

Variants of Schur-Weyl duality and Dirac cohomology



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To the memory of my mum and the strength, support and love of all three
of my parents.

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Abstract

This thesis is divided into the following three parts.

Chapter 1: Realising the projective representations of S_n

We derive an explicit description of the genuine projective representations of the symmetric group S_n using Dirac cohomology and the branching graph for the irreducible genuine projective representations of S_n . In [15] Ciubotaru and He, using the extended Dirac index, showed that the characters of the projective representations of S_n are related to the characters of elliptic graded modules. We derive the branching graph using Dirac theory and combinatorics relating to the cohomology of Borel varieties $\mathcal{B}_e$ of $\mathfrak{g}$. We use Dirac cohomology to construct an explicit model for the projective representations. We also describe Vogan's morphism for Hecke algebras in type A using spectrum data of the Jucys-Murphy elements.

Chapter 2: Dirac cohomology of the Dunkl-Opdam subalgebra

We define a new presentation for the Dunkl-Opdam subalgebra of the rational Cherednik algebra. This presentation uncovers the Dunkl-Opdam subalgebra as a Drinfeld algebra. The Dunkl-Opdam subalgebra is the first natural occurrence of a non-faithful Drinfeld algebra. We use this fact to define Dirac cohomology for the DO subalgebra. We formalise generalised graded Hecke algebras and extend a Langlands classification to generalised graded Hecke algebras. We then describe an equivalence between the irreducibles of the Dunkl-Opdam subalgebra and direct sums of graded Hecke algebras of type A. This equivalence commutes with taking Dirac cohomology.

Chapter 3: Functors relating nonspherical principal series

We define an extension of the affine Brauer algebra called the type B/C affine Brauer algebra. This new algebra contains the hyperoctahedral group,

and it naturally acts on $\text{End}_K(X \otimes V^{\otimes k})$. We study functors $F_{\mu,k}$ from the category of admissible $O(p, q)$ or $Sp_{2n}(\mathbb{R})$ modules to representations of the type B/C affine Brauer algebra. Furthermore, these functors take non-spherical principal series modules to principal series modules for the graded Hecke algebra of type D_k , C_{n-k} or B_{n-k} .

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Introduction

Root systems (Definition 0.1.5) were introduced by Killing [34] to classify semisimple complex Lie algebras. The Weyl group (Definition 0.1.20) is the reflection group acting on a crystallographic root system. Weyl groups are examples of finite real reflection groups which can be classified by, not necessarily crystallographic, root systems. The class of complex reflection groups contains all real reflection groups. These groups were classified by Shephard and Todd [52].

The underlying theme of this thesis is the study of several collections of algebras that are extensions of (complex) reflection groups. Examples include graded Hecke algebras, extended graded Hecke algebras, generalised graded Hecke algebra, Drinfeld algebras (both faithful and non-faithful), symplectic algebras, the rational Cherednik algebra, the Dunkl-Opdam subalgebra, and (affine) Brauer algebras.

The results of this thesis split into three chapters, each one characterised by a distinct question. This introduction highlights the defining questions.

The secondary theme of this thesis is Dirac cohomology, and it is a tool that we will continually use. The first applications of Dirac cohomology were developed for Lie algebras [47],[2]. Barbasch, Ciubotaru, and Trapa [4] defined Dirac cohomology for graded Hecke algebras. An integral component of Dirac cohomology is the Vogan morphism (Theorem 0.4.24), which for graded Hecke algebras links the irreducible projective representations of the Weyl group with central characters of the graded Hecke algebra. Set $\mathbb{H}(W)$ to be the graded Hecke algebra associated to a root system R with Weyl group W . Let X be an $\mathbb{H}(W)$ -module, then the Dirac cohomology $H_D(X)$ is naturally a projective representation of W , that is, a representation of the double cover $\widetilde{W}$ of W . This leads us to our first question:

(Q1) Vershik and Sergeev [56] described the projective representations of the symmetric group S_n using Jucys-Murphy elements. Are we able to rediscover this exposition with Dirac cohomology?

This question forms the basis for Chapter 1. It is much easier to describe the branching graph of the projective modules if we utilise the extended graded Hecke algebra of the symmetric group and the Dirac index [15] of this algebra. One can endow the extended graded Hecke algebra with a superalgebra structure (Definition 0.3.2). In this case, the Dirac index is an interpretation of Dirac cohomology for Hecke algebras in supertheory. We, therefore, study projective supermodules of the symmetric group. One can easily recover the analogous results on the ungraded projective modules. Using the Dirac index and combinatorics in [26], we reconstruct the branching graph of the genuine projective representations of the symmetric group, thus, creating a model for the isomorphism classes of projective representations of the symmetric group. As a corollary to studying the connection between Dirac cohomology and the projective representations of the symmetric group, we can explicitly calculate the Vogan morphism in the case of the symmetric group.

Let G be a finite group with finite complex representation V . Drinfeld [21] introduced the class of Drinfeld algebras (Definition 0.2.8) for a complex representation V of a finite group G and particular skew symmetric forms. Lusztig [39] introduced graded Hecke algebras as the associated graded algebra of a filtration of an affine Hecke algebra. They are of interest since through this association the graded Hecke algebra plays the equivalent role of a Lie algebra to a Lie group for the affine Hecke algebra. Graded Hecke algebras (Definition 0.2.3) are associated to a Weyl group and a parameter function. By employing a different presentation of a graded Hecke algebra, one can show that graded Hecke algebras are examples of Drinfeld algebras. Other notable examples of Drinfeld algebras are symplectic algebras [24] associated to a finite symplectic group (Definition 0.1.31) and within these the rational Cherednik algebras [22] (Definition 0.2.11) associated to a complex reflection group with parameter function.

We note that graded Hecke algebras are defined for Weyl groups and are examples of Drinfeld algebras with orthogonal representations. One could ask, can we extend the

class of graded Hecke algebras to complex reflection groups? Chapter 2 is dedicated to answering the following question.

(Q2) Are there graded Hecke algebras associated to the classical complex reflection groups $G(m, p, n)$ (Definition 0.1.25)?

Faithful Drinfeld algebras are the class of Drinfeld algebras restricted to faithful representations of a finite group $G \subset GL(V)$. Ram and Shepler [49] considered this question for faithful Drinfeld algebras. They prove that there is no Drinfeld algebra for $G(m, p, n)$ with a faithful representation V of dimension n for $m, n \geq 3$. We notice that Drinfeld's original definition allows non-faithful representations. This nuance has largely been forgotten, perhaps because in practice there were no naturally occurring non-faithful Drinfeld algebras. We extend the search for Drinfeld algebras related to $G(m, p, n)$ to include non-faithful representations. Ram and Shepler give a candidate for a 'Drinfeld-esque' algebra associated to $G(m, p, n)$. This algebra is isomorphic to the Dunkl-Opdam subalgebra $\mathbb{H}_{DO}$ (Definition 0.2.15) of the rational Cherednik algebra associated to $G(m, p, n)$. We construct new generators for the Dunkl-Opdam subalgebra which expose $\mathbb{H}_{DO}$ as a non-faithful Drinfeld algebra. Therefore, we simultaneously support Ram and Shepler's intuition and, consequently, discover a naturally occurring non-faithful Drinfeld algebra. Continuing on our sub-theme of Dirac cohomology, we extend Ciubotaru's [13] Dirac cohomology for faithful Drinfeld algebras to non-faithful Drinfeld algebras. As a product, we define Dirac cohomology for $\mathbb{H}_{DO}$.

Inspired by the construction of irreducible representations of the hyperoctahedral group (Example 0.1.4) which utilises induction from symmetric subgroups, we construct an equivalence of irreducibles between the Dunkl-Opdam subalgebra and direct sums of graded Hecke algebras of parabolic symmetric groups. Furthermore, this equivalence behaves well with respect to Dirac cohomology. We can transport the Dirac cohomology of an $\mathbb{H}_{DO}$ -module X to the Dirac cohomology of the corresponding module of the graded Hecke algebra of parabolic symmetric group $\mathbb{H}(S_{a_0}) \otimes \dots \otimes \mathbb{H}(S_{a_{m-1}})$ (Definition 0.2.5).

Dezelée [19] states that the Dunkl-Opdam subalgebra is a generalised graded Hecke algebra. We formalise this idea of generalised graded Hecke algebras into a formal

class of algebras. We extend Evens' [25] Langlands classification for graded Hecke algebras to generalised graded Hecke algebras.

Drinfeld algebras were originally introduced to define a Schur-Weyl duality with the Yangian associated to $\mathfrak{sl}_n$. Schur-Weyl duality, [51, 60], was first discovered for $\mathfrak{gl}_n$ and the symmetric group. This duality pairs $\mathfrak{gl}_n$ and $\mathbb{C}[S_k]$ as double commutator algebras acting on $V^{\otimes k}$ for the matrix module V of $\mathfrak{gl}_n$. This motif was continued; Brauer [8] constructed the Brauer algebra $B_k[m]$ (Definition 0.8.6) to be the commutant of the special orthogonal group $SO_n(\mathbb{R})$, or the symplectic group $Sp_{2n}(\mathbb{R})$, on $V^{\otimes k}$. Arakawa and Suzuki [1] defined functors from category $\mathcal{O}$ of $\mathfrak{gl}_n$ to modules of the graded Hecke algebra of the symmetric group $\mathbb{H}(S_k)$. This utilises a type of affine Schur-Weyl duality on the space $\text{End}_{\mathbb{C}}(X \otimes V^{\otimes k})$, where $X \in \mathcal{O}$ and V is the matrix module. These functors rely on a map $\rho : \mathbb{H}(S_k) \rightarrow \text{End}_{\mathfrak{gl}_n}(X \otimes V^{\otimes k})$. The next step was to study, for $G = SO_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$, the algebra $\text{End}_G(X \otimes V^{\otimes k})$, the affine Schur-Weyl duality for $SO_n(\mathbb{R})$ and $Sp_{2n}(\mathbb{R})$. The analogue of the graded Hecke algebra, in this case, is the affine Brauer algebra, $\mathfrak{B}_k$ (Definition 3.3.2). The Lie groups $SO_{2n+1}(\mathbb{R})$ and $Sp_{2n}(\mathbb{R})$ are associated to root systems of type B_n and C_n respectively, with Weyl group $W(B_n)$ the hyperoctahedral group. Disappointingly, the affine Brauer algebra, $\mathfrak{B}_k$, is an extension of the symmetric group and is not of type B or C . This leads us to our final question; the topic of ongoing work and the basis for Chapter 3.

(Q3) Is there a Schur-Weyl duality related to the real orthogonal and symplectic groups which contains the hyperoctahedral group in its commutator? That is, can we find an endomorphism space for $O(p, q)$ and $Sp_{2n}(\mathbb{R})$ which exhibits a commutator algebra that is an extension of the hyperoctahedral group?

Ciubotaru and Trapa [16] defined functors from spherical principal series modules (Definition 3.6.1) of $O(p, q)$ and $Sp_{2n}(\mathbb{R})$ to representations of the graded Hecke algebra of type B_n , C_n and D_n . Let $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$, with maximal compact subgroup K . In [16], there is implicit use of the algebra $\text{End}_K(X \otimes V^{\otimes k})$. We make this explicit and construct an extension of the affine Brauer algebra, using the Cartan involution θ . This extension is called the type B/C affine Brauer algebra $\mathfrak{B}_k^\theta$ (Definition 3.3.10). This new algebra maps to $\text{End}_K(X \otimes V^{\otimes k})$. The affine type B/C Brauer algebra is an extension of the hyperoctahedral group. This algebra's representations are the natural image of the functors defined in [16]. Furthermore, with this new viewpoint, we can

uniformly construct functors from G -representations to $\mathfrak{B}_k^\theta$ -modules. We extend the results in [16] to non-spherical representations. Non-spherical principal series modules for $O(p, q)$ or $Sp_{2n}(\mathbb{R})$ map to principal series modules of graded Hecke algebras of type D_k , B_{n-k} or C_{n-k} . We also define a star operation on $\mathfrak{B}_k^\theta$ which allows us to study unitary module of the type B/C affine Brauer algebra. Importantly, this star operation descends to a star operation on the graded Hecke algebra via a quotient of $\mathfrak{B}_k^\theta$. We then go on to show that the functors preserve Hermitian forms and unitarity.

Statement of main results

We give an overview of the main results of this thesis.

Chapter 1: Realising the projective representations of S_n

In Chapter 1, we reconstruct the branching graph of the genuine projective super representation of the symmetric group. With this data, we then create an explicit model for every genuine projective representation of S_n .

A partition $\lambda = (\lambda_1, \dots, \lambda_r)$ is strict if $\lambda_i > \lambda_{i+1}$ for all $i = 1, \dots, r-1$. The isomorphism classes of genuine irreducible projective supermodules of the symmetric group are parametrised by strict partitions. We utilise a transversal of these supermodules (Proposition 1.3.8):

$$\{\tilde{\tau}_\lambda = \frac{1}{a_\lambda} X_{-1}(\lambda) \otimes \mathcal{S} : \lambda \text{ is a strict partition}\}.$$

We reconstruct the branching graph for the irreducible genuine projective representation of S_n .

Theorem 1.4.9. *The branching rules of the genuine projective irreducible $\tilde{S}_n$ -supermodules are:*

$$Res_{n-1}^n \tilde{\tau}_\lambda = \begin{cases} 2\tilde{\tau}_{\lambda^{(1)}} \oplus 2\tilde{\tau}_{\lambda^{(2)}} \oplus \dots \oplus 2\tilde{\tau}_{\lambda^{(r-1)}} \oplus \tilde{\tau}_{\lambda^{(r)}} & \text{if } n-r \text{ is odd and } \lambda_r = 1, \\ 2\tilde{\tau}_{\lambda^{(1)}} \oplus 2\tilde{\tau}_{\lambda^{(2)}} \oplus \dots \oplus 2\tilde{\tau}_{\lambda^{(r-1)}} \oplus 2\tilde{\tau}_{\lambda^{(r)}} & \text{if } n-r \text{ is odd and } \lambda_r > 1, \\ \tilde{\tau}_{\lambda^{(1)}} \oplus \tilde{\tau}_{\lambda^{(2)}} \oplus \dots \oplus \tilde{\tau}_{\lambda^{(r-1)}} \oplus \tilde{\tau}_{\lambda^{(r)}} & \text{if } n-r \text{ is even.} \end{cases}$$

Here λ is always a strict partition, $\lambda^{(i)}$ depends on a reduction rule, defined in [26], which gives partitions $\lambda^{(i)}$ from λ .

Following [56] we define two subsets of $\mathbb{N}^n$, one related to the action of the Jucys-Murphy elements:

$$\text{Sspec}(n, \lambda) \text{ (Definition 1.5.5).}$$

The other is related to combinatorial content data on standard shifted tableaux:

$$\text{Scont}(n, \lambda) \text{ (Definition 1.5.7).}$$

The main result of Chapter 1 is the following theorem, originally proved in [56]; we have reconstructed this theorem using Dirac cohomology. We use a new result (Lemma 1.5.12) relating the squares of the Jucys-Murphy elements and the square of the Dirac operator.

Theorem 1.5.13. *The set $\text{Sspec}(n) \subset \mathbb{N}^n$, descending from eigenvalues of Jucys-Murphy elements, is equal to the set $\text{Scont}(n)$, a combinatorial construction from contents of shifted Young tableaux. Furthermore,*

$$\text{Sspec}(n, \lambda) = \text{Scont}(n, \lambda).$$

In section 1.6 we describe how one gets from the spectrum data to an explicit description of the genuine projective supermodules of $\tilde{S}_n$ and the action of $\tilde{S}_n$ in matrix form. In Section 1.7, we give an explicit description of the Vogan morphism for graded Hecke algebras of type A introduced in [4].

Chapter 2: Dirac cohomology of the Dunkl-Opdam subalgebra

In this chapter we focus on a potential candidate for a Drinfeld algebra associated to the complex reflection group $G(m, 1, n)$, the Dunkl-Opdam subalgebra $\mathbb{H}_{DO}$ (Definition 0.2.15). We give a new presentation of the Dunkl-Opdam subalgebra which exhibits it as a non-faithful Drinfeld algebra.

Theorem 2.4.13. *There exists a presentation of $\mathbb{H}_{DO}$ given by elements $\{\tilde{z}_i : i = 1, \dots, n\}$ and generators $\{s_i, g_j : i \leq n-1, 1 \leq j \leq n\}$ in $G(m, 1, n)$ such that:*

$$s_i \tilde{z}_j s_i^{-1} = s_i(\tilde{z}_j),$$

$$g_i \tilde{z}_j = \tilde{z}_j g_i \quad \forall i, j = 1, \dots, n,$$

$$[\tilde{z}_i, \tilde{z}_j] \in \mathbb{C}G(m, 1, n),$$

where $G(m, 1, n) \cong S_n \rtimes (\mathbb{Z}_m)^n$. The group S_n is generated by transpositions s_i , $i = 1, \dots, n-1$ and g_j generates the j^{th} copy of $\mathbb{Z}_m$.

We define a class of algebras, the generalised graded Hecke algebras (Definition 2.3.1), which includes the algebra $\mathbb{H}_{DO}$ and the generalised graded Hecke algebras of type B and D defined in [19]. We extend the Langlands classification, proved by Evens [25] for graded Hecke algebras, to generalised graded Hecke algebras.

Theorem 2.3.8. *Let $\mathbb{GH}(W \rtimes T)$ denote a generalised graded Hecke algebra. A parabolic subalgebra is denoted by $\mathbb{GH}_P$, with $\mathbb{GH}_{P_s}$ denoting the semisimple part of $\mathbb{GH}_P$ (Definition 2.3.2).*

(i) *Every irreducible $\mathbb{GH}$ -module V can be realised as a quotient of $\mathbb{GH}(W \rtimes T) \otimes_{\mathbb{GH}_P} U$, where $U = \hat{U} \otimes \mathbb{C}_\nu$ is such that $\hat{U}$ is an irreducible tempered $\mathbb{GH}_{P_s}$ -module and $\mathbb{C}_\nu$ is a character of $S(\mathfrak{a})$ defined by $\nu \in \mathfrak{a}^{*+}$.*

(ii) *If U is as in (i) then $\mathbb{GH}(W \rtimes T) \otimes_{\mathbb{GH}_P} U$ has a unique irreducible quotient to be denoted $J(P, U)$.*

(iii) *If $J(P, \hat{U} \otimes \mathbb{C}_\nu) \cong J(P', \hat{U}' \otimes \mathbb{C}_{\nu'})$ then $P = P'$, $\hat{U} \cong \hat{U}'$ as $\mathbb{GH}_{P_s}$ -modules and $\nu = \nu'$.*

We show that the semisimplification of the module category of the Dunkl-Opdam subalgebra $\mathbb{H}_{DO}(G(m, 1, n))$ is equivalent to the semisimplification of the module category of a direct sum of graded Hecke algebras for parabolic symmetric groups.

Theorem 2.5.6. *Let $A = \{\underline{a} = (a_0, \dots, a_{m-1}) \in \mathbb{N}^m : \sum_{i=0}^{m-1} a_i = n\}$, note that $\mathbb{N}$ includes zero. The irreducible representations of $\mathbb{H}(G(m, 1, n))$ split into blocks which are induced from products of $\mathbb{H}(S_a)$ -representations:*

$$\text{irr}(\mathbb{H}(G(m, 1, n))) \cong \bigsqcup_{\underline{a} \in A} \text{irr}(\mathbb{H}(S_{a_0})) \otimes \dots \otimes \text{irr}(\mathbb{H}(S_{a_{m-1}})).$$

This bijection of irreducible modules defines an equivalence of the semisimplification of the module categories, the Grothendieck groups, $G_0(\mathbb{H}(G(m, 1, n)))$ and $G_0(\bigoplus_{\underline{a} \in A} \mathbb{H}_{S_{a_0}} \otimes \dots \otimes \mathbb{H}_{S_{a_{m-1}}})$.

The maps F and F^{-1} (Lemma 2.5.4) are functors that exhibit this equivalence. We then show that this equivalence behaves well with respect to Dirac cohomology; we can transport an $\mathbb{H}_{DO}$ -module to its corresponding module under the equivalence and then take its Dirac cohomology. Dirac cohomology of a $\mathbb{H}$ module X is the kernel of a dirac operator $\mathcal{D} \in \mathbb{H} \otimes C(V)$ modulo $\text{Ker } \mathcal{D} \cap \text{Im } \mathcal{D}$.

Theorem 2.6.7. *Given an irreducible $\mathbb{H}_{DO}$ -representation V then $F(V)$ as an $\mathbb{H}_{S_{a_0}} \otimes \dots \otimes \mathbb{H}_{S_{a_{m-1}}}$ -module is isomorphic to $F(V) \cong X_{a_0} \otimes \dots \otimes X_{a_{m-1}}$. The Dirac cohomology of V is*

$$\bigoplus_{c \in C} c \left(H_D(X_{a_0}) \otimes \dots \otimes H_D(X_{a_{m-1}}) \right),$$

where $H_D(X)$ is the type A Dirac cohomology of the $\mathbb{H}_{S_k}$ -module X , P is the parabolic subset associated to $\underline{a}$ and C is a set of coset representatives of S_n/S_P . Let $H_D(\bullet)$ denote the functor taking the relevant module to its Dirac cohomology. We have the following commutative diagram:

$$\begin{array}{ccc} G_0(\mathbb{H}(G(m, 1, n))) & \xrightarrow{F} & G_0(\bigoplus_{\underline{a} \in A} \mathbb{H}_{S_{a_0}} \otimes \dots \otimes \mathbb{H}_{S_{a_{m-1}}}) \\ \downarrow H_D(\bullet) & & \downarrow H_D(\bullet) \\ \widetilde{\mathbb{C}G(m, 1, n)\text{-mod}} & \xleftarrow{\widetilde{F^{-1}}} & \bigoplus_{\underline{a} \in A} \widetilde{\mathbb{C}S_{a_0}} \otimes \dots \otimes \widetilde{\mathbb{C}S_{a_{m-1}}\text{-mod}}. \end{array}$$

Here $G_0(\mathbb{H}(G(m, 1, n)))$ is the Grothendieck group of the isomorphism classes of representations of $\mathbb{H}(G(m, 1, n))$.

Chapter 3: Functors relating non-spherical principal series modules

In this chapter we study Schur-Weyl duality for symplectic and orthogonal real groups. Following on from [16], we study the algebra $\text{End}_K(X \otimes V^{\otimes k})$ for $(\mathfrak{g}, K)$ module X and the reflection representation V . This inspires the formulation of a new algebra, the type B/C affine Brauer algebra $\mathfrak{B}_k^\theta$. From the viewpoint of this algebra, the functors defined in [16] have a natural image category, $\mathfrak{B}_k^\theta\text{-mod}$. This new perspective allows us to highlight ideas used in [16] and extend these results. Fix G to be either $O(p, q)$ or $Sp_{2n}(\mathbb{R})$. Let X be a G -representation, K the maximal compact subgroup of G and let μ be an irreducible K type. We study the functors defined by Ciubotaru and Trapa [16],

$$F_{\mu, k}(X) = \text{Hom}_K(\mu, X \otimes V^{\otimes k}).$$

The extension of the affine Brauer algebra, $\mathfrak{B}_k^\theta$, is defined using operators related to the Cartan involution θ of G . This new algebra $\mathfrak{B}_k^\theta$ acts on $X \otimes V^{\otimes k}$ and commutes with the action of K .

We show that the functors $F_{\mu,k}$ take the category of admissible G -modules to $\mathfrak{B}_k^\theta$ -modules. The algebras $\mathbb{H}_k(c)$ are graded Hecke algebras associated to the hyperoctahedral group of rank k , with parameter function related to $c \in \mathbb{C}$.

Let X_δ^ν (Definition 3.6.1) be a non-spherical principal series module for G . Given a principal series representation X_δ^ν we associate to it characters $\mu, \underline{\mu}$ (Table 3.6.1) with scalars c_μ and $c_{\underline{\mu}}$ (Table 3.7.1). We describe the image of X_δ^ν under the functors $F_{\mu,k}$ and $F_{\underline{\mu},n-k}$. As vector spaces $\mathbb{H}_k(c) \cong S(\mathfrak{a}_k) \otimes W(B_k)$ and $\mathbb{H}_{n-k}(c_{\underline{\mu}}) \cong S(\overline{\mathfrak{a}_{n-k}}) \otimes W(B_{n-k})$. The character ν_k is the restriction of ν to $\mathfrak{a}_k$ and $\overline{\nu_{n-k}}$ is the restriction to $\overline{\mathfrak{a}_{n-k}}$.

Theorem 3.8.14. *For $G = Sp_{2n}(\mathbb{R})$ or $O(p, q)$ $p + q = 2n + 1$, the module $F_{\mu,k}(X_\delta^\nu)$ is isomorphic to the $\mathbb{H}_k(c_\mu)$ principal series module*

$$X(\nu_k) = \text{Ind}_{S(\mathfrak{a}_k)}^{\mathbb{H}_k(c_\mu)} \nu_k.$$

The module $F_{\underline{\mu},n-k}(X_\delta^\nu)$ is isomorphic to the $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ principal series module

$$X(\bar{\nu}_{n-k}) = \text{Ind}_{S(\overline{\mathfrak{a}_{n-k}})}^{\mathbb{H}_k(c_{\underline{\mu}})} \bar{\nu}_{n-k}.$$

For $G = Sp_{2n}(\mathbb{R})$ and $O(n+1, n)$, every principal series module is induced from a character. Hence we can entirely describe the Hecke algebra modules resulting from functors $F_{\mu,k}$ and $F_{\underline{\mu},n-k}$ of any principal series module. Casselman's theorem [12] states that every irreducible representation in $\mathcal{HC}(G)$ is a subrepresentation of a principal series module.

Theorem 3.8.15. *Let G be $O(n+1, n)$ or $Sp_{2n}(\mathbb{R})$, then G is split. Let X be an irreducible G -module. Let X_δ^ν be a principal series representation that contains X , then the $\mathfrak{B}_k^\theta$ and $\mathfrak{B}_{n-k}^\theta$ -modules*

$$F_{\mu,k}(X) \text{ and } F_{\underline{\mu},n-k}(X)$$

are submodules of the $\mathbb{H}_k(c_\mu)$ and $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ -modules

$$X(\nu_k) \text{ and } X(\bar{\nu}_{n-k}).$$

We define two star-operations on $\mathfrak{B}_k^\theta[m]$ which descend to the two star-operations on the graded Hecke algebra. Furthermore, we show that if X is a Hermitian (respectively unitary) module of G then the image of X under the functor $F_{\mu,k}$ is a Hermitian (resp. unitary) module for $\mathfrak{B}_k^\theta[m]$. We also show that the Langlands quotient is preserved.

Theorem 3.9.26. *Let X_δ^ν be a principal series module for $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$. The Langlands quotient $\overline{X_\delta^\nu} = X_\delta^\nu / \text{rad}\langle, \rangle_{X_\delta^\nu}$ is mapped by $F_{\mu, k}$, to the Langlands quotient of the $\mathbb{H}_k(c_\mu)$ -module, $\overline{X(\nu_k)} = X(\nu_k) / \text{rad}\langle, \rangle_{X_{\nu_k}}$. Similarly, X_δ^ν is mapped by $F_{\underline{\mu}, n-k}$, to the $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ -module $\overline{X(\bar{\nu}_{n-k})}$.*

We then give a non-unitary test for non-spherical principal series modules.

Theorem 3.9.29. *[Non-unitary test for principal series modules] If either $\overline{X(\nu_k)}$ or $\overline{X(\bar{\nu}_{n-k})}$ are not unitary, as $\mathbb{H}_k(c_\mu)$ and $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ -modules, then the Langlands quotient of the minimal principal series module $\overline{X_{\delta_k}^\nu}$, for $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$ is not unitary.*

This result gives a functorial proof of the non-unitarity criterion proved with petite K -types by Barbasch, Pantano, Paul and, Salamanca-Riba [3, 46]. We also obtain a non-unitary test for any Harish Chandra module; in the split case one could check unitarity of Hecke algebra modules, but in general, one has to work with Brauer algebras.

Theorem 3.9.30. *[Non-unitary test for Harish-Chandra modules] Let X be a Harish Chandra module. For $G = Sp_{2n}(\mathbb{R})$ or $O(p, q)$ $p+q = 2n+1$, if for any character μ and $k = 1, \dots, n$ the $\mathfrak{B}_k^\theta$ -module $\overline{F_{\mu, k}(X)}$ is not unitary, then the Langlands quotient $\overline{X}$ of X is not a unitary G -module. In the case when G is split then X is a subrepresentation of X_δ^ν and $F_{\mu, k}(X)$, $F_{\underline{\mu}, n-k}(X)$ are Hecke algebra modules. In this case, if either $\overline{F_{\mu, k}(X)}$, $\overline{F_{\underline{\mu}, n-k}(X)}$ is not unitary as a Hecke algebra module then $\overline{X}$ is not unitary as a G -module.*

Chapter 0

Preliminaries

This chapter is entirely preparatory. In Section 0.1 we introduce root systems, Weyl groups, real and complex reflection groups. Section 0.2 defines graded Hecke algebras and Drinfeld algebras. Section 0.3 is a very brief discussion on superalgebras and supermodules. We define Dirac cohomology for Drinfeld algebras in Section 0.4. Sections 0.5, 0.6 and 0.7 introduce the very basics of Lie theory. In Section 0.8 we discuss Schur-Weyl duality and introduce the Brauer algebra.

0.1 Root Systems, Weyl Groups and Coxeter groups

Definition 0.1.1. *Let V be a real Euclidean vector space, i.e. V has a positive definite symmetric bilinear form $(\cdot, \cdot)$. A reflection s is a linear operator on V which sends a non-zero vector $\alpha \in V$ to its negative and fixes the hyperplane H_α orthogonal to α . We denote this reflection by s_α ,*

$$s_\alpha(v) = v - 2 \frac{(v, \alpha)}{(\alpha, \alpha)} \alpha.$$

The reflection s_α is an orthogonal operator on V .

Definition 0.1.2. *A finite reflection group is a finite subgroup of the orthogonal group, $O(V)$, generated by reflections s_α . We will denote a finite reflection group by W .*

Example 0.1.3. *The symmetric group. Set $n \geq 2$, let $V = \mathbb{R}^n$ with basis $e_1, \dots, e_n$, define the orthogonal transformation, s_{ij} , by the permutation which exchanges e_i and e_j and fixes all other basis vectors. The subgroup of $O(V)$ generated by the permutation matrices s_{ij} is the symmetric group S_n . Set $\alpha_{ij} = e_i - e_j$. The transposition s_{ij} is the reflection $s_{\alpha_{ij}}$. The symmetric group is a finite reflection group.*

Example 0.1.4. Again set $V = \mathbb{R}^n$ with basis $e_1, \dots, e_n$. The hyperoctahedral group $W(B_n)$ is the subgroup of $O(V)$ generated by $s_{\alpha_{jk}}$ and s_{e_i} for $i = 1, \dots, n$ and $1 \leq i < j \leq n$. The group $W(B_n)$ is isomorphic to the semidirect product:

$$W(B_n) \cong S_n \rtimes (\mathbb{Z}_2)^n,$$

where S_n acts on $(\mathbb{Z}_2)^n$ by permuting the entries.

Let Φ be a finite subset of V , such that

$$\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\} \text{ for all } \alpha \in \Phi,$$

$$s_\alpha(\Phi) = \Phi \text{ for all } \alpha \in \Phi.$$

Definition 0.1.5. Let Φ be as above and define W to be the subgroup $O(V)$ generated by s_α for all $\alpha \in \Phi$. Then the pair (Φ, W) is a root system and Φ is the set of roots of W .

Note that we do not require a root system to be integral, this will be included in the definition of crystallographic root system. Fix a total ordering $<$ on V . Set the positive roots Φ^+ to be $\{\alpha \in \Phi : 0 < \alpha\}$.

Definition 0.1.6. Given a finite reflection group W with roots Φ we say Δ is a set of simple roots if the vectors in Δ are linearly independent and

$$\Phi^+ = \text{span}_{\mathbb{N}}(\Delta).$$

Definition 0.1.7. For a root system (W, Φ, Δ) define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Theorem 0.1.8. [31, Theorem 1.5] The finite reflection group W , with a simple system $\Delta \subset \Phi$ is generated by the simple reflections

$$\{s_\alpha : \alpha \in \Delta\}.$$

Definition 0.1.9. Let Δ_P be a subset of Δ , we define the parabolic subgroup W_P of W to be the group generated by $s_\alpha, \alpha \in \Delta_P$. Let $\Phi_P = \text{span}_{\mathbb{N}} \Delta_P \cap \Phi$ then W_P, Δ_P, Φ_P is a root system.

Definition 0.1.10. Let W be a finite reflection group, Φ roots for W and simple roots Δ . For a group element $w \in W$ we define the length of $w \in W$ to be:

$$l(w) = \min\{j : w = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_j} \text{ for } \alpha_i \in \Delta\}.$$

Definition 0.1.11. Let w_0 be an element of longest length in W .

The longest element w_0 is unique for any fixed Δ , it is an involution. Define $m(\alpha, \beta)$ to be the order of $s_\alpha s_\beta$ in W .

Theorem 0.1.12. [31, Theorem 1.9] Fix W and $\Delta \subset \Phi$, then W is isomorphic to the abstract group generated by $s_\alpha, \alpha \in \Delta$ subject to the relations

$$(s_\alpha s_\beta)^{m(\alpha, \beta)} = e \text{ for } \alpha, \beta \in \Delta.$$

By Theorem 0.1.12 a given finite reflection group W is parametrised by Δ and $m(\alpha, \beta)$. We associate to this data a diagram. First we define a general Coxeter graph.

Definition 0.1.13. A Coxeter graph Γ is a set of vertices S with edges between s and s' labelled by integers $m(s, s') \geq 3$. If s and $s' \in S$ are not connected by an edge we set $m(s, s') = 2$.

Definition 0.1.14. Let W be a finite reflection group with roots Φ and simple roots Δ . Define the Coxeter graph of W to have vertices $\alpha \in \Delta$ and edges between $\alpha \neq \beta$ when $m(\alpha, \beta) \geq 3$. Label such an edge by $m(\alpha, \beta)$.

Proposition 0.1.15. [31, Proposition 2.2] Let W_1 and W_2 be finite reflection groups with the same Coxeter graphs, then there is isometry of spaces V_1 and V_2 inducing an isomorphism between W_1 and W_2 .

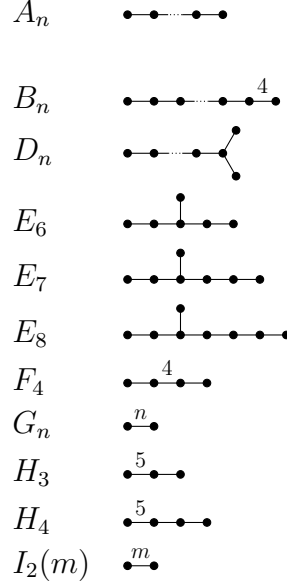
Definition 0.1.16. A Coxeter System (W, Φ, Δ) is irreducible if its associated Coxeter graph is connected.

If S has cardinality n we define a symmetric $n \times n$ matrix A such that

$$A(s, s') = -\cos \frac{\pi}{m(s, s')} \text{ for all } s, s' \in S.$$

Definition 0.1.17. The Coxeter Graph Γ is positive if the associated matrix A is positive definite, that is $x^t A x > 0$ for $x \in \mathbb{R}^n$. The Coxeter graph is semi positive if $x^t A x \geq 0$.

Theorem 0.1.18. [31, Theorem 2.7] *The following graphs are the only connected Coxeter graphs of positive type.*



Definition 0.1.19. *A lattice in a vector space V is the $\mathbb{Z}$ span of a basis. A group $G \subset O(V)$ is said to be crystallographic if it stabilises a lattice in V .*

Definition 0.1.20. *A Weyl group is a reflection group that is also crystallographic.*

Proposition 0.1.21. [31, Proposition 2.8] *If W is crystallographic, then for every $\alpha \neq \beta$, $m(\alpha, \beta)$ is either, 2, 3, 4 or 6.*

A Root system Φ is crystallographic if

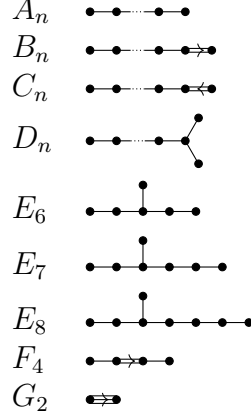
$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Phi.$$

For Weyl groups there are at most two different lengths of roots.

Definition 0.1.22. *A Dynkin diagram is a Coxeter graph which is directed.*

To the associated Coxeter graph Γ of W , a Weyl group, we mandate that the edge $m(\alpha, \beta)$ is directed from β to α if $(\alpha, \alpha) < (\beta, \beta)$. This endows the Coxeter graph of a Weyl group as a Dynkin Diagram.

Theorem 0.1.23. [31, Theorem 2.8] *The following graphs are the only connected Dynkin diagrams.*



The diagrams are in one-to-one correspondence with crystallographic root systems.

Definition 0.1.24. *Let V be a complex Euclidean vector space i.e. V has a positive definite Hermitian form $(\cdot, \cdot)$. A complex reflection s is a linear operator on V which sends a non-zero vector $\alpha \in V$ to $\lambda\alpha$ for a root of unity $\lambda \in \mathbb{C} \setminus 1$ and fixes the hyperplane H_α orthogonal to α . We may denote this complex reflection by $s_{\alpha, \lambda}$, on V :*

$$s_\alpha(v) = v - (1 - \lambda) \frac{(v, \alpha)}{(\alpha, \alpha)} \alpha.$$

Definition 0.1.25. *A complex reflection group is a finite subgroup of $GL(V)$ which is generated by complex reflections.*

Definition 0.1.26. *Let $m, p, n \in \mathbb{N}$ such that $p|m$. Let η be a primitive m^{th} root of unity. Define the abelian group:*

$$A_{p,m} = \{(\eta^{a_1}, \eta^{a_2}, \dots, \eta^{a_n}) : \sum_{i=1}^n a_i = 0 \pmod{p}\}.$$

The symmetric group naturally acts on $(\mathbb{Z}_m)^n$ and $A_{p,m}$ by permuting the coordinates. Define the group $G(m, p, n)$

$$G(m, p, n) = S_n \rtimes A_{p,m}.$$

The group $G(m, 1, n)$ is a complex reflection group acting on $V = \mathbb{C}^n$, generated by S_n and $s_{e_i, \eta}$ for $i = 1, \dots, n$.

Theorem 0.1.27. [52, Table VII] *All irreducible complex reflection groups are of the form $G(m, p, n)$ or one of 34 exceptional types, labelled by Shephard-Todd numbers 4 – 37.*

Remark 0.1.28. The group $G(1, 1, n) \cong S_n$ and $G(2, 1, n)$, $G(2, 2, n)$ are isomorphic to $W(B_n)$ and $W(D_n)$ respectively. The cyclic group $\mathbb{Z}_m \cong G(m, 1, 1)$.

Definition 0.1.29. A $\mathbb{C}$ -vector space V is symplectic if it has a bilinear form $\omega : V \times V \rightarrow \mathbb{C}$ such that:

$$\omega(v', v) = -\omega(v, v') \text{ for all } v, v' \in V,$$

$$\omega(u, v) = 0 \text{ for all } u \in U \text{ implies } v = 0.$$

The symplectic group on V is the group

$$Sp(V) = \{A \in GL(V) : \omega(Av, Av') = \omega(v, v') \text{ for all } v, v' \in V\}.$$

Definition 0.1.30. Let V be a symplectic $\mathbb{C}$ -vector space with symplectic form ω . A symplectic reflection is an element $s \in Sp(V)$ such that $rk(\text{Id}_V - s) = 2$. For a symplectic reflection we can decompose $V = \ker(\text{Id}_V - s) \oplus \text{Im}(\text{Id}_V - s)$ and this decomposition is orthogonal with respect to ω .

Definition 0.1.31. Let V be a symplectic vector space, a finite subgroup $G \subset Sp(V)$ is a symplectic reflection group if it can be generated by symplectic reflections.

Remark 0.1.32. Let $G = G(m, p, n)$ be a classical complex reflection group acting on the vector space $\mathfrak{h} = \mathbb{C}^n$, let $\mathfrak{h}^*$ denote its dual. Set $V = \mathfrak{h} \oplus \mathfrak{h}^*$ and endow V with a symplectic form defined by

$$\omega(x, y) = x^t \begin{bmatrix} 0 & \text{Id}_n \\ -\text{Id}_n & 0 \end{bmatrix} y \text{ for all } x, y \in V = \mathfrak{h} \oplus \mathfrak{h}^*.$$

The group $G(m, p, n)$ acts on V via its actions on $\mathfrak{h}$ and $\mathfrak{h}^*$. The complex reflection group $G(m, n, p)$ is a subgroup of $Sp(V)$ generated by symplectic reflections, rendering $G(m, p, n)$ a symplectic reflection group.

0.2 Graded Hecke Algebras and Drinfeld algebras

Given a $\mathbb{C}$ -vector space U , we define the k^{th} tensor product $T^k(U)$ to be $\overbrace{U \otimes \dots \otimes U}^k$ and the tensor algebra $T(U)$ is

$$T(U) = \bigoplus_{k=0}^{\infty} T^k(U),$$

with multiplication defined by the canonical isomorphism:

$$T^l(U) \otimes T^k(U) \rightarrow T^{k+l}(U).$$

The symmetric algebra $S(U)$ is the quotient of $T(U)$ by the two sided ideal generated by

$$u_1 u_2 - u_2 u_1 \text{ for all } u_1, u_2 \in U.$$

The alternating algebra $\Lambda(U)$ is the quotient of $T(U)$ by the ideal generated by the relations

$$u_1 u_2 + u_2 u_1 \text{ for all } u_1, u_2 \in U.$$

Definition 0.2.1. *Root datum is a tuple $(X, \Phi, \check{X}, \check{\Phi}, \Delta)$ such that X and $\check{X}$ are free abelian groups with a perfect pairing*

$$\langle, \rangle : X \times \check{X} \rightarrow \mathbb{Z}.$$

The roots Φ and coroots $\check{\Phi}$ are finite subsets of X and $\check{X}$ which are in bijection

$$\Phi \leftrightarrow \check{\Phi},$$

$$\alpha \leftrightarrow \check{\alpha},$$

$$\langle \alpha, \check{\alpha} \rangle = 2 \text{ for all } \alpha \in \Phi.$$

The reflection s_α must preserve Φ :

$$s_\alpha(\Phi) = \Phi \text{ for all } \alpha \in \Phi,$$

The set $\Delta \subset \Phi$ is the set of simple roots such that if we defin $\Phi^+ = \text{span}_{\mathbb{N}} \Delta$ and $\Phi^- = \{-\alpha : \alpha \in \Phi^+\}$ then

$$\Phi = \Phi^+ \cup \Phi^-.$$

Define the $\mathbb{C}$ -vector spaces $\mathfrak{t} = X \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathfrak{t}^* = \check{X} \otimes_{\mathbb{Z}} \mathbb{C}$ and extend the pairing $\langle, \rangle$ to $\mathfrak{t} \times \mathfrak{t}^*$. We may also denoted $\mathfrak{t}$ by V . The Weyl group associated to $(X, \Phi, \check{X}, \check{\Phi}, \Delta)$ is the finite subgroup of $GL(\mathfrak{t})$ generated by the reflections s_α for $\alpha \in \Phi$. Using the pairing $\langle, \rangle$ we can describe the reflection s_α as $s_\alpha(x) = x - \langle x, \check{\alpha} \rangle \alpha$.

Definition 0.2.2. *Let W be a Weyl group associated to the root datum $(X, \Phi, \check{X}, \check{\Phi}, \Delta)$. A function $\mathbf{c} : \Phi \rightarrow \mathbb{C}$ is a parameter function if it is W invariant, i.e.*

$$\mathbf{c}(\alpha) = \mathbf{c}(w(\alpha)) \text{ for all } w \in W, \alpha \in \Phi.$$

Definition 0.2.3. [38, Definition 0.1] Let $(X, \Phi, \check{X}, \check{\Phi}, \Delta)$ be root datum with associated Weyl group W and parameter function $\mathbf{c}$. The graded Hecke algebra $\mathbb{H}$ associated to this data is the $\mathbb{C}$ -vector space $S(\mathfrak{t}) \otimes \mathbb{C}[W]$ with a structure of an associative $\mathbb{C}$ -algebra defined by the relations:

$S(\mathfrak{t}) \rightarrow \mathbb{H}$ is an injective algebra homomorphism,

$\mathbb{C}[W] \rightarrow \mathbb{H}$ is an injective algebra homomorphism,

$$s_\alpha \cdot x - s_\alpha(x)s_\alpha = \mathbf{c}(\alpha) \frac{x - s_\alpha(x)}{\alpha}.$$

Remark 0.2.4. One notes that $x - s_\alpha(x) = x - x + \langle x, \check{\alpha} \rangle \alpha = \langle x, \check{\alpha} \rangle \alpha$ is a multiple of α . Hence $\frac{x - s_\alpha(x)}{\alpha} = \langle x, \check{\alpha} \rangle \in \mathbb{C}$ and we can, and will, replace the relation $s_\alpha \cdot x - s_\alpha(x)s_\alpha = \mathbf{c}(\alpha) \frac{x - s_\alpha(x)}{\alpha}$ by

$$s_\alpha \cdot x - s_\alpha(x)s_\alpha = \mathbf{c}(\alpha) \langle x, \check{\alpha} \rangle.$$

Definition 0.2.5. For a parabolic subgroup W_P of W one can define a parabolic subalgebra $\mathbb{H}_P$ of the graded Hecke algebra $\mathbb{H}$. This is the subalgebra generated by $S(\mathfrak{t})$ and the parabolic subgroup W_P .

Proposition 0.2.6. [39, Proposition 4.5] The centre of $\mathbb{H}$ is the W invariant elements of $S(V)$,

$$S(\mathfrak{t})^W.$$

For associative algebras A and B let $A \star B$ be the free product of A and B . The algebra $A \star B$ has a basis of words $a_1 b_1 \dots a_n b_n$, where $\{a_j\}$ is a basis of A and $\{b_k\}$ is a basis of B . For a group algebra $\mathbb{C}[G]$ with representation (U, ρ) we define the semi direct product $\mathbb{C}[G] \rtimes T(U)$ by the quotient of the associative $\mathbb{C}$ -algebra $\mathbb{C}[G] \star T(U)$ by the relations:

$$g \cdot u \cdot g^{-1} = \rho(g)(u) \text{ for all } g \in G, u \in U.$$

Definition 0.2.7. Let G be a finite subgroup of $GL(V)$. Associate to every $g \in G$, a skew symmetric bilinear form $b_g : V \times V \rightarrow \mathbb{C}$. Define the algebra $\mathfrak{H}(b_g)$ associated to G and the set $\{b_g\}$ to be the quotient of the $\mathbb{C}$ -algebra $\mathbb{C}[G] \rtimes T(V)$ by the relations

$$[u, v] = \sum_{g \in G} b_g(u, v)g.$$

We define a filtration on $\mathfrak{h}(b_g)$, let $v \in V$ have degree 1 and $g \in G$ have degree 0, extend this to a filtration on $\mathfrak{H}$.

Definition 0.2.8. Let G be a finite subgroup of $GL(V)$ with skew symmetric forms b_g for $g \in G$. The algebra $\mathfrak{H}(b_g)$ is a faithful Drinfeld algebra if it satisfies the PBW property, that is, the associated graded algebra of $\mathfrak{H}$ is isomorphic to $\mathbb{C}[G] \ltimes S(V)$.

Let $G(b) = \{g \in G : b_g \neq 0\}$ and let $(V^g)^\perp = \{v - g(v) : v \in V\}$.

Proposition 0.2.9. [21],[49, Theorem 1.9] The algebra $\mathfrak{H}$ is a Drinfeld algebra if and only if the following properties hold:

- (1) $b_{h^{-1}gh}(u, v) = b_g(h(u), h(v)),$ for all $h \in G, u, v \in V,$
- (2) $\ker b_g = V^g,$ for all $g \in G(b) \setminus 1,$
- (3) $\dim V^g = \dim V - 2,$ for all $g \in G(b) \setminus 1,$
- (4) $\det(h|_{(V^g)^\perp}) = 1,$ for all $h \in Z_G(g)$

f

Lemma 0.2.10. The graded Hecke algebra associated to $(X, \Phi, \check{X}, \check{\Phi}, \Delta)$ with parameter function $\mathbf{c}$ is isomorphic to the faithful Drinfeld algebra with $W \cong G \subset GL(X)$ with skew symmetric forms:

$$b_w(x_1, x_2) = \sum_{w=s_\alpha s_\beta : \alpha, \beta \in \Phi^+} \mathbf{c}(\alpha)\mathbf{c}(\beta) (\langle x_1, \check{\alpha} \rangle \langle x_2, \check{\beta} \rangle - \langle x_2, \check{\alpha} \rangle \langle x_1, \check{\beta} \rangle).$$

Proof. Denote a degree one element in $T(\mathfrak{t}) \subset \mathfrak{H}$ by $\tilde{x}$ and a degree one element in $S(\mathfrak{t}) \subset \mathbb{H}$ by x . We define an isomorphism from the Drinfeld algebra to the graded Hecke algebra $\phi : \mathfrak{H} \rightarrow \mathbb{H}$ by

$$\phi(\tilde{x}) = x - \frac{1}{2} \sum_{\beta \in R} \mathbf{c}(\beta) \langle x, \check{\beta} \rangle s_\beta \text{ for all } \tilde{x} \in \mathfrak{t} \subset \mathfrak{H},$$

$$\phi(w) = w \text{ for all } w \in W.$$

□

0.2.1 Rational Cherednik Algebras and the Dunkl-Opdam subalgebra

Let $\mathfrak{h}$ be an n dimensional $\mathbb{C}$ -vector space, let $\mathfrak{h}^*$ be its dual. Define $V = \mathfrak{h} \oplus \mathfrak{h}^*$ with symplectic form ω defined in Remark 0.1.32. The complex reflection group $G(m, p, n)$ is a symplectic reflection group inside $Sp(V)$.

For a reflection $s \in G(m, p, n)$ denote by α_s and $\check{\alpha}_s$ non-zero vectors in $\text{Im}(\text{Id} - s)|\mathfrak{h}^*$ and $\text{Im}(\text{Id} - s)|\mathfrak{h}$. There exist $\lambda_s \neq 1$ a root of unity such that

$$s(\check{\alpha}_s) = \lambda_s \check{\alpha}_s, s(\alpha_s) = \lambda_s^{-1} \alpha_s.$$

Let $S \subset G$ be the set of symplectic reflections in G .

Definition 0.2.11. [22] Let $t \in \mathbb{C}$, $V = \mathfrak{h} \oplus \mathfrak{h}^*$, set $G = G(m, p, n) \subset \text{Sp}(V)$, set $\mathbf{c} : S \rightarrow \mathbb{C}$ be a G invariant parameter function. The rational Cherednik algebra $\mathcal{H}_{t, \mathbf{c}}(G(m, p, n))$ is the quotient of the algebra $\mathbb{C}[G(m, p, n)] \rtimes T(V)$ by the relations

$$[y_1, y_2] = 0 = [x_1, x_2] \text{ for all } y_1, y_2 \in \mathfrak{h}, x_1, x_2 \in \mathfrak{h}^*,$$

$$[y, x] = t\omega(y, x) - \sum_{s \in S} \mathbf{c}(s) \frac{\omega(y, \alpha_s)\omega(\check{\alpha}_s, x)}{\omega(\check{\alpha}_s, \alpha_s)} \text{ for all } y \in \mathfrak{h}, x \in \mathfrak{h}^*.$$

Remark 0.2.12. The rational Cherednik algebra is a particular type of symplectic reflection algebra defined in [24]. It is also a Drinfeld algebra by considering $G(m, p, n) \subset \text{GL}(\mathfrak{h} \otimes \mathfrak{h}^*)$ and skew symmetric forms

$$b_g(x, y) = \begin{cases} t\omega(x, y) & \text{if } g = 1, \\ \mathbf{c}(s) \frac{\omega(x, \alpha_s)\omega(\check{\alpha}_s, y)}{\omega(\check{\alpha}_s, \alpha_s)} & \text{if } g = s \in S, \\ 0 & \text{otherwise,} \end{cases}$$

for $x, y \in V = \mathfrak{h} \oplus \mathfrak{h}^*$.

For the rest of the section fix $G = G(m, 1, n)$ and η to be an m^{th} primitive root. We give a different definition for the rational Cherednik algebra of $G(m, 1, n)$. Given $G(m, 1, n)$ acting on $\mathfrak{h}$ let $\{x_i\} \subset \mathfrak{h}$ be a basis such that $w(x_i) = x_{w(i)}$ for $w \in S_n$ and

$$g_i(x_j) = \begin{cases} \eta^{-1}x_j & \text{if } i = j, \\ x_j & \text{otherwise} \end{cases}.$$

Let $\{y_1, \dots, y_n\} \in \mathfrak{h}^*$ be the dual basis to $\{x_1, \dots, x_n\}$.

Remark 0.2.13. The rational Cherednik algebra for $G(m, 1, n)$, $\mathcal{H}_t(G(m, 1, n))$ can be formulated as the quotient of the $\mathbb{C}$ -algebra $\mathbb{C}[G(m, 1, n)] \rtimes T(V \oplus V^*)$ by the relations

$$\begin{aligned} [x_i, x_j] &= [y_i, y_j] = 0, \\ [x_i, y_i] &= \underline{t} - \sum_{l=1}^{m-1} \sum_{i \neq j} \mathbf{c}(\alpha_{ij}) s_{i,j} g_i^{-l} g_j^l - \sum_{l=1}^{m-1} \mathbf{c}(g_i^l) (1 - \eta^{-1}) g_i^l, \\ [x_i, y_j] &= \mathbf{c}(\alpha_{ij}) \sum_{l=1}^{m-1} s_{i,j} g_i^{-l} g_j^l. \end{aligned}$$

The parameter function $\mathbf{c}$ is G invariant and hence is uniform on all α_{ij} . We will write $\mathbf{c}(\alpha)$ for $\mathbf{c}(\alpha_{ij})$.

Definition 0.2.14. [22, Definition 3.7] For $0 < i \leq n$ define Dunkl operators z_i in $\mathcal{H}_t(G(m, 1, n))$

$$\begin{aligned} z_i &= y_i x_i + \mathbf{c}(\alpha) \sum_{l=1}^{m-1} \sum_{i>j} s_{i,j} g_i^{-l} g_j^l + \frac{1}{2} \sum_{l=1}^{m-1} \mathbf{c}(g_i^l) g_i^l + \frac{1}{2} t, \\ &= x_i y_i - \mathbf{c}(\alpha) \sum_{l=1}^{m-1} \sum_{i<j} s_{i,j} g_i^{-l} g_j^l - \frac{1}{2} \sum_{l=1}^{m-1} \mathbf{c}(g_i^l) g_i^l - \frac{1}{2} t. \end{aligned}$$

Definition 0.2.15. The Dunkl-Opdam subalgebra $\mathbb{H}_{DO}(G(m, 1, n))$ of the rational Cherednik algebra is the subalgebra generated by $G(m, 1, n)$ and z_i for $i = 1, \dots, n$.

Remark 0.2.16. [40, Lemma 3.2] The following relations hold in $\mathbb{H}_{DO}(G(m, 1, n))$

$$\begin{aligned} [z_i, z_j] &= 0 && \text{for } i, j = 1, \dots, n, \\ [z_i, g_k] &= 0 && \text{for } i, k = 1, \dots, n, \\ [z_j, s_{i,i+1}] &= 0 && \text{for } j \neq i, i+1, \\ z_i s_{i,i+1} &= s_{i,i+1} z_{i+1} - \mathbf{c}(\alpha) \epsilon_{ij} && \text{for } i = 1, \dots, n-1. \end{aligned}$$

Here $\epsilon_{ij} = \sum_{l=1}^{m-1} g_i^l g_j^{-l}$.

In fact $\mathbb{H}_{DO}(G(m, 1, n))$ is isomorphic to the $\mathbb{C}$ associative algebra generated by z_i and $G(m, 1, n)$ subject to the relations stated in Remark 0.2.16.

0.3 Superalgebras

Definition 0.3.1. A vector superspace is a $\mathbb{Z}_2$ -graded $\mathbb{C}$ -vector space $V = V_0 \oplus V_1$. If $\dim V_0 = n_0$ and $\dim V_1 = n_1$ then $\text{sdim } V = (n_0, n_1)$, $\dim V = n_0 + n_1$.

Elements of V_0 are called even and elements of V_1 are called odd. Even and odd elements are called homogeneous and have degree 0 or 1 respectively. A subspace U of V is a super subspace of V if it is homogenous, that is:

$$U = (U \cap V_0) \oplus (U \cap V_1).$$

Definition 0.3.2. A superalgebra is a vector superspace $\mathcal{A}$ which is an associative unital $\mathbb{C}$ -algebra such that $\mathcal{A}_i \mathcal{A}_j \subset \mathcal{A}_{i+j}$ for $i, j \in \mathbb{Z}_2$.

A superideal of $\mathcal{A}$ is an ungraded ideal which is also a super subspace.

Definition 0.3.3. Let $\mathcal{A}$ be a superalgebra. A (left) $\mathcal{A}$ supermodule is a superspace V which is a left $\mathcal{A}$ -module in the ungraded sense, such that $\mathcal{A}_i V_j \subset V_{i+j}$ for $i, j \in \mathbb{Z}_2$.

Definition 0.3.4. A homomorphism of $\mathcal{A}$ supermodules is a linear map $f : V \rightarrow W$ such that

$$f(av) = (-1)^{\deg(f)\deg(a)}af(v), \text{ for all } a \in \mathcal{A}, v \in V.$$

An isomorphism of supermodules is a bijective homomorphism of supermodules.

0.4 Dirac cohomology for Drinfeld algebras

Let V be a complex vector space and let G be a finite subgroup of $GL(V)$. We must assume there is a G invariant nondegenerate symmetric bilinear form on V , denoted $(,)$. Therefore $G \subset O(V)$ for the corresponding orthogonal group.

Definition 0.4.1. [41, 3.7.1] The Clifford algebra $C(V)$ associated to V and $(,)$ is the quotient of the algebra $T(V)$ by the relations

$$v_1v_2 + v_2v_1 = \varepsilon 2(v_1, v_2).$$

Where ε is either uniformly 1 or -1 .

