

NONLINEAR DYNAMICS FROM DISCRETE TIME TWO-PLAYER STATUS-SEEKING GAMES

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ABSTRACT. We study the dynamics of two-player status-seeking games where moves are made simultaneously in discrete time. For such games, each player's utility function will depend on both non-positional goods and positional goods (the latter entering into "status"). In order to understand the dynamics of such games over time, we sample a variety of different general utility functions, such as CES, composite log-Cobb-Douglas, and King-Plosser-Rebelo utility functions (and their various simplifications). For the various cases considered, we determine asymptotic dynamics of the two-player game, demonstrating the existence of stable equilibria, periodic orbits, or chaos, and we show that the emergent dynamics will depend strongly on the utility functions employed. For periodic orbits, we provide bifurcation diagrams to show the existence or non-existence of period doubling or chaos resulting from bifurcations due to parameter shifts. In cases where multiple feasible solution branches exist at each iteration, we consider both cases where deterministic or random selection criteria are employed to select the branch used, the latter resulting in a type of stochastic game.

1. Introduction. There have been many studies on nonlinear dynamics arising from duopoly games. Dana and Montrucchio [6] considered at periodic and chaotic behaviours that arise in infinite horizon duopoly games with discounting using Markov-perfect equilibrium strategies, when players move alternately and simultaneously. Montrucchio [15] and Boldrin and Montrucchio [2] showed the existence of chaotic paths when the discounting factor is small enough and demonstrated with examples. Rand [20] showed the occurrences of chaos from two general hill-shaped reaction curves in Cournot duopoly. Matsumoto [14] studied methods to control chaos and demonstrated certain firms prefer chaotic market while others prefer a stable one. Papageorgiou [16] gave some specific examples of chaotic dynamics modelled as optimally controlled systems. N -player oligopolistic generalisations have also merited attention. Puu [17, 18] considered three oligopolist Cournot games, and the situation of adding even more players in [19]. Snyder, Van Gorder and Vajravelu [22] studied the dynamics of the continuous-time Cournot adjustment game for many players, and also considered the effects of policies such as taxation or subsidy.

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There are a number of papers focusing on status-seeking games and consumer behaviour, where each player's utility is not only determined by their absolute consumption of goods, but also by their relative expenditure on status-seeking activities. Possible applications of status-seeking games include charitable giving [7, 13], membership of clubs [25], the international demonstration effect [9], and consumer demand in general [4]. Congleton [5] looked at the economics implications of efficient status seeking activities. Brekke, Howarth and Nyborg [3] looked at duopoly status-seeking games and tested their results against the Hirsch hypothesis (that when income rises, a larger portion of expenditure is devoted to status-seeking activities). Ireland [8] studied the effects the income taxation on status-seeking activities. However, there are fewer studies on the categorisation of behaviours that arise from status-seeking games. Rauscher [21] noted the possibility of chaos for two hill-shaped reaction functions for a very specific status game formulation, but aside from this not much exists in the way of a mathematical treatment of status-seeking games.

In this paper, we extend Rauscher's approach [21], and investigate player behaviours systematically for different combinations of utility functions and status functions. This will allow us to better understand the various nonlinear dynamics emergent under such status-seeking games.

This paper is organized as follows. In Section 2, we state the most general model for a N -player discrete-time status-seeking game with the objective of one-period utility maximisation. However, for the remainder of the paper, we shall restrict our attention to the duopoly (two-player) case for a variety of utility functions. In Section 3, we consider possible dynamics of the two-player games when both players adopt a constant elasticity of substitution (CES) utility function. In Section 4, we consider dynamics under composite logarithmic Cobb-Douglas utility functions. One interesting feature is that we obtain non-unique solutions for some parameter values. If the solution branch employed at each timestep is selected randomly, then we obtain a stochastic process, for which case we consider the distribution of consumption for each player. In Section 5, we consider the dynamics under the case where both players adopt King-Plosser-Rebelo (KPR) utility functions. In Section 6 we discuss the appearance of chaos in two-player status-seeking games. Finally, we summarize our findings and make some comparisons and general remarks in Section 7.

2. The model. We adopt the formulation of Rauscher [21]. Assume each player wants to maximise their immediate welfare $u(\cdot, \cdot)$, with social status $s(y, \bar{Y})$, quantity of positional goods y , quantity of non-positional goods x , and average quantity of society's positional goods \bar{Y} . Relevant economic restrictions are let $s_y > 0$, $s_{\bar{Y}} < 0$, $s_{yy} \leq 0$, and $s_{y\bar{Y}} \leq 0$. We assume that all partial derivatives of $s(\cdot, \cdot)$ and $u(\cdot, \cdot)$ exist, and, when convenient, that u is separable in its arguments x and s (although in later sections we shall also consider cases where u is non-separable). The budget constraint is that each player can only purchase a fixed quantity of goods, and we normalize this to $x + y = 1$ with $0 \leq x, y, \bar{Y} \leq 1$. The first-order necessary condition for optimality is $u_x = u_s s_y$.

We focus on discrete time games and assume that both players move simultaneously. Let $u = u(1 - y, s(y, \bar{Y})) = f(y, \bar{Y})$ for some sufficiently smooth function f . Then u is max/minimised with respect to $y \in (0, 1)$ when $f_y = 0$, or at one of the endpoints $y = 0$, $y = 1$. Now assume the separability of variables of f_y ,

rearrange to get $y = h(\bar{Y})$ for some relation h . Note that h can be a multifunction, as multiple branches may be possible when performing the inversion of f_y . Note that the relation need not be defined for all \bar{Y} in those cases where the maximum is at either boundary. If we additionally assume that f is quasi-concave, then it has a unique global maximum, hence there exists a unique y for every \bar{Y} . Including the endpoints, we have that the relation between y - \bar{Y} reads

$$y = \max_{u(\cdot, \bar{Y})} \{0, 1, h(\bar{Y})\} = \arg \max_{y \in \{0, 1, h(\bar{Y})\}} u(\cdot, \bar{Y}). \quad (1)$$

Under the quasi-concave assumption, for each \bar{Y} there is a unique y which maximises u .

Let there be N -players, and label Player i 's variables and functions with subscript i . We have $\bar{Y}_{i,t}$ as a function of $N-1$ variables $y_{1,t}, \dots, y_{i-1,t}, y_{i+1,t}, \dots, y_{N,t}$. Define

$$\bar{Y}_{i,t} = \frac{1}{N-1} \sum_{j \neq i} y_{j,t}. \quad (2)$$

Substituting $\bar{Y}_{i,t}$ into (2) above and iterating once, we get a difference equation involving $y_{i,t+2}$ and $y_{i,t}$. If we additionally assume y - \bar{Y} is hill-shaped (with amplification and discouragement effects acting on different values of \bar{Y} ; see [20, 5]), then the maximum is found in the interior, hence we have the simpler $y = h(\bar{Y})$. Plugging in \bar{Y} , we obtain a system of difference equations

$$y_{i,t+2} = h_i \left(\frac{1}{N-1} \sum_{j \neq i} h_j \left(\frac{1}{N-1} \sum_{k \neq j} y_{k,t} \right) \right). \quad (3)$$

We cannot derive any further implications without giving specific functional forms for h used in Equation (3). Therefore, in the remaining sections, we consider specific examples of the utility functions. We shall focus on two-player games, in which case \bar{Y} is just the y -consumption of the other player. Whenever we need to distinguish between the two players, we use y and Y to denote the consumption of positional goods for Player 1 and Player 2, respectively. Note that in the two-player games, the simultaneous game is effectively the same as two independent games where players move alternatively. We consider the nonlinear dynamics of the adjustment processes of consumption levels that arise from these two-player games under different utility functions.

Throughout this paper, and following the literature on status-seeking, we consider two examples of status functions:

Case 1. Let $s = y/Y$ be the first case, which we shall refer to as ‘‘Case 1’’ henceforth. This measures status as a ratio of the players’ positional goods.

Case 2. Let $s = y - Y$ be the second case, which we shall refer to as ‘‘Case 2’’ henceforth. This measures status as the difference of the players’ positional goods.

These relative and additive status measure were used by Brekke, Howarth and Nyborg [3] and others.

3. CES utility function and its limiting form. We assume both players always adopt the same functional form of their utility function u , while we allow both players to adopt the same or different status functions. In this section, we shall consider the case where u takes the form of a constant elasticity of substitution (CES) utility function.

3.1. General CES utility function. For Player i , let

$$u_i = (\alpha_{i,1}x_i^{\lambda_i} + \alpha_{i,2}s_i^{\lambda_i})^{1/\lambda_i}. \quad (4)$$

We consider the symmetric situation first, and hence assume that taste parameters are the same for each individual. Here α_1, α_2 are non-negative, while $\lambda \in (-\infty, 1) \setminus \{0\}$. Then,

$$u_y = (\alpha_1(1-y)^\lambda + \alpha_2s^\lambda)^{(1-\lambda)/\lambda}(-\alpha_1x^{\lambda-1} + \alpha_2s_y s^{\lambda-1}).$$

When $\alpha_1 = 0$, the optimal choice is $y = 1$, while when $\alpha_2 = 0$, the optimal choice is $y = 0$. In general, optimal consumption will depend on the status functions.

Let us consider the limiting behaviour of the CES utility function with λ . Standard results are

$$\lim_{\lambda \rightarrow 0} u = x^{\alpha_1} s^{\alpha_2}$$

(i.e., utility converges to the Cobb-Douglas utility function),

$$\lim_{\lambda \rightarrow -\infty} u = \min\{x, s\}$$

(i.e., utility converges to a Leontief utility function). We consider general values of λ first, and then we will consider these limiting cases separately. For a detailed analysis of the CES utility function, we refer to Solow [23] and Arrow, Chenery, Minhas and Solow [1].

3.1.1. Both players adopt Case 1 status functions. Assume first that $Y \neq 0$. The budget constraint $x + y = 1$ is equivalent to $x + Ys = 1$. Define $k = \alpha_2/\alpha_1$ and $\tau = (\lambda-1)^{-1}$. The standard solution to CES utility maximisation gives the reaction function

$$y = \left[1 + k^\tau Y^{-(\tau+1)}\right]^{-1}. \quad (5)$$

Consider next the case of $Y = 0$. If $\lambda \in (0, 1)$, then $u = \infty$, giving the reaction function $y = 1$, which is a valid choice. If $\lambda \in (-\infty, 0)$, then $u = kx$ for some non-negative constant k . The reaction function returns $y = 0$ (equivalently $x = 1$), which is the maximising choice. Therefore the reaction function above indeed is suitable for $Y = 0$. Furthermore it is always the case that $y \in (0, 1)$, hence all solutions are interior (no corner solution is needed). We have

$$Y_{t+1} = \left[1 + k^\tau y_t^{-(\tau+1)}\right]^{-1}. \quad (6)$$

When both players adopt the Case 1 status function, we obtain the difference equation

$$y_{t+2} = \left[1 + k^\tau \left(1 + \left(\frac{k}{y_t^\lambda}\right)^{\tau+1}\right)^{\frac{\lambda}{\lambda-1}}\right]^{-1}. \quad (7)$$

The long-run dynamics of this difference equation depends on the four parameters $y(0)$, $Y(0)$, k , and λ . For $k \gg 1$ or $\lambda < 0$, we find that convergence to a stable equilibrium occurs for each player, hence in order to find more exotic dynamics we will focus on k smaller than or around 1 and $\lambda \in (0, 1)$.

