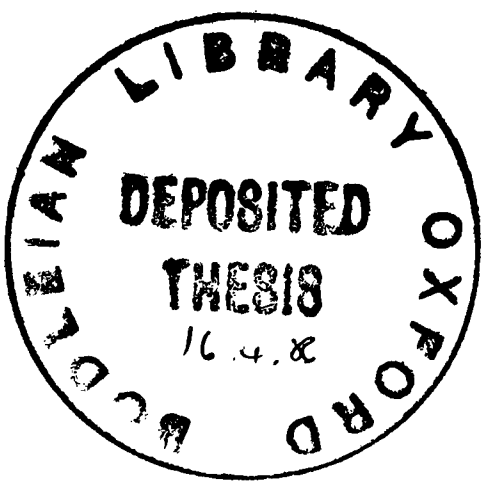
TWISTORS IN CURVED SPACE-TIME



A D. Phil. Thesis by L. J. Mason, Wadham College, 1985.

ABSTRACT

This thesis is concerned with an investigation of twistorial structures present in curved Lorentzian space-times.

Chapter 1 introduces the basic definitions and some theorems that will be used later in the text.

Chapter 2 investigates generalised connections that arise in twistor theory. First the Cartan conformal connection is studied, and some of the geometry underlying it is shown to be that used by Fefferman and Graham [13]. Also a condition that a space-time is conformal to vacuum is given.

Secondly the theory of the Chern connection associated to a C.R. manifold is developed in such a way as to make the calculation of the connection associated to a twistor C.R. manifold straight forward. A new proof of the Chern theorem of existence and uniqueness is given.

The Chern connection of a twistor C.R. manifold is then calculated, and discussed. In particular 5-dimensional C.R. manifolds arising as twistor C.R. manifolds are characterised. Canonical structures peculiar to the twistor case are discussed. Applications of C.R. manifold theory to algebraically special space-times are suggested.

Chapter three analyses how various twistorial structures behave in linearised general relativity. First, deformations of the space of complex null geodesics corresponding to variations of the conformal structure of space-time are shown to be generated by hamiltonians. Those that correspond to variations in the metric satisfying the field equations are given, along with hamiltonians corresponding to different fields and field equations. Generalisations to nonlinear equations are discussed.

These ideas are applied to hypersurface twistors in linearised theory, using flat hypersurfaces and Cech cohomology. Expressions are obtained for the deformation of the complex structure of the spaces and their evolution.

The results are generalised to non flat hypersurfaces using Dolbeault cohomology. It is shown that certain canonically defined forms on the spin bundle are preferred Dolbeault representatives for derivatives of the twistor cohomology classes corresponding to the linearised field.

In chapter four I generalise the results of chapter three to curved space using the Chern connection. In particular twistorial formulations of the constraint equations are given, and a formula for the evolution that satisfies the the vacuum evolution equations is given in terms of an "infinity" twistor and a "time" twistor. This is then discussed.

In chapter five I make some comments on the interpretation of a three form on the spin bundle discovered by G.A.J. Sparling as the gravitational hamiltonian. I then use this to show that one can give an interpretation of Penrose's quasi-local angular momentum twistor in terms of the canonical formalism.

Acknowledgements.

I would like to thank, most of all, my supervisor, Dr. N.M.J. Woodhouse. His constant care and attention throughout my academic career has been essential to my development (such as it is), and has gone far beyond the call of duty. His mathematical attitudes and insights pervade this thesis, although probably somewhat misshapen. I would also like to thank R. Penrose for providing such an interesting field of research, and for his many insightful comments and incisive questions.

I should also thank various other members of the research group in the mathematical institute for stimulating conversations etc..

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Introduction.

Historical comment and motivation:

In [15] R. Penrose suggested that "a link between space-time curvature and quantum processes may be supplied by the use of twistors. Then, roughly speaking, it is the continual slight shifting of the interpretation of quantum twistor operators which result in the curvature of space-time."

The general idea appeared to be that, locally, the geometry of space-time was generated by quantum processes. Essentially the knowledge of the twistor "position operators", Z^{α}, determine the shear structure on the space of null geodesics, which in turn determines which twist free null hypersurfaces focus on space-time events. (In fact the original idea appears to have been motivated by combinatorial approaches to building space-time geometry.)

If one wished to compare two such local geometries, one would have to compare their corresponding quantum systems. One would in general find that their corresponding quantum operators were related by a unitary transformation, $Z^{\alpha} \to \tilde{Z}^{\alpha} = \tilde{Z}^{\alpha}(Z^{\alpha}, \partial/\partial Z^{\alpha})$. This would mean that the shear structure would be altered from region to region. In particular, space-time events in one region would appear "fuzzed out" from the point of view of the geometry defined in another region. However the evolution of the shear structure on the space of null geodesics gives rise to Weyl curvature.

Classically the unitary transformations on the $\hat{Z}^{\alpha}$ operators would be given by symplectic transformations on the the twistor "position" observables Z^{α}. These observables determine the complex structure which in turn determine the shear structure on the space of light rays. So the mechanism above should be reflected classically in a shifting in the complex structure from region to region given in terms of symplectic

transformations.

In order to articulate this point of view in a more mathematically precise manner, R. Penrose introduced the notion of hypersurface twistor space in [16]. This is a curved complex manifold attached to each hypersurface in space-time. The complex structure encodes the information of which null congruences are shear free at the hypersurface.

With the arrival of the nonlinear graviton construction [17] this viewpoint has shifted somewhat. R. Penrose now suggests that a quantum system of particles on a fixed background should be described by a fixed twistor space, a generalisation of the nonlinear graviton to Lorentzian space-times. This can be thought of as a semiclassical approximation. Quantum gravitational effects would then be introduced when one considers backreactions on the geometry. These would hopefully "nonlinearise" quantum theory and provide a mechanism for the collapse of the wave function.

Unfortunately it has been very difficult to find this generalisation of the nonlinear graviton. (Although there have been some promising signs recently.)

Hypersurface twistors remain the best way (by default) of introducing twistors into a curved space-time. The converted should probably think of them as the completely classical residue of an underlying quantised twistor theory. A detailed study of them may yield some insight into this underlying quantised twistor theory.

Both the converted and unconverted can think of them merely as a potentially useful tool, not only for understanding classical general relativity, but also for more conventional approaches to quantising gravity.

The problem with the second point of view has been that very little

was known about hypersurface twistors.

However important advances have been made. Notably C. LeBrun has pointed out that in general, in a nonanalytic space-time, one can only define a real codimension one hypersurface, N, of the hypersurface twistor space. The shear structure is contained in a "C.R. structure" on N. This structure models abstractly the structure a real odd dimensional manifold inherits from a codimension one embedding in a complex manifold. This now appears to be the appropriate object for study.

Another notable advance was made by G.A.J. Sparling, pointing out that there existed an extensive amount of mathematical machinery for the study of C.R. manifolds. In particular the Fefferman metric and the Chern connection. He also calculated the Fefferman metric for N and showed that it had a particularly elegant form.

A resume of the thesis:

The primary purpose of this thesis is to develop the theory of hypersurface twistors to the stage where one can write the evolution equations in a twistorial fashion, and hopefully apply hypersurface twistorial techniques to problems in classical and quantum general relativity. However various other results are presented that are relevant to the study of twistors in curved space-time.

In chapter 1 some of the less accessible background material used in this thesis is presented.

§1.1 presents calculus on the spin bundle, and the various canonically defined structures.

§1.2 deals with extra structures that arise when one restricts to a

space-like hypersurface, and gives the relevant curvature decompositions that will be used later in the thesis.

§1.3 is a brief introduction to the theory of C.R. manifolds. Analytical details are outlined, and Chern Fefferman theory is presented in some detail for comparison to the reformulation in §2.2.

§1.4 gives the basic set up for the study of the space of null geodesics and displays the various canonical structures.

§1.5 contains sketch proofs of some of the theorems used in the thesis.

§1.6 sets up elementary results concerning the behaviour of the various canonical structures under variations of the metric.

Chapter 2 discusses applications of the theory of G-structures and related topics to twistor theory. In particular the Chern connection for hypersurface twistor C.R. manifolds, and the local twistor connection.

In §2.1 the theory of G-structures is reviewed with particular emphasis on the example of the local twistor connection.

The purpose of this is to provide an exposition of the geometry and analysis that underlie G-structures, and present two results concerning the local twistor connection.

The first is that the geometry and analysis that Fefferman uses to study conformal invariants is equivalent to the algebra used in defining the Cartan conformal connection.

The second is the condition on the Cartan conformal connection that the conformal class used to define it is conformal to vacuum. This condition is the existence, at least in the local twistor case, of a covariantly constant infinity twistor. (The use of the infinity twistor to

characterise the field equations will be a consistent theme throughout the thesis.)

Neither of these results are particularly original, but are presented for their own interest and because they will relate to later work in the thesis.

In §2.2 the Chern-Fefferman theory is reformulated in such a way as to make calculations with it on a general twistor C.R. manifold more straightforward and transparent.

The structure of the Chern connection is discussed in considerable detail and a new proof of its existence is presented. The main purpose of this section is to reformulate the theory in such a way as to make the calculation of the twistor Chern connection relatively transparent.

In §2.3 the Chern connection is calculated for a general twistor C.R. manifold. It is then used to characterise those C.R. manifolds arising in twistor theory. Canonical structures peculiar to twistor C.R. manifolds are discussed, the infinity twistor and the origin twistor. Discussion of field equations is put off until §4.

In §2.4 applications of C.R. manifold theory to algebraically special space-times are suggested. It is shown how one may incorporate the information of vacuum type {3,1} and type {4} space-times in terms of a holomorphic form on the C.R. 3-manifold associated to the geodesic congruence. (This form is just the restriction to the congruence of a 1-form constructed from the infinity twistor on a general hypersurface C.R. manifold.) The interesting open problems suitable for treatment with this set up are discussed, but, unfortunately, not solved.

Chapter 3 contains a detailed linearised analysis of the twistorial structures one has available in curved space-time. This is to provide analogies with which to interpret the structures one has available in curved space twistor theory. These calculations motivate the various results in the curved case.

In §3.1 infinitesimal deformations of the complex structure on spaces of null geodesics corresponding to variations in the conformal structure on space-time are shown to be given by hamiltonians. In the flat case the deformations corresponding to variations in the metric satisfying the linearised vacuum equations and linearised Bach equations are presented. (Also hamiltonians for electromagnetism are given.) Strong evidence is provided for a straightforward generalisation to the nonlinear case for the Bach equations. The vacuum equations are not so amenable.

In §3.2 a detailed analysis of the structure and evolution of hypersurface twistor spaces based on flat hypersurfaces in linearised general relativity is performed using the ideas from §3.1. A relatively simple expression for the evolution is obtained.

In §3.3 Dolbeault representatives of the Cech cohomology classes used in §3.1 and §3.2 are calculated. Elementary results concerning these representatives are presented for use both in §3.4 and to provide an analogy which motivates generalisations of the results in §3.4 to curved space-time.

§3.4 generalises the results of §3.2 to nonflat hypersurfaces in terms of the Dolbeault representatives introduced in §3.3.

In chapter 4 the relevant results from §3 are generalised to curved

space-time.

In §4.1 the various structures on hypersurface C.R. manifolds are interpreted in the light of the linearised analysis.

The most important structure is the infinity twistor. It is argued that the infinity twistor should be regarded as a generalisation of one of the Dolbeault representatives discussed in §3. It is shown how both the vacuum field equations and the constraint equations take on a similar form to the one expected from linearised theory.

It is shown that this provides conditions on the curvature of the Chern connection, and it is argued that this should be a characterisation of those hypersurface twistor C.R. manifolds whose corresponding initial data sets are conformal to ones satisfying the constraint equations.

In §4.2 the formula for the evolution from linearised theory is generalised to the curved case. This uses the infinity twistor. One also has to introduce a "time" twistor to describe the lapse.

Moodlo technicalities concerning the time twistor, the first order evolution equations for hypersurface twistors is articulated. It is argued that this has all been done in a manifestly twistorial manner. (By twistorial I mean that one doesn t need to know the location of the space-time events.) Potential generalisations are then discussed.

In chapter 5 I show how the Sparling three form can be interpreted as the gravitational hamiltonian and show that Penrose s quasi-local angular momentum twistor can be interpreted as a set of hamiltonians generating motions of the spanning three surface of the two surface on which the angular momentum twistor was defined. Further problems are discussed.

I shall not indulge in any discussion of the above results here as

this is more than adequately taken care of in the many incoherent remarks interspersed throughout the text.

§1: Background.

This section is not intended so much as a rigorous derivation of the basic theory but more as a brief excursion to facilitate reading the rest of the thesis. It is also intended to fix notation and conventions. These will follow Penrose and Rindler [20], and Penrose and MacCallum [19].

Throughout M will denote a 4-dimensional manifold, either real or complex, with a metric, which will be holomorphic in the complex case. Indices will be assumed to be abstract unless otherwise stated.

§1.1: Working on the spin bundles.

Most of the work in this thesis will be carried out on the spin bundle. There exists a convenient basis of forms and vector fields on the spin bundle which I will use consistently. See Sparling [25] or Woodhouse [28] for an alternative presentation.

In flat space we can choose coordinates $(x^{AA'}, z_{A'}, \bar{z}_{A})$ on S_{A}, the spin bundle. This leads to a convenient basis of spinor indexed forms, $(dx^{AA'}, dz_{A'}, d\bar{z}_{A})$, and vector fields, $(\nabla_{AA'}, \partial/\partial z_{A'} = \partial^{A'}, \partial/\partial \bar{z}_{A} = \bar{\partial}^{A})$. This can be generalised to curved space as follows.

The spin bundle, $S_{A'}$, is fibred over space-time, $p:S_{A'} \to M$, so we can pull back all the spinor and tensor bundles from M to $S_{A'}$. When one pulls back a bundle to its own total space one has a canonical section. At a point v of the bundle this canonical section takes on the value of v. The canonical sections of S_{A} and S_{A} pulled back to $S_{A'}$ will be denoted by z_{A} and $\bar{z}_{A}$ respectively.

When we are working in the complex, $S_{A'}$ complexifies to $S_{A} \oplus S_{A}$ and the corresponding canonical elements will then be denoted $z_{A'}$ and w_{A} respectively.

Clearly if $\lambda_{A'}$ is a section of $S_{A'}$ over M then:

$(\lambda)^{*}z_{A'} = \lambda_{A'}$.

We can extend the ordinary de Rham operator, d, to act on abstractly indexed quantities using the connection on $S_{A'}$. Let $\varepsilon_{\underline{A}'}{}^{A'}$ be a spinor dyad, then, using the Leibnitz property: $\varepsilon_{\underline{A}'}{}^{A'}dz_{A'} = dz_{\underline{A}'} - z_{B'}d\varepsilon_{\underline{A}'}{}^{B'}$

$= dz_{\underline{A}'} - z_{B'}\nabla_{\underline{a}}\varepsilon_{\underline{A}'}{}^{B'}dx^{\underline{a}} = dz_{\underline{A}'} - \gamma_{\underline{a}\underline{A}'}{}^{\underline{B}'}z_{\underline{B}'}dx^{\underline{a}}$.

Where $\gamma_{\underline{a}\underline{A}'}{}^{\underline{B}'}$ are the spin coefficients in the spin frame $\varepsilon_{\underline{A}'}{}^{A'}$, and $dx^{\underline{a}}$ are the frame forms in this frame pulled back from M to $S_{A'}$. ∇_a is the standard covariant derivative operator on M.

If λ_A is the section of the spin bundle referred to above, then we have: $(\lambda)^{*}dz_{A'} = \nabla_b\lambda_{A'}dx^{b}$.

Note that $d^2\neq 0$ on indexed quantities.

To complete $dz_{A'}$ and $d\bar{z}_{A}$ to a coframe of $S_{A'}$, we can pull back the solder form $dx^{AA'}$ from M to $S_{A'}$. The dual frame of vector fields will be denoted by $(\nabla_{AA'},\partial/\partial z_{A'},\partial/\partial\bar{z}_{A})$. Note that this implies that $\nabla_{AA'}$ is the horizontal lift of $\nabla_{AA'}$ on M, ie: $\nabla_{AA'} \lrcorner\, dz_{B'} = \nabla_{AA'}z_{B'} = 0$

(I should apologise for the abuse of notation, with objects on M being confused with their counterparts on $S_{A'}$, however since most calculations will be done on the spin bundle this shouldnt cause any confusion.)

