

On the uniqueness of minimisers of Ginzburg-Landau functionals

Sur l'unicité des minimiseurs de la fonctionnelle de Ginzburg-Landau

Radu Ignat*, Luc Nguyen[†], Valeriy Slastikov[‡] and Arghir Zarnescu^{§¶||}

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Abstract

We provide necessary and sufficient conditions for the uniqueness of minimisers of the Ginzburg-Landau functional for $\mathbb{R}^n$ -valued maps under a suitable convexity assumption on the potential and for $H^{1/2} \cap L^\infty$ boundary data that is non-negative in a fixed direction $e \in \mathbb{S}^{n-1}$. Furthermore, we show that, when minimisers are not unique, the set of minimisers is generated from any of its elements using appropriate orthogonal transformations of $\mathbb{R}^n$. We also prove corresponding results for harmonic maps.

Résumé

Nous montrons des conditions nécessaires et suffisantes pour l'unicité des minimiseurs de la fonctionnelle de Ginzburg-Landau sous une hypothèse de convexité du potentiel et une donnée au bord dans $H^{1/2} \cap L^\infty$ qui est positive dans une direction fixée. De plus, nous prouvons que si le minimiseur n'est pas unique, alors l'ensemble des minimiseurs est généré par une certaine classe de transformations orthogonales appliquées à un minimiseur quelconque. Nous montrons aussi des résultats similaires pour les applications harmoniques à valeurs dans la sphère unité.

*Institut de Mathématiques de Toulouse & Institut Universitaire de France, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France. Email: Radu.Ignat@math.univ-toulouse.fr

[†]Mathematical Institute and St Edmund Hall, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, United Kingdom. Email: luc.nguyen@maths.ox.ac.uk

[‡]School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, United Kingdom. Email: Valeriy.Slastikov@bristol.ac.uk

[§]IKERBASQUE, Basque Foundation for Science, Maria Diaz de Haro 3, 48013, Bilbao, Bizkaia, Spain.

[¶]BCAM, Basque Center for Applied Mathematics, Mazarredo 14, E48009 Bilbao, Bizkaia, Spain. (azarnescu@bcamath.org)

^{||}“Simion Stoilow” Institute of the Romanian Academy, 21 Calea Griviței, 010702 Bucharest, Romania.

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Contents

1	Introduction and main results	2
2	Uniqueness of critical points of E_ε. Proof of Theorem 1.3.	9
3	Uniqueness of minimisers of E_ε. Proof of Theorems 1.4 and 1.7	12
4	Radially symmetric problem. Proof of Theorem 1.1	18
5	Harmonic map problem	22
A	Appendix	24

1 Introduction and main results

We consider the following Ginzburg-Landau type energy functional

$$E_\varepsilon(u) = \int_{\Omega} \left[\frac{1}{2} |\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2) \right] dx, \quad (1.1)$$

where $\varepsilon > 0$, $\Omega \subset \mathbb{R}^m$ ($m \geq 1$) is a bounded domain (i.e., open connected set) with smooth boundary $\partial\Omega$ and the potential $W \in C^1((-\infty, 1], \mathbb{R})$ satisfies

$$W(0) = 0, W(t) > 0 \text{ for all } t \in (-\infty, 1] \setminus \{0\}, W \text{ is strictly convex.} \quad (1.2)$$

We investigate the minimisers of the energy E_ε over the set

$$\mathcal{A} := \{u \in H^1(\Omega; \mathbb{R}^n) : u = u_{bd} \text{ on } \partial\Omega\}, \quad n \geq 1, \quad (1.3)$$

consisting of H^1 maps with a given boundary data (in the sense of $H^{1/2}$ -trace on $\partial\Omega$):

$$u_{bd} \in H^{1/2} \cap L^\infty(\partial\Omega; \mathbb{R}^n).$$

In particular, we are interested in the question of uniqueness (or its failure) for the minimisers of E_ε in $\mathcal{A}$ for all range of $\varepsilon > 0$.

In the case $\varepsilon > 0$ is large enough and (1.2) holds, the energy E_ε is strictly convex (see Remark 3.3); as E_ε is lower semicontinuous in the weak H^1 -topology and coercive over $\mathcal{A}$, there is a unique critical point u_ε of E_ε over $\mathcal{A}$ and this critical point is the global minimiser of the energy E_ε . The case when $\varepsilon > 0$ is “not too large” is much more complex and, in general, one has to impose additional assumptions on the potential W or/and the boundary

data to understand the uniqueness of minimisers and its failure even in the limit $\varepsilon \rightarrow 0$ (i.e. the related case of minimising harmonic maps).

There exists a large literature using various methods that addresses the question of uniqueness of minimisers in the framework of Ginzburg-Landau type models and the related harmonic map problem. See e.g. Béthuel, Brezis and Hélein [2], Mironescu [21], Pacard and Rivière [22], Ye and Zhou [27], Farina and Mironescu [7], Millot and Pisante [20], Pisante [23], Gustafson [9], Jäger and Kaul [16, 17], Sandier and Shafrir [24, 25] and the references therein.

The approach of this work is based on the method that we previously successfully used in the investigation of stability and minimality properties of the critical points in the context of Landau-de Gennes model of nematic liquid crystals [5, 12, 13, 14]. We refer to this tool as the Hardy decomposition technique (see Lemma A.1. in [12]). We show that under the global assumption (1.2) on the potential W (namely its convexity), and an additional assumption on the boundary data (namely its non-negativity in a fixed direction), we can use Hardy decomposition method to provide simple yet quite general proofs of uniqueness of minimisers of energy E_ε and to characterise its failure. In our forthcoming paper [15] we will provide sufficient conditions for uniqueness of minimisers under less restrictive local conditions on the potential, but imposing more restrictive conditions on the boundary data.

We begin the exposition with providing a simple result on the uniqueness and symmetry of minimisers, which will be subsequently extended in a much more general setting. Assume $\Omega \subset \mathbb{R}^2$ is the unit disk, $u : \Omega \rightarrow \mathbb{R}^3$, and the boundary data carries a given winding number $k \in \mathbb{Z} \setminus \{0\}$ on $\partial\Omega$, namely¹

$$u_{bd}(\cos \varphi, \sin \varphi) = (\cos(k\varphi), \sin(k\varphi), 0) \in \mathbb{S}^1 \times \{0\} \subset \mathbb{R}^3, \quad \forall \varphi \in [0, 2\pi). \quad (1.4)$$

By restricting E_ε to a subset of $\mathcal{A}$ (with $n = 3$) consisting of suitable rotationally symmetric maps (see condition (4.1) below), it is not hard to see that E_ε admits critical points of the form

$$u_\varepsilon(r \cos \varphi, r \sin \varphi) := f_\varepsilon(r)(\cos(k\varphi), \sin(k\varphi), 0) \pm g_\varepsilon(r)(0, 0, 1), \quad r \in (0, 1), \quad \varphi \in [0, 2\pi), \quad (1.5)$$

where the couple $(f_\varepsilon, g_\varepsilon)$ of radial profiles solves the system

$$\begin{cases} -f_\varepsilon'' - \frac{1}{r}f_\varepsilon' + \frac{k^2}{r^2}f_\varepsilon &= \frac{1}{\varepsilon^2}f_\varepsilon W'(1 - f_\varepsilon^2 - g_\varepsilon^2) \\ -g_\varepsilon'' - \frac{1}{r}g_\varepsilon' &= \frac{1}{\varepsilon^2}g_\varepsilon W'(1 - f_\varepsilon^2 - g_\varepsilon^2) \end{cases} \quad \text{in } (0, 1), \quad (1.6)$$

subject to the boundary conditions

$$f_\varepsilon(0) = 0, f_\varepsilon(1) = 1, g_\varepsilon'(0) = 0, g_\varepsilon(1) = 0. \quad (1.7)$$

Our first result is the following:

THEOREM 1.1. *Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disk in $\mathbb{R}^2$, $u : \Omega \rightarrow \mathbb{R}^3$, $W \in C^1((-\infty, 1], \mathbb{R})$ satisfy (1.2), and u_{bd} be the boundary data given by (1.4) where $k \in \mathbb{Z} \setminus \{0\}$ is a given integer. Then the following conclusions hold.*

¹We note that the map u_{bd} in (1.4) is non-negative in e_3 -direction.

1. For every $\varepsilon > 0$, any minimiser u_ε of the energy E_ε in the set $\mathcal{A}$ (with $n = 3$) has the representation (1.5) where $f_\varepsilon > 0$, $g_\varepsilon \geq 0$ in $(0, 1)$ and $(f_\varepsilon, g_\varepsilon)$ solves (1.6)-(1.7).
2. There exists $\varepsilon_k > 0$ such that, for every $\varepsilon \in (0, \varepsilon_k)$, (1.6)-(1.7) has a unique solution $(f_\varepsilon, g_\varepsilon)$ with $g_\varepsilon > 0$, and for $\varepsilon \in [\varepsilon_k, \infty)$, no such solution exists.
3. For every $\varepsilon > 0$, there is exactly one solution $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)$ of (1.6)-(1.7) satisfying $\tilde{f}_\varepsilon > 0$, $\tilde{g}_\varepsilon \equiv 0$.
4. For every $\varepsilon \in (0, \varepsilon_k)$, E_ε has exactly two minimisers that are given by $(f_\varepsilon, \pm g_\varepsilon)$ via (1.5) with $g_\varepsilon > 0$, while the critical point of E_ε corresponding to $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon \equiv 0)$ via (1.5) is unstable. For every $\varepsilon \in [\varepsilon_k, \infty)$, E_ε has a unique minimiser which corresponds to $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon \equiv 0)$.

REMARK 1.2. We will prove in Proposition 4.2 that the critical values ε_k in Theorem 1.1 satisfy $\varepsilon_k = \varepsilon_{-k}$ for every $k \geq 1$ and the sequence $(\varepsilon_k)_{k \geq 1}$ is increasing. Moreover, for every $k \in \mathbb{Z} \setminus \{0\}$, the two minimisers of E_ε over $\mathcal{A}$ given by the the solution $(f_\varepsilon, \pm g_\varepsilon)$ of the system (1.6)-(1.7) with $g_\varepsilon > 0$ converge strongly in $H^2(\Omega)$ as $\varepsilon \rightarrow 0$ to the two minimising harmonic maps (see e.g. [1]):

$$(r, \varphi) \in (0, 1) \times [0, 2\pi) \mapsto \left(\frac{2r^k}{1+r^{2k}} \cos(k\varphi), \frac{2r^k}{1+r^{2k}} \sin(k\varphi), \pm \frac{1-r^{2k}}{1+r^{2k}} \right) \in \mathbb{S}^2.$$

The situation presented above indicates that uniqueness of minimisers can be lost in a discrete way (two alternative minimisers). This is due to an additional degree of freedom in the target space and becomes evident in a more general setting, when we work with $\mathbb{R}^n$ -valued maps: if an isometry of $\mathbb{R}^n$ does not change the boundary data, then it transforms a minimiser into a minimiser. This is an important feature of our results that will be presented in the general setting for arbitrary dimensions $m \geq 1$ and $n \geq 1$ of base and target spaces, respectively. The main restriction in our treatment is the assumption that the boundary data is non-negative in a (fixed) direction $e \in \mathbb{S}^{n-1}$, i.e.,

$$u_{bd} \cdot e \geq 0 \quad \mathcal{H}^{m-1}\text{-a.e. in } \partial\Omega. \quad (1.8)$$

Before formulating our main results we need to provide some basic definitions. In the rest of the paper we adopt the following notation: for any integrable $\mathbb{R}^n$ -valued map u on a measurable set ω we denote the essential image of u on ω by

$$u(\omega) = \{u(x) : x \in \omega \text{ is a Lebesgue point of } u\}. \quad (1.9)$$

It is clear that for a continuous function u the above definition coincides with the (standard) range of u on ω . We also define $\text{Span } u(\omega)$ to be the smallest vector subspace of $\mathbb{R}^n$ containing $u(\omega)$.

The following result establishes the minimising property and uniqueness (up to certain isometries) for *critical points* of E_ε that are *positive* in a fixed direction $e \in \mathbb{S}^{n-1}$ inside the domain $\Omega \subset \mathbb{R}^m$.