Note that since $C(V)$ is defined as a $\mathbb{C}$ -algebra then whether ε is 1 or -1 makes no difference on the structure, however it does alter a few equations up to sign. We will be clear in each chapter whether ε equals 1 or -1 . Let $v \in V$ have degree 1 then the Clifford algebra has a filtration $C^n(V) = \{c \in C(V) : \deg(c) \leq n\}$. The associated graded algebra is isomorphic to $\Lambda(V)$. Set

$$C(V)_0 = \{c \in C(V) : \deg(c) = 0 \pmod{2}\}$$

and

$$C(V)_1 = \{c \in C(V) : \deg(c) = 1 \pmod{2}\}.$$

The Clifford algebra has a $\mathbb{Z}_2$ grading

$$C(V) = C(V)_0 \oplus C(V)_1.$$

The Clifford algebra can be considered as a super algebra with this $\mathbb{Z}_2$ grading.

Definition 0.4.2. The transpose of $C(V)$ is the anti-involution defined by

$$^t : C(V) \rightarrow C(V) :$$

$$v^t = -v \text{ for all } v \in V,$$

$$(ab)^t = b^t a^t \text{ for all } a, b \in C(V).$$

Definition 0.4.3. Define an algebra automorphism $\epsilon : C(V) \rightarrow C(V)$ such that $\epsilon = \text{Id}$ on $C(V)_0$ and $\epsilon = -\text{Id}$ on $C(V)_1$.

Definition 0.4.4. [41, Definition 3.8] The *Pin group* is a subgroup of the units of $C(V)$:

$$\text{Pin}(V) = \{a \in C(V)^\times : \epsilon(a) \cdot V \cdot a^{-1} \subset V, a^t = a^{-1}\}.$$

The group $\text{Pin}(V)$ is a double cover of $O(V)$ and the projection $p : \text{Pin}(V) \rightarrow O(V)$ is given by

$$p(a)(v) = \epsilon(a)va^{-1}, a \in \text{Pin}(V), v \in V.$$

Let $W \subset O(V)$ be a reflection group. That is it is generated by reflection s_v for some $v \in V$ such that $s_v(x) = x - 2\frac{(x,v)}{(v,v)}v$. Then

$$p^{-1}(s_v) = \pm \frac{1}{\sqrt{(v,v)}}v \in \text{Pin}(V).$$

Definition 0.4.5. For a subgroup $G \subset O(V)$ define the double cover of G :

$$\tilde{G} = p^{-1}(G) \subset \text{Pin}(V).$$

Example 0.4.6. The symmetric group S_n acts on the vector space $V = \mathbb{C}^n$, which has basis e_i . It is generated by reflections $s_{\alpha_{ij}}$ for $\alpha_{ij} = e_i - e_j$. The group $p^{-1}(S_n)$ is generated by $\frac{1}{2}\alpha_{ij} \in C(V)$. As an abstract group it is the group generated by $\{\alpha_{i,i+1}, z\}$ subject to the relations :

$$z^2 = e, \alpha_{i,i+1}^2 = z, \alpha_{i,i+1}\alpha_{i+1,i+2}\alpha_{i,i+1} = \alpha_{i+1,i+2}\alpha_{i,i+1}\alpha_{i+1,i+2}, z \text{ is central.}$$

There is a second non-trivial double cover of the symmetric group with presentation

$$\{z, \alpha'_{i,i+1} : z^2 = e, \alpha_{i,i+1}'^2 = e, \alpha'_{i,i+1}\alpha'_{i+1,i+2}\alpha'_{i,i+1} = z\alpha'_{i+1,i+2}\alpha'_{i,i+1}\alpha'_{i+1,i+2}z \text{ is central.}\}$$

These groups are not isomorphic but their group algebras over $\mathbb{C}$ are, one can construct an isomorphism by sending $\alpha_{i,i+1}$ to $\sqrt{-1}\alpha'_{i,i+1}$.

Definition 0.4.7. [41, Definition 3.4] Let V be a complex finite-dimensional vector space with non-degenerate symmetric bilinear form $(,)$. A $\mathbb{Z}_2$ -graded (resp. ungraded) spinor module $\mathcal{S}$ over $C(V)$ is a graded (resp. ungraded) module $(\mathcal{S}, \rho)$ such that

$$\rho : C(V) \rightarrow \text{End}(\mathcal{S})$$

is an isomorphism of super (resp. ungraded) algebras.

Theorem 0.4.8. [41, Theorem 3.10] *Let V be an n dimensional complex vector space with non-degenerate symmetric bilinear form. Suppose n is even then there is:*

*a unique isomorphism class of irreducible ungraded $C(V)$ spinor modules,
a unique class of irreducible $\mathbb{Z}_2$ -graded $C(V)$ spinor modules,
two isomorphism classes of irreducible ungraded $C(V)_0$ -modules.*

Suppose n is odd, then there is:

*two isomorphism classes of ungraded $C(V)$ spinor modules,
a unique isomorphism class of $\mathbb{Z}_2$ -graded $C(V)$ spinor modules,
two isomorphism classes of irreducible ungraded $C(V)_0$ spinor modules.*

0.4.1 The Dirac operator

Recall Definition 0.2.8 of a Drinfeld algebra. Let $\mathbb{H}$ denote a Drinfeld algebra associated to $G \subset O(V)$ with skew symmetric forms b_g . Let us extend the automorphism $\epsilon : C(V) \rightarrow C(V)$ to $\epsilon : \mathbb{H} \otimes C(V)$ by setting ϵ to be the identity on $\mathbb{H}$. The Drinfeld algebra is the quotient of the associate algebra $T(V) \rtimes \mathbb{C}[G]$. For $v \in V$ we denote the corresponding element in $T(V)$ (and $\mathbb{H}$) by $\tilde{v}$.

Definition 0.4.9. [13, 2.3] *Let $\{v_i\}$ and $\{v^i\}$ be dual bases of V , with respect to the form $(,)$.*

The Dirac element associated to $\mathbb{H}$ and $C(V)$ is

$$\mathcal{D} = \sum_i \tilde{v}_i \otimes v^i \in \mathbb{H} \otimes C(V).$$

The definition of the Dirac element does not depend on the basis chosen.

Definition 0.4.10. *Define a group homomorphism Δ from the subgroup $\tilde{G}$ of $\text{Pin}(V)$ to $\mathbb{H} \otimes C(V)$:*

$$\Delta(\tilde{g}) = p(\tilde{g}) \otimes \tilde{g}, \tilde{g} \in \tilde{G}.$$

Definition 0.4.11. *Let $\{v_i\}$ and $\{v^i\}$ be dual bases of V . Define*

$$\mathbf{h} = \sum_i \tilde{v}_i \tilde{v}^i \in \mathbb{H}.$$

For every $g \in G(b) = \{g \in G : b_g \neq 0\}$, define

$$\kappa_g = \sum_{i,j} b_g(v_i, v^i) v^i v_j \in C(V).$$

Lemma 0.4.12. [13, Lemma 2.5] In $\mathbb{H} \otimes C(V)$

$$\mathcal{D}^2 = -\mathbf{h} \otimes 1 + \frac{1}{2} \sum_{g \in G(b)} g \otimes \kappa_g.$$

Lemma 0.2.9 stipulated that $\dim V^g = \dim V - 2$ for all $g \in G(b) = \{g \in G : b_g \neq 0\}$.

Lemma 0.4.13. [13, Lemma 2.6] Any element $g \in G(b)$ can be written as a product of two reflection $g = s_\alpha s_\beta$, where $\alpha, \beta \in V^{g^\perp}$ are linearly independent and $(\alpha, \alpha) = (\beta, \beta) = 1$.

Definition 0.4.14. Let $g \in G(b)$, Lemma 0.4.13 states $g = s_\alpha s_\beta$, define

$$\begin{aligned} \tilde{g} &= \alpha\beta \in C(V), \\ c_{\tilde{g}} &= \frac{b_g(\alpha, \beta)}{1 - (\alpha, \beta)^2}, \\ e_g &= \frac{b_g(\alpha, \beta)(\alpha, \beta)}{1 - (\alpha, \beta)^2}. \end{aligned}$$

Furthermore define,

$$\begin{aligned} \Omega_{\mathbb{H}} &= \mathbf{h} - \sum_{g \in G(b)} e_g g \in \mathbb{H}^G, \\ \Omega_{\tilde{G}} &= \sum_{g \in G(b)} c_{\tilde{g}} \tilde{g} \in \mathbb{C}[\tilde{G}]^{\tilde{G}} \subset C(V). \end{aligned}$$

Theorem 0.4.15. [13, Theorem 2.7] The square of the Dirac element equals, in $\mathbb{H} \otimes C(V)$:

$$\mathcal{D}^2 = -\Omega_{\mathbb{H}} + 1 \otimes \frac{1}{2} \kappa_1 + \Delta(\Omega_{\tilde{G}}).$$

0.4.2 Vogan's Dirac morphism

The Clifford algebra $C(V)$ is $\mathbb{Z}_2$ -graded, $C(V) = C(V)_0 \oplus C(V)_1$. The automorphism ϵ is Id on $C(V)_0$ and $-\text{Id}$ on $C(V)_1$. We extend ϵ to be Id on $\mathbb{H}$ creating an automorphism of $\mathbb{H} \otimes C(V)$.

Definition 0.4.16. Define the linear map

$$d : \mathbb{H} \otimes C(V) \rightarrow \mathbb{H} \otimes C(V)$$

by

$$d(a) = \mathcal{D}a - \epsilon(a)\mathcal{D}, a \in \mathbb{H} \otimes C(V).$$

Lemma 0.4.17. [13, Lemma 3.2] $\Delta(\mathbb{C}[\tilde{G}]) \subset \ker d$.

Definition 0.4.18. Recall Definition 0.4.14 sets $\Omega_{\mathbb{H}}$ to be $\mathbf{h} - \sum_{g \in G(b)} e_g g \in \mathbb{H}^G$, then define

$$\tilde{\Omega}_{\mathbb{H}} = \Omega_{\mathbb{H}} \otimes 1 - 1 \otimes \frac{1}{2} \kappa_1 \in (\mathbb{H} \otimes C(V))^{\tilde{G}} = \mathbb{H} \otimes C(V).$$

Define the algebra $\mathbf{A}$ to be the centraliser of $\tilde{\Omega}_{\mathbb{H}}$,

$$\mathbf{A} = Z_{\mathbb{H} \otimes C(V)}(\tilde{\Omega}_{\mathbb{H}}).$$

Remark 0.4.19. If $1 \notin G(b)$, then

$$\mathbf{A} = \mathbb{H} \otimes C(V).$$

Lemma 0.4.20. [13, Lemma 2.4] For every $\tilde{g} \in \tilde{G}$ we have the invariance property:

$$\Delta(\tilde{g})\mathcal{D}\Delta(\tilde{g}^{-1}) = \det(p(\tilde{g}))\mathcal{D}.$$

Remark 0.4.21. The operator $\mathcal{D}$ interchanges the $\tilde{G}$ det-isotypic space and the triv-isotypic space.

Proof. Lemma 0.4.20 proves that $\Delta(\tilde{g})\mathcal{D}\Delta(\tilde{g}^{-1}) = \det(p(\tilde{g}))\mathcal{D}$. Suppose $\mathcal{D}$ is acting on a $\mathbb{H} \otimes C(V)$ -representation U and $u_{\text{triv}} \in U^{\text{triv}}$ a $\tilde{G}$ triv-isotypic space. Then

$$\Delta(\tilde{g})u_{\text{triv}} = u_{\text{triv}},$$

by definition of an isotypic space. Let $\mathcal{D}u_{\text{triv}}$ be the image of $\mathcal{D}$ of u . We show $\mathcal{D}u_{\text{triv}} \in U^{\text{det}}$. Since $u_{\text{triv}} \in U^{\text{triv}}$ then $\Delta(\tilde{g}^{-1})u_{\text{triv}} = u_{\text{triv}}$.

$$\Delta(\tilde{g})\mathcal{D}u_{\text{triv}} = \Delta(\tilde{g})\mathcal{D}\Delta(\tilde{g}^{-1})u_{\text{triv}}$$

then Lemma 0.4.20 shows

$$\Delta(\tilde{g})\mathcal{D}u_{\text{triv}} = \det(p(\tilde{g}))\mathcal{D}u_{\text{triv}}$$

hence $\mathcal{D}u_{\text{triv}} \in U^{\text{det}}$. Suppose $u_{\text{det}} \in U^{\text{det}}$ then $\Delta(\tilde{g}^{-1})u_{\text{det}} = \det(p(\tilde{g}^{-1}))u_{\text{det}}$.

$$\Delta(\tilde{g})\mathcal{D}u_{\text{det}} = \Delta(\tilde{g})\mathcal{D}\Delta(\tilde{g}^{-1})\det p(\tilde{g}^{-1})u_{\text{det}}$$

then Lemma 0.4.20 shows

$$\Delta(\tilde{g})\mathcal{D}u_{\text{det}} = \det(p(\tilde{g}))\det(p(\tilde{g}^{-1}))\mathcal{D}u_{\text{det}} = \mathcal{D}u_{\text{det}}.$$

hence $\mathcal{D}u_{\text{det}} \in U^{\text{triv}}$. □

Remark 0.4.21 shows that $\mathcal{D}$ interchanges the det and triv-isotypic spaces of a $\mathbb{H} \otimes C(V)$ -representation. Therefore $d(a) = \mathcal{D}a - \epsilon(a)\mathcal{D}$ interchanges the det and triv-isotypic spaces.

Definition 0.4.22. Set $\mathbf{A} = Z_{\mathbb{H} \otimes C(V)}(\tilde{\Omega}_{\mathbb{H}})$, then restrict the map d :

$$d_{\text{triv}} : \mathbf{A}^{\text{triv}} \rightarrow \mathbf{A}^{\text{det}}, \quad d_{\text{det}} : \mathbf{A}^{\text{det}} \rightarrow \mathbf{A}^{\text{triv}}.$$

Theorem 0.4.23. [13, Theorem 3.5] The kernel of d_{triv} equals:

$$\ker d_{\text{triv}} = \text{Im } d_{\text{det}} \oplus \Delta(\mathbb{C}[\tilde{G}]^{\tilde{G}}).$$

This is proved for Drinfeld algebras in [13], for graded Hecke algebras in [4].

Theorem 0.4.24 (Vogan's Dirac homomorphism). *The projection $\zeta : \ker d_{\text{triv}} \rightarrow \mathbb{C}[\tilde{G}]^{\tilde{G}}$ defined by Theorem 0.4.23 is an algebra homomorphism.*

Remark 0.4.25. Suppose $\mathcal{B}$ is a finitely generated abelian subalgebra of $\mathbf{A} \cap (\mathbb{H} \otimes C(V)_0)$ such that $\tilde{\Omega}_{\mathbb{H}} \in \mathcal{B} \subset \ker d_{\text{triv}}$. The element $\mathcal{D}$ commutes with every element of $\mathcal{B}$. The homomorphism ζ in Theorem 0.4.24 defines a morphism

$$\zeta^* : \text{Irr}(\tilde{G}) = \text{Spec } \mathbb{C}[\tilde{G}]^{\tilde{G}} \rightarrow \text{Spec } \mathcal{B}.$$

If $G(b)$ does not contain 1 then

$$\zeta^* : \text{irr}(\tilde{G}) = \text{Spec } \mathbb{C}[\tilde{G}]^{\tilde{G}} \rightarrow \text{Spec } S(V) \cong S(V^*)^W.$$

0.4.3 Dirac cohomology

Let (X, π) be an $\mathbb{H}$ -module and let $(\mathcal{S}, \rho)$ be an ungraded spinor module for $C(V)$ (Definition 0.4.7). The Dirac element $\mathcal{D}$ is an element in $\mathbb{H} \otimes C(V)$. Define the Dirac operator, denoted by D , to be the operator $\pi \otimes \rho(\mathcal{D})$ on $X \otimes \mathcal{S}$.

Definition 0.4.26. [4, Definition 4.1] The Dirac cohomology of X (and $\mathcal{S}$) is

$$H^D(X) = \frac{\ker(D)}{\ker(D) \cap \text{Im}(D)}.$$

Remark 0.4.27. Lemma 0.4.20 states that $\Delta(\tilde{g})\mathcal{D}\Delta(\tilde{g}^{-1}) = \det(p(\tilde{g}))\mathcal{D}$. Therefore $\ker D$ is a $\tilde{G}$ sgn-invariant subspace of $X \otimes \mathcal{S}$, similarly for $\text{Im } D \cap \ker D$. The Dirac cohomology $H^D(X)$ can be considered as a $\tilde{G}$ -module.

0.5 Closed Linear Groups

Definition 0.5.1. We denote by $GL(n, \mathbb{R})$ the group of nonsingular n by n real matrices with matrix multiplication. Similarly $GL(n, \mathbb{C})$ denotes the group of nonsingular complex matrices.

The group $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ have topologies defined by identifying them with subsets of $\mathbb{R}^{n^2}$ and $\mathbb{R}^{2n^2}$ respectively. The set of nonsingular matrices is the open subset defined by the polynomial $\det$ being non-zero.

Definition 0.5.2. A closed linear group is a closed subgroup of $GL(n, \mathbb{R})$.

Example 0.5.3. The groups $GL(n, \mathbb{C})$ and $GL(n, \mathbb{R})$ are closed linear groups. For a matrix x write x^t for its transpose and x^* for its conjugate transpose. Some more examples of closed linear groups:

$$\begin{aligned} SO(n) &= \{x \in GL(n, \mathbb{R}) : xx^t = 1, \det x = 1\}, \\ U(n) &= \{x \in GL(n, \mathbb{C}) : xx^* = 1\}, \\ SU(n) &= \{x \in GL(n, \mathbb{C}) : xx^* = 1, \det x = 1\}, \\ SL(n, \mathbb{R}) &= \{x \in GL(n, \mathbb{R}) : \det x = 1\}, \\ SL(n, \mathbb{C}) &= \{x \in GL(n, \mathbb{C}) : \det x = 1\}. \end{aligned}$$

Definition 0.5.4. A real (resp. complex) Lie algebra $\mathfrak{g}$ is a $\mathbb{R}$ (resp. $\mathbb{C}$) algebra with multiplication given by a Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that:

$$[X, X] = 0 \text{ for all } X \in \mathfrak{g},$$

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \text{ for all } X, Y, Z \in \mathfrak{g}.$$

Definition 0.5.5. Let G be a closed linear group. Define a smooth curve on G to be

$$c : \mathbb{R} \rightarrow G,$$

such that $c : \mathbb{R} \rightarrow \text{Im } c$ is continuously differentiable.

Definition 0.5.6. Let G be a closed linear group. We define the linear Lie algebra $\mathfrak{g}$ of G to be

$$\mathfrak{g} = \{c'(0) | c : \mathbb{R} \rightarrow G \text{ is a smooth curve with } c(0) = 1\}.$$

Remark 0.5.7. The set $\mathfrak{g}$ is closed under addition, scalar multiplication by $\mathbb{R}$ and becomes a real Lie algebra with Lie bracket $[X, Y] = XY - YX$ for all $X, Y \in \mathfrak{g}$.

Definition 0.5.8. For G a closed Linear group and $\mathfrak{g}$ its associated Lie algebra, the exponential function $\exp : \mathfrak{g} \rightarrow G$ is:

$$\exp(X) = e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n, \text{ for } X \in \mathfrak{g}.$$

0.6 Lie groups and Lie algebras

Assume a Lie algebra $\mathfrak{g}$ is an algebra over $\mathbb{k}$, where $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$ unless otherwise stated.

For any Lie algebra $\mathfrak{g}$ there is a linear map: $\text{ad} : \mathfrak{g} \rightarrow \text{Der}_{\mathbb{k}} \mathfrak{g}$ by

$$(\text{ad } X)(Y) = [X, Y], \text{ for all } X, Y \in \mathfrak{g}.$$

The subset of derivations of $\mathfrak{g}$ given by $\text{ad } X$ is the set of inner automorphisms $\text{Int } \mathfrak{g}$.

Definition 0.6.1. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. An ideal $\mathfrak{j} \subset \mathfrak{g}$ is a subspace such that $[\mathfrak{j}, \mathfrak{g}] \subset \mathfrak{j}$.

Example 0.6.2. Examples of Ideals:

$$\begin{aligned} Z_{\mathfrak{g}} &= \text{centre of } \mathfrak{g} = \{X : [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}, \\ [\mathfrak{g}, \mathfrak{g}] &= \text{commutator ideal}, \\ \ker \pi &\text{ where } \pi : \mathfrak{g} \rightarrow \mathfrak{h} \text{ is a Lie algebra homomorphism.} \end{aligned}$$

Definition 0.6.3. Let $\mathfrak{g}$ be a Lie algebra, the commutator series of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1] \supseteq \dots$$

A Lie algebra $\mathfrak{g}$ is solvable if $\mathfrak{g}^j = 0$ for some j .

Proposition 0.6.4. [36, Proposition 1.12] If $\mathfrak{g}$ is a finite dimensional Lie algebra then there exists a unique solvable ideal $\text{rad}(\mathfrak{g})$ containing all solvable ideals in $\mathfrak{g}$.

Definition 0.6.5. A finite dimensional Lie algebra $\mathfrak{g}$ is simple if $\mathfrak{g}$ is non-abelian and has no proper nonzero ideals. A finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if $\text{rad}(\mathfrak{g}) = 0$, i.e. it has no nonzero solvable ideals.

Definition 0.6.6. A Lie algebra $\mathfrak{g}$ is reductive if every ideal $\mathfrak{a} \subset \mathfrak{g}$ corresponds to an ideal $\mathfrak{b}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}.$$

Corollary 0.6.7. [36, Corollary 1.56] If $\mathfrak{g}$ is reductive, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z_{\mathfrak{g}}$ with $[\mathfrak{g}, \mathfrak{g}]$ semisimple and $Z_{\mathfrak{g}}$ abelian.

Example 0.6.8. We define some classical Lie algebras, which are all reductive.

$$\begin{aligned}
\mathfrak{so}(n) &= \{X \in \mathfrak{gl}(n, \mathbb{R}) : X + X^t = 0, \operatorname{Tr} X = 0\}, \\
\mathfrak{u}(n) &= \{X \in \mathfrak{gl}(n, \mathbb{C}) : X + X^* = 0\}, \\
\mathfrak{su}(n) &= \{X \in \mathfrak{gl}(n, \mathbb{C}) : X + X^* = 0, \operatorname{Tr} X = 0\}, \\
\mathfrak{sl}(n) &= \{X \in \mathfrak{gl}(n, \mathbb{C}) : \operatorname{Tr} X = 0\}, \\
\mathfrak{so}(n, \mathbb{C}) &= \{X \in \mathfrak{gl}(n, \mathbb{C}) : X + X^t = 0\}, \\
\mathfrak{sp}(n, \mathbb{C}) &= \{X \in \mathfrak{gl}(2n, \mathbb{C}) : X^t J + JX = 0\}, \\
\mathfrak{sl}(n, \mathbb{R}) &= \{X \in \mathfrak{gl}(n, \mathbb{R}) : \operatorname{Tr} X = 0\}, \\
\mathfrak{so}(m, n) &= \{X \in \mathfrak{gl}(m+n, \mathbb{R}) : X^t I_{m,n} + I_{m,n} X = 0\}.
\end{aligned}$$

Here

$$J = \begin{bmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{bmatrix}, \text{ and } I_{m,n} = \begin{bmatrix} \operatorname{Id}_m & 0 \\ 0 & \operatorname{Id}_n \end{bmatrix}.$$

The first three Lie algebras are semisimple when $n > 3$. The middle three are semisimple as complex Lie algebras, furthermore $\mathfrak{g}^{\mathbb{R}}$ is semisimple as a real Lie algebra. The final two are semisimple as real Lie algebras.

0.6.1 Field extensions and Lie algebras

In this section $\mathbb{K}$ will be a field extension of $\mathbb{k}$. In practice $\mathbb{k} = \mathbb{R} \subset \mathbb{C} = \mathbb{K}$.

Definition 0.6.9. Let V be a $\mathbb{k}$ -vector space, we denote the extension to $\mathbb{K}$:

$$V^{\mathbb{K}} = V \otimes_{\mathbb{k}} \mathbb{K}.$$

If U is a $\mathbb{K}$ -vector space then the restriction to $\mathbb{k}$, $U^{\mathbb{k}}$ is U regarded as a $\mathbb{k}$ -vector space.

Remark 0.6.10. There shouldn't be confusion between $V^{\mathbb{K}}$ and $U^{\mathbb{k}}$ since in one case the field is extended and in the other restricted. For $\mathbb{R} \subset \mathbb{C}$ the operations are not inverses:

$$(V^{\mathbb{C}})^{\mathbb{R}} = V \oplus \sqrt{-1}V.$$

Definition 0.6.11. If U is a complex vector space and V is a real vector space such that

$$U^{\mathbb{R}} = V \oplus \sqrt{-1}V,$$

then we say V is a real form of U . A real form defines conjugation on U which is the identity on V and $-\operatorname{Id}$ on $\sqrt{-1}V$.

Remark 0.6.12. Given a real Lie algebra $\mathfrak{g}_0$, the complexification of $\mathfrak{g}_0$ is $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_0^{\mathbb{C}}$. Given a complex Lie algebra $\mathfrak{g}$ a real form of $\mathfrak{g}$ is a real Lie algebra $\mathfrak{g}_0$ such that

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus \sqrt{-1}\mathfrak{g}_0.$$

Proposition 0.6.13. [36, Proposition 1.17] Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{k}$, $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}^{\mathbb{K}}$ is solvable.

Definition 0.6.14. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{k}$ the Killing form is a bilinear form on $\mathfrak{g} \times \mathfrak{g}$:

$$B(X, Y) = \text{trace}(\text{ad } X \text{ ad } Y) \in \mathbb{k}, \text{ for all } X, Y \in \mathfrak{g}.$$

Theorem 0.6.15. Cartan's criterion for semisimple Lie algebras [36, Theorem 1.45] The Lie algebra $\mathfrak{g}$ over $\mathbb{k} \subset \mathbb{C}$ is semisimple if and only if the Killing form of $\mathfrak{g}$ is nondegenerate.

Theorem 0.6.16. The Lie algebra $\mathfrak{g}$ is semisimple if and only if

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_m,$$

with $\mathfrak{g}_j$ ideals that are each simple Lie algebras.

Definition 0.6.17. Let $\mathfrak{g}$ be a complex Lie algebra then the universal enveloping algebra of $\mathfrak{g}$ is

$$U(\mathfrak{g}) = T(\mathfrak{g})/J,$$

where J is the two sided ideal generated by $X \otimes Y - Y \otimes X - [X, Y]$ for $X, Y \in \mathfrak{g}$.

Theorem 0.6.18. Poincaré-Birkhoff-Witt. Let $\{X_i\}_{i \in A}$ be a basis of $\mathfrak{g}$, with a total ordering on A , then the set of monomials

$$(X_{i_1})^{j_1} \dots (X_{i_n})^{j_n}$$

with $i_1 < \dots < i_n$ and $j_k \geq 0$, is a basis of $U(\mathfrak{g})$.

0.6.2 Compact Lie groups

Definition 0.6.19. A compact group is a topological group whose underlying topology is compact Hausdorff. A compact Lie algebra is such that it is a Lie algebra of a compact group.

Definition 0.6.20. A finite dimensional representation of a compact Lie group G is a group homomorphism ρ from G to $GL_{\mathbb{C}}(V)$, where V is a finite dimensional $\mathbb{R}$ -vector space.

A representation of a compact group G on a complex Hilbert space H is a continuous map ρ from G to $GL(H)$.

Definition 0.6.21. If V has a positive definite Hermitian form $\langle, \rangle$ then V is a unitary representation if $\rho(g)$ is unitary with respect to $\langle, \rangle$, that is:

$$\langle \rho(g)v_1, \rho(g)v_2 \rangle = \langle v_1, v_2 \rangle, \text{ for all } v_1, v_2 \in V, g \in G.$$

Theorem 0.6.22. Peter-Weyl Theorem. [36, Theorem 1.12] Let ρ be a unitary representation of a compact group G on a Hilbert space H , then H splits into direct sums of orthogonal irreducible finite dimensional unitary representations of G .

0.6.3 Split, Compact forms and Cartan involutions

In this subsection, we uniformly denote real Lie algebras by $\mathfrak{g}_0$ and drop the subscript for complex Lie algebras. Recall Remark 0.6.12, a real form of a complex Lie algebra $\mathfrak{g}$ is a real Lie algebra $\mathfrak{g}_0$ such that

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus \sqrt{-1}\mathfrak{g}_0.$$

Theorem 0.6.23. [36, Theorem 6.9] Let $\mathfrak{g}$ be a complex semisimple Lie algebra, with $\mathfrak{h}$ a Cartan subalgebra and Φ a set of roots. It is possible to choose root vectors $X_{\alpha} \in \mathfrak{g}_{\alpha}$, $\alpha \in \Phi$, such that

$$[X_{\alpha}, X_{-\alpha}] = H_{\alpha}, \text{ for all } \alpha \in \Phi,$$

$$[X_{\alpha}, X_{\beta}] = N_{\alpha, \beta} X_{\alpha + \beta} \text{ if } \alpha + \beta \in \Phi,$$

$$[X_{\alpha}, X_{\beta}] = 0 \text{ if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \notin \Phi,$$

with constants $N_{\alpha, \beta}$ such that:

$$N_{\alpha, \beta} = -N_{-\alpha, -\beta}.$$

Definition 0.6.24. Let $\mathfrak{g}$ be a complex Lie algebra, with root vectors as in Theorem 0.6.23, define $\mathfrak{h}_0 = \{H \in \mathfrak{h} : \alpha(H) \in \mathbb{R} \text{ for all } \alpha \in \Phi\}$. A real form of $\mathfrak{g}$ that contains $\mathfrak{h}_0$ is called a split real form. The real Lie algebra

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathbb{R}X_\alpha$$

is a split real form.

Definition 0.6.25. A real form of a compact Lie algebra $\mathfrak{g}$ is called a compact real form if $\mathfrak{g}_0$ is compact.

Theorem 0.6.26. [36, Theorem 6.11] If $\mathfrak{g}$ is a complex semisimple Lie algebra, then $\mathfrak{g}$ has a compact real form $\mathfrak{u}_0$

Define root vectors as in Theorem 0.6.23 then a real compact form for $\mathfrak{g}$ is

$$\mathfrak{u}_0 = \sum_{\alpha \in \Phi} \mathbb{R}\sqrt{-1}(H_\alpha) + \sum_{\alpha \in \Phi} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha \in \Phi} \mathbb{R}\sqrt{-1}(X_\alpha + X_{-\alpha}).$$

Recall Definition 0.6.14 of the Killing form $B(X, Y)$.

Definition 0.6.27. Let $\mathfrak{g}_0$ be a real Lie algebra, an involution θ on $\mathfrak{g}_0$ is a Cartan involution if the bilinear symmetric form

$$B_\theta(X, Y) = -B(X, \theta Y)$$

is positive definite.

Proposition 0.6.28. [36, Proposition 6.14] Let $\mathfrak{g}$ be a complex Lie algebra with compact real form $\mathfrak{u}_0$. Let θ be conjugation of $\mathfrak{g}$ defined by the real form $\mathfrak{u}_0$, then θ is a Cartan involution.

Corollary 0.6.29. [36, Corollary 6.18] If $\mathfrak{g}_0$ is real semisimple Lie algebra, then $\mathfrak{g}_0$ has a Cartan involution.

Definition 0.6.30. The Cartan decomposition of a real Lie algebra $\mathfrak{g}_0$ is a decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

such that the Killing form of $\mathfrak{g}_0$ is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$.

A Cartan decomposition determines a Cartan involution θ with $+1$ -eigenspace $\mathfrak{k}_0$ and -1 -eigenspace $\mathfrak{p}_0$. Similarly in the reverse, a Cartan involution determines a Cartan decomposition.

Definition 0.6.31. *Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}_0$, let θ be a fixed Cartan involution with corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, define K to be the analytic subgroup of G , $K = \exp(\mathfrak{k}_0)$, with Lie algebra $\mathfrak{k}_0$.*

Theorem 0.6.32. *[36, Theorem 6.36] Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}_0$, let θ be a fixed Cartan involution with corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, K is an analytic subgroup of G with Lie algebra $\mathfrak{k}_0$. Then*

- (a) *there exists a Lie group automorphism Θ of G with differential θ ,*
- (b) *the subgroup of G fixed by Θ is K ,*
- (c) *the mapping $K \times \mathfrak{p}_0 \rightarrow G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism onto G ,*
- (d) *K is closed,*
- (e) *K contains the centre $Z \subset G$,*
- (f) *K is compact if and only if Z is finite,*
- (g) *when Z is finite, K is a maximal compact subgroup of G .*

0.6.4 Iwasawa Decomposition

In this subsection, real Lie algebras will be denoted by $\mathfrak{g}$, if we mention a complex Lie algebra we will denote it by $\mathfrak{g}^{\mathbb{C}}$. Let G be a semisimple Lie group and let $\mathfrak{g}$ be its Lie algebra. Then $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with corresponding Cartan involution θ . Let B be the Killing form on $\mathfrak{g}$.

Definition 0.6.33. *Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. For $\lambda \in \mathfrak{a}^*$ write*

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : (\text{ad } H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\lambda \neq 0$ and $\lambda \neq 0$, we call λ a restricted root of $\mathfrak{g}$ or a root of $(\mathfrak{g}, \mathfrak{a})$. The set of restricted roots is denoted Σ . Any $\mathfrak{g}_\lambda \neq 0$ is called a restricted root space.

Proposition 0.6.34. *[36, Proposition 6.40] The restricted roots and restricted root spaces have the following properties:*

- (a) *$\mathfrak{g}$ is the orthogonal direct sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$,*
- (b) *$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$,*
- (c) *$\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$, therefore $-\lambda \in \Sigma$ for all $\lambda \in \Sigma$,*
- (d) *$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ orthogonally, where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$.*

Remark 0.6.35. In the restricted root decomposition, $\mathfrak{g}_0$ denotes the 0-eigenspace of $\text{ad } \mathfrak{a}$. Not to be confused with previous sections where $\mathfrak{g}_0$ was a real Lie algebra with complex Lie algebra $\mathfrak{g}$.

Fix an ordering of $\mathfrak{a}^*$ then we define positive restricted roots $\Sigma^+ = \{\lambda \in \Sigma : \lambda > 0\}$.

Definition 0.6.36. For a real Lie algebra $\mathfrak{g}$ define a subspace

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda.$$

The subspace $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$.

Proposition 0.6.37. Iwasawa decomposition of a Lie algebra [36, Proposition 6.43].

With notation as in Proposition 0.6.34, $\mathfrak{g}$ is a vector space direct sum

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Here $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent and $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$, $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.

Theorem 0.6.38. Iwasawa decomposition. [36, Theorem 6.46] Let G be a semisimple Lie group, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G . Let K, A, N be the analytic subgroups of G generated by $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ respectively. Then the multiplication

$$K \times A \times N \rightarrow G$$

$$(k, a, n) \mapsto kan,$$

is a diffeomorphism onto G .

0.7 Real reductive linear groups

We will denote real Lie algebras by $\mathfrak{g}_0$ and the corresponding complex Lie algebra $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathfrak{g}$.

Definition 0.7.1. By a real reductive group, we mean a real Lie group G , a maximal compact subgroup K of G , and a Cartan involution θ of $\mathfrak{g}_0$ the Lie algebra of G ,

satisfying the following conditions.

- (a) The Lie algebra $\mathfrak{g}_0$ is a real reductive Lie algebra.
- (b) For $g \in G$, $\text{Ad } g$ is an inner automorphism of the complex Lie algebra $\mathfrak{g}$.
- (c) The fixed point set of θ is $\mathfrak{k}_0$.
- (d) Write $\mathfrak{p}_0$ for the -1 eigenspace of θ ;
The map $K \times \mathfrak{p}_0 \rightarrow G, (k, X) \mapsto k \cdot \exp(X)$ is a diffeomorphism.
- (e) G has a faithful finite dimensional representation.
- (f) The centralizer H of a Cartan subalgebra $\mathfrak{h}_0$ is abelian.

Remark 0.7.2. A real reductive group need not be connected.

Definition 0.7.3. A representation of a real reductive group G is a pair (π, H) , with H a complex Hilbert space and a continuous homomorphism $\pi : G \rightarrow GL(H)$. The representation (π, H) is irreducible if $H \neq 0$ and there is no proper closed invariant subspaces. It is called unitary if the operators $\pi(g)$ are unitary for all $g \in G$.

Definition 0.7.4. Suppose K is a compact group, hence every irreducible representation is finite dimensional. Define the dual object $\hat{K}$ to be the set of equivalence classes of irreducible representations of H .

Definition 0.7.5. Let (π, H) be a representation of a real reductive group G . A vector $v \in H$ is called K -finite if

$$\dim \text{span}\{\pi(k)v : k \in K\} < \infty.$$

Set

$$H_K = \{v \in H : v \text{ is } K\text{-finite}\}.$$

Fix an irreducible K -representation (δ, V_δ) define the δ K -type of H :

$$H_K(\delta) = \bigcup_{L \in \text{Hom}_K(\delta, \pi)} L(V_\delta).$$

The G -representation (π, H) is admissible if

$$\dim H_K(\delta) < \infty, \text{ for all } \delta \in \hat{K}.$$

In this case the multiplicity of δ in π is $m(\delta, \pi) = \dim \text{Hom}_K(V_\delta, H_K)$.

Theorem 0.7.6. [28][57, Theorem 0.3.5] Let (π, H) be an admissible representation of G . If $v \in H_K$ and $X \in \mathfrak{g}_0$, the limit

$$\lim_{t \rightarrow 0} \frac{1}{t} (\pi(\exp(tX))v - v)$$

exists and call it $\pi(X)v$. Furthermore

$$\pi(X)v \in H_K$$

and this defines H_K as a representation of $\mathfrak{g}_0$.

The above $\mathfrak{g}_0$ -representation can be complexified to a representation of $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$.

Theorem 0.7.7. [28][57, Theorem 0.3.6] Any irreducible unitary representation of a real reductive group G is admissible.

Definition 0.7.8. [57, Definition 0.3.8] A $(\mathfrak{g}, K)$ -module is a pair (π, X) with X a complex vector space and π a map

$$\pi : \mathfrak{g} \cup K \rightarrow \text{End}(X)$$

satisfying:

- (a) The map $\pi|_{\mathfrak{g}}$ is a Lie algebra homomorphism, and $\pi|_K$ is a group representation.
- (b) Every vector $v \in H_K$ is K finite.
- (c) The differential of $\pi|_K$ is the restriction of π to $\mathfrak{k}_0$.
- (d) If $X \in \mathfrak{g}, k \in K$, then $\pi(\text{Ad}(k)X) = \pi(k)\pi(X)\pi(k)^{-1}$.

Definition 0.7.9. The admissible $(\mathfrak{g}, K)$ -module H_K attached to an admissible group representation (π, H) by Theorem 0.7.6 is called the Harish-Chandra module of (π, H) . We denote the category of Harish-Chandra modules by $\mathcal{HC}(G)$.

Definition 0.7.10. Let (π, X) and (π', X') be two $(\mathfrak{g}, K)$ -modules. Define

$$\text{Hom}_{\mathfrak{g}, K}(\pi, \pi') = \{L : X \rightarrow X' : L \text{ is complex linear, and } \pi' L = L \pi\}.$$

Definition 0.7.11. Two $(\mathfrak{g}, K)$ -modules (π, X) and (π', X') are equivalent if there is an invertible $(\mathfrak{g}, K)$ map $L \in \text{Hom}_{\mathfrak{g}, K}(\pi, \pi')$ between them. We say two G -representations are infinitesimally equivalent if their Harish-Chandra modules are equivalent.

Let G be a real reductive group with Lie algebra $\mathfrak{g}_0$, K its maximal compact subgroup. Let $\mathfrak{a}_0$ be a maximal abelian subalgebra of $\mathfrak{p}_0$. We form restricted roots of $\Phi = \Phi(\mathfrak{g}_0, \mathfrak{a}_0)$, and fix an ordering on $\mathfrak{a}^*$ to define positive roots Φ^+ , set $\rho = \frac{1}{2} \sum_{\lambda \in \Phi^+} \lambda$. Let $\mathfrak{n}_0 = \bigoplus_{\lambda \in \Phi^+} \mathfrak{g}_\lambda$. We have an Iwasawa decomposition of the Lie algebra

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0.$$

Let K, A, N be the corresponding Lie subgroups of G to $\mathfrak{k}_0, \mathfrak{a}_0, \mathfrak{n}_0$. Then the Iwasawa decomposition of G is

$$G = KAN.$$

Definition 0.7.12. Let G be a real reductive group, $G = KAN$ an Iwasawa decomposition of G . Set

$$M = Z_K(\mathfrak{a}_0) = \text{centraliser of } \mathfrak{a}_0 \text{ in } K.$$

The minimal parabolic subgroup of G is

$$P = MAN.$$

The subgroup A is abelian with Lie algebra $\mathfrak{a}_0$, we identify the irreducible representations of A with $\mathfrak{a}^*$ by

$$\nu(\exp X) = \exp(\nu(X)) \text{ for } X \in \mathfrak{a}_0, \nu \in \mathfrak{a}^*.$$

Definition 0.7.13. Fix a representation (δ, V_δ) of M and a character $\nu \in \mathfrak{a}^*$. Define a Hilbert space:

$$\begin{aligned} H_{\delta \otimes \nu} = \{ & f : G \rightarrow V_\delta : f \text{ is measurable, } f|_K \text{ is square integrable,} \\ & f(gman) = a^{-(\nu+\rho)} \delta(m^{-1}) f(g) \text{ for all } g \in G, m \in M, a \in A, n \in N \} \end{aligned}$$

with norm defined by

$$\|f\|^2 = \int_K \|f(k)\|^2 dk.$$

The above space $H_{\delta \otimes \nu}$ is invariant by left translation in G . We define the principal series representation with parameters δ and ν by the representation $\pi_{\delta \otimes \nu}$ of G on $H_{\delta \otimes \nu}$:

$$\pi_{\delta \otimes \nu}(g) \cdot f(x) = f(g^{-1}x), \text{ for all } g, x \in G.$$

We write $X(\delta \otimes \nu)$ for the Harish-Chandra $(\mathfrak{g}, K)$ -module of $\pi_{\delta \otimes \nu}$.

Lemma 0.7.14. [57, Lemma 4.1.5] Let G be a real reductive group with maximal compact subgroup K , let $(\delta, V_\delta) \in \hat{M}$. Set

$$H_\delta = \{f : K \rightarrow V_\delta : f \text{ is measurable, } f|_K \text{ is square integrable,} \\ f(km) = \delta(m^{-1})f(k) \text{ for all } k \in K, m \in M\},$$

we denote this space by $\text{Ind}_M^K(\delta)$. Then

$$\text{Res}_K^G(H_{\delta \otimes \nu}) \cong H_\delta$$

and

$$\text{Hom}_K(\mu, H_{\delta \otimes \nu}) = \text{Hom}_M(\mu|_M, \delta) \text{ for any } \mu \in \hat{K}.$$

Theorem 0.7.15. Harish-Chandra's subquotient theorem. [57, Theorem 4.1.9] Every irreducible admissible representation of a real reductive group G is infinitesimally equivalent to a composition factor of a principal series representation of G .

0.8 Schur-Weyl Duality

0.8.1 Schur's double centraliser result.

Let $V = \mathbb{C}^n$. The symmetric group S_k naturally acts on its k fold tensor power $V^{\otimes k}$. This action commutes with the action of $GL_n(\mathbb{C})$ acting by matrix multiplication on V . The vector space $V^{\otimes k}$ is a $GL_n(\mathbb{C}), \mathbb{C}[S_k]$ -bimodule. Denote this structure by:

$$GL_n(\mathbb{C}) \xrightarrow{\rho} \text{End}_{\mathbb{C}}(V^{\otimes k}) \xleftarrow{\sigma} \mathbb{C}[S_k].$$

Theorem 0.8.1. [51] We have the following equalities:

$$\begin{aligned} \rho(GL_n(\mathbb{C})) &= \text{End}_{S_k}(V^{\otimes k}), \\ \sigma(\mathbb{C}[S_k]) &= \text{End}_{GL_n}(V^{\otimes k}). \end{aligned}$$

Furthermore if $k \leq n$ then the representation σ is faithful.

The above result holds for the enveloping algebra of the Lie algebra $\mathfrak{gl}_n$ associated to $GL_n(\mathbb{C})$.

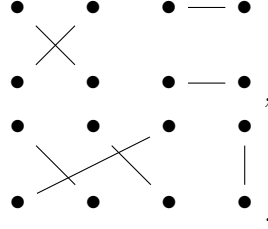
Theorem 0.8.2. Notation as in Theorem 0.8.1, replacing the representation ρ of GL_n with $d\rho : U(\mathfrak{gl}_n) \rightarrow \text{End}(V^{\otimes k})$ leads to an alternative formulation of Schur's result:

$$\begin{aligned} d\rho(\mathfrak{gl}_n) &= \text{End}_{S_k}(V^{\otimes k}), \\ \sigma(\mathbb{C}[S_k]) &= \text{End}_{\mathfrak{gl}_n}(V^{\otimes k}). \end{aligned}$$

0.8.2 Brauer Algebras

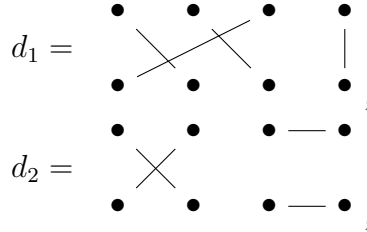
Definition 0.8.3. Define a set of diagrams such that each diagram has k top vertices and k bottom vertices with pairings represented by edges between the pair. Every vertex must be paired with exactly one other vertex.

Example 0.8.4. For $k = 4$, two examples of a diagram of pairings are:

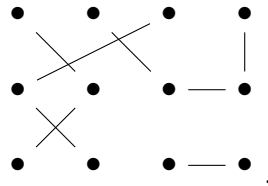


Two diagrams d_1 and d_2 are concatenated by identifying the bottom vertices of d_1 with the top vertices of d_2 . This gives rise to a new diagram which has the top vertices of d_1 and the bottom vertices of d_2 and pairings defined by the connected pairings in the concatenated diagram of $3k$ vertices formed.

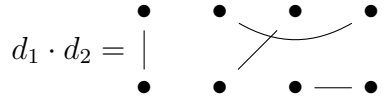
Example 0.8.5. Set



then identifying the bottom vertices of d_1 with the top vertices of d_2 :



Therefore the concatenated diagram is:

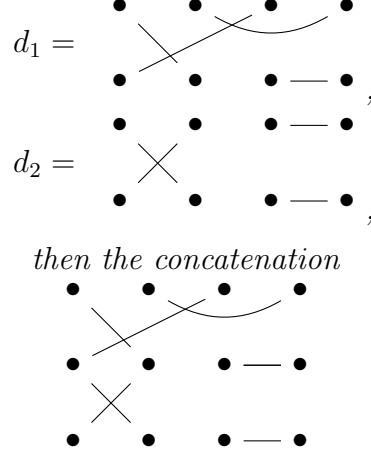


Definition 0.8.6. [8, Section 5 (a)] Let $\mathbb{k}$ be a characteristic zero field, set $k \in \mathbb{N}$ and $\delta \in \mathbb{k}$. The Brauer algebra $Br_k[\delta]$ is the associative unital $\mathbb{k}$ -algebra with basis consisting of the diagrams defined in Definition 0.8.3 of pairings of $2k$ vertices with multiplication defined to be

$$d_1 \cdot_{Br_k[\delta]} d_2 = \delta^{m(d_1, d_2)} d_1 \cdot d_2,$$

where $m(d_1, d_2)$ counts the connected components of the concatenation of d_1 and d_2 which do not appear in $d_1 \cdot d_2$.

Example 0.8.7. Set



has one connected component not occurring in the product $d_1 \cdot d_2$. Therefore the Brauer product $d_1 \cdot_{Br_4[\delta]} d_2$ would be

$$d_1 \cdot_{Br_4[\delta]} d_2 = \delta \left(\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \vdots & & & \\ \bullet & \bullet & \bullet & \bullet \end{array} \right).$$

We give an alternative definition of the Brauer algebra $Br_k[\delta]$

Definition 0.8.8. [8] The rank k Brauer algebra $Br_k[\delta]$, with parameter $\delta \in \mathbb{k}$, is the associative unital $\mathbb{k}$ -algebras generated by elements $t_{i,i+1}$ and $e_{i,i+1}$ for $i = 1, \dots, k-1$, subject to the conditions:

the subalgebra generated by $t_{i,i+1}$ is isomorphic to $\mathbb{C}[S_k]$,

$$e_{i,i+1}^2 = \delta e_{i,i+1},$$

$$t_{i,i+1} e_{i,i+1} = e_{i,i+1} t_{i,i+1} = e_{i,i+1},$$

$$t_{i,i+1} t_{i+1,i+2} e_{i,i+1} t_{i+1,i+2} t_{i,i+1} = e_{i+1,i+2},$$

$$[t_{i+1}, e_{j,j+1}] = 0 \text{ for } j \neq i, i+1.$$

Proposition 0.8.9. Definition 0.8.6 and Definition 0.8.8 define isomorphic $\mathbb{k}$ -algebras.

Proof. One defines an isomorphism on the generators

$$\begin{aligned}
t_{i,i+1} &\mapsto \begin{array}{c} \begin{array}{ccccccc} & & i & & i+1 & & \\ & \bullet & \dots & \bullet & \times & \bullet & \dots & \bullet \\ & \vdots & & & & \vdots & & \\ \bullet & & \bullet & \dots & \bullet & & \bullet & \\ & \vdots & & & & \vdots & & \\ & \bullet & \dots & \bullet & & \bullet & \dots & \bullet \end{array} \end{array}, \\
e_{i,i+1} &\mapsto \begin{array}{c} \begin{array}{ccccccc} & & i & & i+1 & & \\ & \bullet & \dots & \bullet & \text{---} & \bullet & \dots & \bullet \\ & \vdots & & & & \vdots & & \\ \bullet & & \bullet & \dots & \bullet & & \bullet & \\ & \vdots & & & & \vdots & & \\ & \bullet & \dots & \bullet & \text{---} & \bullet & \dots & \bullet \end{array} \end{array}.
\end{aligned}$$

□

Lemma 0.8.10. *If $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ or $\mathfrak{so}_{2n+1}(\mathbb{C})$, with matrix module V , then $V \otimes V$ decomposes as*

$$\Lambda^2 V \oplus S^2 V / 1 \oplus 1 \text{ for } \mathfrak{g} = \mathfrak{so}_{2n+1},$$

$$\Lambda^2 V / 1 \oplus S^2 V \oplus 1 \text{ for } \mathfrak{g} = \mathfrak{sp}_{2n}.$$

Let pr_1 be the projection of $V \otimes V$ onto the trivial submodule 1 in the decomposition above. Let $\text{pr}_{i,i+1}$ be the projection of $V_i \otimes V_{i+1}$ onto the trivial submodule of $V_i \otimes V_{i+1}$.

Given the k fold tensor $V^{\otimes k}$ as mentioned in Theorem 0.8.1 the symmetric group S_k naturally acts on this tensor. Denote by $s_{i,i+1}$ the transpositions $(i, i+1)$ in the symmetric group.

Lemma 0.8.11. [20] *Let $\mathfrak{g}$ be $\mathfrak{sp}_{2n}(\mathbb{C})$ or $\mathfrak{so}_{2n+1}(\mathbb{C})$. Let $V = \mathbb{C}^{2n}$ or $\mathbb{C}^{2n+1}$ be the defining module of $\mathfrak{g}$. Then there exists $\delta \in \mathbb{Z}$ such that there is a homomorphism $\pi : Br_k[\delta] \rightarrow \text{End}(V^{\otimes k})$:*

$$\pi(t_{i,i+1}) = s_{i,i+1},$$

$$\pi(e_{i,i+1}) = id \otimes \dots \otimes id \otimes \delta \text{pr}_{i,i+1} \otimes id \otimes \dots \otimes id.$$

For $\mathfrak{g} = \mathfrak{sp}_{2n}$ then is $\delta = -n$ and if $\mathfrak{g} = \mathfrak{so}_n$, $\delta = n$.

Theorem 0.8.12. [8, Theorem 46] *Let ρ denote the action of $SO_{2n+1}(\mathbb{C})$ or $Sp_{2n}(\mathbb{C})$ on $V^{\otimes k}$ and π denote the action of $Br_k[\delta]$ on $V^{\otimes rk}$.*

We have the following equalities:

$$\begin{aligned}
\rho(G) &= \text{End}_{Br_k}(V^{\otimes r}), \\
\pi(Br_k[\delta]) &= \text{End}_G(V^{\otimes r}).
\end{aligned}$$

Furthermore if $k \leq n$ then the representation π is faithful.

Corollary 0.8.13. *Let $d\rho$ denote the action of $U(\mathfrak{so}_{2n+1}(\mathbb{C}))$ or $U(\mathfrak{sp}_{2n}(\mathbb{C}))$ on $V^{\otimes k}$ and π denote the action of $Br_k[\delta]$ on $V^{\otimes k}$.*

We have the following equalities:

$$\begin{aligned} d\rho(U(\mathfrak{g})) &= \text{End}_{Br_k}(V^{\otimes k}), \\ \pi(Br_k[\delta]) &= \text{End}_{\mathfrak{g}}(V^{\otimes k}). \end{aligned}$$

Chapter 1

Dirac cohomology, the projective supermodules of the symmetric group and the Vogan morphism

Abstract

We derive an explicit description of the genuine projective representations of the symmetric group S_n using Dirac cohomology and the branching graph for the irreducible genuine projective representations of S_n . In [15] Ciubotaru and He, using the extended Dirac index, showed that the characters of the projective representations of S_n are related to the characters of graded elliptic modules. We derive the branching graph using Dirac theory and combinatorics relating to the cohomology of Borel varieties $\mathcal{B}_e$ of $\mathfrak{g}$. We use Dirac cohomology to construct an explicit model for the projective representations. We also describe Vogan's morphism for Hecke algebras in type A using spectrum data of the Jucys-Murphy elements.

1.1 Introduction

Schur (cf. [54]) initially described the characters of the projective representations of the symmetric group. Nazarov [43] produced an orthogonal form for the symmetric group [44]. Okounkov and Vershik [45] developed an approach using Jucys-Murphy elements. This approach was later applied to the projective representations of S_n by Vershik and Sergeev [56] although there appears to be a flaw with their calculation of the spectrum data. We describe the action of the Jucys-Murphy elements using Dirac cohomology, providing an alternative proof for the spectrum calculation. We then describe the genuine projective representations, giving the action of $\tilde{S}_n$ by matrices.

Barbasch, Ciubotaru, and Trapa [4] developed Dirac cohomology for graded Hecke algebras. Ciubotaru [13] defined Dirac cohomology for symplectic reflection algebras. Ciubotaru and He [15] introduced extended Dirac cohomology which relates the tempered modules of the graded Hecke algebra with the irreducible representations of $\tilde{W}$. Notably in [15] the functor taking modules to their extended Dirac index is exact. Combining this with combinatorial results from Garcia and Procesi [26] on the restriction of the cohomology groups of the Lie algebra $\mathfrak{g}$ associated to S_n we are able to deduce the branching graph for the irreducible genuine projective representation of S_n .

Theorem 1.4.9. *The branching rules of the genuine projective irreducible $\tilde{S}_n$ -supermodules are:*

$$Res_{n-1}^n \tilde{\tau}_\lambda = \begin{cases} 2\tilde{\tau}_{\lambda^{(1)}} \oplus 2\tilde{\tau}_{\lambda^{(2)}} \oplus \cdots \oplus 2\tilde{\tau}_{\lambda^{(r-1)}} \oplus \tilde{\tau}_{\lambda^{(r)}} & \text{if } n-r \text{ is odd and } \lambda_r = 1, \\ 2\tilde{\tau}_{\lambda^{(1)}} \oplus 2\tilde{\tau}_{\lambda^{(2)}} \oplus \cdots \oplus 2\tilde{\tau}_{\lambda^{(r-1)}} \oplus 2\tilde{\tau}_{\lambda^{(r)}} & \text{if } n-r \text{ is odd and } \lambda_r > 1, \\ \tilde{\tau}_{\lambda^{(1)}} \oplus \tilde{\tau}_{\lambda^{(2)}} \oplus \cdots \oplus \tilde{\tau}_{\lambda^{(r-1)}} \oplus \tilde{\tau}_{\lambda^{(r)}} & \text{if } n-r \text{ is even.} \end{cases}$$

Here λ is always a strict partition, $\lambda^{(i)}$ depends on a reduction rule, defined in [26], which gives partitions $\lambda^{(i)}$ from λ .

We prove that the Casimir element $\Omega_{\tilde{S}_n}$ introduced in [4] for $\tilde{S}_n$ acts by the second power polynomial in the Jucys-Murphy elements.

Lemma 1.5.12. *On genuine projective representations, $\Omega_{\tilde{S}_n} = \frac{1}{2} \sum_{i=1}^n M_k^2$. Here M_k denotes the Jucys-Murphy elements for $\tilde{S}_n$.*

In section 1.5, we combine this lemma, our branching graph result and a result from [4] to prove inductively that the spectrum data for $\tilde{S}_n$ on projective representations is equivalent to content data on shifted Young diagrams. Therefore we give a different proof of the result claimed in [56], utilising Dirac cohomology and combinatorics from Garcia and Procesi [26].

Theorem 1.5.13. *The set $\text{Spec}(n) \subset \mathbb{N}^n$, descending from eigenvalues of Jucys-Murphy elements, is equal to the set $\text{Scont}(n)$, a combinatorial construction from contents of shifted Young tableaux. Furthermore,*

$$\text{Spec}(n, \lambda) = \text{Scont}(n, \lambda).$$

In section 1.6 we describe how one gets from the spectrum data to an explicit description of the genuine projective supermodules of $\tilde{S}_n$ and the action of $\tilde{S}_n$ in matrix form. In Section 1.7, we use the results from Section 1.5 on the action of the Jucys-Murphy elements to give an explicit description of the Vogan morphism for graded Hecke algebras of type A introduced in [4].