The above structure can be determined by either the exterior derivatives of the forms or the commutators of the vector fields. These are as follows:

$$[\partial/\partial z_{A'},\partial/\partial z_{B'}] = 0 = [\partial/\partial z_{A'},\partial/\partial\bar{z}_{A}] = 0 = [\nabla_{AA'},\partial/\partial z_{A'}]$$

$$[\nabla_{AA'},\nabla_{BB'}] = \varepsilon_{AB}\bar{R}_{A'B'} + \varepsilon_{A'B'}R_{AB}$$

Here R_{AB} and $\bar{R}_{A'B'}$ are the curvature vector fields:

$$R_{AB} = -(\Psi_{ABCD} - 2\Lambda\varepsilon_{D(A}\varepsilon_{B)C})\bar{z}^{C}\partial/\partial\bar{z}_{D} - \Phi_{ABC'D'}z^{C'}\partial/\partial z_{D'}$$

These reflect the usual failure of the horizontal subspaces to

integrate up in curved space. Dually we have, in terms of forms:

$$ddx^{AA'} = 0 \quad \text{(in the absence of torsion)}$$

$$d^2\bar{z}_A = (\Psi_{ABCD} - 2\Lambda\varepsilon_{A(C}\varepsilon_{D)B})\bar{z}^B d^2x^{CD} + \Phi_{A'B'AB}\bar{z}^B d^2x^{A'B'}$$

Where $d^2x^{AB} = \varepsilon_{A'B'}dx^{AA'} \wedge dx^{BB'}$.

Conformal rescalings: Under a conformal rescaling of the metric:

$$g_{ab} \to \Omega^2 g_{ab}$$

$$\nabla_{AA'} \to \nabla_{AA'} + z_{A'}\Upsilon_{AB'}\partial/\partial z_{B'} + \bar{z}_A\Upsilon_{A'B}\partial/\partial\bar{z}_B \quad \text{and:}$$

$dz_{A'} \to dz_{A'} - \Upsilon_{A'B}z_{B'}dx^{BB'}$, all the other objects defined above dont vary. In the above $\Upsilon_a = \nabla_a \log\Omega$.

The twistor distribution: In complex flat space the lifts of α-planes to the spin bundle foliate the spin bundle. The corresponding distribution can be written: $D = \{ z^{A'}\nabla_{AA'}, \partial/\partial w_A \}$

The $\partial/\partial w_A$ is adjoined on the assumption that one is working on $S_A \oplus S_{A'}$, so that when one is working on a real lorentzian space-time we have:

$$D = \{z^{A'}\nabla_{AA'}, \partial/\partial\bar{z}_A\}$$

D can be checked to be conformally invariant. It can also be checked that D is integrable iff $\Psi_{A'B'C'D'} = 0$. For a Lorentzian space-time D has a real part: $D \cap \bar{D} = \{ z^A \bar{z}^A \nabla_{AA} \} = V$

This is the null geodesic spray. The factor space is the space of null geodesics complete with a complex scaling, N. The scaling is that associated to a spinor parallel along the null geodesic whose flag pole is aligned along the geodesic. If D is integrable (ie M conformally flat), it then descends to N as a three complex dimensional distribution where it provides a C.R. structure.

§1.2: Working on the spin bundle restricted to a space-like hypersurface.

In order to perform hypersurface twistor calculations we will need to use the restriction of the connection on M to a spacelike hypersurface, H, rather than the intrinsic connection on the hypersurface. This involves looking at the restriction of the forms defined above. These will only depend on the initial data at H, ie the intrinsic 3-metric and the extrinsic curvature, since the restriction of the spin connection can be represented in terms of these.

On $S_{A'}|_H$ we have some extra structure, the timelike normal which I will normalize to have length $\sqrt{2}$, so that: $T_{AA'}T^{A'}{}_{B} = \epsilon_{AB}$

This can be used to define a conjugation: $\alpha_A \to \check{\alpha}_A = -T_{AA'}\alpha^{A}$

This can be combined with the standard Lorentz conjugation, $\alpha_{A'} \to \bar{\alpha}_{A'}$, to give a Hermitean conjugation, $\alpha_{A'} \to \hat{\alpha}_{A'} = -T_{AA'}\bar{\alpha}^{A}$.

Note that the "$\check{}$" and the "$\hat{}$" conjugations are not preserved by the restriction of the connection from M in the presence of extrinsic curvature. Nevertheless they are important structures to have around. Note that when we complexify we lose both the Lorentzian and the Hermitean conjugations and we are just left with the "$\check{}$" conjugation. The "$\check{}$" conjugation can be interpreted as reflecting the α-plane, with spinor $\alpha_{A'}$ aligned along it, in the hypersurface to obtain the β-plane, with $\check{\alpha}_{A}$ aligned along it.

Since d commutes with restriction, the curvature of the restricted connection is the restriction of the curvature. This however decomposes differently due to the additional structure. We see that:

$$\text{Since } d^2x^{AB}|_H = T^{A}{}_{A'}T^{B}{}_{B'}d^2x^{A'B'}|_H$$

$$d^2\Sigma_A|_H = (\Psi_{ABCD} - 2\Lambda\epsilon_{A(B}\epsilon_{C)D} + \Phi_{BCDA})\bar{\iota}^{D}d^2x^{BC}$$

Where: $\Phi_{ABCD} = \Phi_{A'B'CD} T^{A'}{}_{A} T^{B'}{}_{B}$

We can decompose the Φ term into irreducibles to obtain:

$$d^2\Xi_{A|H} = \{\Xi_{ABCD} + 1/3\ G_{oo}\varepsilon_{A(B}\varepsilon_{C)D} - 1/4\ \varepsilon_{BD}G_{o}{}_{AC} - 1/4\ \varepsilon_{CA}G_{o}{}_{BD}\}\bar{z}^{D}d^2X^{BC}$$

Where: $G_{oo} = 1/2\ G_{ab}T^{a}T^{b}$ is the Hamiltonian constraint,

and $G_{o}{}_{AB} = (1/2)G_{cA'(A}T^{c}T^{A'}{}_{B)}$ are the momentum constraints,

(G_{ab} is the Einstein tensor, $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = -2\Phi_{ab} + 6\Lambda g_{ab}$.)

$\Xi_{ABCD} = \Psi_{ABCD} + \Phi_{ABCD}$. Here $\Phi_{ABCD} = \Phi_{(ABCD)}$ is the trace free part of the projection of the Ricci tensor onto the hypersurface.

Ξ_{ABCD} can be decomposed further:

$$\Xi_{ABCD} = (\Phi_{ABCD} + E_{ABCD}) + iB_{ABCD}$$

where $E_{ABCD} = (1/2)(\Psi_{ABCD} + T_{A}{}^{A'}T_{B}{}^{B'}T_{C}{}^{C'}T_{D}{}^{D'}\Psi_{A'B'C'D'})$ is the electric part of the Weyl spinor and $B_{ABCD} = (i/2)(\Psi_{ABCD} - T_{A}{}^{A}T_{B}{}^{B'}T_{C}{}^{C'}T_{D}{}^{D'}\Psi_{AB'C'D'})$ is the magnetic part. Note that one cannot derive Φ_{ABCD} or E_{ABCD} seperately from the initial data, but one can derive their sum. B_{ABCD}, G_{oo}, and $G_{o}{}_{AB}$, can all be expressed in terms of the initial data.

The restriction of the Bianchi identities yield:

$$D^{AB}\Xi_{ABCD} = 1/3\ D_{CD}G_{oo} + 1/2\ D^{B}{}_{(C}G_{o}{}_{D)B}$$

Where $D_{AB} = T_{(A}{}^{B}\nabla_{B)B'}$. This is the covariant derivative restricted so as to act in directions tangent to H.

The evolution equations are juat $\Phi_{ABCD} = 0$ or:

$$\Xi_{ABCD} = \Psi_{ABCD}$$

(I am grateful to A. Ashtekar for showing me these curvature decompositions. At the time it was thought that the above was the curvature of a special connection concocted by Sen, however, as seen above this is just the curvature of the restricted spin connection.)

§1.3: C.R. manifolds:

A C.R. structure on a manifold PN models abstractly the structure a hypersurface in a complex manifold inherits from the ambient complex structure. I will only be interested in C.R. manifolds of hypersurface type, that is those which model real codimension one hypersurfaces in complex manifolds. Intrinsically we define:

Definition: A 2n+1 dimensional manifold PN is said to have a C.R. structure of hypersurface type if on PN there is a complex integrable n-dimensional distribution, H, $H \cap \bar{H} = 0$.

H models the holomorphic tangent vectors of the ambient manifold which are tangent to PN. The most basic local invariant of a C.R. manifold is the signature of its Levi form.

Definition: The Levi form of a C.R. manifold PN is a line bundle valued hermitean form on H, (or alternatively a conformal equivalence class of hermitean forms). If X and Y are sections of H then $\mathcal{L}(X,Y) = [X,\bar{Y}] \bmod H \oplus \bar{H}$ is the Levi form. It takes values in $\mathbb{C}TPN/H \oplus \bar{H}$.

(It is straightforward to show that this definition is linear over $C^{\infty}PN$, see below.)

The basic example of a C.R. manifold with Levi form of signature (p,q) is the projective hyperquadric Q in $\mathbb{C}P^{n+1}$ given by the zero set of a hermitean form $Z^{\alpha}\bar{Z}_{\alpha}$ of signature (p+1,q+1) on $\mathbb{C}^{n+2}$. Locally a C.R. manifold whose levi form has signature (p,q) can be osculated to low order about a point by such a hyperquadric. Various analytic properties of C.R. manifolds are sensitive only to the signature.

Definition: A C.R. manifold is said to be nondegenerate iff its Levi form is nondegenerate.

In general one is not interested in degenerate C.R. manifolds since they can locally be expressed as a product of a complex manifold and a nondegenerate C.R. manifold.

On a C.R. manifold one can mimic much of the complex analysis that one can do on a complex manifold, with the important differance that, at least in the twistor case, one does not in general have enough C.R. functions, (the analogue of holomorphic functions) to provide coordinates. Put more geometrically, one cannot in general embed a twistor C.R. manifold in a complex manifold. If, however, we assume analyticity, we can complexify the C.R. manifold and factor out by the distribution, $\tilde{H}$, and create a complex manifold with a real slice that is C.R. equivalent to PN.

Definition: C.R. functions are defined as complex functions, f, on PN which satisfy $\bar{X}f = 0$ for all $\bar{X} \in \bar{H}$.

Clearly, if PN is embedded in a complex manifold, the restriction of holomorphic functions will be C.R.. The converse is true if the Levi form of PN has signature (p,q) with p and q non zero, as in the twistor case. (If say q = 0, the C.R. functions are at worst boundary values of holomorphic functions on the convex side of the ambient manifold. These results follow from local applications of Cauchy's integral formula.)

Since in general there do not exist enough C.R. functions to provide coordinates, Cech cohomology will be inappropriate. However there exists a well defined analogue of the Dolbeault theory. This is defined as follows:

The Dolbeault complex:

Let $\Omega^r{}_B = \Lambda^r(\bar{H}^*) = \Omega^r/(\bar{H}^0 \wedge \Omega^{r-1})$, ie r-forms on PN modulo those which vanish when restricted to $\bar{H}$. ($\bar{H}^0$ is the annihilator of $\bar{H}$ in Ω^1.) Since $\bar{H}$ is integrable, the ordinary deRham operator, d, takes r-forms which vanish when restricted to $\bar{H}$ to r+1-forms which vanish when restricted to $\bar{H}$, ie:

$$d: \bar{H}^0 \to \bar{H}^0 \wedge \Omega^1$$

This implies that d descends in a well defined manner to act on $\Omega^*{}_B$, providing an analogue of the $\bar{\partial}$ operator, denoted $\bar{\partial}_B$, which maps r-forms to (r+1)-forms:

$$\bar{\partial}_B: \Omega^r{}_B \to \Omega^{r+1}{}_B$$

So we have a sequence of sheaves:

$$0 \to \mathcal{O}_B \to C^\omega_{\mathbb{C}} \to \Omega^1{}_B \to \Omega^2{}_B \to \cdots\cdots$$

Where the maps are all given by $\bar{\partial}_B$, except the first two which are inclusions. $\mathcal{O}_B$ denotes the sheaf of germs of C.R. functions.

Unlike the case of a complex manifold, this sequence is not exact, even when PN is analytic. It fails to be exact at the p^{th} and q^{th} places when the C.R. manifold has Levi form of signature (p,q). So one has nontrivial sheaves of local cohomology, $H^p{}_B$, and, $H^q{}_B$ (see Andreotti and Hill [2]). When PN is embedded in a complex manifold, local sections of these sheaves can be expressed as differences between boundary values of Dolbeault forms on each side of PN (this is done using a Meyer Vietoris sequance argument).

These sheaves reflect nonanalytic behaviour. In particular, analytic Dolbeaut forms on PN are restrictions of Dolbeaut forms on the ambient complex manifold.

The cohomology theory extends to forms with values in C.R. vector bundles in the usual way. (A C.R. vector bundle, V, is a complex vector

bundle over PN and a $\bar{\partial}_B$ operator that extends the action of the $\bar{\partial}_B$ operator defined above such that $\bar{\partial}_B$: $\Gamma(V) \to \Omega^1{}_B \otimes \Gamma(V)$. We also require $\bar{\partial}_B{}^2 = 0$, and Leibniz):

C.R. vector bundles of interest will be:

(1) the tautological line bundle, which will be defined in §2.2.

(2) For deformation theory we will use the vector bundle $\mathbb{C}TPN/\bar{H}$.

The $\bar{\partial}_B$ operator on $\mathbb{C}TPN/\bar{H}$ is defined by:

$\bar{X} \cdot \bar{\partial}_B Y = [\bar{X}, Y]$ mod $\bar{H}$, where $\bar{X} \in \bar{H}$, and $Y \in \mathbb{C}TPN/\bar{H}$.

Infinitesimal deformations of the C.R. structure are given by elements of $H^1{}_B(PN, \mathbb{C}TPN/\bar{H})$, the cohomology class represents the first order variation of the $\bar{\partial}_B$ operator.

Cartan, Chern, Fefferman theory:

Another important difference between C.R. manifolds and complex manifolds is that the C.R. structure has local invariants associated to it, in much the same way as a conformal structure does. (A complex manifold is locally trivial by definition.)

These have been characterised by Cartan [8], for n = 1, and Chern [10] in general by the use of a generalised Cartan connection whose connection forms take values in the Lie algebra of SU(p+1,q+1).

Fefferman has characterised these invariants as those associated to a conformal structure on a circle bundle over PN, The conformal structure has signature (2p+1,2q+1). These turn out to be equivalent: the Cartan conformal connection of the conformal structure on the circle bundle over PN is also the Chern connection on PN, [7].

As it will be useful to compare the results of §2 to those of Chern, I will give a brief description of the derivation of the connection.

The calculations are done on the total space of a line bundle, $E \to PN$. E is a line subbundle of T^*PN. It is the annihilator of the holomorphic and antiholomorphic tangent spaces, $(H \oplus \bar{H})$. We have a tautological 1-form on E, θ. If θ' is a section of E, then this section determines a conformal factor for the Levi form since it provides a local trivialisation of $[TPN/(H+\bar{H})$. We can write the Levi form with this conformal factor as:

$$d\theta' = ih_{i\bar{i}}\omega'^{i} \wedge \bar{\omega}^{\bar{i}} \bmod \theta'$$

here ω'^{i} are a basis of H^* and annihilate $\bar{H}$, and $h_{i\bar{i}}$ is the hermitean form on H determined by the Levi form.

On the total space of E we can write $\theta = u^2\theta'$, so that:

$$d\theta = ih_{i\bar{i}}u^2\omega'^{i} \wedge \bar{\omega}'^{\bar{i}} + \theta \wedge (-2du/u + \phi') = ig_{i\bar{i}}\omega^{i} \wedge \bar{\omega}^{\bar{i}} + \theta \wedge \phi \qquad (*)$$

Where I have confused the ω'^{i} with their pullbacks to E. We now assume that the forms ω^{i} have been chosen so that the coefficients $g_{i\bar{i}}$ are fixed constants. Indices will be converted from upper barred ones to lower unbarred ones with $g_{i\bar{i}}$.

