7.10), there is an isomorphism $\psi_U : \boldsymbol{\mathcal{P}}{}^bT^*\boldsymbol{X}(U) = {}^b\Omega_{\boldsymbol{C}^\infty(U)} \to \Gamma^\infty({}^bT^*X, U)$. Using the uniqueness of ψ_U for each open $U \subset X$, we can show that ψ_U extends to a map of presheaves $\psi : \boldsymbol{\mathcal{P}}{}^bT^*\boldsymbol{X} \to \Gamma^\infty({}^bT^*X, -)$ that is an isomorphism on stalks.

As ${}^bT^*\boldsymbol{X}$ is the sheafification of $\boldsymbol{\mathcal{P}}{}^bT^*\boldsymbol{X}$, there is a unique map of sheaves $\psi' : {}^bT^*\boldsymbol{X} \to \Gamma^\infty({}^bT^*X, -)$ that is the same as ψ on stalks. Since ψ is an isomorphism on stalks, then ψ' is an isomorphism of sheaves as required. A similar proof shows $T^*\boldsymbol{X} \cong \Gamma^\infty(T^*X, -)$ for X a manifold with corners.

5.7 Corner functor for $\mathbf{LC^\infty RS^c}$

In this section, we define a corner functor, C^{loc}, that describes the boundary and corners of a local C^∞-ringed space with corners. We show this is a right adjoint and show how it relates to the definition of boundary in Gillam and Molcho [28, §4.4] if we consider a local C^∞-ringed space with corners to be a pre-log locally ringed space, as in §5.9.

Definition 5.7.1. Let $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ be a local C^∞-ringed space with corners. As a set, we define

$$C^{\mathrm{loc}}(X) = \{(x, P) : x \in X, P \text{ is a prime ideal in } \mathcal{O}^{\mathrm{ex}}_{X,x}\}.$$

There is a function of sets $\pi : C^{\mathrm{loc}}(X) \to X$, $(x, P) \mapsto x$.

We define a topology on $C^{\mathrm{loc}}(X)$ to be the weakest topology such that π is continuous, so $\pi^{-1}(U)$ is open for all $U \subset X$, and such that for all open $U \subset X$, for all elements

$s' \in \mathcal{O}_X^{\rm ex}(U)$, then

$$U_{s'} = \{(x,P) | x \in U, s'_x \notin P\} \subset \pi^{-1}(U)$$

is both open and closed in $\pi^{-1}(U)$. We denote $\hat{U}_{s'} = \pi^{-1}(U) \setminus U_{s'}$. Then $\pi^{-1}(U) = \hat{U}_0 = U_1$, and $\emptyset = \hat{U}_1 = U_0$ for $0, 1 \in \mathcal{O}_X^{\rm ex}(U)$. The collection $\{U_{s'}, \hat{U}_{s'} : \text{open } U \subset X, s' \in \mathcal{O}_X^{\rm ex}(U)\}$ is a subbase for the topology.

We can pullback the sheaves $\mathcal{O}_X$ and $\mathcal{O}_X^{\rm ex}$ using π to get the sheaves $\pi^{-1}(\mathcal{O}_X)$ and $\pi^{-1}(\mathcal{O}_X^{\rm ex})$ on $C^{\rm loc}(X)$. The identity morphisms induced from π, that is, $\hat{\pi}^\sharp : \pi^{-1}(\mathcal{O}_X) \to \pi^{-1}(\mathcal{O}_X)$ and $\hat{\pi}^\sharp_{\rm ex} : \pi^{-1}(\mathcal{O}_X^{\rm ex}) \to \pi^{-1}(\mathcal{O}_X^{\rm ex})$, give isomorphisms on stalks $\hat{\pi}^\sharp_{\pi(x,P)} : \mathcal{O}_{X,x} \xrightarrow{\sim} (\pi^{-1}(\mathcal{O}_X))_{(x,P)}$ and $\hat{\pi}^\sharp_{{\rm ex},\pi(x,P)} : \mathcal{O}^{\rm ex}_{X,x} \xrightarrow{\sim} (\pi^{-1}(\mathcal{O}_X^{\rm ex}))_{(x,P)}$. Then for each prime P in $\mathcal{O}^{\rm ex}_{X,x}$, we can identify

$$\pi^{-1}(\mathcal{O}_X^{\rm ex})_x \supset \hat{\pi}^\sharp_{{\rm ex},\pi(x,P)}(P) \cong P \subset \mathcal{O}^{\rm ex}_{X,x}$$

and

$$\pi^{-1}(\mathcal{O}_X)_x \supset \hat{\pi}^\sharp_{\pi(x,P)}(\langle \Phi_i(P) \rangle) \cong \langle \Phi_i(P) \rangle \subset \mathcal{O}_{X,x}.$$

Here $\langle \Phi_i(P) \rangle$ is the ideal generated by the image of P under $\Phi_i : \mathcal{O}^{\rm ex}_{X,x} \to \mathcal{O}_{X,x}$.

We define the sheaf of C^∞-rings $\mathcal{O}_{C^{\rm loc}(X)}$ to be the sheafification of the presheaf of C^∞-rings $U \mapsto \pi^{-1}(\mathcal{O}_X)/I$ where

$$I(U) = \{s \in \pi^{-1}(\mathcal{O}_X)(U) : s_{(x,P)} \in \langle \Phi_i(P) \rangle \text{ for all } (x,P) \in U\}.$$

Similarly, define $\mathcal{O}^{\rm ex}_{C^{\rm loc}(X)}$ to be the sheafification of the presheaf $U \mapsto \pi^{-1}(\mathcal{O}_X^{\rm ex})(U)/\sim$ where, for $s'_1, s'_2 \in \pi^{-1}(\mathcal{O}_X^{\rm ex})(U)$, then $s'_1 \sim s'_2$ if for each $(x,P) \in U$ either $s'_{1,x}, s'_{2,x} \in P$, or there is $p \in \langle \Phi_i(P) \rangle$ such that $s'_{1,x} = \Psi_{\exp}(p) s'_{2,x}$. This is a similar process to quotienting the C^∞-ring with corners $(\pi^{-1}(\mathcal{O}_X)(U), \pi^{-1}(\mathcal{O}_X^{\rm ex})(U))$ by a prime ideal in $\pi^{-1}(\mathcal{O}_X^{\rm ex})(U)$, which we described in Example 4.4.4(b), and creates a sheaf of C^∞-rings with corners $\boldsymbol{\mathcal{O}}_{C^{\rm loc}(X)} = (\mathcal{O}_{C^{\rm loc}(X)}, \mathcal{O}^{\rm ex}_{C^{\rm loc}(X)})$.

In Lemma 5.7.2 we show that $\boldsymbol{\mathcal{O}}_{C^{\rm loc}(X)}$ is interior and the stalks at the point (x,P) are local C^∞-rings with corners isomorphic to $\boldsymbol{\mathcal{O}}_{X,x}/{\sim_P}$, using the notation of Example 4.4.4(b). This means that $(C^{\rm loc}(X), \mathcal{O}_{C^{\rm loc}(X)}, \mathcal{O}^{\rm ex}_{C^{\rm loc}(X)}) = C^{\rm loc}(\boldsymbol{X})$ is an interior local C^∞-ringed space with corners.

The continuous function π and the identity morphisms it induces $(\hat{\pi}^\sharp, \hat{\pi}^\sharp_{\rm ex}) : \pi^{-1}(\boldsymbol{\mathcal{O}}_X) \to \pi^{-1}(\boldsymbol{\mathcal{O}}_X)$ can be used to define a canonical morphism in $\mathbf{LC^\infty RS^c}$, $\boldsymbol{\pi} : C^{\rm loc}(\boldsymbol{X}) \to \boldsymbol{X}$. This is equal to π on the topological spaces, and, on the sheaves of C^∞-rings with corners, then $\pi^\sharp(U) : \pi^{-1}(\mathcal{O}_X)(U) \to \mathcal{O}_{C^{\rm loc}(X)}(\pi^{-1}(U))$ sends $s \in \pi^{-1}(\boldsymbol{\mathcal{O}}_X)$ to the image of $\hat{\pi}_\sharp(U)(s) \in \pi^{-1}(\mathcal{O}_X)(\pi^{-1}(U))$ under the quotient map to $\mathcal{O}_{C^{\rm loc}(X)}(\pi^{-1}(U))$, and similarly for the sheaves of monoids map.

Let $\boldsymbol{f} = (f, f^\sharp, f^\sharp_{\rm ex}) : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism in $\mathbf{LC^\infty RS^c}$, we will define a morphism $C^{\rm loc}(\boldsymbol{f}) : C^{\rm loc}(\boldsymbol{X}) \to C^{\rm loc}(\boldsymbol{Y})$. On topological spaces, we define $C^{\rm loc}(f) : C^{\rm loc}(X) \to C^{\rm loc}(Y)$ by $(x, P) \mapsto (f(x), (f^\sharp_{{\rm ex},x})^{-1}(P))$ where $f^\sharp_{{\rm ex},x} : \mathcal{O}^{\rm ex}_{Y,f(x)} \to \mathcal{O}^{\rm ex}_{X,x}$ is the stalk map of $f^\sharp_{\rm ex}$. This is continuous as if $t \in \mathcal{O}^{\rm ex}_Y(U)$ for some open $U \subset Y$ then $C^{\rm loc}(f)^{-1}(U_t) = f^{-1}(U)_{f^{\rm ex}_\sharp(U)(t)}$ is open, and similarly $C^{\rm loc}(f)^{-1}(\hat{U}_t) = \widehat{f^{-1}(U)}_{f^{\rm ex}_\sharp(U)(t)}$ is open. It follows that $\pi \circ C^{\rm loc}(f) = f \circ \pi$.

On sheaves, for $s' \in \mathcal{O}^{\rm ex}_{C^{\rm loc}(Y)}(V)$, then as above, s can be represented by an element $s'' \in \pi^{-1}(\mathcal{O}^{\rm ex}_X)(U)$, and an element $s''' \in \mathcal{O}^{\rm ex}_Y(U)$ for some open $U \subset Y$ with $U \supset \pi(V)$. Then we can map $s' \to \pi^{\rm ex}_\sharp(f^{-1}(U)) \circ f^{\rm ex}_\sharp(U)(s''')|_{\pi^{-1}(V)}$. (Note that this is really a map of the presheaves $\pi^{-1}(\mathcal{O}^{\rm ex}_Y)/\sim \to \pi^{-1}(\mathcal{O}^{\rm ex}_X)/\sim$ but definition of sheafification in Definition 2.3.3 and its universal property give the required morphisms of sheaves.) The restriction to $\pi^{-1}(V)$ means this map is independent of the choice of s''', and the definition of $\mathcal{O}^{\rm ex}_{C^{\rm loc}(Y)}(V)$ as a quotient of $\pi^{-1}(\mathcal{O}^{\rm ex}_X)(U)$ means this map is independent of the choice of s''. This gives a morphism $\mathcal{O}^{\rm ex}_{C^{\rm loc}(Y)}(V) \to \mathcal{O}^{\rm ex}_{C^{\rm loc}(X)}(C^{\rm loc}(f)^{-1}(V))$. A similar definition gives a morphism $\mathcal{O}_{C^{\rm loc}(Y)}(V) \to \mathcal{O}_{C^{\rm loc}(X)}(f^{-1}(V))$. As both of these morphisms behave well with restriction, this gives a morphism of sheaves $C^{\rm loc}(f)_\sharp : \boldsymbol{\mathcal{O}}_{C^{\rm loc}(Y)} \to C^{\rm loc}(f)_*(\boldsymbol{\mathcal{O}}_{C^{\rm loc}(X)})$ adjoint to $C^{\rm loc}(f)^\sharp : C^{\rm loc}(f)^{-1}(\boldsymbol{\mathcal{O}}_{C^{\rm loc}(Y)}) \to \boldsymbol{\mathcal{O}}_{C^{\rm loc}(X)}$ as in (2.4.2), and a morphism in $\mathbf{LC^\infty RS^c}$,

$$C^{\rm loc}(\boldsymbol{f}) : C^{\rm loc}(\boldsymbol{X}) \to C^{\rm loc}(\boldsymbol{Y}).$$

We see that $\boldsymbol{\pi} \circ C^{\rm loc}(\boldsymbol{f}) = \boldsymbol{f} \circ \boldsymbol{\pi}$.

On the stalks, we have $C^{\rm loc}(\boldsymbol{f})^\sharp_{(x,P)} : \boldsymbol{\mathcal{O}}_{Y,f(x)}/\sim_{(f^\sharp_{{\rm ex},x})^{-1}(P)} \to \boldsymbol{\mathcal{O}}_{X,x}/\sim_P$. On the monoid sheaf, if $s' \mapsto 0$ in the stalk, then $s' = [s'']$ for some element $s'' \in \mathcal{O}^{\rm ex}_{Y,f(x)}$, and $f^\sharp_{{\rm ex},(x,P)}(s'')_x \in P$. Then $s''_{f(x)} \in (f^\sharp_{{\rm ex},x})^{-1}(P)$, so $s'' \sim_P 0$ giving that $s' = 0$. Therefore $C^{\rm loc}(\boldsymbol{f})$ is an interior morphism.

Then $C^{\rm loc}(\boldsymbol{f}) = (C^{\rm loc}(f), C^{\rm loc}(f)^\sharp, C^{\rm loc}(f)^\sharp_{\rm ex}) : C^{\rm loc}(\boldsymbol{X}) \to C^{\rm loc}(\boldsymbol{Y})$ is an interior morphism of interior local C^∞-ringed spaces with corners, where $C^{\rm loc}(f)^\sharp, C^{\rm loc}(f)^\sharp_{\rm ex}$ relate to the morphisms $C^{\rm loc}(f)_\sharp, C^{\rm loc}(f)^{\rm ex}_\sharp$ by (2.4.2). One can check that $C^{\rm loc}(\boldsymbol{f} \circ \boldsymbol{g}) = C^{\rm loc}(\boldsymbol{f}) \circ C^{\rm loc}(\boldsymbol{g})$ and hence that $C^{\rm loc} : \mathbf{LC^\infty RS^c} \to \mathbf{LC^\infty RS^c_{in}}$ is a well defined functor.

If we now assume $\boldsymbol{X}$ is interior, then $\{(x, (0)) : x \in X\}$ is contained in $C^{\rm loc}(X)$, and there is an inclusion of sets $\iota_X : X \hookrightarrow C^{\rm loc}(X)$, $x \mapsto (x, (0))$. The image of X under $\iota_X : X \hookrightarrow C^{\rm loc}(X)$ is closed in $C^{\rm loc}(X)$, that is, for all open $U \subset X$, then

$$i_X(U) = \pi^{-1}(U) \setminus \cup_{s' \in \mathcal{O}^{\rm in}_X(U)} \hat{U}_{s'} = \cap_{s' \in \mathcal{O}^{\rm in}_X(U)} U_{s'}$$

is closed in $\pi^{-1}(U)$. Also, i_X is continuous, as the definition of interior implies $i_X^{-1}(U_{s'}) = \{x \in U : s'_x \neq 0 \in \mathcal{O}^{\rm ex}_{X,x}\}$ is open, and $i_X^{-1}(\hat{U}_{s'}) = \{x \in U : s'_x = 0 \in \mathcal{O}^{\rm ex}_{X,x}\}$ is open for all $s' \in \mathcal{O}^{\rm ex}_X(U)$.

Now any element $s \in \mathcal{O}_X(U)$ gives an equivalence class in $\pi^{-1}(\mathcal{O}_X)(\pi^{-1}(U))$, which gives an equivalence class in $\pi^{-1}(\mathcal{O}_X)(\pi^{-1}(U))/\sim$, which gives an equivalence class in $\mathcal{O}_{C^{\mathrm{loc}}(X)}(\pi^{-1}(U))$, which gives an equivalence class in $\iota_X^{-1}(\mathcal{O}_{C^{\mathrm{loc}}(X)})(U)$. So there is a map $\mathcal{O}_X(U) \to \iota_X^{-1}(\mathcal{O}_{C^{\mathrm{loc}}(X)})(U)$ and a similar map can be formed for elements in $\mathcal{O}_X^{\mathrm{ex}}(U)$. These maps respect restriction and form morphisms of sheaves of C^∞-rings with corners. The stalks of $\iota_X^{-1}(\boldsymbol{\mathcal{O}}_{C^{\mathrm{loc}}(X)})$ are isomorphic to the stalks of $\boldsymbol{\mathcal{O}}_{C^{\mathrm{loc}}(X)}$, which are isomorphic to $\boldsymbol{\mathcal{O}}_{X,x}/\sim_{(0)} \cong \boldsymbol{\mathcal{O}}_{X,x}$ by Lemma 5.7.2. This means the inverse image sheaf $\iota_X^{-1}(\boldsymbol{\mathcal{O}}_{C^{\mathrm{loc}}(X)})$ is canonically isomorphic to $\boldsymbol{\mathcal{O}}_X$. So provided $\boldsymbol{X}$ is interior, this isomorphism gives a morphism of local C^∞-ringed spaces with corners $\boldsymbol{\iota_X} : \boldsymbol{X} \to C^{\mathrm{loc}}(\boldsymbol{X})$, which is essentially the inclusion of X into $\{(x,(0)) : x \in X\} \subset C^{\mathrm{loc}}(X)$ and the restriction of the sheaves to this set. This is interior, as the stalk maps are isomorphisms. In addition, $\boldsymbol{\pi_X} \circ \boldsymbol{\iota_X} = \mathbf{id}_{\boldsymbol{X}}$.

Lemma 5.7.2. *The sheaf of C^∞-rings with corners $\boldsymbol{\mathcal{O}}_{C^{\mathrm{loc}}(X)}$ described in Definition 5.7.1 is interior and the stalks at the point (x,P) are local C^∞-rings with corners isomorphic to $\boldsymbol{\mathcal{O}}_{X,x}/\sim_P$, using the notation of Example 4.4.4(b).*

Proof. For each $s' \in \mathcal{O}^{\mathrm{ex}}_{C^{\mathrm{loc}}(X)}(U)$, consider the set

$$K = \{(x,P) \in U : s'_{(x,P)} = 0 \in \mathcal{O}^{\mathrm{ex}}_{C^{\mathrm{loc}}(X),(x,P)}\}.$$

Then there is an open cover $\{U_i\}_{i\in I}$ of U such that $s'|_{U_i}$ corresponds to a $\sim$-equivalence class $[s'_i]$ in $\pi^{-1}(\mathcal{O}_X^{\mathrm{ex}})(U_i)$, and we must have that $s'_i \in P$ at every point $(x,P) \in K \cap U_i$, and $s'_i \notin P$ for all $(x,P) \in (U \setminus K) \cap U_i$. Now the definition of the inverse image sheaf implies there is an open cover $\{W^i_j\}_{j\in J_i}$ of U_i such that $s'_i|_{W_{i,j}} = [s'_{i,j}]$ for some $s'_{i,j} \in \mathcal{O}_X^{\mathrm{ex}}(V_{i,j})$ where we have open $V_{i,j} \subset X$ such that $V_{i,j} \supset \pi(W_{i,j})$. Now we must have $s'_{i,j,x} \in P$ for all $(x,P) \in K \cap W_{i,j}$, and $s'_{i,j,x} \notin P$ for all $(x,P) \in (U \setminus K) \cap W_{i,j}$. Then

$$W_{i,j} \cap (\hat{V_{i,j}})_{s'_{i,j}} = K \cap W_{i,j}$$
$$W_{i,j} \cap (V_{i,j})_{s'_{i,j}} = (U \setminus K) \cap W_{i,j}$$

are both open in X. Taking the union over $i \in I, j \in J_i$, and using that $\cup_{i\in I, j\in J_i} W_{i,j} = U$, we see that K and $U \setminus K$ are both open in U.

The stalks of $(\mathcal{O}_{C^{\mathrm{loc}}(X)}, \mathcal{O}^{\mathrm{ex}}_{C^{\mathrm{loc}}(X)})$ at a point $(x,P) \in C^{\mathrm{loc}}(X)$ are isomorphic to $\mathcal{O}_{X,x}/\langle\Phi_i(P)\rangle$ and $\mathcal{O}^{\mathrm{ex}}_{X,x}/\sim_P$, where $s'_1 \sim_P s'_2$ in $\mathcal{O}^{\mathrm{ex}}_{X,x}$ if $s'_1, s'_2 \in P$ or there is a $p \in \langle\Phi_i(P)\rangle$ such that $\Psi_{\exp}(p)s'_1 = s'_2$, as in Example 4.4.4(b). To see this, consider that the definitions give us the following diagram, where we will show the arrow $\boldsymbol{t}$ exists and is an isomorphism.

$$\begin{array}{ccccc} \boldsymbol{\mathcal{O}}_{X,x} & \xrightarrow[\hat{\boldsymbol{\pi}}_x^\sharp]{\sim} & \pi^{-1}(\boldsymbol{\mathcal{O}}_X)_{(x,P)} & \twoheadrightarrow & \boldsymbol{\mathcal{O}}_{C^{\mathrm{loc}}(X),(x,P)} \\ \downarrow & & & \nearrow{\scriptstyle \boldsymbol{t}} & \\ \boldsymbol{\mathcal{O}}_{X,x}/\sim_P & & & & \end{array} \qquad (5.7.1)$$

To see that $\boldsymbol{t}$ exists at the point (x,P), we use the universal property of $\boldsymbol{\mathcal{O}}_{X,x}/{\sim_P}$ as in Example 4.4.4(b)(**). Let U be an open set in X and take $s' \in \mathcal{O}_X^{\mathrm{ex}}(U)$ such that $s'_x \in P$. Then s' maps to an equivalence class in $\mathcal{O}^{\mathrm{ex}}_{C^{\mathrm{loc}}(X)}(\pi^{-1}(U))$. Consider the open set $\hat{U}'_s = \{(x,P) \in C(X) : x \in U, s'_x \in P\}$, then restricting to this open set, we see that our (x,P) is in $\hat{U}'_s \subset \pi^{-1}(U)$ and so we can restrict the equivalence class of s' to $\hat{U}_{s'}$. In this open set however, $s'_x \in P$ for all $(x,P) \in \hat{U}_{s'}$ so $s' \sim 0$, and so s' is in the kernel of the composition of the top row of (5.7.1). Then the universal property of $\boldsymbol{\mathcal{O}}_{X,x}/{\sim_P}$ says that $\boldsymbol{t}$ must exist and commute with the diagram. Also, $\boldsymbol{t}$ must be surjective as the top line is surjective.