THEOREM 1.3. *Let $m, n \geq 1$, $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary, $W \in C^1((-\infty, 1], \mathbb{R})$ be a potential satisfying (1.2), and $u_{bd} \in H^{1/2} \cap L^\infty(\partial\Omega; \mathbb{R}^n)$ be a boundary data satisfying (1.8) in a fixed direction $e \in \mathbb{S}^{n-1}$. Fix any $\varepsilon > 0$ and let $u_\varepsilon \in H^1 \cap L^\infty(\Omega, \mathbb{R}^n)$ be a critical point of the energy E_ε in the set $\mathcal{A}$, which is positive in the direction e inside Ω :*

$$u_\varepsilon \cdot e > 0 \text{ a.e. in } \Omega. \quad (1.10)$$

Then u_ε is a minimiser of E_ε in $\mathcal{A}$ and we have the following dichotomy:

1. *If there exists a Lebesgue point $x_0 \in \partial\Omega$ of u_{bd} such that $u_{bd}(x_0) \cdot e > 0$ then u_ε is the unique minimiser of E_ε in the set $\mathcal{A}$.*
2. *If $u_{bd}(x) \cdot e = 0$ for $\mathcal{H}^{m-1}$ -a.e. $x \in \partial\Omega$, then all minimisers of E_ε in $\mathcal{A}$ are given by Ru_ε where $R \in O(n)$ is an orthogonal transformation of $\mathbb{R}^n$ satisfying $Rx = x$ for all $x \in \text{Span } u_{bd}(\partial\Omega)$ (within the definition (1.9)).*

Using the above theorem, we prove the following result which completely characterises uniqueness and its failure for minimisers of the energy E_ε . Note that the positivity assumption (1.10) in a direction e will not be enforced anymore on minimisers inside Ω ; however the non-negativity (1.8) of the boundary data u_{bd} is fundamental.

THEOREM 1.4. *Let m, n, Ω, W, u_{bd} be as in Theorem 1.3, $e \in \mathbb{S}^{n-1}$ be such that (1.8) holds and $V \equiv \text{Span } u_{bd}(\partial\Omega)$. Then for every $\varepsilon > 0$ there exists a minimiser u_ε of the energy E_ε on the set $\mathcal{A}$ and this minimiser is unique unless both following conditions hold:*

1. *$u_{bd}(x) \cdot e = 0$ for $\mathcal{H}^{m-1}$ -a.e. $x \in \partial\Omega$,*
2. *the functional E_ε restricted to the set*

$$\mathcal{A}_{res} := \{u \in \mathcal{A} : u(x) \in \text{Span}(V \cup \{e\}) \text{ a.e. in } \Omega\}$$

has a minimiser $\tilde{u}_\varepsilon$ with $\tilde{u}_\varepsilon(\Omega) \not\subset V$.²

Moreover, if uniqueness of minimisers of E_ε in $\mathcal{A}$ does not hold, then all minimisers of E_ε in $\mathcal{A}$ are given by Ru_ε where $R \in O(n)$ is an orthogonal transformation of $\mathbb{R}^n$ satisfying $Rx = x$ for all $x \in V$.

Loosely speaking, the above result asserts that if u_ε is a minimiser of E_ε in $\mathcal{A}$ then

- either $u_\varepsilon(\Omega) \subset \text{Span } u_{bd}(\partial\Omega)$ and u_ε is the unique minimiser of E_ε in $\mathcal{A}$ (a phenomenon that we call “non-escaping” combined with “uniqueness” of minimisers, see the discussion below);

²We will see that any minimiser $\tilde{u}_\varepsilon$ belongs to $C^1(\Omega)$, therefore, $\tilde{u}_\varepsilon(\Omega)$ has the standard meaning. Moreover, if $\tilde{u}_\varepsilon(\Omega) \not\subset V$, then the minimiser $\tilde{u}_\varepsilon$ of E_ε over $\mathcal{A}_{res}$ is unique up to the reflection $\tilde{u}_\varepsilon \cdot e \mapsto -\tilde{u}_\varepsilon \cdot e$ (that keeps u_{bd} unchanged due to point 1.).

- or $u_\varepsilon(\Omega) \not\subset \text{Span } u_{bd}(\partial\Omega)$, and all minimisers of E_ε in $\mathcal{A}$ differ from u_ε by an orthogonal transformation which fixes every point of $\text{Span } u_{bd}(\partial\Omega)$ (that we call “escaping” phenomenon combined with “multiplicity” of minimisers, see below).

Harmonic map problem. We note that Theorem 1.4 is similar to a well-known result of Sandier and Shafrir [24] on the uniqueness of minimising harmonic maps into a closed hemisphere. In fact, our proof of Theorem 1.4 can be adapted to give an alternative proof of their result. Our argument does not assume the smoothness of boundary data and does not use the regularity theory of minimising harmonic maps, which appears to play a role in the argument of [24].

THEOREM 1.5. *Let $m \geq 1$, $n \geq 2$, $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary, $u_{bd} \in H^{1/2}(\partial\Omega; \mathbb{S}^{n-1})$ be a boundary data satisfying (1.8) in a direction $e \in \mathbb{S}^{n-1}$, and $V \equiv \text{Span } u_{bd}(\partial\Omega)$. Then there exists a minimising harmonic map $u \in \mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$ and this minimising harmonic map is unique unless both following conditions hold:*

1. $u_{bd}(x) \cdot e = 0$ for $\mathcal{H}^{m-1}$ -a.e. $x \in \partial\Omega$,
2. the Dirichlet energy $E(w) = \frac{1}{2} \int_\Omega |\nabla w|^2 dx$ restricted to the set

$$\mathcal{A}_{res}^* := \{w \in \mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1}) : w(x) \in \text{Span}(V \cup \{e\}) \text{ a.e. in } \Omega\}$$

has a minimiser $\tilde{u}$ with $\tilde{u}(\Omega) \not\subset V$.

Moreover, if u is not the unique minimising harmonic map in $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$, then all minimising harmonic maps in $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$ are given by Ru where $R \in O(n)$ is an orthogonal transformation of $\mathbb{R}^n$ satisfying $Rx = x$ for all $x \in V$.

Similar to the argument for Theorem 1.4, the proof of Theorem 1.5 uses an analogue of Theorem 1.3 for weakly harmonic maps positive in a given direction e inside Ω ; this result is given in Theorem 5.1.

Relation between uniqueness, stability, and “escaping” phenomenon. We would like to point out some implications of Theorems 1.4 and 1.5 when the boundary data u_{bd} vanishes in a direction e at the boundary, i.e., $\text{codim}_{\mathbb{R}^n} \text{Span } u_{bd}(\partial\Omega) \geq 1$. In this case we have the following observations:

i) The uniqueness of minimisers in Theorems 1.4 and 1.5 is equivalent to the confinement of the range $u(\Omega)$ of minimisers u of E_ε (resp. of E) inside the vector space $V = \text{Span } u_{bd}(\partial\Omega)$, which we will refer to as the “non-escaping” phenomenon.

Uniqueness of minimiser $\iff$ “Non-escaping” phenomenon for minimiser.

The opposite situation when the range $u(\Omega)$ is not confined inside the vector space V (i.e., $u_\varepsilon(\Omega) \not\subset \text{Span } u_{bd}(\partial\Omega)$) - that we call “escaping” phenomenon - often occurs in physical models (e.g. micromagnetic vortices, liquid crystal defects): in order to lower the energy,

the minimisers prefer to escape along a direction e transversal to the natural vector space V where boundary data u_{bd} lives. This “escape” is usually enforced by topological constraints in the boundary data u_{bd} .

We note that the “escaping” phenomenon dictates the loss of uniqueness of a minimiser in the following way: the minimiser $\tilde{u}_\varepsilon$ of E_ε over $\mathcal{A}_{res}$ (resp. $\tilde{u}$ of E over $\mathcal{A}_{res}^*$) is unique up to the reflection about the vector space orthogonal to e (see Remark 3.1). All the other minimisers (if any) of E_ε over $\mathcal{A}$ (resp. of E over $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$) are obtained by orthogonal transformations of $\tilde{u}_\varepsilon$ (resp. $\tilde{u}$) fixing all points of V .

ii) The “escaping” phenomenon is closely related to stability properties of critical points. In particular, in Theorem 1.3 (resp. Theorem 5.1) we show that every “escaping” critical point u (i.e., (1.10) holds inside Ω) of E_ε over $\mathcal{A}$ (resp. of E over $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$) is in fact a minimiser and there are multiple minimisers as one can reflect u about the orthogonal space to the escaping direction. On the contrary, in Propositions 3.2 and 5.2 we show that if a critical point of E_ε over $\mathcal{A}$ (resp. of E over $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$) does not “escape” then its stability is equivalent with its minimality and by point i) it is the unique minimiser.

Therefore, we have the following dichotomy in the behaviour of minimisers:

- either “escaping” of minimisers:

{“escaping critical points”} = {minimisers} with at least two elements;

- or “non-escaping” of minimisers:

{“Non-escaping” stable critical point} = {minimiser} with only one element.

As showed at Theorem 1.1, point 3., one can have at the same time “escaping” minimisers and “non-escaping” *unstable* critical points.

iii) In the case of the harmonic map problem, if V has dimension $m - 1$ and the boundary data u_{bd} is smooth and carries a *nontrivial topological degree* as a map $u_{bd} : \partial\Omega \rightarrow V \cap \mathbb{S}^{n-1}$ with $n > m \geq 2$, then ³ the following holds for minimising harmonic maps in $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$:

“Escaping” phenomenon for minimisers $\iff$ Smoothness for minimisers.

We illustrate the above points on a natural and useful example.

EXAMPLE 1.6. Let $\Omega = B^m$ be the unit ball in $\mathbb{R}^m$, $m \geq 2$, $n \geq m + 1$ and boundary data $u_{bd}(x) = (\frac{x}{|x|}, 0_{\mathbb{R}^{n-m}})$ for every $x \in \partial\Omega = \mathbb{S}^{m-1}$. It is known (see Jäger and Kaul [18]) that the equator map

$$u(x) = \left(\frac{x}{|x|}, 0_{\mathbb{R}^{n-m}}\right) \quad \text{for every } x \in B^m$$

³The fact that “escaping phenomenon” implies smoothness is a consequence of the boundary regularity result of Schoen and Uhlenbeck [26] and the regularity result of Jäger and Kaul [16] for maps with values into $\mathbb{S}^{n-1} \cap \{u_n > \delta\}$ for some $\delta > 0$. The other implication is obvious arguing by contraposition: “non-escaping” smooth maps $u : \Omega \rightarrow V \cap \mathbb{S}^{n-1} \simeq \mathbb{S}^{m-1}$ have zero Jacobian determinant as the partial derivatives $\partial_1 u, \dots, \partial_m u$ are linearly dependent in the tangent space $T_u \mathbb{S}^{m-1}$ and therefore, u carries a zero topological degree at the boundary $\partial\Omega$.

is a minimising $\mathbb{S}^{n-1}$ -valued harmonic map with the boundary data u_{bd} **if and only if** $m \geq 7$.⁴

Let us now explain how our results recover the above statement. We start by illustrating **ii)** of the above discussion: the stability of the equator map (with respect to $\mathbb{S}^{n-1}$ -valued maps) is equivalent to the validity of the following inequality (see (5.7))

$$\int_{B^m} |\nabla \varphi|^2 dx \geq (m-1) \int_{B^m} \frac{|\varphi|^2}{|x|^2} dx \text{ for all } \varphi \in H_0^1(B^m)$$

(because $|\nabla u|^2 = (m-1)/|x|^2$). In view of the sharp Hardy inequality in m dimensions, this is true if and only if

$$\frac{(m-2)^2}{4} \geq m-1 \iff m \geq 7.$$

Invoking Proposition 5.2, we can recover the aforementioned result of Jüger and Kaul [18], i.e., u is a “non-escaping” stable critical point and hence the unique minimising harmonic map in $\mathcal{A}$ if and only if $m \geq 7$. Therefore, the escaping phenomenon occurs in dimensions $2 \leq m \leq 6$ when $\mathbb{S}^{n-1}$ -valued minimising harmonic maps are smooth (see **iii)**), while the “non-escaping” phenomenon occurs in dimensions $m \geq 7$ when the equator map is the unique minimising harmonic map (see **i)**).