The generalised connection will be on the total space of a principal bundle, P, a subbundle of the coframe bundle of E consisting of frames $(\theta, \omega^{i}, \bar{\omega}_{i}, \phi)$ satisfying (*) with the given fixed $g_{i\bar{i}}$.

P is $(p+q+2)^2-1$ dimensional.

Definition: A generalised Cartan connection for P will be a matrix of 1-forms on P, π_{α}^{β}, taking values in the Lie algebra of SU(p+1,q+1).

Let $\mathbf{T}^{\alpha}$ be $\mathbb{C}^{p+q+2}$ together with a Hermitean structure identifying the complex conjugate space with its dual. The structure group of P, G, is the stabilizer of a point $Z^{\alpha} \in \mathbf{T}^{\alpha}$ up to phase, $e^{i\theta}$, where $Z^{\alpha}\bar{Z}_{\alpha} = 0$. The connection form π^{α}_{β} must transform under the adjoint representation of G on the Lie algebra of SU(p+1,q+1) under g^*, where $g:P \to P$ is the action of $g \in G$ on P.

Since the C.R. structure is integrable (θ,ω^i) form a differential ideal. So we can write the structure equations: $d\omega^i = \omega^j \wedge \phi_j{}^i + \phi^i \wedge \theta$ for some forms $\phi_i{}^j$ and ϕ^i.

Taking the exterior derivative of (*) we find that we have the freedom to set:

$$\phi_{i\bar{j}} + \bar{\phi}_{\bar{j}i} - g_{i\bar{j}}\phi = 0$$

and

$$d\phi = i\omega^i \wedge \phi_i + i\phi^i \wedge \bar{\omega}_i + \theta \wedge \psi$$

for some form ψ. We are now left with the following freedom in the forms $\phi_i{}^j$, ϕ^i and ψ:

$$\phi'_i{}^j = \phi_i{}^j + D_i{}^j\theta$$
$$\phi'^i = \phi^i + D_j{}^i\omega^j + E^i\theta$$
$$\psi' = \psi + G\theta + i(\bar{E}_i\omega^i - E^i\bar{\omega}_i)$$

Where G is real and $D_{i\bar{i}} + \bar{D}_{\bar{i}i} = 0$.

In order to eliminate this freedom we must compute the curvature and apply trace conditions on the curvature. There will exist unique choices for D,E and G for which the curvature will satisfy the trace conditions.

To proceed I will first write the connection forms as a matrix in the Lie algebra SU(p+1,q+1). Fix the hermitean form $h_{\alpha\bar{\alpha}}$ on $\mathbb{T}^\alpha \cong \mathbb{C}^{p+q+2}$:

$$h_{\alpha\bar{\alpha}} = \begin{bmatrix} 0 & 0 & -i/2 \\ 0 & g_{i\bar{i}} & 0 \\ i/2 & 0 & 0 \end{bmatrix}$$

Then the above relations imply that $\pi_\beta{}^\alpha$ is in the Lie algebra of SU(p+1,q+1), where $\pi_\beta{}^\alpha$ is given by:

$$\pi_\beta{}^\alpha = \begin{bmatrix} \pi_0{}^0 & \omega^i & 2\theta \\ -i\bar{\phi}_j & \phi_j{}^i + \delta_j{}^i\pi_0{}^0 & 2i\bar{\omega}_j \\ -1/4\ \psi & 1/2\ \phi^i & -\bar{\pi}_0{}^0 \end{bmatrix}$$

where $-(n+2)\pi_0{}^0 = \phi_\alpha{}^\alpha + \phi$.

We can then define the curvature, $\Pi_\beta{}^\alpha$, by:

$$d\pi_\beta{}^\alpha = \pi_\xi{}^\alpha \wedge \pi_\beta{}^\xi + \Pi_\beta{}^\alpha$$

As derived $\Pi_{\alpha}{}^{\alpha} = 0$ automatically. The trace conditions we must apply are on the 2-form part of $\Pi_{\beta}{}^{\alpha}$. This is applied as follows:

For any form Θ, such that:

$$\Theta = a_{i\bar{i}}\omega^{i}\wedge\bar{\omega}^{\bar{i}} + \text{terms quadratic in } \omega^{i} \text{ or } \bar{\omega}^{\bar{i}} \text{ mod } \theta,$$

define $\text{trace}(\Theta) = a_{i}{}^{i}$.

The trace condition on $\Pi_{\beta}{}^{\alpha}$ is then just:

$$\text{trace}(\Pi_{\beta}{}^{\alpha}) = 0$$

This condition then fixes uniquely the remaining freedom in the connection.

This connection is torsion free. In this context the torsion form is definec as the part of the curvature that remains after factoring out by two forms with values in the Lie algebra of G, the fibre group of the principle bundle P. From the definition of the connection we find that $\Pi_{0}{}^{i}$, $\Pi_{0}{}^{n+1}$ and $\Pi_{i}{}^{n+1}$ all vanish. These are the torsion forms.

After playing with the Bianchi identities Webster then finds that $\Pi_{0}{}^{0}$ and $\Pi_{n+1}{}^{n+1}$ also vanish, and the curvature can be written as:

$$\Pi_{\beta}{}^{\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ -i\Phi_{i} & \Phi_{i}{}^{j} & 0 \\ (-1/4)\Psi & (1/2)\Phi^{j} & 0 \end{bmatrix}$$

Written out in terms of irreducibles the curvature forms are:

$$\Phi_{j}{}^{i} = S_{j\,k}{}^{i\,l}\omega^{k}\wedge\bar{\omega}_{l} + V_{j\,k}{}^{i}\omega^{k} - \bar{V}^{i}{}_{j}{}^{k}\bar{\omega}_{k}\wedge\theta$$

$$\Phi^{i} = \bar{V}^{i}{}_{j}{}^{k}\omega^{j}\bar{\omega}_{k} + P_{j}{}^{i}\omega^{j}\wedge\theta + Q^{ji}\bar{\omega}_{j}\wedge\theta$$

$$\Psi = -iP_{j}{}^{l}\omega^{j}\wedge\bar{\omega}_{l} + (R_{i}\omega^{i} + \bar{R}^{\bar{i}}\bar{\omega}_{\bar{i}})\wedge\theta$$

The tensors P,Q,R,S and V are all maximally trace free and symmetric. The S tensor is really the most fundamental piece of the curvature, since all the other parts can be expressed as traces of covariant derivatives of S, using the Bianchi identities.

The Fefferman metric:

Fefferman originally defined the Fefferman metric extrinsically, assuming PN to be embedded in a complex manifold, $\tilde{M}$. The metric was defined as the restriction to $PN \times \mathbb{C}^*$ of the symmetric part of a homogenous Kahler metric on $\tilde{M} \times \mathbb{C}^*$. The Kahler structure is determined by the requirement that the Kahler scalar vanishes on PN and satisfies the Monge-Ampere equations, ie the Kahler connection should be Ricci flat. It should also be homogenous in the $\mathbb{C}^*$ directions.

The metric is degenerate and homogenous along the radial directions in the $\mathbb{C}^*$ and so descends to the corresponding circle bundle as a conformal structure.

Burns Diederich and Schnider, [7], noticed that if one factors out the principle bundle P, constructed by Chern, by the subgroup of G (the fibre group of P) that leaves invariant $Z^\alpha \in T^\alpha$ where $Z^\alpha \bar{Z}_\alpha = 0$, one obtains a circle bundle over E, which descends to PN. Furthermore the following rank two symmetric tensor, f, descends from P to this circle bundle, and if one factors out by E, it descends to the circle bundle over PN up to scale:

$$f = (\pi_0{}^0 - \bar{\pi}_0{}^0)\theta\theta + \omega^i \theta \bar{\omega}_i$$

Webster [27] then showed that this is indeed the Fefferman metric.

Webster also shows that the restriction to PN of the Kahler connection that Fefferman defined on $\mathbb{C}^* \times \tilde{M}$ has the same curvature as the Chern connection.

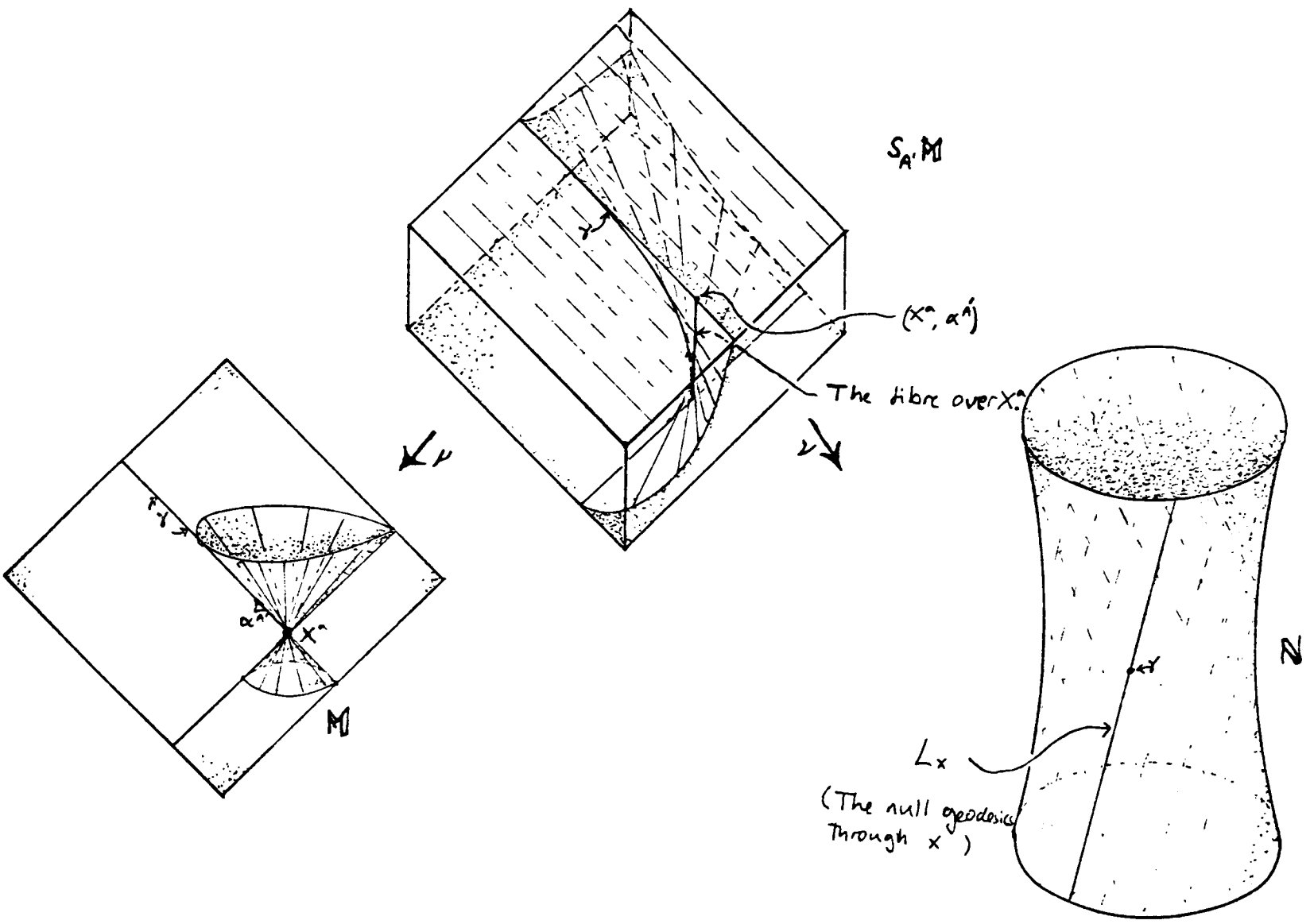
The Chern connection can therefore be thought of as the restriction to a complex line bundle over PN of the Kahler structure defined by Fefferman on $\tilde{M} \times \mathbb{C}^*$.

§1.4: The twistor C.R. structure.

The twistor C.R. structure arises when one wishes to discuss twistor theory in Lorentzian space-times.

In flat space-time, $\mathbb{M}$, the twistor C.R. manifold, $\mathbb{PN}$, arises as the space of α-planes which intersect $\mathbb{M}$. Since α-planes intersect $\mathbb{M}$ in null geodesics, $\mathbb{PN}$ is the space of null geodesics in $\mathbb{M}$.

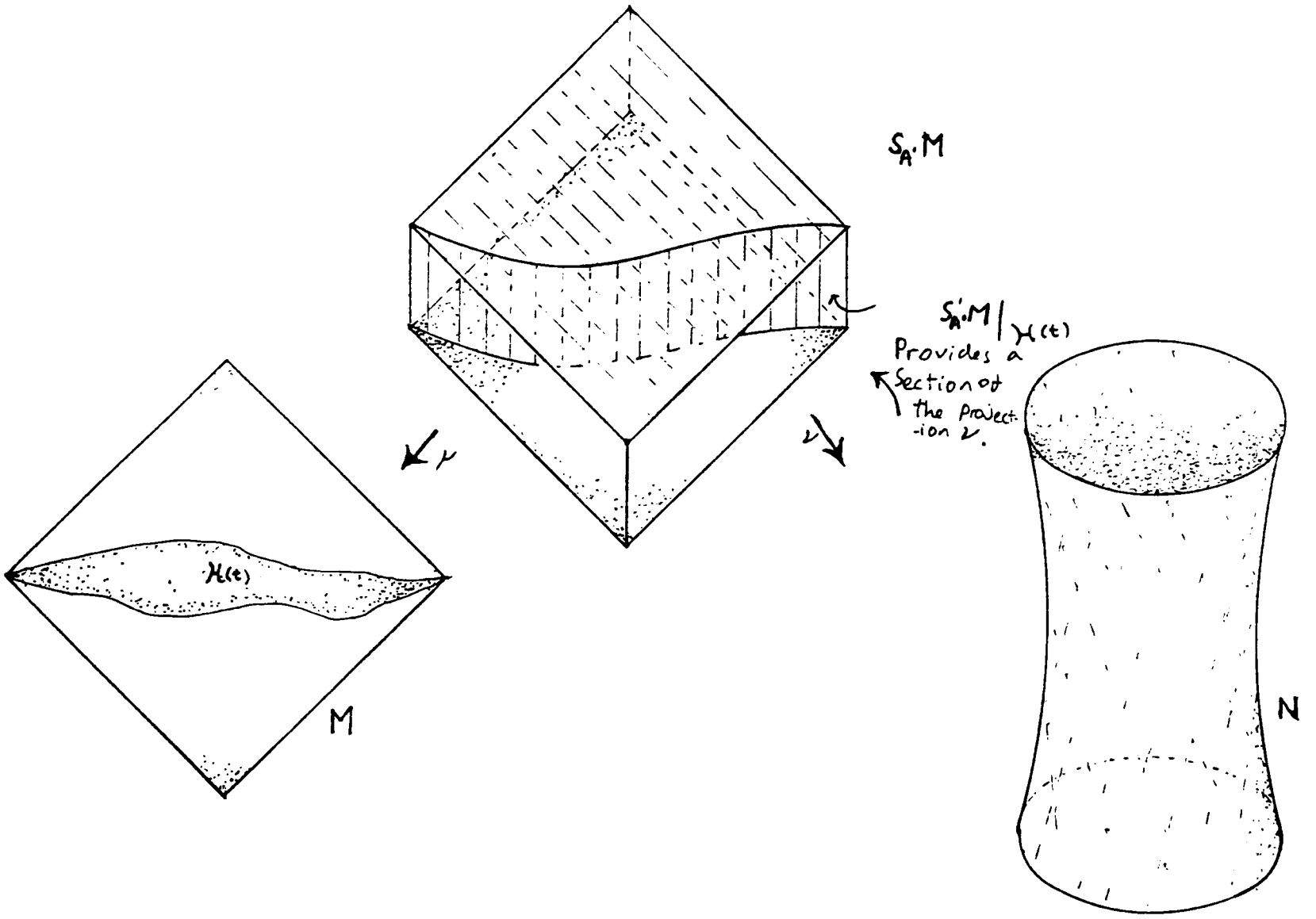
The spin bundle S_A on $\mathbb{M}$ fibres over $\mathbb{PN}$, the fibre being the lift to S_A of the corresponding null geodesic in $\mathbb{M}$ together with its complex scaling. If we consider the space of null geodesics together with their complex scalings, $\mathbb{N}$, then $S_{A'}$ fibres over $\mathbb{N}$ with 1-dimensional fibres.



