



# Profinite and Residual Properties of Fibred Groups



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A thesis submitted for the degree of

*Doctor of Philosophy*

Trinity 2024



## Acknowledgements

Firstly, I would like to thank my D.Phil supervisors, Martin Bridson and Dawid Kielak. I have benefited from your expertise, enthusiasm, great taste in theorems and a fantastic sense of humour. Thank you for being the best supervisors I could have hoped for, and for not making me choose between you – it would have been an impossible task.

I would also like to acknowledge my past mentors; in particular, Ted Wilson, Charles Morgan and Viveka Erlandsson. You have gone beyond what was professionally required of you in helping me achieve my goals – I am indebted to you all.

I have greatly benefited from the company of many talented mathematical peers, most of whom I am fortunate to also call my friends. Thank you to the current and past Oxford crowd: Sam F, Sam H, Misha, Harry, Naomi, Ismael, Shaked, Filippo, Marco, Adele, Nadav and Alice, as well as Lawk, Jean Pierre and Macarena.

I wish to thank my family for their unwavering support and encouragement.

Finally, thank you to Rustin. There is not enough space on this page to list all the ways in which you have helped me. I am deeply grateful for having you in my life.

# Abstract

This thesis focuses on properties of finite quotients in certain families of groups which fibre algebraically.

In the first part of this thesis we study subgroup separability. We show that free-by-cyclic subgroups of free-by-cyclic groups are separable. Moreover, we give a characterisation of subgroup separability for free-by-cyclic groups with polynomially-growing monodromies. Our methods show that many free-by-cyclic groups contain an embedded non-subgroup-separable 3-manifold subgroup.

We also study subgroup separability in random deficiency-one groups. We develop a Brown-type algorithm to deduce when a deficiency-one presentation admits a homomorphism which is contained in the Bieri–Neumann–Strebel invariant  $\Sigma(G)$  of the corresponding group  $G$ , and in its complement  $\Sigma(G)^c$ .

The second part of this thesis is focused on profinite rigidity in groups. We show that many properties of free-by-cyclic groups are invariants of their profinite completion, including admitting a finite-order monodromy and being hyperbolic. In the case of hyperbolic free-by-cyclic groups with first Betti number equal to one, we are able to extract dynamical information about the monodromy map and its inverse. As a consequence, we show that irreducible free-by-cyclic groups with first Betti number equal to one can be distinguished from each other up to finite error using the isomorphism type of their profinite completion. We can also show that generic free-by-cyclic groups are almost profinitely rigid in the class of all free-by-cyclic groups.

The third part of this thesis begins with a chapter on exotic subgroups of hyperbolic groups. We give a general criterion for constructing non-hyperbolic subgroups of hyperbolic groups with strong finiteness properties via fibring, using a criterion of Fisher. We construct an infinite family of quasi-isometry classes of such examples.

The final chapter studies Friedl–Lück’s  $L^2$ -polytopes in the setting of free-by-cyclic groups. We explain how to construct the polytopes using topological representatives of the monodromy. We relate the shape of the  $L^2$ -polytope to cyclic splittings of the corresponding free-by-cyclic group  $G$  and explore the connection between the polytope and the group  $\text{Out}(G)$  of outer automorphisms of  $G$ .

## Statement of Originality

I declare that the work in this thesis is, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known.

Part [II](#) is based on the preprint [\[HK23\]](#) which is joint work with Sam Hughes. The material in Part [I](#) is the subject of the article [\[Kud22\]](#), whilst the material in Part [III](#) is contained in the article [\[Kud23\]](#).

Monika Kudlinska, Oxford, *28<sup>th</sup> of April 2024*

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# Chapter 1

## Introduction

Groups are algebraic objects which encode symmetries. In many areas of science, symmetries are useful for predicting the structure and behaviour of an object or a system on which they act. The motivating idea of *geometric group theory* is to reverse this process; that is, to learn about groups by studying the geometry and topology of the spaces associated to them.

It is generally agreed that some of the most natural topological objects in mathematics are manifolds. A particularly tractable class of manifolds are those which admit the structure of a fibre bundle over the circle. A famous problem posed by William Thurston in the 1980s, which became known as the *Virtual Fibring Conjecture*, asks whether every hyperbolic 3-manifold admits such a structure, possibly after passing to a finite degree cover. The recent resolution of this conjecture by Agol [Ago13] not only contributed to a significant increase in our understanding of the class of 3-manifolds, but also led to a new paradigm for the study of infinite groups which has been prevalent in geometric group theory ever since.

Motivated by this, the main focus of this thesis is a phenomenon known as *algebraic fibring* of groups, which is the group theoretic analogue of fibration over the circle for manifolds. A leitmotif of this work is to exploit the fibring phenomenon in order to reveal hidden structure in groups.

The importance of fibring is well illustrated by the recent groundbreaking work of Italiano–Martelli–Migliorini [IMM23] who produced the first example of a hyperbolic 5-manifold that fibres over the circle. The main consequence of this is the

first construction of a hyperbolic group with an *exotic* subgroup: one that is itself non-hyperbolic, but has strong finiteness properties. The existence of such a group has been a well-known open problem in group theory since the 1980s. Motivated by this work, in Chapter 8 of this thesis we construct the first infinite family of such exotic subgroups of hyperbolic groups, and these arise from algebraic fibring.

A special case of the algebraic fibring construction arises when the fibre is a free group of finite rank; a group which admits a presentation with no relations. In that case, the resulting group is called *free-by-cyclic*. Such groups form a rich and well-studied class which has often been used as a fertile testing ground for conjectures in geometric group theory.

There is a well-documented analogy between  $\text{Out}(F_n)$ , the group of outer automorphisms of the free group, and mapping class groups of surfaces (see, for example, the introduction to [Far06, Chapter 20]). This is often exploited to study outer automorphisms of free groups which are typically harder to understand than elements of the mapping class group. The main idea of this thesis is to naturally extend this analogy to free-by-cyclic and fibred 3-manifold groups; in particular focusing on their *finite quotients*.

The study of *profinite rigidity* of groups aims to determine how much global data about a group can be encoded in its set of finite quotients. It has become a very active area of group theory research in the past 10 years, with a large body of work on the profinite properties of 3-manifolds [BRW17, WZ10, WZ19, Wil18b], as well as the celebrated results concerning *absolute* profinite rigidity of certain groups due to Bridson–McReynolds–Reid–Spitler [BMRS20].

In this thesis, we will study the problem of profinite rigidity within the class of free-by-cyclic groups. We will see that many properties of free-by-cyclic groups are invariants of their profinite completions; that is, can be detected in their finite quotients. This will allow us to conclude that *generic* free-by-cyclic groups are *almost* profinitely rigid, amongst all free-by-cyclic groups. Previous work on this subject by Bridson–Reid–Wilton shows relative profinite rigidity in the special case when the free group has rank two [BRW17]. The work in this thesis is able to bypass the low-rank requirement, by adapting the powerful methods developed by Yi Liu in his recent work on the profinite rigidity of hyperbolic 3-manifolds

[Liu23a]. As such, this work provides a template for showing profinite rigidity results for free-by-cyclic groups, as well as a wider class of fibred groups.

In theme with this subject, the first part of this thesis investigates the problem of *subgroup separability* of groups, which is concerned with deciding whether subgroups of a given group can be ‘separated’ from elements which are not contained in the subgroup, in some finite quotient. This is a key property which has been shown to hold for various families of groups, including many 3-manifold groups. In Chapter 4, we will see a link between the *dynamics* of the map associated to the fibring structure, and the existence of a non-separable subgroup in a free-by-cyclic group. This is the first systematic study of subgroup separability in arbitrary free-by-cyclic groups.

A key difficulty in the study of free-by-cyclic groups, or indeed any groups which admit algebraic fibrations, is that often the fibring structure is **non-unique**. In other words, a given group can be realised as a free-by-cyclic group in multiple different ways, and often it is difficult to decide when two apparently different free-by-cyclic group presentations correspond to isomorphic groups.

The visionary work of William Thurston from the 1980s, shows that different surface bundle structures of a given 3-manifold can be organised and studied using a gadget called the *Thurston polytope* [Thu86]. The Thurston polytope parametrises all the different ways in which a single 3-manifold can fibre over the circle and encodes information about the nature of those fibrings.

Recently, Kielak [Kie20a] showed that it is possible to organise algebraic fibrings of other groups, including free-by-cyclic groups, in a similar fashion. Combined with the earlier work of Friedl–Lück [FL17], Kielak’s results prove that one can study different fibring structures of free-by-cyclic groups using the so-called  *$L^2$ -polytope*, which is an analogue of the Thurston polytope.

The final part of this thesis studies the  $L^2$ -polytope for free-by-cyclic groups. We discuss connections between the  $L^2$ -polytope and graph-of-groups splittings of free-by-cyclic groups, as well as the outer automorphisms of these groups. We compute examples of  $L^2$ -polytopes for certain families of free-by-cyclic groups. We also discuss the conjecture made by Gardam–Kielak in [Obe20], which predicts that

the seminorm on  $\text{Hom}(G; \mathbb{R})$  induced by the  $L^2$ -polytope encodes the complexity of dual splittings of the free-by-cyclic group.

## 1.1 Subgroup separability

A subgroup  $H \leq G$  is *separable* if for any  $g \in G \setminus H$ , there exists a finite quotient  $\pi: G \rightarrow Q$  such that  $\pi(g) \notin \pi(H)$ . We say a group  $G$  is *residually finite* if the trivial subgroup  $1 \leq G$  is separable. A group  $G$  is said to be *subgroup separable* if every finitely generated subgroup of  $G$  is separable.

In this thesis we study the problem of subgroup separability for two families of groups. First, we give a complete classification of subgroup separability for free-by-cyclic groups with polynomially growing monodromies.

**Theorem 4.0.1.** *Let  $\Phi \in \text{Out}(F_n)$  be a polynomially growing outer automorphism. Then  $G = F_n \rtimes_{\Phi} \mathbb{Z}$  is subgroup separable if and only if  $\Phi$  is periodic.*

Subgroup separability is a property which passes to subgroups. We show that subgroup separability in polynomially growing free-by-cyclic groups is characterised by the existence of a “poison” subgroup  $G_{NW}$  given by the presentation

$$G_{NW} = \langle i, j, k, l \mid [i, j], [j, k], [k, l] \rangle.$$

**Theorem 4.1.4.** *Let  $\Phi \in \text{Out}(F_n)$  be a polynomially growing outer automorphism and  $G = F_n \rtimes_{\Phi} \mathbb{Z}$ . Then, the following are equivalent:*

1. *The outer automorphism  $\Phi$  has growth of order  $d = 0$ ;*
2.  *$G$  is virtually a direct product  $F_n \times \mathbb{Z}$ ;*
3.  *$G$  is subgroup separable;*
4.  *$G$  does not contain  $G_{NW}$ .*

On the other hand, we construct many separable subgroups of free-by-cyclic groups:

**Proposition 4.2.1.** *Let  $G$  be a free-by-cyclic group and let  $H \leq G$  be a finitely generated subgroup. If  $H \leq G$  is free-by-cyclic then  $H$  is separable in  $G$ .*

In another direction, we study the problem of subgroup separability of deficiency-one groups.

**Theorem 5.0.1.** *Let  $G$  be a random group of deficiency one with respect to the few-relator model (see Section 5.0.1). Then, with positive asymptotic probability  $G$  is not subgroup separable.*

Our main tool in proving Theorem 5.0.1 is a geometric condition for deciding if a character  $\varphi: G \rightarrow \mathbb{Z}$  of a deficiency-one group is *semi-fibred*; that is  $\varphi$  is an element of the BNS invariant  $\Sigma(G)$ , but  $-\varphi$  is not. Another application of this is a generalisation of a result of Dunfield–Thurston [DT06]:

**Theorem 5.0.2.** *Let  $G$  be a random group of deficiency one with respect to the few-relator model. Then  $G$  is free-by-cyclic with asymptotic probability bounded away from 1.*

## 1.2 Profinite rigidity

Let  $\mathcal{FGRF}$  denote the class of all finitely generated residually finite groups. Let  $\mathcal{C} \subseteq \mathcal{FGRF}$  be a sub-collection. A group  $G$  is *almost profinitely rigid amongst the groups in  $\mathcal{C}$*  if there are at most finitely many isomorphism types of groups in  $\mathcal{C}$  with the same finite quotients as  $G$ .

In this thesis, we study the problem of profinite rigidity within the class of free-by-cyclic groups. A free-by-cyclic group is said to be *irreducible* if it admits irreducible monodromy (see Section 3.1). We prove:

**Theorem 6.6.4.** *Let  $G$  be an irreducible free-by-cyclic group. If  $b_1(G) = 1$  then  $G$  is almost profinitely rigid amongst irreducible free-by-cyclic groups.*

We are also able to prove a stronger profinite rigidity result for *generic* free-by-cyclic groups (see Section 6.1.1):

**Corollary 6.6.6.** *Let  $G$  be a random free-by-cyclic group. Then  $G$  is almost profinitely rigid amongst free-by-cyclic groups with asymptotic probability 1.*

In order to obtain our profinite rigidity results, we show that many properties of free-by-cyclic groups are invariants of their profinite completion:

**Theorem 6.5.7.** *Let  $G = F \rtimes_{\Phi} \mathbb{Z}$  be a free-by-cyclic group with induced character  $\varphi: G \rightarrow \mathbb{Z}$ . If  $b_1(G) = 1$ , then the following properties are determined by the profinite completion  $\widehat{G}$  of  $G$ :*

1. *The rank of  $F$ ;*
2. *The homological stretch factors  $\{\nu_G^+, \nu_G^-\}$ ;*
3. *The characteristic polynomials  $\{\text{Char } \Phi^+, \text{Char } \Phi^-\}$  of the action of  $\Phi$  on  $H_1(F; \mathbb{Q})$ ;*
4. *For each representation  $\rho: G \rightarrow \text{GL}(n, \mathbb{Q})$  factoring through a finite quotient, the twisted Alexander polynomials  $\{\Delta_n^{\varphi, \rho}, \Delta_n^{-\varphi, \rho}\}$  and the twisted Reidemeister torsions  $\{\tau^{\varphi, \rho}, \tau^{-\varphi, \rho}\}$  over  $\mathbb{Q}$ .*

*Moreover, if  $G$  is conjugacy separable, (e.g. if  $G$  is hyperbolic), then  $\widehat{G}$  also determines the Nielsen numbers and the homotopical stretch factors  $\{\lambda_G^+, \lambda_G^-\}$ .*

## 1.3 Structure of fibred groups and polytopes

### 1.3.1 Hyperbolic groups

Let  $G$  be a torsion-free hyperbolic group which fibres algebraically with hyperbolic kernel. Then  $G$  splits as a semidirect product

$$G \simeq (F_n * \Sigma_1 * \dots * \Sigma_k) \rtimes \mathbb{Z},$$

where  $F_n$  is a free group of finite rank  $n$  and each  $\Sigma_i$  is a closed surface group. In particular the cohomological dimension of  $G$  satisfies  $\text{cd}(G) \leq 3$ . In Chapter 8 we exploit this fact to construct exotic subgroups of hyperbolic groups. In particular, we show:

**Corollary 8.0.2.** *There exist infinitely many quasi-isometry classes of finitely generated subgroups of hyperbolic groups which are of type  $\text{FP}(\mathbb{Q})$  and which are not hyperbolic.*

In order to do this we give a general condition for constructing such exotic subgroups, using the fibring criterion of Kielak [Kie20b] and its generalisation due to Fisher [Fis21]:

**Theorem 8.0.1.** *Let  $G$  be a torsion-free hyperbolic virtually special group with  $\text{cd}_{\mathbb{Q}}(G) \geq 4$ . Suppose that the  $L^2$ -Betti numbers of  $G$  satisfy  $b_i^{(2)}(G) = 0$  for all  $i \leq n$ . Then  $G$  contains a non-hyperbolic subgroup  $N \leq G$  of type  $\text{FP}_n(\mathbb{Q})$  and  $\text{cd}_{\mathbb{Q}}(N) \in \{\text{cd}_{\mathbb{Q}}(G) - 1, \text{cd}_{\mathbb{Q}}(G)\}$ . Moreover, if  $\text{cd}_{\mathbb{Q}}(G) \leq n$  then  $\text{cd}_{\mathbb{Q}}(N) = \text{cd}_{\mathbb{Q}}(G) - 1$ .*

### 1.3.2 $L^2$ -polytopes

The  $L^2$ -polytope of a free-by-cyclic group is an invariant which encodes data about algebraic fibrings. It is calculated via Friedl–Lück’s *universal  $L^2$ -torsion*. In the final chapter of this thesis we study the  $L^2$ -polytope of free-by-cyclic groups. We begin by explaining how to calculate the polytope of a free-by-cyclic group from a topological representative of the monodromy. We also study the relationship between cyclic splittings of a free-by-cyclic group and the structure of the polytope, and the action of the outer automorphism group of a free-by-cyclic group  $G$  on various subsets of characters of  $G$ . Finally, we calculate specific examples of polytopes for free-by-cyclic groups and study free-by-cyclic groups which admit polytopes of a particular shape. We study polytopes of free-by-cyclic groups with finite order monodromy, as well as those which admit rank 2 and 3 fibres.

## 1.4 Organisation

This thesis is split into three parts. Part I is concerned with subgroup separability of groups. In Chapter 4 we investigate subgroup separability of free-by-cyclic groups. Chapter 5 studies subgroup separability in random deficiency-one groups.

Part II investigates profinite invariants of groups. In Chapter 6 we consider the problem of profinite rigidity within the class of free-by-cyclic groups. We show that many properties of free-by-cyclic groups are invariants of the profinite completion and use this to deduce profinite rigidity results for certain classes of free-by-cyclic groups.

The final Part [III](#) is about the structure of groups which fibre and the  $L^2$ -polytope. In [Chapter 8](#) we use a characterisation of hyperbolic groups that fibre with hyperbolic kernel to produce exotic subgroups of hyperbolic groups. In [Chapter 9](#) we study the  $L^2$ -polytope of free-by-cyclic groups.

# Chapter 2

## Preliminaries

### 2.1 Bass–Serre theory

A (*combinatorial*) graph  $\Gamma$  consists of a tuple of sets  $(V(\Gamma), E(\Gamma))$ , together with a fixed point free involution  $-: E(\Gamma) \rightarrow E(\Gamma)$  and a pair of maps

$$i, \tau: E(\Gamma) \rightarrow V(\Gamma),$$

such that  $i(\bar{e}) = \tau(e)$  for every  $e \in E$ . We call elements in  $V(\Gamma)$  the *vertices* of  $\Gamma$  and elements in  $E(\Gamma)$  the *oriented edges* of  $\Gamma$ . For each  $e \in E(\Gamma)$ , the vertex  $i(e)$  is called the *initial vertex* of  $e$  and the vertex  $\tau(e)$  is the *terminal vertex*. An *orientation*  $E^+(\Gamma)$  of  $\Gamma$  is a subset of edges  $E^+(\Gamma) \subseteq E(\Gamma)$  such that the sets  $E^+(\Gamma)$  and  $\overline{E^+(\Gamma)}$  together form a partition of the edge set  $E(\Gamma)$ .

A *morphism of graphs*  $f: \Gamma \rightarrow \Gamma'$  from  $\Gamma$  to  $\Gamma'$  consists of a pair of set maps

$$f_V: V(\Gamma) \rightarrow V(\Gamma') \text{ and } f_E: E(\Gamma) \rightarrow E(\Gamma')$$

such that  $f_E \circ i = i \circ f_V$  and  $f_E(\bar{e}) = \overline{f_E(e)}$ , for all  $e \in E(\Gamma)$ . An *isomorphism* of graphs is a morphism  $f: \Gamma \rightarrow \Gamma'$  such that the maps  $f_V$  and  $f_E$  are bijective.

Fix an orientation  $E^+(\Gamma)$  of  $\Gamma$ . The *topological realisation* of the graph  $\Gamma$  is a one-dimensional CW complex  $T(\Gamma)$  constructed as follows. The 0-cells of  $T(\Gamma)$  are identified with the elements in  $V(\Gamma)$ . The 1-cells of  $T(\Gamma)$  are identified with the elements of  $E(\Gamma)^+$  and the attaching maps are determined by the maps

$i, \tau: E(\Gamma) \rightarrow V(\Gamma)$ . Any two choices of orientations of  $\Gamma$  give rise to homeomorphic topological realisations. A graph  $\Gamma$  is said to be a *tree* if its topological realisation  $T(\Gamma)$  is simply connected.

A (*topological*) *graph* is a one-dimensional CW complex. By the discussion in the previous paragraph, any combinatorial graph has a topological representative. Moreover, isomorphic graphs give rise to homeomorphic topological representatives. Conversely, any topological graph is a topological representative of a combinatorial graph and such a graph is well-defined up to isomorphism. As a result, we will use the two points of view interchangeably, omitting the terms “combinatorial” and “topological” when it is clear from the context which type of graph is meant.

A *graph of groups*  $\mathbb{G}$  is a triple  $(\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  where  $\Gamma$  is a graph,  $\mathcal{G}_\bullet$  encodes the assignment of a group  $\mathcal{G}_v$  to every vertex  $v \in V(\Gamma)$  and a group  $\mathcal{G}_e$  to every edge  $e \in E(\Gamma)$  so that  $\mathcal{G}_e = \mathcal{G}_{\bar{e}}$ , and  $\iota_\bullet$  determines monomorphisms  $\iota_e: \mathcal{G}_e \hookrightarrow \mathcal{G}_{\tau(e)}$  for all edges  $e \in E(\Gamma)$ .

Let  $\mathbb{G} = (\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  be a graph of groups. Let  $F(E(\Gamma))$  denote the free group with a free basis identified with the edges of  $\Gamma$  and define  $F_{\mathbb{G}}$  to be the free product

$$F_{\mathbb{G}} = \underset{v \in V(\Gamma)}{*} \mathcal{G}_v * F(E(\Gamma)).$$

Let  $N \trianglelefteq F_{\mathbb{G}}$  be the normal subgroup of  $F_{\mathbb{G}}$  given by the normal closure

$$N = \langle\langle \{e\bar{e}, \forall e \in E(\Gamma)\} \cup \{e\iota_e(a)e^{-1}(\iota_{\bar{e}}(a))^{-1}, \forall e \in E(\Gamma), \forall a \in \mathcal{G}_e\} \rangle\rangle.$$

Finally, let us fix a spanning tree  $\mathcal{T}$  of the graph  $\Gamma$ . Then the *fundamental group of the graph of groups*  $\mathbb{G}$  with respect to the spanning tree  $\mathcal{T}$  is the quotient group

$$\pi_1(\mathbb{G}, \mathcal{T}) = (F_{\mathbb{G}}/N) / \langle\langle \{e \in E(\mathcal{T})\} \rangle\rangle.$$

Note that for any two different choices of spanning trees  $\mathcal{T}$  and  $\mathcal{T}_0$  of the graph  $\Gamma$ , there exists an isomorphism between the corresponding fundamental groups of graphs of groups,  $\pi_1(\mathbb{G}, \mathcal{T}) \simeq \pi_1(\mathbb{G}, \mathcal{T}_0)$ . We will sometimes suppress the choice of spanning tree and write  $\pi_1(\mathbb{G})$  to denote the fundamental group of the graph of

groups  $\mathbb{G}$  with respect to a spanning tree  $\mathcal{T}$ . If  $G = \pi_1(\mathbb{G})$  then we say that  $G$  admits a *graph-of-groups splitting*  $\mathbb{G}$ .

Crucially, for any  $x \in V(\Gamma) \cup E(\Gamma)$  the homomorphism  $\mathcal{G}_x \rightarrow \pi_1(\mathbb{G})$  is injective.

**Example 2.1.1** (HNN extensions). Let  $\mathbb{G} = (\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  be a graph of groups where  $\Gamma$  is a graph which consists of a single vertex and a single edge. Then there is a unique spanning tree  $\mathcal{T}$  of  $\Gamma$  which is given by the unique vertex. If  $G = \pi_1(\mathbb{G}, \mathcal{T})$  then  $G$  admits a presentation of the form

$$G = \langle \mathcal{G}_v, t \mid t \iota_e(a) t^{-1} = \iota_{\bar{e}}(a) \quad \forall a \in \mathcal{G}_e \rangle.$$

We will often identify  $\mathcal{G}_e$  with a subgroup of  $\mathcal{G}_v$  via the monomorphism  $\iota_{\bar{e}}$ , suppress the notation for  $\iota_{\bar{e}}$ , and define a homomorphism

$$\begin{aligned} \theta: \mathcal{G}_e &\rightarrow \mathcal{G}_v \\ a &\mapsto \iota_e(a). \end{aligned}$$

We write  $G = \mathcal{G}_v *_{\mathcal{G}_e, \theta}$  to denote the resulting group,

$$\mathcal{G}_v *_{\mathcal{G}_e, \theta} = \langle \mathcal{G}_v, t \mid t^{-1} a t = \theta(a), \forall a \in \mathcal{G}_e \rangle.$$

We say  $\mathcal{G}_v *_{\mathcal{G}_e, \theta}$  is an *Higman–Neumann–Neumann (HNN) extension* of  $\mathcal{G}_v$  over  $\mathcal{G}_e$ . Moreover, if  $\mathcal{G}_e = \mathcal{G}_v$  then we say that  $\mathcal{G}_v *_{\mathcal{G}_e, \theta}$  is an *ascending HNN extension*, and it is *proper* if the map  $\theta: \mathcal{G}_v \rightarrow \mathcal{G}_v$  is not surjective.

For a group  $G$ , we define a *G-tree*  $T$  to be a tree  $T$  with an action

$$\begin{aligned} G &\rightarrow \text{Aut}(T) \\ g &\mapsto \sigma_g \end{aligned}$$

which is *without inversion*, meaning that for all  $e \in E(\Gamma)$ ,  $\sigma_g(e) \neq \bar{e}$ . We will often abuse notation by using the symbol  $g$  to denote the automorphism  $\sigma_g$  of  $T$  induced by the element  $g \in G$ . For any  $x \in V(T) \cup E(T)$ , we will write  $G_x$  to denote the stabiliser of  $x$  under the action of  $G$ .

An action of a group  $G$  on a tree  $T$  is said to be *minimal* if there are no proper  $G$ -invariant subtrees. If  $g \in G$  is an element which fixes a point in a  $G$ -tree  $T$  then we say that  $g$  is *elliptic*, and otherwise we say that it is *hyperbolic*. If the element  $g$  is hyperbolic then there exists a unique embedded line in  $T$  which is preserved by the cyclic group generated by  $g$ . We call such a line the *axis of  $g$* , and denote it by  $\text{Axis}(g)$ . A hyperbolic element  $g \in G$  acts on its axis by translation.

**Lemma 2.1.2.** *Let  $G$  be a group and  $T$  a  $G$ -tree. Let  $\mathcal{H} \subseteq G$  be the subset of hyperbolic elements in the  $G$ -tree  $T$ . If  $\mathcal{H}$  is non-empty then there exists a unique minimal  $G$ -invariant subtree  $T_0 \subseteq T$  which is the union of the axes of hyperbolic elements in  $G$ ,*

$$T_0 = \bigcup_{h \in \mathcal{H}} \text{Axis}(h).$$

**Lemma 2.1.3.** *Let  $G$  be a finitely generated group and  $T$  a  $G$ -tree without a global fixed point. Then  $G$  contains a hyperbolic element.*

By combining Lemma 2.1.3 and Lemma 2.1.2, we deduce that for any finitely generated group  $G$  and a  $G$ -tree  $T$ , there exists a minimal  $G$ -invariant subtree  $T_0 \subseteq T$ .

**Lemma 2.1.4.** *Let  $G$  be a group and  $T$  a minimal  $G$ -tree. Let  $N$  be a non-trivial normal subgroup of  $G$  and  $G' \leq G$  a finite-index subgroup. Then the induced actions of  $N$  and  $G'$  on  $T$  are minimal.*

Let us also record the following elementary lemma which will be used throughout.

**Lemma 2.1.5.** *Let  $G$  be a group and suppose that  $G$  acts transitively on a set  $X$ . Let  $N \leq G$  be a normal subgroup and write  $\pi: G \rightarrow G/N$  to denote the quotient map. Then for any element  $x \in X$ , the set of orbits of  $x$  under the induced action of  $N$  on the set  $X$  is in bijection with the set of cosets of  $\pi(\text{stab}_G(x))$  in the quotient group  $G/N$ .*

We associate to any  $G$ -tree  $T$  the *quotient graph of groups*  $\mathbb{G}_T = (\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  as follows. Let  $q: T \rightarrow G \backslash T$  be the natural quotient map corresponding to the identification of orbits under the action of  $G$  on  $T$ . Note that since  $G$  acts on  $T$

without inversion, the quotient  $G \backslash T$  is indeed a graph. We let  $\Gamma = G \backslash T$  denote the quotient graph. Now let us fix a spanning tree  $S$  of  $\Gamma$  and let  $f: S \rightarrow T$  be a lift of  $S$  to  $T$ ; explicitly,  $f$  is a morphism such that  $q \circ f = \text{id}_S$ . Fix an orientation  $E^+(\Gamma)$  of  $\Gamma$ . For every  $e \in E^+(\Gamma) \setminus E(S)$ , let  $\hat{e}$  be the unique lift of  $e$  in  $E(T)$  such that  $i(\hat{e}) = f(i(e))$ , and let  $t_e$  be an element of  $G$  such that  $t_e \cdot \tau(\hat{e}) = f(\tau(e))$ . We extend the assignment  $e \mapsto t_e$  to all edges of  $E(\Gamma)$  so that  $t_{\bar{e}} = t_e^{-1}$  and  $t_e = 1$  for all  $e \in E(S)$ . Finally, we extend the map  $f: E(S) \rightarrow E(T)$  to a map  $f: E(\Gamma) \rightarrow E(T)$  so that for every  $e \in E^+(\Gamma)$ ,  $f(e) = \hat{e}$  and such that  $f$  commutes with the involution map.

For each vertex  $v$  of  $\Gamma$ , we define the group  $\mathcal{G}_v$  to be the stabiliser  $\text{stab}_G(f(v))$  of the vertex  $f(v)$  in the  $G$ -tree  $T$ , and for each edge  $e \in E(\Gamma)$  we define  $\mathcal{G}_e$  to be the stabiliser  $G_{f(e)}$  of  $f(e)$ . For any  $e \in E(\Gamma)$ , the edge map  $\iota_e: \mathcal{G}_e \hookrightarrow \mathcal{G}_{\tau(e)}$  is defined by sending  $a \mapsto t_e^{-1} a t_e$ .

By construction, the inclusions  $\mathcal{G}_x \hookrightarrow G$  of the edge and vertex groups extend to a homomorphism  $\pi_1(\mathbb{G}_T, S) \rightarrow G$ . Moreover, one can show that this homomorphism is surjective.

We summarise the above discussion with the following theorem:

**Theorem 2.1.6** (Serre [Ser03, Theorem 13, Section 5.4]). *Let  $G$  be a group and  $T$  a  $G$ -tree. Let  $\mathbb{G}_T = (G \backslash T, \mathcal{G}_\bullet, \iota_\bullet)$  be the quotient graph of groups associated to the action of  $G$  and the choice of spanning tree  $S \subseteq G \backslash T$  (as defined above). Then  $G$  admits a graph-of-groups splitting,*

$$G = \pi_1(\mathbb{G}_T, S).$$

There is a useful converse to the above theorem:

**Theorem 2.1.7** (Serre [Ser03, Section 5.3]). *Let  $G = \pi_1(\mathbb{G}, \mathcal{T})$  be a graph-of-groups decomposition of  $G$ . Then there exists a minimal  $G$ -tree  $T$  such that  $\mathbb{G}$  is the quotient graph of groups for the action of  $G$  on  $T$ .*

We will refer to the tree  $T$  obtained in Theorem 2.1.7 as the *Bass–Serre tree* corresponding to the splitting  $G = \pi_1(\mathbb{G}, \mathcal{T})$ .

Let  $\mathbb{G} = (\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  be a graph of groups. A *subgraph-of-groups*  $\mathbb{G}'$  of  $\mathbb{G}$  is a graph of groups  $(\Gamma', \mathcal{G}'_\bullet, \iota'_\bullet)$  such that  $\Gamma'$  is an induced subgraph of  $\Gamma$ , the assignment

$\mathcal{G}'_\bullet$  is a restriction of  $\mathcal{G}_\bullet$  to the subgraph  $\Gamma'$  and  $\iota'_e = \iota_e$  for all  $e \in E(\Gamma')$ . A subgraph-of-groups is said to be *proper* if the defining graph is a proper subgraph. Let  $\mathcal{T}$  be a spanning tree of  $\Gamma$  such that  $\mathcal{T} \cap \Gamma'$  is a spanning tree of  $\Gamma'$ . Then the fundamental group of the subgraph-of-groups  $\mathbb{G}'$  with respect to the spanning tree  $\mathcal{T} \cap \Gamma'$  naturally embeds in the fundamental group of the graph of groups  $\mathbb{G}$  with respect to  $\mathcal{T}$ .

A graph-of-groups splitting  $G = \pi_1(\mathbb{G})$  is said to be *minimal*, if there does not exist a proper subgraph-of-groups  $\mathbb{G}' \leq \mathbb{G}$  so that  $G \simeq \pi_1(\mathbb{G}')$ .

**Lemma 2.1.8.** *Let  $G$  be a group and  $T$  a  $G$ -tree. If the action of  $G$  on  $T$  is minimal then the corresponding quotient graph of groups is minimal. Conversely, if  $G$  admits a minimal graph-of-groups splitting, then the corresponding Bass–Serre tree is a minimal  $G$ -tree.*

## 2.2 Homological invariants

### 2.2.1 Ring and module theory

**Remark 2.2.1.** Every ring in this thesis is assumed to be associative and unital.

Let  $G$  be a group and  $R$  a ring. The (*untwisted*) *group ring*  $RG$  is defined to be the set of all finite linear combinations of elements in  $G$  with coefficients in  $R$ ,

$$RG := \left\{ \sum_{g \in G} \lambda_g g \mid \text{finitely many } \lambda_g \in R \text{ are non-zero} \right\}.$$

The set  $RG$  admits the structure of a ring with the following addition and multiplication operations

$$\sum_{g \in G} \lambda_g g + \sum_{g \in G} \mu_g g := \sum_{g \in G} (\lambda_g + \mu_g) g \text{ and } \left( \sum_{g \in G} \lambda_g g \right) \cdot \left( \sum_{g \in G} \mu_g g \right) := \sum_{\substack{g \in G \\ g_1 g_2 = g}} (\lambda_{g_1} \cdot \mu_{g_2}) g$$

Let  $M$  be an  $R$ -module. A *projective resolution*  $P_\bullet$  of  $M$  over  $R$ , denoted by  $P_\bullet \rightarrow M$ , is an exact sequence of projective  $R$ -modules

$$\cdots \rightarrow P_{n+1} \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow M \rightarrow 0.$$

The resolution  $P_\bullet$  is said to be *finite* if there exists a finite integer  $n$  such  $P_n \neq 0$  and  $P_i = 0$  for all  $i > n$ , and we call such  $n$  the *length* of the resolution.

**Definition 2.2.2** (Ore condition). Let  $R$  be a ring and  $S \subseteq R$  a multiplicative subset. The ring  $R$  satisfies the *right Ore condition for  $S$* , if for any  $r \in R$  and  $s \in S$ , there exists some  $u \in R$  and  $v \in S$  such that  $rv = su$ . A ring  $R$  is a *right Ore domain* if  $R$  is a domain and satisfies the right Ore condition with respect to  $R - \{0\}$ .

Suppose that  $R$  satisfies the right Ore condition with respect to  $S$ . We define an equivalence relation  $\sim_R$  on the Cartesian product  $R \times S$  so that  $(r, s) \sim_R (r', s')$  if and only if there exist elements  $t, t' \in R$  such that  $rt = r't'$  and  $st = s't'$ . The *Ore localisation of  $R$  with respect to  $S$*  is the set of equivalence classes

$$\text{Ore}(R, S) = (R \times S) / \sim_R$$

When  $R$  is an Ore domain and  $S = R - \{0\}$ , we simply write

$$\text{Ore}(R) := \text{Ore}(R, R - \{0\}).$$

**Definition 2.2.3.** A *division ring*  $R$  is a ring such that for any  $r \in R - \{0\}$ , there exists  $s \in R$  such that  $sr = 1$  and  $rs = 1$ . Note that if such an element  $s$  exists then it is unique. We call  $s$  the *inverse of  $r$*  and denote it by the symbol  $r^{-1}$ .

For a division ring  $R$  and a subset  $S \subseteq R$ , we define the *division closure* of  $S$  in  $R$  to be the smallest subset  $\bar{S} \subseteq R$  such that if  $s \in \bar{S}$  then  $s^{-1} \in \bar{S}$ .

**Lemma 2.2.4.** *If  $R$  is an Ore domain then the Ore localisation  $\text{Ore}(R)$  is a division ring.*

Let  $\pi: G \twoheadrightarrow H$  be a group epimorphism with kernel  $K$ . Let  $\mathbb{Z}K * H$  be the set of all finite linear combinations  $\sum_{h \in H} \lambda_h * h$  of elements in  $H$  with coefficients in the group ring  $\mathbb{Z}K$ . Define addition to be pointwise as above. Fix a set-theoretic section  $s: H \rightarrow G$  of  $\pi$ . Multiplication on  $\mathbb{Z}K * H$  is defined by linearly extending the operation

$$(\lambda_g * g) \cdot (\lambda_h * h) := \lambda_g \text{ad}_{s(g)}(\lambda_h) s(g) s(h) s(gh)^{-1} * gh,$$

where for any  $x \in G$ ,  $\text{ad}_x: \mathbb{Z}K \rightarrow \mathbb{Z}K$  is the function defined by linearly extending the conjugation action by  $x$  on  $K$ ,  $k \mapsto xkx^{-1}$  for all  $k \in K$ .

**Lemma 2.2.5.** *The set  $\mathbb{Z}K * H$  is a ring and it is isomorphic to the (untwisted) group ring  $G$  via the map  $\mathbb{Z}G \rightarrow \mathbb{Z}K * H$  defined by linearly extending*

$$g \mapsto g(s \circ \phi)(g)^{-1} * \pi(g).$$

## 2.2.2 (Co)homology of groups

**Definition 2.2.6.** Let  $G$  a group and let  $P_\bullet$  be a projective resolution of  $R$  over  $RG$ -modules. The *group homology* of  $G$  with coefficients in the left  $RG$ -module  $M$  is defined to be

$$H_n(G; M) := H_n(P_\bullet \otimes_{RG} M).$$

The *group cohomology* of  $G$  with coefficients in the right  $RG$ -module  $N$  is

$$H^n(G; N) := H^n(\text{Hom}_{RG}(P_\bullet, N)).$$

**Remark 2.2.7.** For any two projective resolutions  $P_\bullet \rightarrow M$  and  $Q_\bullet \rightarrow M$  of a given  $S$ -module  $M$ , there exists a chain homotopy equivalence  $P_\bullet \simeq Q_\bullet$ . In particular, the isomorphism types of the homology and cohomology groups defined above are independent of the choice of the projective resolution.

The *cohomological dimension*  $\text{cd}_R(G)$  of a group  $G$  over the ring  $R$  is the smallest integer  $n$  such that there exists an  $RG$ -projective resolution of the trivial module  $R$  of length at most  $n$ . Equivalently, it is the smallest  $n$  such that the cohomology of  $G$  with coefficients in any  $RG$ -module  $M$  vanishes in degree  $i$  for all  $i > n$ . We will sometimes write  $\text{cd}(G)$  to denote the cohomological dimension of  $G$  over the ring  $\mathbb{Z}$ .

There exists a characterisation of groups with cohomological dimension equal to one, which is due to Stallings in the finitely generated case and later Swan in the general case:

**Theorem 2.2.8** (Stallings [Sta68], Swan [Swa69]). *Let  $G$  be a group and  $R$  a ring. Then  $G$  is free if and only if  $\text{cd}_R(G) \leq 1$ .*

A group  $G$  is said to be of type  $\text{FP}_n$  over  $R$ , or of type  $\text{FP}_n(R)$ , for some  $n \in \mathbb{N} \cup \{\infty\}$  if there exists an  $RG$ -projective resolution  $P_\bullet \rightarrow R$  such that the first  $n$  terms  $P_0, \dots, P_{n-1}$  are finitely generated. It is said to be of type  $\text{FP}$  over  $R$ , or  $\text{FP}(R)$ , if there exists a finite  $RG$ -projective resolution of  $R$  such that all the terms are finitely generated. Again, if the ring  $R$  is not specified then the finiteness properties are assumed to be over  $\mathbb{Z}$ .

We will often use the following theorem to compare cohomological dimension of groups in a short exact sequence. The first part of the theorem is a simple application of the Lyndon–Hochschild–Serre spectral sequence, whereas the second part is due to Fel’dman [Fel71].

**Theorem 2.2.9.** *Let  $R$  be a ring. Consider a short exact sequence of groups*

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1.$$

*The cohomological dimensions of the groups in the short exact sequence over  $R$  satisfy*

$$\text{cd}_R(G) \leq \text{cd}_R(N) + \text{cd}_R(Q). \quad (2.1)$$

*Moreover, if  $N$  is of type  $\text{FP}(R)$  and  $H^n(N; RN)$  is  $R$ -free, where  $n = \text{cd}_R(N)$ , and if  $\text{cd}_R(G)$  is finite then*

$$\text{cd}_R(G) = \text{cd}_R(N) + \text{cd}_R(Q).$$

We will now discuss how to compute group cohomology and homology using graph-of-groups splittings. We will be using the same notation as in Section 2.1.

**Theorem 2.2.10** (Chiswell [Chi76]). *Let  $G$  be a group,  $R$  a ring and  $M$  an  $RG$ -module. Suppose that  $G$  splits as a graph of groups  $G \simeq \pi_1(\mathbb{G}, T)$  where  $\mathbb{G} = (\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  and  $T \subseteq \Gamma$  is a spanning tree. Then there are exact sequences in cohomology*

$$\begin{aligned} \cdots &\longrightarrow H^i(G; M) \longrightarrow \prod_{v \in V(\Gamma)} H^i(\mathcal{G}_v; M) \longrightarrow \prod_{e \in E(\Gamma)} H^i(\mathcal{G}_e; M) \\ &\longrightarrow H^{i+1}(G; M) \longrightarrow \cdots \end{aligned}$$

and in homology

$$\begin{aligned} \cdots &\longrightarrow \bigoplus_{e \in E(\Gamma)} H_i(\mathcal{G}_e; M) \longrightarrow \bigoplus_{v \in V(\Gamma)} H_i(\mathcal{G}_v; M) \longrightarrow H_i(G; M) \\ &\longrightarrow \bigoplus_{e \in E(\Gamma)} H_{i-1}(\mathcal{G}_e; M) \longrightarrow \cdots \end{aligned}$$

We will refer to the sequences obtained in Theorem 2.2.10 as the *Mayer–Vietoris exact sequence in cohomology, resp. homology*, for the graph-of-groups splitting  $G \simeq \pi_1(\mathbb{G}, T)$ .

Let  $G$  be a group of type  $\text{FP}(\mathbb{Q})$ . The *Euler characteristic* of  $G$  is defined to be the alternating sum

$$\chi(G) := \sum_{i \geq 0} (-1)^i \cdot \text{rank}_{\mathbb{Q}} H_i(G; \mathbb{Q}).$$

The following proposition follows immediately from the Mayer–Vietoris exact sequence:

**Proposition 2.2.11.** *Suppose that  $G$  is the fundamental group of a finite graph of groups  $(\Gamma, \mathcal{G}_{\bullet}, \iota_{\bullet})$  where the vertex and edge groups are all of type  $\text{FP}(\mathbb{Q})$ . Then  $G$  is of type  $\text{FP}(\mathbb{Q})$  and*

$$\chi(G) = \sum_{v \in V(\Gamma)} \chi(\mathcal{G}_v) - \sum_{e \in E(\Gamma)} \chi(\mathcal{G}_e).$$

**Definition 2.2.12.** The *classifying space*, or a  $K(G, 1)$  complex, for a group  $G$  is a connected CW-complex  $X$  such that  $\pi_1(X) \simeq G$  and  $\pi_i(X) = 1$  for all  $i \geq 2$ .

A group  $G$  is said to be of type  $F_n$  if there exists a classifying space for  $G$  with a finite  $n$ -skeleton. The group  $G$  is of type  $F$  if there exists a finite classifying

space for  $G$ . Any group of type  $F_n$  is also of type  $FP_n(R)$  for any ring  $R$ , however the converse does not hold in general (see [BB97, Section 6]).

### 2.2.3 $L^2$ -invariants, Linnell division ring and the Atiyah conjecture

The aim of this subsection is to give a brief introduction to the theory of  $L^2$ -invariants of groups. In certain settings it is possible to realise  $L^2$ -Betti numbers of groups as ranks of free modules over certain division rings. This is the approach that will be taken in this thesis. As a result, we will not elaborate on the functional analytic definitions. For full details the interested reader is advised to consult Lück's excellent book on the subject [Lüc94].

Let  $G$  be a discrete group. We write  $L^2(G)$  to denote the Hilbert space of square-summable formal sums over  $G$  with coefficients in  $\mathbb{C}$ ,

$$L^2(G) := \left\{ \sum_{g \in G} \lambda_g g, \lambda_g \in \mathbb{C} : \sum |\lambda_g|^2 < \infty \right\}.$$

The *group von Neumann algebra*  $\mathcal{N}(G)$  of  $G$  is the algebra of  $G$ -equivariant bounded operators from  $L^2(G)$  to  $L^2(G)$ . It has a natural  $\mathcal{N}(G)$ - $\mathbb{Z}G$ -bimodule structure.

Let  $X$  be a CW-complex with a cellular  $G$ -action, and let  $(C_\bullet(X), \partial_\bullet)$  denote the corresponding cellular  $\mathbb{Z}G$ -chain complex. Define the  $L^2$ -homology of the complex  $X$  to be

$$H_p^{(2)}(X; G) := H_p(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_\bullet(X)) = \ker \partial_p / \text{cl}(\text{im } \partial_{p+1}),$$

where  $\text{cl}(\text{im } \partial_{p+1})$  denotes the closure of  $\text{im } \partial_{p+1}$  in the Hilbert space  $\ker \partial_p$ .

The  $p$ -th  $L^2$ -Betti number  $b_p^{(2)}(X; G)$  of  $X$  is defined to be

$$b_p^{(2)}(X; G) := \dim_{\mathcal{N}(G)}(H_p^{(2)}(X; G)),$$

Here  $\dim_{\mathcal{N}(G)}(\cdot)$  denotes the *extended von Neumann dimension* of an  $\mathcal{N}(G)$ -module in the sense of Lück [Lüc02, Definition 6.20].

The  $p$ -th  $L^2$ -Betti number of a group  $G$  is defined to be the  $p$ -th  $L^2$ -Betti number of any classifying space  $X$  for  $G$ ,

$$b_p^{(2)}(G) := b_p^{(2)}(X; G).$$

As observed above, elements of  $G$  act as bounded operators  $L^2(G) \rightarrow L^2(G)$  and hence there is a natural embedding  $\mathbb{Q}G \subseteq \mathcal{N}(G)$ . We define the *Linnell ring*  $\mathcal{D}(G)$  of a group  $G$  to be the division closure of  $\mathbb{Q}G$  in the Ore localisation  $\text{Ore}(\mathcal{N}(G), S)$ , where  $S$  is the set of non-zero divisors of  $\mathcal{N}(G)$  (see Section 2.2.1 for the definitions). The Ore localisation  $\text{Ore}(\mathcal{N}(G), S)$  is indeed a division ring as a result of Lemma 2.2.4 combined with the following theorem:

**Theorem 2.2.13** (Lück [Lüc02, Theorem 8.22]). *The group von Neumann algebra  $\mathcal{N}(G)$  satisfies the Ore condition with respect to the set  $S$  of non-zero divisors in  $\mathcal{N}(G)$ .*

The *Atiyah conjecture* predicts rationality of the von Neumann dimension of the kernel of any bounded operator  $L^2(G)^n \rightarrow L^2(G)^m$ . For the sake of brevity, we will omit the precise definition and instead use the following characterisation which follows from the work of Linnell:

**Definition/Theorem 2.2.14** (Linnell [Lin93]). A torsion-free group  $G$  satisfies the *strong Atiyah conjecture over  $\mathbb{Q}$*  if the Linnell ring  $\mathcal{D}(G)$  is a division ring.

We will hereafter abuse terminology and say that a torsion-free group  $G$  satisfies the *Atiyah conjecture* when we mean that  $G$  satisfies the strong Atiyah conjecture over  $\mathbb{Q}$  as in Definition/Theorem 2.2.14.

**Example 2.2.15.** The following classes of groups satisfy the Atiyah conjecture:

1. Extensions of direct unions of free groups by elementary amenable groups (Linnell [Lin93]);
2. Virtually compact special groups in the sense of Haglund–Wise [HW08] (Schreue [Sch14]); and

3. Subgroups of torsion-free groups which satisfy the Atiyah conjecture (Lück [Lüc02, Lemma 10.4]).

**Theorem 2.2.16** (Lück [Lüc02, Lemma 10.28(3)]). *Let  $G$  be a torsion-free group which satisfies the Atiyah conjecture. Then the  $L^2$ -Betti numbers of  $G$  satisfy*

$$b_i^{(2)}(G) = \text{rank}_{\mathcal{D}(G)} H_i(G; \mathcal{D}(G)).$$

We record a useful theorem:

**Theorem 2.2.17** (Lück [Lüc02, Theorem 1.39]). *Let  $G$  be a group which admits a normal subgroup  $N \trianglelefteq G$  which is of type F and such that  $G/N \simeq \mathbb{Z}$ . Then all of the  $L^2$ -Betti numbers of  $G$  vanish.*

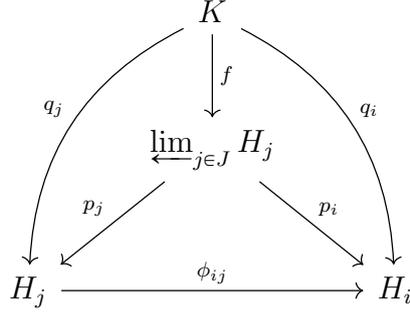
## 2.3 Profinite completions of groups

An *inverse system of groups* consists of a directed poset  $(J, \leq)$ , an indexed system of groups  $(H_i)_{i \in J}$  and a family of homomorphisms  $\phi_{ij}: H_j \rightarrow H_i$  for  $i \leq j$  which satisfy the following properties: for any  $i \in J$ ,  $\phi_{ii}$  is the identity homomorphism, and for any triple  $i, j, k \in J$  with  $i \leq j \leq k$ , we have that  $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ .

The *inverse limit* of an inverse system of groups  $(H_j, \phi_{ij})$ , denoted by  $\varprojlim_{j \in J} H_j$ , is defined to be the set

$$\varprojlim_{j \in J} H_j := \{(h_i)_{i \in J} \mid \phi_{ij}(h_j) = h_i \text{ whenever } i \leq j\} \subseteq \prod_{i \in J} H_i.$$

For each  $i \in J$ , let  $p_i: \varprojlim_{j \in J} H_j \rightarrow H_i$  denote the natural projection onto the  $H_i$ -coordinate. Then, for any group  $K$  which admits a family of surjective homomorphisms  $\{q_i: K \rightarrow H_i\}_{i \in J}$  such that whenever  $i \leq j$ , we have that  $\phi_{ij} \circ q_j = q_i$ , there exists a unique homomorphism  $f: K \rightarrow \varprojlim_{j \in J} H_j$ , so that the following diagram commutes for all  $i \leq j$ ,



We call this the *universal property* of the inverse limit  $\varprojlim_{j \in J} H_j$ .

A *profinite group*  $H$  is a group which can be realised as the inverse limit of an inverse system  $(H_i, \phi_{ij})$  of finite groups. We equip each finite group  $H_i$  with the discrete topology and  $\prod_{i \in J} H_i$  with the product topology. Then the inverse limit  $\varprojlim_{j \in J} H_j$  is endowed with the subspace topology coming from the inclusion

$$\varprojlim_{j \in J} H_j \subseteq \prod_{i \in J} H_i.$$

**Proposition 2.3.1.** *The profinite group  $H$  with the subspace topology coming from the inclusion  $H \subseteq \prod_{i \in J} H_i$  is a compact Hausdorff topological group.*

*Proof.* Since each  $H_i$  is compact, by Tychonoff's theorem we have that the product  $\prod_{i \in J} H_i$  is compact. Moreover,  $\varprojlim_{j \in J} H_j$  is closed since it is the intersection of closed subsets,

$$\varprojlim_{j \in J} H_j = \bigcap_{i \leq j} \{(h_i)_{i \in J} : \phi_{ij}(h_j) = h_i\}.$$

Since  $\varprojlim_{j \in J} H_j$  is a closed subspace of a compact space, it follows that it is compact. Moreover, the product  $\prod_{i \in J} H_i$  is Hausdorff and hence  $\varprojlim_{j \in J} H_j$  is also Hausdorff.

Note that the product  $\prod_{i \in J} H_i$  is a group with binary operation defined by coordinate-wise multiplication. It is trivial to check that  $\varprojlim_{j \in J} H_j$  is non-empty and furthermore it forms a subgroup of  $\prod_{i \in J} H_i$ . Moreover,  $\prod_{i \in J} H_i$  is a topological group and thus so is  $\varprojlim_{j \in J} H_j$ .  $\square$

In practice, when working with the topology on a profinite group  $H$  we will often use the following result:

**Lemma 2.3.2.** *Let  $H = \varprojlim_{i \in J} H_i$  be a profinite group. For any element  $h \in H$  where  $h = (h_i)_{i \in J}$ , there is a basis of open neighbourhoods of  $h$  which is given by  $\{p_i^{-1}(h_i) : i \in J\}$ .*

We also need to record the following fact:

**Lemma 2.3.3.** *Let  $H$  be a profinite group. Then a subgroup of  $H$  is open if and only if it is closed and of finite index.*

In fact, we can say more. A profinite group  $H$  is said to be *topologically finitely generated* if there exists a finite collection of elements  $\gamma_1, \dots, \gamma_n \in H$  such that

$$H = \overline{\langle \gamma_1, \dots, \gamma_n \rangle}.$$

**Theorem 2.3.4** (Nikolov–Segal [NS07]). *Let  $H$  be a topologically finitely generated profinite group. Then every finite-index subgroup of  $H$  is open.*

**Corollary 2.3.5.** *Let  $H$  a topologically finitely generated profinite group and  $K$  a profinite group. Then any group homomorphism  $\phi: H \rightarrow K$  is continuous.*

Let  $G$  be a discrete group and  $\{N_i\}_{i \in J}$  the family of all finite-index normal subgroups of  $G$ . There is a natural poset structure on  $J$  given by  $i \leq j$  whenever  $N_j \leq N_i$ . If  $i \leq j$ , define  $\phi_{ij}: G/N_j \rightarrow G/N_i$  to be the natural quotient. Then  $(G/N_i, \phi_{ij})$  forms an inverse system of groups. The *profinite completion*  $\widehat{G}$  of  $G$  is the inverse limit of the inverse system  $(G/N_i, \phi_{ij})$ ,

$$\widehat{G} := \varprojlim_{i \in J} G/N_i.$$

The universal property of the inverse limit  $\varprojlim_{i \in J} G/N_i$  implies that there is a natural map

$$\begin{aligned} \iota: G &\rightarrow \widehat{G} \\ g &\mapsto (\pi_i(g))_{i \in J}. \end{aligned}$$

**Lemma 2.3.6.** *The image  $\iota(G)$  of the map  $\iota$  is a dense subgroup of  $\widehat{G}$ .*

*Proof.* For contradiction, suppose that there exists an open subset  $U$  of  $\widehat{G}$  which is disjoint from  $\iota(G)$ . By Lemma 2.3.2,  $U$  contains an open subset of the form  $p_i^{-1}(g_i)$ , for some  $g_i \in G/N_i$ . Choose a lift  $g \in G$  of the element  $g_i \in G/N_i$ . Then  $p_i(\iota(g)) = g_i$  and thus  $g \in p_i^{-1}(g_i) \cap \iota(G)$ . This is a contradiction.  $\square$

In particular, we see that if  $G$  is finitely generated then the profinite completion  $\widehat{G}$  is topologically finitely generated.

