SIAM J. NUMER. ANAL.
Vol. 51, No. 4, pp. 2088–2106

DISCONTINUOUS GALERKIN FINITE ELEMENT APPROXIMATION OF NONDIVERGENCE FORM ELLIPTIC EQUATIONS WITH CORDÈS COEFFICIENTS*

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Abstract. Nondivergence form elliptic equations with discontinuous coefficients do not generally possess a weak formulation, thus presenting an obstacle to their numerical solution by classical finite element methods. We propose a new *hp*-version discontinuous Galerkin finite element method for a class of these problems which satisfy the Cordès condition. It is shown that the method exhibits a convergence rate that is optimal with respect to the mesh size h and suboptimal with respect to the polynomial degree p by only half an order. Numerical experiments demonstrate the accuracy of the method and illustrate the potential of exponential convergence under *hp*-refinement for problems with discontinuous coefficients and nonsmooth solutions.

Key words. discontinuous Galerkin, hp-DGFEM, Cordès condition, nondivergence form, discontinuous coefficients, PDEs, finite element methods

AMS subject classifications. 65N30, 65N12, 35J15, 35D35

DOI. 10.1137/120899613

1. Introduction. This work is concerned with boundary-value problems of the form

$$
\begin{aligned}
Lu &= f \qquad \text{in } \Omega,\\
u &= 0 \qquad \text{on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where $f \in L^2(\Omega)$ and L is a second-order elliptic operator in nondivergence form, i.e., the leading term of L is of the form $\sum_{i,j=1}^n a_{ij}\, u_{x_i x_j}$, with coefficients $a_{ij} \in L^\infty(\Omega)$. To keep the exposition clear, we focus on operators of the form $\sum_{i,j=1}^n a_{ij}\, u_{x_i x_j}$ without lower order terms. Nevertheless, the results of this work can be extended to problems with lower order terms by following ideas from [6]. Nonhomogeneous boundary conditions are discussed in section 6 below.

Problem (1.1) arises in many applications from areas such as probability and stochastic processes. These equations also arise as linearizations to fully nonlinear PDEs, as obtained, for instance, from the use of iterative solution algorithms. In such cases, it can rarely be expected that the coefficients of the operator be smooth or even continuous. For example, in applications to Hamilton–Jacobi–Bellman equations [8], the coefficients a_{ij} will usually be merely essentially bounded. The extension of the scheme developed here to these problems is the subject of subsequent work [18].

In contrast to the study of divergence form equations, it is usually not possible to define a notion of weak solution to (1.1) when the coefficients are nonsmooth. In the case of continuous but possibly nondifferentiable coefficients, the Calderon–Zygmund theory of strong solutions [10] establishes the well-posedness of the problem in sufficiently smooth domains. However, without additional hypotheses, the well-posedness of (1.1) is generally lost in the case of discontinuous coefficients; see [14]

*Received by the editors November 20, 2012; accepted for publication (in revised form) April 24, 2013; published electronically July 16, 2013.
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for a comprehensive treatment and various examples; see also the example in [10, p. 185].

Despite these difficulties, well-posedness of solutions in the space $H^2(\Omega) \cap H_0^1(\Omega)$ is recovered for problems in convex domains that satisfy *the Cordès condition* stated in (2.5) below. This condition will play a central role in the numerical analysis of the method proposed in this work. For the analysis and motivation of the Cordès condition, we refer the reader to [14] and the references therein.

Unlike the literature on elliptic equations in divergence form, the literature on the numerical analysis of nondivergence form equations is comparatively sparse. In view of the applications mentioned above, it is important to consider methods that do not assume a priori information about the location of the discontinuities of the coefficients.

Conforming finite element methods for (1.1) require at least H^2-regularity of the approximation; this amounts to a C^1-continuity condition on the finite element space. For instance, [4] proposes a collocation scheme using C^1 splines for nondivergence form equations, but the analysis therein requires at least $C^{1,1}$ regularity of the coefficients. Otherwise, it would appear that the numerical analysis of (1.1) with discontinuous coefficients has remained unexplored.

Discontinuous Galerkin finite element methods (DGFEMs) allow the approximation to be discontinuous between elements, with the continuity conditions being enforced only weakly through the discretized problem. These methods have been analyzed and applied to a large range of problems [1, 13, 16]; see also the book [7]. The ability of DGFEM to handle *hp*-refinement, where one varies both mesh size and polynomial degree, is of significant interest here, in view of the potential loss of higher regularity of the solution near discontinuities of the coefficients. Indeed, *hp*-refinement has been used in the context of continuous Galerkin finite element methods and DGFEMs to obtain exponential convergence for problems with nonsmooth solutions; see [2, 12, 15, 17]. Exponential convergence rates for *hp*-DGFEMs were proved in [20].

This paper proposes and analyses an *hp*-DGFEM for equations in nondivergence form with coefficients satisfying the Cordès condition. A key question addressed here is that of specifying a stable discretization scheme, since it is not possible to use a weak form of the problem to exhibit the underlying coercive structure of the differential operator. Stability is achieved by coupling the residual of the differential equation to terms measuring the lack of H^2-conformity of the numerical solution. The choice of bilinear form draws upon a discrete analogue of an identity that is central to the analysis of well-posedness in $H^2(\Omega)$ of elliptic problems on convex domains [11, 14].

Section 2 defines the problem considered and the notation used in this paper. This is followed by the definition of the scheme and the analysis of consistency in section 3. Stability and well-posedness are proved in section 4, followed by the a priori error analysis in section 5, where it is found that the convergence rates in a broken H^2-type norm are optimal with respect to the mesh size and suboptimal with respect to the polynomial degree by only half an order. Section 6 presents numerical experiments testing the accuracy and robustness of the scheme: the first experiment verifies the predicted convergence rates, and the second experiment gives an example of exponential accuracy under appropriate *hp*-refinement for a problem featuring both discontinuity of the coefficients and nonsmoothness of the solution.

2. Preliminaries. Let Ω be a bounded convex polyhedral domain in $\mathbb{R}^n$, $n \geq 2$. Note that the convexity assumption implies that Ω has a Lipschitz boundary $\partial\Omega$;

see [11]. Let the bounded operator $L\colon H^2(\Omega) \to L^2(\Omega)$ be defined by

$$Lv := \sum_{i,j=1}^{n} a_{ij}\, v_{x_i x_j}, \qquad v \in H^2(\Omega), \tag{2.1}$$

where it is assumed that

$$a_{ij} = a_{ji} \in L^\infty(\Omega) \qquad \forall\, i,j \in \{1,\dots,n\}, \tag{2.2}$$

and that L is uniformly elliptic, i.e., there exist constants $\Lambda, \lambda > 0$ such that

$$\lambda\, |\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\, \xi_j \le \Lambda\, |\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^n, \text{ a.e. in } \Omega. \tag{2.3}$$

We consider the following problem: for a given $f \in L^2(\Omega)$, find a strong solution $u \in H^2(\Omega) \cap H^1_0(\Omega)$ of the boundary-value problem

$$\begin{aligned} Lu &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.4}$$

It is well known that, in the case of discontinuous coefficients a_{ij}, assumptions (2.2) and (2.3) are generally not sufficient to obtain well-posedness of the problem (2.4); see, for instance, the examples in [10, 14]. For this reason, we consider problems that satisfy *the Cordès condition*: there is an $\varepsilon \in (0,1]$ such that

$$\frac{\sum_{i,j=1}^{n} (a_{ij})^2}{\left(\sum_{i=1}^{n} a_{ii}\right)^2} \le \frac{1}{n-1+\varepsilon} \qquad \text{a.e. in } \Omega. \tag{2.5}$$

For problems in two dimensions, it is not hard to show that uniform ellipticity implies the Cordès condition [18]. Let $\gamma \in L^\infty(\Omega)$ be defined by

$$\gamma := \frac{\sum_{i=1}^{n} a_{ii}}{\sum_{i,j=1}^{n} (a_{ij})^2}. \tag{2.6}$$

The uniform ellipticity assumption on the operator L implies that there is a $\gamma_0 > 0$ such that $\gamma \ge \gamma_0$ a.e. in Ω. The Cordès condition implies the following inequality that will be central to the subsequent analysis.

LEMMA 1. *Let the operator L defined by* (2.1) *satisfy* (2.2), (2.3), *and* (2.5), *and let $\gamma \in L^\infty(\Omega)$ be defined by* (2.6). *Then, for any open set $U \subset \Omega$ and $v \in H^2(U)$, we have*

$$|\gamma Lv - \Delta v| \le \sqrt{1-\varepsilon}\, \left|D^2 v\right| \qquad \text{a.e. in } U, \tag{2.7}$$

where $\varepsilon \in (0,1]$ is as in (2.5).

Proof. Let $v \in H^2(U)$. Then,

$$|\gamma Lv - \Delta v| = \left| \sum_{i,j=1}^{n} (\gamma a_{ij} - \delta_{ij})\, v_{x_i x_j} \right| \le \left(\sum_{i,j=1}^{n} |\gamma a_{ij} - \delta_{ij}|^2 \right)^{\frac{1}{2}} \left|D^2 v\right|.$$

Now, by expanding the square and using (2.6) followed by (2.5), we find that

$$\sum_{i,j=1}^{n} |\gamma a_{ij} - \delta_{ij}|^2 = n - \frac{\left(\sum_{i,j=1}^{n} a_{ii}\right)^2}{\sum_{i,j=1}^{n} (a_{ij})^2} \le 1 - \varepsilon.$$

Inequality (2.7) follows immediately. ∎

2.1. Analysis of the PDE. The Cordès condition leads to the well-posedness of (2.4), as the results of this section demonstrate. We follow [14] in naming the following estimate the Miranda–Talenti estimate. See [19] for a proof.