Inspired by Example 1.6, we will prove that the “non-escaping” phenomenon also occurs in the Ginzburg-Landau problem for every $\varepsilon > 0$ in dimensions $n > m \geq 7$ when the equator map is the boundary data

$$u_{bd}(x) = (x, 0_{\mathbb{R}^{n-m}}), \quad x \in \partial B^m.$$

In that situation, the unique minimiser is the “non-escaping” radial critical point of E_ε given by

$$u_\varepsilon(x) = (h_\varepsilon(|x|) \frac{x}{|x|}, 0_{\mathbb{R}^{n-m}}), \quad \text{for all } x \in B^m, \quad (1.11)$$

where the radial profile h_ε is the increasing positive solution of

$$\begin{cases} -h_\varepsilon'' - \frac{m-1}{r} h_\varepsilon' + \frac{m-1}{r^2} h_\varepsilon = \frac{h_\varepsilon}{\varepsilon^2} W'(1 - h_\varepsilon^2) & \text{for every } r \in (0, 1), \\ h_\varepsilon(0) = 0, h_\varepsilon(1) = 1 \end{cases} \quad (1.12)$$

(see e.g. [11]).

THEOREM 1.7. *Let $W \in C^1((-\infty, 1], \mathbb{R})$ be a potential satisfying (1.2). If $n > m \geq 7$, then for every $\varepsilon > 0$, u_ε given in (1.11) is the unique minimiser of E_ε under the boundary data $u_{bd} = (x, 0_{\mathbb{R}^{n-m}})$ for $x \in \partial B^m$.*

⁴However, it is a minimising map among all $\mathbb{S}^{m-1}$ -valued maps for $m \geq 3$ (see Brezis, Coron and Lieb [3] and Lin [19]), where we identify $\mathbb{S}^{m-1} = \mathbb{S}^{m-1} \times \{0_{\mathbb{R}^{n-m}}\}$.

Since the minimiser u_ε has the range included in $\mathbb{R}^m \times \{0_{\mathbb{R}^{n-m}}\}$, Theorem 1.7 also proves the uniqueness of the minimiser of E_ε over the set of maps $u : B^m \rightarrow \mathbb{R}^m$ with the degree-one vortex boundary data $u_{bd}(x) = x$ for $x \in \partial B^m$ for every $m \geq 7$. We will extend this result for general convex potentials W (not necessarily strictly convex as in the current setting) in a forthcoming paper.

REMARK 1.8. *i) Within the hypotheses of Theorem 1.7, a consequence of Theorems 1.3 and 1.7 is the non-existence of “escaping” critical points of E_ε for every $\varepsilon > 0$ if $n > m \geq 7$.*

ii) When $2 \leq m \leq 6$ and $n > m$, the non-existence of “escaping” critical points of E_ε still holds provided that ε is large (in particular, if E_ε is strictly convex, then u_ε in (1.11) is the only critical point of E_ε with the boundary data u_{bd}). However, for small $\varepsilon > 0$, the minimisers of E_ε with the boundary data u_{bd} will “escape” from $\mathbb{R}^m \times \{0_{\mathbb{R}^{n-m}}\}$ (while the “non-escaping” critical point u_ε in (1.11) becomes unstable). This is because any minimiser of E_ε converges in $H^1(B^m)$ as $\varepsilon \rightarrow 0$ to a minimising harmonic map u_ (see e.g. [1]) and by Example 1.6 every minimising harmonic map u_* escapes from the subspace $\mathbb{R}^m \times \{0_{\mathbb{R}^{n-m}}\} = \text{Span } u_{bd}(\partial\Omega)$.*

The paper is organised as follows. In Sections 2 and 3, we prove Theorem 1.3 and Theorems 1.4 and 1.7, respectively. In section 4 we investigate the radially symmetric Ginzburg-Landau problem in its simplest form ($B^2 \rightarrow \mathbb{R}^3$) and prove the uniqueness and symmetry of minimisers formulated in Theorem 1.1. In Section 5 we investigate the harmonic map problem and prove Theorem 5.1 and Theorem 1.5 – the harmonic maps analogues of Theorem 1.3 and Theorem 1.4. Finally, in the Appendix A we discuss the “escape” phenomenon for minimisers of E and E_ε when the boundary data u_{bd} carries a zero topological degree on $\partial\Omega$.

2 Uniqueness of critical points of E_ε . Proof of Theorem 1.3.

Let us fix $\varepsilon > 0$ and $u_{bd} \in H^{1/2} \cap L^\infty(\partial\Omega; \mathbb{R}^n)$. It is clear that $\mathcal{A} \cap L^\infty(\Omega; \mathbb{R}^n) \neq \emptyset$ as it contains the harmonic extension of u_{bd} inside Ω . Assume that $u_\varepsilon \in H^1 \cap L^\infty(\Omega; \mathbb{R}^n)$ is a critical point of the energy E_ε in the set $\mathcal{A}$; then u_ε satisfies the Euler-Lagrange equations of E_ε in the weak sense, i.e.

$$\int_{\Omega} \left[\nabla u_\varepsilon \cdot \nabla v - \frac{1}{\varepsilon^2} W'(1 - |u_\varepsilon|^2) u_\varepsilon v \right] dx = 0 \quad \text{for all } v \in H_0^1(\Omega; \mathbb{R}^n), \quad (2.1)$$

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon W'(1 - |u_\varepsilon|^2) \quad \text{distributionally in } \Omega.$$

Note that (2.1) makes sense since $W'(1 - |u_\varepsilon|^2) \in L^\infty(\Omega)$ (because $u_\varepsilon \in L^\infty(\Omega)$ and $W \in C^1((-\infty, 1])$). Moreover, standard interior elliptic regularity implies $u_\varepsilon \in C^1(\Omega)$.

Proof of Theorem 1.3. For simplicity of the presentation we drop dependence on ε in the indices below. Also due to the invariance of the energy under the transformation $u(x) \mapsto$

$Ru(x)$ for any $R \in O(n)$, we can assume that

$$e := e_n = (0, \dots, 0, 1) \in \mathbb{R}^n, \quad (2.2)$$

and work with a boundary data satisfying (1.8) in e_n direction.

The proof will be done in three steps. In the first step we consider the difference between the energies of a critical point u and an arbitrary competitor $u + v$ and show that this difference is controlled from below by some quadratic energy functional $F(v)$. In the second step, we employ the positivity of $u \cdot e_n$ in (1.10) and apply the Hardy decomposition method in order to show that $F(v) \geq 0$, which proves in particular that u is a minimiser of E . Finally, in the third step, we characterise the situations when this difference is zero and thus completely classify the possible cases of non-uniqueness of minimisers.

Step 1: Lower bound for energy difference. For any $v \in H_0^1(\Omega; \mathbb{R}^n)$ we have

$$\begin{aligned} E(u + v) - E(u) &= \int_{\Omega} \left[\nabla u \cdot \nabla v + \frac{1}{2} |\nabla v|^2 \right] dx \\ &\quad + \frac{1}{2\varepsilon^2} \int_{\Omega} \left[W(1 - |u + v|^2) - W(1 - |u|^2) \right] dx. \end{aligned}$$

Using the strict convexity of W (see (1.2)), we have

$$W(1 - |u + v|^2) - W(1 - |u|^2) \geq -W'(1 - |u|^2)(|u + v|^2 - |u|^2),$$

where equality holds if and only if $|u + v| = |u|$. The last two relations imply that

$$\begin{aligned} E(u + v) - E(u) &\geq \int_{\Omega} \left[\nabla u \cdot \nabla v - \frac{1}{\varepsilon^2} W'(1 - |u|^2) u \cdot v \right] dx \\ &\quad + \int_{\Omega} \left[\frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - |u|^2) |v|^2 \right] dx, \end{aligned} \quad (2.3)$$

with equality holding if and only if $|u(x) + v(x)| = |u(x)|$ a.e. $x \in \Omega$. Moreover, by (2.1), we obtain

$$E(u + v) - E(u) \geq \int_{\Omega} \left[\frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - |u|^2) |v|^2 \right] dx =: F(v) \quad (2.4)$$

for all $v \in H_0^1(\Omega; \mathbb{R}^n)$.

Step 2: Minimality of u and the Hardy decomposition method. Now we want to show the non-negativity of the quadratic energy $F(v)$ defined in (2.4). Recall that $\Phi := u \cdot e_n$ belongs to $C^1(\Omega)$ and $\Phi > 0$ in Ω (by (1.10)) and satisfies weakly the Euler-Lagrange equation in (2.1):

$$-\Delta \Phi - \frac{1}{\varepsilon^2} W'(1 - |u|^2) \Phi = 0 \quad \text{in } \Omega. \quad (2.5)$$

Now, for any smooth map $v \in C_c^\infty(\Omega; \mathbb{R}^n)$ with compact support in Ω , we can define $\psi = \frac{v}{\Phi} \in H_0^1(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ and we claim (by Lemma A.1. in [12]):

$$F(v) = \frac{1}{2} \int_{\Omega} \Phi^2 |\nabla \psi|^2 dx = \frac{1}{2} \int_{\Omega} \sum_{j=1}^n \left| \nabla v_j - v_j \frac{\nabla \Phi}{\Phi} \right|^2 dx \geq 0 \quad \text{for all } v \in C_c^\infty(\Omega; \mathbb{R}^n). \quad (2.6)$$

Indeed, for $1 \leq j \leq n$, integration by parts implies

$$\begin{aligned} \int_{\Omega} \frac{1}{\varepsilon^2} W'(1 - |u|^2) v_j^2 dx &= \int_{\Omega} \left[\frac{1}{\varepsilon^2} W'(1 - |u|^2) \Phi \right] \cdot \left[\Phi \psi_j^2 \right] dx \\ &\stackrel{(2.5)}{=} \int_{\Omega} \nabla \Phi \cdot \nabla \underbrace{(\Phi \psi_j^2)}_{=v_j \psi_j} dx = \int_{\Omega} \left[-\Phi^2 |\nabla \psi_j|^2 + |\nabla v_j|^2 \right] dx \end{aligned}$$

yielding the desired expression (2.6). By density of $C_c^\infty(\Omega; \mathbb{R}^n)$ in $H_0^1(\Omega; \mathbb{R}^n)$ and the continuity of $F(\cdot)$ in strong H^1 topology (as $W'(1 - |u|^2) \in L^\infty(\Omega)$), we have by (2.4) and (2.6) that for all $v \in H_0^1(\Omega; \mathbb{R}^n)$

$$F(v) \geq \frac{1}{2} \int_{\omega} \sum_{j=1}^n \left| \nabla v_j - v_j \frac{\nabla \Phi}{\Phi} \right|^2 dx \geq 0 \quad \text{for any compact } \omega \subset \Omega$$

(recall that $(\nabla \Phi)/\Phi \in L^\infty(\omega)$ for every compact set $\omega \subset \Omega$). As a direct consequence of (2.4), we deduce that u is a minimiser of E over $\mathcal{A}$. In order to determine when $F(v) = 0$, we note that for $v \in H_0^1(\Omega)$ and any open set $\omega \subset \Omega$ compactly supported in Ω :

$$0 = F(v) \geq \frac{1}{2} \int_{\omega} \Phi^2 \left| \nabla \left(\frac{v}{\Phi} \right) \right|^2 dx \geq 0.$$

Thus we have $v(x) = \lambda_{\omega} \Phi(x)$, for a.e. $x \in \omega$ and some constant vector $\lambda_{\omega} \in \mathbb{R}^n$. Since ω is arbitrary we have that there exists a $\lambda \in \mathbb{R}^n$ such that $v(x) = \lambda \Phi(x)$, for a.e. $x \in \Omega$.

Step 3: Characterisation of non-uniqueness of minimisers. We now take another minimiser w of E over $\mathcal{A}$ and denote $v := w - u \in H_0^1$. From Steps 1 and 2 we know that $E(w) = E(v + u) = E(u)$ implies

- there is a pointwise equality $|v + u| = |u|$ a.e. in Ω ,
- $v = \lambda \Phi$ for some $\lambda \in \mathbb{R}^n$ and the scalar function $\Phi = u \cdot e_n$.

Situation 1. $u_{bd}(x_0) \cdot e_n > 0$ at some Lebesgue point $x_0 \in \partial\Omega$ of u_{bd} . We want to show uniqueness of minimisers, i.e., $\lambda = 0$ meaning that $v = 0$ in Ω . Assume by contradiction that $\lambda_j \neq 0$ for some $1 \leq j \leq n$. Then $\Phi = \frac{v_j}{\lambda_j} \in H_0^1(\Omega)$. Since the n -th component $u_{bd,n}$ of u_{bd} is the $H^{1/2}$ -trace of Φ , we have $u_{bd,n}(x) = \frac{v_j(x)}{\lambda_j} = 0$ for $\mathcal{H}^{m-1}$ -a.e. $x \in \partial\Omega$. As x_0 is a Lebesgue point of u_{bd} we would obtain $u_{bd,n}(x_0) = 0$ which contradicts the assumption of Situation 1. Therefore, $\lambda = 0$ and u is the only minimiser of E in $\mathcal{A}$.