In Figure 1, consumption for both players will always be the same, since their utility functions are symmetric. Starting from $y(0) = Y(0) = 1/2, k = 1/2, \lambda = 0$, we find a fix point (equilibrium value for each player). Continuing from the fixed point and varying λ from 0 to 1, we detect a BP (branch point)¹ and PD

¹Definition of BP: Branch point is where two or more FP-curves originate from.

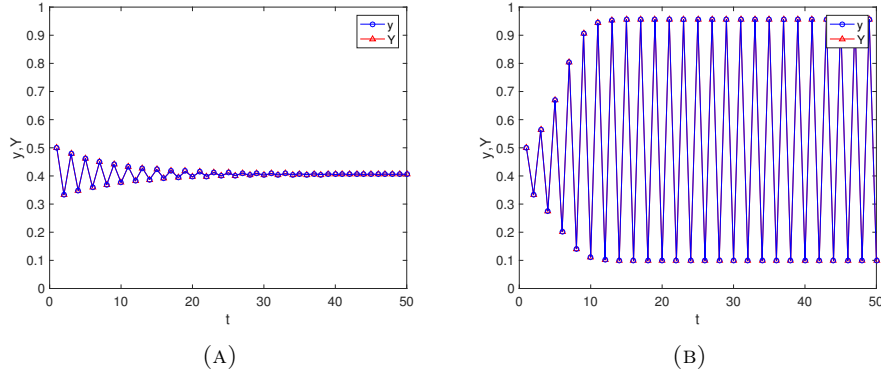


FIGURE 1. Time series for each player from the dynamics in (7) showing examples of convergence to equilibria and period-2 cycles when $y(0) = Y(0) = 0.5$, $k = 0.5$, and λ varies under the CES utility framework. (A) For $\lambda = 0.6$, we observe convergence to equilibrium quantities for both players. (B) For $\lambda = 0.7$, we observe periodic cycles.

(period-doubling)² when $\lambda \approx 0.6$, as shown in Figure 2a. Continue from the PD by two-parameter bifurcation, we detect a series of LPPD (Fold-Flip Bifurcation³) which simultaneous BP points, and a Cusp bifurcation⁴ at the turning point, on the parabolic PD-curve (shown in red), as shown in Figure 2b. We produce the FP-curves (shown in black) by varying λ again from some of these LPPD points, as shown in Figure 2a. For detailed analysis of the properties of different bifurcation points we refer to Kuznetsov [12]. Implications are, reading from Figure 2b, for parameters pairs (k, λ) in the region below or on the PD curve, period-2 oscillations occur. The two long-run fixed points of the oscillations are shown by the FP curves in Figure 2a. We notice period-2 dynamics can only arise for $\lambda > 0.5$ and $k < 1.2$ (given the fixed and symmetric initial condition we choose).

When the initial conditions are no longer the same, regions giving periodic solutions can drastically change, showing the sensitivity of bifurcation regions to initial conditions. For example, compare Figure 1 and Figure 3. With a change in initial conditions, the period-2 dynamics in panel (B) of Figure 1 degenerates into stable equilibria in panel (B) of Figure 3.

Consider now the asymmetric case where both players have the same CES functional form, but the parameters differ. Then the system of difference equations depends on parameters $k_1, k_2, \lambda_1, \lambda_2, y(0)$, and $Y(0)$. We can look at the bifurcation graphs when we vary different pairs of parameters. Similar PD curves with a series of LPPD points and a Cusp are found for each, as shown in Figure 4. Only one region of period-2 oscillations is found for bifurcation diagram. Note again a

²Definition of PD: Also called Flip bifurcation point, the point at which the period of the system is doubled.

³Definition of LPPD: The fold-flip bifurcation occurs if a map has a fixed point with multipliers +1 and -1 simultaneously.

⁴Definition of Cusp: The cusp bifurcation occurs when the critical equilibrium has a zero eigenvalue and the quadratic coefficient for the limit point bifurcation (LP) vanishes. The cusp bifurcation indicates the presence of a hysteresis.

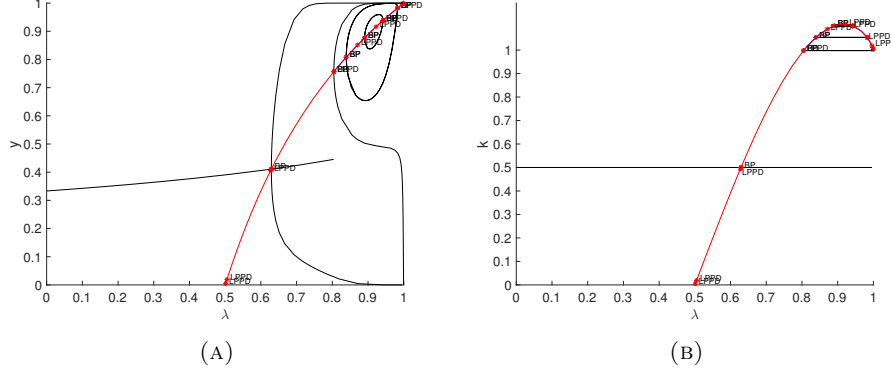


FIGURE 2. Bifurcation diagrams for the dynamics from (7) starting with $y(0) = Y(0) = 1/2, k = 1/2$, and obtained by varying λ first and then a second parameter. (A) One-parameter bifurcation gives the FP-curves (black). The first and only PD occurs at $\lambda \approx 0.6293$. (B) Two-parameter bifurcation gives the PD-curve (red), period-2 arises in the region on and below the PD curve.

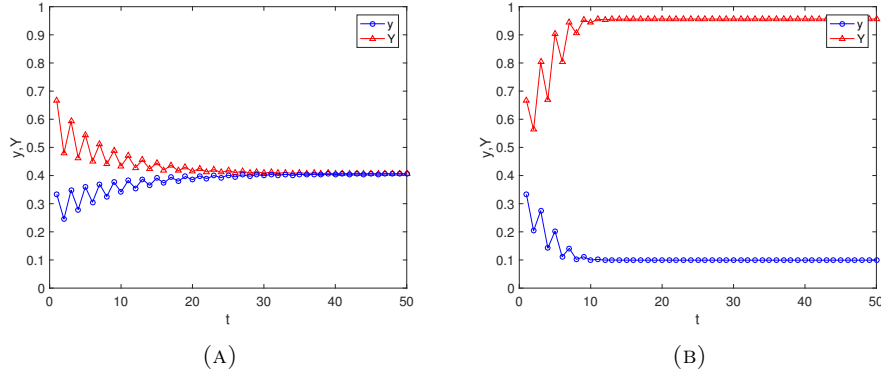


FIGURE 3. Time series plots for each player when we take a symmetric CES utility function giving dynamics (7). We set $y(0) = 1/3, Y(0) = 2/3, k = 0.5$, while taking (A) $\lambda = 0.6$ or (B) $\lambda = 0.7$. Values of fixed points of the system are the same as Figure 1, although the stability of the fixed points in the second panel precludes the period-2 dynamics seen in Figure 1.

change in the initial conditions would shift the regions in the bifurcation diagrams, but would not give rise to any new bifurcations.

3.1.2. Both players adopt Case 2. Now consider the case where both player adopt the Case 2 status functions, $s = y - Y$. We modify u so that $u = (\alpha_1 x^\lambda + \alpha_2(s + 1)^\lambda)^{1/\lambda}$, giving appropriate behavior with respect to s (so that $s + 1 \geq 0$). The

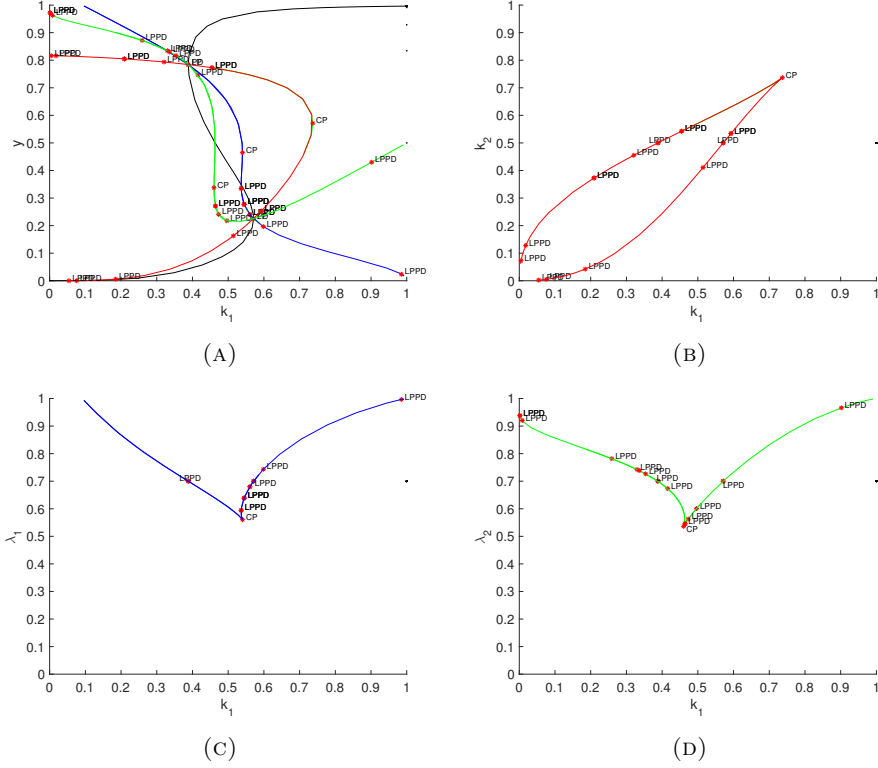


FIGURE 4. Bifurcation diagrams for the dynamics (7), when we have asymmetric CES utility functions for each player. Starting values are $y(0) = Y(0) = 1/2, k_1 = 1, k_2 = 0.5, \lambda_1 = \lambda_2 = 0.7$. There will be one FP-curve and three PD-curves as generated. (A) FP-curve (black) generated by varying k_1 first, detecting two PD points. Three PD-curves (blue, red, green) are generated from them. (B) Two-parameter PD bifurcation are generated by varying k_1, k_2 . (C) Two-parameter PD bifurcations are generated by varying k_1, λ_1 . (D) Two-parameter PD bifurcations are generated by varying k_1, λ_2 .