1.2 Definitions

We fix a real root system $\{V, \Phi, V^\vee, \Phi^\vee\}$, where Φ is the set of roots inside the real vector space V , we have a perfect bilinear pairing $B(,) : V \times V^\vee \rightarrow \mathbb{R}$. The roots are in bijection with the coroots such that $B(\alpha, \alpha^\vee) = 2$. We define reflections

$$s_\alpha : V \rightarrow V, \quad s_\alpha(v) = v - B(v, \alpha^\vee)\alpha.$$

Let W be the subgroup of $GL(V)$ generated by the reflections s_α for all $\alpha \in \Phi$. Fix a W -invariant bilinear form $(,)$ on V , a set of positive (resp. simple) roots Φ^+ (resp. Π). The reflections s_α for $\alpha \in \Pi$ generate W .

Definition 1.2.1. *We let δ denote the automorphism of W given by the conjugation by w_0 on V , where w_0 is the longest element in W . Then let $W_\#$ be the extension of W by δ . Explicitly*

$$W_\# = \langle W, \delta : \delta^2 = 1, \delta w \delta = \delta(w) \text{ for all } w \in W \rangle.$$

Example 1.2.2. Let W be the symmetric group S_n . The roots live in the permutation representation $V = \text{span}_{\mathbb{C}}\{\xi_1, \dots, \xi_n\}$ and we fix the simple roots $\Pi = \{\alpha_i = \xi_i - \xi_{i+1} : i = 1, \dots, n-1\}$, and positive roots $\Phi^+ = \{\xi_i - \xi_j : i < j\}$. The bilinear form is $(\xi_i, \xi_j) = \delta_{ij}$. We have the presentation;

$$S_n = \langle s_1, \dots, s_{n-1} : (s_i s_j)^{m(i,j)} = 1 \rangle.$$

Here $m(i, i) = 1$ and $m(i, j) = (\alpha_i, \alpha_j)(\alpha_j, \alpha_i) + 2$ if $i \neq j$ [42]. The automorphism δ comes from the action of $-w_0$ on Φ hence $\delta(\alpha_i) = \alpha_{n-i}$, and $\delta(s_\alpha) = s_{\delta(\alpha)}$.

Definition 1.2.3. The graded Hecke algebra $\mathbb{H}$ associated to a Weyl group W , V and $(\cdot, \cdot)$ is the associative unital $\mathbb{C}$ -algebra generated by $\{\xi \in V\}$ and $\{w \in W\}$ such that there exist injections, induced by the inclusion of generators,

$$S(V) \hookrightarrow \mathbb{H},$$

$$\mathbb{C}[W] \hookrightarrow \mathbb{H}.$$

Furthermore, for every simple root $\alpha \in \Pi$, there exist cross-relations,

$$\xi \cdot s_\alpha - s_\alpha \cdot s_\alpha(\xi) = (\alpha, \xi).$$

Note that since we are interested in S_n we do not introduce parameters for $\mathbb{H}$. The automorphism δ (Definition 1.2.1) is an involution of W defined by conjugation by w_0 . We extend δ to $\mathbb{H}(W)$ as conjugation by w_0 on all of $\mathbb{H}$. We define $\mathbb{H}_\#$ to be an extension of the graded Hecke algebra by δ . Explicitly,

$$\mathbb{H}_\# = \langle \mathbb{H}, \delta \rangle = \langle \mathbb{H}, \delta : \delta^2 = 1, \delta h \delta = \delta(h) \text{ for all } h \in \mathbb{H} \rangle.$$

As a vector space $\mathbb{H}_\#$ is isomorphic to $\mathbb{H} \oplus \mathbb{H}$.

Recall Definition 0.4.1 of the Clifford algebra $C(V)$ with relations:

$$\xi \xi' + \xi' \xi = 2(\xi', \xi) \text{ for all } \xi, \xi' \in V.$$

Note that this differs from some sources, [4] in particular, where the Clifford algebra is defined with a negative form. The theory is identical over $\mathbb{C}$ as one can just multiply the generators by $\sqrt{-1}$.

Example 1.2.4. For S_n , the vector space V has the Euclidean norm and is of dimension n . In this case we denote the Clifford algebra for the rank n case by $C(n)$.

The Clifford algebra has a natural filtration by $\mathbb{Z}$. Each generator is given degree one and then $C(V)_n$ is the span of all elements of $C(V)$ with degree n or lower. $C(V)$ also has a $\mathbb{Z}/2\mathbb{Z}$ grading,

$$C(V) = C(V)_{\text{even}} \oplus C(V)_{\text{odd}}$$

where $C(V)_{\text{even}}$ is the span of all homogeneous elements of even degree and likewise for $C(V)_{\text{odd}}$. Define the transpose of $C(V)$ to be the antiautomorphism defined by $\xi^t = -\xi$ for all elements of V and let $\epsilon : C(V) \rightarrow C(V)$ be the involution which is the identity on $C(V)_{\text{even}}$ and acts as -1 on $C(V)_{\text{odd}}$.

Recall Definition 0.4.4:

$$\text{Pin}(V) = \{g \in C(V)^\times \mid g^t = g^{-1} \text{ and } \epsilon(g)\xi g^{-1} \in V, \text{ for all } \xi \in V\}.$$

This group is a double cover of the orthogonal group on V with projection $p : \text{Pin}(V) \rightarrow O(V)$,

$$p(g)(\xi) = \epsilon(g)\xi g^{-1}.$$

Since the Weyl group is a subgroup of $O(V)$ we can find a double cover of W in the Pin group. Let $\tilde{W}$ be the preimage of W via p ,

$$\tilde{W} = p^{-1}(W).$$

Theorem 1.2.5. [42, 3.2] *The double cover $\tilde{W}$ of W admits a presentation akin to Coxeter presentations for a Weyl group,*

$$\tilde{W} = \langle z, \tilde{s}_\alpha \text{ for all } \alpha \in \Pi : z^2 = 1, (\tilde{s}_\alpha \tilde{s}_\beta)^{m(\alpha, \beta)} = z^{m(\alpha, \beta)-1}, z \text{ is central} \rangle$$

where $m(\alpha, \alpha) = 1$ and $m(\alpha, \beta) = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle + 2$.

Remark 1.2.6. *Morris also showed that one can define a presentation of $\tilde{W}$ with generators for each positive root [42, 3.2],*

$$\tilde{W} = \langle z, \tilde{s}_\alpha \text{ for all } \alpha \in \Phi^+ : \tilde{s}_\alpha^2 = 1, \tilde{s}_\beta \tilde{s}_\alpha \tilde{s}_\beta = z \tilde{s}_\gamma, \gamma = s_\alpha(\beta), z^2 = 1, z \text{ is central} \rangle.$$

As a subgroup of $O(V)$, we have $W_\# = \langle W, \delta \rangle$ is equal to $\langle W, -1 \rangle$. Hence we can also define $\tilde{W}_\# = p^{-1}(W_\#)$. The non scalar element in the preimage of -1 is ω which can be formulated in terms of an orthonormal basis $\{\xi_1, \dots, \xi_n\}$ of V , as

$$\omega = \xi_1 \xi_2 \dots \xi_n \in \text{Pin}(V).$$

We define $\tilde{W}'$ and $\tilde{W}'_{\#}$ to be

$$\tilde{W}' = \begin{cases} \tilde{W} \cap C(V)_{\text{even}}, & \dim V \text{ even}, \\ \tilde{W}, & \dim V \text{ odd}, \end{cases}$$

and

$$\tilde{W}'_{\#} = \begin{cases} \langle \tilde{W} \cap C(V)_{\text{even}}, \omega \rangle, & \dim V \text{ even}, \\ \tilde{W}_{\#}, & \dim V \text{ odd}. \end{cases}$$

In Section 1.3 we utilise the extended Dirac index, introduced by Ciubotaru and He [15]. In [15] it was crucial that the spinor module considered had a positive and negative part. Since [15] considers ungraded modules, in the even dimensional case, the ungraded spinor module does not have a positive and negative part. Hence to manufacture a module with a positive and negative part, one must restrict the groups $\tilde{W}$ and $\tilde{W}_{\#}$. This is why we have introduced $\tilde{W}'$ and $\tilde{W}'_{\#}$. However, we show in Corollary 1.3.5 that when one considers supermodules, one no longer needs to restrict $\tilde{W}$. In the supertheory case, the $C(V)$ -module $\mathcal{S}$ always has a positive and negative part. We are interested in the representation theory of $\tilde{S}_n$.

Definition 1.2.7. [55, p. 303] Let $\mathcal{T}_n$ be the twisted group algebra of S_n over $\mathbb{C}$. Explicitly $\mathcal{T}_n$ is the unital associative $\mathbb{C}$ -algebra generated by τ_i for $i = 1 \dots n-1$ such that

$$\tau_i^2 = 1, (\tau_i \tau_j)^{m(i,j)} = (-1)^{m(i,j)-1}.$$

Remark 1.2.8. Following the presentation from Theorem 1.2.5 we could label the generators with the simple roots, $\alpha \in \Pi$.

Then $\mathcal{T}_n$ becomes the unital associative $\mathbb{C}$ -algebra with presentation:

$$\langle \tau_{\alpha}, \alpha \in \Pi : (\tau_{\alpha} \tau_{\beta})^{m(\alpha,\beta)} = (-1)^{m(\alpha,\beta)-1} \rangle.$$

Finally we have

$$\mathcal{T}_n = \langle \tau_{\alpha} \text{ for all } \alpha \in \Phi^+ : \tau_{\alpha}^2 = 1, \tau_{\alpha} \tau_{\beta} \tau_{\alpha} = -\tau_{\gamma}, \gamma = s_{\alpha}(\beta) \rangle.$$

By the above presentation, similarly to the symmetric group, one has for each positive root, $\alpha \in \Phi^+$, a ‘pseudo-transposition’ τ_{α} . Note that this τ_{α} is equal to the ‘transposition’ $[ij]$ defined in [9, 3.1]. We switch between labellings of the generators whenever it clarifies the explanation.

The genuine projective irreducible representations of $\tilde{S}_n$ are the irreducible representations that do not descend to a representation of S_n . The group algebra for $\tilde{S}_n$ decomposes as

$$\mathbb{C}[\tilde{S}_n] \cong \mathbb{C}[\tilde{S}_n]/(z-1) \oplus \mathbb{C}[\tilde{S}_n]/(z+1) \cong \mathbb{C}[S_n] \oplus \mathcal{T}_n.$$

If we are interested in the projective representations of $\tilde{S}_n$, i.e. those that do not occur as representations of S_n , then we just need to study the representations of $\mathcal{T}_n$.

Since the supertheory for $\tilde{S}_n$ is cleaner than that of the ungraded representation theory, we focus on the supermodules of $\tilde{S}_n$. The Clifford algebra is a superalgebra, by considering the generators to have degree one. Similarly $\mathcal{T}_n$ is a superalgebra. Again, the generators all have degree one.

Recall the definition 0.3.4 of supermodules and supermodule homomorphisms. When one forgets the grading, all irreducible supermodules are either still irreducible or decompose into two irreducible parts. It is possible to recover the original nongraded representation theory from the super representation theory if we keep track of whether each irreducible supermodule is reducible as an ungraded module.

Definition 1.2.9. *Let U be an A -supermodule. Then U is of type M if, when one forgets the grading, U is an irreducible A ungraded module. The module U is of type Q if it is reducible as an A ungraded module.*

As an ungraded algebra, the Clifford algebra $C(V)$ has either one or two isomorphism classes of irreducible modules, depending on $\dim(V)$. However when we consider supermodules, $C(V)$ is supersimple; it always has exactly one simple supermodule, $\mathcal{S}$ [35, 12.2.4]. When $\dim(V)$ is odd, then $\mathcal{S}$ is of type Q and when $\dim(V)$ is even the supermodule is of type M.

The supertrace on the representation $U = U_0 \oplus U_1$ is defined to be zero on odd elements and, on even elements,

$$\text{str}_V(a) = \text{tr}_0(a) - \text{tr}_1(a)$$

where $\text{tr}_i(a)$ is the trace of the matrix corresponding to a restricted to U_i .

We use three different types of partitions throughout. Let $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{N}^r$. We define λ to be a partition of n if $\sum_{i=1}^r \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$. We denote the set of partitions of n by $\mathcal{P}(n)$. We define the length of a partition, $l(\lambda)$, to be $\#\{\lambda_i : \lambda_i > 0\}$. We say a partition $\lambda \vdash n$ is strict if $\lambda_1 > \lambda_2 > \dots > \lambda_r$.

Definition 1.2.10. For a partition $\lambda \vdash n$ we associate to it a set of boxes forming the Young diagram. For $\lambda = (\lambda_1, \dots, \lambda_r)$, we define the Young diagram to be

$$\lambda = \{(i, j) : 1 \leq i \leq l(\lambda) \text{ and } 1 \leq j \leq \lambda_i\}.$$

Here (i, j) denotes a cell in the i^{th} row and j^{th} column.

Definition 1.2.11. The set of shifted partitions of n is the same as the set of strict partitions of n . However, we consider a different Young diagram for a shifted partition than the associated Young diagram of a normal partition. We write $\lambda \models n$ to denote a shifted partition of n and $\mathcal{SP}(n)$ is the set of shifted partitions of n .

Definition 1.2.12. Given a shifted partition $\lambda = (\lambda_1, \dots, \lambda_r) \models n$ we associate a shifted Young diagram,

$$\lambda = \{(i, j) : 1 \leq i \leq l(\lambda) \text{ and } i \leq j \leq i + \lambda_i - 1\}.$$

A Young tableau of shape λ is a numbering of 1 to n of the Young diagram associated to λ . We write $[\lambda]$ for a Young tableau of shape λ . Similarly $[[\lambda]]$ denotes a numbering of the shifted Young diagram associated to $\lambda \models n$. A standard Young tableau (resp. shifted) is a Young tableau (resp. shifted) such that along each row and column the numbers increase.

Example 1.2.13. The partition $(3, 1) \vdash 4$ has three standard Young tableaux;

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}.$$

When $(3, 1)$ is considered as a shifted partition of 4 it has two standard shifted Young tableaux;

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline & 4 & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline & 3 & \\ \hline \end{array}.$$

Given a partition λ we write λ' for its conjugate partition, which is the reflection of the Young diagram in its main diagonal.

Example 1.2.14. If $\lambda = (3, 1)$ then $\lambda' = (2, 1, 1)$,

$$\lambda = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \lambda' = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}.$$

1.3 Extended Dirac cohomology

In [15] Ciubotaru and He introduced a notion of extended Dirac cohomology as a variation of Dirac cohomology defined in [4]. An important feature is that the functor from an $\mathbb{H}_\#$ -module to its extended Dirac cohomology is exact. Using supertheory, we show the extended Dirac cohomology could be considered as the supertheory analogue of Dirac cohomology.

Let ρ be the diagonal embedding of $\tilde{W}_\#$ into $\mathbb{H}_\# \otimes C(V)$;

$$\rho(w) = p(w) \otimes w, \text{ for all } w \in \tilde{W}_\#.$$

Hence, for an $\mathbb{H}_\#$ -module X and a spinor module $\mathcal{S}$, we can consider $X \otimes \mathcal{S}$ as a $\tilde{W}_\#$ -module via the map ρ .

Recall w_0 is the longest element in W . In our case, with S_n and our choice of generators, w_0 is the element which takes $(1, 2, \dots, n)$ to $(n, n-1, \dots, 1)$.

Definition 1.3.1. *Let $*$ be the linear anti-automorphism on $\mathbb{H}$ defined by*

$$w^* = w^{-1}, \xi^* = w_0 \delta(\xi) w_0, \text{ for all } w \in W \text{ and } \xi \in V.$$

We can extend $*$ to $\mathbb{H}_\#$ by defining it to fix δ . Given ξ , let

$$\tilde{\xi} = \frac{1}{2}(\xi - \xi^*).$$

Definition 1.3.2. *Let $\{\xi_1, \dots, \xi_n\}$ be an orthonormal basis of V . The Dirac element $\mathcal{D}_\#$ in $\mathbb{H}_\# \otimes C(V)$ is defined to be,*

$$\mathcal{D}_\# = \sum_{i=1}^n \tilde{\xi}_i \otimes \xi_i.$$

This is independent of the choice of orthonormal basis. For any $\mathbb{H}_\#$ -module X and $C(V)$ -module $\mathcal{S}$ the Dirac element $\mathcal{D}_\#$ defines an operator $D_\# : X \otimes \mathcal{S} \rightarrow X \otimes \mathcal{S}$.

As a superalgebra we consider $\mathbb{H}_\#$ to be concentrated in even degree and $C(V)$ to have its usual grading. With this grading $\mathbb{H}_\# \otimes C(V)$ is a superalgebra. Furthermore, since $\mathcal{D}_\#$ is odd, the operator $D_\#$ interchanges the even and odd spaces of $X \otimes \mathcal{S}$,

$$D_\#^\pm : X \otimes \mathcal{S}^\pm \rightarrow X \otimes \mathcal{S}^\mp.$$

As a supermodule we consider X to be concentrated in degree 0 and $\mathcal{S}$ splits into homogeneous parts $\mathcal{S}^+, \mathcal{S}^-$. Therefore $X \otimes \mathcal{S}^+$ (resp. $X \otimes \mathcal{S}^-$) is the even (resp. odd) space of $X \otimes \mathcal{S}$.

The extended Dirac cohomology of X (with respect to $\mathcal{S}$) is

$$H_{\#}^D(X) = \ker(D_{\#}) / \ker(D_{\#}) \cap \operatorname{Im}(D_{\#}).$$

This is a $\tilde{W}_{\#}$ -representation since $\tilde{W}_{\#}$ sign-commutes with $\mathcal{D}$. We are also interested in the restriction of $D_{\#}$ to the even and odd spaces. The subspace $X \otimes \mathcal{S}^{\pm}$ is always a $\tilde{W}'_{\#}$ -module. Let,

$$H_{\#}^{D^{\pm}}(X) = \ker(D_{\#}^{\pm}) / \ker(D_{\#}^{\pm}) \cap \operatorname{Im}(D_{\#}^{\mp}).$$

Note that when $\dim V$ is even $X \otimes \mathcal{S}^{\pm}$ is not an $\mathbb{H}_{\#} \otimes C(V)$ -module, just an $\mathbb{H}_{\#} \otimes C(V)_{\text{even}}$ -module. The space $H_{\#}^{D^{\pm}}(X)$ is a $\tilde{W}'_{\#}$ -module.

Definition 1.3.3. *The extended Dirac index is*

$$I_{\#}(X) = H_{\#}^{D^{+}}(X) - H_{\#}^{D^{-}}(X)$$

defined as a virtual $\tilde{W}'_{\#}$ -module.

Proposition 1.3.4. *[15, 5.10] Let X be an $\mathbb{H}_{\#}$ -module. The Dirac index can be expressed as a tensor, $I_{\#}(X) = X \otimes (\mathcal{S}^{+} - \mathcal{S}^{-})$. Hence it is exact. Furthermore for $\tilde{w} \in \tilde{W}'$,*

$$\operatorname{tr}(\tilde{w}, I_{\#}(X)) = \operatorname{tr}(p(\tilde{w}), X) \operatorname{tr}(\tilde{w}, \mathcal{S}^{+} - \mathcal{S}^{-}),$$

$$\operatorname{tr}(\tilde{w}\omega, I_{\#}(X)) = \operatorname{tr}(p(\tilde{w})w_0\delta, X) \operatorname{tr}(\tilde{w}, \mathcal{S}).$$

Note in Proposition 1.3.4 the spinor modules $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$ are ungraded, in the following corollary $\mathcal{S}$ is the $\mathbb{Z}_2$ graded spinor module.

We make the observation that one does not need to split $\tilde{W}_{\#}$ when $\dim V$ is even if one considers the Grothendieck group of supermodules. This leads to a more elegant formulation of Proposition 1.3.4 using supertheory.

Corollary 1.3.5. *Let X be an $\mathbb{H}_{\#}$ -supermodule, concentrated in degree zero, and $\mathcal{S}$ be a $C(V)$ -supermodule. We endow $I_{\#}(X)$ with a superspace structure with even part $H_{\#}^{D^{+}}(X)$ and odd part $H_{\#}^{D^{-}}(X)$. Then $I_{\#}(X)$ is a $\tilde{W}_{\#}$ -supermodule and in the Grothendieck group,*

$$I_{\#}(X) = X \otimes \mathcal{S}.$$

In particular:

$$\operatorname{str}(\tilde{w}, I_{\#}(X)) = \operatorname{str}(p(\tilde{w}), X) \operatorname{str}(\tilde{w}, \mathcal{S}) \text{ for all } \tilde{w} \in \tilde{W}'_{\#}.$$

Proof. From Proposition 1.3.4 we can describe the trace of $w \in \tilde{W}$ on $I_{\#}(X)$ as a product of the trace on X and the difference of the trace on $\mathcal{S}^+$ and $\mathcal{S}^-$. However this is just the product of the supertrace of the modules X and $\mathcal{S}$;

$$\mathrm{tr}(w, I_{\#}(X)) = \mathrm{tr}(p(w), X) \mathrm{tr}(w, S^+ - S^-) = \mathrm{str}(p(w), X) \mathrm{str}(w, S).$$

□

In light of this corollary, one could consider the extended Dirac index as the Dirac supercohomology. When one considers supermodules, as opposed to ungraded modules, the extended Dirac cohomology is the same as considering the supertheory analogue of the original Dirac cohomology defined in [4].

Introducing a certain graded module will be useful. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Weyl group W . Then, for an element e in the nilpotent cone $\mathcal{N}$, we let $\mathcal{B}_e$ denote the variety of Borel subalgebras of $\mathfrak{g}$ containing e . Springer [53] defined a W action on the cohomology groups $H^j(\mathcal{B}_e)$. The cohomology $H^j(\mathcal{B}_e)$ vanishes unless j is even.

Definition 1.3.6. *Let e be an element in the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$, $\mathcal{B}_e$ the variety of Borel subalgebras containing e and d_e be the dimension of $\mathcal{B}_e$. The sign character of W is denoted by sgn . Let $\mathbf{q}$ be a formal symbol. Springer [53] constructed an action of W on $H^j(\mathcal{B}_e)$. Define the $\mathbf{q}$ -graded W -module.*

$$X_{\mathbf{q}}(e) = \sum_{i \geq 0} \mathbf{q}^{d_e - i} H^{2i}(\mathcal{B}_e) \otimes \mathrm{sgn}.$$

From here on we fix W to be S_n and $\mathfrak{g} = \mathfrak{sl}_n$. Because of this, we take results from [15] and restrict them to S_n to avoid surplus definitions. Note that the correspondence from [15, Appendix A], applies to almost any Weyl group, but outside of type A there is an extra complication involving the component group of the centraliser of e . In the full generality of [15] the graded modules involved are labelled by a nilpotent element e and a representation of this component group. However, for S_n this component group is always trivial. The modules $H^j(\mathcal{B}_e)$ and $H^j(\mathcal{B}_{e'})$ are isomorphic if and only if e and e' are in the same conjugacy class.

Remark 1.3.7. *Two $\mathbf{q}$ -graded S_n -modules $X_{\mathbf{q}}(e)$ and $X_{\mathbf{q}}(e')$ are isomorphic if and only if e is in the same conjugacy class as e' . Hence the isomorphism classes of $X_{\mathbf{q}}(e)$ can be labelled by nilpotent orbits, these are equivalent to partitions of n . We write*

the isomorphism class of $X_q(e)$ as $X_q(\lambda)$ where e has Jordan form corresponding to a partition λ .

We can specialize $X_q(e)$ to a $\mathbb{Z}/2\mathbb{Z}$ -graded module:

$$X_{-1}(e) = \sum_{i \geq 0} (-1)^{d_e - i} H^{2i}(\mathcal{B}_e) \otimes \text{sgn}.$$

The W -module $X_{-1}(e)$ can be extended to several different $\mathbb{H}_\#$ -modules. See [15, Section 5] for more details, the $\mathbb{H}_\#$ -modules are denoted $E_{e,h}$. All of these extensions are isomorphic to $X_{-1}(e)$ as a W -module. We fix an extension of $X_{-1}(e)$ to $\mathbb{H}_\#$ such that as ungraded modules $X_{-1}(e) \cong X_\lambda$ (Definition 1.3.15), the standard module parabolically induced from the Steinberg module. We will denote this graded $\mathbb{H}_\#$ module by $X_{-1}(\lambda)$.

Proposition 1.3.8. [15, A.3] *For every strict partition λ of n , let S be an ungraded spinor module and $X_{-1}(\lambda)$ be the graded module of S_n corresponding to λ . Define the $\tilde{S}_n$ -module,*

$$\tilde{\tau}_\lambda = \frac{1}{a_\lambda} X_{-1}(\lambda) \otimes S,$$

where

$$a_\lambda = \begin{cases} 2^{\frac{l(\lambda)}{2}} & n \text{ even and } l(\lambda) \text{ even,} \\ 2^{\frac{l(\lambda)-1}{2}} & l(\lambda) \text{ odd,} \\ 2^{\frac{l(\lambda)-2}{2}} & n \text{ odd and } l(\lambda) \text{ even.} \end{cases}$$

If $n - l(\lambda)$ is even then $\tilde{\tau}_\lambda$ is an irreducible sgn self-dual $\tilde{S}_n$ -module. If $n - l(\lambda)$ is odd then $\tilde{\tau}_\lambda = \tilde{\tau}_\lambda^+ + \tilde{\tau}_\lambda^-$, with $\tilde{\tau}_\lambda^\pm$ irreducible modules which are sgn dual to each other.

Corollary 1.3.9. *For a strict partition λ , when one lets $\mathcal{S}$ be a super simple spinor module, $\tilde{\tau}_\lambda$ is always an irreducible $\tilde{S}_n$ -supermodule. If $n - l(\lambda)$ is even then $\tilde{\tau}_\lambda$ is type M and if $n - l(\lambda)$ is odd then $\tilde{\tau}_\lambda$ is of type Q .*

Remark 1.3.10. *The extended Dirac index of $X_{-1}(\lambda)$, $I_\#(X_{-1}(\lambda))$, is non-zero if and only if λ is strict. This follows from [15, 6.1] and the fact that nilpotent elements that are δ quasi-distinguished have Jordan form associated to a strict partition.*

For ease of explanation later on in this chapter, and motivated by Remark 1.3.10, we set $\tilde{\tau}_\lambda = 0$ if λ is not a strict partition.

Finally, when looking at the action of the Jucys-Murphy elements in Section 1.4 we will need a result about the action of the Casimir elements on the Dirac cohomology.

Definition 1.3.11. *The Casimir element for $\mathbb{H}$ is*

$$\Omega_{\mathbb{H}} = \sum_{i=1}^n \xi_i^2,$$

where $\{\xi_i\}$ is an orthonormal basis for V .

The element $\Omega_{\mathbb{H}}$ is independent of the choice of orthonormal basis, it is central in $\mathbb{H}$ [4, 2.4]. Hence it acts by a scalar for any irreducible representation of X . Furthermore, let (π, X) be an irreducible $\mathbb{H}$ -module. It has an associated central character χ_{ν} which can be defined in terms of $\nu \in \mathbb{H}$. It is shown in [4, 2.5] that

$$\pi(\Omega_{\mathbb{H}}) = (\nu, \nu).$$

Definition 1.3.12. [4, 3.4] *The Casimir element for $\tilde{S}_n$ is*

$$\Omega_{\tilde{S}_n} = \frac{1}{4} \sum_{\substack{\alpha, \beta \in \Phi^+ \\ s_{\alpha}(\beta) < 0}} |\alpha| |\beta| \tau_{\alpha} \tau_{\beta},$$

where τ_{α} is the generator of $\tilde{S}_n$ corresponding to α . In [4] $\Omega_{\tilde{S}}$ is negative but we define it as positive since we have the positive form on $C(V)$.

The Casimir $\Omega_{\tilde{S}_n}$ is central in $\mathbb{C}[\tilde{S}_n]$.

Theorem 1.3.13. [4, 3.5] *As elements of $\mathbb{H} \otimes C(V)$,*

$$\mathcal{D}^2 = \Omega_{\mathbb{H}} \otimes 1 - \rho(\Omega_{\tilde{S}_n}).$$

Recall ρ is the diagonal embedding of $\tilde{W}$ into $\mathbb{H} \otimes C(V)$. Note that this statement has a parity difference from the one in [4], but this is due to the sign difference of the Clifford algebra in Definition 0.4.1.

Definition 1.3.14. [14] *We introduce the one dimensional $\mathbb{H}$ -module, called the Steinberg module St . Here S_n acts on St by the sgn character and $\xi_i \in S(V)$ act by $\frac{-n+1+2i}{2}$.*

Definition 1.3.15. *Let X_{λ} be the parabolically induced $\mathbb{H}$ -module from the Steinberg module for $\mathbb{H}_{\lambda_1} \times \dots \times \mathbb{H}_{\lambda_k}$,*

$$X_{\lambda} = \text{Ind}_{\mathbb{H}_{\lambda_1} \times \dots \times \mathbb{H}_{\lambda_k}}^{\mathbb{H}} (St_{\lambda_1} \otimes \dots \otimes St_{\lambda_k}).$$

In [4, 5.8] it is shown that, inside the Dirac cohomology of X_λ , one can find the genuine projective irreducible representation V_λ of isomorphism class associated to a shifted partition $\lambda \models n$. Since $H_D(X)$ is a quotient of $\ker(D)$, D^2 acts by zero on $H_D(X)$.

Corollary 1.3.16. *On the Dirac cohomology (resp. extended Dirac index), or any submodule found inside it, namely V_λ (resp. $\tilde{\tau}_\lambda$),*

$$\Omega_{\mathbb{H}} \otimes 1 = \rho(\Omega_{\tilde{W}}).$$

Proof. This follows from D^2 acting by zero and the equation given in Theorem 1.3.13. $\square$

1.4 Branching graph for $\tilde{S}_n$

In section 1.5 we need the branching graph of $\tilde{S}_n$. Mainly, we need to understand the modules that occur in the restriction of the irreducible representations $\tilde{\tau}_\lambda$. In this section we provide arguments for the whole branching graph of the genuine projective representations of $\tilde{S}_n$, this is the branching graph of $\mathcal{T}_n$. We derive this graph from Theorem [15, A.4] and a branching result from Garcia and Procesi [26] on certain graded S_n -modules.

We know we can find $\tilde{\tau}_\lambda$ in $\mathcal{S} \otimes X_{-1}(\lambda)$. Tensoring with spinor modules is exact, and since the restriction rules of spinor modules are straightforward, all that is left to understand is the restriction of $X_{-1}(\lambda)$.

Garcia and Procesi [26] studied a very similar graded module. It is the module $X_{\mathbf{q}}(\lambda)$ but with the reverse $\mathbf{q}$ grading.

Definition 1.4.1. [26, I.7] *Let $p^\lambda(\mathbf{q})$ be the character of the graded S_n -module*

$$\sum_{i \geq 0} \mathbf{q}^i H^{2i}(\mathcal{B}_\lambda).$$

In [26] a reduction rule for partitions is defined. Given a partition $\lambda \vdash n$ this rule outputs a set, potentially with multiples, of partitions $\lambda^{(i)} \vdash n - 1$.

Definition 1.4.2. [26, 1.1] *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a fixed partition of n and $\lambda' = (\lambda'_1, \dots, \lambda'_r)$ be its conjugate.*

For $1 \leq i \leq l(\lambda)$ define the integer a_i by the condition

$$\lambda'_{a_i} \geq i > \lambda'_{a_i+1}.$$

Given this integer, let $\lambda^{(i)}$ be the partition created when one removes a block from the bottom of column a_i of the Young diagram associated to λ .

Note that here the inequality differs from [26] but this is due to the fact that Garcia and Procesi define their partitions to increase.

Given $\lambda \vdash n$, let $\Theta_{n-1}^n(\lambda)$ be the set of partitions of $n-1$ which one can create from the Young diagram of λ by removing a single box.

Lemma 1.4.3. *For a strict partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ we have*

$$\lambda^{(i)} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_r).$$

Hence $\{\lambda^{(i)} : i = 1, \dots, l(\lambda)\}$ is equal to the set $\Theta_{n-1}^n(\lambda)$.

Proof. Because λ is strict, the column of λ which is first larger than i , λ'_{a_i} , will always be the column that has a block from $\lambda_1, \dots, \lambda_i$ and no others. So removing a block from row λ_i is equivalent to removing a block from column λ'_{a_i} . Hence $\lambda^{(i)} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_r)$. The second statement follows since we have an explicit description of $\{\lambda^{(i)} : i = 1, \dots, l(\lambda)\}$. $\square$

Given a module or character A of either S_n , $\tilde{S}_n$ or $C(n)$, let $\text{Res}_{n-1}^n(A)$ denote the restriction to the rank $(n-1)$ object, that is the restriction to S_{n-1} , $\tilde{S}_{n-1}$ or $C(n-1)$.

Theorem 1.4.4. [26, 3.3] *The restriction of the S_n $\mathbf{q}$ -graded character $p^\lambda(\mathbf{q})$ to S_{n-1} is a sum of $\mathbf{q}^{i-1}p^{\lambda^{(i)}}(\mathbf{q})$;*

$$\text{Res}_{n-1}^n p^\lambda(\mathbf{q}) = \sum_{i=1}^{l(\lambda)} \mathbf{q}^{i-1} p^{\lambda^{(i)}}(\mathbf{q}).$$

Remark 1.4.5. *If we let $\chi_{\mathbf{q}}(\lambda)$ be the character of $X_{\mathbf{q}}(\lambda)$ then*

$$\chi_{\mathbf{q}}(\lambda) = p^\lambda\left(\frac{1}{\mathbf{q}}\right) \mathbf{q}^{n(\lambda)}$$

where $n(\lambda) = \sum_{i=1}^{l(\lambda)} (i-1)\lambda_i$.

We can combine Remark 1.4.5 with Theorem 1.4.4 to understand the branching rules for $X_{\mathbf{q}}(\lambda)$.

Lemma 1.4.6. *Let λ be a strict partition. The restriction of the S_n -module $X_{\mathbf{q}}(\lambda)$ to S_{n-1} is*

$$\text{Res}_{n-1}^n X_{\mathbf{q}}(\lambda) = \bigoplus_{i=1}^{l(\lambda)} \mathbf{q}^{2i-2} X_{\mathbf{q}}(\lambda^{(i)}).$$

Proof. By Remark 1.4.5 one can write $\chi_{\mathbf{q}}(\lambda) = p^\lambda(\frac{1}{\mathbf{q}})\mathbf{q}^{n(\lambda)}$. Then, we can use the restriction rules given by Theorem 1.4.4 to restrict $\chi_{\mathbf{q}}(\lambda)$ in terms of the characters $p^{\lambda^{(i)}}(\mathbf{q})$. Remark 1.4.5 can be used again to rewrite this in terms of characters $\chi_{\mathbf{q}}(\lambda^{(i)})$. The coefficient one gets is $\mathbf{q}^{i-1+n(\lambda)-n(\lambda^{(i)})}$. Writing out the definition of $n(\lambda)$;

$$n(\lambda) - n(\lambda^{(i)}) = \sum_{j=1}^l (\lambda)(j-1)\lambda_j - \sum_{j=1}^{l(\lambda^{(i)})} (j-1)\lambda_j^{(i)}.$$

However, by Lemma 1.4.3, λ and $\lambda^{(i)}$ differ by only one entry for λ strict. This difference is in the i^{th} entry. Therefore $n(\lambda)$ and $n(\lambda^{(i)})$ differ by $i-1$. Hence $i-1+n(\lambda)-n(\lambda^{(i)}) = 2i-2$. $\square$

Lemma 1.4.7. *Let λ be a strict partition. The restriction of the graded S_n -module $X_{-1}(\lambda)$ is,*

$$\text{Res}_{n-1}^n X_{-1}(\lambda) = \bigoplus_{i=1}^{l(\lambda)} X_{-1}(\lambda^{(i)}).$$

Proof. This is the specialisation of Lemma 1.4.6 to $\mathbf{q} = -1$. $\square$

Recall the definition of $\tilde{\tau}_\lambda$ [4, A.3];

$$\tilde{\tau}_\lambda = \frac{1}{a_\lambda} X_{-1}(\lambda) \otimes \mathcal{S}.$$

The set $\{\tilde{\tau}_\lambda : \lambda \text{ is strict}\}$ is a transversal for the irreducible genuine projective supermodules of S_n .

Remark 1.3.10 states that tensoring with the spinor kills $X_{-1}(\mu)$ if μ is not strict. So we can describe which modules occur in the restriction of $\tilde{\tau}_\lambda$. Since $a_\lambda \tilde{\tau}_\lambda = \mathcal{S} \otimes X_{-1}(\lambda)$

Lemma 1.4.8. *Let λ be a strict partition. For a fixed i , if $\lambda^{(i)}$ is strict then $\tilde{\tau}_\lambda^{(i)}$ is a summand of $\text{Res}_{n-1}^n(\tilde{\tau}_\lambda)$.*

Proof. We note that $a_\lambda \tilde{\tau}_\lambda = X_{-1}(\lambda) \otimes \mathcal{S}$. Here, $\mathcal{S}$ (resp. $\mathcal{S}'$) is the spinor supermodule for the Clifford algebra $C(n)$ (resp. $C(n-1)$). Hence restricting $a_\lambda \tilde{\tau}_\lambda$ to $\tilde{S}_{n-1}$ is equivalent to restricting $X_{-1}(\lambda)$ to S_{n-1} and $\mathcal{S}$ to $C(n-1)$. Therefore

$$\begin{aligned}
a_\lambda \text{Res}_{n-1}^n \tilde{\tau}_\lambda &= \text{Res}_{n-1}^n \mathcal{S} \otimes \text{Res}_{n-1}^n X_{-1}(\lambda) \\
&= c\mathcal{S}' \otimes \bigoplus_{i=1}^{l(\lambda)} X_{-1}(\lambda^{(i)}). \\
&= \bigoplus_{\lambda^{(i)} \text{ which are strict}} c\mathcal{S}' \otimes X_{-1}(\lambda^{(i)}).
\end{aligned}$$

Here, $c \in \mathbb{Z} \setminus \{0\}$ and will be determined in Theorem 1.4.9. Hence, for every strict $\lambda^{(i)}$ the $\tilde{S}_n$ supermodule $\tilde{\tau}_{\lambda^{(i)}}$ occurs as a summand. $\square$

With the information we have, it is possible to describe the explicit branching graph for the supermodules of $\mathcal{T}_n$ and hence the branching graph for genuine projective supermodules of $\tilde{S}_n$. This result could be deduced from the crystal graph theory in [35].

Recall that if λ is not a strict partition we set $\tilde{\tau}_\lambda = 0$.

Theorem 1.4.9. *The branching rules of the genuine projective irreducible $\tilde{S}_n$ -supermodules are:*

$$\text{Res}_{n-1}^n \tilde{\tau}_\lambda = \begin{cases} 2\tilde{\tau}_{\lambda^{(1)}} \oplus 2\tilde{\tau}_{\lambda^{(2)}} \oplus \cdots \oplus 2\tilde{\tau}_{\lambda^{(r-1)}} \oplus \tilde{\tau}_{\lambda^{(r)}} & \text{if } n-r \text{ is odd and } \lambda_r = 1, \\ 2\tilde{\tau}_{\lambda^{(1)}} \oplus 2\tilde{\tau}_{\lambda^{(2)}} \oplus \cdots \oplus 2\tilde{\tau}_{\lambda^{(r-1)}} \oplus 2\tilde{\tau}_{\lambda^{(r)}} & \text{if } n-r \text{ is odd and } \lambda_r > 1, \\ \tilde{\tau}_{\lambda^{(1)}} \oplus \tilde{\tau}_{\lambda^{(2)}} \oplus \cdots \oplus \tilde{\tau}_{\lambda^{(r-1)}} \oplus \tilde{\tau}_{\lambda^{(r)}} & \text{if } n-r \text{ is even.} \end{cases}$$

Here λ is always a strict partition, $\lambda^{(i)}$ depends on a reduction rule, defined in [26], which gives partitions $\lambda^{(i)}$ from λ .

Proof. By Lemma 1.4.7 we already know which summands occur. Hence calculating the multiplicities is all that is required. We know the multiplicities of the restriction of the spinor $\mathcal{S}$; 2 if n is even, 1 otherwise. Also, by Lemma 1.4.7 the multiplicities in the restriction of $X_{-1}(\lambda)$ are always 1. Hence, we can find the multiplicities for $\tilde{\tau}_\lambda$ by comparing $\frac{1}{a_\lambda}$ and $\frac{1}{a_{\lambda^{(i)}}}$ from Proposition 1.3.8. This is a simple calculation on the

eight cases

$$\frac{1}{a_\lambda} = \begin{cases} \frac{1}{2} \frac{1}{a_{\lambda^{(i)}}} & n \text{ even, } l(\lambda) \text{ even and } l(\lambda^{(i)}) = l(\lambda), \\ \frac{1}{2} \frac{1}{a_{\lambda^{(i)}}} & n \text{ even, } l(\lambda) \text{ even and } l(\lambda^{(i)}) = l(\lambda) - 1, \\ \frac{1}{a_{\lambda^{(i)}}} & n \text{ even, } l(\lambda) \text{ odd and } l(\lambda^{(i)}) = l(\lambda), \\ \frac{1}{2} \frac{1}{a_{\lambda^{(i)}}} & n \text{ even, } l(\lambda) \text{ odd and } l(\lambda^{(i)}) = l(\lambda) - 1, \\ 2 \frac{1}{a_{\lambda^{(i)}}} & n \text{ odd, } l(\lambda) \text{ even and } l(\lambda^{(i)}) = l(\lambda), \\ \frac{1}{a_{\lambda^{(i)}}} & n \text{ odd, } l(\lambda) \text{ even and } l(\lambda^{(i)}) = l(\lambda) - 1, \\ \frac{1}{a_{\lambda^{(i)}}} & n \text{ odd, } l(\lambda) \text{ odd and } l(\lambda^{(i)}) = l(\lambda), \\ \frac{1}{a_{\lambda^{(i)}}} & n \text{ odd, } l(\lambda) \text{ odd and } l(\lambda^{(i)}) = l(\lambda) - 1. \end{cases}$$

Note that $l(\lambda^{(i)}) = l(\lambda) - 1$ occurs once if and only if $\lambda_r = 1$, in which case it is the partition $\lambda^{(r)}$ which has shorter length. $\square$

1.5 Spectrum data for $\tilde{S}_n$

In Section 1.6 we construct the genuine projective representations. However, the raw information that we need to do this is the action of the Jucys-Murphy elements squared; this is what we call the spectrum data. We prove that this is equivalent to a function on the contents of the Young tableaux for λ .

As described before $\mathcal{T}_n$ is generated as an associative algebra by τ_α for $\alpha \in \Phi^+$.

Definition 1.5.1. [9, 3.1] *The Jucys-Murphy elements in $\mathcal{T}_n$ for $i = 1, \dots, n$, are:*

$$M_i = \sum_{\alpha \in \Phi^+ : (\alpha, \xi_i) < 0} \tau_\alpha.$$

This is the same construction as the Jucys-Murphy elements for S_n . We have just replaced transpositions with pseudo-transpositions.

Note that Brundan and Kleschev define the Jucys-Murphy elements with different terminology. They use the generators τ_i and replace τ_α with $[ij]$ when $\alpha = \xi_i - \xi_j$ and then define $M_i = \sum_{j < i} [ij]$. These Jucys-Murphy elements are the same just with different labelling of generators.

Remark 1.5.2. *The Jucy-Murphy elements anti-commute [9, 3.1], that is*

$$M_k M_l = -M_l M_k \text{ if } k \neq l.$$

Lemma 1.5.3. [9, 3.2] *The even centre $Z(\mathcal{T}_n)_0$ of $\mathcal{T}_n$ is spanned by the set of symmetric polynomials of the squares of the Jucys-Murphy elements.*

Schur (cf. [54]) defined all of the genuine projective characters for $\tilde{S}_n$ and showed that these correspond to the set of shifted partitions $\lambda \in \mathcal{SP}(n)$.

Definition 1.5.4. *Let V_λ denote any genuine projective $\tilde{S}_n$ -module which has character corresponding to $\lambda \in \mathcal{SP}(n)$.*

It is useful, for notation, to introduce a function $q : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $q(m) = m(m+1)$.

Definition 1.5.5. [56] *We say $\alpha = (\alpha_1, \dots, \alpha_n)$ is in $\text{Sspec}(n)$ if there exists a vector v in an irreducible genuine projective representation V of $\tilde{S}_n$ such that:*

$$M_k^2 v = \frac{1}{2} q(\alpha_k) v \text{ for all } k = 1, \dots, n.$$

$\text{Sspec}(n, \lambda)$ is the restriction of $\text{Sspec}(n)$ by considering vectors in any V_λ , for a fixed $\lambda \in \mathcal{SP}(n)$.

Definition 1.5.6. *Let $\lambda \in \mathcal{SP}(n)$ be a shifted partition. The content of a box (i, j) contained in a shifted Young diagram λ is its distance from the main diagonal.*

$$\text{cont}(i, j) = j - i$$

Definition 1.5.7. *We introduce the set of shifted content vectors $\text{Scont}(n)$. A vector $\beta = (\beta_1, \dots, \beta_n)$ is associated to a standard shifted tableau $[[\lambda]]$ if, for all $i = 1, \dots, n$, β_i is equal to the content of the box labelled i in $[[\lambda]]$. $\text{Scont}(n)$ is the set of vectors which are associated to a standard shifted tableau of size n . Similarly $\text{Scont}(n, \lambda)$, is the set of vectors associated to standard shifted tableaux of shape λ .*

In this section, our goal is to prove that

$$\text{Sspec}(n) = \text{Scont}(n)$$

and $\text{Sspec}(n, \lambda) = \text{Scont}(n, \lambda)$.

Recall the Casimir elements for $\mathbb{H}$ and $\tilde{S}_n$, $\Omega_{\mathbb{H}}$ and $\Omega_{\tilde{S}_n}$ respectively. Our technique for describing the set $\text{Sspec}(n)$ is largely based on using two different descriptions of the action of the Casimir element $\Omega_{\tilde{S}_n}$. The first one, descending from Dirac cohomology,

states that $\Omega_{\tilde{S}_n} = \Omega_{\mathbb{H}}$ on $\tilde{\tau}_\lambda$. The second is linked to the Jucy-Murphy elements. Using these descriptions we are able to show $\text{Spec}(n) = \text{Scont}(n)$ inductively.

Recall the definition of X_λ ;

$$X_\lambda = \text{Ind}_{\mathbb{H}_{\lambda_1} \times \dots \times \mathbb{H}_{\lambda_k}}^{\mathbb{H}} (\text{St}_{\lambda_1} \boxtimes \dots \boxtimes \text{St}_{\lambda_k}).$$

A central character is a map $Z(\mathbb{H}) = S(V)^W \rightarrow \mathbb{C}$. The standard representations X_λ have central characters χ_λ . The vector space $\text{Hom}(S(V)^W, \mathbb{C})$ can be associated with V^*/W via evaluating polynomials in $S(V)/W$ at an element in V^*/W . Hence a central character, χ , corresponds to an element $\nu \in V^*/W$.

Lemma 1.5.8. *Let St_m be the Steinberg module for $\mathbb{H}_m$. Let ν_{St_m} be the element in $S(V)^*/W$ corresponding to χ_{St_m} . Let $e_1, \dots, e_m \in V^*$ be a dual basis of the basis $x_1, \dots, x_m$ of V . The element ν_{St_m} is*

$$\nu_{\text{St}_m} = -\frac{m-1}{2}e_1 + -\frac{m-3}{2}e_2 + \dots + \frac{m-3}{2}e_{m-1} + \frac{m-1}{2}e_m = \sum_{i=1}^m \frac{2i-m}{2}e_i.$$

Corollary 1.5.9. *The element ν_λ defining the central character of X_λ is*

$$\nu = \sum_{i=1}^{l(\lambda)} \nu_{\text{St}_{\lambda_i}}.$$

Lemma 1.5.10. *[4, 2.5] On the representation X_λ of the graded Hecke algebra $\mathbb{H}$,*

$$\Omega_{\mathbb{H}} = (\nu, \nu).$$

Lemma 1.5.11. *On the representation $\tilde{\tau}_\lambda$ corresponding to $\lambda = (\lambda_1, \dots, \lambda_r)$,*

$$\Omega_{\mathbb{H}} = \sum_{i=1}^{l(\lambda)} \sum_{j=1}^{\lambda_i} \left(\frac{2j - \lambda_i}{2} \right)^2.$$

Furthermore, this can be reformulated in terms of the content of λ ,

$$\Omega_{\mathbb{H}} = \frac{1}{4} \sum_{(i,j) \in \lambda} q(\text{cont}(i, j)).$$

Proof. The first part can be proved by noticing that, when one forgets the grading, the module $X_{-1}(\lambda)$ is just the parabolically induced module X_λ , see [7]. Hence, one can use Lemma 1.5.10 to show how $\Omega_{\mathbb{H}}$ acts. The constant (ν, ν) is exactly the sum we stated. The second part follows by a simple induction. One can show $\sum_{j=1}^{\lambda_i} (2j - \lambda_i)^2 = \sum_{j=1}^{\lambda_i} j(j+1) = \sum_{j=1}^{\lambda_k} q(\text{cont}(k, j))$, splitting the cases when λ_i is odd or even. This is covered in more detail in Section 1.7. $\square$

Lemma 1.5.12. *On genuine projective representations, $\Omega_{\tilde{S}_n} = \frac{1}{2} \sum_{i=1}^n M_k^2$. Here M_k denotes the Jucys-Murphy elements for $\tilde{S}_n$.*

Proof.

$$\begin{aligned} \Omega_{\tilde{S}_n} &= \frac{1}{4} \sum_{\substack{\alpha, \beta \in \Phi^+ \\ s_\alpha(\beta) < 0}} |\alpha| |\beta| \tau_\alpha \tau_\beta, \\ &= \frac{1}{4} \sum_{\alpha, \beta \in \Phi^+} |\alpha| |\beta| \tau_\alpha \tau_\beta = \frac{1}{4} \left(\sum_{\alpha \in \Phi^+} |\alpha| \tau_\alpha \right)^2, \\ &= \frac{1}{4} \left(\sum_{k=1}^n |\alpha| M_k \right)^2 = \frac{1}{2} \sum_{i=1}^n M_k^2. \end{aligned}$$

In the second equality, we use $\tau_\alpha \tau_\beta = -\tau_\beta \tau_{s_\alpha(\beta)}$ for α and β such that $s_\alpha(\beta) > 0$. The third equality uses the fact that every $\sum_{i=1}^n M_k = \sum_{\alpha \in \Phi^+} \tau_\alpha$ and the last equality uses the fact that the Jucys-Murphy elements anti-commute in $\mathcal{T}_n$ [9, 3.1]. Hence the square of the sum is equal to the sum of the squares. $\square$

Theorem 1.5.13. *The set $\text{Sspec}(n) \subset \mathbb{N}^n$, descending from eigenvalues of Jucys-Murphy elements, is equal to the set $\text{Scont}(n)$, a combinatorial construction from contents of shifted Young tableaux. Furthermore,*

$$\text{Sspec}(n, \lambda) = \text{Scont}(n, \lambda).$$

Proof. We prove the theorem by induction, the base case being trivial. Suppose for every shifted partition $\mu \in \mathcal{SP}(n-1)$, $\text{Scont}(n-1, \mu) = \text{Sspec}(n-1, \mu)$. Let us fix a shifted partition λ of n , and consider $\tilde{\tau}_\lambda$ the irreducible representation of $\mathcal{T}_n$ associated to λ .

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \text{Sspec}(n, \lambda)$. By definition there exists a vector $v_\alpha \in \tilde{\tau}_\lambda$ such that $M_j^2(v_\alpha) = q(\alpha_j)v_\alpha$ for all j .

The restriction of $\tilde{\tau}_\lambda$ to $\mathcal{T}_{n-1}$ is the direct sum, $\bigoplus_{i=1}^{l(\lambda)} \tilde{\tau}_{\lambda^{(i)}}$. Also v_α is an eigenvector for the Jucy-Murphy elements corresponding to $(\alpha_1, \dots, \alpha_n)$, when restricted to $\mathcal{T}_{n-1}$ v_α corresponds to $(\alpha_1, \dots, \alpha_{n-1})$. We know, by the inductive hypothesis, $(\alpha_1, \dots, \alpha_{n-1}) \in \text{Sspec}(n-1, \lambda^{(j)}) = \text{Scont}(n-1, \lambda^{(j)})$. Explicitly, this means there exists a standard numbering $[[\lambda^{(j)}]]$ of $\lambda^{(j)}$ which corresponds to $(\alpha_1, \dots, \alpha_{n-1})$. Therefore $(\alpha_1, \dots, \alpha_{n-1})$ is in one shape namely $\lambda^{(j)}$. Hence $v_\alpha \in \tilde{\tau}_{\lambda^{(j)}} \subset \tilde{\tau}_\lambda$ for some fixed j .

Now using Lemma 1.5.11, Lemma 1.5.12 and Theorem 1.3.13, we can describe the action of the Casimir elements of $\tilde{S}_n$ and $\tilde{S}_{n-1}$ on v_α . Since v_α is in $\tilde{\tau}_\lambda$,

$$\frac{1}{2} \sum_{i=1}^n M_k^2 v_\alpha = \Omega_{\tilde{S}_n} v_\alpha = \Omega_{\mathbb{H}_n} v_\alpha = \frac{1}{4} \sum_{(i,j) \in \lambda} \text{cont}(i,j)(\text{cont}(i,j) + 1) v_\alpha.$$

However v_α is also contained in $\tilde{\tau}_{\lambda^{(j)}} \subset \tilde{\tau}_\lambda$ so,

$$\frac{1}{2} \sum_{i=1}^{n-1} M_k^2 v_\alpha = \Omega_{\tilde{S}_{n-1}} v_\alpha = \Omega_{\mathbb{H}_{n-1}} v_\alpha = \frac{1}{4} \sum_{(i,j) \in \lambda^{(j)}} \text{cont}(i,j)(\text{cont}(i,j) + 1) v_\alpha.$$

The difference of these two Casimir elements is $\frac{1}{2} M_n^2$. Hence

$$\begin{aligned} M_n^2 v_\alpha &= \left(\frac{1}{2} \sum_{(i,j) \in \lambda} \text{cont}(i,j)(\text{cont}(i,j) + 1) - \frac{1}{2} \sum_{(i,j) \in \lambda^{(j)}} \text{cont}(i,j)(\text{cont}(i,j) + 1) \right) v_\alpha \\ &= \frac{1}{2} \text{cont}(k,l)(\text{cont}(k,l) + 1) v_\alpha, \end{aligned}$$

where (k,l) is the block of λ not included in $\lambda^{(i)}$. Taking the standard shifted tableau $[[\lambda^{(j)}]]_{(\alpha_1, \dots, \alpha_{n-1})}$ associated to $(\alpha_1, \dots, \alpha_{n-1})$ and adding the block (k,l) labelled with the number n creates a standard shifted tableau $[[\lambda]]_\alpha$ of shape λ . By the above argument this standard shifted tableau has content vector equal to $\alpha \in \text{Sspec}(n, \lambda)$. Hence $\alpha \in \text{Scont}(n, \lambda)$.

For the reverse direction, we use an almost identical argument. Suppose $\beta = (\beta_1, \dots, \beta_n) \in \text{Scont}(n, \lambda)$ corresponds to a standard shifted tableau $[[\lambda]]_\beta$. By restriction and inductive hypothesis, there exists a vector $v_{(\beta_1, \dots, \beta_{n-1})} \in \tilde{\tau}_{\lambda^{(j)}} \subset \tilde{\tau}_\lambda$ such that $M_k^2 v_{(\beta_1, \dots, \beta_{n-1})} = q(\beta_k) v_{(\beta_1, \dots, \beta_{n-1})}$. Again using the fact that M_n^2 acts by the difference of the Casimir elements Ω_{S_n} and $\Omega_{S_{n-1}}$, we get,

$$M_n^2 v_{(\beta_1, \dots, \beta_{n-1})} = 2(\Omega_{\tilde{S}_n} - \Omega_{\tilde{S}_{n-1}}) v_{(\beta_1, \dots, \beta_{n-1})} = \frac{1}{2} q(\text{cont}(n)) v_{(\beta_1, \dots, \beta_{n-1})},$$

where $\text{cont}(n)$ is the content of the box labelled by n in $[[\lambda]]_\beta$. Hence $\beta \in \text{Spec}(n, \lambda)$. $\square$

1.6 Explicit representation from spectrum data

This is, in essence, the same as the last part of [56]. We describe how to explicitly construct the genuine irreducible representations from the spectrum data we calculated in the previous section.

Let $\langle M_1, \dots, M_n \rangle$ be the subalgebra of $\mathbb{C}[\tilde{S}_n]$ generated by the Jucys-Murphy elements M_i .

Lemma 1.6.1. *Let V be a genuine irreducible projective representation of $\tilde{S}_n$. Then the restriction of V to $\langle M_1, \dots, M_n \rangle$ is a direct sum of the common eigenspaces of M_i^2 .*

Proof. The M_i^2 all commute so we can separate V into its common eigenspaces. Now M_i fix these spaces so the common eigenspaces are $\langle M_1, \dots, M_n \rangle$ submodules of V . $\square$

Lemma 1.6.2. *Let U be an $\langle M_1, \dots, M_n \rangle$ -supermodule which occurs as a common eigenspace of M_i^2 in some V . Then the representation U of $\langle M_1, \dots, M_n \rangle$ factors through the Clifford algebra of rank r , where $r = |\{M_i^2 \text{ with non zero eigenvalue}\}| = n - l(\lambda)$.*

Proof. The set of Jucys-Murphy elements anti-commute and on U , by Definition 1.5.5, $M_i^2 = q(\alpha_i)$ for some $\alpha = (\alpha_1, \dots, \alpha_n) \in \text{Sspec}(n)$. By sending M_i to $\sqrt{q(\alpha_i)}c_i$ for $\alpha_i \neq 0$, where c_i are the Clifford generators for the Clifford algebra, the action of $\langle M_1, \dots, M_n \rangle$ on U factors through the Clifford algebra of rank $\#\{\alpha_i \neq 0\}$. $\square$

Corollary 1.6.3. *Let $\tilde{\tau}_\lambda$ be the genuine irreducible supermodule of $\tilde{S}_n$, associated to λ . Then the common eigenspaces for M_i^2 are labelled by the spectrum data $\alpha \in \text{Sspec}(n, \lambda)$. Hence, when considered as a $\langle M_1, \dots, M_n \rangle$ -supermodule,*

$$\tilde{\tau}_\lambda = \bigoplus_{\alpha \in \text{Sspec}(n, \lambda)} U_\alpha.$$

If we take any genuine projective irreducible representation V of $\tilde{S}_n$ then when we restrict it to the subalgebra $\langle M_1, \dots, M_n \rangle$, generated by the Jucys-Murphy elements, V decomposes as the eigenspaces of M_i^2 . Furthermore these eigenspaces are irreducible $\langle M_1, \dots, M_n \rangle$ -modules which can be considered as spinor modules for the Clifford algebra of rank $n - l(\lambda)$.