Situation 2. $u_{bd}(x_0) \cdot e_n = 0$ for $\mathcal{H}^{m-1}$ -a.e. $x \in \partial\Omega$. As the energy E is invariant under rotation, we may assume that $V = \mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}$ (for some $0 \leq k \leq n-1$) is the smallest vector space containing the set $u_{bd}(\partial\Omega)$ defined by (1.9). (In the case $n = 1$, then $V = \{0\}$). Note that the map

$$\tilde{u} := (u_1, \dots, u_k, 0, \dots, 0, \sqrt{u_{k+1}^2 + \dots + u_n^2}) \quad (2.7)$$

belongs to $\mathcal{A}$ and does not increase the energy, i.e., $E(u) \geq E(\tilde{u})$. Hence, as u is a minimiser of E on $\mathcal{A}$, $\tilde{u}$ is also a minimiser. Therefore, up to interchanging u and $\tilde{u}$, we may assume

$$\begin{cases} u_{k+1} = \dots = u_{n-1} \equiv 0 \text{ in } \Omega \\ u_n = \Phi \stackrel{(1.10)}{>} 0 \text{ in } \Omega. \end{cases}$$

The relation $|v + u| = |u|$ a.e. in Ω implies $2u \cdot v + |v|^2 = 0$ a.e. in Ω . Since $v = \lambda\Phi$ and $\Phi > 0$ in Ω we obtain

$$2\lambda \cdot u + \Phi|\lambda|^2 = 0 \text{ a.e. in } \Omega. \quad (2.8)$$

In the case $n = 1$, then $\lambda = \lambda_n$, $u = u_n = \Phi$ and $(\lambda + 1)^2 = 1$ yielding $w = (\lambda + 1)u = \pm u$. If $n \geq 2$, we argue as follows: the $H^1(\Omega)$ -functions $2\lambda \cdot u$ and $-\Phi|\lambda|^2$ coincide, therefore they have the same $H^{1/2}$ -trace on $\partial\Omega$. As our assumption states that Φ has vanishing trace on $\partial\Omega$, we deduce that $\lambda \cdot u_{bd}(x) = 0$ for $\mathcal{H}^{m-1}$ -a.e. $x \in \partial\Omega$, in particular, for every Lebesgue point x of u_{bd} on $\partial\Omega$. Since $V = \mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}$ is the smallest vector space containing $u_{bd}(\partial\Omega)$, it means that there are k Lebesgue points $x^1, \dots, x^k \in \partial\Omega$ such that $\{u_{bd}(x^j)\}_j$ is a basis of V and $\lambda \cdot u_{bd}(x^j) = 0$ for every $1 \leq j \leq k$. It follows that $\lambda_1 = \lambda_2 = \dots = \lambda_k = 0$ and therefore, recalling that $u_{k+1} = \dots = u_{n-1} \equiv 0$ in Ω , we have by (2.8):

$$2\lambda_n\Phi + (\lambda_{k+1}^2 + \dots + \lambda_n^2)\Phi = 0 \text{ a.e. in } \Omega.$$

As $\Phi > 0$ in Ω , we obtain

$$\lambda_{k+1}^2 + \dots + \lambda_{n-1}^2 + (\lambda_n + 1)^2 = 1,$$

hence we can find $R \in O(n)$ such that $Rx = x$ for all $x \in V$ and

$$Re_n = (0, \dots, 0, \lambda_{k+1}, \dots, \lambda_{n-1}, \lambda_n + 1).$$

This implies $w = u + v = u + \lambda\Phi = Ru$ as required. The converse statement is obvious: if u is a minimiser of E over $\mathcal{A}$ and $R \in O(n)$ is a transformation fixing all points of V , then Ru is also a minimiser of E over $\mathcal{A}$ (because E is invariant under isometries). $\square$

3 Uniqueness of minimisers of E_ε . Proof of Theorems 1.4 and 1.7

In this section we prove the uniqueness result of Theorem 1.4.

Proof of Theorem 1.4. For simplicity of the presentation we drop dependence on ε in the indices below. Also, by rotation invariance of the energy, we may assume that $e = e_n$ (as in (2.2)). We divide the proof in several steps:

Step 1: Existence of a minimiser of E over $\mathcal{A}$. We apply the direct method in the calculus of variation: if $(u_k)_k$ is a minimising sequence for E in $\mathcal{A}$, then $(\nabla u_k)_k$ is bounded in $L^2(\Omega)$. As $u_k = u_{bd}$ on $\partial\Omega$, by the Poincaré-Wirtinger inequality, we deduce that $(u_k)_k$ is bounded in $H^1(\Omega)$ and we conclude by using lower semi-continuity of E with respect to the weak H^1 -topology (as W is a nonnegative continuous function).

Step 2: Any minimiser u of E over $\mathcal{A}$ belongs to $L^\infty(\Omega)$, namely

$$|u| \leq \max\{1, \sup_{\partial\Omega} |u_{bd}|\} =: \bar{\rho} \quad \text{a.e. in } \Omega. \quad (3.1)$$

Indeed, we start by defining the following functions in Ω : $\rho = |u| \in H^1(\Omega; \mathbb{R}_+)$,

$$v = \begin{cases} u & \text{if } \bar{\rho} \geq \rho, \\ \frac{\bar{\rho}}{\rho} u & \text{if } \bar{\rho} < \rho \end{cases}$$

and $\eta := |v| = \min(\rho, \bar{\rho})$. Then obviously $|\nabla v| = |\nabla u|$ in the set $\{\bar{\rho} \geq \rho\}$. In the complementary set $\tilde{\Omega} := \{\bar{\rho} < \rho\}$, we claim that $|\nabla v| \leq |\nabla u|$ $\mathcal{H}^{m-1}$ -a.e.; for that, we set $\psi := \frac{u}{\rho} \in H^1(\tilde{\Omega})$ (because $\rho > \bar{\rho} \geq 1$ inside $\tilde{\Omega}$). Starting from $u = \rho\psi$ in $\tilde{\Omega}$, we deduce that $\nabla u_j = \psi_j \nabla \rho + \rho \nabla \psi_j$ for every $1 \leq j \leq n$. Since $\sum_{j=1}^n \psi_j^2 = 1$ in $\tilde{\Omega}$, we have

$$\begin{aligned} |\nabla u|^2 &= \sum_{j=1}^n |\nabla u_j|^2 = \sum_{j=1}^n |\nabla \rho \psi_j + \rho \nabla \psi_j|^2 \\ &= \sum_{j=1}^n |\nabla \rho|^2 |\psi_j|^2 + \rho^2 |\nabla \psi_j|^2 + \frac{1}{2} \nabla(\rho^2) \cdot \nabla(\psi_j^2) \\ &= |\nabla \rho|^2 + \rho^2 |\nabla \psi|^2 \geq \bar{\rho}^2 |\nabla \psi|^2 = |\nabla v|^2 \quad \text{in } \tilde{\Omega}. \end{aligned}$$

Now, note that $W'(t) < 0$ for $t < 0$ because W is strictly convex and has a minimum at $t = 0$. Therefore, one has that $W(1 - \eta^2) \leq W(1 - \rho^2)$ with equality if and only if $\eta = \rho$. In particular, $E(u) \geq E(v)$. As u is a minimum of E over $\mathcal{A}$ and $v \in \mathcal{A}$, it follows that the above inequalities become equalities, i.e., $\eta = \rho$ a.e. in Ω which yields (3.1).

Step 3: We show that any minimiser u of E over $\mathcal{A}$ satisfies the following dichotomy

$$\text{either } |u_n| > 0 \text{ or } u_n \equiv 0 \text{ in } \Omega, \quad (3.2)$$

where $u_n := u \cdot e_n$. Indeed, as $u \in L^\infty$ by Step 2, interior elliptic regularity yields $u \in C^1(\Omega)$. Now, note that

$$\hat{u} = (u_1, \dots, u_{n-1}, |u_n|)$$

is also a minimiser of E over $\mathcal{A}$ (here (1.8) is essential). In particular $|u_n| \in C^1(\Omega)$ and satisfies

$$-\Delta|u_n| - \frac{1}{\varepsilon^2}W'(1 - |u|^2)|u_n| = 0 \text{ in } \Omega.$$

As $|u_n| \geq 0$, we thus have by the strong maximum principle that either $|u_n| > 0$ or $|u_n| \equiv 0$ in Ω .

Step 4: Characterisation of (non-)uniqueness of minimisers. Let u be a minimiser of E over $\mathcal{A}$. By Step 3, up to a sign change ⁵, we may assume that $u_n \geq 0$ in Ω .

Situation 1. $u_{bd}(x_0) \cdot e_n > 0$ at some Lebesgue point $x_0 \in \partial\Omega$ of u_{bd} . This assumption implies that u_n cannot be identically 0 in Ω , therefore by Step 3, we must have $u_n > 0$ in Ω (i.e., (1.10) holds). Then by point 1. in Theorem 1.3, u is the unique minimiser of E .

Situation 2. $u_{bd} \cdot e_n = 0$ in $\partial\Omega$. As the functional E is invariant under rotation, we may assume that $V = \mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}$ (for some $0 \leq k \leq n-1$) is the smallest vector subspace of $\mathbb{R}^{n-1} \times \{0\}$ which contains $u_{bd}(\partial\Omega)$. (If $n = 1$, then $V = \{0\}$). Using the transformation (2.7) as in Step 3 of the proof of Theorem 1.3, we can assume without loss of generality that the given minimiser u of E satisfies $u_{k+1} \equiv \dots \equiv u_{n-1} \equiv 0$ and $u_n \geq 0$ in Ω . In particular, we have $u \in \mathcal{A}_{res}$, and so u is also a minimiser of E in $\mathcal{A}_{res}$. We distinguish two cases:

Case A: $u(\Omega) \not\subset V$ (i.e., the point 2. of Theorem 1.4 holds) implies non-uniqueness. Note that $u_n > 0$ in Ω (by (3.2)). It is clear that $(u_1, \dots, u_k, u_{k+1} = 0, \dots, u_{n-1} = 0, -u_n)$ is also a minimiser of E in $\mathcal{A}$ (and $\mathcal{A}_{res}$), i.e. minimisers of E in $\mathcal{A}$ (resp. $\mathcal{A}_{res}$) are not unique. Furthermore, by Theorem 1.3, all minimisers of E in $\mathcal{A}$ are of the form Ru for some $R \in O(n)$ such that $Rx = x$ for all $x \in V$.

Case B: $Span(u(\Omega)) = V$ (i.e., the point 2. of Theorem 1.4 does not hold) implies uniqueness. Note that every other minimiser v of E over $\mathcal{A}_{res}$ also satisfies $Span(v(\Omega)) = V$ (because otherwise, $Span(v(\Omega)) \not\subset V$ for some minimiser v and then Case A would imply that $u = Rv$ for some isometry R fixing the space V which contradicts $Span(u(\Omega)) = V$). In other words, every minimiser u of E over $\mathcal{A}_{res}$ has the form

$$u = (u_1, \dots, u_k, 0, \dots, 0) \quad \text{in } \Omega. \quad (3.3)$$

Then we want to show that u is the unique minimiser of E in $\mathcal{A}$. In the case $V = \{0\}$ (in particular, for $n = 1$), then clearly 0 is the unique minimiser of E . Therefore, in the sequel, we may assume that $V \neq \{0\}$, in particular, $n \geq 2$ and $k \geq 1$. For that, note first that as u is a minimiser of E in $\mathcal{A}_{res}$, we have by our assumption that $u_n \equiv 0$ in Ω , so we cannot directly apply Theorem 1.3. We will argue by contradiction: assume that $w = u + v$ is a different minimiser of E in $\mathcal{A}$ where $0 \neq v \in H_0^1(\Omega, \mathbb{R}^n)$. One can easily check that in this case, $u \neq w$ implies that the map $\tilde{w}$ obtained out of w as in (2.7) satisfies $\tilde{w} \neq u$ and $\tilde{w}$ is a minimiser of E over $\mathcal{A}_{res}$. Therefore, by our assumption, without loss of generality, we assume that the components of index $k+1, \dots, n$ of w (and hence of v) are zero. We

⁵Due to (1.8), the situation $u_n < 0$ inside Ω can occur only if $u_{bd} \cdot e_n = 0$ on $\partial\Omega$; in which case, the sign change $u_n \mapsto -u_n$ is compatible with the boundary data.

will construct a minimiser ξ of E on $\mathcal{A}_{res}$ that has a positive component along e_n which contradicts (3.3).