THEOREM 2 (Miranda–Talenti). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Then, for any $u \in H^2(\Omega) \cap H^1_0(\Omega)$,*

$$|u|_{H^2(\Omega)} \leq \|\Delta u\|_{L^2(\Omega)}, \tag{2.8a}$$

$$\|u\|_{H^2(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)}, \tag{2.8b}$$

where C is a constant depending only on n and $\operatorname{diam}\Omega$.

THEOREM 3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let the operator L defined by* (2.1) *satisfy* (2.2), (2.3), *and* (2.5). *Then, for any given $f \in L^2(\Omega)$, there exists a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ that is a strong solution of* (2.4), *and u satisfies $\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$, where C depends only on n,* $\operatorname{diam}\Omega$, *$\lambda$, Λ, and ε.*

Proof. Let γ be defined by (2.6). Define $H := H^2(\Omega) \cap H^1_0(\Omega)$. Theorem 2 shows that the bilinear form $\langle\cdot,\cdot\rangle_\Delta \colon H \times H \to \mathbb{R}$, $\langle u, v\rangle_\Delta := \int_\Omega \Delta u \, \Delta v \,\mathrm{dx}$, defines an inner product on H, and it follows that $(H, \langle\cdot,\cdot\rangle_\Delta)$ is a Hilbert space. Let $\|\cdot\|_\Delta$ denote the norm induced by the inner product on H. Define the bilinear form $A\colon H \times H \to \mathbb{R}$ by $A(u,v) := \int_\Omega \gamma L u \, \Delta v \,\mathrm{dx}$ for u, $v \in H$. Since $a_{ij} \in L^\infty(\Omega)$, Theorem 2 shows that A is bounded: for all u, $v \in H$, $|A(u,v)| \leq C\|u\|_\Delta \|v\|_\Delta$. We claim that A is coercive on H. Indeed, using (2.7),

$$A(u,u) = \langle u, u\rangle_\Delta - \int_\Omega (\Delta - \gamma L)\, u \, \Delta u \,\mathrm{dx} \geq \|u\|_\Delta^2 - \sqrt{1-\varepsilon}\, |u|_{H^2(\Omega)} \, \|u\|_\Delta.$$

By Theorem 2, $|u|_{H^2(\Omega)} \leq \|u\|_\Delta$, so $A(u,u) \geq \left(1 - \sqrt{1-\varepsilon}\right) \|u\|_\Delta^2$, and hence A is coercive.

Given $f \in L^2(\Omega)$, define $\ell\colon H \to \mathbb{R}$ by $\ell(v) := \int_\Omega \gamma f \, \Delta v \,\mathrm{dx}$ for $v \in H$. Then ℓ is a bounded linear functional on H. The Lax–Milgram theorem shows existence of a unique $u \in H$ such that $A(u,v) = \ell(v)$ for all $v \in H$. We claim that $Lu = f$ pointwise a.e. in Ω. For any $g \in L^2(\Omega)$, there is $v \in H$ such that $\Delta v = g$. So

$$\int_\Omega \gamma L u \, g \,\mathrm{dx} = \int_\Omega \gamma f \, g \,\mathrm{dx} \qquad \forall\, g \in L^2(\Omega).$$

This implies that $\gamma L u = \gamma f$ a.e. in Ω. Since $\gamma > 0$ a.e. in Ω, we deduce that u is a strong solution of (2.4). Finally, we have

$$\|u\|_{H^2(\Omega)} \leq C\|u\|_\Delta \leq C \frac{\|\gamma\|_{L^\infty(\Omega)}}{1 - \sqrt{1-\varepsilon}} \|f\|_{L^2(\Omega)},$$

where the constant C from (2.8b) depends only on n and $\operatorname{diam}\Omega$. ∎

2.2. Finite element spaces. Let $\{\mathcal{T}_h\}_h$ be a sequence of shape-regular meshes on Ω, consisting of simplices or parallelepipeds. For each element $K \in \mathcal{T}_h$, let $h_K := \operatorname{diam} K$. It is assumed that $h = \max_{K\in\mathcal{T}_h} h_K$ for each mesh $\mathcal{T}_h$. Let $\mathcal{F}_h^i$ denote the set of interior faces of the mesh $\mathcal{T}_h$, and let $\mathcal{F}_h^b$ denote the set of boundary faces. The set of all faces of $\mathcal{T}_h$ is denoted by $\mathcal{F}_h^{i,b} := \mathcal{F}_h^i \cup \mathcal{F}_h^b$. Since each element has piecewise flat boundary, the faces may be chosen to be flat.

Mesh conditions. We shall make the following assumptions on the meshes. The meshes are allowed to be irregular; i.e., there may be hanging nodes. We assume that there is a uniform upper bound on the number of faces composing the boundary of any given element; in other words, there is a $c_{\mathcal{F}} > 0$, independent of h, such that

$$\max_{K\in\mathcal{T}_h} \operatorname{card}\left\{F \in \mathcal{F}_h^{i,b} \colon F \subset \partial K\right\} \leq c_{\mathcal{F}} \qquad \forall K \in \mathcal{T}_h,\ \forall h > 0. \tag{2.9}$$

It is also assumed that any two elements sharing a face have commensurate diameters; i.e., there is a $c_{\mathcal{T}} \geq 1$, independent of h, such that

$$\max(h_K, h_{K'}) \leq c_{\mathcal{T}} \min(h_K, h_{K'}) \tag{2.10}$$

for any K and K' in $\mathcal{T}_h$ that share a face. For each h, let $\mathbf{p} := (p_K \colon K \in \mathcal{T}_h)$ be a vector of positive integers. In order to let p_K appear in the denominator of various expressions, we shall assume that $p_K \geq 1$ for all $K \in \mathcal{T}_h$. We make the assumption that $\mathbf{p}$ has *local bounded variation* [13]: there is a $c_{\mathcal{P}} \geq 1$, independent of h, such that

$$\max(p_K, p_{K'}) \leq c_{\mathcal{P}} \min(p_K, p_{K'}) \tag{2.11}$$

for any K and K' in $\mathcal{T}_h$ that share a face.

Function spaces. For each $K \in \mathcal{T}_h$, let $\mathcal{P}_{p_K}(K)$ be either the space of all polynomials with total degree less than or equal to p_K or with partial degree less than or equal to p_K. The discontinuous Galerkin finite element space $V_{h,\mathbf{p}}$ is defined by

$$V_{h,\mathbf{p}} := \left\{v \in L^2(\Omega) \colon\ v|_K \in \mathcal{P}_{p_K}(K)\ \forall K \in \mathcal{T}_h\right\}. \tag{2.12}$$

Let $\mathbf{s} := (s_K \colon K \in \mathcal{T}_h)$ denote a vector of nonnegative real numbers. The broken Sobolev space $H^{\mathbf{s}}(\Omega; \mathcal{T}_h)$ is defined by

$$H^{\mathbf{s}}(\Omega;\mathcal{T}_h) := \left\{v \in L^2(\Omega) \colon\ v|_K \in H^{s_K}(K)\ \forall K \in \mathcal{T}_h\right\}. \tag{2.13}$$

For $s \geq 0$, we set $H^s(\Omega;\mathcal{T}_h) := H^{\mathbf{s}}(\Omega;\mathcal{T}_h)$, where $s_K = s$ for all $K \in \mathcal{T}_h$. The norm $\|\cdot\|_{H^{\mathbf{s}}(\Omega;\mathcal{T}_h)}$ and seminorm $|\cdot|_{H^{\mathbf{s}}(\Omega;\mathcal{T}_h)}$ are defined on $H^{\mathbf{s}}(\Omega;\mathcal{T}_h)$ as

$$\|v\|_{H^{\mathbf{s}}(\Omega;\mathcal{T}_h)} := \left(\sum_{K\in\mathcal{T}_h} \|v\|^2_{H^{s_K}(K)}\right)^{\frac{1}{2}}, \quad |v|_{H^{\mathbf{s}}(\Omega;\mathcal{T}_h)} := \left(\sum_{K\in\mathcal{T}_h} |v|^2_{H^{s_K}(K)}\right)^{\frac{1}{2}}. \tag{2.14}$$

Traces. It will be helpful to briefly review the construction of certain traces that is given in [11]. For each face $F \in \mathcal{F}_h^{i,b}$, let $n_F \in \mathbb{R}^n$ denote a *fixed* choice of a unit normal vector to F. Since F is flat, n_F is constant over F. Let K be an element of $\mathcal{T}_h$ for which $F \subset \partial K$; then n_F is either inward or outward pointing with respect to K. Since n_F is constant over F, n_F extends trivially as a constant vector field over $\overline{K}$. Let $\tau_F \colon H^s(K) \to H^{s-1/2}(F)$, $s > 1/2$, denote the trace operator from K to F. The trace operator τ_F is extended componentwise to vector-valued functions. Then, for $v \in H^s(K)$, $s > 3/2$, the normal derivative of v on F is defined by

$$\tau_F \frac{\partial v}{\partial n_F} := \tau_F\left(\nabla v \cdot n_F\right), \tag{2.15}$$

where we use the fact that, after extending n_F to a constant vector field on $\overline{K}$, the function $\nabla v \cdot n_F \in H^{s-1}(K)$ belongs to the domain of τ_F.

