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Multimodal interpretation of notation

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Abstract: Challenges related to the teaching and learning of formal notation in school mathematics are widely documented, and specifically in relation to the underlying mathematical structures that the notation is intended to convey. In this article, we draw on embodied cognition to examine the interactions among three students working with the software *Grid Algebra*. Embodied cognition emphasises the role of gesture and movement in learning and understanding mathematics. *Grid Algebra* uses movement to direct students' attention to mathematical operations on numbers and numerical expressions within the grid and the structure of these operations, while the software takes care of the formal notation of the numerical expressions that describe these sequences of operations. We analyse how different modes of communication work together to scaffold students' fluency with operations and the formal notation representing these operations and the order in which they are performed. The dynamic between notation, speech, movement and position allows students to educate their interpretation of mathematical notation through the movements and positions that they are very familiar with.

Keywords: Grid Algebra; notation; embodied cognition; multimodal communication

Introduction

This article analyses an episode from a group activity involving three students using the software *Grid Algebra*. The task involves students interpreting formal notation for numerical expressions and interpreting them in terms of mathematical operations and physical journeys on the grid. This analysis focuses on the shifting modes of communication between the three students and the software; that is, the shifts and connections between spoken language, the visual formal notation, and the movements around the grid. These different modes of communication work together to scaffold students' fluency with formal notation and the order of operations.

We begin this article by briefly introducing the theory of embodied cognition that enables us to examine how movements and other modes of communication are involved as the students complete tasks focused on fluency with formal notation and the order of operations within numerical expressions. We then describe the software *Grid Algebra*, outlining some key features that stress numerical expressions as objects in themselves, as well as the neutral feedback offered within the task. We then outline the data extract that we subsequently

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analyse, focusing on the interplay between gestures, spoken words, and interaction with the software.

Embodied cognition

There is now extensive research drawing on embodied cognition showing that including motion and gestures are effective in enhancing students' learning in mathematics (Abrahamson et al., 2020; Khatin-Zadeh et al., 2022). Embodied cognition has been conceptualised in various ways and can be characterised as the idea that the body and bodily experiences are an essential aspect to how we understand the world. As Varela et al. (1991) argue, the ways we think have their roots in bodily experience. As such, we consider it important to attend to the way in which both mind and body are involved within learning. Cognitive activities are constituted in multimodal sensorimotor experiences (Abrahamson et al., 2020). In fact, many fundamental mathematical concepts are or can be understood as motions (Khatin-Zadeh et al., 2022). Consequently, since bodily actions are constitutive of (mathematical) experiences, they are more than just support or scaffolding for thinking (Sinclair, 2023). Indeed, Nathan et al. (2021) found that both mathematics experts and non-experts used gestures when asked to justify geometry conjectures, and that the positive quality of those justifications was associated with the use of gestures. Gestures that both represent the mathematical objects and the associated transformations are strongly predictive of rigorous mathematical reasoning in geometry (Abrahamson et al., 2020; Walkington et al., 2019). It might be argued that geometry particularly lends itself to gesturing compared with less visual aspects of the mathematics curriculum. However, geometry is often represented with static images; it is the use of gestures which turn such static images into motion-based metaphors, and even mental simulation of such hand movements also achieves this (Khatin-Zadeh et al., 2022). Despite the fact that algebra differs from geometry in the sense that it can only be indirectly represented through signs (Radford, 2006, referring to Kant), there can still be significant use of gestures to express algebraic processes (for example, David et al., 2014).

Like Streeck (2009) and Sinclair (2023), we take a broad definition of gestures that includes the handling of, and movement of, objects, digital or physical. Furthermore, drawing on conceptualisations of gestures within embodied cognition research in mathematics education (e.g., Sinclair, 2023), we see gestures as more than representational, iconic or metaphoric, as they are often considered within mathematics education (e.g., Arzarello, 2006; Arzarello et al., 2015; Kimber, 2024), they are part of the meaning-making experience. What makes a difference to learning mathematics is the cognitive relevance of these gestures to the associated mathematical concepts (Walkington et al., 2022). We explore this now in connection to a particular piece of software, called *Grid Algebra* (which can be found at *gridalgebra.com*).

Grid Algebra

Grid Algebra[2] is based upon a grid of numbers where the multiplication tables are shown in rows, initially from table 1 to table 6 (Figure 1).