The C.R. structure on $\mathbb{PN}$ can either be derived by noticing that the twistor distribution D on $S_{A'}$ descends to $\mathbb{PN}$ to provide a C.R. structure, or by noting that $\mathbb{PN}$ is a real codimension one hypersurface in $\mathbb{PT}$, defined by $Z^{\alpha}\bar{Z}_{\alpha} = 0$.

On a conformally curved space-time M, the twistor distribution is no longer integrable. If we foliate M by hypersurfaces, $H(t)$, then $S_{A'}|_{H(t)}$ is a section of the fibration $p: S_{A'} \to N$, where N is the space of scaled complex null geodesics. Denote the intersection of D with the tangent bundle of $S_{A'}|_{H(t)}$ by H_t, then H_t is integrable, and it can be pulled back to N to provide an integrable C.R. structure on N. D can be represented as the distribution made up by H_t together with the null geodesic spray on $S_{A'}$ ie:

$$D = (H_t, V = z^{A'}\bar{z}^{A}\nabla_{AA'}).$$



The nonintegrability of D manifests itself in the fact that $\pounds_V H_t$ no longer lies in D, so that the C.R. structure on N, H_t, evolves with time. The evolution is controled by the self dual part of the Weyl curvature.

The connection between the C.R. structure and null congruences that are shear free at the hypersurface is straightforward. A null congruence at H_t can be specified by a spinor field, $o^{A'}$, on H_t, and is given in N by the vanishing of $z_A o^{A'}$. The corresponding submanifold of N is C.R. if $z_A o^A$ is a C.R. function at its zero set. This gives rise to the equation:

$$z^{A'} z^{B'} T_B{}^A{}_{AA'} (z_C o^{C'})\big|_{(z^{A'}=o^{A'})} = o^{A'} o^{B'} o^C T_{B'}{}^A{}_{AA'} o_C = 0$$

Which is precisely the restriction of the geodesic shear free equations on $o^{A'}$ to the hypersurface.

Basic structures on the space of null geodesics.

The following structures on S_A are conformally invariant and descend from S_A to N:

The Euler vector fields: $Y = z_A \partial/\partial z_A$, and $\bar{Y} = \bar{z}_A \partial/\partial \bar{z}_A$

The canonical 1-form: $\theta = z_A \bar{z}_A dx^{AA}$

The symplectic form: $\omega = d\theta = z_A d\bar{z}_A dx^{AA'} + \bar{z}_A dz_A dx^{AA'}$

On $\mathbb{M}$ these can be written in terms of twistor coordinates:

$$Y = Z^\alpha \partial/\partial Z^\alpha, \quad \theta = i\bar{Z}_\alpha dZ^\alpha, \text{ and } \omega = id\bar{Z}_\alpha dZ^\alpha.$$

These all complexify naturally, and so exist on the space of complex null geodesics as well. They do not depend on the C.R. structure; however they can be derived from it; they are therefore compatible with it. In particular θ annihilates $H \oplus \bar{H}$.

§1.5: Some useful theorems

I will only give very brief outlines of the proofs of the following.

Theorem (C. LeBrun [TN9]): The hypersurface C.R. structure determines and is determined by the conformal initial data on the hypersurface.

Proof: Clearly the C.R. structure is determined by the conformal initial data only, the interesting part of the proof is the converse.

A hypersurface C.R. manifold, PN, is necessarily foliated by holomorphic CP^1's. The factor space is the hypersurface, H, we started with. The horizontal part of any vector in the holomorphic tangent space at a point of PN is a null vector, and as you move up and down the CP^1, the null vector sweeps out the (complex) null cone of the corresponding point of H. So the push down of the holomorphic tangent vectors provides the conformal structure on H. Furthermore the CP^1 corresponding to a point $x \in H$ can be identified with the projective spin space at x, since the null vector corresponding to a point of the CP^1 depends quadratically on homogenous coordinates on the CP^1.

We must then find the trace free part of the extrinsic curvature. To do this we construct the Levi-Civita connection for some metric in the conformal class, $D_{A'B'}$, (I am here using SU(2) spinors). Denote also by $D_{A'B'}$ its horizontal lift to the spin bundle. If the extrinsic curvature vanished, $z^{A'} z^{B'} D_{A'B'}$ would be antiholomorphic. If $K_{ab} \neq 0$, the corresponding antiholomorphic vector is:

$$z^{A'} z^{B'} D_{A'B'} - z^{A'} z^{B'} K_{A'B'C'D'} z^{C'} \partial/\partial z_{D'}$$

So the amount one needs to project $z^{A'} z^{B'} D_{A\ B}$ along $\partial/\partial z_A$ determines the (trace free part of the) extrinsic curvature.□

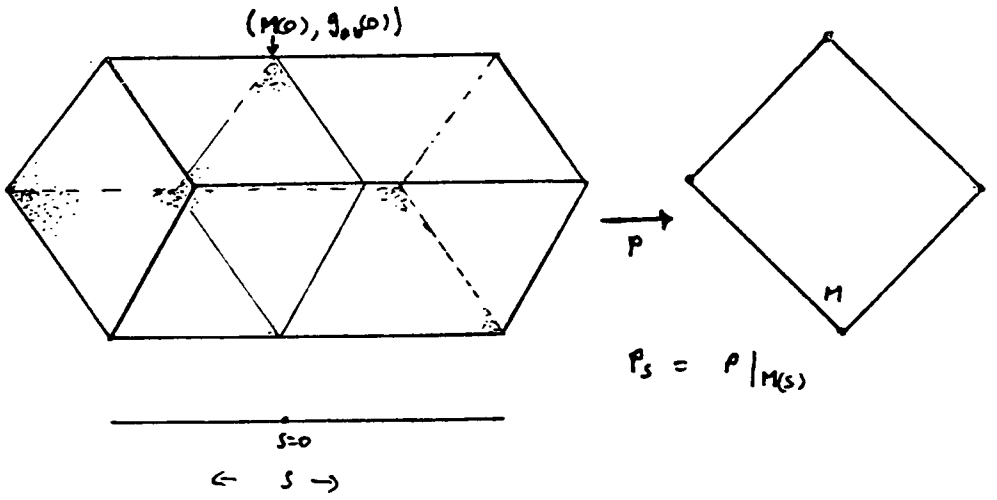
Remark: Not all 5 dimensional C.R. manifolds that are foliated by CP^1s are twistor C.R. manifolds. This follows from the calculations in §2 (see also R. Penrose [ ]). In the above argument it can be seen that it is essential that the holomorphic tangent vector transverse to the CP^1s vary quadratically with the homogenous coordinates on CP^1 in order to obtain a quadratic conformal structure. This is clearly special. In §2.3 the appropriate class of C.R. manifolds will be characterised.

Theorem (C. LeBrun [4]): Let $\mathbb{C}N$ be the space of complex null geodesics in a complex 4-manifold M with holomorphic conformal structure and conformal connection. Then the complex structure on $\mathbb{C}N$ determines M together with its conformal structure and connection. (The connection will in general have conformal torsion). This correspondance is preserved under deformations of the complex structure of $\mathbb{C}N$. The conformal torsion vanishes iff the symplectic potential θ descends to $\mathbb{C}N$.

Proof: The celestial quadrics, ie the set of light rays through points of M are compact complex manifolds, in fact quadrics. Kodaira theory then gives most of the rest of the proof. The condition for the vanishing of the conformal torsion follows from the expression for the Lie drag of θ on the space of null vectors along the null geodesic spray in the presence of torsion. □

§1.6: Perturbations of the metric.

In §3 and §5 I will make use of variations in the metric. When spinors are used in such circumstances a certain amount of care is required. Consider a 1-parameter family of space-times, M(s), with the same background manifold M, all identified with M.



To identify spinors of M(s) with those of M(0) we must identify the null vectors and null two forms on M(s) with those on M(0) (Lie dragging along ∂/∂s will fail to do this). So consider the following map between $T_x M(s)$ and $T_x M(0)$:

$$L_s(V^a) = V^a + s h^a{}_b V^b + O(s^2)$$

The condition that this map takes null vectors to null vectors is:

$$g_{ab}(s) L_s(V^a) L_s(V^b) = g_{ab}(0) V^a V^b + O(s^2) \quad \text{where } V^a \text{ is null.}$$

This implies the trace free part of: $h_{(ab)} = -(1/2)(\partial/\partial s) g_{ab}|_{s=0}$

The skew part of h corresponds to the freedom of performing a SO(1,3) rotation at the same time, and the trace corresponds to the possibility of performing a dilation which will not affect the spinors. It is simplest to set: $h_{ab} = -(1/2)(\partial/\partial s) g_{ab}|_{s=0}$.

Extending L_s to tensors it can be seen that it takes null vectors to null vectors of the varied metric and null two forms to null two forms of the varied metric.

In order to calculate how the various vector fields on the spin bundle

vary we would like to calculate $(\partial/\partial s)[\nabla_{AA'}\alpha^{B'}]$ With the above identification of the spinors. Standard formulae yield:

$$\partial/\partial s(\nabla_a)V^b\Big|_{s=0} = -(2\nabla_{(a}h_{c)}{}^{b} - \nabla^b h_{ac})V^c$$

So: $$(\partial/\partial s)[\nabla_a L_s(V^b)]\Big|_{s=0} = 2(\nabla^{[b}h^{c]}{}_{a})V_c$$

Decomposing this spinorially we find:

$$(\partial/\partial s)[L_s(\nabla_{AA'}\alpha^{B'})]\Big|_{s=0} = h_a{}^c\nabla_c\alpha^{B'} + \{\nabla_C{}^{(C'}h^{B')C}{}_{a}\}\alpha_{C'}$$

So the vector fields ∇_a on the spin bundle vary according to:

$$(\partial/\partial s)\{L_s(\nabla_a)\} = h_a{}^c\nabla_c + \chi_{aB}{}^{C'}z_{C'}\partial/\partial z_B + \bar{\chi}_{aB}{}^{C}\bar{z}_C\partial/\partial\bar{z}_b$$

Where $\bar{\chi}_{aBC} = \nabla_{(B}{}^{C'}h_{C)C'a}$. The variations in the other structures on the spin bundle can be calculated easily from this.

Chapter 2: G-Structures and the C.R. structure on PN.

In this chapter I will be concerned with the theory of G-structures and its application to conformal structures and C.R. structures. In §2.1 the general theory of G-structures is briefly reviewed so as to highlight the geometrical origin of the various ideas. In particular, the example of the local twistor connection (that is the spinor double cover of the Cartan conformal connection) is studied. It is shown that the geometry underlying this is precisely that used by Fefferman and Graham [13] in their discussion of conformal invariants. Also an old result concerning space-times conformal to vacuum is presented.

In §2.2 I will derive various geometrical structures on general C.R. manifolds and use these and the ideas of §2.1 to derive the standard structures on C.R. manifolds, the Fefferman metric and the Chern connection. This presentation simplifies the original proofs to a certain extent. (The presentation is constructed so as to make the calculation of the twistor Chern connection relatively straightforward.) Following S.M. Webster, we will see that the geometry underlying the Chern connection is that used by Fefferman to introduce the Fefferman metric.

In §2.3 I will use the ideas of §2.2 to calculate explicitly the structures on PN. I will also discuss some elementary applications: in particular I will present a characterisation of C.R. manifolds that arise as twistor C.R. manifolds. I shall put off the discussion of the constraint equations and evolution until §4.

In §2.4 I will discuss the C.R. submanifold of PN that arises when the space-time has a geodesic shearfree congruence.

§2.1 G-structures and the Cartan conformal connection:

§2.1.1: G-structures:

Definition: A G-structure is defined as a subbundle, P, of the coframe bundle on M^n with structure group $G \leq GL(n,\mathbb{R})$.

Cartan's theory of G-structures provides an algebraic algorithm for deciding when two manifolds with the G-structures are locally equivalent.

I will consider three examples:

1) A Riemannian manifold, (M,g). Here P is the orthonormal frame bundle, and G = SO(n).

2) A C.R. manifold PN^{2n+1}, of hypersurface type whose Levi form has signature (p,q). Here P is the bundle of coframes of the form $(\theta, \theta^i, \bar{\theta}^{\bar{i}})$ where i takes values from 1 to n, θ annihilates the holomorphic tangent space, and θ^i span the holomorphic forms such that:

$d\theta = \theta^i \wedge \bar{\theta}^{\bar{i}} h_{i\bar{i}} \mod \theta$, for some fixed pseudo hermitean matrix $h_{i\bar{i}}$.

In this case $G = \mathbb{R}^+ \times U(p,q) \times \mathbb{C}^n$.

3) For a manifold M with conformal structure [g] of signature (p,q), P consists of coframes (θ^a) such that $g_{ab}\theta^a\theta^b \in [g]$. Here $G = \mathbb{R}^+ \times SO(p,q)$.

I will first discuss the general case. In the following M^n will be a general manifold with G-structure consisting of a subbundle P of the coframe bundle.

The idea is to reduce the problem of equivalence for general G, to the problem for the trivial G-structure (ie G = Id) on a larger manifold. This will provide a unique coframe of the larger manifold, ie a unique basis of the cotangent bundle at each point. I will first explain how to obtain the

larger manifold with the trivial G-structure and then explain how to solve the equivalence problem and construct generalised connections using the larger manifold with its trivial G-structure.

Reduction to the trivial G-structure.

If G = Id, we are already done. This is just the "Fern-parallelism" of Einstein and Cartan.

If G ≠ Id we can "prolong" by working on P and attempt to find a unique coframe of P. To start with we have the tautological frame forms on P, θ^i, whose value at p ∈ P is just the pullback to P from M of the frame determined by p∈P. (The i index takes on values from 1 to n = dim M.) We then attempt to extend this to a coframe of P. To do this introduce new forms $\theta^i{}_j$ by taking the exterior derivative of the tautological frame forms on P, θ^i:

$$d\theta^i = \theta^i{}_j \wedge \theta^j \qquad (*)$$

This defines the forms $\theta^i{}_j$ up to the equivalence:

$$\theta^i{}_j \to \theta^i{}_j + A^i{}_{jk}\theta^k, \quad A^i{}_{jk} = A^i{}_{(jk)}.$$

We wish to fix the freedom in the definition of $\theta^i{}_j$ such that $(\theta^i, \theta^i{}_j)$ is a coframe of P.

We first restrict the number of the forms $\theta^i{}_j$ by demanding that $\theta^i{}_j$ is in the lie algebra of G. Then the number of forms in $(\theta^i, \theta^i{}_j)$ is the same as dim P. In the case of a Riemannian structure, we can also use "Cartan s lemma" to show that this fixes the remaining freedom in the above definition of $\theta^i{}_j$ so long as (*) still holds. (The condition that (*) holds for the adjusted $\theta^i{}_j$ is the torsion free condition.)

Note that any $(\theta^i, \theta^i{}_j)$ as defined above will be a coframe, since θ^i are nondegenerate on vectors in P transverse to the fibre by definition, and G equivariance of θ^i on P ensures that $\theta^i{}_j$ are

nondegenerate on vertical vectors.

More generally one can demand various conditions on the exterior derivatives of these forms ie conditions on the torsion and curvature. (torsion and curvature will be defined below.) The method of restricting the frames must only depend on G.

If the frame is only determined up to equivalence under some new group G^1 we prolong again to the subbundle P^1 of the bundle of coframes of P determined by this equivalence. One proceeds until some P^r is framed. (This may never happen, as for example with a complex structure, in this case this is because all complex structures are locally isomorphic. There is a general criterion for this, that is that the P^r will be framed for some r iff the lie algebra of G does not contain a rank one element. There are in fact very few cases for which the procedure does stop.)

In our first example, ie a Riemannian structure, we can stop at P, whereas for the other two examples we must go up to P^1. In general if the procedure hasn't stopped by r = 2 it will never stop.

Assume from now on that we have succesefully completed the prolongation procedure.

The problem of equivalence.