optimisation formula is unchanged, and we have the reaction function

$$y = \begin{cases} \frac{Y + (\frac{\alpha_1}{\alpha_2})^{\frac{1}{\lambda-1}} - 1}{1 + (\frac{\alpha_1}{\alpha_2})^{\frac{1}{\lambda-1}}}, & \text{for } Y \geq 1 - (\frac{\alpha_1}{\alpha_2})^{\frac{1}{\lambda-1}}, \\ 0, & \text{for } Y < 1 - (\frac{\alpha_1}{\alpha_2})^{\frac{1}{\lambda-1}}. \end{cases} \quad (8)$$

Defining the parameter $\kappa = (\frac{\alpha_1}{\alpha_2})^{\frac{1}{\lambda-1}} = (\frac{\alpha_1}{\alpha_2})^\tau$, we obtain the difference equation

$$y_{t+2} = \begin{cases} \frac{y_t + (\kappa-1)(\kappa+2)}{(\kappa+1)^2}, & \text{for } y_t \geq (1-\kappa)(\kappa+2), \\ 0, & \text{for } y_t < (1-\kappa)(\kappa+2). \end{cases} \quad (9)$$

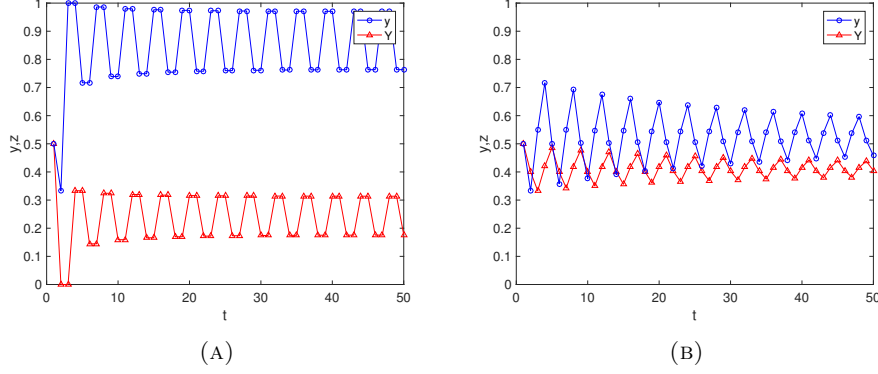


FIGURE 5. Time series for both players from the dynamics (10)-(11), when parameters are fixed at $y(0) = Y(0) = 1/2, k_1 = 1/2, \lambda_1 = 0.8$ while k_2 varies. (a) When $k_2 = 0.5$, we find gradual convergence to ‘period-2 with width-2’ dynamics. Since $k_2 < 1$, the corner solution is employed for this case. (b) When $k_2 = 1.5$, gradual convergence to period-4 dynamics is detected.

When $\kappa \geq 1$, we have $\lim_{t \rightarrow \infty} y_t = 1 - \frac{1}{\kappa}$. When $\kappa < 1$, we have that y_t will inevitably reach the second case ($y_t = 0$) and stays at zero thereafter. Therefore, the Case 2 status function choice yields equilibrium dynamics.

3.1.3. Asymmetric choice of status function. Let us now consider what happens when Player 1 adopts the Case 1 status function while Player 2 adopts the Case 2 status function (with an appropriately modified choice of u), to break the symmetry seen above. Since the status functions are different, we allow different parameters in utility functions. Define $k_1 = \frac{\alpha_{1,2}}{\alpha_{1,1}}$ and $k_2 = \frac{\alpha_{2,1}}{\alpha_{2,2}} \frac{1}{\lambda_2^{-1}}$. By the respective reaction functions defined earlier (Equations (5), (8)), we have the difference equation for Player 1 given by

$$y_{t+2} = \begin{cases} \left[1 + k_1^{\frac{1}{\lambda_1-1}} \left(\frac{1+k_2}{y_t+k_2-1} \right)^{\frac{\lambda_1}{\lambda_1-1}} \right]^{-1}, & \text{for } y_t > 1 - k_2, \\ 1, & \text{for } y_t < 1 - k_2, \lambda_1 \in (0, 1), \\ 0, & \text{for } y_t < 1 - k_2, \lambda_1 \in (-\infty, 0), \end{cases} \quad (10)$$

and the difference equation for Player 2 given by

$$Y_{t+2} = \begin{cases} \frac{\left[1 + k_1^{\frac{1}{\lambda_1-1}} (Y_t)^{\frac{-\lambda_1}{\lambda_1-1} + k_2 - 1} \right]^{-1}}{k_2 + 1} + k_2 - 1, & \text{for } Y_t > \left(\frac{1}{k_2} - 1 \right)^{1 - \frac{1}{\lambda_1}} k_1^{\frac{1}{\lambda_1}}, \\ 0, & \text{for } Y_t < \left(\frac{1}{k_2} - 1 \right)^{1 - \frac{1}{\lambda_1}} k_1^{\frac{1}{\lambda_1}}. \end{cases} \quad (11)$$

The dynamics will depend on the parameters k_1, k_2, λ_1 . We detect periodic behaviours, as shown in Figure 5.

Consider $k_2 \geq 1$ so that no corner solution occurs (hence we have a smooth map). Starting at $y(0) = Y(0) = 1/2, k_1 = 0.5, k_2 = 1.5, \lambda_1 = 0$ we find its fixed point. Continuing from the fix point and varying λ_1 from 0 to 1, we detect a

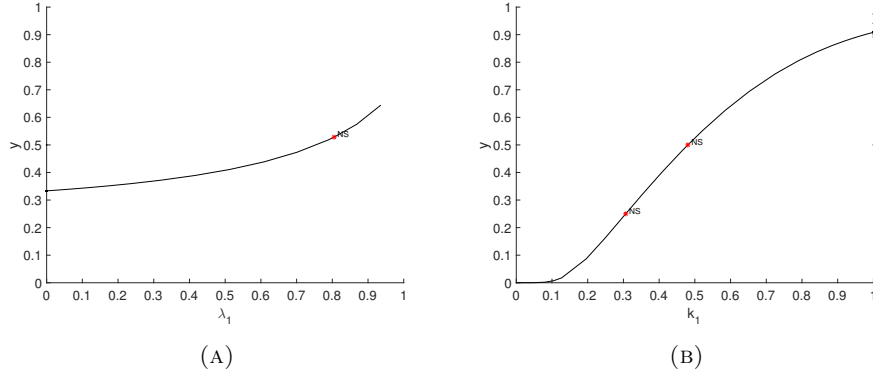


FIGURE 6. One-parameter bifurcation diagrams for the dynamics (10)-(11) starting at $y(0) = Y(0) = 1/2, k_2 = 1.5$. (a) We fix $k_1 = 0.5$, and find an NS bifurcation at $\lambda_1 = 0.80491$. (b) We fix $\lambda_1 = 0.8$, and find NS bifurcations at both $k_1 = 0.306389$ and $k_1 = 0.48045$.

Neimark-Sacker(NS) bifurcation point⁵, as shown in Figure 6a. It is a supercritical NS point (stable) with normal form coefficient $\approx -1.633 < 0$. Similarly, starting at $y(0) = Y(0) = 1/2, k_1 = 1, k_2 = 1.5, \lambda_1 = 0.8$, we find a fixed point and vary k_1 instead, obtaining two supercritical NS points, as shown in Figure 6b. Varying k_2 , or changing initial conditions, give similar diagrams. Therefore, only two regions that give stable period-4 dynamics are found.

3.2. Log utility function. For Player i , let $u_i = \alpha_{i,1} \ln(x_i) + \alpha_{i,2} \ln(s_i)$. Again let both players have the same taste parameters. Assume that $\alpha_1, \alpha_2 \geq 0$. We then have $u_y = \frac{\alpha_2 s_y}{s} - \frac{\alpha_1}{1-y}$. Utility will be maximised when $u_y = 0$. If $\alpha_1 = 0$, utility is completely comparative and optimal consumption is $(x, y) = (0, 1)$, while if $\alpha_2 = 0$, utility is independent of status, and the optimal consumption is $(x, y) = (1, 0)$.

Consider when both players have Case 1 status functions. The reaction function is then $y = \frac{\alpha_2}{\alpha_1 + \alpha_2}$, which is a constant. This also covers the case when $Y = 0$, since any nonzero y gives the same s (infinite). When both players adopt Case 1, consumption is static. On the other hand, consider when both players have Case 2 status functions. Assuming $x \neq 0, s \neq -1$, we obtain reaction function $y = \frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2} + \frac{\alpha_1}{\alpha_1 + \alpha_2} Y$ (given an interior solution) or $y = 0$ (when $Y < 1 - \frac{\alpha_2}{\alpha_1}$). Let $k = \frac{\alpha_2}{\alpha_1}$, and we have exactly the same behaviour as in the difference equation (9). The interpretation is that when there is more bias toward status, consumption for positional goods y converge to a positive limit, whereas when there is more bias toward x , consumption for y tends to zero. For the asymmetric case where one player adopts the Case 1 status function and the other player adopts the Case 2 status function, consumption will again be static.

3.3. Cobb-Douglas utility function. For Player i , let $u_i = x_i^{\alpha_{i,1}} s_i^{\alpha_{i,2}}$. Assume again that the utility function is the same for both players, and that $x, s \geq 0$, while

⁵Definite of NS: Neimark-Sacker bifurcation is when a closed invariant curve arises from a fix point, when the fix point changes its stability via a pair of complex eigenvalues with unit modulus. It can be stable or unstable, depending on the sign of the first Lyapunov exponent.

the parameters satisfy $\alpha_1, \alpha_2 \geq 0$. By taking \ln on both sides of the definition of the utility function we have $\ln(u) = \alpha_1 \ln(x) + \alpha_2 \ln(s)$. Since \ln is a strictly increasing function defined on the positive reals, the optimisation solution for choices of (x, y) for a Cobb-Douglas utility function, will be the same as that for the equivalent logarithmic utility function considered in Section 3.2. Hence, all dynamics obtained from using a Cobb-Duglass utility function can be recovered from those studied in Section 3.2.