Figure 1

The multiplication table basis of the structure of the grid

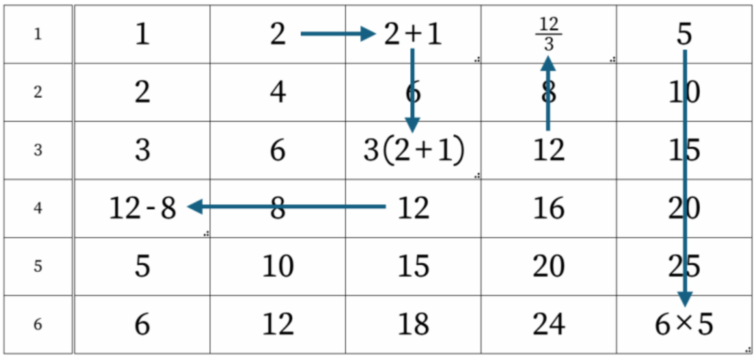
1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20
5	5	10	15	20	25
6	6	12	18	24	30

A key aspect of the software is that any number can be picked up and dragged either horizontally or vertically. Addition is a movement to the right with the inverse, subtraction, to the left. Multiplication is a movement down with its inverse, division, up. The notation within the cell containing the dragged number changes according to the operation which is being carried out by the movement (Figure 2). When number '2' in row 1 is dragged one cell to the right, it will now say $2+1$. If '12' in row 4 is dragged two cells to the left, the dragged cell will change from 12 to $12-4$ and then $12-8$ (this being multiplication table 4). If the new $2+1$ is dragged from row 1 to row 3, this is a multiplication by 3 and the notation in the moving cell will show $2+1$, then $2(2+1)$ and finally $3(2+1)$. Dragging a single number down is multiplying that number, as with 6×5. The design decision to represent this as 6×5 rather than 5×6 is so that this is consistent with where the multiplier is placed when an expression, or a letter, is being multiplied.

[2] *Freely available at gridalgebra.com*

Figure 2

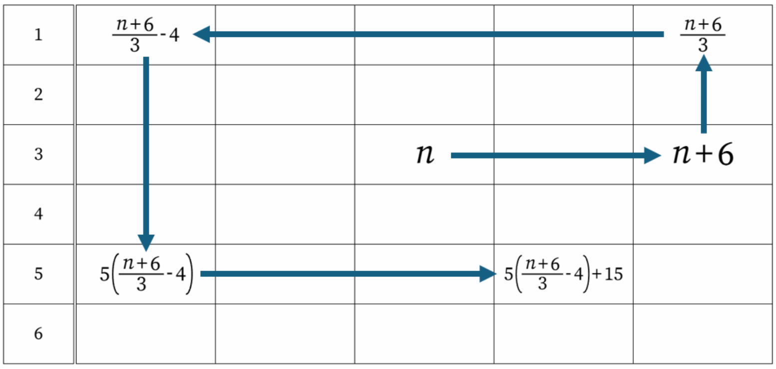
Some of the movements which can be made on the grid



The last movement shown in Figure 2 is taking the 12 in row 3 and moving it up to row 1. This is a division by 3 and is shown in the notational form of $\frac{12}{3}$ rather than 12 ÷ 3. The reason is so that students will become used to that notation, as it is that notation which is used to express algebraic expressions formally.

Figure 3

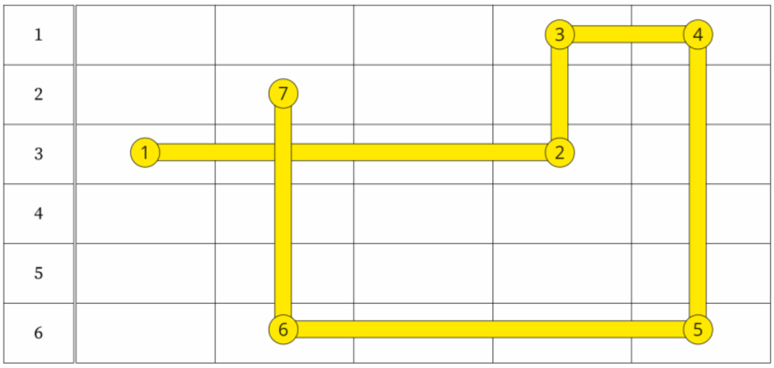
An example of a letter or number being taken on a journey



As well as numbers, a letter can be placed in a cell and dragged around the grid to produce an expression representing the journey which has been made (Figure 3). A route can also be created on the screen, indicating a journey around the grid (Figure 4).

Figure 4

A route is created indicating a journey which might be made around the grid



The software is designed to address many number, arithmetic and algebraic areas within the curriculum, including work towards students becoming confident with letters and formal algebraic notation, as well solving linear equations (Hewitt, 2012).

Symbolic representation might be viewed as being static. However, algebraic symbols can be viewed as structured physical objects that can be touched and moved (Ottmar & Landy, 2017) and reasoning with these symbols can be highly visual (Landy & Goldstone, 2007). *Grid Algebra* acts as a field of promoted action (Abrahamson & Trninic, 2014) where students engage in movement-based activities around the grid in ways that support them to act according to the underlying mathematical structures. That is to say, it is an example of what is now described as action-based embodied design where physical movement is used in mapping the journey towards a goal in the form of a numerical expression (such as $\frac{2\times4+8}{2} -$ 1). Actions in the grid are multimodal as they involve movement alongside visual and symbolic modes. Numeric operations and equations are also represented in a variety of modes, including symbolic, verbal or motion.