To see that $\boldsymbol{t}$ is injective is straightforward. For example, in the monoid case, if $[s'_{1,x}], [s'_{2,x}] \in \mathcal{O}^{\mathrm{ex}}_{X,x}/{\sim_P}$ with representatives $s'_1, s'_2 \in \mathcal{O}^{\mathrm{ex}}_X(U)$, and if $t_{\mathrm{ex}}([s'_{1,x}]) = t_{\mathrm{ex}}([s'_{2,x}])$ then $s'_{1,x} \sim s'_{2,x}$ for all $(x,P) \in V$ for some open $V \subset \pi^{-1}(U)$. This means at every $(x,P) \in V$ then $s'_{1,x} \sim_P s'_{2,x}$, so this must be true at our (x,P), so that $[s'_{1,x}] = [s'_{2,x}] \in \mathcal{O}^{\mathrm{ex}}_{X,x}/{\sim_P}$ as required.

Now $\mathcal{O}^{\mathrm{ex}}_{X,x}/{\sim_P}$ is interior as the complement of $P \in \mathcal{O}^{\mathrm{ex}}_{X,x}$ has no zero-divisors. It is also local, as the unique morphism $\mathcal{O}_{X,x} \to \mathbb{R}$ must have P in its kernel so it factors through the morphism $\mathcal{O}_{X,x} \to \mathcal{O}^{\mathrm{ex}}_{X,x}/{\sim_P}$ giving a unique morphism $\mathcal{O}^{\mathrm{ex}}_{X,x}/{\sim_P} \to \mathbb{R}$ with the correct properties to be local.

This means that $(C^{\mathrm{loc}}(X), \mathcal{O}_{C^{\mathrm{loc}}(X)}, \mathcal{O}^{\mathrm{ex}}_{C^{\mathrm{loc}}(X)}) = C^{\mathrm{loc}}(\boldsymbol{X})$ is an interior local C^∞-ringed space with corners. $\square$

Theorem 5.7.3. *The corner functor* $C^{\mathrm{loc}} : \mathbf{LC^\infty RS^c} \to \mathbf{LC^\infty RS^c_{in}}$ *is right adjoint to the inclusion functor* $i : \mathbf{LC^\infty RS^c_{in}} \to \mathbf{LC^\infty RS^c}$*. Thus we have natural, functorial isomorphisms* $\mathrm{Hom}_{\mathbf{LC^\infty RS^c_{in}}}(C^{\mathrm{loc}}(\boldsymbol{X}), \boldsymbol{Y}) \cong \mathrm{Hom}_{\mathbf{LC^\infty RS^c}}(\boldsymbol{X}, i(\boldsymbol{Y}))$*.*

Proof. We describe the unit $\boldsymbol{\eta} : \mathrm{id} \Rightarrow Ci$ and counit $\boldsymbol{\epsilon} : iC \Rightarrow \mathrm{id}$ of the adjunction. Here $\boldsymbol{\epsilon_X} = \boldsymbol{\pi_X}$ for $\boldsymbol{X}$ a C^∞-scheme with corners, and $\boldsymbol{\eta_X} = \boldsymbol{\iota_X}$, for $\boldsymbol{X}$ an interior C^∞-scheme with corners. That $\boldsymbol{\epsilon}$ is a natural transformation follows directly from the definition of $C(\boldsymbol{f})$ for each morphism of C^∞-schemes with corners $\boldsymbol{f}$.

To show $\boldsymbol{\eta}$ is a natural transformation, we need to show $\boldsymbol{\eta_Y} \circ \boldsymbol{f} = Ci(\boldsymbol{f}) \circ \boldsymbol{\eta_X}$ for all $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y} \in \mathbf{C^\infty Sch^c_{in}}$. On topological spaces we have $x \mapsto f(x) \mapsto (f(x), (0))$ under $\boldsymbol{\eta_Y} \circ \boldsymbol{f}$, and $x \mapsto (x,(0)) \mapsto (f(x), (f^\sharp_{x,\mathrm{ex}})^{-1}(0))$ under $Ci(\boldsymbol{f}) \circ \boldsymbol{\eta_X}$. As $\boldsymbol{f}$ is interior, then $(f^\sharp_{x,\mathrm{ex}})^{-1}(0) = (0)$ so we have equality on topological spaces.

On the sheaves of C^∞-rings with corners, elements of $\mathcal{O}_{C^{\mathrm{loc}}(Y)}(U)$ are equivalence classes $[[[s]]]$ where $s \in \mathcal{O}_Y(V)$ for some open set $V \supset \pi(U)$, $[s] \in \pi^{-1}(\mathcal{O}_Y)(U)$, $[[s]] \in \pi^{-1}(\mathcal{O}_Y)(U)/{\sim}$ and $[[[s]]] \in \mathcal{O}_{C^{\mathrm{loc}}(Y)}(U)$. The map $\iota_{Y,\sharp}(U)$ sends $[[[s]]]$ to $s|_{\iota^{-1}(U)}$. So $\boldsymbol{\eta_Y} \circ \boldsymbol{f}$ sends $[[[s]]]$ to $f_\sharp(\iota^{-1}(U))(s|_{\iota^{-1}(U)})$, and $Ci(\boldsymbol{f}) \circ \boldsymbol{\eta_X}$ sends $[[[s]]]$ to $[[[f_\sharp(V)(s)]]]$ to

$f_\sharp(V)(s)|_{\iota^{-1}(U)}$, which is equal to $f_\sharp(\iota^{-1}(U))(s|_{\iota^{-1}(U)})$ as required. (Again, this is actually a map of the underlying presheaves, but the properties of the sheafification functor from Definition 2.3.3 implies this gives a map of the corresponding sheaves.) The same occurs for the sheaves of monoids, hence $\boldsymbol{\eta}$ is a natural transformation.

Finally, to show naturality, we need to show that $C \Rightarrow CiC \Rightarrow C$ is the identity natural transformation and $i \Rightarrow iCi \Rightarrow i$ is the identity natural transformation. However, both of these follow as $\pi_{\boldsymbol{X}} \circ \boldsymbol{\iota_X} = \mathbf{id}_{\boldsymbol{X}}$. □

Remark 5.7.4. If a functor $F : C \to D$ from a category C to a category D has a right adjoint $G : D \to C$, then one of the equivalent definitions of adjoint (cf. Awodey [5, §9]) says that for elements $d \in D$ there is a morphism $f_d : FG(d) \to d$, such that for all $c \in C$ and $d \in D$, and for all morphisms $g : c \mapsto G(d)$, there is a unique morphism $h : c \mapsto FG(d)$ with $f_d \circ h = g$.

For the corner functor $C^{\text{loc}} : \mathbf{LC^\infty RS^c} \to \mathbf{LC^\infty RS^c_{in}}$, this means if $\boldsymbol{X}$ is a local C^∞-ringed space with corners, then we could instead define $C^{\text{loc}}(\boldsymbol{X})$ to be the (unique up to isomorphism) interior local C^∞-ringed space with corners with morphism $\boldsymbol{\pi_X} : C^{\text{loc}}(\boldsymbol{X}) \to \boldsymbol{X}$ that satisfies the following universal property: for all interior local C^∞-ringed spaces with corners $\boldsymbol{Y}$ and morphisms $\boldsymbol{f} : \boldsymbol{Y} \to \boldsymbol{X}$, there is a unique interior morphism $\hat{\boldsymbol{f}} : \boldsymbol{Y} \to C^{\text{loc}}(\boldsymbol{X})$ such that $\boldsymbol{\pi_X} \circ \hat{\boldsymbol{f}} = \boldsymbol{f}$. For any morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{LC^\infty RS^c}$, we can then define $C^{\text{loc}}(\boldsymbol{f}) = \widehat{\boldsymbol{f} \circ \pi_{\boldsymbol{X}}}$.

For all interior local C^∞-ringed spaces with corners $\boldsymbol{X}$, then the identity morphism $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ gives a unique interior morphism $\hat{\mathbf{id}}_{\boldsymbol{X}} : \boldsymbol{X} \to C^{\text{loc}}(\boldsymbol{X})$, such that $\boldsymbol{\pi_X} \circ \hat{\mathbf{id}}_{\boldsymbol{X}} = \mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$. We define $\boldsymbol{\iota_X} = \hat{\mathbf{id}}_{\boldsymbol{X}}$, and by uniqueness we see $\hat{\boldsymbol{f}} = C^{\text{loc}}(\boldsymbol{f}) \circ \boldsymbol{\iota_X}$ for all morphisms $\boldsymbol{f} : \boldsymbol{Y} \to \boldsymbol{X}$ for interior $\boldsymbol{Y}$. This is consistent with our previous definition.

5.7.1 Boundary

We now consider defining a notion of boundary of local C^∞-ringed spaces with corners. For this, we need the following:

Definition 5.7.5. A prime ideal P in a non-trivial monoid M (under multiplication with a zero element 0) is a *minimal non-trivial prime ideal* if $P \neq \{0\}$ and there are no prime ideals P' with $P' \subset P$.

Definition 5.7.6. If $\boldsymbol{X}$ is a local C^∞-ringed space with corners, then the *boundary* $\partial^{\text{loc}}\boldsymbol{X}$ of $\boldsymbol{X}$ is a sub-local C^∞-ringed space with corners of $C^{\text{loc}}(\boldsymbol{X})$. It is the restriction of $C^{\text{loc}}(\boldsymbol{X})$ to the set

$$\partial^{\text{loc}} X = \{(x, P) \in C^{\text{loc}}(X) : x \in X, P \text{ is a minimal prime ideal of } \mathcal{O}^{\text{ex}}_{X,x}\}.$$

That is, if we consider the inclusion of topological spaces $\partial^{\mathrm{loc}} X \to C^{\mathrm{loc}}(X)$ then the inverse image sheaf of $\boldsymbol{\mathcal{O}}_{C^{\mathrm{loc}}(X)}$ under this map gives the sheaf of C^∞-rings with corners on $\partial^{\mathrm{loc}} X$.

Note that we would not expect $\boldsymbol{X} \mapsto \partial^{\mathrm{loc}}\boldsymbol{X}, \boldsymbol{X} \in \mathbf{LC^\infty RS^c}$ to be a functor.

Remark 5.7.7. Gillam and Molcho [28, §4.4] define a notion of the boundary $\Delta\boldsymbol{X}$ of a pre-log locally ringed space $\boldsymbol{X}$, discussed in §5.9. In Remark 5.9.4 we explain that our local C^∞-ringed spaces with corners are indeed pre-log locally ringed spaces. Then for $\boldsymbol{X} \in \mathbf{LC^\infty RS^c}$ we can compare $\Delta\boldsymbol{X}$ and $\partial^{\mathrm{loc}}\boldsymbol{X}$. Considering their definition of boundary $\Delta\boldsymbol{X}$, we see the underlying sets of the topological spaces are identical, however the topology on $\partial^{\mathrm{loc}}\boldsymbol{X}$ is finer than the topology on $\Delta\boldsymbol{X}$, so there is a morphism of topological spaces $\partial^{\mathrm{loc}}\boldsymbol{X} \to \Delta\boldsymbol{X}$. Notably, the sets $U_{s'}, \hat{U}_{s'}$ defined in Definition 5.7.1 form a subbase for the topology on $\partial^{\mathrm{loc}}\boldsymbol{X}$, however only the sets $U_{s'}$ form a base for the topology on $\Delta\boldsymbol{X}$.

The sheaves on $\partial^{\mathrm{loc}}\boldsymbol{X}, \Delta\boldsymbol{X}$ are constructed in identical ways, so there is actually a morphism $\partial^{\mathrm{loc}}\boldsymbol{X} \to \Delta\boldsymbol{X}$ of pre-log locally ringed spaces (in fact, both are local C^∞-ringed spaces with corners whenever $\boldsymbol{X}$ is), and the stalks are isomorphic. Then $\partial^{\mathrm{loc}}\boldsymbol{X} \to \Delta\boldsymbol{X}$ is an isomorphism whenever the topology on $\Delta\boldsymbol{X}$ is as fine as the topology on $\partial^{\mathrm{loc}}\boldsymbol{X}$.

5.8 Corner functor for $\mathbf{C^\infty Sch^c_{fi}}$

In this section, we define a corner functor for firm C^∞-schemes with corners and compare it to the corner functor for local C^∞-ringed spaces with corners. We first consider the prime ideals necessary for this construction.

Definition 5.8.1. For a C^∞-ring with corners $\mathfrak{C}$, define the following two sets of prime ideals

$$\mathrm{Pr}_{\mathfrak{C}} = \{P \subset \mathfrak{C}_{\mathrm{ex}} | P \text{ is a prime ideal}\}$$

and

$$\mathrm{Pr}'_{\mathfrak{C}} = \{P \subset \mathfrak{C}_{\mathrm{ex}} | P = \pi_{x,\mathrm{ex}}^{-1}(P') \text{ for some } x \in X \text{ for a prime ideal } P' \subset \mathcal{O}_{X,x}^{\mathrm{ex}}\}.$$

Here, $\pi_{x,\mathrm{ex}} : \mathfrak{C}_{\mathrm{ex}} \to \mathfrak{C}_{x,\mathrm{ex}}$ is the surjective localisation morphism. While $\mathrm{Pr}'_{\mathfrak{C}} \subset \mathrm{Pr}_{\mathfrak{C}}$, Example 5.8.3 shows these two sets are not always equal. In the following lemma we will show some facts about prime ideals including that any $P \in \mathrm{Pr}_{\mathfrak{C}} \setminus \mathrm{Pr}'_{\mathfrak{C}}$ are such that $\mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}/{\sim_P})$ is the empty set with the zero-sheaf.

Lemma 5.8.2. (a) *If we consider any* $\mathbb{R}$*-point* $x : \mathfrak{C} \to \mathbb{R}$ *and the localisation* $\boldsymbol{\pi} : \mathfrak{C} \to \mathfrak{C}_x$*, then* $\pi_{x,\mathrm{ex}}$ *takes prime ideals in* $\mathfrak{C}_{\mathrm{ex}}$ *to prime ideals in* $\mathfrak{C}_{x,\mathrm{ex}}$ *(or to the entire monoid).*

(b) *For a C^∞-ring with corners $\mathfrak{C}$ if $P \in \mathrm{Pr}_{\mathfrak{C}} \setminus \mathrm{Pr}'_{\mathfrak{C}}$ then $\mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}/\sim_P)$ is the empty set with the zero-sheaf. Also, an $\mathbb{R}$-point $x : \mathfrak{C} \to \mathbb{R}$ factors through $\mathfrak{C}/\sim_P$ whenever $P \subseteq x^{-1}(0)$, which implies $\pi_{x,\mathrm{ex}}(P) \neq \mathfrak{C}_{\mathrm{ex},x}$. So we have $P \in \mathrm{Pr}_{\mathfrak{C}} \setminus \mathrm{Pr}'_{\mathfrak{C}}$ if and only if for all $\mathbb{R}$-points $x : \mathfrak{C} \to \mathbb{R}$ then $\pi_{x,\mathrm{ex}}(P) = \mathfrak{C}_{x,\mathrm{ex}}$ under the localisation $\boldsymbol{\pi} : \mathfrak{C} \to \mathfrak{C}_x$.*

(c) *If we have $P \in \mathrm{Pr}_{\mathfrak{C}}$ and an element $c \in \mathfrak{C}_{\mathrm{ex}}$ such that $c \notin P$ then for each $\mathbb{R}$-point $x : \mathfrak{C} \to \mathbb{R}$ we have $\pi_{x,\mathrm{ex}}(c) \in \pi_{x,\mathrm{ex}}(P)$ if and only if $\pi_{x,\mathrm{ex}}(P) = \mathfrak{C}_{x,\mathrm{ex}}$. This implies if we have two prime ideals $P_1, P_2 \in \mathrm{Pr}_{\mathfrak{C}}$ with $P_1 \subsetneq P_2$, then for all $x \in X$, $\pi_{x,\mathrm{ex}}(P_1) \subseteq \pi_{x,\mathrm{ex}}(P_2)$ with equality only occurring when $\pi_{x,\mathrm{ex}}(P_1) = \pi_{x,\mathrm{ex}}(P_2) = \mathfrak{C}_{x,\mathrm{ex}}$.*

Proof. Firstly, if P is a prime ideal in $\mathfrak{C}_{\mathrm{ex}}$ and we have a morphism of C^∞-rings with corners $\boldsymbol{\phi} : \mathfrak{D} \to \mathfrak{C}$ then it follows that $\phi_{\mathrm{ex}}^{-1}(P)$ is a prime ideal in $\mathfrak{D}_{\mathrm{ex}}$. We now prove (a).

(a) Take a prime ideal $P \subset \mathfrak{C}_{\mathrm{ex}}$. To see that $\pi_{x,\mathrm{ex}}(P)$ take any $c'_x \in \pi_{x,\mathrm{ex}}(P)$ then as $\pi_{x,\mathrm{ex}}$ is surjective there is $c' \in \mathfrak{C}_{\mathrm{ex}}$ is such that $\pi_{x,\mathrm{ex}}(c') = c'_x$ and we have

$$c'_x \pi_{x,\mathrm{ex}}(P) = \pi_{x,\mathrm{ex}}(c)\pi_{x,\mathrm{ex}}(P) = \pi_{x,\mathrm{ex}}(cP) \subset \pi_{x,\mathrm{ex}}(P).$$

Now say $c'_x d'_x \in \pi_{x,\mathrm{ex}}(P)$, then there are $c', d' \in \mathfrak{C}_{\mathrm{ex}}$ such that $\pi_{x,\mathrm{ex}}(c') = c'_x$ and $\pi_{x,\mathrm{ex}}(d') = d'_x$. So $\pi_{x,\mathrm{ex}}(c'd') \in \pi_{x,\mathrm{ex}}(P)$. So there are $a', b' \in \mathfrak{C}_{\mathrm{ex}}$ such that $a'c'd' = b'p'$ for some $p \in P$, and with $x \circ \Phi_i(a') \neq 0$ and $\Phi_i(a') - \Phi_i(b') \in I$ as in Lemma 4.6.9. However, $p' \in P$ so $b'p' \in P$ and so $a'c'd' \in P$. This means either $c'd' \in P$, in which case, as P is prime, then either $c' \in P$ or $d' \in P$ and so either c'_x or d'_x are in $\pi_{x,\mathrm{ex}}(P)$; or we have that $a' \in P$. However, this $a' \in P$ is such that $x \circ \Phi_i(a') \neq 0$, which means that $\pi_{x,\mathrm{ex}}(a') \in \mathfrak{C}_{\mathrm{ex}}$ is invertible, which would imply that $\pi_{x,\mathrm{ex}}(a') \in \pi_{x,\mathrm{ex}}(P) = \mathfrak{C}_{x,\mathrm{ex}}$. In this case, both c'_x and d'_x will be in $\pi_{x,\mathrm{ex}}(P)$. Hence $\pi_{x,\mathrm{ex}}(P)$ is either prime in $\mathfrak{C}_{x,\mathrm{ex}}$ or equal to all of $\mathfrak{C}_{x,\mathrm{ex}}$.

(b) Take $P \in \mathrm{Pr}_{\mathfrak{C}} \setminus \mathrm{Pr}'_{\mathfrak{C}}$, this means that for all $x : \mathfrak{C} \to \mathbb{R}$ then $P \subsetneq \pi_{x,\mathrm{ex}}^{-1}(\pi_{x,\mathrm{ex}}(P))$. So for each x there is a $c' \in \mathfrak{C}_{\mathrm{ex}}$ such that $c' \notin P$ but $\pi_{x,\mathrm{ex}}(c') = \pi_{x,\mathrm{ex}}(p')$ for some $p' \in P$. However, similar to the above, we must then have $a', b' \in \mathfrak{C}_{\mathrm{ex}}$ with $x \circ \Phi_i(a') \neq 0$ and $\Phi_i(a') - \Phi_i(b') \in I$ and such that $a'c' = b'p'$. As $p' \in P$, then $b'p' \in P$ so $a'c' \in P$ and as $c' \notin P$ then $a' \in P$. However, again $\pi_{x,\mathrm{ex}}(a') \in \mathfrak{C}_{\mathrm{ex}}$ is invertible, which would imply that $\pi_{x,\mathrm{ex}}(P) = \mathfrak{C}_{x,\mathrm{ex}}$ for all $x : \mathfrak{C} \to \mathbb{R}$. $P \in \mathrm{Pr}_{\mathfrak{C}} \setminus \mathrm{Pr}'_{\mathfrak{C}}$ implies $\pi_{x,\mathrm{ex}}(P) = \mathfrak{C}_{x,\mathrm{ex}}$ for all $\mathbb{R}$-points x. The reverse implication follows from the definition of $\mathrm{Pr}'_{\mathfrak{C}}$.

If $\pi_{x,\mathrm{ex}}(P) = \mathfrak{C}_{x,\mathrm{ex}}$ for all $\mathbb{R}$-points x then there are no $\mathbb{R}$-points $y : \mathfrak{C}/\sim_P \to \mathbb{R}$, as if there was such an $\mathbb{R}$-point then composing it with the projection $\boldsymbol{\phi} : \mathfrak{C} \to \mathfrak{C}/\sim_P$ gives an $\mathbb{R}$-point $x = y \circ \phi : \mathfrak{C} \to \mathbb{R}$. Then taking the a' corresponding to this $\mathbb{R}$-point as in the previous paragraph implies that $x \circ \Phi_i(a') \neq 0$, however $a' \in P$ so $\phi_{\mathrm{ex}}(a') = 0 \in \mathfrak{C}_{\mathrm{ex}}/\sim_P$. This gives

$$0 \neq x \circ \Phi_i(a') = y \circ \Phi_i \circ \phi_{\mathrm{ex}}(a') = 0,$$

which is a contradiction to the existence of y.

Hence $\mathrm{Spec^c}(\mathfrak{C}/{\sim_P})$ is the empty set with the zero-sheaf. Notably, if $\mathfrak{C}/{\sim_P}$ is not the zero ring, then this will give an example of C^∞-rings with corners that is not semi-complete.