We build a genuine irreducible representation V_λ of $\tilde{S}_n$ which is isomorphic to $\tilde{\tau}_\lambda$. Since the set $\text{Sspec}(n)$ is a subset of $\mathbb{N}^n$ then S_n naturally acts on this set, further,

S_n preserves $\text{Spec}(n, \lambda)$. We understand how M_i act on an irreducible $\mathcal{T}_n$ -module corresponding to λ . Motivated by studying how τ_i acts on $\tilde{\tau}_\lambda$, we will add in the action of τ_i on V_λ .

Lemma 1.6.4. *Using the decomposition in Corollary 1.6.3, the subspace $(\tau_i - \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})})U_\alpha$ of $\tilde{\tau}_\lambda$ is fixed by $\langle M_j \rangle$ and*

$$\left(\tau_i - \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})} \right) U_\alpha = U_{s_i(\alpha)}.$$

Proof. One can see that for every $u \in U_\alpha$,

$$M_j \left(\tau_i - \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})} \right) u = - \left(\tau_i - \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})} \right) M_{s_i(j)} u.$$

Therefore we can conclude that $\left(\tau_i - \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})} \right) U_\alpha$ is the subspace we labelled $U_{s_i(\alpha)}$. Note that if $s_i(\alpha) \notin \text{Spec}(n, \lambda)$ then $\left(\tau_i - \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})} \right) U_\alpha = 0$ hence, in this case, $\tau_i = \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})}$ on U_α . $\square$

Definition 1.6.5. *If α and $s_i(\alpha)$ are in $\text{Spec}(n, \lambda)$ then define $P_{s_i} : V_\alpha \rightarrow V_{s_i(\alpha)}$ to be the vector space isomorphism such that*

$$M_{s_i(j)}^2(P_{s_i}(v)) = P_{s_i} M_j v \text{ for all } v \in V_\alpha.$$

Note that P_{s_i} is not an $\langle M_1, \dots, M_n \rangle$ isomorphism since it interchanges the action of different Jucys-Murphy elements. Now we can describe V_λ as a $\tilde{S}_n$ -module.

Definition 1.6.6. *Let $V_\lambda = \sum_{\alpha \in \text{Spec}(n, \lambda)} V_\alpha$ as supermodules of $\langle M_1, \dots, M_n \rangle \subset \tilde{S}_n$. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \text{Spec}(n, \lambda)$. The action of τ_i on V_α is defined in the following way. If $s_i(\alpha) \in \text{Spec}(n)$ then,*

$$\tau_i = \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})} - \left(1 - \frac{q(a_i) + q(a_{i+1})}{(q(a_i) - q(a_{i+1}))^2} \right)^{-\frac{1}{2}} P_{s_i}.$$

If $s_i(\alpha) \notin \text{Spec}(n, \lambda)$ then $(1 - \frac{q(a_i) + q(a_{i+1})}{(q(a_i) - q(a_{i+1}))^2}) = 0$ and on V_α ,

$$\tau_i = \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})}.$$

Theorem 1.6.7. *For the set of shifted tableaux $\mathcal{SP}(n)$, $\{V_\lambda : \lambda \in \mathcal{SP}(n)\}$ is a full set of representatives of the irreducible genuine projective supermodules of $\tilde{S}_n$.*

Proof. All we need to show is that the supermodules V_λ are isomorphic to $\tilde{\tau}_\lambda$. Corollary 1.6.3 shows that they are isomorphic as $\langle M_1, \dots, M_n \rangle$ -supermodules. The following arguments show that the action of τ_i agree on both V_λ and $\tilde{\tau}_\lambda$. If $s_i(\alpha) \notin \text{Spec}(n, \lambda)$ then Lemma 1.6.4 shows that τ_i acts on both supermodules identically. In the case that $(\tau_i - \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})})$ interchanges V_α and $V_{s_i(\alpha)}$ one observes, that applying this operator twice gives

$$\left(\tau_i - \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})} \right)^2 = \left(1 - \frac{q(a_i) + q(a_{i+1})}{(q(a_i) - q(a_{i+1}))^2} \right) \text{Id}$$

on V_α . Let $p = 1 - \frac{q(a_i) + q(a_{i+1})}{(q(a_i) - q(a_{i+1}))^2}$. Furthermore, again by Lemma 1.6.4

$$\tau_i - \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})} = -\frac{1}{\sqrt{p}} P_{s_i}.$$

Rearranging this equation gives the result,

$$\tau_i = \frac{M_i - M_{i+1}}{q(a_i) - q(a_{i+1})} - \left(1 - \frac{q(a_i) + q(a_{i+1})}{(q(a_i) - q(a_{i+1}))^2} \right)^{-\frac{1}{2}} P_{s_i}.$$

So again the action of τ_i is identical on both modules. $\square$

Lemma 1.6.8. [50, 1.2] *The hook length formula for shifted tableaux gives an explicit description for the number of standard shifted tableaux for a shifted partition $\lambda = (\lambda_1, \dots, \lambda_r) \models n$,*

$$\#\{\text{standard shifted tableaux of shape } \lambda\} = \frac{n!}{\lambda_1! \dots \lambda_r!} \prod_{i=1}^r \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

Corollary 1.6.9. *The dimension of V_λ is*

$$2^{\lceil \frac{n-l(\lambda)}{2} \rceil} \frac{n!}{\lambda_1! \dots \lambda_r!} \prod_{i=1}^r \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

Proof. We built V_λ as a direct sum of simple supermodules $\mathcal{S}_{n-l(\lambda)}$, which have dimension $2^{\lceil \frac{n-l(\lambda)}{2} \rceil}$. The number of $\mathcal{S}_{n-l(\lambda)}$ in the sum is equal to the number of standard shifted tableaux, which is $\frac{n!}{\lambda_1! \dots \lambda_r!} \prod_{i=1}^r \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}$. $\square$

Corollary 1.6.9 shows that the ungraded representations $\tilde{\sigma}_\lambda$ in [15] have dimension

$$2^{\lceil \frac{n-l(\lambda)}{2} \rceil} \frac{n!}{\lambda_1! \dots \lambda_r!} \prod_{i=1}^r \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

This is stated but not proved in [15, Example 6.9].

We illustrate, with an example, how one can create an explicit model of a representation corresponding to a shifted partition λ .

Example 1.6.10. Take $\tilde{S}_4$ and the shifted tableau $\lambda = (3, 1) =$

. There are two standard shifted tableaux, namely,

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline & & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline & & 3 \\ \hline \end{array}.$$

with content vectors

$$\alpha_1 = (0, 1, 2, 0), \alpha_2 = (0, 1, 0, 2).$$

In this case $n - l(\lambda) = 2$ so our building blocks are $C(2)$ -supermodules. Let $\mathcal{S}$ be the $(1, 1)$ dimensional supermodule of $C(2) = \langle c_1, c_2 : c_1 c_2 = -c_2 c_1, c_i^2 = 1 \rangle$ where

$$c_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } c_2 = \begin{bmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{bmatrix}.$$

As a superspace $V_{(3,1)} = \mathcal{S}^1 \oplus \mathcal{S}^2$, where $\mathcal{S}^j \cong \mathcal{S}$ as a $C(2)$ -module. For the action of the Jucys-Murphy elements, $M_1 = 0$, $M_2 = (\sqrt{2}c_1, \sqrt{2}c_1)$, $M_3 = (\sqrt{6}c_2, 0)$ and $M_4 = (0, \sqrt{6}c_2)$. Finally, to calculate the action of τ_i , when $i = 1, 2$, $s_i(\alpha_j) \notin \text{Sspec}(4, (3, 1))$ therefore τ_1 and τ_2 fix $\mathcal{S}^1$ and $\mathcal{S}^2$. Explicitly, using Definition 1.6.6:

$$\tau_1 = \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix},$$

$$\tau_2 = \begin{bmatrix} 0 & \frac{1}{2}(\sqrt{-6} - \sqrt{2}) & 0 & 0 \\ -\frac{1}{2}(\sqrt{-6} + \sqrt{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix}.$$

Now τ_3 interchanges α_1 and α_2 , hence again using Definition 1.6.6

$$\tau_3 = \begin{bmatrix} 0 & \frac{\sqrt{-6}}{3} & \sqrt{\frac{2}{3}} & 0 \\ -\frac{\sqrt{-6}}{3} & 0 & 0 & \sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & 0 & 0 & \frac{\sqrt{-6}}{3} \\ 0 & \sqrt{\frac{2}{3}} & -\frac{\sqrt{-6}}{3} & 0 \end{bmatrix}.$$

This gives an explicit realisation of the supermodule $V_{(3,1)}$ on the four dimensional space spanned by $\{v_1, v_2, v_3, v_4\}$, by the above matrices. Note that the even space is $V_{(3,1)}^0 = \text{span}\{v_1, v_4\}$ and the odd space is $V_{(3,1)}^1 = \text{span}\{v_2, v_3\}$.

1.7 Description of Vogan's morphism

This section is dedicated to describing Vogan's morphism in type A. We use the description of the action of $Z(\mathcal{T}_n)_0$ on V_λ from Section 1.5 and the central character χ_λ of X_λ describing the actions of $Z(\mathbb{H})$ on X_λ .

Vogan's morphism for graded Hecke algebras is the algebra homomorphism $\zeta : Z(\mathbb{H}) \rightarrow Z(\mathbb{C}[\tilde{S}_n])$ defined such that $\zeta(z)$ for $z \in Z(\mathbb{H})$ is the unique element in $Z(\tilde{S}_n)$ such that

$$z \otimes 1 = \rho(\zeta(z)) + Da + aD$$

as elements of $\mathbb{H} \otimes C(V)$, for some $a \in \mathbb{H} \otimes C(V)$. This morphism was introduced in [4] and is inspired by Vogan's morphism for real reductive groups. It occurs naturally in Dirac theory for graded Hecke algebras but as far as the author is aware it has not been described unlike in Dirac theory for real reductive groups. We will describe

$$\underline{\zeta} : Z(\mathbb{H}) \rightarrow Z(\mathcal{T}_n).$$

This is ζ composed with the projection from $\mathbb{C}[\tilde{S}_n]$ to $\mathcal{T}_n$.

The following lemma is well known.

Lemma 1.7.1. *Let A be a semisimple finite dimensional $\mathbb{C}$ -algebra and let a and $b \in Z(A)$. Suppose for all irreducible representations σ of A ,*

$$\sigma(a) = \sigma(b),$$

then $a = b$.

Since $\mathcal{T}_n$ is a semisimple finite dimensional algebra over $\mathbb{C}$, we use Lemma 1.7.1 and our description of how the centres act on V_λ to describe ζ .

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition. Recall the central character χ_λ of the $\mathbb{H}$ -module X_λ is a function $\chi_\lambda : Z(\mathbb{H}) \rightarrow \mathbb{C}$. Since $Z(\mathbb{H}) = S(V)^{S_n}$ any central character can be defined by a S_n orbit of V^* . Let $\{y_i\}$ be a dual basis of $\{x_i\}$ then the element defining χ_λ is

$$\nu_\lambda = \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^{\lambda_i} (-\lambda_i + 2j - 1) y_{\lambda_{i+1} + j}.$$

Example 1.7.2. Take $\lambda = (5, 3, 2)$, then the coefficients of ν can be encoded in the following labelled diagram similar to a Young diagram for $(5, 3, 2)$ but translated to be vertically symmetrical.

$$\begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \\ & -1 & 0 & 1 & \\ & & -\frac{1}{2} & \frac{1}{2} & \end{array}$$

Definition 1.7.3. The dual map $\underline{\zeta}^*$ takes the irreducible representations of $\mathcal{T}_n$ to central characters of $\mathbb{H}$. Let $\sigma \in \text{irr}(\mathcal{T}_n)$ and let $p \in Z(\mathbb{H})$.

The central character $\underline{\zeta}^*(\sigma) : Z(\mathbb{H}) \rightarrow \mathbb{C}$ is

$$\underline{\zeta}^*(\sigma)(p) = \sigma(\underline{\zeta}(p)).$$

We can describe $\underline{\zeta}^*$ explicitly.

Lemma 1.7.4. Let $(V_\lambda, \sigma_\lambda)$ be the $\mathcal{T}_n$ -representation described in Section 1.6 corresponding to a strict partition λ , let χ_λ be the central character of X_λ . Then as described

$$\underline{\zeta}^*(V_\lambda) = \chi_\lambda.$$

This is a corollary of [4, 4.4].

Lemma 1.7.5. Let $P_{\mathbb{H}} \in Z(\mathbb{H})$ and $P_{\mathcal{T}_n} \in Z(\mathcal{T}_n)$. If, for all λ strict,

$$V_\lambda(P_{\mathcal{T}_n}) = \chi_\lambda(P_{\mathbb{H}})$$

then $\underline{\zeta}(P_{\mathbb{H}}) = P_{\mathcal{T}_n}$.

Proof. The set of V_λ for shifted partitions is a full set of irreducible representations of $\mathcal{T}_n$. Also $\underline{\zeta}^*(V_\lambda) = \chi_\lambda$. So the assumption is equivalent to, for all $\sigma \in \text{irr}(\mathcal{T}_n)$, $\sigma(P_{\mathcal{T}_n}) = (\sigma)(\underline{\zeta}(P_{\mathbb{H}}))$. Then applying Lemma 1.7.1, $\underline{\zeta}(P_{\mathbb{H}}) = P_{\mathcal{T}_n}$. □

We define polynomials in $S(V)^{S_n} = Z(\mathbb{H})$ and elements in $Z(\mathcal{T}_n)$ which have the same action. Then we use Lemma 1.7.5 to describe $\underline{\zeta}$. Recall that the even centre of $\mathcal{T}_n$ is spanned by symmetric polynomials in the squares of the Jucy-Murphy elements M_i .

Definition 1.7.6. Let M_i be the Jucys-Murphy elements for $\mathcal{T}_n$. We define,

$$P_{\mathcal{T}_n}^{2m} = \sum_{i=1}^n (2M_i)^{2m} \in Z(\mathcal{T}_n)_{\text{even}}.$$

The set $\{x_1, \dots, x_n\}$ is an orthogonal basis of V . Define

$$P_{\mathbb{H}}^{2m} = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j} \sum_{i=1}^n (2x_i)^{2m-2j} \in Z(\mathbb{H})_{\text{even}}.$$

Theorem 1.7.7. For $m, n \in \mathbb{N}$ and for all shifted λ ,

$$\sigma_\lambda(P_{\mathcal{T}_n}^{2m}) = \chi_\lambda(P_{\mathbb{H}}^{2m}).$$

We delay the proof of this theorem and first state the consequences.

Corollary 1.7.8. We have

$$\underline{\zeta}(P_{\mathbb{H}}^{2m}) = P_{\mathcal{T}_n}^{2m}.$$

Proof. This follows from Lemma 1.7.5 and Theorem 1.7.7. $\square$

Hence we can describe how $\underline{\zeta}$ acts on $P_{\mathbb{H}}^{2m}$. Note that this set of polynomials spans the even centre of $\mathbb{H}$ so we have described half of $\underline{\zeta}$.

Lemma 1.7.9. For all λ strict and $n, m \in \mathbb{N}$,

$$\chi_\lambda\left(\sum_{i=1}^n x_i^{2m+1}\right) = 0.$$

Proof. The character χ_λ is defined by ν_λ which for every positive coefficient has a negative coefficient of the same magnitude. Hence any symmetric odd polynomial evaluated on ν_λ is zero. $\square$

This gives a description of $\underline{\zeta}$ on the odd part of the centre, $Z(\mathbb{H})_{\text{odd}}$. Hence we have described the action of $\underline{\zeta}$ on $Z(\mathbb{H})$.

Corollary 1.7.10. The map $\underline{\zeta}$ surjects onto the even part of the centre of $\mathcal{T}_n$. Also $\ker(\underline{\zeta}) \supset S(V)_{\text{odd}}$.

Proof. The first statement is clear since $Z(\mathcal{T})_{\text{even}}$ is generated by the symmetric polynomials in the squares of the Jucys-Murphy elements. The second statement follows from the fact that every symmetric homogeneous polynomial of odd degree is generated by monomials of power polynomials where there is at least one odd power polynomial. Hence $\underline{\zeta}$, since it is a homomorphism, will kill any homogeneous symmetric polynomial of odd degree. $\square$

Note that Kleschev and Brundan [9, 3.2] provide a basis for $Z(\mathcal{T}_n)_{\text{even}}$ in the form of products of power polynomials of the M_i^2 . The set $\{\Pi_{i=1}^k P_{\mathcal{T}_n}^{2m_i} : \sum_{i=1}^k 2m_i + 1 \leq n\}$ is a basis for $Z(\mathcal{T}_n)_{\text{even}}$. Therefore

$$\{\underline{\zeta}(\Pi_{i=1}^k P_{\mathbb{H}}^{2m_i}) : \sum_{i=1}^k 2m_i + 1 \leq n\}$$

is a basis of $Z(\mathcal{T}_n)$. For a particular n it is possible to find the even elements in the kernel of $\underline{\zeta}$, but this must be done on a case-by-case basis comparing the action on each V_λ and showing it to be zero.

1.7.1 Proof of Theorem 1.7.7

Definition 1.7.11. Define $\mathcal{P}^1$ to be the set of partitions, of any number, of length 1. $\mathcal{P}_n^1$ is the singleton subset of $\mathcal{P}^1$, consisting of the partition $\lambda = (n)$.

Our first step is to show Theorem 1.7.7 can be proved just by considering partitions of length one.

Theorem 1.7.12. Let $\mu \in \mathcal{P}^1$ be a partition of length one then, for all $m \in \mathbb{N}$,

$$\sigma_\mu(P_{\mathcal{T}_n}^{2m}) = \chi_\mu(P_{\mathbb{H}}^{2m}).$$

Lemma 1.7.13. Theorem 1.7.12 implies Theorem 1.7.7, that is, it is enough to show the result on modules corresponding to the partition of length one.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a shifted partition. We are studying the action of power polynomials. Let $Q_{\mathbb{H}}^j$ be the j^{th} power polynomial in the x_i 's, $\sum_{i=1}^n x_i^j$. Let $\chi_{\lambda_1 \times \dots \times \lambda_k}$ be the central character of the $\mathbb{H}_{\lambda_1} \times \dots \times \mathbb{H}_{\lambda_k}$ -module $St_{\lambda_1} \otimes \dots \otimes St_{\lambda_k}$ and χ_{λ_l} be the central character of the $\mathbb{H}_{\lambda_l}$ -module St_{λ_l} . We consider $\mathbb{H}_{\lambda_l}$ to be embedded in $\mathbb{H}$ such that $x_{\lambda_{l-1}+1}, \dots, x_{\lambda_l}$ are in the image of $\mathbb{H}_{\lambda_l}$. Since $Q_{\mathbb{H}}^j \in \mathbb{H}_{\lambda_1} \times \dots \times \mathbb{H}_{\lambda_k}$, then

$$\chi_\lambda(Q_{\mathbb{H}}^j) = \chi_{\lambda_1 \times \dots \times \lambda_k}(Q_{\mathbb{H}}^j) = \sum_{l=1}^k \chi_{\lambda_l}(Q_{\mathbb{H}_{\lambda_l}}^j).$$

Similarly, for $Q_{\mathcal{T}_n}^j = \sum_{i=1}^n M_i^j$,

$$\sigma_\lambda(Q_{\mathcal{T}_n}^j) = \sum_{l=1}^k \sigma_{\lambda_l}(Q_{\mathcal{T}_{\lambda_l}}^j).$$

Therefore if we can prove the result of the theorem for every $\mu = (\lambda_l) \in \mathcal{P}^1$, then using the above decomposition of both $\chi_\lambda(Q_{\mathbb{H}}^j)$ and $\sigma_\lambda(Q_{\mathcal{T}_n}^j)$ we can extend this to σ_λ and χ_λ for every strict λ . $\square$

Lemma 1.7.14. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \vdash n$ be a shifted partition, with associated shifted Young diagram $\lambda = \{(i, j) : i = 1, \dots, l(\lambda) \text{ and } j = i - 1, \dots, i - 1 + \lambda_i\}$. Then*

$$\sigma_\lambda(P_{\mathcal{T}_n}^{2m}) = \sigma_\lambda\left(\sum_{i=1}^n (2M_i)^{2m}\right) = \sum_{i=1}^n (\text{cont}(i))^m (\text{cont}(i) + 1)^m.$$

Proof. This is a restatement of Theorem 1.5.13 where we described the action of M_i^2 on V_λ . Recall that the result showed that on a certain subspace M_i^2 acted by $\text{cont}(i)(\text{cont}(i) + 1)$. $\square$

Corollary 1.7.15. *Fix a $t \in \mathbb{Z}$ and let $\mu = (\mu_i) = (t) \vdash t$. Then*

$$\sigma_\mu\left(\sum_{i=1}^t (2M_i)^{2m}\right) = \sum_{i=1}^t (i - 1)^m (i)^m.$$

Proof. This follows from Lemma 1.7.14 since for μ , $\text{cont}(i) = i - 1$. $\square$

Lemma 1.7.16. *Fix a $t \in \mathbb{Z}$. Let $\mu = (\mu_1) = (t) \vdash t$. Then*

$$\chi_\mu\left(\sum_{i=1}^t (2x_i)^{2m}\right) = \sum_{i=1}^t (2(i - 1) - (t - 1))^{2m}.$$

Proof. The central character χ_μ is defined by $\nu_\mu = \sum_{i=1}^t (2(i - 1) - (t - 1))y_i$. The result follows by evaluating $\sum_{i=1}^t (2x_i)^{2m}$ at ν_μ . $\square$

Proof of Theorem 1.7.12. Fix $m \in \mathbb{N}$. We prove this statement by induction on n with steps of length two. Note that for $n = 0$ and $n = 1$, all operators act by zero. Hence the base cases are trivial. Suppose the result is true for $n - 2$, so:

$$\sigma_{(n-2)}(P_{\mathcal{T}_{n-2}}^{2m}) = \chi_{(n-2)}(P_{\mathbb{H}}^{2m}).$$

Now, considering that $\sigma_{(n-2)}$ is the restriction of $\sigma_{(n)}$ to $\mathcal{T}_{n-2}$,

$$\sigma_{(n)}(P_{\mathcal{T}_n}^{2m}) - \sigma_{(n-2)}(P_{\mathcal{T}_{n-2}}^{2m}) = \sigma_{(n)}((2M_{n-1})^{2m} + (2M_n)^{2m}).$$

By Corollary 1.7.15 this is equal to

$$\begin{aligned}
\sum_{i=1}^t (i-1)^m (i)^m - \sum_{i=1}^{t-2} (i-1)^m (i)^m &= (t-2)^m (t-1)^m + (t-1)^m (t)^m \\
&= (t-1)^m ((t-1+1)^m + (t-1-1)^m) \\
&= (t-1)^m \left(\sum_{l=0}^m \binom{m}{l} (t-1)^{m-l} (1^l + (-1)^l) \right) \\
&= (t-1)^m \sum_{\substack{l \in 0, \dots, m \\ \text{and even}}} 2 \binom{m}{l} (t-1)^{m-l} \\
&= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} 2 \binom{m}{2j} (t-1)^{2m-2j}.
\end{aligned}$$

We know that

$$\chi_{(t)}(P_{\mathbb{H}_t}^{2m}) - \chi_{(t-2)}(P_{\mathbb{H}_{t-2}}^{2m}),$$

is just the action of the x_t and x_{t-1} . Expanding out $P_{\mathbb{H}_t}^{2m}$ and explicitly writing $\chi_{(t)}(P_{\mathbb{H}_t}^{2m})$ using Lemma 1.7.16, we have

$$\chi_{(t)}(P_{\mathbb{H}_t}^{2m}) - \chi_{(t-2)}(P_{\mathbb{H}_{t-2}}^{2m}) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} 2 \binom{m}{2j} (t-1)^{2m-2j}.$$

Therefore we have shown that the inductive step holds. Hence

$$\sigma_{(n)}(P_{\mathcal{T}_{n-2}}^{2m}) = \chi_{(n)}(P_{\mathbb{H}}^{2m})$$

by induction. □

We have proved Theorem 1.7.12 and hence have proved Theorem 1.7.7.

Chapter 2

Dirac cohomology and simple modules of the Dunkl-Opdam subalgebra via inherited Drinfeld properties

Abstract

We define a new presentation for the Dunkl-Opdam subalgebra of the rational Cherednik algebra. This presentation uncovers the Dunkl-Opdam subalgebra as a Drinfeld algebra. The Dunkl-Opdam subalgebra is the first natural occurrence of a non-faithful Drinfeld algebra. We use this fact to define Dirac cohomology for the DO subalgebra. We formalise generalised graded Hecke algebras and extend a Langlands classification to generalised graded Hecke algebras. We then describe an equivalence between the irreducibles of the Dunkl-Opdam subalgebra and direct sums of graded Hecke algebras of type A . This equivalence commutes with taking Dirac cohomology.

2.1 Introduction

We study the Dunkl-Opdam subalgebra, $\mathbb{H}_{DO}$ [22], of the rational Cherednik algebra associated to $G(m, 1, n) = S_n \rtimes (\mathbb{Z}_m)^n$. This subalgebra of the rational Cherednik algebra $\mathcal{H}_t(G(m, 1, n))$ (Definition 2.4.1) is independent of the parameter t . In this chapter, we take a closer look at $\mathbb{H}_{DO}$ and notice that it is similar to both a graded Hecke algebra and a Drinfeld algebra. We extend several results for Hecke algebras and faithful Drinfeld algebras to include the Dunkl-Opdam subalgebra.

We construct a new presentation of $\mathbb{H}_{DO}$:

Theorem 2.4.13. *There exists a presentation of $\mathbb{H}_{DO}$ given by elements $\{\tilde{z}_i : i = 1, \dots, n\}$ and generators $\{s_i, g_j : i \leq i \leq n-1, 1 \leq j \leq n\}$ in $G(m, 1, n)$ such that:*

$$s_i \tilde{z}_j s_i^{-1} = s_i(\tilde{z}_j),$$

$$g_i \tilde{z}_j = \tilde{z}_j g_i \quad \forall i, j = 1, \dots, n,$$

$$[\tilde{z}_i, \tilde{z}_j] \in \mathbb{C}G(m, 1, n),$$

where $G(m, 1, n) \cong S_n \rtimes (\mathbb{Z}_m)^n$. The group S_n is generated by transpositions s_i , $i = 1, \dots, n-1$ and g_j generates the j^{th} copy of $\mathbb{Z}_m$.

This presentation exposes $\mathbb{H}_{DO}$ as a Drinfeld algebra. Drinfeld [21] initially defined these algebras (Definition 2.2.1) with the potential to have non-faithful representations. In the literature this has been largely forgotten, perhaps because there appeared to be no natural examples of a non-faithful Drinfeld algebra. The Dunkl-Opdam subalgebra is a naturally occurring non-faithful Drinfeld algebra. Ciubotaru [13] defined Dirac cohomology for faithful Drinfeld algebras and we extend this to non-faithful Drinfeld algebras.

Dez  lee introduced the idea of generalised graded Hecke algebra to look at the Dunkl-Opdam subalgebra. We concretely define a class of algebras, which contains Dez  lee's examples. We extend Evens' [25] Langlands classification to generalised graded Hecke algebras.

Theorem 2.3.8. *Let $\mathbb{G}\mathbb{H}(W \rtimes T)$ denote a generalised graded Hecke algebra. A parabolic subalgebra is denoted by $\mathbb{G}\mathbb{H}_P$, with $\mathbb{G}\mathbb{H}_{P_S}$ denoting the semisimple part of $\mathbb{G}\mathbb{H}_P$ (Definition 2.3.2).*

- (i) Every irreducible $\mathbb{GH}$ -module V can be realised as a quotient of $\mathbb{GH}(W \rtimes T) \otimes_{\mathbb{GH}_P} U$, where $U = \hat{U} \otimes \mathbb{C}_\nu$ is such that $\hat{U}$ is an irreducible tempered $\mathbb{GH}_{P_s}$ -module and $\mathbb{C}_\nu$ is a character of $S(\mathfrak{a})$ defined by $\nu \in \mathfrak{a}^{*+}$.
- (ii) If U is as in (i) then $\mathbb{GH}(W \rtimes T) \otimes_{\mathbb{GH}_P} U$ has a unique irreducible quotient to be denoted $J(P, U)$.
- (iii) If $J(P, \hat{U} \otimes \mathbb{C}_\nu) \cong J(P', \hat{U}' \otimes \mathbb{C}_{\nu'})$ then $P = P'$, $\hat{U} \cong \hat{U}'$ as $\mathbb{GH}_{P_s}$ -modules and $\nu = \nu'$.

The Dunkl-Opdam subalgebra has a commutative subalgebra $\mathbb{CT} \cong (\mathbb{Z}_m)^n$. We decompose representations into weight spaces (Definition 2.3.4), which then defines weights of a representation. The weights of an $\mathbb{H}_{DO}$ -representation come in orbits (Lemma 2.3.10). We use these weights to highlight that irreducible representations of $\mathbb{H}_{DO}$ are pullbacks of $\mathbb{H}(S_{a_i})$ -representations via specific quotients (Lemma 2.5.5).

Theorem 2.5.6. *Let $A = \{\underline{a} = (a_0, \dots, a_{m-1}) \in \mathbb{N}^m : \sum_{i=0}^{m-1} a_i = n\}$, note that $\mathbb{N}$ includes zero. The irreducible representations of $\mathbb{H}(G(m, 1, n))$ split into blocks which are induced from products of $\mathbb{H}(S_{a_i})$ -representations:*

$$\text{irr}(\mathbb{H}(G(m, 1, n))) \cong \bigsqcup_{\underline{a} \in A} \text{irr}(\mathbb{H}(S_{a_0})) \otimes \dots \otimes \text{irr}(\mathbb{H}(S_{a_{m-1}})).$$

This bijection of irreducible modules defines an equivalence of the semisimplification of the module categories, the Grothndieck groups, $G_0(\mathbb{H}(G(m, 1, n)))$ and $G_0(\bigoplus_{\underline{a} \in A} \mathbb{H}_{S_{a_0}} \otimes \dots \otimes \mathbb{H}_{S_{a_{m-1}}})$.

We use this equivalence to describe the Dirac cohomology of an $\mathbb{H}_{DO}$ -module X in terms of Dirac cohomology of the associated $\mathbb{H}(S_{a_i})$ -modules. Let F and F^{-1} be functors displaying this equivalence.

Theorem 2.6.7. *Given an irreducible $\mathbb{H}_{DO}$ -representation V then $F(V)$ as an $\mathbb{H}_{S_{a_0}} \otimes \dots \otimes \mathbb{H}_{S_{a_{m-1}}}$ -module is isomorphic to $F(V) \cong X_{a_0} \otimes \dots \otimes X_{a_{m-1}}$. The Dirac cohomology of V is*

$$\bigoplus_{c \in C} c(H_D(X_{a_0}) \otimes \dots \otimes H_D(X_{a_{m-1}})),$$

where $H_D(X)$ is the type A Dirac cohomology of the $\mathbb{H}_{S_k}$ -module X , P is the parabolic subset associated to $\underline{a}$ and C is a set of coset representatives of S_n/S_P . Let $H_D(\bullet)$ denote the functor taking the relevant module to its Dirac cohomology. We have the following commutative diagram:

$$\begin{array}{ccc}
G_0(\mathbb{H}(G(m, 1, n))) & \xrightarrow{F} & G_0(\bigoplus_{\underline{a} \in A} \mathbb{H}_{S_{a_0}} \otimes \dots \otimes \mathbb{H}_{S_{a_{m-1}}}) \\
\downarrow H_D(\bullet) & & \downarrow H_D(\bullet) \\
\widetilde{\mathbb{C}G(m, 1, n)}\text{-mod} & \xleftarrow{\widetilde{F^{-1}}} & \bigoplus_{\underline{a} \in A} \widetilde{\mathbb{C}S_{a_0}} \otimes \dots \otimes \widetilde{\mathbb{C}S_{a_{m-1}}}\text{-mod}.
\end{array}$$

Here $G_0(\mathbb{H}(G(m, 1, n)))$ is the Grothendieck group of the isomorphism classes of representations of $\mathbb{H}(G(m, 1, n))$.

In Section 2.2 we study Drinfeld algebras, we focus on the fact that these algebras can be defined with a non-faithful representation. We extend Dirac cohomology defined in [13] to the non-faithful case. In Section 2.3, we introduce the class of generalised graded Hecke algebras, and we extend Evens' [25] Langlands classification to this class. In Section 2.4 we introduce the Dunkl-Opdam subalgebra. We highlight that it is a generalised graded Hecke algebra and by introducing a new presentation show that it is also a non-faithful Drinfeld algebra. Section 2.5 defines an equivalence between the irreducible modules of the Dunkl-Opdam subalgebra and direct sums of graded Hecke algebras associated to parabolic symmetric groups. In Section 2.6 we combine results on Dirac cohomology (Section 2.2) and the equivalence (Section 2.5) to describe Dirac cohomology of an $\mathbb{H}_{DO}$ -module with Dirac cohomology of its associated module under this equivalence. This highlights that this equivalence of irreducibles behaves well with respect to Dirac cohomology.

2.2 Drinfeld algebras

In this section, we will define Drinfeld algebras as introduced by Drinfeld, that is, with a potentially non-faithful representation. Ciubotaru [13] defined Dirac cohomology for faithful Drinfeld algebras, and we will extend Dirac cohomology to non-faithful Drinfeld algebras. The results in this section follow almost verbatim from the proofs in [13], so we will not write them here.

Given a finite group G , antisymmetric bilinear forms b_g for $g \in G$ and a representation (ρ, V) of G , we construct an algebra

$$\mathbb{H} = \mathbb{C}[G] \rtimes T(V)/R.$$

Here R is the two sided ideal of $\mathbb{C}[G] \rtimes T(V)$ generated by the relations,

$$g^{-1}vg = \rho(g)(v) \text{ for all } g \in G \text{ and } v \in V,$$

and

$$[u, v] = \sum_{g \in G} b_g(u, v)g \text{ for all } v, u \in V.$$

We define a filtration on the algebra $\mathbb{C}[G] \rtimes T(V)/R$, a vector v has degree 1 and a group element $g \in G$ has degree 0.

Definition 2.2.1. [21] *An algebra of the form $\mathbb{H} = \mathbb{C}[G] \rtimes T(V)/R$ is a Drinfeld algebra if it satisfies a PBW criterion. That is the associated graded algebra is naturally isomorphic to*

$$\mathbb{C}[G] \rtimes S(V).$$

Here $\rtimes$ denotes the semi-direct product with the natural action of G on V .

We state the conditions on the bilinear forms b_g such that $\mathbb{H}$ is a Drinfeld algebra. This was originally stated in [21], and explained for the faithful case in [49]. Define $G(b) = \{g \in G : b_g \neq 0\}$.

Theorem 2.2.2. [21][49, Theorem 1.9] *The algebra $\mathbb{H}$ is a Drinfeld algebra if and only if:*

- (1) *For every $g \in G$, $b_{g^{-1}hg}(u, v) = b_h(\rho(g)(u), \rho(g)(v))$ for every $u, v \in V$,*
- (2) *For every $g \in G(b) \setminus \text{Ker } \rho$, $\text{Ker } b_g = V^{\rho(g)}$ and $\dim(V^{\rho(g)}) = \dim V - 2$,*
- (3) *For every $g \in G(b) \setminus \text{Ker } \rho$ and $h \in Z_G(g)$, $\det(h|_{V^{\rho(g)}^\perp}) = 1$, where $V^{\rho(g)}^\perp = \{v - \rho(g)(v) : v \in V\}$.*

The above statements follow immediately from the proofs given in [49] for the faithful case. The only variation is that the set $\text{Ker } \rho$ replaces 1 in [49].

2.2.1 Non-faithful Drinfeld algebras

In the recent literature, Drinfeld algebras have predominately been considered with G a subgroup of $GL(V)$. However, Drinfeld originally expressed them with a potentially non-faithful representation. To address this disparity and to avoid confusion we will say that a Drinfeld algebra is a faithful Drinfeld algebra if the representation involved is faithful and we will state that a Drinfeld algebra is non-faithful if the representation is non-faithful. The class of Drinfeld algebras includes both faithful and non-faithful Drinfeld algebras.

2.2.2 The Dirac operator for (non-faithful) Drinfeld algebras

If V has a G -invariant symmetric bilinear form, then one can define a Dirac operator $\mathcal{D}$. In [13] Dirac cohomology is defined for any faithful Drinfeld algebra. Furthermore, an equation involving the square of the Dirac operator is proved [16, Theorem 2.7]. The extension of these theorems to the case of non-faithful representations is clear from the proofs. We will, however, give the equivalent formulation of the theorems in the non-faithful case. In this section, we will denote a Drinfeld algebra by $\mathbb{H}$.

2.2.2.1 The Clifford algebra

Let $\langle, \rangle$ be a G -invariant non-degenerate symmetric bilinear form on V . The Clifford algebra $C(V)$ associated to V and $\langle, \rangle$ is the quotient of the tensor algebra $T(V)$ by the relations

$$v \cdot v' + v' \cdot v = -2\langle v, v' \rangle.$$

The Clifford algebra has a filtration by degrees and a $\mathbb{Z}/2\mathbb{Z}$ -grading by parity of degrees. In this grading $C(V) = C(V)_0 \oplus C(V)_1$. We define an automorphism $\epsilon : C(V) \rightarrow C(V)$ which is the identity on $C(V)_0$ and minus the identity on $C(V)_1$. Let us extend ϵ to be an automorphism of $\mathbb{H} \otimes C(V)$ by defining ϵ be the identity on $\mathbb{H}$. We define an anti-automorphism, the transpose of $C(V)$: $v^t = -v$ for all $v \in V$. The Pin group is:

$$\text{Pin}(V) = \{a \in C(V)^\times : \epsilon(a)Va^{-1} \subset V, a^t = a^{-1}\}.$$

Let (ρ, V) be a complex representation of G with G -invariant form. This establishes $\rho(G)$ as a subgroup of $O(V)$. The Pin group is a double cover of $O(V)$ with surjection $p : \text{Pin}(V) \rightarrow O(V)$. We define the pin double cover of $\rho(G) \subset O(V)$ as

$$\widetilde{\rho(G)} := p^{-1}(\rho(G)) \subset \text{Pin}(V).$$

Note that $\widetilde{\rho(G)}$ is not a double cover of G but it is a double cover of $\rho(G)$. We construct a cover of G .

Definition 2.2.3. Consider the algebra $\mathbb{C}[G] \otimes C(V)$, we define $\tilde{G}$ to be the subgroup generated by elements $g \otimes p^{-1}(\rho(g))$ for all $g \in G$. That is

$$\tilde{G} = \{g \otimes h : g \in G, h = p^{-1}(\rho(g)) \forall g \in G\}.$$

If $\text{Ker } \rho$ is abelian then $\tilde{G}$ is the semi-direct product $\text{Ker } \rho \rtimes \widetilde{\rho(G)}$ with cross multiplication:

$$(h, \tilde{g}) \cdot (h', \tilde{g}') = (hp(\tilde{g}^{-1})h'p(\tilde{g}), \tilde{g}\tilde{g}'), \quad \text{for all } \tilde{g}, \tilde{g}' \in \widetilde{\rho(G)} \text{ and } h, h' \in \text{Ker } \rho.$$

By construction $\tilde{G}$ embeds in $\mathbb{H} \otimes C(V)$ via

$$\Delta : \tilde{G} \rightarrow \mathbb{H} \otimes C(V).$$

For more information on the Clifford algebra see [29] and [41].

2.2.2.2 The Dirac element

Given any basis $\{v_i\}$ of V and dual basis $\{v^i\}$ with respect to $\langle, \rangle$ we define the Dirac element

$$\mathcal{D} = \sum_i v_i \otimes v^i \in \mathbb{H} \otimes C(V).$$

We give a formula for $\mathcal{D}^2$. This is equivalent to [13, Theorem 2.7]. The only variation being that $\text{Ker } \rho$ replaces 1. For every $g \in G(b)$ set,

$$\mathbf{k}_g = \sum_{i,j} b_g(v_i, v^j) v^i v_j \in C(V),$$

and

$$\mathbf{h} = \sum_i v_i v^i \in \mathbb{H}.$$

The commutation relation defined for a Drinfeld algebra shows:

$$\mathcal{D}^2 = -\mathbf{h} \otimes 1 + \frac{1}{2} \sum_{g \in G(b)} g \otimes \mathbf{k}_g.$$

This result is [13, Lemma 2.5]. Recall $G(b) = \{g \in G : b_g \neq 0\}$, we write $\tilde{G}(b)$ for the cover of this subset.

Lemma 2.2.4. *Similarly to [13, Lemma 2.6] every element g in $G(b) \setminus \text{Ker } \rho$ can be expressed as a product of two reflections. Every element in $G(b) \setminus \text{Ker } \rho$ can be written as a coset representative of $G(b)/\text{Ker } \rho$ conjugated by an element in $\text{Ker } \rho$. Therefore given $g \in G(b) \setminus \text{Ker } \rho$, there exists an $h \in \text{Ker } \rho$ and $\alpha, \beta \in V$ such that $g = h^{-1} s_\alpha s_\beta h$ and the roots α, β span the space $(V^{\rho(g)})^\perp$. We scale α and β such that $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1$. This gives α and β up to sign, we choose all α and β for every g to be such that $\tilde{g}_1 \tilde{g}_2 = g_1 \tilde{g}_2$ for all g_1 and $g_2 \in G/\text{Ker } \rho$. There are two ways to do this due to the $(G/\tilde{\text{Ker } \rho})$ being a double cover, we fix a particular choice.*

Proof. See proof of [13, Lemma 2.6] with G replaced by $G/\text{Ker}(\rho)$. $\square$

For every coset representative $g \in G(b)/\text{Ker}(\rho)$ define

$$\tilde{g} = \alpha\beta \in C(V), \quad c_{\tilde{g}} = \frac{b_g(\alpha, \beta)}{1 - \langle \alpha, \beta \rangle^2} \in \mathbb{C}, \quad e_g = \frac{b_g(\alpha, \beta) \langle \alpha\beta \rangle}{1 - \langle \alpha, \beta \rangle^2} \in \mathbb{C}.$$

Every $x \in \tilde{G}(b)$ can be written as $h^{-1}gh$ where g is a coset representative of $\tilde{G}(b)/\text{Ker} \rho$ and $h \in \text{Ker} \rho$. Lemma 2.2.4 gives $g = s_\alpha s_\beta$ and $\tilde{g} = \alpha\beta \in C(V)$. We define, for $x = h^{-1}gh \in \tilde{G}$,

$$\tilde{x} = \tilde{g} = \alpha\beta \in C(V), \quad c_{\tilde{x}} = c_{\tilde{g}}, \quad e_x = h e_{g_C} h^{-1}.$$

Let us define the Casimir elements, $\Omega_{\mathbb{H}}$ in $\mathbb{H}$ and $\Omega_{\tilde{G}}$ in $\tilde{G}$.

$$\Omega_{\mathbb{H}} = \mathbf{h} - \sum_{g \in G(b)/\text{Ker} \rho} e_g g \in \mathbb{H}^G,$$

$$\Omega_{\tilde{G}} = \sum_{\substack{h \in \text{Ker} \rho \\ g \in G(b)/\text{Ker} \rho}} h^{-1} \tilde{g} h c_{\tilde{g}} \in \mathbb{C}[\tilde{G}]^{\tilde{G}}.$$

Theorem 2.2.5. [13, c.f. Theorem 2.7] *The square of the Dirac element can be expressed as a sum of the two Casimir elements plus terms from the kernel;*

$$\mathcal{D}^2 = -\Omega_{\mathbb{H}} \otimes 1 + \Delta(\Omega_{\tilde{G}}) + \frac{1}{2} \otimes \sum_{g \in \text{ker} \rho} \mathbf{k}_g.$$

2.2.2.3 Vogan's Morphism

Let $\tilde{\Omega}_{\mathbb{H}} = \Omega_{\mathbb{H}} - \frac{1}{2} \otimes \sum_{h \in \text{Ker} \rho} \mathbf{k}_h$, define

$$\mathbf{A} = Z_{\mathbb{H} \otimes C(V)}(\tilde{\Omega}_{\mathbb{H}}) \subset (\mathbb{H} \otimes C(V))^{\tilde{G}}.$$

If $\text{Ker} \rho \cap G(b) = \emptyset$ then $\mathbf{A} = \mathbb{H} \otimes C(V)$. Define a derivation

$$d : \mathbb{H} \otimes C(V) \rightarrow \mathbb{H} \otimes C(V),$$

$$d(a) = \mathcal{D}a - \epsilon(a)\mathcal{D}.$$

The Dirac operator $\mathcal{D}$ interchanges the trivial and $\det \tilde{G}$ - isotypic spaces of $\mathbf{A}$. We define d_{triv} and d_{det} to be the restriction of d to the trivial and $\det \tilde{G}$ - isotypic spaces. We state the theorems in [13] but note that the proofs apply verbatim to this case.

Theorem 2.2.6. [13, c.f. Theorem 3.5] *The kernel of d_{triv} equals:*

$$\text{Ker } d_{\text{triv}} = \text{Im } d_{\text{det}} \oplus \Delta(\mathbb{C}[\tilde{G}]^{\tilde{G}}).$$

Since d is a derivation then $\text{Ker } d_{\text{triv}}$ is an algebra. The following theorem is the statement about Vogan's Dirac homomorphism in the non-faithful Drinfeld case

Theorem 2.2.7. [13, c.f. Theorem 3.8] *The projection $\zeta : \text{Ker } d_{\text{triv}} \rightarrow \mathbb{C}[\tilde{G}]^{\tilde{G}}$ defined in Theorem 2.2.6 is an algebra homomorphism.*

Note that since the image of ζ is an abelian algebra the morphism must factor through the abelianisation of $\text{Ker } d_{\text{triv}}$. Recall that when $b_g = 0$ for all $g \in \text{Ker } \rho$ then $Z(\mathbb{H}) \otimes 1$ is contained in $\text{Ker } d_{\text{triv}}$. With this extra condition we can consider the dual of ζ which relates the representations of $\tilde{G}$ with characters of $Z(\mathbb{H})$.

$$\zeta^* : \text{irr}(\tilde{G}) = \text{Spec } \mathbb{C}[\tilde{G}]^{\tilde{G}} \rightarrow \text{Spec } Z(\mathbb{H}).$$

Here Spec denotes the algebra of characters on a given algebra.

2.3 Generalised graded Hecke algebras

Ram and Shepler [49] show that there does not exist a faithful Drinfeld algebra associated to the complex reflection group $G(m, 1, n) = S_n \rtimes (\mathbb{Z}_m)^n$. However, they define a candidate for an algebra that is similar to a graded Hecke algebra. Dez  lee [19] introduced the term generalized graded Hecke algebra. In Section 2.4 we show that these algebras are non-faithful Drinfeld algebras. We define a larger group of algebras denoted generalised graded Hecke algebras or GGH for short.

Set $A \star B$ to be the free product of unital associative complex algebras.

Definition 2.3.1. *Let W be a Weyl group generated by simple reflections $s_\alpha, \alpha \in \Pi$. The group W acts on a finite commutative group T , $\mathfrak{t}$ is a faithful complex W -representation. The form $\langle, \rangle$ is a W -invariant pairing between the vector spaces $\mathfrak{t}^*$ and $\mathfrak{t}$. We define a parameter function*

$$\tilde{c} : \Pi \rightarrow \mathbb{C}[T]$$

which is invariant under the action of W . The generalised graded Hecke algebra $\text{GGH}(W \rtimes T)$ is the quotient of the algebra

$$\mathbb{C}[W \rtimes T] \star S(\mathfrak{t})$$

by the relations

$$s_\alpha t = s_\alpha(t)s_\alpha + \langle \alpha, t \rangle \tilde{c}(\alpha), \quad \forall t \in \mathfrak{t}, \alpha \in \Pi$$

$$[h, t] = 0 \quad \forall t \in \mathfrak{t}, h \in T.$$

When T is the trivial group then $\mathbb{GH}(W \rtimes 1)$ is a graded Hecke algebra. In this form, the relations look very similar to the graded Hecke algebras except that the parameter function takes values in $\mathbb{C}[T]$ instead of $\mathbb{C}$. In the following section, we will prove a Langlands classification for generalised graded Hecke algebras. This proof follows Evens' [25] proof of the Langlands classification for graded Hecke algebras.

2.3.1 Preliminaries for the Langlands classification

Let $\{X, R, Y, \check{R}, \Pi\}$ be root datum, where X and Y are free finitely generated abelian groups and there exists a perfect bilinear pairing between them. The roots $R \subset X$ and coroots $\check{R} \subset Y$ are finite subsets with a bijection between them. Let Π denote the simple roots $\{\alpha_1, \dots, \alpha_l\}$. Positive roots R_+ (resp. $\check{R}^+$) are the $\mathbb{N}$ span of Π (respectively $\check{\alpha}$ for $\alpha \in \Pi$). Let $\mathfrak{t} = X \otimes \mathbb{C}$ and $\mathfrak{t}^* = Y \otimes \mathbb{C}$ be dual vector spaces, similarly let $\mathfrak{t}_{\mathbb{R}}, \mathfrak{t}_{\mathbb{R}}^*$ be the real spans of X and Y . Let T be a finite abelian group such that W acts on T . Let $\tilde{c}$ be a function from Π to $\mathbb{C}T$ which is invariant on conjugacy classes, $\tilde{c}(g^{-1}hg) = g^{-1}\tilde{c}(h)g = g(\tilde{c}(h))$.

We will denote a generalised graded Hecke algebra by $\mathbb{GH}$. In the case of the generalised graded Hecke algebra associated to $G(m, 1, n)$, we set $W = S_n$, $T = (\mathbb{Z}_m)^n$. The function $\tilde{c}$ from Π to $\mathbb{C}T$ is defined by $\tilde{c}(\epsilon_{i+1} - \epsilon_i) = \sum_{l=0}^{m-1} g_i^l g_{i+1}^{-l}$. The element g_i is the i^{th} generator of $(\mathbb{Z}_m)^n$.

Definition 2.3.2. *Given a subset Π_P of Π we can define a parabolic subgroup W_P of W generated by s_α for $\alpha \in \Pi_P$. The corresponding parabolic subalgebra of the generalised graded Hecke algebra $\mathbb{GH}_P$ is generated by s_α for $\alpha \in \Pi$ and $\mathbb{C}[T] \otimes S(\mathfrak{t})$. This is the generalised graded Hecke algebra associated to $W_P \rtimes T$.*

Definition 2.3.3. *Define $\mathfrak{a}$ to be the vector space $\{x \in \mathfrak{t} : \check{\alpha}(x) = 0, \alpha \in \Pi\}$. Given any parabolic subalgebra $\mathbb{GH}_P$, set $\mathfrak{a}_P = \{x \in \mathfrak{t} : \check{\alpha}(x) = 0, \alpha \in \Pi_P\}$. Let $\mathfrak{a}_s$ be the perpendicular subspace to $\mathfrak{a}_P^*$ under the pairing of $\mathfrak{t}$ and $\mathfrak{t}^*$.*

Then $\mathbb{GH}_P \cong \mathbb{GH}_{P_s}(W_P \rtimes T) \otimes S(\mathfrak{a}_P)$, where $\mathbb{GH}_{P_s}(W_P \rtimes T)$ is constructed (Definition 2.3.1) as the quotient of the following algebra:

$$\mathbb{GH}_{P_s}(W_P \rtimes T) \cong \mathbb{C}[W_P \rtimes T] \star S(\mathfrak{a}_s).$$

The commutative subalgebra $\mathcal{A} = \mathbb{C}T \otimes S(\mathfrak{t})$ is a subalgebra of all parabolic subalgebras. For every $\mathcal{A}$ -module V we can consider a weight space decomposition.

Definition 2.3.4. Let $\mathcal{A} = \mathbb{C}T \otimes S(\mathfrak{t})$ and $\mathcal{A}^*$ denote characters on this algebra. Given an $\mathcal{A}$ -module V , on which $S(\mathfrak{t})$ acts locally finitely, and a character $\mu \otimes \lambda \in \mathcal{A}^* = \mathbb{C}T^* \otimes S(\mathfrak{t})^*$ define the subspace:

$$V_{\mu \otimes \lambda} = \{v \in V : y \otimes x(v) = \mu(y) \otimes \lambda(x)(v) \text{ for all } y \otimes x \in \mathcal{A}\}.$$

We can decompose V into weight spaces:

$$V = \bigoplus_{\lambda \otimes \mu \in \mathcal{A}^*} V_{\mu \otimes \lambda}.$$

The weights of V are the $\mu \otimes \lambda \in \mathcal{A}^*$ such that $V_{\mu \otimes \lambda}$ is non-zero.

Definition 2.3.5. Given simple roots $\alpha_1, \dots, \alpha_n$, the fundamental coweights $x_i \in \mathfrak{t}$ are such that

$$\check{\alpha}_j(x_i) = \delta_{ij} \text{ and } \nu(x_i) = 0 \text{ for all } \nu \in \mathfrak{a}^*.$$

Example 2.3.6. Let $W = S_n$, $\mathfrak{t} = \text{span}\{\epsilon_1, \dots, \epsilon_n\}$, $\mathfrak{t}^* = \text{span}\{e_1, \dots, e_n\}$, $\mathfrak{a}^* = \text{span}\{e_1 + e_2 + \dots + e_n\}$. Let the simple roots be $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \dots, n-1$. Then the fundamental coweights are

$$x_i = \sum_{j \leq i} \epsilon_j - \frac{i}{n} (\epsilon_1 + \epsilon_2 + \dots + \epsilon_n).$$

The modification by $\epsilon_1 + \dots + \epsilon_n$ is required so that $\mathfrak{a}^*$ is perpendicular to x_i . If we had defined $\mathfrak{t}$ to be $\text{span}\{\epsilon_1, \dots, \epsilon_n\} / \text{span}\{\epsilon_1 + \dots + \epsilon_n\}$ this would not be required as $\mathfrak{a}^* = 0$.

Definition 2.3.7. An irreducible $\mathbb{GH}$ -module V is essentially tempered if for all weights $\mu \otimes \lambda : \mathbb{C}T \otimes S(\mathfrak{t}) \rightarrow \mathbb{C}$ of V , $\text{Re}(\lambda(x_i)) \leq 0$, for all fundamental coweights x_i . The module V is tempered if V is essentially tempered and $\text{Re}(\lambda|_{\mathfrak{a}_{\mathbb{R}}}) = 0$. Here $\mathfrak{a}_{\mathbb{R}}$ is the real span of $x \in X$ perpendicular to the coroots.

Let

$$\mathfrak{a}_P^{*+} = \{\nu \in \mathfrak{a}_P^* : \text{Re}(\nu(\alpha)) > 0, \alpha \in \Pi - \Pi_P\}.$$

2.3.2 The Langlands classification for generalized graded Hecke algebras

Theorem 2.3.8. *Let $\mathbb{GH}(W \rtimes T)$ denote a generalised graded Hecke algebra. A parabolic subalgebra is denoted by $\mathbb{GH}_P$, with $\mathbb{GH}_{P_s}$ denoting the semisimple part of $\mathbb{GH}_P$ (Definition 2.3.2).*

(i) *Every irreducible $\mathbb{GH}$ -module V can be realised as a quotient of $\mathbb{GH}(W \rtimes T) \otimes_{\mathbb{GH}_P} U$, where $U = \hat{U} \otimes \mathbb{C}_\nu$ is such that $\hat{U}$ is an irreducible tempered $\mathbb{GH}_{P_s}$ -module and $\mathbb{C}_\nu$ is a character of $S(\mathfrak{a})$ defined by $\nu \in \mathfrak{a}^{*+}$.*

(ii) *If U is as in (i) then $\mathbb{GH}(W \rtimes T) \otimes_{\mathbb{GH}_P} U$ has a unique irreducible quotient to be denoted $J(P, U)$.*

(iii) *If $J(P, \hat{U} \otimes \mathbb{C}_\nu) \cong J(P', \hat{U}' \otimes \mathbb{C}_{\nu'})$ then $P = P'$, $\hat{U} \cong \hat{U}'$ as $\mathbb{GH}_{P_s}$ -modules and $\nu = \nu'$.*

Firstly, we state a couple of lemmas of Langlands and a technical lemma about orbits of weights. Let Z be a real inner product space of dimension n . Let $\{\check{\alpha}_1, \dots, \check{\alpha}_n\}$ be a basis such that $(\check{\alpha}_i, \check{\alpha}_j) \leq 0$ whenever $i \neq j$. Let $\{\beta_1, \dots, \beta_n\}$ be a dual basis. For a subset F of Π , let

$$S_F = \left\{ \sum_{j \notin F} c_j \beta_j - \sum_{i \in F} d_i \check{\alpha}_i : c_j > 0, d_i \leq 0 \right\}.$$

Lemma 2.3.9. *[6, IV, 6.11] Let $x \in Z$. Then $x \in S_F$ for a unique subset $F = F(x)$.*

If $x \in Z$ then let $x_0 = \sum_{j \notin F} c_j \beta_j$, where $x \in S_F$ and $x = \sum_{j \notin F} c_j \beta_j - \sum_{i \in F} d_i \check{\alpha}_i$. It is clear that if $x_0 = y_0$ then $F(x) = F(y)$. Define a partial order on Z by setting $x \geq y$ if $x - y = \sum_{t_i \geq 0} t_i \check{\alpha}_i$.