For a group  $G$ , the *profinite topology* on  $G$  is the topology induced by the map  $\iota: G \rightarrow \widehat{G}$ . Concretely, it is the coarsest topology on  $G$  so that the map  $\iota: G \rightarrow \widehat{G}$  is continuous. It is not difficult to see that the profinite topology admits a basis of open sets given by the set of cosets of finite-index normal subgroups of  $G$ . Let  $N \trianglelefteq G$  be a finite-index normal subgroup and  $\{x_1, \dots, x_k\}$  a collection of coset representatives. Then each coset

$$x_i N = G - \bigcup_{j \neq i} x_j N$$

is the complement of an open subgroup, and thus is closed.

**Remark 2.3.7.** Note that for any subgroup  $H$  of  $G$  of finite index, there exists a normal subgroup of finite index  $N \trianglelefteq G$  which is contained in  $H$ . Hence, the profinite topology on  $G$  admits a basis of open sets given by left cosets of finite-index subgroups of  $G$ .

**Lemma 2.3.8.** *Let  $G$  be a group and  $H \leq G$  a subgroup. Then, the following are equivalent.*

1.  $H$  is separable in  $G$ .
2.  $H$  is the intersection of normal finite-index subgroups.
3.  $H$  is the intersection of finite-index subgroups.

**Definition 2.3.9.** The group  $G$  is said to be *residually finite* if the trivial subgroup  $1 \leq G$  is closed in the profinite topology on  $G$ .

**Lemma 2.3.10.** *The map  $\iota$  is injective if and only if  $G$  is residually finite.*

*Proof.* If  $g \in G$  is such that  $\iota(g) = 1$  then  $g$  is contained in the intersection of all finite-index normal subgroups of  $G$ . On the other hand, the trivial subgroup  $1 \leq G$  is closed in the profinite topology exactly when it is the intersection of a sub-collection of finite-index normal subgroups.  $\square$

Let  $G$  be a residually finite, finitely generated group. There is a one-to-one correspondence

$$\begin{aligned} \{H \leq_f G \text{ finite-index subgroup}\} &\rightarrow \{K \leq_o \widehat{G} \text{ open subgroup}\} \\ H &\mapsto \overline{\iota(H)} \end{aligned} \quad (2.2)$$

so that

1.  $H \trianglelefteq G$  is a normal subgroup if and only if  $\overline{\iota(H)} \trianglelefteq \widehat{G}$ ;
2. the index of  $H$  in  $G$  satisfies  $[G : H] = [\widehat{G} : \overline{\iota(H)}]$ ; and
3. if  $H$  is normal in  $G$ , then  $G/H \simeq \widehat{G}/\overline{\iota(H)}$ .

We say a subgroup  $H$  of  $G$  is *separable* in  $G$  if it is closed in the profinite topology on  $G$ . A subgroup  $H$  is *fully separable* in  $G$ , if every finite-index subgroup  $H' \leq H$  is closed in the profinite topology on  $G$ .

**Proposition 2.3.11.** *Let  $G$  be residually finite and suppose that  $H \leq G$  is a fully separable subgroup. Then there exists an isomorphism*

$$\overline{\iota(H)} \simeq \widehat{H}.$$

*Proof.* Since  $G$  is residually finite, the map  $\iota$  is injective and thus we may identify  $H$  with its image  $\iota(H)$ . Hence  $\overline{\iota(H)}$  is equal to the inverse limit

$$\overline{\iota(H)} = \varprojlim_{j \in J} p_j(H).$$

Thus there exists an epimorphism

$$\widehat{H} \twoheadrightarrow \overline{\iota(H)}$$

given by projecting onto the coordinates which correspond to the finite-index normal subgroups of  $H$  of the form  $H/(H \cap N_i)$ , where  $N_i \leq G$  is a finite-index normal subgroup of  $G$ . Clearly the epimorphism is injective if for every finite-index normal subgroup  $H_i \leq H$ , there exists a finite-index normal subgroup  $N_i \leq G$  such that  $H \cap N_i \leq H_i$ . But  $H_i$  is closed in the profinite topology on  $G$  and thus it is equal to the intersection of some collection of finite-index subgroups of  $G$ .  $\square$

Finally, we encode the following useful lemma:

**Lemma 2.3.12.** *If  $G$  and  $H$  are finitely generated abelian groups and  $\widehat{G} \simeq \widehat{H}$  then  $G \simeq H$ .*

**Corollary 2.3.13.** *Let  $G$  and  $H$  be finitely generated groups and suppose that  $\widehat{G} \simeq \widehat{H}$ . Then  $G_{\text{ab}} \simeq H_{\text{ab}}$  and in particular  $b_1(G) = b_1(H)$ .*

# Chapter 3

## $\text{Out}(F_n)$ and free-by-cyclic groups

Let  $F_n$  denote the free group of rank  $n$ . An automorphism of  $F_n$  is said to be *inner* if it acts by conjugation by a fixed element of  $F_n$ . The set of all inner automorphisms  $\text{Inn}(F_n)$  forms a normal subgroup of  $\text{Aut}(F_n)$  called the *inner automorphism group*. The *outer automorphism group* of  $F_n$  is the quotient

$$\text{Out}(F_n) = \text{Aut}(F_n)/\text{Inn}(F_n).$$

The group  $\text{Out}(F_n)$  of outer automorphisms of the free group  $F_n$  is a natural and widely studied object in geometric group theory. A range of powerful tools has been developed to study  $\text{Out}(F_n)$ . As a result, much is known about its geometry, cohomological properties, subgroup structure and actions on measure spaces, see e.g. [CV86, BFH00, BFH05, GH21b].

In this thesis we will be studying *free-by-cyclic* groups, which are mapping tori of elements in  $\text{Out}(F_n)$ . We will investigate how the structure of a free-by-cyclic group is impacted by the properties of its monodromy maps. To that end, we will begin this chapter by recalling the definitions of dynamical properties of elements in  $\text{Out}(F_n)$ , such as growth and irreducibility, as well as the theory of their topological representatives including relative train tracks. In the second part of this chapter, we will discuss known facts about free-by-cyclic groups which will be used throughout the thesis.

### 3.1 Growth and irreducibility

Let  $X$  be a free basis of the free group  $F_n$ . For any  $g \in F_n$ , we denote by  $|g|_X$  the length of the reduced word representative of  $g$ . We write  $|\bar{g}|_X$  to denote the minimal length of a cyclically reduced word representing a conjugate of  $g$ .

An outer automorphism  $\Phi \in \text{Out}(F_n)$  acts on the set of conjugacy classes of elements in  $F_n$ . Given a conjugacy class  $\bar{g}$  of an element  $g \in F_n$ , we say that  $\bar{g}$  *grows polynomially of degree  $d$  under the iteration of  $\Phi$* , if there exist constants  $A, B > 0$  such that for all  $k \geq 1$

$$Ak^d \leq |\Phi^k(\bar{g})|_X \leq Bk^d.$$

We say  $g$  *grows exponentially* if there exists a constant  $\mu > 1$  such that for all  $k \geq 1$

$$\mu^k \leq |\Phi^k(\bar{g})|.$$

For any two free generating sets  $X$  and  $Y$  of  $F_n$ , the word metrics with respect to  $X$  and  $Y$  are bi-Lipschitz equivalent. It follows that the growth of a conjugacy class under  $\Phi$  does not depend on the specific choice of free basis for  $F_n$ .

We say the outer automorphism  $\Phi \in \text{Out}(F_n)$  *grows polynomially of degree  $d$*  if there exists an element  $g \in F_n$  whose conjugacy class grows polynomially of degree  $d$  and such that every other conjugacy class grows polynomially of degree  $\leq d$ . We say  $\Phi$  *grows exponentially* if there exists an exponentially-growing conjugacy class.

**Proposition 3.1.1.** *Let  $\Phi \in \text{Out}(F_n)$  be an outer automorphism which grows polynomially of degree  $d$ . Then, the following hold:*

- $d \leq n - 1$ ;
- $d = 0$  if and only if  $\Phi$  has finite order in  $\text{Out}(F_n)$ .

We say an outer automorphism  $\Phi \in \text{Out}(F_n)$  is *atoroidal* if there does not exist a non-trivial conjugacy class of elements of  $F_n$  which is periodic or, equivalently, grows polynomially of degree 0 under the iterations of  $\Phi$ .

**Definition 3.1.2.** A *free factor system* of  $F_n$  is a non-empty collection of conjugacy classes  $\{[[F^1]], \dots, [[F^k]]\}$  of subgroups of  $F_n$  such that  $F^1 * \dots * F^k$  is a free factor of  $F_n$ . If each  $F^i$  is non-trivial then it is said to be a non-trivial free factor system. A free factor system is said to be *proper* if it is not equal to  $\{[[F_n]]\}$ .

Given an outer automorphism  $\Phi \in \text{Out}(F_n)$  and a free factor system  $\mathcal{F} = \{[[F^1]], \dots, [[F^k]]\}$ , we say that  $\mathcal{F}$  is  $\Phi$ -*invariant* if  $\Phi$  permutes the conjugacy classes in  $\mathcal{F}$ .

An outer automorphism  $\Phi$  is said to be *reducible* if there exists a non-trivial proper  $\Phi$ -invariant free factor system. If  $\Phi$  is not reducible then it is *irreducible*. An automorphism  $\Phi$  is *fully irreducible* if  $\Phi^k$  is irreducible for all  $k \geq 1$ .

**Theorem 3.1.3** (Dowdall–Kapovich–Leininger [DKL15, Corollary B.4]). *Let  $\Phi \in \text{Out}(F_n)$  be atoroidal. Then  $\Phi$  is irreducible if and only if  $\Phi$  is fully irreducible.*

## 3.2 Topological representatives for elements in $\text{Out}(F_n)$

Let  $R_n$  denote the graph which consists of a single vertex and  $n$  edges, and call it the *rose with  $n$  petals*. We identify the fundamental group of  $R_n$  with the free group  $F_n$  by fixing an orientation on edges and labelling the edges by the elements of the free basis  $X$  of  $F_n$ . Then, an automorphism  $\phi \in \text{Aut}(F_n)$  defines a homotopy equivalence  $\rho: R_n \rightarrow R_n$  which fixes the unique vertex of  $R_n$  and expands each edge labelled by the generator  $x \in X$  to an edge path labelled by the reduced word in  $X$  representing  $\phi(x)$ . We call such  $\rho$  the *standard topological representative* of  $\phi$ .

A *topological representative* of an outer automorphism  $\Phi \in \text{Out}(F_n)$  is a tuple  $(f, \Gamma)$ , where  $\Gamma$  is a connected graph and  $f: \Gamma \rightarrow \Gamma$  is a homotopy equivalence which *induces*  $\Phi$ . More precisely, there exists a homotopy equivalence  $\alpha: R_n \rightarrow \Gamma$  such that the following diagram commutes up to homotopy,

$$\begin{array}{ccc} R_n & \xrightarrow{\rho} & R_n \\ \downarrow \alpha & & \downarrow \alpha \\ \Gamma & \xrightarrow{f} & \Gamma \end{array}$$

Here,  $\rho: R_n \rightarrow R_n$  is the standard topological representative of an automorphism  $\phi$  which represents  $\Phi$ .

We will also always assume that  $f$  preserves the set of vertices of  $\Gamma$  and is locally injective on the interiors of the edges of  $\Gamma$ .

## Irreducible representatives

Fix an orientation  $E^+(\Gamma)$  of  $\Gamma$  and an ordering of the edges in  $E^+(\Gamma)$ . The *incidence matrix*  $A$  of  $f$  is the matrix with entries  $a_{ij}$ , where  $a_{ij}$  is the number of occurrences of the unoriented edge  $e_j$  in the edge-path  $f(e_i)$ . Recall that a non-negative integral  $n$ -by- $n$  square matrix  $M$  is said to be *irreducible* if for any  $i, j \leq n$ , there exists some  $k \geq 1$  such that the  $(i, j)$ -th entry of  $M^k$  is positive. We say that a topological representative  $(f, \Gamma)$  is *irreducible* if its incidence matrix  $A$  is an irreducible matrix.

By the Perron–Frobenius Theorem (see Chapter 2 in [Sen06]), if the topological representative  $(f, \Gamma)$  is irreducible then the spectral radius  $\rho(A)$  of its incidence matrix  $A$  is equal to an eigenvalue  $\lambda$  of  $A$ , which is known as the *Perron–Frobenius eigenvalue*. Furthermore,  $\rho(A) \geq 1$  and equality holds exactly when  $A$  is a permutation matrix. We will call  $\lambda$  the *stretch factor* of  $f$ .

We say that a subgraph of a graph is *non-trivial* if it has a component which is not a vertex.

**Proposition 3.2.1** (Bestvina–Handel [BH92]). *An outer automorphism  $\Phi \in \text{Out}(F_n)$  is irreducible if and only if every topological representative  $(f, \Gamma)$  of  $\Phi$ , where  $\Gamma$  has no valence-one vertices and no non-trivial  $f$ -invariant forests, is irreducible.*

The *stretch factor* of an irreducible outer automorphism  $\Phi$  is the infimum of the stretch factors of the irreducible topological representatives of  $\Phi$ . By the proof of Theorem 1.7 in [BH92], the infimum is realised. We will write  $\lambda(\Phi)$  to denote the stretch factor of  $\Phi$ .

**Lemma 3.2.2.** *Let  $n \geq 2$  and  $C > 1$ . There exist at most finitely many conjugacy classes of irreducible elements in  $\text{Out}(F_n)$  with stretch factor at most  $C$ .*

*Proof.* We mimic the proof of the analogous result in the setting of mapping class groups of surfaces from [FM12, Theorem 14.9].

Let  $\text{CV}_n$  denote the Culler–Vogtmann Outer space. For  $\epsilon > 0$ , write  $\text{CV}_n(\epsilon)$  to denote the *thick part* of  $\text{CV}_n$ , which is defined as the set of all metric graphs  $\Gamma$  in  $\text{CV}_n$  such that the length of every loop  $\alpha$  in  $\Gamma$  satisfies  $\ell_\Gamma(\alpha) \geq \epsilon$ . We consider  $\text{CV}_n$  as a metric space with the Lipschitz metric.

Let  $\{\Phi_i\}_{i \in I}$  be a collection of irreducible elements in  $\text{Out}(F_n)$  which are non-pairwise conjugate, and such that  $\lambda(\Phi_i) \leq C$  for each  $i \in I$ . Suppose first that  $\lambda(\Phi_i) = 1$  for all  $i \in I$ . Then each  $\Phi_i$  has finite order in  $\text{Out}(F_n)$ . Every finite order element in  $\text{Out}(F_n)$  is induced by a periodic automorphism of a graph with no valence-one and valence-two vertices. In particular, there are finitely many finite order elements in  $\text{Out}(F_n)$  and hence  $I$  is finite.

Suppose now that some  $\Phi_i$  has infinite order. Without loss of generality, we may assume that every  $\Phi_i$  has infinite order. Let  $\epsilon = 1/((3n-3)(C+1)^{3n-2})$ . By [FMS21, Proposition 2.14], each axis of  $\Phi_i$  is contained in the  $\epsilon$ -thick part  $\text{CV}_n(\epsilon)$ .

Since action of  $\text{Out}(F_n)$  on the thick part  $\text{CV}_n(\epsilon)$  is cocompact, there exists some compact subset  $K \subseteq \text{CV}_n(\epsilon)$  such that  $\bigcup_{g \in \text{Out}(F_n)} g \cdot K = \text{CV}_n(\epsilon)$ . Thus, for each  $i \in I$  there is an element  $\Psi_i \in \text{Out}(F_n)$  which is conjugate to  $\Phi_i$  and such that  $\text{Axis}(\Psi_i) \cap K \neq \emptyset$ . Let  $N_{\log C}(K)$  denote the  $(\log C)$ -neighbourhood of  $K$  in  $\text{CV}_n(\epsilon)$ . Then,  $\Psi_i \cdot N_{\log C}(K) \cap N_{\log C}(K) \neq \emptyset$  for all  $i \in I$ . Since the thick part  $\text{CV}_n(\epsilon)$  is proper, we have that  $N_{\log C}(K)$  is a compact subset. Hence, since the action of  $\text{Out}(F_n)$  on  $\text{CV}_n(\epsilon)$  is proper, it must be the case that  $I$  is finite. □

## The general case

Let  $(f, \Gamma)$  be a topological representative. A *filtration of length  $l$*  of  $(f, \Gamma)$  is a sequence of subgraphs

$$\emptyset = \Gamma_0 \subseteq \Gamma_1 \subseteq \dots \subseteq \Gamma_l = \Gamma, \tag{3.1}$$

so that  $f(\Gamma_i) \subseteq \Gamma_i$  for all  $i$ . The closure  $S_i = \text{Cl}(\Gamma_i \setminus \Gamma_{i-1})$  is called the  *$i$ -th stratum* of the filtration. We reorder the edges in  $E^+(\Gamma)$  so that whenever  $i < j$ , the edges in  $\Gamma_i$  precede the edges in  $\Gamma_j$ . The filtration is said to be *maximal* if the square submatrix  $A_i$  of the incidence matrix  $A$  which corresponds to the  $i$ -th stratum is

either the zero matrix, or it is irreducible. It is a standard fact that any topological representative admits a maximal filtration which is unique up to reordering of the strata. Note that  $(f, \Gamma)$  is irreducible if and only if it admits a maximal filtration of length one.

A non-zero stratum  $S_i$  of a filtration is said to be an *exponentially-growing stratum* if the Perron–Frobenius eigenvalue of the incidence matrix  $A_i$  of  $f$  restricted to  $S_i$  is strictly greater than one, and otherwise it is called a *non-exponentially-growing stratum*. For a topological representative  $(f, \Gamma)$ , we write  $\lambda_f$  (or  $\lambda$  if there is no risk of confusion) to denote the maximal Perron–Frobenius eigenvalue taken over all the non-zero strata of the maximal filtration of  $(f, \Gamma)$ , and we call it the *(homotopical) stretch factor* of  $f$ .

For an outer automorphism  $\Phi$ , we define the stretch factor  $\lambda(\Phi)$  of  $\Phi$  to be the infimum, taken over all the *bounded* topological representatives  $(f, \Gamma)$  of  $\Phi$ , of  $\lambda_{\max}$ , where  $\lambda_{\max}$  denotes the maximum stretch factor of the non-zero strata in a maximal filtration of  $(f, \Gamma)$ . A *bounded* topological representative  $(f, \Gamma)$  of  $\Phi \in \text{Out}(F_n)$  is a topological representative where the number of exponentially-growing strata is bounded by  $3n - 3$ , and each exponential stratum has stretch factor which is the Perron–Frobenius eigenvalue of an irreducible square matrix of dimensions bounded above by  $3n - 3$ . This condition ensures that, for any constant  $C$ , there are finitely many values of the maximal stretch factor  $\lambda_{\max} \leq C$  which can be realised by a bounded topological representative. It follows that the infimum  $\lambda(\Phi)$  is realised, and moreover it is realised by a bounded *relative train track* representative  $(f, \Gamma)$ .

## (Relative) train tracks

A topological representative  $(f, \Gamma)$  is said to be a *train track* if every positive power of  $f$  is locally injective on the interiors of edges.

**Theorem 3.2.3** (Bestvina–Handel [BH92]). *Every irreducible outer automorphism  $\Phi \in \text{Out}(F_n)$  admits a train track representative.*

Define a map  $Df$  on the set of (oriented) edges of  $\Gamma$ , so that for each edge  $e \in E(\Gamma)$ ,  $Df(e)$  is the first oriented edge in the edge-path  $f(e)$ . A *turn* in  $\Gamma$

is a set of edges  $\{e_1, e_2\}$  in  $\Gamma$  such that the initial vertices of  $e_1$  and  $e_2$  coincide. The turn  $\{e_1, e_2\}$  is *degenerate* if  $e_1 = e_2$  (as oriented edges), and *non-degenerate* otherwise. The map  $Df$  induces a map on the set of turns defined by

$$Df\{e_1, e_2\} = \{Df(e_1), Df(e_2)\}.$$

A turn is said to be *legal* if its image under  $Df^k$  is non-degenerate for every  $k \geq 0$ . A locally injective path  $\gamma: I \rightarrow \Gamma$  is said to be *legal*, if for any two consecutive oriented edges  $(e_1, e_2)$  in the image of  $\gamma$ , we have that  $\{e_1^{-1}, e_2\}$  is a legal turn.

Let  $\{S_i\}$  denote the strata of a maximal filtration of  $(f, \Gamma)$  as in (3.1). The topological representative  $(f, \Gamma)$  is a *relative train track* if the following conditions are satisfied for every exponentially-growing stratum  $S_i$ :

1. For every edge  $e$  in  $S_i$ ,  $Df(e)$  is also an edge of  $S_i$ .
2. For any locally injective path  $\gamma$  in  $\Gamma_{i-1}$  with endpoints in  $\Gamma_{i-1} \cap S_i$ , the path  $f \circ \gamma$  also has endpoints in  $\Gamma_{i-1} \cap S_i$  and  $f \circ \gamma$  cannot be homotoped relative its endpoints to the trivial path.
3. For any locally injective legal path  $\gamma$  in  $H_i$ ,  $f \circ \gamma$  is locally injective and does not contain illegal turns in  $H_i$ .

A useful consequence of the definition of relative train tracks is the following, which is Lemma 5.10 in [BH92].

**Lemma 3.2.4.** *Let  $(f, \Gamma)$  be a relative train track map. Fix an exponentially-growing stratum  $S_i$  of  $(f, \Gamma)$  and let  $\lambda_i$  denote its stretch factor. Then there exists a map defined on the edge set of  $\Gamma$ ,*

$$L_i: E(\Gamma) \rightarrow \mathbb{R}_{\geq 0},$$

*such that for every edge  $e$  in  $S_i$ ,  $L_i(e) > 0$  and  $L_i(f(e)) = \lambda_i \cdot L_i(e)$ .*

A *periodic Nielsen path* in  $(f, \Gamma)$  is a locally injective path  $\gamma: I \rightarrow \Gamma$  such that for some  $k > 0$ , the map  $f^k \circ \gamma$  is homotopic to  $\gamma$  relative endpoints. The least such integer  $k$  is called the *period* of  $\gamma$ . A periodic Nielsen path is *indivisible* if it cannot be realised as a concatenation of non-trivial periodic Nielsen paths.

**Theorem 3.2.5** (Bestvina–Handel [BH92]). *Every outer automorphism  $\Phi \in \text{Out}(F_n)$  admits a topological representative which is a relative train track.*

Moreover, it is possible to obtain greater control of the Nielsen paths if one allows passing to a power of the outer automorphism:

**Theorem 3.2.6** (Bestvina–Feighn–Handel [BFH00]). *For every outer automorphism  $\Phi \in \text{Out}(F_n)$ , there exists integer  $k > 0$  such that  $\Phi^k$  admits a topological representative which is a relative train track and such that every periodic Nielsen path has period one, and each exponentially-growing stratum intersects at most one indivisible Nielsen path.*

## Representatives for polynomially growing automorphisms

We call a topological representative  $(f, \Gamma)$  *upper triangular* if  $f$  fixes every vertex of  $\Gamma$  and if  $(f, \Gamma)$  admits a maximal filtration

$$\emptyset = \Gamma_0 \subseteq \Gamma_1 \subseteq \dots \subseteq \Gamma_l = \Gamma,$$

such that the following two properties hold:

1. Each graph  $\Gamma_i$  is obtained from  $\Gamma_{i-1}$  by adding a single edge  $e_i$ .
2. For each  $i$ , the map  $f$  sends  $e_i$  to the concatenation  $e_i p_i$ , where  $p_i$  is either trivial or an immersed loop contained in  $\Gamma_{i-1}$ .

Moreover, we will assume that  $f$  fixes each vertex of  $\Gamma$ .

Recall that a matrix  $A \in \text{GL}_n(R)$  is *unipotent* if it is conjugate in  $\text{GL}_n(R)$  to an upper triangular matrix with 1's on the diagonal.

**Definition 3.2.7.** An outer automorphism  $\Phi \in \text{Out}(F_n)$  is *unipotent polynomially growing (UPG)* if it is polynomially growing and the induced automorphism  $\Phi^{\text{ab}}$  in  $\text{GL}_n(\mathbb{Z})$  is unipotent.

**Theorem 3.2.8** (Bestvina–Feighn–Handel [BFH00]). *Any UPG outer automorphism  $\Phi \in \text{Out}(F_n)$  admits a relative train track representative which is upper triangular.*

**Lemma 3.2.9.** *Any polynomially growing outer automorphism  $\Phi \in \text{Out}(F_n)$  admits a positive power  $\Phi^k$  which is UPG.*

### 3.3 Free-by-cyclic groups

A group  $G$  is said to be *free-by-cyclic* if it fits into a short exact sequence

$$1 \rightarrow F \rightarrow G \xrightarrow{\varphi} Z \rightarrow 1, \quad (3.2)$$

where  $F$  is a non-trivial free group of **finite rank** and  $Z$  is an infinite cyclic group.

Abusing notation, we identify  $F$  with its image under the monomorphism in the short exact sequence (3.2). Let us fix a free generating set of the subgroup  $F$ ,  $F = \langle x_1, \dots, x_n \rangle$ , and let  $t \in G$  be an element which is mapped to the positive generator of  $Z$  under  $\varphi$ . The conjugation action of  $t$  on  $F \leq G$  induces an automorphism  $\phi \in \text{Aut}(F(x_1, \dots, x_n))$ , and  $G$  admits the structure of a semidirect product

$$G \simeq F \rtimes_{\phi} \langle t \rangle := \langle x_1, \dots, x_n, t \mid x_i^t = \phi(x_i) \ \forall i \rangle. \quad (3.3)$$

We will call  $\phi$  the *monodromy*,  $F$  the *fibre* and  $t \in G$  the *stable letter* associated to the splitting (3.3). We call the map  $\varphi: G \rightarrow \mathbb{Z}$  the *standard fibring* associated to the splitting.

Note that for any two elements  $t_1, t_2 \in \varphi^{-1}(1)$ , the monodromies of the corresponding semidirect product splittings  $F \rtimes \langle t_1 \rangle$  and  $F \rtimes \langle t_2 \rangle$  differ by an inner automorphism. Conversely, if  $\phi, \psi \in \text{Aut}(F_n)$  are such that  $\phi = \text{ad}_x \circ \psi$  for some inner automorphism  $\text{ad}_x \in \text{Aut}(F_n)$ , then there is an isomorphism  $F_n \rtimes_{\phi} \langle t_1 \rangle \rightarrow F_n \rtimes_{\psi} \langle t_2 \rangle$  which acts as the identity on the fibre and sends  $t_1 \mapsto t_2 x$ . Moreover, the standard fibrings corresponding to the two splittings are equal. Thus, for an element  $\Phi \in \text{Out}(F_n)$ , we write  $G_{\Phi} = F_n \rtimes_{\Phi} \mathbb{Z}$  to denote the isomorphism class of the group  $G_{\phi}$ , for any representative  $\phi$  of  $\Phi$ .

Moreover, different choices of free generating sets of the fibre  $F$  give rise to conjugate monodromies. We summarise the previous observations with the following lemma:

**Lemma 3.3.1.** *Let  $\Phi, \Psi \in \text{Out}(F_n)$  be outer automorphisms with representatives  $\phi \in \Phi$ ,  $\psi \in \Psi$ . Let  $G_\phi = F_\phi \rtimes_\phi \langle t \rangle$  and  $G_\psi = F_\psi \rtimes_\psi \langle s \rangle$  and  $\chi_\phi: G_\phi \rightarrow \mathbb{Z}$ ,  $\chi_\psi: G_\psi \rightarrow \mathbb{Z}$  be the standard characters. Then the following are equivalent:*

1. *The outer automorphisms  $\Phi$  and  $\Psi$  are conjugate in  $\text{Out}(F_n)$ .*
2. *There exists an isomorphism  $f: G_\phi \rightarrow G_\psi$  such that  $\chi_\phi = \chi_\psi \circ f$ .*
3. *There exists an isomorphism  $f: G_\phi \rightarrow G_\psi$  such that  $f(F_\phi) = F_\psi$  and  $f(t) \in sF_\psi$ .*

*Proof.* (1)  $\Rightarrow$  (2) Let  $\Phi$  and  $\Psi$  be conjugate elements of  $\text{Out}(F_n)$  with representatives  $\phi$  and  $\psi$ , respectively. Let  $\omega \in \text{Aut}(F_n)$  and  $x \in F_n$  be such that  $\phi^\omega = \text{ad}_x \psi$ . We define a map  $f: G_\phi \rightarrow G_\psi$  such that  $f(t) = sx$  and  $f(a) = \omega^{-1}(a)$  for all  $a \in F_\phi$ . Then

$$f(t^{-1}at) = x^{-1}s^{-1}\omega^{-1}(a)sx = \text{ad}_x \psi \omega^{-1}(a) = \omega^{-1}\phi(a) = f(\phi(a)),$$

for all  $a \in F_\phi$ . Hence  $f$  is a homomorphism and furthermore it is clear that  $f$  is bijective.

(2)  $\Rightarrow$  (3) We have that  $f(\ker \chi_\phi) = \ker \chi_\psi$  and thus  $f(F_\phi) = F_\psi$ . Also

$$\chi_\psi \circ f(t) = \chi_\phi(t) = 1.$$

Hence  $f(t) \in \chi_\psi^{-1}(1) = sF_\psi$ .

(3)  $\Rightarrow$  (1) Let  $x \in F_\psi$  be such that  $f(t) = sx$ . Then

$$f(\phi(a)) = f(t^{-1}at) = \text{ad}_x \psi f(a)$$

for all  $a \in F_\phi$ . After identifying  $F_n \simeq F_\psi$  and  $F_n \simeq F_\phi$ , the map  $f$  when restricted to  $F_\phi$  induces an element of  $\text{Aut}(F_n)$ .  $\square$

We will record some basic facts about free-by-cyclic groups which will be used throughout this thesis:

**Lemma 3.3.2.** *If  $G$  is free-by-cyclic then  $G$  has the following properties:*

1. *It is torsion free;*
2. *It is one ended;*
3. *It is of type F;*
4. *The cohomological dimension of  $G$  is equal to two.*

*Proof. Item 1.* Note that if  $G$  is free-by-cyclic then it is an HNN extension of a torsion free group and thus by the normal form theorem for HNN extensions it is torsion free.

*Item 2.* Suppose that  $G$  is not one ended. If  $G$  is two ended then  $G$  is virtually infinite cyclic. This contradicts the fact that  $G$  contains an infinite normal subgroup  $F$  of infinite index. Hence  $G$  must be infinitely ended. Since  $G$  is torsion free by Item 1, it follows that  $G$  splits as a free product  $G = G_1 * G_2$ , where  $G_1$  and  $G_2$  are non-trivial groups. Let  $T$  be the Bass–Serre tree corresponding to the free splitting. Let  $F \trianglelefteq G$  be the fibre of a free-by-cyclic splitting of  $G$ . The action of  $G$  on  $T$  is minimal, and hence since  $F$  is a normal subgroup of  $G$ , Lemma 2.1.4 implies that the induced action of  $F$  on  $T$  is also minimal. Since  $F$  is finitely generated, it must be the case that the quotient graph  $F \backslash T$  is finite. In particular, the induced action of  $F$  on  $T$  has finitely many orbits of edges. However, since the  $G$ -stabiliser of each edge is trivial, the number of orbits of edges under the action of  $F$  must be infinite by Lemma 2.1.5. This is a contradiction.

*Item 3.* Fix a free-by-cyclic splitting  $G = F \rtimes_{\phi} \mathbb{Z}$  of  $G$  for some  $\phi \in \text{Aut}(F)$ . Let  $\rho: R_n \rightarrow R_n$  be the standard topological representative of  $\phi$ . Recall that the mapping torus of  $\rho$  is defined as the quotient space

$$M_{\rho} := \frac{R_n \times [0, 1]}{(\rho(x), 0) \sim (x, 1)}.$$

Applying Seifert–Van Kampen’s Theorem, we see that  $\pi_1(M_{\rho}) \simeq F_n \rtimes_{\phi} \mathbb{Z}$ . Moreover, the universal cover  $\widetilde{M}_{\rho}$  of  $M_{\rho}$  deformation retracts onto a point. Hence,  $M_{\rho}$  is a finite classifying space for  $G$  and thus it is of type F.

*Item 4.* By the proof of Item 3, it follows that  $G$  admits a two-dimensional classifying space and thus  $\text{cd}(G) \leq 2$ . Since  $G$  is one ended by Item 2 and hence not free, by the Stallings–Swan Theorem 2.2.8 it follows that  $\text{cd}(G) > 1$ .  $\square$

**Lemma 3.3.3.** *Let  $G$  be a free-by-cyclic group and  $\varphi: G \rightarrow \mathbb{Z}$  a homomorphism with finitely generated kernel. Then  $\ker \varphi$  is free.*

*Proof.* By the Stallings–Swan Theorem 2.2.8 it suffices to prove that  $\ker \varphi$  has cohomological dimension equal to one,  $\text{cd}_R(\ker \varphi) = 1$ , for some ring  $R$ .

We start by showing that  $\ker \varphi$  is of type  $\text{FP}_2$ . This follows by Feighn–Handel’s Theorem 3.3.5, which shows that all finitely generated subgroups of a free-by-cyclic groups are finitely presented and thus of type  $\text{FP}_2$ . We choose to take a more direct approach here, inspired by the work of Kochloukova [Koc06]. For the convenience of the reader we sketch the proof of [Koc06, Theorem 1] in our particular case.

By Sikorav’s Theorem 5.0.5, in order to show that  $\ker \varphi$  is of type  $\text{FP}_2$  it suffices to prove that the Novikov homology (see Section 5.0.2) vanishes in degree two,

$$H_2(G, \widehat{\mathbb{Z}G}^{\pm\varphi}) = 0.$$

Since  $\ker \varphi$  is finitely generated, by Sikorav’s Theorem we have that for  $i = 0, 1$ ,

$$H_i(G, \widehat{\mathbb{Z}G}^{\pm\varphi}) = 0.$$

By the proof of Lemma 3.3.2, the presentation complex of  $G$  with respect to any free-by-cyclic presentation gives rise to a free resolution of  $\mathbb{Z}$  over  $\mathbb{Z}G$ -modules, which is of the form

$$0 \rightarrow \mathbb{Z}G^n \xrightarrow{\partial_2} \mathbb{Z}G^{n+1} \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\partial_0} \mathbb{Z}$$

for some  $n \geq 1$ . After tensoring the chain complex by the Novikov ring  $\widehat{\mathbb{Z}G}^\varphi$ , we obtain a chain complex

$$0 \rightarrow (\widehat{\mathbb{Z}G}^\varphi)^n \xrightarrow{\text{id} \otimes \partial_2} (\widehat{\mathbb{Z}G}^\varphi)^{n+1} \xrightarrow{\text{id} \otimes \partial_1} \widehat{\mathbb{Z}G}^\varphi \xrightarrow{\text{id} \otimes \partial_0} 0,$$

which is exact in dimensions 0 and 1. It then follows that  $\text{id} \otimes \partial_2$  maps onto an  $n$ -dimensional direct summand of the module  $(\widehat{\mathbb{Z}G}^\varphi)^{n+1}$ . Hence,

$$\text{id} \otimes \partial_2: (\widehat{\mathbb{Z}G}^\varphi)^n \rightarrow \text{Im}(\text{id} \otimes \partial_2)$$

can be represented by a square matrix  $M$  with coefficients in the ring  $\widehat{\mathbb{Z}G}^\varphi$  which is onto. Then by [Koc06, Theorem 3],  $M$  is also injective. It follows that

$$H_2(G, \widehat{\mathbb{Z}G}^{\pm\varphi}) = \ker(\text{id} \otimes \partial_2) = 0.$$

Since the cohomological dimension of a free-by-cyclic group is equal to 2 by Item 4 in Lemma 3.3.2, and  $\ker \varphi$  is of type  $\text{FP}_2$ , by Fel'dman's Theorem 2.2.9 it follows that  $\text{cd}_{\mathbb{Q}}(\ker \varphi) = 1$  and thus  $\ker \varphi$  is free.  $\square$

### 3.3.1 Subgroups of free-by-cyclic groups

We start with a simple observation:

**Lemma 3.3.4.** *Let  $G$  be a free-by-cyclic group. Then any finite-index subgroup  $G' \leq G$  is also free-by-cyclic.*

*Proof.* Let  $\varphi: G \rightarrow \mathbb{Z}$  be the standard fibring of  $G$  corresponding to a free-by-cyclic splitting. There is an induced homomorphism from  $G'$  to  $\mathbb{Z}$  obtained by restricting  $\varphi$  to the subgroup  $G' \leq G$ . Let  $k \in \mathbb{Z}$  denote the positive generator of  $\varphi(G') \leq \mathbb{Z}$ . Define a homomorphism  $\varphi': G' \rightarrow \mathbb{Z}$  by sending  $x \mapsto \frac{1}{k} \cdot \varphi(x)$  for every  $x \in G'$ . Then the map  $\varphi'$  is surjective and

$$\ker \varphi' = \ker \varphi \cap G' \leq \ker \varphi.$$

Since  $G' \leq G$  is a finite-index subgroup, it follows that  $\ker \varphi' \leq \ker \varphi$  is a finite-index subgroup and thus it is free of finite rank.  $\square$

In [FH99], Feighn–Handel determine the structure of all finitely generated subgroups of ascending HNN extensions of free groups. The following theorem summarises their results, which we state only for free-by-cyclic groups.

**Theorem 3.3.5** (Feighn–Handel [FH99]). *Let  $G = F \rtimes_{\phi} \mathbb{Z}$  be a free-by-cyclic group and let  $\varphi: G \rightarrow \mathbb{Z}$  be the character associated to this fibring, and  $t$  the stable letter. Let  $H$  be a finitely generated subgroup of  $G$ .*

1. *If  $H \leq \ker \varphi$  then  $H$  is free.*

2. Otherwise, the cyclic subgroup  $\varphi(H)$  is generated by some  $k > 0$  and there exist finite subsets  $A, B \subseteq F$  and  $x \in F$  such that  $\text{ad}_x \phi^k(A) \leq \langle A, B \rangle$ , and

$$H = \langle A, B, s \mid s^{-1}as = \text{ad}_x \phi^k(a), \forall a \in A \rangle,$$

where  $s = xt^k$ .

The Feighn–Handel structure theorem implies that free-by-cyclic groups satisfy a strong form of coherence:

**Corollary 3.3.6.** *If  $G$  is free-by-cyclic then every finitely generated subgroup  $H \leq G$  is of type F.*

Free-by-cyclic groups are  $L^2$ -acyclic by Theorem 2.2.17, and thus by the work of Gaboriau–Nous [GN21] every infinite-index subgroup of a free-by-cyclic group has vanishing second  $L^2$ -Betti number. Combining this with the characterisation of Feighn–Handel, we get the following dichotomy for subgroups of free-by-cyclic groups:

**Proposition 3.3.7.** *Let  $G$  be a free-by-cyclic group and  $H$  a finitely generated subgroup. Then either,*

1.  $\chi(H) = 0$  and  $b_1^{(2)}(H) = 0$ , in which case  $H$  is free-by-cyclic or infinite cyclic; or
2.  $\chi(H) < 0$  and  $b_1^{(2)}(H) > 0$ , in which case  $H$  is an HNN extension of a finite rank free group over a proper free factor.

### 3.3.2 Cyclic splittings

In this subsection we will record some facts about splittings of free-by-cyclic groups. We start with the following general lemma. We say a graph of groups  $(\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  is *admissible* if it satisfies the following.

1. For any edge  $e$  in  $\Gamma$  with distinct endpoints, the edge inclusions  $\iota_e$  and  $\iota_{\bar{e}}$  are non-surjective.

2. If  $\Gamma$  has a single (oriented) edge  $e$  then the edge inclusions  $\iota_e$  and  $\iota_{\bar{e}}$  are non-surjective.

**Lemma 3.3.8** (Cashen–Levitt [CL16, Proposition 2.5]). *Let  $G$  be a group and  $\varphi: G \rightarrow \mathbb{Z}$  a homomorphism with finitely generated kernel. Suppose that  $G$  admits an admissible graph-of-groups splitting  $G = \pi_1(\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$ . Then  $\varphi(\mathcal{G}_e) \neq 0$  for every edge  $e \in E(\Gamma)$ .*

Let  $G$  be free-by-cyclic and suppose that  $G$  is the fundamental group of an admissible graph of groups  $\mathbb{G} = (\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  with infinite cyclic edge groups. It follows that each vertex group is finitely generated and by Corollary 3.3.6, it is of type F. Hence by Proposition 2.2.11 the Euler characteristic of  $G$  is equal to

$$\chi(G) = \sum_{v \in V(\Gamma)} \chi(\mathcal{G}_v).$$

By Proposition 3.3.7, each vertex group satisfies  $\chi(\mathcal{G}_v) \leq 0$ . Then since  $\chi(G) = 0$ , it must be the case that the Euler characteristic of every vertex group vanishes. Thus Proposition 3.3.7 implies that each vertex group is infinite cyclic or free-by-cyclic.

Now write  $G = F \rtimes \langle t \rangle$  and let  $\varphi: G \rightarrow \mathbb{Z}$  be the fibering corresponding to this splitting. Consider the induced action of the fibre  $F$  on the Bass–Serre tree  $T$  corresponding to the graph of groups  $\mathbb{G} = (\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  with infinite cyclic edge groups above. The action is minimal by Lemma 2.1.4. The fibre  $F$  intersects each edge group trivially by Lemma 3.3.8. Hence the action of  $F$  on  $T$  induces a free splitting of  $F$ . Moreover, the quotient group  $G/F$  acts on the quotient  $F \backslash T$  and the resulting quotient is the graph  $\Gamma$ ,

$$(G/F) \backslash (F \backslash T) \simeq \Gamma.$$

The following theorem summarises the discussion above:

**Theorem 3.3.9.** *Let  $G$  be a free-by-cyclic group with an admissible graph-of-groups splitting  $G = \pi_1(\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  where the edge groups are infinite cyclic. Let  $\varphi: G \rightarrow \mathbb{Z}$  be a homomorphism with finitely generated kernel  $\ker \varphi$  and monodromy  $\Phi \in \text{Out}(\ker \varphi)$ . Then the following hold.*

1. Every edge group  $\mathcal{G}_e$  intersects the kernel  $\ker \varphi$  trivially.
2. The collection  $\{[\ker \varphi \cap \mathcal{G}_v] \mid v \in V(\Gamma)\}$  forms a  $\Phi$ -invariant free factor system for  $\ker \varphi$ .

We end this chapter with the following simple observation:

**Proposition 3.3.10.** *Let  $G$  be a free-by-cyclic group which admits an admissible graph-of-groups splitting  $(\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  where each edge group is infinite cyclic. Then for every edge  $e \in E(\Gamma)$  and vertex  $v \in V(\Gamma)$ , the maps*

$$H_1(\mathcal{G}_e, R) \rightarrow H_1(G, R) \text{ and } H_1(\mathcal{G}_v, R) \rightarrow H_1(G, R)$$

*induced by subgroup inclusion  $\mathcal{G}_e \rightarrow G$  and  $\mathcal{G}_v \rightarrow G$  are non-trivial, for  $R \in \{\mathbb{Z}, \mathbb{R}\}$ .*

### 3.3.3 Polynomially growing monodromy

In Section 3.2, we saw that unipotent polynomially growing outer automorphisms admit relative train track representatives which are upper triangular. This has strong consequences on the algebraic structure of the corresponding free-by-cyclic groups.

Indeed, using the notation from Section 3.2, let  $f: \Gamma \rightarrow \Gamma$  be an upper triangular representative of  $\Phi \in \text{Out}(F)$  with corresponding maximal filtration

$$\emptyset = \Gamma_0 \subseteq \Gamma_1 \subset \dots \subseteq \Gamma_l = \Gamma,$$

such that  $\Gamma_i = \Gamma_{i-1} \cup \{e_i\}$  and  $f(e_i) = e_i p_i$ .

Suppose that the topmost edge  $e_l$  is separating. Then  $\Gamma_{l-1}$  is partitioned into  $\Gamma_{l-1} = \Gamma_{n-1}^1 \cup \Gamma_{n-1}^2$  and  $f(\Gamma_{l-1}^i) \subseteq \Gamma_{l-1}^i$  for each  $i$ . We have that  $\pi_1(\Gamma) \simeq F$  and the fundamental group of each  $\Gamma_{l-1}^i$  can be identified with a free factor  $F^i$  of  $F$  such that  $F = F^1 * F^2$ . We write  $f_i$  to denote the restriction of  $f$  to  $\Gamma_{n-1}^i$ , and  $\Phi_i$  the restriction of  $\Phi$  to  $F^i$ . Let  $M_f$  and  $M_{f_i}$  denote the mapping tori of  $f$  and  $f_i$ . Let  $b_i$  be the endpoint of the topmost edge  $e_l$  contained in  $\Gamma_{l-1}^i$ , for each  $i$ , and let  $t_i$  be the unique edge in  $M_f$  which is based at  $b_i$  and which is not contained in  $\Gamma$ .

By Seifert-Van Kampen's Theorem,

$$\begin{aligned}\pi_1(M_f, b_1) &\simeq \pi_1(M_{f_1}, b_1) *_{\langle t_1 \rangle \sim \langle t_2 p_n^{-1} \rangle} \pi_1(M_{f_2}, b_2) \\ &\simeq (F^1 \rtimes_{\Phi_1} \langle t \rangle) *_{\langle t \rangle \sim \langle s \rho^{-1} \rangle} (F^2 \rtimes_{\Phi_2} \langle s \rangle),\end{aligned}\tag{3.4}$$

where we fix a path joining  $b_1$  to  $b_2$  and thus identify  $\pi_1(M_{f_2}, b_2)$  with a subgroup of  $\pi_1(M_f, b_1)$ , so that  $\rho$  is the image of  $p_n$  under this identification.

Similarly, if  $e_n$  is non-separating then  $\Gamma_{n-1}$  is connected and  $f(\Gamma_{n-1}) \subseteq \Gamma_{n-1}$ . Let  $b_1$  and  $b_2$  be the endpoints of  $e_n$ , and  $t_1$  and  $t_2$  be the edges in  $M_f \setminus \Gamma$  based at  $b_1$  and  $b_2$ , respectively. Fix a path  $\gamma$  joining  $b_1$  to  $b_2$  in  $\Gamma_{n-1}$  and let  $\rho$  be the element of  $\pi_1(M_f, b_1)$  corresponding to the concatenation of paths  $\gamma p_n f(\gamma)^{-1}$ . Then  $F_n \simeq F^1 * \langle s \rangle$  with  $\Phi(F^1) = F^1$ , and

$$\begin{aligned}\pi_1(M_f, b_1) &\simeq \pi_1(M_{f_1}, b_1) *_{\langle t_1 \rangle \sim \langle t_2 p_n^{-1} \rangle} \\ &\simeq (F^1 \rtimes_{\Phi_1} \langle t \rangle) *_{\langle t \rangle \sim \langle t \rho^{-1} \rangle}.\end{aligned}\tag{3.5}$$

Let  $\mathcal{C}$  be a collection of groups. A  $\mathcal{C}$ -hierarchy for a group  $G$  is an iterated sequence of graph-of-groups splittings such that the edge groups are in  $\mathcal{C}$ . We say a group  $G$  is in  $\mathcal{VZH}$  if it has a subgroup of finite index which admits a finite  $\mathcal{Z}$ -hierarchy which terminates in groups from  $\mathcal{Z}$ , where  $\mathcal{Z}$  is the class of infinite cyclic subgroups of  $G$ .

**Theorem 3.3.11.** *Let  $G$  be a free-by-cyclic group. Then  $G$  admits polynomially growing monodromy  $\Phi$  if and only if  $G$  is contained in  $\mathcal{VZH}$ .*

*Proof.* If  $G$  admits polynomially growing monodromy, then by Lemma 3.2.9 there exists a finite-index subgroup  $G' \leq G$  which is a free-by-cyclic group with unipotent polynomially growing monodromy. Then, by the discussion above,  $G'$  admits a  $\mathcal{Z}$ -hierarchy.

Conversely, suppose that  $G$  is free-by-cyclic and  $G' \leq G$  is a finite-index subgroup which admits a  $\mathcal{Z}$ -hierarchy. By Lemma 3.3.4,  $G'$  is also free-by-cyclic. Let  $F \trianglelefteq G'$  be the fibre of a fibration and  $\phi \in \text{Aut}(F)$  an automorphism so that  $G' = F \rtimes_{\phi} \mathbb{Z}$ .

By the discussion in Section 3.3.2, at each level of the hierarchy, the vertex groups  $\mathcal{G}_v$  are infinite cyclic or free-by-cyclic and of the form

$$\mathcal{G}_v = (F \cap \mathcal{G}_v) \rtimes_{\text{ad}_x \cdot \phi^k} \mathbb{Z},$$

for some  $x \in F$  and  $k \neq 0$ .

By restricting the action of  $G$  on the Bass–Serre tree corresponding to the  $\mathcal{Z}$ -splitting at each level of the  $\mathcal{Z}$ -hierarchy, we obtain a hierarchy over the trivial subgroup for the fibre  $F$ . The automorphism  $\phi$  acts periodically on the conjugacy classes of the vertex groups at each level, and thus induces a topological representative  $f$  which permutes the edges and the vertices of the graph of groups at each level of the hierarchy.

Hence, there is a graph  $\Gamma$  with  $\pi_1(\Gamma) \simeq F$ , and a topological representative  $f: \Gamma \rightarrow \Gamma$ , together with a filtration of  $\Gamma$  coming from the hierarchy, so that for a sufficiently high iterate  $f^N$  of  $f$ , we have that  $(f^N, \Gamma)$  is upper triangular in the sense of Section 3.2.

Note that the outer automorphism  $\Phi$  is polynomially growing if and only if  $\Phi^N$  is polynomially growing. Moreover, induction on the length of the filtration shows that any outer automorphism with upper triangular topological representative is polynomially growing. □

In particular, Theorem 3.3.11 shows that if there exist outer automorphisms  $\Phi \in \text{Out}(F_n)$  and  $\Psi \in \text{Out}(F_m)$  such that

$$F_n \rtimes_{\Phi} \mathbb{Z} \simeq F_m \rtimes_{\Psi} \mathbb{Z},$$

then  $\Phi$  is polynomially growing if and only if  $\Psi$  is polynomially growing. More generally, Macura shows that the growth of the monodromy is an invariant of the quasi-isometry class of a free-by-cyclic group [Mac02].

### 3.4 Geometry of free-by-cyclic groups

Let  $G$  be a group with a finite generating set  $S$ . Recall that  $G$  is said to be *hyperbolic*, if the Cayley graph of  $G$  with respect  $S$  has *slim triangles*; that is, there exists some number  $\delta \geq 0$  such that for any triangle  $\tau$ , each side of  $\tau$  is contained in the  $\delta$ -neighbourhood of the union of the remaining sides.

Let  $\mathcal{H}$  be a collection of subgroups of  $G$ . The *coned-off Cayley graph*  $\text{Cay}(G, S; \mathcal{H})$  of  $G$  with respect to  $\mathcal{H}$ , is defined as

$$\text{Cay}(G, S; \mathcal{H}) := \text{Cay} \left( G, S \cup \bigcup_{H \in \mathcal{G}} H \right).$$

A group  $G$  is *relatively hyperbolic* with respect to a peripheral system of subgroups  $\mathcal{H}$ , if the coned-off Cayley graph  $\text{Cay}(G, S; \mathcal{H})$  is hyperbolic and satisfies the *bounded coset penetration property* (see [Far98, Section 3.3]).

The geometry of the free-by-cyclic group  $G_\Phi := F_n \rtimes_\Phi \mathbb{Z}$  is intimately linked to the dynamics of the monodromy  $\Phi$ , as illustrated by the following theorem which combines results by multiple authors:

**Theorem 3.4.1** (Brinkmann [Bri00a], Dahmani–Li [DL20], Hagen [Hag19], Genevois–Horbez [GH21a]). *Let  $\Phi \in \text{Out}(F_n)$  be an outer automorphism.*

1. *The mapping torus  $G_\Phi$  is hyperbolic if and only if  $\Phi$  is atoroidal.*
2. *The mapping torus  $G_\Phi$  is relatively hyperbolic if and only if  $\Phi$  is exponentially growing.*
3. *The mapping torus  $G_\Phi$  is acylindrically hyperbolic if and only if  $\Phi$  has infinite order in  $\text{Out}(F_n)$ .*

We say an outer automorphism  $\Phi \in \text{Out}(F_n)$  is *geometric* if it is induced by a homeomorphism of a surface of finite type.

**Theorem 3.4.2** (Mutanguha [Mut21, Theorem A.4], Dicks–Ventura [DV93]). *Suppose that  $\Phi \in \text{Out}(F_n)$  is an irreducible outer automorphism and that  $\Phi$  is not atoroidal. Then the mapping torus  $G_\Phi$  is isomorphic to the fundamental group of a compact 3-manifold  $M$  with boundary. If  $\Phi$  has infinite order in  $\text{Out}(F_n)$  then  $M$  is hyperbolic and otherwise it is Seifert fibred.*

**Theorem 3.4.3** (Mutanguha [Mut21]). *An element  $\Phi \in \text{Out}(F_n)$  is atoroidal and irreducible if and only if  $G$  has no infinite-index free-by-cyclic subgroups.*

An important corollary of the above theorem which will be used throughout this thesis is the following:

**Corollary 3.4.4** (Mutanguha [Mut21]). *Let  $\Phi \in \text{Out}(F_n)$  and  $\Psi \in \text{Out}(F_m)$  be outer automorphisms such that  $F_n \rtimes_{\Phi} \mathbb{Z} \simeq F_m \rtimes_{\Psi} \mathbb{Z}$ . Then  $\Phi$  is irreducible if and only if  $\Psi$  is irreducible.*

The following table encodes the results discussed in this section. Any two properties in a given row are equivalent to each other and  $G = F_n \rtimes_{\Phi} \mathbb{Z}$  is assumed to be a free-by-cyclic group throughout.

Geometry	Dynamics	Algebra
$\text{Cay}(G)$ is hyperbolic	$\Phi$ is atoroidal	$G$ contains no $\mathbb{Z}^2$ subgroups
$\text{Cay}(G)$ is relatively hyperbolic	$\Phi$ is exponentially growing	$G \notin \mathcal{VZH}$
$G$ admits a non-elementary acylindrical action on a hyperbolic space	$\Phi$ has infinite order in $\text{Out}(F_n)$	$G$ has trivial center
$G$ admits a geometric action on a CAT(0) space	?	?
?	$\Phi$ is irreducible and atoroidal	$G$ contains no infinite-index free-by-cyclic subgroups
$G$ is the fundamental group of a geometric 3-manifold	$\Phi$ is irreducible and not atoroidal	?

Figure 3.1: Table encoding the relationships between the geometry and algebra of a free-by-cyclic group and the dynamics of its monodromy.

# Part I

## Subgroup separability

## Chapter 4

# Subgroup separability of free-by-cyclic groups

A group  $G$  is said to be *subgroup separable* if every finitely generated subgroup  $H \leq G$  is closed in the profinite topology on  $G$ . Subgroup separability initially gained prominence through its applications to low-dimensional topology and specifically 3-manifold theory, as it allows for certain immersions to be lifted to embeddings in finite degree covers. It has since become useful in a much wider group-theoretic setting, in particular in studying profinite rigidity of groups. For instance, Hughes–Kielak [HK22] showed that algebraic fibering is a profinite invariant of subgroup separable groups.

