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Note

# A doubly cyclic channel assignment problem

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**Abstract**

A standard model for radio channel assignment involves a set $V$ of sites, the set $\{0,1,2,\ldots\}$ of channels, and a constraint matrix $(w(u,v))$ specifying minimum channel separations. An assignment $f: V \to \{0,1,2,\ldots\}$ is feasible if the distance $|f(u)-f(v)| \geqslant w(u,v)$ for each pair of sites $u$ and $v$. The aim is to find the least $k$ such that there is a feasible assignment using only the $k$ channels $0,1,\ldots,k-1$, and to find a corresponding optimal assignment.

We consider here a related problem involving also two cycles. There is a given cyclic order $\tau$ on the sites, and feasible assignments $f$ must also satisfy $f(\tau v) \geqslant f(v)$ for all except one site $v$. Further, the channels are taken to be evenly spaced around a circle, so that if the $k$ channels $0,1,\ldots,k-1$ are available then the distance between channels $i$ and $j$ is the minimum of $|i-j|$ and $k-|i-j|$. We show how to find a corresponding optimal channel assignment in $O(|V|^3)$ steps.

Suppose that we are given a set $V$ of $n$ sites (or customers or transmitters), each of which is to be assigned one of a number of channels or colours, which may be taken to be the non-negative integers. Minimum channel separations are specified by a graph $G=(V,E)$ on the vertex set $V$, together with a non-negative integer length $w(e)$ on each its $m$ edges $e$. Thus, we are using a 'constraint matrix' approach – see, for example [2].

We think of the available channels as being evenly spaced around a circle, with distance between them measured around the circle. Thus, if there are the $k$ channels $0,1,\ldots,k-1$ available, the distance $d_k(i,j)$ between channels $i$ and $j$ is taken to be the minimum of $|i-j|$ and $k-|i-j|$. A *cyclic $k$-colouring of the pair $G,w$* is a function $f$ from $V$ to the set $\{0,1,\ldots,k-1\}$ of colours, such that for each edge $e=\{u,v\}$ we have $d_k(f(u),f(v)) \geqslant w(e)$. The *cyclic span* $\sigma_c(G,w)$ is the least $k$ such that there is a cyclic $k$-colouring of the pair $G,w$. It is easy to see that, if each of the edge-lengths $w(e)=1$, then $\sigma_c(G,w)$ equals the chromatic number $\chi(G)$; and thus it is NP-hard to determine the cyclic span.

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Observe that if $f$ is a cyclic $k$-colouring, then so is $g$, where $g(v) = f(v) + c \pmod{k}$ for each vertex $v$. The fact that these colourings can be 'rotated' is one reason why we are interested in them. Also, if we do not assign to each site a single channel $i$ but rather a 'channel-set', consisting of the integers (in a certain range) which are congruent to $i \bmod k$, then indeed it is natural to work with the distance function $d_k$ – see [4, 5, 7].

The second 'cyclic' in the problem name arises because we assume that there is also a given cyclic order $\tau$ on the set $V$, so that we can think of the customer sites as being located around a circle. Call a function $f$ from $V$ to the non-negative integers *$\tau$-monotonic with leading vertex* $v_0$, if when we write $\tau$ in cycle notation as $(v_0 v_1 \cdots v_{n-1})$ (so that $v_1 = \tau v_0$, $v_2 = \tau v_1$ and so on), then we have

$$f(v_0) \leqslant f(v_1) \leqslant \cdots \leqslant f(v_{n-1}).$$

A *cyclic $k$-colouring of the triple* $G, w, \tau$ is a cyclic $k$-colouring of the pair $G, w$ which is $\tau$-monotonic with leading vertex $v_0$ for some choice of $v_0$. The *cyclic span* $\sigma_c(G, w, \tau)$ is the least $k$ such that there is a cyclic $k$-colouring of the triple $G, w, \tau$.

We are interested here in the value of the cyclic span $\sigma_c(G, w, \tau)$ and in how to calculate it, and further in how to find a corresponding optimal cyclic colouring, i.e., a cyclic colouring using this number of colours (channels). This is the *doubly cyclic channel assignment problem*, which arose out of discussions amongst the authors of [7].