3.4. Leontief utility function. For Player i , let $u_i = \min\{\alpha_{i,1}x_i, \alpha_{i,2}s_i\}$, where $\alpha_1, \alpha_2 > 0$. The maximal u always occurs when $\alpha_1 x = \alpha_2 s$.

If both players have Case 1 status functions, we obtain the difference equation

$$y_{t+2} = \frac{\alpha_1^2 y_t}{\alpha_2^2 + \alpha_1 y_t (\alpha_2 + \alpha_1)}. \quad (12)$$

This difference equation has the fixed point $y = 1 - \frac{\alpha_2}{\alpha_1}$ for $\alpha_2 < \alpha_1$. When $\alpha_2 \geq \alpha_1$, optimal output converges to 0.

Assume both players have Case 2 status functions. The reaction function is $y = \frac{Y + \alpha_1/\alpha_2 - 1}{\alpha_1/\alpha_2 + 1}$ or $y = 0$ if $Y < 1 - \frac{\alpha_1}{\alpha_2}$. When both players adopt Case 2, this is again the same as studied in Equation (9) (by putting $k = \frac{\alpha_1}{\alpha_2}$, and noting that the roles of α_1 and α_2 are switched).

In the asymmetric case where Player 1 adopts the Case 1 status function and Player 2 adopts the Case 2 status function, let $k_1 = \frac{\alpha_{1,2}}{\alpha_{1,1}}$ and $k_2 = \frac{\alpha_{2,1}}{\alpha_{2,2}}$. We obtain the difference equation for Player 1,

$$y_{t+2} = \begin{cases} \frac{y_t + k_2 - 1}{y_t + k_1 k_2 + k_1 + k_2 - 1}, & \text{for } y_t > 1 - k_2, \\ 0, & \text{for } y_t < 1 - k_2. \end{cases} \quad (13)$$

Note the similarity with Equation (10). When $k_2 \geq 1$, the boundary solution never occurs. Solving the first case we obtain a unique fixed point. Hence y_t converges to a stable equilibrium value. When $k_2 < 1$, y_t either converges to a positive equilibrium value, or hits and stays at $y = 0$ at some finite time. Player 2 has the difference equation

$$Y_{t+2} = \begin{cases} \frac{\frac{Y_t}{k_1 + Y_t} + k_2 - 1}{k_2 + 1}, & \text{for } Y_t > k_1(\frac{1}{k_2} - 1), \\ 0, & \text{for } Y_t < k_1(\frac{1}{k_2} - 1). \end{cases} \quad (14)$$

Again note the similarity with Equation (11). However, since Player 1 has a stable equilibrium solution, so will Player 2.

4. Composite Log-Cobb-Douglas utility. From the previous section, we saw that only the only general CES provides possibility of non-equilibrium dynamics. In this section, we demonstrate that a composite of logarithmic and Cobb-Douglas utility functions introduce richer dynamics to the two-player status game. To begin with, let

$$u_i = \alpha_{i,1} \ln(x_i) + \alpha_{i,2} \ln(s_i) + \alpha_{i,3} x_i^{\beta_{i,1}} s_i^{\beta_{i,2}}, \quad (15)$$

where $\alpha_{i,1}, \alpha_{i,2}, \alpha_{i,3}, \beta_{i,1}, \beta_{i,2} \geq 0$ for both players. Again we partition the results based on choice of status function.

4.1. Both players adopt Case 1 status functions. For a player with a Case 1 status function, $s = y/Y$, we have the inverse reaction function (for and interior solution) obtained by taking $u_s/u_x = \text{MRS} = \text{price ratio}$,

$$Y = \frac{(1-y)^{\frac{\beta_{i,1}}{\beta_{i,2}}} y [y \alpha_{i,3} (\beta_{i,1} + \beta_{i,2}) - \alpha_{i,3} \beta_{i,2}]^{\frac{1}{\beta_{i,2}}}}{[\alpha_{i,2} - (\alpha_{i,1} + \alpha_{i,2}) y]^{\frac{1}{\beta_{i,2}}}}. \quad (16)$$

Without further assumptions it is difficult to proceed. Therefore we assume both players put the same weight on x and s . Let $\alpha_{i,1} = \alpha_{i,2} = \alpha_i, \beta_{i,1} = \beta_{i,2} = \beta_i$ for added simplicity. The reaction function for Player i now becomes

$$Y = (-1)^{\frac{1}{\beta_i}} \left(\frac{\alpha_{i,3} \beta_i}{\alpha_i} \right)^{\frac{1}{\beta_i}} (1-y)y. \quad (17)$$

Only for certain values of β_i will this give a correct interior solution. Assume further that $\beta_1 = \beta_2 = 1/2$, then for Player i this reaction function becomes

$$Y = \left(\frac{\alpha_{i,3}}{2\alpha_i} \right)^2 (1-y)y.$$

Let $r_i = \left(\frac{\alpha_{i,3}}{2\alpha_i} \right)^2$. Then the *backward* reaction function becomes the standard logistic map $Y = r_i(1-y)y$, equivalently

$$y = \frac{r_i \pm (r_i(r_i - 4Y))^{1/2}}{2r_i}. \quad (18)$$

Recall that discrete time logistic maps such as this were the motivation for chaos in status-seeking games [21] (which we shall discuss further in Section 6, in the case of *forward* reaction functions of quadratic form). Assume further that $r_i \geq 4$ (equivalently, $\alpha_{i,3} \geq a_i$) so that there is no complex root (at the same time, $y \in (0,1)$). When $r_i < 4$, there is no real root which means corner solution is always optimal. The dynamics then depend on how the players choose between each of the two optimal consumption branches. We shall label these branches as the ‘+’ branch and the ‘-’ branch. There are essentially two ways of choosing between them. One is a deterministic selection, and one is to select between the optimal branches at random. Without looking at the time difference equation (since little algebraic simplification can be done), we simulate the dynamics directly from their reaction function formulations, in each of these two possible cases.

If both players react with fixed choice, convergence to their respective equilibrium occurs, since there is a pair of linear reaction functions. If one player has a fixed choice at each iteration, and the other has some non-fixed yet deterministic rule (e.g., select the ‘+’ branch every n iterations), then behaviour dynamics can be observed; see Figure 7. Note that changing initial configurations $y(0), Y(0)$ makes no impact on the long-run behaviour. When both players have non-fixed deterministic solution branch choices, similar albeit more complicated periodic dynamics are still observed; see Figure 8.

If both players choose their branch at each iteration randomly, there will still be structure to the solutions. The set of numerical values of consumption falls in specific regions. We can also see repetition of patterns in some small time sections. We give time series in Figure 9 and the distribution of consumption levels in Figure 10.

Let us now consider the asymmetric case where Player 1 makes a deterministic choice of the optimal branch, while Player 2 makes random choices. We give one example of this in Figure 11, and give a corresponding phase portrait in Figure 12.

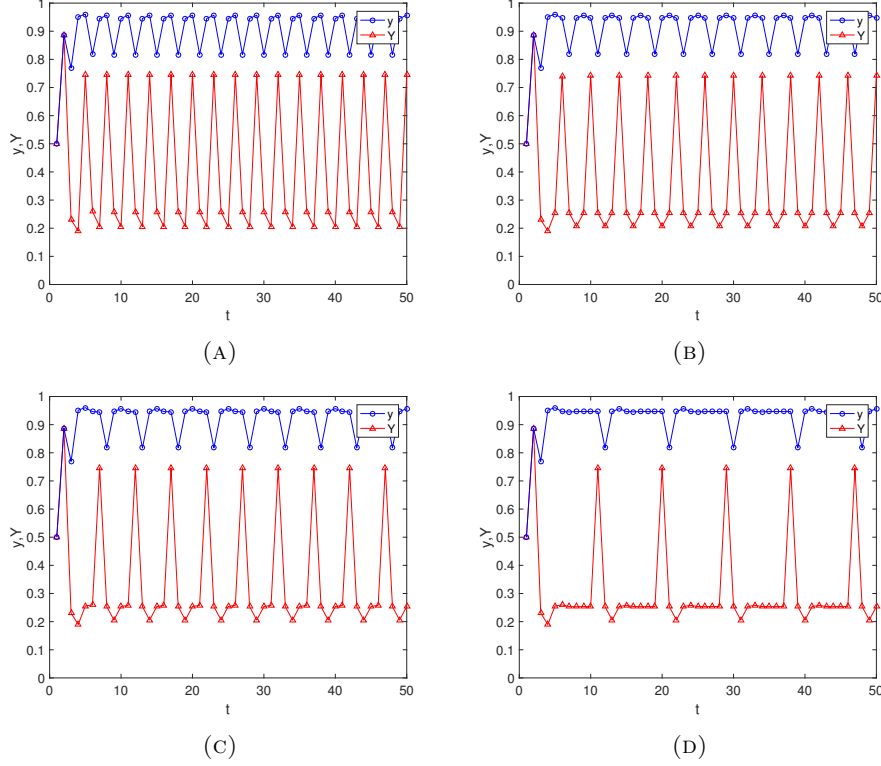


FIGURE 7. Time series of the dynamics from reaction curves of type (18) when $r_1 = r_2 = 5$, $y(0) = Y(0) = 0.5$. Player 1 picks the + branch at each iteration. (a) Player 2 picks the + branch every 3 iterations. (b) Player 2 picks the + branch every 4 iterations. (c) Player 2 picks the + branch every 5 iterations. (d) Player 2 picks the + branch every 9 iterations.

We observe regions of strong concentration or return (red dots) for different strategies of Player 1. This suggests that order arise as the strategy of Player 1 strategy gets closer to a fixed choice. We consider the situation where Player 1 always picks the same branch, while Player 2 still makes random choices, in Figure 13.