Two manifolds M and M' with a G-structures defined by subbundles of their bundles of coframes P and P' respectively are said to be equivalent if there exists a diffeomorphism $f:M \to M'$ which pulls back P' to P. Since the prolongation procedure depends on G alone, f will necessarily induce an equivalence of P^r and P'^r with their trivial G-structures. Conversely an equivalence of P^r and P'^r implies an equivalence of M and M'.

So we need only concern ourselves with the problem of equivalence for trivial G-structures.

Let θ^A be the frame forms on P^r and θ'^A the frame forms on P'^r. Then in order to have an equivalence between P^r and P'^r given by a map f from P^r to P'^r, we must have $f^*\theta'^A = \theta^A$. Equivalently if we pull back the frame forms to $P^r \times P'^r$, then $\theta^A - \theta'^A$ must vanish when restricted to the graph of f. This reduces the equivalence problem to a Frobenius problem on $P^r \times P'^r$, finding integral submanifolds annihilated by $\theta^A - \theta'^A$.

In order to study this Frobenius problem define C^A_{BC} by:

$$d\theta^A = C^A_{BC}\theta^B \wedge \theta^C.$$

There are two extreme cases of interest.

1) The C^A_{BC} are all constant.

2) The C^A_{BC} contain a coordinate system of P^r.

In case 1) it can be seen that the structural equations give P^r the structure of a lie group, L, the frame forms θ^A being Maurer-Cartan forms; the index A denotes membership of some lie algebra l, and the C^A_{BC} are the structure constants. So M is endowed with the structure of a homogenous space, L/H, where H is the stabilizer of a point of M in P^r.

In case 2) the $C^A_{BC} - C'^A_{BC}$ provide the obstruction to integrability, so we must restrict to the submanifold on which $C^A_{BC} - C'^A_{BC}$ vanishes. If $\theta^A - \theta'^A$ vanishes on this submanifold, then this submanifold is the graph of an equivalence between P^r and P'^r. If it doesn't vanish then there is no possible equivalence.

Intermediate cases are handled similarly.

Generalized connections:

If for a given G there exists a canonical example of type 1 (ie one with maximal symmetry), we can go a good deal further in our analysis of the general case.

Since the prolongation procedure depends only on $G \subset GL(n,\mathbb{R})$, not on M

itself, we can regard the frame form as a form taking values in the Lie algebra, l, of L. If, furthermore, l splits naturally as a direct sum $l = h \oplus k$, where h is the Lie algebra of H, and this splitting is appropriately H-equivariant, then the frame form provides a connection on P^r as a principal bundle over M. The horizontal subspaces are just the kernel of the part of the frame form that lies in h. The curvature is defined in the usual way by:

$$\Omega = d\theta + [\theta,\theta].$$

The connection is said to be torsion-free if Ω restricted to k vanishes. It is worth noting that the original tautological forms, θ^i, on P pull back to P^r to become an analogue of what, in the Riemannian case would be solder forms. The index i then refers to θ^i taking values in the Lie algebra k. The torsion free condition can then be stated as $D\theta^i = 0$, where D is the covariant derivative of the connection.

In the case of C.R. structures we have the hyperquadric Q in $\mathbb{CP}^{n+1}$. The above ideas applied to this yield $Q = SU(p+1,q+1)/H$ as a homogenous space, where H is the stabiliser of a point of Q. The Lie algebra of H is $h = \mathbb{R}^+ \oplus u(p,q) \oplus k$, where k is the Lie algebra of the "pseudo" Heisenberg group K, which is, in fact, isomorphic with Q\(point). Furthermore the Lie algebra of SU(p+1,q+1) splits as required for the existence of a connection, $su(p+1,q+1) = h \oplus k$.

For a manifold with conformal structure, the canonical example is that of the projective light cone, M, in $\mathbb{R}^{n+2}$ with a metric of signature (p+1,q+1).

Geometrical interpretation:

When we have such a canonical example of the G-structure, a homogenous space Q, we can regard the procedure for obtaining a connection in the

general case as osculating the manifold with Q using elements of H. To be more precise, take a point $q \in Q$ and $m \in M$, then we will try to construct a map f between a neighbourhood of q in Q and m in M by matching the G-structure to various orders, ie the prolongations. Now clearly the framing on the final prolongation, P^r, will do the trick. That is, there is a canonical first order osculation of P^r by L, which provides an $(r+1)^{th}$ osculation of M by Q. The connection will then allow us to continue this osculation along a curve, but the curvature will then provide the obstruction to extending this to a 2-surface. If it vanishes we can extend this map everywhere.

This notion can be used to pass over local structures defined to low order about a point of Q to M, and also distinguished curves, ie null geodesics in the case of a conformal strucure or, in the C.R. case, chains.

Canonical structures.

We now come to the major points of this somewhat extensive digression.

The first is that one can, at least in our two examples, avoid the complications, and resulting obscurity of the prolongation procedure by noticing that P^r is a subbundle of the bundle of frames of the total space of a canonically defined line bundle. On the total spaces of these line bundles the derivation of the connections has a somewhat more transparent geometrical interpretation. These line bundles will be described below, the case of a C.R. manifold will be put off until §2.2.

The second point is that one has a canonical element of the associated L/H bundle at every point of M. This can be defined as the element of the associated L/H bundle that is invariant under the action of the fibre group at $m \in M$. This element will be useful when we try to understand the structure of the connection. It can be thought of as the element of the

homogenous space L/H about which we are osculating L/H to M.

§2.1.2: The Cartan conformal connection:

As an example I will consider the Cartan conformal connection on an n dimensional manifold, M, with conformal structure of signature (p,q). The standard presentation of this connection can be found in [25] or [7]; the presentation in these references is very algebraic and not very transparent so I hope the reader will bear with me if it remains undefined as yet.

The algorithms described above will lead to an SO(p+1,q+1) connection on some principal bundle over M. The fibre is the stabilizer of a point of $\mathbb{M}$ under the action of the conformal group, where $\mathbb{M}$ is the canonical example which is thought of as the projective light cone of a metric of signature (p+1,q+1) in $\mathbb{R}^{p+q+2}$.

As mentioned above, the final prolongation can be thought of as a subbundle of the frame bundle of the total space of a line bundle over M.

Canonical structures:

The construction is as follows. The conformal structure [g] provides a line subbundle, of $\odot^2 T^*M$ over M (which I will assume is oriented). Denote its square root by $p:E \to M$.

On the total space of E we have two structures of interest, firstly the Euler vector field Y up the fibres and also a degenerate metric, g, which is the tautological element of $\pi^* E^2$ that exists since E is pulled back to itself.

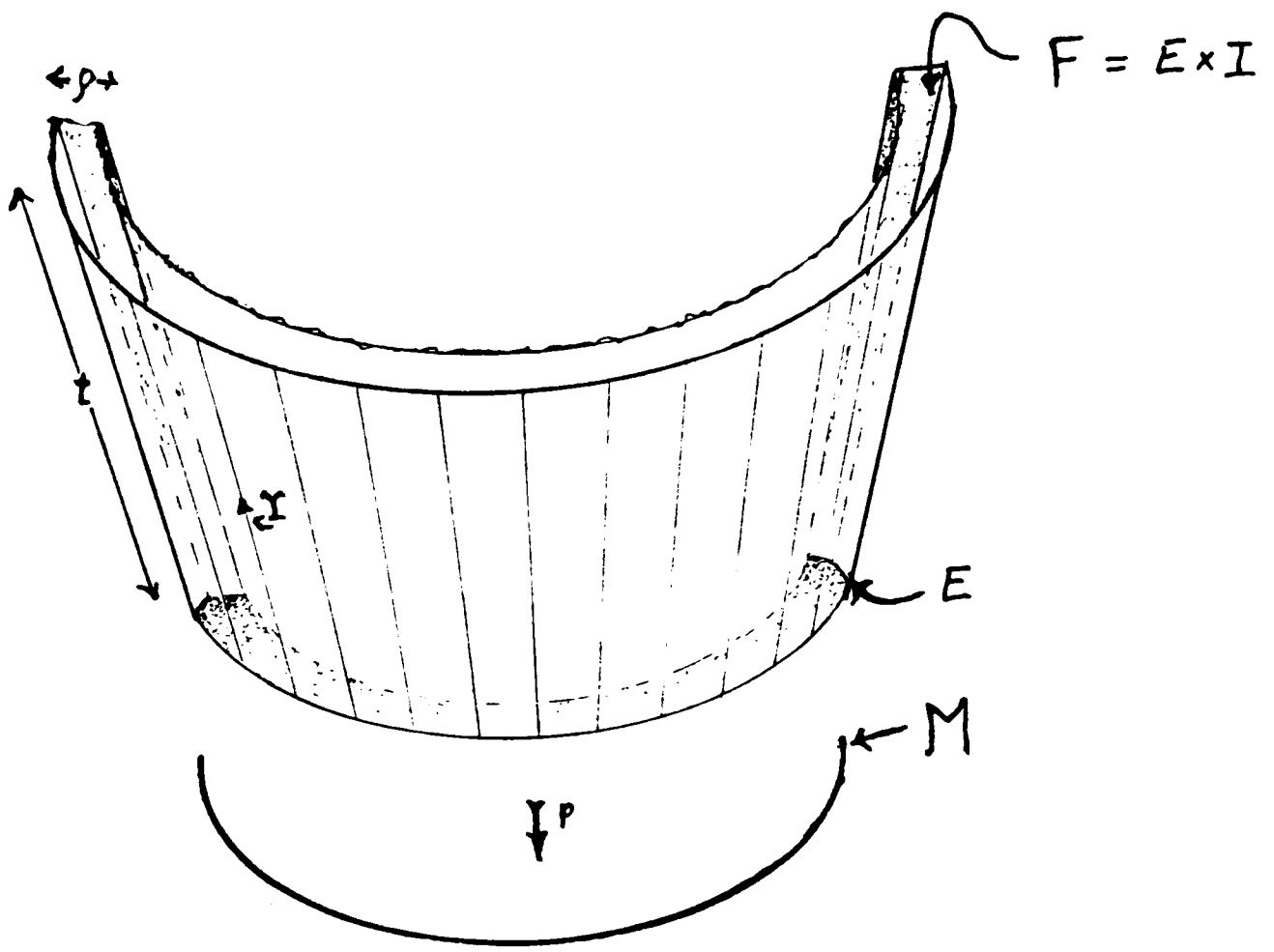
Note that $g(Y,) = 0$, and $\pounds_Y g = 2g$. We will see that E has the structure of a null hypersurface, and the vector field Y defines an affine parameter up the generators. Also Y is conformal Killing. If M were conformally flat, E would be the light cone of a metric of signature (p+1,q+1) in $\mathbb{R}^{p+q+2}$. In general E has the structure of a "light cone" in the sense that at the zero section of E, g vanishes, and as you go up the fibres g grows with the square of the affine parameter.

Following Fefferman and Graham [13], we can use the structures on E as initial data for a Ricci-flat (p+q+2)-dimensional space-time, F. More formally, we wish to construct a metric, $\tilde{g}$, on a neighbourhood of Ex(0) in ExI, where I is some open interval, satisfying the following conditions:

(1) $\pounds_Y \tilde{g} = 2\tilde{g}$.

(2) $\tilde{g}|_{Ex(0)} = g$.

(3) $\tilde{g}$ is Ricci flat.



Such a $\tilde{g}$ is guaranteed to 2^{nd} order by the algebra proving existence and uniqueness of the Cartan conformal connection. Fefferman and Graham have done somewhat better with the following theorem:

Theorem: a) Dimension of M is odd: Up to a diffeomorphism preserving Y and fixing E, there is a unique formal power series solution $\tilde{g}$

to (1)-(3). If the conformal structure on M is real analytic, then this formal power series converges, so that $\tilde{g}$ exists on a neighbourhood of Ex(0) in ExI.

(b) Dimension of M is even: There exists an obstruction to the existence of a formal power series expansion for $\tilde{g}$. The obstruction, B_{ab}, is a nontrivial conformally invariant trace free symmetric 2-tensor of weight (n-2)/2. B_{ab} involves derivatives of the metric up to n^{th} order, and generalises the Bach tensor when n = 4. $B_{ab}=0$ whenever M is conformal to vacuum. When $B_{ab} = 0$, we have a unique power series expansion for $\tilde{g}$, which converges on a neighbourhood of Ex(0) in ExI when M is analytic.

When $B_{ab} \neq 0$, we only have a formal power series for $\tilde{g}$ defined up to $(n/2)^{th}$ order, so that the Ricci tensor of $\tilde{g}$ vanishes only to $(n-4)/2^{th}$ order, the components tangent to E vanishing to $(n - 2)/2^{th}$ order.□

Sketch of the proof: More detail can be found in [13].

One chooses a metric g(x) in the conformal equivalence class given on M. This gives a trivialisation of E, so that one can introduce a coordinate t "up the fibres" of E. Introduce a coordinate ρ on the interval I.

By making gauge choices on $\tilde{g}$ (ie $\partial/\partial\rho$ is geodesic), and applying some of the field equations we can restrict $\tilde{g}$ to have the form:

$$\tilde{g} = t^2 g_{ab}(x,\rho)dx^a dx^b + 2\rho dt^2 + 2tdtd\rho.$$

Where x^a are some coordinates on M. The field equations then reduce to:

$$\rho g_{ab,\rho\rho} - \rho g^{cd} g_{ac,\rho} g_{bd,\rho} + (1/2)\rho g^{cd} g_{cd,\rho} g_{ab,\rho} + [(2-n)/2]g_{ab,\rho} - (1/2)g^{cd} g_{cd,\rho} g_{ab} + Ric(g)_{ab} = 0$$

One then attempts to produce a power series expansion for the solution of this equation. When n is even log terms can arise, these are proportional to the Bach tensor. These are the only obstruction, if the

Bach tensor vanishes then there is no problem. When n is odd there is no problem.□

The relationship with the Cartan conformal connection.

The restriction to E of the Levi-Civita connection of $\tilde{g}$ on F can be seen to descend to M. This arises as follows.

Let V^i be the vector bundle on M whose fibre, at a point $x \in M$, consists of vector fields v^i (not necessarily tangent to E) defined on the fibre of E over x, that are covariantly constant up the fibre. Since the metric $\tilde{g}$ is homogenous along Y, a short calculation shows that the vector Y when contracted into the curvature yields zero. So parallel transport along a path Γ in M can be defined by parallel transport along any lift of Γ to $\tilde{\Gamma}$ in E such that $\tilde{\Gamma}$ projects to Γ.

One can check explicitly that this connection is the Cartan Conformal connection, however the derivation of the Cartan conformal connection is very messy, see e.g. [25], so I shall omit this calculation.

(In [7], Burns, Diederich and Schnider derive the Cartan conformal connection on the total space of E. With this derivation the fact that the above provides the Cartan conformal connection is transparent. They calculate all the connections that preserve the metric on E, and then observe that the vanishing Ricci curvature condition singles out a unique one. This is the Cartan conformal connection.)

Remark: The above shows that the conformal invariants derived by Fefferman and Graham are precisely those that one can derive from the Cartan conformal connection.

Local twistors.

I shall now restrict to the case of a 4-dimensional space-time, M, with conformal structure of signature (1,3).

On F we have the standard SO(4,2) correspondance between twistors and vectors, that is, a vector is given by a skew 2-index twistor. So denote Y by $X^{\alpha\beta} = X^{[\alpha\beta]}$. The metric on F is $\varepsilon_{\alpha\beta\xi\delta}$ the alternating tensor on $\mathbb{T}^{\alpha}$. Since Y is null $X^{\alpha\beta}$ is simple.

Denote the covariant derivative by D. The solder form is then $DX^{\alpha\beta}$ since if D is the covariant derivative of the connection, since it is torsion free, and the solder form has homogeneity 1 along Y:

$DX^{\alpha\beta} = \pounds_{Y} DX^{\alpha\beta} = Y \lrcorner D(DX^{\alpha\beta}) + D(Y \lrcorner DX^{\alpha\beta}) = D(Y \lrcorner DX^{\alpha\beta})$ as required.

$X^{\alpha\beta}$ is only defined up to scale on M since $(Y \lrcorner D)X^{\alpha\beta} = X^{\alpha\beta}$.