By a classical result of M. Hall [Hal49], finitely generated free groups are known to be subgroup separable. More recently, D. Wise [Wis00] showed that if  $G$  is the fundamental group of a finite graph of finite rank free groups with cyclic edge groups, then  $G$  is subgroup separable if and only if it is *balanced*; that is, there does not exist a non-trivial element  $g \in G$  such that  $g^n$  is conjugate to  $g^m$ , for some  $n \neq \pm m$ . Any free-by-cyclic group is balanced, and furthermore, if it admits a linearly growing UPG monodromy then it can be realised as a mapping torus of a cyclic splitting of a finite-rank free group [AM22, Proposition 5.2.2]. It is therefore tempting to conjecture that such free-by-cyclic groups are subgroup separable. In this chapter we will show that this is almost never true.

**Theorem 4.0.1.** *Let  $\Phi \in \text{Out}(F_n)$  be a polynomially growing outer automorphism. Then  $G = F_n \rtimes_{\Phi} \mathbb{Z}$  is subgroup separable if and only if  $\Phi$  is periodic.*

Since the property of being subgroup separable passes to subgroups, Theorem 4.0.1 combined with standard results on polynomial subgroups of free-by-cyclic groups (see e.g. [Lev09, Proposition 1.4]), implies the following corollary:

**Corollary 4.0.2.** *Let  $\Phi \in \text{Out}(F_n)$  and let  $G = F_n \rtimes_{\Phi} \mathbb{Z}$ .*

1. *If  $\Phi$  acts periodically on every conjugacy class of elements in  $F_n$  (equivalently, if  $\Phi$  is a finite-order outer automorphism) then  $G$  is subgroup separable.*
2. *If there exists a conjugacy class  $\bar{g}$  in  $F_n$  which grows polynomially of order  $d > 0$  under the action of  $\Phi$  then  $G$  is not subgroup separable.*

Leary–Niblo–Wise construct examples of hyperbolic free-by-cyclic groups which are not subgroup separable, by realising such groups as ascending, non-descending HNN extensions of finitely generated free groups [LNW99]. It is interesting to note that whilst the failure of subgroup separability in the Leary–Niblo–Wise examples is due to the non-symmetric nature of the BNS invariant, free-by-cyclic groups with polynomially growing monodromies have symmetric BNS invariants [CL16].

In the other direction, we will see that there is a class of subgroups of free-by-cyclic groups which are always separable.

**Proposition 4.2.1.** *Let  $G$  be a free-by-cyclic group and  $H \leq G$  a free-by-cyclic subgroup. Then  $H$  is separable in  $G$ .*

## 4.1 Polynomially growing automorphisms

Recall that an outer automorphism  $\Phi \in \text{Out}(F_n)$  is said to be *unipotent polynomially growing* (abbreviated to *UPG*), if it is polynomially growing and its image in  $\text{GL}(n, \mathbb{Z})$  is unipotent. It is a well-known fact that every polynomially growing  $\Phi \in \text{Out}(F_n)$  has a positive power  $\Phi^k$  which is UPG.

The aim of this section is to prove that mapping tori of polynomially growing outer automorphisms are non-subgroup separable, unless the outer automorphism has finite order. We will reduce the problem to studying *linearly growing UPG*

outer automorphisms. The mapping tori of such automorphisms are analogous to fundamental groups of graph manifolds. More precisely,

**Proposition 4.1.1.** *Let  $\Phi \in \text{Out}(F_n)$  be a linearly growing UPG element and  $G = F_n \rtimes_{\Phi} \mathbb{Z}$ . There exists a simplicial  $G$ -tree  $T$  with maximal  $\mathbb{Z}^2$ -edge stabilisers and maximal  $F_m \times \mathbb{Z}$  vertex stabilisers, where  $2 \leq m \leq n$ .*

The above proposition follows from the Parabolic Orbits Theorem in [CL95, CL99] (see also [AM22, Theorem 2.4.9] and the discussion which follows it).

Inspired by the work of Niblo–Wise [NW01] on subgroup separability of graph manifolds, we will show that the mapping torus of every linearly growing UPG element  $\Phi \in \text{Out}(F_n)$  contains a non-subgroup separable “poison” subgroup. Since subgroup separability is a property which passes to subgroups, this will be enough to conclude that  $F_n \rtimes_{\Phi} \mathbb{Z}$  is not subgroup separable.

Our poison subgroup arises as the fundamental group of a link complement given by the presentation

$$G_{NW} = \langle i, j, k, l \mid [i, j], [j, k], [k, l] \rangle.$$

Niblo–Wise proved that it is not subgroup separable in [NW01, Theorem 1.2], building on the work of Burns–Karrass–Solitar [BKS87].

The proof of the following proposition is strongly inspired by the argument used to prove Lemma 4.1 and Theorem 2.1 in [NW01].

**Proposition 4.1.2.** *Let  $\Phi \in \text{Out}(F_n)$  be a linearly growing UPG element and  $G = F_n \rtimes_{\Phi} \mathbb{Z}$ . The group  $G_{NW}$  embeds as a subgroup of  $G$ .*

*Proof.* Let  $T$  be the simplicial tree obtained in Proposition 4.1.1 and identify  $G$  with the fundamental group of the quotient graph of groups,  $G \simeq \pi_1(T/G, v)$ . Note that since  $\Phi$  has growth of order greater than zero, it must be the case that the vertex  $v$  admits at least one incident edge  $e$ .

Suppose first that  $e = [v, w]$  has distinct endpoints  $v \neq w$ . Note that the stabilisers  $\mathcal{G}_v$  and  $\mathcal{G}_w$  of the vertices  $v$  and  $w$ , respectively, have infinite cyclic centers. Let  $t_v \in \mathcal{G}_v$  and  $t_w \in \mathcal{G}_w$  be the generators of the centers of  $\mathcal{G}_v$  and  $\mathcal{G}_w$ , respectively.

Suppose that  $t_v \in Z(\mathcal{G}_w)$ . Then  $t_v = t_w^k$  for some integer  $k \neq 0$ . It follows that the subgroup of  $G$  generated by  $\mathcal{G}_v$  and  $\mathcal{G}_w$  has infinite cyclic center generated by  $t_v$ . Since every subgroup of a free-by-cyclic group is free-by-cyclic, where the free fibre possibly has infinite rank, and since  $\langle \mathcal{G}_v, \mathcal{G}_w \rangle$  is finitely generated, we deduce that it is of the form  $F \times \mathbb{Z}$  where  $F$  is finite-rank free. But this contradicts the maximality of  $\mathcal{G}_v$ . Hence  $t_v$  is not contained in the center of  $\mathcal{G}_w$ , and so there exists an element  $g_w \in \mathcal{G}_w$  which does not commute with  $t_v$ . Similarly, there exists an element  $g_v \in \mathcal{G}_v$  which does not commute with  $t_w$ .

Let  $K = \langle g_v, t_v, t_w, g_w \rangle \leq G$ . We have that  $[g_v, t_v] = [t_v, t_w] = [t_w, g_w] = 1$ , and so there exists an epimorphism

$$\begin{aligned} G_{NW} &\rightarrow K \\ i &\mapsto g_v, j \mapsto t_v, k \mapsto t_w, l \mapsto g_w. \end{aligned}$$

Normal forms show that the epimorphism is injective.

Suppose now that the endpoints of the edge  $e$  coincide. Let  $s \in G$  be the element of  $G$  corresponding to the loop  $e$ . Then, the subgroup of  $G$  generated by the stabiliser  $\mathcal{G}_v$  of  $v$  and its conjugate  $\mathcal{G}_v^s := s^{-1}\mathcal{G}_v s$ , is isomorphic to  $\langle \mathcal{G}_v, \mathcal{G}_v^s \rangle \simeq \mathcal{G}_v *_{\mathcal{G}_e} \overline{\mathcal{G}_v}$ , where  $\overline{\mathcal{G}_v} \simeq \mathcal{G}_v^s$  denotes a copy of  $\mathcal{G}_v$ . Observe now that the same argument as above proves that  $G_{NW}$  embeds into  $\langle \mathcal{G}_v, \mathcal{G}_v^s \rangle$ , and thus into  $G$ .  $\square$

The following result will be used to reduce to the case of linearly growing outer automorphisms. It exists in the literature in various forms (see e.g. [Mac02], [Hag19]). The proof of the exact statement below can be found in [AHK22, Proposition 2.5].

**Proposition 4.1.3.** *Let  $\Phi \in \text{Out}(F_n)$  be UPG with growth of order  $d \geq 2$ . Then  $G = F_n \rtimes_{\Phi} \mathbb{Z}$  splits as the fundamental group of a graph of groups with vertex groups isomorphic to mapping tori  $F_k \rtimes_{\Psi} \mathbb{Z}$  with  $\Psi \in \text{Out}(F_k)$  a UPG element with growth of order at most  $d - 1$ ; moreover, there exists a vertex group with polynomially growing monodromy of order exactly  $d - 1$ .*

We are now ready to state and prove our main theorem:

**Theorem 4.1.4.** *Let  $\Phi \in \text{Out}(F_n)$  be a polynomially growing outer automorphism and  $G = F_n \rtimes_{\Phi} \mathbb{Z}$ . Then, the following are equivalent:*

1. The outer automorphism  $\Phi$  has growth of order  $d = 0$ ;
2.  $G$  is virtually a direct product  $F_n \times \mathbb{Z}$ ;
3.  $G$  is subgroup separable;
4.  $G$  does not contain  $G_{NW}$ .

*Proof.* The equivalence of (1) and (2) follows from Proposition 3.1.1. The implication (2)  $\Rightarrow$  (3) follows from the well-known fact that direct products of the form  $F_n \times \mathbb{Z}$  are subgroup separable (see e.g. [AG73]), and the fact that subgroup separability is a commensurability invariant. Niblo–Wise show that  $G_{NW}$  is not subgroup separable in [NW01, Theorem 1.2], and thus any subgroup which contains  $G_{NW}$  is not subgroup separable, which gives the implication (3)  $\Rightarrow$  (4).

It remains to show (4)  $\Rightarrow$  (1). Suppose that  $\Phi$  has growth of order  $d > 0$ . We recall that for every outer automorphism  $\Phi \in \text{Out}(F_n)$  there exists some integer  $k > 0$  such that  $\Phi^k$  is UPG. The element  $\Phi^k$  has the same polynomial growth rate as  $\Phi$ , and  $G' := F_n \rtimes_{\Phi^k} \mathbb{Z}$  is isomorphic to a finite-index subgroup of  $G = F_n \rtimes_{\Phi} \mathbb{Z}$ . Since subgroup separability is a commensurability invariant, it follows that there is no loss of generality in assuming that  $\Phi$  is UPG.

The proof now follows by induction on the degree of growth. The base case  $d = 1$  is exactly Proposition 4.1.2. For the inductive step, apply Proposition 4.1.3 to deduce that if  $\Phi$  has growth of order  $d \geq 2$ , then it splits as a graph of groups where some vertex group is a mapping torus of an UPG outer automorphism with growth of order  $d - 1$ .  $\square$

## 4.2 Separable subgroups of free-by-cyclic groups

**Proposition 4.2.1.** *Let  $G$  be a free-by-cyclic group and let  $H \leq G$  be a finitely generated subgroup. If  $H \leq G$  is free-by-cyclic then  $H$  is separable in  $G$ .*

*Proof.* Fix a fibred character  $\varphi: G \rightarrow \mathbb{Z}$  of  $G$ . Let  $F = \ker \varphi$  be the fibre,  $t \in \varphi^{-1}(1)$  the stable letter and let  $\phi \in \text{Aut}(F)$  be the automorphism corresponding to the conjugation action of  $t$  on  $F$  in  $G$ . Let  $H \leq G$  be a free-by-cyclic subgroup and  $g \in G \setminus H$  an element in the complement.

By Theorem 3.3.5, there exist a finitely generated subgroup  $A \leq F$ , an element  $y \in F$  and a positive integer  $k$  such that  $\phi^k(A) = yAy^{-1}$ , and

$$H = \langle A, t^k y \rangle \simeq A \rtimes \langle t^k y \rangle.$$

The element  $g$  can be expressed as  $g = bt^m$ , for some  $b \in F$  and  $m \in \mathbb{Z}$ . Suppose that  $m$  is not a multiple of  $k$ . Let  $C_k$  denote the finite group of order  $k$  and consider the epimorphism  $\pi: G \rightarrow C_k$ , which maps each element of  $F$  to 0, and which sends  $t$  to a generator of  $C_k$ . It follows that  $\pi(H) = 0$  and  $\pi(g) \neq 0$ .

Suppose now that  $m = kl$  for some  $l \in \mathbb{Z}$ . Then  $g = b'(t^k y)^l$ , for some  $b' \in F$ , and since  $g \notin H$  it must be that  $b' \notin A$ . By Marshall Hall's theorem, there exists a finite-index subgroup  $F' \leq F$  such that  $b' \notin F'$  and  $A \leq F'$ . Let  $N = [F : F']$ . Let  $\text{ad}_y$  denote the inner automorphism of  $F$  which acts by conjugation with  $y$ . Note that  $\text{ad}_y \cdot \phi^k: F \rightarrow F$  permutes the subgroups of  $F$  of index  $N$ . Hence there exists some positive integer  $M$  such that  $(\text{ad}_y \cdot \phi^k)^M(F') = F'$ . Let  $F'' = \bigcap_{i=0}^{M-1} (\text{ad}_y \cdot \phi^k)^i(F')$ . Then  $\text{ad}_y \cdot \phi^k(F'') = F''$  and  $A \leq F''$ . Furthermore, since  $F'' \leq F'$ , we have that  $b' \notin F''$ . Thus  $G' = \langle F'', t^k y \rangle \cong F'' \rtimes \langle t^k y \rangle$  is a finite-index subgroup of  $G$ , such that  $g \notin G'$  and  $H \leq G'$ . The result now follows.  $\square$

**Corollary 4.2.2.** *Let  $G$  be a free-by-cyclic group. If  $H \leq G$  is a free-by-cyclic subgroup then  $H$  is fully separable in  $G$ . In particular  $\bar{H}$ , the closure of  $H$  in  $\hat{G}$ , is isomorphic to  $\hat{H}$ .*

*Proof.* Every finite-index subgroup of  $G$  is free-by-cyclic by Lemma 3.3.4 and thus separable by Proposition 4.2.1. The result now follows from Proposition 2.3.11.  $\square$

We will need the following proposition later. It is a special case of [BR20, Lemma 2.2] but we include a proof for completeness.

**Proposition 4.2.3.** *Let  $G$  be a finitely generated, residually finite group and  $\varphi: G \rightarrow \mathbb{Z}$  an epimorphism. If  $N = \ker \varphi$  is finitely generated, then  $N$  is fully separable in  $G$ .*

*Proof.* By Proposition 2.3.11, it suffices to prove that every finite-index subgroup  $N' \leq_f N$  of  $N$  is closed in the profinite topology on  $G$ . Note that since  $N$  is finitely generated, for any such subgroup  $N' \leq_f N$  there exists a finite-index subgroup

$G' \leq_f G$  such that  $\varphi$  induces an epimorphism  $\varphi': G' \rightarrow \mathbb{Z}$  with  $N' = \ker \varphi'$ . Since  $G'$  has finite index in  $G$ , every subset of  $G'$  which is closed in the profinite topology on  $G'$  is also closed in the profinite topology on  $G$ . Hence it suffices to show that  $N$  is closed in the profinite topology on  $G$ . To that end, fix  $t \in \varphi^{-1}(1)$  and let  $g \in G \setminus N$ . Then  $\varphi(g) = k$ , for some  $k \neq 0$ . Let  $C_{|k|+1}$  denote the cyclic group of order  $|k| + 1$ . Define a homomorphism  $\pi: G \rightarrow C_{|k|+1}$  which sends  $t$  to a generator of  $C_{|k|+1}$  and all the elements of  $N$  to 0. Then  $\pi(g) \neq 0$ , whereas  $N \leq \ker \pi$ .  $\square$

# Chapter 5

## Generic behaviour of deficiency-one groups

This chapter studies the problem of subgroup separability in random deficiency-one groups. It is a general fact that if a group  $G$  has an integral character  $\varphi: G \rightarrow \mathbb{Z}$  which is contained in the Bieri–Neumann–Strebel (BNS) invariant, and  $-\varphi$  is not contained in the BNS invariant, then  $G$  is not subgroup separable. We will leverage this fact to prove that many random groups which admit deficiency-one presentations are not subgroup separable:

**Theorem 5.0.1.** *Let  $G$  be a random group of deficiency one with respect to the few-relator model (see Section 5.0.1). Then, with positive asymptotic probability  $G$  is not subgroup separable.*

Kielak–Kropholler–Wilkes show that a random few-relator deficiency-one group is free-by-cyclic with positive asymptotic probability [KKW22]. Our methods for proving Theorem 5.0.1 imply that such a group is *not* free-by-cyclic with positive probability, generalising a result of Dunfield–Thurston [DT06] who prove this for 2-generator 1-relator groups.

**Theorem 5.0.2.** *Let  $G$  be a random group of deficiency one with respect to the few-relator model. Then  $G$  is free-by-cyclic with asymptotic probability bounded away from 0 and 1.*

A random 2-generator 1-relator group is virtually free-by-cyclic, almost surely [KKW22, Corollary 2.10]. The analogous result in the deficiency-one case is not known.

The key technical tool in this chapter is Lemma 5.0.9 which characterises the maps  $\varphi: G \rightarrow \mathbb{Z}$  which are *not* contained in the BNS invariant of  $G$ , in terms of the minima of  $\varphi$  evaluated at the suffixes of the relators. This approach is similar in flavour to that of Brown's algorithm [Bro87], a classical tool used to calculate the BNS invariant of 2-generator 1-relator groups. However, as we are (in general) no longer in the realm of 1-relator groups, the characterisation that we obtain is less clean than that in [Bro87], and the methods used to prove it are completely different.

### 5.0.1 Random groups

Let  $k \in \mathbb{Z}$ . A *deficiency  $k$  presentation* is a presentation of the form

$$\langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle,$$

where  $m - n = k$ , and  $r_1, \dots, r_n$  are non-empty reduced words in the alphabet  $\{x_1^\pm, \dots, x_m^\pm\}$ . A group  $G$  is said to be of *deficiency  $k$*  if it admits a deficiency  $k$  presentation and it does not admit a deficiency  $k'$  presentation, for any  $k' \geq k$ .

In this article, we will use the few-relator model for random groups. After fixing  $n \geq 1$  and  $m \geq 1$ , and for every  $l \geq 1$ , we write  $\mathcal{R}_l$  to denote the set of group presentations of the form  $\langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$ , where each  $r_i$  is a cyclically reduced non-empty word in the alphabet  $\{x_1^\pm, \dots, x_m^\pm\}$  of length  $\leq l$ . For any given property  $P$  of groups, we say that a presentation *satisfies the property  $P$*  if the corresponding group satisfies it. The property  $P$  is said to hold with *asymptotic probability  $p$* , for some  $0 \leq p \leq 1$ , if

$$\frac{\#\{\text{presentations in } \mathcal{R}_l \text{ which satisfy } P\}}{\#\mathcal{R}_l} \rightarrow p \text{ as } l \rightarrow \infty.$$

The property  $P$  is said to hold with *positive asymptotic probability* if

$$\liminf_{l \rightarrow \infty} \frac{\#\{\text{presentations in } \mathcal{R}_l \text{ which satisfy } P\}}{\#\mathcal{R}_l} > 0.$$

The probability is said to be *bounded away from 1* if it holds with asymptotic probability  $p < 1$ . Finally, we say that the property  $P$  holds *almost surely* if it holds with asymptotic probability  $p = 1$ .

A random presentation on  $m$  generators and  $n$  relators, with  $m - n = k$ , will correspond to a deficiency  $k$  group, almost surely [Wil19b]. Hence, it makes sense to talk of a random deficiency  $k$  group.

## 5.0.2 Bieri–Neumann–Strebel invariants and the Novikov ring

**Definition 5.0.3** (Bieri–Neumann–Strebel [BNS87]). The *Bieri–Neumann–Strebel invariant* (also known as the *BNS invariant*)  $\Sigma(G)$  of a group  $G$ , is the set of non-zero homomorphisms  $\varphi: G \rightarrow \mathbb{R}$  such that the monoid  $\{g \in G \mid \varphi(g) \geq 0\}$  is finitely generated.

Let  $G$  be a group and  $\varphi: G \rightarrow \mathbb{Z}$  a homomorphism. The *Novikov ring*  $\widehat{\mathbb{Q}G}^\varphi$  of  $G$  with respect to  $\varphi$ , is the set of all formal sums  $x = \sum_{g \in G} \lambda_g g$  where  $\lambda_g \in \mathbb{Q}$ , such that for any  $r \in \mathbb{R}$ , the intersection  $\text{supp}(x) \cap \varphi^{-1}((-\infty, r])$  is a finite set. Multiplication and addition in  $\widehat{\mathbb{Q}G}^\varphi$  are defined in the obvious way, so that the natural inclusion  $\mathbb{Q}G \leq \widehat{\mathbb{Q}G}^\varphi$  is an embedding of rings.

**Lemma 5.0.4.** *Let  $G$  be group and  $\varphi: G \rightarrow \mathbb{Z}$  a homomorphism. Then, for infinite-order elements  $g \in G$  and  $\alpha \in \mathbb{Q}^\times$ ,  $g - \alpha$  is a unit in  $\widehat{\mathbb{Q}G}^\varphi$  if and only if  $\varphi(g) \neq 0$ .*

*Proof.* Suppose  $\varphi(g) \neq 0$ . If  $\varphi(g) > 0$  then the formal sum

$$h = \alpha^{-1} \cdot \sum_{i=0}^{\infty} (\alpha^{-1}g)^i$$

is an element of  $\widehat{\mathbb{Q}G}^\varphi$ , and  $(\alpha - g)h = h(\alpha - g) = 1$ . If  $\varphi(g) < 0$ , then  $\varphi(g^{-1}) > 0$  and since  $g$  is a unit in  $\widehat{\mathbb{Q}G}^\varphi$ , it follows that  $g - \alpha = \alpha g(\alpha^{-1} - g^{-1})$  is also a unit.

Suppose that  $\varphi(g) = 0$ . For contradiction, assume that there exists some  $h \in \widehat{\mathbb{Q}G}^\varphi$  such that  $(g - \alpha)h = 1$ . Write  $h = \sum_{k \in G} \lambda_k k$ , where the coefficients  $\lambda_k \in \mathbb{Q}$  are such that for any  $r \in \mathbb{R}$ , there are only finitely many elements  $k \in G$  with  $\varphi(k) \leq r$  and  $\lambda_k \neq 0$ . Since  $(g - \alpha)h = 1$ , we have that for all  $n > 0$ ,  $\lambda_{g^n} = \alpha^{-n} \cdot \lambda_{1_G}$  and  $\lambda_{g^{-n}} = \alpha^{n-1}(\alpha \cdot \lambda_{1_G} + 1)$ . Hence  $\lambda_{g^n} \neq 0$  for all  $n > 0$ , or  $\lambda_{g^{-n}} \neq 0$  for all  $n > 0$ . However  $\varphi(g^n) = n \cdot \varphi(g) = 0$  for all  $n \in \mathbb{Z}$ . Since  $g \in G$  has infinite order, it follows that  $\text{supp}(h) \cap \varphi^{-1}((-\infty, 0])$  is infinite. This is a contradiction.  $\square$

The significance of the Novikov ring lies in its relation to the Bieri–Neumann–Strebel invariant of a group  $G$ . The original version of the following theorem, where the coefficient ring is equal to  $\widehat{\mathbb{Z}G}^\varphi$ , is attributed to Sikorav and can be found in his PhD thesis [Sik87]. The proof of the result over  $\mathbb{Q}$  is outlined in [Kie20b, Theorem 3.11].

**Theorem 5.0.5.** *Let  $G$  be a finitely generated group and  $\varphi: G \rightarrow \mathbb{Z}$  an epimorphism. Then  $\varphi$  is an element of the BNS invariant  $\Sigma(G)$  of  $G$  if and only if  $H_1(G; \widehat{\mathbb{Q}G}^\varphi) = 0$ .*

### 5.0.3 A Brown-type algorithm

Let  $R$  be a ring and  $t$  a formal symbol. We write  $R((t))$  to denote the set of Laurent power series over  $R$  with a single variable  $t$ ,

$$R((t)) = \left\{ \sum_{i \geq k} a_i t^i \mid a_i \in R, k \in \mathbb{Z} \right\}.$$

Let  $\alpha$  be an automorphism of  $R$ . The ring of *twisted Laurent series* is the set  $R((t))$ , with the obvious summation and multiplication defined by linearly extending

$$r_1 t^{n_1} \cdot r_2 t^{n_2} := r_1 \alpha^{n_1}(r_2) t^{n_1+n_2},$$

for all  $r_1, r_2 \in R$  and  $n_1, n_2 \in \mathbb{Z}$ . The  $t$ -order of a Laurent series  $f \in R((t))$ , denoted  $\text{ord}_t(f)$ , is the lowest power of  $t$  with a non-zero coefficient in the expansion of  $f$ . We define  $\text{ord}_t(0) = \infty$ .

Let  $G$  be a group and  $\varphi: G \rightarrow \mathbb{Z}$  a homomorphism. Let  $t \in G$  be an element such that  $\varphi(t)$  generates  $\mathbb{Z}$ . Let  $K = \ker(\varphi)$  and let  $\mathbb{Q}K((t))$  denote the ring of twisted Laurent series, where the twisting automorphism  $\alpha$  is obtained by extending the automorphism of  $K$  induced by the conjugation action of  $t$  on  $K$  in  $G$ . Then there is a natural identification  $\widehat{\mathbb{Q}G}^\varphi \simeq \mathbb{Q}K((t))$ . Given a subset  $S \subseteq \mathbb{Z}$ , we say that  $x \in \widehat{\mathbb{Q}G}^\varphi$  is *supported over*  $S$  if  $x = \sum_{i \in S} a_i t^i$ , for some  $a_i \in \mathbb{Q}K$ .

**Lemma 5.0.6.** *Let  $G$  be a group and  $\varphi: G \rightarrow \mathbb{Z}$  a homomorphism. Then, the group ring  $\mathbb{Q}G$  is a domain if and only if the Novikov ring  $\widehat{\mathbb{Q}G}^\varphi$  is a domain.*

*Proof.* Since  $\mathbb{Q}G$  embeds as a subring of  $\widehat{\mathbb{Q}G}^\varphi$ , it is clear that if  $\widehat{\mathbb{Q}G}^\varphi$  is a domain then so is  $\mathbb{Q}G$ .

Suppose now that there exist non-zero elements  $x, y \in \widehat{\mathbb{Q}G}^\varphi$  such that  $xy = 0$ . Let  $K = \ker(\varphi)$  and let  $t \in G$  be such that  $\varphi(t) = 1$ . Let  $x = \sum_{i \geq k} a_i t^k$  and  $y = \sum_{j \geq l} b_j t^j$  where  $a_i, b_j \in \mathbb{Q}K$ , and  $a_k \neq 0, b_l \neq 0$ . Then the coefficient of  $t^{k+l}$  in  $xy$  is  $a_k b_l$ . Since  $xy = 0$ , we must have that  $a_k b_l = 0$ . Hence  $\mathbb{Q}G$  is not a domain.  $\square$

**Lemma 5.0.7.** *Let  $G$  be a group such that the group ring  $\mathbb{Q}G$  has no zero-divisors. Let  $B$  and  $P$  be  $n \times n$  matrices over  $\mathbb{Q}G$ . Suppose that  $B = \text{diag}(k_1 t^{\rho_1}, \dots, k_n t^{\rho_n})$ , where  $k_i \in \mathbb{Q}K$  and  $\rho_i \in \mathbb{Z}$  for every  $1 \leq i \leq n$ . Assume that  $k_i \in K$  for  $i > 1$  and that  $k_1$  is not a unit in  $\widehat{\mathbb{Q}G}^\varphi$ . Suppose that all the elements in the  $i^{\text{th}}$  row of  $P$  are supported over  $\mathbb{Z} \cap [\rho_i + 1, \infty)$ . Then the matrix  $A = B + P$  is not invertible over  $\widehat{\mathbb{Q}G}^\varphi$ .*

*Proof.* Since  $t^{\rho_1}$  and  $k_i t^{\rho_i}$  for  $i > 1$  are units in  $\mathbb{Q}G$ , it follows that they are also units in  $\widehat{\mathbb{Q}G}^\varphi$  and the matrix

$$M = \text{diag}(t^{\rho_1}, k_2 t^{\rho_2}, \dots, k_n t^{\rho_n})$$

is invertible over  $\widehat{\mathbb{Q}G}^\varphi$ . Hence  $A$  is invertible if and only if  $A' = M^{-1}A$  is invertible. The diagonal elements of  $A'$  other than the element in the first row are of the form  $1 + p_{ii}$ , for some  $p_{ii} \in \widehat{\mathbb{Q}G}^\varphi$  supported over a positive subset of the integers. Such elements are invertible over  $\widehat{\mathbb{Q}G}^\varphi$  and the inverse  $(1 + p_{ii})^{-1}$  is an element supported over non-negative integers. Hence by applying elementary column operations over

$\widehat{\mathbb{Q}G}^\varphi$ , we may transform  $A'$  into an upper triangular matrix  $A''$  where the first element on the diagonal is given by  $k_1 + p'_{11}$ , with  $k_1 \in \mathbb{Q}K$  a non-unit, and  $p'_{11} \in \widehat{\mathbb{Q}G}^\varphi$ , an element supported over the positive integers. Since elementary column operations are invertible, again  $A''$  is invertible if and only if  $A'$  is invertible.

Suppose now that  $A''$  is invertible over  $\widehat{\mathbb{Q}G}^\varphi$  and let  $C = (c_{ij})$  be the inverse. Then  $c_{11}(k_1 + p'_{11}) = 1$ . Since  $\mathbb{Q}G$  does not have non-trivial zero-divisors, neither does  $\widehat{\mathbb{Q}G}^\varphi$  by Lemma 5.0.6. Hence for any elements  $p, q \in \widehat{\mathbb{Q}G}^\varphi$   $\text{ord}_t(pq) = \text{ord}_t(p) + \text{ord}_t(q)$ . Suppose that  $\text{ord}_t(c_{11}p'_{11}) > 0$ . Then  $\text{ord}_t(c_{11}k_1) = \text{ord}_t(1 - c_{11}p'_{11}) = 0$ . Hence

$$0 = \text{ord}_t(c_{11}k_1) = \text{ord}_t(c_{11}) + \text{ord}_t(k_1) = \text{ord}_t(c_{11}).$$

Let  $d \in \mathbb{Q}K$  be the coefficient of the  $t^0$  term in  $c_{11}$ . Note that  $d \neq 0$ . Then  $dk_1 = 1$  and thus  $k_1$  is a unit. Hence  $\text{ord}_t(c_{11}p'_{11}) \leq 0$ .

Suppose that  $\text{ord}_t(c_{11}p'_{11}) < 0$ . Then

$$\text{ord}_t(c_{11}k_1) = \text{ord}_t(1 - c_{11}p'_{11}) = \text{ord}_t(c_{11}p'_{11}).$$

Hence  $\text{ord}_t(c_{11}k_1) = \text{ord}_t(c_{11}p'_{11})$ . Thus

$$0 = \text{ord}_t(k_1) = \text{ord}_t(p'_{11}) > 0.$$

Hence, it must be the case that  $\text{ord}_t(c_{11}p'_{11}) = 0$ . But then  $\text{ord}_t(c_{11}) < 0$  and thus  $\text{ord}_t(c_{11}k_1) < 0$ . But then  $\text{ord}_t(1 - c_{11}p'_{11}) < 0$ , which is impossible since  $\text{ord}_t(c_{11}p'_{11}) = 0$ . In all cases we get a contradiction, and thus  $A''$  is not invertible.  $\square$

Given a cyclically reduced word  $w = w_1 \cdots w_m$  in the alphabet  $\{x_1^\pm, \dots, x_{n+1}^\pm\}$  and  $k \leq |w|$ , we let  $[w]_k = w_1 \dots w_k$  be the prefix of  $w$  of length  $k$ . Let  $C_w$  denote the cyclic graph of length  $|w|$ , with a marked vertex  $*$  and labelled edges, such that consecutive edges of  $C_w$ , starting at the vertex  $*$  and moving in the clockwise direction, spell out the word  $w$ . Assign labels to vertices of  $C_w$  so that the vertex  $v$  is labelled by the word which is spelled out by the embedded path joining  $*$  to  $v$ , in the clockwise direction. Let  $\varphi: F(x_1, \dots, x_{n+1}) \rightarrow \mathbb{Z}$  be a homomorphism. There is an induced map  $\tilde{\varphi}: C_w \rightarrow \mathbb{R}$  defined by linearly extending the map  $\varphi$

from the labels of the vertices to the whole graph. We define the *lower section* of  $w$  to be the preimage

$$L_\varphi(w) = \tilde{\varphi}^{-1}(\min\{\tilde{\varphi}(p) \mid p \in C_r\}).$$

Let  $(r_1, \dots, r_n)$  be a collection of cyclically reduced words in the alphabet  $\{x_1^\pm, \dots, x_{n+1}^\pm\}$ . Let  $\varphi: F(x_1, \dots, x_{n+1}) \rightarrow \mathbb{Z}$  be a homomorphism. The tuple  $((r_1, \dots, r_n), \varphi)$  is said to satisfy the *unique minimum condition* if, after possible re-ordering, the following conditions are satisfied.

1. We have that  $\varphi(x_i) \geq 0$  for each  $i \leq n$  and  $\varphi(x_{n+1}) < 0$ .
2. The homomorphism  $\varphi$  vanishes on each  $r_i$ .
3. The lower section  $L_\varphi(r_i)$  consists of exactly one of the following:
  - A single vertex such that one of the adjacent edges is labelled by  $x_i^\pm$  and the other is labelled by  $x_{n+1}^\pm$ .
  - A single edge labelled by  $x_i^\pm$  such that the adjacent edges are labelled by  $x_{n+1}^\pm$ .

The tuple  $((r_1, \dots, r_n), \varphi)$  satisfies the *repeated minimum condition* if it satisfies the unique minimal condition, except at a single relator  $r_j$ , for some  $1 \leq j \leq m$ , where  $L_\varphi(r_j)$  consists of two occurrences of a vertex, or two occurrences of an edge as in the unique minimum condition. In that case, we call  $r_j$  the *relator with a repeated minimum*.

Let  $G$  be a group given by the deficiency-one presentation

$$G = \langle x_1, \dots, x_{n+1} \mid r_1, \dots, r_n \rangle.$$

Let  $\varphi: G \rightarrow \mathbb{Z}$  be a homomorphism with kernel  $K$ , and  $t \in G$  an element such that  $\varphi(t)$  generates  $\mathbb{Z}$ .

**Lemma 5.0.8.** *Suppose that  $((r_1, \dots, r_n), \varphi)$  satisfies the repeated minimum condition, where  $r_1$  is the relator with a repeated minimum. Then for each  $i \leq n$ , there*

exists some integer  $P_i \in \mathbb{Z}$ , and for every  $j \leq n$  and  $k \geq P_i$ , there exist elements  $u_{ij,k} \in \mathbb{Q}K$ , such that the Fox derivatives of  $r_i$  are of the form

$$\frac{\partial r_i}{\partial x_j} = \sum_{k \geq P_i} u_{ij,k} t^k,$$

such that for any  $i \neq j$ , the element  $u_{ij,P_i} = 0$ , and  $u_{ii,P_i} \in K$  for  $i \neq 1$ , and  $u_{11,P_1}$  is a non-unit in  $\widehat{\mathbb{Q}G}^\varphi$ .

*Proof.* For every relator  $r_i$  and generator  $x_j$ , the partial derivative  $\frac{\partial r_i}{\partial x_j}$  is the sum of prefixes of  $r_i$  of the form  $ux_j^{-1}$  and  $v$ , where  $v$  immediately precedes an instance of  $x_j$  in  $r_i$ . For each  $i$ , let  $P_i = \tilde{\varphi}(L_\varphi(r_i)) \in \mathbb{Z}$ . Hence, for every summand  $u$  of  $\frac{\partial r_i}{\partial x_j} \in \mathbb{Z}G$ , we have that  $\varphi(u) \geq P_i$  and  $\varphi(u) = P_i$  if and only if  $u$  is the label of a vertex of  $C_{r_i}$  contained in  $L_\varphi(r_i)$ . Any such vertex has adjacent edges labelled by  $x_i^\pm$  and  $x_{n+1}^\pm$ . In particular, either the prefix  $u$  has  $x_i^\pm$  as its last letter and is followed by  $x_{n+1}^\pm$  in  $r_i$ , or the same holds but with the roles of  $x_i$  and  $x_{n+1}$  reversed. This implies that for every summand  $u$  of  $\frac{\partial r_i}{\partial x_j}$ , if  $i \neq j$  then  $\varphi(u) > P_i$ .

Now suppose that  $i > 1$ . Let  $\mathcal{A} = \{u_\alpha\}$  be the collection of summands of  $\frac{\partial r_i}{\partial x_i}$  such that  $\varphi(u_\alpha) = P_i$ . Each element of  $\mathcal{A}$  must be the label of a vertex in  $L_\varphi(r_i)$ . Suppose that  $L_\varphi(r_i)$  is a single vertex with label  $u$ . Since each  $\varphi(x_i) \geq 0$ , either  $u$  is followed by  $x_i$  in  $r_i$ , or the final letter of  $u$  is  $x_i^{-1}$ . In either case,  $u \in \mathcal{A}$  and thus  $\mathcal{A}$  contains exactly one element. Suppose instead that  $L_\varphi(r_i)$  consists of two vertices  $u$  and  $ux_i^\pm$ . Exactly one of these words is a summand of  $\frac{\partial r_i}{\partial x_i}$ , depending on whether we choose  $x_i$  or  $x_i^{-1}$ . Hence, it follows in this case also that  $\mathcal{A}$  contains exactly one element, and this element can be expressed as  $kt^{P_i}$ , for some  $k \in K$ .

Finally we consider  $\frac{\partial r_1}{\partial x_1}$ . Defining  $\mathcal{A}$  as above,  $\mathcal{A}$  has exactly two elements given by the reduced words  $u$  and  $uv$ , where  $\varphi(u) = P_1$  and  $\varphi(v) = 0$ , where  $u$  is the label of the path joining the marked vertex  $*$  to the first minimum vertex which is a summand of  $\frac{\partial r_1}{\partial x_1}$ , and  $v$  is the label of the path joining the two minima. Then  $u = kt^{P_1}$  and  $uv = kv't^{P_1}$ , for some  $k, v' \in K$ . Note that the element  $1 + v' \in \mathbb{Z}G$  is not invertible over  $\widehat{\mathbb{Q}G}^\varphi$  by Lemma 5.0.4, and thus  $k(1 - v')$  is not a unit in  $\widehat{\mathbb{Q}G}^\varphi$ .  $\square$

**Lemma 5.0.9.** *Let  $G$  be a group given by the deficiency-one presentation*

$$G = \langle x_1, \dots, x_{n+1} \mid r_1, \dots, r_n \rangle.$$

*Suppose that  $\mathbb{Q}G$  has no non-trivial zero-divisors. Let  $\varphi: G \rightarrow \mathbb{Z}$  be a homomorphism.*

1. *If  $((r_1, \dots, r_n), \varphi)$  satisfies the unique minimum condition then  $\varphi \in \Sigma(G)$ .*
2. *If  $((r_1, \dots, r_n), \varphi)$  satisfies the repeated minimum condition then  $\varphi \notin \Sigma(G)$ .*

*Proof.* The first statement follows from [KKW22, Theorem 3.4].

For the second statement, by Theorem 5.0.5 it suffices to show that  $H_1(G; \widehat{\mathbb{Q}G}^\varphi)$  is non-trivial whenever  $((r_1, \dots, r_n), \varphi)$  satisfies the repeated minimum condition. To that end, consider the chain complex of  $\mathbb{Q}G$ -modules

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0. \tag{5.1}$$

Here, the  $\mathbb{Q}G$ -module  $C_2$  is the free module of rank  $n$  with an ordered basis identified with the relators  $(r_1, \dots, r_n)$ . The  $\mathbb{Q}G$ -module  $C_1$  is the free module of rank  $n + 1$  with an ordered basis identified with the generators  $(x_1, \dots, x_{n+1})$  and  $C_0 = \mathbb{Q}G$ . The boundary map  $\partial_1$  is given by the column vector with entries  $x_i - 1$ , for  $1 \leq i \leq n + 1$ , and the boundary map  $\partial_2$  is the matrix  $A$  of Fox derivatives  $\left(\frac{\partial r_i}{\partial x_j}\right)$ . After possible re-ordering, we may assume that  $r_1$  is the relator with the repeated minimum. We tensor the chain complex (5.1) with  $\widehat{\mathbb{Q}G}^\varphi$  and let  $(e_1, \dots, e_{n+1})$  be the resulting free generating set of  $C_1 \otimes \widehat{\mathbb{Q}G}^\varphi$ . We write  $A'$  to denote the matrix obtained from  $A$  by restricting the image of the boundary map to the subspace spanned by  $\{e_1, \dots, e_n\}$ . We claim that

$$H_1(G, \widehat{\mathbb{Q}G}^\varphi) = \text{coker}(A').$$

Since  $\mathbb{Q}G$  has no non-trivial zero-divisors, the element  $x_{n+1}$  has infinite order in  $G$ . By the definition of the repeated minimum condition,  $\varphi(x_{n+1}) \neq 0$  and thus

by Lemma 5.0.4, the element  $x_{n+1} - 1$  is invertible over  $\widehat{\mathbb{Q}G}^\varphi$ . Let us define a map

$$C_1 \otimes \widehat{\mathbb{Q}G}^\varphi \rightarrow \ker(\partial_1 \otimes \text{id})$$

$$\sum_i^{n+1} \lambda_i e_i \mapsto \sum_{i=1}^n \lambda_i e_i + \lambda'_{n+1} e_{n+1},$$

where  $\lambda'_{n+1} = -\sum_{i=1}^n \lambda_i (x_i - 1)(x_{n+1} - 1)^{-1}$ . This map is clearly onto and every element of  $\text{im}(A')$  is sent to  $\text{im}(A)$ . This proves the claim.

Combining Lemma 5.0.8 with Lemma 5.0.7 shows that  $A'$  is non-invertible over  $\widehat{\mathbb{Q}G}^\varphi$ . Thus  $H_1(G, \widehat{\mathbb{Q}G}^\varphi) \neq 0$ .  $\square$

We are now ready to prove the key lemma of the section, inspired by [KKW22, Proposition 5.1].

**Lemma 5.0.10.** *Let  $G$  be a random group of deficiency one. Then, with positive probability, there exists a character  $\varphi: G \rightarrow \mathbb{Z}$  such that  $\varphi$  satisfies the unique minimum condition and  $-\varphi$  satisfies the repeated minimum condition.*

*Proof.* For each positive integer  $l$ , let  $\mathcal{R}_l$  denote the set of  $n$ -tuples  $(r_1, \dots, r_n)$  of cyclically reduced words in the alphabet  $\{x_1^\pm, \dots, x_{n+1}^\pm\}$  of positive length  $\leq l$ . We define  $\mathcal{R}'_l$  to be the subset of  $n$ -tuples  $(r_1, \dots, r_n)$  in  $\mathcal{R}_l$ , such that the group  $G = \langle x_1, \dots, x_{n+1} \mid r_1, \dots, r_n \rangle$  has first Betti number equal to 1. We let  $\mathcal{T}$  denote the set of all deficiency-one presentations such that the resulting group admits a homomorphism to  $\mathbb{Z}$  which satisfies the hypotheses. To prove the lemma, it suffices to construct an injective map  $f: \mathcal{R}'_l \rightarrow \mathcal{T} \cap \mathcal{R}'_{l+12}$ . Then,

$$\frac{|\mathcal{T} \cap \mathcal{R}'_{l+12}|}{|\mathcal{R}'_{l+12}|} \geq \frac{|\mathcal{R}'_l|}{|\mathcal{R}'_{l+12}|} > \varepsilon,$$

where  $\varepsilon > 0$  depends only on  $n$ . The result follows by noting that  $|\mathcal{R}'_l|/|\mathcal{R}_l| \rightarrow 1$  as  $l \rightarrow \infty$ .

Now for each  $n$ -tuple  $(r_1, \dots, r_n) \in \mathcal{R}'_l$ , there exists a non-trivial map  $\varphi: F(x_1, \dots, x_{n+1}) \rightarrow \mathbb{Z}$  such that  $\varphi(r_i) = 0$  for every  $i$ . After possibly re-ordering and inverting the generators  $x_i$ , we can assume that  $\varphi(x_i) \geq 0$  for all  $i$ , and  $\varphi(x_{n+1}) < 0$ . Our strategy for defining  $f$  is to alter the  $n$ -tuple  $(r_1, \dots, r_n)$  so that  $\varphi$  still vanishes on each element of the tuple, and such that it has a unique minimum and repeated

maxima. We do this by inserting commutators. We follow the convention where  $[x, y] = xyx^{-1}y^{-1}$ .

For each relator  $r_i$ , form a new relator  $r'_i$  by inserting a commutator  $[x_{n+1}, x_i^\epsilon]$  at the first  $\varphi$ -minimal vertex along  $C_{r_i}$ , where  $\epsilon = -1$  if  $\varphi(x_i) > 0$  and  $\epsilon = 1$  otherwise. Now for each  $i > 1$ , form a new relator  $r''_i$  by inserting the commutator  $[x_{n+1}^{-1}, x_i^{-\epsilon}]$  at the first  $\varphi$ -maximal vertex along  $C_{r'_i}$ . Form  $r''_1$  by inserting the square  $[x_{n+1}^{-1}, x_1^{-\epsilon}]^2$  of the commutator at the first  $\varphi$ -maximal vertex along  $C_{r'_1}$ . The lower section  $L_\varphi(r''_i)$  of each  $r''_i$  consists of a single vertex or an edge labelled by the element  $x_i$ . The upper section  $U_\varphi(r''_1)$  of  $r''_1$  consists of two vertices or two edges labelled by  $x_1$ , and for  $i > 1$  the upper section  $U_\varphi(r''_i)$  of  $r''_i$  consists of a single vertex or edge labelled by  $x_i$ . Hence  $((r''_1, \dots, r''_n), \varphi)$  satisfies the unique minimum condition and  $((r''_1, \dots, r''_n), -\varphi)$  satisfies the repeated minimum condition. The map  $f$  is injective since there exists a left inverse  $g: \text{im}(f) \rightarrow \mathcal{R}_l$  of  $f$  which acts by removing the commutators at the  $\varphi$ -minimal and  $\varphi$ -maximal vertices or edges of the  $r''_i$ . This completes the proof.  $\square$

**Theorem 5.0.11.** *Let  $G$  be a random group of deficiency one. Then with positive asymptotic probability,  $\Sigma(G) \cap H^1(G; \mathbb{Z})$  is non-symmetric.*

*Proof.* A random deficiency-one presentation satisfies the  $C''(\frac{1}{6})$  condition, almost surely [Gro93]. Combining the work of Wise [Wis04] and Agol [Ago13], it follows that such a group is virtually special and thus satisfies the Atiyah conjecture by [Sch14]. Hence  $\mathbb{Q}G$  has no non-trivial zero-divisors. By Lemma 5.0.10, a random deficiency-one group admits a character  $\varphi$  such that  $\varphi$  satisfies the unique minimum condition and  $-\varphi$  satisfies the repeated minimum condition, with positive asymptotic probability. Thus by Lemma 5.0.9,  $\varphi \in \Sigma(G)$  and  $-\varphi \notin \Sigma(G)$ .  $\square$

The non-symmetric nature of the BNS invariant provides a useful criterion for detecting when a group is not subgroup separable.

**Lemma 5.0.12.** *If the set  $\Sigma(G) \cap H^1(G; \mathbb{Z})$  of integral characters of  $G$  which are contained in the BNS invariant is non-symmetric under the antipodal involution, then  $G$  is not subgroup separable.*

*Proof.* Let  $\varphi \in \Sigma(G) \cap H^1(G; \mathbb{Z})$  be such that  $\varphi \notin -\Sigma(G)$ . Proposition 4.1 in [BNS87] implies that there exists a finitely generated subgroup  $A \leq G$  and an injective, non-surjective endomorphism  $\theta: A \rightarrow A$ , such that  $G \simeq A *_{\theta}$ . A standard argument (see e.g. [LNW99, Proposition 4]) shows that if  $G$  contains subgroups  $B \leq A$  which are conjugate in  $G$ , then  $B$  cannot be separated from any  $g \in A \setminus B$  in any finite quotient of  $G$ . Hence  $\text{im}(\theta)$  is a non-separable subgroup of  $G$ .  $\square$

**Corollary 5.0.13** (Theorem 5.0.1). *Let  $G$  be a random group of deficiency one. Then with positive asymptotic probability,  $G$  is not subgroup separable.*

*Proof.* Combine Theorem 5.0.11 with Lemma 5.0.12.  $\square$

**Corollary 5.0.14** (Theorem 5.0.2). *Let  $G$  be a random group of deficiency one. Then  $G$  is free-by-cyclic with asymptotic probability that is positive and bounded away from 1.*

*Proof.* A random deficiency-one group has first Betti number  $b_1(G)$  equal to 1, almost surely. Hence  $\text{Hom}(G, \mathbb{Z}) \simeq \mathbb{Z} \simeq \langle \varphi \rangle$ . By Theorem 5.0.11,  $\Sigma(G) \cap H^1(G; \mathbb{Z})$  is non-empty and non-symmetric, with positive asymptotic probability. Hence  $\Sigma(G) = \{\lambda\varphi \mid \lambda \in \mathbb{R}_{>0}\}$  or  $\Sigma(G) = \{\lambda\varphi \mid \lambda \in \mathbb{R}_{<0}\}$ . In particular  $\Sigma(G) \cap -\Sigma(G) = \emptyset$  and thus  $G$  does not fibre algebraically. Hence the asymptotic probability that a random deficiency-one group is free-by-cyclic is bounded away from 1. The fact that it is greater than 0 follows from [KKW22, Theorem A].  $\square$

## Part II

### Profinite invariants

## Chapter 6

# Profinite invariants of free-by-cyclic groups

There exists a large body of work investigating profinite rigidity of 3-manifold groups. For example, deep work of Bridson–McReynolds–Reid–Spitler shows that there are hyperbolic 3-manifolds which are profinitely rigid amongst all finitely generated residually finite groups [BMRS20], with more examples constructed in [CW22] and [BR22]. On the other hand, there exist Anosov torus bundles and periodic closed surface bundles with non-isomorphic but profinitely isomorphic fundamental groups [Ste72, Fun13, Hem14].

Significant progress has been made on the problem of profinite rigidity *within* the class of 3-manifolds. A key step in showing that various classes and properties of 3-manifolds are invariants of the profinite completion is to establish the profinite invariance of fibring. In this vein, and in order to deduce results about the profinite completion of knot groups, Bridson–Reid studied profinite invariants of compact 3-manifolds with boundary and first Betti number equal to one, in particular showing that fibring and the rank of the fibre is a profinite invariant of such 3-manifolds [BR20]. At the same time, Boileau–Friedl tackled the problem of profinite invariants of knot groups by showing that fibring is an invariant of 3-manifolds whose profinite completions are related by a particular type of isomorphism, called a *regular isomorphism* [BF20]. Finally, Jaikin-Zapirain showed that being fibred is a profinite invariant of all 3-manifold groups [JZ20], and this

was generalised to all subgroup separable groups in [HK22].

Another crucial element is the work Wilton–Zalesskii on profinite detection of Thurston geometries [WZ17a] and of Wilkes and Wilton–Zalesskii on profinite invariance of various decompositions of 3-manifolds [Wil18a, Wil18b, WZ19]. The case of Seifert fibred manifolds was entirely solved by Wilkes, who proved that these are almost profinitely rigid in the class of all 3-manifold groups [Wil17]. Graph manifolds have received much attention too [WZ10, Wil18b, Wil19a]. Most recently, Liu proved the spectacular theorem that finite volume hyperbolic 3-manifold groups are almost profinitely rigid [Liu23a]. Other results have also been obtained, e.g. [BRW17, WZ17b, BF20, Zal22, Liu23b].

The goal of this chapter is to investigate profinite rigidity amongst free-by-cyclic groups. The study of profinite invariants of free-by-cyclic groups saw its inception in the work of Bridson–Reid [BR20]. Although the aim of their work was to prove results about fibred knot complements, their methods apply more generally and are later used by Bridson–Reid–Wilton [BRW17] to show profinite rigidity amongst the groups of the form  $F_2 \rtimes \mathbb{Z}$ .

Whilst we draw inspiration from the results in the 3-manifold setting, the problem for free-by-cyclic groups is significantly more subtle. This stems in part from the lack of a sufficient  $\text{Out}(F_n)$ -analogue of the Nielsen–Thurston decomposition for homeomorphisms of finite-type surfaces. One artefact of this is that we frequently have to restrict our attention to the class of *irreducible* free-by-cyclic groups, that is free-by-cyclic groups which admit irreducible monodromy. Recall that by the work of Mutanguha [Mut21], for any two realisations of  $G$  as a free-by-cyclic group,  $G \simeq F_n \rtimes_{\Phi} \mathbb{Z} \simeq F_m \rtimes_{\Psi} \mathbb{Z}$ , the monodromy  $\Phi$  is irreducible if and only if  $\Psi$  is.

Our first result is analogous to Liu’s theorem with the additional hypotheses that  $b_1(G) = 1$  and restricting to the class of irreducible free-by-cyclic groups. The first hypothesis is due to the fact that we do not have a method to establish  $\widehat{\mathbb{Z}}$ -regularity (see Section 6.3 for a definition) without an analogous result to the main theorems in [FV08, FV11] — this is one of the main technical steps in Jaikin-Zapirain’s and Liu’s results. The second hypothesis arises since, although we can show that hyperbolicity of free-by-cyclic groups is a profinite invariant, we are currently unable to show the same holds true for irreducibility.

**Theorem 6.6.4.** *Let  $G$  be an irreducible free-by-cyclic group. If  $b_1(G) = 1$ , then  $G$  is almost profinitely rigid amongst irreducible free-by-cyclic groups.*

The next theorem is somewhat more technical. The definitions of the invariants can be found in Section 3.1. Note that the result actually holds in the more general setting of a  $\widehat{\mathbb{Z}}$ -regular isomorphism (the specific results stated throughout the paper comprising Theorem 6.5.7 are stated in this generality). We point out the general fact that the first Betti number of any discrete group is an invariant of its profinite completion.

**Theorem 6.5.7.** *Let  $G = F \rtimes_{\Phi} \mathbb{Z}$  be a free-by-cyclic group with induced character  $\varphi: G \rightarrow \mathbb{Z}$ . If  $b_1(G) = 1$ , then the following properties are determined by the profinite completion  $\widehat{G}$  of  $G$ :*

1. *the rank of  $F$ ;*
2. *the homological stretch factors  $\{\nu_G^+, \nu_G^-\}$ ;*
3. *the characteristic polynomials  $\{\text{Char } \Phi^+, \text{Char } \Phi^-\}$  of the action of  $\Phi$  on  $H_1(F; \mathbb{Q})$ ;*
4. *for each representation  $\rho: G \rightarrow \text{GL}(n, \mathbb{Q})$  factoring through a finite quotient, the twisted Alexander polynomials  $\{\Delta_n^{\varphi, \rho}, \Delta_n^{-\varphi, \rho}\}$  and the twisted Reidemeister torsions  $\{\tau^{\varphi, \rho}, \tau^{-\varphi, \rho}\}$  over  $\mathbb{Q}$ .*

*Moreover, if  $G$  is conjugacy separable, (e.g. if  $G$  is hyperbolic), then  $\widehat{G}$  also determines the Nielsen numbers and the homotopical stretch factors  $\{\lambda_G^+, \lambda_G^-\}$ .*

We note that Item 1 of our Theorem 6.5.7 was already known by the work of Bridson–Reid [BR20, Lemma 3.1].

The reason for obtaining a set of invariants corresponding to  $\Phi$  and  $\Phi^{-1}$  is that the dynamics of  $\Phi$  and  $\Phi^{-1}$  can be different. Indeed, this is somewhat a feature of free-by-cyclic groups rather than a bug. A large technical hurdle in this work was overcoming this phenomenon which cannot occur for surface bundles.