Pedagogic design of *Grid Algebra*

For many years there has been considerable development of virtual programs and tools that are aimed at assisting students' learning of algebra. For example, Ottmar et al. (2015) developed a program (FH2T) where students could manipulate and transform numeric and algebraic expressions by clicking and dragging parts of the expression to a different place within the expression. Students were given tasks which involved trying to transform an initial expression into a given target expression which was equivalent to the initial expression. This had certain similarities with the microworld developed by Jones and Pratt (2007) that focused on the equals sign, and importantly the distinction between a substitutive-relational conception and a sameness-relational conception of the equals sign (Jones et al., 2013). The use of movement in Jones et al.'s software made it possible for students to work on the substitutive-relational conception without relying on the sameness-relational conception of the equals sign. Similarly, the students in Ottmar et al.'s study showed strong learning gains in their algebraic fluency when working, which was partly attributed to the development of strong visual-motor routines. Many of these programs focus on associating particular actions with specific algebraic transformations. For example, with Ottmar et al.'s study, a focus on the commutative property was made through the ability to drag one number to the other side of the other number in an expression such as 2 + 3. Upon doing this, the expression would change to 3 + 2. The focus of Ottmar et al.'s study was specifically on symbolic manipulation. In Jones et al.'s study, a focus on the substitutive conception of the equals sign was made through the ability to drag different equality relations and substitute and combine them in order to decompose number problems such as 31 + 67.

In contrast, *Grid Algebra* makes arithmetic operations the focus of attention, as the movements determine the operations taking place. This shift of focus from numbers, and their position in the grid, to movements between numbers in the grid, shifts the focus from number to operations. We argue this shift is essentially one from number/arithmetic to algebra since, at school level, the latter concerns what can be said about operations rather than the carrying out of any calculations. Knowing about order of operations and inverse operations allows students access to such curriculum items as multiplying out brackets, factorising, equivalent expressions, simplifying expressions, substitution, changing the subject of an equation and solving equations. *Grid Algebra* was designed specifically to enable learners to work directly with algebraic notation (Hewitt, 2016) and has shown that students become confident with seeing the order of operations within that notation (Hewitt, 2012). There are specific socially agreed conventions around how we use mathematical notation, which are arbitrary, meaning that students will need to be informed of these conventions (Hewitt, 1999). The *Grid Algebra* software looks after the symbols (Tatha, 1989) by providing the related notation, so that students can focus on the operations.

Students already know about making journeys, they walk from their home to somewhere else, and know that in order to return home, they walk the same distance but in the inverse direction. They also know that the last road they walked along in getting to their destination will be the first road they walk along on the way back. This existing awareness is utilised within the software, with inverse journeys being associated with inverse operations, and the order reversal of a backward journey being associated with the order of operations being reversed. This results in the mathematical awarenesses involved with solving an equation being intuitive as it matches the visual and bodily awarenesses they already have from their life experiences of making inverse journeys (Hewitt, 2016).

The journeys make visible the order of operations as each step represents an operation, and the sequence of horizontal and vertical steps represents the order of operations carried out. A different order would result in a different route. Thus, the notation reveals the history of movements which have taken place. The notation has a correspondence between the number of movements made with the number of operations, and the direction and length of those movements with the mathematical operations and the magnitude of each operation.

The dragging of expressions (whether by mouse or with a finger on the screen), from an initial number or letter, builds up an expression gradually, a step at a time. This is done by the software requiring someone to release the mouse (or pressure with their finger) before they can drag that new expression on the next stage of the journey. Thus, there is a physical, as well as visual, experience of carrying out a separate movement for each operation. The number of operations within an expression will have been felt physically by the person building up that expression. Alongside this, they also see an expression being built up notationally. For example, they see the expression $\frac{2(n+3)-12}{6}+4$ being created in the following way:

- n
- $n+3$
- $2(n+3)$
- $2(n+3)-12$
- $\frac{2(n+3)-12}{6}$
- $\frac{2(n+3)-12}{6}+4$

The way in which the expression is built is not a left-to-right order, but follows the order of operations. This helps students to read the final expression in the order in which it was created: the order of operations. As such, the order of operations does not have to be 'taught' as a separate topic. Instead, it is inherent within the construction of the notational expressions as well as within the visual construction of the physical journey. Indeed, no explicit instruction needs to be given at all; students become very confident with the notation and order of operations through their engagement with the computer-generated tasks built into the software.