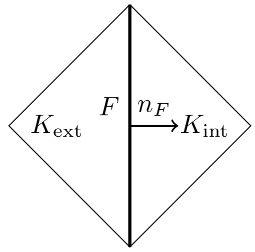

FIG. 1. *Diagram for the notation of jump and average operators. For a face $F \in \mathcal{F}_h^i$, and a chosen normal vector n_F, K_{int} is the element for which n_F is inward pointing, and K_{ext} is the element for which n_F is outward pointing.*

Jump and average operators. The jump and average operators on faces of a mesh are defined as follows. For each face F with corresponding unit normal vector n_F, define $[\![\cdot]\!]$, the jump operator over F, by

$$[\![\phi]\!] := \begin{cases} \tau_F\left(\phi|_{K_{\mathrm{ext}}}\right) - \tau_F\left(\phi|_{K_{\mathrm{int}}}\right) & \text{if } F \in \mathcal{F}_h^i, \\ \tau_F\left(\phi|_{K_{\mathrm{ext}}}\right) & \text{if } F \in \mathcal{F}_h^b, \end{cases} \tag{2.16}$$

and define $\{\cdot\}$, the average operator over F, by

$$\{\phi\} := \begin{cases} \frac{1}{2}\left(\tau_F\left(\phi|_{K_{\mathrm{ext}}}\right) + \tau_F\left(\phi|_{K_{\mathrm{int}}}\right)\right) & \text{if } F \in \mathcal{F}_h^i, \\ \tau_F\left(\phi|_{K_{\mathrm{ext}}}\right) & \text{if } F \in \mathcal{F}_h^b, \end{cases} \tag{2.17}$$

where ϕ is a sufficiently regular scalar- or vector-valued function, and K_{ext} and K_{int} are the elements to which F is a face, i.e., $F = \partial K_{\mathrm{ext}} \cap \partial K_{\mathrm{int}}$. Here, the labeling is chosen so that for any $x \in \operatorname{int} F$, $x - \lambda n_F \in K_{\mathrm{ext}}$ for small $\lambda > 0$ and $x + \lambda n_F \in K_{\mathrm{int}}$ for small $\lambda > 0$; see Figure 1. Using this notation, the jump and average of scalar-valued functions, respectively, vector-valued functions, are scalar-valued, respectively, vector-valued.

For two matrices $A, B \in \mathbb{R}^{n\times n}$, we set $A : B := \sum_{i,j=1}^n A_{ij} B_{ij}$. For an element K, we define the bilinear form $\langle\cdot,\cdot\rangle_K$ by

$$\langle u, v\rangle_K := \begin{cases} \int_K u\, v \,\mathrm{dx} & \text{if } u, v \in L^2(K), \\ \int_K u\cdot v\,\mathrm{dx} & \text{if } u, v \in L^2(K;\mathbb{R}^n), \\ \int_K u : v\,\mathrm{dx} & \text{if } u, v \in L^2(K;\mathbb{R}^{n\times n}). \end{cases} \tag{2.18}$$

The abuse of notation will be resolved by the arguments of the bilinear form. The bilinear forms $\langle\cdot,\cdot\rangle_{\partial K}$ and $\langle\cdot,\cdot\rangle_F$, $F \in \mathcal{F}_h^{i,b}$, are defined in a similar way.

Tangential differential operators. For $F \in \mathcal{F}_h^{i,b}$, denote the space of H^s-regular tangential vector fields on F by $H_{\mathrm{T}}^s(F) := \{v \in H^s(F)^n : v\cdot n_F = 0 \text{ on } F\}$. We define below the tangential gradient $\nabla_{\mathrm{T}} \colon H^s(F) \to H_{\mathrm{T}}^{s-1}(F)$ and the tangential divergence $\operatorname{div}_{\mathrm{T}} \colon H_{\mathrm{T}}^s(F) \to H^{s-1}(F)$, where $s \geq 1$, following [11]. Let $\{t_i\}_{i=1}^{n-1} \subset \mathbb{R}^n$ be an orthonormal coordinate system on F. Then, for $u \in H^s(F)$ and $v = \sum_{i=1}^{n-1} v_i\, t_i$, with $v_i \in H^s(F)$ for $i = 1, \dots, n-1$, we define

$$\nabla_{\mathrm{T}}\, u := \sum_{i=1}^{n-1} t_i \frac{\partial u}{\partial t_i}, \qquad \operatorname{div}_{\mathrm{T}}\, v := \sum_{i=1}^{n-1} \frac{\partial v_i}{\partial t_i}. \tag{2.19}$$

The next lemma implies that traces and tangential differential operators commute.

LEMMA 4. *Let Ω be a bounded polytopal domain, and let $\mathcal{T}_h$ be a mesh on Ω consisting of simplices or parallelepipeds. Then, for each $K \in \mathcal{T}_h$ and each face $F \subset \partial K$, the following identities hold:*

$$\tau_F(\nabla v) = \nabla_{\mathrm{T}}(\tau_F v) + \left(\tau_F \frac{\partial v}{\partial n_F}\right) n_F \qquad \forall\, v \in H^s(K),\ s > \tfrac{3}{2}, \tag{2.20}$$

$$\tau_F(\Delta v) = \mathrm{div}_{\mathrm{T}} \nabla_{\mathrm{T}}(\tau_F v) + \tau_F \frac{\partial}{\partial n_F}(\nabla v \cdot n_F) \qquad \forall\, v \in H^s(K),\ s > \tfrac{5}{2}. \tag{2.21}$$

Proof. First, observe that the terms in (2.20) and (2.21) are independent of the choice of n_F, since a reversal in the sign of n_F leaves the right-hand sides of these equations unchanged. Recall that F is flat; so, after a suitable change of coordinate system, we may assume without loss of generality that $K \subset \mathbb{R}^n_- := \{(x,x') \colon x \in \mathbb{R}^{n-1},\ x' \leq 0\}$ and that $F \subset \partial\mathbb{R}^n_- = \{(x,0) \colon x \in \mathbb{R}^{n-1}\}$. Since the identities (2.20) and (2.21) are independent of the choice of unit normal n_F, we may assume that $n_F = e_n = (0,\dots,0,1)^\top$.

Let $s > 3/2$; for $i \in \{1,\dots,n-1\}$ we have the identity

$$\tau_F \frac{\partial v}{\partial x_i} = \frac{\partial}{\partial x_i}(\tau_F v) \qquad \forall\, v \in H^s(K). \tag{2.22}$$

Indeed, this identity is valid for a smooth function v and thus extends to general $v \in H^s(K)$, $s > 3/2$, by construction of the trace operator. So, $v \in H^s(K)$ satisfies

$$\nabla v = \sum_{i=1}^{n-1} \frac{\partial v}{\partial x_i} e_i + \frac{\partial v}{\partial x_n} e_n = \nabla_{\mathrm{T}} v + (\nabla v \cdot n_F)\, n_F \qquad \text{in } K,$$

and we use the linearity of the trace operator with (2.22) to obtain

$$\tau_F(\nabla v) = \nabla_{\mathrm{T}}(\tau_F v) + \left(\tau_F \frac{\partial v}{\partial n_F}\right) n_F,$$

thus establishing (2.20). Similarly, for $v \in H^{s+1}(K)$, we write

$$\Delta v = \sum_{i=1}^{n-1} \frac{\partial^2 v}{\partial x_i^2} + \frac{\partial^2 v}{\partial x_n^2} = \mathrm{div}_{\mathrm{T}} \nabla_{\mathrm{T}} v + n_F \cdot \nabla(\nabla v \cdot n_F) \qquad \text{in } K,$$

where the last equality follows from the fact that n_F is constant over K. Then, (2.21) is found by applying the trace operator to both sides of the previous identity and repeatedly applying (2.22) to v and its first tangential derivatives. ∎

Extensions of the results of this work to meshes with curved elements may make use of generalizations of Lemma 4 found in [11, p. 136].

3. Numerical scheme. To define the numerical scheme, we use the following auxiliary bilinear forms. First, let $B_{h,*} \colon V_{h,\mathbf{p}} \times V_{h,\mathbf{p}} \to \mathbb{R}$ be defined by

$$\begin{aligned} B_{h,*}(u_h, v_h) := &\sum_{K \in \mathcal{T}_h} \langle D^2 u_h, D^2 v_h \rangle_K \\ &+ \sum_{F \in \mathcal{F}_h^i} \left[\langle \mathrm{div}_{\mathrm{T}} \nabla_{\mathrm{T}} \{u_h\}, [\![\nabla v_h \cdot n_F]\!] \rangle_F + \langle \mathrm{div}_{\mathrm{T}} \nabla_{\mathrm{T}} \{v_h\}, [\![\nabla u_h \cdot n_F]\!] \rangle_F \right] \\ &- \sum_{F \in \mathcal{F}_h^{i,b}} \left[\langle \nabla_{\mathrm{T}} \{\nabla u_h \cdot n_F\}, [\![\nabla_{\mathrm{T}} v_h]\!] \rangle_F + \langle \nabla_{\mathrm{T}} \{\nabla v_h \cdot n_F\}, [\![\nabla_{\mathrm{T}} u_h]\!] \rangle_F \right], \end{aligned}$$

where u_h, v_h will denote functions in $V_{h,\mathbf{p}}$ throughout this work, and $D^2 u_h$ denotes the broken Hessian of u_h. Then, for face-dependent quantities $\mu_F > 0$ and $\eta_F > 0$ to be specified later, let the jump stabilization term J_h be defined by

$$J_h(u_h, v_h) := \sum_{F\in\mathcal{F}_h^i} \mu_F \langle [\![\nabla u_h \cdot n_F]\!], [\![\nabla v_h \cdot n_F]\!]\rangle_F + \sum_{F\in\mathcal{F}_h^{i,b}} \left[\mu_F \langle [\![\nabla_{\mathrm{T}}\, u_h]\!], [\![\nabla_{\mathrm{T}}\, v_h]\!]\rangle_F + \eta_F \langle [\![u_h]\!], [\![v_h]\!]\rangle_F\right]. \tag{3.1}$$

For each $\theta \in [0, 1]$, define the bilinear form $B_{h,\theta} \colon V_{h,\mathbf{p}} \times V_{h,\mathbf{p}} \to \mathbb{R}$ by

$$B_{h,\theta}(u_h, v_h) = \theta B_{h,*}(u_h, v_h) + (1-\theta) \sum_{K\in\mathcal{T}_h} \langle \Delta u_h, \Delta v_h\rangle_K + J_h(u_h, v_h). \tag{3.2}$$

The bilinear form $A_h \colon V_{h,\mathbf{p}} \times V_{h,\mathbf{p}} \to \mathbb{R}$ is defined by

$$A_h(u_h, v_h) := \sum_{K\in\mathcal{T}_h} \langle \gamma L u_h, \Delta v_h\rangle_K + B_{h,1/2}(u_h, v_h) - \sum_{K\in\mathcal{T}_h} \langle \Delta u_h, \Delta v_h\rangle_K. \tag{3.3}$$

The numerical scheme for approximating the solution of (2.4) is to find $u_h \in V_{h,\mathbf{p}}$ such that

$$A_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} \langle \gamma f, \Delta v_h\rangle_K \qquad \forall\, v_h \in V_{h,\mathbf{p}}. \tag{3.4}$$

If $\mathbf{p} \equiv 1$, i.e., $p_K = 1$ for all $K \in \mathcal{T}_h$, then all terms in $A_h(u_h, v_h)$ vanish except for the jump stabilization terms of $J_h(u_h, v_h)$. In this case, the numerical solution is $u_h \equiv 0$. This suggests that at least quadratic polynomials ought to be employed. Nevertheless, this still compares favorably with conforming elements, because, for instance, Argyris elements require at least polynomials of degree five on simplicial meshes in two dimensions [5].