4.2. Both players adopt Case 2 status functions. If both players adopts the Case 2 status function, we need to modify our utility function by changing s to $s + 1$. We can only obtain an implicit algebraic relation for an interior solution, given by

$$\begin{aligned} \alpha_{i,2}(1-y) + \alpha_{i,3}\beta_{i,2}(1-y)^{\beta_{i,1}+1}(y+Y+1)^{\beta_{i,2}} \\ = \alpha_{i,1}(y+Y+1) + \alpha_{i,3}\beta_{i,1}(y+Y+1)^{\beta_{i,2}+1}(1-y)^{\beta_{i,1}}. \end{aligned} \quad (19)$$

We make similar assumptions to those made in Section 4.1. Let $\alpha_{i,1} = \alpha_{i,2} = \alpha_i$ and $\beta_{i,1} = \beta_{i,2} = \beta_i$. We obtain the reaction function

$$y = \frac{-Y \pm (Y^2 + 4Y + 4 - 4(\frac{-\alpha_i}{\alpha_{i,3}\beta_i})^{\frac{1}{\beta_i}})^{1/2}}{2}. \quad (20)$$

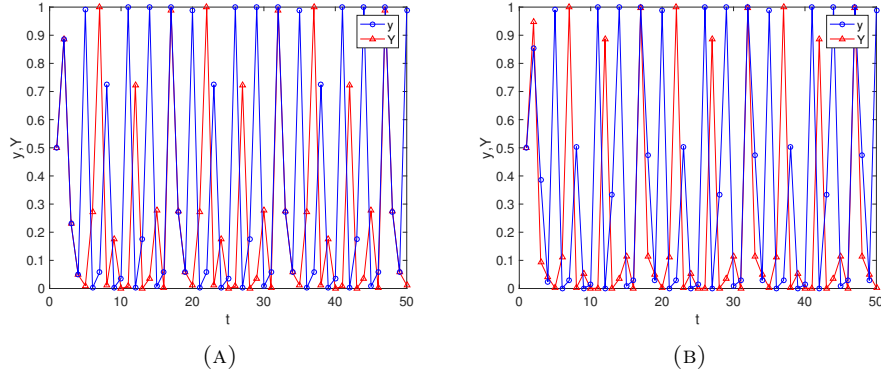


FIGURE 8. Time series of the dynamics from reaction curves of type (18). Player 1 chooses the + branch every 3 iterations, Player 2 chooses the + branch every 5 iterations. We observe period-15 dynamics. For the parameter values, we take (a) $r_1 = 5, r_2 = 5, y(0) = Y(0) = 1/2$, (b) $r_1 = 4, r_2 = 10, y(0) = Y(0) = 1/2$.

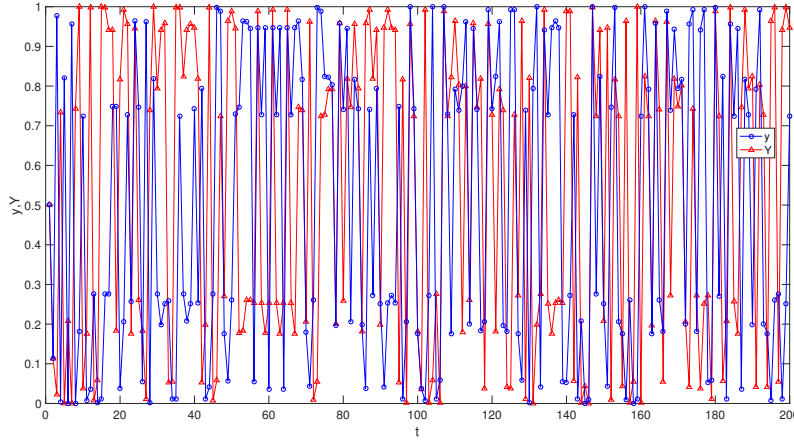


FIGURE 9. Time series of the dynamics from reaction curves of type (18) when we take $r_1 = r_2 = 5, y(0) = Y(0) = 0.5$. Both players choose between the two optimal branches randomly at each iteration, with the probability of selecting either branch equal to 0.5.

The negative branch can be discarded. Assume for simplicity that $\beta_1 = \beta_2 = 1/2$. For the square root to be real for all Y , let $r_i = \frac{\alpha_i}{\alpha_{i,3}}$ and assume $r_i \leq 2$. The reaction function now becomes

$$y = \frac{-Y + (Y^2 + 4Y + 4 - r_i^2)^{1/2}}{2}. \quad (21)$$

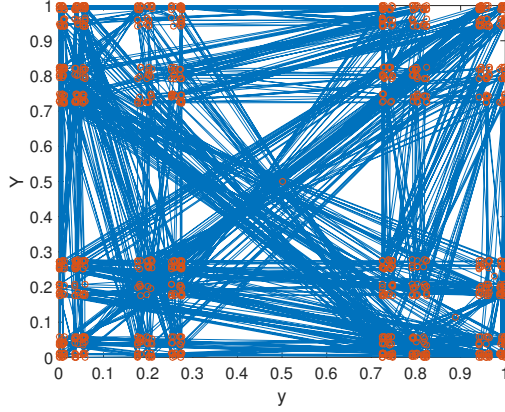


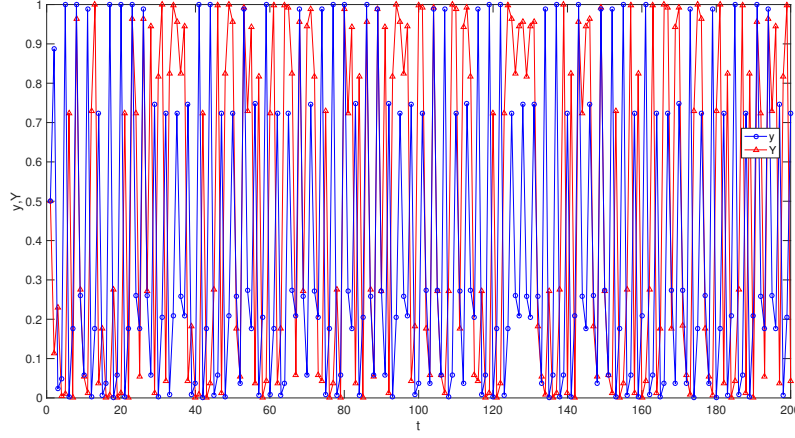
FIGURE 10. Phase portrait (y, Y) of 1000 iterations of the dynamics from reaction curves of type (18), given $r_1 = r_2 = 5, y(0) = Y(0) = 1/2$. Clear structure can be observed. Only a small concentration region (note there are not exactly 4, or 16, or 64 points) of (y, Y) is hit (these are denoted by the red dots). Consumptions fall into those regions immediately, and jump between them frequently (as indicated by the cobweb lines in blue).

Note that $y \in [0, 1]$, hence all solutions are interior. Without looking at form of the difference equation corresponding to this reaction function, we can assert optimal consumption converges for both players. Since $dy/dY > 0$ for both players, the iteration converges quickly to the intersection of the reaction functions, regardless of initial conditions, as in the example shown in Figure 14.

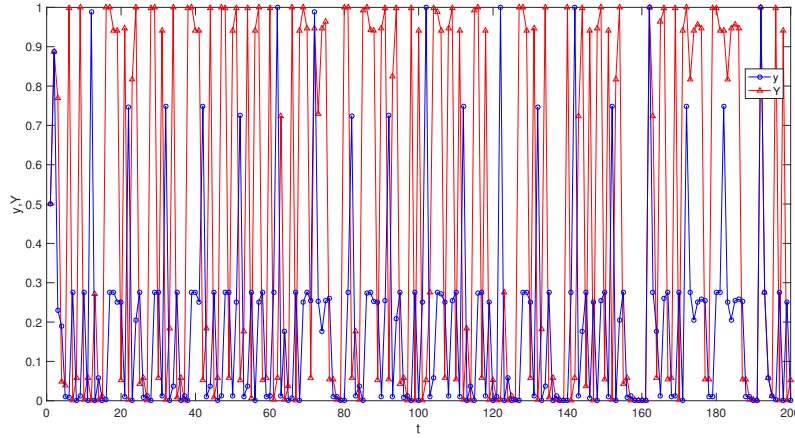
4.3. Asymmetric status function. Now assume that Player 1 chooses a Case 1 status function, while Player 2 chooses a Case 2 status function (with appropriately modified utility function) and hence Player 2 always has a fixed reaction function branch as in (21). We the assumptions $\alpha_{i,1} = \alpha_{i,2} = \alpha_i$, $\beta_{i,1} = \beta_{i,2} = \beta_i$, $\beta_i = 1/2$ for simplify, and take $r_1 \geq 4, r_2 \in [0, 2]$ to avoid boundary solutions. Limited simplifications can be done on their difference equations, hence we simulate the system from the reaction functions. Similar behaviours to those observed in Section 4.1.1 are seen. If Player 1 always picks one branch, rapid convergence to an equilibrium value occurs since there is a pair of linear reaction functions in this case. If Player 1 picks a different branch in a deterministic manner, then periodic dynamics are observed. Note that unlike what was seen in Figure 7, the number of values obtainable by the players is not bounded; however, within a period, both players have linear reaction functions, and convergence occurs, as shown in Figure 15.

If Player 1 picks branches randomly (with a probability of 0.5 for each branch), then we observe similar behaviour to what was seen in Figure 13. We give two examples with different parameters in Figures 16 and 17.

5. King-Plosser-Rebelo utility. King-Plosser-Rebelo (KPR) preferences is another utility measure ([10, 11]), defined over consumption and leisure. For our purposes, we replace leisure with status. A general KPR utility function takes the



(A)



(B)

FIGURE 11. Time series of the dynamics from reaction curves of type (18) when $r_1 = r_2 = 5, y(0) = Y(0) = 0.5$. Player 1 selects optimal branches deterministically, while Player 2 selects branches randomly with probability 0.5 for each. (a) Player 1 selects the + branch every 3 iterations, (b) Player 1 selects the + branch every 10 iterations.

form

$$u_i = \frac{1}{1 - \alpha_i} x_i^{1 - \alpha_i} v_i(s_i), \quad (22)$$

where v_i is increasing and concave if $\alpha_i \in (0, 1)$ or decreasing and convex if $\alpha_i > 1$. As a decreasing function v on s doesn't make sense in the economics context we are interested in, we only consider $\alpha_i \in (0, 1)$.

One example of an increasing and concave function v is $v = s^\lambda$, for $\lambda \in (0, 1)$. The utility function then becomes $(1 - \alpha)u = x^{1 - \alpha} s^\lambda$, which is just Cobb-Douglas.

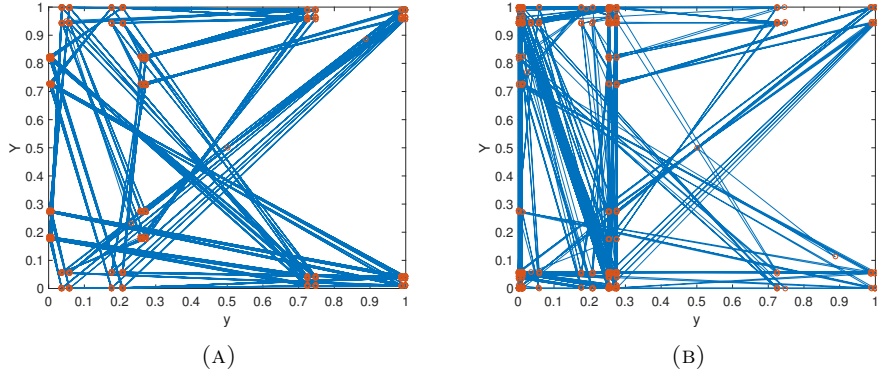


FIGURE 12. Phase portraits of the dynamics from reaction curves of type (18) with 500 iterations, given parameter values $r_1 = r_2 = 5, y(0) = Y(0) = 0.5$. Player 2 selects branches at random (with probability 0.5), while (a) Player 1 selects the + branch every 3 iterations, (b) Player 1 selects the + branch every 10 iterations.