Construction of the minimiser ξ . Let u, w and $v = w - u$ be as above. Recall that by (2.4) in Step 1 of the proof of the Theorem 1.3, we have:

$$0 = E(u + v) - E(u) \geq \int_{\Omega} \left[\frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - |u|^2) |v|^2 \right] dx =: F(v), \quad (3.4)$$

with equality if and only if $|u + v| = |u|$ a.e. in Ω . Note that we cannot use Hardy decomposition trick anymore to show the non-negativity of $F(v)$ as any component $u_1, \dots, u_k$ of u could change sign. However, since u is a minimiser of E in $\mathcal{A}_{res}$, we have for any scalar test function $\varphi \in H_0^1(\Omega)$ that ⁶

$$0 \leq \frac{d^2}{dt^2} \Big|_{t=0} E(u + t\varphi e_n) = \int_{\Omega} [|\nabla \varphi|^2 - \frac{1}{\varepsilon^2} W'(1 - |u|^2) \varphi^2] dx. \quad (3.5)$$

Taking $\varphi = v_j$ for $1 \leq j \leq n$ and summing over all j we obtain

$$0 \stackrel{(3.4)}{\geq} \int_{\Omega} \left[\frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - |u|^2) |v|^2 \right] dx \stackrel{(3.5)}{\geq} 0.$$

So equality holds in (3.4) yielding

$$|u| = |u + v| = |w| \text{ in } \Omega.$$

We are now ready to construct a competitor. We define

$$\xi = u + \frac{1}{2}v + \frac{1}{2}|v|e_n \in \mathcal{A}_{res}$$

and note that $\xi_n = \frac{1}{2}|v| > 0$, i.e., $\xi_n \not\equiv 0$ (because $u \cdot e_n = v \cdot e_n = 0$) yielding

$$|\xi|^2 = |u + \frac{1}{2}v|^2 + \frac{1}{4}|v|^2 = \frac{1}{2}(|u|^2 + |u + v|^2) = |u|^2 = |w|^2 \text{ in } \Omega,$$

$$|\nabla \xi|^2 = |\nabla(u + \frac{1}{2}v)|^2 + \frac{1}{4}|\nabla|v||^2 \leq |\nabla(u + \frac{1}{2}v)|^2 + \frac{1}{4}|\nabla v|^2 = \frac{1}{2}(|\nabla u|^2 + |\nabla w|^2).$$

It follows that

$$E(\xi) \leq \frac{1}{2}(E(u) + E(w)) = E(w) = E(u),$$

and so ξ is also a minimiser of E in $\mathcal{A}_{res}$. But since $\xi \cdot e_n \not\equiv 0$, this is a contradiction with our preliminary observation that the range on Ω of all minimisers of E over $\mathcal{A}_{res}$ stays inside V . This completes the proof. $\square$

⁶Although W is only C^1 , one can easily check that $E(u + t\varphi e_n)$ is twice differentiable with respect to t at $t = 0$ because $u \cdot e_n = 0$ in Ω .

REMARK 3.1. In the context of Theorem 1.4, if points 1 and 2 hold then the minimiser $\tilde{u}_\varepsilon$ of E_ε over $\mathcal{A}_{res}$ is unique up to the reflection $\mathcal{I}$ mapping $\tilde{u}_\varepsilon \cdot e \mapsto -\tilde{u}_\varepsilon \cdot e$ w.r.t. V inside $\text{Span}(V \cup \{e\})$. Indeed, up to a rotation, we may assume that $e = e_n$ (as in (2.2)) and $V = \mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}$ for some $0 \leq k \leq n-1$ and by (2.7) and (3.2) we may assume that $\tilde{u}_\varepsilon \cdot e_n > 0$ in Ω ; then point 2. in Theorem 1.3 yields the conclusion.

Using Theorem 1.4 we can prove the following result, which is of independent interest: “non-escaping” stable critical points of E_ε are minimisers and moreover, they are unique.

PROPOSITION 3.2. Let $m, n \geq 1$, $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary. We consider a potential $W \in C^1((-\infty, 1], \mathbb{R})$ satisfying (1.2) and a boundary data $u_{bd} \in H^{1/2} \cap L^\infty(\partial\Omega; e^\perp)$ for a direction $e \in \mathbb{S}^{n-1}$, where

$$e^\perp := \{v \in \mathbb{R}^n : v \cdot e = 0\}.$$

For any fixed $\varepsilon > 0$, if u_ε is a critical point of E_ε in $\mathcal{A}$, $u_\varepsilon \in L^\infty(\Omega; e^\perp)$ and is stable in direction e , i.e.

$$\frac{d^2}{dt^2} \Big|_{t=0} E_\varepsilon(u_\varepsilon + t\varphi e) = \int_\Omega \left[|\nabla \varphi|^2 - \frac{1}{\varepsilon^2} W'(1 - |u_\varepsilon|^2) \varphi^2 \right] dx \geq 0 \text{ for all } \varphi \in H_0^1(\Omega),$$

then u_ε is a minimiser of E_ε in $\mathcal{A}$. Moreover, if $u_\varepsilon(\Omega) \subset \text{Span } u_{bd}(\partial\Omega)$, then u_ε is the unique minimiser of E_ε in $\mathcal{A}$.

Proof of Proposition 3.2. Indeed, for any $v \in H_0^1(\Omega, \mathbb{R}^n)$ we have $E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq F(v)$ thanks to (2.4). Employing the stability assumption in the direction e and taking $\varphi = v_j$ for $1 \leq j \leq n$ we obtain $F(v) \geq 0$ for every $v \in H_0^1(\Omega, \mathbb{R}^n)$, yielding that u_ε is a minimiser of E_ε over $\mathcal{A}$. (In the case $n = 1$, then $e^\perp = \{0\}$ so that $u_{bd} = 0$ on $\partial\Omega$ as well as $u_\varepsilon = 0$ in Ω). If $u_\varepsilon(\Omega) \subset \text{Span } u_{bd}(\partial\Omega)$, then the uniqueness follows via Theorem 1.4 (as the “non-escaping” case holds, i.e., point 2 fails). $\square$

REMARK 3.3. For a given Lipschitz bounded domain $\Omega \subset \mathbb{R}^m$, under the assumption (1.2) for the potential $W \in C^1((-\infty, 1])$, there exists $\varepsilon_0 > 0$ (depending only on $W'(1)$ and Ω) such that E_ε is strictly convex over $\mathcal{A}$ for every $\varepsilon \geq \varepsilon_0$ and every boundary data $u_{bd} \in H^{1/2} \cap L^\infty(\partial\Omega, \mathbb{R}^n)$; in this situation ($\varepsilon \geq \varepsilon_0$), there exists a unique critical point⁷ of E_ε over $\mathcal{A}$. Indeed, following the ideas in [2] (see Theorem VIII.7), if $\lambda = \lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta)$ on Ω (with zero Dirichlet data), then the functional

$$G : v \mapsto \int_\Omega |\nabla v|^2 - \lambda |v|^2 dx$$

is convex⁸ on $H_0^1(\Omega)$ which entails that G is convex over $\mathcal{A}$ (as we can write $\mathcal{A} = u_0 + H_0^1(\Omega, \mathbb{R}^n)$ for the harmonic extension $u_0 \in \mathcal{A}$ of u_{bd} inside Ω). Moreover, by (1.2) we

⁷This critical point is the global minimiser of E_ε over $\mathcal{A}$ as a minimiser always exists (see Step 1 in the proof of Theorem 1.4).

⁸For $v, w \in H_0^1(\Omega)$ and $\alpha \in (0, 1)$, we have $\alpha G(v) + (1-\alpha)G(w) - G(\alpha v + (1-\alpha)w) = \alpha(1-\alpha)G(v-w) \geq 0$ by the Poincaré inequality.

deduce that $t \in [0, \infty) \mapsto \frac{1}{\varepsilon^2}W(1-t) + \lambda t$ is strictly convex and non-decreasing for $\varepsilon \geq \varepsilon_0 = (|W'(1)|/\lambda)^{1/2}$, so that the functional

$$u \in \mathcal{A} \mapsto \int_{\Omega} \frac{1}{\varepsilon^2}W(1-|u|^2) + \lambda|u|^2 dx$$

is strictly convex.

As a consequence of Proposition 3.2, we prove Theorem 1.7:

Proof of Theorem 1.7. By Proposition 3.2, it is enough to prove the stability of u_ε given in (1.11), i.e.,

$$F_\varepsilon(v) = \frac{1}{2} \int_{B^m} |\nabla v|^2 - \frac{1}{\varepsilon^2}W'(1-h_\varepsilon^2)v^2 dx \geq 0, \quad \forall v \in H_0^1(B^m).$$

As a point has zero H^1 capacity in $\mathbb{R}^m$, by a standard density argument, it is enough to prove the above inequality for $v \in C_c^\infty(B^m \setminus \{0\})$. We consider the operator

$$L_\varepsilon := \nabla_{L^2} F_\varepsilon = -\Delta - \frac{1}{\varepsilon^2}W'(1-h_\varepsilon^2).$$

Using the decomposition $v = h_\varepsilon w$ for $v \in C_c^\infty(B^m \setminus \{0\})$, we have (see e.g. [12, Lemma A.1]):

$$\begin{aligned} 2F_\varepsilon(v) &= \int_{B^m} L_\varepsilon v \cdot v dx = \int_{B^m} w^2 L_\varepsilon h_\varepsilon \cdot h_\varepsilon dx + \int_{B^m} h_\varepsilon^2 |\nabla w|^2 dx \\ &= \int_{B^m} h_\varepsilon^2 \left(|\nabla w|^2 - \frac{m-1}{r^2} w^2 \right) dx =: G_\varepsilon(w), \end{aligned}$$

because (1.12) yields $L_\varepsilon h_\varepsilon \cdot h_\varepsilon = -\frac{m-1}{r^2} h_\varepsilon^2$ in B^m . To prove that $G_\varepsilon(w) \geq 0$ for every $w \in C_c^\infty(B^m \setminus \{0\})$, we use the decomposition $w = \varphi g$ with $\varphi = |x|^{-\frac{m-2}{2}}$ being the first eigenfunction of the Hardy's operator $-\Delta - \frac{(m-2)^2}{4|x|^2}$ in $\mathbb{R}^m \setminus \{0\}$ and $g \in C_c^\infty(B^m \setminus \{0\})$. We compute

$$|\nabla w|^2 = |\nabla \varphi|^2 g^2 + |\nabla g|^2 \varphi^2 + \frac{1}{2} \nabla(\varphi^2) \cdot \nabla(g^2).$$

As φ^2 is harmonic in $B^m \setminus \{0\}$ and $|\nabla \varphi|^2 = \frac{(m-2)^2}{4|x|^2} \varphi^2$, integration by parts yields

$$\begin{aligned} G_\varepsilon(w) &= \int_{B^m} h_\varepsilon^2 \left(|\nabla g|^2 \varphi^2 + \frac{(m-2)^2}{4r^2} \varphi^2 g^2 - \frac{m-1}{r^2} \varphi^2 g^2 \right) dx - \frac{1}{2} \int_{B^m} \nabla(\varphi^2) \cdot \nabla(h_\varepsilon^2) g^2 dx \\ &\geq \int_{B^m} h_\varepsilon^2 |\nabla g|^2 \varphi^2 + \left(\frac{(m-2)^2}{4} - (m-1) \right) \int_{B^m} \frac{h_\varepsilon^2}{r^2} \varphi^2 g^2 dx \geq 0, \end{aligned}$$

where we have used $m \geq 7$ and $\frac{1}{2} \nabla(\varphi^2) \cdot \nabla(h_\varepsilon^2) = 2\varphi\varphi'h_\varepsilon h'_\varepsilon \leq 0$ in $B^m \setminus \{0\}$. $\square$

4 Radially symmetric problem. Proof of Theorem 1.1

Theorem 1.1 is an immediate consequence of the following two results.