We also obtain a complete geometric picture à la Wilton–Zalesskii in the case of hyperbolic free-by-cyclic groups.

**Theorem 6.1.6.** *Let  $G_A$  and  $G_B$  be profinitely isomorphic free-by-cyclic groups. Then  $G_A$  is Gromov hyperbolic if and only if  $G_B$  is Gromov hyperbolic.*

Next we apply Theorem 6.6.4, Theorem 6.5.7, and Theorem 6.1.6 in order to obtain strong profinite rigidity phenomena for various classes of free-by-cyclic groups.

We say that a free-by-cyclic group  $G$  is *super irreducible*, if  $G \simeq F_n \rtimes_{\Phi} \mathbb{Z}$  and the matrix  $M: H_1(F_n; \mathbb{Q}) \rightarrow H_1(F_n; \mathbb{Q})$  representing the action of  $\Phi$  on  $H_1(F_n; \mathbb{Q})$  satisfies the property that no positive power of  $M$  maps a proper subspace of  $H_1(F_n; \mathbb{Q})$  into itself. Note that this implies  $G$  is irreducible by [GS91, Theorem 2.5] and that  $b_1(G) = 1$ .

An example of a super irreducible free-by-cyclic group is whenever the characteristic polynomial of  $M$  is a *Pisot–Vijayaraghavan polynomial*, namely, it is monic, it has exactly one root (counted with multiplicity) with absolute value strictly greater than one, and all other roots have absolute value strictly less than one [GS91].

**Corollary 6.6.5.** *Let  $G$  be a super irreducible free-by-cyclic group. Then, every free-by-cyclic group profinitely isomorphic to  $G$  is super irreducible. In particular,  $G$  is almost profinitely rigid amongst free-by-cyclic groups.*

The key observation is that super irreducibility is a *generic* phenomenon amongst elements of  $\text{Out}(F_n)$  (see Section 6.1.1 for the relevant definitions of genericity and random free-by-cyclic groups). As a consequence of Corollary 6.6.5, we obtain the following result:

**Corollary 6.6.6.** *Let  $G$  be a random free-by-cyclic group. Then, asymptotically almost surely  $G$  is almost profinitely rigid amongst free-by-cyclic groups.*

When the fibre of the free-by-cyclic group has rank two or three we are able to obtain rigidity statements within the class of all free-by-cyclic groups.

**Corollary 6.6.7.** *Let  $G = F_3 \rtimes \mathbb{Z}$ . If  $G$  is hyperbolic and  $b_1(G) = 1$ , then  $G$  is almost profinitely rigid amongst free-by-cyclic groups.*

Note in the next statement we see that  $G$  is uniquely determined.

**Corollary 6.6.8.** *Let  $G = F_2 \rtimes \mathbb{Z}$ . If  $b_1(G) = 1$ , then  $G$  is profinitely rigid amongst free-by-cyclic groups.*

Finally, we investigate conjugacy in  $\text{Out}(\widehat{F}_n)$ . We say two outer automorphisms  $\Psi$  and  $\Phi$  of  $F_n$  are *profinutely conjugate* if they induce a conjugate pair of outer automorphisms in  $\text{Out}(\widehat{F}_n)$ . In this setting we have no assumption on the action of  $\Psi$  or  $\Phi$  on the homology of  $F_n$ .

**Theorem 6.7.2.** *Let  $\Psi \in \text{Out}(F_n)$  be atoroidal. If  $\Phi \in \text{Out}(F_n)$  is profinitely conjugate to  $\Psi$ , then  $\Phi$  is atoroidal and  $\{\lambda_\Psi, \lambda_{\Psi^{-1}}\} = \{\lambda_\Phi, \lambda_{\Phi^{-1}}\}$ . In particular, if  $\Psi$  is additionally irreducible, then there are only finitely many  $\text{Out}(F_n)$ -conjugacy classes of irreducible automorphisms which are conjugate with  $\Psi$  in  $\text{Out}(\widehat{F}_n)$ .*

## 6.0.1 Structure

In Section 6.1 we recall the necessary background on free group automorphisms and free-by-cyclic groups and prove a number of results we will need throughout the chapter.

Section 6.2 establishes a number of facts about twisted Alexander polynomials. In Section 6.3 we recall the notion of a matrix coefficient module and a  $\widehat{\mathbb{Z}}$ -regular isomorphism. The main reason for this section is to allow us to work in the generality of a  $\widehat{\mathbb{Z}}$ -regular isomorphism. This means that if one established a positive answer to Question 10.0.5 then one could apply the results in this chapter without any further modifications.

In Section 6.4 we set out to prove profinite invariance of Reidemeister torsion over  $\mathbb{Q}$  twisted by representations of finite quotients for  $G_A$  and  $G_B$ . Our strategy is parallel to that of Liu [Liu23a, Section 7], however due to the extra complexity of free-by-cyclic groups we have to invoke extra results about twisted Alexander polynomials of free-by-cyclic groups established in Section 6.2.

In Section 6.5, under the assumption of conjugacy separability of  $G_A$  and  $G_B$  we prove that the homotopical stretch factors  $\{\lambda_A, \lambda_A^-\}$  and  $\{\lambda_B, \lambda_B^-\}$  are equal.

Again our strategy is largely motivated by [Liu23a, Section 8]. The key difference is that for a fibred character  $\varphi$  on a finite volume hyperbolic 3-manifold the stretch factors of  $\varphi$  and  $\varphi^{-1}$  are the same. This is not true for free-by-cyclic groups where we must deal with both directions at once and so our main work is resolving this issue. Combining the major results up to this point proves Theorem 6.5.7.

In Section 6.6 we prove Theorem 6.6.4 and then go on to deduce Corollaries 6.6.5-6.6.8. Section 6.7 is concerned with Theorem 6.7.2 on detecting profinite conjugacy classes.

## 6.1 Preliminaries

### 6.1.1 Generic elements of $\text{Aut}(F_n)$

Fix a finite generating set  $S$  of  $\text{Aut}(F_n)$ . For each  $l \geq 1$ , let  $\mathcal{W}_{l,n}$  denote the set of reduced words of length  $l$  in  $S$ . We say that a *random element of  $\text{Aut}(F_n)$  satisfies property  $P$  with probability  $p$* , if

$$\frac{\#\{w \in \mathcal{W}_{l,n} \mid w \text{ satisfies } P\}}{\#\mathcal{W}_{l,n}} \rightarrow p \text{ as } l \rightarrow \infty.$$

We say that a *generic element in  $\text{Aut}(F_n)$  has property  $P$* , if a random element satisfies property  $P$  with probability  $p = 1$ .

An automorphism  $\phi \in \text{Aut}(F_n)$  is said to be *super irreducible* if no positive power of the induced map  $\phi_{\text{ab}} \in \text{GL}(n, \mathbb{Q})$  maps a proper subspace of  $H_1(F_n; \mathbb{Q})$  into itself. A free-by-cyclic group  $G$  is *super irreducible* if there exists some splitting  $G \simeq F_n \rtimes_{\phi} \mathbb{Z}$  such that  $\phi$  is super irreducible.

The following theorem is a consequence of the results in Section 7 of [Riv08], which hold verbatim after replacing  $\text{SL}(n, \mathbb{Z})$  by  $\text{GL}(n, \mathbb{Z})$  in all the statements.

**Theorem 6.1.1** ([Riv08]). *A generic element in  $\text{Aut}(F_n)$  is super irreducible.*

**Proposition 6.1.2.** *For a generic element in  $\text{Aut}(F_n)$ , the first Betti number of its mapping torus is equal to one.*

*Proof.* Write  $\phi_{\text{ab}}$  to denote the image of  $\phi$  under the natural map induced by the action on the abelianisation of  $F_n$ ,

$$\begin{aligned} \text{Aut}(F_n) &\rightarrow \text{GL}(n, \mathbb{Z}) \\ \phi &\mapsto \phi_{\text{ab}}. \end{aligned}$$

The free abelianisation of  $F_n \rtimes_{\phi} \mathbb{Z}$  is isomorphic to  $\mathbb{Z}$  if and only if  $\phi_{\text{ab}}$  has no eigenvalue equal to 1 [BMV07, Theorem 2.4]. By Theorem 6.1.1, for a generic element in  $\text{Aut}(F_n)$  which represents the automorphism  $\phi$ ,  $\phi_{\text{ab}}$  has characteristic polynomial that is irreducible over  $\mathbb{Q}$ . Hence the result follows.  $\square$

Write  $\mathcal{H}_{l,n}$  to denote the set of free-by-cyclic presentations

$$\mathcal{H}_{l,n} := \{ \langle x_1, \dots, x_n, t \mid t^{-1}x_it = \phi(x_i), 1 \leq i \leq n \rangle \mid \phi \in \mathcal{W}_{l,n} \}.$$

We say that a *generic  $F_n$ -by-cyclic group satisfies property  $P$  with probability  $p$* , if

$$\frac{\#\{G \in \mathcal{H}_{l,n} \mid G \text{ satisfies } P\}}{\#\mathcal{H}_{l,n}} \rightarrow 1 \text{ as } l \rightarrow \infty.$$

**Proposition 6.1.3.** *A generic  $F_n$ -by-cyclic group has first Betti number equal to one and is super irreducible.*

## 6.1.2 Nielsen fixed point theory

Let  $X$  be a connected compact polyhedral complex and  $f: X \rightarrow X$  a continuous self-map. An  *$m$ -periodic point*  $p \in X$  is a fixed point of the map  $f^m$ . Let  $\text{Orb}_m(f)$  be the set of orbits of  $m$ -periodic points of  $X$  under the action of  $f$ . Each orbit  $\mathcal{O} \in \text{Orb}_m(f)$  determines a free homotopy class of loops in the mapping torus  $M_f$ , and thus a conjugacy class in  $\pi_1(M_f)$ , which we denote by  $\text{cd}(\mathcal{O})$ . Furthermore, every  $\mathcal{O} \in \text{Orb}_m(f)$  admits an index  $\text{ind}_m(f; \mathcal{O}) \in \mathbb{Z}$ , which is the fixed point index of  $f^m$  at any point  $p \in \mathcal{O}$  (see [Jia96, Section 1.3]). A periodic orbit of a point in  $X$  under the action of  $f$  is said to be *essential* if it has non-zero index. Note that if  $X$  is a graph then an isolated fixed point  $x$  has index zero, if and only if  $f$  is not locally injective at  $x$ .

**Definition 6.1.4.** The  *$m$ -th Nielsen number* of  $f$ , denoted by  $N_m(f)$ , is the number of essential  $m$ -periodic orbits of  $f$ .

It is a standard fact from Nielsen fixed point theory (see e.g. [Jia83, Chapter 1] and [Jia96]), that each Nielsen number is independent of the choice of topological representative of  $\Phi \in \text{Out}(\pi_1(X))$ . Hence, we may write  $N_\infty(\Phi)$  to denote

$$N_\infty(\Phi) = \limsup_{m \rightarrow \infty} N_m(f)^{1/m},$$

where  $(f, \Gamma)$  is any topological representative of  $\Phi$ .

**Proposition 6.1.5.** *Let  $\Phi \in \text{Out}(F_n)$  be an outer automorphism with stretch factor  $\lambda > 1$ . Then  $N_\infty(\Phi)$  is equal to  $\lambda$ .*

*Proof.* By Theorem 3.2.6, there exists a positive integer  $k$  such that  $\Phi^k$  admits a topological representative  $(f, \Gamma)$  which is a relative train track and such that every periodic Nielsen path has period one and each exponentially-growing stratum intersects at most one indivisible Nielsen path. Let  $A$  be the corresponding incidence matrix and fix a maximal filtration of  $\Gamma$ . Let  $\{S_i\}_{i \in I}$  be the set of exponentially-growing strata of  $\Gamma$  and write  $A_i^m$  to denote the submatrix of  $A^m$  spanned by the edges of  $S_i$ .

For each exponentially-growing stratum  $S_i$ , there exists a length assignment

$$L: E(\Gamma) \rightarrow \mathbb{R}_{\geq 0}$$

on the edges of  $\Gamma$ , such that  $L(e) > 0$  and  $L(f(e)) = \lambda_i \cdot L(e)$ , for every edge  $e$  in  $S_i$ . Hence the number of fixed points of  $f^m$  contained in the interior of the edge  $e$  of  $\Gamma_i$  is given by the number of times the edge path  $f^m(e)$  crosses the edge  $e$  in either direction. This is precisely the element on the diagonal of the matrix  $A^m$  corresponding to the edge  $e$ .

The number of Nielsen fixed point classes which intersect the non-exponentially-growing strata non-trivially or which contain a vertex is uniformly bounded as  $m$  goes to infinity. Hence, there exists some constant  $C$  such that

$$\limsup_{m \rightarrow \infty} N_m(f)^{1/m} = \limsup_{m \rightarrow \infty} \left( C + \sum_{i \in I} \text{tr}(A_i^m) \right)^{1/m}.$$

For each matrix  $A_i$ , let  $n_i$  denote the order of  $A_i$  and let  $\lambda_{i,j}$  be its eigenvalues, for  $1 \leq j \leq n_i$ . Then

$$\text{tr}(A_i^m) = \sum_{1 \leq j \leq n_i} \lambda_{i,j}^m.$$

Let  $\lambda$  be the stretch factor of  $\Phi$ . Then  $|\lambda_{i,j}/\lambda|^m \leq 1$  for each  $i \in I$ ,  $j \leq n_i$  and

$m \in \mathbb{N}$ , and

$$\begin{aligned} \limsup_{m \rightarrow \infty} N_m(f)^{1/m} &= \lambda \cdot \limsup_{m \rightarrow \infty} \left( C/\lambda^m + \sum_{i \in I} \sum_{1 \leq j \leq n_i} (\lambda_{i,j}/\lambda)^m \right)^{1/m} \\ &= \lambda \cdot 1 \end{aligned}$$

where the second equality follows from the fact that  $C/\lambda^m \rightarrow 0$  as  $m \rightarrow \infty$  since  $\lambda > 1$ . Thus,  $N_\infty(\Phi^k)$  is equal to the stretch factor of  $\Phi^k$ .

By [FM21, Corollary 7.14], if  $\lambda$  is the maximal stretch factor of a relative train track representative of  $\Phi^k$ , then  $\lambda^{1/k}$  is the maximal stretch factor associated to  $\Phi$ . Note also that  $N_\infty(\Phi^k) = N_\infty(\Phi)^k$ . The result follows by combining the arguments in the previous paragraphs.  $\square$

### 6.1.3 Detecting atoroidal monodromy

In this section we will prove that hyperbolicity (equivalently the property of admitting atoroidal monodromy) is determined by the profinite completion.

**Theorem 6.1.6.** *Let  $G_A$  and  $G_B$  be profinitely isomorphic free-by-cyclic groups. Then  $G_A$  is Gromov hyperbolic if and only if  $G_B$  is Gromov hyperbolic.*

*Proof.* Let  $G_A$  and  $G_B$  be free-by-cyclic groups such that  $\widehat{G}_A \simeq \widehat{G}_B$ . Suppose that  $G_A$  is Gromov hyperbolic. By [HW15],  $G_A$  is a cocompactly cubulated and thus virtually special. Hence we may apply [WZ17a, Theorem D] to deduce that  $\widehat{\mathbb{Z}}^2$  is not a subgroup of  $\widehat{G}_A$ . By Corollary 4.2.2, the  $\mathbb{Z}^2$  subgroups of  $G_B$  are fully separable and since  $\widehat{G}_B$  contains no  $\widehat{\mathbb{Z}}^2$  subgroups, it follows  $G_B$  contains no  $\mathbb{Z}^2$  subgroups. In particular, by [Bri00a, Theorem 1.2]  $G_B$  is Gromov hyperbolic.

Suppose conversely that  $G_A$  is not Gromov hyperbolic. Then by [Bri00a],  $G_A$  has a  $\mathbb{Z}^2$  subgroup and so by Corollary 4.2.2,  $\widehat{G}_A$  contains a  $\widehat{\mathbb{Z}}^2$  subgroup. Suppose now  $G_B$  is not Gromov hyperbolic, then by the argument in the previous paragraph  $\widehat{G}_B$  does not contain  $\widehat{\mathbb{Z}}^2$  subgroups. This contradiction completes the proof.  $\square$

### 6.1.4 Goodness

A group  $G$  is said to be *cohomologically good*, or *good* for short, if for every  $n \geq 0$  and every finite  $\mathbb{Z}G$ -module  $M$ , the map

$$H^n(\widehat{G}; M) \rightarrow H^n(G; M)$$

induced by the natural homomorphism of  $G$  into its profinite completion  $\widehat{G}$  is an isomorphism.

The following is a special case of [Lor08, Corollary 2.9]:

**Lemma 6.1.7.** *If  $G$  is free-by-cyclic then  $G$  is cohomologically good.*

## 6.2 Some properties of twisted Alexander polynomials and Reidemeister torsion

In this section we will collect a number of facts about twisted Alexander polynomials and twisted Reidemeister torsion that we will use later on. Our main contribution is a complete computation of the zeroth Alexander polynomials twisted by representations factoring through finite groups over fields of characteristic zero (Lemma 6.2.6).

**Remark 6.2.1.** Let  $R$  be a unique factorisation domain. Unless otherwise stated, we will use the dotted equality symbol  $\doteq$  to denote equality of rational functions over  $R$  up to monomial factors with coefficients in  $\text{Frac}(R)^\times$ .

**Definition 6.2.2** (Alexander modules and polynomials). Let  $R$  be a unique factorisation domain and let  $G$  be a finitely generated group. Let  $\varphi$  be a non-trivial primitive class in  $H^1(G; \mathbb{Z})$  considered as a homomorphism  $G \rightarrow \mathbb{Z}$  and let  $\rho: G \rightarrow \text{GL}_n(R)$  be a representation. We define a representation  $\rho^\phi: G \rightarrow \text{GL}_n(R[t])$  by  $g \mapsto t^{\phi(g)} \cdot \rho(g)$ . Now we consider  $R^n[t^{\pm 1}]$  equipped with the  $RG$ -bimodule structure given by

$$g.x = \rho^\phi(g)x, \quad x.g = x\rho^\phi(g)$$

for  $g \in G, x \in R^n[t^{\pm 1}]$ . For  $n \in \mathbb{Z}$ , we define the  $k$ th twisted Alexander module of  $\varphi$  and  $\rho$  to be  $H_k(G; R^n[t^{\pm 1}])$ , where  $R^n[t^{\pm 1}]$  has the right  $RG$ -module structure described above. Observe that  $H_k(G; R^n[t^{\pm 1}])$  is a left  $R[t^{\pm 1}]$ -module. If  $G$  is of type  $\text{FP}_k(R)$ , then the  $k$ th twisted Alexander module is a finitely generated  $R[t^{\pm 1}]$ -module. Moreover, it is zero whenever  $k < 0$  or  $k$  is greater than the cohomological dimension of  $G$  over  $R$ .

Since  $R$  is UFD so is  $R[t^{\pm 1}]$ . Let  $M$  be an  $R[t^{\pm 1}]$ -module. The *order* of  $M$  is the greatest common divisor of all maximal minors in a presentation matrix of  $M$  with finitely many columns. The order of  $M$  is well-defined up to a unit of  $R[t^{\pm 1}]$  and depends only on the isomorphism type of  $M$ .

Suppose that  $G$  is of type  $\text{FP}_k(R)$ . The  $k$ th twisted Alexander polynomial  $\Delta_{k,R}^{\varphi,\rho}(t)$  over  $R$  with respect to  $\varphi$  and  $\rho$  is defined to be the order of the  $k$ th twisted (homological) Alexander module of  $\varphi$  and  $\rho$ , treated as a left  $R[t^{\pm 1}]$ -module.

We will now collect a number of facts about twisted Alexander polynomials.

Let  $R$  be a unique factorisation domain. Given any polynomial  $c(t) \in R[t^{\pm 1}]$  where  $c(t) = \sum_{i=0}^r c_i t^i$  we write  $c^\star(t)$  for the polynomial  $\sum_{i=0}^r c_{r-i} t^i$ .

The following lemma is a triviality.

**Lemma 6.2.3.** *Let  $G$  be a group of type  $\text{FP}_n(R)$ , let  $\varphi: G \twoheadrightarrow \mathbb{Z}$ , and let  $\rho, \sigma: G \rightarrow \text{GL}_n(R)$  be representations of  $G$  over a UFD  $R$ . If  $\rho$  and  $\sigma$  are conjugate representations, then*

$$\Delta_n^{\varphi,\rho}(t) \doteq \Delta_n^{\varphi,\sigma}(t).$$

**Lemma 6.2.4.** *Let  $G$  be a group of type  $\text{FP}_n(R)$ , let  $\varphi: G \twoheadrightarrow \mathbb{Z}$ , and let  $\rho, \sigma: G \rightarrow \text{GL}_n(R)$  be representations of  $G$  over a UFD  $R$ . Then,*

$$\Delta_n^{\varphi,\rho \oplus \sigma}(t) \doteq \Delta_n^{\varphi,\rho}(t) \times \Delta_n^{\varphi,\sigma}(t).$$

*Proof.* This follows from the fact that homology commutes with taking direct sums of coefficient modules. □

The following lemma is a triviality

**Lemma 6.2.5.** *Let  $R$  be a UFD. Let  $G$  be a group, let  $\varphi: G \rightarrow \mathbb{Z}$ , and let  $\rho: G \rightarrow \mathrm{GL}_k(R)$  be a representation. Then,*

$$(\Delta_n^{\varphi, \rho})^\star(t) \doteq \Delta_n^{-\varphi, \rho}(t) \doteq \Delta_n^{\varphi, \rho}(t^{-1})$$

*up to monomial factors with coefficients in  $R^\times$ .*

The next lemma will be a key step in proving profinite rigidity of twisted Reidemeister torsion for our class of free-by-cyclic groups. For a  $G$ -module  $M$  being acted on via  $\alpha: G \times M \rightarrow M$  we write  $M_\alpha$  when we wish to make clear the  $G$ -module structure.

**Lemma 6.2.6.** *Let  $G$  be a finitely generated group, let  $\varphi: G \rightarrow \mathbb{Z}$  be algebraically fibred, and let  $\rho: G \rightarrow Q \rightarrow \mathrm{GL}_k(\mathbb{Q})$  be a representation factoring through a finite group. Then,*

$$\Delta_0^{\varphi, \rho}(t) \doteq (1-t)^m P(t),$$

*where  $m \geq 0$  and  $P(t)$  is a product of cyclotomic polynomials, up to multiplication by monomials with coefficients in  $\mathbb{Q}^\times$ . In particular,*

$$\Delta_0^{\varphi, \rho}(t) \doteq \Delta_0^{\varphi, \rho}(t^{-1}).$$

*Proof.* Let  $F$  denote the kernel of  $\varphi$ . We need to compute  $M := H_0(G; \mathbb{Q}^k[t^{\pm 1}])$  which is naturally isomorphic to the coinvariants  $(\mathbb{Q}^k[t^{\pm 1}])_G$ .

By Maschke's Theorem we may write the representation  $\rho$  of  $Q$  as a sum  $\bigoplus_{i=1}^{\ell} \rho_i: Q \rightarrow \prod_{i=1}^{\ell} \mathrm{GL}_{k_i}(\mathbb{Q})$ , where  $\sum_{i=1}^{\ell} k_i = k$ , of irreducible  $\mathbb{Q}$ -representations of  $L$ . We may now write

$$M = \bigoplus_{i=1}^{\ell} (\mathbb{Q}^{k_i}[t^{\pm 1}])_G.$$

For each  $i$  there are three possibilities:

**Case 1:**  $\rho_i(Q) \neq \{1\}$  but  $\rho_i(F) = \{1\}$ .

In this case  $\rho_i$  has image a non-trivial finite cyclic group  $L$ . We quickly recap the  $\mathbb{Q}$ -representation theory of  $\mathbb{Z}/n$  for  $n \geq 2$ . Recall that  $\mathbb{Q}[\mathbb{Z}/n] = \mathbb{Q}[X]/(X^n - 1)$  so the irreducible representations of  $\mathbb{Z}/n$  are exactly the cyclotomic fields  $\mathbb{Q}(\chi_d)$

for each  $d$  dividing  $n$ . These representations are exactly given by the quotient map  $\pi_d: \mathbb{Q}[\mathbb{Z}/n] \rightarrow \mathbb{Q}(\chi_d)$ . Note that in this case for a generator  $g$  of  $\mathbb{Z}/n$  the characteristic polynomial of  $\pi_d(g)$  is the cyclotomic polynomial  $\chi_d$ .

Since  $\rho_i$  is irreducible it follows that we are in the situation of a cyclotomic representation. Consider the tail end of the standard resolution for  $\mathbb{Z}$  over  $\mathbb{Z}G$

$$C_1 \xrightarrow{\partial} C_0 = a_0\mathbb{Z}G \oplus \cdots \oplus a_{m-1}\mathbb{Z}G \oplus t\mathbb{Z}G \xrightarrow{\partial} \mathbb{Z}G$$

where  $a_0, \dots, a_{m-1}$  is a generating set for  $F$ , where  $t$  is the generator of  $\mathbb{Z}$  viewing  $G = F \rtimes \mathbb{Z}$ , and where

$$\partial = \left[ 1 - a_0, \dots, 1 - a_{m-1}, 1 - t \right]. \quad (6.1)$$

We need to compute the order of the presentation matrix

$$\partial \otimes_{\mathbb{Z}G} \text{id}_{\mathbb{Q}[\Phi_d][t^{\pm 1}]} = \left[ 0, \dots, 0, \text{id} - \rho_i(t)t \right].$$

But this is the same as computing an order of the square matrix  $\text{id} - \rho_i(t)t$ .

Now,

$$\text{ord}(\text{id} - \rho_i(t)t) \doteq \det(\text{id}t^{-1} - \rho_i(t)t \cdot t^{-1})t^{p-1} \doteq \det(\text{id}t^{-1} - \rho_i(t)) \quad (6.2)$$

but this is exactly the characteristic polynomial of  $\rho_i(t)$  with respect to  $t^{-1}$ . Namely, it is the cyclotomic polynomial  $\chi_d(t^{-1})$  but this is palindromic of even degree,  $t - 1$ , or  $t + 1$  so we have that  $\Delta_0^{\varphi, \rho_i}(t) \doteq \chi_d(t)$ .  $\blacklozenge$

**Case 2:**  $\rho_i(F) \neq \{1\}$ .

We start by again by viewing  $G$  as  $F \rtimes \mathbb{Z}$ . In particular, we have a differential  $\partial$  as in (6.1) such that  $\Delta_0^{\varphi, \rho_i}$  is given by an order of

$$D_i := \partial \otimes_{\mathbb{Z}G} \text{id}_{\mathbb{Q}^{k_i}[t^{\pm 1}]} = [\text{id} - \rho_i(a_0), \dots, \text{id} - \rho_i(a_{m-1}), \text{id} - \rho_i(t)].$$

To this end we define  $D$  to be the set of cofactors of  $D_i$ . So  $\Delta_0^{\varphi, \rho_i} \doteq \text{gcd } D$ .

We first conjugate  $\rho_i$  so that  $\rho_i(t)$  is in block diagonal form. Since the image of  $t$  is cyclic, say of order  $n$ , we obtain a block structure where the non-identity blocks

are matrices corresponding to non-trivial  $\mathbb{Q}$ -representations of various subgroups  $H \leq \mathbb{Z}/n$ . Thus, arguing as in (6.2) we see that

$$(1-t)^{n'} \cdot \prod_{j=1}^{\ell} \chi_{n_j}(t) \in D,$$

where  $n'$  is dimension of the fixed subspace of  $\rho_i(t)$  and  $\chi_{n_j}(t)$  is the cyclotomic polynomial of order  $n_j$  such that  $n_j$  divides  $n$ .

Now,  $\Delta_0^{\varphi, \rho_i}$  divides every element of  $D$  and is a polynomial defined over  $\mathbb{Q}[t]$  (up to multiplication by  $t^\ell$  for some  $\ell \geq 0$ ) and  $\chi_{n_j}(t)$  is the minimal polynomial for all primitive  $n_j$ th roots of unity. In particular, any non-trivial polynomial dividing and not equal to  $\chi_{n_j}(t)$  is not defined over  $\mathbb{Q}[t^{\pm 1}]$ . It follows that  $\Delta_0^{\varphi, \rho_i} = P_i(t) \cdot (1-t)^{n''}$  where  $P_i(t)$  is a product of cyclotomic polynomials and  $n''$  is a non-negative integer less than or equal to  $k_i$ .  $\blacklozenge$

**Case 3:**  $\rho_i(G) = \{1\}$ .

In this case we are computing  $(\mathbb{Q}[t^{\pm 1}])_G$  where  $G$  acts trivially on  $\mathbb{Q}$ . Clearly, this is isomorphic to  $\mathbb{Q}[t^{\pm 1}]/(1-t)$  which is additively isomorphic to  $\mathbb{Q}$ .  $\blacklozenge$

By Lemma 6.2.4 we have that  $\Delta_0^{\varphi, \rho}(t) \doteq \prod_{i=1}^{\ell} \Delta_0^{\varphi, \rho_i}(t) \doteq (1-t)^n P(t)$  where  $n$  is some non-negative integer and  $P(t)$  is a product of cyclotomic polynomials.

The ‘‘in particular’’ now follows from the fact cyclotomic polynomials are palindromic (provided  $d \neq 2$ ) or equal to  $t-1$  and an easy computation: Write  $P(t) = (t-1)^{m'} P'(t)$  where  $m'$  is the number of  $(t-1)$  factors of  $P(t)$ . Let  $\delta$  denote the degree of  $P(t)$  and let  $\epsilon = 1$  if exactly one of  $m$  and  $m'$  are non-zero, and let  $\epsilon = 0$  otherwise. Now,

$$\begin{aligned} \Delta_0^{\varphi, \rho}(t^{-1}) &\doteq (-1)^{\epsilon} t^{m+m'+\delta} (1-t^{-1})^m (t^{-1}-1)^{m'} P'(t^{-1}) \\ &\doteq (1-t)^m (t-1)^{m'} P'(t) \\ &\doteq \Delta_0^{\varphi, \rho}(t). \end{aligned} \quad \square$$

**Remark 6.2.7.** The previous lemma easily generalises to any field  $\mathbb{F}$  of characteristic zero with the modified conclusion that  $\Delta_0^{\varphi, \rho}(t) \doteq Q(t)P(t)$ , where  $Q(t)$  is a

product of polynomials  $(1 - \zeta_i t)$  such that  $\zeta_i$  is some root of unity in  $\mathbb{F}$ , and where  $P(t)$  is a product of cyclotomic polynomials whose roots do not lie in  $\mathbb{F}$ .

Let  $R$  be a unique factorisation domain. A polynomial  $c(t) \in R[t^{\pm 1}]$  is *palindromic* if  $c(t) = \sum_{i=0}^r c_i t^i$  and  $c_i = c_{r-i}$ . Recall that given any polynomial  $c(t) \in R[t^{\pm 1}]$  where  $c(t) = \sum_{i=0}^r c_i t^i$  we write  $c^\star(t)$  for the polynomial  $\sum_{i=0}^r c_{r-i} t^i$ . Note that  $c(t) \cdot c^\star(t)$  is palindromic.

For a group  $G$  we let  $\mathbf{1}$  denote the trivial homomorphism  $G \rightarrow \{1\}$ .

The following lemma is well known to experts. We include a proof for completeness.

**Lemma 6.2.8.** *Let  $\mathbb{F}$  be a field. Let  $G$  be a group of type  $\text{FP}_n(\mathbb{F})$ . If  $\varphi: G \rightarrow \mathbb{Z}$  is an  $\text{FP}_n(\mathbb{F})$ -fibring, then*

$$\deg \Delta_{G,n}^{\varphi,\mathbf{1}}(t) = b_n(\ker \varphi; \mathbb{F}),$$

where the Alexander polynomial is taken over  $\mathbb{F}$ .

*Proof.* We may write  $G$  as  $\ker \varphi \rtimes \langle t \rangle$  and  $\Delta_{G,n}^{\varphi,\mathbf{1}}(t)$  as the characteristic polynomial of the  $\mathbb{F}$ -linear transformation  $T_n: H_n(\ker \varphi; \mathbb{F}) \rightarrow H_n(\ker \varphi; \mathbb{F})$  and  $T^n: H^n(\ker \varphi; \mathbb{F}) \rightarrow H^n(\ker \varphi; \mathbb{F})$ , where  $T$  is the induced map of  $t$  on (co)homology. Hence,

$$H^1(\ker \varphi; \mathbb{F}) \simeq \mathbb{F}[t^{\pm 1}] / (\Delta_{G,n}^{\varphi,\mathbf{1}}(t)). \quad \square$$

**Lemma 6.2.9.** *Let  $R$  be a UFD. Let  $G$  be a group of type  $\mathbf{F}$  admitting a compact  $K(G, 1)$  of dimension  $n$ , let  $\varphi: G \rightarrow \mathbb{Z}$ , and let  $\rho: G \rightarrow \text{GL}_k(R)$  be a representation. If  $\Delta_n^{\varphi,\rho} \neq 0$  over  $R$ , then  $\Delta_n^{\varphi,\rho} \doteq 1$ .*

*Proof.* Consider the head end of the cellular chain complex for  $G$ , namely,

$$0 \longrightarrow C_n \xrightarrow{\partial_{n-1}} C_{n-1} \longrightarrow \cdots$$

tensoring with  $R^k[t^{\pm 1}]$  and taking homology we see that  $H_n(G; R^k[t^{\pm 1}]) = \ker \partial_{n-1} \otimes \text{id}_{R^k[t^{\pm 1}]}$ . In particular, it is a submodule of a free  $R[t^{\pm 1}]$ -module and so cannot be  $R[t^{\pm 1}]$ -torsion unless it is 0. But since  $\Delta_n^{\varphi,\rho} \neq 0$  by assumption, we have that  $H_n(G; R^k[t^{\pm 1}])$  is  $R[t^{\pm 1}]$ -torsion. The result follows.  $\square$

We now wish to define *the twisted Reidemeister torsion*  $\tau_{G,R}^{\varphi,\rho}(t)$  of  $\varphi$  twisted by  $\rho$  over  $R$ . Rather than give the original definition which we will not need, we instead use the following lemma which recasts the invariant in terms of twisted Alexander polynomials as our definition. The lemma can be deduced by standard methods, for example, it is an immediate corollary of [Tur86, Lemma 2.1.1].

**Lemma 6.2.10.** *Let  $R$  be a UFD. Let  $G$  be a group of type  $F$ , let  $\varphi: G \rightarrow \mathbb{Z}$  have kernel of type  $F$ , and let  $\rho: G \rightarrow \mathrm{GL}_k(R)$  be a representation. Then,*

$$\tau_{G,R}^{\varphi,\rho}(t) \doteq \prod_{n \geq 0} (\Delta_{G,n}^{\varphi,\rho}(t))^{(-1)^{n+1}}$$

*up to monomial factors with coefficients in  $\mathrm{Frac}(R)^\times$ .*

This allows us to easily compute the Reidemeister torsion of free-by-cyclic groups.

**Proposition 6.2.11.** *Let  $R$  be a UFD. Let  $G = F_n \rtimes_{\varphi} \mathbb{Z}$  be a free-by-cyclic group and let  $\rho: G \rightarrow \mathrm{GL}_k(R)$  be a representation. Then,*

$$\tau_{G,R}^{\varphi,\rho}(t) = \frac{\Delta_{G,1}^{\varphi,\rho}(t)}{\Delta_{G,0}^{\varphi,\rho}(t)}$$

*up to monomial factors with coefficients in  $\mathrm{Frac}(R)^\times$ .*

*Proof.* This follows from Lemma 6.2.9 and Lemma 6.2.10. □

The final well known lemma is elementary.

**Lemma 6.2.12.** *Let  $R$  be a UFD. Let  $G$  be a group of type  $F$  admitting a character  $\varphi: G \rightarrow \mathbb{Z}$  which has kernel of type  $F$ . If  $\rho_1$  and  $\rho_2$  are conjugate representations of  $G$  into  $\mathrm{GL}_k(R)$ , then  $\tau_{G,R}^{\varphi,\rho_1}(t) \doteq \tau_{G,R}^{\varphi,\rho_2}(t)$ .*

### 6.3 Regularity

In this section we will introduce the definition of a  $\widehat{\mathbb{Z}}$ -regular isomorphism. We will prove that in the case where  $G$  has  $b_1(G) = 1$  every profinite isomorphism is  $\widehat{\mathbb{Z}}$ -regular and deduce some consequences.

**Definition 6.3.1** (Matrix coefficient modules). Let  $H_A$  and  $H_B$  be a pair of finitely generated  $\mathbb{Z}$ -modules. Let  $\Theta: \widehat{H}_A \rightarrow \widehat{H}_B$  be a continuous homomorphism of the profinite completions. We define the *matrix coefficient module*

$$\text{MC}(\Theta; H_A, H_B)$$

(or simply  $\text{MC}(\Theta)$  if there is no chance of confusion) for  $\Theta$  with respect to  $H_A$  and  $H_B$  to be the smallest  $\mathbb{Z}$ -submodule  $L$  of  $\widehat{\mathbb{Z}}$  such that  $\Theta(H_A)$  lies in the submodule  $H_B \otimes_{\mathbb{Z}} L$  of  $\widehat{H}_B$ . We denote by

$$\Theta^{\text{MC}}: H_A \rightarrow H_B \otimes_{\mathbb{Z}} \text{MC}(\Theta)$$

the homomorphism uniquely determined by the restriction of  $\Theta$  to  $H_A$ .

For a finitely generated group  $G$  let  $G^{\text{fab}}$  denote the free part of the abelianisation  $G^{\text{ab}}$ . That is, the quotient of the abelianisation of  $G$  by its torsion elements.

Given groups  $G_A$  and  $G_B$  and a continuous homomorphism  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$  we have an induced continuous homomorphisms  $\Theta_*: \widehat{G}_A^{\text{fab}} \rightarrow \widehat{G}_B^{\text{fab}}$  and  $\Theta^*: H^1(G_B, \mathbb{Z}) \rightarrow H^1(G_A, \mathbb{Z})$ . We define  $\text{MC}(\Theta) := \text{MC}(\Theta_*, G_A^{\text{fab}}, G_B^{\text{fab}})$ .

**Definition 6.3.2** ( $\widehat{\mathbb{Z}}$ -regular isomorphism). The isomorphism  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$  is  $\widehat{\mathbb{Z}}$ -regular, if there exists a unit  $\mu \in \widehat{\mathbb{Z}}^\times$  and an isomorphism  $\Xi: G_A^{\text{fab}} \rightarrow G_B^{\text{fab}}$  such that  $\Theta_*$  is the profinite completion of the map given by the composite

$$G_A^{\text{fab}} \xrightarrow{\Xi} G_B^{\text{fab}} \xrightarrow{\cdot \times \mu} \widehat{G}_B^{\text{fab}}. \quad (6.3)$$

We sometimes write  $\Theta_*^{1/\mu}: G_A^{\text{fab}} \rightarrow G_B^{\text{fab}}$  to denote the map  $\Xi$  in (6.3) and  $\Theta_{1/\mu}^*: H^1(G_B, \mathbb{Z}) \rightarrow H^1(G_A, \mathbb{Z})$  to denote its dual. Let  $\varphi \in H^1(G_B; \mathbb{Z})$  and  $\psi \in H^1(G_A; \mathbb{Z})$ . We say  $\psi$  is the *pullback of  $\varphi$  via  $\Theta$* , if  $\psi = \Theta_{1/\mu}^*(\varphi)$ .

We say a pair  $(G, \psi)$  is a  $\mathcal{P}$ -by- $\mathbb{Z}$  group for some group property  $\mathcal{P}$  if  $G$  admits an epimorphism  $\psi: G \rightarrow \mathbb{Z}$  such that the kernel has property  $\mathcal{P}$ .

**Proposition 6.3.3** ( $\widehat{\mathbb{Z}}$ -regularity). *Let  $G_A$  and  $G_B$  be  $\{\text{type FP}_\infty\}$ -by- $\mathbb{Z}$  groups satisfying  $b_1(G_A) = b_1(G_B) = 1$ . If  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$  is an isomorphism, then there exists a unit  $\mu \in \widehat{\mathbb{Z}}^\times$  such that  $\text{MC}(\Theta) = \mu\mathbb{Z}$ .*

*Proof.* By [Liu23a, Proposition 3.2(1)], the  $\mathbb{Z}$ -module  $\text{MC}(\Theta)$  is a non-zero finitely generated free  $\mathbb{Z}$ -module spanned by the single entry of the  $1 \times 1$  matrix  $(\mu)$  over  $\widehat{\mathbb{Z}}$ . By [Liu23a, Proposition 3.2(2)] we obtain a homomorphism  $\Xi: G_A^{\text{fab}} \rightarrow G_B^{\text{fab}}$  such that  $\Psi_* = \mu\widehat{\Xi}$ . Moreover,  $\mu$  is a unit because  $\Theta$  is an isomorphism. Hence,  $\text{MC}(\Theta_*) = \mu\mathbb{Z}$ .  $\square$

**Proposition 6.3.4** (Fibre closure isomorphisms). *Let  $(L_A, \psi)$  and  $(L_B, \varphi)$  be  $\{\text{type } \text{FP}_\infty\}$ -by- $\mathbb{Z}$  groups. Suppose  $\Theta: \widehat{L}_A \rightarrow \widehat{L}_B$  is a  $\widehat{\mathbb{Z}}$ -regular isomorphism and  $\psi$  is the pullback of  $\varphi$  via  $\Theta$  with unit  $\mu$ . If  $F_A$  is the fibre subgroup of  $L_A$ , then  $\overline{F}_A$  projects isomorphically onto  $\overline{F}_B$ , the closure of the fibre subgroup of  $L_B$ , under  $\Theta$ .*

*Proof.* Our proof essentially follows [Liu23a, Corollary 6.2]. Write  $L_A = F_A \rtimes_\psi Z_A$  and  $L_B = F_B \rtimes_\varphi Z_B$  with  $Z_A \simeq Z_B \simeq \mathbb{Z}$ . Identify,  $H_A$  with  $G_A^{\text{fab}}$  and  $H_B$  with  $G_B^{\text{fab}}$ . By hypothesis the map  $\Theta_*$  is the completion of an isomorphism  $\Theta_\mu: H_A \rightarrow H_B$  followed by multiplication by  $\mu$  in  $\widehat{H}_B = H_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ . Thus,  $\psi$  is the composite

$$H_A \xrightarrow{\Theta_\mu \otimes \mu} H_B \otimes_{\mathbb{Z}} \mu\widehat{\mathbb{Z}} \xrightarrow{1 \otimes \mu^{-1}} H_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \xrightarrow{=} H_B \xrightarrow{\varphi|_{Z_B}} \mathbb{Z}.$$

We obtain that  $\Theta_*(\ker \psi_*) = \mu F_\mu(\ker \varphi_*) = \mu \ker(\varphi_*)$  in  $\widehat{H}_B$ . Since  $\ker \varphi_*$  is a  $\mathbb{Z}$ -submodule of  $H_B$ , the closure of  $\widehat{H}_B$  is invariant under multiplication by a unit. Hence,  $\Theta_* \overline{\ker \psi_*} = \overline{\mu \ker \varphi_*} = \mu \overline{\ker \varphi_*} = \overline{\ker \varphi_*}$ . This completes the proof of the first case.  $\square$

Note that the following proposition would be trivial if the unit  $\mu$  equalled 1. However, the definition of pullback we are using (Definition 6.3.2) only assumes the existence of a unit.

**Proposition 6.3.5** (Isomorphism of fibre subgroups). *Let  $(G_A, \psi)$  and  $(G_B, \varphi)$  be free-by-cyclic groups. Suppose  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$  is a  $\widehat{\mathbb{Z}}$ -regular isomorphism. If  $\psi$  is the pullback of  $\varphi$  via  $\Theta$ , then the fibre subgroup  $F_A$  of  $G_A$  and the fibre subgroup  $F_B$  of  $G_B$  are isomorphic.*

*Proof.* We will show that the degree of the first Alexander polynomials of  $G_A$  and  $G_B$  are equal. By Lemma 6.2.8 this computes the rank of the  $\mathbb{F}_p$ -homology of

$F_A$  and  $F_B$  which determines their rank. Since  $F_A$  and  $F_B$  are free groups this determines them up to isomorphism.

Let  $\psi_n$  and  $\varphi_n$  denote the modulo  $n$  reduction of  $\psi: G_A \rightarrow \mathbb{Z}$  and  $\varphi: G_B \rightarrow \mathbb{Z}$  respectively, namely the composites

$$G_A \xrightarrow{\psi} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n \quad \text{and} \quad G_B \xrightarrow{\varphi} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n.$$

We endow  $M_{A,n} := \mathbb{F}_p[\mathbb{Z}/n]$  with the  $G_A$ -module structure given by  $\psi_n$  and  $M_{B,n} := \mathbb{F}_p[\mathbb{Z}/n]$  with the  $G_B$ -module given by  $\varphi_n$ . Since  $G_A$  and  $G_B$  are cohomologically good (Lemma 6.1.7), by [BF20, Proposition 4.2] we have isomorphisms  $H_k(G_A; M_{A,n}) \simeq H_k(G_B; M_{B,n})$  for all  $k, n \geq 0$ . In particular,  $\dim_{\mathbb{F}_p} H_k(G_A; M_{A,n}) = \dim_{\mathbb{F}_p} H_k(G_B; M_{B,n})$ . Now, by applying [BF20, Proposition 3.4] twice we get

$$\begin{aligned} \deg \Delta_{G_A,1}^{\psi,1}(t) &= \max_{n \in \mathbb{N}} \{ \dim_{\mathbb{F}_p} H_1(G_A; M_{A,n}) - \dim_{\mathbb{F}_p} H_0(G_A; M_{A,n}), \} \\ &= \max_{n \in \mathbb{N}} \{ \dim_{\mathbb{F}_p} H_1(G_B; M_{B,n}) - \dim_{\mathbb{F}_p} H_0(G_B; M_{B,n}), \} \\ &= \deg \Delta_{G_B,1}^{\varphi,1}(t). \end{aligned} \quad \square$$

## 6.4 Profinite invariance of twisted Reidemeister torsion

The goal of this section is to establish profinite invariance of twisted Reidemeister torsion (Corollary 6.4.9) for free-by-cyclic groups with first Betti number equal to one. We do this by first establishing invariance of the twisted Alexander polynomials in a more general setting. Finally, in Section 6.4.3 we establish profinite invariance of homological stretch factors.

### 6.4.1 Twisted Alexander polynomials

**Definition 6.4.1** (Corresponding quotients). Let  $G_A$  and  $G_B$  be residually finite groups. Suppose there exists an isomorphism  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$ . Let  $Q$  be a finite group. A pair of quotient maps  $\gamma_A: G_A \rightarrow Q$  and  $\gamma_B: G_B \rightarrow Q$  is said to be

$\Theta$ -corresponding, if  $\gamma_A$  is given by the composite

$$G_A \xrightarrow{i} \widehat{G}_A \xrightarrow{\Theta} \widehat{G}_B \xrightarrow{\widehat{\gamma}_B} Q \quad (6.4)$$

Here,  $i: G_A \rightarrow \widehat{G}_A$  denotes the natural inclusion and  $\widehat{\gamma}_B$  denotes the (profinite) completion of  $\gamma_B$ .

**Proposition 6.4.2** (Profinite invariance of twisted Alexander polynomials). *Let  $(G_A, \psi_A)$  and  $(G_B, \varphi_B)$  be residually finite  $\{\text{good type F}\}$ -by- $\mathbb{Z}$  groups. Let  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$  be a  $\widehat{\mathbb{Z}}$ -regular isomorphism and suppose  $\psi_A$  is the pullback of  $\varphi_B$  via  $\Theta$  with unit  $\mu$ . Let  $\psi_B \in H^1(G_B, \mathbb{Z})$  be a primitive fibred class. Let  $\psi_A \in H^1(G_A, \mathbb{Z})$  be the fibred class  $\Theta_\mu^*(\psi_B)$ . Fix a  $\Theta$ -corresponding pair of finite quotients  $\gamma_A: G_A \rightarrow Q$  and  $\gamma_B: G_B \rightarrow Q$ . Suppose  $\rho: Q \rightarrow \mathrm{GL}(k, \mathbb{Q})$  is a representation and  $\rho_A: G_A \rightarrow \mathrm{GL}(k, \mathbb{Q})$  and  $\rho_B: G_B \rightarrow \mathrm{GL}(k, \mathbb{Q})$  the pullbacks. Then,*

$$\Delta_{G_A, n}^{\psi_A, \rho_A}(t) \cdot \Delta_{G_A, n}^{\psi_A, \rho_A}(t^{-1}) \doteq \Delta_{G_B, n}^{\varphi_B, \rho_B}(t) \cdot \Delta_{G_B, n}^{\varphi_B, \rho_B}(t^{-1})$$

holds in  $\mathbb{Q}[t^{\pm 1}]$  up to monomial factors with coefficients in  $\mathbb{Q}^\times$ .

Before proving Proposition 6.4.2 we will collect a number of facts. The following criterion is due to Ueki [Uek18, Lemma 3.6].

**Theorem 6.4.3** (Ueki). *Let  $a(t), b(t) \in \mathbb{Z}[t]$  be a pair of palindromic polynomials and  $\mu \in \widehat{\mathbb{Z}}$  be a unit. If the principal ideals  $(a(t^\mu))$  and  $(b(t))$  of the completed group algebra  $\widehat{\mathbb{Z}}[[t^{\widehat{\mathbb{Z}}}]$  are equal, then  $a(t) \doteq b(t)$  holds in  $\mathbb{Z}[t^{\pm 1}]$ .*

**Definition 6.4.4** ( $\mu$ -powers). Let  $G$  be a profinite group, let  $g \in G$ , and let  $\mu \in \widehat{\mathbb{Z}}$ . We define the  $\mu$ -power of  $g$  to be  $g^\mu = \varprojlim_N g^n \pmod{N}$  where  $N$  ranges over the inverse system of open normal subgroups of  $G$  and  $n \in \mathbb{Z}$  is congruent to  $\mu$  modulo  $|G/N|$ . Note that  $hg^\mu h^{-1} = (hgh^{-1})^\mu$  for all  $h \in G$ .

The following fact is classical, for convenience we cite Liu.

**Lemma 6.4.5.** [Liu23a, Lemma 7.6] *Let  $L$  be a finite group. If  $\rho: L \rightarrow \mathrm{GL}_k(\mathbb{Q})$  is a representation, then  $\rho$  is conjugate to the representation  $\sigma_{\mathbb{Q}}$  over  $\mathbb{Q}$  given by extension of scalars of some representation  $\sigma: L \rightarrow \mathrm{GL}_k(\mathbb{Z})$ .*

**Remark 6.4.6.** Combining Lemma 6.4.5 and Lemma 6.2.3 we may assume without loss of generality that the representation  $\rho$  is equal to the extension of scalars of some integral representation  $\sigma: Q \rightarrow \mathrm{GL}_k(\mathbb{Z})$ . We denote by  $\sigma_A: G_A \rightarrow \mathrm{GL}_k(\mathbb{Z})$  the pullback  $\gamma_A^*(\sigma)$  and similarly write  $\sigma_B$  for  $\gamma_B^*(\sigma)$ .

By Proposition 6.3.4 and Proposition 4.2.3 we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
1 & \twoheadrightarrow & F_A & \twoheadrightarrow & G_A & \xrightarrow{\psi_A} \twoheadrightarrow & \mathbb{Z} \longrightarrow \twoheadrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \twoheadrightarrow & \widehat{F}_A & \twoheadrightarrow & \widehat{G}_A & \xrightarrow{\widehat{\psi}_A} \twoheadrightarrow & \widehat{\mathbb{Z}} \longrightarrow 1 \\
& & \downarrow \Theta_F & & \downarrow \Theta & & \downarrow \mu \\
1 & \twoheadrightarrow & \widehat{F}_B & \twoheadrightarrow & \widehat{G}_B & \xrightarrow{\widehat{\varphi}_B} \twoheadrightarrow & \widehat{\mathbb{Z}} \longrightarrow 1 \\
& & \uparrow & & \uparrow & & \uparrow \\
1 & \twoheadrightarrow & F_B & \twoheadrightarrow & G_B & \xrightarrow{\varphi_B} \twoheadrightarrow & \mathbb{Z} \longrightarrow 1,
\end{array} \tag{6.5}$$

where  $\Theta_F = \Theta|_{\overline{F}_A}$  and  $\Theta_F$ ,  $\Theta$ , and  $\mu$  are isomorphisms.

We now write  $G_A = F_A \rtimes \langle t_A \rangle$  with  $\psi_A(t_A) = 1$  and  $G_B = F_B \rtimes \langle t_B \rangle$  with  $\varphi_B(t_B) = 1$ . Now (6.5) implies that  $\Theta(t_A)$  is conjugate to the  $\mu$ -power  $t_B^\mu$  of  $t_B$  in  $\widehat{G}_B$ . Let  $M_A$  be  $\mathbb{Z}^k$  equipped with the  $F_A$ -module structure given by  $\sigma_A|_{F_A}$  and similarly for  $M_B$ . Note that  $\psi_A$  and  $\varphi_B$  induce automorphisms  $\Psi_A$  of  $F_A$  and  $\Phi_B$  of  $F_B$  (up to choosing an inner automorphism). Moreover,  $\Psi_A$  induces a  $\mathbb{Z}$ -linear isomorphism  $\psi_{A,n}: H_n(F_A; M_A) \rightarrow H_n(F_A; M_A)$ . We note that the choices made here for picking group automorphisms  $\Psi_A$  and  $\Phi_B$  only depend on the outer automorphism class. This is sufficient for us since these induce the same action on  $H_n(F_A; -)$  resp.  $H_n(F_B; -)$ . It follows that  $\psi_{A,n}$  only depends on  $\sigma$  and  $\psi_A$ . We obtain a commutative diagram of  $\mathbb{Z}$ -modules with exact rows

$$\begin{array}{ccccccc}
0 & \twoheadrightarrow & H_n(F_A; M_A)_{\mathrm{tors}} & \twoheadrightarrow & H_n(F_A; M_A) & \longrightarrow \twoheadrightarrow & H_n(F_A; M_A)_{\mathrm{free}} \longrightarrow \twoheadrightarrow 0 \\
& & \downarrow \psi_{A,n}^{\mathrm{tors}} & & \downarrow \psi_{A,n} & & \downarrow \psi_{A,n}^{\mathrm{free}} \\
0 & \twoheadrightarrow & H_n(F_A; M_A)_{\mathrm{tors}} & \twoheadrightarrow & H_n(F_A; M_A) & \longrightarrow \twoheadrightarrow & H_n(F_A; M_A)_{\mathrm{free}} \longrightarrow \twoheadrightarrow 0.
\end{array} \tag{6.6}$$

Note that after fixing bases we may consider  $\psi_{A,n}^{\text{free}}$  as a matrix in  $\text{GL}(H_n(F_A; M_A)_{\text{free}})$ . Define

$$P_{A,n}(t) := \det_{\mathbb{Z}[t^{\pm 1}]}(\mathbf{1} - t \cdot \psi_{A,n}^{\text{free}}) \quad (6.7)$$

and

$$P_{B,n}(t) := \det_{\mathbb{Z}[t^{\pm 1}]}(\mathbf{1} - t \cdot \varphi_{B,n}^{\text{free}}). \quad (6.8)$$

The following lemma is [Liu23a, Lemma 7.7]. The proof goes through verbatim once one assumes the kernels of  $\psi_A$  and  $\varphi_B$  are type F.

**Lemma 6.4.7.** [Liu23a, Lemma 7.7] *Adopt the notation from Proposition 6.4.2, Remark 6.4.6, (6.7), and (6.8). We have  $\Delta_{G_A,n}^{\psi_A, \rho_B}(t) \doteq P_{A,n}(t)$  and  $\Delta_{G_B,n}^{\varphi_B, \rho_B}(t) \doteq P_{B,n}(t)$  in  $\mathbb{Q}[t^{\pm 1}]$  up to monomials with coefficients in  $\mathbb{Q}^\times$ .*

The following lemma is [Liu23a, Lemma 7.8]. The proof goes through verbatim once one assumes that the kernels of  $\psi_A$  and  $\varphi_B$  are type F, that  $F_A$  and  $F_B$  are fully separable in  $G_A$  and  $G_B$  respectively (this is given by Proposition 4.2.3), and that  $F_A$  and  $F_B$  are good.