Observe that $\sigma_c(G, w)$ is the minimum over all cyclic orders $\tau$ of $\sigma_c(G, w, \tau)$. In practical problems there may be some natural candidates for the cyclic order $\tau$, and then an optimal cyclic colouring for a corresponding triple $G, w, \tau$ may be a good cyclic colouring for $G, w$. Observe also that $\sigma_c(G, w, \tau)$ is at least the sum of the lengths $w(\{v, \tau v\})$ around $\tau$. Thus, we may obtain lower bounds on $\sigma_c(G, w)$ from lower bounds on the length of a travelling salesman tour in $G, w$, and such bounds have proved useful in practice – see [3, 7]. Perhaps further understanding of $\sigma_c(G, w, \tau)$ will allow us to obtain other lower bounds?

Suppose that we are given a doubly cyclic channel assignment problem instance $G, w, \tau$ as above. We may assume, without loss of generality, that the graph $G$ contains all the edges in the cycle $\tau$ – i.e., all edges of the form $\{v, \tau v\}$ – perhaps with length 0. Given an oriented (simple) cycle $C$ in $G$, the 'number of turns' of $C$ is the number of times we move around the cycle $\tau$ when we move once around the cycle $C$. More formally, choose an arbitrary vertex $v_0$ and write the cycle $\tau$ in cycle notation as $(v_0 v_1 \cdots v_{n-1})$. Then the *number of turns* of $C$ is the number of times the cycle $C$ follows an edge from $v_i$ to $v_j$ for some $i > j$. (This number does not depend on the choice of $v_0$, as we shall see below.) Define $\mu = \mu(G, w, \tau)$ to be the maximum over all oriented cycles $C$ of the ratio of the total length of the edges in $C$ to the number of turns of $C$.

**Theorem 1.** *As above, let $G$ be a graph on the set $V$ of $n$ vertices, with a non-negative integer length $w(e)$ on each its $m$ edges $e$; let $\tau$ be a given cyclic order on $V$; and assume that $G$ contains each of the edges in the cycle $\tau$. Then the cyclic span satisfies*

$$\sigma_c(G, w, \tau) = \lceil \mu(G, w, \tau) \rceil,$$

*and we can determine this number and find a corresponding optimal cyclic colouring in* $O(n^2 + nm)$ *steps.*

Before we consider how to solve the doubly cyclic problem and thus prove this theorem, let us briefly consider a related 'singly cyclic' problem. Suppose temporarily that the channels $0, 1, 2, \ldots$ are arranged linearly rather than in a cycle, so that the distance between channels $i$ and $j$ is just $|i-j|$. Then it is easy to find the corresponding 'linear span' and optimal labelling of $G, w, \tau$. For it suffices to consider separately each possible choice of leading vertex $v_0$; and then we may simply work through the vertices in order, assigning to each the smallest available channel; see also the 'Chilean problem' in [7].

Let us return to the doubly cyclic problem. The key to solving the problem is to consider a slight extension of the maximum mean cycle problem, which is still a special case of the maximum cost-to-time ratio cycle problem – see, for example, [1]. Let $D=(V,A)$ be a directed graph, with vertex set $V$ and with an integer length $w(a)$ on each arc $a \in A$. Also, let $R$ be a set of (red) arcs such that every cycle in $D$ contains at least one arc in $R$. Call the triple $D, w, R$ the *network* $N$. We let $\nu = \nu(N)$ be the maximum over all cycles in $D$ of the ratio of the length of the cycle to the number of red arcs in it. (In the maximum mean cycle problem all arcs are in $R$.)

Given a problem instance $G, w, \tau$ as above (containing all edges in the cycle $\tau$) we construct a network $N$ as follows. Replace each edge by a pair of oppositely oriented arcs each with the same length as the edge. Choose a vertex $v_0$ arbitrarily. Let $\tau$ be given in cycle notation as $(v_0 v_1 \cdots v_{n-1})$, and let $R$ be the set of arcs of the form $v_i v_j$ in the digraph for some $i>j$. It follows directly from the definitions that $\mu(G, w, \tau) = \nu(N)$.

**Lemma 1.** *For any triple* $G, w, \tau$, *the number* $\mu$ *does not depend on the choice of the vertex* $v_0$; *and if there is a cyclic* $k$*-colouring then* $k \geqslant \mu$.