PROPOSITION 4.1. *Under the hypotheses of Theorem 1.1, we have for every $\varepsilon > 0$:*

1. *Any minimiser u_ε of the energy E_ε in the set $\mathcal{A}$ (with $n = 3$) has the representation (1.5) where $f_\varepsilon > 0$, $g_\varepsilon \geq 0$ in $(0, 1)$ and $(f_\varepsilon, g_\varepsilon)$ solves (1.6)-(1.7).*
2. *There is at most one solution $(f_\varepsilon, g_\varepsilon)$ of (1.6)-(1.7) satisfying $g_\varepsilon > 0$ and there is exactly one solution $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)$ of (1.6)-(1.7) satisfying $\tilde{f}_\varepsilon > 0$, $\tilde{g}_\varepsilon \equiv 0$.*
3. *If the solution $(f_\varepsilon, g_\varepsilon > 0)$ of (1.6)-(1.7) exists, then E_ε has exactly two minimisers that are given by $(f_\varepsilon, \pm g_\varepsilon)$ via (1.5), while the critical point of E_ε corresponding to $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon \equiv 0)$ via (1.5) is unstable.*

PROPOSITION 4.2. *Under the hypotheses of Theorem 1.1, there exists some $\varepsilon_k > 0$ such that a solution $(f_\varepsilon, g_\varepsilon)$ of (1.6)-(1.7) with $g_\varepsilon > 0$ exists if and only if $\varepsilon \in (0, \varepsilon_k)$. Moreover, $\varepsilon_k = \varepsilon_{-k}$ for every $k \geq 1$ and the sequence $(\varepsilon_k)_{k \geq 1}$ is increasing.*

Proof of Proposition 4.1. Let us fix $\varepsilon > 0$. By Theorem 1.4 there exists a minimiser $u_\varepsilon \in C^1 \cap L^\infty(\Omega)$ of E_ε over $\mathcal{A}$. Using the transformation (2.7) we can assume that $u_\varepsilon \cdot e_3 \geq 0$ in Ω . By (3.2), we know that:

- either $u_\varepsilon \cdot e_3 > 0$ in Ω and by Theorem 1.3, E_ε has exactly two minimisers, namely u_ε and $\mathcal{I}u_\varepsilon$ where $\mathcal{I}$ is the reflection matrix about the (x_1, x_2) -plane containing $u_{bd}(\partial\Omega)$;
- or $u_\varepsilon \cdot e_3 \equiv 0$ in Ω and u_ε is the unique minimiser of E_ε (as point 2 in Theorem 1.4 does not hold).

Now, we consider the rotation

$$R_k(\theta) = \begin{pmatrix} \cos k\theta & -\sin k\theta & 0 \\ \sin k\theta & \cos k\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for every } \theta \in [0, 2\pi)$$

and define the $\mathbb{S}^1$ -group action:

$$u_{\varepsilon, \theta}(x) = R_k(\theta)^{-1} u_\varepsilon \left(\Pi(R_1(\theta) \cdot (x, 0)) \right), \quad x \in \Omega, \quad (4.1)$$

where $\Pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is the projection $\Pi(x_1, x_2, x_3) = (x_1, x_2)$. Then $u_{\varepsilon, \theta}$ is also a minimiser of E_ε over $\mathcal{A}$. Thanks to the above discussion on the uniqueness of u_ε , it follows that

$$u_\varepsilon \equiv u_{\varepsilon, \theta} \text{ for all } \theta \in [0, 2\pi)$$

(because $u_{\varepsilon, \theta} \cdot e_3$ and $u_\varepsilon \cdot e_3$ have the same sign). This invariance implies the existence of three scalar functions $f_\varepsilon, \hat{f}_\varepsilon, g_\varepsilon \in C^1(0, 1)$ such that

$$u_\varepsilon(r \cos \varphi, r \sin \varphi) = f_\varepsilon(r)(\cos k\varphi, \sin k\varphi) + \hat{f}_\varepsilon(r)(-\sin k\varphi, \cos k\varphi) + g_\varepsilon(r)e_3,$$

for every $r \in (0, 1)$ and $\varphi \in [0, 2\pi)$ (see e.g. proof of Proposition 2.1 in [13]). Moreover, the Euler-Lagrange equation (2.1) of u_ε combined with the boundary condition $u_\varepsilon = u_{bd}$ on $\partial\Omega$ imply that $\hat{f}_\varepsilon \equiv 0$ in $(0, 1)$ (i.e., u_ε is represented as in (1.5)) and the couple $(f_\varepsilon, g_\varepsilon)$ satisfies the system (1.6)-(1.7) (see e.g. the proof of Proposition 2.3 in [13]).

Clearly, $g_\varepsilon \geq 0$ (by the assumption at the beginning of the proof). We want to prove that $f_\varepsilon > 0$ in $(0, 1)$. First, note that $u^\circ \in \mathcal{A}$ corresponding to $(|f_\varepsilon|, g_\varepsilon)$ via (1.5) has the same energy as u_ε , i.e., $E_\varepsilon(u^\circ) = E_\varepsilon(u_\varepsilon)$, so by the uniqueness of minimisers (see the beginning of the proof) we deduce that $f_\varepsilon = |f_\varepsilon| \geq 0$ in $(0, 1)$. Moreover, by Step 2 in the proof of Theorem 1.4 we know that $|u_\varepsilon| \leq 1$ yielding $f_\varepsilon^2 + g_\varepsilon^2 \leq 1$. In particular, by (1.2), we deduce that the right hand side in (1.6) is non-negative. By the strong maximum principle applied to $f_\varepsilon \in H^1(B^2)$ satisfying $-\Delta f_\varepsilon + \frac{k^2}{r^2} f_\varepsilon \geq 0$ in B^2 and $f_\varepsilon = 1$ on ∂B^2 , we conclude that $f_\varepsilon > 0$ in $(0, 1)$. Also, the strong maximum principle applied to $g_\varepsilon \in H^1(B^2)$ satisfying $-\Delta g_\varepsilon \geq 0$ in B^2 and $g_\varepsilon \geq 0$ in B^2 implies that either $g_\varepsilon > 0$ in B^2 , or $g_\varepsilon \equiv 0$ in B^2 .

It remains to prove the stated uniqueness of solutions of the system (1.6)-(1.7) satisfying $g_\varepsilon \geq 0$ in $(0, 1)$.

Case 1. If $(f_\varepsilon, g_\varepsilon > 0)$ is a C^1 solution of (1.6)-(1.7), then the map u_ε defined by (1.5) with the “+” sign is a critical point of E_ε with $u_\varepsilon \cdot e_3 = g_\varepsilon > 0$ (as one can easily verify (2.1)) and $f_\varepsilon, g_\varepsilon \in L^\infty$ implies that $u_\varepsilon \in L^\infty$. Thus, by Theorem 1.3, u_ε is the unique minimiser of E_ε in the set

$$\mathcal{A}_+ = \{u \in \mathcal{A} : u \cdot e_3 \geq 0 \text{ a.e. in } \Omega\}.$$

This proves that there is at most one solution of (1.6)-(1.7) satisfying $g_\varepsilon > 0$.

Case 2. If $(\tilde{f}_\varepsilon > 0, \tilde{g}_\varepsilon \equiv 0)$ solves (1.6)-(1.7), then $\tilde{f}_\varepsilon$ satisfies

$$\begin{cases} -\tilde{f}_\varepsilon'' - \frac{1}{r}\tilde{f}_\varepsilon' + \frac{k^2}{r^2}\tilde{f}_\varepsilon = \frac{1}{\varepsilon^2}\tilde{f}_\varepsilon W'(1 - \tilde{f}_\varepsilon^2), \\ \tilde{f}_\varepsilon(0) = 0, \tilde{f}_\varepsilon(1) = 1, \end{cases} \quad (4.2)$$

which is known to enjoy existence and uniqueness for every $\varepsilon > 0$ (see e.g. [4, 6, 10, 11]). Moreover, if the solution $(f_\varepsilon, g_\varepsilon > 0)$ exists (yielding uniqueness - up to the reflection $\mathcal{I}$ - of the minimiser u_ε corresponding to $(f_\varepsilon, g_\varepsilon > 0)$ via (1.5)), we will show that $\tilde{u}_\varepsilon$ corresponding to $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon \equiv 0)$ via (1.5) is unstable for E_ε over $\mathcal{A}$. For that, assume by contradiction that $\tilde{u}_\varepsilon$ is stable (in the direction e_3). As $\tilde{u}_\varepsilon$ is “non-escaping”, then Proposition 3.2 would imply that $\tilde{u}_\varepsilon$ should be the unique minimiser of E_ε over $\mathcal{A}$ which is a contradiction with the existence of u_ε . $\square$

Proof of Proposition 4.2. Let I denote the set of $\varepsilon > 0$ such that a solution $(f_\varepsilon, g_\varepsilon)$ of (1.6)-(1.7) satisfying $g_\varepsilon > 0$ exists and let $J = (0, \infty) \setminus I$. Recall that $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)$ is the unique solution of (1.6)-(1.7) satisfying $\tilde{g}_\varepsilon \equiv 0$. By Proposition 4.1, $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)$ is a stable critical point of E_ε if and only if $\varepsilon \in J$. By the definition of stability of the solution $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)$ (see Proposition 3.2) and the continuity of the map $\varepsilon \in (0, \infty) \mapsto \tilde{f}_\varepsilon \in C^0([0, 1])$ (see e.g. [11]), it follows that J is closed and I is open.

Step 1. We show that there is some $\tilde{\varepsilon}_k > 0$ such that $(0, \tilde{\varepsilon}_k) \subset I$. Indeed, we note that, as $\varepsilon \rightarrow 0$, minimisers of E_ε over $\mathcal{A}$ converge strongly in $H^2(\Omega)$ to the following $\mathbb{S}^2$ -valued

minimising harmonic maps with boundary values given by (1.4) (see e.g. [1]):

$$(r, \varphi) \in (0, 1) \times [0, 2\pi) \mapsto \left(\frac{2r^k}{1+r^{2k}} \cos(k\varphi), \frac{2r^k}{1+r^{2k}} \sin(k\varphi), \pm \frac{1-r^{2k}}{1+r^{2k}} \right) \in \mathbb{S}^2.$$

In view of Proposition 4.1, the conclusion of Step 1 follows.

Step 2. We show that there is some $\hat{\varepsilon}_k > \tilde{\varepsilon}_k$ such that $[\hat{\varepsilon}_k, \infty) \not\subset I$. Indeed, if $\varepsilon > 0$ is large enough, then the energy E_ε is strictly convex (see Remark 3.3) and thus, the unique critical point of E_ε over $\mathcal{A}$ is given by the solution $(\tilde{f}_\varepsilon > 0, \tilde{g}_\varepsilon \equiv 0)$ of the system (1.6)-(1.7) via (1.5). The conclusion of Step 2 follows.

Step 3. We prove the desired dichotomy with respect to the critical value

$$\varepsilon_k = \sup\{\varepsilon > 0 : (0, \varepsilon) \subset I\} \in [\tilde{\varepsilon}_k, \hat{\varepsilon}_k].$$

Indeed, we start by noting that $\varepsilon_k \in J$ because I is open. Stability then implies that

$$\int_{\Omega} \left[|\nabla \varphi|^2 - \frac{1}{\varepsilon_k^2} W'(1 - |\tilde{f}_{\varepsilon_k}|^2) \varphi^2 \right] dx \geq 0 \quad \text{for all } \varphi \in H_0^1(\Omega). \quad (4.3)$$

To conclude, we only need to show that, for every $\varepsilon > \varepsilon_k$,

$$\int_{\Omega} \left[|\nabla \varphi|^2 - \frac{1}{\varepsilon^2} W'(1 - |\tilde{f}_\varepsilon|^2) \varphi^2 \right] dx \geq 0 \quad \text{for all } \varphi \in H_0^1(\Omega). \quad (4.4)$$

(This would mean that $(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)$ is stable and so $\varepsilon \in J$ for all $\varepsilon > \varepsilon_k$.) Define

$$\bar{f}_\varepsilon(x) = \tilde{f}_\varepsilon(\varepsilon x) \text{ for } x \in \Omega_\varepsilon = \{|x| < \varepsilon^{-1}\}.$$

Then, by (4.3), we have

$$\int_{\Omega_{\varepsilon_k}} \left[|\nabla \varphi|^2 - W'(1 - |\bar{f}_{\varepsilon_k}|^2) \varphi^2 \right] dx \geq 0 \text{ for all } \varphi \in H_0^1(\Omega_{\varepsilon_k}).$$

By the comparison principle [11, Proposition 3.5], we have $1 \geq \bar{f}_\varepsilon \geq \bar{f}_{\varepsilon_k} \geq 0$ in Ω_ε for $\varepsilon > \varepsilon_k$, which implies that

$$W'(1 - |\bar{f}_\varepsilon|^2) \leq W'(1 - |\bar{f}_{\varepsilon_k}|^2) \text{ in } \Omega_\varepsilon.$$

Extending $\bar{f}_\varepsilon$ to Ω_{ε_k} by setting $\bar{f}_\varepsilon = 1$ in $\Omega_{\varepsilon_k} \setminus \Omega_\varepsilon$, since $W'(0) = 0$, we thus obtain

$$\int_{\Omega_{\varepsilon_k}} \left[|\nabla \varphi|^2 - W'(1 - |\bar{f}_\varepsilon|^2) \varphi^2 \right] dx \geq 0 \text{ for all } \varphi \in H_0^1(\Omega_{\varepsilon_k}).$$

This implies the stability inequality (4.4) and concludes Step 3.