**Lemma 6.4.8.** [Liu23a, Lemma 7.8] *Adopt the notation from Proposition 6.4.2, Remark 6.4.6, (6.7), and (6.8). For all  $n$  we have an equality of principal ideals  $(P_{A,n}(t^\mu)) = (P_{B,n}(t))$  in  $\widehat{\mathbb{Z}}[[t^{\widehat{\mathbb{Z}}}]$ .*

*Proof of Proposition 6.4.2.* This follows from Lemmas 6.4.7 and 6.4.8 and Theorem 6.4.3 after observing that the polynomials  $\Delta_{G_A,n}^{\psi_A, \rho_A}(t) \cdot \Delta_{G_A,n}^{\psi_A, \rho_A}(t^{-1})$  and  $\Delta_{G_B,n}^{\varphi_A, \rho_B}(t) \cdot \Delta_{G_B,n}^{\varphi_B, \rho_B}(t^{-1})$  are palindromic by Lemma 6.2.5.  $\square$

## 6.4.2 Twisted Reidemeister torsion

We now prove profinite invariance of twisted Reidemeister torsion for free-by-cyclic groups with first Betti number equal to one.

**Corollary 6.4.9** (Profinite invariance of twisted Reidemeister torsion). *Let  $(G_A, \psi_A)$  and  $(G_B, \varphi_B)$  be free-by-cyclic groups. Let  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$  be a  $\widehat{\mathbb{Z}}$ -regular isomorphism. Let  $\varphi_B \in H^1(G_B; \mathbb{Z})$  be a primitive fibred class and suppose  $\psi_A$  is the pullback of  $\varphi_B$  via  $\Theta$ . Fix a  $\Theta$ -corresponding pair of finite quotients  $\gamma_A: G_A \rightarrow Q$*

and  $\gamma_B: G_B \rightarrow Q$ . Suppose  $\rho: Q \rightarrow \mathrm{GL}(k, \mathbb{Q})$  is a representation and  $\rho_A: G_A \rightarrow \mathrm{GL}(k, \mathbb{Q})$  and  $\rho_B: G_B \rightarrow \mathrm{GL}(k, \mathbb{Q})$  the pullbacks. Then,

$$\{\tau_{G_A}^{\psi_A, \rho_A}(t), \tau_{G_B}^{-\psi_A, \rho_A}\} = \{\tau_{G_B}^{\varphi_B, \rho_B}(t), \tau_{G_B}^{-\varphi_B, \rho_B}\}.$$

*Proof.* By Proposition 6.4.2, unique factorisation in  $\mathbb{Q}[t^{\pm 1}]$ , and Lemma 6.2.5 we obtain

$$S_{A,n} = \{\Delta_{G_A,n}^{\psi_A, \rho_A}(t), \Delta_{G_A,n}^{-\psi_A, \rho_A}(t)\} = \{\Delta_{G_B,n}^{\varphi_B, \rho_B}(t), \Delta_{G_B,n}^{-\varphi_B, \rho_B}(t)\} = S_{B,n}.$$

By Proposition 6.2.11 the relevant Alexander polynomials are concentrated in degrees 0 and 1. By Lemma 6.2.6 the sets  $S_{A,0}$  and  $S_{B,0}$  contain exactly one element up to  $\doteq$ -equivalence. Finally, the result follows from Proposition 6.2.11.  $\square$

### 6.4.3 Profinite invariance of homological stretch factors

**Theorem 6.4.10** (Profinite invariance of homological stretch factors). *Let  $(G_A, \psi)$  and  $(G_B, \varphi)$  be free-by-cyclic groups. If  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$  is a  $\widehat{\mathbb{Z}}$ -regular isomorphism and  $\psi$  is the pullback of  $\varphi$  via  $\Theta$ , then  $\{\nu_\psi^+, \nu_\psi^-\} = \{\nu_\varphi^+, \nu_\varphi^-\}$ .*

*Proof.* Denote the non-trivial primitive characters of  $G_A$  by  $\psi_A^\pm$  and the non-trivial primitive characters of  $G_B$  by  $\varphi_B^\pm$ . By Proposition 6.4.2 we have

$$\Delta_{G_A,1}^{\psi_A^+, \mathbf{1}}(t) \cdot \Delta_{G_A,1}^{\psi_A^-, \mathbf{1}}(t) \doteq \Delta_{G_B,1}^{\varphi_B^+, \mathbf{1}}(t) \cdot \Delta_{G_B,1}^{\varphi_B^-, \mathbf{1}}(t)$$

over  $\mathbb{Q}[t^{\pm 1}]$ . Normalise the polynomials so that every term is a non-negative power of  $t$  and the lowest term is 1, and note that each of the four terms has the same degree. Now, by unique factorisation in  $\mathbb{Q}[t^{\pm 1}]$  we obtain the equality of sets

$$S_A = \{\Delta_{G_A,1}^{\psi_A^+, \mathbf{1}}(t), \Delta_{G_A,1}^{\psi_A^-, \mathbf{1}}(t)\} = \{\Delta_{G_B,1}^{\varphi_B^+, \mathbf{1}}(t), \Delta_{G_B,1}^{\varphi_B^-, \mathbf{1}}(t)\} = S_B.$$

Now, since we are working over  $\mathbb{Q}$  the set  $S_A$  [resp.  $S_B$ ] is the set of characteristic polynomials for  $(\psi_A^\pm)_1$  [resp.  $(\varphi_B^\pm)_1$ ], that is, the set of characteristic polynomials for the induced maps on degree 1 homology of the respective fibres. In particular,

the sets

$$\{\nu_\psi^+, \nu_\psi^-\} \text{ and } \{\nu_\varphi^+, \nu_\varphi^-\}$$

can be computed by taking the modulus of the largest root of the Alexander polynomials in  $S_A$  and  $S_B$ . The desired equality follows.  $\square$

## 6.5 Profinite invariance of Nielsen numbers

Let  $X$  be a connected, compact topological space that is homeomorphic to a finite-dimensional cellular complex, with a finite number of cells in each dimension, and let  $f: X \rightarrow X$  be a self-map. Recall from Section 6.1.2 the definitions of the fixed point index  $\text{ind}_m(f; \mathcal{O})$  of  $f^m$  at any point  $p \in \mathcal{O}$ , and the  $m$ -th Nielsen number  $N_m(f)$  of  $f$ .

We will write  $M_f$  to denote the mapping torus

$$M_f = \frac{X \times [0, 1]}{(f(x), 0) \sim (x, 1)}.$$

Let  $x_0 \in X$  and fix a path  $\alpha$  from  $f(x_0)$  to  $x_0$  in  $X$ . We identify  $X$  with the fibre  $X \times \{0\}$  in  $M_f$  and write  $\bar{x}_0$  denote the image of  $x_0$  in  $M_f$ . We define  $t \in \pi_1(M_f, \bar{x}_0)$  to be the loop obtained by concatenation of paths  $\eta \cdot \alpha$ , where  $\eta_s = (x_0, s)$  for  $s \in [0, 1]$ . The *induced character*  $\varphi: \pi_1(M_f) \rightarrow \mathbb{Z}$  maps every loop in  $X$  based at  $x_0$  to zero, and  $\varphi(t) = 1$ .

Let  $\zeta: \pi_1(M_f) \rightarrow \mathbb{Q}$  be any map that is constant on conjugacy classes. Then the  $m$ -th *twisted Lefschetz number* of  $f$  with respect to  $\zeta$  is

$$L_m(f; \zeta) = \sum_{\mathcal{O} \in \text{Orb}_m(f)} \zeta(\text{cd}(\mathcal{O})) \cdot \text{ind}_m(f; \mathcal{O}). \quad (6.9)$$

For a finite-dimensional representation  $\rho: \pi_1(M_f) \rightarrow \text{GL}(k, R)$  of  $\pi_1(M_f)$ , let  $\chi_\rho: \pi_1(M_f) \rightarrow R$  denote the trace map. We write  $\exp(\cdot)$  to denote the formal power series,

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

**Theorem 6.5.1** ([Jia96], [Liu23a, Lemma 8.2]). *Let  $\varphi: \pi_1(M_f) \rightarrow \mathbb{Z}$  denote the induced character. Suppose that  $\mathbb{F}$  is a commutative field of characteristic 0 and  $\rho: \pi_1(M_f) \rightarrow \mathrm{GL}(k, \mathbb{F})$  a finite-dimensional linear representation of  $\pi_1(M_f)$ . Then*

$$\tau_{\pi_1(M_f), \mathbb{F}[t^{\pm 1}]^k}^{\rho, \varphi} \doteq \exp \sum_{m \geq 1} L_m(f; \chi_\rho) \frac{t^m}{m},$$

where the equality holds as rational functions in  $t$  over  $\mathbb{F}$ , up to multiplication by monomial factors with coefficients in  $\mathbb{F}^\times$ .

Let  $Q$  be a finite group. We say two elements  $g_1$  and  $g_2$  in  $Q$  are  $\widehat{\mathbb{Z}}$ -conjugate if the cyclic groups  $\langle g_1 \rangle$  and  $\langle g_2 \rangle$  are conjugate in  $Q$  (note that this is equivalent to the notion of  $\widehat{\mathbb{Z}}$ -conjugacy defined in [Liu23a]). This gives rise to an equivalence relation on the set  $\mathrm{Orb}(Q)$  of conjugacy classes of  $Q$ . We write  $\Omega(Q)$  to denote the resulting set of equivalence classes. For  $\omega \in \Omega(Q)$ , we let  $\chi_\omega: \mathrm{Orb}(Q) \rightarrow \mathbb{Q}$  denote the characteristic function of  $\omega$ .

**Lemma 6.5.2** ([Liu23a, Lemma 8.5]). *Fix  $m \in \mathbb{N}$ . Let  $\gamma: \pi_1(M_f) \rightarrow Q$  be a quotient of  $\pi_1(M_f)$  onto a finite group  $Q$ . Then,*

$$N_m(f) \geq \#\{\omega \in \Omega(Q) \mid L_m(f; \gamma^* \chi_\omega) \neq 0\}.$$

Note that by (8.1), for every  $\omega \in \Omega(Q)$  such that  $L_m(f; \gamma^* \chi_\omega) \neq 0$ , there exists some  $\mathcal{O} \in \mathrm{Orb}_m(f)$  such that  $\mathrm{ind}_m(f, \mathcal{O}) \neq 0$  and

$$\begin{aligned} \gamma^* \chi_\omega(\mathrm{cd}(\mathcal{O})) &= \chi_\omega \circ \gamma(\mathrm{cd}(\mathcal{O})) \\ &\neq 0, \end{aligned}$$

which holds if and only if  $\gamma(\mathrm{cd}(\mathcal{O})) \in \omega$ . Hence the number of such elements in  $\Omega(Q)$  is bounded above by the number of essential  $m$ -periodic orbits of  $f$ , which is exactly  $N_m(f)$ .

The following lemma is a strengthening of Lemma 8.6 in [Liu23a], however the proof follows from Liu's proof with only a slight modification. We provide a sketch for the convenience of the reader.

**Lemma 6.5.3.** *Suppose that  $\pi_1(M_f)$  is conjugacy separable. Then, for any  $m \in \mathbb{N}$  there exists a finite quotient  $Q_m$  of  $\pi_1(M_f)$  such that*

$$\begin{aligned} N_m(f) &= \{\omega \in \Omega(Q_m) \mid L_m(f; \gamma^* \chi_\omega) \neq 0\}, \text{ and} \\ N_m(f^{-1}) &= \{\omega \in \Omega(Q_m) \mid L_m(f^{-1}; \gamma^* \chi_\omega) \neq 0\}. \end{aligned} \tag{6.10}$$

*Proof.* Let  $G = \pi_1(M_f)$  and write  $\varphi: G \rightarrow \mathbb{Z}$  to denote the induced character,  $t \in G$  the stable letter and  $K = \ker \varphi$  the fibre subgroup as before. Since  $G$  is conjugacy separable, for each  $m \geq 1$  there exists a finite quotient  $\tilde{\pi}_m: G \rightarrow \tilde{Q}_m$ , such that for all  $m$ -periodic orbits of  $f$  and  $f^{-1}$ , the corresponding distinct conjugacy classes in  $G$  are mapped to distinct conjugacy classes in  $\tilde{Q}_m$ .

By the discussion directly following the statement of Lemma 6.5.2, the inequality provided by Lemma 6.5.2 is achieved when the conjugacy classes corresponding to the essential  $m$ -periodic orbits of  $f$  are mapped to distinct  $\widehat{\mathbb{Z}}$ -conjugacy classes in the finite quotient. Hence, it suffices to find a finite quotient  $\pi_m: G \rightarrow Q_m$  such that  $\tilde{\pi}_m$  factors through  $\pi_m$ , and which satisfies the following property. If  $x_1$  and  $x_2$  are two elements of  $G$  which correspond to  $m$ -periodic orbits of  $f$ , or of  $f^{-1}$ , and if  $\langle \pi_m(x_1) \rangle$  and  $\langle \pi_m(x_2) \rangle$  are conjugate in  $Q_m$ , then in fact the elements  $\pi_m(x_1)$  and  $\pi_m(x_2)$  are conjugate in  $Q_m$ . This will then imply that  $\tilde{\pi}_m(x_1)$  and  $\tilde{\pi}_m(x_2)$  are conjugate in  $\tilde{Q}_m$ , since  $\tilde{\pi}_m$  factors through  $\pi_m$ . Hence  $x_1$  and  $x_2$  are conjugate in  $G$ , showing that the required property holds for  $\pi_m$ .

To construct  $Q_m$  note that the  $m$ -periodic orbits of  $f$  correspond to elements in the coset  $Kt^m$  of  $G$ , and the  $m$ -periodic orbits of  $f^{-1}$  to the elements in the coset  $Kt^{-m}$ . If  $\bar{K}$  and  $\bar{t}$  are the images of  $K$  and  $t$  in a finite quotient of  $G$ , then the coset  $\bar{K}\bar{t}^m$  is invariant under conjugation by elements in the quotient group. Hence, it suffices to find  $Q_m$  such that that the cyclic subgroups generated by  $\bar{x}_1$  and  $\bar{x}_2$ , for any  $x_1, x_2 \in Kt^m$ , intersect  $\bar{K}\bar{t}^m$  exactly at  $\bar{x}_1$  and  $\bar{x}_2$ , respectively. It will then follow that if  $\langle \bar{x}_1 \rangle$  and  $\langle \bar{x}_2 \rangle$  are conjugate, then  $\bar{x}_1$  and  $\bar{x}_2$  are conjugate. The details of this construction are spelled out in the proof of Lemma 8.6 in [Liu23a].  $\square$

We will also need the following proposition from representation theory of finite groups (see e.g. [Ser77, Section 12.4]). We refer the reader to [Liu23a, Lemma 8.4] for the proof of this result rephrased in the language of  $\widehat{\mathbb{Z}}$ -conjugacy classes.

**Proposition 6.5.4.** *Let  $K$  be a finite group. The set of irreducible finite-dimensional characters of  $K$  over  $\mathbb{Q}$  forms a basis for the space of maps  $\text{Orb}(K) \rightarrow \mathbb{Q}$  which are constant on  $\widehat{\mathbb{Z}}$ -conjugacy classes of  $K$ .*

Let  $X_A$  and  $X_B$  be topological spaces as before, with self-maps  $f_A: X_A \rightarrow X_A$  and  $f_B: X_B \rightarrow X_B$ . We write  $G_A = \pi_1(M_{f_A})$  and  $G_B = \pi_1(M_{f_B})$ , and let  $\psi_A: G_A \rightarrow \mathbb{Z}$  and  $\varphi_B: G_B \rightarrow \mathbb{Z}$  be the induced characters.

**Lemma 6.5.5.** *Suppose that  $G_A$  and  $G_B$  are conjugacy separable. Let  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$  be an isomorphism such that for any finite group  $Q$ , some  $\Theta$ -corresponding pair of quotients  $\gamma_B: G_B \twoheadrightarrow Q$  and  $\gamma_A: G_A \twoheadrightarrow Q$  (see Definition 6.4.1), and some representation  $\rho: Q \rightarrow \text{GL}(k, \mathbb{Q})$ , we have*

$$\{\tau_{G_A}^{\psi_A \cdot \rho \gamma_A}, \tau_{G_B}^{-\psi_A \cdot \rho \gamma_A}\} = \{\tau_{G_B}^{\varphi_B \cdot \rho \gamma_B}, \tau_{G_B}^{-\varphi_B \cdot \rho \gamma_B}\}.$$

Then, for every  $m \in \mathbb{N}$ ,

$$\{N_m(f_A), N_m(f_A^{-1})\} = \{N_m(f_B), N_m(f_B^{-1})\}.$$

*Proof.* Let  $m \in \mathbb{N}$ . Invoke Lemma 6.5.3 to obtain a finite quotient  $\gamma_B: G_B \rightarrow Q_m$  such that

$$N_m(f_B^\pm) = \#\{\omega \in \Omega(Q_m) \mid L_m(f_B^\pm; \gamma_B^* \chi_\omega) \neq 0\}.$$

By Proposition 6.5.4, for every  $\omega \in \Omega(Q_m)$ ,  $\chi_\omega$  can be expressed uniquely as a  $\mathbb{Q}$ -linear combination  $\chi_\omega = \sum_i \lambda_i \chi_{\rho_i}$ , where each  $\rho_i: Q_m \rightarrow \text{GL}(k_i, \mathbb{Q})$  is an irreducible representation, and  $\lambda_i \in \mathbb{Q}$ . Hence

$$L_m(f_B; \gamma_B^* \chi_\omega) = \sum_i \lambda_i L_m(f_B; \gamma_B^* \chi_{\rho_i}).$$

Let  $\gamma_A$  be the map obtained by composing

$$G_A \xrightarrow{\iota} \widehat{G}_A \xrightarrow{\widehat{\gamma}_B} Q,$$

where  $\iota: G_A \rightarrow \widehat{G}_A$  is the natural inclusion. In particular,  $\gamma_A$  and  $\gamma_B$  are  $\Theta$ -corresponding, and thus by our assumption, for every representation  $\rho_i: Q_m \rightarrow$

$\mathrm{GL}(k_i, \mathbb{Q})$  we have that

$$\{\tau_{G_A}^{\psi_A, \rho_i \gamma_A}, \tau_{G_A}^{-\psi_A, \rho_i \gamma_A}\} = \{\tau_{G_B}^{\varphi_B, \rho_i \gamma_B}, \tau_{G_B}^{-\varphi_B, \rho_i \gamma_B}\}.$$

By Theorem 6.5.1 it follows that, up to multiplication by monomials in  $t$ ,

$$\tau_{G_A}^{\psi_A, \rho_i \gamma_A}(t) \doteq 1 + L_1(f_A; \gamma_A^* \chi_\omega) t + \sum_{i=2}^{\infty} a_i t^i,$$

where for every  $i \geq 2$ , the coefficient  $a_i$  is of the form

$$a_i = \frac{1}{i} L_i(f_A; \gamma_A^* \chi_\omega) + C_i,$$

with  $C_i$  a constant term obtained from the numbers  $L_k(f_A; \gamma_A^* \chi_\omega)$ ,  $k < i$ . Similarly,

$$\tau_{G_A}^{-\psi_A, \rho_i \gamma_A}(t) \doteq 1 + L_1(f_A^{-1}; \gamma_A^* \chi_\omega) t + \sum_{i=2}^{\infty} b_i t^i,$$

$$b_i = \frac{1}{i} L_i(f_A^{-1}; \gamma_A^* \chi_\omega) + D_i,$$

and each  $D_i$  is a constant term which only depends on the numbers  $L_k(f_A^{-1}; \gamma_A^* \chi_\omega)$ ,  $k < i$ . Note that the coefficients  $a_i$  and  $b_j$  are non-zero for only finitely many values of  $i$  and  $j$ . Furthermore, the analogous equalities hold true for  $\tau_{G_B}^{\varphi_B, \rho_i \gamma_B}$  and  $\tau_{G_B}^{-\varphi_B, \rho_i \gamma_B}$ .

Hence, by comparing the coefficients of the powers of  $t$  in the expansions of the Redemeister torsions, it follows that for each  $\rho_i$ ,

$$\{L_m(f_B; \gamma_B^* \chi_{\rho_i}), L_m(f_B^{-1}; \gamma_B^* \chi_{\rho_i})\} = \{L_m(f_A; \gamma_A^* \chi_{\rho_i}), L_m(f_A^{-1}; \gamma_A^* \chi_{\rho_i})\}.$$

Thus,

$$L_m(f_B; \gamma_B^* \chi_\omega) + L_m(f_B^{-1}; \gamma_B^* \chi_\omega) = L_m(f_A; \gamma_A^* \chi_\omega) + L_m(f_A^{-1}; \gamma_A^* \chi_\omega), \text{ and}$$

$$L_m(f_B; \gamma_B^* \chi_\omega) L_m(f_B^{-1}; \gamma_B^* \chi_\omega) = L_m(f_A; \gamma_A^* \chi_\omega) L_m(f_A^{-1}; \gamma_A^* \chi_\omega).$$

Solving the above equations, we obtain

$$\{L_m(f_B; \gamma_B^* \chi_\omega), L_m(f_B; \gamma_B^* \chi_\omega)\} = \{L_m(f_A; \gamma_A^* \chi_\omega), L_m(f_A^{-1}; \gamma_A^* \chi_\omega)\}.$$

Now,

$$\begin{aligned} N_m(f_B) + N_m(f_B^{-1}) &= \#\{\omega \in \Omega(Q_m) : L_m(f_B, \gamma_B^* \chi_\omega) \neq 0\} \\ &\quad + \#\{\omega \in \Omega(Q_m) : L_m(f_B^{-1}, \gamma_B^* \chi_\omega) \neq 0\} \\ &= \#\{\omega \in \Omega(Q_m) : L_m(f_A, \gamma_A^* \chi_\omega) \neq 0\} \\ &\quad + \#\{\omega \in \Omega(Q_m) : L_m(f_A^{-1}, \gamma_A^* \chi_\omega) \neq 0\} \\ &\leq N_m(f_A) + N_m(f_A^{-1}), \end{aligned}$$

where the last inequality follows from Lemma 6.5.2. The same argument shows that  $N_m(f_A) + N_m(f_A^{-1}) \leq N_m(f_B) + N_m(f_B^{-1})$ . Hence  $N_m(f_A) + N_m(f_A^{-1}) = N_m(f_B) + N_m(f_B^{-1})$ . Similarly, we get that  $N_m(f_B) \cdot N_m(f_B^{-1}) = N_m(f_A) \cdot N_m(f_A^{-1})$ . It follows that

$$\{N_m(f_A), N_m(f_A^{-1})\} = \{N_m(f_B), N_m(f_B^{-1})\}. \quad \square$$

Combining Corollary 6.4.9 with Lemma 6.5.5 and Proposition 6.1.5, we obtain the following theorem.

**Theorem 6.5.6** (Profinite invariance of Nielsen numbers and stretch factors). *Let  $G_A$  and  $G_B$  be conjugacy separable free-by-cyclic groups with a  $\widehat{\mathbb{Z}}$ -regular isomorphism  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$ . Let  $\varphi_B \in H^1(G_B, \mathbb{Z})$  be primitive and fibred, and let  $\psi_A \in H^1(G_A, \mathbb{Z})$  be the primitive fibred class which is the pullback of  $\varphi_B$  via  $\Theta$ . Let  $(f_A^\pm, \Gamma_A)$  and  $(f_B^\pm, \Gamma_B)$  be the corresponding relative train track representatives with stretch factors  $\lambda_{f_A^\pm}$  and  $\lambda_{f_B^\pm}$ , respectively. Then, for all  $m \in \mathbb{N}$ ,*

$$\begin{aligned} \{N_m(f_A), N_m(f_A^{-1})\} &= \{N_m(f_B), N_m(f_B^{-1})\}, \text{ and} \\ \{\lambda_{f_A}, \lambda_{f_A^{-1}}\} &= \{\lambda_{f_B}, \lambda_{f_B^{-1}}\}. \end{aligned}$$

We now have everything we need to prove Theorem 6.5.7.

**Theorem 6.5.7.** *Let  $G = F \rtimes_{\Phi} \mathbb{Z}$  be a free-by-cyclic group with induced character  $\varphi: G \rightarrow \mathbb{Z}$ . If  $b_1(G) = 1$  then following properties are determined by the profinite completion  $\widehat{G}$  of  $G$ :*

1. *the rank of  $F$ ;*
2. *the homological stretch factors  $\{\nu_G^+, \nu_G^-\}$ ;*
3. *the characteristic polynomials  $\{\text{Char } \Phi^+, \text{Char } \Phi^-\}$  of the action of  $\Phi$  on  $H_1(F; \mathbb{Q})$ ;*
4. *for each representation  $\rho: G \rightarrow \text{GL}(n, \mathbb{Q})$  factoring through a finite quotient, the twisted Alexander polynomials  $\{\Delta_n^{\varphi, \rho}, \Delta_n^{-\varphi, \rho}\}$  and the twisted Reidemeister torsions  $\{\tau^{\varphi, \rho}, \tau^{-\varphi, \rho}\}$  over  $\mathbb{Q}$ .*

*Moreover, if  $G$  is conjugacy separable, (e.g. if  $G$  is hyperbolic), then  $\widehat{G}$  also determines the Nielsen numbers and the homotopical stretch factors  $\{\lambda_G^+, \lambda_G^-\}$ .*

*Proof.* Let  $G_A$  be a free-by-cyclic group and let  $G_B := G$ . Suppose that there exists a profinite isomorphism  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$ . Then the first Betti number of  $G_A$  is  $b_1(G_A) = 1$  and hence by Proposition 6.3.3, the profinite isomorphism  $\Theta$  is  $\widehat{\mathbb{Z}}$ -regular.

Let  $\psi: G_A \rightarrow \mathbb{Z}$  be a surjective character. It follows that  $\psi$  is the pullback of  $\varphi$  or  $\varphi^{-1}$  via  $\Theta$ . With this setup we have that Item 1 is given by Proposition 6.3.5; Item 2 is given by Theorem 6.4.10; Item 3 follows from (4) and the fact that we can identify  $\text{Char } \Phi^{\pm}$  with  $\Delta_1^{\pm\varphi, 1}$ ; Item 4 is given by Proposition 6.4.2. The final statement follows by Theorem 6.5.6.  $\square$

## 6.6 Almost profinite rigidity for free-by-cyclic groups

The aim of this section is to prove Theorem 6.6.4. We reproduce the statement below. Before we prove the theorem we collect some facts.

**Lemma 6.6.1.** *Let  $G_A$  and  $G_B$  be free-by-cyclic groups with finite and infinite order monodromies respectively. Then,  $\widehat{G}_A$  is not isomorphic to  $\widehat{G}_B$ .*

*Proof.* Suppose for contradiction that such an isomorphism exists. Note that since the monodromy of  $G_B$  has infinite order, the center  $Z(G_B)$  of  $G_B$  is trivial. Let  $G_A = F_m \rtimes_{\phi} \mathbb{Z}$  where  $\phi$  represents a finite order outer automorphism. Clearly  $m \geq 2$ , otherwise  $G_A$  is virtually abelian and  $G_B$  is a virtually abelian free-by-cyclic group, which contradicts the fact that  $G_B$  has trivial center.

Let  $G'_A \leq G_A$  be a finite-index subgroup of  $G_A$  so that  $G'_A \simeq F_m \times \mathbb{Z}$ . Then  $\widehat{G}'_A \simeq \widehat{F}_m \times \widehat{\mathbb{Z}}$ . Let  $H$  be the image of  $\widehat{G}'_A$  under the isomorphism  $\widehat{G}_A \simeq \widehat{G}_B$ . Then,  $H \simeq \widehat{G}'_B \simeq \widehat{G}'_B$ , for some finite-index subgroup  $G'_B \leq G_B$ . Since  $Z(G'_B) = \{1\}$  we have  $Z(\widehat{G}'_B)/\overline{Z(G'_B)} = Z(\widehat{G}'_A) \simeq \widehat{\mathbb{Z}}$ . By [Lüc94, Theorem 7.2] we have  $b_1^{(2)}(G'_B) = b_1^{(2)}(G'_A) = b_1^{(2)}(F_m \times \mathbb{Z}) = 0$ , where  $b_1^{(2)}$  denotes the first  $\ell^2$ -Betti number. It follows that the dense projection  $\pi$  of  $G'_B$  to  $\widehat{F}_m \leq \widehat{G}'_A$  is not injective. Indeed, otherwise, by [BCR16, Corollary 3.3], we have

$$0 = b_1^{(2)}(G'_B) \geq b_1^{(2)}(F_m) = m - 1 \geq 1,$$

which is a contradiction. It follows that  $G'_B$  intersects  $\ker \pi \leq Z(\widehat{G}'_B)$  non-trivially. But then,  $Z(G'_B) \neq \{1\}$  contradicting our original hypothesis.  $\square$

**Proposition 6.6.2.** *Let  $G$  be a free-by-cyclic group with finite order monodromy and  $b_1(G) = 1$ . Then,  $G$  is almost profinitely rigid amongst free-by-cyclic groups and every free-by-cyclic group in the profinite genus of  $G$  has finite order monodromy.*

*Proof.* Let  $G_A$  be a free-by-cyclic group with finite order monodromy and first Betti number equal to one, and suppose  $G_B$  is a free-by-cyclic group profinitely isomorphic to  $G_A$ . By Lemma 6.6.1 we may assume  $G_B$  has finite order monodromy. Note  $b_1(G_B) = 1$ . Now, Theorem 6.5.7(1) implies that the (uniquely defined) fibre subgroups of  $G_A$  and  $G_B$  have the same rank — say  $n$ . Since, by [CV86],  $\text{Out}(F_n)$  has only finitely many conjugacy classes of torsion subgroups, there are only finitely many possibilities for the isomorphism type of  $G_B$ .  $\square$

Recall, an outer automorphism  $\Phi \in \text{Out}(F_n)$  is said to be *atoroidal* if there does not exist a non-trivial element  $x \in F_n$  and  $n \geq 1$  such that  $\Phi^n$  preserves the conjugacy class of  $x$ .

The following proposition is a folklore result which can be traced back to the work of Bestvina–Handel, who proved it for fully irreducible elements of  $\text{Out}(F_n)$  [BH92, Theorem 4.1]. A careful proof in the more general setting of expanding free group endomorphisms can be found in the paper of Mutanguha [Mut21, Theorem A.4].

**Proposition 6.6.3.** *Let  $\Phi \in \text{Out}(F_n)$  be an outer automorphism of  $F_n$ . Suppose that  $\Phi$  is infinite-order irreducible and not atoroidal. Then  $\Phi$  is induced by a pseudo-Anosov homeomorphism of a once-punctured surface.*

**Theorem 6.6.4.** *Let  $G$  be an irreducible free-by-cyclic group. If  $b_1(G) = 1$ , then  $G$  is almost profinitely rigid amongst irreducible free-by-cyclic groups.*

*Proof.* Let  $G_A$  be a free-by-cyclic group with  $b_1(G_A) = 1$  and irreducible monodromy  $\Phi$ . Let  $G_B$  be another free-by-cyclic group with irreducible monodromy  $\Psi$  and suppose that  $\widehat{G}_A \simeq \widehat{G}_B$ . If the monodromy  $\Psi$  has finite order, then we are done by Proposition 6.6.2.

Assume  $\Psi$  has infinite order. Note that by Theorem 6.1.6,  $\Phi$  is atoroidal if and only if  $\Psi$  is atoroidal.

If  $\Psi$  is not atoroidal, then by Proposition 6.6.3, both  $\Phi$  and  $\Psi$  are induced by pseudo-Anosov homeomorphisms of compact surfaces. Thus,  $G_A$  and  $G_B$  are fundamental groups of compact hyperbolic 3-manifolds and the result holds by [Liu23a, Theorem 9.1].

Finally, suppose that  $\Phi$  is atoroidal. Hence  $G_A$  and  $G_B$  are Gromov hyperbolic free-by-cyclic groups. By [HW15],  $G_A$  and  $G_B$  are virtually compact special, and thus by [Min06] they are conjugacy separable. Furthermore,  $b_1(G_B) = 1$  since Betti numbers are invariants of profinite completions. Thus by Proposition 6.3.3, the isomorphism  $\widehat{G}_A \rightarrow \widehat{G}_B$  is  $\widehat{\mathbb{Z}}$ -regular. Hence by Theorem 6.5.6, the sets of stretch factors  $\{\lambda_\Phi, \lambda_{\Phi^{-1}}\}$  of  $\Phi^{\pm 1}$  and  $\{\lambda_\Psi, \lambda_{\Psi^{-1}}\}$  of  $\Psi^{\pm 1}$  are equal. The result now follows from Lemma 3.2.2.  $\square$

## 6.6.1 Applications

We conclude this section with the applications of Theorem 6.6.4, Theorem 6.5.7 and Theorem 6.1.6.

**Corollary 6.6.5.** *Let  $G$  be a super irreducible free-by-cyclic group. Then, every free-by-cyclic group profinitely isomorphic to  $G$  is super irreducible. In particular,  $G$  is almost profinitely rigid amongst free-by-cyclic groups.*

*Proof.* Let  $H$  be a free-by-cyclic group and suppose  $\widehat{H} \simeq \widehat{G}$ . As explained in [GS91, Section 2]  $G$  being super irreducible is a property of the characteristic polynomial of the matrix  $M: H_1(F_n; \mathbb{Q}) \rightarrow H_1(F_n; \mathbb{Q})$  representing the action of  $\Phi$  on  $H_1(F_n; \mathbb{Q})$ . Thus, by Theorem 6.5.7 we see  $H$  is super irreducible. The result follows from Theorem 6.6.4.  $\square$

**Corollary 6.6.6.** *Let  $G$  be a random free-by-cyclic group. Then, asymptotically almost surely  $G$  is almost profinitely rigid amongst free-by-cyclic groups.*

*Proof.* By Proposition 6.1.3, every generic free-by-cyclic group  $G$  is super irreducible and has  $b_1(G) = 1$ . The result follows from Corollary 6.6.5.  $\square$

**Corollary 6.6.7.** *Let  $G = F_3 \rtimes \mathbb{Z}$ . If  $G$  is hyperbolic and  $b_1(G) = 1$ , then  $G$  is almost profinitely rigid amongst free-by-cyclic groups.*

*Proof.* We first prove  $G$  is irreducible. Suppose that this is not the case. Then  $G$  has a subgroup isomorphic to either  $\mathbb{Z} \times \mathbb{Z}$  or  $F_2 \rtimes \mathbb{Z}$ . But both possibilities would imply  $G$  contains a  $\mathbb{Z}^2$  subgroup contradicting hyperbolicity. Now let  $H$  be a free-by-cyclic group and suppose that  $\widehat{H} \simeq \widehat{G}$ . By Theorem 6.1.6 we see  $H$  is hyperbolic and by Theorem 6.5.7 we see that  $H$  splits as  $F_3 \rtimes \mathbb{Z}$ . Thus, the previous paragraph implies  $H$  is irreducible. The result follows from Theorem 6.6.4.  $\square$

**Corollary 6.6.8.** *Let  $G = F_2 \rtimes \mathbb{Z}$ . If  $b_1(G) = 1$ , then  $G$  is profinitely rigid amongst free-by-cyclic groups.*

*Proof.* Let  $H$  be a free-by-cyclic group and suppose  $\widehat{H} \simeq \widehat{G}$ . By Theorem 6.5.7 we see that  $H \simeq F_2 \rtimes \mathbb{Z}$ . But each  $F_2 \rtimes \mathbb{Z}$  is profinitely rigid amongst groups of the form  $F_2 \rtimes \mathbb{Z}$  by [BRW17].  $\square$

**Remark 6.6.9.** In fact, Theorems A - C apply within a wider class of groups than stated in the hypothesis; namely, we can consider the class of mapping tori of (possibly infinite rank) free group automorphisms (imposing irreducibility if the fibre is finitely generated). The key point is that by [FH99] any finitely generated group  $G$  in this class is finitely presented and has  $\chi(G) \leq 0$  with equality if and only if the fibre subgroup is finitely generated. Now,  $\chi(G) < 0$  if and only if  $b_1^{(2)}(G) > 0$  by [Lüc02, Theorem 6.80], but the first  $\ell^2$ -Betti number is a profinite invariant amongst finitely presented groups [BCR16, Corollary 3.3]. It follows no {infinitely generated free}-by-cyclic group  $G$  is profinitely isomorphic to a {finitely generated free}-by-cyclic group.

## 6.7 Profinite conjugacy in $\text{Out}(F_n)$

In this section we show that the stretch factors of atoroidal elements of  $\text{Out}(F_n)$  are profinite conjugacy invariants.

**Definition 6.7.1** (Profinutely conjugate). Let  $\Psi, \Phi \in \text{Out}(F_n)$ . We say  $\Psi$  and  $\Phi$  are *profinutely conjugate* if they induce a pair of conjugate outer automorphisms in  $\text{Out}(\widehat{F}_n)$ .

**Theorem 6.7.2.** *Let  $\Psi \in \text{Out}(F_n)$  be atoroidal. If  $\Phi \in \text{Out}(F_n)$  is profinitely conjugate to  $\Psi$ , then  $\Phi$  is atoroidal and  $\{\lambda_\Psi, \lambda_{\Psi^{-1}}\} = \{\lambda_\Phi, \lambda_{\Phi^{-1}}\}$ . In particular, if  $\Psi$  is additionally irreducible, then there are only finitely many  $\text{Out}(F_n)$ -conjugacy classes of irreducible automorphisms which are conjugate with  $\Psi$  in  $\text{Out}(\widehat{F}_n)$*

*Proof.* The first result follows from applying Theorem 6.1.6, Theorem 6.5.7, and Proposition 6.7.4, the latter of which is proved below. The “in particular” then follows from Lemma 3.2.2.  $\square$

**Definition 6.7.3** (Aligned isomorphism). Let  $\Psi, \Phi \in \text{Out}(F_n)$ . Write  $G_A = F_n \rtimes_\Psi \mathbb{Z}$  and  $G_B = F_n \rtimes_\Phi \mathbb{Z}$  and let  $\psi: G_A \rightarrow \mathbb{Z}$  and  $\psi: G_B \rightarrow \mathbb{Z}$  be the induced

characters. We say that an isomorphism  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$  is *aligned* if the following diagram commutes

$$\begin{array}{ccc} \widehat{G}_A & \xrightarrow{\widehat{\psi}} & \widehat{\mathbb{Z}} \\ \downarrow \Theta & & \downarrow \text{id} \\ \widehat{G}_B & \xrightarrow{\widehat{\varphi}} & \widehat{\mathbb{Z}}. \end{array}$$

Note that an aligned isomorphism realises  $\psi$  as the pullback of  $\varphi$  with respect to  $\Theta$  with unit 1 in the sense that  $\Theta_*(\varphi) = \psi$ .

The following proposition follows [Liu23b, Proposition 3.7].

**Proposition 6.7.4.** *Let  $\Phi, \Psi \in \text{Out}(F_n)$ . The following are equivalent:*

1. *the profinite completions of the free-by-cyclic groups  $G_A = F_n \rtimes_{\Psi} \mathbb{Z}$  and  $G_B = F_n \rtimes_{\Phi} \mathbb{Z}$  are aligned isomorphic;*
2. *the outer automorphisms  $\Phi$  and  $\Psi$  are profinitely conjugate.*

*Proof.* In constructing  $G_A$  and  $G_B$  we have implicitly picked lifts of  $\Phi$  and  $\Psi$  to  $\text{Aut}(F_n)$  which abusing notation we have also denoted by  $\Phi$  and  $\Psi$ . Write  $G_A = F_n \rtimes_{\Psi} \langle t_A \rangle$  and  $G_B = F_n \rtimes_{\Phi} \langle t_B \rangle$ . Denote the images of  $t_A$  and  $t_B$  in  $\text{Out}(F_n)$  by  $\tau_A$  and  $\tau_B$ . Note  $\widehat{G}_A = \widehat{F}_n \rtimes \widehat{\langle t_A \rangle}$  and similarly for  $G_B$ . Denote the images of  $\tau_A$  and  $\tau_B$  in  $\text{Aut}(\widehat{F}_n)$  by  $\widehat{\tau}_A$  and  $\widehat{\tau}_B$  respectively.

We now prove that (1) implies (2). Suppose there is an aligned isomorphism  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$  and denote its restriction to  $\widehat{F}_n$  by  $\Theta_F$ . We have  $\Theta(t_A) = t_B h$  for some  $h \in \widehat{F}_n$ . Since  $g t_A = t_A t_A^{-1} g t_A = t_A \widehat{\tau}_A(g)$  we have  $\Theta_F(g) t_B h = t_B h \Theta_0(\widehat{\tau}_A(g))$ . Let  $I_h$  denote the inner automorphism given by conjugation by  $h$ . We have  $\Theta_F(g) t_B = t_B I_h(\Theta_F(\widehat{\tau}_A(g)))$ , and hence,  $t_B \widehat{\tau}_B(\Theta_F(g)) = t_B I_h(\Theta_F(\widehat{\tau}_A(g)))$  for all  $g \in \widehat{F}_n$ . Hence,  $\widehat{\tau}_B = I_h \Theta_F \widehat{\tau}_A \Theta^{-1}$ . It follows that  $\widehat{\tau}_A$  and  $\widehat{\tau}_B$  are conjugate when projected to  $\text{Out}(\widehat{F}_n)$ . Hence,  $\Phi$  and  $\Psi$  are profinitely conjugate.

To show (2) implies (1) we reverse the previous calculation to obtain a group isomorphism  $\widehat{G}_A \rightarrow \widehat{G}_B$ . □

## Chapter 7

# Profinite rigidity of {universal Coxeter}-by-cyclic groups

Let  $n \geq 2$  be an integer. The *universal Coxeter group of rank  $n$*  is the free product  $W_n$  of  $n$  copies of  $\mathbb{Z}/2$ ,

$$W_n = \bigast_{i=1}^n \mathbb{Z}/2.$$

The group  $\text{Out}(W_n)$  of outer automorphisms of the universal Coxeter group  $W_n$  is related to the better-understood groups of outer automorphisms of free groups  $\text{Out}(F_n)$  and the general linear groups  $\text{GL}(n, \mathbb{Z})$ , with isomorphisms between all of these groups in the lowest ranks

$$\text{Out}(W_{n+1}) \simeq \text{Out}(F_n) \simeq \text{GL}(n, \mathbb{Z}) \text{ for } n \in \{1, 2\}.$$

There has recently been increased interest in studying the properties of  $\text{Out}(W_n)$  in analogy with the group  $\text{Out}(F_n)$  of outer automorphisms of free groups, including the work of Guerch on the automorphisms and commensurations of  $\text{Out}(W_n)$  [Gue20, Gue21] and Gaboriau–Guerch–Horbez on computing the  $L^2$ -Betti numbers of  $\text{Out}(W_n)$  [GGH22].

In this chapter we study elements of  $\text{Out}(W_n)$  one at a time by considering their *mapping tori*, which we also refer to as {*universal Coxeter*}-by-cyclic groups,

$$G = W_n \rtimes_{\Phi} \mathbb{Z} \text{ for } \Phi \in \text{Out}(W_n).$$

Such groups are virtually free-by-cyclic and thus residually finite. In this chapter we study the *profinite rigidity* within the class of such groups. We say that a universal Coxeter group is *fully irreducible*, if it admits fully irreducible monodromy (see Section 7.0.2 for definitions).

**Theorem 7.0.10.** *Let  $G$  be a fully irreducible {universal Coxeter}-by-cyclic group. Then  $G$  is almost profinitely rigid amongst the class of fully irreducible {universal Coxeter}-by-cyclic groups.*

Let  $K \leq W_n$  be the unique torsion-free subgroup of index 2. For any choice of free basis for  $W_n$ ,  $K$  is the kernel of the homomorphism  $W_n \rightarrow \mathbb{Z}/2$  which maps every free generator of  $W_n$  to 1. We note that  $K$  is characteristic and it is isomorphic to the free group of rank  $n - 1$ .

Fix a free basis of the free group  $F_n$  of rank  $n$ , and let  $\iota \in \text{Aut}(F_n)$ , denote the automorphism which inverts each basis element. Let  $[\iota]$  be the image of  $\iota$  in  $\text{Out}(F_n)$ . Following [BF18], we define the group of *hyperelliptic outer automorphisms* of  $F_n$ , denoted by  $\text{HOut}(F_n)$ , to be the centraliser of  $[\iota]$  in  $\text{Out}(F_n)$ .

**Theorem 7.0.9.** *Let  $G = W_n \rtimes \mathbb{Z}$  be a {universal Coxeter}-by-cyclic group. Then the rank  $n$  of the fibre  $W_n$  is an invariant of  $\hat{G}$ .*

*Suppose that all free-by-cyclic groups with monodromy in  $\text{HOut}(F_{n-1})$  are conjugacy separable. Then  $\hat{G}$  determines the stretch factors  $\{\lambda^+, \lambda^-\}$  associated to the monodromy of the splitting  $W_n \rtimes \mathbb{Z}$ .*

## 7.0.1 More on graphs of groups

Recall in Section 2.1 we defined a graph of groups  $\mathbb{G} = (\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  to be a graph  $\Gamma$ , an assignment  $\mathcal{G}_\bullet$  of groups to vertices and edges of  $\Gamma$ , and a collection of monomorphisms  $\iota_e: \mathcal{G}_e \hookrightarrow \mathcal{G}_{\tau(e)}$ . For the remainder of this section, we will assume that all the edge groups  $\mathcal{G}_e$  are trivial. The vertex  $v$  is said to be *essential* if  $\mathcal{G}_v$  is non-trivial.

To every graph of groups with trivial edge groups  $\mathbb{G} = (\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  we associate a graph of spaces  $X_{\mathbb{G}}$  constructed by attaching a  $K(\mathcal{G}_v, 1)$  with a unique vertex  $v_0$  to the corresponding vertex  $v$  of  $\Gamma$ .

A *morphism*  $F$  between graphs of groups  $\mathbb{G} = (\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  and  $\mathbb{H} = (\Lambda, \mathcal{H}_\bullet, \kappa_\bullet)$  consists of a pair of maps  $(f, f_X)$  with the following properties. The first map  $f: \Gamma \rightarrow \Lambda$  sends vertices to vertices, and edges to edge paths. The second map  $f_X: X_{\mathbb{G}} \rightarrow X_{\mathbb{H}}$  is a map of spaces such that the following diagram commutes,

$$\begin{array}{ccc} X_{\mathbb{G}} & \xrightarrow{f_X} & X_{\mathbb{H}} \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{f} & \Lambda \end{array}$$

The vertical maps are the retractions obtained by collapsing the vertex spaces to their basepoints.

A *homotopy* from the morphism  $(f, f_X): \mathbb{G} \rightarrow \mathbb{H}$  to  $(f', f'_X): \mathbb{G} \rightarrow \mathbb{H}$  is a collection of morphisms

$$\{(f_s, f_{X,s}): \mathbb{G} \rightarrow \mathbb{H} : s \in [0, 1]\},$$

such that  $\{f_s\}$  is a homotopy from  $f$  to  $f'$ , and  $\{f_{X,s}\}$  is a homotopy from  $f_X$  to  $f'_X$ .

A morphism  $F: \mathbb{G} \rightarrow \mathbb{H}$  is a *homotopy equivalence*, if there exists a morphism  $F': \mathbb{H} \rightarrow \mathbb{G}$  such that  $F \circ F'$  and  $F' \circ F$  are homotopic to the identity morphisms. Any homotopy equivalence  $H: \mathbb{G} \rightarrow \mathbb{H}$  induces an isomorphism  $H_*: \pi_1(\mathbb{G}) \rightarrow \pi_1(\mathbb{H})$ .

For further detail and careful proofs of the claims made in this section, the interested reader is referred to [Lym22b].

## 7.0.2 Topological representatives for elements in $\text{Out}(W_n)$ and Nielsen numbers

For each  $n \geq 2$ , define the *thistle with  $n$  prickles* to be the graph of groups  $\mathcal{T}_n$ , where the underlying graph is a tree with one vertex of degree  $n$  and  $n$  vertices of degree 1, and where each edge and the central vertex are labelled by the trivial group, and where the leaves are labelled by  $\mathbb{Z}/2$ . Once and for all, fix the basepoint  $*$  of  $\mathcal{T}_n$  to be the central vertex. Then, there is a natural identification  $\pi_1(\mathcal{T}_n, *) \simeq W_n$ , so that each standard generator of  $W_n$  is identified with the path in  $\mathcal{T}_n$  given by

the concatenation  $e \cdot x \cdot \bar{e}$ , where  $e$  is an edge in  $\mathcal{T}_n$  with  $i(e) = *$  and  $x$  is the generator of the group associated to the vertex  $\tau(e)$ .

Let  $\Phi \in \text{Out}(W_n)$ . The *standard topological representative* of  $\Phi$  is the homotopy equivalence  $\rho: (\mathcal{T}_n, *) \rightarrow (\mathcal{T}_n, *)$  determined by  $\Phi$  and the identification  $\pi_1(\mathcal{T}_n, *) \simeq W_n$  as above. A *topological representative* of  $\Phi$  is a pair  $(F, \mathbb{G})$  where  $\mathbb{G}$  is a graph of groups together with a homotopy equivalence  $\alpha: \mathcal{T}_n \rightarrow \mathbb{G}$ , and  $F: \mathbb{G} \rightarrow \mathbb{G}$  is a homotopy equivalence, such that the following diagram commutes up to homotopy

$$\begin{array}{ccc} \mathcal{T}_n & \xrightarrow{\rho} & \mathcal{T}_n \\ \downarrow \alpha & & \downarrow \alpha \\ \mathbb{G} & \xrightarrow{F} & \mathbb{G} \end{array}$$

where  $\rho: \mathcal{T}_n \rightarrow \mathcal{T}_n$  is the standard representative of  $\Phi$ . We assume that  $f$  is locally injective on the interiors of the edges of  $\Gamma$ . When we talk of the *transition matrix*, *maximal filtration* and *exponential strata* of  $(F, \mathbb{G})$ , we are referring to those objects associated to the underlying graph map  $(f, \Gamma)$  (see Section 3.2). In particular, the topological representative  $(F, \mathbb{G})$  is said to be *irreducible* if the maximal filtration of the underlying graph map  $(f, \Gamma)$  has length one.

Let  $(F, \mathbb{G})$  be a topological representative of  $\Phi \in \text{Out}(W_n)$ . An *invariant forest* is a subgraph  $\Gamma_0$  of the underlying graph  $\Gamma$  such that each component  $C$  of  $\Gamma_0$  is a tree, and the fundamental group of the sub-graph of groups corresponding to  $C$  acts with a global fixed point on its Bass–Serre tree. A forest is said to be *non-trivial* if it contains at least one edge.

The outer automorphism  $\Phi \in \text{Out}(W_n)$  is said to be *irreducible*, if every topological representative  $(F, \mathbb{G})$  of  $\Phi$ , where the underlying graph  $\Gamma$  has no inessential valence-one vertices and no invariant non-trivial forests, is irreducible. The *stretch factor* of  $\Phi$  is the infimum of the stretch factors of irreducible topological representatives of  $\Phi$ . The outer automorphism  $\Phi$  is *fully irreducible* if  $\Phi^k$  is irreducible for every  $k \geq 1$ .

There exists a theory of (improved) relative train track representatives for elements of  $\text{Out}(W_n)$  [Lym22a] (see also [CT94], [FM15] and [Lym22b] for earlier results on train tracks on graphs of groups), which is completely analogous to that

for elements in  $\text{Out}(F_n)$ . As in the case of  $\text{Out}(F_n)$ , the stretch factor of an irreducible outer automorphism  $\Phi \in \text{Out}(W_n)$ , as defined in the previous paragraph, coincides with the stretch factor of any train track representative. The stretch factor of a general element  $\Phi \in \text{Out}(W_n)$  is defined to be the stretch factor of any relative train track representative.

The proof of the following lemma is completely analogous to the proof of Proposition 6.1.5.

**Lemma 7.0.1.** *Let  $\Phi \in \text{Out}(W_n)$  be an outer automorphism of  $W_n$  with stretch factor  $\lambda$ . Let  $(F, \mathbb{G})$  be a topological representative of  $\Phi$ , with underlying graph map  $f$ . Then*

$$\lambda = \limsup_{m \rightarrow \infty} N_m(f)^{1/m}.$$

Before proceeding further, we take a detour to discuss irreducibility of matrices and graphs.

Let  $A \in M_n(\mathbb{Z})$  be a matrix with non-negative integer entries  $a_{ij}$ . We construct a (directed) graph  $\Gamma_A$  associated to  $A$ , so that  $\Gamma_A$  has  $n$  vertices  $\{v_1, \dots, v_n\}$  and there exist  $a_{ij}$  (directed) edges from  $v_i$  to  $v_j$ , for every  $i, j \leq n$ . The graph  $\Gamma_A$  is said to be *irreducible*, if for any two vertices  $u$  and  $v$  of  $\Gamma_A$ , there exists a directed path from  $u$  to  $v$ . The following is an elementary exercise.

**Lemma 7.0.2.** *The non-negative integer matrix  $A$  is irreducible if and only if the associated graph  $\Gamma_A$  is irreducible.*

We now prove a crucial lemma on the irreducibility of degree-two covers of directed graphs. In what follows, when we say *path* from  $u$  to  $v$ , we will always mean a directed path. Given an oriented edge  $e$  in an oriented graph  $\Gamma$ , we write  $i(e)$  to denote the initial vertex of  $e$  in  $\Gamma$  and  $t(e)$  the terminal vertex.

**Lemma 7.0.3.** *Let  $\Gamma$  be a directed graph on  $n$  vertices, and let  $\Gamma'$  be a degree-two cover of  $\Gamma$ . If  $\Gamma$  is irreducible then either  $\Gamma'$  is irreducible, or it has two connected components and each is isomorphic to  $\Gamma$ .*

*Furthermore, if  $\Gamma'$  is irreducible then the Perron–Frobenius eigenvalues of  $A_{\Gamma'}$  and  $A_{\Gamma}$  are equal.*

*Proof.* Let  $\{v_1, \dots, v_n\}$  be the vertex set of  $\Gamma$ . Let  $v_i^1$  and  $v_i^2$  be the two lifts of  $v_i$  in  $\Gamma'$ , and write  $V_1 = \{v_i^1 \mid 1 \leq i \leq n\}$  and  $V_2 = \{v_i^2 \mid 1 \leq i \leq n\}$ . Let  $N$  be the number of edges  $e$  in  $\Gamma'$  such that  $i(e) \in V_1$  and  $t(e) \in V_2$ . We call such edges *special*. We prove our result by induction on  $N$ .

If  $N = 0$  then the lemma is clearly true, since  $\Gamma'$  has two connected components and each is isomorphic to  $\Gamma$ .

Let  $N \geq 1$  and suppose the lemma is true whenever the number of special edges is at most  $N - 1$ . Let  $\Gamma' \rightarrow \Gamma$  be a degree-two cover with  $N$  special edges. Note that since  $\Gamma$  is irreducible, for any vertices  $v_i$  and  $v_j$  of  $\Gamma$ , there exists a path  $\gamma$  from  $v_i$  to  $v_j$ . This path has two lifts  $\gamma_1$  and  $\gamma_2$  in  $\Gamma'$  such that either

- i)  $\gamma_1$  joins  $v_i^1$  to  $v_j^1$  and  $\gamma_2$  joins  $v_i^2$  to  $v_j^2$ ; or
- ii)  $\gamma_1$  joins  $v_i^1$  to  $v_j^2$  and  $\gamma_2$  joins  $v_i^2$  to  $v_j^1$ .

Hence to prove the lemma it suffices to show that there exists a path in  $\Gamma'$  from  $v_k^1$  to  $v_k^2$ , and a path from  $v_k^2$  to  $v_k^1$ , for all  $k$ .