(c) If we have $P \in \mathrm{Pr}_{\mathfrak{C}}$ and an element $c \in \mathfrak{C}_{\mathrm{ex}}$ such that $c \notin P$ and say we have $\pi_{x,\mathrm{ex}}(c) \in \pi_{x,\mathrm{ex}}(P)$ for some $\mathbb{R}$-point $x : \mathfrak{C} \to \mathbb{R}$. Then there is a $p \in P$ such that $\pi_{x,\mathrm{ex}}(c) = \pi_{x,\mathrm{ex}}(p)$. Then, as above, there are $a, b \in \mathfrak{C}_{\mathrm{ex}}$ satisfying certain conditions such that $ac = bp$. However, as $c \notin P$ and P is prime, then $a \in P$. Yet $\pi_{x,\mathrm{ex}}(a)$ is invertible in $\mathfrak{C}_{x,\mathrm{ex}}$, so $\pi_{x,\mathrm{ex}}(P) = \mathfrak{C}_{x,\mathrm{ex}}$ for this $\mathbb{R}$-point x. □

Example 5.8.3. If $\mathfrak{C} = \boldsymbol{C^\infty}(\mathbb{R} \amalg \mathbb{R} \amalg \mathbb{R})$ then we will show $\mathrm{Pr}_{\mathfrak{C}} \setminus \mathrm{Pr}'_{\mathfrak{C}}$ is non-empty. Label each $\mathbb{R}$ as $\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3$ then the prime ideals in $\mathfrak{C}_{\mathrm{ex}}$ are

$$P_i = \{f : \mathbb{R}_1 \amalg \mathbb{R}_2 \amalg \mathbb{R}_3 \to [0,\infty) : f(\mathbb{R}_i) = 0\}$$

for $i = 1, 2, 3$ and

$$P_4 = \langle P_1, P_2\rangle, P_5 = \langle P_1, P_3\rangle, P_6 = \langle P_2, P_3\rangle, P_7 = \langle P_1, P_2, P_3\rangle.$$

If we take elements $c'_1, c'_2, c'_3 \in \mathfrak{C}_{\mathrm{ex}}$ such that $c'_i = 0$ in $\mathbb{R}_i$ and $c'_i = 1$ in $\mathbb{R}_j$ for $j \neq i$ for $i, j = 1, 2, 3$, then these prime ideals are finitely generated with

$$P_1 = \langle c'_1\rangle, P_2 = \langle c'_2\rangle, P_3 = \langle c'_3\rangle, P_4 = \langle c'_1, c'_2\rangle, P_5 = \langle c'_2, c'_3\rangle, P_6 = \langle c'_1, c'_3\rangle, P_7 = \langle c'_1, c'_2, c'_3\rangle.$$

Then $\mathfrak{C}/{\sim_{P_i}} \cong \boldsymbol{C^\infty}(\mathbb{R}_i)$ is semi-complete for $i = 1, 2, 3$, and $\mathfrak{C}/{\sim_{P_7}} = 0$. In every stalk $\pi_{x,\mathrm{ex}}(P_i) = \mathfrak{C}_{x,\mathrm{ex}}$ for $i = 4, 5, 6, 7$ for each $x : \mathfrak{C} \to \mathbb{R}$ so the primes P_4, P_5, P_6, P_7 are in $\mathrm{Pr}_{\mathfrak{C}} \setminus \mathrm{Pr}'_{\mathfrak{C}}$. As for example, c'_3 maps to a non-zero element under the quotient morphism $\mathfrak{C} \to \mathfrak{C}/{\sim_{P_4}}$, the $\mathfrak{C}/{\sim_{P_i}}$ for $i = 4, 5, 6$ are non-zero but are not semi-complete.

Proposition 5.8.4. *Let $\boldsymbol{X} = \mathrm{Spec^c}(\mathfrak{C})$ with $\mathfrak{C}$ a firm C^∞-ring with corners. Then $C^{\mathrm{aff}} : \boldsymbol{X} \mapsto \coprod_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathrm{Spec^c}(\mathfrak{C}/{\sim_P})$ is a functor $C^{\mathrm{aff}} : \mathbf{AC^\infty Sch^c_{fi}} \to \mathbf{C^\infty Sch^c_{fi,in}}$. The points of the topological space of $C^{\mathrm{aff}}(\boldsymbol{X})$ are in a one-to-one correspondence with pairs (x, P'), for some $x \in X$ and prime ideal $P' \subset \mathfrak{C}_{x,\mathrm{ex}}$. The stalks at a point (x, P') are isomorphic to*

$$\mathfrak{C}_x/{\sim_{P'}} \cong \left(\mathfrak{C}/{\sim_{\pi^{-1}_{x,\mathrm{ex}}(P')}}\right)_x. \tag{5.8.1}$$

Proof. Firstly, let $\mathfrak{C}$ be any firm C^∞-ring with corners and $P \subset \mathfrak{C}_{\mathrm{ex}}$ prime. Let $c'_1, c'_2 \in \mathfrak{C}_{\mathrm{ex}}$ and $[c'_1], [c'_2]$ be their equivalence classes in $\mathfrak{C}_{\mathrm{ex}}/{\sim_P}$. Then if $[c'_1][c'_2] = 0 \in \mathfrak{C}_{\mathrm{ex}}/{\sim_P}$, the explicit description of quotient from Example 4.4.4(b) implies $c'_1 c'_2 \in P$. As P is prime, then, without loss of generality, $c_1' \in P$. However, then $[c_1'] = 0 \in \mathfrak{C}_{\rm ex}/{\sim_P}$. Hence there are no zero divisors in $\mathfrak{C}_{\rm ex}/{\sim_P}$, and $\boldsymbol{\mathfrak{C}}/{\sim_P}$ is interior and firm, so $C^{\rm aff}(\boldsymbol{X})$ is an interior firm C^∞-scheme with corners.

If $\boldsymbol{X} \cong \operatorname{Spec^c}(\boldsymbol{\mathfrak{C}})$ with $\boldsymbol{\mathfrak{C}}$ firm, then the elements of the topological space of $C^{\rm aff}(\boldsymbol{X})$ are the disjoint unions of the topological spaces of $\operatorname{Spec^c}(\boldsymbol{\mathfrak{C}}/{\sim_P})$ for $P \in \Pr_{\mathfrak{C}}$. Note that for any $P \in \Pr_{\mathfrak{C}} \setminus \Pr'_{\mathfrak{C}}$ we have $\operatorname{Spec^c}(\boldsymbol{\mathfrak{C}}/{\sim_P})$ is the empty set with zero-sheaf by Lemma 5.8.2, so $\coprod_{P\in\Pr_{\mathfrak{C}}} \operatorname{Spec^c}(\boldsymbol{\mathfrak{C}}/{\sim_P}) \cong \coprod_{P\in\Pr'_{\mathfrak{C}}} \operatorname{Spec^c}(\boldsymbol{\mathfrak{C}}/{\sim_P})$. As we have the surjective quotient morphism $\boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{C}}/{\sim_P}$ then any $\mathbb{R}$-point, y of $\boldsymbol{\mathfrak{C}}/{\sim_P}$ comes from an $\mathbb{R}$-point of $\boldsymbol{\mathfrak{C}}$ by composition with this morphism. However, the only $\mathbb{R}$-points x of $\boldsymbol{\mathfrak{C}}$ that factor through this composition must have $P \subset x^{-1}(0)$, so these are precisely the $\mathbb{R}$-points of $\boldsymbol{\mathfrak{C}}/{\sim_P}$. Primes $P \subset x^{-1}(0) \in \Pr_{\mathfrak{C}}$ must be in $\Pr'_{\mathfrak{C}}$ by Lemma 5.8.2, so they are in one-to-one correspondence with primes $P' = \pi_{x,\rm ex}(P) \subset \mathfrak{C}_{x,\rm ex}$. So an $\mathbb{R}$-point of $C^{\rm aff}(X)$ is determined uniquely by a $\mathbb{R}$-point $x \in X$ and a prime $P' \subset \mathfrak{C}_{x,\rm ex}$ (equivalently a prime $P \in \Pr'_{\mathfrak{C}}$ with $P \subset x^{-1}(0) \in \mathfrak{C}_{\rm ex}$).

To show that (5.8.1) holds, note that the coproduct of C^∞-schemes with corners has sheaf of C^∞-rings with corners that is the product of the sheaves of C^∞-rings with corners. This means the stalks at a point (x, P') are isomorphic to the localisations $(\boldsymbol{\mathfrak{C}}/{\sim_P})_x$ for $\pi_{x,\rm ex}^{-1}(P') = P$, as in Lemma 5.4.14. It is then an exercise in universal properties of quotients and localisations to see that this is isomorphic to $\boldsymbol{\mathfrak{C}}_x/{\sim_{P'}}$.

We will use the notation in the following commutative diagram to define the functor on morphisms. Note that $\boldsymbol{\pi}_P : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{C}}/{\sim_P}$ gives a morphism $\boldsymbol{g}_P = \operatorname{Spec^c} \boldsymbol{\pi}_P : \operatorname{Spec^c}(\boldsymbol{\mathfrak{C}}/{\sim_P}) \to \operatorname{Spec^c}(\boldsymbol{\mathfrak{C}})$. The universal property of coproduct allows these to be amalgamated to give $\boldsymbol{g} : \coprod_{P\in\Pr_{\mathfrak{C}}} \operatorname{Spec^c}(\boldsymbol{\mathfrak{C}}/{\sim_P}) \to \operatorname{Spec^c}(\boldsymbol{\mathfrak{C}})$.

$$\begin{array}{ccccc} & & \boldsymbol{\mathfrak{C}} & & \\ & \swarrow^{\boldsymbol{\pi}_x} & & \searrow^{\boldsymbol{\pi}_P} & \\ \boldsymbol{\mathfrak{C}}_x & & & & \boldsymbol{\mathfrak{C}}/{\sim_P} \\ {\scriptstyle \boldsymbol{\pi}_{P',x}}\downarrow & & & & \downarrow{\scriptstyle \boldsymbol{\pi}_{x,P}} \\ \boldsymbol{\mathfrak{C}}_x/{\sim_{P'}} & & \cong & & (\boldsymbol{\mathfrak{C}}/{\sim_P})_x \\ & \searrow & & \swarrow & \\ & & \mathbb{R} & & \end{array} \tag{5.8.2}$$

Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism of firm affine C^∞-schemes with corners with $\boldsymbol{X} \cong \operatorname{Spec^c}(\boldsymbol{\mathfrak{C}})$ and $\boldsymbol{Y} \cong \operatorname{Spec^c}(\boldsymbol{\mathfrak{D}})$ with $\boldsymbol{\mathfrak{C}}$ and $\boldsymbol{\mathfrak{D}}$ firm. We define a morphism $C^{\rm aff}(\boldsymbol{f}) : C^{\rm aff}(\boldsymbol{X}) \to C^{\rm aff}(\boldsymbol{Y})$. On topological spaces, it takes (x, P') to $(f(x), (f^\sharp_{f(x),\rm ex})^{-1}(P'))$ and we show this is continuous in Lemma 5.8.5. On the sheaves of C^∞-rings, take an element $s \in \mathcal{O}_{C^{\rm aff}(Y)}(U)$ and we show how to define $C^{\rm aff}(f)_\sharp(U) : \mathcal{O}_{C^{\rm aff}(Y)}(U) \to$

$\mathcal{O}_{C^{\mathrm{aff}}(X)}(C^{\mathrm{aff}}(f)^{-1}(U))$ for open $U \subset C^{\mathrm{aff}}(Y)$. The same method will show how to define $C^{\mathrm{aff}}(f)^{\mathrm{ex}}_{\sharp}(U)$ for the sheaves of monoids.

For all points $(y, Q) \in U$, there is an open set $U_{y,Q} \subset U$ such that $s(y', Q') = \pi_{y',Q'}(d_y)$ for some $d_y \in \mathfrak{D}/{\sim_Q}$ and all $(y', Q') \in U_{y,Q}$. We can restrict this open set further around (y, Q) by requiring $Q' = Q$. Then there is an element $\hat{d}_y \in \mathfrak{D}$ such that $\pi_Q(\hat{d}_y) = d_y$. Then there is an $\hat{s}_{\hat{d}_y} \in \mathcal{O}_Y(Y)$ such that $\hat{s}_{\hat{d}_y,y'} = \pi_{y'}(\hat{d})$ for all $y' \in Y$. Then we can apply $f_\sharp(Y)$ to this element to get an element $\hat{t} \in \mathcal{O}_X(X)$.

Now we reverse the process: for all $x' \in X$ there is an open set $V_{x'}$ such that $\hat{t}(x) = \pi_x(\hat{c}_{x'})$ for some $\hat{c}_{x'} \in \mathfrak{C}$ and all $x \in V_{x'}$. Then we can define $t_{x',P'} \in \mathcal{O}_{C(X)}(g^{-1}(V_x))$ by $t_{x',P'}(x, P) = \pi_{x,P} \circ \pi_P(\hat{c}_{x'})$. On overlaps $(x, P) \in g^{-1}(V_{x'_1}) \cap g^{-1}(V_{x'_2})$, then $\pi_{x,P} \circ \pi_P(\hat{c}_{x'_1}) = \pi_{P',x} \circ \pi_x(\hat{c}_{x'_1}) = \pi_{P',x} \circ \pi_x(\hat{c}_{x'_2}) = \pi_{x,P} \circ \pi_P(\hat{c}_{x'_2})$, with the first and last equality holding as diagram (5.8.2) is commutative, and the second equality from $\hat{t}$ being a well defined element of $\mathcal{O}_X(X)$. So $t_{x',P'}$ agree on overlaps, and we can glue to an element $t \in \mathcal{O}_{C^{\mathrm{aff}}(X)}(C^{\mathrm{aff}}(X))$.

Now this t was created from s restricted to an open set $U_{y,Q}$, so there is a t for each open set $U_{y,Q}$ and we need to restrict each t to $C^{\mathrm{aff}}(f)^{-1}(U_{y,Q})$, to create a collection of sections $t_{y,Q} \in \mathcal{O}_{C^{\mathrm{aff}}(X)}(C^{\mathrm{aff}}(f)^{-1}(U_{y,Q}))$. Finally we need to show that these agree on overlaps. Say $(x, P) \in C^{\mathrm{aff}}(f)^{-1}(U_{y_1,Q_1}) \cap C^{\mathrm{aff}}(f)^{-1}(U_{y_2,Q_2})$, then $Q_1 = Q_2 = Q = (f^{\sharp}_{x,\mathrm{ex}})^{-1}(P)$ and $C^{\mathrm{aff}}(f)(x, P) = (y, Q) \in U_{y_1,Q} \cap U_{y_2,Q}$. Then $s(y, Q) = \pi_{y,Q}(d_{y_1}) = \pi_{y,Q}(d_{y_2})$. We want to show that in the stalks, we have $g^{\sharp}_{g(x,P)} \circ f^{\sharp}_{f(x)}(\hat{s}_{\hat{d}_{y_1}}) = g^{\sharp}_{g(x,P)} \circ f^{\sharp}_{f(x)}(\hat{s}_{\hat{d}_{y_2}})$. However, this follows directly from Remark 4.4.5.

Finally, we can glue the sections $t_{y,Q} \in \mathcal{O}_{C^{\mathrm{aff}}(X)}(C^{\mathrm{aff}}(f)^{-1}(U_{y,Q}))$ to create a section $t \in \mathcal{O}_{C^{\mathrm{aff}}(X)}(C^{\mathrm{aff}}(f)^{-1}(U))$, which gives the map $C^{\mathrm{aff}}(f)_\sharp(U)$. By construction, it behaves well with respect to restriction, so this defines $C^{\mathrm{aff}}(f)_\sharp$. The same construction on the monoid sheaves gives $C^{\mathrm{aff}}(f)^{\mathrm{ex}}_{\sharp}$. This gives a morphism $(C^{\mathrm{aff}}(f), C^{\mathrm{aff}}(f)^{\sharp}, C^{\mathrm{aff}}(f)^{\sharp}_{\mathrm{ex}}) : C^{\mathrm{aff}}(\boldsymbol{X}) \to C^{\mathrm{aff}}(\boldsymbol{Y})$ using (2.4.2). This construction respects compositions of morphism, so C^{aff} is a functor from affine C^∞-schemes with corners to interior C^∞-schemes with corners. □

Lemma 5.8.5. *The morphism $C^{\mathrm{aff}}(f) : C^{\mathrm{aff}}(X) \to C^{\mathrm{aff}}(Y), (x, P) \mapsto (f(x), (f^{\sharp}_{x,\mathrm{ex}})^{-1}(P))$ in the previous proof is continuous provided $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a morphism of firm affine C^∞-schemes with corners.*

Proof. As in the proof of Proposition 5.8.4, we have $\boldsymbol{f} : \boldsymbol{X} \cong \mathrm{Spec}^{\mathrm{c}}(\boldsymbol{\mathfrak{C}}) \to \boldsymbol{Y} \cong \mathrm{Spec}^{\mathrm{c}}(\boldsymbol{\mathfrak{D}})$ a morphism of firm affine C^∞-schemes with corners with $\boldsymbol{\mathfrak{C}}$ and $\boldsymbol{\mathfrak{D}}$ firm. We will use the following commutative diagram, where we have prime ideals $P \subset \mathfrak{C}_{\mathrm{ex}}, Q \subset \mathfrak{D}_{\mathrm{ex}}$ and points

$x \in X$, $y = f(x) \in Y$.

$$\begin{array}{ccccccc} \mathfrak{C}_{\mathrm{ex}}/{\sim_P} & \xleftarrow{\pi_P^{\mathrm{ex}}} & \mathfrak{C}_{\mathrm{ex}} & & \mathfrak{D}_{\mathrm{ex}} & \xrightarrow{\pi_Q^{\mathrm{ex}}} & \mathfrak{D}_{\mathrm{ex}}/{\sim_Q} \\ & {\scriptstyle \pi_{x,\mathrm{ex}}} & \downarrow & & \downarrow & {\scriptstyle \pi_{y,\mathrm{ex}}} & \\ & & \mathcal{O}_X^{\mathrm{ex}}(X) & \xleftarrow[f_\sharp^{\mathrm{ex}}(Y)]{} & \mathcal{O}_Y^{\mathrm{ex}}(Y) & & \\ & & \downarrow & & \downarrow & & \\ & & \mathfrak{C}_{x,\mathrm{ex}} \cong \mathcal{O}_{X,x}^{\mathrm{ex}} & \xleftarrow[f_{x,\mathrm{ex}}^\sharp]{} & \mathcal{O}_{Y,y}^{\mathrm{ex}} \cong \mathfrak{D}_{y,\mathrm{ex}} & & \end{array} \qquad (5.8.3)$$

The topology of $C^{\mathrm{aff}}(\boldsymbol{Y}) = \coprod_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathrm{Spec^c}(\boldsymbol{\mathfrak{C}}/{\sim_P})$ is generated by open sets $U_{\pi_Q(d)} \subset \mathrm{Spec^c}(\boldsymbol{\mathfrak{D}}/{\sim_Q})$ for $d \in \mathfrak{D}$ and prime ideal $Q \in \mathrm{Pr}_{\boldsymbol{\mathfrak{D}}}$ where

$$U_{\pi_Q(d)} = \{(y, Q') : (y, Q')(\pi_Q(d)) = y(d) \neq 0, Q' = \pi_{y,\mathrm{ex}}(Q)\}.$$

To show $C^{\mathrm{aff}}(f)$ is continuous then it is enough to show $C^{\mathrm{aff}}(f)^{-1}(U_{\pi_Q(d)})$ is open in $C^{\mathrm{aff}}(\boldsymbol{X})$.

Firstly, we can consider $f^{-1}(U_d)$ where U_d is the basic open set of $\mathrm{Spec^c}(\boldsymbol{\mathfrak{D}})$ with

$$U_d = \{y : \mathfrak{D} \to \mathbb{R} : y(d) \neq 0\} \subset \mathrm{Spec^c}(\boldsymbol{\mathfrak{D}}).$$

We can then take open set

$$V = \mathrm{Spec^c}(\boldsymbol{\pi}_P)^{-1}(f^{-1}(U_d)) \subset \mathrm{Spec^c}(\boldsymbol{\mathfrak{C}}/{\sim_P}) \subset C^{\mathrm{aff}}(X)$$

for P a prime ideal of $\mathfrak{C}_{\mathrm{ex}}$ that we need to determine. Here $\boldsymbol{\pi}_P$ is as in (5.8.3).

Consider that

$$\begin{aligned} C^{\mathrm{aff}}(f)^{-1}(U_{\pi_Q(d)}) &= \{(x, P') : (f(x), (f^\sharp_{x,\mathrm{ex}})^{-1}(P')) \in U_d\} \\ &= \{(x, P') : f(x)(d) \neq 0, (f^\sharp_{x,\mathrm{ex}})^{-1}(P') = \pi_{x,\mathrm{ex}}(Q)\}. \end{aligned}$$

So we need prime ideals $P' \subset \mathfrak{C}_{x,\mathrm{ex}} = \mathcal{O}_{X,x}^{\mathrm{ex}}$ such that $(f^\sharp_{x,\mathrm{ex}})^{-1}(P') = \pi_{x,\mathrm{ex}}(Q)$. Fix $(x', P') \in C^{\mathrm{aff}}(f)^{-1}(U_{\pi_Q(d)})$, let $C^{\mathrm{aff}}(f)(x', P') = (y', Q') \in U_{\pi_Q(d)}$, and let $P = \pi_{x',\mathrm{ex}}(P') \subset \mathfrak{C}_{\mathrm{ex}}$. We will show there is an open set $W \subset \mathrm{Spec^c}(\boldsymbol{\mathfrak{C}}/{\sim_P})$ such that

$$W \cap V \subset \mathrm{Spec^c}(\boldsymbol{\mathfrak{C}}/{\sim_P}) \subset C^{\mathrm{aff}}(X)$$

contains (x', P') and is contained in $C^{\mathrm{aff}}(f)^{-1}(U_{\pi_Q(d)})$.

To do this, we need to use that $\boldsymbol{\mathfrak{D}}$ is firm. Take generators $d_1, \ldots, d_n \in \mathfrak{D}_{\mathrm{ex}}$ that generate the sharpening $\mathfrak{D}^\sharp_{\mathrm{ex}}$. Order them so that $d_1, \ldots, d_k$ generate Q and $d_{k+1}, \ldots, d_n$ are not in Q. Then $\pi_{y',\mathrm{ex}}(d_i)$ generate Q' for $i = 1, \ldots, k$ and $\pi_{y',\mathrm{ex}}(d_i) \notin Q'$ by Lemma 5.8.2(c). As $(f^\sharp_{x',\mathrm{ex}})^{-1}(P') = \pi_{x',\mathrm{ex}}(Q) = Q'$, then there are elements $c_1, \ldots, c_n \in \mathfrak{C}_{\mathrm{ex}}$ with

$$\pi_{x',\mathrm{ex}}(c_1), \ldots, \pi_{x',\mathrm{ex}}(c_k) \in P' = \pi_{x',\mathrm{ex}}(P), \text{ and } \pi_{x',\mathrm{ex}}(c_{k+1}), \ldots, \pi_{x',\mathrm{ex}}(c_n) \notin P' = \pi_{x',\mathrm{ex}}(P),$$

which implies $c_1, \dots, c_k \in P$ and $c_{k+1}, \dots, c_n \notin P$ by Lemma 5.8.2(c), and are such that $f^\sharp_{x',\mathrm{ex}}(\pi_{y',\mathrm{ex}}(d_i)) = \pi_{x',\mathrm{ex}}(c_i)$ for each $i = 1, \dots, n$.