$X^{\alpha\beta}$ is the canonical element referred to above in the section on canonical structures associated to generalised connections. One can think of it as the origin twistor. At the point $x \in M$, $X^{\alpha\beta}$ defines the line in $\mathbb{T}^{\alpha}$ that corresponds to the point of $\mathbb{M}$ about which we are osculating M to $\mathbb{M}$.

I will just recall the standard representation of the local twistor connection in order to prove the last proposition in this section.

As usual we can identify $Z^{\alpha} \in \mathbb{T}^{\alpha}$ such that $X_{\alpha\beta} Z^{\alpha} = 0$ with the primed spin space at x, and $Z^{\alpha} / \{Z^{\alpha} \mid Z^{\alpha} X_{\alpha\beta} = 0\}$ with the unprimed spin space at x. This leads to the standard conformally invariant exact sequence:

$S_{A'} \to \mathbb{T}^{\alpha} \to S^{A}$ on M.

Breaking conformal invariance:

The choice of a conformal factor corresponds to a section of E over M. Consider the form on E that annihilates this section, $I_{\alpha\beta} DX^{\alpha\beta}$. Assume it to be normalized against Y so that $I_{\alpha\beta} X^{\alpha\beta} = 1$. $I_{\alpha\beta}$ is the infinity

twistor. This splits the above exact sequence:

$$\mathbf{T}^{\alpha} = S^{A} \otimes S_{A'}, \text{ ie } Z^{\alpha} = (\omega^{A}, \pi_{A'}).$$

This implies that the origin and infinity twistors have been represented in the form:

$$X_{\alpha\beta} = \begin{bmatrix} \varepsilon^{AB} & 0 \\ 0 & 0 \end{bmatrix}, \quad I_{\alpha\beta} = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon^{A'B'} \end{bmatrix}$$

With respect to this decomposition the differance between D and the standard spin connection on M is given by (d denotes the standard spinor covariant derivative):

$$D = d + \begin{bmatrix} 0 & i dx^{AB'} \\ i P_{A'B} & 0 \end{bmatrix}$$

$P_a = P_{ab} dx^b$, $P_{ab} = \Phi_{ab} - \Lambda g_{ab}$, and $-2\Phi_{ab}$ is the trace free part of the Ricci tensor.

When the space-time M is conformal to vacuum we have the following simplification of the above construction. (This result is in fact known to twistorians. It also appears in a paper of Sasaki's [24]):

Proposition: A 4-dimensional manifold with lorentzian conformal structure is conformal to a space-time with vanishing trace free Ricci tensor, $R_{ab} - (1/4)Rg_{ab}$, iff there exists a covariantly constant skew, twistor, $I_{\alpha\beta}$.

Proof: The proof goes by straightforward calculation:

(a) Using the representation above, $DI_{\alpha\beta} = \begin{bmatrix} 0 & iP_{A}^{\ B'} \\ -iP^{A'}_{\ B} & 0 \end{bmatrix}$

Where $P_a = P_{ab} dx^b$.

This implies that $DI_{\alpha\beta} = 0$ is equivalent to $P_{ab} = 0$, that is, the vacuum equations. If there is a cosmological constant, Λ, $DI_{\alpha\beta} = \frac{1}{6}\Lambda DX_{\alpha\beta}$. So if the space-time has vanishing trace free Ricci tensor, $I_{\alpha\beta} - \frac{\Lambda}{6} X_{\alpha\beta}$ provides such a covariantly constant skew twistor. Conversely, given such a

covariantly constant skew twistor, we can subtract off multiples of $X_{\alpha\beta}$ to make a simple infinity twistor, $I_{\alpha\beta}$. The above expression provides the metric in the conformal class, which will be vacuum by virtue of the expression for $DI_{\alpha\beta}$.□

We can see how the Fefferman-Graham geometry simplifies when $R_{ab} = 0$ as follows:

Let g' be the vacuum metric on M, then we can define the coordinate, t, up the fibre of E by setting $g = t^2 g'$. Then $\Upsilon = t\partial/\partial t$. Let ρ be the coordinate on the interval I. We can then write down the metric $\tilde{g}$ on ExI: $\tilde{g} = 2tdt.d\rho + 2\rho dt.dt + t^2 g'$

It is easily checked that g' vacuum implies that $\tilde{g}$ thus defined is also vacuum, and is therefore the unique Ricci flat extension of g by Feffermans theorem above. $I_{\alpha\beta}$ clearly then corresponds to the covariantly constant vector field $\frac{1}{t}\partial/\partial\rho$.

Remarks:

This result is true in all dimensions.

There appear to be two obvious geometrical interpretations of this result. Firstly $DI_{\alpha\beta} = 0$ implies that the section of E determined by the corresponding conformal factor has no extrinsic curvature. Secondly, osculate the conformal structure of M with that of $\mathbb{M}$, along some curve in M. When M is conformal to vacuum, one can perform this osculation using only Poincare motions, rather than the whole conformal group. The significance of these interpretations is unclear.

This result underlies C. LeBrun's recent characterisation of the Einstein equations on ambitwistor space [5]. His characterisation is effectively the Penrose transform onto ambitwistor space of the above, modulo various technicalities.

It will be seen later that $I_{\alpha\beta}$, in the context of the Chern connection, plays a prominent role in twistorial formulations of the field equations. These formulations appear to be independant of the above.

This formulation is not twistorial in the sense that it is local on M, and we need to know the location of the space-time events. The use of twistors was entirely unnecessary.

§2.2 The Chern connection and Fefferman metric.

The case of a C.R. manifold, PN, is to a certain extent analagous to that of a manifold with conformal structure. Again P^1 is a subbundle of the bundle of coframes over a line bundle, $N \to PN$. This line bundle is, however, complex. N then has the structure of a real null hypersurface in a Kahler manifold with homogeneous Kahler scalar, (except that the restriction of the Kahler metric is not uniquely defined until we start to build the connection). N also has the analogue of the restriction of a holomorphic volume form.

We can then attempt to construct a torsion free connection preserving these structures. The requirement that the connection preserve both the volume form and the "restriction" of the Kahler structure can only be satisfied by one of the Kahler metrics in the equivalence class so far determined. This fixes the freedom in the symmetric part of the Kahler structure, which is then the Fefferman metric.

At this stage there is still one component of the connection to be determined. To obtain a unique connection we must regard N as initial data for a Kahler Einstein manifold and apply the appropriate trace condition on the curvature. This corresponds to the use of the Monge-Ampere equations by Fefferman. The restriction of the corresponding unique SU(p+1,q+1) connection is the Chern connection.

The canonical element is again defined by the Euler vector field up the fibres of the line bundle. It manifests itself in the form of a null element of the representation space, $Z^\alpha \in \mathbb{T}^\alpha = \mathbb{C}^{n+2}$, defined up to scale, on which the SU(p+1,q+1) connection acts. When working on N, Z^α has a well defined scale.

It will be seen that if we restrict attention to just the Fefferman

metric on N, N has the structure of a null hypersurface, and the metric is homogenous of degree two along the real part of the Euler vector field. This vector field is also orthogonal to N. So the Chern connection is just a special Cartan conformal connection determined by the conformal structure on N factored out by the real part of the Euler vector field.

I will now present a derivation of the Chern-Moser connection and Fefferman metric based on the above.

PN will be a general nondegenerate C.R. manifold of dimension 2n+1 of hypersurface type, whose Levi form has signature (p,q). The C.R. structure is given by an integrable distribution, H the holomorphic tangent space, which is a subbundle of the complexified tangent bundle, $\mathbb{C}$TPN.

§2.2.1 The tautological line bundle.

First of all I will introduce a C.R. line bundle $\hat{N} \to PN$ which reduces to the restriction of the tautological O(-1) bundle on $\mathbb{C}P^n$ when PN is embedded in $\mathbb{C}P^n$:

Define: $N^{n+2} = (\Lambda^{n+1}\bar{H}^0)$ where H^0 is the annihilator of H in $\mathbb{C}T^*PN$. (Note that there are global obstructions to taking the $(n+2)^{th}$ root, and that when these vanish there are possible ambiguities. However in the application we are considering N is God given, otherwise we can follow Chern etc. where the whole construction and hence N is factored out by Z_{n+2} thus avoiding these difficulties.)

Lemma: N has the following structures,it has:

a) a C.R. structure.

b) an Euler vector field, Y, (and complex conjugate).

c) a canonically defined real 1-form $\theta \in (H+\bar{H})^0$, where H is the holomorphic tangent space on N. Furthermore $\omega = d\theta$ is nondegenerate except

in the Y-$\bar{Y}$ direction. θ and ω are homogenous of degree (1,1) along the Euler vector fields.

d) a canonically defined n+1-form, ν, orthogonal to $\bar{H}$, such that $d\nu$ is also orthogonal to $\bar{H}$.

furthermore:

e) N is isomorphic to the restriction of $O(-1)$ on $\mathbb{C}P^n$ when PN is embedded in $\mathbb{C}P^n$, when more generally embedded it is the $(n+2)^{th}$ root of the canonical bundle on the ambient manifold.

f) Intrinsically it is the C.R. line bundle obtained from the Chern connection if one factors out the principal bundle by the subgroup of $SU(p+1,q+1)$ that leaves invariant a null point $Z^\alpha \in \mathbb{C}^{p+q+2}$.

Note: e) proves a conjecture of G.A.J. Sparling that the section of the first local cohomology defined by the line bundle N is the first obstruction to embeddability of PN since if N is the restriction of a line bundle on the ambient space, then the local cohomology class of N must vanish.

Proof of lemma: a) Let ν be a section of $\Lambda^{n+1}\bar{H}^0$, then ν is holomorphic iff $d\nu = 0$, just as for the canonical bundle on a complex manifold. This then implies e), since $\Lambda^{n+1}\bar{H}^0$ will indeed be the restriction of the canonical bundle on an ambient complex manifold when PN is embedded.

b) This follows from the fact that N is a line bundle.

c) On PN we have a line subbundle of T^*PN, the annihilator of $H+\bar{H}$. I will first show that $N \oslash \bar{N} = (H+\bar{H})^0$.

Now $((H+\bar{H})^0)^n = (\Lambda^{2n}(H+\bar{H}))^{-1}$ since if θ is a section of $(H+\bar{H})^0$ then $(d\theta)^n$ is dual to $\Lambda^{2n}(H+\bar{H})$ and non zero by virtue of nondegeneracy of the C.R. structure. This correspondence can be seen to be linear over $C^\infty(PN)$,

as is the following:

$$\{(H+\bar{H})^\circ\}^{n+1} = \Lambda^{2n+1}PN \text{ where } \theta \to \theta \hat{} (d\theta)^n.$$

We need a further identity which follows from $T^*PN = H^\circ + H^*$:

$$\Lambda^{2n+1}T^*PN = \Lambda^{n+1}H^\circ \otimes \Lambda^n H^* = \Lambda^{n+1}H^\circ \otimes (\Lambda^n H)^{-1}$$

Putting these together we find:

$$\begin{aligned}(N\times\bar{N})^{n+2} &= \Lambda^{n+1}\bar{H}^\circ \otimes \Lambda^{n+1}H^\circ \\ &= (\Lambda^{2n+2}T^*PN)^2 \otimes \Lambda^n\bar{H}\otimes\Lambda^n H \\ &= (\Lambda^{2n+2}T^*PN)^2 \otimes \Lambda^{2n}(H+\bar{H}) = \{(H+\bar{H})^\circ\}^{2n+2}\otimes\{(H+\bar{H})^\circ\}^{-n} \\ &= \{(H+\bar{H})^\circ\}^{n+2}\end{aligned}$$

So $N\otimes\bar{N} = (H+\bar{H})^\circ$. So when we pull back $(H+\bar{H})^\circ$ to $N\otimes\bar{N}$ we have a canonical section as is always the case when one pulls back a bundle to its total space. Call this section θ and pull it back to N.

Since PN is nondegenerate $d\theta$ is nondegenerate on vectors on N which push down to $H+\bar{H}$ in PN. Furthermore, the homogeneity of θ gives:

$$\mathfrak{L}_Y\theta = \theta = \mathfrak{L}_{\bar{Y}}\theta$$

So, since $Y\lrcorner\theta = 0$, $Y\lrcorner d\theta = \theta$. This shows that the only direction $\omega = d\theta$ is degenerate in is $Y-\bar{Y}$.

d) ν is likewise defined as the canonical section of $\Lambda^{n+1}\bar{H}^\circ$ that exists when pulled back to itself. Note that it is also orthogonal to Y, and necessarily C.R., ie $d\nu$ also annihilates $\bar{H}$, from integrability of the C.R. structure on N.

f) We can see this by examining the local dolbeault representative for the C.R. structure on N given by a section of $\Lambda^{n+1}\bar{H}^\circ$ constructed from a Chern frame, $\{\theta,\theta_i,\bar{\theta}^i\}$. Let this section be $\nu = \theta\hat{}\,\theta^1\hat{}\,..\theta^n$. Then from the structural equations we find $d\nu = \pi_0{}^0\hat{}\,\nu$ where $\pi_0{}^0 = (n+2)^{-1}(\phi_\alpha{}^\alpha + \phi)$. So with this trivialisation $\pi_0{}^0$ is the dolbeaut representative of the C.R. structure on N.

[See §1.3 or Chern-Moser for definitions and notation, except that in

Chern-Moser capital roman indices are used instead of greek and $Z^{\alpha} = (1,0,..,0)$].

However $\pi_0{}^0$ is the only part of the connection that acts on Z^{α} which does not necessarily annihilate antiholomorphic vector fields. Note that it is $\bar{\partial}_B$ closed by virtue of the structural equations.□

§2.2.2 The Fefferman metric:

The structures that I have just defined can be used to define the Fefferman metric on N. This proceeds by analogy with the construction of the symmetric part of a Kahler structure when N is embedded in a complex Kahler manifold with homogeneous Kahler scalar k of degree (1,1) that vanishes on N. The Kahler structure must also be Einstein, ie have vanishing trace free Ricci curvature. This corresponds to the Monge-Ampere equations on k as used by Fefferman. This condition is equivalent to the reduction of the holonomy of the connection from U(p+1,q+1) to SU(p+1,q+1), that is, to the existence of a covariantly constant holomorphic (n+2)-form.

When N is embedded as above, $\theta = i\partial k$, and $\omega = i\partial\bar{\partial}k$. The symmetric form f is then defined on the ambient manifold by:

$$f(X,Y) = \omega(X,JY),$$

where J is the complex structure on the ambient manifold and X and Y are arbitrary vector fields.

So on N we can define f as above for X general and Y in $H+\bar{H}$. We can also demand $f(X,Y) = f(Y,X)$. So we are left with just one component undefined, $f(X,X)$, when X points out of the holomorphic tangent space. That is f is determined modulo addition of multiples of $\theta\theta\bar{\theta}$; ie we have the freedom $f \to f + \lambda\theta\theta\bar{\theta}$. We can restrict λ by requiring that f has homogeneity (1,1), so λ must have homogeneity (-1,-1). This last component

is very tricky to specify. There is a unique choice for which one can construct a connection preserving $h = f + i\omega$ and dv that is torsion free.

In summary: given a nondegenerate C.R. manifold PN with Levi form of signature (p,q), we can produce a C.R. manifold N with degenerate Levi form of signature (p,q,0) which is a complex line bundle over PN, with the following canonically defined structures:

1) Euler vector fields, Y, and $\bar{Y}$.

2) A 1-form θ homogenous of degree (1,1) orthogonal to the holomorphic tangent space on N.

3) A 2-form $\omega = d\theta$ homogenous of degree (1,1) which is degenerate only in the direction $Y-\bar{Y}$.

4) A symmetric tensor f of rank 2 which is homogenous of degree (1,1) which is degenerate only in the direction $Y+\bar{Y}$ and, $h = f+i\omega$ is nondegenerate and real (Note that f is so far defined only up to the addition of $\lambda\theta\otimes\theta$).

5) An n+1 form v such that v and dv annhilate antiholomorphic vectors.

§2.2.3: The Chern connection.

Notation and definitions: The vector bundle $\mathbb{C}TN/\bar{H}$, and its dual, $\bar{H}^0$, will be the bundles which the Chern connection "connects", so I will denote the first by $\mathbb{T}^\alpha$ and the second by $\mathbb{T}_\alpha$.