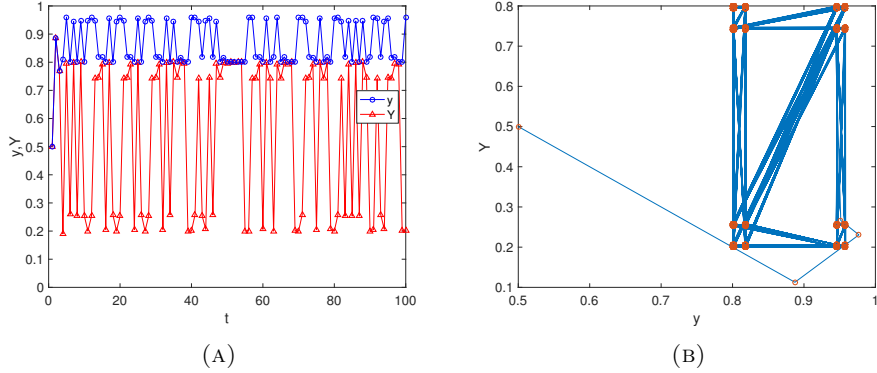


FIGURE 13. Time series (a) and bifurcation diagram after 100 iterations (b) of the dynamics from reaction curves of type (18) when $r_1 = r_2 = 5, y(0) = Y(0) = 0.5$. Player 1 always picks the + branch, while Player 2 chooses branches at random with probability 0.5 for each iteration. Due to the consistency in Player 1's choices, the concentration clusters (red dots) narrow compared to what was seen in Figure 12.

Consider, as another example, $v = \ln(s)$. Clearly v is an increasing and concave function for $s \geq 0$. Let both players use Case 1 status functions. We obtain an implicit reaction relation for Player i ,

$$1 - y = (1 - \alpha_i)y(\ln y - \ln Y). \quad (23)$$

We rearrange to obtain an explicit inverse reaction function,

$$Y = \exp\left(\ln(y) - \frac{1 - y}{y(1 - \alpha_i)}\right). \quad (24)$$

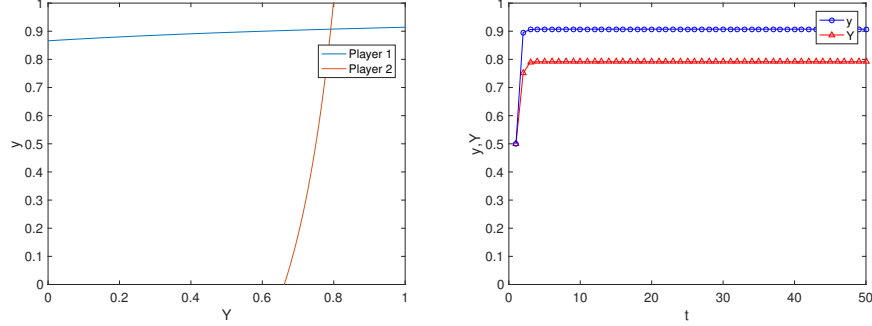


FIGURE 14. An example of (a) two reaction curves of type (21) for both players and (b) time series showing convergence to the fixed point corresponding to the point of intersection along the reaction curves. Parameter values are $y(0) = Y(0) = 1/2, r_1 = 1, r_2 = 1.5$.

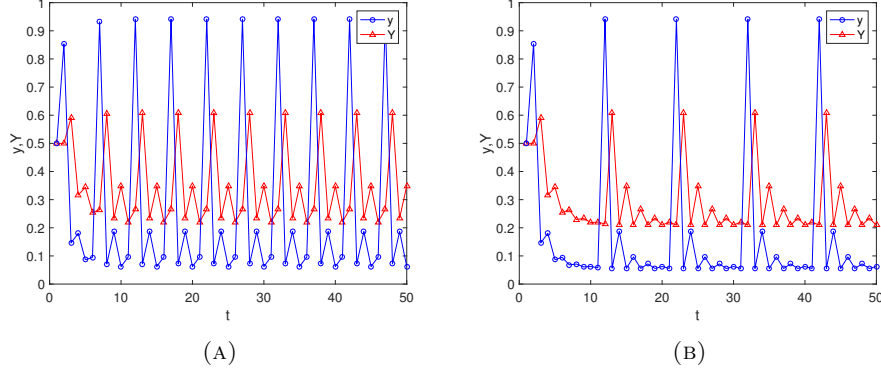


FIGURE 15. Time series of the dynamics from reaction curves of type (18) with asymmetric status choice and parameter values $r_1 = 4, r_2 = 2, y(0) = Y(0) = 1/2$. (a) Player 1 picks the + branch every 5 iterations. (b) Player 1 picks the + branch every 10 iterations. Player 2 always faces a single solution branch (21).

Looking at the right-hand side, $\ln(y) - \frac{1-y}{y(1-\alpha_i)}$ is a strictly increasing function of y . Therefore the reaction function is indeed strictly increasing ($dy/dY > 0$). When both players adopt case 1, we have convergence to equilibrium values. On the other hand, assume that both players use Case 2 status functions, and modify v so that $v = \ln(s+1)$. We obtain an implicit reaction relation for Player i ,

$$1 - y = (1 - \alpha_i)(y - Y + 1) \ln(y - Y + 1). \quad (25)$$

We can obtain an explicit inverse reaction function

$$Y = 1 + y - \exp\left(W\left(\frac{1-y}{1-\alpha_i}\right)\right), \quad (26)$$

where W is the principle branch of the Lambert W function. We observe the right-hand side of the equation, as a function of y , is monotonically increasing. It is also

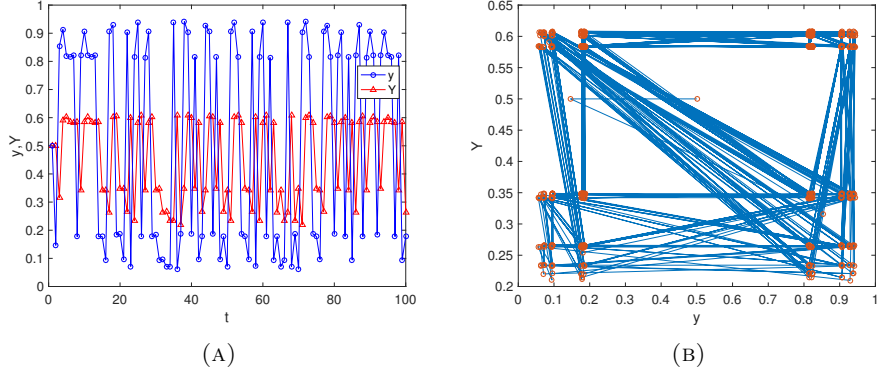


FIGURE 16. Time series (a) and bifurcation diagram (b) of the dynamics from reaction curves of type (18) with asymmetric status choice, given parameter values $r_1 = 4, r_2 = 2, y(0) = Y(0) = 1/2$. Player 1 picks branches randomly (with a probability of 0.5 for each branch) at each iteration, while Player 2 always faces a single solution branch (21).

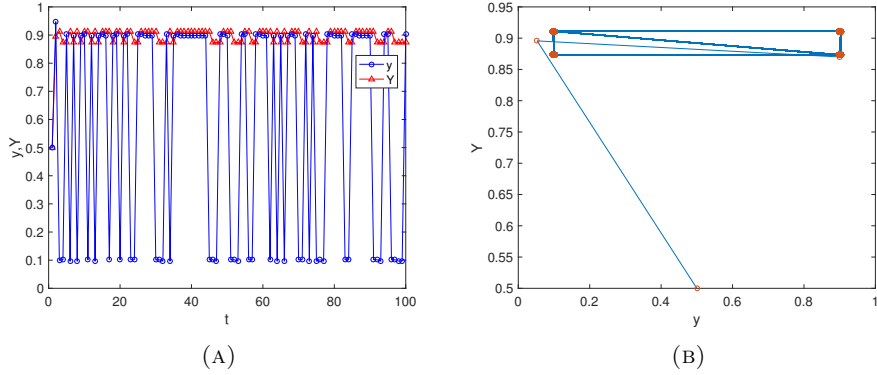


FIGURE 17. Time series (a) and bifurcation diagram (b) of the dynamics from reaction curves of type (18) with asymmetric status choice, given parameter values $r_1 = 10, r_2 = 1, y(0) = Y(0) = 1/2$. Player 1 picks branches randomly (with a probability of 0.5 for each branch) at each iteration, while Player 2 always faces a single solution branch (21).

a one-to-one function. Therefore y is an increasing function of Y (i.e. $dy/dY > 0$), and we have convergence to equilibrium values. Other specific values of $v(s)$ give similar results.

Let us return to the more general formulation (22). The optimality condition gives the identity

$$0 \equiv -v_i(s_i) + \frac{1-y}{1-\alpha_i} v'_i(s_i) \frac{\partial s_i}{\partial y}. \quad (27)$$

Consider the Case 2 status function for both players, and let us differentiate this identity with respect to Y . We obtain

$$0 = -v'_i(y - Y) \left(\frac{dy}{dY} - 1 \right) - \frac{1}{1 - \alpha_i} v'_i(y - Y) \frac{dy}{dY} + \frac{1 - y}{1 - \alpha_i} v''_i(y - Y) \left(\frac{dy}{dY} - 1 \right). \quad (28)$$

Assuming $\alpha_i \in (0, 1)$ and hence that $v_i(\cdot), v'_i(\cdot) > 0$ while $v''_i(\cdot) < 0$, we have that

$$\frac{dy}{dY} = \frac{v'_i(y - Y) - \frac{1-y}{1-\alpha_i} v''_i(y - Y)}{\left(1 + \frac{1}{1-\alpha_i}\right) v'_i(y - Y) - \frac{1-y}{1-\alpha_i} v''_i(y - Y)} > 0, \quad (29)$$

hence $dy/dY > 0$. As such, even for general functional forms of v_i which are increasing and concave, we have convergence to equilibrium values rather than non-equilibrium dynamics. Similar, yet more messy, computations may be carried out for the Case 1 status function. In that case, one can not always sign the derivative dy/dY , so more care must be taken. Still, for specific functional forms of s we have considered, we find $dy/dY > 0$ for the Case 1 status function, as well.