Some of the computer-generated tasks give students a final expression and they are asked to re-create the journey which would make that expression. This requires students to read the target expression in terms of order of operations, so that they know which movement to carry out first, second, etc. As they make movements, the expression builds up according to the movements they make. This process of feeding back the consequences of their actions in a notational form, allows students to see whether they are progressing correctly. This is because they can compare what their expression looks like with what the target expression looks like. For example, if the target expression is $\frac{t+6}{2} - 4$ and a student thinks that after carrying out the add six, they should subtract four before dividing by two, then their expression will look like $\frac{t+6-4}{2}$. The visual difference between the two expressions indicates that a mistake must have been made. Students can go back to $t + 6$, which is the same in both their expression and the target expression, and try doing the dividing by two next instead of the subtracting four. In this way, the students never need to be explicitly told whether they are correct or not since this is something which is clear to them through just checking what is the same or what is different between their expression and the target expression.

Methodology

The focus of this article is on the interaction between students and between students and the *Grid Algebra* software. The analysis has its basis in ethnomethodology (Garfinkel, 1967), exploring how the students make sense of the mathematics and the grid in interaction (Ingram & Elliott, 2019). The analysis draws on widely documented interactional features involved in a fine-grained analysis of interaction (see Sidnell & Stivers, 2013), such as the one we present below. This action-based approach focuses on how language and gestures are used to achieve intersubjectivity, that is, what students are doing to make sense of the task, the mathematics and the interaction that is visible in their multimodal communication. In the context of this article, this involves examining what it means to add, subtract, multiply and divide in the context of solving increasingly complex calculations using the *Grid Algebra* software. This approach is ideally suited to researching the use of *Grid Algebra* because of the design focus on operations rather than numbers.

Data

The analysis below is based on an extract from a data set where students worked in groups of two or three to complete a set of tasks built into the *Grid Algebra* software. The clip involved three students (12–13 years old) from New Zealand, whom we call Alpha, Beta and Delta. Their work on the task was captured using videoconferencing software, which included capturing the screen, including how they moved the mouse, as well as a video of the three students. Unfortunately, the students' physical gestures in relation to the screen that did not involve the mouse were largely not captured unless they occurred close to the video camera which was placed at the top of the screen. The task they were given was to draw the route which must have been made to create a given expression. This was done by them clicking on where the journey started and then clicking on the cell at each stage of the journey which must have been made to create the final expression. Each stage of the journey should correspond with each operation within an expression, in the order that the operations were carried out. Although we know that these students had experience of working with *Grid*

Algebra before the video recorded session, we do not know the precise nature of those experiences.

Data analysis

Both authors repeatedly viewed the video, both together and separately. In the early stages of the data analysis, the two authors met to discuss what we noticed in the videos before identifying a short clip on which to focus a more detailed analysis. As the focus of this article was on the relationship between the actions on the grid and symbolic notation used in mathematics, a clip where there initially appeared to be some sort of difficulty in the connection between these actions and the notation was chosen for more detailed analysis.

The subsequent data analysis included a careful transcription of the sequence before each author independently analysed it systematically, working turn-by-turn, considering the turn-taking, sequence organisation, and turn design (ten Have, 2007) not only considering the spoken communication, but also the movements of the mouse and the visual changes through their interaction with the grid. The task focused on operations, the symbolic notation for these operations, and the mathematical conventions around the order of operations were also included in this turn-by-turn analysis, incorporating both the spoken and gestural aspects that were visible in the video recording.

The transcripts below are presented using three lines for each spoken turn where there is an accompanying interaction with the grid. The first line is what was being verbally spoken and was transcribed using Jefferson transcription (Jefferson, 1984), the second line is a description of the actions involving gestures such as clicking the mouse. Unfortunately, pointing to specific parts of the screen was not captured by the camera. The third line indicates changes on the screen that usually, but not exclusively, accompanied the mouse clicks described on the second line. Some aspects of the Jefferson transcription system are not included in the extract presented below in order to make the transcript more readable. The transcription conventions are detailed at the end of the article, but it is worth noting that punctuation reflects the intonation of the spoken turns and not the grammatical structure.

The analysis was data driven, and the validity of the claims we make in our analysis are achieved through the transparency in which we present both the transcription and our analytical claims. The analysis aims to present a single case as the interactions around the software, mathematical notation and mathematical content are newly described (White, 2019).