Step 4. We prove that there exists a positive function $\varphi_k \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \left[|\nabla \varphi_k|^2 - \frac{1}{\varepsilon_k^2} W'(1 - |\tilde{f}_{\varepsilon_k}|^2) \varphi_k^2 \right] dx = 0.$$

Indeed, for $\varepsilon > 0$ and $\tilde{f}_{\varepsilon}$ the solution of (4.2) for a fixed $k \in \mathbb{Z} \setminus \{0\}$, let

$$\lambda_{1,k}(\varepsilon) := \inf_{\varphi \in H_0^1(\Omega), \|\varphi\|_{L^2(\Omega)}=1} \int_{\Omega} \left[|\nabla \varphi|^2 - \frac{1}{\varepsilon^2} W'(1 - |\tilde{f}_{\varepsilon}|^2) \varphi^2 \right] dx.$$

As $W'(1 - |\tilde{f}_{\varepsilon}|^2) \in L^{\infty}(\Omega)$, standard arguments show that the infimum is attained by some positive and smooth $\varphi_{\varepsilon,k} \in H_0^1(\Omega)$ which satisfies

$$-\Delta \varphi_{\varepsilon,k} - \frac{1}{\varepsilon^2} W'(1 - |\tilde{f}_{\varepsilon}|^2) \varphi_{\varepsilon,k} = \lambda_{1,k}(\varepsilon) \varphi_{\varepsilon,k} \quad \text{in } \Omega.$$

Furthermore, by elliptic estimates, $\tilde{f}_{\varepsilon}$ and $\varphi_{\varepsilon,k}$ depend continuously on ε . On the other hand, by the dichotomy, $\lambda_{1,k}(\varepsilon) \geq 0$ for all $\varepsilon \geq \varepsilon_k$ and $\lambda_{1,k}(\varepsilon) < 0$ for $\varepsilon < \varepsilon_k$. Hence $\lambda_{1,k}(\varepsilon_k) = 0$. Letting $\varphi_k = \varphi_{\varepsilon_k,k}$ yields Step 4.

Step 5. We have $\varepsilon_k = \varepsilon_{-k}$ for every $k \geq 1$ and the sequence $(\varepsilon_k)_{k \geq 1}$ is increasing. Indeed, by the transformation $u = (u_1, u_2, u_3) \mapsto (u_1, -u_2, u_3)$, the minimisation problem corresponding to the boundary data u_{bd} with winding number k is equivalent with the one of winding number $-k$. Therefore, $\varepsilon_k = \varepsilon_{-k}$ for every $k \geq 1$. Let us now prove that $\varepsilon_{k-1} < \varepsilon_k$ for every $k \geq 2$. To make it clearer as we are now working with different winding numbers, let us now rename $\tilde{f}_{\varepsilon}$ to $\tilde{f}_{\varepsilon,k}$.

Now, note that $\tilde{f}_{\varepsilon_k, k-1}$ is the solution of the problem (4.2) for $\varepsilon = \varepsilon_k$ and the winding number $k-1$, i.e.

$$\begin{cases} -f'' - \frac{1}{r} f' + \frac{(k-1)^2}{r^2} f = \frac{1}{\varepsilon_k^2} f W'(1 - f^2), \\ f(0) = 0, f(1) = 1. \end{cases} \quad (4.5)$$

On the other hand, $\tilde{f}_{\varepsilon_k, k}$ is a subsolution to (4.5) with $\lim_{r \rightarrow 0} \frac{\tilde{f}_{\varepsilon_k, k}}{r^{k-1}} = 0$. The comparison principle [11, Proposition 3.5] applied for the problem (4.5) implies that $0 < \tilde{f}_{\varepsilon_k, k} < \tilde{f}_{\varepsilon_k, k-1} < 1$ in $(0, 1)$. This yields $W'(1 - |\tilde{f}_{\varepsilon_k, k}|^2) > W'(1 - |\tilde{f}_{\varepsilon_k, k-1}|^2)$ in $(0, 1)$. Since by Step 3, $(\tilde{f}_{\varepsilon_k, k}, 0)$ is stable for E_{ε_k} (see (4.3)), we deduce that $(\tilde{f}_{\varepsilon_k, k-1}, 0)$ is also stable for E_{ε_k} . As ε_{k-1} is the smallest ε for which $(\tilde{f}_{\varepsilon, k-1}, 0)$ is stable for E_{ε} , we deduce that $\varepsilon_k \geq \varepsilon_{k-1}$. On the other hand, if $\varepsilon_k = \varepsilon_{k-1}$, then, by Step 4,

$$\begin{aligned} 0 &= \int_{\Omega} \left[|\nabla \varphi_{k-1}|^2 - \frac{1}{\varepsilon_{k-1}^2} W'(1 - |\tilde{f}_{\varepsilon_{k-1}, k-1}|^2) \varphi_{k-1}^2 \right] dx \\ &= \int_{\Omega} \left[|\nabla \varphi_{k-1}|^2 - \frac{1}{\varepsilon_k^2} W'(1 - |\tilde{f}_{\varepsilon_k, k-1}|^2) \varphi_{k-1}^2 \right] dx \\ &> \int_{\Omega} \left[|\nabla \varphi_{k-1}|^2 - \frac{1}{\varepsilon_k^2} W'(1 - |\tilde{f}_{\varepsilon_k, k}|^2) \varphi_{k-1}^2 \right] dx, \end{aligned}$$

which contradicts the stability of $(\tilde{f}_{\varepsilon_k, k}, 0)$ for E_{ε_k} . We conclude that $\varepsilon_k > \varepsilon_{k-1}$. $\square$

5 Harmonic map problem

In this section, we consider $\mathbb{S}^{n-1}$ -valued maps. Recall that a map $u \in H^1(\Omega, \mathbb{S}^{n-1})$ is called a weakly harmonic map in Ω if it satisfies in the weak sense the harmonic map equation

$$-\Delta u = |\nabla u|^2 u \text{ in } \Omega.$$

This is the Euler-Lagrange equation for critical points of the Dirichlet energy

$$E(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 dx. \quad (5.1)$$

A weakly harmonic map is called a minimising harmonic map in Ω if there holds

$$E(u) \leq E(w) \text{ for all } w \in H^1(\Omega, \mathbb{S}^{n-1}) \text{ such that } w|_{\partial\Omega} = u|_{\partial\Omega}.$$

We start by proving the analogue of Theorem 1.3 for weakly harmonic maps that are positive in a direction $e \in \mathbb{S}^{n-1}$ inside Ω .

THEOREM 5.1. *Let $m \geq 1$, $n \geq 2$, $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary, and $u_{bd} \in H^{1/2}(\partial\Omega, \mathbb{S}^{n-1})$ be a given boundary data. Assume that $u \in \mathcal{A} \cap H^1(\Omega, \mathbb{S}^{n-1})$ is a weakly harmonic map satisfying*

$$u \cdot e > 0 \text{ a.e. in } \Omega \quad (5.2)$$

in a (fixed) direction $e \in \mathbb{S}^{n-1}$. Then u is a minimising harmonic map and we have the following dichotomy:

1. *If there exists some Lebesgue point $x_0 \in \partial\Omega$ of u_{bd} such that $u_{bd}(x_0) \cdot e > 0$ then u is the unique minimising harmonic map in $\mathcal{A} \cap H^1(\Omega, \mathbb{S}^{n-1})$.*
2. *If $u_{bd}(x) \cdot e = 0$ for $\mathcal{H}^{m-1}$ -a.e. $x \in \partial\Omega$, then all minimising harmonic maps in $\mathcal{A} \cap H^1(\Omega, \mathbb{S}^{n-1})$ are given by Ru , where $R \in O(n)$ is an orthogonal transformation of $\mathbb{R}^n$ satisfying $Rx = x$ for all $x \in \text{Span } u_{bd}(\partial\Omega)$.*

Proof of Theorem 5.1. Up to a rotation, we can assume that $e = e_n$ (as in (2.2)). It is clear that $\Phi = u \cdot e_n > 0$ satisfies

$$-\Delta \Phi = |\nabla u|^2 \Phi \text{ in } \Omega. \quad (5.3)$$

We consider the perturbation $u + v$ with $v \in H_0^1(\Omega, \mathbb{R}^n)$ and $|u + v| = 1$ a.e. in Ω (in particular, $|v| \leq 2$ in Ω). Then

$$2u \cdot v + |v|^2 = 0 \quad \text{a.e. in } \Omega. \quad (5.4)$$

Using the harmonic map equation, we obtain

$$2 \int_{\Omega} \nabla u \cdot \nabla v = 2 \int_{\Omega} |\nabla u|^2 u \cdot v dx \stackrel{(5.4)}{=} - \int_{\Omega} |\nabla u|^2 |v|^2 dx,$$

which leads to the second variation $\mathcal{D}$ of E at u :

$$\int_{\Omega} |\nabla(u+v)|^2 dx - \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} |\nabla v|^2 - |\nabla u|^2 |v|^2 dx =: \mathcal{D}(v). \quad (5.5)$$

(For simplicity, we still called $\mathcal{D}$ the second variation of E at u even if we do not ask that v is orthogonal to u). To show that u is minimising, we show that $\mathcal{D}(v) \geq 0$ for all $v \in H_0^1 \cap L^\infty(\Omega; \mathbb{R}^n)$ (note that this is a class larger than what we need, as we do not require that v satisfy the pointwise constraint (5.4)). To this end we take an arbitrary map $\bar{v} \in C_c^\infty(\Omega; \mathbb{R}^n)$ and decompose it as $\bar{v} = \Phi \psi$ in Ω . By (5.2) and (5.3), we note that Φ is a superharmonic function (i.e., $-\Delta \Phi \geq 0$ in Ω) that belongs to $H^1(\Omega)$. Thus, the weak Harnack inequality (see e.g. [8, Theorem 8.18]) implies that on the support ω of v (that is compact in Ω), we have $\Phi \geq \delta > 0$ for some δ (which may depend on ω). Thus, one obtains $\psi \in H_0^1 \cap L^\infty(\Omega; \mathbb{R}^n)$ and then, for $1 \leq j \leq n$, integration by parts yields:

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 \bar{v}_j^2 dx &= \int_{\Omega} |\nabla u|^2 \Phi \Phi \psi_j^2 dx \\ &\stackrel{(5.3)}{=} \int_{\Omega} \nabla \Phi \cdot \nabla (\Phi \psi_j^2) dx \\ &= \int_{\Omega} \left[-\Phi^2 |\nabla \psi_j|^2 + |\nabla \bar{v}_j|^2 \right] dx. \end{aligned}$$

It follows that

$$\mathcal{D}(\bar{v}) = \int_{\Omega} \Phi^2 |\nabla \frac{\bar{v}}{\Phi}|^2 dx \geq 0 \text{ for all } \bar{v} \in C_c^\infty(\Omega; \mathbb{R}^n).$$

It is well known that for every $v \in L^\infty \cap H_0^1(\Omega; \mathbb{R}^n)$, there exists a sequence $v_k \in C_c^\infty(\Omega; \mathbb{R}^n)$ such that $v_k \rightarrow v$ and $\nabla v_k \rightarrow \nabla v$ in L^2 and a.e. in Ω and $|v_k| \leq \|v\|_{L^\infty} + 1$ in Ω . In particular, by dominated convergence theorem, we have $\mathcal{D}(v_k) \rightarrow \mathcal{D}(v)$ thanks to (5.5). Thus, we deduce that for every compact $\omega \subset \Omega$,

$$\mathcal{D}(v) = \lim_{k \rightarrow \infty} \mathcal{D}(v_k) \geq \liminf_{k \rightarrow \infty} \int_{\omega} \Phi^2 |\nabla \frac{v_k}{\Phi}|^2 dx \geq \int_{\omega} \Phi^2 |\nabla \frac{v}{\Phi}|^2 dx \geq 0$$

for all $v \in H_0^1 \cap L^\infty(\Omega; \mathbb{R}^n)$, where we used Fatou's lemma. In particular, u is a minimising harmonic map by (5.5). We can now argue as in the Step 3 of the proof of Theorem 1.3 to obtain the rest of the result. We omit the details. $\square$