Now take sections $s_1, \dots, s_n \in \mathcal{O}^{\mathrm{ex}}_X(X), t_1, \dots, t_n \in \mathcal{O}^{\mathrm{ex}}_Y(Y)$ such that $(s_i)_x = \pi_{x,\mathrm{ex}}(c_i)$ for all $x \in X$ and $(t_i)_x = \pi_{y',\mathrm{ex}}(d_i)$ for all $y \in Y$. The previous paragraph implies that for each $i = 1, \dots, n$,

$$f^{\mathrm{ex}}_\sharp(Y)(t_i)_x = f^\sharp_{x,\mathrm{ex}}(\pi_{y,\mathrm{ex}}(d_i)) = \pi_{x,\mathrm{ex}}(c_i) = (s_i)_x \tag{5.8.4}$$

when $x = x' \in X$, and that this must be true for all x in an open neighbourhood W_i of x' in X. Let

$$W' = \cap_{i=1}^k W_i \subset X, \qquad W'' = \cap_{i=k+1}^n W_i \subset X,$$
$$W = \mathrm{Spec^c}(\boldsymbol{\pi}_P)^{-1}(W' \cap W'') \subset \mathrm{Spec^c}(\mathfrak{C}/{\sim_P}) \subset C^{\mathrm{aff}}(X).$$

We see that $x' \in W' \cap W''$ so that $(x', P') \in W$.

We now show that $W \cap V \subset C^{\mathrm{aff}}(f)^{-1}(U_{\pi_Q(d)})$. Take $(x'', P'') \in W \cap V$. As $(x'', P'') \in V$ then $f(x'')(d) \neq 0$ and $P'' = \pi_{x'',\mathrm{ex}}(P)$, so P'' contains $\pi_{x'',\mathrm{ex}}(c_1), \dots, \pi_{x'',\mathrm{ex}}(c_1)$ and does not contain $\pi_{x'',\mathrm{ex}}(c_{k+1}), \dots, \pi_{x'',\mathrm{ex}}(c_n)$. As $(x'', P'') \in W$ then $x'' \in W'$, which implies that

$$(f^\sharp_{x'',\mathrm{ex}})^{-1}(P'') \supset (f^\sharp_{x'',\mathrm{ex}})^{-1}(\pi_{x'',\mathrm{ex}}(c_i)) \ni \pi_{y'',\mathrm{ex}}(d_i) \in Q'$$

for all $i = 1, \dots, k$, and as these d_i generate Q, then $\pi_{y'',\mathrm{ex}}(Q) = Q' \subseteq (f^\sharp_{x'',\mathrm{ex}})^{-1}(P'')$.

Now say that $Q' \neq (f^\sharp_{x'',\mathrm{ex}})^{-1}(P'')$, so that there is $d \in \mathfrak{D}_{\mathrm{ex}}$ such that $\pi_{y'',\mathrm{ex}}(d) \in (f^\sharp_{x'',\mathrm{ex}})^{-1}(P'')$ but $\pi_{y'',\mathrm{ex}}(d) \notin Q'$, which means $d \notin Q$. Then d is a product of some of the d_j for $j = k+1, \dots, n$, and as $(f^\sharp_{x'',\mathrm{ex}})^{-1}(P'')$ is a prime ideal in $\mathfrak{D}_{x,\mathrm{ex}}$ then $\pi^{-1}_{y,\mathrm{ex}}((f^\sharp_{x'',\mathrm{ex}})^{-1}(P''))$ is prime in $\mathfrak{D}_{\mathrm{ex}}$, so a least one of the d_j is such that $\pi_{y'',\mathrm{ex}}(d_j) \in (f^\sharp_{x'',\mathrm{ex}})^{-1}(P'')$ but $\pi_{y'',\mathrm{ex}}(d_j) \notin Q'$ for some $j = k+1, \dots, n$. However, as $(x'', P'') \in W$ then $x'' \in W''$, and we know that $f^\sharp_{x'',\mathrm{ex}}(\pi_{y'',\mathrm{ex}}(d_j)) = \pi_{x'',\mathrm{ex}}(c_j) \notin P''$ by (5.8.4), which implies a contradiction to the existence of such a d.

Hence $\pi_{y'',\mathrm{ex}}(Q) = Q' = (f^\sharp_{x'',\mathrm{ex}})^{-1}(P'')$ for all $(x'', P'') \in W \cap V$, so $W \cap V \subset C^{\mathrm{aff}}(f)^{-1}(U_{\pi_Q(d)})$ as required. So $C^{\mathrm{aff}}(f)^{-1}(U_{\pi_Q(d)})$ is open, and $C^{\mathrm{aff}}(f)$ is continuous. □

Corollary 5.8.6. *The image of* $C^{\mathrm{aff}} : \mathbf{AC^\infty Sch^c_{fi}} \to \mathbf{C^\infty Sch^c_{fi,in}}$ *is an affine firm* C^∞*-scheme with corners, and we can consider it as a functor* $C^{\mathrm{aff}} : \mathbf{AC^\infty Sch^c_{fi}} \to \mathbf{AC^\infty Sch^c_{fi,in}}$.

Proof. For any firm C^∞-ring with corners $\mathfrak{C}$ the prime ideals of $\mathfrak{C}_{\mathrm{ex}}$ are finitely generated and there are finitely many of them. Then $\mathrm{Pr}_{\mathfrak{C}}$ has finite cardinality, so this is a consequence of Proposition 5.4.15 and Proposition 5.8.4. □

Remark 5.8.7. The only place the condition firm is used in the proof of Proposition 5.8.4 and Lemma 5.8.5 is in showing the morphism $C^{\rm aff}(f)$ on topological spaces is continuous. The intuition for the lemma is similar to Proposition 5.4.10, where we showed that locally a morphism between affine C^∞-schemes with corners with firm target comes from a morphism of the C^∞-rings with corners. By Remark 4.4.5, a morphism $\boldsymbol{\phi}:\boldsymbol{\mathfrak{C}}\to\boldsymbol{\mathfrak{D}}$ of C^∞-rings with corners gives morphisms $\boldsymbol{\mathfrak{C}}/{\sim_P}\to\boldsymbol{\mathfrak{D}}/{\sim_Q}$ for all prime ideals $P\in\mathrm{Pr}_{\boldsymbol{\mathfrak{C}}}$ and $Q\in\mathrm{Pr}_{\boldsymbol{\mathfrak{D}}}$ such that $P=\phi_{\rm ex}^{-1}(Q)$, here $\pi_P(c)\mapsto\pi_Q(\phi(c)),\pi_P^{\rm ex}(c')\mapsto\pi_Q^{\rm ex}(\phi_{\rm ex}(c'))$ for $c\in\mathfrak{C},c'\in\mathfrak{C}_{\rm ex}$. Then this gives a morphism

$$\prod_{P\in\mathrm{Pr}_{\boldsymbol{\mathfrak{C}}}}\boldsymbol{\mathfrak{C}}/{\sim_P}\to\prod_{Q\in\mathrm{Pr}_{\boldsymbol{\mathfrak{D}}}}\boldsymbol{\mathfrak{D}}/{\sim_D},$$
$$(\pi_P(c_P))_{P\in\mathrm{Pr}_{\boldsymbol{\mathfrak{C}}}}\mapsto(\pi_Q(\phi(c_P)))_{Q\in\mathrm{Pr}_{\boldsymbol{\mathfrak{D}}},P=\phi_{\rm ex}^{-1}(Q)}$$
$$(\pi_P^{\rm ex}(c'_P))_{P\in\mathrm{Pr}_{\boldsymbol{\mathfrak{C}}}}\mapsto(\pi_Q^{\rm ex}(\phi_{\rm ex}(c'_P)))_{Q\in\mathrm{Pr}_{\boldsymbol{\mathfrak{D}}},P=\phi_{\rm ex}^{-1}(Q)},$$

where $c_P\in\mathfrak{C}$ and $c'_P\in\mathfrak{C}_{\rm ex}$ for each $P\in\mathrm{Pr}_{\boldsymbol{\mathfrak{C}}}$. Taking $\mathrm{Spec}^{\rm c}$ of this morphism gives the morphism $C^{\rm aff}(f)=(x,P)\mapsto(f(x),(f_{x,\rm ex}^\sharp)^{-1}(P))$ on topological spaces.

In the following example we show that the construction for $C^{\rm aff}(f)$ can fail to be continuous if $\boldsymbol{\mathfrak{C}}$ is not firm. Specifically, we have $\boldsymbol{X}=\mathrm{Spec}^{\rm c}(\boldsymbol{\mathfrak{C}})$ with $\boldsymbol{\mathfrak{C}}$ firm but we have a representation $\boldsymbol{\phi}:\boldsymbol{X}\cong\mathrm{Spec}^{\rm c}(\boldsymbol{\mathfrak{D}})$ with $\boldsymbol{\mathfrak{D}}$ not firm, and we show the C^∞-scheme with corners $\coprod_{P\in\mathrm{Pr}_{\boldsymbol{\mathfrak{D}}}}\mathrm{Spec}^{\rm c}(\boldsymbol{\mathfrak{D}}/{\sim_P})$ is not in general isomorphic to $C^{\rm aff}(\boldsymbol{X})=\coprod_{P\in\mathrm{Pr}_{\boldsymbol{\mathfrak{C}}}}\mathrm{Spec}^{\rm c}(\boldsymbol{\mathfrak{C}}/{\sim_P})$, and $C^{\rm aff}(\phi)$ is not continuous.

Example 5.8.8. Consider the set $X=\{(0,1/n):n\in\mathbb{Z}\setminus\{0\}\}\cup\{(0,0)\}\subset\mathbb{R}^2$ with the induced topology from $\mathbb{R}^2$. While this is not a manifold with corners, we can still consider the smooth maps from it to $\mathbb{R}$ and $[0,\infty)$, which make $\boldsymbol{X}$ into a C^∞-scheme with corners that is isomorphic to both $\mathrm{Spec}^{\rm c}(C^\infty(X),\mathrm{Ex}(X))$ and $\mathrm{Spec}^{\rm c}(C^\infty(X),\mathrm{In}(X)\amalg\{0\})$. Here C^∞, Ex, In are as defined in Definition 4.1.1. Let $\boldsymbol{\phi}:\mathrm{Spec}^{\rm c}(C^\infty(X),\mathrm{In}(X)\amalg\{0\})\to\mathrm{Spec}^{\rm c}(C^\infty(X),\mathrm{Ex}(X)$ be the isomorphism.

Now $(C^\infty(X),\mathrm{In}(X)\amalg\{0\})=\boldsymbol{\mathfrak{C}}$ is a firm C^∞-ring with corners, where the only non-invertible element in $\mathfrak{C}_{\rm ex}$ is 0. We can use this to determine the corners of $\boldsymbol{X}$ as in Proposition 5.8.4. As the sharpening is $\{0\}$ the only prime ideal is $\{0\}$ and we have

$$C^{\rm aff}(\boldsymbol{X})=\coprod_{P\in\mathrm{Pr}_{\boldsymbol{\mathfrak{C}}}}\mathrm{Spec}^{\rm c}(\boldsymbol{\mathfrak{C}}/{\sim_P})=\mathrm{Spec}^{\rm c}(\boldsymbol{\mathfrak{C}}/{\sim_{\{0\}}})=\mathrm{Spec}^{\rm c}(\boldsymbol{\mathfrak{C}})=\boldsymbol{X}.$$

This is the same as $C^{\rm loc}(\boldsymbol{X})\in\mathbf{LC^\infty RS^c}$. As $C^{\rm loc}$ is a right adjoint it respects limits, so we can also calculate the corners as the fibre product of $f:\mathbb{R}\to\mathbb{R}^2,x\mapsto(0,x)$ and the inclusion $g:Y\to\mathbb{R}^2$ where $Y=\{(e^{-y^2}\sin(\pi/y),y):y\in\mathbb{R}\}\subset\mathbb{R}^2\}$.

However, $(C^\infty(X), \mathrm{Ex}(X)) = \mathfrak{D}$ is not firm. An example of generators of $\mathrm{Ex}(X)$ are the smooth maps $f_n : X \to [0,\infty)$ for $n \in \mathbb{Z} \setminus \{0\}$ with $f_n(0, 1/n) = 0$, and equal to 1 elsewhere, along with $g_m : X \to [0,\infty)$ for $m \in \mathbb{Z} \setminus \{0\}$ with $g_m(0, 1/n) = 0$ for all $n \geqslant m$, $g_m(0,0) = 0$, and g_m equal to 1 elsewhere. Of this collection of generators, we see that while we can remove many of the g_m, we must have at least infinitely many g_m.

The prime ideals of $\mathrm{Ex}(X)$ are

$$P_n = \langle f_n \rangle = \{f : X \to [0,\infty) \text{ smooth } : f(0, 1/n) = 0\} \text{ for } n \in \mathbb{Z} \setminus \{0\},$$
$$P_0 = \langle g_m \rangle_{m \in \mathbb{Z} \setminus \{0\}} = \{f : X \to [0,\infty) \text{ smooth } : f(0,0) = 0\}.$$

Quotienting $(C^\infty(X), \mathrm{Ex}(X))$ by the P_n for $n \in \mathbb{Z}$ gives $(\mathbb{R}, [0,\infty))$ where $\mathrm{Spec^c}(\mathbb{R}, [0,\infty)) \cong (0, 1/n)$ or $(0,0)$ as expected. Then $\coprod_{P \in \mathrm{Pr}_{\mathfrak{D}}} \mathrm{Spec^c}(\mathfrak{D}/\!\sim_P)$ is isomorphic to $\boldsymbol{X}$ with the discrete topology, not $C^{\mathrm{aff}}(\boldsymbol{X}) = \boldsymbol{X}$ with the induced topology from $\mathbb{R}^2$. Also $C^{\mathrm{aff}}(\phi) : X \to X$ is the identity function of sets but is not continuous on the topological spaces.

Theorem 5.8.9. *The functor* $C^{\mathrm{aff}} : \mathbf{AC^\infty Sch^c_{fi}} \to \mathbf{AC^\infty Sch^c_{fi,in}}$ *is right adjoint to the inclusion* $i^{\mathrm{aff}} : \mathbf{AC^\infty Sch^c_{fi,in}} \to \mathbf{AC^\infty Sch^c_{fi}}$*. The functor* C^{aff} *can be used to define a functor* $C : \mathbf{C^\infty Sch^c_{fi}} \to \mathbf{C^\infty Sch^c_{fi,in}}$ *with* $C|_{\mathbf{AC^\infty Sch^c_{fi}}} = C^{\mathrm{aff}}$ *such that* C *is right adjoint to the inclusion functor* $i : \mathbf{C^\infty Sch^c_{fi,in}} \to \mathbf{C^\infty Sch^c_{fi}}$*. Hence* i, i^{aff} *respect colimits and* C, C^{aff} *respect limits.*

Remark 5.8.10. Before we prove Theorem 5.8.9, let us remark that the maps $F^{(\mathrm{in})} : \mathfrak{C} \mapsto \prod^{(\mathrm{in})}_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathfrak{C}/\!\sim_P$ are functors from C^∞-rings with corners to (interior) C^∞-rings with corners. Note that when we take the interior product, we still take the quotient as non-interior C^∞-rings with corners, even if $\mathfrak{C}$ is interior.

Here, if $\boldsymbol{\phi} : \mathfrak{C} \to \mathfrak{D}$ is a morphism of C^∞-rings with corners, then by Remark 4.4.5 there is a morphism $\boldsymbol{\phi}_Q : \mathfrak{C}/\!\sim_{\phi_{\mathrm{ex}}^{-1}(Q)} \to \mathfrak{D}/\!\sim_Q$ for each prime $Q \in \mathrm{Pr}_{\mathfrak{D}}$. The definition of quotient gives non-interior morphisms $\boldsymbol{\pi}_P : \mathfrak{C} \to \mathfrak{C}/\!\sim_P$ for each prime P. If $\phi_{\mathrm{ex},Q}(c') = 0$ then for any $\hat{c}' \in \pi^{-1}_{\mathrm{ex},\phi_{\mathrm{ex}}^{-1}(Q)}(c') \subset \mathfrak{C}_{\mathrm{ex}}$, then $\phi_{\mathrm{ex}}(\hat{c}') \in Q$, which implies $\hat{c}' \in P$ so $c' = 0$. Hence each $\boldsymbol{\phi}_Q$ is interior.

Then the functors $F^{(\mathrm{in})}$ take $\boldsymbol{\phi}$ to $F^{(\mathrm{in})}(\boldsymbol{\phi}) : \prod^{(\mathrm{in})}_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathfrak{C}/\!\sim_P \to \prod^{(\mathrm{in})}_{Q \in \mathrm{Pr}_{\mathfrak{D}}} \mathfrak{D}/\!\sim_Q$. These take an element $(\boldsymbol{c}_P)_{P \in \mathrm{Pr}_{\mathfrak{C}}} \in \prod^{(\mathrm{in})}_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathfrak{C}/\!\sim_P$ to $(\boldsymbol{d}_Q)_{Q \in \mathrm{Pr}_{\mathfrak{D}}}$, where $\boldsymbol{d}_Q = \boldsymbol{\phi}_Q(\boldsymbol{c}_{\phi_{\mathrm{ex}}^{-1}(Q)})$, which is an interior morphism in both cases. In the non-interior case, the morphisms $\boldsymbol{\pi}_P$ can be amalgamated to give a projection $\boldsymbol{\pi} : \mathfrak{C} \to \prod_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathfrak{C}/\!\sim_P$ by the universal property of product, and then $F(\boldsymbol{\phi})$ commutes with these projections.

However, in general there is no interior morphism $\boldsymbol{\pi} : \mathfrak{C} \to \prod^{\mathrm{in}}_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathfrak{C}/\!\sim_P$. For example, if $\mathfrak{C} = (\{0\}, \{0\})$ then its image under F^{in} is $(\{0\}, \{0,1\})$ but there is no morphism from $(\{0\},\{0\})$ to $(\{0\},\{0,1\})$. This is important as it means $F^{\rm in}$ is not a left adjoint to the forgetful functor $i : \mathbf{C^\infty Rings^c_{in}} \to \mathbf{C^\infty Rings^c}$. Specifically, it means there is no unit for the adjunction, so this does not contradict Theorem 4.3.7(b) which tells us a left adjoint cannot exist. This discussion is the same if we restrict to firm C^∞-rings with corners.

Proof of Theorem 5.8.9. Let $\boldsymbol{X} = \mathrm{Spec^c}(\mathfrak{C})$ for $\mathfrak{C}$ a firm C^∞-ring with corners and take $C^{\rm aff}(\boldsymbol{X}) = \coprod_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathrm{Spec^c}(\mathfrak{C}/\sim_P)$. We have morphisms $\boldsymbol{\pi_X} : C^{\rm aff}(\boldsymbol{X}) \to \boldsymbol{X}$, and $\boldsymbol{\iota_X} : \boldsymbol{X} \to C^{\rm aff}(\boldsymbol{X})$ for interior $\boldsymbol{X}$. Here we take $\boldsymbol{g_P} = \mathrm{Spec^c}(\boldsymbol{\pi_P}) : \mathrm{Spec^c}(\mathfrak{C}/\sim_P) \to \mathrm{Spec^c}(\mathfrak{C})$ as in Proposition 5.8.4, and then amalgamate these maps using the universal property of coproduct to get $\boldsymbol{\pi_X} = \boldsymbol{g} : C^{\rm aff}(\boldsymbol{X}) = \coprod_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathrm{Spec^c}(\mathfrak{C}/\sim_P) \to \boldsymbol{X}$. If $\mathfrak{C}$ is interior, then $\langle 0 \rangle$ is a prime ideal of $\mathfrak{C}_{\rm ex}$ and $\mathrm{Spec^c}(\mathfrak{C}/\sim_{\langle 0 \rangle}) \cong \mathrm{Spec^c}(\mathfrak{C})$, so there is an inclusion $\boldsymbol{\iota_X} : \boldsymbol{X} \to C^{\rm aff}(\boldsymbol{X}) = \coprod_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathrm{Spec^c}(\mathfrak{C}/\sim_P)$.

In fact, as $C^{\rm aff}(\boldsymbol{X}) = \coprod_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathrm{Spec^c}(\mathfrak{C}/\sim_P) \cong \mathrm{Spec^c}(\prod_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathfrak{C}/\sim_P)$, then we can write $\boldsymbol{\pi_X} = \mathrm{Spec^c}(\mathfrak{C} \to \prod_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathfrak{C}/\sim_P)$, and $\boldsymbol{\iota_X} = \mathrm{Spec^c}(\prod_{P \in \mathrm{Pr}_{\mathfrak{C}}} \mathfrak{C}/\sim_P \to \mathfrak{C})$ where $\mathfrak{C}$ and $\boldsymbol{X}$ are interior for the latter. Showing these are natural transformations, commute with morphisms $\boldsymbol{f} : \boldsymbol{X} \to \mathrm{Spec^c}(\mathfrak{D}) \cong \boldsymbol{Y}$ and $C^{\rm aff}(\boldsymbol{f}) : C^{\rm aff}(\boldsymbol{X}) \to C^{\rm aff}(\boldsymbol{Y})$, and that these form an adjoint pair follows from the definitions, where $\boldsymbol{\pi_X}$ is the unit of the adjunction and $\boldsymbol{\iota_X}$ is the counit.