Let  $e_1$  be a special edge and suppose that  $i(e_1) = v_i^1$  and  $t(e_1) = v_j^2$ , for some  $i$  and  $j$ . Then  $\Gamma'$  contains an edge  $e_2$  such that  $i(e_2) = v_i^2$  and  $t(e_2) = v_j^1$ . Construct a graph  $\Gamma''$  from  $\Gamma'$  by replacing  $e_1$  with the edge  $e'_1$  which joins  $v_i^1$  to  $v_j^1$ , and replacing  $e_2$  with the edge  $e'_2$  which joins  $v_i^2$  to  $v_j^2$ . Note that  $\Gamma''$  is a degree-two cover of  $\Gamma$  with  $N - 1$  special edges.

Suppose first that  $N = 1$  and fix index  $k \leq n$ . Since  $\Gamma$  is irreducible, there exists a path in  $\Gamma$  from  $v_k$  to  $v_i$ . Let  $\gamma$  be a shortest such path. Then  $\gamma$  has two lifts  $\gamma_1$  and  $\gamma_2$  in  $\Gamma''$ . Since  $\Gamma''$  has zero special edges,  $\gamma_1$  only crosses edges with both endpoints in  $V_1$  and  $\gamma_2$  only crosses edges with both endpoints in  $V_2$  (possibly after swapping  $\gamma_1$  and  $\gamma_2$ ). Also by minimality of the length of  $\gamma$ , the lifts of  $\gamma$  do not cross the edges  $e'_1$  and  $e'_2$ . Hence the path  $\gamma_1$  descends to a path in  $\Gamma'$  joining  $v_k^1$  to  $v_i^1$ . Similarly one constructs a path from  $v_j^2$  to  $v_k^2$  in  $\Gamma'$ . The concatenation of these two paths and the edge  $e_1$  gives a path from  $v_k^1$  to  $v_k^2$ .

Now assume  $N \geq 2$ . Then  $\Gamma''$  is irreducible and thus there exists a shortest path  $\eta_1$  in  $\Gamma''$  from  $v_k^1$  to  $v_i^1$ , and a shortest path  $\eta_2$  from  $v_j^2$  to  $v_k^2$ . Since  $i(e'_1) = v_i^1$ , any shortest path from  $v_k^1$  to  $v_i^1$  does not contain  $e'_1$ . Similarly, any shortest path from  $v_j^2$  to  $v_k^2$  does not contain  $e'_2$ . Hence  $\eta_1$  and  $\eta_2$  descend to paths in  $\Gamma'$ . The

concatenation of these paths, together with the edge  $e_1$  give rise to a path from  $v_k^1$  to  $v_k^2$ . Similarly, one constructs a path from  $v_k^2$  to  $v_k^1$ . Hence the statement holds for  $\Gamma'$ . This proves the first part of the lemma.

To prove the statement about equality of Perron–Frobenius eigenvalues, suppose that  $\Gamma'$  is irreducible. Relabel the vertices of  $\Gamma'$  so that for each  $i \leq n$ , the vertices labelled by  $i$  and  $i+n$  in  $\Gamma'$  are the two lifts of the  $i^{\text{th}}$  vertex of  $\Gamma$ . Let  $a_{ij}$  and  $a'_{ij}$  denote the  $(i, j)^{\text{th}}$  elements of  $A_\Gamma$  and  $A_{\Gamma'}$ , respectively. Since  $\Gamma'$  is a degree-two cover of  $\Gamma$ , it follows that for every  $i, j \leq n$ ,

$$a_{ij} = a'_{ij} + a'_{i(j+n)} = a'_{(i+n)j} + a'_{(i+n)(j+n)}. \quad (7.1)$$

Let  $v_{pf}$  denote the Perron–Frobenius eigenvector of  $A_\Gamma$  and let  $\lambda$  be the Perron–Frobenius eigenvalue. Let  $v'_{pf}$  be the vector obtained by concatenating two copies of  $v_{pf}$ . Then by (7.1),

$$A_{\Gamma'} v'_{pf} = \lambda \cdot v'_{pf}.$$

Hence the Perron–Frobenius eigenvalue of  $A_{\Gamma'}$  is  $\lambda$ . □

Let  $W_n$  be the universal Coxeter group with a free basis  $\{a_1, \dots, a_n\}$ . There exists a homomorphism  $W_n \twoheadrightarrow \mathbb{Z}/2$  which maps each generator  $a_i$  to the non-trivial element of  $\mathbb{Z}/2$ . The kernel  $K \leq W_n$  is the unique torsion-free index-two subgroup of  $W_n$  and thus it is independent of the choice of the free basis. Moreover,  $K$  is isomorphic to the free group of rank  $n-1$ .

Fix a preferred free basis  $X$  of the free group  $F_{n-1}$ . Let  $\iota_X \in \text{Aut}(F_{n-1})$  denote the automorphism which acts by inverting each element of  $X$ . We call  $\iota_X$  the *hyperelliptic involution* of  $F_{n-1}$  with respect to  $X$ . We will write  $\iota$  to denote  $\iota_X$  when  $X$  is clear from the context. Let  $[\iota]$  be the image of  $\iota$  in  $\text{Out}(F_{n-1})$ .

**Remark 7.0.4.** For any two choices of free generating sets  $X$  and  $Y$  of the free group  $F$ , the outer classes of the hyperelliptic involutions  $[\iota_X]$  and  $[\iota_Y]$  are conjugate in  $\text{Out}(F)$  [BF18, Lemma 6.1].

**Definition 7.0.5** ([BF18]). The *hyperelliptic automorphism group*  $\text{HAut}(F_{n-1})$  is the centraliser of  $\iota$  in  $\text{Aut}(F_{n-1})$ . The *hyperelliptic outer automorphism group*  $\text{HOut}(F_{n-1})$  of  $F_{n-1}$  is the centraliser of  $[\iota]$  in  $\text{Out}(F_{n-1})$ .

There is a homomorphism  $\rho: \text{Aut}(W_n) \rightarrow \text{Aut}(F_{n-1})$  induced by restricting each automorphism of  $W_n$  to the characteristic subgroup  $K \leq W_{n-1}$ . By [Krs92, Section 2 ], the map  $\rho$  restricts to an isomorphism

$$\rho: \text{Aut}(W_n) \rightarrow x^{-1} \text{HAut}(F_{n-1}) x,$$

for some  $x \in \text{Aut}(F_{n-1})$ . Furthermore, the image of the subgroup  $\text{Inn}(W_n)$  of inner automorphisms of  $W_n$  under  $\rho$  is contained in the subgroup  $\text{Inn}(F_{n-1}) \cdot \langle \iota \rangle \cap \text{HAut}(F_{n-1})$ . Hence there is an isomorphism

$$\text{Aut}(F_{n-1})/\text{Inn}(F_{n-1}) \rightarrow \text{HAut}(F_{n-1})/(\text{Inn}(F_{n-1}) \cdot \langle \iota \rangle \cap \text{HAut}(F_{n-1}))$$

Moreover, it is easy to see that  $\text{HAut}(F_{n-1}) \cap \text{Inn}(F_{n-1}) = 1$ , and hence there is an injective map

$$\text{Out}(W_n) \hookrightarrow \text{HOut}(F_{n-1})/\langle [\iota] \rangle.$$

It follows that each outer automorphism  $\Phi$  in  $\text{Out}(W_n)$  defines a coset  $\bar{\Phi} \cdot \langle [\iota] \rangle$  in the quotient  $\text{Out}(F_{n-1})/\langle [\iota] \rangle$ . Hence, there is a well-defined map  $\text{Out}(W_n) \rightarrow \text{Out}(F_{n-1})$  which sends  $\Phi$  to the outer automorphism  $\bar{\Phi}^2$ , which we label by  $\Phi_K \in \text{Out}(F_{n-1})$ , and call the outer automorphism of  $F_{n-1}$  *induced* by  $\Phi \in \text{Out}(W_n)$ .

**Proposition 7.0.6.** *Let  $n \geq 3$  and  $\Phi \in \text{Out}(W_n)$  be an outer automorphism with stretch factor  $\lambda(\Phi)$ . Then, the stretch factor of the induced outer automorphism  $\Phi_K \in \text{Out}(F_{n-1})$  is equal to  $\lambda(\Phi)^2$ .*

*Furthermore, if  $\Phi$  is fully irreducible then so is  $\Phi_K$ .*

*Proof.* Let  $(F, \mathbb{G})$  be a bounded relative train track representative of  $\Phi^2 \in \text{Out}(W_n)$ , where  $\mathbb{G} = (\Gamma, \mathcal{G})$  is a graph of groups as before, with the vertex  $v_0$  in  $\Gamma$  acting as a basepoint, and  $F = (f, f_X)$ . Let  $\{a_1, \dots, a_n\}$  be a free basis of  $W_n$  so that each vertex of the underlying graph  $\Gamma$  of  $\mathbb{G}$  is labelled by some  $\langle a_i \rangle \simeq \mathbb{Z}/2$  or the trivial group. Note that  $\Gamma$  is simply connected. Let  $K = \langle a_1 a_2, a_1 a_3, \dots, a_1 a_n \rangle$ .

As before, let  $X_{\mathbb{G}}$  denote the graph of spaces associated to  $\mathbb{G}$ . In particular, we identify  $W_n$  with  $\pi_1(X_{\mathbb{G}}, v_0, \Gamma)$ . Let  $\pi: Y \rightarrow X_{\mathbb{G}}$  be the cover of  $X_{\mathbb{G}}$  corresponding to the subgroup  $K$ . Let  $\tilde{X}$  be a connected lift of  $X_{\mathbb{G}}$  to  $Y$  with  $\tilde{v}_0 \in \tilde{X}$  a lift of the basepoint  $v_0$ .

Since  $K$  is a characteristic subgroup, there is a lift of the map  $f_X$  to a map  $f_Y: Y \rightarrow Y$  which represents the induced outer automorphism  $\Phi_K$ .

Since each  $a_i$  is not an element of  $K$ , the unique length-one loop in  $X_{\mathbb{G}}$  contained in the free homotopy class of  $a_i \in \pi_1(X_{\mathbb{G}})$  lifts to an edge in  $Y$  with distinct endpoints. The endpoints are the two vertices of  $Y$  which project down to the essential vertex labelled by  $a_i$ .

Note that the morphism  $f$  preserves the set of essential vertices. Let  $Y'$  be the space obtained from  $Y$  by collapsing the edges which join the two lifts of each essential vertex, and the lifts of the two-cells. Then  $Y'$  is homotopy equivalent to  $Y$ , and there is a map  $f_{Y'}: Y' \rightarrow Y'$  which is homotopic to  $f_Y$ . It follows that  $(f_{Y'}, Y')$  is a topological representative of  $\Phi_K \in \text{Out}(F_{n-1})$ . Then,  $Y'$  is a (combinatorial) graph which is obtained by doubling the underlying graph  $\Gamma$  of  $\mathbb{G}$  along the essential vertices. In particular, the incidence matrix of  $f_{Y'}$  gives rise to a directed graph which is an index-two cover of the directed graph associated to the incidence matrix of  $f$ .

The relative train track structure of  $f$  lifts to a relative train track structure of  $f_{Y'}$ . If  $S$  is a non-zero stratum of  $\mathbb{G}$  with stretch factor  $\lambda$ , then by Lemma 7.0.3, its lift to  $Y'$  is either an irreducible stratum with stretch factor  $\lambda$  or two irreducible strata, each with stretch factor  $\lambda$ . Then  $\lambda(\Phi_K) = \lambda(\Phi^2) = \lambda(\Phi)^2$ .

Suppose now that  $\Phi$  is fully irreducible. For contradiction, suppose that  $\Phi_K$  is not fully irreducible. Then there exists some positive integer  $N$  and a free splitting  $F_{n-1} = A * B$  such that  $A \neq 1$  and  $B \neq 1$ , and  $\Phi^N$  preserves the conjugacy class of  $A$ .

Then it is possible to choose  $f$  to be an irreducible train track. Then either  $f_{Y'}$  is irreducible, or  $Y' = \Gamma \cup \Gamma^*$  is the double of a simply-connected graph  $\Gamma$ , and  $f_{Y'}$  preserves  $\Gamma$  and  $\Gamma^*$ . Let  $Y''$  be the graph obtained from  $Y'$  by collapsing the subgraph  $\Gamma^* \subseteq Y'$  to a single point. Then  $Y''$  is homotopy equivalent to  $Y'$  and there is an induced map  $f_{Y''}: Y'' \rightarrow Y''$  which is homotopic to  $f_{Y'}$ . Then since  $f_{Y''}$  is a train track map, it follows that  $\Phi_K$  is irreducible. Similar argument works for showing irreducibility of  $\Phi_K^i$  for all  $i \geq 1$ .  $\square$

**Corollary 7.0.7.** *Let  $n \geq 3$  and  $C > 1$ . There exists at most finitely many conjugacy classes of elements in  $\text{Out}(W_n)$  with stretch factor at most  $C$ .*

*Proof.* This is an immediate consequence of Proposition 7.0.6 and Lemma 3.2.2.  $\square$

### 7.0.3 Profinite invariants and almost rigidity of {universal Coxeter}-by-cyclic groups

A group  $G$  is said to be {*universal Coxeter*}-by-cyclic if it fits into the short exact sequence

$$1 \rightarrow W_n \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1.$$

For the remainder of this section, we let  $(G_A, \varphi)$  and  $(G_B, \psi)$  denote {universal Coxeter}-by-cyclic groups with fibred characters  $\varphi: G_A \rightarrow \mathbb{Z}$  and  $\psi: G_B \rightarrow \mathbb{Z}$ . We write  $G_A = W_n \rtimes_{\varphi} \mathbb{Z}$  and  $G_B = W_m \rtimes_{\psi} \mathbb{Z}$  to denote the splittings of  $G_A$  and  $G_B$  induced by the characters, and let  $K_A \leq G_A$  and  $K_B \leq G_B$  be the unique torsion-free index-two subgroups of the fibres. Recall that there is a well-defined map  $\text{Out}(W_n) \rightarrow \text{Out}(F_{n-1})$  which sends an outer automorphism class  $\Phi$  represented by  $\phi \in \text{Aut}(W_n)$ , to the the outer automorphism class of  $\phi^2|_K$ , where  $K \leq W_n$  is the unique torsion-free index-two subgroup. We write  $\Phi_K$  to denote the image of  $\Phi$  under this map, and call it the outer automorphism of  $F_{n-1}$  induced by  $\Phi$ .

Fix some  $t \in \varphi^{-1}(1)$  and  $s \in \psi^{-1}(1)$ , and let

$$\begin{aligned} H_A &= \langle K_A, t^2 \rangle_{G_A} \simeq K_A \rtimes_{\Phi_{K_A}} \mathbb{Z}, \\ H_B &= \langle K_B, s^2 \rangle_{G_B} \simeq K_B \rtimes_{\Psi_{K_B}} \mathbb{Z}. \end{aligned} \tag{7.2}$$

We write  $\bar{\varphi}$  to denote the character  $\varphi: G_A \rightarrow \mathbb{Z}$  restricted to the subgroup  $H_A$ , and define  $\bar{\psi}$  similarly. We note that the characters  $\bar{\varphi}$  and  $\bar{\psi}$  induce the splittings (7.2).

For a group  $G$  and prime  $p$  we denote its *pro- $p$  completion* by  $\widehat{G}^p$ . Note this is exactly the inverse limit of the system of finite quotients of order a power of  $p$ .

**Proposition 7.0.8.** *Let  $(G_A, \varphi)$  and  $(G_B, \psi)$  be {universal Coxeter}-by-cyclic groups, and suppose  $\Theta: \widehat{G}_A \rightarrow \widehat{G}_B$  is an isomorphism. The following conclusions hold:*

1.  $\Theta$  is  $\widehat{\mathbb{Z}}$ -regular;

2.  $G_A$  and  $G_B$  have isomorphic fibres;
3. *the profinite isomorphism  $\Theta$  restricts to a  $\widehat{\mathbb{Z}}$ -regular isomorphism  $\Theta|_{\widehat{H}_A}: \widehat{H}_A \rightarrow \widehat{H}_B$  and  $\bar{\varphi}$  is the pullback of  $\bar{\psi}$  via  $\Theta|_{\widehat{H}_A}$*
4.  $G_A$  and  $G_B$  are good.

*Proof.* It is easy to see that  $G_A$  and  $G_B$  satisfy  $b_1(G_A) = b_1(G_B) = 1$ . Thus, Item 1 follows from Proposition 6.3.3. Note that  $b_1(W_n; \mathbb{F}_2) = n$ . We may prove Item 2 by an identical argument to Proposition 6.3.5 but taking the twisted Alexander polynomials over  $\mathbb{F}_2$  instead of an arbitrary prime.

The subgroups  $H_A \leq G_A$  and  $H_B \leq G_B$  have finite index in their respective overgroups, and are free-by-cyclic. Since goodness passes to finite-index overgroups this proves Item 4.

Now, the group  $H_A$  is the kernel of a map  $\alpha: G_A \rightarrow \mathbb{Z}/2$ . We see that  $H_A$  is torsion-free and so its pro-2 completion has finite cohomological dimension, whereas  $G_A$  has 2-torsion so  $\text{cd}_2(\widehat{G}_A^2) = \infty$  (see [Wil98, Section 1.1. and Proposition 11.1.5] for the definition of  $\text{cd}_2$  and the relevant facts). Completing the map  $\alpha$  to  $\widehat{G}_A$  we obtain an induced map  $\widehat{G}_B \rightarrow \mathbb{Z}/2$  and hence a map  $\beta: G_B \rightarrow \mathbb{Z}/2$ . Now  $\ker \beta$  is torsion-free since  $\ker \widehat{\beta} \simeq \ker \widehat{\alpha}$  and  $\text{cd}_2(\ker \widehat{\alpha}^2)$  is finite. We have shown that  $H_A$  and  $H_B$  are profinitely isomorphic free-by-cyclic groups. Since  $\Theta$  is  $\widehat{\mathbb{Z}}$ -regular by Item 1, it follows that  $\bar{\varphi}$  is the pullback of  $\bar{\psi}$  via  $\Theta|_{\widehat{H}_A}$ . This proves Item 3.  $\square$

**Theorem 7.0.9.** *Suppose that all free-by-cyclic groups with monodromy contained in  $\text{HOut}(F_n)$  (see Definition 7.0.5) for some  $n$ , are conjugacy separable.*

*Let  $(G_A, \varphi)$  and  $(G_B, \psi)$  be profinitely isomorphic {universal Coxeter}-by-cyclic groups. Let  $\{\lambda_A^+, \lambda_A^-\}$  and  $\{\lambda_B^+, \lambda_B^-\}$  be the stretch factors of  $(G_A, \varphi)$  and  $(G_B, \psi)$ , respectively. Then*

$$\{\lambda_A^+, \lambda_A^-\} = \{\lambda_B^+, \lambda_B^-\}.$$

*Proof.* The groups  $(G_A, \varphi)$  and  $(G_B, \psi)$  have isomorphic fibres by Proposition 7.0.8 Item 2. By Proposition 7.0.8 Item 3, the character  $\bar{\varphi}: H_A \rightarrow \mathbb{Z}$  is the pullback of  $\bar{\psi}: H_B \rightarrow \mathbb{Z}$  under a profinite isomorphism  $\widehat{H}_A \rightarrow \widehat{H}_B$ . Also, by assumption,

$(H_A, \bar{\varphi})$  and  $(H_B, \bar{\psi})$  are conjugacy separable free-by-cyclic groups. Hence by Theorem 6.5.6, the stretch factors associated to  $(H_A, \bar{\varphi})$  and  $(H_B, \bar{\psi})$  are equal. Thus by Proposition 7.0.6 the stretch factors of  $(G_A, \varphi)$  and  $(G_B, \psi)$  are equal.  $\square$

A {universal Coxeter}-by-cyclic group is said to be *fully irreducible* if it admits monodromy  $\Phi$  such that  $\Phi^k$  is irreducible for all  $k \geq 1$ .

**Theorem 7.0.10.** *Let  $G$  be a fully irreducible {universal Coxeter}-by-cyclic group. Then  $G$  is almost profinitely rigid amongst the class of fully irreducible {universal Coxeter}-by-cyclic groups.*

*Proof.* Let  $G_A$  and  $G_B$  be fully irreducible {universal Coxeter}-by-cyclic groups with index-two free-by-cyclic groups  $H_A \leq G_A$  and  $H_B \leq G_B$  as before. Suppose that there exists a profinite isomorphism  $\hat{G}_A \rightarrow \hat{G}_B$ , and for a fixed character  $\psi: G_B \rightarrow \mathbb{Z}$ , let  $\varphi: G_A \rightarrow \mathbb{Z}$  be the pullback of  $\psi$ . By Proposition 7.0.8, there exists an isomorphism  $\hat{H}_A \simeq \hat{H}_B$ , and the induced character  $\bar{\varphi}: H_A \rightarrow \mathbb{Z}$  is the pullback of  $\bar{\psi}: H_B \rightarrow \mathbb{Z}$ . Hence by Theorem 6.1.6,  $H_A$  is hyperbolic if and only if  $H_B$  is hyperbolic.

Suppose first that  $H_A$  and  $H_B$  are hyperbolic. Then they are virtually special by [HW15] and thus conjugacy separable by [Min06]. Now arguing as in the proof of Theorem 7.0.9, the sets of stretch factors of  $(G_A, \varphi)$  and  $(G_B, \psi)$  coincide. Thus by Corollary 7.0.7,  $G_B$  can take on at most finitely many different isomorphism types.

Assume now that  $H_A = K_A \rtimes_{\Phi_{K_A}} \mathbb{Z}$ , and thus also  $H_B = K_B \rtimes_{\Psi_{K_B}} \mathbb{Z}$ , are not hyperbolic. Then  $\Phi_{K_A}$  and  $\Psi_{K_B}$  are not atoroidal. By Proposition 7.0.6,  $\Phi_{K_A}$  and  $\Psi_{K_B}$  are irreducible. Now apply the same argument as in the proof of Theorem 6.6.4.  $\square$

## Part III

# Structure of fibred groups and polytopes

## Chapter 8

# Exotic subgroups of hyperbolic groups

A finitely generated group  $G$  is said to be *hyperbolic*, if the Cayley graph of  $G$  with respect to a finite generating set  $S$  has *slim triangles*; that is, there exists some number  $\delta \geq 0$  such that for any triangle  $\tau$ , each side of  $\tau$  is contained in the  $\delta$ -neighbourhood of the remaining two sides. Since their introduction by Gromov in [Gro87], hyperbolic groups have been of fundamental importance in geometric group theory, providing a template to study wider classes of groups [Far98, Osi16, BHS17].

It is well known that the class of hyperbolic groups is not closed with respect to taking subgroups. Indeed, the free group of rank two  $F_2$  contains a copy of the free group of infinite rank  $F_\infty$ , which is not hyperbolic since it is not finitely generated. Note that every torsion-free hyperbolic group admits a finite classifying space given by the Rips complex [BH99, p.468]. Hence, a natural question to ask is whether the only obstruction for a subgroup of a hyperbolic group  $G$  to be hyperbolic arises from its failure to satisfy sufficiently strong finiteness properties (see Section 2.2.2 for definitions).

The study of this question has a long history, starting with the work of Rips who constructed the first examples of non-hyperbolic subgroups of hyperbolic groups which are finitely generated [Rip82]. Then Brady produced a non-hyperbolic subgroup of a hyperbolic group which is finitely presented but not of type  $FP_3(\mathbb{Z})$

[Bra99]. The existence of non-hyperbolic subgroups of hyperbolic groups with higher finiteness properties was established by Isenrich–Martelli–Py [IMP21] and Fisher [Fis21]. Most recently, Isenrich–Py, using methods from complex geometry, produced non-hyperbolic subgroups of hyperbolic groups of type  $F_n$  but not  $F_{n+1}$ , for every  $n \geq 1$  [IP22, Corollary 3].

A natural question that follows is whether one can obtain non-hyperbolic subgroups of hyperbolic groups of type  $FP(R)$  for some ring  $R$ . Italiano–Martelli–Migliorini recently obtained the first such known example, which is moreover of type  $F$  [IMM23]. To do so, they constructed a topological fibring of a finite-volume hyperbolic 5-manifold  $M$ ; the resulting fibre is a finite-volume 4-manifold  $N$  such that the outer automorphism group  $\text{Out } \pi_1(N)$  is infinite. They argue further that one can fill the cusps in  $M$  to obtain a negatively curved pseudo-manifold  $M'$  which still fibres, and such that the new fibre  $N'$  is aspherical and has  $\text{Out } \pi_1(N')$  infinite. If  $\pi_1(N')$  were to be hyperbolic, Paulin’s theorem [Pau91] would imply that  $\pi_1(N')$  splits over an infinite cyclic subgroup. However, a simple Mayer–Vietoris argument shows that  $\pi_1(N')$  cannot split over  $\mathbb{Z}$ .

In the first part of this chapter, we give a general criterion for constructing non-hyperbolic subgroups of hyperbolic groups which satisfy strong finiteness properties. A group  $G$  is said to be *special* if it is the fundamental group of a special cube complex in the sense of Haglund–Wise [HW08]. A group  $G$  is  *$L^2$ -acyclic* if its  $L^2$ -Betti numbers  $b_i^{(2)}(G)$  vanish in every degree.

**Theorem 8.0.1.** *Let  $G$  be a torsion-free hyperbolic virtually special group which is  $L^2$ -acyclic and such that  $\text{cd}_{\mathbb{Q}}(G) \geq 4$ . Then  $G$  contains a non-hyperbolic subgroup  $N \leq G$  of type  $FP(\mathbb{Q})$ .*

The proof of the criterion crucially uses the main result of Fisher in [Fis21] which shows how to obtain homomorphisms to  $\mathbb{Z}$  with kernels that have strong finiteness properties. We note that it was already observed by Fisher [Fis21] and Isenrich–Martelli–Py [IMP21] that such an approach can be used to construct non-hyperbolic subgroups of hyperbolic groups of type  $FP_n(\mathbb{Q})$  but not  $FP_{n+1}(\mathbb{Q})$ .

The main application of the criterion in Theorem 8.0.1 is the construction of infinitely many non-hyperbolic subgroups of hyperbolic groups which are of type  $FP(\mathbb{Q})$ :

**Theorem 8.2.1.** *For every integer  $n \geq 2$ , there exists a non-hyperbolic subgroup  $N$  of a torsion-free hyperbolic group, such that  $N$  is of type  $\text{FP}(\mathbb{Q})$  and  $\text{cd}_{\mathbb{Q}}(N) = 2n$ .*

Let  $\mathcal{F}$  denote the family of discrete countable groups with finite cohomological dimension over  $\mathbb{Z}$ . By the work of Sauer [Sau06, Theorem 1.2], cohomological dimension over  $\mathbb{Z}$  is a quasi-isometry invariant amongst groups in  $\mathcal{F}$ . Hence we get the following corollary:

**Corollary 8.0.2.** *There exist infinitely many quasi-isometry classes of finitely generated subgroups of hyperbolic groups which are of type  $\text{FP}(\mathbb{Q})$  and which are not hyperbolic.*

Recall that a group  $G$  is said to be *locally hyperbolic* if every finitely generated subgroup of  $G$  is hyperbolic. As another application of our methods, we show that local hyperbolicity and algebraic fibering are mutually exclusive phenomena in groups of higher cohomological dimension:

**Corollary 8.0.3.** *Let  $G$  be a finitely generated, locally hyperbolic, torsion-free group. Suppose that  $G$  fibres algebraically. Then*

$$G \simeq (F_n * \Sigma_1 * \dots * \Sigma_k) \rtimes \mathbb{Z},$$

where  $F_n$  is a free group of rank  $n$ , and each  $\Sigma_i$  is a closed surface group. In particular,  $\text{cd}_{\mathbb{Z}}(G) \leq 3$ .

Furthermore, if  $\text{cd}_{\mathbb{Q}}(G) = 3$  then  $G$  contains a closed surface subgroup.

## 8.1 A criterion for exotic subgroups of hyperbolic groups

The aim of this section is to prove a general criterion for constructing exotic subgroups of hyperbolic groups:

**Theorem 8.1.1.** *Let  $G$  be a torsion-free hyperbolic virtually special group with  $\text{cd}_{\mathbb{Q}}(G) \geq 4$ . Suppose that the  $L^2$ -Betti numbers of  $G$  satisfy  $b_i^{(2)}(G) = 0$  for all  $i \leq n$ . Then  $G$  contains a non-hyperbolic subgroup  $N \leq G$  of type  $\text{FP}_n(\mathbb{Q})$ .*

Moreover,  $\text{cd}_{\mathbb{Q}}(N) \in \{\text{cd}_{\mathbb{Q}}(G) - 1, \text{cd}_{\mathbb{Q}}(G)\}$  and if  $\text{cd}_{\mathbb{Q}}(G) \leq n$  then  $\text{cd}_{\mathbb{Q}}(N) = \text{cd}_{\mathbb{Q}}(G) - 1$ .

Note that if a group  $N$  is of type  $\text{FP}_n(\mathbb{Q})$  and its cohomological dimension satisfies  $\text{cd}_{\mathbb{Q}}(N) \leq n$ , then  $N$  is of type  $\text{FP}(\mathbb{Q})$ . Hence Theorem 8.1.1 implies Theorem 8.0.1 from the introduction to this chapter.

We delay the proof of Theorem 8.1.1 to Section 8.1.2. First in Section 8.1.1 we study the structure of hyperbolic kernels of infinite quotients  $G \twoheadrightarrow Q$ , where  $G$  is a hyperbolic group. As an application, we prove Corollary 8.0.3 on the structure of locally hyperbolic groups which fibre.

### 8.1.1 Hyperbolic subgroups of hyperbolic groups

**Lemma 8.1.2.** *Let  $H$  be a one-ended torsion-free hyperbolic group. If  $H$  is not the fundamental group of a surface then  $H \rtimes \mathbb{Z}$  contains a subgroup isomorphic to  $\mathbb{Z}^2$ .*

*Proof.* Suppose that  $H$  is one-ended torsion-free hyperbolic and that it is not the fundamental group of a closed surface. Paulin's theorem [Pau91] (see also [BS94]) shows that if  $\text{Out}(H)$  is infinite then  $H$  acts on an  $\mathbb{R}$ -tree with cyclic arc stabilisers. It then follows from the work of Rips that  $H$  admits such an action on a simplicial tree, i.e. it splits over  $\mathbb{Z}$ ; see [BF95].

Hence if  $H$  does not split over  $\mathbb{Z}$  then  $\text{Out}(H)$  is finite. In particular, there exists  $m \geq 1$  such that  $\phi^m$  is an inner automorphism of  $H$ . It follows that there exists some  $x \in G$  such that  $\langle H, xt^m \rangle \simeq H \times \mathbb{Z}$ , and so  $G$  contains a subgroup isomorphic to  $\mathbb{Z}^2$ .

Suppose now that  $H$  splits over  $\mathbb{Z}$ . Since  $H$  is not a surface group, it admits a non-trivial JSJ decomposition [Sel97, Theorem 1.7]. The JSJ splitting is canonical, and thus  $\phi$  permutes the conjugacy classes of edge stabilisers. Hence, for every edge of the splitting, there exists some  $x \in G$  and  $m \geq 1$  such that the generator of the edge stabiliser commutes with  $t^m x$ , and thus  $G$  contains a  $\mathbb{Z}^2$  subgroup.  $\square$

**Proposition 8.1.3.** *Let  $G$  be a group with no  $\mathbb{Z}^2$ -subgroups. Suppose that  $G$  fits into the short exact sequence of groups*

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1,$$

where  $Q$  is infinite. If  $H$  is torsion-free hyperbolic, then  $H$  splits as a free product of finitely many fundamental groups of compact hyperbolic surfaces.

In particular, the cohomological dimension of  $H$  over  $\mathbb{Z}$  satisfies  $\text{cd}_{\mathbb{Z}}(H) \leq 2$ .

*Proof.* The quotient  $Q$  is hyperbolic and thus contains an element of infinite order by [Mos96, Theorem A]. Let  $\gamma$  denote the lift of such an element to  $G$ . Then  $G$  contains a subgroup  $\langle H, \gamma \rangle \simeq H \rtimes \mathbb{Z}$ . If  $H$  is one-ended, then by Lemma 8.1.2  $H$  is the fundamental group of a closed hyperbolic surface and  $\text{cd}_{\mathbb{Z}}(H) = 2$ .

If  $H$  is two-ended then it is isomorphic to the infinite cyclic group and  $G \simeq \mathbb{Z} \rtimes \mathbb{Z}$ . But then  $G$  is virtually  $\mathbb{Z}^2$  and thus it cannot be hyperbolic.

Suppose that  $H$  is infinitely-ended. Let  $H = H_1 * \dots * H_n * F_k$  be the Grushko decomposition of  $H$ . Fix a lift  $t \in G$  of a generator of the infinite cyclic group. Let  $\phi \in \text{Aut}(H)$  be the automorphism of  $H$  induced by the conjugation action of  $t$  on  $H$  in  $G$ . Then  $\phi$  permutes the conjugacy classes of the subgroups  $H_i$ . Hence for all  $i$ , there exists  $x_i \in G$  such that

$$\langle H_i, t^m x_i \rangle \simeq H_i \rtimes_{\text{Ad}_{x_i} \phi^m} \mathbb{Z},$$

where  $\text{Ad}_{x_i}$  denotes the inner automorphism of  $G$  which acts by conjugation with  $x_i$ . Note that each  $H_i$  is one-ended, torsion-free hyperbolic. If there exists some  $H_i$  which is not a surface group, then by Lemma 8.1.2  $G$  is not hyperbolic. Moreover, if  $H = H_1 * \dots * H_n * F_k$  then the cohomological dimension of  $H$  satisfies

$$\text{cd}_{\mathbb{Z}}(G) \leq \max\{\text{cd}_{\mathbb{Z}}(H_i), \text{cd}_{\mathbb{Z}}(F_k)\} \leq 2.$$

□

We note that the special case of Proposition 8.1.3 when  $Q \simeq \mathbb{Z}$  was observed by Peter Brinkmann in his thesis [Bri00b]. The author would like to thank Daniel Groves for pointing this out.

We end this subsection by outlining the proof of Corollary 8.0.3. Let  $G$  be a locally hyperbolic torsion-free group which fibres algebraically. Then the kernel  $H$  of the fibration of  $G$  is hyperbolic and thus by Proposition 8.1.3, it is a free product of closed surface groups and a free group. Arguing as in the proof of Proposition 8.1.3,  $\text{cd}_{\mathbb{Z}}(H) \leq 2$  and by (2.1),

$$\text{cd}_{\mathbb{Z}}(G) \leq \text{cd}_{\mathbb{Z}}(H) + \text{cd}_{\mathbb{Z}}(\mathbb{Z}) \leq 3.$$

This proves the first part of Corollary 8.0.3.

For the second part, note that  $H$  is of type  $\text{FP}(\mathbb{Q})$  and thus by Fel'dman's Theorem 2.2.9,  $\text{cd}_{\mathbb{Q}}(G) = \text{cd}_{\mathbb{Q}}(H) + 1$ . The cohomological dimension of  $H$  over  $\mathbb{Q}$  is bounded above by 2, and it is exactly equal to 2 when at least one of the components of the free product decomposition of  $H$  is a closed surface group. This proves the second part of Corollary 8.0.3.

## 8.1.2 Proof of the criterion

The key tool for proving the criterion is the following theorem of Fisher:

**Theorem 8.1.4** (Fisher [Fis21]). *Let  $G$  be a group of type  $\text{FP}_n(\mathbb{Q})$ . Suppose that  $G$  is RFRS and its  $L^2$ -Betti numbers satisfy  $b_i^{(2)}(G) = 0$  for  $i \leq n$ . Then there exists a finite-index subgroup  $G' \leq G$  and an epimorphism  $\varphi: G' \rightarrow \mathbb{Z}$  such that  $\ker \varphi$  is of type  $\text{FP}_n(\mathbb{Q})$ .*

*Proof of Theorem 8.1.1.* If  $G$  is torsion-free hyperbolic, then  $G$  has a finite classifying space given by the Rips complex (see [BH99, p.468]). In particular,  $G$  is of type  $\text{FP}_n(\mathbb{Q})$  for all  $n \geq 1$ . Hence by Fisher's Theorem 8.1.4, the hypothesis on the vanishing of the  $L^2$ -Betti numbers of  $G$  implies that there exists a finite-index subgroup  $G' \leq G$  such that  $G'$  fibres algebraically with kernel  $N$  of type  $\text{FP}_k(\mathbb{Q})$  for  $k \leq n$ .

Consider the Lyndon–Hochschild–Serre spectral sequence for the short exact sequence

$$1 \rightarrow N \rightarrow G' \rightarrow G'/N \rightarrow 1.$$

It follows that

$$\mathrm{cd}_{\mathbb{Q}}(G') \leq \mathrm{cd}_{\mathbb{Q}}(N) + \mathrm{cd}_{\mathbb{Q}}(G'/N).$$

Since  $G'/N \simeq \mathbb{Z}$ , we have that  $\mathrm{cd}_{\mathbb{Q}}(G'/N) = 1$ . Moreover since  $G' \leq G$  is a finite-index subgroup, [Bie81, Proposition 5.7] implies that the cohomological dimensions of  $G$  and  $G'$  over  $\mathbb{Q}$  agree.

Combining these facts, we obtain

$$\begin{aligned} \mathrm{cd}_{\mathbb{Q}}(N) &\geq \mathrm{cd}_{\mathbb{Q}}(G') - 1 \\ &= \mathrm{cd}_{\mathbb{Q}}(G) - 1 \geq 3. \end{aligned} \tag{8.1}$$

Hence  $\mathrm{cd}_{\mathbb{Z}}(N) \geq \mathrm{cd}_{\mathbb{Q}}(N) \geq 3$  and thus by Proposition 8.1.3 the subgroup  $N$  is not hyperbolic.

If  $\mathrm{cd}_{\mathbb{Q}}(G)$  is bounded above by  $n$ , then

$$\mathrm{cd}_{\mathbb{Q}}(N) \leq \mathrm{cd}_{\mathbb{Q}}(G) \leq n.$$

Since  $N$  is of type  $FP_n(\mathbb{Q})$  and satisfies  $\mathrm{cd}_{\mathbb{Q}}(N) \leq n$ , we conclude that  $N$  is of type  $FP(\mathbb{Q})$ . Thus by Fel'dman's Theorem 2.2.9 the first inequality in (8.1) is an equality.  $\square$

## 8.2 Explicit examples

In this section we will construct explicit examples of non-hyperbolic subgroups of hyperbolic groups with strong finiteness properties.

**Theorem 8.2.1.** *For every integer  $n \geq 2$ , there exists a non-hyperbolic subgroup  $N$  of a torsion-free hyperbolic group, such that  $N$  is of type  $FP(\mathbb{Q})$  and  $\mathrm{cd}_{\mathbb{Q}}(N) = 2n$ .*

The non-hyperbolic subgroups in Theorem 8.2.1 will arise as subgroups of fundamental groups of hyperbolic manifolds. Dodziuk shows that it is possible to extract information about the  $L^2$ -cohomology of a manifold  $M$  from studying the  $L^2$ -harmonic forms on  $M$  [Dod77]. His explicit calculation of the  $L^2$ -harmonic forms for hyperbolic manifolds yields the following result about the  $L^2$ -Betti numbers of such manifolds.

**Theorem 8.2.2** (Dodziuk [Dod79]). *Let  $n \geq 1$  be odd. If  $G$  is the fundamental group of a closed oriented hyperbolic  $n$ -manifold, then for all  $i \geq 0$ ,*

$$b_i^{(2)}(G) = 0.$$

Our criterion in Theorem 8.1.1 combined with Dodziuk’s Theorem 8.2.2 shows that any odd-dimensional closed hyperbolic  $n$ -manifold with  $n \geq 4$  whose fundamental group is virtually special, will contain a subgroup as in Theorem 8.2.1.

Such examples can be obtained from the work of Bergeron–Haglund–Wise, who prove that in every dimension  $n$  there exist cocompact arithmetic lattices in  $SO(n, 1)$  which are virtually special [BHW11, Theorem 1.10]. After passing to a subgroup of finite index, we may assume that the lattice  $G$  is torsion-free and special.

Note that in this case there is an alternate way to see that the kernel  $K$  of an algebraic fibration is non-hyperbolic. Mainly, if  $K$  is of type  $FP_{n-1}(\mathbb{Q})$  then by a theorem of Hillman [Hil02, Theorem 1.19] it is an  $(n - 1)$ -dimensional Poincaré Duality group (over  $\mathbb{Q}$ ). Thus, a Mayer–Vietoris argument shows that it cannot split over  $\mathbb{Z}$  or over the trivial group. Assuming that  $K$  is hyperbolic, it follows by Paulin’s theorem [Pau91] and the Rips machine [BF95] that the outer automorphism group of  $K$  is finite. Then, arguing as in the proof of Lemma 8.1.2, we conclude that  $G$  virtually splits as a direct product  $H \times \mathbb{Z}$  and thus cannot be hyperbolic.

Finally, we note that it would be interesting to construct such examples which do not arise from the world of manifolds. One potential avenue to do so is through the work of Arenas in [Are22], whose cubical Rips construction is a new tool for constructing higher dimensional Gromov hyperbolic groups which fibre algebraically.

# Chapter 9

## The $L^2$ -polytope

Following the introduction of the Thurston polytope which controls the fibering of 3-manifolds, there has been much interest in finding larger classes of groups for which the Bieri–Neumann–Strebel (BNS) invariant is controlled by an integral polytope. This has now been shown to be true for many families of groups, including right-angled Artin groups [MV95], finitely generated nilpotent or metabelian groups [BG10], pure braid groups [KMM15] and 2-generator, 1-relator groups [Bro87]. Recent work of Kielak using novel group ring techniques shows that many classes of groups, including free-by-cyclic groups, admit BNS invariants which are controlled by integral polytopes [Kie20a].

In this chapter we study the  $L^2$ -polytopes of free-by-cyclic groups. Defined by Friedl–Lück in [FL17], the  $L^2$ -polytope is constructed from the so-called *universal  $L^2$ -torsion* of an  $L^2$ -acyclic  $\mathbb{Z}G$ -chain complex. A priori, the  $L^2$ -polytope is a formal difference of two polytopes. Kielak’s work in [Kie20a] shows that if  $G$  is an ascending HNN extension of a finite rank free group, and in particular a free-by-cyclic group, then the  $L^2$ -polytope is indeed a polytope and the construction is independent of the choice of the chain complex (and thus an invariant of the group).

The  $L^2$ -polytope of a group  $G$  is a translation-invariant class of integral polytopes in the finite-dimensional vector space  $V$ , where

$$V = H_1(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^{b_1(G)}.$$

It encodes information about the  $L^2$ -Euler characteristic of the kernels of characters  $G \rightarrow \mathbb{Z}$ . In particular, when  $G$  is free-by-cyclic and the epimorphism  $\varphi: G \rightarrow \mathbb{Z}$  has a finitely generated free kernel, then the polytope detects the rank of  $\ker \varphi$ .

In this chapter we investigate the basic properties of the  $L^2$ -polytope for free-by-cyclic groups. We show how to calculate the polytope from a topological representative of the monodromy map. We study the structure of the polytope when the corresponding group  $G$  splits as a graph of groups with cyclic edge groups. We also study the action of  $\text{Out}(G)$  on the integral characters  $G \rightarrow \mathbb{Z}$  with kernel of finitely generated fixed rank using the polytope. Finally, we calculate the  $L^2$ -polytope in certain specific cases.

## 9.1 Universal $L^2$ -torsion

The *universal  $L^2$ -torsion* was defined by Friedl–Lück in [FL17] as a way of encompassing a number of important invariants from the world of 3-manifold theory and knot theory, including the  $L^2$ -torsion and the *Thurston norm polytope*. Friedl–Lück use the universal  $L^2$ -torsion to define the  $L^2$ -polytope, which coincides with the dual Thurston norm polytope in the case of 3-manifolds. We will be applying their tools in the setting of free-by-cyclic groups.

To motivate the definition of the universal  $L^2$ -torsion we will begin by defining the classical torsion for an acyclic chain complex. The universal  $L^2$ -torsion is then the  $L^2$ -analogue of the classical notion.

**Definition 9.1.1** ( $K_1(G)$  and  $\text{Wh}(G)$ ). We define  $\text{GL}_\infty(G)$  to be the direct limit of the sequence

$$\text{GL}(1, \mathbb{Z}G) \hookrightarrow \text{GL}(2, \mathbb{Z}G) \hookrightarrow \dots$$

where each inclusion  $\text{GL}(k, \mathbb{Z}G) \hookrightarrow \text{GL}(k+1, \mathbb{Z}G)$  is defined by extending each  $\alpha \in \text{GL}(k, \mathbb{Z}G)$  to an element of  $\text{GL}(k+1, \mathbb{Z}G)$  given by

$$\alpha \oplus \text{id}_{\mathbb{Z}G}: (\mathbb{Z}G)^{k+1} \rightarrow (\mathbb{Z}G)^{k+1}.$$

The  $K_1$ -group of  $G$  is the abelianisation of  $\mathrm{GL}_\infty(G)$ ,

$$K_1(G) := \mathrm{GL}_\infty(G) / [\mathrm{GL}_\infty(G), \mathrm{GL}_\infty(G)].$$

The *Whitehead group*  $\mathrm{Wh}(G)$  of  $G$  is the quotient of  $K_1(G)$  by the classes of elements  $[\pm g]$  for  $g \in G$ ,

$$\mathrm{Wh}(G) := K_1(G) / \{[\pm g] \mid g \in G\}.$$

Let  $C_\bullet$  be a finite based chain complex of finitely generated free  $\mathbb{Z}G$ -modules,

$$\cdots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots$$

If  $C_\bullet$  is acyclic then there exists a *chain contraction*, which is a map  $h_\bullet: C_\bullet \rightarrow C_{\bullet+1}$  such that  $h\partial + \partial h = \mathrm{id}$ . Define  $C_{\mathrm{even}} = \bigoplus_n C_{2n}$  and  $C_{\mathrm{odd}} = \bigoplus_n C_{2n+1}$ . Then the map

$$\begin{aligned} \partial + h: C_{\mathrm{even}} &\rightarrow C_{\mathrm{odd}} \\ (\dots, c_0, c_2, c_4, \dots) &\mapsto (\dots, \partial_2(c_2) + h_0(c_0), \partial_4(c_4) + h_2(c_2), \dots) \end{aligned}$$

is an isomorphism.

Let  $\tilde{K}_1(G)$  be the quotient  $K_1(G) / \{[\pm 1]\}$ . The (*classical*) *torsion* of  $C_\bullet$ , denoted by  $\rho(C_\bullet)$ , is defined to be the image of  $\partial + h$  under the composition

$$\mathrm{GL}_m(\mathbb{Z}G) \hookrightarrow \mathrm{GL}_\infty(G) \twoheadrightarrow \tilde{K}_1(G).$$

It is an easy exercise to show that the torsion does not depend on the choice of bases of the terms in  $C_\bullet$ , nor on the contraction  $h_\bullet$ .

Recall that we write  $\mathcal{N}(G)$  to denote the group von Neumann algebra associated to  $G$  (see Section 2.2.3 for the definitions). An endomorphism of finite Hilbert  $\mathcal{N}(G)$ -modules is a *weak isomorphism* if it is injective and has dense image. The *weak  $K_1$ -group*  $K_1^w(G)$  of a group  $G$  is the abelian group generated by endomorphisms  $\mathbb{Z}G^k \rightarrow \mathbb{Z}G^k$  which become weak isomorphisms upon tensoring by  $L^2(G)$ , with relations given by the usual relations in  $K_1$ . Note that if  $G$  is torsion free and satisfies the Atiyah conjecture, then an endomorphism  $M: \mathbb{Z}G^k \rightarrow \mathbb{Z}G^k$  is a weak

isomorphism upon tensoring by  $L^2(G)$  if and only if  $M$  becomes an isomorphism upon tensoring by the Linnell division ring  $\mathcal{D}(G)$  (see Section 2.2.3). As before, the group  $\tilde{K}_1^w(G)$  is the quotient  $K_1^w(G)/\{\pm 1\}$ .

Let  $(C_\bullet, \partial_\bullet)$  be a based finite chain complex of finitely generated free  $\mathbb{Z}G$ -modules as before. The chain complex  $C_\bullet$  is said to be  $L^2$ -acyclic if the homology groups  $H_i(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_\bullet)$  are trivial for all  $i$ . The *universal  $L^2$ -torsion* of an  $L^2$ -acyclic chain complex  $C_\bullet$  is obtained from a so-called *weak chain contraction* (see [FL17, Definition 1.4]) of  $C_\bullet$ . It is denoted by  $\rho^u(C_\bullet)$  and is an element of the weak  $K_1$ -group  $\tilde{K}_1^w(G)$ .

In practice we will not be working with the definition of the universal  $L^2$ -torsion and instead we will be using the following key properties:

**Lemma 9.1.2** (Friedl–Lück [FL17, Lemma 1.9]). *Let  $B, C$  and  $D$  be finite chain complexes of free  $\mathbb{Z}G$ -modules. Suppose that  $0 \rightarrow B \rightarrow C \rightarrow D \rightarrow 0$  is a short exact sequence of chain complexes. If at least two of  $B \otimes L^2(G), C \otimes L^2(G)$  and  $D \otimes L^2(G)$  are  $L^2$ -acyclic then all three are  $L^2$ -acyclic and*

$$\rho_u^2(C) = \rho_u^2(B) + \rho_u^2(D).$$

**Lemma 9.1.3.** *Let  $C_\bullet$  be chain complex of  $\mathbb{Z}G$  modules which is  $L^2$ -acyclic and of the form*

$$\dots \rightarrow 0 \rightarrow C_k \xrightarrow{\partial_k} C_{k-1} \rightarrow 0 \rightarrow \dots \rightarrow 0.$$

$$\text{Then } \rho^{(u)}(C_\bullet) = (-1)^{k+1}[\partial_k].$$

**Example 9.1.4** (The circle and the torus). Let  $G = \langle t \rangle$  be infinite cyclic. Let  $X \simeq \mathbb{S}^1$  be the CW-complex with a single 0-cell and a single 1-cell. Let  $\tilde{X}$  be the universal cover of  $X$  with a free  $G$ -action. Then the corresponding  $\mathbb{Z}G$ -chain complex  $C_\bullet(\tilde{X})$  is of the form

$$0 \rightarrow \mathbb{Z}G \xrightarrow{(1-t)} \mathbb{Z}G \rightarrow 0.$$

The element  $(1-t)$ , considered as an element of  $\mathcal{D}(G)$ , is non-zero and hence invertible over  $\mathcal{D}(G)$ . Thus  $(1-t)$  corresponds to an element of  $\tilde{K}_1^w(G)$  and we have that  $\rho^{(u)}(C_\bullet) = [(1-t)]$ .

Now let  $G$  be free abelian of rank 2,

$$G = \langle a, t \mid [t, a] \rangle.$$

Let  $X \simeq \mathbb{T}$  be the 2-torus with a cellular structure coming from the identification of the opposite sides of a square. The  $\mathbb{Z}G$ -chain complex  $C_\bullet(\tilde{X})$  of the universal cover  $\tilde{X}$  is

$$0 \rightarrow \mathbb{Z}G \xrightarrow{\partial_2} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \rightarrow 0.$$

After fixing appropriate bases for the  $\mathbb{Z}G$ -modules in the chain complex  $C_\bullet$ , the boundary maps  $\partial_2$  and  $\partial_1$  are represented by the vectors

$$\partial_2 = (1 - t, a - 1) \text{ and } \partial_1 = \begin{pmatrix} a - 1 \\ t - 1 \end{pmatrix}$$

We restrict the boundary map  $\partial_2$  so that its image lies in the first coordinate. Then  $C_\bullet$  fits into an exact sequence of chain complexes  $0 \rightarrow B \rightarrow C \rightarrow D \rightarrow 0$ . We have that  $B_\bullet$  is the chain complex given by

$$0 \rightarrow \mathbb{Z}G \xrightarrow{(t-1)} \mathbb{Z}G \rightarrow 0 \rightarrow 0$$

and  $D_\bullet$  is

$$0 \rightarrow 0 \rightarrow \mathbb{Z}G \xrightarrow{(t-1)} \mathbb{Z}G \rightarrow 0.$$

It follows by Lemma 9.1.2 that

$$\begin{aligned} \rho^{(u)}(C_\bullet) &= \rho^u(B) + \rho^u(D) \\ &= [t - 1] - [t - 1] \\ &= 0. \end{aligned}$$

## 9.2 The $L^2$ -polytope

### 9.2.1 Background on polytopes

Let  $H$  be a free abelian group of finite rank  $n$ . Define  $V := H \otimes_{\mathbb{Z}} \mathbb{R}$  and identify  $H$  with the standard integer lattice in  $V$ .

A *polytope*  $P$  in  $V$  is defined to be the convex hull of finitely many points in  $V$ . A polytope is said to be *integral* if its vertices are contained in the integer lattice  $H \subseteq V$ . We say  $P$  is of *full rank* if no translate of  $P$  is contained in a proper subspace  $W \subseteq V$ . The *codimension of  $P$*  is the maximal integer  $k$ , such that a translate of  $P$  is contained in a subspace  $W \subseteq V$  and  $k = \text{rank}(V) - \text{rank}(W)$ .

A *face* of a polytope  $P \subseteq V$  is a subset of  $P$  of the form

$$F_\varphi(P) := \{p \in P \mid \varphi(p) = \min_{q \in P} \varphi(q)\},$$

for some  $\varphi \in V^*$ .

If  $F$  is a face of  $P$ , the *duals of  $F$*  are the connected components of

$$\{\varphi \in V^* \setminus \{0\} \mid F_\varphi(P) = F\} \subseteq V^*.$$

We define a *marking* to be a function of the dual space  $m: V^* \setminus \{0\} \rightarrow \{0, 1\}$ , such that  $m^{-1}(0)$  is a closed subspace. We say a character  $\varphi \in V^* \setminus \{0\}$  is *marked* if  $\varphi \in m^{-1}(0)$ . For a polytope  $P \subseteq V$ , we say  $m$  is a *marking of  $P$*  if it is constant on duals of faces. A marking  $m$  of  $P$  is said to be a *marking of vertices of  $P$*  if  $m^{-1}(0)$  consists only of duals of vertices.

The *Minkowski sum* of two polytopes  $P, Q \subseteq V$  is defined to be the polytope

$$P + Q := \{p + q \mid p \in P, q \in Q\}.$$

The set of non-empty integral polytopes forms an abelian monoid with respect to the Minkowski sum. We define an equivalence relation on pairs of non-empty integral polytopes such that  $(P_1, Q_1) \sim (P_2, Q_2)$  whenever  $P_1 + Q_2 = P_2 + Q_1$ . Let  $\mathfrak{P}(V)$  be the resulting set of equivalence classes. Then  $\mathfrak{P}(V)$  is an abelian group with respect to the operation  $+$  defined as

$$[(P_1, Q_1)] + [(P_2, Q_2)] := [(P_1 + P_2, Q_1 + Q_2)].$$

We call  $\mathfrak{P}(V)$  the *Grothendieck group of polytopes in  $V$* , or *polytope group* for short. An element of  $\mathfrak{P}(V)$  is said to be a *single polytope* if it is represented by a pair  $(P, Q)$  where  $Q$  is a singleton.

Two elements  $\mathcal{P}_1$  and  $\mathcal{P}_2$  of  $\mathfrak{P}(V)$  are *translation equivalent* if there exists representatives  $(P_1, Q_1)$  of  $\mathcal{P}_1$  and  $(P_2, Q_2)$  of  $\mathcal{P}_2$ , and a singleton  $\{r\} \subseteq V$ , such that  $P_1 + Q_2 = P_2 + Q_1 + \{r\}$ . Note that if this notion is independent of the choice of representatives. We define  $\mathfrak{P}_T(V)$  to be the set of equivalence classes of polytopes in  $V$  under the equivalence relation of translation. Note that the property of being a single polytope is invariant under translation and thus can be extended to elements of  $\mathfrak{P}_T(V)$ . An element of  $\mathfrak{P}_T(V)$  is said to be *trivial* if it is represented by a singleton.