Lemma 2.3.10. *[6, IV, 6.13] If $x, y \in Z$ and $x \geq y$ then $x_0 \geq y_0$.*

Lemma 2.3.11. *Given an irreducible $\mathbb{GH}_P$ module V , the set of weights $\{\lambda \otimes \mu\}$ are all in the same W_P orbit.*

Proof. The group W_P is the only part of $\mathbb{GH}_P$ which does not act by eigenvalues on $V_{\mu \otimes \lambda}$. For any $\mathbb{GH}_P$ module U and a weight $\mu \otimes \lambda$ the subspace

$$\bigoplus_{w \in W_P} U_{w(\lambda) \otimes w(\mu)}$$

is a $\mathbb{GH}_P$ submodule of U . □

Proof of (i). The simple coroots $\check{\alpha}_1, \dots, \check{\alpha}_n$ will have a dual basis $\beta_1, \dots, \beta_n$ in $\mathfrak{t}_{\mathbb{R}}$, relative to the Killing form. Let V be an irreducible $\mathbb{G}\mathbb{H}$ -representation. Let $\mu \otimes \lambda$ be an $\mathcal{A}$ weight of V which is maximal among $Re(\lambda)$. Let $\Pi_P = F = F(Re(\lambda))$. Let $\mathfrak{a}_s$ (respectively $\mathfrak{a}_s^*$) be the elements of $\mathfrak{t}$ (respectively $\mathfrak{t}^*$) perpendicular to $\mathfrak{a}_P^*$ (respectively $\mathfrak{a}_P$). The space $\mathfrak{t}^*$ splits

$$\mathfrak{t}^* = \mathfrak{a}_P^* \oplus \mathfrak{a}_s^*.$$

We can restrict characters of $\mathfrak{t}$ to $\mathfrak{a}_P$ (respectively $\mathfrak{a}_s$) by considering the projection of the character in $\mathfrak{t}^*$ to $\mathfrak{a}_P^*$ (respectively $\mathfrak{a}_s^*$).

Let $\nu = \lambda|_{\mathfrak{a}_P}$. Since λ was considered maximal then by construction $\nu \in \mathfrak{a}_P^{*+}$. Let U be an irreducible representation of $\mathbb{G}\mathbb{H}_P$ appearing in V such that $S(\mathfrak{a}_P)$ acts by ν . Let $\mu \otimes \phi$ be a $\mathbb{C}T \otimes S(\mathfrak{a}_s)$ weight of U . Since $\mathbb{C}T \otimes S(\mathfrak{t}) \cong \mathbb{C}T \otimes S(\mathfrak{a}_s) \otimes S(\mathfrak{a}_P)$ then $\mu \otimes \phi \otimes \nu = \mu \otimes (\phi + \nu)$ is a $\mathbb{C}T \otimes S(\mathfrak{a})$ weight of V .

$$Re(\phi + \nu) = \sum_{j \notin F} c_j \beta_j - \sum_{j \in F} z_i \check{\alpha}_i, c_j > 0,$$

while

$$Re(\lambda) = \sum_{j \notin F} c_j \beta_j - \sum_{j \in F} d_i \check{\alpha}_i, c_j > 0, d_i \geq 0.$$

To prove U is a tempered representation of $\mathbb{G}\mathbb{H}_{P_s}$ it is sufficient to prove that $z_i \geq 0$.

Let $F_2 = \{i \in F : z_i < 0\}$ and $F_1 = F - F_2$. Then $Re(\phi + \nu) \geq \sum_{j \notin F} c_j \beta_j - \sum_{i \in F_1} z_i \check{\alpha}_i$. Thus by Lemma 2.3.10 $Re(\phi + \nu)_0 \geq \sum_{j \notin F} c_j \beta_j = Re(\lambda)_0$. But $Re(\lambda) \geq Re(\phi + \nu)$, hence $Re(\lambda)_0 = Re(\phi + \nu)_0$. Therefore $F(Re(\lambda)_0) = F(Re(\phi + \nu)_0)$. Thus $\phi + \nu$ is in S_F and $z_i \geq 0$ for all i . The inclusion of $\mathbb{G}\mathbb{H}_P$ -modules $U \subset V$ induces a nonzero map $\pi : \mathbb{G}\mathbb{H} \otimes_{\mathbb{G}\mathbb{H}_P} U \rightarrow V$ given by $\pi(h \otimes w) = h.w$. Since V is irreducible, V is a quotient of $\mathbb{G}\mathbb{H} \otimes_{\mathbb{G}\mathbb{H}_P} U$. $\square$

This argument is very similar to the argument given by Evens for the case of graded Hecke algebras. Note that this argument implies that every weight $\mu' \otimes \lambda'$ of U has $F(Re(\lambda')) = F$.

Proof of (ii). The space U is naturally embedded in $\mathbb{G}\mathbb{H} \otimes_{\mathbb{G}\mathbb{H}_P} U$. U is a $\mathbb{G}\mathbb{H}_P$ -module therefore it is invariant under W_P . Lemma 2.3.11 implies that the weights of $\mathbb{G}\mathbb{H} \otimes_{\mathbb{G}\mathbb{H}_P} U$ are $w(\mu) \otimes w(\lambda)$ where $w \in W$ and $\mu \otimes \lambda$ is a weight of U . Considering the weights of $(\mathbb{G}\mathbb{H} \otimes_{\mathbb{G}\mathbb{H}_P} U)/U$, these are $w(\mu) \otimes w(\lambda)$ where $w \neq 1$ and is a coset representative of W/W_P , alternatively $w \in W^P = \{w \in W : w(R_P^+) \subset R^+\}$. Note that for all $w \in W$ one can write w as a product of $w^P \in W^P$ and $w_P \in W_P$.

Let $\mu \otimes \lambda$ be a weight of U , and write

$$Re(\lambda) = \sum_{j \notin F} c_j \beta_j - \sum_{j \in F} d_j \check{\alpha}_j, c_j > 0, d_j \geq 0.$$

Then if $w \in W^P$, $Re(w\lambda) = \sum_{j \notin F} c_j w\beta_j - \sum_{j \in F} d_j w\check{\alpha}_j$. Define $\rho : \mathfrak{t} \rightarrow \mathbb{C}$ by $\rho(\check{\alpha}) = 1, \check{\alpha} \in \check{\Pi}$. Since $w : \Pi_P \rightarrow R^+$ then $\rho(w(\check{\alpha}_i)) \geq \rho(\check{\alpha}_i)$, for $i \in F$. Since β_j is a fundamental weight, $w(\beta_j) \leq \beta_j$, with equality if and only if each expression of w as a product of simple reflections is such that each simple reflection fixes β_j . If we make this requirement for all $j \notin F$ then this implies $w \in W_P$. Hence $w \in W_P \cap W^P = \{1\}$, therefore $w = 1$. Thus we can assume that if $w \in W^P \setminus 1$ then $\rho(Re(w(\lambda))) < \rho(Re(\lambda))$.

Fix a weight $\mu \otimes \lambda$ such that $\rho(Re(\lambda))$ is maximal, then $\mu \otimes \lambda$ can not occur as a weight of $(\mathbb{G}\mathbb{H} \otimes_{\mathbb{G}\mathbb{H}_P} U)/U$. This implies that if a submodule Z of $\mathbb{G}\mathbb{H} \otimes_{\mathbb{G}\mathbb{H}_P} U$ contains $\mu \otimes \lambda$ then Z contains U and hence is $\mathbb{G}\mathbb{H} \otimes_{\mathbb{G}\mathbb{H}_P} U$. Define I_{max} to be the sum of all submodules of $\mathbb{G}\mathbb{H} \otimes_{\mathbb{G}\mathbb{H}_P} U$ which do not contain $\mu \otimes \lambda$ then I_{max} is maximal and $(\mathbb{G}\mathbb{H} \otimes_{\mathbb{G}\mathbb{H}_P} U)/I_{max}$ is the unique irreducible quotient. □

Proof of (iii). Suppose $\pi : J(P, U) \cong J(P', U')$. Let $\mu \otimes \lambda$ (respectively $\mu' \otimes \lambda'$) be a weight of U (respectively U') which is maximal with respect to ρ . Suppose $F(Re(\lambda)) \neq F(Re(\lambda'))$. Then it follows that $\mu \otimes \lambda$ is not a weight of U' and $\mu' \otimes \lambda'$ is not a weight of U . Therefore $\mu \otimes \lambda$ is a weight of $(\mathbb{G}\mathbb{H} \otimes_{\mathbb{G}\mathbb{H}_P} U')/U'$ which suggests that $\rho(Re(\lambda)) < \rho(Re(\lambda'))$. However exchanging λ with λ' suggests $\rho(Re(\lambda')) < \rho(Re(\lambda))$, which can not be the case. Hence $F(Re(\lambda)) = F(Re(\lambda'))$ and $P = P'$.

Since $J(P, U) \cong J(P', U')$ is irreducible Lemma 2.3.11 implies there exists a $w \in W$ such that $w(\mu) \otimes w(\lambda) = \mu' \otimes \lambda'$. If we suppose that $w \notin W_P$ then w has part of its decomposition in W^P , by the proof of (ii) this suggests that

$$\rho(Re(w(\lambda))) = \rho(Re(\lambda')) < \rho(Re(\lambda)).$$

However if this is the case then λ is not maximal with respect to ρ . Therefore $w(\mu) = \mu'$ where $w \in W_P$. $\pi(U) = U'$ since U (respectively U') is the unique $\mathbb{G}\mathbb{H}_P$ submodule which has a weight $\mu_1 \otimes \lambda_1$ such that $\rho(Re(\lambda)) = \rho(Re(\lambda_1))$ and μ is in the same W_P orbit as μ_1 . Similarly $\rho(Re(\lambda')) = \rho(Re(\lambda_1))$ and μ' is in the same W_P orbit as μ_1 . Hence $U \cong U'$ as $\mathbb{G}\mathbb{H}_P$ submodules. □

2.4 Dunkl-Opdam subalgebra

In this section, we study the Dunkl-Opdam subalgebra $\mathbb{H}_{DO}$, defined in [22]. The algebra $\mathbb{H}_{DO}$ is a subalgebra of the rational Cherednik algebra associated to the complex reflection group $G(m, 1, n)$. This subalgebra exists for any parameter $\underline{t}$ and its existence is independent of the parameters $\mathbf{c}_1, \dots, \mathbf{c}_{m-1}$. We show that $\mathbb{H}_{DO}$ is a naturally occurring example of a non-faithful Drinfeld algebra. Section 2.2.2 endows $\mathbb{H}_{DO}$ with a Dirac operator. From the defining presentation given by [22], this subalgebra is a generalized Hecke algebra. Therefore we have a Langlands classification for $\mathbb{H}_{DO}$. This then sets up Section 2.5, in which we describe the representation theory of $\mathbb{H}_{DO}$ as blocks corresponding to multi-partitions and representations of the graded Hecke algebra of type A.

2.4.1 The rational Cherednik algebra

Dunkl and Opdam [22] introduced the rational Cherednik algebra. Let $G \subset GL(V)$ be a complex reflection group with reflections $\mathcal{S}$. Let $\langle, \rangle$ be the natural pairing of V and V^* . Let $\alpha_s \in V$ be a λ eigenvector for $s \in \mathcal{S}$ and let $v_s \in V^*$ be a λ^{-1} eigenvectors for $s \in \mathcal{S}$ such that $\lambda \neq 1$, and $\langle \alpha_s, v_s \rangle = 1$. For every reflection $s \in \mathcal{S}$ introduce the parameters $\underline{t}, \mathbf{c}_s \in \mathbb{C}$ such that $\mathbf{c}_s = \mathbf{c}_{s'}$ if s' and s are in the same conjugacy class. The rational Cherednik algebra is defined as the quotient of the associative $\mathbb{C}$ -algebra

$$T(V \oplus V^*) \rtimes \mathbb{C}[G]$$

by the relations

$$\begin{aligned} [x, x'] &= [y, y'] = 0, & \text{for all } x, x' \in V, y, y' \in V^*, \\ [x, y] &= \underline{t}\langle x, y \rangle - \sum_{s \in \mathcal{S}} \mathbf{c}_s \frac{\langle \alpha_s, y \rangle \langle x, v_s \rangle}{\langle \alpha_s, v_s \rangle} s & \forall x \in V, y \in V^*, \\ g^{-1}vg &= g(v), & \forall v \in V \text{ or } V^*. \end{aligned}$$

If one restricts to rational Cherednik algebras associated to classical complex groups $G(m, p, n)$ then [22] shows there is a set of commuting operators inside the rational Cherednik algebra. The main part of these operators is quadratic on a special basis of V and V^* . We give a particular presentation of the rational Cherednik algebra associated to $G(m, 1, n) \cong S_n \rtimes (\mathbb{Z}_M)^n$.

Define a generating set for $G(m, 1, n)$ consisting of the reflections $\{s_{i,i+1} : i = 1, \dots, n-1\}$ in S_n and the reflections $\{g_i : i = 1, \dots, n\}$ which have order m , we may write s_i for $s_{i,i+1}$. Let η be a primitive m^{th} root of unity. Given $G(m, 1, n)$ acting on V let $x_i \in V$ be the vectors such that $w(x_i) = x_{w(i)}$ for $w \in S_n$ and

$$g_i(x_j) = \begin{cases} \eta^{-1}x_j & \text{if } i = j, \\ x_j & \text{otherwise.} \end{cases}$$

Let $\{y_1, \dots, y_n\} \in V^*$ be the dual basis to $\{x_1, \dots, x_n\}$. For $G = G(m, 1, n)$ there are $m+1$ conjugacy classes of reflections, for reflections in the conjugacy class of $s_{1,2}$ then let $\mathbf{k} \in \mathbb{C}$ denote their parameter. For reflections conjugate to g_1^l denote the parameter by $\mathbf{c}_l \in \mathbb{C}$.

Definition 2.4.1. *The rational Cherednik algebra for $G(m, 1, n)$ and parameters $\mathbf{k}, \mathbf{c}_l, \underline{t}$, $\mathcal{H}_t(G(m, 1, n))$ is the quotient of the $\mathbb{C}$ -algebra $T(V \oplus V^*) \rtimes \mathbb{C}[G(m, 1, n)]$ by the relations*

$$\begin{aligned} [x_i, x_j] &= [y_i, y_j] = 0, \\ [x_i, y_i] &= \underline{t} - \mathbf{k} \sum_{l=1}^{m-1} \sum_{i \neq j} s_{i,j} g_i^{-l} g_j^l - \sum_{l=1}^{m-1} \mathbf{c}_l g_i^l, \\ [x_i, y_j] &= \mathbf{k} \sum_{l=1}^{m-1} s_{i,j} g_i^{-l} g_j^l, \\ g^{-1}vg &= g(v), \text{ for all } v \in V \text{ or } V^*. \end{aligned}$$

2.4.2 Dunkl-Opdam quadratic operators

Definition 2.4.2. *For $i \leq n$ define elements in $\mathcal{H}_t(G(m, 1, n))$*

$$\begin{aligned} z_i &= y_i x_i + \mathbf{k} \sum_{l=1}^{m-1} \sum_{i > j} s_{i,j} g_i^{-l} g_j^l + \frac{1}{2} \sum_{l=1}^{m-1} \mathbf{c}_l g_i^l + \frac{1}{2} \underline{t}, \\ &= x_i y_i - \mathbf{k} \sum_{l=1}^{m-1} \sum_{i < j} s_{i,j} g_i^{-l} g_j^l - \frac{1}{2} \sum_{l=1}^{m-1} \mathbf{c}_l g_i^l - \frac{1}{2} \underline{t}. \end{aligned}$$

These operators were defined in [22] and also appeared in [27] for the specialization at $\underline{t} = 1$. Martino [40] used them for general $\underline{t}$ to study the blocks of the rational Cherednik algebra.

Definition 2.4.3. *The Dunkl-Opdam subalgebra $\mathbb{H}_{DO}(G(m, 1, n))$ of the rational Cherednik algebra is the subalgebra generated by $G(m, 1, n)$ and z_i for $i = 1, \dots, n$.*

Remark 2.4.4. *The following relations hold in $\mathbb{H}_{DO}(G(m, 1, n))$*

$$\begin{aligned} [z_i, z_j] &= 0 && \text{for } i, j = 1, \dots, n, \\ [z_i, g_k] &= 0, && \forall i, k = 1, \dots, n, \\ [z_j, s_{i,i+1}] &= 0 && \text{for } j \neq i, i+1, \\ z_i s_{i,i+1} &= s_{i,i+1} z_{i+1} - \mathbf{k} \epsilon_{ij}. \end{aligned}$$

Here $\epsilon_{ij} = \sum_{l=0}^{m-1} g_i^l g_j^{-l}$.

In fact $\mathbb{H}_{DO}(G(m, 1, n))$ is isomorphic to the associative $\mathbb{C}[\mathbf{k}]$ - algebra generated by z_i and $G(m, 1, n)$ subject to the relations stated in Remark 0.2.16.

2.4.3 Dunkl-Opdam subalgebra admits a non-faithful Drinfeld presentation

In this section, we derive a new presentation of $\mathbb{H}_{DO}$ which demonstrates that $\mathbb{H}_{DO}$ is a non-faithful Drinfeld algebra. Thus, simultaneously showing that one can associate a Drinfeld algebra to $G(m, 1, n)$, and also uncovering a natural example of a non-faithful Drinfeld algebra.

We introduce Jucys-Murphy elements for $G(m, 1, n)$, these are well known. However, the tool that we use here is that we consider two different sets of Jucys-Murphy elements.

Definition 2.4.5. *We define Jucys-Murphy elements for $G(m, 1, n)$.*

$$\begin{aligned} M_i &= \sum_{k < i} \sum_{s=0}^{m-1} s_{k,i} g_k^{-s} g_i^s, \\ \underline{M}_i &= \sum_{k > i} \sum_{s=0}^{m-1} s_{i,k} g_i^{-s} g_k^s. \end{aligned}$$

The commutator $[M_i, M_j] = 0 = [\underline{M}_i, \underline{M}_j]$ by a standard argument using the fact that $\sum_{j \leq i} M_i$ is in the centralizer of the subgroup generated by $\{s_{k-1,k}, g_k : k \leq i\}$. It should be noted that $[M_i, \underline{M}_j] \neq 0$ for $i > j$. Furthermore

$$s_i M_i = M_{i+1} s_i - \sum_{s=0}^{m-1} g_i^{-s} g_{i+1}^s$$

and

$$s_i \underline{M}_i = \underline{M}_{i+1} s_i + \sum_{s=0}^{m-1} g_i^{-s} g_{i+1}^s.$$

If we adjust z_i by $-\mathbf{k}M_i$ or $\mathbf{k}\underline{M}_i$, $\hat{z}_i = z_i - \mathbf{k}M_i$ ($\hat{z}_i = z_i + \mathbf{k}\underline{M}_i$ respectively) the elements $\hat{z}_i$ satisfy the relations $s_i \hat{z}_i s_i = \hat{z}_{i+1}$ and $g_j \hat{z}_i g_j^{-1} = \hat{z}_i$. Hence we obtain an action of $G(m, 1, n)$ on the set $\hat{z}_i$. However the set $\{\hat{z}_i\}$ no longer commutes. This presentation was given in [22, Corollary 3.6] where the symbol $T_i x_i$ denotes $\hat{z}_i$. We provide an exposition of the presentation with $\{\hat{z}_i\}$ using a simple automorphism of $\mathbb{H}_{DO}$.

Corollary 2.4.6. [22, Corollary 3.6] Let $\hat{z}_i = z_i - \mathbf{k}M_i$ then $\hat{z}_i$ and G generate $\mathbb{H}_{DO}$ and the following relations define $\mathbb{H}_{DO}(G(m, 1, n))$:

$$\begin{aligned} s_j \hat{z}_i &= \hat{z}_{s_j(i)} s_j, \\ [g_i, \hat{z}_j] &= 0, \\ [\hat{z}_i, \hat{z}_j] &= \mathbf{k}(\hat{z}_i - \hat{z}_j) \sum_{s=0}^{m-1} s_{s_i, j} g_i^{-s} g_j^s. \end{aligned}$$

Lemma 2.4.7. Let $\Phi : \mathbb{H}_{DO} \rightarrow \mathbb{H}_{DO}$ such that

$$\begin{aligned} \Phi(z_i) &= -z_{n+1-i}, \\ \Phi(s_i) &= s_{n-i}, \\ \Phi(g_i) &= g_{n+1-i}. \end{aligned}$$

The map Φ is an automorphism of $\mathbb{H}_{DO}$. Furthermore $\Phi(M_i) = \underline{M}_{n+1-i}$.

Proof.

$$\Phi(s_i z_i - z_{i+1} s_i - \mathbf{k}\epsilon_{i, i+1}) = -s_{n-i} z_{n+1-i} + z_{n-i} s_{n-i} - \mathbf{k}\epsilon_{n-i, n+1-i}.$$

We formally define Φ as a map from $\mathbb{C}G \rtimes S(V) \rightarrow \mathbb{H}_{DO}$ then Φ takes the set of defining relations in $\mathbb{H}_{DO}$ to itself. Φ is surjective since it takes generators to generators. Hence we can define Φ as an automorphism of $\mathbb{H}_{DO}$. $\square$

Using Φ we define the presentation of $\mathbb{H}_{DO}$ with generators $\{\hat{z}_i\}$.

Lemma 2.4.8. *Let $\hat{z}_i = z_i + \mathbf{k}\underline{M}_i$, then the set $\hat{z}_i$ and G generate $\mathbb{H}_{DO}$. Furthermore, the following relations hold:*

$$[\hat{z}_i, \hat{z}_j] = -\mathbf{k}(\hat{z}_i - \hat{z}_j) \sum_{s=0}^{m-1} s_{s_{i,j}} g_i^{-s} g_j^s,$$

$$s_i \hat{z}_i = \hat{z}_{i+1} s_i,$$

$$[s_i, \hat{z}_j] = 0, \forall j \neq i, i+1,$$

$$[g_i, \hat{z}_j] = 0.$$

Proof. $\Phi(\hat{z}_i) = \Phi(z_i - \mathbf{k}M_i) = -z_{n+1-i} - \mathbf{k}\underline{M}_{n+1-i} = -\hat{z}_{n+1-i}$. Hence $\hat{z}_i$ and G generate $\mathbb{H}_{DO}$ since they are images of a generating set under the automorphism Φ . From the definition of Φ ,

$$[\hat{z}_i, \hat{z}_j] = \Phi([- \hat{z}_{n+1-i}, - \hat{z}_{n+1-j}]) = \Phi([\hat{z}_{n+1-i}, \hat{z}_{n+1-j}])$$

Expanding $[\hat{z}_{n+1-i}, \hat{z}_{n+1-j}]$ with Corollary 2.4.6 one obtains:

$$\begin{aligned} [\hat{z}_i, \hat{z}_j] &= \Phi \left(\mathbf{k}(\hat{z}_{n+1-i} - \hat{z}_{n+1-j}) \left(\sum_{s=0}^{m-1} s_{(n+1-i, n+1-j)} g_{n+1-i}^{-s} g_{n+1-j}^s \right) \right) \\ &= -\mathbf{k}(\hat{z}_i - \hat{z}_j) \left(\sum_{s=0}^{m-1} s_{s_{i,j}} g_i^{-s} g_j^s \right). \end{aligned}$$

Similarly

$$\begin{aligned} s_i \hat{z}_i &= -\Phi(s_{(n+1-i, n-i)} \hat{z}_{n+1-i}) = -\Phi(\hat{z}_{n-i} s_{(n+1-i, n-i)}) \\ &= \hat{z}_{i+1} s_i. \end{aligned}$$

The relations we gave are images of relations in the second presentation under Φ . Furthermore, the generators and relations are exactly the images of the generators and relations of a presentation. Hence 2.4.8 gives another presentation of $\mathbb{H}_{DO}$. $\square$

We will now work towards a fourth presentation of $\mathbb{H}_{DO}$. We are aiming for a Drinfeld presentation of $\mathbb{H}_{DO}$. We require the commutator $[\hat{z}_i, \hat{z}_j]$ to be an element of the group algebra. We now work through several Lemmas to prove that we can alter z_i by $\frac{\mathbf{k}}{2}(M_i - \underline{M}_i)$ to give the Drinfeld presentation we expect.

We observe that the commutators of the set $\{\hat{z}_i\}$, and similarly $\{\hat{z}_j\}$ can be expressed as commutators of the Jucys-Murphy elements and z_i .

Lemma 2.4.9. *The commutator of the operators $\hat{z}_i$ and $\hat{\underline{z}}_i$ are such that:*

$$[\hat{z}_i, \hat{z}_j] = \mathbf{k}([z_j, M_i] - [z_i, M_j]).$$

Similarly for $\hat{\underline{z}}_i$ and $\underline{M}_i$.

$$[\hat{\underline{z}}_i, \hat{\underline{z}}_j] = \mathbf{k}([z_i, \underline{M}_j] - [z_j, \underline{M}_i]).$$

Proof.

$$\begin{aligned} [\hat{z}_i, \hat{z}_j] &= [z_i - \mathbf{k}M_i, z_j - \mathbf{k}M_j] = [z_i, z_j] - \mathbf{k}[z_i, M_j] + \mathbf{k}[z_j, M_i] + \mathbf{k}^2[M_i, M_j], \\ &= \mathbf{k}([z_j, M_i] - [z_i, M_j]), \end{aligned}$$

since $[M_i, M_j] = [\underline{M}_i, \underline{M}_j] = [z_i, z_j] = 0$.

□

Lemma 2.4.10. *For operators $\hat{z}_i$ and $\hat{\underline{z}}_j$,*

$$[\hat{z}_i, \hat{z}_j] + [\hat{\underline{z}}_i, \hat{\underline{z}}_j] \in \mathbb{C}G.$$

Proof. Using Corollary 2.4.6 and Lemma 2.4.8 we can expand the commutators:

$$\begin{aligned} &[\hat{z}_i, \hat{z}_j] + [\hat{\underline{z}}_i, \hat{\underline{z}}_j] \\ &= (\hat{z}_i - \hat{z}_j) \sum_{s=0}^{m-1} s_{i,j} g_i^{-s} g_j^s - (\hat{\underline{z}}_i - \hat{\underline{z}}_j) \sum_{s=0}^{m-1} s_{s_i,j} g_i^{-s} g_j^s. \end{aligned}$$

Writing out $\hat{z}_i$ and $\hat{\underline{z}}_i$ in terms of the commuting operators z_i one obtains

$$= (z_i - \mathbf{k}M_i - z_j + \mathbf{k}M_j) \sum_{s=0}^{m-1} s_{i,j} g_i^{-s} g_j^s - (z_i + \mathbf{k}\underline{M}_i - z_j - \mathbf{k}\underline{M}_j) \sum_{s=0}^{m-1} s_{s_i,j} g_i^{-s} g_j^s.$$

Cancelling out the operators z_i and z_j we arrive at the element of the group algebra

$$[\hat{z}_i, \hat{z}_j] + [\hat{\underline{z}}_i, \hat{\underline{z}}_j] = \mathbf{k}(M_j + \underline{M}_j - M_i - \underline{M}_i) \sum_{s=0}^{m-1} s_{i,j} g_i^{-s} g_j^s \in \mathbb{C}G.$$

□

Lemma 2.4.11. *The commutators of $z_i - \frac{\mathbf{k}}{2}M_i + \frac{\mathbf{k}}{2}\underline{M}_i$ are in the group algebra, explicitly:*

$$[z_i - \frac{\mathbf{k}}{2}M_i + \frac{\mathbf{k}}{2}\underline{M}_i, z_j - \frac{\mathbf{k}}{2}M_j + \frac{\mathbf{k}}{2}\underline{M}_j] \in \mathbb{C}G.$$

Proof. Expanding out the commutator linearly:

$$\begin{aligned}
& [z_i - \frac{\mathbf{k}}{2}M_i + \frac{\mathbf{k}}{2}\underline{M}_i, z_j - \frac{\mathbf{k}}{2}M_j + \frac{\mathbf{k}}{2}\underline{M}_j] \\
&= [z_i, z_j] + \frac{\mathbf{k}}{2} ([z_j, M_i] - [z_i, M_j] + [z_i, \underline{M}_j] - [z_j, \underline{M}_i]) \\
&+ \left(\frac{\mathbf{k}}{2}\right)^2 ([M_i, M_j] - [M_i, \underline{M}_j] - [\underline{M}_i, M_j] + [\underline{M}_i, \underline{M}_j])
\end{aligned}$$

Using Lemma 2.4.9:

$$\begin{aligned}
&= \frac{1}{2} ([\hat{z}_i, \hat{z}_j] + [\hat{z}_i, \hat{z}_j]) - \left(\frac{\mathbf{k}}{2}\right)^2 ([M_i, \underline{M}_j] + [\underline{M}_i, M_j]) \in \mathbb{C}G, \\
&= \frac{\mathbf{k}}{2} (M_j + \underline{M}_j - M_i - \underline{M}_i) \sum_{s=0}^{m-1} s_{s_{i,j}} g_i^{-s} g_j^s - \left(\frac{\mathbf{k}}{2}\right)^2 ([M_i, \underline{M}_j] + [\underline{M}_i, M_j]) \in \mathbb{C}G.
\end{aligned}$$

□

Lemma 2.4.12. $s_j(z_i - \frac{\mathbf{k}}{2}M_i + \frac{\mathbf{k}}{2}\underline{M}_i)s_j^{-1} = (z_{s_j(i)} - \frac{\mathbf{k}}{2}M_{s_j(i)} + \frac{\mathbf{k}}{2}\underline{M}_{s_j(i)}).$

Proof. The result follows from compiling these three relations:

$$s_i z_i = z_{i+1} s_i + \mathbf{k} \epsilon_{i,i+1},$$

$$s_i M_i = M_{i+1} s_i + \epsilon_{i,i+1},$$

$$s_i \underline{M}_i = \underline{M}_{i+1} s_i - \epsilon_{i,i+1}.$$

□

Theorem 2.4.13. *There exists a presentation of $\mathbb{H}_{DO}$ given by elements $\{\tilde{z}_i : i = 1, \dots, n\}$ and generators $\{s_i, g_j : i \leq i \leq n-1, 1 \leq j \leq n\}$ in $G(m, 1, n)$ such that:*

$$s_i \tilde{z}_j s_i^{-1} = s_i(\tilde{z}_j),$$

$$g_i \tilde{z}_j = \tilde{z}_j g_i \quad \forall i, j = 1, \dots, n,$$

$$[\tilde{z}_i, \tilde{z}_j] \in \mathbb{C}G(m, 1, n),$$

where $G(m, 1, n) \cong S_n \rtimes (\mathbb{Z}_m)^n$. The group S_n is generated by transpositions s_i , $i = 1, \dots, n-1$ and g_j generates the j^{th} copy of $\mathbb{Z}_m$.

Proof. Let $\tilde{z}_i = \frac{1}{2}(\hat{z}_i + \hat{z}_j) = z_i - \frac{k}{2}(M_i - \underline{M}_i)$ then the first two relations follows from Lemma 2.4.12 and by Lemma 2.4.11 their commutant is in $\mathbb{C}G$. One may be worried that we have defined an algebra that surjects onto $\mathbb{H}_{DO}$ but does not procure an injection. However performing the above arguments in reverse setting $z_i = \tilde{z}_i + \frac{k}{2}(M_i - \underline{M}_i)$ shows that the original relations follow from these relations. $\square$

Definition 2.4.14. Give V a basis $\{v_i\}$ and recall that S_n act on this basis by permutations. Let π be the homomorphism of $G(m, 1, n)$ onto S_n . (V, ϕ) is the standard representation of S_n , now define (V, ρ) to be the representation of G via the projection onto S_n , that is, $\rho : G \rightarrow GL(V)$ via $\rho(g) = \phi(\pi(g))$. We define skew-symmetric forms on V for elements in $G(m, 1, n)$:

$$b_{s_{ij}s_{jk}}(v_p, v_q) = \underline{k}^2 (\langle \epsilon_i - \epsilon_j, v_p \rangle \langle \epsilon_j - \epsilon_k, v_q \rangle - \langle \epsilon_i - \epsilon_j, v_q \rangle \langle \epsilon_j - \epsilon_k, v_p \rangle),$$

for $0 < i < j < k \leq n$,

$$b_{s_{ij}g_i^l g_j^{-l} s_{jk} g_j^{l'} l g_k^{-l'}} = b_{s_{ij}s_{jk}} \text{ for all } l, l' = 0, \dots, m-1$$

$b_g = 0$ otherwise.

Theorem 2.4.15. The algebra $\mathbb{H}_{DO}$ is a Drinfeld algebra. More concretely $\mathbb{H}_{DO}$ is isomorphic to $\mathbb{C}G \rtimes T(V)$ with the relations:

$$[u, v] = \sum_{g \in G} b_g(u, v)g \quad \forall u, v \in V,$$

where b_g are skew-symmetric forms on V defined in Definition 2.4.14.

Proof. Conjugating b_g by g_i must fix b_g since g_i acts trivially on V . Taking the quotient of $\mathbb{H}_{DO}(G(m, 1, n))$ by the two sided ideal generated by the elements $\{g_i - 1 : 1 \leq i \leq n\}$ produces an algebra isomorphic to the graded Hecke algebra of type A, $\mathbb{H}(S_n)$. Hence the forms b_g must agree with, under the quotient, the forms that construct the Drinfeld presentation of the graded Hecke algebra for S_n . The forms $b_{s_{ij}s_{jk}}$ descend to the forms, labelled by the same element, defining the graded Hecke algebra as a Drinfeld algebra. Conjugating by various g_i^l gives the forms $b_{s_{ij}g_i^l g_j^{-l} s_{jk} g_j^{l'} l g_k^{-l'}}$ above. Since $b_1 = 0$ in $\mathbb{H}(S_n)$ then $b_k = 0$ for all $k \in \ker \rho$. There are no other elements of $G(m, 1, n)$ such that $\dim V^\rho(g) = \dim V - 2$, therefore the rest of the $b_g = 0$ for all g not mentioned above. $\square$

2.4.4 Dunkl-Opdam subalgebra is a generalised graded Hecke algebra

Recall Definition 2.3.1 of the generalised graded Hecke algebra associated to the root system of W , T and parameter function $\tilde{c}$. Let $\epsilon_{ij} = \sum_{l=1}^{m-1} g_i^l g_j^{-l} \in \mathbb{C}G(m, 1, n)$. The algebra $\mathbb{H}_{DO}$ is isomorphic to, as a vector space, $\mathbb{C}[G(m, 1, n)] \otimes S(V)$, with multiplication such that $\mathbb{C}[G(m, 1, n)]$ and $S(V)$ are subalgebras and the following cross relations hold;

$$\begin{aligned} [z_i, g_k] &= 0, & \forall i, k = 1, \dots, n, \\ [z_j, s_{i,i+1}] &= 0 & \text{for } j \neq i, i+1, \\ z_i s_{i,i+1} &= (i, i+1) z_{i+1} - \mathbf{k} \epsilon_{ij}. \end{aligned}$$

If we substitute $G(m, 1, n) \cong S_n \rtimes (\mathbb{Z}/m\mathbb{Z})^n$ for $W \rtimes T$ then $\mathbb{H}_{DO}$ is a generalised Hecke algebra with parameter function

$$\tilde{c}(s_{ij}) = \mathbf{k} \epsilon_{ij} = \sum_{l=0}^{m-1} g_j^l g_j^{-l} \in \mathbb{C}[(\mathbb{Z}_m)^n].$$

Since $\mathbb{H}_{DO}$ is a generalised graded Hecke algebra we can apply the Langlands classification from Section 2.3.2. Therefore we can construct every representation of $\mathbb{H}_{DO}$ as a quotient of the module induced from a tempered module of a parabolic subalgebra.

Corollary 2.4.16. *Let $U = \check{U} \otimes \mathbb{C}_\nu$ be such that $\check{U}$ is an irreducible tempered $\mathbb{H}_{DO}$ -module and ν is a character of $\mathfrak{a}^{*+}$. Every irreducible representation of $\mathbb{H}_{DO}$ can be constructed as a quotient of a tempered module of an irreducible parabolic subalgebra $\mathbb{H}_P$. That is it is a quotient of*

$$\mathbb{H}_{DO} \otimes_{\mathbb{H}_P} U.$$

2.5 Constructing the representations of $\mathbb{H}(G(m, 1, n))$ from $\mathbb{H}(S_n)$

In this section we prove the irreducible representations of $\mathbb{H}(G(m, 1, n))$ can be built up from blocks of irreducible representations of the graded Hecke algebras associated to the symmetric group. We then get a Functor between the semisimplification of the category of $\mathbb{H}(G(m, 1, n))$ modules and the semisimplification of the category of $\bigoplus_{a \in A} \bigotimes \mathbb{H}_{s_{a_i}}$. Let $K_0(\mathbb{H}(G(m, 1, n)))$ denote the Grothendieck group of representation

of $\mathbb{H}(G(m, 1, n))$, this is the smisimplification of th category of $\mathbb{H}(G(m, 1, n))$ modules. This is very similar to how one can build the representations of $W(B_n)$ from the pullback of two representations of symmetric groups, S_a and S_b , where $a + b = n$

We denote the usual graded Hecke algebra of type A_{k-1} by $\mathbb{H}(S_k)$, η denotes a fixed primitive m^{th} root of unity.

We define $\mathbb{N}$ to include zero. Let $A \subset \mathbb{N}^m$ be the set of vectors such that the coordinates sum to n . Then let $\underline{a} = (a_0, \dots, a_{m-1})$ be a vector in A , explicitly $\sum_{i=0}^{m-1} a_i = n$. We define the character $\mu_{\underline{a}} \in \mathbb{C}[T]^*$ by

$$\mu_{\underline{a}}(g_j) = \eta^j$$

where $\sum_{k=0}^{i-1} a_k < j \leq \sum_{k=0}^i a_k$.

This character takes the first a_0 reflections to 1 it then takes the following a_1 reflections to η then the following a_2 to η^2 and continues in this way. Finally it takes the last a_{m-1} reflections to η^{m-1} . The set A will become a parametrising set.

Example 2.5.1. Let $n = 5$ and $m = 3$, define ω to be a primitive 3rd root of unity. The character of $\mathbb{C}[(\mathbb{Z}/3)^5]$ associated to the vector $(1, 1, 3)$ is such that;

$$\begin{aligned} \mu_{(1,1,3)}(g_1) &= 1, & \mu_{(1,1,3)}(g_2) &= \omega, \\ \mu_{(1,1,3)}(g_3) &= \mu_{(1,1,2)}(g_4) = \mu_{(1,1,3)}(g_5) &= \omega^2. \end{aligned}$$

If we take the S_n orbits of $\mathbb{C}[T]^*$ then a representative set of these orbits is

$$\{\mu_{\underline{a}} | \underline{a} \in A\}.$$

Let $\text{irr}(\mathbb{H}(G(m, 1, n)))$ be the set of isomorphism classes of irreducible modules for $\mathbb{H}(G(m, 1, n))$. We define $\text{irr}(\mathbb{H}(G(m, 1, n)) | \mu_{\underline{a}})$ to be the subset of $\text{irr}(\mathbb{H}(G(m, 1, n)))$ consisting of representations that have a weight $\mu_{\underline{a}} \otimes \lambda$ for any $\lambda \in S(V)^*$. Similarly, we will denote the set of irreducible representations of a complex algebra B by $\text{irr}(B)$.

Lemma 2.5.2. The irreducible representations of $\mathbb{H}(G(m, 1, n))$ split into disjoint sets labelled by A ,

$$\text{irr}(\mathbb{H}(G(m, 1, n))) = \bigsqcup_{\underline{a} \in A} \text{irr}(\mathbb{H}(G(m, 1, n)) | \mu_{\underline{a}}).$$

Proof. Since every irreducible representation of $\mathbb{H}(G(m, 1, n))$ has at least one $\mathbb{C}[T]$ weight then by Lemma 2.3.11 it must contain one and only one S_n orbit, hence it must contain exactly one $\mu_{\underline{a}}$. Therefore every irreducible representation V is in exactly one of the sets $\text{irr}(\mathbb{H}(G(m, 1, n))|\mu_{\underline{a}})$. $\square$

Let $S_{a_0} \times S_{a_1} \times \dots S_{a_{m-1}}$ be the parabolic subgroup of S_n generated by s_{α} for

$$\Pi_{\underline{a}} = \{\epsilon_i - \epsilon_{i+1} \mid \sum_{k=0}^{j-1} a_k \leq i < \sum_{k=0}^j a_k \text{ for some } j\} \subset \Pi.$$

Fix $\underline{a} \in A$. The stabiliser, $\text{stab}(\mu_{\underline{a}}) \subset \mathbb{H}(G(m, 1, n))$, of the character $\mu_{\underline{a}}$ is generated by $\mathbb{C}[T]$, $S(V)$ and $s_i \in S_{a_0} \times S_{a_1} \times \dots S_{a_{m-1}} \subset S_n$. The set $\Pi_{\underline{a}}$ is equivalent to the set of simple roots $\epsilon_i - \epsilon_{i+1}$ such that $\mu_{\underline{a}}(g_i) = \mu_{\underline{a}}(g_{i+1})$. This is the parabolic subalgebra associated to the subset $\Pi_{\underline{a}} \subset \Pi$ defined in Definition 2.3.2.

Lemma 2.5.3. *The subalgebra $\text{stab}(\mu_{\underline{a}})$ which stabilises the character $\mu_{\underline{a}}$ is isomorphic to $\mathbb{H}(G(a_0, 1, m)) \otimes \mathbb{H}(G(a_1, 1, m)) \otimes \dots \otimes \mathbb{H}(G(a_{m-1}, 1, m))$.*

Proof. The subalgebra generated by $S_{a_0} \times S_{a_1} \times \dots S_{a_{m-1}}$, $\mathbb{C}[T]$ and $S(V)$ certainly contains $\mathbb{H}(G(a_i, 1, m))$ for every $i = 0, \dots, m-1$. The algebra $\mathbb{H}(G(a_i, 1, m))$ consists, as a vector space of $S(V_i) \otimes \mathbb{C}[T_i] \otimes S_{a_i}$ where V_i is the span of ϵ_j , and $\mathbb{C}[T_i]$ is generated by g_j such that $\sum_{k=0}^{i-1} a_k < j \leq \sum_{k=0}^i a_k$. We have $V_0 \oplus \dots \oplus V_{m-1} = V$ hence $S(V_0) \otimes \dots \otimes S(V_{m-1}) = S(V_0 \oplus \dots \oplus V_{m-1}) = S(V)$. Similarly $\mathbb{C}[T_0] \otimes \dots \otimes \mathbb{C}[T_{m-1}] = \mathbb{C}[T]$, and $\mathbb{C}[S_{a_0}] \otimes \dots \otimes \mathbb{C}[S_{a_{m-1}}] = \mathbb{C}[S_{a_0} \times \dots \times S_{a_{m-1}}]$. Hence as a vector space:

$$\text{stab}(\mu_{\underline{a}}) = \bigotimes_{i=0}^{m-1} S(V_i) \otimes \mathbb{C}[T_i] \otimes \mathbb{C}[S_{a_i}].$$

Each $\mathbb{H}(G(a_i, 1, m))$ is a subalgebra and as vector spaces we have equality, one just needs to check that each subalgebra commutes with the other subalgebras. We already know S_{a_i} and S_{a_j} commute, for $i \neq j$, and S_{a_i} commutes with $\mathbb{C}[T_j]$ because S_{a_i} fixes T_j . Similarly S_{a_i} fixes V_j so $s_{\alpha_i} \in S_{a_i}$ commutes with $\epsilon_j \in S(V_j)$. $\square$

Lemma 2.5.4. *The set of irreducible representations $\text{irr}(\mathbb{H}(G(m, 1, n))|\mu_{\underline{a}})$ is in natural one-to-one correspondence with $\text{irr}(\text{stab}(\underline{a})|\mu_{\underline{a}})$. The bijection F is defined by*

$$F^{-1} : \text{irr}(\text{stab}(\underline{a})|\mu_{\underline{a}}) \rightarrow \text{irr}(\mathbb{H}(G(m, 1, n))|\mu_{\underline{a}}),$$

$$F^{-1}(W) = \text{Ind}_{\text{stab}(\underline{a})}^{\mathbb{H}(G(m, 1, n))} W,$$

and

$$F : \text{irr}(\mathbb{H}(G(m, 1, n)) | \mu_{\underline{a}}) \rightarrow \text{irr}(\text{stab}(\underline{a}) | \mu_{\underline{a}}),$$

$$F(U) = \text{Unique irreducible submodule of } \text{Res}_{\text{stab}(\underline{a})}^{\mathbb{H}(G(m, 1, n))} U \text{ with weight } \mu_{\underline{a}}.$$

For an irreducible module U in $\text{irr}(\mathbb{H}(G(m, 1, n)) | \mu_{\underline{a}})$, $F^{-1}(U)$ is the $\mu_{\underline{a}}$ -weight space of U .

Proof. Let P be the corresponding partition defined by the subset $\Pi_{\underline{a}} \subset \Pi$ and set C to be the set of shortest length coset representatives of the parabolic group S_P in S_n . For a $\text{stab}(\underline{a})$ -module W the module W^c , for $c \in C$, is isomorphic to W as a vector space with the action

$$b \cdot W^c = c^{-1}bcW \text{ for all } b \in \text{stab}(\underline{a}).$$

We must check that $F^{-1}(W)$ for $W \in \text{irr}(\text{stab}(\mu_{\underline{a}}))$ is an irreducible $\mathbb{H}(G(m, 1, n))$ -module, the rest follows. Here C is a set of shortest length coset representative of the parabolic group S_P in S_n . Since C becomes a basis for the quotient vector space $\mathbb{H}(G(m, 1, n)) / \text{stab}(\underline{a})$ then as a vector space $\text{Ind}_{\text{stab}(\underline{a})}^{\mathbb{H}} W$ is $\bigoplus_{c \in C} cW$. The space W is a $\text{stab}(\underline{a})$ -submodule which can be taken as the first composition factor. Then cW with c of shortest length can be taken as the next submodule in the quotient $\text{Ind } W / W$. Continuing in this manner each composition factor is isomorphic to W^c for the labeling set C . This statement about composition factors is also proved in [19, Proposition 3.5]. The $\text{stab}(\underline{a})$ -composition series of $\text{Ind}_{\text{stab}(\underline{a})}^{\mathbb{H}(G(m, 1, n))} W$ consists of the $\text{stab}(\underline{a})$ -modules W^c where $c \in C$. We must show that $\text{Ind}_{\text{stab}(\underline{a})}^{\mathbb{H}(G(m, 1, n))} W$ is irreducible. The module W is an irreducible $\text{stab}(\underline{a})$ -module. If W is an irreducible $G(m, 1, a_0) \times \dots \times G(m, 1, a_{m-1})$ -module then utilising Mackey's criterion for finite groups we need to show that W^c are not isomorphic. However by construction W has only weights containing $\mu_{\underline{a}}$, and W^c will only have weights containing $c(\mu_{\underline{a}})$, and since S_P is the stabiliser of $\mu_{\underline{a}}$ in S_n then for all $c \neq 1$ we have $c(\mu_{\underline{a}}) \neq \mu_{\underline{a}}$. Therefore each W^c has a different set of weights and hence are not isomorphic. Hence if W is an irreducible $G(m, 1, n)$ -module then using Mackey's irreducibility criterion $\text{Ind}_{\text{stab}(\underline{a})}^{\mathbb{H}(G(m, 1, n))} W$ is an irreducible $G(m, 1, n)$ -module and hence $F(W)$ is irreducible as an $\mathbb{H}(G(m, 1, n))$ -module.

The module W decomposes as a direct sum of irreducibles $W = \bigoplus V_i$ as irreducible modules of $G(m, 1, a_0) \times \dots \times G(m, 1, a_{m-1})$. By the same argument as above the induction of each of these is an irreducible $G(m, 1, n)$ -module. We have

$$\text{Ind}_{\text{stab}(\underline{a})}^{\mathbb{H}(G(m, 1, n))} W = \bigoplus \text{Ind}_{G(m_1, 1, n) \times \dots \times G(m_{m-1}, 1, n)}^{G(m, 1, n)} V_i$$

as a $G(m, 1, n)$ -module. Suppose that $Ind_{\text{stab}(\underline{a})}^{\mathbb{H}(G(m, 1, n))} W$ is not irreducible as an $\mathbb{H}(G(m, 1, n))$ -module then some direct sum of $Ind V_i$'s is a submodule. Suppose $\bigoplus_{i \in I} Ind V_i$ is a $\mathbb{H}(G(m, 1, n))$ -submodule, then $F(\bigoplus_{i \in I} Ind V_i)$ is a $\text{stab}(\underline{a})$ -submodule of W . Therefore $\bigoplus_{i \in I} V_i$ is a $\text{stab}(\underline{a})$ -submodule of W . Since W is irreducible as a $\text{stab}(\underline{a})$ -module then I is the whole indexing set, i.e. $W = \bigoplus_{i \in I} V_i$. Therefore the only irreducible non-trivial submodule is the whole module. Hence $Ind_{\text{stab}(\underline{a})}^{\mathbb{H}(G(m, 1, n))} W$ is irreducible.

It is easy to verify that $F^{-1} \cdot F(V) = V$ using the universal property of induced modules and similarly $F \cdot F^{-1}(W) = W$. $\square$

Given a representation of $(V, \pi) \in \text{irr}(\text{stab}(\underline{a})|\mu_{\underline{a}})$ we can explicitly describe how $g_i \in G(m, 1, n)$ acts. Since this algebra stabilises $\mu_{\underline{a}}$ this is the only $\mathbb{C}[T]$ weight occurring in V . Therefore

$$g_i = \mu_{\underline{a}}(g_i) \text{Id}.$$

Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$, if we study the relation $s_{\alpha_i} \alpha_i = s_{\alpha_i}(\alpha_i) s_{\alpha_i} + \sum_{l=0}^{m-1} g_i^l g_{i+1}^{-l}$ in $\mathbb{H}(G(m, 1, n))$, on (V, π) , the element $\sum_{l=0}^{m-1} g_i^l g_{i+1}^{-l}$ is equal to $\pi(\sum_{l=0}^{m-1} g_i^l g_{i+1}^{-l}) = \sum_{l=0}^{m-1} \mu_{\underline{a}}(g_i)^l \mu_{\underline{a}}(g_{i+1})^{-l} \text{Id}$. This then equals m if $\mu_{\underline{a}}$ is constant on g_i and g_{i+1} and $\sum_{l=0}^{m-1} g_i^l g_{i+1}^{-l}$ is zero if $\mu_{\underline{a}}(g_i) \neq \mu_{\underline{a}}(g_{i+1})$. One can summarise, on any representation in $\text{irr}(\text{stab}(\underline{a})|\mu_{\underline{a}})$

$$\sum_{l=0}^{m-1} g_i^l g_{i+1}^{-l} = \begin{cases} m & \text{if } s_{\epsilon_i - \epsilon_{i+1}} \in \text{stab}(\mu_{\underline{a}}), \\ 0 & \text{otherwise.} \end{cases}$$

Recall from Lemma 2.5.3 that $\text{stab}(\underline{a})$ is isomorphic to $\mathbb{H}(G(a_0, 1, m)) \otimes \dots \otimes \mathbb{H}(G(a_{m-1}, 1, m))$.

We have shown that if $(V, \pi) \in \text{irr}(\text{stab}(\underline{a})|\mu_{\underline{a}})$ then the relations $s_{\alpha_i} \alpha_i = s_{\alpha_i}(\alpha_i) s_{\alpha_i} + \mathbf{k} \sum_{l=0}^{m-1} g_i^l g_{i+1}^{-l}$ become $s_{\alpha_i} \alpha_i = s_{\alpha_i}(\alpha_i) s_{\alpha_i} + \mathbf{k}m$ on V for all $\alpha_i \in \Pi_{\underline{a}}$ and the relations $s_{\alpha_i} \alpha_i = s_{\alpha_i}(\alpha_i) s_{\alpha_i} + \mathbf{k} \sum_{l=0}^{m-1} g_i^l g_{i+1}^{-l}$ become $s_{\alpha_i} \alpha_i = s_{\alpha_i}(\alpha_i) s_{\alpha_i} + 0$ on V for all $\alpha_i \notin \Pi_{\underline{a}}$. Hence we can conclude that if $(V, \pi) \in \text{irr}(\text{stab}(\underline{a})|\mu_{\underline{a}})$ then this representation factors through the algebra

$$\mathbb{H}(S_{a_0}) \otimes \dots \otimes \mathbb{H}(S_{a_{m-1}}),$$

via the quotient by the ideal $I_{\underline{a}} = \langle g_i - \mu_{\underline{a}}(g_i) \text{Id} \rangle$. Explicitly

$$\pi : \mathbb{H}(G(a_0, 1, m)) \otimes \dots \otimes \mathbb{H}(G(a_{m-1}, 1, m)) \twoheadrightarrow \mathbb{H}(S_{a_0}) \otimes \dots \otimes \mathbb{H}(S_{a_{m-1}}) \rightarrow GL(V).$$

Where $\mathbb{H}(S_n)$ is the usual graded Hecke algebra associated to S_n , with parameter $c(\alpha) = m\mathbf{k}$.

Lemma 2.5.5. *The set $\text{irr}(\mathbb{H}(G(a_0, 1, m)) \otimes \dots \otimes \mathbb{H}(G(a_{m-1}, 1, m)) | \mu_{\underline{a}})$ is in one-to-one correspondence with $\text{irr}(\mathbb{H}(S_{a_0})) \otimes \dots \otimes \text{irr}(\mathbb{H}(S_{a_{m-1}}))$.*

Proof. The irreducible representations in $\text{irr}(\mathbb{H}(G(a_0, 1, m)) \otimes \dots \otimes \mathbb{H}(G(a_{m-1}, 1, m)) | \mu_{\underline{a}})$ all occur as pullbacks of the irreducible representations of $\mathbb{H}(S_{a_0}) \otimes \dots \otimes \mathbb{H}(S_{a_{m-1}})$ via the specific quotient of $\mathbb{H}(G(a_0, 1, m)) \otimes \dots \otimes \mathbb{H}(G(a_{m-1}, 1, m)) | \mu_{\underline{a}}$ onto $\mathbb{H}(S_{a_0}) \otimes \dots \otimes \mathbb{H}(S_{a_{m-1}})$, with the ideal

$$I_{\underline{a}} = \langle g_i - \mu_{\underline{a}}(g_i) \text{Id} \mid i = 1, \dots, n \rangle.$$

Furthermore given a representation U of $\mathbb{H}(S_{a_0}) \otimes \dots \otimes \mathbb{H}(S_{a_{m-1}})$ one can create a representation in $\text{irr}(\mathbb{H}(G(a_0, 1, m)) \otimes \dots \otimes \mathbb{H}(G(a_{m-1}, 1, m)) | \mu_{\underline{a}})$ by pulling back the representation U from the quotient of $I_{\underline{a}}$. $\square$

Theorem 2.5.6. *Let $A = \{\underline{a} = (a_0, \dots, a_{m-1}) \in \mathbb{N}^m : \sum_{i=0}^{m-1} a_i = n\}$, note that $\mathbb{N}$ includes zero. The irreducible representations of $\mathbb{H}(G(m, 1, n))$ split into blocks which are induced from products of $\mathbb{H}(S_a)$ -representations:*

$$\text{irr}(\mathbb{H}(G(m, 1, n))) \cong \bigsqcup_{\underline{a} \in A} \text{irr}(\mathbb{H}(S_{a_0})) \otimes \dots \otimes \text{irr}(\mathbb{H}(S_{a_{m-1}})).$$

This bijection of irreducible modules defines an equivalence of the semisimplification of the module categories, the Grothendieck groups, $G_0(\mathbb{H}(G(m, 1, n)))$ and $G_0(\bigoplus_{\underline{a} \in A} \mathbb{H}_{S_{a_0}} \otimes \dots \otimes \mathbb{H}_{S_{a_{m-1}}})$.

Since we have a bijection between irreducible module of $\mathbb{H}(G(m, 1, n))$ and irreducible modules of $\bigoplus_{\underline{a} \in A} \bigotimes \mathbb{H}_{S_{a_i}}$ then we have an equivalence of the Grothendieck groups of each algebra.

When we consider tempered $\mathbb{H}(G(m, 1, n))$ modules, that is V such that the $\mathbb{C}[T] \otimes S(\mathfrak{t})$ weights $\mu \otimes \lambda$ are such that $\text{Re}(\lambda(x_i)) \leq 0$ for all fundamental coweights and that $\text{Re}(\lambda|_{\mathfrak{a}_{\mathbb{R}}}) = 0$, we note this condition is only dependent on the $S(\mathfrak{t})$ weight λ . Therefore a tempered $\mathbb{H}(G(m, 1, n))$ -module corresponds to a tempered $\mathbb{H}(S_n)$ -module with weight λ . Every tempered $\mathbb{H}(S_n)$ -module V will correspond to m different tempered $\mathbb{H}(G(m, 1, n))$ -modules. Each tempered $\mathbb{H}(G(m, 1, n))$ -module will be a pullback of the module V . However the difference between the modules is that the short reflections g_i will act by η^j for fixed $j = 1, \dots, m$.

This gives a method to parametrise the Langlands data for an irreducible $\mathbb{H}(G(m, 1, n))$ -module via tempered modules of $\mathbb{H}(S_a)$. Recall that every irreducible $\mathbb{H}(G(m, 1, n))$ -module can be realized as a quotient of

$$\mathbb{H}(G(m, 1, n)) \otimes_{\mathbb{H}(G(m, 1, n))_P} \check{U} \otimes \mathbb{C}_\nu.$$

If we fix an irreducible module X then using the above realization, we associate to it Langlands data (P, U) .

Fix $P = (p_0, \dots, p_{m-1})$ a partition of n with at most m parts. The tempered $\mathbb{H}(G(m, 1, n))$ -modules are the pullbacks of tempered $\mathbb{H}(S_{a_0}) \otimes \dots \otimes \mathbb{H}(S_{a_{m-1}})$ -modules.

Recall that the size of a partition $\lambda = \{x_1, \dots, x_j\}$ is $\sum_{i=1}^j x_i$. The tempered modules of graded Hecke algebra with real central character correspond to partitions (e, ψ) where ψ is nilpotent [17, 3.3], [33]. In the case of $W = S_n$, e is always 1 and ψ is characterized by it's Jordan form and hence corresponds to a partition. Hence the tempered modules of $\mathbb{H}(S_{p_0}) \otimes \dots \otimes \mathbb{H}(S_{p_{m-1}})$ with real central character will correspond to a set of m partitions $\{\lambda_0, \dots, \lambda_{m-1}\}$ such that the sum of the sizes of the partitions λ_i is a_i .