6. Chaos in status-seeking games. Up to this point we have found a range of equilibrium and non-equilibrium dynamics for a variety of economically relevant utility functions. We found chaotic time series when random selection was employed to select the optimal solution branch in the case of multi-valued relations, but for all single branch cases or scenarios where one of multiple branches were selected in a deterministic manner, we observed various periodic dynamics for non-equilibrium cases rather than chaos. This is in large part due to the form of the utility functions selected. Still, chaos was shown to be possible in Rauscher [21], so we shall now explore this feature in more detail.

Unlike the utility functions used in Sections 3 - 5, the utility function employed to give chaos in Rauscher [21] is perhaps a bit contrived, rather than economically relevant, as the main purpose of the utility function was to demonstrate that chaos could indeed be possible in two-player status-seeking games. The utility functions employed by Rauscher [21] are of the form

$$u_1(x, s) = x + s - \frac{a}{2} s^2, \quad (30)$$

$$u_2(x, s) = 4x - x^2 + 2s - s^2, \quad (31)$$

where a is a taste parameter. Under the assumption that Player 1 has a Case 1 status function and Player 2 has a Case 2 status function, one obtains the difference equation

$$y_{t+2} = \frac{1}{2a} y_t \left(1 - \frac{1}{2} y_t \right). \quad (32)$$

If we rescale like $z_t = \frac{1}{2} y_t$, then we obtain

$$z_{t+2} = \frac{1}{2a} z_t (1 - z_t), \quad (33)$$

which is precisely the difference equation obtained in Rauscher [21].

6.1. Generalisation of Rauscher's chaotic dynamics. Let us now consider a natural generalisation of Rauscher's chaotic dynamics given in [21]. Assume that Player 1 has the utility function

$$u_1(x, s) = x + As - p(s) \quad (34)$$

for some differentiable function p and constant $A > 0$, while Player 2 has the utility function

$$u_2(z, s) = Bx + Bs - Cs^2 \quad (35)$$

for constants $B, C > 0$. Player 1 will chase a Case 1 status function, while Player 2 will chose a Case 2 status function. The optimality condition on Player 2 gives $Y = y$ and hence $Y_{t+1} = y_t$. The optimality condition for Player 1 is equivalent to

$$Y - A + p' \left(\frac{y}{Y} \right) = 0. \quad (36)$$

Inverting relation (36) will give the needed reaction curve. Noting $Y_t = y_{t-1}$, we then obtain the implicit difference equation

$$y_t - A + p' \left(\frac{y_{t+2}}{y_t} \right) = 0. \quad (37)$$

To obtain any explicit difference equation, we need to specify the functional form of p . First consider the case where

$$p(s) = P_0 s^{1+\sigma^{-1}}, \quad (38)$$

where $P_0 > 0$ and $\sigma > 0$ are parameters. Then, we recover from (36) the reaction curve

$$y = \left(\frac{\alpha - Y}{P_0 (1 + \frac{1}{\sigma})} \right)^\sigma Y \quad (39)$$

and then the difference equation

$$y_{t+2} = \left(\frac{\alpha - y_t}{P_0 (1 + \frac{1}{\sigma})} \right)^\sigma y_t. \quad (40)$$

While this difference equation involves α , P_0 , and σ , we can introduce the change of function $z_t = \frac{y_t}{\alpha}$ and the parameter $\mu = \left(\frac{\alpha}{P_0 (1 + \frac{1}{\sigma})} \right)^\sigma > 0$ so that (40) becomes

$$z_{t+2} = \mu(1 - z_t)^\sigma z_t. \quad (41)$$

Note that (41) is a natural generalisation of the difference equation considered in Rauscher's chaotic dynamics. Indeed, setting $\sigma = 1$ and $\mu = \frac{1}{2a}$ we recover (33). This difference equation is valid for arbitrary power-law functions (with power law greater than one) in the utility function for Player 1. In addition to the case of $\sigma = 1$ (corresponding to a power-law of two), we find that chaos is common in such equations for various choices of σ . In Figure 18 we demonstrate bifurcations and chaos due to μ for several values of the power-law parameter σ . This demonstrates that chaos within status-seeking games, given utility functions for Player 1 of the form (34), is somewhat ubiquitous.

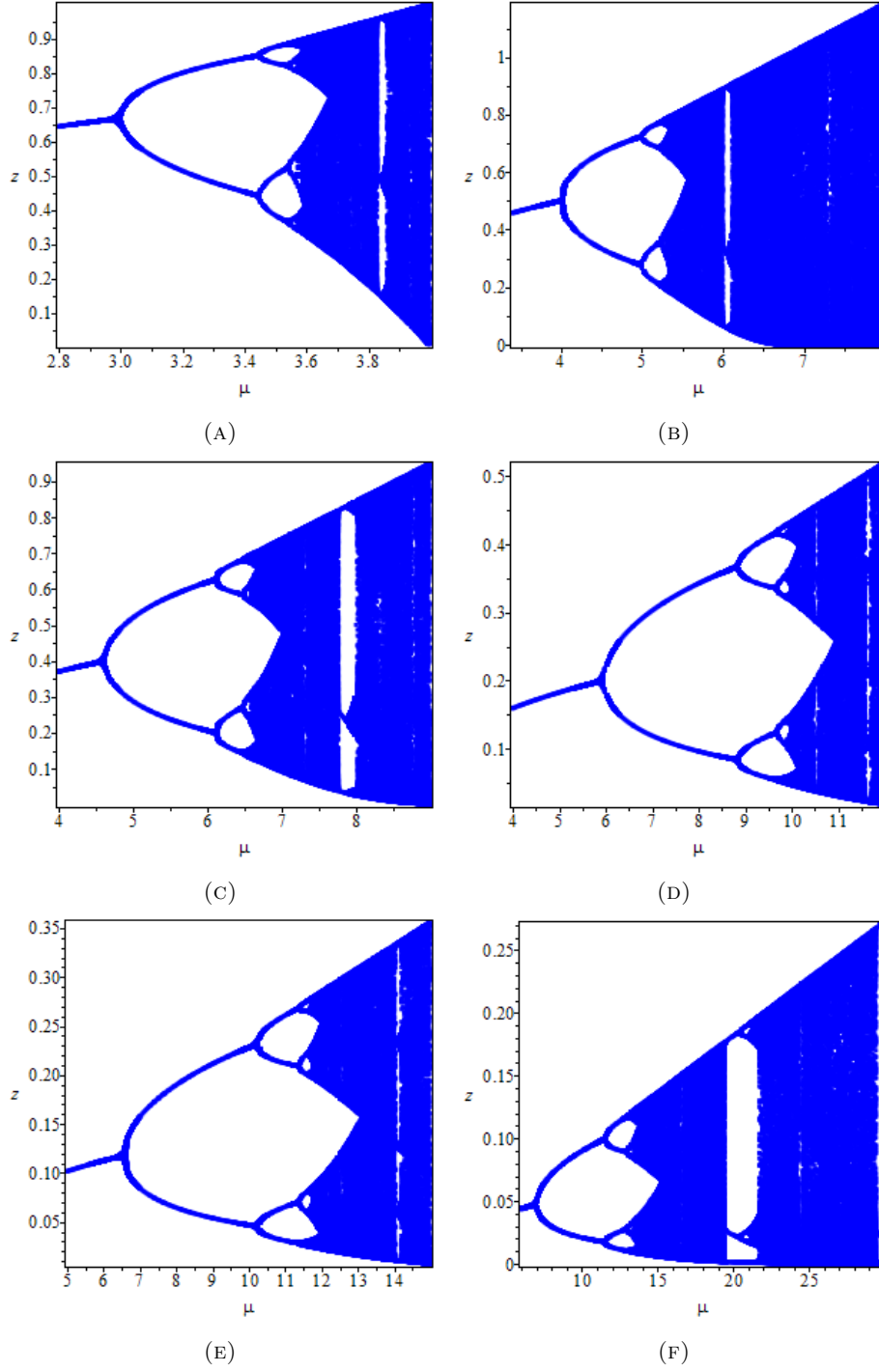


FIGURE 18. Bifurcation diagrams showing routes to chaos for the difference equation (41) for (a) $\sigma = 1$, (b) $\sigma = 2$, (c) $\sigma = 3$, (d) $\sigma = 8$, (e) $\sigma = 15$, (f) $\sigma = 40$. Panel (a) corresponds to the dynamics of the system found in Rauscher [21].