3.1. Consistency. We turn to the question of consistency of the scheme (3.4) with respect to the original problem (2.4). It will be seen below that a discrete analogue of the identity of [11, Theorem 3.1.1.1] is central to the analysis of the numerical scheme. The following proposition establishes the broken form of this identity.

LEMMA 5. *Let Ω be a bounded Lipschitz polytopal domain, and let $\mathcal{T}_h$ be a simplicial or parallelepipedal mesh. Let $w \in H^s(\Omega; \mathcal{T}_h) \cap H^2(\Omega) \cap H^1_0(\Omega)$, with $s > 5/2$. Then, for every $v_h \in V_{h,\mathbf{p}}$, we have the identities*

$$B_{h,*}(w, v_h) := \sum_{K\in\mathcal{T}_h} \langle \Delta w, \Delta v_h\rangle_K \quad \text{and} \quad J_h(w, v_h) = 0. \tag{3.5}$$

Proof. Let w satisfy the above assumptions, and let $v_h \in V_{h,\mathbf{p}}$. The second statement in (3.5) is trivial. Let $K \in \mathcal{T}_h$, let $\bar{n}$ be the piecewise constant outward normal on ∂K, and momentarily assume that $w \in H^3(K)$. Then, for $1 \le i, j \le n$, integration by parts gives

$$\begin{aligned} \int_K w_{x_i x_j} (v_h)_{x_i x_j} \mathrm{dx} &= \int_{\partial K} w_{x_i x_j} \bar{n}_i (v_h)_{x_j} \mathrm{ds} - \int_K w_{x_i x_j x_i} (v_h)_{x_j} \mathrm{dx} \\ &= \int_K w_{x_i x_i} (v_h)_{x_j x_j} \mathrm{dx} - \int_{\partial K} \left[w_{x_i x_i} \bar{n}_j (v_h)_{x_j} - w_{x_i x_j} \bar{n}_i (v_h)_{x_j} \right] \mathrm{ds}. \end{aligned} \tag{3.6}$$

Summing (3.6) over i, j and using the fact that $\bar{n}$ is piecewise constant over ∂K, we obtain

$$(3.7)\qquad \langle D^2 w, D^2 v_h\rangle_K + \langle \Delta w, \nabla v_h \cdot \bar{n}\rangle_{\partial K} - \langle \nabla(\nabla w\cdot \bar{n}), \nabla v_h\rangle_{\partial K} = \langle \Delta w, \Delta v_h\rangle_K.$$

A density argument shows that (3.7) holds for $w \in H^s(K)$, $s > 5/2$. Note that for each face $F \subset \partial K$, $\bar{n} = \pm n_F$ on F. Also, for each face $F \subset \partial K$, identity (2.20) gives

$$(3.8)\qquad \tau_F\big(\nabla(\nabla w\cdot n_F)\big)\cdot \tau_F(\nabla v_h) \\ = \nabla_{\mathrm{T}}\big(\tau_F\left(\nabla w\cdot n_F\right)\big)\cdot \nabla_{\mathrm{T}}\left(\tau_F v_h\right) + \left(\tau_F \frac{\partial}{\partial n_F}\left(\nabla w \cdot n_F\right)\right)\left(\tau_F\frac{\partial v_h}{\partial n_F}\right).$$

For each face $F \subset \partial K$, identity (2.21) gives

$$(3.9)\qquad \tau_F(\Delta w)\,\tau_F(\nabla v_h\cdot n_F) = \left(\operatorname{div}_{\mathrm{T}}\nabla_{\mathrm{T}}(\tau_F w) + \tau_F\frac{\partial}{\partial n_F}(\nabla w\cdot n_F)\right)\left(\tau_F\frac{\partial v_h}{\partial n_F}\right).$$

Substituting (3.8) and (3.9) into (3.7) and summing over all elements shows that

$$(3.10)\qquad \sum_{K\in\mathcal{T}_h}\langle D^2 w, D^2 v_h\rangle_K + \sum_{F\in\mathcal{F}_h^{i,b}}\int_F [\![(\operatorname{div}_{\mathrm{T}}\nabla_{\mathrm{T}} w)\,(\nabla v_h\cdot n_F) - \nabla_{\mathrm{T}}(\nabla w\cdot n_F)\cdot\nabla_{\mathrm{T}} v_h]\!]\mathrm{ds} \\ = \sum_{K\in\mathcal{T}_h}\langle \Delta w, \Delta v_h\rangle_K.$$

For $F \in \mathcal{F}_h^i$ and $w \in H^s(\Omega;\mathcal{T}_h)\cap H^2(\Omega)$, we use the facts that the trace operator commutes with tangential differential operators and that $[\![w]\!] = 0$ on F to obtain

$$[\![\operatorname{div}_{\mathrm{T}}\nabla_{\mathrm{T}} w]\!] = \operatorname{div}_{\mathrm{T}}\nabla_{\mathrm{T}}[\![w]\!] = 0 \qquad \text{on } F.$$

Furthermore, $w \in H^2(\Omega)$ implies $[\![\nabla w]\!] = 0$ on F; therefore,

$$[\![\nabla_{\mathrm{T}}(\nabla w\cdot n_F)]\!] = \nabla_{\mathrm{T}}[\![\nabla w\cdot n_F]\!] = 0 \qquad \text{on } F.$$

So, for any $F \in \mathcal{F}_h^i$, it is found that

$$[\![(\operatorname{div}_{\mathrm{T}}\nabla_{\mathrm{T}} w)\,(\nabla v_h\cdot n_F) - \nabla_{\mathrm{T}}(\nabla w\cdot n_F)\cdot\nabla_{\mathrm{T}} v_h]\!] \\ = (\operatorname{div}_{\mathrm{T}}\nabla_{\mathrm{T}}\{w\})\,[\![\nabla v_h\cdot n_F]\!] - \nabla_{\mathrm{T}}\{\nabla w\cdot n_F\}\cdot[\![\nabla_{\mathrm{T}} v_h]\!].$$

For $F \in \mathcal{F}_h^b$, $\tau_F w = 0$ on F because $w \in H_0^1(\Omega)$; hence $\operatorname{div}_{\mathrm{T}}\nabla_{\mathrm{T}} w = 0$ on F. As a result,

$$[\![(\operatorname{div}_{\mathrm{T}}\nabla_{\mathrm{T}} w)\,(\nabla v_h\cdot n_F) - \nabla_{\mathrm{T}}(\nabla w\cdot n_F)\cdot\nabla_{\mathrm{T}} v_h]\!] = -\nabla_{\mathrm{T}}\big(\tau_F(\nabla w\cdot n_F)\big)\cdot\nabla_{\mathrm{T}}\left(\tau_F v_h\right) \\ = -\nabla_{\mathrm{T}}\{\nabla w\cdot n_F\}\cdot[\![\nabla_{\mathrm{T}} v_h]\!].$$

Substituting the above simplifications into (3.10) shows that

$$(3.11)\qquad \sum_{K\in\mathcal{T}_h}\langle D^2 w, D^2 v_h\rangle_K + \sum_{F\in\mathcal{F}_h^i}\langle \operatorname{div}_{\mathrm{T}}\nabla_{\mathrm{T}}\{w\}, [\![\nabla v_h\cdot n_F]\!]\rangle_F \\ - \sum_{F\in\mathcal{F}_h^{i,b}}\langle \nabla_{\mathrm{T}}\{\nabla w\cdot n_F\}, [\![\nabla_{\mathrm{T}} v_h]\!]\rangle_F = \sum_{K\in\mathcal{T}_h}\langle \Delta w, \Delta v_h\rangle_K.$$

It follows from the hypotheses on w that $[\![\nabla w \cdot n_F]\!]$ vanishes on any interior face and that $[\![\nabla_{\mathrm{T}} w]\!]$ vanishes on any face. Therefore,

$$\sum_{F\in\mathcal{F}_h^i} \langle \operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}} \{v_h\}, [\![\nabla w \cdot n_F]\!]\rangle - \sum_{F\in\mathcal{F}_h^{i,b}} \langle \nabla_{\mathrm{T}} \{\nabla v_h \cdot n_F\}, [\![\nabla_{\mathrm{T}} w]\!]\rangle_F = 0. \tag{3.12}$$

Identity (3.5) then follows from (3.11) and (3.12). □

Recalling the definition of $B_{h,\theta}$ in (3.2), it is clear that if a function w satisfies the hypotheses of Lemma 5, then for any $\theta \in [0,1]$ we have

$$B_{h,\theta}(w, v_h) = \sum_{K\in\mathcal{T}_h} \langle \Delta w, \Delta v_h\rangle_K \qquad \forall\, v_h \in V_{h,\mathbf{p}}. \tag{3.13}$$

Recalling the definition of A_h in (3.3), we obtain the following consistency result.