Now, take $\boldsymbol{X} \in \mathbf{C^\infty Sch^c_{fi}}$, so that there is a cover of $\boldsymbol{X}$ by open affines $\{U_i\}_{i \in I}$, with $\boldsymbol{X}|_{U_i} \cong \mathrm{Spec^c}(\mathfrak{C}_i)$ with $\mathfrak{C}_i$ firm C^∞-schemes with corners. Then $C^{\rm aff}(\boldsymbol{X}|_{U_i})$ makes sense for each i. We would like to use Remark 5.4.13 to glue these together to define $C(\boldsymbol{X})$, so that $C(\boldsymbol{X})$ is the colimit of all $C^{\rm aff}(\boldsymbol{X}|_{U_i})$ for $\boldsymbol{U}_i = \boldsymbol{X}|_{U_i}$ affine in $\boldsymbol{X}$. We will show that

$$C^{\rm aff}(\boldsymbol{X}|_{U_i})|_{\pi_i^{-1}(U_i \cap U_j)} \cong C^{\rm aff}(\boldsymbol{X}|_{U_j})|_{\pi_j^{-1}(U_i \cap U_j)}$$

where $\boldsymbol{\pi}_i : C(\boldsymbol{X}|_{U_i}) \to \boldsymbol{X}$ is the unit of the adjunction defined above, which will imply that the properties in Remark 5.4.13 hold. To do this, we need only show that the isomorphisms $\mathrm{Spec^c}(\mathfrak{C}_i)|_V \cong \mathrm{Spec^c}(\mathfrak{C}_j)|_V$ for open $V \subset U_i \cap U_j \subset X$ give isomorphisms $C^{\rm aff}(\boldsymbol{X}|_{U_i})|_{\pi_i^{-1}(V)} \cong C^{\rm aff}(\boldsymbol{X}|_{U_j})|_{\pi_j^{-1}(V)}$.

This is straightforward. In both cases the topological spaces are isomorphic to

$$\{(x,P) : x \in V, P \subset \mathcal{O}^{\rm ex}_{X,x} \text{ prime ideal}\},$$

where we use that $\mathcal{O}^{\rm ex}_{X,x} \cong \mathfrak{C}_{i,{\rm ex},x} \cong \mathfrak{C}_{j,{\rm ex},x}$. This creates a morphism of the topological spaces. If $s \in \mathcal{O}_{C^{\rm aff}(\boldsymbol{U_i})}(\pi_i(V))$ then $s_x = \pi_{x,P} \circ \pi_P(\hat{c})$ for some $\hat{c} \in \mathfrak{C}$. Then take $\hat{s} \in \mathcal{O}_{U_i}(V)$ defined by $\hat{s}_x = \pi_x(c)$ for all c. Apply the isomorphism $\mathrm{Spec^c}(\mathfrak{C}_i)|_V \cong \mathrm{Spec^c}(\mathfrak{C}_j)|_V$

to get an element of $\hat{t} \in \mathcal{O}_{U_j}(V)$. Then apply π_j to this element. This gives a morphism of the sheaves of C^∞-rings, and a similar process gives a morphism of the sheaves of monoids.

To check these morphisms of sheaves are well defined, respect restrictions and do not depend on choices follows similarly to the proof of the functor in Lemma 5.4.16, and we see that we get a morphism $C^{\text{aff}}(\boldsymbol{X}|_{U_i})|_{\pi_i^{-1}(V)} \to C^{\text{aff}}(\boldsymbol{X}|_{U_j})|_{\pi_i^{-1}(V)}$ that commutes with each $\boldsymbol{\pi}_i, \boldsymbol{\pi}_j$. As $\mathfrak{C}_{i,\text{ex},x} \cong \mathfrak{C}_{j,\text{ex},x}$, then $(\mathfrak{C}_i/\sim_{P_i})_{(x,P_i)} \cong (\mathfrak{C}_i/\sim_{P_j})_{(x,P_j)}$, so this is an isomorphism of sheaves of C^∞-rings with corners. Hence these $C^{\text{aff}}(\boldsymbol{U_i})$ are isomorphic on overlaps.

This means we can glue these $C^{\text{aff}}(\boldsymbol{U_i})$ to form $C(\boldsymbol{X})$ using Remark 5.4.13. We now show this is a functor. Take a morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of firm C^∞-schemes with corners. Then take an open cover of $\boldsymbol{Y}$ by affine firm open sets $\{V_i\}_{i\in I}$. Then take an affine firm open cover of $f^{-1}(V_i)$ in $\boldsymbol{X}$, say $\{U_{i,j}\}_{j\in J}$, so we have morphisms through $\boldsymbol{f}$ from each $\boldsymbol{U}_{i,j}$ to $\boldsymbol{V}_i$ and then inclusions to $\boldsymbol{Y}$. The functor $C^{\text{aff}} : \mathbf{AC^\infty Sch^c_{fi}} \to \mathbf{C^\infty Sch^c_{fi,in}}$ gives morphisms $C^{\text{aff}}(\boldsymbol{U}_{i,j}) \to C^{\text{aff}}(\boldsymbol{V}_i)$ and the definition of colimit gives morphisms $C^{\text{aff}}(\boldsymbol{V}_i) \to C(\boldsymbol{Y})$. The universal property of $C(\boldsymbol{X})$ as a colimit allows us to amalgamate these morphisms to a unique morphism $C(\boldsymbol{f}) : C(\boldsymbol{X}) \to C(\boldsymbol{Y})$ that commutes with all the other morphisms. This defines $C : \mathbf{C^\infty Sch^c_{fi}} \to \mathbf{C^\infty Sch^c_{fi,in}}$ on morphisms.

To show $C : \mathbf{C^\infty Sch^c_{fi}} \to \mathbf{C^\infty Sch^c_{fi,in}}$ is a right adjoint to the inclusion $i : \mathbf{C^\infty Sch^c_{fi,in}} \to \mathbf{C^\infty Sch^c_{fi}}$, we use the $\boldsymbol{\pi_X}$ and $\boldsymbol{\iota_X}$ defined above for affine firm C^∞-schemes with corners to give morphisms $C^{\text{aff}}(\boldsymbol{U}) \to \boldsymbol{X}|_U$ and $\boldsymbol{X}|_U \to C(\boldsymbol{X})|_{C^{\text{aff}}(\boldsymbol{U})}$ on affine firm neighbourhoods U. We again amalgamate these morphisms to $\boldsymbol{\pi_X}$ and $\boldsymbol{\iota_X}$ defined on all of $C(\boldsymbol{X})$ and $\boldsymbol{X}$, using that any firm C^∞-scheme with corners is the colimit of any (firm) affine cover and the definition of $C(\boldsymbol{X})$ as a colimit of affine (firm) covers. That these are natural transformations and the unit and counit of the adjunction follows from the definitions. □

Remark 5.8.11. As in Remark 5.7.4 we can actually define the corners $C(\boldsymbol{X})$ of a firm C^∞-scheme with corners $\boldsymbol{X}$ to be the unique (up to isomorphism) firm C^∞-scheme with corners with morphism $\boldsymbol{\pi_X} : C(\boldsymbol{X}) \to \boldsymbol{X}$ that satisifies the following universal property: for all interior firm C^∞-schemes with corners $\boldsymbol{Y}$ and morphisms $\boldsymbol{f} : \boldsymbol{Y} \to \boldsymbol{X}$, there is a unique interior morphism $\hat{\boldsymbol{f}} : \boldsymbol{Y} \to C(\boldsymbol{X})$ such that $\boldsymbol{\pi_X} \circ \hat{\boldsymbol{f}} = \boldsymbol{f}$. For any morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{C^\infty Sch^c_{fi}}$, we can then define $C(\boldsymbol{f}) = \widehat{\boldsymbol{f} \circ \pi_{\boldsymbol{X}}}$.

For all interior C^∞-schemes with corners $\boldsymbol{X}$, then the identity morphism $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ gives a unique interior morphism $\hat{\mathbf{id}}_{\boldsymbol{X}} : \boldsymbol{X} \to C(\boldsymbol{X})$, such that $\boldsymbol{\pi_X} \circ \hat{\mathbf{id}}_{\boldsymbol{X}} = \mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$. We define $\boldsymbol{\iota_X} = \hat{\mathbf{id}}_{\boldsymbol{X}}$, and by uniqueness we see $\hat{\boldsymbol{f}} = C(\boldsymbol{f}) \circ \boldsymbol{\iota_X}$ for all morphisms $\boldsymbol{f} : \boldsymbol{Y} \to \boldsymbol{X}$ for interior $\boldsymbol{Y}$. This is consistent with our previous definition.

Example 5.8.12. If X is a manifold without boundary, then there is an isomorphism

$C^{\mathrm{aff}}(F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X)) \cong F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X)$ with $\boldsymbol{\pi}_X = \mathbf{id}$. This occurs because any morphism $\boldsymbol{Y} \to F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X)$, where $\boldsymbol{Y}$ is an interior C^∞-scheme with corners, is interior, as the monoid elements of the stalks of $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X)$ are either invertible or zero.

In general, if X is a manifold with corners (possible with mixed dimension), then the functors C and $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}$ commute. That is, there is an isomorphism $C(F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X)) \cong F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(C(X))$, with $\boldsymbol{\pi_X} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(i_X)$ and $C : \check{\mathbf{Man}}^{\mathbf{c}} \to \check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}}$ from Definition 3.3.6. As $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}(X) = \boldsymbol{X}$ is interior, then we have $\boldsymbol{\iota_X} : \boldsymbol{X} \hookrightarrow C(\boldsymbol{X})$, which includes $\boldsymbol{X}$ in $C(\boldsymbol{X})$ by $x \mapsto (x, (0))$. Here (0) is the zero prime ideal in each stalk. The sheaves of C^∞-rings with corners are isomorphisms over the image $\boldsymbol{\iota_X}(\boldsymbol{X})$.

The corners functor of manifolds with g-corners also behaves like this. This implies the following corollary, which appears to be new and tells us that the corners of manifolds with (g-)corners satisfy a universal property.

Corollary 5.8.13. *The corners functor of manifolds with (g-)corners with mixed dimension* $C : \check{\mathbf{Man}}^{\mathbf{c}} \to \check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}}$ *(*$C : \check{\mathbf{Man}}^{\mathbf{gc}} \to \check{\mathbf{Man}}^{\mathbf{gc}}_{\mathbf{in}}$*) in Definition 3.3.2 is a right adjoint to the inclusion* $\check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}} \to \check{\mathbf{Man}}^{\mathbf{c}}$ *(*$\check{\mathbf{Man}}^{\mathbf{gc}}_{\mathbf{in}} \to \check{\mathbf{Man}}^{\mathbf{gc}}$*).*

Proposition 5.8.14. *For* $\boldsymbol{X} \in \mathbf{C^\infty Sch^c_{fi}}$ *there is a morphism* $C(\boldsymbol{X}) \to C^{\mathrm{loc}}(\boldsymbol{X})$ *in* $\mathbf{LC^\infty RS^c_{in}}$ *commuting with the projections to* $\boldsymbol{X}$*. This is an isomorphism.*

Proof. As $C(\boldsymbol{X})$ is interior then we have the following commutative diagram in $\mathbf{LC^\infty RS^c}$.

$$\begin{array}{ccc} C(\boldsymbol{X}) & \xrightarrow{\boldsymbol{\pi_X}} & \boldsymbol{X} \\ \boldsymbol{\pi}_{C(\boldsymbol{X})} \uparrow \downarrow \boldsymbol{\iota}_{C(\boldsymbol{X})} & & \uparrow \boldsymbol{\pi_X} \\ C^{\mathrm{loc}}(C(\boldsymbol{X})) & \xrightarrow{C^{\mathrm{loc}}(\boldsymbol{\pi_X})} & C^{\mathrm{loc}}(\boldsymbol{X}) \end{array} \tag{5.8.5}$$

Then the composition $C^{\mathrm{loc}}(\boldsymbol{\pi_X}) \circ \iota_{C(\boldsymbol{X})} : C(\boldsymbol{X}) \to C^{\mathrm{loc}}(\boldsymbol{X})$ is the required interior morphism. From the definition of C and C^{aff} as described in Proposition 5.8.4 and Theorem 5.8.9, then elements of $C(\boldsymbol{X})$ can be described as pairs $(x, P) \in C(\boldsymbol{X})$ where P is a prime ideal of $\mathcal{O}_{X,x}$. This composition of morphisms then sends (x, P) first to $((x, P), (0))$ where (0) is the zero ideal of $\mathcal{O}_{X,x}/{\sim_P}$, and then to $(\pi(x, P), \pi^{\sharp,-1}_{X,\pi_X((x,P)),\mathrm{ex}}((0))$. Here $\pi(x, P) = x$ and $\pi^{\sharp,-1}_{X,\pi_X((x,P)),\mathrm{ex}}$ is the quotient morphism $\mathcal{O}^{\mathrm{ex}}_{X,x} \to \mathcal{O}^{\mathrm{ex}}_{X,x}/{\sim_P}$, so $\pi^{\sharp,-1}_{X,\pi_X((x,P)),\mathrm{ex}}((0)) = P$, and the composition is an isomorphism of underlying sets of the topological spaces.

Now say $\boldsymbol{X} \cong \mathrm{Spec^c}(\mathfrak{C}) \in \mathbf{AC^\infty Sch^c_{fi}}$ with $\mathfrak{C}$ firm. Then $\mathfrak{C}^\sharp_{\mathrm{ex}}$ is generated by $c'_1, \ldots, c'_n$. The prime ideals $P \in \mathrm{Pr}_{\mathfrak{C}}$ are generated by subsets of this set of generators. We want to check $C^{\mathrm{loc}}(\boldsymbol{\pi_X}) \circ \iota_{C(\boldsymbol{X})} : C(\boldsymbol{X}) \to C^{\mathrm{loc}}(\boldsymbol{X})$ is an isomorphism. We already know that it is a continuous morphism on the topological spaces and we can check that as follows.

First we have that $\pi^{-1}(U)$ in $C^{\mathrm{loc}}(\boldsymbol{X})$ is isomorphic to $\pi^{-1}(U)$ in $C^{\mathrm{aff}}(\boldsymbol{X})$. We will show that for any $(x, P) \in U_{s'}$ there is an open set $V \in C^{\mathrm{aff}}(\boldsymbol{X})$ with $V \subset U_{s'}$ and $(x, P) \in V$. For $(x', P') \in U_{s'}$, then $s'_x = \pi_{x,\mathrm{ex}}(c')$ for some $c' \in \mathfrak{C}_{\mathrm{ex}}$ for all $x \in W \subset U$ with $x' \in W$, with $\pi_{x',\mathrm{ex}}(c') \notin P'$. Then $\pi_{x,\mathrm{ex}}(c') \notin \pi_{x,\mathrm{ex}}(\pi_{x,\mathrm{ex}}^{-1}(P'))$ for all $x \in X$. So

$$(x', P') \in \pi^{-1}(V) \cap \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}/\sim_{\pi_{x,\mathrm{ex}}^{-1}(P')}) \subset U_{s'}.$$

Similarly, for $(x', P') \in \hat{U}_{s'}$, then

$$(x', P') \in \pi^{-1}(V) \cap \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}/\sim_{\pi_{x,\mathrm{ex}}^{-1}(P')}) \subset \hat{U}_{s'}.$$

As the underlying set of $\mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}/\sim_{\pi_{x,\mathrm{ex}}^{-1}(P')})$ is open in $C^{\mathrm{aff}}(X)$, then these sets are open in $C^{\mathrm{aff}}(X)$. So we have a continuous map of topological spaces $C^{\mathrm{aff}}(\boldsymbol{X}) \to C^{\mathrm{loc}}(\boldsymbol{X})$, $(x, P) \mapsto (x, P)$. To show this a homeomorphism, we will use the firm assumption to check that the inverse map is continuous.

Let U be an open set in $C(\boldsymbol{X})$, so it is the union of open sets in $\mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}/\sim_P)$ for some $P \in \mathrm{Pr}_{\mathfrak{C}}$. Pick a particular $(x', P') \in U$. Then there is a basic open set

$$U_{\pi_P(c)} = \{(x, P) \in \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}/\sim_{\pi_{x,\mathrm{ex}}^{-1}(P')}) : (x, P)(\pi_P(c)) \neq 0\},$$

in $\mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}/\sim_P)$ with $U_{\pi_P(c)} \subset U$ and $(x', P') \in U$. Here, we have projection $\pi_P : \mathfrak{C} \to\in \mathfrak{C}/\sim_{\pi_{x,\mathrm{ex}}^{-1}(P')}$ and $c \in \mathfrak{C}$, so $(x, P)(\pi_P(c)) = x(c)$, and any $(x, P) \in U_{\pi_P(c)}$ must have $P = P'$. We must then have a basic open set

$$U_c = \{x \in X : x(c) \neq 0\},$$

in X.

As P' is a prime ideal in $\mathfrak{C}_{x,\mathrm{ex}}$ it is finitely generated by subsets of the images of the $c'_1, \ldots, c'_n$ under $\pi_{x,\mathrm{ex}} : \mathfrak{C}_{\mathrm{ex}} \to \mathfrak{C}_{x,\mathrm{ex}}$. Say the subset is $\{c'_1, \ldots, c'_k\}$. Then in $C^{\mathrm{loc}}(\boldsymbol{X})$ there is an open set

$$V = \pi^{-1}(U_c) \cap \Big(\bigcap_{i=1}^{k} (\hat{U}_c)_{s'_i}\Big) \cap \Big(\bigcap_{i=k+1}^{n} (U_c)_{s'_i}\Big),$$

where $s'_i \in \mathcal{O}_X(U_{\pi_P(c)})$ with $s'_{i,x} = \pi_{x,\mathrm{ex}}(c'_i)$ for each $i \in 1, \ldots, n$. Here, $(\hat{U}_c)_{s'_i}$ and $(U_c)_{s'_i}$ are the open sets corresponding to $U_c \subset X$ from Definition 5.7.1. As $\mathfrak{C}$ is firm, each prime ideal is finitely generated and the set V is an intersection of only finitely many open sets, so it is open. Also V non-empty as it contains (x', P'). If $(x, P) \in V$ then P must contain $\pi_{x,\mathrm{ex}}(c'_i)$ for $i = 1, \ldots, k$ and must not contain $\pi_{x,\mathrm{ex}}(c'_i)$ for $i = k + 1, \ldots, n$, so P must be equal to P'. We must also have $x \in U_c$. This implies any $(x, P) \in V$ is also in U. As we can do this for all point $(x', P') \in U$ then the image of U under $C^{\mathrm{aff}}(\boldsymbol{X}) \to C^{\mathrm{loc}}(\boldsymbol{X})$,

$(x, P) \mapsto (x, P)$ is open so the inverse map is continuous and we have an isomorphism of topological spaces.

Finally we check the morphisms of sheaves. However, we already know that the stalks of the sheaves of $C^{\mathrm{aff}}(\boldsymbol{X})$ and $C^{\mathrm{loc}}(\boldsymbol{X})$ at the point (x, P) are both isomorphic to $\mathcal{O}^{\mathrm{ex}}_{X,x}/\sim_P$ and it is straightforward to check that the morphisms of the sheaves respect this isomorphism, so are isomorphisms themselves and $C^{\mathrm{loc}}(\boldsymbol{\pi_X}) \circ \iota_{C(\boldsymbol{X})} : C(\boldsymbol{X}) \to C^{\mathrm{loc}}(\boldsymbol{X})$ is an isomorphism for firm affine C^∞-schemes with corners.

For $\boldsymbol{X} \in \mathbf{C^\infty Sch^c_{fi}}$, this result follows as $C(\boldsymbol{X})|_{\pi^{-1}(U)}$ is isomorphic to $C^{\mathrm{aff}}(\boldsymbol{X}|_U)$ on affine open subsets $\boldsymbol{U}$ of $\boldsymbol{X}$. □

In the following example, we will take an affine C^∞-scheme with corners $\boldsymbol{X}$ and show $C^{\mathrm{loc}}(\boldsymbol{X})$ is not a C^∞-scheme with corners.

Example 5.8.15. Take interior C^∞-ring with corners $\mathfrak{C} = (\mathbb{R}, \mathfrak{C}_{\mathrm{in}} \amalg \{0\})$ with $\mathfrak{C}_{\mathrm{in}}$ a subset of the monoid $(0,\infty) \times \mathbb{N}^{\mathbb{N}}$ (with $(0,\infty)$ a monoid under multiplication and $\mathbb{N}^{\mathbb{N}}$ a monoid under addition) such that all elements have only finite support. That is, if we write $(\delta_i)_{j\in\mathbb{N}}$ for the element of $\mathbb{N}^{\mathbb{N}}$ with a 1 in the ith position and 0 elsewhere, then every element of $\mathfrak{C}_{\mathrm{in}}$ is of the form $a\sum_{i\in I\subset\mathbb{N}}(\delta_i)_{j\in\mathbb{N}}$ for I a finite subset of $\mathbb{N}$, $a \in (0,\infty)$.

The C^∞-operations are defined as follows. For each smooth function $f : \mathbb{R}^n_k \to \mathbb{R}$, then $\Phi_f(c_1', \ldots, c_k', c_{k+1}, \ldots, c_n) = f(a_1 \cdot 0^{\sum_{i\in I_1}\delta_i}, \ldots, a_k \cdot 0^{\sum_{i\in I_k}\delta_i'}, c_{k+1}, \ldots, c_n)$, where $c_l' = a_l \sum_{i\in I_l}(\delta_i)_{j\in\mathbb{N}}$. For each smooth function $f : \mathbb{R}^n_k \to [0,\infty)$, then either $f = 0$ and $\Psi_f = 0$, or f is of the form $f(x_1, \ldots, x_n) = x_1^{b_1} \cdots x_1^{b_k} F(x_1, \ldots, x_n)$ with F strictly positive, so that there is a smooth function $g = \log \circ F : \mathbb{R}^n_k \to \mathbb{R}$ with $F = \exp \circ g$. Then $\Psi_f(c_1', \ldots, c_k', c_{k+1}, \ldots, c_n) = (c_1')^{b_1} \cdots (c_k')^{b_k} \Psi_{\exp} \circ \Phi_g(c_1', \ldots, c_k', c_{k+1}, \ldots, c_n)$ and we define $\Psi_{\exp}(c) = \exp(c) \in (0,\infty) \subset \mathfrak{C}_{\mathrm{in}}$ for any $c \in \mathfrak{C} = \mathbb{R}$. This gives a C^∞-ring with corners with one $\mathbb{R}$-point, so that $\boldsymbol{X} = \mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}) \cong (*, \mathcal{O}_X, \mathcal{O}^{\mathrm{ex}}_X)$ with $\boldsymbol{\mathcal{O}}_X(*) = \mathfrak{C}$.