If this definition seems a little obscure, one can think of N as being embedded in a complex manifold. Then $\mathbb{C}TN/\bar{H}$ models holomorphic vector fields at N, and $\bar{H}^0$ is the restriction of holomorphic forms to N.

Note that we have a canonical $\mathbb{T}^\alpha$ valued 1-form, $\theta^\alpha \in \mathbb{T}^\alpha \otimes \bar{H}^0$, which takes a vector into its counterpart in $\mathbb{T}^\alpha$.

Note that a choice of h, as defined above in 4, determines and is determined by a hermitean form $h_{\bar{\alpha}\alpha}$ on $\mathbb{T}^\alpha$, by setting $h = h_{\bar{\alpha}\alpha}\theta^\alpha\bar{\theta}^{\bar{\alpha}}$. Similarly dv determines a totally skew tensor, $\varepsilon_{\alpha\beta\cdots\delta}$, by $dv = \varepsilon_{\alpha\beta\cdots\delta}\theta^\alpha \wedge \theta^\beta \wedge \cdots \theta^\delta$. (It can be shown that any $h_{\alpha\bar{\alpha}}$ in the equivalence class has constant determinant with respect to $\varepsilon_{\alpha\beta\cdots\delta}$.)

Theorem: (Cartan, Tanaka, Chern, Moser, Webster etc. [8,10,26,27])

Let PN be a 2n+1 dimensional nondegenerate C.R. manifold of hypersurface type with Levi form of signature (p,q). On the total space of the tautological line bundle, N, of PN, there exists a unique connection acting on vector fields in $TN/\bar{H} = \mathbb{T}^\alpha$, or dually, on $\bar{H}^0 = \mathbb{T}_\alpha$, with the following properties:

1) The connection is torsion free, ie, if D is the covariant derivative, $D\theta^\alpha = 0$.

2) $D\varepsilon_{\alpha\beta\cdots\delta} = 0$.

3) $Dh_{\bar{\alpha}\alpha} = 0$ for some h in the equivalence class defined above.

It is only possible to satisfy 1-3 for a unique h in the equivalence class defined above. This h is the Fefferman metric. These conditions fix

all but one component of the connection.

4) In order to fix the last component of the connection the curvature must satisfy a trace free condition which will be specified in the proof.

Proof:

The canonical element of $Z^\alpha \in \mathbb{T}^\alpha$ is defined by $Y \in \mathbb{C}TN$.

From now on we will fix a basis of $\mathbb{T}^\alpha$. For the rest of the proof all indices will be concrete.

Fix a hermitean form on $\mathbb{C}^{n+2}$, $h_{\alpha\bar{\alpha}}$ and a compatible volume form, $\varepsilon_{\alpha\beta..\delta}$. Choose a basis of $\mathbb{T}^\alpha$ and let θ^α be the corresponding dual basis of $\bar{H}^0$ such that: $h = h_{\alpha\bar{\alpha}}\theta^\alpha \otimes \bar{\theta}^{\bar{\alpha}}$ and $dv = \varepsilon_{\alpha\beta..\delta}\theta^\alpha \wedge \theta^\beta \wedge ... \theta^\delta$.

where $h = f + i\omega$ and dv are defined above. (The fact that h and dv are compatible in this way follows from a tensor calculation using the isomorphisms in the lemma in §2.2.1.)

Note that since h has homogeneity $(1,1)$, we can choose θ^α of homogeneity $(1,0)$, assume this from now on. Since h is only defined modulo the addition of $\theta\otimes\theta$, our choice of frame is not yet constrained to be $SU(p+1,q+1)$.

Indices will be raised and lowered with $h_{\alpha\bar{\alpha}}$.

Because h is not yet fully determined, we have the following extra freedom in our choice of frame over and above the $SU(p+1,q+1)$ freedom:

$$\theta^\alpha \to \theta^\alpha + \lambda Z^\alpha \bar{Z}_\beta \theta^\beta = \theta'^\alpha. \qquad (*)$$

This induces $h \to h + 2\lambda\theta\otimes\theta$. Since we require h to have homogeneity degree $(1,1)$, λ must have homogeneity degree $(-1,-1)$, ie $Y\lambda = \bar{Y}\lambda = -\lambda$.

Integrability of the C.R. structure implies: $d\theta^\alpha = \pi^\alpha{}_\beta \wedge \theta^\beta$

This equation determines $\pi^\alpha{}_\beta$ up to:

$$\pi^\alpha{}_\beta \to \pi^\alpha{}_\beta + \Gamma_{\gamma\ \beta}^{\ \alpha}\theta^\gamma = \pi'^\alpha{}_\beta \text{ where } \Gamma_{\gamma\ \beta}^{\ \alpha} = \Gamma_{(\gamma\ \beta)}^{\ \alpha} \qquad (**)$$

Note that $\pi^\alpha{}_\beta$ has homogeneity $(0,0)$.

Write $D = d - \pi^{\alpha}{}_{\beta}$ for the covariant derivative acting on $\mathbf{T}^{\alpha}$, and extend in the usual way to act on tensors over $\mathbf{T}^{\alpha}$.

Conditions 1-4 are restrictions on $\pi^{\alpha}{}_{\beta}$. It remains to show that they can be satisfied and that they fix the freedoms (*) and (**) uniquely. Conditions 1-3 can now be stated in the form:

1) $D\theta^{\alpha} = 0$.

2) $D\varepsilon_{\alpha\beta..\delta} = \pi^{\xi}{}_{\xi}\varepsilon_{\alpha\beta..\delta} = 0$, ie $\pi^{\alpha}{}_{\alpha} = 0$.

3) $Dh_{\alpha\bar{\alpha}} = \pi_{\alpha\bar{\alpha}} + \bar{\pi}_{\bar{\alpha}\alpha} = 0$.

Note that condition 1) will be satisfied trivially if we only use (*) and (**) to adjust $\pi^{\alpha}{}_{\beta}$ so that 2) and 3) hold.

That 2) can be implemented using (**) follows from $d^2v = 0$ since this implies that $\pi^{\alpha}{}_{\alpha} dv = 0$. It is straightforward linear algebra to show that $\pi^{\alpha}{}_{\alpha}$ can be set to zero using the freedom (**). This however constrains the (**) freedom to $\Gamma_{\gamma}{}^{\alpha}{}_{\beta}$ such that $\Gamma_{\gamma}{}^{\alpha}{}_{\alpha} = 0$.

We now come to attempting to set $\pi_{\bar{\alpha}\alpha} + \bar{\pi}_{\alpha\bar{\alpha}} = 0$. The condition for being able to use the constrained (**) freedom to do this would follow from

$$d\omega = (\pi_{\bar{\alpha}\alpha} + \bar{\pi}_{\alpha\bar{\alpha}})\,\theta^{\alpha} \wedge \bar{\theta}^{\bar{\alpha}} = 0$$

if θ^{α} and $\bar{\theta}^{\bar{\alpha}}$ were linearly independant; however, $Z^{\alpha}\bar{\theta}_{\alpha} + \bar{Z}_{\alpha}\theta^{\alpha} = 0$. So we have to be careful with our linear algebra.

What we will find is that $d\omega = 0$ will only provide consistency using the unconstrained (**) freedom and we will have to use (*) as well if we are to satisfy conditions 2 and 3 together.

Set $\pi^{\alpha}{}_{\beta} = \pi_{\gamma}{}^{\alpha}{}_{\beta}\theta^{\gamma} + \pi^{\gamma\alpha}{}_{\beta}\bar{\theta}_{\gamma}$, then $d\omega = 0$ implies:

$$(\pi_{\gamma}{}^{\alpha}{}_{\beta} + \bar{\pi}_{\gamma\beta}{}^{\alpha})\theta^{\gamma} \wedge \theta^{\beta} \wedge \bar{\theta}_{\alpha} + \text{complex conjugate} = 0.$$

Restricting this expression to $H \wedge H \wedge TN$ we find that the second term automatically vanishes leaving:

$$\pi_{[\gamma}{}^{\alpha}{}_{\beta]} + \bar{\pi}_{[\gamma\beta]}{}^{\alpha} = 0 \text{ mod } \bar{Z}_{\gamma}, \bar{Z}_{\beta}$$

$$= \bar{Z}_{[\gamma}\mu_{\beta]}{}^{\alpha} \text{ for some functions } \mu_{\beta}{}^{\alpha}.$$

Inserting this back into $d\omega = 0$ we find:

$$\theta \wedge (\mu_\beta{}^\alpha - \bar{\mu}^\alpha{}_\beta)\theta^\beta \wedge \bar{\theta}_\alpha = 0,$$

where I have used $\theta = i\bar{Z}_\alpha\theta^\alpha = -Z^\alpha\bar{\theta}_\alpha$. So:

$$\mu_\beta{}^\alpha - \bar{\mu}^\alpha{}_\beta = 0 \text{ mod } Z^\alpha, \bar{Z}_\beta,$$
$$= (Z^\alpha\bar{V}_\beta - V^\alpha\bar{Z}_\beta) \text{ for some functions } V^\alpha.$$

So we can set: $\pi_\gamma{}^\alpha{}_\beta + \bar{\pi}_{\gamma\beta}{}^\alpha = \bar{Z}_\gamma\mu_\beta{}^\alpha + \phi_{\gamma\beta}{}^\alpha$,

where $\phi_{\gamma\beta}{}^\alpha = \phi_{(\gamma\beta)}{}^\alpha$. This leads to:

$$\pi^\alpha{}_\beta + \bar{\pi}_\beta{}^\alpha = \phi_{\gamma\beta}{}^\alpha\theta^\gamma + \text{complex conjugate} + i\theta(\bar{\mu}^\alpha{}_\beta - \mu_\beta{}^\alpha)$$
$$= (\phi_{\gamma\beta}{}^\alpha + \bar{Z}_\gamma\bar{Z}_\beta V^\alpha)\theta^\gamma + \text{complex conjugate}.$$

Therefore we can use the freedom (**) to set $\pi_{\bar{\alpha}\alpha} + \bar{\pi}_{\alpha\bar{\alpha}} = 0$. However we will not in general be able to do so with the constrained (**) freedom. This follows by examining $\Gamma_\gamma{}^\alpha{}_\alpha$ and where $\Gamma_\gamma{}^\alpha{}_\beta\theta^\gamma$ is the adjustment we would need to make to $\pi^\alpha{}_\beta$ to ensure that $\pi_{\bar{\alpha}\alpha} + \bar{\pi}_{\alpha\bar{\alpha}} = 0$.

Now $\Gamma_\gamma{}^\alpha{}_\beta = -\phi_{\gamma\beta}{}^\alpha - \bar{Z}_\gamma\bar{Z}_\beta V^\alpha$ so $\Gamma_\gamma{}^\alpha{}_\alpha = -\phi_{\gamma\alpha}{}^\alpha - \bar{Z}_\gamma\bar{Z}_\alpha V^\alpha$.

But $\pi^\alpha{}_\alpha = 0$ implies that $\phi_{\gamma\alpha}{}^\alpha + \bar{Z}_\gamma\mu = 0$, where $\mu = \mu_\alpha{}^\alpha$.

So $\Gamma_\gamma{}^\alpha{}_\alpha = \bar{Z}_\gamma(\mu - \bar{Z}_\alpha V^\alpha)$.

So problems arise if $\mu + \bar{Z}_\alpha V^\alpha \neq 0$. Note that $\mu + \bar{Z}_\alpha V^\alpha$ is real and has homogeneity $(-1,-1)$. Now we must examine the variation in this quantity under the variation of θ^α given by (*).

Under the frame change (*) $\pi^\alpha{}_\beta$ transforms as follows:

$$\pi^\alpha{}_\beta \to \pi^\alpha{}_\beta + d\lambda Z^\alpha\bar{Z}_\beta + \lambda DZ^\alpha\bar{Z}_\beta + \lambda Z^\alpha D\bar{Z}_\beta = \pi'^\alpha{}_\beta.$$

Set $d\lambda = \lambda_\alpha\theta^\alpha + \lambda^\alpha\bar{\theta}_\alpha$.

Since λ has homogeneity $(-1,-1)$, $Z^\alpha\lambda_\alpha = -\lambda$ and $\bar{Z}_\alpha\lambda^\alpha = -\lambda$. We obtain, after a small calculation:

$(\mu_\beta{}^\alpha) \to (\mu_\beta{}^\alpha + 2\lambda_\beta Z^\alpha + \lambda\delta_\beta{}^\alpha$ which implies $V^\alpha \to (V^\alpha + 2\bar{\lambda}^\alpha)$, and:

$\mu + \bar{Z}_\alpha V^\alpha \to \mu + \bar{Z}_\alpha V^\alpha - (n+2)2\lambda.$

We see that adjusting the offending term to be zero fixes the (*) freedom. This fixes the symmetric part of h as claimed.

We still need to worry about the remaining (**) freedom. This is given by $\Gamma_{\gamma\ \beta}^{\ \alpha}$ satisfying $\Gamma_{\gamma\ \alpha}^{\ \alpha} = 0$ and $\Gamma_{\gamma\ \beta}^{\ \alpha}\theta^{\gamma} + \bar{\Gamma}^{\gamma}_{\ \beta}{}^{\alpha}\bar{\theta}_{\gamma} = 0$. The second condition forces: $\Gamma_{\gamma\ \beta}^{\ \alpha} = 0 \mod \bar{Z}_{\gamma}$

$$= \bar{Z}_{\gamma}\rho^{\alpha}_{\ \beta} \quad \text{for some } \rho^{\alpha}_{\ \beta}.$$

Symmetry on the lower two indices forces $\rho^{\alpha}_{\ \beta} = \bar{Z}_{\beta}U^{\alpha}$ for some U^{α}. Inserting this back into the condition we find: $\theta(\bar{Z}_{\beta}U^{\alpha} - \bar{U}_{\beta}Z^{\alpha}) = 0$. So we must have $U^{\alpha} = \sigma Z^{\alpha}$ where σ is real. To fix this last component we must put a trace condition on the curvature. The curvature is given by:

$$\Pi^{\alpha}_{\ \beta} = d\pi^{\alpha}_{\ \beta} + \pi^{\alpha}_{\ \gamma}\wedge\pi^{\gamma}_{\ \beta}.$$

We can see that since $\pi^{\alpha}_{\ \alpha} = 0$, $\Pi^{\alpha}_{\ \alpha} = 0$.

$D^2h_{\alpha\bar{\alpha}} = 0$ implies: $\Pi^{\alpha}_{\ \beta} + \bar{\Pi}^{\ \ }_{\beta}{}^{\alpha} = 0.$

Also the torsion free condition implies $D^2\theta^{\alpha} = \Pi^{\alpha}_{\ \beta}\theta^{\beta} = 0$.

Define $\Pi^{\alpha\ \gamma}_{\ \beta\ \delta}$ by: $\Pi^{\alpha}_{\ \beta} = \Pi^{\alpha\ \gamma}_{\ \beta\ \delta}\bar{\theta}_{\gamma}\theta^{\delta}$

then $\Pi^{\alpha\ \gamma}_{\ \beta\ \delta}$ is defined modulo terms of the form $a^{\alpha}_{\ \beta}Z^{\gamma}\bar{Z}_{\delta}$.

Using the above conditions we can restrict $a^{\alpha}_{\ \beta}$ to be trace free and orthogonal to Z. (Then remaining freedom can be fixed, see the 2^{nd} remark below.)

The torsion free condition tells us that Π is symmetric in its upstairs and downstairs indices modulo Z^{α} and $\bar{Z}_{\alpha}$ respectively (as in the calculation of $d\omega$ in terms of $\pi^{\alpha}_{\ \beta}$ above). Eventually we find after some manipulation that: $\Pi^{\alpha\ \gamma}_{\ \beta\ \delta} = \Omega^{\alpha\gamma}_{\beta\delta} + Z^{\alpha}\bar{Z}_{\beta}\Xi^{\gamma}_{\ \delta}$,

where $\Omega^{\alpha\beta}_{\gamma\delta}$ is trace free and symmetric and Ω and Ξ are both real.