If  $G$  is a group, we define  $\mathfrak{P}(G) := \mathfrak{P}(H_1(G; \mathbb{R}))$ . A group homomorphism  $i: H \rightarrow G$  induces a homomorphism  $i_*: \mathfrak{P}(H) \rightarrow \mathfrak{P}(G)$  of the associated polytope groups.

### 9.2.2 Polytope of a matrix

Let  $H$  be a finitely generated free abelian group and fix a free basis  $\{x_1, \dots, x_n\}$  of  $H$  so that

$$H = x_1\mathbb{Z} \oplus x_2\mathbb{Z} \oplus \cdots \oplus x_n\mathbb{Z}.$$

Identify  $H$  with the standard integer lattice  $\mathbb{Z}^n$  in  $V = \mathbb{R}^n$ .

Suppose that  $R$  is a ring and let  $RH$  be a (possibly twisted) group ring with no zero divisors. Let  $x \in RH$  be a non-zero element. Then  $x = \sum_{h \in H} x_h h$ , with at most finitely many  $x_h \in R$  non-zero. The *polytope of the element*  $x \in RH$  is defined to be the convex hull in  $V$  of the elements  $h \in H$  such that  $x_h \neq 0$ ,

$$\mathcal{P}(x) := \text{hull}\{h \mid x_h \neq 0\} \subseteq V.$$

We define  $\mathcal{P}(0)$  to be the empty set.

Suppose that  $RH$  is an Ore domain, and let  $\text{Ore}(RH)$  be its Ore localisation (see Section 2.2.1 for the definitions). Then for any  $z \in \text{Ore}(RH)$ , we can realise  $z$  as  $z = xy^{-1}$ , for some  $x, y \in RH$ . We define

$$\mathcal{P}(z) = \mathcal{P}(xy^{-1}) := \mathcal{P}(x) - \mathcal{P}(y) \in \mathfrak{P}(H).$$

Recall that  $\text{Ore}(RH)$  is a division ring by Lemma 2.2.4.

**Lemma 9.2.1** ([Kie20a, Lemma 3.12]). *The map  $\mathcal{P}$  is well-defined and induces a homomorphism*

$$\mathcal{P} : \text{Ore}(RH)_{\text{ab}}^{\times} \rightarrow \mathfrak{P}(H).$$

Moreover, the homomorphism in Lemma 9.2.1 induces a homomorphism to the translation-invariant polytope group

$$\mathcal{P} : \text{Ore}(RH)_{\text{ab}}^{\times} / \{\pm h \mid h \in H\} \rightarrow \mathfrak{P}_T(H).$$

We will use the term *polytope homomorphism* to describe both maps.

Let  $\mathcal{D} = \text{Ore}(RH)$  and let  $M$  be a square matrix with coefficients in  $\mathcal{D}$ . We will now describe the procedure of assigning to  $M$  an element  $\mathcal{P}(M)$  of the polytope group  $\mathfrak{P}(H)$ . To do so, we need to first describe the definition of a determinant of a matrix over a division ring.

The *Dieudonné determinant* [Die43] over  $\mathcal{D}$  is a multiplicative map

$$\det_D : M_n(\mathcal{D}) \rightarrow \mathcal{D}^{\times} / [\mathcal{D}^{\times}, \mathcal{D}^{\times}] \sqcup \{0\},$$

defined as follows.

We start by defining a function  $\det : M_n(\mathcal{D}) \rightarrow \mathcal{D}$  which assigns to each matrix  $A \in M_n(\mathcal{D})$  and element of the division ring  $\mathcal{D}$ . We do so by induction on the dimension  $n$ .

When  $n = 1$  and  $A = (a_{11})$ , we define  $\det(A) := a_{11}$ . Now let  $n > 1$  and suppose that  $\det$  has already been defined for  $(n - 1)$ -by- $(n - 1)$  matrices. Let  $A \in M_n(\mathcal{D})$ . If the rightmost column is zero then we set  $\det(A) = 0$ . Suppose now that  $a_{nn} \neq 0$ . For each  $i$ , let  $R_i$  denote the  $i$ -th row of  $A$ . Let  $A'$  be the matrix obtained from  $A$  by replacing  $R_i$  with  $R_i - a_{in}a_{nn}^{-1}R_n$ . Let  $A'_{(n,n)}$  be the  $(n, n)$ -minor of  $A$ . Then  $\det(A') := \det(A'_{(n,n)}) \cdot a_{nn}$ . Finally, if  $a_{nn} = 0$ , pick an index  $i$  such that  $a_{in} \neq 0$  and let  $A'$  be the matrix obtained from  $A$  by swapping rows  $i$  and  $n$ . Then  $\det(A) := (-1)^{\text{sgn}(\sigma)} \det(A')$ , where  $\sigma$  is the transposition  $(i, n)$ .

**Definition 9.2.2** (Dieudonné determinant). The *Dieudonné determinant of  $A$* , denoted  $\det_D(A)$ , is the image of  $\det(A)$  under the natural abelianisation map

$$\mathcal{D} \rightarrow \mathcal{D}^{\times} / [\mathcal{D}^{\times}, \mathcal{D}^{\times}] \sqcup \{0\}.$$

We are now ready to define the Newton polytope:

**Definition 9.2.3.** Let  $A \in M_n(\mathcal{D})$  be a square matrix. Then the *Newton polytope* of  $A$  is

$$\mathcal{P}(A) := \mathcal{P}(\det_D A).$$

### 9.2.3 Definition of the $L^2$ -polytope

Let  $G$  be a torsion free group which satisfies the Atiyah conjecture. Then, the group ring  $\mathbb{Z}G$  embeds into the Linnell division ring  $\mathcal{D}(G)$  by Definition/Theorem 2.2.14.

If  $K$  is the kernel of the natural quotient map  $G \rightarrow G_{\text{fab}}$ , then by Lemma 2.2.5 the group ring  $\mathbb{Z}G$  is isomorphic to the twisted group ring  $\mathbb{Z}K * G_{\text{fab}}$ . The group ring  $\mathbb{Z}K$  embeds into the Linnell division ring  $\mathcal{D}(K)$  of  $K$ , which induces a map  $\mathbb{Z}K * G_{\text{fab}} \rightarrow \mathcal{D}(K) * G_{\text{fab}}$ . The twisted group ring  $\mathcal{D}(K) * G_{\text{fab}}$  satisfies the Ore condition. Moreover, it can be identified with a subring of  $\mathcal{D}(G)$ , and the inclusion induces an isomorphism

$$T: \text{Ore}(\mathcal{D}(K) * G_{\text{fab}}) \rightarrow \mathcal{D}(G).$$

**Definition 9.2.4** (The  $L^2$ -polytope). Let  $G$  be a torsion free group of type F which satisfies the Atiyah conjecture. Let  $X$  be a finite  $K(G, 1)$ -complex which is  $L^2$ -acyclic. Let  $A$  be a matrix representative of  $-\rho^u(X)$  where  $\rho^u(X) \in K_1^w(G)$  is the universal  $L^2$ -torsion of  $X$ . Then the  $L^2$ -polytope of  $X$  is

$$\mathcal{P}(X) := \mathcal{P}(T^{-1}(\det_{\mathcal{D}(G)} A \otimes \mathcal{D}(G))) \in \mathfrak{P}_T(G_{\text{fab}}).$$

**Theorem 9.2.5** (Kielak [Kie20a]). *Let  $G$  be an ascending HNN extension of a finite rank free group and fix a free basis of  $G_{\text{fab}}$ . Let  $X$  and  $Y$  be two finite  $L^2$ -acyclic classifying spaces for  $G$ . Then the following holds:*

1.  $\mathcal{P}(X) = \mathcal{P}(Y)$ , and
2.  $\mathcal{P}(X)$  is a single polytope (see Section 9.2.1).

Hence, when  $G$  is an ascending HNN extension of a finite rank free group, and in particular a free-by-cyclic group, then up to the action of  $\text{Aut}(G_{\text{fab}})$  the  $L^2$ -polytope is an invariant of the group. We write

$$\mathcal{P}(G) := \mathcal{P}(X),$$

where  $X$  is any finite  $L^2$ -acyclic classifying space for  $G$ . Moreover,  $\mathcal{P}(G)$  is represented by a translation-invariant class of a single polytope in  $V$ .

By the work of Kielak–Sun in [KS21], there is an alternative construction of the polytope which avoids twisted group rings and which we will briefly outline below.

Let  $\sigma: G \rightarrow \mathcal{D}(G)$  be the natural embedding. Let  $q: G \rightarrow \text{Ore}(\mathbb{Q}G_{\text{fab}})$  be the map obtained by composing the natural quotient  $G \rightarrow G_{\text{fab}}$  with the inclusion of  $G_{\text{fab}}$  into the Ore localisation of its group ring. Let  $\mathcal{D}(G)G_{\text{fab}}$  denote the (untwisted) group ring of  $G_{\text{fab}}$  over  $\mathcal{D}(G)$ . Note that this is an Ore domain and there is a map

$$\begin{aligned} \sigma \otimes_{\mathbb{Q}} q: G &\rightarrow \text{Ore}(\mathcal{D}(G)G_{\text{fab}}) \\ g &\mapsto \sigma(g)q(g). \end{aligned}$$

Again, let  $X$  be a finite classifying space for  $G$  and let  $A$  be a matrix representative of the negative universal  $L^2$ -torsion  $-\rho^u(X)$ . Then the *Kielak–Sun polytope* is defined to be

$$\mathcal{P}_{KS}(X) := \mathcal{P}(\det_{\mathcal{D}(G)G_{\text{fab}}}(A \otimes \mathcal{D}(G)G_{\text{fab}})).$$

**Proposition 9.2.6.** *Let  $G$  be an ascending HNN extension of a free group. Then the  $L^2$ -polytope and the Kielak–Sun polytope are equal in  $\mathfrak{P}_T(G_{\text{fab}})$ .*

**Example 9.2.7** (The polytope of  $\mathbb{Z}$  and  $\mathbb{Z}^2$ ). Let  $G$  be a free abelian group of rank  $k$  and let  $X = \mathbb{T}^k$  be the  $k$ -torus with the cellular structure coming from identifying opposite faces of the standard  $k$ -cube. Then  $X$  is a  $K(G, 1)$ -complex. Let  $C_{\bullet}(\tilde{X})$  be the free  $\mathbb{Z}G$ -chain complex induced by the action of  $G$  on the universal cover  $\tilde{X}$  of  $X$ .

When  $k = 2$ , we saw in Example 9.1.4 that the universal  $L^2$ -torsion of  $C_\bullet(\tilde{X})$  vanishes. Hence  $\mathcal{P}(\mathbb{Z}^2) = \mathcal{P}(1_G)$  is the translation-invariant class of a single point in  $H_1(G, \mathbb{R}) \simeq \mathbb{R}^2$ .

Suppose now that  $G = \langle t \rangle$  is infinite cyclic. By Example 9.1.4, the universal  $L^2$ -torsion of the chain complex  $C_\bullet(\tilde{X})$  is  $\rho^{(u)}(C_\bullet) = [t - 1]$ . Hence

$$\mathcal{P}(G) = -\mathcal{P}([t - 1]) \equiv_T -\text{hull}\{1, t\}.$$

### 9.3 Constructing the polytope from topological representatives

In this section we will show how to calculate the  $L^2$ -polytope of a free-by-cyclic group  $F_n \rtimes_\phi \mathbb{Z}$  from a topological representative of the monodromy  $\phi \in \text{Aut}(F_n)$ .

Let  $\phi \in \text{Aut}(F_n)$  and let  $(\Gamma, f)$  be a topological representative of  $\phi$  such that  $f$  permutes the vertices of  $\Gamma$ . Let  $G = F_n \rtimes_\phi \mathbb{Z}$  be the corresponding free-by-cyclic group. Recall that the mapping torus  $M_f$  of the map  $f$  is defined to be

$$M_f := \Gamma \times [0, 1] / \{(f(x), 0) \sim (x, 1), \forall x \in \Gamma\}.$$

As noted before, the mapping torus  $M_f$  is the classifying space for the free-by-cyclic group  $G$ .

We let  $E_h$  denote the set of edges of  $M_f$  which are edges of the graph  $\Gamma$ , and we call these the *horizontal edges*. The remaining edges of  $M_f$  are denoted by  $E_v$  and are called the *vertical edges*. Let  $h$  denote the number of horizontal edges, which is equal to the cardinality of the edge set of  $\Gamma$ . Let  $v$  denote the number of vertical edges which also equals the cardinality of the vertex set of  $\Gamma$ .

Let  $C_\bullet = C_\bullet(\tilde{M}_f)$  denote the cellular chain complex of the universal cover  $\tilde{M}_f$  of  $M_f$ . For each  $n \in \{0, 1, 2\}$ , the group  $G$  acts freely on the  $n$ -cells of  $\tilde{M}_f$  and the orbits of the action are in one-to-one correspondence with the  $n$ -cells in  $M_f$ . Hence we get an exact sequence of free  $\mathbb{Z}G$ -modules

$$0 \rightarrow \mathbb{Z}G^h \rightarrow \mathbb{Z}G^{v+h} \rightarrow \mathbb{Z}G^v \rightarrow 0. \quad (9.1)$$

Now fix a lift of  $M_f$  in  $\widetilde{M}_f$ . Choose bases of the  $\mathbb{Z}G$ -modules in (9.1) coming from the cells in the lift. Let  $\{\rho_1, \dots, \rho_h\}$  denote the generators coming from the 2-cells,  $\{\sigma_1, \dots, \sigma_h\}$  the generators coming from the horizontal 1-cells,  $\{\tau_1, \dots, \tau_v\}$  the generators from the vertical 1-cells, and  $\{u_1, \dots, u_v\}$  the generators from the 0-cells. We fix an orientation on all the cells. If necessary, relabel the horizontal 1-cells so that the oriented cell  $\tau_i$  originated at the 0-cell  $u_i$ .

Let  $C_1^h$  denote the subspace of  $C_1$  generated by the horizontal edges and  $C_1^v$  the subspace generated by vertical edges. Let  $A$  be the matrix obtained by restricting the boundary map  $C_1 \rightarrow C_0$  to  $C_1^v$ . Let  $N$  be the matrix corresponding to the  $\mathbb{Z}G$ -linear map obtained by composing the boundary map  $C_2 \rightarrow C_1$  with the projection  $C_1 \rightarrow C_1^h$ . Let  $B_\bullet$  be the chain complex

$$0 \rightarrow 0 \rightarrow C_1^v \xrightarrow{A} C_0 \rightarrow 0,$$

and  $D_\bullet$  the chain complex

$$0 \rightarrow C_2 \xrightarrow{N} C_1^h \rightarrow 0 \rightarrow 0. \quad (9.2)$$

Then there is a short exact sequence of  $\mathbb{Z}G$ -chain complexes

$$0 \rightarrow B_\bullet \rightarrow C_\bullet \rightarrow D_\bullet \rightarrow 0.$$

Before proceeding further, we need to prove the following lemma:

**Lemma 9.3.1.** *Let  $G$  be a group such that  $\mathbb{Z}G$  embeds into a division ring  $\mathcal{D}$ . Let  $T = \text{diag}(t_1, \dots, t_v)$ , where  $t_1, \dots, t_v \in G$  are such that no product of elements in  $\{t_1, \dots, t_v\}$  is equal to the identity element in  $G$ . Let  $P$  be a  $v \times v$  permutation matrix. Then the matrix  $A := TP - I$  is invertible over  $\mathcal{D}$ .*

*Proof.* We proceed by induction on  $v$ . When  $v = 1$ ,  $A = (t_1 - 1)$  is invertible over  $\mathcal{D}$  since  $(t_1 - 1) \in \mathcal{D}$  is non-zero. Suppose  $v > 1$  and denote the  $ij$ -th element of  $A$  by  $a_{ij}$ . If  $a_{11} = t_1 - 1$  then since  $P$  is a permutation matrix, the remaining elements of the first row and column of  $A$  are zero. Hence  $A$  is a block diagonal matrix with two blocks. The first block is the  $1 \times 1$  matrix  $(t_1 - 1)$  and the second block is a  $(v-1) \times (v-1)$  matrix  $A'$  of the form  $A' = T'P' - I$ , where  $T' = \text{diag}(t_2, \dots, t_v)$ ,  $P'$

is a permutation matrix and  $I$  is the identity. The element  $(t_1 - 1) \in \mathcal{D}$  is non-zero and thus invertible, and  $A'$  is invertible by induction. Hence  $A$  is invertible.

Suppose now that  $a_{11} \neq t_1 - 1$ . Hence  $P$  is a permutation matrix which acts by sending the first column to the  $p^{\text{th}}$ -column and the  $q^{\text{th}}$ -column to the first column, for some  $p > 1$  and  $q > 1$ . Hence  $A$  is of the form

$$A = \begin{bmatrix} -1 & 0 & \dots & 0 & t_1 & 0 & \dots \\ 0 & & & & & & \\ \vdots & & \ddots & & & & \\ 0 & & & M & & & \\ t_q & & & & & \ddots & \\ 0 & & & & & & \\ \vdots & & & & & & \end{bmatrix}$$

where  $t_1$  is the  $p^{\text{th}}$ -entry of the first row.

Let  $A_1$  be the matrix obtained from  $A$  by first adding  $t_1$  times the first column to the  $p^{\text{th}}$ -column and then adding  $t_i$  times the first row to the  $q^{\text{th}}$ -row (note that the column operation acts by multiplying on the right). The first operation has the effect of modifying the first row so that it is equal to the row vector  $(-1, 0, \dots, 0)$  and modifying the  $p^{\text{th}}$ -column so that all of its entries are zero except for the  $p^{\text{th}}$ -entry which is equal to  $-1$ , and the  $q^{\text{th}}$ -entry which is given by  $t_q t_1$ . The second operation has the effect of modifying the first column so that it is of the form  $(-1, 0, \dots, 0)^T$ , and it acts by identity on the rest of the matrix.

Hence  $A_1$  is a block diagonal matrix with two blocks. The first block is the  $1 \times 1$  matrix  $(-1)$ . The second block is a  $(v - 1) \times (v - 1)$  matrix  $A'_1$  of the form  $A'_1 = T'_1 P' - I$ , where  $P'$  is a permutation matrix and  $T'_1 = \text{diag}(t_2, \dots, t_i t_1, \dots, t_v)$ . By the original hypothesis, no product of elements in  $\{t_2, \dots, t_i t_1, \dots, t_v\}$  is equal to the trivial element in  $G$ . Hence by induction,  $A'_1$  is invertible and thus so is  $A_1$ . Since  $A_1$  was obtained from  $A$  by applying elementary row and column operations, it follows that  $A$  is also invertible.  $\square$

Fix a base vertex  $x_0$  and a spanning tree  $T_0$  in the graph  $\Gamma$ , which we think of as being embedded in the mapping torus  $M_f$ . For each  $i$  with  $1 \leq i \leq v$ , let  $t_i$  denote the element of  $G$  which corresponds to the horizontal loop  $\tau_i$ .

The map  $f$  permutes the vertices of  $\Gamma$  and thus  $A = TP - I$ , where  $P$  is the corresponding permutation matrix and  $T = \text{diag}(t_1, \dots, t_v)$ . Let  $\varphi: G \rightarrow \mathbb{Z}$  be the natural homomorphism to  $\mathbb{Z}$  induced by the mapping torus structure of  $M_f$ . Then  $\varphi$  does not vanish on any of the  $t_i$ . It follows that no product of the elements in  $\{t_1, \dots, t_v\}$  is equal to the identity. Hence by Lemma 9.3.1,  $A$  is invertible. This means that  $B_\bullet$  is exact upon tensoring by  $\mathcal{D}(G)$  and thus it is  $L^2$ -acyclic. Since  $C_\bullet$  is  $L^2$ -acyclic by Theorem 2.2.17, by Lemma 9.1.2 we get that  $D_\bullet$  is also  $L^2$ -acyclic and

$$\begin{aligned}\rho_u^2(C) &= \rho_u^2(B) + \rho_u^2(D) \\ &= -[A] + [N].\end{aligned}$$

Hence  $\mathcal{P}(G) = \mathcal{P}(\det_D N) - \mathcal{P}(\det_D A)$ .

Let  $\emptyset = \Gamma_0 \subseteq \Gamma_1 \subseteq \dots \subseteq \Gamma_s = \Gamma$  be a maximal filtration of  $(\Gamma, f)$  and let  $\Lambda_i = \text{cl}(\Gamma_i \setminus \Gamma_{i-1})$ . Since  $f(\Gamma_i) \subseteq \Gamma_i$ ,  $N$  is an upper block triangular matrix with the matrices  $N_1, \dots, N_s$  on the diagonal, where  $N_i$  is the submatrix of  $N$  corresponding to edges of the  $i$ -th stratum  $\Lambda_i$ . Then

$$\mathcal{P}(G) = \sum_{i=1}^s \mathcal{P}(\det_D N_i) - \mathcal{P}(\det_D A).$$

Note that if  $\Lambda_i$  is a zero stratum then  $N_i = I$  and thus  $\mathcal{P}(\det_D N_i)$  is a vertex.

## Algebraic construction

Let  $\phi \in \text{Aut}(F_n)$  and let  $(\rho, R_n)$  be the standard topological representative of  $\phi$  where  $R_n$  is the rose. In this case, we will recover the algebraic construction of the  $L^2$ -polytope as in [FK18, Section 3].

Since the mapping torus  $M_\rho$  has a single vertex, the matrix  $A$  is one-dimensional and is given by  $A = (t - 1)$  where  $t$  is the element of  $G$  (and  $G_{\text{fab}}$ ) corresponding to the unique vertical loop based at the unique vertex of  $M_\rho$ . The matrix  $N$  from (9.2) describes the boundary map from the 2-cells of  $M_\rho$  to the horizontal one-cells of  $M_\rho$ , which are identified with the edges of the rose under the obvious embedding  $R_n \subseteq M_f$ . After identifying the horizontal one-cells of  $M_\rho$  with an ordered basis

$\{x_1, \dots, x_n\}$  of  $F_n$ , we see that the  $ij$ -entry  $n_{ij}$  of the matrix  $N$  is of the form

$$n_{ij} = \delta_{ij} - t \frac{\partial \phi(x_i)}{\partial x_j},$$

where  $\delta_{ij}$  is the Kroenecker delta symbol, and  $\frac{\partial \phi(x_i)}{\partial x_j}$  the *Fox derivative* of the element  $\phi(x_i) \in F_n$  with respect to  $x_j$ .

Let  $\text{Fox}(\phi)$  denote the matrix of Fox derivatives of  $\phi$ ,

$$\text{Fox}(\phi) := \left( \frac{\partial \phi(x_i)}{\partial x_j} \right)_{1 \leq i, j \leq v}$$

Hence, by the above discussion we have that

$$\begin{aligned} \mathcal{P}(G) &= \mathcal{P}(\det_D(I - t \cdot \text{Fox}(\phi))) - \mathcal{P}(1 - t) \\ &= \mathcal{P}(\det_D(I - t \cdot \text{Fox}(\phi))) - [0, \bar{t}], \end{aligned}$$

where  $[0, \bar{t}]$  denotes the line segment in  $H_1(G; \mathbb{R})$  joining the origin to the image of  $t$  in the abelianisation of  $G$

## Polytopes for UPG monodromy

We will now compute the polytope in the case when  $\phi \in \text{Aut}(F_n)$  admits an upper triangular topological representative  $(f, \Gamma)$  (see Section 3.2).

Recall, this means that  $f$  fixes all the vertices of  $\Gamma$ , and the maximal filtration of  $(f, \Gamma)$  is such that each stratum  $\Lambda_i$  consists of a single edge, which we label by  $\sigma_i$  in order to remain consistent with the notation introduced at the beginning of this section. Furthermore, we have that  $f(\sigma_i) = \sigma_i p_i$ , where  $p_i$  is either trivial or an immersed path in  $\Gamma$  which only traverses edges that are strictly lower in the filtration of  $\Gamma$ .

As before, let  $t_i$  denote the element of  $G$  corresponding to the vertical loop based at the vertex  $u_i$  (after fixing a choice of basepoint and spanning tree in the 1-skeleton of the mapping torus  $M_f$ ). The matrix  $A$  is given by  $A = \text{diag}(t_1 -$

$1, \dots, t_v - 1$ ). Hence the Dieudonné determinant of  $A$  is simply the product

$$\det_D(A) = (1 - t_1) \cdots (1 - t_v).$$

Each 2-cell  $\rho_k$  in  $M_f$  has a boundary path of the form  $\sigma_k \tau_{t(\sigma_k)} p_k^{-1} \sigma_k^{-1} \tau_{i(\sigma_k)}^{-1}$ , where  $p_i$  traverses edges in  $\{\sigma_1, \dots, \sigma_{k-1}\}$ . Hence the matrix  $N$  is upper triangular with diagonal entries of the form  $n_{kk} = 1 - t_{i(\sigma_k)}$ .

Note that each  $1 - t_j \in \mathbb{Z}G$  is an invertible element of  $\mathcal{D}(G)$  and hence the Dieudonné determinant is equal to

$$\det_D(N) = \prod_{1 \leq j \leq v} (1 - t_j)^{d_j},$$

where  $d_j$  is the number of horizontal 1-cells in  $M_f$  which originate at the vertex  $u_j$ .

It follows that

$$\begin{aligned} \mathcal{P}(G) &= \mathcal{P} \left( \prod_{1 \leq j \leq v} (1 - t_j)^{d_j} \right) - \mathcal{P}((1 - t_1) \cdots (1 - t_v)) \\ &= \mathcal{P} \left( \prod_{1 \leq j \leq v} (1 - t_j)^{d_j - 1} \right) \\ &= \sum_{j=1}^v (d_j - 1) \mathcal{P}(1 - t_j). \end{aligned}$$

Each  $\mathcal{P}(1 - t_j)$  is represented by the line joining the origin to the point in  $H_1(G; \mathbb{R})$  corresponding to the image of  $t_j$  in the abelianisation of  $G$ .

## 9.4 Link to the BNS invariants and the Thurston seminorm

For the remainder of this chapter, we will assume that  $G$  is a free-by-cyclic group. We fix a choice of free basis of  $G_{\text{fab}}$  and let  $\mathcal{P}(G)$  be the corresponding  $L^2$ -polytope. Recall that  $\mathcal{P}(G)$  is a translation class of integral polytopes in  $G_{\text{fab}} \otimes_{\mathbb{Z}} \mathbb{R}$ . We fix a polytope  $P_G$  which represents  $\mathcal{P}(G)$ .

**Theorem 9.4.1** (Kielak [Kie20a, Theorem 5.29]). *There exists a marking on the vertices of  $P_G$  such that for an element  $\varphi \in H^1(G; \mathbb{R})$ ,  $\varphi$  is contained in  $\Sigma(G)$  if and only if  $\varphi$  is dual to a marked vertex.*

**Definition 9.4.2** (Thurston (semi)norm). The *Thurston seminorm* is a map  $\|\cdot\|: H^1(G; \mathbb{R}) \rightarrow \mathbb{R}$  such that for all  $\varphi \in H^1(G; \mathbb{R})$ ,

$$\|\varphi\| = \max\{|\varphi(p) - \varphi(q)| : p, q \in P_G\}.$$

Note that the value of the seminorm does not depend on the choice of representative  $P_G$  of  $\mathcal{P}(G)$ . We will sometimes write  $\|\cdot\|_{\mathcal{P}(G)}$  to emphasise that we are evaluating characters on  $\mathcal{P}(G)$ .

The work of Funke–Kielak in [FK18, Corollary 3.5] confirms that  $\|\cdot\|$  is indeed a seminorm.

As remarked in [FLT19, Section 6], the results in [FL19] imply the following:

**Proposition 9.4.3.** *Let  $G$  be free-by-cyclic and  $\varphi: G \rightarrow \mathbb{Z}$  be an epimorphism. Then*

$$\|\varphi\| = -\chi^{(2)}(\ker \varphi).$$

Note that if  $G$  is free-by-cyclic and  $\varphi: G \rightarrow \mathbb{Z}$  is an epimorphism with finitely generated kernel, then the kernel of  $\varphi$  is a finite rank free group. Moreover, in that case the  $L^2$ -Euler characteristic and the standard Euler characteristic agree [Lüc94, Theorem 1.35]. Hence, we have

$$\|\varphi\| = -\chi^{(2)}(\ker \varphi) = -\chi(\ker \varphi) = \text{rank}(\ker \varphi) - 1.$$

More generally, if  $\varphi$  fibres and  $\text{Im}(\varphi) = k\mathbb{Z} \leq \mathbb{Z}$  for some positive integer  $k$ , then

$$\|\varphi\| = k(\text{rank}(\ker \varphi) - 1). \tag{9.3}$$

## 9.5 Polytopes and group splittings

Let  $G$  be a group and suppose that  $G$  splits as an amalgamated free product  $G = A *_C B$ , or as an HNN extension  $G = A *_C$ . Let  $i_A: A \rightarrow G, i_B: B \rightarrow G$

and  $i_C: C \rightarrow G$  be the inclusion maps. The following combination theorem of Friedl–Lück allows us to construct the polytopes inductively.

**Theorem 9.5.1** (Friedl–Lück [FL17]). *Let  $A, B, C$  be torsion free  $L^2$ -acyclic groups, and suppose that  $A *_C B$  (resp.  $A *_C$ ) satisfies the Atiyah conjecture. Then  $A *_C B$  (resp.  $A *_C$ ) is  $L^2$ -acyclic and*

$$\begin{aligned} \mathcal{P}(A *_C B) &= i_{A*} \mathcal{P}(A) + i_{B*} \mathcal{P}(B) - i_{C*} \mathcal{P}(C), \\ \text{resp. } \mathcal{P}(A *_C) &= i_{A*} \mathcal{P}(A) - i_{C*} \mathcal{P}(C). \end{aligned}$$

A graph of groups splitting of a free-by-cyclic groups over infinite cyclic subgroups impacts the structure of the  $L^2$ -polytopes:

**Proposition 9.5.2.** *Let  $G$  be a free-by-cyclic group that admits a graph of groups decomposition  $G = \pi_1(\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  where each vertex groups is free-by-cyclic and not virtually abelian, and each edge group is infinite cyclic. Then  $\mathcal{P}(G)$  is the Minkowski sum of  $|V(\Gamma)| + |E^+(\Gamma)|$  non-trivial polytopes.*

*Proof.* Let  $G = \pi_1(\Gamma, \mathcal{G}_\bullet, \iota_\bullet)$  be as in the statement. Applying Theorem 9.5.1 inductively, we obtain

$$\mathcal{P}(G) = \sum_{v \in V(\Gamma)} i_* \mathcal{P}(G_v) - \sum_{e \in E^+(\Gamma)} i_* \mathcal{P}(G_e),$$

where each  $i: \mathcal{G}_\bullet \rightarrow G$  denotes the inclusion map.

By Example 9.2.7, if  $\mathcal{G}_e = \langle z \rangle \leq G$  then  $i_* \mathcal{P}(\mathcal{G}_e) = i_*(-\mathcal{P}(1 - z)) = -[0, \bar{z}]$ , where  $\bar{z}$  denotes the image of  $z$  in  $H_1(G; \mathbb{R})$ , which is non-trivial by the discussion in Section 3.3.2. Hence each edge corresponds to a non-trivial line segment in the Minkowski sum.

Fix a free-by-cyclic splitting  $G = F \rtimes \langle t \rangle$ . Each vertex group is free-by-cyclic and of the form  $\mathcal{G}_v = F_v \rtimes \langle xt^k \rangle$  for some  $x \in F$ ,  $k > 0$  and  $F_v \leq F$ . By assumption,  $\mathcal{G}_v$  is not isomorphic to  $\mathbb{Z}^2$  or the fundamental group of a Klein bottle, and thus the rank of  $F_v$  is at least 2. By the discussion in Section 9.3 it follows that the polytope  $\mathcal{P}(\mathcal{G}_v)$  is non-trivial. It remains to show that  $i_* \mathcal{P}(\mathcal{G}_v)$  is also non-trivial.

Indeed, let  $\varphi: G \rightarrow \mathbb{Z}$  be the homomorphism sending each element of  $F$  to 0 and such that  $\varphi(t) = 1$ . Then the restriction of  $\varphi$  to  $\mathcal{G}_v$  has a finitely generated

kernel equal to  $F_v$  and its image in  $\mathbb{Z}$  is generated by  $k$ . It follows that the polytope norm of  $\varphi|_{\mathcal{G}_v}$  evaluated on  $\mathcal{P}(\mathcal{G}_v)$  is

$$\|\varphi|_{\mathcal{G}_v}\|_{\mathcal{P}(\mathcal{G}_v)} = k \cdot (\text{rank}(F_v) - 1) > 0.$$

Hence the kernel  $\ker \|\cdot\|_{\mathcal{P}(\mathcal{G}_v)}$  of the polytope norm of the group  $\mathcal{G}_v$  intersects  $H^1(G; \mathbb{R})$  in a proper subset. It follows that  $i_*\mathcal{P}(\mathcal{G}_v)$  is non-trivial.  $\square$

**Lemma 9.5.3.** *Let  $G$  be a torsion free finitely generated group which satisfies the Atiyah conjecture and splits as an HNN extension  $G = A *_C$  with  $A$  and  $C$  finitely generated and  $L^2$ -acyclic. Then the polytope  $\mathcal{P}(G)$  has codimension strictly greater than 0.*

*Proof.* Let  $G = A *_C$  be as in the statement. Let  $s \in G$  be the stable letter of the HNN extension. Note that the image  $\bar{s}$  of  $s$  in the free abelianisation of  $G$  is non-trivial. Let  $i: H_1(A; \mathbb{R}) \rightarrow H_1(G; \mathbb{R})$  be the map induced by the inclusion  $A \rightarrow G$ . Then  $H_1(G; \mathbb{R}) \simeq i(H_1(A; \mathbb{R})) \oplus \bar{s}\mathbb{R}$ . By Lemma 9.5.1,  $\mathcal{P}(G) = i_{A*}\mathcal{P}(A) - i_{C*}\mathcal{P}(C) \subseteq i(H_1(A; \mathbb{R}))$ .  $\square$

**Lemma 9.5.4.** *Let  $G$  be a free-by-cyclic group. Suppose that  $G = A *_C B$ , where  $C \simeq \mathbb{Z}$  and  $A \not\simeq \mathbb{Z}$ ,  $B \not\simeq \mathbb{Z}$ . Let  $\varphi: G \rightarrow \mathbb{Z}$  be a primitive character with finitely generated kernel. Then  $\varphi|_A: A \rightarrow \mathbb{Z}$  and  $\varphi|_B: B \rightarrow \mathbb{Z}$  have finitely generated kernels and*

$$\text{rank}(\ker \varphi) - 1 = a(\text{rank}(\ker \varphi|_A) - 1) + b(\text{rank}(\ker \varphi|_B) - 1) + c,$$

where  $a, b$  and  $c$  are positive integers which generate the subgroups  $\varphi(A)$ ,  $\varphi(B)$  and  $\varphi(C)$ , respectively.

*Proof.* Let  $G = A *_C B$  and  $\varphi: G \rightarrow \mathbb{Z}$  be as in the statement of the lemma. By Theorem 3.3.9,  $\varphi|_A: A \rightarrow \mathbb{Z}$  and  $\varphi|_B: B \rightarrow \mathbb{Z}$  have finitely generated kernels. By Theorem 9.5.1,  $\mathcal{P}(G) = i_{A*}\mathcal{P}(A) + i_{B*}\mathcal{P}(B) + i_{C*}\mathcal{P}(1 - \bar{z}_1)$ , where  $z_1$  generates the image of  $C$  in  $G$ . Hence,

$$\|\varphi\|_{\mathcal{P}(G)} = \|\varphi\|_{i_{A*}\mathcal{P}(A)} + \|\varphi\|_{i_{B*}\mathcal{P}(B)} + |\varphi(\bar{z}_1)|.$$

Since the induced maps  $H_1(A; \mathbb{R}) \rightarrow H_1(G; \mathbb{R})$  and  $H_1(B; \mathbb{R}) \rightarrow H_1(G; \mathbb{R})$  are injective and the images of the homologies are direct summands in the image, it follows that  $i_{A*}\mathcal{P}(A) \simeq \mathcal{P}(A)$  and  $i_{B*}\mathcal{P}(B) \simeq \mathcal{P}(B)$ . Hence  $\|\varphi\|_{i_{A*}\mathcal{P}(A)} = \|\varphi|_A\|_{\mathcal{P}(A)}$  and  $\|\varphi\|_{i_{B*}\mathcal{P}(B)} = \|\varphi|_B\|_{\mathcal{P}(B)}$ . The result follows by (9.3).  $\square$

**Lemma 9.5.5.** *Let  $G$  be free-by-cyclic and suppose  $G = A*_C$ , where  $C$  is infinite cyclic and  $A \not\cong \mathbb{Z}$ . If  $\varphi: G \rightarrow \mathbb{Z}$  is a primitive character with finitely generated kernel, then  $\varphi|_A: A \rightarrow \mathbb{Z}$  has finitely generated kernel and*

$$\text{rank}(\ker\varphi) - 1 = a(\text{rank}(\ker\varphi|_A) - 1) + |\varphi(z)|,$$

where  $a$  is the positive integer which generates  $\varphi(A)$ .

*Proof.* Let  $G = \langle A, s \mid z_1^s = z_2 \rangle$ . As before, if  $\varphi: G \rightarrow \mathbb{Z}$  is a character with finitely generated kernel then

$$\|\varphi\|_{\mathcal{P}(G)} = \|\varphi\|_{i_{A*}\mathcal{P}(A)} + |\varphi(\bar{z}_1)|.$$

Let  $t \in \varphi^{-1}(1)$  and  $F = \ker(\varphi)$ . By Theorem 3.3.9,  $A$  is conjugate to the subgroup  $\langle F_A, yt^k \rangle$  for  $F_A \leq F$ ,  $y \in F$  and  $k \neq 0$ . Note that  $z = z_1 z_2^{-1}$  is conjugate into  $F_A$ . Hence,

$$H_1(G; \mathbb{R}) = \frac{H_1(A; \mathbb{R})}{\langle \bar{z} \rangle} \oplus s\mathbb{R}.$$

Now let  $a, b \in \mathcal{P}(A) \subseteq H^1(A; \mathbb{R})$  be such that  $\|\varphi|_A\|_{\mathcal{P}(A)} = |\varphi(a) - \varphi(b)|$ . Since  $\mathcal{P}(A)$  is an integral polytope and the points  $a, b \in \mathcal{P}(A)$  may be taken to be vertices of the polytope, it follows that  $a$  and  $b$  lift to elements of  $A$ . In particular,  $a = \alpha t^m$  and  $b = \beta t^n$  for some  $\alpha, \beta \in F$  and  $m, n \in \mathbb{Z}$ . Hence  $|\varphi(a) - \varphi(b)| = |m - n| = |\varphi(i_{A*}(a)) - \varphi(i_{A*}(b))|$ .  $\square$

## 9.6 Links between the polytope and the outer automorphism group

In this section we study the action of  $\text{Out}(G)$  on characters of a free-by-cyclic group  $G$ . Let  $G_{\text{ab}}$  be the abelianisation of  $G$ . There is a natural identifica-

tion  $\text{Hom}(G, \mathbb{R}) \simeq \text{Hom}(G_{\text{ab}}, \mathbb{R})$ . The group  $\text{Aut}(G)$  acts on  $\text{Hom}(G, \mathbb{R})$  by pre-composition. The inner automorphisms act trivially and thus the action descends to an action of  $\text{Out}(G)$ .

Let  $\Psi \in \text{Out}(F_n)$  and choose a representative  $\psi$  of  $\Psi$ . Let  $\varphi \in \text{Hom}(G_\psi, \mathbb{Z})$  be the character associated to the fibring  $G_\psi = F_n \rtimes_\psi \langle t \rangle$ . Write  $\text{stab}(\varphi) = \text{stab}_{\text{Out}(G_\psi)}(\varphi)$  to denote the stabiliser of  $\varphi$  under the action of  $\text{Out}(G_\psi)$ . Let  $C_{\text{Out}(F_n)}(\Psi)$  be the centraliser of  $\Psi$  in  $\text{Out}(F_n)$ .

Each  $\Phi \in C_{\text{Out}(F_n)}(\Psi)$  induces an outer automorphism class  $\bar{\Phi} \in \text{Out}(G_\psi)$ , as follows. Let  $\phi$  be a representative of  $\Phi$ . Then there exists some  $x \in F_n$  such that  $[\phi, \psi] = \iota_x$ . Let  $\bar{\Phi}$  be the outer automorphism class of the map  $G_\psi \rightarrow G_\psi$  which sends  $t \mapsto tx$  and  $a \mapsto \phi(a)$  for all  $a \in F_n$ . This map is well-defined, since for any two choices  $\phi'$  and  $\phi$  of automorphisms representing  $\Phi$ , the resulting automorphisms of  $G_\psi$  will differ by an inner automorphism.

**Lemma 9.6.1.** *Let  $G_\psi = F_n \rtimes_\psi \langle t \rangle$  and let  $\varphi: G_\psi \rightarrow \mathbb{Z}$  be the character corresponding to the splitting. Then there is a short exact sequence*

$$1 \rightarrow Z \rightarrow C_{\text{Out}(F_n)}([\psi]) \rightarrow \text{stab}_{\text{Out}(G_\psi)}(\varphi) \rightarrow 1,$$

where  $Z \simeq \langle [\psi] \rangle \leq C_{\text{Out}}([\psi])$ .

*Proof.* Let  $\Phi \in C_{\text{Out}(F_n)}([\psi])$  and fix a representative  $\phi$  of  $\Phi$ . It is clear that the automorphism  $\Psi \mapsto \bar{\Phi}$  defined above preserves the fibre and thus is an element of  $\text{stab}_{\text{Out}(G_\psi)}(\varphi)$ . It is also clear that this map is a homomorphism.

Now let  $\Theta \in \text{stab}_{\text{Out}(G_\psi)}(\varphi)$  and let  $\theta$  be an automorphism representative of  $\Theta$ . Since  $\varphi \circ \theta = \varphi$ ,  $\theta(\ker \varphi) = \ker \varphi$  and  $\theta(t) = tx$  for some  $x \in \ker \varphi$ . Hence  $\theta$  restricts to an automorphism of  $\ker \varphi = F_n$ . Furthermore, since  $\theta: G_\psi \rightarrow G_\psi$  is an automorphism, for any  $a \in \ker \varphi$ ,

$$\theta\psi(a) = \theta(t^{-1}at) = x^{-1}t^{-1}\theta(a)tx = \iota_x\psi\theta(a).$$

Hence the restriction of  $\theta$  to  $\ker \varphi$  represents an element of  $C_{\text{Out}(F_n)}([\psi])$  which maps to  $\Theta$ . Thus the map is surjective.

Denote by  $K$  the kernel of the map  $C_{\text{Out}(F_n)}([\psi]) \rightarrow \text{stab}_{\text{Out}(G_\psi)}(\varphi)$ , which sends  $\Phi \mapsto \bar{\Phi}$ . It is clear that  $\langle [\psi] \rangle \leq K$ . Let  $\Phi \in C_{\text{Out}(F_n)}([\psi])$  be such that

$\bar{\Phi} = \text{id}_{\text{Out}(G_\psi)}$ . Hence each representative of  $\bar{\Phi}$  is an inner automorphism. Let  $\phi$  be a representative of  $\bar{\Phi}$  and  $y \in G_\psi$  be such that  $\bar{\phi} = \iota_y$ . Then  $y = t^l z$  for some  $l \in \mathbb{Z}$  and  $z \in F_n$ , and

$$\phi(a) = \iota_y(a) = \iota_z \psi^l(a),$$

for all  $a \in F_n$ . Hence  $\phi$  is contained in the outer automorphism class of  $\psi^l$  in  $\text{Out}(F_n)$ . Thus  $\bar{\Phi} \in \langle [\psi] \rangle$  and  $K = \langle [\psi] \rangle$ .  $\square$

Let  $G$  be a free-by-cyclic group and let  $X_n$  be the set of homomorphisms  $\varphi: G \rightarrow \mathbb{Z}$  with finitely generated kernel of rank  $n$ . The action of  $\text{Out}(G)$  preserves the sets  $X_n$ .

**Proposition 9.6.2.** *Suppose that  $\psi \in \text{Aut}(F_n)$  and  $G = F_n \rtimes_\psi \langle t \rangle$ . Assume that  $\mathcal{P}(G_\psi)$  has codimension 0 in  $H_1(G; \mathbb{R})$ . Then the quotient  $C_{\text{Out}(F_n)}([\psi]) / \langle [\psi] \rangle$  is isomorphic to a finite-index subgroup of  $\text{Out}(G)$ .*

*Hence, for any  $\Phi \in \text{Out}(F_n)$  such that  $G_\Phi \simeq G_\psi$ ,  $C_{\text{Out}(F_n)}([\psi]) / \langle [\psi] \rangle$  and  $C_{\text{Out}(F_n)}(\Phi) / \langle \Phi \rangle$  are commensurable.*

*Proof.* Suppose that  $\psi \in \text{Aut}(F_n)$  and  $G = F_n \rtimes_\psi \langle t \rangle$ . Assume that  $\mathcal{P}(G_\psi)$  has codimension 0 in  $H_1(G; \mathbb{R})$ . Let  $\varphi: G \rightarrow \mathbb{Z}$  be the homomorphism associated to the splitting  $G = F_n \rtimes_\psi \langle t \rangle$ . Then the set  $X_n$  of homomorphisms  $G \rightarrow \mathbb{Z}$  with finitely generated kernel of rank  $n$  is non-empty and finite, and  $\varphi \in X_n$ . Hence by the Orbit-Stabiliser Theorem,

$$[\text{Out}(G) : \text{stab}_{\text{Out}(G)}(\varphi)] = |\text{Orb}(\varphi)| < |X_n| < \infty.$$

The result now follows by Lemma 9.6.1.  $\square$

We can also say something about the general case when the polytope has codimension  $\geq 0$ . Fix a basis of  $H_1(G; \mathbb{R})$  and take the dual basis for  $H^1(G; \mathbb{R})$ . Let  $V \subseteq H^1(G; \mathbb{R})$  denote the subspace of characters of  $G$  which are constant when evaluated on the polytope  $\mathcal{P}(G)$ . Since  $\text{Out}(G)$  preserves  $\mathcal{P}(G)$ , it also preserves  $V$ . We let  $\bar{X}_n$  be the set of elements of the form  $\varphi + V$  where  $\varphi \in V^\perp$  is a character of  $G$  and  $\ker \varphi$  is finitely generated and of rank  $n$ . Then  $\bar{X}_n$  is a finite set which admits an action of  $\text{Out}(G)$ . Hence for any character  $\varphi$  of  $G$  with finitely generated kernel, the stabiliser  $\text{stab}(\varphi + V)$  of its coset is a finite-index subgroup of  $\text{Out}(G)$ .

We now return to considering the action of  $\text{Out}(G)$  on the whole set  $X_n$  of characters of  $G$  with finitely generated kernel of rank  $n$ . If the codimension of the polytope  $\mathcal{P}(G)$  is greater than zero then  $X_n$  is infinite. By Lemma 3.3.1, the orbits of elements in  $X_n$  under the action of  $\text{Out}(G)$  correspond to conjugacy classes of outer automorphisms  $\Phi \in \text{Out}(F_n)$  such that  $G = F_n \rtimes_{\Phi} \mathbb{Z}$ .

**Question 9.6.3.** *When is it the case that there are finitely many orbits of  $X_n$  under the action of  $\text{Out}(G)$ ?*

Question 9.6.3 has a positive answer whenever  $G$  is a free-by-cyclic group whose polytope is codimension zero, since in that case  $X_n$  is finite for all  $n$ , as remarked above. We are also able to give a positive answer in certain special cases. To that end, given a group  $G$  and a collection  $\mathcal{H}$  of subgroups of  $G$ , the *McCool group*  $\text{Mc}(G; \mathcal{H})$  of  $G$  with respect to  $\mathcal{H}$ , is the group of outer automorphisms of  $G$  which admit a representative which restricts to identity on each element of  $\mathcal{H}$ .

**Lemma 9.6.4.** *Let  $G$  be a free-by-cyclic group and suppose that  $G$  splits as an HNN extension  $G = \langle G_1, s \mid z_1^s = z_1 \rangle$ . Suppose that for each  $k \geq 1$ ,  $G_1$  has finitely many characters with finitely generated kernel of rank  $k$ , up to the action of  $\text{Mc}(G_1; \{\langle z_1 \rangle, \langle z_1^s \rangle\})$ . Then  $G$  has finitely many characters with finitely generated kernel of fixed rank  $k$ , for each  $k$ , up to the action of  $\text{Out}(G)$ .*

*Proof.* Let  $G$  be as in the statement. Then  $H_1(G; \mathbb{R}) = i_* H_1(G_1; \mathbb{R}) \oplus s\mathbb{R}$ , where  $i: G_1 \rightarrow G$  is the inclusion map and  $\mathcal{P}(G) = i_* \mathcal{P}(G_1) + \mathcal{P}(s - 1) \subseteq i_* H_1(G_1; \mathbb{R})$ .

Fix  $k \geq 1$ . For any integral character  $\varphi \in H^1(G; \mathbb{R})$ ,  $\varphi = \varphi_1 + ms^*$  where  $\varphi_1 \in H^1(G_1; \mathbb{Z})$  and  $m \in \mathbb{Z}$ . Thus  $\varphi$  evaluated on points in the polytope  $\mathcal{P}(G)$  is equal to  $\varphi_1$ . Suppose that  $\varphi$  is a primitive integral character with finitely generated kernel. By Lemma 9.5.5,  $\varphi_1$  is an integral character of  $G_1$  with finitely generated kernel such that

$$\text{rank}(\ker \varphi) - 1 = c(\text{rank}(\ker \varphi_1) - 1) + |\varphi(u)|,$$

where  $c$  is the positive generator of  $\varphi_1(G_1)$  in  $\mathbb{Z}$ . In particular, if  $\text{rank}(\ker \varphi_1) \geq k + 1$  then  $\text{rank}(\ker \varphi) \geq k + 1$ .

Let  $X_k$  be the collection of primitive integral characters of  $G$  with finitely generated kernel of rank  $k$ . For each  $i \leq k$ , let  $Y_i$  be the collection of primitive integral characters of  $G_1$  with finitely generated kernel of rank  $i$ . By the above,

$$X_k \subseteq \bigcup_{i=1}^k \{\varphi_1 + ms^* \mid \varphi_1 \in Y_i, m \in \mathbb{Z}\}.$$

Define an automorphism  $\theta$  of  $G$  which acts as the identity on  $G_1$  and sends  $s \mapsto z_1s$ . Then for any character  $\varphi_1$  of  $G_1$  and for any  $g \in G$ ,

$$\varphi_1 \circ \theta(s) = \varphi_1(z_1) \text{ and } \varphi_1 \circ \theta(g) = \varphi_1(g).$$

Hence  $\varphi_1 \circ \theta^k = \varphi_1 + k \cdot \varphi_1(z_1) \cdot s^*$  for each  $k \in \mathbb{Z}$ .

By Theorem 3.3.9,  $z_1$  is not contained in the fibre of any fibring of  $G$ . It follows that  $\langle \theta \rangle$  acts on  $\{\varphi_1 + ms^* \mid m \in \mathbb{Z}\}$  with finitely many orbits. Furthermore, each outer automorphism in  $\text{Mc}(G_1; \{\langle z_1 \rangle, \langle z_1 \rangle\})$  extends to an outer automorphism of  $G$ . Hence the subgroup of  $\text{Out}(G)$  generated by  $\text{Mc}(G_1; \{\langle z_1 \rangle, \langle z_1 \rangle\}) \cup \langle [\theta] \rangle$  acts on  $\bigcup_{i=1}^k \{\varphi_1 + ms^* \mid \varphi_1 \in Y_i, m \in \mathbb{Z}\}$  with finitely many orbits.  $\square$

**Lemma 9.6.5.** *Let  $G$  be a free-by-cyclic group such that  $G$  splits as an HNN extension  $G = \langle G_1, s \mid z_1^s = z_2 \rangle$ . Suppose that  $i_*\mathcal{P}(G)$  has full rank in  $i_*H_1(G_1; \mathbb{R})$ . Then for every  $k$ ,  $G$  has at most finitely many characters with finitely generated kernel of rank  $k$ , up to the action of  $\text{Out}(G)$ .*

*Furthermore, the number of orbits of characters  $\varphi$  of  $G$  with finitely generated kernel and such that  $\varphi(z_1) = 1$ , is at most the number of such characters contained in the subspace  $H^1(i_*H_1(G_1; \mathbb{R}); \mathbb{R})$ .*

*Proof.* Let  $G$  be as in the statement. As above, we have that  $H_1(G; \mathbb{R}) = i_*H_1(G_1; \mathbb{R}) \oplus s\mathbb{R}$  and  $\mathcal{P}(G) = i_*\mathcal{P}(G_1) + \mathcal{P}(\bar{z}_1 - 1) \subseteq i_*H_1(G_1; \mathbb{R})$ . Hence every integral character  $\varphi \in H^1(G; \mathbb{R})$  is of the form  $\varphi = \varphi_1 + ms^*$  where  $\varphi_1 \in H^1(i_*H_1(G_1; \mathbb{R}); \mathbb{R})$  and  $m \in \mathbb{Z}$ .

Let  $X_k$  be the collection of primitive integral characters of  $G$  with finitely generated kernel of rank  $k$ . For each  $i \leq k$ , let  $Y_i$  be the collection of primitive characters of  $G$  with finitely generated kernel of rank  $i$  and which are contained

in  $H^1(i_*H_1(G_1; \mathbb{R}); \mathbb{R})$ . Since  $\mathcal{P}(G)$  has full dimension in  $i_*(H_1(G_1; \mathbb{R}))$ , each  $Y_i$  is finite. By the same argument as in the proof of Lemma 9.6.4, we have that

$$X_k \subseteq \bigcup_{i=1}^k \{\varphi_1 + ms^* \mid \varphi_1 \in Y_i, m \in \mathbb{Z}\}.$$

The automorphism  $\theta$  of  $G$  defined in the proof of Lemma 9.6.4 acts on each  $\{\varphi_1 + ms^* \mid \varphi_1 \in Y_i, m \in \mathbb{Z}\}$  with finitely many orbits.

Let  $\varphi$  be a primitive integral character of  $G$  with finitely generated kernel and such that  $\varphi(z_1) = 1$ . Then there exists a unique character  $\varphi_1 \in H^1(i_*H_1(G_1; \mathbb{R}); \mathbb{R})$  such that  $\varphi = \varphi_1 + ms^*$  for some  $m \in \mathbb{Z}$  and  $\varphi(z_1) = \varphi_1(z_1) = 1$ . Note that  $\varphi_1$  is necessarily primitive and integral. Also  $\varphi_1 \circ \theta^m = \varphi_1 + ms^*$ .

□

## 9.7 Examples

In this section, we calculate examples of polytopes for free-by-cyclic groups.

### 9.7.1 Free-by-cyclic groups whose polytopes are lines

We start with a simple but important observation:

**Proposition 9.7.1.** *Let  $G$  be a free-by-cyclic group. Suppose that there exists a free-by-cyclic group  $G'$  which is isomorphic to a finite-index subgroup of  $G$ . If  $\mathcal{P}(G')$  is a line then  $\mathcal{P}(G)$  is also a line.*

*Proof.* Firstly, note that if  $P \subseteq \mathbb{R}^n$  is a polytope which is a line then for any automorphism  $A \in \text{GL}(n, \mathbb{R})$ , the image  $A \cdot P$  is also a line. Hence if  $G$  is a free-by-cyclic group with polytope  $\mathcal{P}(G)$  represented by a line, and  $H$  is isomorphic to  $G$ , then  $\mathcal{P}(H)$  is also represented by a line.