Theorem 2.5.7. *Let P be a parabolic set associated to $\underline{a} = (a_0, \dots, a_m)$. We associate to P , a parabolic subalgebra $\mathbb{H}_P \subset \mathbb{H}(G(m, 1, n))$. The tempered modules of the parabolic algebra $\mathbb{H}_P$ are built up from tempered modules of each parabolic part $\mathbb{H}(G(m, 1, a_i))$. By above a tempered module of $\mathbb{H}(G(m, 1, a_i))$ corresponds to a tempered module of $\mathbb{H}(S_{a_i})$. The tempered modules of $\mathbb{H}(S_{a_i})$ with real central character are labelled by partitions of a_i . Hence tempered modules of $\mathbb{H}_P(G(m, 1, n))$ with real central character are labelled by multipartitions $\{\lambda_0, \dots, \lambda_{m-1}\}$ with m partitions such that the size of λ_i equals a_i . Furthermore one can construct these tempered modules via the pullback of*

$$\mathbb{H}(G(m, 1, n)) \xrightarrow{\phi} \mathbb{H}(S_{a_0}) \otimes \dots \otimes \mathbb{H}(S_{a_{m-1}}) \rightarrow GL(V_{\lambda_0} \otimes \dots \otimes V_{\lambda_{m-1}}).$$

Where V_{λ_i} is the tempered module of $\mathbb{H}(S_{a_i})$ corresponding to the partition λ_i and ϕ is the quotient by the ideal $I_{\underline{a}} = \langle g_i - \mu_{\underline{a}}(g_i) \text{Id} \mid i = 1, \dots, n \rangle$.

2.6 Dirac cohomology of the Dunkl-Opdam subalgebra

In this section, we will use the description of irreducible representations from Section 2.5 to describe how the Dirac operator for the Dunkl-Opdam subalgebra acts on irreducible modules. We show that the Dirac operator $\mathcal{D}_{DO}$ for $\mathbb{H}_{DO}$ descends to a relevant Dirac operator for a tensor of type A graded Hecke algebras. Then, we describe the Dirac operator in terms of Dirac operators for type A parabolic algebras.

Let A be a Drinfeld algebra $T(V) \rtimes \mathbb{C}[G]/R$. We have an associated Clifford algebra $C(V)$, with respect to the G -invariant symmetric product $\langle, \rangle$. Given a basis B of V with dual basis B^* with respect to $\langle, \rangle$, we defined the Dirac operator to be

$$\sum_{b \in B} b \otimes b^* \in A \otimes C(V).$$

We have two presentations of the Dunkl-Opdam subalgebra, one producing the Lusztig presentation Definition 0.2.16 with commuting basis elements and the Drinfeld presentation used in Theorem 2.4.13 which shows that $\mathbb{H}(G(m, 1, n))$ is a Drinfeld algebra. We used the Lusztig presentation to show the equivalence of the irreducibles of the Dunkl-Opdam subalgebra to irreducibles of a sum of tensors of type A graded Hecke algebras. This utilised parabolic subalgebras. However, the Dirac theory developed for the Dunkl-Opdam subalgebra uses the Drinfeld presentation. This Drinfeld presentation does not admit parabolic subalgebras.

Let us recall that to transform from the Lusztig presentation to the Drinfeld presentation one takes the standard basis $\{z_1, \dots, z_n\}$ of the reflection representation of S_n which along with $G(m, 1, n)$ gives the Lusztig presentation. Then to obtain the Drinfeld presentation we use the generators:

$$\tilde{z}_i = z_i + \frac{\mathbf{k}}{2} \sum_{i < j} s_{i,j} \sum_{l=1}^{m-1} g_i^{-l} g_j^l - \frac{\mathbf{k}}{2} \sum_{j < i} s_{i,j} \sum_{l=1}^{m-1} g_i^{-l} g_j^l = z_i + \frac{\mathbf{k}}{2} (\underline{M}_i - M_i).$$

Recall M_i and $\underline{M}_i$ are Jucys-Murphy elements of $G(m, 1, n)$ with reverse orderings. The Dirac element in terms of the Drinfeld presentation is:

$$\mathcal{D}_{DO} = \sum_{i=1}^n \tilde{z}_i \otimes z_i^*.$$

In terms of the Lusztig presentation $\{z_i\}$ the Dirac element is

$$\mathcal{D}_{DO} = \sum_{i=1}^n \left(z_i + \frac{\mathbf{k}}{2} \sum_{i < j} s_{i,j} \sum_{l=1}^{m-1} g_i^{-l} g_j^l - \frac{\mathbf{k}}{2} \sum_{j < i} s_{i,j} \sum_{l=1}^{m-1} g_i^{-l} g_j^l \right) \otimes z_i^*.$$

Definition 2.6.1. Given an $\mathbb{H}$ -module X and a spinor S of $C(V)$, then $\mathcal{D}_{DO} : X \otimes S \rightarrow X \otimes S$. The Dirac cohomology of X with respect to S is defined to be

$$\text{Ker}(\mathcal{D}_{DO}) / \text{Im}(\mathcal{D}_{DO}) \cap \text{Ker}(\mathcal{D}_{DO}).$$

Since $\mathcal{D}_{DO}$ sgn-commutes with the group $\tilde{G}$ then the Dirac cohomology is naturally a $\tilde{G}$ -module.

From Section 2.5, if an irreducible V contains a weight $\mu_{\underline{a}}$ corresponding to $\underline{a} = \{a_0, \dots, a_{m-1}\}$ then we will write $V_{\mu_{\underline{a}}}$ for the $\mathbb{C}[T]$ -weight space corresponding to the weight $\underline{a}$. We can decompose V into $\mathbb{C}[T]$ weight spaces,

$$V = \bigoplus_{\underline{a} \in A} V_{\mu_{\underline{a}}} = \bigoplus_{c \in C} V_{\mu_{\underline{a}}}^c.$$

Lemma 2.5.4 shows that $V_{\mu_{\underline{a}}}$ is in the image of the functor F . It is a $\text{stab}(\mu_{\underline{a}})$ -module and is the pullback of a tensor of $\mathbb{H}_{S_{a_i}}$ -modules. A problem that occurs is that the Dirac operator $\mathcal{D}_{DO}$ does not sit in the subalgebra $\text{stab}(\mu_{\underline{a}}) \cong \bigotimes_{i=1}^{m-1} \mathbb{H}_{DO}(G(m, 1, a_i))$. We will look at the Dirac operators already given for the standard type A graded Hecke algebra.

Definition 2.6.2. [4] For the graded Hecke algebra $\mathbb{H}(S_k)$ the Dirac operator is

$$D_{S_k} = \sum_{i=1, \dots, k} \left(z_i + \frac{m\mathbf{k}}{2} \sum_{i < j} s_{i,j} - \frac{m\mathbf{k}}{2} \sum_{j < i} s_{i,j} \right) \otimes z_i^*.$$

Remark 2.6.3. We abuse notation here as z_i in this context denotes the same basis as we have used in the definition of $\mathbb{H}_{DO}$ but of course it is not in the same algebra. We justify this since all surjections of $\mathbb{H}_{DO}$ onto $\mathbb{H}_{S_n}$ preserve this notation. We have used the parameter $m\mathbf{k}$ as opposed to $\mathbf{k}$ for $\mathbb{H}_{S_n}$ because naturally our map sends $\mathbb{H}_{DO}$ to $\mathbb{H}_{S_n}$ with parameter $m\mathbf{k}$.

Recall that the weight space $V_{\mu_{\underline{a}}}$ is naturally a $\bigotimes_{i=1}^{m-1} \mathbb{H}_{S_{a_i}}$ -module, via the functor F defined in the Lemma 2.5.4. We extend Definition 2.6.2 to define a Dirac operator for $\bigotimes_{i=1}^{m-1} \mathbb{H}_{S_{a_i}}$:

$$\mathcal{D}_{S_{a_0} \times \dots \times S_{a_{m-1}}} = \mathcal{D}_{S_{a_0}} \otimes \dots \otimes \mathcal{D}_{S_{a_{m-1}}}.$$

Written out explicitly this is

$$\mathcal{D}_{S_{a_i} \times \dots \times S_{a_{m-1}}} = \sum_{i=0}^{m-1} \sum_{j=a_{i-1}}^{j=a_i} \left(z_j + \frac{m\mathbf{k}}{2} \sum_{j < k \leq a_i} s_{j,k} - \frac{m\mathbf{k}}{2} \sum_{a_{i-1} < k < j} s_{j,k} \right) \otimes z_j^*.$$

Here we have associated $\bigotimes_{i=1}^{m-1} S(V_i)$ with $S(\oplus V_i)$. Similarly we have substituted $C(\oplus V_i) = \bigotimes C(V_i)$, using the graded tensor product. Initially this looks like the Dirac operator for $\mathbb{H}_{S_n}$, however one should notice that not all of the reflections are involved in this Dirac operator. We highlight this with an example.

Example 2.6.4. *Let $n=3$. The Dirac operator for $\mathbb{H}_{S_3}$ is*

$$\begin{aligned} & \left(z_1 - \frac{m\mathbf{k}}{2}(1,2) - \frac{m\mathbf{k}}{2}(1,3) \right) \otimes z_1^* + \left(z_2 + \frac{m\mathbf{k}}{2}(1,2) - \frac{m\mathbf{k}}{2}(2,3) \right) \otimes z_2^* \\ & + \left(z_3 + \frac{m\mathbf{k}}{2}(1,3) + \frac{m\mathbf{k}}{2}(2,3) \right) \otimes z_3^*. \end{aligned}$$

However the Dirac operator for $\mathbb{H}_{S_1} \otimes \mathbb{H}_{S_2} \subset \mathbb{H}_{S_3}$ is

$$z_1 \otimes z_1^* + \left(z_2 - \frac{m\mathbf{k}}{2}(2,3) \right) \otimes z_2^* + \left(z_3 + \frac{m\mathbf{k}}{2}(2,3) \right) \otimes z_3^*.$$

One can see that there are four reflections in the $\mathcal{D}_{\mathbb{H}(S_3)}$ not involved in $\mathcal{D}_{\mathbb{H}(S_1)} \times \mathcal{D}_{\mathbb{H}(S_2)}$.

Viewing $V_{\mu_{\underline{a}}}$ as a $\text{stab}(\mu_{\underline{a}})$ -module, V is a sum of twists of $F(V)$. Let us look at the $\mathbb{C}[T]$ -invariant element of $\text{stab}(\mu_{\underline{a}})$ which maps to $\mathcal{D}_{S_{a_0} \times \dots \times S_{a_{m-1}}}$, this is:

$$\sum_{i=0}^{m-1} \sum_{j=a_{i-1}}^{j=a_i} \left(z_j + \frac{\mathbf{k}}{2} \sum_{l=1}^{m-1} g_j^{-l} g_k^l \sum_{j < k \leq a_i} s_{j,k} - \frac{\mathbf{k}}{2} \sum_{l=1}^{m-1} g_k^{-l} g_j^l \sum_{a_{i-1} < k < j} s_{j,k} \right) \otimes z_j^*.$$

Written this way one notices that this looks very similar to the $\mathbb{H}_{DO}$ Dirac operator however it excludes the reflections that are not in the parabolic subgroup that stabilises $\mu_{\underline{a}}$. The following lemma shows that the difference vanishes on $V_{\mu_{\underline{a}}}$.

Lemma 2.6.5. *Given an irreducible module $V_{\underline{a}}$ with $\mathbb{C}[T]$ weight $\underline{a}$ then on the subspace $F(V_{\underline{a}})$ the Dirac operator for $\mathbb{H}_{DO}$ acts by the Dirac operator $\mathcal{D}_{S_{a_0} \times \dots \times S_{a_{m-1}}}$.*

Proof. Recall that since $V_{\mu_{\underline{a}}}$ only has one $\mathbb{C}[T]$ weight, namely $\underline{a}$ we can explicitly describe how $\sum_{l=0}^{m-1} g_i^{-l} g_j^l$ acts on this subspace.

$$\sum_{l=0}^{m-1} g_i^{-l} g_j^l = \begin{cases} m \text{ Id} & \text{if } \mu_{\underline{a}}(g_i) = \mu_{\underline{a}}(g_j), \\ 0 & \text{if } \mu_{\underline{a}}(g_i) \neq \mu_{\underline{a}}(g_j). \end{cases}$$

This parametrisation of pairs $\{i, j\}$ can be described in another way. If the transposition $s_{i,j}$ stabilises the character $\mu_{\underline{a}}$ then $\sum_{l=0}^{m-1} g_i^{-l} g_j^l = m$. However if $s_{i,j}$ is not in $\text{stab}(\mu_{\underline{a}})$ then $\sum_{l=0}^{m-1} g_i^{-l} g_j^l = 0$ on the $\mu_{\underline{a}}$ -weight space. Ultimately this means that the Dirac operator $\mathcal{D}_{DO} \in \mathbb{H}_{DO} \otimes CL(V)$ preserves the subspace $F^{-1}(V) \otimes \mathcal{S}$ since

the transpositions included in $\mathcal{D}_{DO}$ which do not preserve $V_{\mu_{\underline{a}}}$ are preceded by the element $\sum_{l=0}^{m-1} g_i^{-l} g_j^l$ which acts by zero in this case. Finally since $\mathcal{D}_{DO}$ preserves $V_{\mu_{\underline{a}}}$ it equals an element inside $\text{stab}(\underline{a}) \otimes \mathcal{S}$. This is the pull back of $\mathcal{D}_{S_{a_0} \times \dots \times S_{a_{m-1}}}$ and hence $\mathcal{D}_{DO}$ agrees with $\mathcal{D}_{S_{a_0} \times \dots \times S_{a_{m-1}}}$ on the $\mu_{\underline{a}}$ -weight space of V . $\square$

We have described how the Dirac operator acts on the $\underline{a}$ weight space of V . Since $\mathcal{D}_{DO}$ is $G(m, 1, n)$ invariant we can describe how it acts on the rest of the weight spaces. As discussed in Lemma 2.5.4, the other weight spaces are twists of this space by the coset representatives, $c \in C$ of the parabolic subgroup S_P in S_n . The group S_P fixes the $\underline{a}$ weight space. Therefore if $\mathcal{D}_{DO}$ acts by $\mathcal{D}_{S_{a_0} \times \dots \times S_{a_{m-1}}}$ on $V_{\mu_{\underline{a}}}$ then $\mathcal{D}$ acts by $c\mathcal{D}_{S_{a_0} \times \dots \times S_{a_{m-1}}}c^{-1}$ on $cV_{\mu_{\underline{a}}}$. Hence $\text{Ker}(\mathcal{D}_{DO}) \subset V \otimes \mathcal{S}$ is

$$\bigoplus_{c \in C} c \text{Ker } \mathcal{D}_{S_{a_0} \times \dots \times S_{a_{m-1}}}.$$

Similarly since $\mathcal{D}_{DO}$ acts by $\mathcal{D}_{S_{a_0} \times \dots \times S_{a_{m-1}}}$ on $\text{stab}(\underline{a}) \otimes \mathcal{S}$ then

$$\text{Im } \mathcal{D}_{DO} = \bigoplus_{c \in C} c \text{Im } \mathcal{D}_{S_{a_0} \times \dots \times S_{a_{m-1}}}.$$

We can describe the Dirac cohomology of an irreducible module X in terms of the Dirac cohomology of its corresponding $\mathbb{H}_{S_{a_0}} \otimes \dots \otimes \mathbb{H}_{S_{a_{m-1}}}$ -module.

Definition 2.6.6. *The group $\tilde{G}$ for $\mathbb{H}(G(m, 1, n))$ is isomorphic to $\tilde{S}_n \ltimes (\mathbb{Z}_n)^m$. Since this is a semidirect product irreducible representations of $\tilde{S}_n \ltimes (\mathbb{Z}_n)^m$ decompose into blocks corresponding to characters $\mu_{\underline{a}} \in (\mathbb{C}(\mathbb{Z}_n)^m)^*$ and under this correspondence*

$$\text{irr}(\tilde{S}_n \ltimes (\mathbb{Z}_n)^m | \mu_{\underline{a}}) \cong \text{irr}(\tilde{S}_{a_0} \times \dots \times \tilde{S}_{a_{m-1}}).$$

There is a corresponding equivalence between the semisimplification of the module category of $\tilde{G}$ and $\bigoplus_{\underline{a} \in A} \bigotimes \mathbb{C}\tilde{S}_{a_0} \otimes \dots \otimes \mathbb{C}\tilde{S}_{a_{m-1}}$. Define $\tilde{F}$ and $\tilde{F}^{-1}$ to be the functors exhibiting this equivalence between $\widetilde{G(m, 1, n)}$ and $\bigoplus_{\underline{a} \in A} \bigoplus_{\underline{a} \in A} \widetilde{\mathbb{C}\tilde{S}_{a_0}} \otimes \dots \otimes \widetilde{\mathbb{C}\tilde{S}_{a_{m-1}}}$, similarly to Lemma 2.5.4.

Theorem 2.6.7. *Given an irreducible $\mathbb{H}_{DO}$ -representation V then $F(V)$ as an $\mathbb{H}_{S_{a_0}} \otimes \dots \otimes \mathbb{H}_{S_{a_{m-1}}}$ -module is isomorphic to $F(V) \cong X_{a_0} \otimes \dots \otimes X_{a_{m-1}}$. The Dirac cohomology of V is*

$$\bigoplus_{c \in C} c (H_D(X_{a_0}) \otimes \dots \otimes H_D(X_{a_{m-1}})),$$

where $H_D(X)$ is the type A Dirac cohomology of the $\mathbb{H}_{S_k}$ -module X , P is the parabolic subset associated to $\underline{a}$ and C is a set of coset representatives of S_n/S_P . Let $H_D(\bullet)$ denote the functor taking the relevant module to its Dirac cohomology. We have the following commutative diagram:

$$\begin{array}{ccc}
G_0(\mathbb{H}(G(m, 1, n))) & \xrightarrow{F} & G_0(\bigoplus_{\underline{a} \in A} \mathbb{H}_{S_{a_0}} \otimes \dots \otimes \mathbb{H}_{S_{a_{m-1}}}) \\
\downarrow H_D(\bullet) & & \downarrow H_D(\bullet) \\
\widetilde{\mathbb{C}G(m, 1, n)\text{-mod}} & \xleftarrow{\widetilde{F^{-1}}} & \bigoplus_{\underline{a} \in A} \widetilde{\mathbb{C}S_{a_0}} \otimes \dots \otimes \widetilde{\mathbb{C}S_{a_{m-1}}\text{-mod}}.
\end{array}$$

Here $G_0(\mathbb{H}(G(m, 1, n)))$ is the Grothendieck group of the isomorphism classes of representations of $\mathbb{H}(G(m, 1, n))$.

Chapter 3

Duality between non-spherical principal series representations

Abstract

We define an extension of the affine Brauer algebra called the type B/C affine Brauer algebra. This new algebra contains the hyperoctahedral group and it naturally acts on $\text{End}_K(X \otimes V^{\otimes k})$. We study functors $F_{\mu,k}$ from the category of admissible $O(p, q)$ or $Sp_{2n}(\mathbb{R})$ -modules to representations of the type B/C affine Brauer algebra. Furthermore these functors take non-spherical principal series modules to principal series modules for the graded Hecke algebra of type D_k , C_{n-k} or B_{n-k} .

3.1 Introduction

Let G be an odd real orthogonal group or symplectic group whose complexification is $G_{\mathbb{C}}$, G is $O(p, q)$ for $p + q = 2n + 1$ or $Sp_{2n}(\mathbb{R})$. Let K denote a maximal compact subgroup of G . Let $\mathfrak{g}_0$ be the real Lie algebra of G . Define its complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Let X be an admissible G -module and let V be the defining matrix module of the linear group G . The papers [1, 59, 18, 10, 23] study the $\mathbb{C}$ -algebra $\text{End}_G(X \otimes V^{\otimes k})$ of operators on $X \otimes V^{\otimes k}$ commuting with G . For $G = Sp_{2n}(\mathbb{R}), O(p, q)$ there is a homomorphism [20] of the affine Brauer algebra to $\text{End}_G(X \otimes V^{\otimes k})$. In this chapter we focus on the larger algebra

$$\text{End}_K(X \otimes V^{\otimes k}).$$

We define an extension of the affine Brauer algebra, $\mathfrak{B}_k^{\theta}$, by operators related to the Cartan involution θ of G . This new algebra $\mathfrak{B}_k^{\theta}$ acts on $X \otimes V^{\otimes k}$ and commutes with the action of K . It is the analogue of the affine Brauer algebra for operators commuting with K . The extension contains the Weyl group of type B/C , the hyperoctahedral group. This new algebra's module category is a natural image for the functors defined by Ciubotaru and Trapa [16]:

$$F_{\mu, k}(X) = \text{Hom}_K(\mu, X \otimes V^{\otimes k}).$$

We show that the functors $F_{\mu, k}$ take the category of admissible $O(p, q)$ or $Sp_{2n}(\mathbb{R})$ -modules to $\mathfrak{B}_k^{\theta}$ -modules. Unlike previous functors, for $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$, both categories are related to the hyperoctahedral group. Let $G = KAN$ be the Iwasawa decomposition of G , and $P = MAN$ be the minimal parabolic subgroup. For characters δ of M and e^{ν} of A , the principal series representation X_{δ}^{ν} (Definition 3.6.1) is:

$$X_{\delta}^{\nu} = \text{Ind}_{MAN}^G(\delta \otimes e^{\nu} \otimes 1).$$

For split real orthogonal or symplectic groups this covers all of the principal series modules. When $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$ then $M = (\mathbb{Z}_2)^n$ or $O(p - q) \times (\mathbb{Z}_2)^q$. Denote the character of M which is *triv* (or *det*) on $O(p - q)$, -1 on the first k generators and 1 on the remaining $n - k$ or $q - k$ by δ_{triv}^k (resp. δ_{det}^k). For $Sp_{2n}(\mathbb{R})$ we drop the subscript *det* and *triv*. The graded Hecke algebra $\mathbb{H}_k(c)$ (Definition 3.4.1) is the graded Hecke algebra associated to the hyperoctahedral group $W(B_k)$ with a certain parameter function related to $c \in \mathbb{R}$. For $G = Sp_{2n}(\mathbb{R})$, the functors $F_{\text{triv}, k}$ and

$F_{\det, n-k}$ take principal series modules $X_{\delta^k}^\nu$ to principal series modules for the graded Hecke algebra $\mathbb{H}_k(0)$ and $\mathbb{H}_{n-k}(1)$ respectively. For $G = O(p, q)$ the functors $F_{\text{triv} \otimes \det, k}$ and $F_{\det \otimes \text{triv}, q-k}$ take principal series modules $X_{\delta_{\text{triv}}^k}^\nu$ to principal series modules for the graded Hecke algebra $\mathbb{H}_k(0)$ and $\mathbb{H}_{q-k}(1)$ respectively. A similar result holds for $X_{\delta_{\det}^k}^\nu$ and functors $F_{\text{triv} \otimes \text{triv}, k}$ and $F_{\det \otimes \det, q-k}$. Given a particular character δ of M we associate to it K -characters μ , and $\underline{\mu}$ (Table 3.6.1) with scalars c_μ and $c_{\underline{\mu}}$ (Table 3.7.1). We prove that under these conditions principal series representations map to principal series representations.

Theorem 3.8.14. *For $G = Sp_{2n}(\mathbb{R})$ or $O(p, q)$ $p + q = 2n + 1$, the module $F_{\mu, k}(X_\delta^\nu)$ is isomorphic to the $\mathbb{H}_k(c_\mu)$ principal series module*

$$X(\nu_k) = \text{Ind}_{S(\mathfrak{a}_k)}^{\mathbb{H}_k(c_\mu)} \nu_k.$$

The module $F_{\underline{\mu}, n-k}(X_\delta^\nu)$ is isomorphic to the $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ principal series module

$$X(\bar{\nu}_{n-k}) = \text{Ind}_{S(\bar{\mathfrak{a}}_{n-k})}^{\mathbb{H}_k(c_{\underline{\mu}})} \bar{\nu}_{n-k}.$$

This extends the results of Ciubotaru and Trapa [16] to non-spherical principal series modules. Importantly, if G is a split real orthogonal or symplectic group, we can describe the Hecke algebra module of the image of every principal series modules resulting from functors $F_{\mu, k}$ and $F_{\underline{\mu}, n-k}$. Furthermore using Casselman's subrepresentation theorem, for these split groups we have a correspondence of irreducible Harish-Chandra modules of G and graded Hecke algebra modules.

Theorem 3.8.15. *Let G be $O(n + 1, n)$ or $Sp_{2n}(\mathbb{R})$, then G is split. Let X be an irreducible G -module. Let X_δ^ν be a principal series representation that contains X , then the $\mathfrak{B}_k^\theta$ and $\mathfrak{B}_{n-k}^\theta$ -modules*

$$F_{\mu, k}(X) \text{ and } F_{\underline{\mu}, n-k}(X)$$

are submodules of the $\mathbb{H}_k(c_\mu)$ and $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ -modules

$$X(\nu_k) \text{ and } X(\bar{\nu}_{n-k}).$$

We define two anti-involutions on $\mathfrak{B}_k^\theta$ which descend to the usual anti-involutions on the graded Hecke algebra [5]. Furthermore we show that if X is a Hermitian (resp. unitary) module of $G = Sp_{2n}(\mathbb{R})$ then the image of X under the functor $F_{\mu, k}$ is a Hermitian (resp. unitary) module for $\mathfrak{B}_k^\theta[m]$. We also show that the Langlands quotient is preserved.

Theorem 3.9.26. *Let X_δ^ν be a principal series module for $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$. The Langlands quotient $\overline{X_\delta^\nu} = X_\delta^\nu / \text{rad}\langle, \rangle_{X_\delta^\nu}$ is mapped by $F_{\mu, k}$, to the Langlands quotient of the $\mathbb{H}_k(c_\mu)$ -module, $\overline{X(\nu_k)} = X(\nu_k) / \text{rad}\langle, \rangle_{X_{\nu_k}}$. Similarly, X_δ^ν is mapped by $F_{\underline{\mu}, n-k}$, to the $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ -module $\overline{X(\bar{\nu}_{n-k})}$.*

We then give a non-unitary test for principal series modules.

Theorem 3.9.29. *[Non-unitary test for principal series modules] If either $\overline{X(\nu_k)}$ or $\overline{X(\bar{\nu}_{n-k})}$ are not unitary, as $\mathbb{H}_k(c_\mu)$ and $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ -modules, then the Langlands quotient of the minimal principal series module $\overline{X_{\delta^k}^\nu}$, for $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$ is not unitary.*

This result gives a functorial proof of the nonunitarity criterion proved by Barbasch, Pantano, Paul and Salamanca-Riba [3, 46]. We also obtain a non-unitary test for any Harish Chandra module; in the split case one could check unitarity of Hecke algebra modules however in the non-split case one would have to work with affine type B/C Brauer algebra modules.

Theorem 3.9.30. *[Non-unitary test for Harish-Chandra modules] Let X be a Harish Chandra module. For $G = Sp_{2n}(\mathbb{R})$ or $O(p, q)$ $p+q = 2n+1$, if for any character μ and $k = 1, \dots, n$ the $\mathfrak{B}_k^\theta$ -module $\overline{F_{\mu, k}(X)}$ is not unitary, then the Langlands quotient $\overline{X}$ of X is not a unitary G -module. In the case when G is split then X is a subrepresentation of X_δ^ν and $F_{\mu, k}(X)$, $F_{\underline{\mu}, n-k}(X)$ are Hecke algebra modules. In this case, if either $\overline{F_{\mu, k}(X)}$, $\overline{F_{\underline{\mu}, n-k}(X)}$ is not unitary as a Hecke algebra module then $\overline{X}$ is not unitary as a G -module.*

In Section 3.3, we define the type B/C affine Brauer algebra $\mathfrak{B}_k^\theta$ and show that it acts on $X \otimes V^{\otimes k}$ and commutes with the action of K . Section 3.4 defines particular quotients of $\mathfrak{B}_k^\theta$ isomorphic to the graded Hecke algebras $\mathbb{H}_k(c)$. In Section 3.5, we introduce the functors, defined in [16], $F_{\mu, k} : \mathcal{HC}(G) \rightarrow \mathfrak{B}_n^\theta\text{-mod}$. These functors naturally create $\mathfrak{B}_k^\theta$ -modules. In Section 3.7, we show that the functors restricted to principal series modules define Hecke algebra modules. Section 3.8 describes the isomorphism classes of $F_{\mu, k}(X_{\delta^k}^\nu)$ and $F_{\underline{\mu}, n-k}(X_{\delta^k}^\nu)$ as principal series modules of graded Hecke algebras $\mathbb{H}_k(c_\mu)$ and $\mathbb{H}_{n-k}(c_{\underline{\mu}})$. In Section 3.9, we prove that functors $F_{\mu, k}$ preserve unitarity and invariant Hermitian forms.

3.2 Preliminaries

Let G be $O(p, q)$, $p + q = 2n + 1$ or $Sp_{2n}(\mathbb{R})$. Let $\mathfrak{g}_0$ be its Lie algebra, with complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. We uniformly denote a real Lie algebra by $\mathfrak{g}_0$, for a complex Lie algebra we drop the subscript. We fix a Cartan involution θ of $\mathfrak{g}_0$ and extend to $\mathfrak{g}$, let Θ be the corresponding involution of G . A maximal compact subgroup of G is K , the fixed space of Θ . The Lie algebra $\mathfrak{g}_0$ decomposes as $\mathfrak{k}_0 \oplus \mathfrak{p}_0$. The subspace $\mathfrak{p}_0$ is the -1 eigenspace of θ , the subalgebra $\mathfrak{k}_0$ is the $+1$ eigenspace of θ and the Lie algebra of K . Similarly, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}_0$ be a maximal commutative Lie subalgebra of $\mathfrak{p}_0$. Let M be the centralizer of $\mathfrak{a}_0$ in K under the adjoint action. Define the Lie algebra of M to be $\mathfrak{m}_0$.

Example 3.2.1. For $A \in GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$ let A^t denote its transpose. Let $J_n = \begin{bmatrix} 0 & \text{Id}_n \\ -\text{Id}_n & 0 \end{bmatrix}$ and set

$$Sp_{2n}(\mathbb{R}) = \{A \in GL_{2n}(\mathbb{R}) : A^t J_n A = J_n\}.$$

The algebra

$$\mathfrak{sp}_{2n}(\mathbb{R}) = \{g \in M_{2n}(\mathbb{R}) : g^t J_n + J_n g = 0\},$$

is the Lie algebra of $Sp_{2n}(\mathbb{R})$. It is a split real form of its complexification

$$\mathfrak{sp}_{2n}(\mathbb{C}) = \{g \in M_{2n}(\mathbb{C}) : g^t J_n + J_n g = 0\}.$$

The maximal compact Lie group K of $Sp_{2n}(\mathbb{R})$ is $U(n)$; the real Lie algebra $\mathfrak{k}_0$ of K is $\mathfrak{u}_n$ embedded into $\mathfrak{sp}_{2n}(\mathbb{R})$ as

$$\left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} : A + iB \in \mathfrak{u}_n \right\}.$$

The complexification of $\mathfrak{sp}_{2n}(\mathbb{R})$ is $\mathfrak{sp}_{2n}(\mathbb{C})$, the corresponding complex Lie algebra of $\mathfrak{u}_n$ is $\mathfrak{gl}_n(\mathbb{C})$. For the complex Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ the complex Lie subalgebra $\mathfrak{k}$ is $\mathfrak{gl}_n(\mathbb{C})$ embedded into $\mathfrak{sp}_{2n}(\mathbb{C})$ as

$$\left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} : A, B \in \mathfrak{gl}_n(\mathbb{C}), A^t = -A, B^t = B \right\}.$$

The subspace $\mathfrak{p}$ for $\mathfrak{sp}_{2n}(\mathbb{C})$ is the set of matrices

$$\left\{ \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \text{ such that } B \in \mathfrak{gl}_n(\mathbb{C}) \text{ and } B^t = B \right\}.$$

In this form, the Cartan involution of $\mathfrak{sp}_{2n}(\mathbb{C})$ is equivalent to conjugation by the involutory matrix

$$\xi = \begin{bmatrix} 0 & i \text{Id}_n \\ -i \text{Id}_n & 0 \end{bmatrix}.$$

When a specific basis and formulation of $\mathfrak{sp}_{2n}(\mathbb{C})$ is needed, we will use the above example. We will also denote $\mathfrak{sp}_{2n}(\mathbb{C})$ by $\mathfrak{sp}_{2n}$.

Example 3.2.2. Let $I_{p,q}$ be a diagonal matrix with p 1's followed by q -1 's, we fix $p \geq q$ and $p + q = 2n + 1$. We define the real orthogonal group $O(p, q)$ to be

$$O(p, q) = \{A \in GL_n(\mathbb{R}) : A^t I_{p,q} A = I_{p,q}\}.$$

Definition 3.2.3. For G equal to $O(p, q)$ or $Sp_{2n}(\mathbb{R})$ write V_0 for the defining matrix module. That is $\rho : G \rightarrow GL(V_0)$ is the injection defining G as a linear group. Write $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ for the complexification of V_0 .

If $G = Sp_{2n}(\mathbb{R})$ then $V = \mathbb{C}^{2n}$ and if $G = O(p, q)$ then $V = \mathbb{C}^{2n+1}$. When $G = Sp_{2n}(\mathbb{R})$, let $e_1, \dots, e_{2n}$ be the standard matrix basis of V , then define a new basis $f_i = e_i + e_{n+i}$ for $i = 1, \dots, n$ and $f'_i = e_i - e_{n+i}$ for $i = 1, \dots, n$. We also label f_i by f_i^1 and f'_i by f_i^{-1} . When $G = O(p, q)$ then V has basis $e_1, \dots, e_{2n+1}$, we let $f_i = e_{p-i+1} + e_{p+i}$ and $f'_i = e_{p-i+1} - e_{p+i}$.

Following [39, Section 1.1], let $\{R, X, \hat{R}, \hat{X}, \Delta\}$ be root datum where R is the set of roots, $\hat{R}$ is the set of coroots and X and $\hat{X}$ are free groups that contain R and $\hat{R}$ respectively. There is a perfect pairing $\langle, \rangle$ between X and $\hat{X}$ which defines a pairing between R and $\hat{R}$. The simple roots Δ are a subset of R . Let $\mathfrak{t}$ equal the complexification of X , and similarly $\hat{\mathfrak{t}}$ is the complexification of $\hat{X}$.

The Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian Lie subalgebra of $\mathfrak{p}$. The restricted roots Σ of $\mathfrak{g}$ are given by the eigenvalues of $\mathfrak{a}$ acting on $\mathfrak{g}$. The nilpotent Lie subalgebra $\mathfrak{n}$ is the sum of positive root spaces of the restricted roots of $\mathfrak{a}$.

Definition 3.2.4. [36, Proposition 6.46], [32] The Iwasawa decomposition of the complex vector space $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

The Iwasawa decomposition of G [36, Theorem 6.46] is

$$G = KAN.$$

Let M be the centralizer of $\mathfrak{a}$ in K and denote by $N_K(\mathfrak{a})$ the normalizer of $\mathfrak{a}$ in K . Let $\mathfrak{m}_0$ be the Lie algebra of M with complexification $\mathfrak{m}$. The Weyl group associated to G is the group

$$W_G = N_K(\mathfrak{a})/M.$$

Example 3.2.5. For $G = Sp_{2n}(\mathbb{R})$, a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ is

$$\mathfrak{a} = \left\{ \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} : B \text{ is diagonal, } B \in \mathfrak{gl}_n(\mathbb{C}) \right\}.$$

The algebra $\mathfrak{a}$ has dimension n , this is the real rank of $Sp_{2n}(\mathbb{R})$. Let $E_{i,j}$ be the matrix with 1 in the (i, j) position and zero elsewhere. Let $k \in \{0, \dots, n\}$. The subspace $\mathfrak{a}_k$ is the span of $E_{i,n+i} + E_{n+i,i}$ for $i = 1, \dots, k$. The subspace $\bar{\mathfrak{a}}_{n-k} \subset \mathfrak{a}$ is the span of the vectors $E_{k+i,n+k+i} + E_{n+k+i,k+i}$ for $i = 1, \dots, n-k$. Note that

$$\mathfrak{a} = \mathfrak{a}_k \oplus \bar{\mathfrak{a}}_{n-k}.$$

Define a basis of $\mathfrak{a}$ as $\varepsilon_i = E_{i,n+i} + E_{n+i,i}$ and let ϵ_i be the dual basis in $\mathfrak{a}^*$ [31]. The roots of $Sp_{2n}(\mathbb{R})$ are

$$\{\pm 2\epsilon_i, \pm\epsilon_j \mp \epsilon_k, \pm\epsilon_j \pm \epsilon_k : \text{ for } i, j, k = 1, \dots, n\} \subset \mathfrak{a}^*.$$

Fix the simple roots

$$\Delta = \{2\epsilon_n, \epsilon_i - \epsilon_{i+1} : \text{ for } i = 1, \dots, n-1\}.$$

In this example

$$\begin{aligned} R &= \{\pm 2\epsilon_i, \pm\epsilon_j \mp \epsilon_k, \pm\epsilon_j \pm \epsilon_k : \text{ for } i, j, k = 1, \dots, n\}, \\ X &= \{\sum_{\lambda_i \in \mathbb{Z}} \lambda_i \epsilon_i \in \mathfrak{a}^*\}, \\ \mathfrak{t} &= X \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{a}^*, \\ \hat{R} &= \{\pm \varepsilon_i, \pm \varepsilon_j \mp \varepsilon_k, \pm \varepsilon_j \pm \varepsilon_k : \text{ for } i, j, k = 1, \dots, n\}, \\ \hat{X} &= \{\sum_{\lambda_i \in \mathbb{Z}} \lambda_i \varepsilon_i \in \mathfrak{a}\}, \\ \hat{\mathfrak{t}} &= \hat{X} \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{a}, \end{aligned}$$

The Dynkin classification [30] labels this the root system of type C_n . For $G = Sp_{2n}(\mathbb{R})$ the subgroup M of K is the diagonal matrices with 1 or -1 on the diagonal. This group is generated by n 2-cycles,

$$M \cong (\mathbb{Z}_2)^n.$$

By the i^{th} generator of M we mean the diagonal matrix with -1 in the i^{th} position and 1 on all the others:

$$M = \langle M_1, \dots, M_n \rangle,$$

$$M_i = \text{Diag}(1, 1, \dots, 1, \overbrace{-1}^i, 1, \dots, 1).$$

A character for M can be represented by an ordered set in $(\mathbb{Z}_2)^n$. The trivial character would be denoted $\{1, 1, \dots, 1\}$ and the determinant character restricted to M would correspond to the sgn character $\{-1, -1, \dots, -1\}$. Let δ^k denote the character that takes the first k generators of M to -1 and the last $n - k$ generators to 1. The character δ^k corresponds to the ordered set

$$\{\overbrace{-1, -1, \dots, -1}^{k \text{ times}}, \overbrace{1, 1, \dots, 1}^{n-k \text{ times}}\}$$

For $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$, $\mathfrak{g}$ is $\mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$. It is well known that the associated Weyl group is isomorphic to the hyperoctahedral group,

$$W_{O(p,q)} \cong W_{Sp_{2n}} \cong S_n \ltimes (\mathbb{Z}_2)^n.$$

Let us denote the hyperoctahedral group of rank $n \in \mathbb{N}$ by $W(B_n)$, i.e.

$$W(B_n) = S_n \ltimes (\mathbb{Z}_2)^n.$$

Definition 3.2.6. Given a finite dimensional complex Lie algebra $\mathfrak{g}$ with basis B and dual basis B^* with respect to the Killing form, we define the Casimir element in the enveloping algebra $U(\mathfrak{g})$ to be

$$C^{\mathfrak{g}} = \sum_{b \in B} bb^* \in U(\mathfrak{g}).$$

For a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ we denote the Casimir element of $\mathfrak{h}$ in $\mathfrak{g}$ by $C^{\mathfrak{h}} = \sum_{b \in B \cap \mathfrak{h}} bb^*$ where the basis B is taken such that $B \cap \mathfrak{h}$ is a basis of $\mathfrak{h}$.

Definition 3.2.7. Let A be a group or a $\mathbb{C}$ -algebra with a complex representation U . We define the commutator of A in $\text{End}(U)$ as

$$\text{End}_A(U) = \{f \in \text{End}_{\mathbb{C}}(U) : [f, a] = 0 \text{ for all } a \in A\}.$$

For a complex vector space Y we denote by $T(Y)$ and $S(Y)$ the tensor algebra and symmetric algebra of Y .

3.3 Brauer Algebras

For a given $\mathfrak{g}$ -module X and the matrix module V , the endomorphism ring $\text{End}_{\mathbb{C}}(X \otimes V^{\otimes k})$ has been thoroughly studied. Most attention ([59, 18, 20, 10, 23]) has been on understanding the subalgebra of $\text{End}_{\mathbb{C}}(X \otimes V^{\otimes k})$ commuting with G :

$$\text{End}_G(X \otimes V^{\otimes k}).$$

In the case of $\mathfrak{g} = \mathfrak{gl}_n$, $\text{End}_{\mathfrak{gl}_n}(X \otimes V^{\otimes k})$ has a map from the graded Hecke algebra associated to the symmetric group [1]. However in Type B [20], with $\mathfrak{g} = \mathfrak{so}_{2n+1}$, the relevant algebra is the affine Brauer algebra with parameter n . In Type C, with $\mathfrak{g} = \mathfrak{sp}_{2n}$, the corresponding algebra is the affine Brauer algebra with parameter $-n$.

In this section, we define the affine type B/C Brauer algebra as an extension of the affine Brauer algebra. We endow it with a natural action on $X \otimes V^{\otimes k}$ and prove that it commutes with the action of K .

Definition 3.3.1. [8] *The rank k Brauer algebra $B_k[m]$, with parameter $m \in \mathbb{C}$, is the associative $\mathbb{C}$ -algebra generated by elements $t_{i,i+1}$ and $e_{i,i+1}$ for $i = 1, \dots, k-1$, subject to the conditions:*

the subalgebra generated by $t_{i,i+1}$ is isomorphic to $\mathbb{C}[S_n]$,

$$e_{i,i+1}^2 = m e_{i,i+1},$$

$$t_{i,i+1} e_{i,i+1} = e_{i,i+1} t_{i,i+1} = e_{i,i+1},$$

$$t_{i,i+1} t_{i+1,i+2} e_{i,i+1} t_{i+1,i+2} t_{i,i+1} = e_{i+1,i+2},$$

$$[t_{i,i+1}, e_{j,j+1}] = 0 \text{ for } j \neq i, i+1.$$

Definition 3.3.2. *Let U be a vector space with basis $x_1, \dots, x_k$. The rank k affine Brauer algebra $\mathfrak{B}_k[m]$, with parameter $m \in \mathbb{C}$ is as a vector space equal to*

$$\mathfrak{B}_k[m] \cong B_k[m] \otimes S(U).$$

The multiplication satisfies the following conditions:

$$t_{i,i+1} x_i - x_{i+1} t_{i,i+1} = 1 + e_{i,i+1},$$

$$[t_{i,i+1}, x_j] = 0, j \neq i, i+1,$$

$$e_{i,i+1}(x_i + x_{i+1}) = 0 = (x_i + x_{i+1})e_{i,i+1},$$

$$[e_{i,i+1}, x_j] = 0, j \neq i, i+1,$$

$$[x_i, x_j] = 0,$$

the subalgebra generated by $t_{i,i+1}$, $e_{i,i+1}$ is isomorphic to $B_k[m]$.

Let us consider X and V as $U(\mathfrak{g})$ -modules then $X \otimes V^{\otimes k}$ has a $U(\mathfrak{g})^{\otimes k+1}$ -module structure. We define operators that form the action of the Brauer algebra.

Definition 3.3.3. Given the action of $U(\mathfrak{g})^{\otimes k+1}$ on $X \otimes V^{\otimes k}$ we write $(g)_i$ for the action of g on the $i+1^{st}$ tensor in $U(\mathfrak{g})^{\otimes k+1}$,

$$(g)_i = \overbrace{id \otimes \dots \otimes id}^{i \text{ times}} \otimes g \otimes \overbrace{id \otimes \dots \otimes id}^{k-i \text{ times}}.$$

By construction we start counting from zero. Hence $(g)_0 = g \otimes id \otimes \dots \otimes id \in U(\mathfrak{g})^{\otimes k+1}$.

Definition 3.3.4. Fix a basis B such that $B = (B \cap \mathfrak{k}) \cup (B \cap \mathfrak{p})$. Let B^* be the dual basis with respect to the Killing form of $\mathfrak{g}$. For $0 \leq i < j \leq k$, define Ω_{ij} to be the operator

$$\Omega_{ij} = \sum_{b \in B} (b)_i \otimes (b^*)_j \in U(\mathfrak{g})^{\otimes k+1}.$$

Similarly we define $\Omega_{ij}^{\mathfrak{k}}$ and $\Omega_{ij}^{\mathfrak{p}}$ as

$$\Omega_{ij}^{\mathfrak{k}} = \sum_{b \in B \cap \mathfrak{k}} (b)_i \otimes (b^*)_j \in U(\mathfrak{g})^{\otimes k+1},$$

$$\Omega_{ij}^{\mathfrak{p}} = \sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (b^*)_j \in U(\mathfrak{g})^{\otimes k+1}.$$

Lemma 3.3.5. The operators $\Omega_{ij}, \Omega_{ij}^{\mathfrak{k}}$ and $\Omega_{ij}^{\mathfrak{p}}$ are independent of the choice of basis of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ respectively.

Proof. It is sufficient to prove the statement for $\Omega_{12} \in U(\mathfrak{g})^2$. Let $C^{\mathfrak{g}} = \sum_{b \in B} bb^* \in U(\mathfrak{g})$ be the Casimir element and $\Delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \times U(\mathfrak{g})$ denote comultiplication. The operator Ω_{12} can be written as:

$$\Omega_{12} = \Delta(C^{\mathfrak{g}}) - C^{\mathfrak{g}} \otimes 1 - 1 \otimes C^{\mathfrak{g}}.$$

The Casimir element $C^{\mathfrak{g}}$ is independent of the choice of basis therefore Ω_{12} is also independent. Similarly replacing $\mathfrak{g}$ with the Lie subalgebra $\mathfrak{k}$, $\Omega_{12}^{\mathfrak{k}}$ is independent of choice of basis. Finally $\Omega_{12}^{\mathfrak{p}}$ is independent as it is the difference of the other two,

$$\Omega_{ij} - \Omega_{ij}^{\mathfrak{k}} = \Omega_{ij}^{\mathfrak{p}}.$$

□

Let the symmetric group on k elements S_k act on $X \otimes V^{\otimes k}$ by permuting the factors of V . Explicitly the simple reflection $s_{i,i+1}$ acts by

$$s_{i,i+1}(x_0 \otimes v_1 \otimes \dots \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_k) = x_0 \otimes v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_k.$$

Lemma 3.3.6. *If $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n+1}$ then $V \otimes V$ decomposes as*

$$\Lambda^2 V \oplus S^2 V / 1 \oplus 1 \text{ for } \mathfrak{so}_{2n+1},$$

$$\Lambda^2 V / 1 \oplus S^2 V \oplus 1 \text{ for } \mathfrak{sp}_{2n}.$$

Here 1 denotes the trivial module of $\mathfrak{g}$.

Let pr_1 be the projection of $V \otimes V$ onto the trivial submodule 1 in the decomposition above. Let $\text{pr}_{i,i+1}$ be the projection onto the trivial submodule of $V_i \otimes V_{i+1}$.

Lemma 3.3.7. *[23, Theorem A] Let G be $O(p, q)$ or $Sp_{2n}(\mathbb{R})$. Let X be a complex G -representation and V the defining matrix module of G . Then there exists $m \in \mathbb{N}$ such that there is a homomorphism $\pi : \mathfrak{B}_k[m] \rightarrow \text{End}(X \otimes V^{\otimes k})$:*

$$\pi(x_i) = \sum_{j < i} \Omega_{ji},$$

$$\pi(t_{i,i+1}) = s_{i,i+1},$$

$$\pi(e_{i,i+1}) = \text{id} \otimes \dots \otimes \text{id} \otimes m \text{pr}_{i,i+1} \otimes \text{id} \otimes \dots \otimes \text{id}.$$

For $G = Sp_{2n}(\mathbb{R})$ the parameter is $m = -n$ and if $G = O(p, q)$ then $m = \lfloor \frac{p+q}{2} \rfloor$.

Theorem 3.3.8. *For $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$, the affine Brauer algebra with the action on $X \otimes V^{\otimes k}$ defined in Lemma 3.3.7 commutes with the action of $U(\mathfrak{g})$ on $X \otimes V^{\otimes k}$.*

Lemma 3.3.9. *[16, Lemma 2.3.1] Let $0 < i < j \leq k$ and $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$. As operators on $X \otimes V^{\otimes k}$*

$$\Omega_{ij} = s_{ij} + m \text{pr}_{i,i+1},$$

where $m = \lfloor \frac{p+q}{2} \rfloor$ or $-n$ respectively.

Proof. One only needs to consider the operator Ω_{12} on $V \otimes V$. By Lemma 0.8.10 $V \otimes V$ decomposes as

$$\Lambda^2 V \oplus S^2 V / 1 \oplus 1 \text{ for } \mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_{2n+1}(\mathbb{R}),$$

$$\Lambda^2 V / 1 \oplus S^2 V \oplus 1 \text{ for } \mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}_{2n}(\mathbb{C}).$$

On $V \otimes V$ $s_{12} = pr_{S^2 V} - pr_{\Lambda^2 V}$. Then using the fact that $\Omega_{12} = \Delta(C) - C \otimes 1 - 1 \otimes C$ we find the operators

$$\Omega_{12} \text{ and } s_{12} + me_{12},$$

act by the same scalars on the irreducible decomposition of $V \otimes V$. $\square$

For $G = GL_n$ the commutator $\text{End}_{GL_n}(X \otimes V^{\otimes k})$ contains the same type Weyl group, the symmetric group ([1]). One might expect that in type B and C this may be the case too. However $\text{End}_{Sp_{2n}(\mathbb{R})}(X \otimes V^{\otimes k})$, $\text{End}_{O(p,q)}(X \otimes V^{\otimes k})$ and the affine Brauer algebra, do not contain a copy of the hyperoctahedral group. We look to establish a theory that has this type symmetry reflected in the commutator.

We introduce the type B/C affine Brauer algebra which acts on $X \otimes V^{\otimes k}$ and commutes with the action of K for $G = Sp_{2n}(\mathbb{R})$ or $O(p, q)$. Crucially the type B/C affine Brauer algebra contains the Weyl group of Type B/C, $W(B_k)$. Recall the hyperoctahedral group is generated by simple reflections $s_{\epsilon_i - \epsilon_{i+1}}$ and s_{ϵ_k} .

Definition 3.3.10. *The type B/C affine Brauer algebra $\mathfrak{B}_k^\theta[m]$ is generated by the affine Brauer algebra $\mathfrak{B}_k[m]$ and reflections θ_j , for $j = 1, \dots, k$, such that the subalgebra generated by $t_{i,i+1}$, for $i = 1, \dots, k-1$ and θ_j is isomorphic to the group algebra of the k^{th} hyperoctahedral group $\mathbb{C}[W(B_k)]$ and the following relations hold;*

$$[e_{i,i+1}, \theta_j] = 0 \text{ for all } j,$$

$$e_{i,i+1}\theta_i\theta_{i+1} = e_{i,i+1} = \theta_i\theta_{i+1}e_{i,i+1} \text{ for } i = 1, \dots, k-1,$$

$$[\theta_n, x_j] = 0 \text{ for } j \neq k.$$

The Lie algebra $\mathfrak{g}$ decomposes as eigenspaces of a Cartan involution θ that is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. For $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$ there is a semisimple involutive $\xi \in \mathfrak{g}$ such that θ is equal to conjugation by ξ . For $G = Sp_{2n}(\mathbb{R})$, ξ is given in Example 3.2.1. For $O(p, q)$ ξ is given by the matrix $I_{p,q}$.

Lemma 3.3.11. *The type B/C affine Brauer algebra $\mathfrak{B}_k^\theta[m]$ acts on $X \otimes V^{\otimes k}$. This action is defined by extending the action π of the affine Brauer algebra to the extra generators θ_i . The generators θ_i act by $(\xi)_i \in U(\mathfrak{g})^{\otimes k+1}$. Extend π to $\mathfrak{B}_k^\theta[m]$ by $\pi(\theta_i) = (\xi)_i \in U(\mathfrak{g})^{\otimes k+1} \subset \text{End}(X \otimes V^{\otimes k})$. That is*

$$\pi : \mathfrak{B}_k^\theta[m] \longrightarrow \text{End}_K(X \otimes V^{\otimes k}),$$

$$\pi(\theta_i) = (\xi)_i.$$

Explicitly, $(\xi)_i = \overbrace{id \otimes \dots \otimes id}^i \otimes \xi \otimes \overbrace{id \otimes \dots \otimes id}^{k-i} \in U(\mathfrak{g})^{\otimes k+1}$. The constant m equals $\lfloor \frac{p+q}{2} \rfloor$ when $G = O(p, q)$ and $m = -n$ if $G = Sp_{2n}(\mathbb{R})$.

Proof. Since we know that the affine Brauer algebra $\mathfrak{B}_k[m]$ acts on $X \otimes V^{\otimes k}$ we only need to check the action of θ_j and the relations involving it in Definition 3.3.10. First, we show that $(\xi)_j$ and s_{ij} generate the hyperoctahedral group. Since ξ is an involution then $(\xi)_j^2 = id$. For any $g \in \mathfrak{g}$ we have that

$$s_{ij}(g)_i = (g)_j s_{ij} \text{ for } s_{ij} \in S_k, (g)_i \in U(\mathfrak{g})^{\otimes k+1} \text{ and } i, j \neq 0.$$

Hence $s_{ij}(\xi)_k s_{ij} = (\xi)_{s_{ij}(k)}$. Therefore $(\xi)_j$ and s_{ij} generate the hyperoctahedral group. If $i \neq j$, then $(g)_i$ and $(h)_j$ commute in $U(\mathfrak{g})^{k+1}$. Definition 3.3.7 states $\pi(x_k) = \sum_{i < k} \Omega_{ik}$, hence:

$$[x_k, (\xi)_n] = \sum_{i < k} [\Omega_{ik}, (\xi)_n] = 0 \text{ for } k < n.$$

Also,

$$[\theta_j, e_{i,i+1}] = 0 \text{ for all } j \neq i, i+1.$$

We show the relations involving $e_{i,i+1}$ and θ_i, θ_{i+1} on the kernel and image of $e_{i,i+1}$ separately. Recall that $e_{i,i+1}$ acts by $m \text{pr}_{i,i+1}$; the projection onto the trivial submodule of $V_i \otimes V_{i+1}$. The image of $\text{pr}_{i,i+1}$ is a $U(\mathfrak{g})$ trivial submodule, $(\xi)_i(\xi)_{i+1} \in U(\mathfrak{g})^{k+1}$ act by the identity on this module. The operator $\text{pr}_{i,i+1}$ is an idempotent, hence on the trivial module $\text{pr}_{i,i+1} = id$. On this submodule the relations

$$e_{i,i+1}\theta_i = e_{i,i+1}\theta_{i+1} = e_{i,i+1} = \theta_i e_{i,i+1} = \theta_{i+1} e_{i,i+1}$$

become $id = id$. On the kernel of $e_{i,i+1}$, by definition $e_{i,i+1}$ is the zero operator and the relations are immediately true.

This verifies the relations in Definition 3.3.10 of $\mathfrak{B}_k^\theta[m]$. Hence π is a well defined action:

$$\pi : \mathfrak{B}_k^\theta[m] \rightarrow \text{End}(X \otimes V^{\otimes k}).$$

□

Theorem 3.3.12. *Let $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$ and X a Harish-Chandra module. The type B/C Brauer algebra $\mathfrak{B}_k^\theta[m]$ acts on $X \otimes V^{\otimes k}$ and commutes with the action of K on $X \otimes V^{\otimes k}$.*

Proof. The action of $\mathfrak{B}_k[m]$ commutes with $\mathfrak{g}$ and by restriction with K . The algebra $\mathfrak{B}_k^\theta[m] = \langle \mathfrak{B}_k[m], \theta_j : \text{for } j = 1, \dots, k \rangle$. Therefore, to verify that $\mathfrak{B}_k^\theta[m]$ commutes with the action of K , one only needs to check that $\pi(\theta_j) = (\xi)_j$ commutes with the action of K . Conjugation by ξ is the Cartan involution: $\xi^{-1}K\xi = \Theta(K)$. By definition, Θ is the identity on K . Hence $\xi k - k\xi = 0$ for $k \in \mathfrak{k}$. Therefore:

$$[(\xi)_i, k] = \sum_{j=0}^{k+1} (\xi)_i - (k)_j (\xi)_i (k)_j = 0.$$

Hence the action of $(\xi)_i$ and K commute. □

3.4 Quotients of the affine type B/C Brauer algebra $\mathfrak{B}_k^\theta[m]$

In Section 3.5 we introduce functors, defined in [16], from the category $\mathcal{HC}(G)\text{-mod}$ to the category of $\mathfrak{B}_k^\theta[m]$ modules. However, we are aiming at graded Hecke algebra modules. In this section, we look at particular ideals and quotients of $\mathfrak{B}_k^\theta[m]$ which are isomorphic to graded Hecke algebras. This will set up Section 3.6 in which we focus on principal series modules and show that via the quotients defined in this section, the functors defined in Section 3.5 descend to take principal series modules to graded Hecke algebra modules.

Recall that $W(R)$ denotes the Weyl group associated to a root datum $(R, X, \hat{R}, \hat{X}, \Delta)$ and $\langle, \rangle : X \times \hat{X} \rightarrow \mathbb{C}$ is the pairing between dual spaces. Define the $\mathbb{C}$ -spaces $\mathfrak{t} = X \otimes_{\mathbb{Z}} \mathbb{C}$, $\mathfrak{t}^* = \hat{X} \otimes_{\mathbb{Z}} \mathbb{C}$.