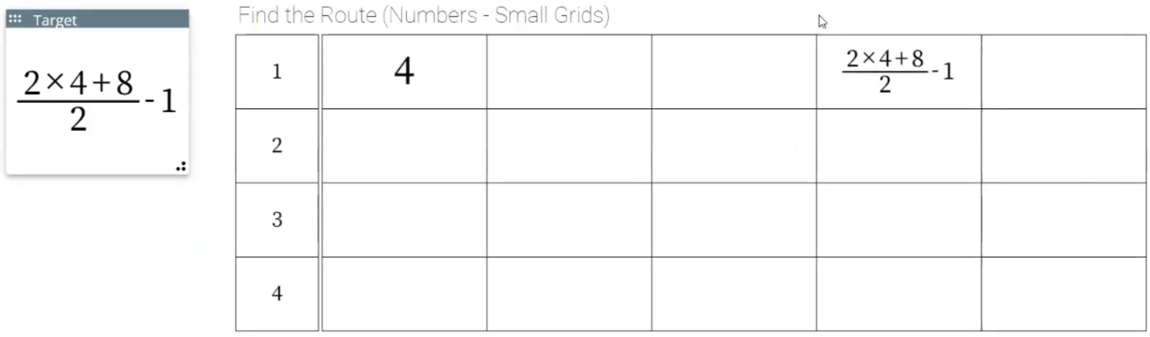
Throughout the video clip, the students are sometimes distracted by something off-screen. The analysis does not include these diversions, but they are indicated in the transcript where they coincide with actions associated with the task.

The interaction

The three students were given the task shown in Figure 5, where they had to draw the route taken from the initial number to the final expression. The extract focuses on the initial actions of the students when working on the task and covered a period of just one minute and ten seconds.

Figure 5

The starting task and grid



Excerpt 1

Alpha: two times four.
Beta clicks on the cell showing 4
point one of a route appears
Beta: no:
Alpha: ye-
Beta: -no (1.4) shush
Alpha: son of a digger
Beta clicks on the cell in the fourth row and first column
point two of the route appears
Beta: no:! (0.9) plus eight

In this opening extract, Alpha states two times four, which could be interpreted as simply reading the target expression, but is interpreted by Beta as an instruction for what to do given his negative evaluation. Yet Beta's actions of moving to and clicking on the cell in the fourth row indicate a sense of multiplying by four, not two. Mathematically, the commutativity of multiplication means that two times four and four times two can often be considered to be the same, but the movements within the grid and the given position of the four in the grid require the four to be multiplied by two. Consequently, Beta's two steps in the constructed route represent four times four. However, after the first placement of a point on the route, this starting 4 on the grid is no longer visible as the 'point one' marker of the route now lays over this.

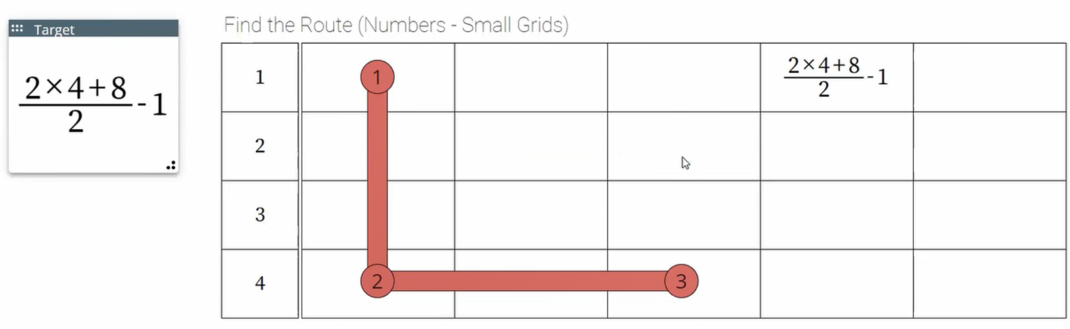
Excerpt 2

Alpha: no: li-t-the monster truck thing.
Beta moves the mouse to the cell two to the right
Beta: wouldn't that be there.
Alpha: oh I don't even know how to spell
Beta clicks on the cell he's hovered over
point three of the route appears

As they continue, Beta moves the mouse two cells to the right (Figure 6). As they are moving in the fourth row, this action does represent adding eight. Beta has followed the next action required by the target expression.

Figure 6

The first route constructed by the students



Excerpt 3

Beta: and then:, minus one. (3.2) is that div-
[[Group are looking at something off camera and they all laugh]]
Beta: is that correct
[[Group are looking at something off camera]]
Beta: Alpha,
[[Alpha is still looking at something off camera]]
Beta: is this correct.

In the next step Beta attempts to minus one, which is not possible in the grid as they are currently in the fourth row (multiplication table 4), and is also not the next operation to carry out in the target expression. Delta moves forward in his chair seemingly to look more carefully at the screen. Beta is aware that there is a problem as indicated initially by the long pause, the beginning of a question potentially about dividing by two, and subsequently by his questions to Alpha about whether it is correct. In the next turn, Beta moves the mouse back to the target expression, but the slight change in the vertical position of the -1 in relation to the rest of the expression does not appear to have stopped Beta from reading the expression from left to right and Beta does not mention the division by two, though the cut-off after the long pause, "div-" suggests that Beta has considered this.