The prime ideals in $\mathfrak{C}_{\mathrm{ex}}$ are either the zero ideal $\langle 0\rangle$ or of the form $\langle(\delta_i)_{j\in\mathbb{N}}\rangle_{i\in J\subset\mathbb{N}}$ for any subset $J \subset \mathbb{N}$, that is, they are generated by the $(\delta_i)_{j\in\mathbb{N}}$. Then the topological space of $C^{\mathrm{loc}}(X)$ as a set is isomorphic to the collection of these prime ideals, but has a coarse topology. For example, $\langle 0\rangle$ is a closed point but it is not open. Its complement is the union $\cup_{i\in\mathbb{N}}\hat{U}_{(\delta_i)_{j\in\mathbb{N}}}$, which is open but not closed.

In fact, as a local C^∞-ringed space with corners, $C^{\mathrm{loc}}(\boldsymbol{X})$ is not a C^∞-scheme with corners. Suppose it were, then around every point, there must be an open set such $C^{\mathrm{loc}}(\boldsymbol{X})$ is affine on this open set. So by Joyce [40, Lem. 4.15] the topology must be Hausdorff on this open set. Consider that any open set around the point $\langle 0\rangle$ must contain the open set $U_{s'} = \{(*, P) : s_x' \notin P\}$ for some $s' \in \mathcal{O}_X(*)$. So $s' = a\sum_{i\in I\subset\mathbb{N}}(\delta_i)_{j\in\mathbb{N}}$ for I a finite subset of $\mathbb{N}$, $a \in (0,\infty)$. Consider that the prime ideals $\langle(\delta_i)_{j\in\mathbb{N}}\rangle_{i\in 2\mathbb{N}\setminus I}$ and $\langle(\delta_i)_{j\in\mathbb{N}}\rangle_{i\in 2\mathbb{N}+1\setminus I}$ are contained in $U_{s'}$, however there are no open sets in X or in $U_{s'}$ that separate these two points. That is, $U_{s'}$ is not Hausdorff, and no open set containing $U_{s'}$ is Haursdorff. So $C^{\rm loc}(\boldsymbol{X})$ is not affine in a neighbourhood of $\langle 0\rangle$ and is not a C^∞-scheme with corners.

Note that if $\mathfrak{C}_{\rm in} = (0,\infty)\times\mathbb{N}^{\mathbb{N}}$, not just the finitely supported ones, $C^{\rm loc}(\boldsymbol{X})$ can be shown to be a C^∞-scheme with corners.

5.8.1 Boundary

Finally we can consider the boundary of a firm C^∞-scheme with corners, where we use the definition of minimal prime ideal from §5.7.1.

Definition 5.8.16. If $\boldsymbol{X}$ is a firm C^∞-scheme with corners, then the *boundary* $\partial\boldsymbol{X}$ of $\boldsymbol{X}$ is a sub-C^∞-scheme with corners of $C(\boldsymbol{X})$. If locally $\boldsymbol{X}|_U \cong \mathrm{Spec}^{\rm c}\,\boldsymbol{\mathfrak{C}}$ for an open set $U\subset X$ and $\boldsymbol{\mathfrak{C}}$ firm, then $\partial\boldsymbol{X}$ is locally isomorphic to $\coprod_P \mathrm{Spec}(\boldsymbol{\mathfrak{C}}/{\sim_P})$ where the coproduct is over prime ideals P such that there is an $x\in X$ with $\pi_{x,\rm ex}(P)$ a minimal prime ideal in $\mathcal{O}^{\rm ex}_{X,x}\cong\mathfrak{C}_{x,\rm ex}$.

Note that we would not expect $\boldsymbol{X}\mapsto\partial\boldsymbol{X}, \boldsymbol{X}\in\mathbf{C^\infty Sch^c_{fi}}$ to be a functor.

This gives a corollary to Proposition 5.8.14 that the two notions of boundary, $\partial\boldsymbol{X}$ and $\partial^{\rm loc}\boldsymbol{X}$ from §5.7.1 are the same for firm C^∞-schemes with corners.

Corollary 5.8.17. *If $\boldsymbol{X}$ is a firm C^∞-scheme with corners, then $\partial\boldsymbol{X}\cong\partial^{\rm loc}\boldsymbol{X}$.*

Proof. Proposition 5.8.14 explains that for any firm C^∞-scheme with corners $\boldsymbol{X}$ there is an isomorphism $C(\boldsymbol{X})\to C^{\rm loc}(\boldsymbol{X})$. As $\boldsymbol{X}$ is firm, then $\partial^{\rm loc}\boldsymbol{X}$ is an open subset and sub-local C^∞-ringed space with corners of $C^{\rm loc}(\boldsymbol{X})$, and that the image of $\partial\boldsymbol{X}\subset C(\boldsymbol{X})$ under the isomorphism lies in $\partial^{\rm loc}\boldsymbol{X}$. This gives a morphism $\boldsymbol{p}:\partial\boldsymbol{X}\to\partial^{\rm loc}\boldsymbol{X}$. From Definition 5.7.1 and Proposition 5.8.4 we see that the topological spaces as sets are both isomorphic to the collection of pairs

$$\{(x,P): x\in X, P \text{ minimal prime ideal in } \mathcal{O}^{\rm ex}_{X,x}\},$$

and that $\boldsymbol{p}$ respects this. As $C(\boldsymbol{X})\to C^{\rm loc}(\boldsymbol{X})$ is an isomorphism, then $\boldsymbol{p}$ must be an isomorphism. □

Remark 5.8.18. For a firm C^∞-scheme with corners $\boldsymbol{X}$ we can also compare ∂X to $\Delta\boldsymbol{X}$, where the latter is Gillam and Molcho's [28, §4.4] definition of boundary as in Remark 5.7.7. Then there are morphisms $\partial\boldsymbol{X}\to\partial^{\rm loc}\boldsymbol{X}\to\Delta\boldsymbol{X}$. The first morphism is an isomorphism by Corollary 5.8.17.

The second can also be shown to be an isomorphism using a similar proof to Proposition 5.8.14. All that needs to be done is to show that the topology of $\Delta\boldsymbol{X}$ is as fine as the topologies of $\partial\boldsymbol{X}, \partial^{\mathrm{loc}}\boldsymbol{X}$. To do this, if $\mathrm{Spec}^{\mathrm{c}}(\mathfrak{C}) \cong \boldsymbol{X}|_U$ for some open $U \subset X$ and $\mathfrak{C}$ firm, then take $c_1, \ldots, c_n \in \mathfrak{C}_{\mathrm{ex}}$ that generate the sharpening of $\mathfrak{C}_{\mathrm{ex}}$. Take sections $s_i \in \mathcal{O}_X^{\mathrm{ex}}(U)$ such that $s_i(x) = \pi_x^{\mathrm{ex}}(c_i)$ for $i = 1, \ldots, n$. Then for a section $s \in \mathcal{O}_X(U)$, around each point $x \in U$ we have that there is an open set $V \ni x$ such that $s = F s_1^{a_1} \cdots s_n^{a_n}$ for a_i non-negative integers. Let I be the subset of $\{1, \ldots, n\}$ such that for $i \in I$ then $a_i \neq 0$, and take $J = \{1, \ldots, n\} \setminus I$. Then consider that

$$V_s = \{(x, P) : x \in V, P \text{ minimial non-trivial prime in } \mathcal{O}_{X,x}^{\mathrm{ex}}, s(x) \notin P\}$$

is a basic open set in $\partial\boldsymbol{X}, \partial^{\mathrm{loc}}\boldsymbol{X}, \Delta\boldsymbol{X}$, however

$$\hat{V}_s = \{(x, P) : x \in V, P \text{ minimial non-trivial prime ideal in } \mathcal{O}_{X,x}^{\mathrm{ex}}, s(x) \in P\}$$

is a basic open set in $\partial\boldsymbol{X}, \partial^{\mathrm{loc}}\boldsymbol{X}$ but not in $\Delta\boldsymbol{X}$. However, we can show that

$$\hat{V}_s = \cap_{j \in J} V_{s_j}.$$

As J is a finite set, then $\hat{V}_s$ is open in $\Delta\boldsymbol{X}$. Using this, we can see that the topology on $\Delta\boldsymbol{X}$ is as fine as $\partial\boldsymbol{X}, \partial^{\mathrm{loc}}\boldsymbol{X}$ so that $\partial\boldsymbol{X} \to \partial^{\mathrm{loc}}\boldsymbol{X} \to \Delta\boldsymbol{X}$ is an isomorphism.

Finally, we could also consider how our definitions of corners give a stratification that aligns with the stratification of a manifold with (g-)corners from Definition 3.3.2. To do this, let $P^\times$ be the collection of non-units of a monoid P, which is a prime ideal.

Definition 5.8.19. Let P be a monoid. The *dimension* of a monoid is the maximal length d (or ∞ if there is no maximum length) of a chain of prime ideals

$$\emptyset = Q_0 \subset Q_1 \ldots \subset Q_d = P^\times.$$

If Q is a prime ideal of P, then the *codimension* or *height* of Q is the maximal length d of a chain of prime ideals

$$Q = Q_0 \supset Q_1 \ldots \supset Q_d = \emptyset.$$

If F is the corresponding face to Q, then the dimension of the quotient monoid P/F is the same as the height of Q.

Then our boundary definitions correspond to the codimension 1 prime ideals, and the stratification of $C(\boldsymbol{X})$ and $C^{\mathrm{loc}}(\boldsymbol{X})$ into the k-corners can occur by considering the codimension k prime ideals in an analogous way to Definitions 5.8.16 and 5.7.6. For example, if $\boldsymbol{X}$ is in the image of a manifold with corners under $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}} : \mathbf{Man^c} \to \mathbf{C^\infty Sch^c_{fi}}$ from Definition 5.5.1, then locally $\boldsymbol{X} \cong \mathrm{Spec^c}(C^\infty(\mathbb{R}^n_k), \mathrm{Ex}(\mathbb{R}^n_k))$. The prime ideals in $\mathrm{Ex}(\mathbb{R}^n_k)$ are generated by different choices of the coordinate functions $x_1, \ldots, x_k : \mathbb{R}^n_k \to [0, \infty)$. If we consider a prime ideal $P = \langle x_{i_1}, \ldots, x_{i_m} \rangle$ which will have codimension $m \leqslant k$ where $i_1, \ldots i_m$ are distinct integers in $\{1, \ldots, k\}$, then

$$(C^\infty(\mathbb{R}^n_k), \mathrm{Ex}(\mathbb{R}^n_k))/{\sim_P} \cong (C^\infty(\mathbb{R}^{n-m}_{k-m}), \mathrm{Ex}(\mathbb{R}^{n-m}_{k-m}))$$

and taking $\mathrm{Spec^c}$ of this gives the appropriate m-corner component in $C_m(X)$ (defined in §3.3) locally. Hence, these k-corners will correspond to the images of the k-corners (defined in §3.3) under $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch^c_{fi}}}$.

5.9 Log geometry and log schemes

Log geometry was originally used to understand a certain type of cohomology theory for schemes, such as in Kato [53], who was influenced by Fontaine and Illusie. It was further developed to deal with issues of degeneration and non-compactness of schemes, particularly when considering moduli spaces; Ogus [78] has a comprehensive introduction to log geometry, and a survey paper by Abramovich et al. [1] details how it is used in the context of moduli schemes.

Our C^∞-schemes with corners in Section 5.1 are related to the 'positive log differentiable spaces' of Gillam and Molcho [28].

We define pre-log rings as in Gillam [26], which form a category that are related to the category of pre C^∞-rings with corners. We then explain log rings, (pre-)log schemes, log differentiable spaces and positive log differentiable spaces, and their relation to our C^∞-rings with corners and C^∞-schemes with corners. In this section, we assume all rings and monoids are commutative.

Definition 5.9.1. A *pre-log ring* (R, M, α), is a ring, $(R, +, \cdot)$ and a monoid $(M, *)$ with a morphism of monoids $\alpha : M \to R$, where we consider R a monoid under the operation '$\cdot$'. A *morphism of pre-log rings* $(f, f') : (R_1, M_1, \alpha_1) \to (R_2, M_2, \alpha_2)$ is a morphism of rings $f : R_1 \to R_2$ and a morphism of monoids $f' : M_1 \to M_2$ such that $\alpha_2 \circ f' = f \circ \alpha_1$. We call a pre-log ring (R, M, α) a *pre-log structure* on the ring R. Pre-log rings and their morphisms form a category.

A pre-log ring (R, M, α) is a *log ring* if the morphism α induces an isomorphism $\alpha^{-1}(R^\times) \cong R^\times$, where $R^\times$ is the group of units of R. If (R_1, M_1, α_1) and (R_2, M_2, α_2) are log rings and $(f, f') : (R_1, M_1, \alpha_1) \to (R_2, M_2, \alpha_2)$ is a morphism of pre-log rings, then we call (f, f') a *morphism of log rings.* We call a log ring (M, R, α) a *log structure* on the ring R. Log rings and their morphisms form a category.

Ogus [78, p. 274–275] uses 'log-rings' for our notion of pre-log rings. Note that $\alpha^{-1}(R^\times) \cong R^\times$ implies that $\alpha_i|_{M^\times} : M^\times \to R^\times$ is an isomorphism, but the converse may not be true in general.

If R is a ring, then the *trivial log structure* on R is the log ring $(R, R^\times, i)$ where i is the inclusion map. This is the initial object in the category of log rings with ring R, and gives a left adjoint to the forgetful functor $(R, M, \alpha) \to R$. There is also a final object, (R, R, id), which gives a right adjoint to the forgetful functor $(R, M, \alpha) \to R$. These adjoints imply colimits and limits commute with the forgetful functor.

For any pre-log ring (R, M, α), there is a log ring associated to it (R, M', α'). Here, M' is the monoid pushout $M' = R^\times \oplus_{\alpha, \alpha^{-1}(\mathbb{R}^\times), i} M$ and $\alpha' : M' \to R$ is defined using the universal property of pushouts. This defines a left adjoint to the inclusion of the category of log rings into pre-log rings.

Definition 5.9.2. A *pre-log structure* on a scheme $(X, \mathcal{O}_X)$ is a sheaf of monoids M_X and a morphism of sheaves of monoids $\alpha_X : M_X \to \mathcal{O}_X$, where $\mathcal{O}_X$ is considered a sheaf of monoids under multiplication from the rings. We call $(X, \mathcal{O}_X, M_X)$ a *pre-log scheme.* A morphism of pre-log schemes $(X_1, \mathcal{O}_{X_1}, M_{X_1})$ and $(X_2, \mathcal{O}_{X_2}, M_{X_2})$ is a morphism of schemes $f = (f, f^\sharp) : (X_1, \mathcal{O}_{X_1}) \to (X_2, \mathcal{O}_{X_2})$ with a morphism of sheaves of monoids $f^\flat : M_{X_2} \to f_*(M_{X_1})$ such that

$$f^\sharp \circ \alpha_{X_2} = f_*(\alpha_{X_1}) \circ f^\flat. \tag{5.9.1}$$

A *log structure* on a scheme $(X, \mathcal{O}_X)$ is a pre-log scheme $(X, \mathcal{O}_X, M_X)$ such that α_X induces an isomorphism $\alpha_X^{-1}(\mathcal{O}_X^\times) \cong \mathcal{O}_X^\times$ where $\mathcal{O}_X^\times$ is the sheaf of units of $\mathcal{O}_X$. Then $(X, \mathcal{O}_X, M_X)$ is called a *log scheme.* Morphisms of log schemes are morphisms of the underlying pre-log schemes.

The *trivial log structure* on a scheme $(X, \mathcal{O}_X)$ is given by $(X, \mathcal{O}_X, \mathcal{O}_X^\times)$, with α_X the inclusion morphism. This is the final object in the category of log structures on a scheme $(X, \mathcal{O}_X)$. This induces an inclusion functor from the category of schemes to the category of log schemes, right adjoint to the forgetful functor $(X, \mathcal{O}_X, \mathcal{O}_X^\times) \to (X, \mathcal{O}_X)$, realising schemes as a full subcategory of log schemes. The log scheme $(X, \mathcal{O}_X, \mathcal{O}_X)$, with α_X the identity morphism, is the initial object in the category of log structures on $(X, \mathcal{O}_X)$, which gives a left adjoint to the forgetful functor $(X, \mathcal{O}_X, \mathcal{O}_X^\times) \to (X, \mathcal{O}_X)$.

Definition 5.9.3. To any pre-log ring $\alpha : M \to R$, Ogus [78, p. 275] and Gillam [26, p. 76] give a description of a spectrum functor $\mathrm{Spec} : (\alpha : M \to R) \mapsto (X, \mathcal{O}_X, M_X)$

where $(X, \mathcal{O}_X) = \mathrm{Spec}(R)$ is the usual spectrum of a ring, and M_X is the log structure associated to the pre-log structure $M \to \mathcal{O}_X$ which is induced by $M \to R$. Precisely, $R \cong \mathcal{O}_X(X)$ so restriction gives a morphism $M \to R \cong \mathcal{O}_X(X) \to \mathcal{O}_X(U)$ for each open $U \subset X$. Then we define the presheaf of monoids $\hat{M}_X$ such that $\hat{M}_X(U)$ is the pushout $M \otimes_{\alpha^{-1}(\mathcal{O}_X(U)^\times)} \mathcal{O}_X(U)^\times$. Here $\mathcal{O}_X(U)^\times$ is the group of invertible elements of $\mathcal{O}_X(U)$. Note that the presheaf restriction property is satisfied using the universal property of pushouts, and that this universal property also gives morphisms $\hat{M}_X(U) \to \mathcal{O}_X(U)$. Then let M_X be the sheafification of this presheaf, so we have that $(X, \mathcal{O}_X, M_X)$ is a log scheme. Here, universal properties show that $M_{X,x}$ is isomorphic to the pushout $M \otimes_{\alpha^{-1}(\mathcal{O}_{X,x}^\times)} \mathcal{O}_{X,x}^\times$.

This spectrum construction is very similar to the notion of a *chart* for a log scheme, (see for example [28, p. 33]) where if $(X, \mathcal{O}_X, M_X)$ is a log scheme, then a chart is a morphism of monoids $P \to M_X(X)$ such that the *associated log structure*, $(X, \mathcal{O}_X, P_X)$ is isomorphic to $(X, \mathcal{O}_X, M_X)$. Here P_X is the sheafification of the presheaf $\hat{P}_X$ where $\hat{P}_X(U)$ is the pushout $P \otimes_{\alpha^{-1}(\mathcal{O}_X(U)^\times)} \mathcal{O}_X(U)^\times$ using the composition $P \to M_X(X) \to O_X(X) \to \mathcal{O}_X(U)$. If $\alpha : M \to R$ is a pre-log ring then $M \to M_X(X)$ is a chart for $\mathrm{Spec}(\alpha : M \to R) = (X, \mathcal{O}_X, M_X)$. In [28, p. 33] a log scheme is called *quasi-coherent* if there exists charts locally. It is straight forward to show that a log scheme with a chart (or with local charts) is locally the spectrum of a pre-log ring, however the notion of chart applies more generally for log locally ringed spaces (as in the remark below) whereas the spectrum functor does not.

Remark 5.9.4. In Definition 5.9.2, we can change the word 'scheme' to 'locally ringed space' to define (pre-)log locally ringed spaces. Our local C^∞-ringed spaces with corners are examples of pre-log locally ringed spaces. Gillam and Molcho [28, §4.4] define a notion of boundary on pre-log locally ringed spaces which related to our notions of corners, as we discuss in §5.7.1.

There are some similarities in the definitions of C^∞-schemes with corners and log schemes. However, log schemes are based on ordinary schemes, not C^∞-schemes. Also, for a log scheme $(X, \mathcal{O}_X, M_X)$ we have $(X, \mathcal{O}_X)$ locally isomorphic to the spectrum of a ring, however $(X, \mathcal{O}_X, M_X)$ is not required to locally be isomorphic to the spectrum of a log ring. In this sense, C^∞-schemes with corners are analogous to quasi-coherent log schemes.

The categories of differentiable spaces, log differentiable spaces, and positive log differentiable spaces consider ways to construct log structures for manifolds and manifolds with corners. These are based on the notion of a differentiable algebra, which we recall the following definitions from Navarro González and Sancho de Salas [76], and are closely related to C^∞-rings.

Definition 5.9.5. A *differentiable algebra* $\mathfrak{D}$ is an $\mathbb{R}$-algebra isomorphic to a quotient of $C^\infty(\mathbb{R}^n)$ by an ideal $\mathfrak{a}$ such that $\mathfrak{a}$ is closed in the C^∞-Whitney topology of $C^\infty(\mathbb{R}^n)$. A reference for the C^∞-Whitney topology can be found in [34, p. 35–36]. Morphisms of differentiable algebras are morphisms of $\mathbb{R}$-algebras.

A *locally ringed* $\mathbb{R}$*-space* is a topological space X equipped with a sheaf $\mathcal{O}_X$ of $\mathbb{R}$-algebras, such that the stalks are local rings. Morphisms of locally ringed $\mathbb{R}$-spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are of the form $(f, f^\sharp)$ with $f : X \to Y$ a continuous map of topological spaces, and $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ a morphism of sheaves.

The spectrum, $\operatorname{Spec}\mathfrak{D}$, of a differentiable algebra $\mathfrak{D}$, is a locally ringed $\mathbb{R}$-space $(X, \mathcal{O}_X)$. The topological space X is the set of all maximal ideals $\mathfrak{m}$ of $\mathfrak{D}$ with residue field $\mathbb{R} \cong \mathfrak{D}/\mathfrak{m}$, equipped with the Gelfand topology. This is the smallest topology such that for all $d \in \mathfrak{D}$, the corresponding morphism $d^* : \operatorname{Spec}\mathfrak{D} \to \mathbb{R}$, that sends $\mathfrak{m} \in \operatorname{Spec}\mathfrak{D}$ to the value of d under the isomorphism $\mathbb{R} \cong \mathfrak{D}/\mathfrak{m}$, is continuous.

The sheaf is defined by $\mathcal{O}_X(U) = \mathfrak{D}_U$ for open $U \subset X$. Here $\mathfrak{D}_U$ is the localisation of $\mathfrak{D}$ at U, that is, the $\mathbb{R}$-algebra of (equivalence classes) of fractions d/m with $d, m \in \mathfrak{D}$, and with m^* non-zero on U.

An *affine differentiable space* is a locally ringed $\mathbb{R}$-space that is isomorphic to $\operatorname{Spec}\mathfrak{D}$ for some differentiable algebra $\mathfrak{D}$. A *differentiable space* is a locally ringed $\mathbb{R}$-space $(X, \mathcal{O}_X)$ that is locally isomorphic to an affine differentiable space.

Morphisms of differentiable spaces are morphisms of locally ringed $\mathbb{R}$-spaces.