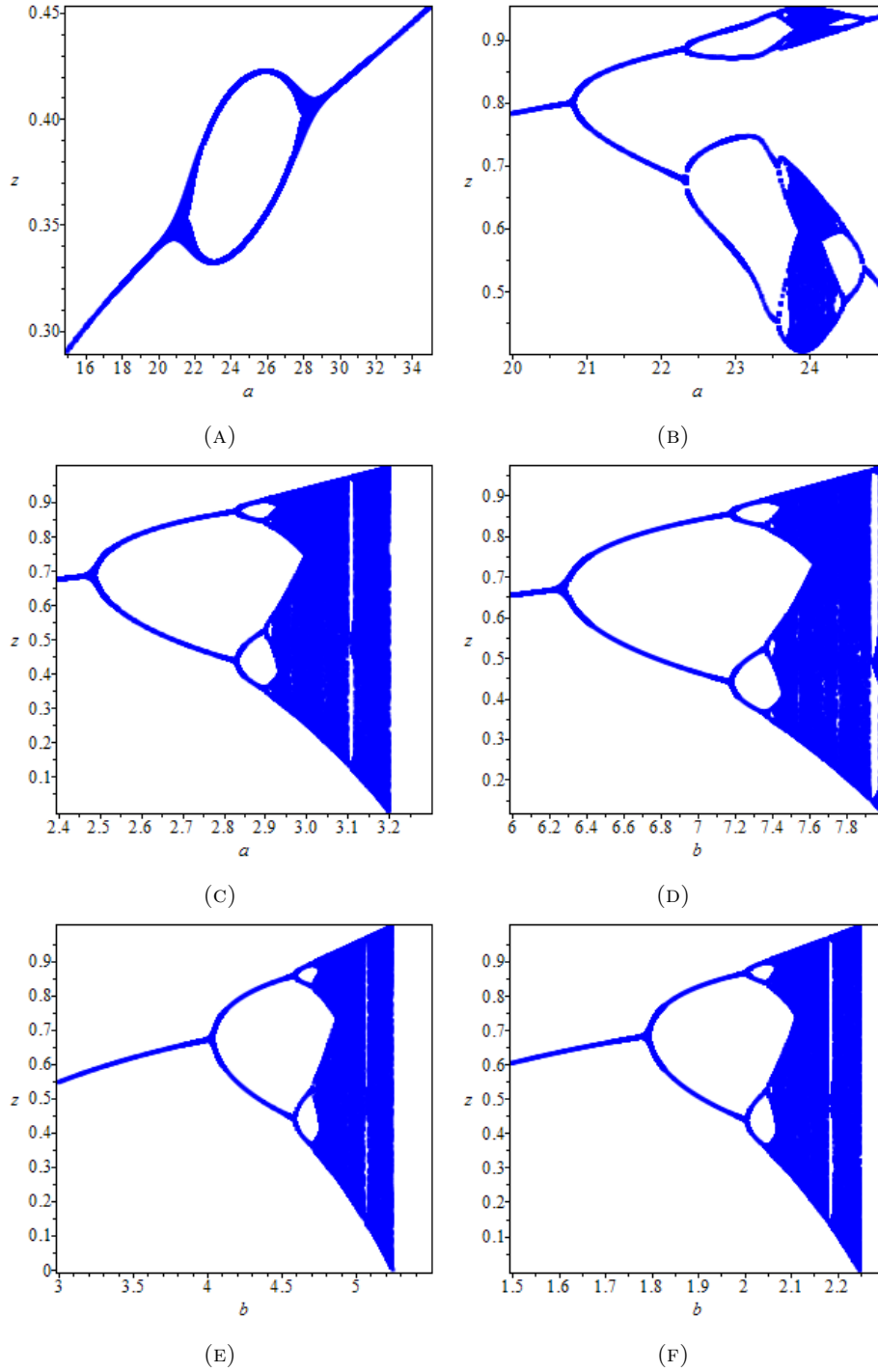


FIGURE 19. Bifurcation diagrams showing routes to chaos for the difference equation (46) for (a) $b = 0.3$, (b) $b = 0.4$, (c) $b = 1.5$, (d) $a = 0.5$, (e) $a = 0.8$, (f) $a = 2.0$.

6.2. Both players have Case 1 status functions. In the case where both players have Case 1 status functions and adopt utility functions of the form (34), we can destabilise the dynamics even further. Assume both players adopt utility functions

$$u_i(x, s) = x + A_i s - B_i s^2 \quad (42)$$

for $A_i, B_i > 0$, while also adopting Case 1 status functions. We obtain the reaction curves

$$y = \frac{1}{2B_1}(A_1 - Y)Y \quad \text{and} \quad Y = \frac{1}{2B_2}(A_2 - y)y. \quad (43)$$

Combining into one difference equation, we find

$$y_{t+2} = \frac{1}{4B_1B_2} \left(A_1 - \frac{1}{2B_2}(A_2 - y_t)y_t \right) (A_2 - y_t)y_t. \quad (44)$$

Changing variable $z_t = \frac{y_t}{A_2}$, and defining constants

$$a = \frac{A_2^3}{8B_1B_2^2} \quad \text{and} \quad b = \frac{2A_1B_2}{A_2^2}, \quad (45)$$

we transform the difference equation (44) into the more elegant scaled difference equation

$$z_{t+2} = a(b - (1 - z_t)z_t)(1 - z_t)z_t. \quad (46)$$

In Figure 19 we demonstrate bifurcations and chaos in this difference equation due to changes in parameters a and b .

Again, chaos is found fairly frequently, with values of a giving chaos for fixed b , and values of b giving chaos for fixed a . Other choices of utility functions can likely be defined which give rise to chaotic dynamics, as we have done here for utility functions of the form (34).

7. Conclusion. Motivated by Rauscher's comments on status-seeking games [21], we have considered the dynamics of a fairly general class of discrete time two-player status-seeking games. We have considered at four main categories of utility functions and a wide range of dynamics are observed. Utility functions commonly used in economics, such as constant elasticity of substitution (CES), Cobb-Douglas, Leontief, King-Plosser-Rebelo (KPR), and logarithmic utility functions, have been considered, and generally show equilibrium or tame periodic dynamics. Composite utility functions involving the superposition of Cobb-Douglas and logarithmic utility functions (which still satisfies desired properties of utility functions) give more interesting dynamics, as the optimisation problem leads to multiple-branch reaction functions.

We are able to show that for CES utility functions (and various limiting cases), only stable equilibria and period-2 oscillations are possible. For logarithmic utility functions or for the classical Cobb-Douglas utility function, we always observe time series which converge to equilibrium. Similarly, for the KPR utility functions involving increasing and concave functions of status, $v(s)$, solutions will tend to stable equilibrium values.

We have also shown that for composite log-Cobb-Douglas utility functions, a player's reaction curve can be a multifunction, resulting in the need to make a choice of which branch is taken at each iteration. If one player alternates their selected branch after a fixed number of iterations, then this can induce periodic dynamics. On the other hand, if a player chooses branches at random on each iteration, this will introduce stochasticity to the problem, resulting in less regular

dynamics. Still, even with this randomness, we observe some order in the phase portraits for such dynamics. Therefore, due to the added complexity of having to select among multiple solution branches, the use of composite log-Cobb-Douglas utility functions results in a richer set of nonlinear dynamics from the two-player status-seeking game.

When one or both utility functions were quadratic in status (or, more generally, the difference of a linear function and a power-law function in the status function), we were able to demonstrate the existence of chaotic dynamics via continued period doubling (a period doubling cascade) as the dynamics resulted in difference equations of the form $z_{t+2} = \mu(1 - z_t)^\sigma z_t$. In the case where $\sigma = 1$, the utility function is quadratic in status, and we recover the difference equation of Rauscher [21]. However, we find that the generalisation $s \neq 1$ still gives chaos, and that chaos is fairly ubiquitous in status-seeking games of this variety. Another generalization of the choice of status function results in difference equations of the form $z_{t+2} = a(b - (1 - z_t)z_t)(1 - z_t)z_t$, which again permits a period doubling cascade and chaos for appropriate parameter values.

The extension of two-player status-seeking games to the general N -player Cournot game, as discussed in Section 2, could result in less regular dynamics. Indeed, the stability that exists for dynamics corresponding to many of the utility function choices may diminish, as dependencies on different player consumption becomes more complex, and we anticipate less regular dynamics should occur. While we have considered two-player status-seeking games under the mechanism of Cournot duopoly, modelling the games as a Stackelberg duopoly [24] (a leader-follower game) could provide different but equally sensible player dynamics, and hence would be one interesting extension. A Stackelberg N -player game would add another layer of complexity to the decision making process [26]. Other rules, such as information asymmetry in the status functions, could yield interesting generalisations to the problem we study.

REFERENCES

- [1] K.J. Arrow, H.B. Chenery, B.S. Minhas, R.M. Solow, Capital-labor substitution and economic efficiency, *The Review of Economics and Statistics* 43 (1961) 225-250.
- [2] M. Boldrin, L. Montrucchio, The emergence of dynamic complexities in models of optimal growth: the role of impatience, Rochester Center for Economic Research, Working Paper 7 (1985).
- [3] K.A. Brekke, R.B. Howarth, K. Nyborg, Status-seeking and material affluence: evaluation the Hirsch hypothesis, *Ecological Economics* 45 (2003) 29-39.
- [4] A. Chao, J.B. Schor, Empirical tests of status consumption: Evidence from women's cosmetics, *Journal of Economic Psychology* 19(1) (1998) 107-131.
- [5] R.D. Congleton, Efficient status seeking: externalities and the evolution of status games, *Journal of Economic Behavior and Organization* 11 (1989) 175-190.
- [6] R.-A. Dana, L. Montrucchio, Dynamic Complexity in Duopoly Games, *Journal of Economic Theory* 40 (1986) 40-56.
- [7] A. Glazer, K.A. Konrad, A signaling explanation for charity, *The American Economic Review* 86(4) (1996) 1019-1028.
- [8] N.J. Ireland, Status-seeking, income taxation and efficiency, *Journal of Public Economics* 70 (1998) 99-113.
- [9] J. James, Positional goods, conspicuous consumption and the international demonstration effect reconsidered, *World Development* 15(4) (1987) 449-462.
- [10] R.G. King, C.I. Plosser, S.T. Rebelo, Production, growth and business cycles, *Journal of Monetary Economics* 21 (1988) 195-232.
- [11] R.G. King, C.I. Plosser, S.T. Rebelo, Production, growth and business cycles: technical appendix, *Computational Economics* 20 (2002) 87-116.

- [12] Y.A. Kuznetsov, Elements of applied bifurcation theory, Springer-Verlag New York (1995).
- [13] C.S. Kumru, L. Vesterlund, The effect of status on charitable giving, *Journal of Public Economic Theory* 12(4) (2010) 709-735.
- [14] A. Matsumoto, Controlling the Cournot-nash chaos, *Journal of Optimization Theory and Applications* 128 (2006) 379-392.
- [15] L. Montrucchio, Optimal decisions over time and strange attractors: An analysis by the Bellman principle, *Mathematical Modelling* 7 (1986) 341-352.
- [16] M. Papageorgiou, Technical note: chaos may be an optimal plan, *Journal of Optimization Theory and Applications* 119 (2003) 387-393.
- [17] T. Puu, Complex dynamics with three oligopolists, *Chaos, Solitons and Fractals* 7 (1996) 2075-2081.
- [18] T. Puu, The chaotic duopolists revisited, *Journal of Economics Behavior and Organization* 33 (1998) 385-394.
- [19] T. Puu, On the stability of Cournot equilibrium when the number of competitors increases, *Journal of Economics Behavior and Organization* 66 (2008) 445-456.
- [20] D. Rand, Exotic phenomena in games and duopoly models, *Journal of Mathematical Economics* 5 (1978) 173-184.
- [21] M. Rauscher, Keeping up with the Joneses: chaotic patterns in a status game, *Economics Letters* 40 (1992) 287-290.
- [22] B.C. Snyder, R.A. Van Gorder, K. Vajravelu, Continuous-time dynamic games for the cournot adjustment process for competing oligopolists, *Applied Mathematics and Computation* 219 (2013) 6400-6409.
- [23] R.M. Solow, A contribution to the theory of economic growth, *The Quarterly Journal of Economics* 70 (1956) 65-94.
- [24] H.V. Stackelberg, Market structure and equilibrium, Springer-Verlag Berlin Heidelberg (2011).
- [25] A. van Ackere, C. Haxholdt, Clubs as status symbol: would you belong to a club that accepts you as a member?, *Socio-Economic Planning Sciences* 36(2) (2002) 93-107.
- [26] R.A. Van Gorder, M.R. Caputo, Envelope theorems for locally differentiable open-loop Stackelberg equilibria of finite horizon differential games, *Journal of Economic Dynamics and Control* 34(6) (2010) 1123-1139.

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