Excerpt 4

Alpha: two ti[mes four]
Beta hovers the mouse over the Target box
Beta: [I got to-]
Alpha: plus eight, divided by two,
Beta moves the mouse back to the 3 on the route
Alpha: is um that two [[Alpha appears to be pointing to the screen]]
Beta moves the mouse two cells up
Beta: n-no but what about the one
Beta moves the mouse to the target expression

Alpha begins from the start, reading the target expression as two times four plus eight, but follows this with "divided by two". Beta notices the difference between what Alpha has said and the operations he was attempting to carry out, asking "what about the one". Alpha also appears to be aware that there is a source of difficulty in the interaction but does not respond directly to Beta's question. Instead, Alpha directs attention to the intended final destination of the route as given in the grid, with the change of state token 'ah':

Excerpt 5

Alpha: ah it has to end up there [[Alpha appears to be pointing]]
Beta moves the mouse to two cells above the third route marker
Alpha: so maybe it goes right to the top
Beta moves the mouse to the top row
Alpha: I don't know [[laugh]]
Beta: I did something wrong
Beta clicks on the START AGAIN button
The marked route disappears

This shift in attention to the intended end of the route also focuses attention on the students' route on the grid rather than the operations in the target expression. This does not resolve the difficulty and Beta restarts the task. This act of restarting clears the route markers and the original 4 is visible once again. We note that a teacher has not informed the students that they have done something wrong. The visual nature of the grid and how movements are associated with the mathematical operations means that they are able to recognise for themselves that something had gone wrong.

Excerpt 6

Alpha: ah it goes to there. [[Alpha is pointing at the screen]]
Beta clicks on the 4
Point 1 of a route appears
Alpha: two time- two
Delta: no:?

Beta: no.

Beta makes a very quick move of the mouse to the target expression, a little to the left of the number 2 in the top part of that expression

Alpha: two!

Beta moves the mouse to the cell under the 4

Beta: there. no but it's-

Beta makes another quick movement of the mouse to the target expression, towards the 2x4 part of the expression

Beta: oh::

Beta clicks on the cell under the 4
Point 2 of a route appears

Alpha then begins to restate the first operation as two times (presumably four) but cuts this off and emphasises the two, repeating it more enthusiastically after the other two students' negative evaluation in the ambiguous turn "two time- two". Twice, Beta moves the mouse from the cell under the 4 to the 2×4 part of the target expression and back to the cell again. This movement between the grid and the notation of the target expression seems to show a link being made between the position on the grid and the notation. Furthermore, Beta's action of moving the mouse to the particular cell under the 4 makes the connection between the operation of multiplying by two and the appropriate action on the grid (moving down from row 1 to row 2). Initially this is guided by Alpha, and Beta remains unconvinced as indicated by the "no but" before there is a clear change in state "oh::" and Beta adds in point two of the new route.

In what follows Alpha initially repeats the goal of heading for the given expression in the grid. Beta continues by skip counting out the steps in the two row, moving the mouse one cell at a time, for adding eight. However, Beta started counting in the cell where point 2 of the route was labelled, meaning that although he has spoken eight, point 3 of the route is at the point that represents adding six (see Figure 7). This may or may not be as a consequence of the number 2 appearing in the start of his count, though this 2 represents the second stage of the route, rather than the number two. Alpha repeats the operations he suggested earlier, but recognises that there is a problem, though he does not articulate what this problem is. However, it does come after saying "divided by two", and if they divided by two on the grid they would end up in the final cell with the expression in it, yet they still had a subtract one to do before arriving at the cell.

Excerpt 7

Alpha: you've just got to head for this ((now clue))

Beta: (.) four six eight

Beta steps the mouse as they count to the right of point 2

Alpha: plus eight divided by two.

Beta clicks on the cell below the expression

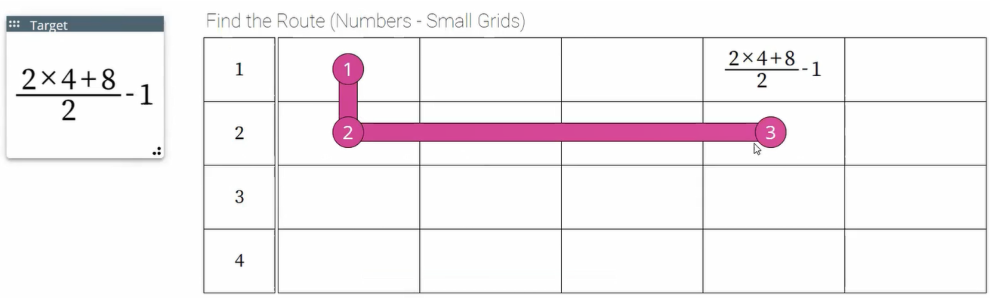
Point 3 on the route appears

Alpha: wait no it can't be

During this time, Delta moves his chair closer to the screen but seems to have something in his eye which he tries to rub out

Figure 7

Students' second route



At this point Alpha takes control of the mouse, clears the route and then clicks the correct route described by the expression which the software accepts. During this, the group are discussing the speed of the mouse.