Corollary 6. *Let Ω be a bounded convex polytopal domain, let $\mathcal{T}_h$ be a simplicial or parallelepipedal mesh, and let $u \in H^2(\Omega)\cap H_0^1(\Omega)$ be the unique solution of* (2.4). *If $u \in H^s(\Omega;\mathcal{T}_h)$, $s > 5/2$, then u satisfies*

$$A_h(u, v_h) = \sum_{K\in\mathcal{T}_h} \langle \gamma f, \Delta v_h\rangle_K \qquad \forall\, v_h \in V_{h,\mathbf{p}}. \tag{3.14}$$

4. Well-posedness of the numerical scheme.

For a positive constant c_*, independent of h and to be specified later, and $\theta \in [0,1]$, define the family of functionals $\|\cdot\|_{\mathrm{DG}(\theta)} \colon V_{h,\mathbf{p}} \to \mathbb{R}$ by

$$\|v_h\|_{\mathrm{DG}(\theta)}^2 = \sum_{K\in\mathcal{T}_h} \left[\theta \|D^2 v_h\|_{L^2(K)}^2 + (1-\theta)\|\Delta v_h\|_{L^2(K)}^2\right] + c_* J_h(v_h, v_h). \tag{4.1}$$

For each $\theta \in [0,1]$, $\|\cdot\|_{\mathrm{DG}(\theta)}$ defines a norm on $V_{h,\mathbf{p}}$. Indeed, homogeneity and the triangle inequality are clear. If $\|v_h\|_{\mathrm{DG}(\theta)} = 0$, then $v_h \in H^2(\Omega)\cap H_0^1(\Omega)$ since $[\![\nabla v_h]\!] = 0$ for all $F \in \mathcal{F}_h^i$, and $[\![v_h]\!] = 0$ for all $F \in \mathcal{F}_h^{i,b}$. Moreover, $\Delta v_h \equiv 0$. Uniqueness of solutions to Poisson's equation in $H^2(\Omega)\cap H_0^1(\Omega)$ on convex domains implies that $v_h = 0$.

For each face $F \in \mathcal{F}_h^{i,b}$, define

$$\tilde{h}_F := \begin{cases} \min(h_K, h_{K'}) & \text{if } F \in \mathcal{F}_h^i, \\ h_K & \text{if } F \in \mathcal{F}_h^b, \end{cases} \qquad \tilde{p}_F := \begin{cases} \max(p_K, p_{K'}) & \text{if } F \in \mathcal{F}_h^i, \\ p_K & \text{if } F \in \mathcal{F}_h^b, \end{cases} \tag{4.2}$$

where K and K' are such that $F = \partial K \cap \partial K'$ if $F \in \mathcal{F}_h^i$ or $F \subset \partial K \cap \partial\Omega$ if $F \in \mathcal{F}_h^b$. The assumptions on the mesh and the polynomial degrees, in particular (2.10) and (2.11), show that if F is a face of K, then

$$h_K \le c_{\mathcal{T}}\, \tilde{h}_F \quad \text{and} \quad \tilde{p}_F \le c_{\mathcal{P}}\, p_K. \tag{4.3}$$

Lemma 7. *Let Ω be a bounded convex polytopal domain, and let $\{\mathcal{T}_h\}_h$ be a shape-regular sequence of simplicial or parallelepipedal meshes satisfying* (2.9). *Then, for each constant $\kappa > 1$, there exist positive constants c_{stab} and c_*, independent of h, $\mathbf{p}$, and θ, such that*

$$\|v_h\|_{\mathrm{DG}(\theta)}^2 \le \kappa B_{h,\theta}(v_h, v_h) \qquad \forall\, v_h \in V_{h,\mathbf{p}},\ \forall\, \theta \in [0,1], \tag{4.4}$$

whenever

$$\mu_F \geq c_{\mathrm{stab}} \frac{\tilde{p}_F^2}{\tilde{h}_F} \quad \text{and} \quad \eta_F > 0. \tag{4.5}$$

Proof. Let $v_h \in V_{h,\mathbf{p}}$. For some $\delta > 0$ to be chosen below, the Cauchy–Schwarz inequality with a parameter gives

$$\begin{aligned} I_1 &:= \left| 2 \sum_{F \in \mathcal{F}_h^i} \langle \operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}} \{v_h\}, [\![\nabla v_h \cdot n_F]\!]\rangle_F \right| \\ &\leq 2 \left(\sum_{F \in \mathcal{F}_h^i} \frac{\delta \tilde{h}_F}{\tilde{p}_F^2} \|\operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}} \{v_h\}\|_{L^2(F)}^2 \right)^{\frac{1}{2}} \left(\sum_{F \in \mathcal{F}_h^i} \frac{\tilde{p}_F^2}{\delta \tilde{h}_F} \|[\![\nabla v_h \cdot n_F]\!]\|_{L^2(F)}^2 \right)^{\frac{1}{2}} \\ &\leq \delta \sum_{F \in \mathcal{F}_h^i} \frac{\tilde{h}_F}{\tilde{p}_F^2} \|\operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}} \{v_h\}\|_{L^2(F)}^2 + \sum_{F \in \mathcal{F}_h^i} \frac{\tilde{p}_F^2}{\delta \tilde{h}_F} \|[\![\nabla v_h \cdot n_F]\!]\|_{L^2(F)}^2. \end{aligned}$$

Since the tangential differential operators commute with the trace operator, for each face $F = \partial K \cap \partial K'$, Young's inequality yields

$$\|\operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}} \{v_h\}\|_{L^2(F)}^2 \leq \tfrac{1}{2} \|\operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}} v_h|_K\|_{L^2(F)}^2 + \tfrac{1}{2} \|\operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}} v_h|_{K'}\|_{L^2(F)}^2.$$

Therefore, the trace and inverse inequalities give

$$\delta \sum_{F \in \mathcal{F}_h^i} \frac{\tilde{h}_F}{\tilde{p}_F^2} \|\operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}} \{v_h\}\|_{L^2(F)}^2 \leq \delta \sum_{F \in \mathcal{F}_h^i} \frac{\tilde{h}_F}{\tilde{p}_F^2} \sum_{\substack{K \\ F \subset \partial K}} C_{\mathrm{Tr}} \frac{p_K^2}{h_K} C(n) \|D^2 v_h\|_{L^2(K)}^2,$$

where $C(n)$ is a constant depending only on n and C_{Tr} is the constant of the trace and inverse inequality. Since each element has at most $c_{\mathcal{F}}$ faces (see (2.9)), a counting argument shows that

$$\sum_{F \in \mathcal{F}_h^i} \sum_{\substack{K \\ F \subset \partial K}} \|D^2 v_h\|_{L^2(K)}^2 \leq c_{\mathcal{F}} \sum_{K \in \mathcal{T}_h} \|D^2 v_h\|_{L^2(K)}^2.$$

We then use the definitions of $\tilde{p}_F$ and $\tilde{h}_F$ from (4.2) to obtain

$$I_1 \leq \delta C(n) C_{\mathrm{Tr}} c_{\mathcal{F}} \sum_{K \in \mathcal{T}_h} \|D^2 v_h\|_{L^2(K)}^2 + \sum_{F \in \mathcal{F}_h^i} \frac{\tilde{p}_F^2}{\delta \tilde{h}_F} \|[\![\nabla v_h \cdot n_F]\!]\|_{L^2(F)}^2. \tag{4.6}$$

A similar analysis shows that

$$\begin{aligned} I_2 &:= \left| 2 \sum_{F \in \mathcal{F}_h^{i,b}} \langle \nabla_{\mathrm{T}} \{\nabla v_h \cdot n_F\}, [\![\nabla_{\mathrm{T}} v_h]\!]\rangle_F \right| \\ &\leq \delta C(n) C_{\mathrm{Tr}} c_{\mathcal{F}} \sum_{K \in \mathcal{T}_h} \|D^2 v_h\|_{L^2(K)}^2 + \sum_{F \in \mathcal{F}_h^{i,b}} \frac{\tilde{p}_F^2}{\delta \tilde{h}_F} \|[\![\nabla_{\mathrm{T}} v_h]\!]\|_{L^2(F)}^2, \end{aligned} \tag{4.7}$$

where $C(n)$ is a constant depending only on n. Inequalities (4.6) and (4.7) imply that $B_{h,\theta}(v_h, v_h) \geq \sum_{i=1}^{5} A_i$, where

$$A_1 := \theta(1 - 2\delta C(n) C_{\mathrm{Tr}} c_{\mathcal{F}}) \sum_{K\in\mathcal{T}_h} \|D^2 v_h\|^2_{L^2(K)}, \quad A_2 := (1-\theta) \sum_{K\in\mathcal{T}_h} \|\Delta v_h\|^2_{L^2(K)},$$

$$A_3 := \sum_{F\in\mathcal{F}_h^i} \left(\mu_F - \frac{\theta \tilde{p}_F^2}{\delta \tilde{h}_F}\right) \|[\![\nabla v_h \cdot n_F]\!]\|^2_{L^2(F)}, \qquad A_4 := \sum_{F\in\mathcal{F}_h^{i,b}} \eta_F \|[\![v_h]\!]\|^2_{L^2(F)},$$

$$A_5 := \sum_{F\in\mathcal{F}_h^{i,b}} \left(\mu_F - \frac{\theta \tilde{p}_F^2}{\delta \tilde{h}_F}\right) \|[\![\nabla_{\mathrm{T}}\, v_h]\!]\|^2_{L^2(F)}.$$