Proof of Theorem 1.5. Up to a rotation, we may assume that $e_n = e$ with u_{bd} satisfying the non-negativity assumption (1.8) in direction e_n . First we note that $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1}) \neq \emptyset$. Indeed, one can consider the harmonic extension u_0 of u_{bd} inside Ω implying that $u_0(\Omega) \subset \overline{B^n}_+ := \overline{B^n} \cap \{x_n \geq 0\}$; then set u to be the retraction of u_0 onto the hemisphere $\mathbb{S}_+^{n-1}$ using the pole $-e_n$, namely $u = \Psi \circ u_0$, where $\Psi : \overline{B^n}_+ \rightarrow \mathbb{S}_+^{n-1}$ is defined as

$$\Psi(x', x_n) = (tx', -1 + t(x_n + 1)) \text{ with } t = \frac{2(x_n + 1)}{|x'|^2 + (x_n + 1)^2}, |x'|^2 + x_n^2 \leq 1 \text{ and } x_n \geq 0.$$

Therefore, $u \in \mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$. By the direct method of the calculus of variation, we deduce the existence of a minimising harmonic map inside $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$. We claim that any minimising harmonic map v in $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$ satisfies

$$\text{either } |v_n| > 0 \text{ or } v_n \equiv 0 \text{ in } \Omega. \quad (5.6)$$

To see this, first note that, in view of (1.8), $\hat{v} = (v_1, \dots, v_{n-1}, |v_n|)$ is also a minimising harmonic map in $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$. Then, by the harmonic map equation, $|v_n| \in H^1(\Omega)$ is super-harmonic in Ω . Thus, if the (essential) infimum of $|v_n|$ on some ball B with $\bar{B} \subset \Omega$ is zero, then the weak Harnack inequality (see e.g. [8, Theorem 8.18]) implies that $|v_n|$ must be identically zero in B ; moreover, by a continuation argument on every path passing through B , we conclude that $|v_n|$ vanishes in Ω (as Ω is an open connected set, so path-connected). This implies the claim (5.6). The rest of the argument is similar to Step 4 in the proof of Theorem 1.4. We omit the details. $\square$

To conclude this section, we note that the same methods yield the following analogue of Proposition 3.2 for harmonic maps.

PROPOSITION 5.2. *Let $m \geq 1$, $n \geq 2$, $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary, $e \in \mathbb{S}^{n-1}$. If $u \in H^1(\Omega; \mathbb{S}^{n-1} \cap e^\perp)$ is a weakly harmonic map and u is stable in the direction e , i.e.*

$$\frac{d^2}{dt^2} \Big|_{t=0} E\left(\frac{u + t\varphi e}{|u + t\varphi e|}\right) = \int_{\Omega} \left[|\nabla \varphi|^2 - |\nabla u|^2 \varphi^2 \right] dx \geq 0 \text{ for all } \varphi \in H_0^1(\Omega; \mathbb{R}), \quad (5.7)$$

then u is a minimising harmonic map. Moreover, if $u(\Omega) \subset \text{Span } u(\partial\Omega)$, then u is the unique minimising harmonic map within its boundary data $u|_{\partial\Omega}$.

A Appendix

In the introduction we discussed the “escaping” and “non-escaping” phenomena of minimisers of E_ε and E considering specific examples in Theorem 1.1 and Example 1.6. In both cases the boundary data carried a *non-zero topological degree* (as a map $u_{bd} : \partial\Omega \rightarrow u_{bd}(\partial\Omega)$), in addition to the assumption that the co-dimension of $\text{Span } u_{bd}(\partial\Omega)$ in the target space $\mathbb{R}^n$ is nonzero. These examples illustrated the effect of “escaping” and “non-escaping” phenomena on minimisers, in particular, the relation between “escape” and smoothness of minimising harmonic maps.

In this section we would like to give a simple example of “escape” and “non-escape” phenomena for minimisers with a boundary data carrying a *zero topological degree* on $\partial\Omega$. In particular, we would like to point out that in this case there exist smooth and “non-escaping” minimising harmonic maps. More precisely, let $\Omega \subset \mathbb{R}^2$ be a bounded smooth simply connected domain, $n \geq 3$ and for some fixed $a \in \mathbb{R}$, we consider the “horizontal” harmonic map

$$u_a(x_1, x_2) = (\cos(ax_1), \sin(ax_1), 0, \dots, 0) \in \mathbb{R}^n \quad \text{for every } x \in \mathbb{R}^2. \quad (\text{A.1})$$

Restricting u_a to the boundary $\partial\Omega$ we define $u_{bd} : \partial\Omega \rightarrow \mathbb{S}^1 = \mathbb{S}^1 \times \{0_{\mathbb{R}^{n-2}}\}$ (i.e., $u_{bd} \equiv u_a|_{\partial\Omega}$). It is clear that u_{bd} carries a zero winding number on $\partial\Omega$ and can be smoothly lifted by $\varphi_{bd}(x) = ax_1$ on $\partial\Omega$. We will show that with the boundary data u_{bd} the appearance of “escaping” phenomenon for minimisers of harmonic map problem and Ginzburg-Landau problem strongly depends on the value $|a|$.

Harmonic map problem.

PROPOSITION A.1. *Let u_{bd} be defined as above and $\lambda_1(\Omega)$ be the first eigenvalue of $(-\Delta)$ on Ω with zero Dirichlet data. Then*

1. **“Non-escaping”**: *For $a^2 \leq \lambda_1(\Omega)$, the map u_a defined in (A.1) is the unique minimising harmonic map in $\mathcal{A} \cap H^1(\Omega, \mathbb{S}^{n-1})$.*
2. **“Escaping”**: *For $a^2 > \lambda_1(\Omega)$ every minimising harmonic map $u \in \mathcal{A} \cap H^1(\Omega, \mathbb{S}^{n-1})$ “escapes” from the horizontal plane $\mathbb{R}^2 \times \{0_{\mathbb{R}^{n-2}}\}$ and u is unique up to orthogonal transformations $R \in O(n)$ fixing every point of the horizontal plane. Moreover, the harmonic map u_a is unstable.*

Proof. We first note that for any $a \in \mathbb{R}$ the set $\mathcal{A} \cap H^1(\Omega, \mathbb{S}^{n-1}) \neq \emptyset$ since u_a belongs to this set and by the direct methods there exists a minimising harmonic map $u \in \mathcal{A} \cap H^1(\Omega, \mathbb{S}^{n-1})$. Moreover, we observe that $u_a = (\cos \varphi_a, \sin \varphi_a, 0, \dots, 0)$ with $\varphi_a(x) = ax_1$ for every $x \in \mathbb{R}^2$; in particular, φ_a is harmonic and hence u_a is a weakly $\mathbb{S}^{n-1}$ -valued harmonic map, i.e.,

$$-\Delta u_a = |\nabla u_a|^2 u_a \text{ in } \Omega.$$

Taking any $v \in H_0^1(\Omega, \mathbb{R}^n)$ such that $|u_a + v| = 1$ and following the proof of the Theorem 5.1 we obtain

$$\int_{\Omega} |\nabla(u_a + v)|^2 dx - \int_{\Omega} |\nabla u_a|^2 dx = \int_{\Omega} |\nabla v|^2 - \underbrace{|\nabla u_a|^2}_{=|\nabla \varphi_a|^2 = a^2} |v|^2 dx =: \mathcal{D}(v). \quad (\text{A.2})$$

1. If $a^2 \leq \lambda_1(\Omega)$, then Poincaré’s inequality and (A.2) yield $\mathcal{D}(v) \geq 0$ for every $v \in H_0^1(\Omega, \mathbb{R}^n)$; in particular, u_a is a minimising harmonic map. Moreover, by Theorem 1.5, we deduce that u_a is the unique minimising harmonic map in $\mathcal{A} \cap H^1(\Omega, \mathbb{S}^{n-1})$.

2. Let us take $a^2 > \lambda_1(\Omega)$. Assume by contradiction that u does not “escape” from the horizontal plane, i.e., $u \in H^1(\Omega, \mathbb{S}^1 \times \{0_{\mathbb{R}^{n-2}}\})$. In particular, u is a minimiser of E restricted to “horizontal” configurations $\mathcal{A} \cap H^1(\Omega, \mathbb{S}^1 \times \{0_{\mathbb{R}^{n-2}}\})$. Since Ω is a simply connected domain, then any map $w \in \mathcal{A} \cap H^1(\Omega, \mathbb{S}^1 \times \{0_{\mathbb{R}^{n-2}}\})$ can be represented as $w = (\cos \varphi, \sin \varphi, 0, \dots, 0)$ with the lifting $\varphi \in H^1(\Omega)$ and $\varphi = \varphi_{bd}$ on $\partial\Omega$. In particular, if φ is the lifting of u , then φ is harmonic (because u is a harmonic map); as $\varphi = \varphi_{bd}$ on $\partial\Omega$, one obtains that $\varphi(x) = \varphi_a(x) = ax_1$ in Ω . Therefore $u = u_a$ in Ω , i.e., u_a is a minimising harmonic map. We take $v \in H_0^1(\Omega)$ and compute the second variation of E at the point u_a in the direction ve_3 (that is orthogonal to u_a) to obtain

$$\frac{d^2}{dt^2} \Big|_{t=0} E\left(\frac{u_a + tve_3}{|u_a + tve_3|}\right) = \int_{\Omega} |\nabla v|^2 - \underbrace{|\nabla u_a|^2}_{=a^2} |v|^2 dx.$$

Since $a^2 > \lambda_1(\Omega)$ we can find $v \in H_0^1(\Omega)$ making the second variation negative (in the direction ve_3) which contradicts the minimality of u_a over $\mathcal{A} \cap H^1(\Omega, \mathbb{S}^{n-1})$. This proves in particular that the (unique) “horizontal” critical point u_a of E is unstable (when vertical directions are admissible in the target space $\mathbb{R}^n$, i.e., $n \geq 3$). The uniqueness of minimising harmonic maps is then given by Theorem 1.5. $\square$

Ginzburg-Landau problem. We can transfer the above ideas to the case of the Ginzburg-Landau problem. In particular, we have:

- “Escaping” phenomenon for $a^2 > \lambda_1(\Omega)$ and small enough $\varepsilon \leq \varepsilon_a$. Similar to minimising harmonic maps, in the case $a^2 > \lambda_1(\Omega)$ the “escaping” phenomenon happens for minimisers of E_ε over $\mathcal{A}$ within the boundary data u_{bd} , provided $\varepsilon \leq \varepsilon_a$ with ε_a depending on a . The proof uses the fact that as $\varepsilon \rightarrow 0$ the minimisers u_ε of E_ε over $\mathcal{A}$ converge in $H^1(\Omega)$ to minimising harmonic maps u , the key point is H^1 regularity of the minimising harmonic maps u (see e.g. [1]). The uniqueness holds up to isometries $R \in O(n)$ keeping every point of the horizontal plane fixed.

- “Non-escaping” phenomenon for any $a \in \mathbb{R}$ and large enough $\varepsilon \geq \varepsilon_0$. Using Remark 3.3 we know that E_ε is strictly convex over $\mathcal{A}$ for large enough $\varepsilon \geq \varepsilon_0$ (here ε_0 depends only on $W'(1)$ and $\lambda_1(\Omega)$ and is independent of a). Therefore there exists a unique critical point (that is the global minimiser) u_ε of E_ε over $\mathcal{A}$. Moreover, u_ε is “non-escaping” for every $\varepsilon \geq \varepsilon_0$ and for every $a \in \mathbb{R}$ since by Theorem 1.4 the existence of an “escaping” minimiser u_ε of E_ε over $\mathcal{A}$ would break the uniqueness (as the reflected map $\mathcal{I}u_\varepsilon \neq u_\varepsilon$ w.r.t. the horizontal plane would also be a minimiser of E_ε over $\mathcal{A}$).

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