The category of differentiable algebras has all finite colimits, and the category of differentiable spaces has all finite limits, these commute with the forgetful functor to topological spaces.

We now define log differentiable spaces and positive log differentiable spaces following Gillam and Molcho [28].

Definition 5.9.6. A *log differentiable space* $(X, \mathcal{O}_X, M_X)$ is a differentiable space $(X, \mathcal{O}_X)$ equipped with a log structure. That is, M_X is a sheaf of monoids on X, and there is a morphism of sheaves $\alpha_X : M_X \to \mathcal{O}_X$ such that α induces an isomorphism $\alpha_X^{-1}(\mathcal{O}_X^\times) \cong \mathcal{O}_X^\times$.

A *positive log differentiable space*, $(X, \mathcal{O}_X, M_X)$, is a differentiable space $(X, \mathcal{O}_X)$, with a sheaf of monoids M_X on X, and a morphism of sheaves $\alpha_X : M_X \to \mathcal{O}_X^{\geqslant 0}$ such that α_X induces an isomorphism $\alpha_X^{-1}(\mathcal{O}_X^{>0}) \cong \mathcal{O}_X^{>0}$. Here $\mathcal{O}_X^{>0}$ and $\mathcal{O}_X^{\geqslant 0}$ are the sheaves of monoids such that for any open $U \subset X$, then

$$\mathcal{O}_X^{>0}(U) = \{s \in \mathcal{O}_X(U) : s(x) > 0 \in \mathbb{R} \text{ for all } x \in U\}$$

and

$$\mathcal{O}_X^{\geqslant 0}(U) = \{s \in \mathcal{O}_X(U) : s(x) \geqslant 0 \in \mathbb{R} \text{ for all } x \in U\}.$$

In this definition, $s(x)$ is the value of s under the composition of the following maps

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbb{R},$$

where $\mathfrak{m}_x$ is the maximal ideal in the local differentiable algebra $\mathcal{O}_{X,x}$.

Morphisms of (positive) log differentiable spaces are composed of a morphism of differentiable spaces and a morphism of sheaves of monoids, such that the morphisms behave well with respect to each α, as in (5.9.1).

A positive log differentiable space $(X, \mathcal{O}_X, M_X)$ gives the data of a log differentiable space $(X, \mathcal{O}_X, M_X^*)$. We have that $\alpha_X : M_X \to \mathcal{O}_X^{\geqslant 0} \hookrightarrow \mathcal{O}_X$ is a morphism of sheaves, and we can take the log structure associated to this morphism, which is given by the pushout $M_X^* = M_X \amalg_{\mathcal{O}_X^{\geqslant 0}} \mathcal{O}_X^\times$, where $\mathcal{O}_X^\times$ is the sheaf of invertible elements on X. In fact there is an isomorphism to the coproduct $M_X^* \cong M_X \otimes \underline{\mathbb{Z}/(2\mathbb{Z})}$, where $\underline{\mathbb{Z}/(2\mathbb{Z})}$ is the locally constant sheaf with stalk $\mathbb{Z}/(2\mathbb{Z})$. This gives a faithful but not full functor from positive log differentiable spaces to log differentiable spaces (c.f [28, p. 50]).

Similarly, a log differentiable space $(X, \mathcal{O}_X, M_X)$ with morphism α_X gives the data of a positive log differentiable space $(X, \mathcal{O}_X, M_X^{\geqslant 0})$ where $M_X^{\geqslant 0}$ is the sheaf of monoids such that

$$M_X^{\geqslant 0}(U) = \{s' \in M_X(U) | s'(x) \geqslant 0 \in \mathbb{R} \text{ for all } x \in U\}.$$

Here $\alpha_X|_{M_X^{\geqslant 0}}$ gives the required morphism. This is full but not faithful, and a left adjoint to the previous functor. The composition of both of these functors is the identity functor on positive log differentiable spaces, but not on log differentiable spaces.

The categories of log differentiable spaces and positive log differentiable spaces have all finite limits and all coproducts.

5.9.1 Comparison to C^∞-algebraic geometry

We now describe the relations of these logarithmic geometry constructions to C^∞-algebraic geometry.

A (pre) C^∞-ring with corners $(\mathfrak{C}, \mathfrak{C}_{\text{ex}})$ is a pre-log ring with morphism $\alpha = \Phi_i : \mathfrak{C}_{\text{ex}} \to \mathfrak{C}$. However, a C^∞-ring with corners $(\mathfrak{C}, \mathfrak{C}_{\text{ex}})$ is not a log ring, as our Definition 4.3.2, does not mean that $\Phi_i^{-1}(\mathfrak{C}^\times) \cong \mathfrak{C}^\times$. In fact, it is never an isomorphism, as if we take a C^∞-ring with corners, $(\mathfrak{C}, \mathfrak{C}_{\text{ex}})$, one can show that there is no element $a \in \mathfrak{C}_{\text{ex}}^\times$ such that $\Phi_i(a) = -1_{\mathfrak{C}} \in \mathfrak{C}^\times$, where $1_{\mathfrak{C}}$ is the identity element of $\mathfrak{C}$.

In the following Lemma and its corollaries, we see that for a semi-complete C^∞-ring with corners the invertible elements of $\mathfrak{C}_{\rm ex}$ correspond only to the 'positive' elements of $\mathfrak{C}$.

Lemma 5.9.7. *Let $\mathfrak{C}$ be a complete C^∞-ring. Then for each $c \in \mathfrak{C}$ such that $x \circ c > 0$ for each $\mathbb{R}$-point $x : \mathfrak{C} \to \mathbb{R}$, there is $d \in \mathfrak{C}$ such that $\Phi_{\exp}(d) = c$. Define the monoid $\mathfrak{C}_{>0} = \{c \in \mathfrak{C} : x(c) \neq 0$ for each $\mathbb{R}$-point $x : \mathfrak{C} \to \mathbb{R}\}$, then this is equal to $\Phi_{\exp}(\mathfrak{C})$ and element is invertible.*

Proof. Let $\mathfrak{C}$ be complete and take $c \in \mathfrak{C}$ such that $x(c) > 0$ for all $\mathbb{R}$-points $x : \mathfrak{C} \to \mathbb{R}$. Fix an $\mathbb{R}$-point $\hat{x}$, let $\epsilon_{\hat{x}} = \hat{x}(c)$ and take the open set $U_{\hat{x}} = \{x : \mathfrak{C} \to \mathbb{R} \mid x(c) > \epsilon_{\hat{x}}/2\} \ni \hat{x}$ which is open in $\operatorname{Spec}\mathfrak{C}$. Choose a smooth function $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(t) = \log(t)$ for $t \in [1/n, \infty)$. Then $\Phi_{f_n}(c)$ exists for all $n \in (0, \infty)$. For each $\hat{x}$ take $n_{\hat{x}}$ such that $n_{\hat{x}} > 2/\epsilon_{\hat{x}}$.

Now, say $\hat{x} \in U_x$ for some $\mathbb{R}$-point x. Then we claim $\Phi_{\exp} \circ \Phi_{f_{n_x}}(c)$ is equal to c in the stalk at $\hat{x}$. As $\mathfrak{C}$ is complete and $\Phi_{\exp}$ is injective by Proposition 4.3.1(a), then this will tell us the $\Phi_{f_{n_x}}(c)$ are equal in every stalk in appropriate $U_{\hat{x}}$, and there is an element d such that $d|_{U_{\hat{x}}} = \Phi_{f_{n_x}}(c)|_{U_{\hat{x}}}$ and $\Phi_{\exp}(d) = c$.

For this $\hat{x}$ we know that $\hat{x} \circ c > 1/n_x$. So choose smooth $g : \mathbb{R} \to \mathbb{R}$ such that $g(t) = 0$ for all $t \in (-\infty, 1/n_x]$ and $t > 0$ otherwise, so that $g(\exp \circ f_{n_x}(t) - \mathrm{id}(t)) = 0$ for all $t \in \mathbb{R}$, and $\hat{x} \circ (\Phi_g(c)) = g(\hat{x}(c)) \neq 0$. Then letting $e = \Phi_g(c)$ we have that $e(\Phi_{\exp} \circ \Phi_{f_{n_x}}(c) - \Phi_{\rm id}(c)) = 0$, so Proposition 2.1.15 implies $\Phi_{\exp} \circ \Phi_{f_{n_x}}(c)$ is equal to c in the stalk at $\hat{x}$. So such an element d exists and we must have $\mathfrak{C}_{>0} \subseteq \Phi_{\exp}(\mathfrak{C})$.

Also, if $d \in \mathfrak{C}$ then $x \circ \Phi_{\exp}(d) = \exp(x(d)) > 0$ for all $x : \mathfrak{C} \to \mathbb{R}$, so $\Phi_{\exp}(\mathfrak{C}) = \mathfrak{C}_{>0}$. In addition, elements of $\Phi_{\exp}(\mathfrak{C})$ are invertible, as $\Phi_{\exp}(d)$ has inverse $\Phi_{\exp}(-d)$ for all $d \in \mathfrak{C}$. □

Corollary 5.9.8. *If $\mathfrak{C}$ is semi-complete C^∞-ring with corners, then $\Phi_i : \mathfrak{C}_{\rm ex} \to \mathfrak{C}$ induces an isomorphism $\Phi_i : \mathfrak{C}_{\rm ex}^\times \to \mathfrak{C}_{>0} = \Phi_{\exp}(\mathfrak{C})$.*

Proof. If $c \in \mathfrak{C}_{>0}$ then by Lemma 5.9.7 there is $d \in \mathfrak{C}$ such that $\Phi_{\exp}(d) = c$. Then $\Psi_{\exp}(d) \in \mathfrak{C}_{\rm ex}^\times$ and $\Phi_i \circ \Psi_{\exp}(d) = \Phi_{\exp}(d) = c$. So there is a map

$$\mathfrak{C}_{>0} \to \mathfrak{C}_{\rm ex}^\times, \Phi_{\exp}(d) \mapsto \Psi_{\exp}(d).$$

As $\Phi_{\exp}$ is injective by Proposition 4.3.1(a) and $\Psi_{\exp} : \mathfrak{C} \to \mathfrak{C}_{\rm ex}^\times$ is a bijection by Definition 4.3.2(iii), then $\mathfrak{C}_{>0} \to \mathfrak{C}_{\rm ex}^\times$ is a bijection, with inverse $\Phi_i|_{\mathfrak{C}_{\rm ex}^\times}$. Also if $a, b \in \mathfrak{C}_{>0}$, $a = \Phi_{\exp}(c)$, $b = \Phi_{\exp}(d)$ then $ab = \Phi_{\exp}(c + d) \mapsto \Psi_{\exp}(cd) = \Psi_{\exp}(c)\Psi_{\exp}(d)$, so $\mathfrak{C}_{>0} \to \mathfrak{C}_{\rm ex}^\times$ is an isomorphism of monoids (and in fact an isomorphism of abelian groups.) □

Corollary 5.9.9. *For* $\mathfrak{C} \in \mathbf{C^\infty Rings^c}$*,* Φ_i *induces an isomorphism* $\Phi_i^{-1}(\mathfrak{C}_{>0}) \to \mathfrak{C}_{>0}$.

Proof. Corollary 5.9.8 implies there is canonical isomorphism of monoids (or groups) $\Phi_i|_{\mathfrak{C}_{\mathrm{ex}}^\times} : \mathfrak{C}_{\mathrm{ex}}^\times \to \mathfrak{C}_{>0}$, so $\Phi_i^{-1}(\mathfrak{C}_{>0})$ contains $\mathfrak{C}_{\mathrm{ex}}^\times$. As elements of $\Phi_i^{-1}(\mathfrak{C}_{>0})$ are isomorphic to $\Phi_i^{-1}(\Phi_{\exp}(\mathfrak{C}))$, then $\Phi_i^{-1}(\mathfrak{C}_{>0})$ is equal to $\Psi_{\exp}(\mathfrak{C})$, which is equal to $\mathfrak{C}_{\mathrm{ex}}^\times$ by Definition 4.3.2(iii). This implies $\Phi_i|_{\mathfrak{C}_{\mathrm{ex}}^\times} : \mathfrak{C}_{\mathrm{ex}}^\times \to \mathfrak{C}_{>0}$ is equal to $\Phi_i|_{\Phi_i^{-1}(\mathfrak{C}_{>0})} : \Phi_i^{-1}(\mathfrak{C}_{>0}^\times) \to \mathfrak{C}_{>0}$ and so the latter is an isomorphism. □

Remark 5.9.10. In Lemma 5.9.7, we did not need to assume that $\mathfrak{C}$ was a complete C^∞-ring, but we just needed to assume that $\mathfrak{C}$ was some C^∞-ring that is isomorphic to (a subring of) $\mathcal{O}_X(U)$ where $\mathcal{O}_X$ is some sheaf of C^∞-rings on a topological space X and open $U \subset X$. This property is equivalent to the property of 'germ determined' defined in Moerdijk and Reyes [72, Def. 4.1] for C^∞-rings.

Similarly, in Corollary 5.9.8 and Corollary 5.9.9 we did not need to assume that $(\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ is a semi-complete C^∞-ring with corners, but that it is isomorphic to (a subring of) $\boldsymbol{\mathcal{O}}_X(U)$ where $\boldsymbol{\mathcal{O}}_X$ is some sheaf of C^∞-rings with corners on a topological space X and open $U \subset X$. Joyce [40] defined a notion of *fair* C^∞-ring, which is equivalent to germ determined and finitely generated, and are called 'C^∞-rings of finite type presented by an ideal of local character' in Dubuc [20, 21]. Kalashnikov [51, §. 4.8] extended this notion of fair to pre C^∞-rings with corners, so C^∞-rings with corners that satisfy Kalashnikov's notion of fair would be sufficient for these corollaries, although we can weaken this as we do not need to require finitely generated.

We would like to compare the notions of spectrum of a log scheme to our notion of spectrum of C^∞-rings with corners. Consider the following definition along the lines of a spectrum of (pre-)log rings as in [78], [26] and [28] but instead for C^∞-rings with corners.

Definition 5.9.11. Let $\boldsymbol{\mathfrak{C}}$ be a C^∞-ring with corners then define $\mathrm{Spec^c}(\Phi_i : \mathfrak{C}_{\mathrm{ex}} \to \mathfrak{C})$ to be the local C^∞-ringed space with corners $(X, \mathcal{O}_X, M_X^{\mathrm{ex}})$ with $(X, \mathcal{O}_X) \cong \mathrm{Spec}(\mathfrak{C})$ an affine C^∞-scheme and M_X^{ex} is the sheafification of the presheaf $\hat{M}_X^{\mathrm{ex}}$, where for each $U \subset X$ we let $\hat{M}_X^{\mathrm{ex}}(U)$ be the pushout $\mathfrak{C}_{\mathrm{ex}} \amalg_{\Phi_i^{-1}(\mathcal{O}_X^{>0}(U))} \mathcal{O}_X^{>0}(U)$.

We will show that this definition matches our definition of $\mathrm{Spec^c}$ in Definition 5.2.1. This definition would also work for pre C^∞-rings with corners, in which case this definition of $\mathrm{Spec^c}$ would match our definition of $\mathrm{Spec^c}$ applied to $\Pi_{\mathrm{pre}\,C^\infty}^{C^\infty}(\boldsymbol{\mathfrak{C}})$ where $\Pi_{\mathrm{pre}\,C^\infty}^{C^\infty}$ is defined in Proposition 4.3.5.

Proposition 5.9.12. *If* $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ *is a* C^∞*-ring with corners then* $\mathrm{Spec^c}(\Phi_i : \mathfrak{C}_{\mathrm{ex}} \to \mathfrak{C}) = (X, \mathcal{O}_X, M_X)$ *from Definition 5.9.11 is isomorphic to* $\mathrm{Spec^c}\,\boldsymbol{\mathfrak{C}} = (X, \mathcal{O}_X, \mathcal{O}_X^{\mathrm{ex}})$ *in Definition 5.2.1.*

Proof. We can assume $\mathfrak{C}$ is semi-complete by Lemma 5.3.3. By definition they are isomorphic on topological spaces and as sheaves of C^∞-rings. Corollary 5.9.8 gives a map $\mathcal{O}_X^{\geq 0}(U) \to \mathcal{O}_X^{\mathrm{ex}}(U)$ for each open $U \subset X$. As there is a map $\mathfrak{C}_{\mathrm{ex}} \to \mathcal{O}_X^{\mathrm{ex}}(U)$, the universal property of $\mathfrak{C}_{\mathrm{ex}} \amalg_{\Phi_i^{-1}(\mathcal{O}_X^{\geq 0}(U))} \mathcal{O}_X^{\geq 0}(U)$ gives a unique map $\hat{M}_X(U) \to \mathcal{O}_X^{\mathrm{ex}}(U)$, which respects restriction and descends to a morphism of stalks. So there is a morphism of sheaves of monoids $M_X \to \mathcal{O}_X^{\mathrm{ex}}$.

On the stalks, we can show we have isomorphisms $M_{X,x} \cong \mathfrak{C}_{\mathrm{ex}} \amalg_{\Phi_i^{-1}(\mathfrak{C}_x^{\geq 0})} \mathfrak{C}_x^{\geq 0} \cong \mathfrak{C}_{x,\mathrm{ex}}$ using the universal properties of $M_{X,x}$ as a colimit, $\mathfrak{C}_{\mathrm{ex}} \amalg_{\Phi_i^{-1}(\mathfrak{C}_x^{\geq 0})} \mathfrak{C}_x^{\geq 0}$ as a pushout, and $\mathfrak{C}_{x,\mathrm{ex}}$ as a localisation. This means $M_X \to \mathcal{O}_X^{\mathrm{ex}}$ is an isomorphism on stalks. This gives an isomorphism $\mathrm{Spec^c}\,\mathfrak{C} \to \mathrm{Spec^c}(\Phi_i : \mathfrak{C}_{\mathrm{ex}} \to \mathfrak{C})$ which is the identity on the topological spaces and the sheaves of C^∞-rings, and an isomorphism on the sheaves of monoids. □

This suggests that C^∞-schemes with corners are related to quasi-coherent log schemes. Gillam [26, p. 76] points out that it can be difficult to determine the log scheme corresponding to $\mathrm{Spec}(M \to R)$. This sentiment aligns with our observations: in general $\mathcal{O}_X^{\mathrm{ex}}(X)$ is not isomorphic to $\mathfrak{C}_{\mathrm{ex}}$ when $\boldsymbol{X} \cong \mathrm{Spec^c}\,\mathfrak{C}$. We now see that our C^∞-schemes with corners are more closely related to positive log differentiable spaces.

The $\mathbb{R}$-algebra structure of a differentiable algebra can be extended to a unique C^∞-ring structure by Definition 2.1.7. In particular, using the Whitney embedding theorem, $C^\infty(X)$ is a differentiable algebra for all manifolds X. Morphisms of differentiable algebras are $\mathbb{R}$-algebra morphisms, which respect the unique C^∞-ring structures of the differentiable algebras, as shown below in Lemma 5.9.13. This realises the category of differentiable algebras as a full subcategory of $\mathbf{C^\infty Rings}$, and similarly the category of (affine) differentiable spaces is a full subcategory of ($\mathbf{AC^\infty Sch}$) $\mathbf{C^\infty Sch}$.

Lemma 5.9.13. *Morphisms of differentiable algebras respect the C^∞-operations from the unique C^∞-ring structures corresponding to the differentiable algebras. Hence morphisms of differentiable spaces correspond to morphisms of C^∞-schemes.*

Proof. Let $A = C^\infty(\mathbb{R}^n)/\mathfrak{a}$ and $B = C^\infty(\mathbb{R}^m)/\mathfrak{b}$ be differentiable algebras, and $\phi : A \to B$ a morphism of differentiable algebras, that is, an $\mathbb{R}$-algebra morphism. Let $\tau_1 : C^\infty(\mathbb{R}^n) \to A$ and $\tau_2 : C^\infty(\mathbb{R}^m) \to B$ be the quotient maps. As $\mathfrak{a}$ and $\mathfrak{b}$ are closed in the Whitney topology of $C^\infty(\mathbb{R}^n)$, then by Navarro González and Sancho de Salas [76, Cor. 2.21], there is a morphism of $\mathbb{R}$-algebras $\psi : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)$ such that $\tau_2 \circ \psi = \phi \circ \tau_1$.

By Moerdijk and Reyes [72, Cor. 3.7], all $\mathbb{R}$-algebra morphisms $\psi : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)$ come from smooth maps $F : \mathbb{R}^m \to \mathbb{R}^n$, and by Moerdijk and Reyes [72, Th. 2.8], then ψ is a morphism of C^∞-rings. As quotient maps are C^∞-ring morphisms and are surjective, then ϕ is a C^∞-ring morphism.

If we choose a different presentation of $A \cong C^\infty(\mathbb{R}^{n_1})/\mathfrak{a}_1$, then because $C^\infty(\mathbb{R}^{n_1})/\mathfrak{a}_1 \cong C^\infty(\mathbb{R}^n)/\mathfrak{a}$ as $\mathbb{R}$-algebras, the above implies they are isomorphic as C^∞-rings. The same holds for different choices of presentation for B. Hence, morphisms of differential spaces are precisely morphisms of C^∞-rings. □

This result gives a partial answer to the question of whether C^∞-ring morphisms are just morphisms of the underlying $\mathbb{R}$-algebras; if there are $\mathbb{R}$-algebra morphisms of C^∞-rings that are not C^∞-ring morphisms, at least one of the C^∞-rings must not be isomorphic to a differentiable algebra. Lemma 5.9.13 is also a corollary of Kainz et al. [50, Th. 2.4 (3)], and Navarro González and Sancho de Salas [76, Cor. 2.22].

Next we will show that a local C^∞-ringed space with corners, $(X, \mathcal{O}_X, \mathcal{O}_X^{\rm ex})$, for which $(X, \mathcal{O}_X)$ is a differentiable space, is a positive log differentiable space. In fact, there is a containment of subcategories, as all positive log differentiable spaces can be considered as local C^∞-ringed spaces with corners.

Also, a C^∞-scheme with corners $(X, \mathcal{O}_X, \mathcal{O}_X^{\rm ex})$, for which $(X, \mathcal{O}_X)$ a differentiable space, is positive log differentiable space. However, there is no containment of subcategories here, as not all C^∞-schemes are differentiable spaces, and C^∞-schemes are locally required to be $\mathrm{Spec}^{\rm c}\,\mathfrak{C}$ for a C^∞-ring with corners, which is stronger than the definition of a positive log differentiable space.