**Proof.** Let $v'_i = v_{i+1}$ for $i=0,\ldots,n-2$ and let $v'_{n-1} = v_0$, so that $\tau$ may be written in cycle notation as $(v'_0 v'_1 \cdots v'_{n-1})$. Let $R'$ be the set of arcs of the form $v'_i v'_j$ for some $i>j$. Let $C$ be a cycle in the network, thought of as a set of arcs. To prove the first part of the lemma, it suffices to show that $|C \cap R| = |C \cap R'|$.

Denote the vertex $v_0 = v'_{n-1}$ by $x$. If $x$ is not on the cycle $C$ then $C \cap R = C \cap R'$. If $x$ is on $C$ then $(C \cap R) \setminus (C \cap R')$ consists of the unique arc in $C$ arriving at $x$, and $(C \cap R') \setminus (C \cap R)$ consists of the unique arc in $C$ leaving $x$. This completes the proof that $\mu$ does not depend on the choice of $v_0$.

Suppose now that there is a cyclic $k$-colouring $f$. We must show that $k \geqslant \mu$. By the first part of the lemma, we may assume that $f$ is $\tau$-monotonic with leading vertex $v_0$. Consider a simple cycle $C = (v_{i_0}, v_{i_1}, \ldots, v_{i_j})$ in $N$ where $i_j = i_0$ (so that $C$ has $j$ arcs). Consider an edge $e = \{v_{i_t}, v_{i_{t+1}}\}$. If $i_t < i_{t+1}$ then $f(v_{i_{t+1}}) - f(v_{i_t}) \geqslant w(e)$; and if $i_t > i_{t+1}$ (corresponding to a red edge) then $f(v_{i_{t+1}}) - f(v_{i_t}) \geqslant w(e) - k$. Adding these inequalities, we find that $0 = f(v_{i_j}) - f(v_{i_0})$ is at least the length of the cycle $C$ minus $k$ times the number of red arcs in $C$. Thus, $k$ is at least the ratio of the length of $C$ to the number of red arcs in $C$. It follows that $k \geqslant \mu$, as required. □

**Lemma 2.** *Suppose that $k \geqslant \mu$. Subtract $k$ from the length of each red arc in $N$, so that there is no positive length cycle. For each vertex $v$, let $f(v)$ be the maximum length of a $v_0$–$v$ path, and let $g(v)=0$ if $f(v)=k$ and $g(v)=f(v)$ if not. Then $g$ is a cyclic $k$-colouring for $G, w, \tau$.*

**Proof.** Note that $f(v_0)=0$. Consider a vertex $v \neq v_0$. Of course, $f(v) \geqslant 0$. Let $b$ denote the maximum length of a $v$–$v_0$ path. Then $b \geqslant -k$, since the $v$–$v_0$ path following $\tau$ has length $-k$. Also, there is a cycle (perhaps not simple) with length $f(v)+b$, so $f(v) \leqslant -b \leqslant k$. Thus, $f$ takes values in $\{0,1,\ldots,k\}$, and so $g$ takes values in $\{0,1,\ldots,k-1\}$.

Extend the domain of $d_k$ to $\{0,1,\ldots,k\}$ with the same definition as before, so that $d_k(0,i)=d_k(k,i)$ for each $i$. Consider an edge $e=\{v_i,v_j\}$ in the graph $G$, with length $w(e)$, where $i<j$. We can add the (non-red) arc $v_iv_j$ onto any $v_0$–$v_i$ path in $N$, and so $f(v_j)-f(v_i) \geqslant w(e) \geqslant 0$. Similarly, we can add the (red) arc $v_jv_i$ onto the end of any $v_0$–$v_j$ path, hence $f(v_i)-f(v_j) \geqslant w(e)-k$, and so $k-(f(v_j)-f(v_i)) \geqslant w(e)$. Thus,

$$d_k(g(v_i),g(v_j))=d_k(f(v_i),f(v_j)) \geqslant w(e).$$

By considering the arcs $v_iv_{i+1}$, we see from the above that $f$ is $\tau$-monotonic with leading vertex $v_0$. If $f(v)<k$ for each vertex $v$, then $g=f$ and so $g$ is $\tau$-monotonic with leading vertex $v_0$. If there is a vertex $v$ with $f(v)=k$, then the set of such vertices consists of all vertices from some vertex $u_0$ to $\tau^{-1}v_0$ along $\tau$, by the monotonicity of $f$; and then $g$ is $\tau$-monotonic with leading vertex $u_0$. □

We complete the proof of the theorem by showing that we can calculate the quantities $v$ and $f(v)$ (and hence $g(v)$) sufficiently quickly for any network $N$.