Now suppose that  $G'$  is a finite-index subgroup of  $G$ . A primitive integral character  $\varphi: G \rightarrow \mathbb{R}$  of  $G$  induces an integral character  $\varphi': G' \rightarrow \mathbb{R}$  defined by restriction  $\varphi' = \varphi|_{G'}$ . Using the multiplicative property of  $L^2$ -Euler characteristic,

it follows that

$$\begin{aligned}\|\varphi'\|_{\mathcal{P}(G')} &= -\frac{1}{N}\chi^{(2)}(\ker \varphi') = -\frac{1}{N}\chi^{(2)}(\ker \varphi) [\ker \varphi : \ker \varphi'] \\ &= \frac{1}{N}\|\varphi\|_{\mathcal{P}(G)}[\ker \varphi : \ker \varphi'],\end{aligned}\tag{9.4}$$

where  $N$  denotes the positive generator of the image of  $\varphi'$  in  $\mathbb{Z}$ .

Hence,  $\|\varphi\|_{\mathcal{P}(G)} = 0$  if and only if  $\|\varphi'\|_{\mathcal{P}(G')} = 0$ . Since  $\mathcal{P}(G')$  is a line joining the origin to an integral point  $x$  in  $H_1(G'; \mathbb{R})$  which corresponds to an element of the free abelianisation of  $G'$ , the kernel  $\ker \|\cdot\|_{\mathcal{P}(G')}$  is the annihilator of the subspace of  $H_1(G'; \mathbb{R})$  spanned by  $x$ . Hence  $\ker \|\cdot\|_{\mathcal{P}(G)}$  is the annihilator of the subspace of  $H_1(G; \mathbb{R})$  spanned by the image of  $x$  in  $H_1(G; \mathbb{R})$ . Note that  $G$  is free-by-cyclic and thus admits an integral character with finitely generated kernel. In particular,  $\ker \|\cdot\|_{\mathcal{P}(G)}$  is a proper subspace of  $H^1(G; \mathbb{R})$  and the image of  $x$  in  $H_1(G; \mathbb{R})$  is non-trivial. Thus  $\mathcal{P}(G)$  is also a line (and not a point).  $\square$

**Corollary 9.7.2.** *If  $\Phi \in \text{Out}(F_n)$  has finite order then  $\mathcal{P}(F_n \rtimes_{\Phi} \mathbb{Z})$  is a line.*

**Definition 9.7.3.** Let  $F$  be a free group of rank  $n$  and  $\phi \in \text{Aut}(F)$  an automorphism of  $F$ . A free basis  $\{x_1, \dots, x_n\}$  of  $F$  is said to be *right layered* for  $\phi$  if

- $\phi(x_1) = x_1$ , and
- $\phi(x_i) = x_i w_i$  for every  $i > 1$ , where  $w_i \in \langle x_1, \dots, x_{i-1} \rangle$ .

**Lemma 9.7.4.** *If  $G = F_n \rtimes_{\phi} \mathbb{Z}$  is free-by-cyclic such that  $\phi \in \text{Aut}(F_n)$  admits a right layered basis, then  $\mathcal{P}(G)$  is a line.*

*Proof.* Suppose that  $\phi \in \text{Aut}(F_n)$  admits a right layered basis. Then the matrix  $F$  of Fox derivatives of  $\phi$  with respect to this basis is upper triangular with each diagonal element equal to 1.

Let  $G = F_n \rtimes_{\phi} \mathbb{Z}$  and let  $t \in G$  be a stable letter. By the discussion in Section 9.3,

$$\begin{aligned}\mathcal{P}(G) &= \mathcal{P}(\det_D(I - tF)) - \mathcal{P}(1 - t) \\ &= \mathcal{P}((1 - t)^n) - \mathcal{P}(1 - t) \\ &= (n - 1) \cdot [0, t].\end{aligned}\tag{9.4}$$

**Lemma 9.7.5.** *Let  $\phi \in \text{Aut}(F_n)$  be such that the rank of the fixed subgroup of  $\phi$  is equal to  $n$ . Then  $\mathcal{P}(G)$  is a line.*

*Proof.* By the work of Collins–Turner [CT96] every automorphism  $\phi \in \text{Aut}(F_n)$  with maximal rank fixed subgroup admits a right layered basis. The result then follows by Lemma 9.7.4.  $\square$

At the other extreme, there exist hyperbolic free-by-cyclic groups with non-cyclic free abelianisation whose polytopes are lines.

**Example 9.7.6.** Let  $n \geq 3$  and  $F = F(a_0, a_1, \dots, a_{n-1})$  be the free group of rank  $n$ . Define an automorphism  $\phi \in \text{Aut}(F)$  which acts on the generators in the following way:

$$a_0 \mapsto a_1 \mapsto \dots \mapsto a_{n-1} \mapsto a_0 a_1.$$

Then  $\phi$  is atoroidal by the work of Gersten–Stallings [GS91]. Let  $G_\phi = F \rtimes_\phi \langle t \rangle$ . Then  $G_\phi^{\text{fab}} = t\mathbb{Z}$  and  $G_{\phi^k}^{\text{fab}} = t\mathbb{Z}$  for all  $k \geq 1$ . Hence the polytope  $\mathcal{P}(G_{\phi^k})$  is a line.

For  $k$  large enough, the automorphism  $\phi$  and the letter  $a_0$  satisfy the conditions of [Bri02, Proposition 2.1]. Hence, if we let  $F' = F(a_0, \dots, a_{n-1}, b)$  and  $\psi \in \text{Aut}(F')$  be given by  $\psi(a_i) = \phi(a_i)$  for all  $i$ , and  $\psi(b) = ba_0$  then  $G_\psi$  is atoroidal. Moreover,  $G_\psi$  splits as an HNN extension

$$G_\psi = G_\phi *_{\langle t \rangle}.$$

Hence the polytope  $\mathcal{P}(G)$  is equal to

$$\mathcal{P}(G_\psi) = i_*(\mathcal{P}(G_\phi)) + \mathcal{P}(1 - t).$$

## 9.7.2 Low rank cases

Given an outer automorphism  $\Phi \in \text{Out}(F_n)$ , let  $\Phi_{\text{ab}} \in \text{GL}(n, \mathbb{Z})$  denote the induced map on the abelianisation. The case of free-by-cyclic groups with fibre of rank 2 is considerably easier to study by the following useful fact:

**Theorem 9.7.7** (Nielsen). *The map induced by abelianisation,*

$$\begin{aligned} \text{Out}(F_2) &\rightarrow \text{GL}(2, \mathbb{Z}) \\ \Phi &\mapsto \Phi_{\text{ab}}, \end{aligned}$$

*is an isomorphism.*

**Proposition 9.7.8.** *Let  $\Phi$  be an element of  $\text{Out}(F_2)$ . Then the induced map  $\Phi_{\text{ab}}$  satisfies exactly one of the following.*

1.  $\Phi$  is the trivial outer automorphism and  $\beta_1(G_\Phi) = 3$ ;
2.  $\Phi_{\text{ab}}$  is conjugate to  $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$  for some  $k \in \mathbb{Z} \setminus 0$ , and  $\beta_1(G_\Phi) = 2$ ;
3.  $\Phi_{\text{ab}}$  is conjugate to  $\begin{pmatrix} -1 & k \\ 0 & 1 \end{pmatrix}$  for some  $k \in \mathbb{Z}$ , and  $\beta_1(G_\Phi) = 2$ ; or
4.  $\Phi_{\text{ab}}$  does not have 1 as an eigenvalue and  $\beta_1(G_\Phi) = 1$ .

*Furthermore, the classification preserves conjugacy classes.*

**Corollary 9.7.9.** *If  $\Phi \in \text{Out}(F_2)$  then  $\mathcal{P}(G_\Phi)$  is a line.*

*Proof.* It suffices to calculate the polytope for one representative of each conjugacy class in  $\text{Out}(F_2)$ . We use the classification in Proposition 9.7.8. Let  $G = F_2 \rtimes_\phi \langle t \rangle$ . Let  $F$  be the matrix of Fox derivatives of  $\phi$ .

Suppose first that  $\phi$  falls under Item 1 or Item 2 of Proposition 9.7.8. By Theorem 9.7.7,  $\phi$  is conjugate to the outer automorphism represented by  $\psi \in \text{Aut}(F(a, b))$ , where  $\psi$  acts by sending  $a \mapsto a, b \mapsto ba^i$ , for some  $i \in \mathbb{Z}$ . The matrix of Fox derivatives of  $\psi$  is upper triangular with 1s on the diagonal. Hence  $\det_D(I - tF) = (1 - t)^2$ . Thus

$$\begin{aligned} \mathcal{P}(G_\psi) &= \mathcal{P}(\det_D(I - tF)) - \mathcal{P}(1 - t) \\ &= \mathcal{P}(1 - t) \\ &\equiv_T [0, t]. \end{aligned}$$

If  $\phi$  falls under Item 3 then  $\phi$  is conjugate to  $\psi \in \text{Aut}(F(a, b))$ , where  $\psi$  sends  $a \mapsto a^{-1}, b \mapsto ba^i$  for some  $i \in \mathbb{Z}$ . The matrix of Fox derivatives is upper triangular with  $-a^{-1}$  and 1 on the diagonal. Since  $a$  vanishes in the free abelianisation of  $G_\psi$ , it follows that  $\mathcal{P}(G_\psi) = [0, t]$ , also.

Finally, if  $\phi$  falls under Item 4 of Proposition 9.7.8 then  $H_1(G; \mathbb{R}) = t\mathbb{R}$ , and thus  $\mathcal{P}(G)$  is necessarily also a line.  $\square$

**Lemma 9.7.10.** *Let  $G = F_n \times \mathbb{Z}$ . Then there are finitely many orbits of integral characters of  $G$  with kernel of a given finite rank under the action of  $\text{Out}(G)$ .*

*Proof.* Let  $G$  be a group given by the presentation

$$G = \langle a_1, \dots, a_n, t \mid a_i^t = a_i \ \forall i \rangle.$$

The abelianisation of  $G$  is of the form

$$G_{\text{ab}} = \bar{a}_1\mathbb{Z} \oplus \dots \oplus \bar{a}_n\mathbb{Z} \oplus \bar{t}\mathbb{Z}.$$

The polytope  $\mathcal{P}(G)$  is represented by a line joining the origin to the vertex corresponding to the element  $(n-1) \cdot \bar{t}$  in  $H_1(G; \mathbb{R})$ . Hence, for any primitive integral character  $\varphi$  of  $G$ , the character  $\varphi$  fibres if and only if  $\varphi(t) \neq 0$ . In that case,

$$\text{rank}(\ker \varphi) = (n-1) \cdot |\varphi(t)| + 1.$$

Thus, the integral characters of rank  $k$  are exactly the integral characters in the hyperplanes

$$H_k^\pm = \left\{ \varphi \in \text{Hom}(G; \mathbb{R}) \mid \varphi(t) = \pm \frac{k-1}{n-1} \right\} \subseteq H^1(G; \mathbb{R}).$$

For every  $i \leq n$ , define the automorphism  $\theta_i \in \text{Aut}(G)$  which maps  $a_i \mapsto ta_i$  and fixes the remaining generators. Then the subgroup of  $\text{Aut}(G)$  generated by the  $\theta_i$  acts on each hyperplane  $H_k^\pm$  and there are finitely many orbits under the action.  $\square$

**Lemma 9.7.11.** *If  $\phi \in \text{Aut}(F_2)$  then there are finitely many orbits of primitive characters  $\varphi: G_\phi \rightarrow \mathbb{Z}$  with finitely generated kernel of a given rank, up to the action of  $\text{Out}(G_\phi)$ .*

*Proof.* Let  $G = F \rtimes_\phi \langle t \rangle$  where  $F = F(a, b)$  is a free group of rank 2. We treat each case from Proposition 9.7.8 separately. If  $\phi$  falls under Item 4 then  $G_\phi$  admits exactly two characters so there is nothing to check.

Suppose  $\phi$  falls under Item 2 or Item 3. We may assume that  $\phi(a) = a^{\pm 1}$  and  $\phi(b) = ba^i$ , with  $i \in \mathbb{Z} \setminus \{0\}$ . Then  $G_\phi$  splits as an HNN extension,  $G_\phi = \langle a, t \rangle *_{t=ta^{-i}}$ . Note that in both cases  $a$  vanishes in the free abelianisation of  $G_\phi$  and thus  $\mathcal{P}(\langle a, t \rangle) + \mathcal{P}(1 - t) = [0, t]$  is full dimensional in  $i^*H_1(\langle a, t \rangle; \mathbb{R}) \simeq t\mathbb{R}$ . Hence we can apply Lemma 9.6.5.

If  $\phi$  belongs to Case 1 then the result follows by Lemma 9.7.10. □

In [Bri02], Brinkmann classifies the free-by-cyclic groups which split over  $\mathbb{Z}$ . Whilst in the general rank case the picture is quite complicated, when the fibre has rank 3 there are only three types of situations:

**Theorem 9.7.12** (Brinkmann [Bri02]). *Let  $\phi \in \text{Aut}(F_3)$  and suppose that  $G_\phi = F \rtimes_\phi \langle t \rangle$  splits over  $\mathbb{Z}$ . Then at least one of the following is true.*

1. *The group  $G_\phi$  splits as an amalgamated free product,  $G_\phi = G_1 *_\mathbb{Z} G_2$ , and thus  $F = F^1 * F^2$ , where  $\text{rank}(F^1) = 2$ ,  $\text{rank}(F^2) = 1$  and for each  $i = 1, 2$ ,  $\phi(F^i) = F^i$  and  $G_i = F^i \rtimes \langle t_i \rangle$ , for some  $t_i \in Ft$ .*
2. *The group  $G_\phi$  splits as an HNN extension,  $G_\phi = G_1 *_\mathbb{Z}$ , and*

(a)  *$F = F^1 * \langle a_2 \rangle$ , where  $\phi(F^1) = F^1$  and  $\phi(a_2) = wa_2v^{-1}$  for some  $w, v \in F^1$ ,  $G_1 = F^1 \rtimes_\phi \langle t \rangle$ , and*

$$G = \langle G_1, s \mid (tw)^s = tv \rangle, \text{ or}$$

(b)  *$F = \langle a_0, a_1, a_2 \rangle$ , where  $\phi(a_0) = a_1^{\varepsilon_0}$ ,  $\phi(a_1) = a_2^{-1}a_0^{\varepsilon_1}a_2$ ,  $\phi(a_2) = a_0^ka_2$  for some  $\varepsilon_0, \varepsilon_1 \in \{0, 1\}$  and  $k \in \mathbb{Z}$ . Also,  $G_1 = \langle a_0, t^2a_2^{-1} \rangle$ , and*

$$G = \langle G_1, s \mid (t^2a_2^{-1})^s = t^2a_2^{-1}a_0^{-k} \rangle.$$

Note that in general there exist hyperbolic free-by-cyclic groups which split as HNN extensions and amalgamated free products over  $\mathbb{Z}$  (see Example 9.7.6).

**Corollary 9.7.13.** *If  $\phi \in \text{Aut}(F_3)$  is atoroidal, then  $G_\phi$  does not split over  $\mathbb{Z}$ .*

*Proof.* In Cases 1 and 2a, the map  $\phi$  preserves a subgroup of  $F$  isomorphic to the free group of rank 2. Hence  $\phi$  restricts to an automorphism of  $F_2$ , which always preserves the conjugacy class of a commutator of a pair of generators of  $F_2$ .

In Case 2b,  $\Phi^2(a_0)$  is conjugate to  $a_0$  or  $a_0^{-1}$ . □

**Corollary 9.7.14.** *If  $\phi \in \text{Aut}(F_3)$  is atoroidal, then  $\text{Out}(G_\phi)$  is finite.*

*Proof.* This follows by Corollary 9.7.13 and Paulin's theorem [Pau91]. □

**Corollary 9.7.15.** *If  $\phi \in \text{Aut}(F_3)$  is such that  $G_\phi$  splits over  $\mathbb{Z}$  then there exists a subgroup  $H \leq G_\phi$  of index at most 2 such that  $\mathcal{P}(H)$  is less than full dimensional.*

*Proof.* By Lemma 9.5.3, it suffices to check the case where  $G_\phi$  splits as an amalgamated free product. By Theorem 9.7.12 this happens exactly when the fibre admits a  $\phi$ -invariant splitting  $F \simeq F^1 * F^2$  where  $F^2 \simeq \mathbb{Z}$ . Hence

$$\mathcal{P}(F \rtimes_\phi \langle t \rangle) = \mathcal{P}(F^1 \rtimes_\phi \langle t \rangle) + \mathcal{P}(F^2 \rtimes_\phi \langle t \rangle) + \mathcal{P}(1 - t).$$

Note that the polytope  $\mathcal{P}(F^2 \rtimes \mathbb{Z})$  is trivial since  $F^2$  is infinite cyclic. If  $\phi$  acts as the identity on a generator  $s$  of  $F^2$  then the homology of  $G_\phi$  splits as  $H_1(G_\phi, \mathbb{R}) = i_*(H_1(F^1 \rtimes_\phi \mathbb{Z})) \oplus s\mathbb{R}$  and  $\mathcal{P}(F \rtimes_\phi \langle t \rangle) \subseteq i_*(H_1(F^1 \rtimes_\phi \mathbb{Z}))$ . If  $\phi$  maps  $s \mapsto s^{-1}$  then replace  $G_\phi$  by the index-2 subgroup  $H = F \rtimes_\phi \langle t^2 \rangle \leq G_\phi$ . □

**Proposition 9.7.16.** *Suppose that  $\phi \in \text{Aut}(F_3)$  is exponentially growing and  $G_\phi$  splits over  $\mathbb{Z}$ . Then there are finitely many orbits of characters  $\chi: G_\phi \rightarrow \mathbb{Z}$  with kernel of a fixed finite rank under the action of  $\text{Out}(G_\phi)$ . There is a single orbit of characters with kernel of rank 3.*

*Proof.* We use Lemma 9.5.1 and analysis of polytopes for free-by-cyclic groups with fibres of rank 2 in Corollary 9.7.9 to calculate the polytopes in this case.

Suppose first that  $G_\phi$  belongs to Case 1 of Theorem 9.5.1. The factor  $F^2$  is infinite cyclic so  $\mathcal{P}(G_2)$  is a vertex and the abelianisation is either  $\langle t \rangle$  if  $G_2$

is the Klein bottle  $K$  or  $\mathbb{Z}^2$  if  $G_2 \simeq \mathbb{Z}^2$ . Since  $\phi$  is exponentially growing, it follows that  $\phi$  restricted to  $F^1$  is exponential and thus it belongs to Case 2 of Proposition 9.7.8 and  $G_1^{\text{ab}} = \langle t \rangle$ . So either  $G_2 = K$  and the free abelianisation of  $G$  is one-dimensional, or  $G_2 \simeq \mathbb{Z}^2$ , the free abelianisation of  $G$  is freely generated by  $t$  and  $a_2$  and  $\mathcal{P}(G_\phi)$  is a line of length 2 in the  $t$ -axis. In the latter case, the polytope is not full dimensional but the characters with finitely generated kernels of constant rank are partitioned into finitely many orbits under the action of the automorphism  $\varphi \in \text{Aut}(G_\phi)$  given by  $\varphi: t \mapsto t, a_2 \mapsto a_2 t$  and  $\varphi|_{F^1} = \text{id}_{F^1}$ . Note that  $\varphi$  is not inner. Furthermore, there is a single orbit of characters  $\chi$  with  $\chi(t) = 1$ .

Suppose now that  $G$  belongs to Case 2a. Again  $\phi|_{F^1}$  is exponential and so  $G_1^{\text{ab}} \simeq \langle t \rangle$  and  $\mathcal{P}(G_1)$  is a line of unit length in the  $t$ -axis. Then  $\mathcal{P}(G) = \mathcal{P}(G_1) - \mathcal{P}(\langle tw \rangle)$ , where  $w \in F^1$ . But  $w$  is mapped to the trivial element in the abelianisation of  $G_\phi$  and thus  $\mathcal{P}(G) = \mathcal{P}(G_1) + \mathcal{P}(t - 1)$  is a line of length 2 in the  $t$ -axis. We define an automorphism  $\varphi \in \text{Aut}(G_\phi)$  so that  $a_2 \mapsto a_2 t v$  and  $\varphi$  acts as the identity on  $G_1$ . It is clear that  $\varphi$  is not an inner automorphism and the characters with fibre of the same rank again fall into finitely many orbits under the action of  $\langle \varphi \rangle$ .

If  $\phi$  belongs to Case 2b then it is polynomially growing. □

**Corollary 9.7.17.** *Let  $\Phi, \Psi \in \text{Out}(F_3)$  and suppose  $\Phi$  is exponentially growing and  $G_\Phi$  splits over  $\mathbb{Z}$ . If  $G_\Phi \simeq G_\Psi$  then  $\Phi$  and  $\Psi$  are conjugate in  $\text{Out}(F_3)$ .*

# Chapter 10

## Questions and further work

### Subgroup separability and quasiconvex subgroups

The results in Chapter 4 show that subgroup separability is a rare property for free-by-cyclic groups. In fact, the only free-by-cyclic groups which are known to be subgroup separable arise as fundamental groups of geometric 3-manifolds. This suggests the following question:

**Question 10.0.1.** *Let  $G$  be a non-hyperbolic free-by-cyclic group which is subgroup separable. Then is it the case that  $G$  is commensurable to the fundamental group of a geometric 3-manifold?*

If  $G$  is a hyperbolic free-by-cyclic group then  $G$  acts properly and cocompactly on a CAT(0) cube complex by the work of Hagen–Wise [HW15]. Hence  $G$  is virtually special in the sense of Haglund–Wise [HW08], and thus every quasiconvex subgroup  $H \leq G$  is separable in  $G$ . Hence, understanding quasiconvex subgroups of hyperbolic groups is a key step towards the classification of separable subgroups.

Note that when  $G$  is the fundamental group of a hyperbolic 3-manifold of finite volume, then any finitely generated subgroup  $H \leq G$  is either a virtual fibre of  $G$  or is *geometrically finite* and hence quasiconvex. We propose an analogous classification for subgroups of hyperbolic free-by-cyclic groups. We say a subgroup  $A \leq G$  is a *virtual semi-fibre* if there exists a finite-index subgroup  $G' \leq G$  such that  $G'$  is a proper ascending HNN extension over  $G' \cap A$ .

**Conjecture 10.0.2.** *Let  $G$  be a hyperbolic free-by-cyclic group and suppose that  $H \leq G$  is a finitely generated subgroup. Then  $H$  is quasiconvex unless it is a virtual fibre or a virtual semi-fibre of a free-by-cyclic subgroup  $G_0 \leq G$ .*

If  $G$  is hyperbolic and irreducible then by the work of Mutanguha any free-by-cyclic subgroup  $G_0 \leq G$  has finite index. In that case, Conjecture 10.0.2 predicts that the only possible non-separable subgroups arise from homomorphisms  $\varphi: G' \rightarrow \mathbb{Z}$  of finite-index subgroups  $G' \leq G$  such that  $\varphi: G' \rightarrow \mathbb{Z}$  is contained in the BNS invariant and  $-\varphi$  is not (as in the argument in Lemma 5.0.12).

**Conjecture 10.0.3.** *Let  $G$  be an irreducible and hyperbolic free-by-cyclic group. Then  $G$  is subgroup separable if and only if for every finite-index subgroup  $G' \leq G$  the BNS invariant  $\Sigma(G)$  is symmetric.*

**Problem 10.0.4.** *Classify hyperbolic free-by-cyclic groups with a symmetric BNS invariant.*

## Profinite rigidity

Whilst the results in Chapter 6 make some progress towards the problem of distinguishing free-by-cyclic groups through finite quotients, many questions in this area remain open.

Recall that for any two groups  $G$  and  $H$ , a profinite isomorphism  $\Theta: \widehat{G} \rightarrow \widehat{H}$  is said to be  $\widehat{\mathbb{Z}}$ -regular, if the induced isomorphism on the completions of the free abelianisations

$$\Theta_{\text{fab}}: \widehat{G}_{\text{fab}} \rightarrow \widehat{H}_{\text{fab}}$$

is the composition of the completion of an isomorphism  $\Psi: G_{\text{fab}} \rightarrow H_{\text{fab}}$  and multiplication by a profinite integer in  $\widehat{\mathbb{Z}}^\times$ .

In broad terms, the existence of a  $\widehat{\mathbb{Z}}$ -regular profinite isomorphism allows us to show that certain homological information about a group is preserved under profinite isomorphisms. It is not too difficult to show that when  $G$  is a free-by-cyclic group with first Betti number equal to one, then all profinite isomorphisms are  $\widehat{\mathbb{Z}}$ -regular. In general, we ask:

**Question 10.0.5.** *If  $G$  and  $H$  are free-by-cyclic groups which are profinitely isomorphic, does there exist a  $\widehat{\mathbb{Z}}$ -regular profinite isomorphism  $\widehat{G} \rightarrow \widehat{H}$ ?*

One may hope to answer the previous question as in [Liu23a] using the  $L^2$ -polytope (see Chapter 9) in place of the Thurston polytope. The key issue is that we do not have the  $\text{TAP}_1$  property for free-by-cyclic groups (for 3-manifolds this is a deep result of Friedl–Vidussi [FV08, FV11]). The reader is referred to [HK22, Definition 3.1] for the definition due to its technical nature.

**Question 10.0.6.** *Is every free-by-cyclic group  $G$  in  $\text{TAP}_1(\mathbb{F})$  for  $\mathbb{F} \in \{\mathbb{Q}, \mathbb{F}_p\}$  with  $p$  prime?*

The key step in proving the profinite rigidity results in Chapter 6 involved showing that many properties of free-by-cyclic groups can be detected in the profinite completion [HK23, Theorem B]. Hence, we ask

**Question 10.0.7.** *Which of the following properties can be detected in the profinite completion of a free-by-cyclic group  $G = F_n \rtimes_{\Phi} \mathbb{Z}$ :*

1. *the (full) irreducibility of  $\Phi$ ;*
2. *whether  $\Phi$  is polynomially or exponentially growing, and the degree of polynomial growth;*
3. *whether  $G$  splits over  $\mathbb{Z}$ ; more generally, the JSJ decomposition of a free-by-cyclic group;*
4. *the structure of the BNS invariant and more generally the marked  $L^2$ -polytope of  $G$ .*

## The $L^2$ -polytope

Let  $G$  be a group and  $\varphi: G \rightarrow \mathbb{Z}$  an epimorphism. We say that  $(G, \varphi)$  *splits with associated group*  $B \leq G$ , if there exists a finitely generated subgroup  $A \leq \ker \varphi$  which contains  $B$ , and a monomorphism  $\theta: B \rightarrow A$ , such that  $G$  splits as an HNN extension

$$G \simeq A *_B \theta .$$

**Definition 10.0.8.** The *splitting complexity* of the pair  $(G, \varphi)$  is

$$c(G, \varphi) = \min\{-\chi(B) \mid B \text{ is of type } F \text{ and } G \text{ splits with associated subgroup } B\}.$$

**Remark 10.0.9.** If  $G$  is the mapping torus of a free group monomorphism, in particular a free-by-cyclic group, then by the work of Feighn–Handel [FH99], every finitely generated subgroup of  $G$  is of type  $F$ . Hence, in Definition 10.0.8 of splitting complexity for a free-by-cyclic group, we consider all finitely generated subgroups  $B$  such that  $(G, \varphi)$  splits with associated subgroup  $B$ .

The following conjecture is due to Gardam–Kielak and appears in [Obe20] and [GK].

**Conjecture 10.0.10** (The splitting complexity conjecture). *Let  $G$  be a free-by-cyclic group and  $\varphi: G \rightarrow \mathbb{Z}$  an epimorphism. Then*

$$c(G, \varphi) = -\chi^{(2)}(\ker \varphi) = \|\varphi\|,$$

where  $\|\cdot\|$  is the Thurston seminorm as in Definition 9.4.2.

**Remark 10.0.11.** The argument used by Henneke–Kielak in [HK20, Theorem 6.4] can be used to show that if  $G$  is free-by-cyclic, then for every epimorphism  $\varphi: G \rightarrow \mathbb{Z}$ ,

$$c(G, \varphi) \geq \|\varphi\|.$$

A key consequence of the splitting complexity conjecture is the following:

**Theorem 10.0.12** ([GK]). *If Conjecture 10.0.10 is true then the first BNS invariant of a free-by-cyclic group  $G$  is computable.*

Note that the work of Cavallo–Delgado–Kahrobaei–Vantura in [CDKV17] implies that the BNS invariant is not computable in the class of all finitely presented groups.

The following special case of the splitting complexity conjecture itself has a number of interesting consequences.

**Conjecture 10.0.13** (Vanishing norm conjecture). *Let  $G$  be free-by-cyclic and  $\varphi: G \rightarrow \mathbb{Z}$  an epimorphism such that  $\chi^{(2)}(\ker \varphi) = 0$ . Then  $(G, \varphi)$  splits with an associated finitely generated group  $B \leq G$  such that  $\chi(B) = 0$ .*

From now until the end of this section, we will be working under the assumption that Conjecture 10.0.13 is true.

**Corollary 10.0.14.** *Let  $G$  be free-by-cyclic. Then the following are equivalent.*

1. *The polytope  $\mathcal{P}(G)$  has codimension greater than 0;*
2. *The group  $G$  splits as an HNN extensions over an infinite cyclic or a free-by-cyclic subgroup.*

*Proof.* For (1)  $\Rightarrow$  (2), suppose that  $\mathcal{P}(G)$  has codimension greater than 0. Then, there exists a character  $\varphi: G \rightarrow \mathbb{Z}$  with  $\|\varphi\|_{\mathcal{P}} = 0$ . Conjecture 10.0.13 implies that  $G$  splits as an HNN extension over a finitely generated subgroup  $B$  with  $\chi(B) = 0$ . Then by Proposition 3.3.7,  $B$  is infinite cyclic or free-by-cyclic.

The implication (2)  $\Rightarrow$  (1) is Lemma 9.5.3. □

**Corollary 10.0.15.** *Let  $\Phi \in \text{Out}(F_n)$  be a non-periodic irreducible outer automorphism and  $G = F_n \rtimes_{\Phi} \mathbb{Z}$ . Then for every finite-index subgroup  $G' \leq G$ , the polytope  $\mathcal{P}(G')$  has codimension 0.*

*Proof.* Suppose that  $\Phi \in \text{Out}(F_n)$  is not atoroidal. By Theorem 3.4.2,  $\Phi$  is induced by a pseudo-Anosov homeomorphism of a punctured surface and thus  $G$  is the fundamental group of a hyperbolic 3-manifold  $M$ . In particular, every finite-index subgroup  $G' \leq G$  corresponds to a finite degree cover  $M' \rightarrow M$  which is also a hyperbolic 3-manifold. Since the Thurston seminorm is a norm for hyperbolic 3-manifolds, it follows that the polytope  $\mathcal{P}(\pi_1(M'))$ , which has a representative given by the unit ball of the Thurston norm, has codimension 0.

Now suppose that  $\Phi \in \text{Out}(F_n)$  is atoroidal and let  $G = F_n \rtimes_{\Phi} \mathbb{Z}$ . Note that every finite-index subgroup  $G' \leq G$  has atoroidal and irreducible monodromy by [Mut21]. Towards a contradiction, assume that there is a finite-index subgroup  $G' \leq G$  such that the polytope  $\mathcal{P}(G')$  has codimension greater than 0. By Corollary 10.0.14,  $G'$  splits as an HNN extension over the infinite cyclic or free-by-cyclic

subgroup. Then by the work of Brinkmann [Bri02],  $G'$  admits a monodromy  $\Phi$  which is not fully irreducible. But then by Theorem 3.1.3, the monodromy of  $G'$  is not irreducible. This is a contradiction.  $\square$

# Bibliography

- [AG73] R. B. J. T. Allenby and R. J. Gregorac. On locally extended residually finite groups. In *Conference on Group Theory (Univ. Wisconsin-Parkside, Kenosha, Wis., 1972)*, Lecture Notes in Math., Vol. 319, pages 9–17. Springer, Berlin, 1973.
- [Ago13] Ian Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning.
- [AHK22] Naomi Andrew, Sam Hughes, and Monika Kudlinska. Torsion homology growth of polynomially growing free-by-cyclic groups, 2022. arXiv:2211.04389 [math.GR].
- [AM22] Naomi Andrew and Armando Martino. Free-by-cyclic groups, automorphisms and actions on nearly canonical trees. *J. Algebra*, 604:451–495, 2022.
- [Are22] Macarena Arenas. A cubicalrips construction, 2022. arXiv:2202.01048 [math.GR].
- [BB97] Mladen Bestvina and Noel Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3):445–470, 1997.
- [BCR16] Martin R. Bridson, Marston Conder, and Alan Reid. Determining Fuchsian groups by their finite quotients. *Israel J. Math.*, 214(1):1–41, 2016.
- [BF95] Mladen Bestvina and Mark Feighn. Stable actions of groups on real trees. *Invent. Math.*, 121(2):287–321, 1995.

- [BF18] Corey Bregman and Neil J. Fullarton. Hyperelliptic graphs and the period mapping on outer space. *J. Topol.*, 11(1):221–256, 2018.
- [BF20] Michel Boileau and Stefan Friedl. The profinite completion of 3-manifold groups, fiberedness and the Thurston norm. In *What’s next?—the mathematical legacy of William P. Thurston*, volume 205 of *Ann. of Math. Stud.*, pages 21–44. Princeton Univ. Press, Princeton, NJ, 2020.
- [BFH00] Mladen Bestvina, Mark Feighn, and Michael Handel. The Tits alternative for  $\text{Out}(F_n)$ . I. Dynamics of exponentially-growing automorphisms. *Ann. of Math. (2)*, 151(2):517–623, 2000.
- [BFH05] Mladen Bestvina, Mark Feighn, and Michael Handel. The Tits alternative for  $\text{Out}(F_n)$ . II: A Kolchin type theorem. *Ann. Math. (2)*, 161(1):1–59, 2005.
- [BG10] Martin R. Bridson and Daniel Groves. The quadratic isoperimetric inequality for mapping tori of free group automorphisms. *Mem. Amer. Math. Soc.*, 203(955):xii+152, 2010.
- [BH92] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Ann. of Math. (2)*, 135(1):1–51, 1992.
- [BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.
- [BHS17] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups. *Geom. Topol.*, 21(3):1731–1804, 2017.
- [BHW11] Nicolas Bergeron, Frédéric Haglund, and Daniel T. Wise. Hyperplane sections in arithmetic hyperbolic manifolds. *J. Lond. Math. Soc. (2)*, 83(2):431–448, 2011.

- [Bie81] Robert Bieri. *Homological dimension of discrete groups*. Queen Mary College, Department of Pure Mathematics, London,, second edition, 1981.
- [BKS87] Robert G. Burns, Abraham Karrass, and Donald Solitar. A note on groups with separable finitely generated subgroups. *Bull. Austral. Math. Soc.*, 36(1):153–160, 1987.
- [BMRS20] Martin R. Bridson, David B. McReynolds, Alan W. Reid, and Ryan Spitler. Absolute profinite rigidity and hyperbolic geometry. *Ann. Math. (2)*, 192(3):679–719, 2020.
- [BMV07] Oleg Bogopolski, Armando Martino, and Eric Ventura. The automorphism group of a free-by-cyclic group in rank 2. *Comm. Algebra*, 35(5):1675–1690, 2007.
- [BNS87] Robert Bieri, Walter D. Neumann, and Ralph Strebel. A geometric invariant of discrete groups. *Invent. Math.*, 90(3):451–477, 1987.
- [BR20] Martin R. Bridson and Alan W. Reid. Profinite rigidity, fibering, and the figure-eight knot. In *What’s next?—the mathematical legacy of William P. Thurston*, volume 205 of *Ann. of Math. Stud.*, pages 45–64. Princeton Univ. Press, Princeton, NJ, 2020.
- [BR22] Martin R. Bridson and Alan W. Reid. Profinite rigidity, Kleinian groups, and the cofinite Hopf property. *Michigan Math. J.*, 72:25–49, 2022.
- [Bra99] Noel Brady. Branched coverings of cubical complexes and subgroups of hyperbolic groups. *J. London Math. Soc. (2)*, 60(2):461–480, 1999.
- [Bri00a] Peter Brinkmann. Hyperbolic automorphisms of free groups. *Geom. Funct. Anal.*, 10(5):1071–1089, 2000.
- [Bri00b] Peter Brinkmann. *Mapping tori of automorphisms of hyperbolic groups*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)—The University of Utah.

- [Bri02] Peter Brinkmann. Splittings of mapping tori of free group automorphisms. *Geom. Dedicata*, 93:191–203, 2002.
- [Bro87] Kenneth S. Brown. Trees, valuations, and the Bieri-Neumann-Strebel invariant. *Invent. Math.*, 90(3):479–504, 1987.
- [BRW17] Martin R. Bridson, Alan W. Reid, and Henry Wilton. Profinite rigidity and surface bundles over the circle. *Bull. Lond. Math. Soc.*, 49(5):831–841, 2017.
- [BS94] Martin R. Bridson and Gadde A. Swarup. On Hausdorff-Gromov convergence and a theorem of Paulin. *Enseign. Math. (2)*, 40(3-4):267–289, 1994.
- [CDKV17] Bren Cavallo, Jordi Delgado, Delaram Kahrobaei, and Enric Ventura. Algorithmic recognition of infinite cyclic extensions. *J. Pure Appl. Algebra*, 221(9):2157–2179, 2017.
- [Chi76] Ian M. Chiswell. Exact sequences associated with a graph of groups. *J. Pure Appl. Algebra*, 8(1):63–74, 1976.
- [CL95] Marshall M. Cohen and Martin Lustig. Very small group actions on  $\mathbf{R}$ -trees and Dehn twist automorphisms. *Topology*, 34(3):575–617, 1995.
- [CL99] Marshall M. Cohen and Martin Lustig. The conjugacy problem for Dehn twist automorphisms of free groups. *Comment. Math. Helv.*, 74(2):179–200, 1999.
- [CL16] Christopher H. Cashen and Gilbert Levitt. Mapping tori of free group automorphisms, and the Bieri-Neumann-Strebel invariant of graphs of groups. *J. Group Theory*, 19(2):191–216, 2016.
- [CT94] D. J. Collins and E. C. Turner. Efficient representatives for automorphisms of free products. *Michigan Math. J.*, 41(3):443–464, 1994.
- [CT96] D. J. Collins and E. C. Turner. All automorphisms of free groups with maximal rank fixed subgroups. *Math. Proc. Cambridge Philos. Soc.*, 119(4):615–630, 1996.

- [CV86] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.
- [CW22] Tamunonye Cheetham-West. Absolute profinite rigidity of some closed fibered hyperbolic 3-manifolds, 2022. arXiv:2205.08693 [math.GT].
- [Die43] Jean Dieudonné. Les déterminants sur un corps non commutatif. *Bull. Soc. Math. France*, 71:27–45, 1943.
- [DKL15] Spencer Dowdall, Ilya Kapovich, and Christopher J. Leininger. Dynamics on free-by-cyclic groups. *Geom. Topol.*, 19(5):2801–2899, 2015.
- [DL20] François Dahmani and Ruoyu Li. Relative hyperbolicity for automorphisms of free products and free groups. *Journal of Topology and Analysis*, page 1–38, Oct 2020.
- [Dod77] Jozef Dodziuk. de Rham-Hodge theory for  $L^2$ -cohomology of infinite coverings. *Topology*, 16(2):157–165, 1977.
- [Dod79] Jozef Dodziuk.  $L^2$  harmonic forms on rotationally symmetric Riemannian manifolds. *Proc. Amer. Math. Soc.*, 77(3):395–400, 1979.
- [DT06] Nathan M. Dunfield and Dylan P. Thurston. A random tunnel number one 3-manifold does not fiber over the circle. *Geom. Topol.*, 10:2431–2499, 2006.
- [DV93] Warren Dicks and Enric Ventura. Irreducible automorphisms of growth rate one. *J. Pure Appl. Algebra*, 88(1-3):51–62, 1993.
- [Far98] Benson Farb. Relatively hyperbolic groups. *Geom. Funct. Anal.*, 8(5):810–840, 1998.
- [Far06] Benson Farb, editor. *Problems on mapping class groups and related topics*, volume 74 of *Proceedings of Symposia in Pure Mathematics*. American Mathematical Society, Providence, RI, 2006.

- [Fel71] Grigory L. Fel'dman. On the homological dimension of group algebras of solvable groups. *Mathematics of The Ussr-izvestiya*, 6:1231–1244, 1971.
- [FH99] Mark Feighn and Michael Handel. Mapping tori of free group automorphisms are coherent. *Ann. of Math. (2)*, 149(3):1061–1077, 1999.
- [Fis21] Sam P. Fisher. Improved algebraic fibrings, 2021. arXiv:2112.00397 [math.GR].
- [FK18] Florian Funke and Dawid Kielak. Alexander and Thurston norms, and the Bieri-Neumann-Strebel invariants for free-by-cyclic groups. *Geom. Topol.*, 22(5):2647–2696, 2018.
- [FL17] Stefan Friedl and Wolfgang Lück. Universal  $L^2$ -torsion, polytopes and applications to 3-manifolds. *Proc. Lond. Math. Soc. (3)*, 114(6):1114–1151, 2017.
- [FL19] Stefan Friedl and Wolfgang Lück.  $L^2$ -Euler characteristics and the Thurston norm. *Proc. Lond. Math. Soc. (3)*, 118(4):857–900, 2019.
- [FLT19] Stefan Friedl, Wolfgang Lück, and Stephan Tillmann. Groups and polytopes. In *Breadth in contemporary topology*, volume 102 of *Proc. Sympos. Pure Math.*, pages 57–77. Amer. Math. Soc., Providence, RI, 2019.
- [FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.
- [FM15] Stefano Francaviglia and Armando Martino. Stretching factors, metrics and train tracks for free products. *Illinois J. Math.*, 59(4):859–899, 2015.
- [FM21] Stefano Francaviglia and Armando Martino. Displacements of automorphisms of free groups I: Displacement functions, minpoints and train tracks. *Trans. Amer. Math. Soc.*, 374(5):3215–3264, 2021.

- [FMS21] Stefano Francaviglia, Armando Martino, and Dionysios Syrigos. The minimally displaced set of an irreducible automorphism of  $F_N$  is co-compact. *Arch. Math. (Basel)*, 116(4):369–383, 2021.
- [Fun13] Louis Funar. Torus bundles not distinguished by TQFT invariants. With an appendix by Louis Funar and Andrei Rapinchuk. *Geom. Topol.*, 17(4):2289–2344, 2013.
- [FV08] Stefan Friedl and Stefano Vidussi. Symplectic  $S^1 \times N^3$ , subgroup separability, and vanishing Thurston norm. *J. Amer. Math. Soc.*, 21(2):597–610, 2008.
- [FV11] Stefan Friedl and Stefano Vidussi. Twisted Alexander polynomials detect fibered 3-manifolds. *Ann. of Math. (2)*, 173(3):1587–1643, 2011.
- [GGH22] Damien Gaboriau, Yassine Guerch, and Camille Horbez. On the homology growth and the  $\ell^2$ -Betti numbers of  $\text{Out}(w_n)$ , 2022.
- [GH21a] Anthony Genevois and Camille Horbez. Acylindrical hyperbolicity of automorphism groups of infinitely ended groups. *J. Topol.*, 14(3):963–991, 2021.
- [GH21b] Vincent Guirardel and Camille Horbez. Measure equivalence rigidity of  $\text{Out}(f_n)$ , 2021. arXiv 2103.03696.
- [GK] Giles Gardam and Dawid Kielak. Computing fibering of free-by-cyclic groups. In preparation.
- [GN21] Damien Gaboriau and Camille Noûs. On the top-dimensional  $\ell^2$ -Betti numbers. *Ann. Fac. Sci. Toulouse Math. (6)*, 30(5):1121–1137, 2021.
- [Gro87] Mikhael Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.
- [Gro93] Mikhael Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London*

*Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

- [GS91] Stephen M. Gersten and John R. Stallings. Irreducible outer automorphisms of a free group. *Proc. Amer. Math. Soc.*, 111(2):309–314, 1991.
- [Gue20] Yassine Guerch. The symmetries of the outer space of a universal coxeter group, 2020.
- [Gue21] Yassine Guerch. Commensurations of the outer automorphism group of a universal coxeter group, 2021.
- [Hag19] Mark Hagen. A remark on thickness of free-by-cyclic groups. *Illinois J. Math.*, 63(4):633–643, 2019.
- [Hal49] Marshall Hall, Jr. Coset representations in free groups. *Trans. Amer. Math. Soc.*, 67:421–432, 1949.
- [Hem14] John Hempel. Some 3-manifold groups with the same finite quotients, 2014. arXiv:1409.3509 [math.GT].
- [Hil02] Jonathan A. Hillman. *Four-manifolds, geometries and knots*, volume 5 of *Geometry & Topology Monographs*. Geometry & Topology Publications, Coventry, 2002.
- [HK20] Fabian Henneke and Dawid Kielak. The agrarian polytope of two-generator one-relator groups. *J. Lond. Math. Soc. (2)*, 102(2):722–748, 2020.
- [HK22] Sam Hughes and Dawid Kielak. Profinite rigidity of fibring, 2022. arXiv:2206.11347 [math.GR].
- [HK23] Sam Hughes and Monika Kudłinska. On profinite rigidity amongst free-by-cyclic groups I: the generic case, 2023. arXiv 2303.16834 [math.GR].
- [HW08] Frédéric Haglund and Daniel T. Wise. Special cube complexes. *Geom. Funct. Anal.*, 17(5):1551–1620, 2008.

- [HW15] Mark F. Hagen and Daniel T. Wise. Cubulating hyperbolic free-by-cyclic groups: the general case. *Geom. Funct. Anal.*, 25(1):134–179, 2015.
- [IMM23] Giovanni Italiano, Bruno Martelli, and Matteo Migliorini. Hyperbolic 5-manifolds that fiber over  $S^1$ . *Invent. Math.*, 231(1):1–38, 2023.
- [IMP21] Claudio Llosa Isenrich, Bruno Martelli, and Pierre Py. Hyperbolic groups containing subgroups of type  $\mathcal{F}_3$  not  $\mathcal{F}_4$ , 2021.
- [IP22] Claudio Llosa Isenrich and Pierre Py. Subgroups of hyperbolic groups, finiteness properties and complex hyperbolic lattices, 2022. arXiv:2204.05788 [math.GR].
- [Jia83] Boju Jiang. *Lectures on Nielsen fixed point theory*, volume 14 of *Contemporary Mathematics*. American Mathematical Society, Providence, R.I., 1983.
- [Jia96] Boju Jiang. Estimation of the number of periodic orbits. *Pacific J. Math.*, 172(1):151–185, 1996.
- [JZ20] Andrei Jaikin-Zapirain. Recognition of being fibered for compact 3-manifolds. *Geom. Topol.*, 24(1):409–420, 2020.
- [Kie20a] Dawid Kielak. The Bieri-Neumann-Strebel invariants via Newton polytopes. *Invent. Math.*, 219(3):1009–1068, 2020.
- [Kie20b] Dawid Kielak. Residually finite rationally solvable groups and virtual fibering. *J. Amer. Math. Soc.*, 33(2):451–486, 2020.
- [KKW22] Dawid Kielak, Robert Kropholler, and Gareth Wilkes.  $\ell^2$ -Betti numbers and coherence of random groups. *J. Lond. Math. Soc. (2)*, 106(1):425–445, 2022.
- [KMM15] Nic Koban, Jon McCammond, and John Meier. The BNS-invariant for the pure braid groups. *Groups Geom. Dyn.*, 9(3):665–682, 2015.

- [Koc06] Dessislava H. Kochloukova. On a conjecture of E. Rapaport Strasser about knot-like groups and its pro- $p$  version. *J. Pure Appl. Algebra*, 204(3):536–554, 2006.
- [Krs92] Sava Krstić. Finitely generated virtually free groups have finitely presented automorphism group. *Proc. London Math. Soc. (3)*, 64(1):49–69, 1992.
- [KS21] Dawid Kielak and Bin Sun. Agrarian and  $\ell^2$ -betti numbers of locally indicable groups, with a twist, 2021. arXiv 2112.07394 [math.GT].
- [Kud22] Monika Kudłinska. On subgroup separability of free-by-cyclic and deficiency 1 groups, 2022. arXiv 2211.05752 [math.GR].
- [Kud23] Monika Kudłinska. A note on arithmetic lattices in  $\text{isom}(\mathbb{H}^n)$  and exotic subgroups of hyperbolic groups, 2023. 2303.11218 [math.GR].
- [Lev09] Gilbert Levitt. Counting growth types of automorphisms of free groups. *Geom. Funct. Anal.*, 19(4):1119–1146, 2009.
- [Lin93] Peter A. Linnell. Division rings and group von Neumann algebras. *Forum Math.*, 5(6):561–576, 1993.
- [Liu23a] Yi Liu. Finite-volume hyperbolic 3-manifolds are almost determined by their finite quotient groups. *Invent. Math.*, 231(2):741–804, 2023.
- [Liu23b] Yi Liu. Mapping classes are almost determined by their finite quotient actions. *Duke Math. J.*, 172(2), 2023.
- [LNW99] Ian J. Leary, Graham A. Niblo, and Daniel T. Wise. Some free-by-cyclic groups. In *Groups St. Andrews 1997 in Bath, II*, volume 261 of *London Math. Soc. Lecture Note Ser.*, pages 512–516. Cambridge Univ. Press, Cambridge, 1999.
- [Lor08] Karl Lorensen. Groups with the same cohomology as their profinite completions. *J. Algebra*, 320(4):1704–1722, 2008.

- [Lüc94] Wolfgang Lück.  $L^2$ -Betti numbers of mapping tori and groups. *Topology*, 33(2):203–214, 1994.
- [Lüc02] Wolfgang Lück.  $L^2$ -invariants: theory and applications to geometry and  $K$ -theory, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.
- [Lym22a] Rylee Alanza Lyman. CTs for free products, 2022. arXiv:2203.08868 [math.GR].
- [Lym22b] Rylee Alanza Lyman. Train track maps on graphs of groups. *Groups, Geom. Dyn.*, 16(4):1389–1422, 2022.
- [Mac02] Nataša Macura. Detour functions and quasi-isometries. *Q. J. Math.*, 53(2):207–239, 2002.
- [Min06] Ashot Minasyan. On residual properties of word hyperbolic groups. *J. Group Theory*, 9(5):695–714, 2006.
- [Mos96] Lee Mosher. Hyperbolic extensions of groups. *J. Pure Appl. Algebra*, 110(3):305–314, 1996.
- [Mut21] Jean Pierre Mutanguha. Irreducibility of a free group endomorphism is a mapping torus invariant. *Comment. Math. Helv.*, 96(1):47–63, 2021.
- [MV95] John Meier and Leonard VanWyk. The Bieri-Neumann-Strebel invariants for graph groups. *Proc. London Math. Soc. (3)*, 71(2):263–280, 1995.
- [NS07] Nikolay Nikolov and Dan Segal. On finitely generated profinite groups. I. Strong completeness and uniform bounds. *Ann. of Math. (2)*, 165(1):171–238, 2007.
- [NW01] Graham A. Niblo and Daniel T. Wise. Subgroup separability, knot groups and graph manifolds. *Proc. Amer. Math. Soc.*, 129(3):685–693, 2001.

- [Obe20] Geometric structures in group theory (hybrid meeting). *Oberwolfach Rep.*, 17(2-3):877–918, 2020. Abstracts from the conference held June 21–27, 2020, Organized by Martin Bridson, Cornelia Druțu, Linus Kramer, Bertrand Rémy and Petra Schwer.
- [Osi16] Denis Osin. Acylindrically hyperbolic groups. *Trans. Amer. Math. Soc.*, 368(2):851–888, 2016.
- [Pau91] Frédéric Paulin. Outer automorphisms of hyperbolic groups and small actions on  $\mathbf{R}$ -trees. In *Arboreal group theory (Berkeley, CA, 1988)*, volume 19 of *Math. Sci. Res. Inst. Publ.*, pages 331–343. Springer, New York, 1991.
- [Rip82] Eliyahu Rips. Subgroups of small cancellation groups. *Bull. London Math. Soc.*, 14(1):45–47, 1982.
- [Riv08] Igor Rivin. Walks on groups, counting reducible matrices, polynomials, and surface and free group automorphisms. *Duke Math. J.*, 142(2):353–379, 2008.
- [Sau06] Roman Sauer. Homological invariants and quasi-isometry. *Geom. Funct. Anal.*, 16(2):476–515, 2006.
- [Sch14] Kevin Schreve. The strong Atiyah conjecture for virtually cocompact special groups. *Math. Ann.*, 359(3-4):629–636, 2014.
- [Sel97] Zlil Sela. Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II. *Geom. Funct. Anal.*, 7(3):561–593, 1997.
- [Sen06] Eugene Seneta. *Non-negative matrices and Markov chains*. Springer Series in Statistics. Springer, New York, 2006. Revised reprint of the second (1981) edition [Springer-Verlag, New York; MR0719544].
- [Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Graduate Texts in Mathematics, Vol. 42. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott.

- [Ser03] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.
- [Sik87] Jean-Claude Sikorav. *Homologie de Novikov associée à une classe de cohomologie réelle de degré un*. PhD thesis, Université Paris-Sud, 1987.
- [Sta68] John R. Stallings. On torsion-free groups with infinitely many ends. *Ann. of Math. (2)*, 88:312–334, 1968.
- [Ste72] Peter F. Stebe. Conjugacy separability of groups of integer matrices. *Proc. Am. Math. Soc.*, 32:1–7, 1972.
- [Swa69] Richard G. Swan. Groups of cohomological dimension one. *J. Algebra*, 12:585–610, 1969.
- [Thu86] William P. Thurston. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.*, 59(339):i–vi and 99–130, 1986.
- [Tur86] Vladimir G. Turaev. Reidemeister torsion in knot theory. *Uspekhi Mat. Nauk*, 41(1(247)):97–147, 240, 1986.
- [Uek18] Jun Ueki. The profinite completions of knot groups determine the Alexander polynomials. *Algebr. Geom. Topol.*, 18(5):3013–3030, 2018.
- [Wil98] John S. Wilson. *Profinite groups*, volume 19 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1998.
- [Wil17] Gareth Wilkes. Profinite rigidity for Seifert fibre spaces. *Geom. Dedicata*, 188:141–163, 2017.
- [Wil18a] Gareth Wilkes. Profinite completions, cohomology and JSJ decompositions of compact 3-manifolds. *N. Z. J. Math.*, 48:101–113, 2018.
- [Wil18b] Gareth Wilkes. Profinite rigidity of graph manifolds and JSJ decompositions of 3-manifolds. *J. Algebra*, 502:538–587, 2018.

- [Wil19a] Gareth Wilkes. Profinite rigidity of graph manifolds. II: Knots and mapping classes. *Isr. J. Math.*, 233(1):351–378, 2019.
- [Wil19b] Gareth Wilkes. Virtually abelian quotients of random groups, 2019. arXiv:1902.02152 [math.GR].
- [Wis00] Daniel T. Wise. Subgroup separability of graphs of free groups with cyclic edge groups. *Q. J. Math.*, 51(1):107–129, 2000.
- [Wis04] Daniel T. Wise. Cubulating small cancellation groups. *Geom. Funct. Anal.*, 14(1):150–214, 2004.
- [WZ10] Henry Wilton and Pavel Zalesskii. Profinite properties of graph manifolds. *Geom. Dedicata*, 147:29–45, 2010.
- [WZ17a] Henry Wilton and Pavel Zalesskii. Distinguishing geometries using finite quotients. *Geom. Topol.*, 21(1):345–384, 2017.
- [WZ17b] Henry Wilton and Pavel Zalesskii. Pro- $p$  subgroups of profinite completions of 3-manifold groups. *J. Lond. Math. Soc., II. Ser.*, 96(2):293–308, 2017.
- [WZ19] Henry Wilton and Pavel Zalesskii. Profinite detection of 3-manifold decompositions. *Compos. Math.*, 155(2):246–259, 2019.
- [Zal22] Pavel Zalesskii. The profinite completion of relatively hyperbolic virtually special groups, 2022. arXiv:2205.13201 [math.GR].