Definition 3.4.1. [39] *The graded Hecke algebra $\mathbb{H}^R(\mathbf{c})$ associated to the root system $(R, X, \hat{R}, \hat{X}, \Delta)$ and parameter function $\mathbf{c}$ from Δ to $\mathbb{C}$, is as a vector space*

$$\mathbb{H}^R(\mathbf{c}) \cong S(\mathfrak{t}) \otimes \mathbb{C}[W(R)],$$

such that as an algebra $S(\mathfrak{t})$ and $\mathbb{C}[W(R)]$ are subalgebras and the following cross relations hold,

$$s_\alpha \epsilon - s_\alpha(\epsilon) s_\alpha = \mathbf{c}(\alpha) \langle \alpha, \hat{\epsilon} \rangle, \text{ for } \epsilon \in \mathfrak{t} \text{ and } \alpha \in \Delta.$$

If the parameter function $\mathbf{c} : \Delta \rightarrow \mathbb{C}$ is taken to uniformly be 1, then in this case the graded Hecke algebra is entirely defined by the root system (W, R, Δ) . For a Hecke algebra determined by the uniform parameter we denote it by $\mathbb{H}^{R_k}$ where R_k is the root system. For example $\mathbb{H}^{D_k}$ denotes the graded Hecke algebra associated to the root system D_k with the parameter function $\mathbf{c} : \delta \rightarrow \mathbb{C}$ such that $\mathbf{c}(\alpha) \equiv 1$.

We fix the set of simple reflections of the hyperoctahedral group $W(B_k)$ to be $\{s_{i,i+1}, \theta_k : i = 1, \dots, k-1\}$. We also associate to the hyperoctahedral group a k dimensional vector space $\mathfrak{t}$ with basis $\epsilon_1, \dots, \epsilon_k$ and subset $\Delta = \{\epsilon_i - \epsilon_{i+1} \text{ and } \epsilon_k : i = 1, \dots, k-1\}$. Then for $c \in \mathbb{C}$ we define the parameter $c : \Delta \rightarrow \mathbb{C}$ as

$$c(\alpha) = \begin{cases} 1 & \text{if } \alpha = \epsilon_i - \epsilon_{i+1}, \\ c & \text{if } \alpha = \epsilon_k. \end{cases}$$

We denote the graded Hecke algebra associated to the Weyl group $W(B_k)$ with the parameter c as $\mathbb{H}_k(c)$.

Lemma 3.4.2. *The graded Hecke algebra of type B_k (resp. type C_k) is isomorphic to $\mathbb{H}_k(1)$ (resp. $\mathbb{H}_k(\frac{1}{2})$) and the algebra $\mathbb{H}_k(0)$ is isomorphic to an extension of the Hecke algebra of type D_k ,*

$$\mathbb{H}_k(0) \cong \mathbb{H}^{D_k} \rtimes \mathbb{Z}_2.$$

Proof. The isomorphism of $\mathbb{H}_k(1)$ and the graded Hecke algebra $\mathbb{H}^{B_k}$ is apparent from the definitions. The space $\mathfrak{t}_{D_k}$ is equal to the space $\mathfrak{t}$ in $\mathbb{H}_k(0)$. The Weyl group $W(D_k)$ is naturally a subgroup of $W(B_k)$. The generator $t \in \mathbb{Z}_2$ acts on $\mathbb{H}^{D_k}$ by interchanging roots $\epsilon_{k-1} - \epsilon_k$ and $\epsilon_{k-1} + \epsilon_k$ and acts by conjugation by $s_{\epsilon_k} \in W(B_k)$ on $W(D_k) \subset W(B_k)$.

□

We define two ideals in the type B/C affine Brauer algebra $\mathfrak{B}_k^\theta[m]$. We then show that the quotient of $\mathfrak{B}_k^\theta[m]$ by these ideals is isomorphic to a graded Hecke algebra.

Definition 3.4.3. Let I_e be the two sided ideal in $\mathfrak{B}_k^\theta[m]$ generated by the idempotents,

$$\{e_{i,i+1} : \text{for } i = 1, \dots, k-1\}.$$

Let $c \in \mathbb{C}$ and $r \in \mathbb{Z}$, define I_c^r to be the two sided ideal,

$$I_c^r = \langle \theta_k x_k + x_k \theta_k - 2c + 2r\theta_k \rangle.$$

The ideal I_e can be generated by any idempotent since they are all in the same S_k conjugation orbit. By using $c \in \mathbb{C}$ we have abused notation; however the two occurrences of c will correspond to the same constant.

Lemma 3.4.4. The quotient of the algebra $\mathfrak{B}_k^\theta[m]$ by the ideal generated by I_e and I_c^r is isomorphic to the graded Hecke algebra

$$\mathfrak{B}_k^\theta[m] / \langle I_e, I_c^r \rangle \cong \mathbb{H}_k(c).$$

Proof. Consider the presentation in Definition 3.3.10 with generators

$$x_i, \theta_j, t_{i,i+1}, e_{i,i+1}$$

and relations

$$\theta_j^2 = 1, s_{i,i+1}^2 = 1, (s_{i,i+1} s_{i+1,i+2})^3 = 1, (s_{k-1,k} \theta_k)^4 = 1,$$

$$t_{i,i+1} x_i - x_{i+1} t_{i,i+1} = 1 + e_{i,i+1},$$

$$[t_{i,i+1}, x_j] = 0, j \neq i, i+1,$$

$$e_{i,i+1}(x_i + x_{i+1}) = 0 = (x_i + x_{i+1})e_{i,i+1},$$

$$[e_{i,i+1}, x_j] = 0, j \neq i, i+1,$$

$$[x_i, x_j] = 0,$$

$$[e_{i,i+1}, \theta_j] = 0 \text{ for all } j,$$

$$e_{i,i+1} \theta_i \theta_{i+1} = e_{i,i+1} = \theta_i \theta_{i+1} e_{i,i+1} \text{ for } i = 1, \dots, k-1,$$

$$[\theta_n, x_j] = 0 \text{ for } j \neq k.$$

Under the quotient by I_e and I_c^r the generators $e_{i,i+1}$ and the relations $e_{i,i+1} = 0$ cancel out. Furthermore we add another relation: $x_k \theta_k + \theta_k x_k - 2c + 2r\theta_k$. Hence the presentation has generators

$$x_i, \theta_j, t_{i,i+1}$$

with relations

$$\theta_j^2 = 1, s_{i,i+1}^2 = 1, (s_{i,i+1}s_{i+1,i+2})^3 = 1, (s_{k-1,k}\theta_k)^4 = 1,$$

$$t_{i,i+1}x_i - x_{i+1}t_{i,i+1} = 1,$$

$$[t_{i,i+1}, x_j] = 0, j \neq i, i+1,$$

$$[x_i, x_j] = 0,$$

$$[\theta_n, x_j] = 0 \text{ for } j \neq k,$$

$$x_k\theta_k + \theta_kx_k - 2c + 2r\theta_k.$$

This is a presentation of the Hecke algebra $\mathbb{H}_k(c)$; it is the modification of the presentation in Definition 3.4.1 by $\epsilon_i \mapsto x_i + r$. Since we have shown that the presentation of $\mathfrak{B}_k^\theta[m]/\langle I_e I_c^r \rangle$ is identical to the presentation of $\mathbb{H}_k(c)$ then these algebras are isomorphic.

□

Remark 3.4.5. *We could have chosen to quotient by the ideal generated by $\theta_kx_k + x_k\theta_k - c$ without the $2r\theta_k$ part. This quotient would also be isomorphic to $\mathbb{H}_k(c)$ with ϵ_i mapping to x_i . However, we need the modification of the affine parts by the scalar r to enable our results regarding images of principal series modules descending to Hecke algebra modules. One can think of this modification by r as an analogue of the ρ shift.*

3.5 Functors from $\mathcal{HC}(G)$ -mod to $\mathfrak{B}_k^\theta$ -mod.

In this section, we introduce functors, defined in [16]. We show these functors take Harish-Chandra modules to modules of the $\mathfrak{B}_k^\theta$ algebra.

Definition 3.5.1. [16, (2.8)] *Let n be the real rank of G . If $G = Sp_{2n}(\mathbb{R})$ the real rank is n . If $G = O(p, q)$ then $n = q = \min(p, q)$. Let μ be an irreducible K -module, fix an integer $k \leq n$. The space V is the matrix module of G . We define the functor $F_{\mu,k}$ to be:*

$$F_{\mu,k} : \mathcal{HC}(G)\text{-mod} \longrightarrow \mathfrak{B}_k^\theta\text{-mod}$$

$$X \mapsto \text{Hom}_K(\mu, X \otimes V^{\otimes k}),$$

and on morphisms $f : X \rightarrow Y$ and $g \in \text{Hom}_k(\mu, X \otimes V^{\otimes k})$,

$$F_{\mu,k}f(g) : \mu \rightarrow Y \otimes V^{\otimes k},$$

$$F_{\mu,k}f(g)(\mu) = f \otimes \text{id}^{\otimes k} g(\mu).$$

Remark 3.5.2. Lemma 3.3.11 gives an action of $\mathfrak{B}_k^\theta$ on $X \otimes V^{\otimes k}$. Since this action commutes with the action of K then $\mathfrak{B}_k^\theta$ naturally acts on $\text{Hom}_K(\mu, X \otimes V^{\otimes k})$ from the inherited action on $X \otimes V^{\otimes k}$.

Lemma 3.5.3. For any irreducible K -module μ and $k \leq n$, the functor $F_{\mu,k}$ defined in Definition 3.5.1 is exact.

Proof. Tensoring with a finite dimensional module is exact. The module $V^{\otimes k}$ is finite dimensional hence the functor taking X to $X \otimes V^{\otimes k}$ is exact. Furthermore, μ is an irreducible K -module. Therefore the functor which takes Y to $\text{Hom}_K(\mu, Y)$ is exact. The functor $F_{\mu,k}$ is the composition of these two exact functors, hence the result follows. $\square$

3.6 Restricting functors to principal series modules

The functors (Definition 3.5.1) take any Harish-Chandra module to a $\mathfrak{B}_k^\theta$ -module. In this section, given a principal series module we give a basis for the image of the functors $F_{\mu,k}$ and $F_{\underline{\mu},n-k}$ for particular characters $\mu, \underline{\mu}$ depending on the principal series modules.

Let $G = Sp_{2n}(\mathbb{R})$. As in Example 3.2.1 $K \cong U(n)$, $M \cong (\mathbb{Z}_2)^n$. The Cartan involution θ is equal to conjugation by the matrix

$$\xi = \begin{bmatrix} 0 & i \text{Id}_n \\ -i \text{Id}_n & 0 \end{bmatrix}.$$

The subspace $\mathfrak{a}$ has dimension n with basis ε_i and corresponds to the subgroup A under the exponential map. We label a character of $\mathfrak{a}$ by $\nu \in \mathfrak{a}^*$ and characters of A by e^ν . The matrix module $V \cong \mathbb{C}^{2n}$ has two bases: $\{e_1, \dots, e_{2n}\}$ and $\{f_1^1, \dots, f_n^1, f_1^{-1}, \dots, f_n^{-1}\}$, where $f_i^\eta = e_i + \eta e_{n+i}$.

Recall that the Iwasawa decomposition of G is

$$G = KAN,$$

also, that M is the centraliser of $\mathfrak{a}_0$ in K , which is isomorphic to $\mathbb{Z}_2^n$. The character δ^k is defined to be the character of M which takes the first k generators of $\mathbb{Z}_2^n$ to -1 and the last $n - k$ to 1 . We write 1 for the trivial character of N .

If $G = O(p, q)$ then $K \cong O(p) \times O(q)$, $M = O(p - q) \times O(1)^q$ embedded into $O(p, q)$ as the block matrix

$$(O(p - q), x_1, x_2, \dots, x_q, x_q, \dots, x_1)$$

where $x_i \in O(1)$. We denote characters of M , δ_{triv}^k and δ_{det}^k to be

$$\delta_{\text{triv}}^k = \text{triv} \otimes (\text{sgn}^k) \otimes \text{triv}^{q-k} \text{ on } O(p - q) \otimes O(1)^q,$$

$$\delta_{\text{det}}^k = \det \otimes (\text{sgn}^k) \otimes \text{triv}^{q-k} \text{ on } O(p - q) \otimes O(1)^q.$$

The Cartan involution θ is equal to conjugation by the matrix

$$\xi = \begin{bmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{bmatrix}.$$

Definition 3.6.1. [58] Let $G = KAN$ (resp. $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$) be the Iwasawa decomposition of G (resp. $\mathfrak{g}_0$) and let M be the centraliser of $\mathfrak{a}_0$ in K . Given a character e^ν of A and the character δ of M we define the minimal principal series representation;

$$X_\delta^\nu = \text{Ind}_{MAN}^G(\delta \otimes e^\nu \otimes 1).$$

In the non-split case principal series representations may be induced from irreducible representations of M which are not one dimensional. In this chapter we will only study principal series modules that are induced from a character of M . We write $\mathbb{1}_\delta^\nu$ for the vector spanning the representation space of the character $\delta \otimes e^\nu \otimes 1$. Hence

$$X_\delta^\nu = \text{Ind}_{MAN}^G \mathbb{1}_\delta^\nu.$$

For $G = Sp_{2n}(\mathbb{R})$, we can calculate the dimension of $F_{\text{triv},k}(X_{\delta^k}^\nu)$ and $F_{\text{det},n-k}(X_{\delta^k}^\nu)$. Note that if we want to describe the trivial isotypic component we must take $F_{\text{triv},k}$ and if we wish to look at the det isotypic component then we must take the functor $F_{\text{det},n-k}$.

For $G = O(p, q)$, we can calculate the dimension of $F_{\text{triv} \otimes \text{sgn},k}$ and $F_{\text{triv} \otimes \text{triv},q-k}$. Similarly for $X_{\delta_{\text{det}}^k}^\nu$, we take the functors $F_{\text{sgn} \otimes \text{triv},k}$ and $F_{\text{sgn} \otimes \text{sgn},q-k}$.

To enable us to succinctly discuss all of the above cases we will associate a character μ and $\underline{\mu}$ to each principal series modules. Note δ is a K -character and $\mu, \underline{\mu}$ are characters of M .

$G = Sp_{2n}(\mathbb{R})$	$X_\delta^\nu, \delta = (\text{triv})^k \otimes (\text{sgn})^{n-k}$	$\mu = \text{triv}$	$\underline{\mu} = \det$
$G = O(p, q)$	$X_\delta^\nu, \delta = \text{triv}_{p-q} \otimes (\text{triv})^k \otimes (\text{sgn})^{q-k}$	$\mu = \text{triv} \otimes \det$	$\underline{\mu} = \text{triv} \otimes \text{triv}$
$G = O(p, q)$	$X_\delta^\nu, \delta = \det_{p-q} \otimes (\text{triv})^k \otimes (\text{sgn})^{q-k}$	$\mu = \det \otimes \text{triv}$	$\underline{\mu} = \text{sgn} \otimes \text{sgn}$

Table 3.6.1: Characters $\mu, \underline{\mu}$ associated to particular principal series module.

Lemma 3.6.2. *Let $G = Sp_{2n}(\mathbb{R})$ or $G = O(p, q)$. If X_δ^ν is a minimal principal series module, then $F_{\mu,k}(X_\delta^\nu)$ and $F_{\underline{\mu},n-k}(X_\delta^\nu)$ are finite dimensional. with dimensions:*

$$\dim(F_{\mu,k}(X_\delta^\nu)) = k!2^k = |W(B_k)|,$$

Similarly,

$$\dim(F_{\underline{\mu},n-k}(X_\delta^\nu)) = (n-k)!2^{n-k} = |W(B_{n-k})|.$$

This is an extension of [16, Lemma 2.5.1] to non-spherical principal series modules and we use the same arguments.

Proof. We explicitly calculate a basis for

$$F_{\mu,k}(X_\delta^\nu) = \text{Hom}_K(\mu, X_\delta^\nu \otimes V^{\otimes k}).$$

Since X_δ^ν is an induced module from $\mathbb{1}_\delta^\nu$ and K is a compact group, by Frobenius reciprocity this is equal to,

$$F_{\mu,k}(X_\delta^\nu) = \text{Hom}_M(\mu|_M, \mathbb{1}_\delta^\nu \otimes V|_M^{\otimes k}).$$

One can tensor by μ^* to get a space fixed by M , hence

$$F_{\mu,k}(X_\delta^\nu) = (\mu^* \otimes \mathbb{1}_\delta^\nu \otimes V^{\otimes k})^M.$$

We first prove the result for $G = Sp_{2n}(\mathbb{R})$. The module V has basis $\{f_i^{n_i} : i = 1, \dots, n \text{ and } n_i = \pm 1\}$ and the j^{th} generator of M acts by $-1^{\delta_{ij}}$ on $f_i^{n_i}$. Therefore if we require M to act trivially on $u \in X_{\delta_k}^\nu \otimes V^{\otimes k}$ the generators $M_1, \dots, M_n$ must act by 1. Let us first calculate all of the elementary tensors in $X_{\delta_k}^\nu \otimes V^{\otimes k}$ which are fixed by M . The generators $M_1, \dots, M_k$ act by -1 on $\mathbb{1}_{\delta_k}^\nu$, hence must act by -1 on the tensor part contributed by $V^{\otimes k}$. To satisfy this we need to have f_i^1 or f_i^{-1} feature in the tensor of u , for every $i = 1, \dots, k$. Since there can only be k elements tensored together in $V^{\otimes k}$ then the contribution of u from $V^{\otimes k}$ must be $f_1^{n_1}, \dots, f_k^{n_k}$ in some order. The

set of elementary tensors in $V^{\otimes k}$ which feature all the required f_i is the S_k orbit of $f_1 \otimes \dots \otimes f_k$. Considering not necessarily elementary tensors in $v \in X_{\delta^k}^\nu \otimes V^{\otimes k}$,

$$v = \sum x_0 \otimes v_1 \otimes \dots \otimes v_k,$$

where $v_i \in \{f_l^{n_l} : l = 1, \dots, n \text{ and } n_l = \pm 1\}$. The j^{th} generator of M , M_j , acts by $-1^{\delta^{lj}}$ on f_l . Since every elementary tensor in this basis is an eigenvector of the action of M then if M fixes $v = \sum x_0 \otimes v_1 \otimes \dots \otimes v_k$ then M fixes each elementary tensor in v . Hence every M fixed vector in $X_{\delta^k}^\nu \otimes V^{\otimes k}$ is in the subspace

$$\text{span} \left\{ \sum_{w \in S_k} \mathbb{1}_{\delta^k}^\nu \otimes f_{w(1)}^{n_1} \otimes \dots \otimes f_{w(k)}^{n_k} : n_i = \pm 1 \right\}.$$

The size of the basis is $|S_k| \times 2^k = k!2^k = |W(B_k)|$. The proof is almost identical for $\dim(F_{\det, n-k}(X_{\delta^k}^\nu))$. One needs to note that all of the generators of M must act by -1 on the $\det$ isotypic space, since $\det|_M = \text{sgn}$. Using Frobenius reciprocity one can show,

$$F_{\det, n-k}(X_{\delta^k}^\nu) = \text{Hom}_M(\text{sgn}, \delta_k \otimes V^{\otimes n-k}),$$

which has a basis:

$$F_{\det, n-k}(X_{\delta^k}^\nu) = \text{span} \left\{ \sum_{w \in S_{n-k}} \mathbb{1}_{\delta^k}^\nu \otimes f_{w(k+1)}^{n_{k+1}} \otimes \dots \otimes f_{w(n)}^{n_n} : n_i = \pm 1 \right\}.$$

For $G = O(p, q)$ note that $V|_M = V_{p-q} \oplus_{i=1}^q \text{triv} \otimes \dots \otimes \overbrace{\text{sgn}}^{i^{\text{th}}} \otimes \dots \otimes \text{triv}$ and $\mu|_M = \text{triv}_{p-q} \otimes \text{sgn}^q$. Recall the notation $f_i^{n_i} = e_{p-i+1} + n_i e_{p+i}$, the vectors f_i^1 and f_i^{-1} are the two eigenvectors of M with character $\text{triv} \otimes \text{triv} \dots \otimes \overbrace{\text{sgn}}^{i^{\text{th}}} \otimes \dots \otimes \text{triv}$. I.e. the i^{th} generator of $O(1)^q$ in M acts by -1 .

We will prove that $F_{\text{triv} \otimes \text{sgn}, k}(X_{\delta_{\text{triv}}^k}^\nu)$ has basis

$$\left\{ \sum_{w \in S_k} \mathbb{1}_{\delta_{\text{triv}}^k}^\nu \otimes f_{w(k)}^{n_1} \otimes \dots \otimes f_{w(k)}^{n_k} : n_i = \pm 1 \right\}.$$

The other four calculations are almost identical. Note that this is equivalent to giving a basis for

$$((\text{triv} \otimes \text{sgn})|_M \otimes \mathbb{1}_{\delta_{\text{triv}}^k}^\nu \otimes V^k)^M$$

which is equal to, as a vector space,

$$(\mathbb{1}_{\text{triv}_{p-q} \otimes \text{sgn}^q} \otimes \mathbb{1}_{\delta_{\text{triv}}^k}^\nu \otimes (V_{p-q} \bigoplus \text{triv} \dots \otimes \text{sgn} \otimes \dots \text{sgn})^k)^M.$$

The vector $\mathbb{1}_{\text{triv}_{p-q} \otimes \text{sgn}^q} \otimes \mathbb{1}_{\delta_{\text{triv}}^k}^\nu \otimes f_1 \otimes \dots \otimes f_q$ is fixed by M since $O(p-q)$ acts trivially on each tensor. Furthermore for $i = 1, \dots, k$ the i^{th} generator of $O(1)^q$ in M acts by -1 on $\mathbb{1}_{\text{triv}_{p-q} \otimes \text{sgn}^q}$, 1 on $\mathbb{1}_{\delta_{\text{triv}}^k}^\nu$, and -1 on $f_1 \otimes \dots \otimes f_q$. For $i = k+1, \dots, q$ the i^{th} generator of $O(1)^q$ in M acts by -1 on $\mathbb{1}_{\text{triv}_{p-q} \otimes \text{sgn}^q}$, -1 on $\mathbb{1}_{\delta_{\text{triv}}^k}^\nu$ and 1 on $f_1 \otimes \dots \otimes f_q$. Hence every generator of M acts by 1 . An identical argument shows that the orbit of $\mathbb{1}_{\text{triv}_{p-q} \otimes \text{sgn}^q} \otimes \mathbb{1}_{\delta_{\text{triv}}^k}^\nu \otimes f_1 \otimes \dots \otimes f_q$ by $W(B_q)$ is also fixed. Any elementary tensor fixed by M must be of this form; if it is not, one of the generators will act by -1 . Finally suppose that another vector v is fixed by M , then v is a sum of elementary tensors which are all eigenvalues for $O(1)^q$, hence every elementary tensor involved must be fixed. This concludes that v is in the span of the vectors

$$\left\{ \sum_{w \in S_k} \mathbb{1}_{\delta_{\text{triv}}^k}^\nu \otimes f_{w(1)}^{n_1} \otimes \dots \otimes f_{w(k)}^{n_k} : n_i = \pm 1 \right\}.$$

We state the basis for $F_{\mu,k}$ and $F_{\mu,n-k}$. Let $G = O(p, q)$

$$\begin{aligned} F_{\text{triv} \otimes \text{det}, k}(X_{\delta_{\text{triv}}^k}^\nu) &= \text{span} \left\{ \sum_{w \in S_k} \mathbb{1}_{\delta_{\text{triv}}^k}^\nu \otimes f_{w(1)}^{n_1} \otimes \dots \otimes f_{w(k)}^{n_k} : n_i = \pm 1 \right\}, \\ F_{\text{triv} \otimes \text{triv}, q-k}(X_{\delta_{\text{triv}}^k}^\nu) &= \text{span} \left\{ \sum_{w \in S_{q-k}} \mathbb{1}_{\delta_{\text{triv}}^k}^\nu \otimes f_{w(k+1)}^{n_{k+1}} \otimes \dots \otimes f_{w(q)}^{n_q} : n_i = \pm 1 \right\}, \\ F_{\text{det} \otimes \text{triv}, k}(X_{\delta_{\text{det}}^k}^\nu) &= \text{span} \left\{ \sum_{w \in S_k} \mathbb{1}_{\delta_{\text{det}}^k}^\nu \otimes f_{w(1)}^{n_1} \otimes \dots \otimes f_{w(k)}^{n_k} : n_i = \pm 1 \right\}, \\ F_{\text{det} \otimes \text{det}, q-k}(X_{\delta_{\text{det}}^k}^\nu) &= \text{span} \left\{ \sum_{w \in S_{q-k}} \mathbb{1}_{\delta_{\text{det}}^k}^\nu \otimes f_{w(k+1)}^{n_{k+1}} \otimes \dots \otimes f_{w(q)}^{n_q} : n_i = \pm 1 \right\}. \end{aligned}$$

□

3.7 Images of principal series modules

We write the Type B/C affine Brauer algebra as $\mathfrak{B}_k^\theta$ and omit m .

We show that on minimal principal series representations the functors (Definition 3.5.1) which take admissible $O(p, q)$ or Sp_{2n} -modules to $\mathfrak{B}_k^\theta$ -modules naturally descend to graded Hecke algebra $\mathbb{H}_k(c)$ -modules, for c equal to 0 , 1 or $\frac{p-q}{2}$.

In Section 3.4 Lemma 3.4.4, we proved that the type B/C affine Brauer algebra has quotients isomorphic to the Hecke algebra $\mathbb{H}_k(c)$ with parameter $c \in \mathbb{R}$. This quotient was defined by the relations $e_{i,i+1} = 0$ and $\theta_k x_k + x_k \theta_k = 2c - 2r\theta_k$. Hence to show that $F_{\mu,k}(X_\delta^\nu)$ descends to an $\mathbb{H}_k(c_\mu)$ -module we must prove $e_{i,i+1} = 0$ and

$\theta_k x_k + x_k \theta_k = 2c_\mu - 2r_\mu \theta_k$ as operators on $F_{\mu,k}(X_\delta^\nu)$. Similarly to show $F_{\underline{\mu},n-k}(X_\delta^\nu)$ is an $\mathbb{H}_{n-k}(r_{\underline{\mu}})$ -module then we must show $e_{i,i+1} = 0$ and $\theta_{n-k} x_{n-k} + x_{n-k} \theta_{n-k} = 2c_{\underline{\mu}} - 2r_{\underline{\mu}} \theta_{n-k}$ on $F_{\underline{\mu},n-k}(X_\delta^\nu)$. The scalars r_μ and c_μ will be defined in Table 3.7.1. The arguments of this section are inspired and very similar to [16, Proposition 2.4.5, Lemma 2.7.2]. We extend these results to non-spherical principal series modules. We also utilise an approach from the Brauer algebra perspective not used in [16].

Lemma 3.7.1. *c.f. [16, 2.4.5] On the $\mathfrak{B}_k^\theta$ (resp. $\mathfrak{B}_{n-k}^\theta$) module $F_{\mu,k}(X_\delta^\nu)$ (resp. $F_{\underline{\mu},n-k}(X_\delta^\nu)$) the idempotents $e_{i,i+1}$ uniformly act by zero.*

Proof. Lemma 3.6.2 states that the basis of $F_{\mu,k}(X_\delta^\nu)$ is given by $\mathbb{1}_\delta^\nu \otimes f_{w(1)}^{n_1} \otimes \dots \otimes f_{w(k)}^{n_k}$ for $w \in S_k$. The idempotents $e_{i,i+1}$ act by the projection onto the trivial component of $V_i \otimes V_{i+1}$. The trivial component of $V \otimes V$ is one dimensional with spanning vector $\sum_{i=1}^n f_i \wedge f'_i$. The vector $\mathbb{1}_\delta^\nu \otimes f_{w(1)}^{n_1} \otimes \dots \otimes f_{w(k)}^{n_k}$ is in the subspace perpendicular to $\sum_{i=1}^n f_i \wedge f'_i$ given in Lemma 0.8.10. Therefore it is in the kernel of the projection $\text{pr}_{i,i+1}$. $\square$

Recall Definition 3.3.4, $\Omega_{i,j} = \sum_{b \in B} (b)_i \otimes (b^*)_j \in U(g)^{k+1}$, and $\Omega_{i,j}^\mathfrak{k} = \sum_{b \in B \cap \mathfrak{k}} (b)_i \otimes (b^*)_j$. Lemma 3.3.7 gives $x_k = \Omega_{0,k} + \Omega_{1,k} + \dots + \Omega_{k-1,k}$.

As operators on $F_{\mu,k}(X_\delta^\nu)$:

$$\begin{aligned} \theta_k x_k + x_k \theta_k &= \theta_k \sum_{i < k} \Omega_{i,k} + \sum_{i < k} \Omega_{i,k} \theta_k \\ &= (\xi)_k \sum_{i < k} \sum_{b \in B} (b)_i \otimes (b^*)_k + \sum_{i < k} \sum_{b \in B} (b)_i \otimes (b^*)_k (\xi)_k \\ &= \sum_{i < k} \sum_{b \in B} (b)_i \otimes (\xi b^* + b^* \xi)_k. \end{aligned}$$

Conjugating by ξ is the Cartan involution. Therefore

$$\xi b^* + b^* \xi = \begin{cases} 0 & \text{if } b \in \mathfrak{p}, \\ 2\xi b^* & \text{if } b \in \mathfrak{k}. \end{cases}$$

Hence,

$$\begin{aligned} \theta_k x_k + x_k \theta_k &= 2 \sum_{i < k} \sum_{b \in B \cap \mathfrak{k}} (b)_i \otimes (\xi b)_k \\ &= 2\theta_n \sum_{i < k} \Omega_{i,k}^\mathfrak{k}. \end{aligned}$$

As operators on $F_{\mu,k}(X_\delta^\nu)$

$$\theta_k x_k + x_k \theta_k = 2\theta_k \sum_{i < k} \Omega_{i,k}^\mathfrak{k}.$$

Similarly on $F_{\underline{\mu},n-k}(X_\delta^\nu)$

$$\theta_{n-k} x_{n-k} + x_{n-k} \theta_{n-k} = 2\theta_{n-k} \sum_{i < n-k} \Omega_{i,n-k}^\mathfrak{k}.$$

Lemma 3.7.2. *c.f. [16, 2.7.2] On the $\mathfrak{B}_k^\theta$ -module $F_{\mu,k}(X_\delta^\nu)$,*

$$\theta_k x_k + x_k \theta_k = 2\xi \left(\sum_{b \in B \cap \mathfrak{z}} \mu(b) b^* - C^\mathfrak{k} \right)_k,$$

where $\mathfrak{z}$ is the centre of $\mathfrak{g}$.

Proof. Recall Definition 3.3.4, $\Omega_{ij} = \sum_{b \in B \cap \mathfrak{k}} (b)_i \otimes (\xi b)_k$. Writing $\theta_k x_k + x_k \theta_k$ as operators on $F_{\mu,k}(X_\delta^\nu)$,

$$\begin{aligned} \theta_k x_k + x_k \theta_k &= 2\theta_k \sum_{i < k} \Omega_{i,k}^\mathfrak{k}, \\ &= 2 \sum_{i < k} \sum_{b \in B \cap \mathfrak{k}} (b)_i \otimes (\xi b)_k. \end{aligned}$$

An element $g \in \mathfrak{g}$ acts on the tensor of two modules, $U \otimes W$, as $g \otimes 1 + 1 \otimes g$. Extending this, we can write the action of $b \in U(\mathfrak{g})$ as $\sum_{j=1}^{k+1} (b)_j$ on $X \otimes V^{\otimes k}$. This gives

$$\theta_k x_k + x_k \theta_k = 2\theta_k \sum_{b \in B \cap \mathfrak{k}} (b^*)_k b - \sum_{b \in B \cap \mathfrak{k}} (bb^*)_k.$$

By definition $F_{\mu,k}(X_\delta^\nu)$ is the μ isotypic component of X_δ^ν , hence

$$\theta_k x_k + x_k \theta_k = 2\theta_k \sum_{b \in B \cap \mathfrak{k}} (b^*)_k \mu(b) - \sum_{b \in B \cap \mathfrak{k}} (bb^*)_k.$$

The operator $\sum_{b \in \mathfrak{k}} (bb^*)_k$ is the Casimir operator $C^\mathfrak{k}$ on the k^{th} tensor V . We have $\mu(b) = 0$ unless b is in the centre of $U(\mathfrak{k})$ for any character μ . Let $\mathfrak{z}$ denote the centre of $\mathfrak{g}$. Therefore,

$$\begin{aligned} \theta_k x_k + x_k \theta_k &= 2\theta_k \left(\sum_{b \in B \cap \mathfrak{z}} \mu(b) (b^*)_k - (C^\mathfrak{k})_k \right), \\ &= 2 \left(\xi \left(\sum_{b \in B \cap \mathfrak{z}} \mu(b) b^* - C^\mathfrak{k} \right) \right)_k. \end{aligned}$$

□

In order to calculate the action of $\theta_k x_k + x_k \theta_k$ we must understand the operator

$$Q_\mu = 2\xi \left(\sum_{b \in B \cap \mathfrak{z}} \mu(b) b^* - C^\mathfrak{k} \right)$$

acting on the k^{th} tensor of V .

Lemma 3.7.3. *On the $\mathfrak{B}_{n-k}^\theta$ -module $F_{\underline{\mu},n-k}(X_\delta^\nu)$;*

$$\theta_{n-k} x_{n-k} + x_{n-k} \theta_{n-k} = 2 \left(\xi \left(\sum_{b \in B \cap \mathfrak{z}} \underline{\mu}(b) b^* - C^\mathfrak{k} \right) \right)_{n-k}.$$

$G = Sp_{2n}(\mathbb{R})$	$\mu = \text{triv}$	$r_{\text{triv}} = 0$	$c_{\text{triv}} = -n$
$G = Sp_{2n}(\mathbb{R})$	$\underline{\mu} = \text{det}$	$r_{\text{det}} = 1$	$c_{\text{det}} = -n$
$G = O(p, q)$	$\mu = \text{triv} \otimes \text{det}$	$r_{\mu} = \frac{p+q}{2}$	$c_{\mu} = \frac{p-q}{2}$
$G = O(p, q)$	$\mu = \text{triv} \otimes \text{triv}$	$r_{\mu} = \frac{p+q}{2}$	$c_{\mu} = \frac{p-q}{2}$
$G = O(p, q)$	$\underline{\mu} = \text{det} \otimes \text{triv}$	$r_{\underline{\mu}} = \frac{p+q}{2}$	$c_{\underline{\mu}} = \frac{p-q}{2}$
$G = O(p, q)$	$\underline{\mu} = \text{det} \otimes \text{det}$	$r_{\underline{\mu}} = \frac{p+q}{2}$	$c_{\underline{\mu}} = \frac{p-q}{2}$

Table 3.7.1: Values of c_{μ} and r_{μ} for particular K -characters μ .

Replacing μ with $\underline{\mu}$, this follows the same way as Lemma 3.7.2.

Lemma 3.7.4. *On the module V the operator $Q_{\mu} = 2\xi \left(\sum_{b \in B \cap \mathfrak{g}} \mu(b)b^* - C^{\mathfrak{k}} \right)$ (resp. $Q_{\underline{\mu}} = 2\xi \left(\sum_{b \in B \cap \mathfrak{g}} \underline{\mu}(b)b^* - C^{\mathfrak{k}} \right)$) is equal to $2r_{\mu} + 2c_{\mu}\xi$ (resp. $2r_{\underline{\mu}} + 2c_{\underline{\mu}}\xi$), where r_{μ} and c_{μ} are scalars given below. In fact for $G = O(p, q)$, r_{μ} and c_{μ} are independent of μ .*

Recall Lemma 3.4.2, we have isomorphisms: $\mathbb{H}_k(1) \cong \mathbb{H}^{B_k}$, $\mathbb{H}_k(\frac{1}{2}) \cong \mathbb{H}^{C_k}$ and $\mathbb{H}_k(0)$ is congruent to an extension of the type D graded Hecke algebra $\mathbb{H}^{D_k}$. Hence when G is split, that is $G = O(n+1, n)$ or $Sp_{2n}(\mathbb{R})$ then $c_{\mu} = 1, \frac{1}{2}$ or 0 and we obtain correspondences between principal series modules of split real orthogonal Lie groups with graded Hecke algebras of type C and split real symplectic groups with graded Hecke algebras of type B and D .

Proof. We prove the result first for $G = Sp_{2n}(\mathbb{R})$, in this case $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $\mathfrak{k} = \mathfrak{gl}_n$. The Casimir $C^{\mathfrak{k}}$ acts by the scalar n on V . The character triv is zero uniformly on $\mathfrak{k}$ hence $\text{triv}(b) = 0$ for all b and there is no contribution from $\sum_{b \in B \cap \mathfrak{g}} \text{triv}(b)b^*$. For the operator $\sum_{b \in B \cap \mathfrak{g}} \det(b)b^*$, we note that the centre of $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{C})$ is the span of the identity matrix, also the character $\det$ of $U(n)$ differentiated to $\mathfrak{k}$ is the trace character of $\mathfrak{gl}_n$. Taking the spanning vector Id_n of the centre $\mathfrak{z}$ of $\mathfrak{gl}_n$ then on V , $\sum_{b \in B \cap \mathfrak{g}} \det(b)b^*$ is equal to

$$\begin{aligned}
\sum_{b \in B \cap \mathfrak{g}} \det(b)b^* &= \text{trace}(\text{Id}_n) \text{Id}_n^* \\
&= n \frac{1}{n} \text{Id}_n, \\
&= \text{Id}_n.
\end{aligned}$$

Referring to Example 3.2.1, since Id_n is symmetric, the identity matrix in $U(\mathfrak{k})$ embedded into $\mathfrak{g}$ is

$$\begin{bmatrix} 0 & i \text{Id}_n \\ -i \text{Id}_n & 0 \end{bmatrix}.$$

The matrix ξ , defined by the Cartan involution of $Sp_{2n}(\mathbb{R})$ is equal to

$$\xi = \begin{bmatrix} 0 & i \text{Id}_n \\ -i \text{Id}_n & 0 \end{bmatrix}.$$

Hence

$$\sum_{b \in B \cap \mathfrak{z}} \det(b) b^* = \xi,$$

as operators on V . □

Now let $G = O(p, q)$ $p + q = 2n + 1$, then $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{k} = \mathfrak{so}_p \oplus \mathfrak{so}_q$.

Any character μ of K differentiated and then restricted to $\mathfrak{z}$ is zero. Hence for any μ ,

$$\sum_{b \in \mathfrak{z}} \mu(b) b^* = 0.$$

We are left to calculate $C^\mathfrak{k}$ on V . $C^\mathfrak{k}$ acts by

$$\begin{bmatrix} p \text{Id}_p & 0 \\ 0 & q \text{Id}_q \end{bmatrix}.$$

For $G = O(p, q)$ the semisimple element defining θ is

$$\xi = \begin{bmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{bmatrix}.$$

Hence for $G = O(p, q)$

$$Q_\mu = 2\xi \left(\sum_{b \in \mathfrak{z}} \mu(b) b^* - C^\mathfrak{k} \right) = 2\xi \left(-\frac{p+q}{2} \text{Id} - \frac{p-q}{2} \xi \right) = (q-p) \text{Id}_n - (p+q) \xi.$$

Corollary 3.7.5. *For $G = O(p, q)$ or Sp_{2n} , consider the principal series module X_δ^ν for particular μ and $\underline{\mu}$ given in Table 3.6.1. On the $\mathfrak{B}_k^\theta$ -module $F_{\mu,k}(X_\delta^\nu)$, the following equality holds:*

$$\theta_{n-k} x_{n-k} + x_{n-k} \theta_{n-k} = 2r_\mu - 2c_\mu \theta_{n-k}.$$

Hence by Lemma 3.4.4, $F_{\mu,k}(X_\delta^\nu)$ is an $\mathbb{H}_k(c_\mu)$ -module via the quotient defined by the relations $e_{i,i+1} = 0$ and $\theta_{n-k} x_{n-k} + x_{n-k} \theta_{n-k} = 2r_\mu + 2c_\mu \theta_{n-k}$. Similarly $F_{\underline{\mu},n-k}(X_\delta^\nu)$ is an $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ -module.

We have shown that the image of X_δ^ν under the functor $F_{\mu,k}$ naturally descends to a module of the graded Hecke algebra $\mathbb{H}_k(c_\mu)$.

Theorem 3.7.6. *Let X_δ^ν be a minimal principal series module of $G = Sp_{2n}(\mathbb{R})$ or $O(p, q)$. Let μ and $\underline{\mu}$ be the particular characters in Table 3.6.1 and r_μ, c_μ be particular scalars in Table 3.7.1. Let π denote the homomorphism from $\mathfrak{B}_k^\theta[m]$ to $\text{End}(F_{\mu,k}(X_\delta^\nu))$ in Lemmas 3.3.7 and 3.3.11. The graded Hecke algebra $\mathbb{H}_k(c_\mu)$ acts on $F_{\mu,k}(X_\delta^\nu)$, by the homomorphism,*

$$\begin{aligned}\psi : \mathbb{H}_k(c_\mu) &\rightarrow \text{End}(F_{\mu,k}(X_\delta^\nu)), \\ \epsilon_i &\mapsto \pi(x_i - r_\mu), \\ s_{i,i+1} &\mapsto \pi(s_{i,i+1}), \\ s_{\epsilon_i} &\mapsto \pi(\theta_i).\end{aligned}$$

Hence $F_{\mu,k}(X_\delta^\nu)$, can be considered as an $\mathbb{H}_k(c_\mu)$ -module.

Let $\underline{\pi}$ denote the homomorphism from $\mathfrak{B}_{n-k}^\theta[m]$ to $\text{End}(F_{\underline{\mu},n-k}(X_\delta^\nu))$. The graded Hecke algebra $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ acts on $F_{\underline{\mu},n-k}(X_\delta^\nu)$, by the homomorphism,

$$\begin{aligned}\underline{\psi} : \mathbb{H}_{n-k}(c_{\underline{\mu}}) &\rightarrow \text{End}(F_{\underline{\mu},n-k}(X_\delta^\nu)), \\ \epsilon_i &\mapsto \underline{\pi}(x_i - r_{\underline{\mu}}), \\ s_{i,i+1} &\mapsto \underline{\pi}(s_{i,i+1}), \\ s_{\epsilon_i} &\mapsto \underline{\pi}(\theta_i).\end{aligned}$$

Hence $F_{\underline{\mu},n-k}(X_\delta^\nu)$, can be considered an $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ -module.

It should also be noted that as a $\mathfrak{B}_k^\theta$ -module $F_{\mu,k}(X_\delta^\nu)$ is essentially an $\mathbb{H}_k(c_\mu)$ -module. That is, there is no element in $\mathfrak{B}_k^\theta$ that has a non-trivial action on $F_{\text{triv},k}(X_\delta^\nu)$ that does not correspond to an element in the Hecke algebra.

For $G = O(n+1, n)$ or $Sp_{2n}(\mathbb{R})$, every principal series module is induced from a character on M . Therefore for split real orthogonal or symplectic groups we can entirely describe the Hecke algebra modules resulting from functors $F_{\mu,k}$ and $F_{\underline{\mu},n-k}$ on principal series modules. Casselman [12] states that every irreducible representation in $\mathcal{HC}(G)$ is a subrepresentation of a principal series module. Therefore if X is a subrepresentation of X_δ^ν then $F_{\mu,k}(X)$ also descends to a Hecke algebra module.

Theorem 3.7.7. *Let G be a split real Lie group of type B or C . Let X be an irreducible Harish-Chandra G -module. Hence X is a subrepresentation of a principal series module X_δ^ν , then the $\mathfrak{B}_k^\theta$ and $\mathfrak{B}_{n-k}^\theta$ -modules*

$$F_{\mu,k}(X) \text{ and } F_{\underline{\mu},n-k}(X)$$

naturally descend to $\mathbb{H}_k$ and $\mathbb{H}_{n-k}$ -modules.

Proof. Let X be an irreducible Harish-Chandra module. Casselman's theorem shows that X is a submodule of some principal series module, let X_δ^ν be such a principal series modules containing X as a submodule. Note that this principal series module may not be unique. Then since $F_{\mu,k}(X)$ is exact and X is a submodule of X_δ^ν then $F_{\mu,k}(X)$ is a submodule of $F_{\mu,k}(X_\delta^\nu)$ which is a $\mathbb{H}_k$ module. Therefore $F_{\underline{\mu},k}(X)$ is a $\mathbb{H}_k$ module. Similarly for $\underline{\mu}$ and $n-k$. $\square$

Therefore for every Harish Chandra module of $O(n+1, n)$ and $Sp_{2n}(\mathbb{R})$ we can define two corresponding Hecke algebra modules.

3.8 Principal series modules map to principal series modules

In this section we take a closer look at the $\mathbb{H}(c_\mu)$ -modules obtained from X_δ^ν under the functors $F_{\mu,k}$ and $F_{\underline{\mu},n-k}$. We fully classify these as graded Hecke algebra principal series representations related to ν .

Recall that $\mathbb{H}_k(c)$, defined in 3.4.1 is the graded Hecke algebra associated to $W(B_k)$ with parameter function $\mathbf{c} : \Delta \rightarrow \mathbb{C}$ such that

$$\mathbf{c}_{\epsilon_i - \epsilon_{i+1}} = 1 \text{ and } \mathbf{c}_{2\epsilon_i} = 2c.$$

The algebra $\mathbb{H}_k(c)$ contains the group algebra, $\mathbb{C}[W(B_k)]$, of the hyperoctahedral group. Recall the labeling of vectors in $X \otimes V^{\otimes k}$; we label the tensor product starting at zero. A general elementary tensor in $X \otimes V^{\otimes k}$ would be written $x_0 \otimes v_1 \otimes v_2 \otimes \dots \otimes v_k$. We begin by restricting to the action of the Weyl group $W(B_k)$ inside $\mathbb{H}(c)$ and computing the resulting $\mathbb{C}[W(B_k)]$ -modules isomorphism class. Fix a M -character δ and recall the K -characters μ and $\underline{\mu}$ depending on δ from Table 3.6.1.

Lemma 3.8.1. *As a $\mathbb{C}[W(B_k)]$ -module*

$$F_{\mu,k}(X_\delta^\nu) \cong \mathbb{C}[W(B_k)],$$

and as a $\mathbb{C}[W(B_{n-k})]$ -module

$$F_{\underline{\mu},n-k}(X_\delta^\nu) \cong \mathbb{C}[W(B_{n-k})].$$

Proof. From Lemma 3.6.2 we have an explicit basis of $F_{\mu,k}(X_\delta^\nu)$;

$$\text{Hom}_K(\mu, X_{\delta^k} \otimes V^{\otimes k}) = \text{span}\left\{ \sum_{w \in S_k} \mathbb{1}_\delta^\nu \otimes f_{w(1)}^{n_1} \otimes \dots \otimes f_{w(k)}^{n_k} \right\}.$$

The symmetric group $\mathbb{C}[S_k] \subset \mathbb{C}[W(B_k)]$ acts by permuting the tensor product. The reflections in $\mathbb{C}[W(B_k)]$ related to $2\epsilon_i$ act by $id \otimes \dots \otimes \theta_i \otimes \dots \otimes id$. They take f_i to f'_i on the i^{th} factor of the tensor product.

Take the vector $\mathbb{1}_\delta^\nu \otimes f_1 \otimes \dots \otimes f_k$, the $\mathbb{C}[W(B_k)]$ submodule of $F_{\text{triv},k}(X_{\delta^k}^\nu)$ generated by $\mathbb{1}_\delta \otimes f_1 \otimes \dots \otimes f_k$ is the subspace spanned by

$$\{\mathbb{1}_\delta^\nu \otimes f_{w(1)}^{n_1} \otimes \dots \otimes f_{w(k)}^{n_k} : w \in \mathbb{C}[S_k]\},$$

The only group element of $\mathbb{C}[W(B_k)]$ that fixes $\mathbb{1}_\delta^\nu \otimes f_1 \otimes \dots \otimes f_k$ is the identity, hence this module has dimension equal to $k!2^k$, the dimension of $\mathbb{C}[W(B_k)]$. The dimension is equal to the dimension of $F_{\text{triv},k}(X_\delta^\nu)$, therefore we have equality. An isomorphism between the $\mathbb{C}[W(B_k)]$ -module $\mathbb{C}[W(B_k)]$ and $F_{\text{triv},k}(X_\delta^\nu)$ can be defined by sending the identity element $e \in \mathbb{C}[W(B_k)]$ to $\mathbb{1}_\delta^\nu \otimes f_1 \otimes \dots \otimes f_k$.

The decomposition of $F_{\underline{\mu},n-k}(X_\delta^\nu)$ follows in exactly the same way, sending $e \in \mathbb{C}[W(B_{n-k})]$ to $\mathbb{1}_\delta^\nu \otimes f_{k+1} \otimes \dots \otimes f_n$. $\square$

We have a description of $F_{\mu,k}(X_{\delta^k}^\nu)$ as a $\mathbb{C}[W(B_k)]$ -module. We would like to describe it as an $\mathbb{H}(c_\mu)$ -module. The algebra $\mathbb{H}(c_\mu)$ is generated by $\mathbb{C}[W(B_k)]$ and the affine operators $\epsilon_1, \dots, \epsilon_k$. Our calculation reduces to calculating the action of the affine operators ϵ_i . The operators $\epsilon_i \in S(\mathfrak{a}_k)$ act on $X_\delta^\nu \otimes V^{\otimes k}$ by

$$\sum_{0 < j < i \leq n} \Omega_{ji} + r_\mu.$$

We define principal series representations for $\mathbb{H}_k(c)$. Then we show that the image of X_δ^ν is isomorphic to a principal series representation defined by a particular character.

The subspace $\mathfrak{a}_k \subset \mathfrak{a}$ defined in Example 3.2.5 is a dimension k subspace of $\mathfrak{a}$.

Definition 3.8.2. [37] Let λ be a character for $S(\mathfrak{a}_k) \subset \mathbb{H}_k(c_\mu)$, we define a principal series representation $X(\lambda)$ for $\mathbb{H}_k(c_\mu)$:

$$X(\lambda) = \text{Ind}_{S(\mathfrak{a}_k)}^{\mathbb{H}_k(c)} \lambda.$$

We write $\mathbb{1}_\lambda$ for a fixed vector in the image of the character $\lambda : S(\mathfrak{a}) \rightarrow \mathbb{C}$. The symmetric algebra $S(\mathfrak{a}_k)$ is generated by the affine operators $\epsilon_1, \dots, \epsilon_k$. The principal series representation can be described as a representation generated by, $\mathbb{1}_\lambda$, a $\mathbb{C}[W(B_k)]$ -cyclic vector on which ϵ_i acts by the scalar $\lambda(\epsilon_i)$. We prove that the $\mathbb{C}[W(B_k)]$ -module, $F_{\mu,k}(X_{\delta^k}^\nu)$ is as a $\mathbb{H}_k(c_\mu)$ -module isomorphic to a principal series module for the correct character λ .

We fix a specific basis for $\mathfrak{sp}_{2n}$ and $\mathfrak{so}(p, q)$. Since the operators $\Omega_{ij} \in U(\mathfrak{g})^{k+1}$ are defined in terms of, although independent of, a basis for $\mathfrak{g}$. This basis allows us to explicitly calculate Ω_{0j} . It should be emphasized that the following basis is a decomposition of $\mathfrak{g}$ into reduced root spaces under the adjoint action of $\mathfrak{a}$. Recall that $\mathfrak{a} \subset \mathfrak{sp}_{2n}(\mathbb{R})$ is

$$\left\{ \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} : B \text{ is diagonal} \right\}.$$

Definition 3.8.3. Recall the decomposition of the Lie algebra $\mathfrak{g}_0$ as

$$\mathfrak{g}_0 = \mathfrak{n}_0^+ \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0^-,$$

where $\mathfrak{a}_0$ is the maximal abelian subalgebra of $\mathfrak{p}_0$ and $\mathfrak{n}_0^+$ is the span of the positive root spaces with respect to the restricted root decomposition. Let $B_{\mathfrak{n}^+}, B_{\mathfrak{n}^-}, B_{\mathfrak{a}}$ be fixed bases for $\mathfrak{n}_0^+, \mathfrak{n}_0^-$ and $\mathfrak{a}_0$. The restricted roots Σ are $\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_j$. We will denote a vector in the positive root space $\lambda \in \Sigma^+$ by n_λ and the negative root space will be $\hat{n}_\lambda$. For example $n_{\epsilon_i - \epsilon_j}$ for $i < j$ is in $\mathfrak{n}^+$. And $\hat{n}_{\epsilon_i - \epsilon_j} \in \mathfrak{n}_0^-$. We will scale $\hat{n}_\lambda$ such that

$$\hat{n}_\lambda = n_{-\lambda} = \theta(n_\lambda).$$

Hence $n_\lambda + \hat{n}_\lambda$ is θ -invariant and hence in $\mathfrak{k}$.

Definition 3.8.4. For $1 \leq s, t \leq n$, the matrix $E_{s,t}$ is the matrix with a 1 in the s, t position and zero elsewhere. Let $i < j$. Set

$$\begin{aligned} n_{\epsilon_i - \epsilon_j} &= E_{i,j} + E_{i,n+j} - E_{j,i} + E_{j,n+i} + E_{n+i,j} + E_{n+i,n+j} + E_{n+j,i} - E_{n+j,n+i}, \\ \hat{n}_{\epsilon_i - \epsilon_j} &= -E_{i,j} + E_{i,n+j} + E_{j,i} + E_{j,n+i} + E_{n+i,j} - E_{n+i,n+j} + E_{n+j,i} + E_{n+j,n+i}, \\ n_{\epsilon_i + \epsilon_j} &= -E_{i,j} + E_{i,n+j} - E_{j,i} + E_{j,n+i} - E_{n+i,j} + E_{n+i,n+j} - E_{n+j,i} + E_{n+j,n+i}, \\ \hat{n}_{\epsilon_i + \epsilon_j} &= E_{i,j} + E_{i,n+j} + E_{j,i} + E_{j,n+i} - E_{n+i,j} - E_{n+i,n+j} - E_{n+j,i} - E_{n+j,n+i}, \\ n_{\epsilon_i} &= E_{i,i} - E_{i,n+i} + E_{n+i,i} - E_{n+i,n+i}, \\ \hat{n}_{\epsilon_i} &= -E_{i,i} - E_{i,n+i} + E_{n+i,i} + E_{n+i,n+i}, \\ a_{\epsilon_i} &= E_{i,n+1} + E_{n+i,i}. \end{aligned}$$

These vectors give a reduced root space decomposition for $\mathfrak{sp}_{2n}(\mathbb{R}) = \mathfrak{n}_0^+ \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0^-$ where $a_{\epsilon_i} \in \mathfrak{a}_0$, $n \in \mathfrak{n}_0^+$ and $\hat{n} \in \mathfrak{n}_0^-$.

Example 3.8.5. Let $\mathfrak{g} = \mathfrak{sp}_4$. We give the basis given in Definition 3.8.4 for $\mathfrak{n}^+$,

$$n_{\epsilon_1 - \epsilon_2} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix},$$

$$n_{\epsilon_1 + \epsilon_2} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix},$$

$$n_{\epsilon_1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$n_{\epsilon_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}.$$

Definition 3.8.6. Let $\mathfrak{g}_0 = \mathfrak{so}(p, q)$ we follow [36, VI, pg. 371 Example $\mathfrak{so}(p, q)$].

$$n_{\epsilon_i - \epsilon_j} = E_{p-j+1, p-i+1} + E_{p-j+1, p+i} - E_{p-i+1, p-j+1} + E_{p-i+1, p+j} \\ - E_{p+i, p-j+1} - E_{p+i, p+j} - E_{p+j, p-i+1} + E_{p+j, p+i},$$

$$n_{\epsilon_i + \epsilon_j} = E_{p-j+1, p-i+1} - E_{p-j+1, p+i} - E_{p-i+1, p-j+1} + E_{p-i+1, p+j} \\ - E_{p+i, p-j+1} + E_{p+i, p+j} + E_{p+j, p-i+1} - E_{p+j, p+i},$$

$$\hat{n}_{\epsilon_i - \epsilon_j} = E_{p-j+1, p-i+1} - E_{p-j+1, p+i} - E_{p-i+1, p-j+1} - E_{p-i+1, p+j} \\ + E_{p+i, p-j+1} - E_{p+i, p+j} + E_{p+j, p-i+1} + E_{p+j, p+i},$$

$$n_{\epsilon_i + \epsilon_j} = E_{p-j+1, p-i+1} + E_{p-j+1, p+i} - E_{p-i+1, p-j+1} - E_{p-i+1, p+j} \\ + E_{p+i, p-j+1} + E_{p+i, p+j} - E_{p+j, p-i+1} - E_{p+j, p+i}.$$

The root space for ϵ_i is $p - q$ dimensional. Let $l = 1, \dots, p - q$ then

$$n_{\epsilon_i}^l = E_{l, p-i+1} - E_{l, p+i} - E_{p-i+1, l} - E_{p+i, l}.$$

Finally

$$a_{\epsilon_i} = E_{p-i+1, p+i} + E_{p+i, p-i+1}.$$

Example 3.8.7. Let $\mathfrak{g}_0 = \mathfrak{so}(3, 2)$. We give the basis given in Definition 3.8.6 for $\mathfrak{n}_0^+$,

$$n_{\epsilon_1 - \epsilon_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix},$$

$$n_{\epsilon_1 + \epsilon_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix},$$

$$n_{\epsilon_1} = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$n_{\epsilon_2} = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Lemma 3.8.8. *For $G = Sp_{2n}$, recall the basis $f_i = e_i + e_{n+i}$, $f'_i = e_i - e_{n+i}$ of $V = \mathbb{C}^{2n}$. For $G = O(p, q)$ we recall that $f_i = e_{p-i+1} + e_{p+i}$, $f'_i = e_{p-i+1} - e_{p+i}$. Then by left multiplication of the given matrix in Definitions 3.8.4 and 3.8.6 we can calculate the following actions on f_i :*

$$n_{\epsilon_i + \epsilon_j}(f_k) = 0 \text{ for all } k,$$

$$n_{\epsilon_i + \epsilon_j}(f'_k) = \begin{cases} 2f'_j & \text{if } f'_k = f'_j, \\ 0 & \text{otherwise.} \end{cases}$$

$$n_{\epsilon_i}(f_k) = 0 \text{ for all } k,$$

$$n_{\epsilon_i}(f'_k) = \begin{cases} 2f_k & \text{if } f'_k = f'_i, \\ 0 & \text{otherwise,} \end{cases}$$

$$n_{\epsilon_i - \epsilon_j}(f_k) = \begin{cases} 2f_i & \text{if } f_k = f_j, \\ 0 & \text{otherwise,} \end{cases}$$

$$n_{\epsilon_i - \epsilon_j}(f'_k) = \begin{cases} 2f'_j & \text{if } f'_k = f_i, \\ 0 & \text{otherwise,} \end{cases}$$

$$(n_{\epsilon_i - \epsilon_j} + \hat{n}_{\epsilon_i - \epsilon_j})(f_k) = \begin{cases} f_i & \text{if } f_k = f_j, \\ -f_j & \text{if } f_k = f_i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows from left multiplication of the elements of $\mathfrak{sp}_{2n}$ and $\mathfrak{so}(p, q)$ on the defining module V with elements f_i and f'_i in the basis of V . $\square$

To prove that the $\mathbb{C}[W(B_k)]$ -module is in fact isomorphic to a principal series $\mathbb{H}_k(c_\mu)$ -module we need to find a $\mathbb{C}[W(B_k)]$ cyclic vector such that the ϵ_i act by scalars on this cyclic vector. The cyclic vector is $\mathbb{1}_\delta^\nu \otimes f_1 \otimes \dots \otimes f_k$.