Let $D=(V,A)$ be a directed graph, with vertex set $V$ and with an integer length $w(a)$ on each arc $a$. Suppose that $|V|=n$ and $|A|=m$. For convenience we shall assume that the digraph $D$ is strongly connected. Let $R$ be a set of (red) arcs such that every (directed) cycle in $D$ contains at least one arc in $R$. A minor extension of a method of Karp [6] (see also [1]) for the minimum mean cycle problem will determine $v$ in $O(nm)$ arithmetic steps, and allow us to calculate all the values $f(v)$ in a further $O(n^2)$ steps.

We proceed as follows. First, number the vertices $1,2,\ldots,n$ so that for each arc not in $R$ we have $i<j$. Let $h(i,t)$ be the maximum length of a walk from vertex 1 to vertex $i$ which uses exactly $t$ arcs in $R$. We may calculate the quantities $h(i,t)$ as follows. To start with, note that $h(1,0)=0$. Set $h(i,-1)=-\infty$ for $i=1,2,\ldots,n$. For $t=0,1,\ldots,n$ and $i=1,\ldots,n$ other than $(i,t)=(1,0)$, we may calculate the $h(i,t)$ in turn from

$$h(i,t)=\max\left(\max_{x:xi\in R}(h(x,t-1)+w(xi)),\ \max_{x:xi\in A\setminus R}(h(x,t)+w(xi))\right),$$

since if $xi \in A\setminus R$ then $x<i$. All this may be done in $O(nm)$ arithmetic steps. Now we may calculate $v$ quickly, using the following result, which we prove below.

**Lemma 3.**

$$v = \max_{1 \leqslant i \leqslant n} \min_{0 \leqslant t < n} \left( \frac{h(i,n) - h(i,t)}{n-t} \right).$$

Finally, let $k$ be an integer at least $v$. Subtract $k$ from the length of each arc in $R$, so that no cycle has strictly positive length. Then the maximum length $f(i)$ of a path from vertex 1 to vertex $i$ may be calculated by

$$f(i) = \max_{0 \leqslant t < n} (h(i,t) - tk). \qquad \square$$

**Proof of Lemma 3.** If we subtract a constant $c$ from the length of each arc in $R$, then both sides of the equation asserted in the lemma decrease by $c$. Therefore, it suffices to consider the case when $v = 0$, which we now assume. Thus, no cycle has strictly positive length but there exists a cycle $C$ with zero length.

Let $d_i$ denote the maximum length of a path from vertex 1 to vertex $i$, and define adjusted arc lengths by letting $\tilde{w}(e) = w(e) + d_i - d_j$ for each arc $e = ij$. Then $\tilde{w}(e) \leqslant 0$ for each arc $e$, and $\tilde{w}(e) = 0$ for each arc $e$ in $C$ and for each arc $e$ in any longest path from vertex 1. Let $\tilde{h}(i,t)$ be defined as for $h(i,t)$ except using the adjusted lengths $\tilde{w}(e)$ rather than the $w(e)$. Then $\tilde{h}(i,t) = h(i,t) - d_i$. So the right-hand side in the lemma is unchanged if we use the adjusted lengths. Now, for each vertex $i$,

$$\min_{0 \leqslant t < n} \{\tilde{h}(i,n) - \tilde{h}(i,t)\} \leqslant 0,$$

since for some $0 \leqslant t < n$, $\tilde{h}(i,t)$ is the maximum length of a walk from vertex 1 to vertex $i$. Hence, to complete the proof it suffices to show that there is some vertex $i_0$ such that the left-hand side here is non-negative, and this will follow if $\tilde{h}(i_0, n) = 0$. To find such a vertex $i_0$, pick a vertex on the cycle $C$, follow a longest path from vertex 1 to this vertex, then follow around $C$ until the walk uses a total of exactly $n$ arcs in $R$, and let $i_0$ be the final vertex. Then $\tilde{h}(i_0, n) = 0$, as required. $\square$

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