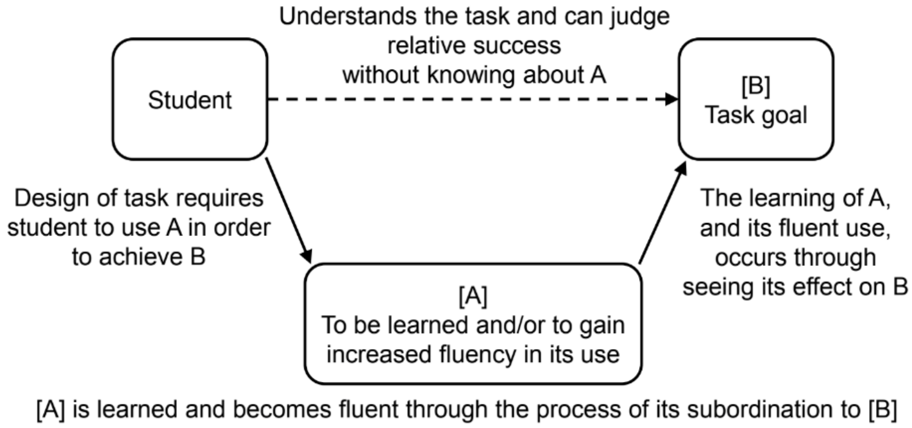
Discussion

The aim of our analysis of this group activity was to show how different modes of communication work together to scaffold students' fluency with operations, the formal notation representing these operations, and the order in which they were performed. The different modes we drew upon include the words spoken by the three students, their movements and clicks of the mouse as they interact with the *Grid Algebra* software, their gestures such as pointing at different parts of the grid, as well as the communication modes of the software itself through its use of a structured visual environment and designed feedback. In what follows, we discuss the role of these different modes in the students' work with notation and mathematical conventions around operations, simultaneously and sequentially, and how shifting between and within these modes highlights the intertwining of notation, movements and gestures, as well as speech in the design of the *Grid Algebra* software.

The notation acts in a subordinate role (Hewitt, 1996) to the aim of the activity: finding the correct route from one cell to another. The subordinate role has three key features. Firstly, they have no choice but to try to interpret the notation in interaction; thus such interpretation is required. Secondly, their interpretation is transformed into physical movements within the grid, which they had come to know through previous activities with the software. Thirdly, they understand the aim of the activity, moving from one cell to another, and can judge their success through what is happening with those movements: are they getting them from the start cell to the final cell? This means that the movements, which they can relate to as they have been making movements throughout their lives, informs their current understanding of the notation, which they may not initially be confident about. The dynamic between notation, movement and position allows students to educate their interpretation of mathematical notation through something with which they are very familiar with – movements and position (see Figure 8).

Figure 8

How students' learning of notation is being subordinated to the visual task of drawing a route



The software connects the notation to the movements on the grid, that is the notation to the operations, and is thus taking care of the symbols (Tatha, 1989). Hewitt (2023) gives a similar example where a teacher can use speech to draw attention to the order of operations while reading notation, by reading the notation in the order of the operations, rather than from left to right.

In addition, this connection between the notation and movements on the grid gives the students feedback about their own journeys that reveal where there are problems with the journey, without indicating the source of these problems. In the first instance, multiplying by four rather than by two, and later adding six rather than eight. The students thus need to consider this connection to identify the cause of the problems they encounter; the notation itself does not offer any explanation. The diagram in Figure 8 can represent ways in which students become aware of a problem with their thinking and actions, and also give a sense of some of the dynamics between different modes of communication. For example, in Excerpt 5, Alpha states that "it has to end up there". This is a visual awareness of what is required to be successful. This is the dynamic identified by the dotted arrow from Student to Task goal in Figure 8. At the same time, Beta makes a movement with the mouse which is consistent with dividing by two, which was the next operation in the given expression. This is an indication of the way Beta is interpreting the notation and applying that to movements within the grid. This is represented by the arrows going through the box marked A in Figure 8. We interpret

Beta's comment of "I did something wrong" as becoming aware that what is understood through visual awareness from the dotted line in Figure 8 is in conflict with the notational and physical movement awareness from the solid lines in Figure 8. This dynamic is augmented with the interchange between Alpha and Beta, with Beta's mouse movements following Alpha's comment of "so maybe it goes right to the top".