Lemma 3.8.9. *On the vector $\mathbb{1}_\delta^\nu \otimes f_1 \otimes \dots \otimes f_k$ the operator Ω_{0l} acts by*

$$\nu(\epsilon_l) - \sum_{t < l} (s_{tl} + id) - \sum_{t > l} id.$$

Proof. Recall that Ω_{0l} is defined to be $\sum_{b \in B} (b)_0 \otimes (b^*)_l$ for a given basis B of $\mathfrak{g}_0$. We choose to use the fixed basis defined in Definition 3.8.3.

The subspace $\mathfrak{a}$ is the Lie algebra of the subgroup $A \subset G$. The basis of $\mathfrak{a}$ defined in 3.8.4 and 3.8.6 is such that $a_{\epsilon_i}(f_j) = \delta_{ij}f_j$. Furthermore a_{ϵ_i} acts on the cyclic vector $\mathbb{1}_\delta^\nu$ of X_δ^ν by $\nu(x_i)$. Therefore the contribution from $\mathfrak{a} \subset \mathfrak{g}$ is

$$(a_{\epsilon_i})_0 \otimes (a_{\epsilon_i})_l = \delta_{il} \nu(x_{i_l}).$$

The module X_δ^ν is induced from the character $\delta \otimes e^\nu \otimes 1$ of MAN , which is the trivial character on N . The space $\mathfrak{n}_0^+$ is the Lie algebra of N . The differential of the trivial character to $\mathfrak{n}_0^+$ is zero. Therefore $(n)_0$ acts by zero on $\mathbb{1}_\delta^\nu$ for all $n \in \mathfrak{n}_0^+$. Hence the contribution from $\mathfrak{n}^+$ is:

$$(n)_0 \otimes (n)_l^* = 0, \text{ for } n \in \mathfrak{n}^+.$$

Since $n \in \mathfrak{n}_0^+$ annihilates $\mathbb{1}_\delta^\nu$ then $(n)_0 \otimes (b)_l = 0$ for any $b \in \mathfrak{g}_{2n}$, $n \in \mathfrak{n}^+$, a fact we will use later in this proof. The operator $(\hat{n}_{\epsilon_i + \epsilon_j})_l^*$ is equal to $\frac{1}{2}(n_{\epsilon_i + \epsilon_j})_l$ which is zero on any f_k hence;

$$(\hat{n}_{\epsilon_i + \epsilon_l})_0 \otimes (\hat{n}_{\epsilon_i + \epsilon_j})_l^* = 0.$$

Similarly n_{ϵ_i} is zero on any f_k therefore;

$$(\hat{n}_{\epsilon_i})_0 \otimes (\hat{n}_{\epsilon_i}^*)_l = 0.$$

The only remaining basis elements to consider are those of the form $n_{\epsilon_i - \epsilon_j}$ from $\mathfrak{n}_0^- \subset \mathfrak{g}_0$. We utilise the trick that as a K -module $F_{\mu,k}(X_{\delta^k}^\nu)$ is just the μ isotypic component of $X_{\delta^k}^\nu \otimes V^{\otimes k}$. The contribution from $\hat{n}_{\epsilon_i - \epsilon_j}$ is:

$$(\hat{n}_{\epsilon_i - \epsilon_j})_0 \otimes (\hat{n}_{\epsilon_i - \epsilon_j}^*)_l.$$

We can add the operator $(n_{\epsilon_i - \epsilon_j})_0 \otimes (\hat{n}_{\epsilon_i - \epsilon_j}^*)_l$ which since $n_{\epsilon_i - \epsilon_j} \in \mathfrak{n}^+$, by above, acts by zero. Therefore we are not modifying the original operator,

$$(\hat{n}_{\epsilon_i - \epsilon_j})_0 \otimes (\hat{n}_{\epsilon_i - \epsilon_j}^*)_l = \frac{1}{2}(\hat{n}_{\epsilon_i - \epsilon_j} + n_{\epsilon_i - \epsilon_j})_0 \otimes (n_{\epsilon_i - \epsilon_j})_l.$$

The vector $\hat{n}_{\epsilon_i - \epsilon_j} + n_{\epsilon_i - \epsilon_j}$ is θ -invariant, hence is in $\mathfrak{k}$. Recall that for $k \in \mathfrak{k}$ acting on the tensor $X \otimes V^{\otimes k}$ that $k = \sum_{i=0}^k (k)_i$. Since we are working with the μ -isotypic space, we replace $\hat{n}_{\epsilon_i - \epsilon_j} + n_{\epsilon_i - \epsilon_j} \in \mathfrak{k}$ by $\mu(\hat{n}_{\epsilon_i - \epsilon_j} + n_{\epsilon_i - \epsilon_j})$ and subtract the difference to find,

$$\begin{aligned} & (\hat{n}_{\epsilon_i - \epsilon_j})_0 \otimes (\hat{n}_{\epsilon_i - \epsilon_j}^*)_l \\ &= \frac{1}{2}\mu(\hat{n}_{\epsilon_i - \epsilon_j} - n_{\epsilon_i - \epsilon_j}) \otimes (n_{\epsilon_i - \epsilon_j})_l - \frac{1}{2} \sum_{m>0} (\hat{n}_{\epsilon_i - \epsilon_j} - n_{\epsilon_i - \epsilon_j})_m \otimes (n_{\epsilon_i - \epsilon_j})_l. \end{aligned}$$

The character μ (or $\underline{\mu}$) differentiated to $\mathfrak{a}$ is zero (or the trace character) hence $\mu(\hat{n}_{\epsilon_i - \epsilon_j} + n_{\epsilon_i - \epsilon_j}) = 0$. Lemma 3.8.8 gives the explicit action of $n_{\epsilon_i - \epsilon_j}$ on f_k , using this one can determine the action;

$$(\hat{n}_{\epsilon_i - \epsilon_j})_0 \otimes (\hat{n}_{\epsilon_i - \epsilon_j}^*)_l = \begin{cases} -s_{il} - id & \text{if } f_{i_t} = f_i \text{ and } f_{i_l} = f_j, \\ -id & \text{if } f_{i_l} = f_i, \\ 0 & \text{otherwise.} \end{cases}$$

The only non-zero terms are contributed by a_{ϵ_l} , and $\hat{n}_{\epsilon_i - \epsilon_l}$ and $\hat{n}_{\epsilon_l - \epsilon_i}$. Which act, on the cyclic vector, by $\nu(\epsilon_l)$, $-s_{il} - id$ and $-id$ respectively. Summing these up gives,

$$\Omega_{0l} = \nu(\epsilon_l) - \sum_{t<l} (s_{tl} + id) - \sum_{t>l} id,$$

on the $\mathbb{C}[W(B_k)]$ cyclic vector $\mathbb{1}_\delta^\nu \otimes f_1 \otimes \dots \otimes f_k$. □

The equivalent statement for $F_{\underline{\mu}, n-k}(X_\delta^\nu)$ is below. It follows from the proof of Lemma 3.8.9.

Lemma 3.8.10. *On the vector $\mathbb{1}_\delta^\nu \otimes f_{k+1} \otimes \dots \otimes f_n$ the operator Ω_{0l} acts by*

$$\nu(\epsilon_{k+l}) - \sum_{t < l} (s_{k+t, k+l} + id) - \sum_{t > l} id,$$

for $l = 1, \dots, n - k$.

Corollary 3.8.11. *The operator $\epsilon_l = \sum_{i < l} \Omega_{il} + n$ acts by the scalar $\nu(\epsilon_l)$ on the vector $\mathbb{1}_\delta^\nu \otimes f_1 \otimes \dots \otimes f_k$.*

Proof. This follows from the fact that Ω_{0l} acts by $\nu(\epsilon_l) - n - \sum_{t < l} s_{tl}$ and, by Lemma 3.3.9, $\sum_{t=1}^{l-1} \Omega_{tl}$ acts by $\sum_{t < l} s_{tl}$ on $\mathbb{1}_\delta^\nu \otimes f_1 \otimes \dots \otimes f_k$. $\square$

Corollary 3.8.12. *The operator $\epsilon_l = \sum_{i < l} \Omega_{il} + n$ acts by the scalar $\nu(\epsilon_{k+l})$ on the vector $\mathbb{1}_\delta^\nu \otimes f_{k+1} \otimes \dots \otimes f_n$.*

Definition 3.8.13. *Example 3.2.5 defines subspaces $\mathfrak{a}_k$ and $\bar{\mathfrak{a}}_{n-k}$ of $\mathfrak{a}$ such that*

$$\mathfrak{a} = \mathfrak{a}_k \oplus \bar{\mathfrak{a}}_{n-k}.$$

Let ν be a character of $\mathfrak{a}$. Define ν_k to be the restricted character

$$\nu|_{\mathfrak{a}_k}$$

and $\bar{\nu}_{n-k}$ to be $\nu|_{\bar{\mathfrak{a}}_{n-k}}$.

For a principal series module X_δ^ν we have shown that as a $W(B_k)$ -module $F_{\mu,k}(X_\delta^\nu)$ is isomorphic to $\mathbb{C}[W(B_k)]$ and as a Hecke algebra module it is a principal series module induced from a character of $S(V) \subset \mathbb{H}_k(c_\mu)$.

Theorem 3.8.14. *For $G = Sp_{2n}(\mathbb{R})$ or $O(p, q)$ $p + q = 2n + 1$, the module $F_{\mu,k}(X_\delta^\nu)$ is isomorphic to the $\mathbb{H}_k(c_\mu)$ principal series module*

$$X(\nu_k) = \text{Ind}_{S(\mathfrak{a}_k)}^{\mathbb{H}_k(c_\mu)} \nu_k.$$

The module $F_{\underline{\mu}, n-k}(X_\delta^\nu)$ is isomorphic to the $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ principal series module

$$X(\bar{\nu}_{n-k}) = \text{Ind}_{S(\bar{\mathfrak{a}}_{n-k})}^{\mathbb{H}_k(c_{\underline{\mu}})} \bar{\nu}_{n-k}.$$

For spherical principal series representations, this recovers the results of [16, Theorem 3.0.4].

Proof. One defines an isomorphism by taking the given cyclic vector $\mathbb{1}_\delta^\nu \otimes f_1 \otimes \dots \otimes f_k \in F_{\text{triv},k}(X_{\delta^k}^\nu)$ to the cyclic vector $\mathbb{1}_{\nu_k}$ of $X(\nu_k)$. Both vectors are $\mathbb{C}[W(B_k)]$ cyclic. By Corollary 3.8.11 the affine operators ϵ_i act on both vectors by $\nu_k(\epsilon_i)$, therefore this is a well-defined isomorphism.

Lemma 3.6.2 gives a basis of $F_{\text{det},n-k}(X_{\delta^k}^\nu)$:

$$\{\mathbb{1}_\delta^\nu \otimes f_{w(1)+k}^{n_1} \otimes \dots \otimes f_{w(n-k)+k}^{n_{n-k}} : w \in S_{n-k}\}.$$

For $F_{\underline{\mu},k}(X_\delta^\nu)$ and $X(\bar{\nu}_{n-k})$, both modules are $\mathbb{C}[W(B_{n-k})]$ cyclic and Corollary 3.8.12 shows that the affine operators ϵ_i for $i = 1, \dots, n-k$, act by the same scalar on the cyclic vector $\mathbb{1}_\delta^\nu \otimes f_{k+1} \otimes \dots \otimes f_n$ and $\mathbb{1}_{\bar{\nu}_{n-k}}$, respectively. $\square$

Casselman's theorem [12] states that every irreducible representation in $\mathcal{HC}(G)$ is a subrepresentation of a principal series module. If G is a split real orthogonal or symplectic group then M is abelian and every principal series module is induced from a character.

Theorem 3.8.15. *Let G be $O(n+1, n)$ or $Sp_{2n}(\mathbb{R})$, then G is split. Let X be an irreducible G -module. Let X_δ^ν be a principal series representation that contains X , then the $\mathfrak{B}_k^\theta$ and $\mathfrak{B}_{n-k}^\theta$ -modules*

$$F_{\underline{\mu},k}(X) \text{ and } F_{\underline{\mu},n-k}(X)$$

are submodules of the $\mathbb{H}_k(c_\mu)$ and $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ -modules

$$X(\nu_k) \text{ and } X(\bar{\nu}_{n-k}).$$

3.9 Hermitian forms

In this section we define two star operations on $\mathfrak{B}_k^\theta$. Through the quotients defined in Lemma 3.4.4 these star operations descend to the usual star operations on the graded Hecke algebras $\mathbb{H}_k(c)$ [5]. We then show that a Harish-Chandra module $X \in \mathcal{HC}(G)$ with invariant Hermitian form gets mapped, by $F_{\underline{\mu},k}$, to a $\mathfrak{B}_k^\theta$ -module with invariant Hermitian form. This extends the results in [16] to any Harish-Chandra module. Furthermore, if X is a unitary module, then it maps to a unitary module for $\mathfrak{B}_k^\theta$. In this section we assume that μ is a character of K .

Definition 3.9.1. Let G be $O(p, q)$ $p+q = 2n+1$ or $Sp_{2n}(\mathbb{R})$, let $\mathfrak{g}_0$ be its Lie algebra, with complexification $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Conjugation $^-: \mathfrak{g} \rightarrow \mathfrak{g}$ is defined by the real form $\mathfrak{g}_0$. Define the star operation as the conjugate linear map $^*: \mathfrak{g} \rightarrow \mathfrak{g}$ such that:

$$g^* = -\bar{g} \text{ for all } g \in \mathfrak{g}.$$

Define the operation $^\bullet: \mathfrak{g} \rightarrow \mathfrak{g}$ by:

$$p^\bullet = \bar{p} \text{ for all } p \in \mathfrak{p}.$$

$$k^\bullet = -\bar{k} \text{ for all } k \in \mathfrak{k}.$$

Recall Definition 3.4.1 of the Hecke algebra $\mathbb{H}_k(c)$. We define the Drinfeld presentation of $\mathbb{H}_k(c)$.

Definition 3.9.2. Let R be a root system with pairing $\langle, \rangle: V \times V \rightarrow \mathbb{C}$, simple roots δ , and a parameter function $\mathbf{c}: \Delta \rightarrow \mathbb{C}$. Denote the Weyl group of R by $W(R)$. The Drinfeld Hecke algebra $\mathfrak{H}^R(\mathbf{c})$ is a quotient of the algebra

$$\mathbb{C}[W(R)] \rtimes T(V),$$

by the relations

$$\begin{aligned} w\tilde{\alpha}w^{-1} &= \widetilde{w(\alpha)} \text{ for all } w \in W(R), \alpha \in V, \\ [\tilde{\alpha}, \tilde{\beta}] &= \sum_{\gamma, \delta \in \Delta} \mathbf{c}(\gamma)\mathbf{c}(\delta)(\langle \tilde{\alpha}, \gamma \rangle \langle \tilde{\beta}, \delta \rangle - \langle \tilde{\beta}, \gamma \rangle \langle \tilde{\alpha}, \delta \rangle) s_\gamma s_\delta. \end{aligned}$$

Lemma 3.9.3. The Drinfeld Hecke algebra and the graded Hecke algebra are defined by a root system and a parameter on simple roots. If the defining root systems and parameters are equal then these algebras are isomorphic.

Proof. One defines an isomorphism $\phi: \mathbb{H}^R(c)$ to $\mathfrak{H}^R(\mathbf{c})$ by

$$\begin{aligned} \phi\left(\alpha - \frac{1}{2} \sum_{\gamma \in \Delta} \mathbf{c}(\gamma) \langle \gamma, \alpha \rangle s_\gamma\right) &= \tilde{\alpha}, \\ \phi(w) &= w, \quad \forall w \in W(R). \end{aligned}$$

□

Given that the graded Hecke algebra and the Drinfeld Hecke algebra are isomorphic we omit the different notation and denote the graded Hecke algebra by $\mathbb{H}^R(\mathbf{c})$. We uniformly denote a generator in the Drinfeld presentation by $\tilde{\alpha}$ and α denotes a generator in the Lusztig presentation (Definition 3.4.1).

Definition 3.9.4. Let $*$: $\mathbb{H}_k(c) \rightarrow \mathbb{H}_k(c)$ be the antihomomorphism such that:

$$\tilde{\alpha}^* = \overline{-\tilde{\alpha}} \text{ for all } \alpha \in \tilde{\mathfrak{t}},$$

$$g^* = g^{-1} \text{ for all } g \in W(B_k).$$

Let $\bullet$: $\mathbb{H}_k(c) \rightarrow \mathbb{H}_k(c)$ be the antihomomorphism such that:

$$\alpha^\bullet = \bar{\alpha} \text{ for all } \alpha \in \mathfrak{t} \text{ (equivalently } \tilde{\alpha}^\bullet = \overline{\tilde{\alpha}}),$$

$$g^\bullet = g^{-1} \text{ for all } g \in W(B_k).$$

Here $\bar{v}$ is the complex conjugate of v .

Let w_0 be the longest element in $W(B_k)$, it is an involution and is generated by k^2 simple reflections. It is in the centre of $W(B_k)$. On the space of roots w_0 acts by -1 .

Lemma 3.9.5. The longest element w_0 can be written as

$$w_0 = \theta_1 \theta_2 \dots \theta_k.$$

It is well known that the longest element w_0 relates the two star operations $*$ and $\bullet$ in $\mathbb{H}_k(c)$.

Lemma 3.9.6.

$$h^* = w_0 h^\bullet w_0 \text{ for all } h \in \mathbb{H}_k(c).$$

Lemma 3.9.7. The longest element w_0 is central in the finite Brauer algebra $Br_k[m]$.

Proof. The element w_0 is central in $W(B_k)$, therefore it is sufficient to prove that w_0 commutes with the idempotents $e_{i,i+1}$. The reflections θ_l commute with $e_{i,j}$

$$[e_{i,j}, \theta_l] = 0 \text{ for all } i, j, l.$$

We have,

$$w_0 e_{i,j} w_0 = \theta_1 \dots \theta_k e_{i,j} \theta_k \dots \theta_1 = e_{i,j}.$$

Hence w_0 is central in the finite Brauer algebra.

□

Since $w_0 = \theta_1 \theta_2 \dots \theta_k$ then as an operator on $X \otimes V^{\otimes k}$

$$\pi(w_0) = (\xi)_1 (\xi)_2 \dots (\xi)_k = id \otimes \xi \otimes \xi \dots \otimes \xi.$$

We calculate how w_0 and $\Omega_{ij}, \Omega_{ij}^{\mathfrak{t}}, \Omega_{ij}^{\mathfrak{p}}$ interact.

Lemma 3.9.8. *As operators on $X \otimes V^{\otimes k}$,*

$$w_0(\Omega_{ij}^{\mathfrak{k}})w_0 = \Omega_{ij}^{\mathfrak{k}} \text{ for all } 0 \leq i < j \leq n,$$

$$w_0(\Omega_{ij}^{\mathfrak{p}})w_0 = \begin{cases} \Omega_{ij}^{\mathfrak{p}} & \text{for all } 0 < i < j \leq n, \\ -\Omega_{0j}^{\mathfrak{p}} & \text{when } i = 0. \end{cases}$$

Proof. Recall $\xi g \xi = \begin{cases} g & \text{if } g \in \mathfrak{k}, \\ -g & \text{if } g \in \mathfrak{p}. \end{cases}$ Therefore one finds that $\pi(w_0) = id \otimes \xi \otimes \dots \xi$ commutes with $\Omega_{ij}^{\mathfrak{k}} = \sum_{b \in B \cap \mathfrak{k}} (b)_i \otimes (b^*)_j$. For $\Omega_{ij}^{\mathfrak{p}}$ we have:

$$\begin{aligned} w_0(\Omega_{ij}^{\mathfrak{p}})w_0 &= (id \otimes \xi \otimes \xi \otimes \dots \otimes \xi) \Omega_{ij}^{\mathfrak{p}} (id \otimes \xi \otimes \dots \otimes \xi), \\ &= (id \otimes \xi \otimes \xi \otimes \dots \otimes \xi) \sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (b^*)_j (id \otimes \xi \otimes \dots \otimes \xi), \\ &= \begin{cases} \sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (\xi b \xi)_j & \text{if } i = 0, \\ \sum_{b \in B \cap \mathfrak{p}} (\xi b \xi)_i \otimes (\xi b \xi)_j & \text{if } i \neq 0, \end{cases} \\ &= \begin{cases} \sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (-b)_j & \text{if } i = 0, \\ \sum_{b \in B \cap \mathfrak{p}} (-b)_i \otimes (-b)_j & \text{if } i \neq 0, \end{cases} \\ &= \begin{cases} -\sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (b)_j & \text{if } i = 0, \\ \sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (b)_j & \text{if } i \neq 0. \end{cases} \end{aligned}$$

□

Definition 3.9.9. Let $\bullet : \mathfrak{B}_k^\theta \rightarrow \mathfrak{B}_k^\theta$ be the conjugate linear antihomomorphism defined on the generators as follows:

$$\begin{aligned} z_i^\bullet &= z_i \\ g^\bullet &= g^{-1} \text{ for } g \in W(B_k) \\ e_{i,i+1}^\bullet &= e_{i,i+1}. \end{aligned}$$

Remark 3.9.10. To check this antihomomorphism is well defined one must just check that the relations in Definition 3.3.10 are fixed.

Definition 3.9.11. Let $*$: $\mathfrak{B}_k^\theta \rightarrow \mathfrak{B}_k^\theta$ by the antihomomorphism such that,

$$b^* = w_0 b^\bullet w_0.$$

Remark 3.9.12. Since w_0 is central in the finite Brauer algebra then $g^* = g^{-1}$ for $g \in W(B_k)$ and $e_{i,j}^* = e_{i,j}$.

Lemma 3.9.13. *Under the quotients in Lemma 3.4.4 the antihomomorphisms $*$: $\mathfrak{B}_k^\theta \rightarrow \mathfrak{B}_k^\theta$ and $\bullet$: $\mathfrak{B}_k^\theta \rightarrow \mathfrak{B}_k^\theta$ descend to the antihomomorphisms $*$: $\mathbb{H}_k(c) \rightarrow \mathbb{H}_k(c)$ and $\bullet$: $\mathbb{H}_k(c) \rightarrow \mathbb{H}_k(c)$ respectively.*

Proof. The operation $\bullet$ fixes $e_{i,i+1}$ and

$$\theta_k z_k + z_k \theta_k = 2c - 2r\theta_k.$$

Therefore $\bullet$ on $\mathfrak{B}_k^\theta$ descends to $\mathbb{H}_k(c)$. On the generators of $\mathbb{H}_k(c)$ it fixes the affine generators and is the inverse antihomomorphism on the group $W(B_k)$. Hence the operation $\bullet$ on $\mathfrak{B}_k^\theta$ descends to the antihomomorphism $\bullet$ on $\mathbb{H}_k(c)$. Since

$$h^* = w_0 h^\bullet w_0,$$

in both $\mathfrak{B}_k^\theta$ and $\mathbb{H}_k(c)$ then the star operation $*$ on $\mathfrak{B}_k^\theta$ descends to $*$ on $\mathbb{H}_k(c)$. $\square$

We give a new set of generators for $\mathfrak{B}_k^\theta$.

Definition 3.9.14. *Define*

$$\tilde{z}_i = \frac{z_i - w_0 z_i w_0}{2}, \text{ for } i = 1, \dots, k,$$

then

$$\mathfrak{B}_k^\theta \cong \langle \tilde{z}_i, s_{j,j+1}, e_{j,j+1}, \theta_i \rangle.$$

The operators $\tilde{z}_i$ form a Drinfeld type presentation for $\mathfrak{B}_k^\theta$, they descend to the Drinfeld presentation of $\mathbb{H}_k(c)$ under the quotients defined in 3.4.4. As operators on $X \otimes V^{\otimes k}$:

$$\begin{aligned} \pi(\tilde{z}_i) &= \frac{1}{2} \pi(z_i - w_0 z_i w_0) \\ &= \frac{1}{2} \left(\sum_{j < i} \Omega_{ij} - (\xi)_1 (\xi)_2 \dots (\xi)_k \sum_{j < i} \Omega_{ij} (\xi)_1 (\xi)_2 \dots (\xi)_k \right), \\ &= \frac{1}{2} \left(\sum_{j < i} \Omega_{ij}^{\mathfrak{e}} + \Omega_{ij}^{\mathfrak{p}} - (\xi)_1 (\xi)_2 \dots (\xi)_k \sum_{j < i} \Omega_{ij}^{\mathfrak{e}} + \Omega_{ij}^{\mathfrak{p}} (\xi)_1 (\xi)_2 \dots (\xi)_k \right), \\ &= \frac{1}{2} \left(\sum_{j < i} \Omega_{ij}^{\mathfrak{e}} + \Omega_{ij}^{\mathfrak{p}} + \Omega_{0i}^{\mathfrak{p}} - \Omega_{0i}^{\mathfrak{e}} \sum_{0 < j < i} \Omega_{ij}^{\mathfrak{e}} + \Omega_{ij}^{\mathfrak{p}} \right), \\ &= \Omega_{0i}^{\mathfrak{p}}. \end{aligned}$$

Remark 3.9.15. *With this presentation of $\mathfrak{B}_k^\theta$ the operation $*$ is defined as*

$$\begin{aligned} \tilde{z}_i^* &= -\tilde{z}_i, \\ g^* &= g^{-1} \text{ for all } g \in w(B_k), \\ e_{i,i+1}^* &= e_{i,i+1}. \end{aligned}$$

Definition 3.9.16. Let X be a complex vector space, a Hermitian form $\langle, \rangle_X$ on X is a map $\langle, \rangle_X : X \times X \rightarrow \mathbb{C}$ such that

$$\begin{aligned}\langle \lambda_1 x_1 + \lambda_2 x_2, x' \rangle_X &= \lambda_1 \langle x'_1, x' \rangle_X + \lambda_2 \langle x'_2, x' \rangle_X \text{ for all } x_1, x_2, x' \in X, \lambda_1, \lambda_2 \in \mathbb{C}, \\ \langle x, \lambda_1 x'_1 + \lambda_2 x'_2 \rangle_X &= \bar{\lambda}_1 \langle x, x'_1 \rangle_X + \bar{\lambda}_2 \langle x, x'_2 \rangle_X \text{ for all } x'_1, x'_2, x \in X, \lambda_1, \lambda_2 \in \mathbb{C}.\end{aligned}$$

Definition 3.9.17. Let X be a $\mathcal{HC}(G)$ -module. A Hermitian form $\langle, \rangle_X$ is $*$ -invariant if:

$$\langle g(x_1), x_2 \rangle_X = \langle x_1, g^*(x_2) \rangle, \text{ for all } x_1, x_2 \in X \text{ and } g \in \mathfrak{g}.$$

Definition 3.9.18. Let U be an $\mathbb{H}_k(c)$ -module. A Hermitian form $\langle, \rangle_U$ on U is invariant with respect to $*$ if:

$$\langle h(x_1), x_2 \rangle_X = \langle x_1, h^*(x_2) \rangle, \text{ for all } x_1, x_2 \in U \text{ and } h \in \mathbb{H}_k(c).$$

Similarly for U a $\mathfrak{B}_k^\theta$ -module, a Hermitian form $\langle, \rangle_U$ on U is $*$ -invariant if

$$\langle b(x_1), x_2 \rangle_X = \langle x_1, b^*(x_2) \rangle, \text{ for all } x_1, x_2 \in U \text{ and } b \in \mathfrak{B}_k^\theta.$$

Definition 3.9.19. A $\mathcal{HC}(G)$ -module X is unitary if there exists a positive definite invariant Hermitian form on X .

Similarly, an $\mathbb{H}_k(c)$ -module U is unitary if U has an invariant positive definite Hermitian form and a $\mathfrak{B}_k^\theta$ -module is unitary if it has a positive definite invariant Hermitian form.

Recall V is the defining matrix module of G . Let $\langle, \rangle_V$ be a non-degenerate Hermitian form on V such that

$$\langle kv_1, v_2 \rangle = \langle v_1, k^{-1}v_2 \rangle \text{ for all } v_1, v_2 \in V, k \in K,$$

$$\langle pv_1, v_2 \rangle = \langle v_1, pv_2 \rangle \text{ for all } v_1, v_2 \in V, p \in \mathfrak{p}.$$

This makes V unitary with respect to $\bullet$.

Definition 3.9.20. c.f. [16, (4.4)] Let X be in $\mathcal{HC}(G)$ with a $*$ -invariant Hermitian form $\langle, \rangle_X$ then we endow $X \otimes V^{\otimes k}$ with a Hermitian form defined on elementary tensors by

$$\langle x_0 \otimes v_1 \otimes \dots \otimes v_k, x'_0 \otimes v'_1 \otimes \dots \otimes v'_k \rangle_{X \otimes V^{\otimes k}} = \langle x_0, x'_0 \rangle_X \langle v_1, v'_1 \rangle_V \dots \langle v_k, v'_k \rangle_V,$$

then extended to a Hermitian form. For μ a character of K , define a Hermitian form on $F_{\mu,k}(X) = \text{Hom}_K(\mu, X \otimes V^{\otimes k})$ by:

$$\langle \phi, \psi \rangle_{F_{\mu,k}} = \langle \phi(1), \psi(1) \rangle_{X \otimes V^{\otimes k}}, \text{ for all } \phi, \psi \in \text{Hom}_K(\mu, X \otimes V^{\otimes k}).$$

Remark 3.9.21. If X is a unitary space then $\langle, \rangle_{X \otimes V^{\otimes k}}$ endows $X \otimes V^k$ as a unitary space.

Lemma 3.9.22. Let V be the complex matrix module of $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$ and pr_{12} be the projection of $V \otimes V$ onto its trivial G submodule. Define $\langle, \rangle_{V \otimes V}$ on $V \otimes V$ by

$$\langle v_1 \otimes v_2, v'_1 \otimes v'_2 \rangle_{V \otimes V} = \langle v_1, v'_1 \rangle_V \langle v_2, v'_2 \rangle_V,$$

and extend to a Hermitian form. Then

$$\langle pr_{12}(v_1 \otimes v_2), v'_1 \otimes v'_2 \rangle_{V \otimes V} = \langle v_1 \otimes v_2, pr_{12}(v'_1 \otimes v'_2) \rangle_{V \otimes V}.$$

Proof. It is sufficient to prove that the trivial submodule in $V \otimes V$ and its complement are orthogonal with the form $\langle, \rangle_{V \otimes V}$. The Peter-Weyl Theorem [48, Theorem 1.12] states that a unitary module of a compact group decomposes as an orthogonal direct sum of irreducibles. Considering $V \otimes V$ as a $\bullet$ -unitary K -module, we have that the trivial submodule of $V \otimes V$ is orthogonal to its complement with respect to $\langle, \rangle_V$. $\square$

Lemma 3.9.23. Suppose $X \in \mathcal{HC}(G)$ with a $*$ -invariant Hermitian form $\langle, \rangle_X$ then $F_{\mu,k}(X) \in \mathfrak{B}_k^\theta\text{-mod}$ has a $*$ -invariant Hermitian form $\langle, \rangle_{F_{\mu,k}}$. Furthermore if X is unitary then $F_{\mu,k}(X)$ is unitary.

Ciubotaru and Trapa [16] prove this result for spherical principal series modules mapping to graded Hecke algebras. We extend this to any Harish-Chandra module which requires considering the image as a Type B/C affine Brauer algebra module.

Proof. We show that the Hermitian form is invariant under the generators $\tilde{z}_i, s_{i,i+1}, \theta_j$ and $e_{i,i+1}$. For $\tilde{z}_i, \tilde{z}_i^* = -\tilde{z}_i$ and $\pi(\tilde{z}_i) = \Omega_{0i}^p$. The form $\langle, \rangle_X$ is a $*$ -invariant form on X and $\langle, \rangle_V$ is a $\bullet$ -invariant form on V . Let $\phi, \psi \in F_{\mu,k}(X) = \text{Hom}_K(\mu, X \otimes V^{\otimes k})$, then

$$\langle \pi \tilde{z}_i(\phi), \psi \rangle_{F_{\mu,k}(X)} = \langle \pi(\tilde{z}_i)(\phi(1)), \psi(1) \rangle_{X \otimes V^{\otimes k}}.$$

Since $\pi(\tilde{z}_i) = \Omega_{0i}^{\mathfrak{p}} = \sum_{b \in B \cap \mathfrak{p}} (b)_0 \otimes (b^*)_i$,

$$\begin{aligned} \langle \pi \tilde{z}_i^*(\phi), \psi \rangle_{F\mu, k(X)} &= \langle (\Omega_{0i}^{\mathfrak{p}})^* \phi(1), \psi(1) \rangle_{X \otimes V^{\otimes k}}, \\ &= -\langle \Omega_{0i}^{\mathfrak{p}} \phi(1), \psi(1) \rangle_{X \otimes V^{\otimes k}}, \\ &= -\left\langle \left(\sum_{b \in B \cap \mathfrak{p}} (b)_0 \otimes (b)_i \right) \phi(1), \psi(1) \right\rangle_{X \otimes V^{\otimes k}}. \end{aligned}$$

Denote $\phi(1)$ by $\sum x_0 \otimes v_1 \otimes \dots \otimes v_k$ and $\psi(1)$ by $\sum x'_0 \otimes v'_1 \otimes \dots \otimes v'_k$ substituting in the definition of $\langle, \rangle_{X \otimes V^{\otimes k}}$ then

$$\langle \pi \tilde{z}_i(\phi), \psi \rangle_{F\mu, k(X)} = \sum_{b \in B \cap \mathfrak{p}} \sum \langle -(b)_0(x_0), x'_0 \rangle_X \langle v_1, v'_1 \rangle_V \dots \langle (b)_i v_i, v'_i \rangle_V \dots \langle v_k, v'_k \rangle_V.$$

The form $\langle, \rangle_X$ is $*$ -invariant for $\mathfrak{g}$ and $\langle, \rangle_V$ is $\bullet$ -invariant for $\mathfrak{g}$, hence $\langle -bx_0, x'_0 \rangle_X = \langle x_0, bx'_0 \rangle_X$ and $\langle bv_i, v'_i \rangle_V = \langle v_i, bv'_i \rangle_V$ for all $b \in \mathfrak{p}$:

$$\langle \pi \tilde{z}_i(\phi), \psi \rangle_{F\mu, k(X)} = \sum_{b \in B \cap \mathfrak{p}} \sum \langle (x_0), (b)_0 x'_0 \rangle_X \langle v_1, v'_1 \rangle_V \dots \langle v_i, (b)_i v'_i \rangle_V \dots \langle v_k, v'_k \rangle_V.$$

Reversing through the definitions, we show

$$\langle \pi(\tilde{z}_i)^* \phi, \psi \rangle_{F\mu, k} = \langle \phi, \pi(\tilde{z}_i) \psi \rangle_{F\mu, k}.$$

The element θ_j acts by $(\xi)_j$ where $\xi \in \mathfrak{k}$, hence $\langle \xi v, v' \rangle_V = \langle v, \xi v' \rangle_V$. Therefore

$$\langle x_0, x'_0 \rangle_X \langle v_1, v'_1 \rangle_V \dots \langle (\xi) v_j, v'_j \rangle_V \dots \langle v_k, v'_k \rangle_V = \langle x_0, x'_0 \rangle_X \langle v_1, v'_1 \rangle_V \dots \langle v_j, (\xi) v'_j \rangle_V \dots \langle v_k, v'_k \rangle_V.$$

Similarly for $s_{i,i+1}$

$$\begin{aligned} &\langle s_{i,i+1}(x_0 \otimes v_1 \otimes \dots \otimes v_k), x'_0 \otimes v'_1 \otimes \dots \otimes v'_k \rangle_{X \otimes V^{\otimes k}} \\ &= \langle x_0 \otimes v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_k, x'_0 \otimes v'_1 \otimes \dots \otimes v'_k \rangle_{X \otimes V^{\otimes k}} \\ &= \langle x'_0 \otimes v_1 \otimes \dots \otimes v_k, x_0 \otimes v'_1 \otimes \dots \otimes v'_{i+1} \otimes v'_i \otimes \dots \otimes v_k \rangle_{X \otimes V^{\otimes k}} \\ &= \langle x_0 \otimes v_1 \otimes \dots \otimes v_k, s_{i,i+1}(x'_0 \otimes v'_1 \otimes \dots \otimes v'_k) \rangle_{X \otimes V^{\otimes k}}. \end{aligned}$$

The projection $e_{i,i+1}$ acts on elementary tensors $x_0 \otimes v_1 \otimes \dots \otimes v_k$ by

$$e_{i,i+1}(x_0 \otimes v_1 \otimes \dots \otimes v_k) = m x_0 \otimes v_1 \otimes \dots \otimes v_{i-1} \otimes \text{pr}_{12}(v_i \otimes v_{i+1}) \otimes \dots \otimes v_k.$$

Then

$$\begin{aligned} &\langle e_{i,i+1}(x_0 \otimes v_1 \otimes \dots \otimes v_k), x'_0 \otimes v'_1 \otimes \dots \otimes v'_k \rangle_{X \otimes V^{\otimes k}}, \\ &= m \langle x_0 \otimes v_1 \otimes \dots \otimes v_{i-1} \otimes \text{pr}_{12}(v_i \otimes v_{i+1}) \otimes \dots \otimes v_k, x'_0 \otimes v'_1 \otimes \dots \otimes v'_k \rangle_{X \otimes V^{\otimes k}}, \\ &= \langle x_0, x'_0 \rangle_X \langle v_1, v'_1 \rangle_V \dots \langle \text{pr}_{12}(v_i \otimes v_{i+1}), v'_i \otimes v'_{i+1} \rangle_{V \otimes V} \dots \langle v_k, v'_k \rangle_V. \end{aligned}$$

Using Lemma 3.9.22 ,

$$= \langle x_0 \otimes v_1 \otimes \dots \otimes v_k, x'_0 \otimes v'_1 \otimes \dots \otimes v'_{i-1} \otimes \text{pr}_{12}(v'_i \otimes v'_{i+1}) \otimes \dots \otimes v'_k \rangle_{X \otimes V^{\otimes k}},$$

Therefore

$$\langle e_{i,i+1}(\phi), \psi \rangle_{F_{\mu,k}} = \langle \phi, e_{i,i+1}(\psi) \rangle_{F_{\mu,k}}.$$

The module $F_{\mu,k}(X)$ has induced Hermitian form $\langle, \rangle_{F_{\mu,k}}$ which is $*$ -invariant on the generators of $\mathfrak{B}_k^\theta$. Hence $\langle, \rangle_{F_{\mu,k}}$ is a $*$ -invariant form. If X is unitary then $\langle, \rangle_{X \otimes V^{\otimes k}}$ is positive definite. Hence $\langle, \rangle_{F_{\mu,k}}$ is a positive definite invariant Hermitian form and $F_{\mu,k}(X)$ is unitary. □

Definition 3.9.24. Let $X \in \mathcal{HC}(G)$, $\mathfrak{B}_k^\theta$ -mod or $\mathbb{H}_k(c)$ -mod module with invariant Hermitian form $\langle, \rangle_X$. We define the Langlands quotient $\overline{X}$ to be:

$$\overline{X} = X / \text{rad}\langle, \rangle_X,$$

where $\text{rad}\langle, \rangle$ is the radical of the form $\langle, \rangle$.

Lemma 3.9.25. Let X be in $\mathcal{HC}(G)$ -mod with Hermitian invariant form $\langle, \rangle_X$ and Langlands quotient $\overline{X}$. The form $\langle, \rangle_{F_{\mu,k}}$ is the endowed hermitian form of $F_{\mu,k}(X)$ from Definition 3.9.20 then

$$F_{\mu,k}(\overline{X}) = \overline{F_{\mu,k}(X)} = F_{\mu,k}(X) / \text{rad}\langle, \rangle_{F_{\mu,k}}.$$

Proof. One can construct an exact sequence:

$$0 \longrightarrow \text{rad}\langle, \rangle_X \longrightarrow X \longrightarrow \overline{X} \longrightarrow 0.$$

Exactness of the functors $F_{\mu,k}$ is given by Lemma 3.5.3. Hence there is an exact sequence:

$$0 \longrightarrow F_{\mu,k}(\text{rad}\langle, \rangle_X) \longrightarrow F_{\mu,k}(X) \longrightarrow F_{\mu,k}(\overline{X}) \longrightarrow 0.$$

For the result it is sufficient to prove that $F_{\mu,k}(\text{rad}\langle, \rangle_X) = \text{rad}\langle, \rangle_{F_{\mu,k}}$. Since $\langle, \rangle_{F_{\mu,k}}$ is an invariant form on $F_{\mu,k}(X)$ and a non-degenerate form on $\overline{F_{\mu,k}(X)}$ then $F_{\mu,k} \text{rad}\langle, \rangle_X = \text{rad}\langle, \rangle_{F_{\mu,k}}$. □

Theorem 3.9.26. *Let X_δ^ν be a principal series module for $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$. The Langlands quotient $\overline{X_\delta^\nu} = X_\delta^\nu / \text{rad}\langle, \rangle_{X_\delta^\nu}$ is mapped by $F_{\mu, k}$, to the Langlands quotient of the $\mathbb{H}_k(c_\mu)$ -module, $\overline{X(\nu_k)} = X(\nu_k) / \text{rad}\langle, \rangle_{X_{\nu_k}}$. Similarly, X_δ^ν is mapped by $F_{\underline{\mu}, n-k}$, to the $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ -module $\overline{X(\bar{\nu}_{n-k})}$.*

Definition 3.9.27. *We define subsets of $\mathfrak{a}^*$:*

$$\mathcal{U}_\delta = \{\nu \in \mathfrak{a}^* : \overline{X_\delta^\nu} \text{ is a unitary Harish-Chandra module}\}.$$

Similarly define

$$\mathcal{U}_k(\mathbb{1}) = \{\lambda \in \mathfrak{a}_k^* : \overline{X_\lambda} \text{ is a unitary } \mathbb{H}_k(c_\mu) \text{ module}\}$$

and

$$\mathcal{U}_{n-k}(\mathbb{1}) = \{\bar{\lambda} \in \bar{\mathfrak{a}}_{n-k}^* : \overline{X(\bar{\lambda})} \text{ is a unitary } \mathbb{H}_{n-k}(c_{\underline{\mu}}) \text{ module}\}.$$

Since $\mathfrak{a} = \mathfrak{a}_k \oplus \bar{\mathfrak{a}}_{n-k}$ we can associate a pair $(\lambda_k, \bar{\lambda}_{n-k}) \in \mathfrak{a}^* \times \bar{\mathfrak{a}}_{n-k}^*$ to a character of $\mathfrak{a}$ via:

$$(\lambda_k, \bar{\lambda}_{n-k}) : \mathfrak{a} \rightarrow \mathbb{C}$$

$$(\lambda_k, \bar{\lambda}_{n-k})(\mathfrak{a}_k) = \lambda_k(\mathfrak{a}_k),$$

$$(\lambda_k, \bar{\lambda}_{n-k})(\bar{\mathfrak{a}}_{n-k}) = \bar{\lambda}_{n-k}(\bar{\mathfrak{a}}_{n-k}).$$

Theorem 3.9.26 shows that the Langlands quotients of X_δ^ν map under $F_{\mu, k}$ and $F_{\underline{\mu}, n-k}$ to Langlands quotients of principal series modules for $\mathbb{H}_k(c_\mu)$ and $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ hence we can formulate the following non-unitary test.

Lemma 3.9.28. *We have an inclusion of sets:*

$$\mathcal{U}_\delta \subseteq \mathcal{U}_k(\mathbb{1}) \times \mathcal{U}_{n-k}(\mathbb{1}).$$

This inclusion of sets states that if we take a minimal principal series module X and find that, under the functor $F_{\mu, k}$, the Langlands quotient of the image is not unitary then the Langlands quotient of X is not unitary.

Theorem 3.9.29. *[Non-unitary test for principal series modules] If either $\overline{X(\nu_k)}$ or $\overline{X(\bar{\nu}_{n-k})}$ are not unitary, as $\mathbb{H}_k(c_\mu)$ and $\mathbb{H}_{n-k}(c_{\underline{\mu}})$ -modules, then the Langlands quotient of the minimal principal series module $\overline{X_{\delta^\nu}^\nu}$, for $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$ is not unitary.*

This provides a functorial proof of the results of Barbasch, Pantano, Paul and Salamanca-Riba [3, 46] on unitarity of principal series module using petite K -types. The above working does not require the image to be a Hecke algebra module. Therefore, we have also proved the following theorem.

Theorem 3.9.30. *[Non-unitary test for Harish-Chandra modules] Let X be a Harish Chandra module. For $G = Sp_{2n}(\mathbb{R})$ or $O(p, q)$ $p+q = 2n+1$, if for any character μ and $k = 1, \dots, n$ the $\mathfrak{B}_k^\theta$ -module $\overline{F_{\mu,k}(X)}$ is not unitary, then the Langlands quotient $\overline{X}$ of X is not a unitary G -module. In the case when G is split then X is a subrepresentation of X_δ^ν and $F_{\mu,k}(X)$, $F_{\mu,n-k}(X)$ are Hecke algebra modules. In this case, if either $\overline{F_{\mu,k}(X)}$, $\overline{F_{\mu,n-k}(X)}$ is not unitary as a Hecke algebra module then $\overline{X}$ is not unitary as a G -module.*

We finish with a toy example.

Example 3.9.31. *Let $G = Sp_2(\mathbb{R}) \cong SL_2(\mathbb{R})$. Then principal series modules are labelled by $\delta = \pm 1$ and $\nu \in \mathbb{C}$. The principal series modules X_δ^ν are unitary exactly when $\nu = ib$ for $b \in \mathbb{R}$, that is ν is entirely imaginary.*

In this case all principal series modules are spherical principal series modules. The root system associated to Sp_2 has one root ϵ and the Weyl group is $\mathbb{Z}_2$ which acts by -1 on ϵ . Here $\mathbb{H}(c)$ will be the graded Hecke algebra associated to type B_1 with parameter c . The algebra $\mathbb{H}(c)$ is generated by ϵ and $s \in \mathbb{Z}_2$ such that $s\epsilon = -\epsilon s + c$. We note that $s^* = s$ and $\epsilon^* = -\epsilon + cs$.

Our theorem gives that

$$F_{\text{triv},1}(X_1^\nu) \cong \text{Ind}_{\mathbb{C}}^{\mathbb{H}(c)} \mathbb{1}_\nu.$$

Note that $\text{Ind}_{\mathbb{C}}^{\mathbb{H}(c)} \mathbb{1}_\nu$ is two dimensional with basis $\{\mathbb{1}_\nu, s\mathbb{1}_\nu\}$ we will denote the module $\text{Ind}_{\mathbb{C}}^{\mathbb{H}(c)} \mathbb{1}_\nu$ by Y_ν . Let $\langle, \rangle$ be a hermitian form on Y_ν , for Y_ν to be unitary we require

$$\langle s(u), v \rangle = \langle u, s^*(v) \rangle = \langle u, s(v) \rangle$$

and

$$\langle \epsilon(u), v \rangle = \langle u, \epsilon^*(v) \rangle = \langle u, [-\epsilon + cs](v) \rangle.$$

Letting $u = \mathbb{1}_\nu$ and $v = \mathbb{1}_\nu$, then the above requirement implies

$$\nu \langle \mathbb{1}_\nu, \mathbb{1}_\nu \rangle = \langle \epsilon(\mathbb{1}_\nu), \mathbb{1}_\nu \rangle = \langle \mathbb{1}_\nu, [-\epsilon + cs]\mathbb{1}_\nu \rangle = -\bar{\nu} \langle \mathbb{1}_\nu, \mathbb{1}_\nu \rangle + \langle \mathbb{1}_\nu, s\mathbb{1}_\nu \rangle.$$

For the above equation to hold $\nu = -\bar{\nu}$ and $\langle \mathbb{1}_\nu, s\mathbb{1}_\nu \rangle = 0$. Hence for Y_ν to be unitary ν must be purely imaginary. Furthermore if ν is purely imaginary then we can construct a Hermitian non-degenerate form on Y_ν such that it is a unitary form. Therefore in the case of $Sp_2(\mathbb{R})$ our non-unitary test becomes an equivalence:

X_δ^ν is unitary if and only if $F_{\text{triv},1}(X_\delta^\nu) \cong \text{Ind}_{\mathbb{C}}^{\mathbb{H}(c)} \mathbb{1}_\nu$ is unitary.

Bibliography

- [1] T. Arakawa and T. Suzuki. Duality between $sl_n(\mathbb{C})$ and the degenerate affine Hecke algebra. *Journal of Algebra*, 209(1):288 – 304, 1998.
- [2] M. Atiyah and W. Schmid. A geometric construction of the discrete series for semisimple Lie groups. *Invent. Math.*, 42:19–37, 1977.
- [3] D. Barbasch, D. Ciubotaru, and A. Pantano. Unitarizable minimal principal series of reductive groups. *arXiv preprint arXiv:0804.1937*, 2008.
- [4] D. Barbasch, D. Ciubotaru, and P. Trapa. Dirac cohomology for graded affine Hecke algebras. *Acta Math.*, 209 (2):197–227, 2012.
- [5] D. Barbasch and D. M. Ciubotaru. Star operations for affine Hecke algebras. *ArXiv:1504.04361*, 2015.
- [6] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*. Princeton University Press, 1980.
- [7] W. Borho and R. MacPherson. Partial resolutions of nilpotent varieties. *Asterisque*, 101:23–74, 1997.
- [8] R. Brauer. On algebras which are connected with the semisimple continuous groups. *Annals of Mathematics*, 38(4):857–872, 1937.
- [9] J. Brundan and A. Kleshchev. Representation theory of symmetric groups and their double covers. *Groups, combinatorics & geometry (Durham, 2001)*, pages 31–53, 2003.
- [10] J. Brundan and C. Stroppel. Gradings on walled Brauer algebras and Khovanov arc algebra. *Advances in Mathematics*, 231(2):709 – 773, 2012.

- [11] K. Calvert. Dirac cohomology, the projective supermodules of the symmetric group and the Vogan morphism. *The Quarterly Journal of Mathematics*, 2018.
- [12] W. Casselman. Jacquet modules for real reductive groups. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, volume 557563, 1978.
- [13] D. Ciubotaru. Dirac cohomology for symplectic reflection algebras. *Selecta Math.*, 22.1:111–144, 2016.
- [14] D. Ciubotaru. Tutorial on graded affine Hecke algebras. Lecture notes for new developments in Representations theory NUS 2016, 2016.
- [15] D. Ciubotaru and X. He. Green polynomials of Weyl groups, elliptic pairings, and the extended index. *Advances in Mathematics*, 283:1–50, 2015.
- [16] D. Ciubotaru and P. E. Trapa. Functors for unitary representations of classical real groups and affine Hecke algebras. *Advances in Mathematics*, 227(4):1585 – 1611, 2011.
- [17] D. M. Ciubotaru and P. E. Trapa. Characters of Springer representations on elliptic conjugacy classes. *Duke Mathematical Journal*, 162(2):201–223, 2013.
- [18] A. Cox, M. De Visscher, S. Doty, and P. Martin. On the blocks of the walled brauer algebra. *Journal of Algebra*, 320(1):169–212, 2008.
- [19] C. Dez  lee. Generalized graded Hecke algebras of types B and D. *Comm. Algebra*, 34(6):2105–2128, 2006.
- [20] R. Dipper, S. Doty, and J. Hu. Brauer algebras, symplectic Schur algebras and Schur-Weyl duality. *Transactions of the American Mathematical Society*, 360(1):189–213, 2008.
- [21] V. G. Drinfel’d. Degenerate affine Hecke algebras and Yangians. *Functional Analysis and Its Applications*, 20(1):58–60, 1986.
- [22] C. F. Dunkl and E. M. Opdam. Dunkl operators for complex reflection groups. *Proceedings of the London Mathematical Society*, 86(1):70–108, 2003.

- [23] M. Ehrig and C. Stroppel. Schur–weyl duality for the brauer algebra and the ortho-symplectic lie superalgebra. *Mathematische Zeitschrift*, 284(1-2):595–613, 2016.
- [24] P. Etingof and V. Ginzburg. Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Inventiones mathematicae*, 147(2):243–348, 2002.
- [25] S. Evens. The Langlands classification for graded Hecke algebras. *Proceedings of the AMS*, 124(4):1285–1290, 1996.
- [26] A. M. Garcia and C. Procesi. On certain graded S_n -modules and the q -Kostka polynomials. *Advances in Mathematics*, 94:82–138, 1992.
- [27] S. Griffeth. Towards a combinatorial representation theory for the rational Cherednik algebra of type $G(r, p, n)$. *Proceedings of the Edinburgh Mathematical Society*, 53(2):419–445, 2010.
- [28] Harish-Chandra. Representations of semi-simple Lie groups I. *Trans. Amer. Math. Soc.*, 75:185–243, 1953.
- [29] J. S. Huang and P. Pandzic. *Dirac operators in representation theory*. Mathematics: Theory and Applications. Birkäuser Boston, MA, 2006.
- [30] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Birkhäuser, 1972.
- [31] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1990.
- [32] K. Iwasawa. On some types of topological groups. *Annals of Mathematics*, 50(3):507–558, 1949.
- [33] D. Kazhdan and G. Lusztig. Proof of the Deligne-Langlands conjecture for Hecke algebras. *Inventiones mathematicae*, 87(1):153–215, 1987.
- [34] Killing. Die zusammensetzung der stetigen endlichen transformationsgruppen. *Mathematische Annalen*, 31:252–290, 1888.

- [35] A. Kleshchev. *Linear and projective representations of the symmetric group*. Cambridge University Press, 2005.
- [36] A. Knapp. *Lie Groups Beyond an Introduction*. Progress in Mathematics. Birkhäuser Boston, 2002.
- [37] C. Krilloff and A. Ram. Representations of graded Hecke algebras. *Representation Theory of the American Mathematical Society*, 6(2):31–69, 2002.
- [38] G. Lusztig. Cuspidal local systems and graded Hecke algebras, I. *Publications Mathématiques de l’IHS*, 67:145–202, 1988.
- [39] G. Lusztig. Affine Hecke algebras and their graded version. *Journal of the American Mathematical Society*, 2(3):599–635, 1989.
- [40] M. Martino. Blocks of restricted rational Cherednik algebras for $G(m, d, n)$. *Journal of Algebra*, 397:209 – 224, 2014.
- [41] E. Meinrenken. *Clifford algebra and Lie theory*. Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics. Springer, Heidelberg, 2013.
- [42] A. Morris. Projective characters of exceptional Weyl groups. *Journal of Algebra*, 29(3):567 – 586, 1974.
- [43] M. Nazarov. Young’s orthogonal form of irreducible projective representations of the symmetric group. *J. Lond Math. Soc.*, 42:437–451, 1990.
- [44] M. Nazarov. Young’s symmetrizers for projective representations of the symmetric group. *Advances in Mathematics*, 127(2):190 – 257, 1997.
- [45] A. Okounkov and A. Vershik. A new approach to representation theory of symmetric groups. *Selecta Mathematica*, 2(4):581–605, 1996.
- [46] A. Pantano, A. Paul, and S. Salamanca-Riba. Unitary genuine principal series of the metaplectic group. *Representation Theory of the American Mathematical Society*, 14(5):201–248, 2010.
- [47] R. Parthasarathy. Dirac operator and the discrete series. *Ann. of Math.*, 96:1–30, 1972.

- [48] F. Peter and H. Weyl. Die vollständigkeit der primitiven darstellungen einer geschlossenen kontinuierlichen gruppe. *Mathematische Annalen*, 97(1):737–755, Dec 1927.
- [49] A. Ram and A. Shepler. Classification of graded Hecke algebras for complex reflection groups. *Comment. Math. Helv.*, 78(2):308–334, 2003.
- [50] B. Sagan. On selecting a random shifted Young tableau. *J. Algorithms*, 1:213–234, 1980.
- [51] I. Schur. Über die rationalen darstellungen der allgemeinen linearen gruppe. *Sitzungsberichte Akad. Berlin*, (JMF 53.0108.05):5875, 1986.
- [52] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canad. J. Math*, 6(2):274–301, 1954.
- [53] T. Springer. A construction of representations of Weyl groups. *Invent. Math.*, 44:279–293, 1978.
- [54] J. R. Stembridge. Shifted tableaux and the projective representations of symmetric groups. *Advances in Mathematics*, 74(1):87 – 134, 1989.
- [55] M. Suzuki. *Group theory I*. Spring-Verlag, 1982.
- [56] A. M. Vershik and A. N. Sergeev. A new approach to the representation theory of the symmetric group, IV. $\mathbb{Z}_2$ -graded groups and algebras; projective representations of the group S_n . *Moscow Mathematical Journal*, 8(4):813–842, 2008.
- [57] D. Vogan. Representations of real reductive Lie groups. *Progr. Math.*, 15, 1981.
- [58] D. Vogan. *Linear and projective representations of the symmetric group*. Birkhuser Basel, 2015.
- [59] H. Wenzl. On the structure of Brauer’s centralizer algebras. *Annals of Mathematics*, 128(1):173–193, 1988.
- [60] H. Weyl. *The Classical Groups. Their Invariants and Representations*. Princeton University Press, 1939.