Let $\kappa > 1$ be given. Then, since $\kappa^{-1} < 1$, there exists a $\delta > 0$ sufficiently small such that $(1 - 2\delta C(n) C_{\mathrm{Tr}} c_{\mathcal{F}}) > \kappa^{-1}$. Then, we choose $c_{\mathrm{stab}} = 2\delta^{-1}$, $c_* = \kappa/2$, and $\mu_F \geq c_{\mathrm{stab}} \tilde{p}_F^2 / \tilde{h}_F$. Therefore, for any $\theta \in [0, 1]$,

$$\begin{aligned}
A_3 &\geq \tfrac{1}{2} \sum_{F\in\mathcal{F}_h^i} \mu_F \|[\![\nabla v_h \cdot n_F]\!]\|^2_{L^2(F)} = \kappa^{-1} c_* \sum_{F\in\mathcal{F}_h^i} \mu_F \|[\![\nabla v_h \cdot n_F]\!]\|^2_{L^2(F)},\\
A_5 &\geq \tfrac{1}{2} \sum_{F\in\mathcal{F}_h^{i,b}} \mu_F \|[\![\nabla_{\mathrm{T}}\, v_h]\!]\|^2_{L^2(F)} = \kappa^{-1} c_* \sum_{F\in\mathcal{F}_h^{i,b}} \mu_F \|[\![\nabla_{\mathrm{T}}\, v_h]\!]\|^2_{L^2(F)},\\
A_4 &\geq \tfrac{1}{2} A_4 = \kappa^{-1} c_* \sum_{F\in\mathcal{F}_h^{i,b}} \eta_F \|[\![v_h]\!]\|^2_{L^2(F)}.
\end{aligned}$$

So, we obtain the following inequality which completes the proof of (4.4):

$$\kappa B_{h,\theta}(v_h, v_h) \geq \sum_{K\in\mathcal{T}_h} \left[\theta \|D^2 v_h\|^2_{L^2(K)} + (1-\theta)\|\Delta v_h\|^2_{L^2(K)}\right] + c_* J_h(v_h, v_h). \qquad \square$$

Lemma 7 ensures that it is possible to choose c_{stab} and c_* such that (4.4) holds with $\kappa < (1-\varepsilon)^{-1/2}$, because $(1-\varepsilon)^{-1/2} > 1$.

THEOREM 8. *Under the hypotheses of Lemma* 7*, let* c_{stab} *and* c_*, μ_F*, and* η_F *be chosen so that* (4.4) *and* (4.5) *hold with* $\kappa < (1-\varepsilon)^{-1/2}$*. Let the operator* L *satisfy* (2.2), (2.3)*, and the Cordès condition* (2.5)*. Then, the bilinear form* A_h *is coercive on* $V_{h,\mathbf{p}}$ *with respect to the norm* $\|\cdot\|_{\mathrm{DG}(1)}$*. In particular, for any* $v_h \in V_{h,\mathbf{p}}$*, there holds*

$$\|v_h\|^2_{\mathrm{DG}(1)} \leq \frac{2\kappa}{1 - \kappa^2(1-\varepsilon)} A_h(v_h, v_h). \tag{4.8}$$

Therefore, there exists a unique solution $u_h \in V_{h,\mathbf{p}}$ *of the numerical scheme* (3.4)*. Moreover,* u_h *satisfies*

$$\|u_h\|_{\mathrm{DG}(1)} \leq \frac{2\kappa\sqrt{n}\|\gamma\|_{L^\infty(\Omega)}}{1 - \kappa^2(1-\varepsilon)} \|f\|_{L^2(\Omega)}. \tag{4.9}$$

Proof. Let $v_h \in V_{h,\mathbf{p}}$, and note that (2.7) implies that

$$\begin{aligned}
\langle \gamma L v_h, \Delta v_h\rangle_K - \langle \Delta v_h, \Delta v_h\rangle_K &= \langle (\gamma L - \Delta)\, v_h, \Delta v_h\rangle_K\\
&\leq \|(\gamma L - \Delta)\, v_h\|_{L^2(K)} \|\Delta v_h\|_{L^2(K)}\\
&\leq \sqrt{1-\varepsilon}\|D^2 v_h\|_{L^2(K)} \|\Delta v_h\|_{L^2(K)}.
\end{aligned}$$

We use the Cauchy–Schwarz inequality with a parameter, together with the fact that (4.4) holds with $\kappa < (1-\varepsilon)^{-1/2}$, to get

$$\begin{aligned} A_h(v_h, v_h) &\geq \kappa^{-1}\|v_h\|^2_{\mathrm{DG}(1/2)} - \sum_{K\in\mathcal{T}_h} \sqrt{1-\varepsilon}\|D^2 v_h\|_{L^2(K)}\|\Delta v_h\|_{L^2(K)} \\ &\geq \kappa^{-1}\|v_h\|^2_{\mathrm{DG}(1/2)} - \sum_{K\in\mathcal{T}_h}\left[\frac{\kappa(1-\varepsilon)}{2}\|D^2 v_h\|^2_{L^2(K)} + \frac{\kappa^{-1}}{2}\|\Delta v_h\|^2_{L^2(K)}\right] \\ &\geq \frac{\kappa^{-1}-\kappa(1-\varepsilon)}{2}\sum_{K\in\mathcal{T}_h}\|D^2 v_h\|^2_{L^2(K)} + \kappa^{-1}c_* J_h(v_h,v_h) \\ &\geq \frac{1-\kappa^2(1-\varepsilon)}{2\kappa}\|v_h\|^2_{\mathrm{DG}(1)}. \end{aligned}$$

The previous inequality implies (4.8), which in turn proves that there exists a unique solution $u_h \in V_{h,\mathbf{p}}$ of (3.4). Then, applying (4.8) to the numerical solution u_h shows that

$$\|u_h\|^2_{\mathrm{DG}(1)} \leq \frac{2\kappa}{1-\kappa^2(1-\varepsilon)} A_h(u_h,u_h) \leq \frac{2\kappa}{1-\kappa^2(1-\varepsilon)}\left|\sum_{K\in\mathcal{T}_h}\langle \gamma f, \Delta u_h\rangle_K\right|.$$

Since $\|\Delta u_h\|_{L^2(K)} \leq \sqrt{n}\|D^2 u_h\|_{L^2(K)}$, it is found that (4.9) follows from

$$\|u_h\|^2_{\mathrm{DG}(1)} \leq \frac{2\kappa\sqrt{n}\|\gamma\|_{L^\infty(\Omega)}}{1-\kappa^2(1-\varepsilon)}\|f\|_{L^2(\Omega)}\|u_h\|_{\mathrm{DG}(1)}. \qquad \square$$

5. Error analysis. In the following analysis, for $a, b \in \mathbb{R}$, we shall write $a \lesssim b$ to signify that there exists a constant C such that $a \leq C\, b$, with C independent of $\mathbf{h} = (h_K : K \in \mathcal{T}_h)$, $\mathbf{p}$, and u but otherwise possibly dependent on the shape-regularity constants of $\mathcal{T}_h$, $c_{\mathcal{F}}$, $c_{\mathcal{P}}$, $c_{\mathcal{T}}$, $\mathbf{s}$, etc.

THEOREM 9. *Let Ω be a bounded convex polytopal domain, and let the shape-regular sequence of simplicial or parallelepipedal meshes $\{\mathcal{T}_h\}_h$ satisfy* (2.9) *and* (2.10), *with $\mathbf{p}$ satisfying* (2.11) *for each h. Let the operator L satisfy* (2.2), (2.3), *and the Cordès condition* (2.5), *and let $u \in H^2(\Omega)\cap H^1_0(\Omega)$ be the unique solution of* (2.4). *Assume that $u \in H^{\mathbf{s}}(\Omega;\mathcal{T}_h)$, with $s_K > 5/2$ for each $K \in \mathcal{T}_h$. Let c_{stab}, c_*, and μ_F be chosen as in Theorem* 8, *and choose $\eta_F \lesssim \tilde{p}_F^4/\tilde{h}_F^3$ for all $F \in \mathcal{F}_h^{i,b}$. Then, there exists a constant $C > 0$, independent of h, $\mathbf{p}$, and u, but depending on $\max_K s_K$, such that*

$$\|u-u_h\|^2_{\mathrm{DG}(1)} \leq C\sum_{K\in\mathcal{T}_h}\frac{h_K^{2t_K-4}}{p_K^{2s_K-5}}\|u\|^2_{H^{s_K}(K)}, \tag{5.1}$$

where $t_K = \min(p_K+1, s_K)$ for each $K \in \mathcal{T}_h$.

Note that for the special case of quasi-uniform meshes and uniform polynomial degrees, the a priori error bound (5.1) simplifies to

$$|u-u_h|_{H^2(\Omega;\mathcal{T}_h)} \leq \|u-u_h\|_{\mathrm{DG}(1)} \leq C\frac{h^{\min(p+1,s)-2}}{p^{s-2.5}}\|u\|_{H^s(\Omega)}.$$

Thus it is seen that the rates are optimal with respect to the mesh size and suboptimal in the polynomial degree only by half an order. Note that the standard analysis of the symmetric interior penalty method for divergence form elliptic equations leads to a similar suboptimality and that optimal rates were recovered in [9] through considerations of regularity of the solution in augmented Sobolev spaces.