The embodied nature of the activity around this task has key features that seem to contribute to the students' progress with this task and the other tasks recorded. Their interpretation of the notation informed the movements they made on the grid, and the position on the grid of the target expression in relation to their resultant position following those movements enabled students to judge whether their interpretation of the notation was correct or not. The interpretation of the notation had a subordinate role in the activity in the sense that it was required to be interpreted in deciding how to move on the grid. The students kept shifting attention from the notation to the movements and position in the grid, as shown by Beta's movement of the mouse in Excerpt 6 between the two whilst trying to verbally agree whether there was a multiplication by two. Speech, movement and notation all had a role to play within their eventual correct interpretation of the notation.

There were three students in the group working on this task, yet the third student has only briefly appeared in our analysis above, as he only utters one word within the extract we focused on, though he does contribute to the discussion about the mouse speed shortly before the conclusion of the task. This student does, at times, appear to be engaged in what is happening on the screen. They adjust their body position to move closer at some points during the interaction. They shift their eye gaze between the activity on the grid and the off-screen activity that also occasionally engages Alpha and Beta. Yet there is no communication through their actions specific to the mathematics involved in the task, including the interpretation of the notation and the structure of the grid. Silent participation is still a form of participation, and we found it interesting that later on in this session, he very efficiently created the route for $4\left(\frac{15-6+3}{3}-2\right)+12$ with no errors.

What is distinct from other research focusing on these aspects of the software (e.g., Hewitt, 2012) is the additional dimension of the students working on the task as a group, where students' speech and gestures with the mouse and through pointing to locations on the grid can support this connection between the movements and the notation. There were several instances where there was some form of trouble in the interaction, often as a result of the feedback from the software, where the movements of the mouse, the route marked on the grid, and the spoken reading of the notation did not correspond. It is this relationship between the different modes of communication that highlights the connection between the notation and the movement around the grid. Beta repeatedly refers to the notation in the target expression, but this is only visible through his mouse movements. Both Alpha and Beta indicate steps on the journey through their gestures, whether pointing or clicking the mouse, while their spoken communication emphasises the operations as indicated through the notation.

Conclusion

This analysis shows students focusing specifically on the order of operations within the given notation. The mathematical structure within the grid is such that there is a unique route going from the start number to the cell with the final expression. No explanation of order of operations is given through the software or by a teacher. Yet students were able to judge whether their movements were correct or not through the dynamic between the notation, movements on the grid, resultant positions on the grid and what was said (a mixture of their interpretation of the notation and how they might move on the grid). Thus, the structural mathematical connection between the notation and the grid itself provided the stimulus for their need to talk about the notation and consider the order of the operations. The resultant position of each step of the route in relation to the start number and target expression provided sufficient feedback for students to judge the correctness of their moves. The students went on to fluently create routes for more complex expressions, such as $\frac{\frac{4+8}{2}+2}{2}+1-2$. This raises issues and questions for us about ways in which students engage with tasks related to notation. Often this may concern questions where the students may not know whether they are right or wrong until having their work marked by a teacher. This may be some time after they finished answering the question. In contrast, these students were able to check the correctness of what they were doing as they were proceeding with the task, and without any explicit feedback about the correctness. How can mathematics classrooms involve more activities where students can be their own judges of the correctness of their work whilst carrying out that work?

This activity involved a dynamic between something which they know about (movements) with something they are less familiar with (notation). The former informing their interpretation of the latter, as indicated by the subordination model in Figure 8. This enables them to see the consequences of their actions within a context with which they are familiar and within which they can judge the correctness of their actions. How can more activities be such that there is such a strong mathematical structural link between something students have strong embodied experiences within their lives, which can guide them as to the way they carry out the related mathematical tasks?

The whole body involvement in this task – static notation, dynamic physical movement of the mouse, visual structure, movement and positioning within the grid – generated the need to discuss through different modal forms: looking, gesturing, pointing, moving and talking.

How can we make mathematics activity involve such a variety of modal forms, each of which has a fundamental mathematical role?

Transcript notation (Jefferson, 1984)

Symbol	Definition and use
[yeah]	
	[okay] Overlapping talk
(.)	Brief interval, usually between 0.08 and 0.2 seconds
(1.4)	Time (in absolute seconds) between end of a word and beginning of next.
Word	Underlining indicates emphasis.
wo::rd	Colon indicates prolonged vowel or consonant.
.	Final falling intonation
,	Slight rising intonation
?	Sharp rising intonation
°word°	Degree sign indicate syllables or words distinctly quieter than surrounding speech by the same speaker

Pre-positioned left carat indicates a hurried start of a word, typically at TCU beginning

word-	A dash indicates a cut-off. In phonetic terms this is typically a glottal stop
(( ))	Double parentheses contain analyst comments or descriptions

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