Proposition 5.9.14. *A positive log differentiable space is equivalent to the data of a local C^∞-ringed space with corners, $(X, \mathcal{O}_X, \mathcal{O}_X^{\rm ex})$, such that $(X, \mathcal{O}_X)$ is a differentiable space.*

Proof. Let $(X, \mathcal{O}_X, \mathcal{O}_X^{\rm ex})$ be as in the statement of the proposition. We first want to show this is a positive log differentiable space. To do this, we need only show that Φ_i induces an isomorphism $\Phi_i^{-1}(\mathcal{O}_X^{>0}) \cong \mathcal{O}_X^{>0}$. However, for each open set $U \subset X$, then Remark 5.9.10 implies we can apply Corollary 5.9.9 to $(\mathcal{O}_X(U), \mathcal{O}_X^{\rm ex}(U))$ to deduce this.

Now let $(X, \mathcal{O}_X, M_X)$ be a positive log differentiable space, with morphism $\alpha_X : M_X \to \mathcal{O}_X$. We need to show that $(\mathcal{O}_X, M_X)$ has a sheaf of C^∞-rings with corners structure with local stalks. To do this, we first show that the differentiable algebra and monoid structures extend uniquely to a C^∞-ring with corners structure.

Firstly, $\mathcal{O}_X$ is a sheaf of differentiable algebras, so it extends to a unique C^∞-ring sheaf, with the usual C^∞-operations. The operation $\Phi_i : M_X \to \mathcal{O}_X$ corresponds to α_X. Take open $U \subset X$ and $s'_1, \ldots, s'_k \in M_X(U)$ and $s_{k+1}, \ldots, s_n \in \mathcal{O}_X(U)$. For any smooth $f : \mathbb{R}^n_k \to \mathbb{R}$, extend f to a smooth function $\hat{f} : \mathbb{R}^n \to \mathbb{R}$, and define

$$\Phi_f(s'_1, \ldots, s'_k, s_{k+1}, \ldots, s_n) = \Phi_{\hat{f}}(\alpha_X(s'_1), \ldots, \alpha_X(s'_k), s_{k+1}, \ldots, s_n),$$

where the right hand side $\Phi_{\hat{f}}$ is the operation on the C^∞-ring $\mathcal{O}_X(U)$. To check this is well defined, let $\hat{g}$ be a different extension of f, then we need to check that

$$\Phi_{\hat{f}-\hat{g}}(\alpha_X(s_1'),\ldots,\alpha_X(s_k'),s_{k+1},\ldots,s_n)=0.$$

We know that $\hat{f}-\hat{g}|_{\mathbb{R}^n_k}=0$. However, as $\mathcal{O}_X(U)$ is a differentiable algebra, then $\mathcal{O}_X(U)\cong C^\infty(\mathbb{R}^m)/\mathfrak{a}$ for some non-negative integer m. Then any $s\in\mathcal{O}_X(U)$ is an equivalence class represented by a function $h\in C^\infty(\mathbb{R}^m)$, and we can write $h=\Phi_h(\pi_1,\ldots,\pi_m)$ for $\pi_i:\mathbb{R}^m\to\mathbb{R}$ the projection onto the ith factor. Then

$$\Phi_{\hat{f}-\hat{g}}(\alpha_X(s_1'),\ldots,\alpha_X(s_k'),s_{k+1},\ldots,s_n)=\Phi_{(\hat{f}-\hat{g})(h_1,\ldots,h_n)}([\pi_1],\ldots,[\pi_m]),$$

where $[\pi_i]$ is the equivalence class of π_i in $\mathcal{O}_X(U)\cong C^\infty(\mathbb{R}^m)/\mathfrak{a}$, and h_i is the function representing s_i' for $i=1,\ldots,k$ and s_i for $i=k+1,\ldots,n$. As $\alpha_X(s_1')$ are non-negative, then h_i are non-negative for $i=1,\ldots,k$. However as $\hat{f}-\hat{g}|_{\mathbb{R}^n_k}=0$, then $(\hat{f}-\hat{g})(h_1,\ldots,h_n)=0$ and therefore

$$\Phi_{\hat{f}-\hat{g}}(\alpha_X(s_1'),\ldots,\alpha_X(s_k'),s_{k+1},\ldots,s_n)=0,$$

as required.

Now define the operation $\Psi_{\exp}:\mathcal{O}_X(U)\to M_X(U)$ by $\Psi_{\exp}(s)=\alpha_X^{-1}(\Phi_{\exp}(s))$ for $s\in\mathcal{O}_X(U)$. Note that $\Phi_{\exp}(s)\in\mathcal{O}_X^{>0}(U)$, and that α_X induces an isomorphism $\alpha_X^{-1}(\mathcal{O}_X^{>0}(U))\cong\mathcal{O}_X^{>0}(U)\cong M_X^\times(U)$, so $\Psi_{\exp}$ is well defined.

For any non-zero smooth $g:\mathbb{R}^n_k\to[0,\infty)$, then we have

$$g(x_1,\ldots,x_n)=x_1^{a_1}\cdots x_k^{a_k}G(x_1,\ldots,x_n)$$

where $G:\mathbb{R}^n_k\to(0,\infty)$ is smooth and positive. Here a_i are non-negative integers. Then define

$$\Psi_g(s_1',\ldots,s_k',s_{k+1},\ldots,s_n)=\\ s_1'^{a_k}\cdots s_k'^{a_k}\Psi_{\exp}(\Phi_{\log\circ G}(\alpha_X(s_1'),\ldots,\alpha_X(s_k'),s_{k+1},\ldots,s_n)).$$

Here $s_i'^{a_i}$ means applying the monoid operation a_i times to s_i'. For the zero function, $0:\mathbb{R}^n_k\to[0,\infty)$ then define $\Psi_0(s_1',\ldots,s_k',s_{k+1},\ldots,s_n)=0$. A subtlety here is that $M_X(U)$ may not have a zero, so we may need to add a zero to each $M_X(U)$ and then sheafify to do this, and also extend α_X to send $0\in M_X$ to 0 in $\mathcal{O}_X$.

Direct calculation shows that this gives a pre C^∞-ring with corners structure to $(\mathcal{O}_X(U),M_X(U))$. Also, $\Phi_i|_{M_X^\times(U)}:M_X^\times(U)\to\mathcal{O}_X$ is injective as its image is $\mathcal{O}_X^{>0}(U)$, on which is it an isomorphism as it is equal to α_X. Hence $(\mathcal{O}_X,M_X)$ is a sheaf of C^∞-rings with corners. Each stalk $(\mathcal{O}_{X,x}, M_{X,x})$ is local, as we know $\mathcal{O}_{X.x}$ is local, and that elements in $M_{X,x}$ are invertible if and only if they are invertible under the morphism $\alpha_X = \Phi_i$ to $\mathcal{O}_{X,x}$. Hence $(X, \mathcal{O}_X, M_X)$ is a local C^∞-ringed space with corners. □

Gillam and Molcho [28] have a different notion of manifold with corners than we do, "a positive log smooth differentiable space with free log structure". If $(X, \mathcal{O}_X, M_X)$ is a positive log differentiable space then a free log structure means for all $x \in X$, there is an open set $U \subset X$, such that the sharpening of $M_X(U)$ is isomorphic to free finitely generated monoid, that is, $\mathbb{N}^k$ for some non-negative integer k that may depend upon X and U. This implies the sharpening of the stalk $M_{X,x}^\sharp$ is also isomorphic to $\mathbb{N}^l$ for some non-negative integer l. Here $\mathbb{N}^k, \mathbb{N}^l$ are considered as monoids under addition. This condition does not require the underlying topological space to be Hausdorff, nor of constant dimension, nor second countable.

If X is a manifold with corners as in our Definition 3.2.2, then

$$(X, \boldsymbol{C}^\infty(\cdot)) = (X, C^\infty(\cdot), \mathrm{Ex}(\cdot))$$

is a positive log differentiable space. As in Example 4.5.5, $(X, \boldsymbol{C}^\infty(\cdot))$ is such that for a coordinate neighbourhood $U \subset X$, we have that

$$\mathrm{Ex}(U) \cong (\mathbb{N}^k \times C^\infty(U)) \amalg \{0\} \cong \mathrm{In}(U) \amalg \{0\}$$

for some non-negative integer k, where $C^\infty(U)$ represents the invertible functions. Then the sharpening is isomorphic to $\mathbb{N}^k \amalg \{0\}$. This means $(X, \boldsymbol{C}^\infty(\cdot))$ does not have a free log structure, as we have an additional 0 appearing. However, while $(X, C^\infty(\cdot), \mathrm{In}(\cdot))$ is not a C^∞-scheme with corners, it is a positive log smooth differentiable space with free log structure, that is, we can remove the '0' because a manifold with corners is an interior C^∞-scheme with corners. In this sense, manifold with corners from [28] correspond to interior C^∞-schemes with corners without the '0', and our manifolds with corners correspond to constant dimension, Hausdorff, second countable, positive log smooth differentiable spaces with free log structure from [28] with the '0'.

Appendix A

Additional Material

A.1 Fibre products of manifolds

We describe the two facts needed for the details in the example of §1.1.1 where the fibre product of manifolds does not exist. Here we will write $|X|$ for the underlying set of a manifold X. These results are referred to in Joyce [49, 2.37].

Lemma A.1.1. *The fibre product of manifolds, if it exists, has set equal to the fibre product of its underlying sets.*

Proof. Take manifolds X, Y, Z with smooth maps $g : X \to Z$ and $f : Y \to Z$. Assume the fibre product exists in category of manifolds and denote it $X \times_Z Y$. We know the fibre products of sets exists and is equal to $|X| \times_{|Z|} |Y| = \{(x, y) : f(x) = g(y)\}$.

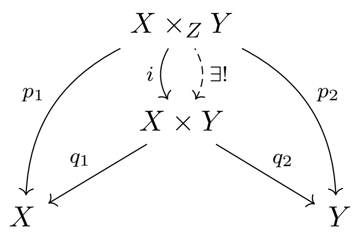
Then the fibre product $X \times_Z Y$ induces the following isomorphism of sets

$$\mathrm{Hom}(U, X \times_Z Y) \cong \mathrm{Hom}(U, X) \times_{\mathrm{Hom}(U,Z)} \mathrm{Hom}(U, Y),$$

where U is a manifold, $\mathrm{Hom}(A, B)$ is the set of smooth maps from A to B, and the left hand side is the fibre product of sets. If we consider U to be the 0-manifold $*$, then the right hand side involves all maps from $*$ to X and Y that commute to Z, which is equivalent to picking points $x \in X, y \in Y$ with $f(x) = g(y)$. So the right hand side is isomorphic as a set to $|X| \times_{|Z|} |Y|$. Any element of the left hand side is equivalent to picking a point of $X \times_Z Y$, so the left hand side is isomorphic as a set to $|X \times_Z Y|$. This implies $|X| \times_{|Z|} |Y| = |X \times_Z Y|$ as required. □

Lemma A.1.2. *For manifolds X, Y, Z with smooth maps $g : X \to Z$ and $f : Y \to Z$, then the topology of their fibre product $X \times_Z Y$ is at least as coarse as the topology induced on $|X| \times_{|Z|} |Y| = |X \times_Y Z|$ coming from $X \times Y = |X| \times |Y|$.*

Proof. The universal property of the product $X \times Y$ gives a unique smooth morphism $X \times_Z Y \to X \times Y$, and there is also the inclusion of sets $i : X \times_Z Y \to X \times Y$. As both morphisms commute with the other maps in the diagram, the explicit descriptions of $X \times Y$ and $X \times_Z Y$ as sets implies these two morphisms are equal, that is the inclusion $i : X \times_Z Y \to X \times Y$ is smooth.



□

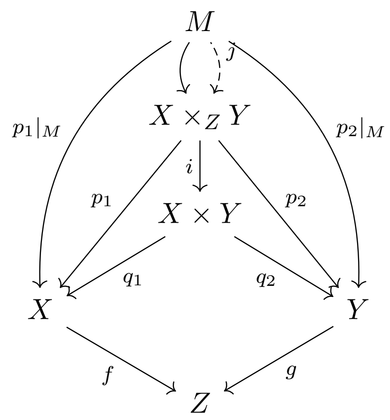
Lemma A.1.3. *Take manifolds X, Y, Z with smooth maps $g : X \to Z$ and $f : Y \to Z$. For all subsets M of the fibre product $X \times_Z Y$ (assuming this exists) that under the inclusion $i : X \times_Z Y \to X \times Y$ are a submanifold of $X \times Y$, then the topology from $X \times_Z Y$ on M is the same as the topology from $X \times Y$ on $i(M)$.*

Proof. Take $M \subseteq X \times_Z Y$ and assume $i(M)$ is a submanifold of $X \times Y$. Then we have

$$\begin{array}{ccc} M & \subseteq & X \times_Z Y \\ \big\downarrow{\scriptstyle i|_M} & & \big\downarrow{\scriptstyle i} \\ i(M) & \subseteq & X \times Y, \end{array}$$

where i is a smooth inclusion by Lemma A.1.2, and the bottom inclusion $i(M) \subseteq X \times Y$ is also smooth.

However, M is a submanifold of $X \times Y$ and the restriction of $p_1 : X \times_Z Y \to X$ and $p_2 : X \times_Z Y \to Y$ to M give morphisms from M to X and Y that commute with the morphisms $q_1 : X \times Y \to X$, $q_2 : X \times Y \to Y$. This means that $p_1|_M$ and $p_2|_M$ are smooth, so the universal property of $X \times_Z Y$ gives a smooth morphism $j : M \to X \times_Z Y$ that commutes with the following diagram (where only q_1 and q_2 do not commute with f and g). However, as j commutes with p_1 and p_2, and we know $|X \times_Z Y| = |X| \times_{|Z|} |Y|$, we see that j must be the inclusion $M \to X \times_Z Y$, so the inclusion must be smooth. This implies the topology on M is the same as the topology on $i(M)$.



So there is homeomorphism from $i(M)$ with topology from $X \times Y$ and M with topology from $X \times_Z Y$. □

Example A.1.4. Assume fibre product of manifolds $\mathbb{R} \times_{x^2,\mathbb{R},y^2} \mathbb{R}$ exists, then by Lemma A.1.1 it is equal as a set to

$$\mathbb{R} \times_{x^2,\mathbb{R},y^2} \mathbb{R} = \{(x, \pm x) \in \mathbb{R}^2\}.$$

Applying Lemma A.1.3 to $\{(x, x) \in \mathbb{R}^2\}$, $\{(x, -x) \in \mathbb{R}^2\}$ and $\{(x, \pm x) \in \mathbb{R}^2\} \setminus \{0\}$ we see the topology on fibre product must be induced from the topology on $\mathbb{R}^2$.

However, if X were a submanifold, then around the point $(0, 0)$, it would have to be locally homeomorphic to $\mathbb{R}^k$ for some integer k. Let U be a connected open neighbourhood of X containing $(0, 0)$, and remove the point $(0, 0)$. Then there are four remaining connected components. However, removing a point from any connected open set of $\mathbb{R}^k$ gives one connected component if $k > 1$, two connected components when $k = 1$, the empty set if $k = 0$, and it never gives four connected components. So there can be no homeomorphism to $\mathbb{R}^k$, and X cannot be a submanifold of $\mathbb{R}^2$. This means this fibre product cannot exist in the category of manifolds.

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Glossary

$\mathbf{Sets}$ Sets, Definition 2.1.1

$\mathbf{Euc}$ Euclidean Spaces, Definition 2.1.1

$\mathbf{Euc^c}$ Euclidean Spaces with corners, Definition 5.4.1

$\mathbf{Euc^c_{in}}$ Euclidean Spaces with corners with interior maps, Definition 5.4.1

$\mathbf{Man}$ Manifolds, Definition 2.1.1

$\mathbf{Man^b}$ Manifolds with boundary, Definition 5.4.1

$\mathbf{Man^c}$ Manifolds with corners, Definition 5.4.1

$\mathbf{Man^{gc}}$ Manifolds with g-corners, Definition 5.4.1

$\mathbf{Man^c_{in}}$ Manifolds with corners with interior maps, Definition 5.4.1

$\mathbf{Man^{gc}_{in}}$ Manifolds with g-corners with interior maps, Definition 5.4.1

$\mathbf{\check{M}an^c}$ Manifolds with corners with mixed dimension, Definition 5.4.1

$\mathbf{\check{M}an^c_{in}}$ Manifolds with corners with mixed dimension with interior maps, Definition 5.4.1

$\mathbf{\check{M}an^{gc}}$ Manifolds with g-corners with mixed dimension, Definition 5.4.1

$\mathbf{\check{M}an^{gc}_{in}}$ Manifolds with g-corners with mixed dimension with interior maps, Definition 5.4.1

$\mathbf{Mon}$ Monoids, Definition 5.4.1

$\mathbf{C^\infty Rings}$ C^∞-rings Definition 2.1.2

$\mathbf{CC^\infty Rings}$ Categorical C^∞-rings, Definition 2.1.1

$\mathbf{C^\infty Rings^{co}}$ Complete C^∞-rings, Definition 2.4.13

$\mathbf{CPC^\infty Rings^c}$ Categorical pre C^∞-rings with corners, Definition 4.1.2

$\mathbf{CPC^\infty Rings^c_{in}}$ Categorical interior pre C^∞-rings with corners, Definition 4.1.2

$\mathbf{PC^\infty Rings^c}$ Pre C^∞-rings with corners, Definition 4.2.1

$\mathbf{PC^\infty Rings^c_{in}}$ Interior pre C^∞-rings with corners, Definition 4.2.6

$\mathbf{C^\infty Rings^c}$ C^∞-rings with corners Definition 4.3.2

$\mathbf{C^\infty Rings^c_{in}}$ Interior C^∞-rings with corners Definition 4.3.2

$\mathbf{C^\infty Rings^c_{sc}}$ Semi-complete C^∞-rings with corners, Definition 5.3.4

$\mathbf{C^\infty Rings^c_{fi}}$ Firm C^∞-rings with corners, Definition 4.5.1

$\mathbf{C^\infty RS}$ C^∞-ringed spaces, Definition 2.4.1

$\mathbf{LC^\infty RS}$ Local C^∞-ringed spaces, Definition 2.4.1

$\mathbf{C^\infty RS^c}$ C^∞-ringed spaces with corners, Definition 5.1.1

$\mathbf{LC^\infty RS^c}$ Local C^∞-ringed spaces with corners, Definition 5.1.1

$\mathbf{C^\infty RS^c_{in}}$ Interior C^∞-ringed spaces with corners,, Definition 5.1.3

$\mathbf{LC^\infty RS^c_{in}}$ Interior local C^∞-ringed spaces with corners, Definition 5.1.3

$\mathbf{AC^\infty Sch}$ Affine C^∞-schemes, Definition 2.4.10

$\mathbf{C^\infty Sch}$ C^∞-schemes, Definition 2.4.10

$\mathbf{AC^\infty Sch^c}$ Affine C^∞-schemes with corners, Definition 5.4.1

$\mathbf{AC^\infty Sch^c_{fi}}$ Firm affine C^∞-schemes with corners, Definition 5.4.1

$\mathbf{AC^\infty Sch^c_{in}}$ Interior affine C^∞-schemes with corners, Definition 5.4.3

$\mathbf{AC^\infty Sch^c_{fi,in}}$ Firm interior affine C^∞-schemes with corners, Definition 5.4.3

$\mathbf{C^\infty Sch^c}$ C^∞-schemes with corners, Definition 5.4.1

$\mathbf{C^\infty Sch^c_{fi}}$ Firm C^∞-schemes with corners, Definition 5.4.1

$\mathbf{C^\infty Sch^c_{in}}$ Interior C^∞-schemes with corners, Definition 5.4.1

$\mathbf{C^\infty Sch^c_{fi,in}}$ Firm interior C^∞-schemes with corners, Definition 5.4.1

Spec Spectrum for C^∞-rings, Definition 2.4.4

$\mathrm{Spec}^{\mathrm{c}}$ Spectrum for C^∞-rings with corners, Definition 5.2.1

$\mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}}$ Spectrum for interior C^∞-rings with corners, Definition 5.2.4

Γ Global sections of C^∞-ringed spaces, Definition 2.4.7

Γ^{c} Global sections of C^∞-ringed spaces with corners, Definition 5.2.5

$\Gamma^{\mathrm{c}}_{\mathrm{in}}$ Global sections of interior C^∞-ringed spaces with corners, Definition 5.2.7

$\mathfrak{C}$-mod Modules over the C^∞-ring $\mathfrak{C}$, Definition 2.2.1

$\boldsymbol{\mathfrak{C}}$-mod Modules over the C^∞-ring with corners $\boldsymbol{\mathfrak{C}}$ Definition 4.7.1

$\mathcal{O}_X$-mod Sheaves of modules over the sheaf of C^∞-rings $\mathcal{O}_X$, Definition 2.5.1

$\boldsymbol{\mathcal{O}}_X$-mod Sheaves of modules over the sheaf of C^∞-rings with corners $\boldsymbol{\mathcal{O}}_X$, Definition 5.6.2

$\Omega_{\mathfrak{C}}$ Cotangent module of a C^∞-ring $\mathfrak{C}$, Definition 2.2.4

$\Omega_{\boldsymbol{\mathfrak{C}}}$ Cotangent module of a C^∞-ring with corners, Definition 4.7.3

${}^b\Omega_{\boldsymbol{\mathfrak{C}}}$ b-cotangent module of a C^∞-ring with corners, Definition 4.7.6

$\mathrm{Pr}_{\boldsymbol{\mathfrak{C}}}$ Prime ideals in $\mathfrak{C}_{\mathrm{ex}}$ of a C^∞-ring with corners $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$, Definition 5.8.1

C^{loc} Corners functor for local C^∞-ringed spaces with corners, Definition 5.7.1

C^{aff} Corners functor for affine firm C^∞-schemes with corners, Proposition 5.8.4

C Corners functor for firm C^∞-schemes with corners, Theorem 5.8.9

Φ_i C^∞-operation corresponding to the inclusion $i : [0,\infty) \to \mathbb{R}$, Definition 4.2.5

$\Psi_{\exp}$ C^∞-operation corresponding to the exponential map $\exp : \mathbb{R} \to [0,\infty)$, Definition 4.2.5

$\Phi_{\exp} = \Phi_i \circ \Psi_{\exp}$ C^∞-operation corresponding to the exponential map $\exp : \mathbb{R} \to \mathbb{R}$, Definition 4.2.5