Proof. Since the sequence of meshes is shape-regular, it follows from the results of [3] that there are a $z_h \in V_{h,\mathbf{p}}$ and a constant C independent of u, h_K, and p_K, but dependent on $\max_K s_K$, such that, for all $0 \le q \le 2$,

$$\|u - z_h\|_{H^q(K)} \le C \frac{h_K^{t_K - q}}{p_K^{s_K - q}} \|u\|_{H^{s_K}(K)}, \tag{5.2}$$

$$\|D^\beta (u - z_h)\|_{L^2(\partial K)} \le C \frac{h_K^{t_K - q - 1/2}}{p_K^{s_K - q - 1/2}} \|u\|_{H^{s_k}(K)} \qquad \forall \beta\colon\ |\beta| = q. \tag{5.3}$$

Since u satisfies the hypotheses of Corollary 6, (3.14) holds. Now, set $\psi_h = z_h - u_h$ and $\xi_h = z_h - u$. Then, coercivity of A_h from (4.8) implies that

$$\begin{aligned} \text{(by (3.14))} \quad \|z_h - u_h\|_{\mathrm{DG}(1)}^2 &\lesssim A_h(z_h, \psi_h) - A_h(u_h, \psi_h) \\ &= A_h(z_h, \psi_h) - \sum_{K \in \mathcal{T}_h} \langle \gamma f, \Delta \psi_h \rangle_K = A_h(\xi_h, \psi_h). \end{aligned}$$

Therefore, $\|z_h - u_h\|_{\mathrm{DG}(1)}^2 \lesssim \sum_{i=1}^{8} E_i$, where

$$\begin{aligned} E_1 &:= \sum_{K \in \mathcal{T}_h} \left| \langle D^2 \xi_h, D^2 \psi_h \rangle_K \right|, & E_5 &:= \sum_{F \in \mathcal{F}_h^{i,b}} \left| \langle \nabla_{\mathrm{T}} \left\{ \nabla \xi_h \cdot n_F \right\}, [\![\nabla_{\mathrm{T}} \psi_h]\!] \rangle_F \right|, \\ E_2 &:= \sum_{K \in \mathcal{T}_h} \left| \langle (\gamma L - \Delta) \xi_h, \Delta \psi_h \rangle_K \right|, & E_6 &:= \sum_{F \in \mathcal{F}_h^{i}} \left| \langle \operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}} \left\{ \xi_h \right\}, [\![\nabla \psi_h \cdot n_F]\!] \rangle_F \right|, \\ E_3 &:= \sum_{K \in \mathcal{T}_h} \left| \langle \Delta \xi_h, \Delta \psi_h \rangle_K \right|, & E_7 &:= \sum_{F \in \mathcal{F}_h^{i}} \left| \langle \operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}} \left\{ \psi_h \right\}, [\![\nabla \xi_h \cdot n_F]\!] \rangle_F \right|, \\ E_4 &:= \left| J_h(\xi_h, \psi_h) \right|, & E_8 &:= \sum_{F \in \mathcal{F}_h^{i,b}} \left| \langle \nabla_{\mathrm{T}} \left\{ \nabla \psi_h \cdot n_F \right\}, [\![\nabla_{\mathrm{T}} \xi_h]\!] \rangle_F \right|. \end{aligned}$$

It is then deduced that

$$E_1 + E_2 + E_3 \lesssim \left(\sum_{K \in \mathcal{T}_h} \frac{h_K^{2t_K - 4}}{p_K^{2s_K - 4}} \|u\|_{H^{s_K}(K)}^2 \right)^{\frac{1}{2}} \|\psi_h\|_{\mathrm{DG}(1)}. \tag{5.4}$$

Now,

$$E_4 \le J_h(\xi_h, \xi_h)^{\frac{1}{2}} J_h(\psi_h, \psi_h)^{\frac{1}{2}} \le (e_1 + e_2 + e_3)^{\frac{1}{2}} \|\psi_h\|_{\mathrm{DG}(1)},$$

where

$$e_1 := \sum_{F \in \mathcal{F}_h^{i}} \mu_F \|[\![\nabla \xi_h \cdot n_F]\!]\|_{L^2(F)}^2, \qquad e_2 := \sum_{F \in \mathcal{F}_h^{i,b}} \mu_F \|[\![\nabla_{\mathrm{T}} \xi_h]\!]\|_{L^2(F)}^2,$$

$$e_3 := \sum_{F \in \mathcal{F}_h^{i,b}} \eta_F \|[\![\xi_h]\!]\|_{L^2(F)}^2.$$

Recalling (4.3) and (4.5), we use (5.3) to obtain

$$e_1 \lesssim \sum_{F\in\mathcal{F}_h^i} \frac{\tilde{p}_F^2}{\tilde{h}_F} \sum_{\substack{K\\ F\subset\partial K}} \|\nabla \xi_h\|_{L^2(\partial K)}^2 \lesssim \sum_{K\in\mathcal{T}_h} \frac{h_K^{2t_K-4}}{p_K^{2s_K-5}} \|u\|_{H^{s_K}(K)}^2.$$

Similarly, we use the hypothesis $\eta_F \lesssim \tilde{p}_F^4/\tilde{h}_F^3$ to find that

$$e_2 + e_3 \lesssim \sum_{K\in\mathcal{T}_h} \frac{h_K^{2t_K-4}}{p_K^{2s_K-5}} \|u\|_{H^{s_K}(K)}^2.$$

Therefore,

$$E_4 \lesssim \left(\sum_{K\in\mathcal{T}_h} \frac{h_K^{2t_K-4}}{p_K^{2s_K-5}} \|u\|_{H^2 s_K(K)}^2 \right)^{\frac{1}{2}} \|\psi_h\|_{\mathrm{DG}(1)}.$$

It is found that

$$\begin{aligned} E_5 + E_6 &\lesssim \left(\sum_{F\in\mathcal{F}_h^{i,b}} \frac{\tilde{h}_F}{\tilde{p}_F^2} \sum_{\substack{K\\ F\subset\partial K}} \|D^2 \xi_h\|_{L^2(\partial K)}^2 \right)^{\frac{1}{2}} \|\psi_h\|_{\mathrm{DG}(1)} \\ &\lesssim \left(\sum_{K\in\mathcal{T}_h} \frac{h_K^{2t_k-4}}{p_K^{2s_K-3}} \|u\|_{H^{s_K}(K)}^2 \right)^{\frac{1}{2}} \|\psi_h\|_{\mathrm{DG}(1)}. \end{aligned}$$

It follows from the inverse and trace inequalities that

$$E_7 + E_8 \lesssim (e_1+e_2)^{\frac{1}{2}} \|\psi_h\|_{\mathrm{DG}(1)} \lesssim \left(\sum_{K\in\mathcal{T}_h} \frac{h_K^{2t_k-4}}{p_K^{2s_k-5}} \|u\|_{H^{s_K}(K)}^2 \right)^{\frac{1}{2}} \|\psi_h\|_{\mathrm{DG}(1)}.$$

The above inequalities imply that

$$\|u-z_h\|_{\mathrm{DG}(1)} + \|z_h-u_h\|_{\mathrm{DG}(1)} \lesssim \left(\sum_{K\in\mathcal{T}_h} \frac{h_K^{2t_K-4}}{p_K^{2s_K-5}} \|u\|_{H^{s_K}(K)}^2 \right)^{\frac{1}{2}}.$$

The triangle inequality $\|u-u_h\|_{\mathrm{DG}(1)} \leq \|u-z_h\|_{\mathrm{DG}(1)} + \|z_h-u_h\|_{\mathrm{DG}(1)}$ and the above inequalities complete the proof of the error bound (5.1). ∎

5.1. A bound for problems with minimal regularity. Theorem 9 involved a regularity assumption on the solution u of (2.4). In comparison, Proposition 10 below provides an a priori estimate on the error that is valid for problems with minimal regularity, namely, $u \in H^2(\Omega)\cap H_0^1(\Omega)$. Furthermore, the result below shows that the scheme proposed here is at least as accurate in H^2-type norms as a method using H^2-conforming elements with the same polynomial degrees on the same mesh.

PROPOSITION 10. *Let Ω be a bounded convex polytopal domain, and let the shape-regular sequences of simplicial or parallelepipedal meshes $\{\mathcal{T}_h\}_h$ satisfy* (2.9) *and* (2.10)*, with $\mathbf{p}$ satisfying* (2.11) *for each h. Let the operator L satisfy* (2.2), (2.3), *and the Cordès condition* (2.5)*, and let $u \in H^2(\Omega)\cap H_0^1(\Omega)$ be the unique solution* *of* (2.4)*. Let* c_{stab}*,* c_**,* μ_F*, and* η_F *be chosen as in Theorem* 8*. Then, there exists a constant* C*, independent of* h *and* $\mathbf{p}$*, such that*

$$\|u - u_h\|_{\mathrm{DG}(1)} \le C \inf\{|u - z_h|_{H^2(\Omega;\mathcal{T}_h)} : z_h \in V_{h,\mathbf{p}} \cap H^2(\Omega) \cap H_0^1(\Omega)\}. \tag{5.5}$$

Proof. If $z_h \in V_{h,\mathbf{p}} \cap H^2(\Omega) \cap H_0^1(\Omega)$, then Lemma 5 applies to z_h, because z_h is a piecewise polynomial, and thus $z_h \in H^s(\Omega;\mathcal{T}_h)$ for $s > 5/2$. Setting $\psi_h = z_h - u_h \in V_{h,\mathbf{p}}$, coercivity of A_h gives

$$\begin{aligned}\|z_h - u_h\|^2_{\mathrm{DG}(1)} \lesssim A_h(z_h - u_h, \psi_h) &= \sum_{K\in\mathcal{T}_h} \langle \gamma L z_h, \Delta\psi_h\rangle_K - \sum_{K\in\mathcal{T}_h} \langle \gamma f, \Delta\psi_h\rangle_K \\ &= \sum_{K\in\mathcal{T}_h} \langle \gamma L(z_h - u), \Delta\psi_h\rangle_K \lesssim |u - z_h|_{H^2(\Omega;\mathcal{T}_h)} \|z_h - u_h\|_{\mathrm{DG}(1)}.\end{aligned}$$

Thus $\|z_h - u_h\|_{\mathrm{DG}(1)} \lesssim |u - z_h|_{H^2(\Omega;\mathcal{T}_h)}$. Since $u \in H^2(\Omega) \cap H_0^1(\Omega)$, it follows that $[\![\nabla u]\!] = 0$ and $[\![u]\!] = 0$ for all interior faces and that $\nabla_{\mathrm{T}}(\tau_F u) = 0$ and $\tau_F u = 0$ on all boundary faces. Therefore, $\|u - z_h\|_{\mathrm{DG}(1)} = |u - z_h|_{H^2(\Omega;\mathcal{T}_h)}$. So, the triangle inequality gives

$$\|u - u_h\|_{\mathrm{DG}(1)} \le \|u - z_h\|_{\mathrm{DG}(1)} + \|z_h - u_h\|_{\mathrm{DG}(1)} \lesssim |u - z_h|_{H^2(\Omega;\mathcal{T}_h)}.$$

Since z_h was arbitrary, taking the infimum over all $z_h \in V_{h,\mathbf{p}} \cap H^2(\Omega) \cap H_0^1(\Omega)$ completes the proof. ∎

6. Numerical experiments. In the first numerical experiment, we demonstrate the convergence rates predicted by Theorem 9, and in the second experiment, we test the scheme under hp-refinement on a problem with a singular solution.

6.1. First experiment. Consider the following problem:

$$\begin{aligned}\sum_{i,j=1}^{n} (1+\delta_{ij}) \frac{x_i}{|x_i|}\frac{x_j}{|x_j|} u_{x_i x_j} &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega.\end{aligned} \tag{6.1}$$

Here, $\Omega = (-1,1)^2$ and f is chosen so that the solution of (6.1) is

$$u(x,y) = \left(x\,\mathrm{e}^{1-|x|} - x\right)\left(y\,\mathrm{e}^{1-|y|} - y\right). \tag{6.2}$$

For the problem (6.1), observe that the Cordès condition (2.5) holds with $\varepsilon = 3/5$ and that the coefficients of the differential operator are discontinuous across the set $D = \{(x,y) \in \Omega\colon x = 0 \text{ or } y = 0\}$.

We apply the numerical scheme (3.4) to problem (6.1) with meshes obtained by regular subdivision of Ω into uniform quadrilateral meshes $\mathcal{T}_h$ with mesh sizes $h = 2^{-k}$, $2 \le k \le 6$. It follows that $u \in H^s(\Omega;\mathcal{T}_h)$ for all $s > 5/2$. The finite element spaces $V_{h,\mathbf{p}}$ are defined by employing the space of polynomials of fixed total degree p on each element. For the choice of penalty parameter, we set $c_{\mathrm{stab}} = 10$ and set $\eta_F = c_{\mathrm{stab}}\,\tilde{p}_F^4/\tilde{h}_F^3$. Figure 2 plots the errors measured in the broken H^2 norm for various choices of polynomial degrees p, $2 \le p \le 5$. The expected optimal rates $\|u - u_h\|_{H^2(\Omega;\mathcal{T}_h)} = \mathrm{O}\left(h^{p-1}\right)$ are observed, in accordance with Theorem 9.

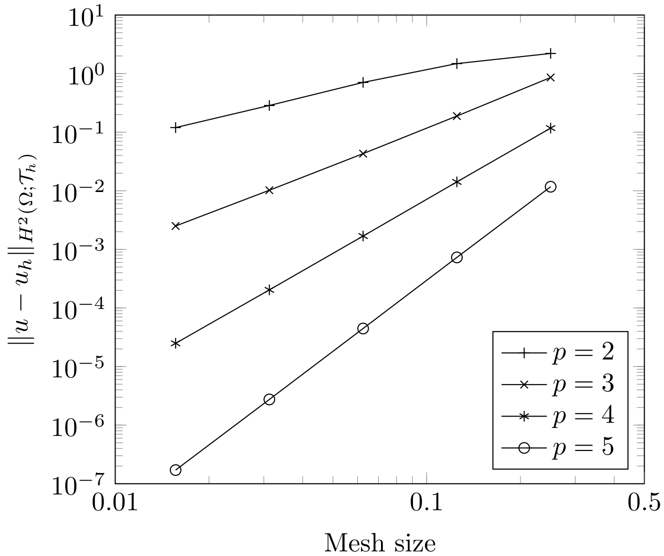

FIG. 2. *Convergence rates for the numerical scheme applied to problem* (6.1)*. The error* $\|u-u_h\|_{H^2(\Omega;\mathcal{T}_h)}$ *is plotted against mesh size* h *for various polynomial degrees* p*. The optimal convergence rates* $\|u-u_h\|_{H^2(\Omega;\mathcal{T}_h)} = \mathsf{O}\left(h^{p-1}\right)$ *are observed.*

6.2. Second experiment. In this example, we demonstrate the robustness of the scheme by illustrating exponential accuracy for a problem that involves both nonsmoothness of the solution and discontinuity of the coefficients at a corner of the domain. We also show how to apply the numerical scheme to problems with nonhomogeneous boundary conditions.

It can be verified that for $\alpha > 1$, $u = |x|^{\alpha}$, $x \in \Omega = (0,1)^2$, solves

$$\sum_{i,j=1}^{n}\left(\delta_{ij} + \frac{x_i\, x_j}{|x|^2}\right) u_{x_i x_j} = c_\alpha\, |x|^{\alpha-2} =: f \qquad \text{in } \Omega, \tag{6.3}$$

where c_α is a suitable constant depending only on α. Notice that the term $x_i\, x_j / |x|^2$ fails to be continuous at the origin when $i \neq j$. This example draws upon the examples in [10, 14] that illustrate the possibility of ill-posedness of the problem when the Cordès condition fails. However, the operator in (6.3) satisfies the Cordès condition (2.5) with $\varepsilon = 4/5$. In the following, we take $\alpha = 1.6$, so $u \in H^{2.6-\delta}(\Omega)$ for arbitrarily small δ.

In order to extend the numerical scheme (3.4) to problems with nonhomogeneous boundary conditions, the right-hand side must be suitably modified as follows. Let g be the restriction of u on $\partial\Omega$. Then the numerical scheme for problem (6.3) is to find $u_h \in V_{h,\mathbf{p}}$ such that for every $v_h \in V_{h,\mathbf{p}}$, there holds

$$\begin{aligned} A_h(u_h, v_h) = &\sum_{K\in\mathcal{T}_h} \langle \gamma\, f, \Delta v_h\rangle_K + \sum_{F\in\mathcal{F}_h^b}\left[\mu_F \langle \nabla_{\mathrm{T}}\, g, \nabla_{\mathrm{T}}\, v_h\rangle_F + \eta_F \langle g, v_h\rangle_F\right] \\ &- \frac{1}{2}\sum_{F\in\mathcal{F}_h^b}\left[\langle \operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}}\, g, \nabla v_h \cdot n_F\rangle_F + \langle \nabla_{\mathrm{T}}\left(\nabla v_h\cdot n_F\right), \nabla_{\mathrm{T}}\, g\rangle_F\right]. \end{aligned} \tag{6.4}$$

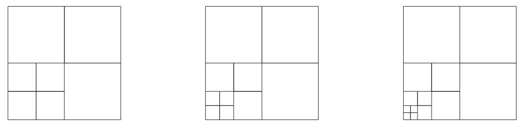

Fig. 3. *Sequence of geometrically graded meshes used for the solution of* (6.3)*. The polynomial degrees are chosen to be linearly increasing away from the origin, which is located at the bottom left corner of each diagram. So, for example, in the coarsest mesh pictured here, the degree on the element closest to the origin is two, it is three on the neighbors of the latter element, and it is four on the remaining elements. The sequence of meshes is continued by uniform refinement of the element closest to the origin.*

Following [17], we construct a sequence of finite element spaces on geometrically refined meshes by increasing the elemental polynomial degrees linearly away from the origin; see Figure 3 for further details.

Table 1 reports the errors obtained by applying the scheme on nine successively refined meshes. Figure 4 plots the errors in the broken H^1 norm and H^2 seminorm against $\sqrt[3]{N}$, where N is the number of degrees of freedom, and shows that a convergence rate of at least $\mathsf{O}(\exp(-c\sqrt[3]{N}))$ is achieved [20].

Table 1

Errors of the approximations to the solution of problem (6.3) *on geometrically graded meshes. Exponential convergence is observed, with faster convergence rates in lower order norms.*

Elements	DoF	$\Vert u-u_h\Vert_{L^2(\Omega)}$	$\Vert u-u_h\Vert_{H^1(\Omega;\mathcal{T}_h)}$	$\vert u-u_h\vert_{H^2(\Omega;\mathcal{T}_h)}$
4	36	2.349e-03	2.829e-02	4.799e-01
7	81	4.346e-04	9.439e-03	3.176e-01
10	144	8.166e-05	3.132e-03	2.096e-01
13	228	1.491e-05	1.036e-03	1.383e-01
16	336	2.743e-06	3.426e-04	9.124e-02
19	471	4.954e-07	1.131e-04	6.020e-02
22	636	9.840e-08	3.737e-05	3.972e-02
25	834	1.949e-08	1.233e-05	2.620e-02
28	1068	4.799e-09	4.072e-06	1.729e-02

7. Conclusion. We have introduced a new hp-DGFEM for nondivergence form elliptic equations with discontinuous coefficients that satisfy the Cordès condition. Convergence rates were shown to be optimal with respect to the mesh size h and suboptimal with respect to the polynomial degree p by only half an order. The robustness and accuracy of the scheme was further evidenced by the numerical experiments. As a result, this method permits the effective numerical solution of a broad class of nondivergence form elliptic equations.

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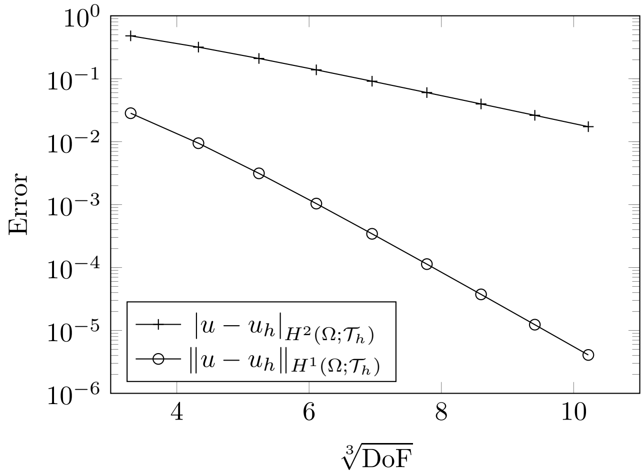

FIG. 4. *Exponential accuracy of the numerical scheme for* (6.3) *on geometrically graded meshes. The errors in the broken H^1-norm and H^2-seminorm are plotted against the cube root of the number of degrees of freedom.*

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