So finally we come to fixing the remaining (**) freedom. There exists a unique choice for the remaining component of the connection for which $\Xi^{\alpha}_{\ \beta}$ is trace free. The proof goes by studying the variation of $\Xi^{\alpha}_{\ \alpha}$ under the

remaining (**) freedom; the details are analogous to those fixing the (*) freedom, and so I will omit them.□

Remarks: 1) We can tidy up the notation a little by using the fact that θ^α is homogenous of degree (1,0). Since D is torsion free we have:

$$\pounds_Y\theta^\alpha = D(Y\lrcorner\theta^\alpha) + Y\lrcorner D\theta^\alpha = D(Y\lrcorner\theta^\alpha) = DZ^\alpha$$
$$= \theta^\alpha.$$

So we can build all the canonical structures out of Z^α and D. Note that the torsion free condition now translates to $D^2Z^\alpha = \Pi^\alpha{}_\beta Z^\beta = 0$. So the curvature is orthogonal to Z^α. Furthermore from the fact that $\Pi^\alpha{}_\beta$ has homogeneity degree zero and the Bianchi identities etc it is possible to show that $Y\lrcorner\Pi^\alpha{}_\beta = 0$ so that Ω and Ξ are both orthogonal to Z^α.

2) Fixing ambiguities and the Bianchi identities:

As seen above, given a form, Φ, there is an ambiguity in writing it as a tensor over $\mathbb{T}^\alpha$ arising from the fact that:

$$\bar{Z}_\alpha\theta^\alpha + Z^\alpha\bar{\theta}_\alpha = 0.$$

In certain cases it is possible to remove this ambiguity. A case of interest is when we have taken the derivative of a tensor, say $I_{\alpha\beta}$. We can write:

$$DI_{\alpha\beta} = I_{\alpha\beta;\gamma}DZ^\gamma + I_{\alpha\beta}{}^{;\bar{\gamma}}D\bar{Z}_{\bar{\gamma}}$$

Where $I_{\alpha\beta;\gamma}$ is defined mod $a_{\alpha\beta}\bar{Z}_\gamma$, and $I_{\alpha\beta}{}^{;\bar{\gamma}}$ mod $a_{\alpha\beta}Z^\gamma$. This ambiguity can be removed by imagining that we are extending $I_{\alpha\beta}$ off N into some ambient complex manifold by demanding that it satisfy $\Box I_{\alpha\beta} = 0$, where $\Box$ is the Kahler Laplacian. This is equivalent to requiring:

$$I_{\alpha\beta;\bar{\gamma}}{}^{\bar{\gamma}} = 0$$

There is no obstruction to this as $I_{\alpha\beta;\bar{\gamma}}{}^{\bar{\gamma}} - I_{\alpha\beta}{}^{;\bar{\gamma}}{}_{\bar{\gamma}} = 0$ as a result of the trace free condition on the two form part of the curvature. One can put a similar condition on the curvature tensor, $\Pi^\alpha{}_\beta{}^\gamma{}_\delta$ to remove the ambiguities in its definition. The condition is:

$$\Pi^{\alpha\ \gamma\ ;\delta}_{\beta\ \delta} = 0.$$

There is no obstruction to this since the Bianchi identity gives:

$$\Pi^{\alpha\ [\gamma\ ;\epsilon]}_{\beta\ \ \delta} DZ^{\delta} \wedge D\bar{Z}_{\gamma} \wedge D\bar{Z}_{\epsilon} = 0.$$

The trace of this is the obstruction.

3) Comparison with Chern-Moser:

First we must show that the above connection descends to PN. As with the Cartan conformal connection this follows from the fact that the Euler vector field kills the curvature. At a point $p \in PN$, $\mathbf{T}^{\alpha}$ is defined by the sections of $\mathbf{T}^{\alpha}$ on the fibre of N over p that are covariantly constant. The connection descends to PN in the obvious fashion.

Chern actually constructs the connection on $N \oplus \bar{N}$.

The comparison is made by decomposing the curvature into irreducible pieces under the action of the stabilizer of Z^{α}. This is slightly messy since Z^{α} is null. We have the exact sequences:

$$0 \to \{(\bar{Z}_{\alpha})^{0}\} \to \mathbf{T}^{\alpha} \to \mathbf{C} \to 0 \quad \text{and} \quad 0 \to \{Z^{\alpha}\} \to \{(\bar{Z}_{\alpha})^{0}\} \to H \to 0$$

The last map in the first sequence being contraction with $\bar{Z}^{\alpha}$, and H is just the quotient of $\{(\bar{Z}_{\alpha})^{0}\}$ by Z^{α}. Note that $\{(\bar{Z}_{\alpha})^{0}\} \cong H$ where H is the holomorphic tangent space on PN.

This decomposition can be articulated less covariantly but more coveniently by choosing a splitting of the above sequences by choosing a vector X^{α} such that $X^{\alpha}\bar{Z}_{\alpha} = 1$, $X^{\alpha}\bar{X}_{\alpha} = 0$. Then choose a basis of $\mathbf{T}^{\alpha}$ such that $Z^{\alpha} = (1,0,..,0)$ and $X^{\alpha} = (0,..,1)$, and let P^{i}_{α} be the projection onto the orthogonal complement of the span of X^{α} and Z^{α}. The lower case Roman indices here correspond to the greek indices in Chern. The Chern Hermitean form is then just: $P^{\alpha}_{i} P_{\bar{i}\alpha}$.

And conversely: $h_{\bar{\alpha}\alpha} = \bar{X}_{\alpha} Z_{\bar{\alpha}} + \bar{Z}_{\alpha} X_{\bar{\alpha}} + \bar{P}_{\alpha}^{\ i} P_{\bar{\alpha} i}$.

A Chern frame is given by: $\{\omega, \omega^{i}, \phi\} = \{i\bar{Z}_{\alpha}\theta^{\alpha},\ P^{i}_{\ \alpha}\theta^{\alpha},\ \bar{X}_{\alpha}\theta^{\alpha} + X^{\alpha}\bar{\theta}_{\alpha}\}$.

Further $\bar{X}_\alpha \theta^\alpha = (n+2)^{-1}(\phi^i{}_i + \phi)$. It is now straightforward to compare the various components of the connection. (See §1.3 for a detailed description of all these Chern quantities.)

The curvature forms compare as follows:

$\Phi^i{}_j = P^i{}_\alpha \bar{P}^\beta{}_j \Pi^\alpha{}_\beta$, $\Phi_i = P^\beta{}_i \bar{X}_\alpha \Pi^\alpha{}_\beta$, and $\Psi = X^\beta \bar{X}_\alpha \Pi^\alpha{}_\beta$.

Then we have: $S^{ik}_{jl} = \mathrm{Proj}(\Pi^\alpha{}_\beta{}^\gamma{}_\delta) = \mathrm{Proj}(\Omega^\alpha{}_\beta{}^\gamma{}_\delta)$ where proj denotes projection onto the orthogonal complement of Z^α and X^α. Furthermore:

$V_j{}^i{}_k = \mathrm{Proj}(\bar{X}_\gamma \Omega^\alpha{}_\beta{}^\gamma{}_\delta)$, $P^i{}_j = \mathrm{Proj}(\Xi^\alpha{}_\beta)$, $\mathbb{Q}^{ij} = \mathrm{Proj}(X^\beta X^\delta \Omega^\alpha{}_\beta{}^\gamma{}_\delta)$ and $R^i = \mathrm{Proj}(X^\beta \Xi^\alpha{}_\beta)$.

The trace conditions in Chern can now be seen to be a consequence of the trace conditions on Ω and Ξ and their orthogonality to Z^α.

The trace condition reffered to in the statement of this theorem corresponds to $P^i{}_i = 0$ in Chern-Moser.

The various irreducible pieces of the curvature are more covariantly to be regarded as determining sections of various powers of N tensored with tensor bundles over $\{(\bar{Z}_\alpha)^\circ\}/\{Z^\alpha\}$.

Note that the trace of the Bianchi identity, $\Pi^\alpha{}_\beta{}^\gamma{}_\delta{}^{;\delta} = 0$, when written out in terms of irreducibles, expresses V, Q, P and R as successive divergences of S. This follows from the fact that $DZ^\alpha = \delta^\alpha{}_\beta \theta^\beta$.

4) It should be noted that this approach is distinct from that of Chern in that the curvature trace conditions are mostly implied by the derivation of the connection, only one component of the connection is determined by the trace condition. In Chern there remains $n^2 + 2n + 1$ components of the connection to be determined before the curvature trace conditions are to be applied. The essential difference comes from the use of the C.R. volume form, dv, and the fact that we are working on the total space of a complex line bundle over PN instead of a real one.

It is unfortunate that this presentation is still very messy, nevertheless it is an order of magnitude less so than Chern's original one.

5) When working on the complex line bundle the analogy with the restriction of a homogenous Kahler connection which preserves a holomorphic volume form becomes precise. Indeed the calculation of the connection in such a case is almost identical to that above except one doesnt have the irritations caused by $Z^{\alpha}\bar{\theta}_{\alpha} + \bar{Z}_{\dot{\alpha}}\theta^{\dot{\alpha}} = 0$.

6) It can be seen that the Chern connection is the Cartan conformal connection of the Fefferman metric when one regards the Fefferman metric as defining a conformal class on the manifold N factored out by $Y + \bar{Y}$. This is analogous to regarding a Kahler connection as the Levi-Civita connection of the symmetric part of the Kahler form.

7) The chains on PN are just the projection to PN from N of the null geodesics of the Kahler structure, or equivalently of the Fefferman metric. It can be seen that they project down cleanly since $Y + \bar{Y}$ is conformal killing and $Y-\bar{Y}$ is killing. So the null geodesics remain null geodesics when Lie dragged along Y.

§2.3: The twistor Chern connection.

Setting up the basic structures:

In order to calculate the twistor Chern connection we must first identify N. If we represent PN as the projective primed spin space restricted to the hypersurface, $PN = PS_{A'}|_H$, we have an obvious candidate, the non projective primed spin space restricted to the hypersurface:

$$N = S_{A'}|_H.$$

To show this we must show that sections of its 4^{th} power can be identified with 3-forms that annhilate the antiholomorphic tangent space. This can be done by using the conformally invariant canonically defined 3-form, $v = z^{A'} dz_{A'} \wedge z_B dz_C d^4x^{B\,C}$. This has homogeneity degree 4, whereas sections of N^4 have homogeneity degree -4 so that v does indeed provide the required correspondence.

Note that dv also annihilates the antiholomorphic tangent space on N.

To proceed we must also identify θ, $\omega = d\theta$, and Υ. These are well known in twistor theory: $\theta = z_A \bar{z}_A dx^{AA'}$,

$\omega = dz_A \wedge \bar{z}_a dx^{AA'} - z_A dx^{AA'} \wedge d\bar{z}_A$, and $\Upsilon = z_A \partial/\partial z_A$.

G.A.J. Sparling has identified the Fefferman metric [25]. I shall proceed with his formula and the fact that no inconsistencies arise will prove it to be correct. With Sparling's formula we find:

$$h = f + i\omega = i dz_A \otimes \bar{z}_A dx^{AA'} - i z_A dx^{AA} \otimes dz_{\bar{A}}.$$

A basis for $\bar{H}^c$ is given by $\theta^\alpha = (iz_A\, dx^{AA}, dz_A)$. This suggests that it will be convenient to represent the connection as acting on the direct sum of the primed and the unprimed spin bundles pulled back to N. This follows by analogy with the local twistor connection when one has chosen a conformal factor. So set $\mathbf{T}^\alpha = S^A \otimes S_{A'}$. Then we immediately have:

$$Z^{\alpha} = (0, z_{A'}) = \Upsilon \rfloor \theta^{\alpha} \text{ since } \Upsilon = z_{A'} \partial/\partial z_{A'}.$$

Also the twistor Hermitean form has the usual representation in terms of spinor primary and secondary parts. Note also that the primary and secondary parts transform appropriately under conformal rescalings.

Calculation of the connection:

We will now try to calculate the difference between the spin connection, d, and the Chern connection, D, when acting on T^{α}. Note that the spin connection is already SU(2,2) so that the difference must also be SU(2,2). So we have:

$$D = d + \begin{bmatrix} \pi^{A}{}_{B} & i\lambda^{AB'} \\ i\pi_{A'B} & \bar{\pi}_{A'}{}^{B'} \end{bmatrix} = d + \pi^{\alpha}{}_{\beta}$$

Where, since $\pi^{\alpha}{}_{\beta}$ is in the lie algebra of SU(2,2), $\pi_{A'B}$ and $\lambda^{AB'}$ are distinct and real (indexed) 1-forms, $\pi^{A}{}_{B}$ and $\bar{\pi}_{A}{}^{B'}$ are complex conjugate adjoints of each other and the sum of their traces vanish.

So far this implies that D preserves $\varepsilon_{\alpha\beta\gamma\delta}$ and the hermitean metric $h_{\alpha\bar{\alpha}}$. Applying the torsion free condition will then fix all but one component of $\pi^{\alpha}{}_{\beta}$.

After a short calculation we find:

$\lambda^{AB'} = dx^{AB'}$, $\pi^{A}{}_{B} = 0 = \bar{\pi}_{A'}{}^{B'}$, and $\pi_{A'B} = \pi_{A'Bc}dx^{c}$ where π_{ab} must satisfy: $\pi_{AA'b}dx^{b} \wedge z_{C'}dx^{AC'} = d^2 z_{A'} = R_{A'B'C'D'}z^{B'}d^2x^{C'D'}$.

After a somewhat longer calculation it is possible to deduce:

$$2\pi_{ab} = P_{ab} + E_{ab} - B_{a}{}^{c}N^{d}T^{e}\varepsilon_{bcde}$$
$$= P_{ab} + E_{ab} + iB_{aDB}N^{D}{}_{B'} - iB_{aD'B'}N^{D'}{}_{B}$$

$P_{ab} = \Phi_{ab} - \Lambda g_{ab}$ is the trace adjusted Ricci tensor, E_{ab} is the electric Weyl curvature, B_{ab} is the magnetic Weyl curvature and:

$$N^a = z^{A'}\bar{z}^A/(z_A.\bar{z}_A T^{AA'}).$$

(Note that only the combination $(P_{ab} + E_{ab})dx^b|_H$ can be constructed out of initial data.)

The vector N^a at $n \in PN$ is the null vector pointing along the null geodesic corresponding to the point $n \in PN$.

(This last calculation can be made a little more transparent by writing $\pi_{ab}dx^b = \eta_{a\beta}\theta^\beta + \eta_a{}^{\dot\beta}\bar\theta_{\dot\beta}$ I will come back to this later.)

We still have the freedom to add terms of the form $\lambda z_{A'}\bar{z}_{B'}\theta$ to $\pi_{A'B'}$. To check that this component is specified correctly, we must show that the curvature satisfies the trace free condition mentioned in the last section.

The curvature decomposes into spinor parts as follows:

$$\Pi^\alpha{}_\beta = \begin{bmatrix} \Pi^A{}_B & 0 \\ i\Pi_{AB} & \bar\Pi_{A'B'} \end{bmatrix} = d^2 + d\pi^\alpha{}_\beta + \pi^\alpha{}_\gamma \wedge \pi^\gamma{}_\beta$$

Where d^2 is understood to be acting on $S_A \oplus S_{A'}$, and the various spinor parts of $\Pi^\alpha{}_\beta$ are given by:

$$\Pi^A{}_B = -2N^{AB'}B_{bc}\theta^c{}_{def}T^d dx^e \wedge dx^f, \qquad \Pi_A{}^{B'} = \bar\Pi^{B'}{}_A, \text{ and:}$$

$$\Pi_a = \nabla_b \pi_{ac} dx^b \wedge dx^c - 2iB_{ab}z^B z_C dx^{BC'} \wedge \bar{z}^C d\bar{z}_{C'} + 2iB_{ab}\bar{z}^B\bar{z}_C dx^{CB'} \wedge z^C dz_{C'}.$$

The trace condition is one on the two form part of the curvature. To define it we must write the curvature as a tensor, $\Pi^\alpha{}_\beta = \Pi^{\alpha\ \gamma\dot\delta}{}_{\beta\ \delta}\theta^\delta \wedge \bar\theta_{\dot\gamma}$, and check that $\Pi^{\alpha\ \gamma\dot\delta}{}_{\beta\ \gamma} = 0$. We therefore have to check that each spinor part of $\Pi^\alpha{}_\beta$, as an indexed two-form, vanishes when traced in this way over its two-form indices. It is straightforward to see that the forms $dx^a \wedge dx^b$ vanish when traced in this fashion so the only parts that provide any difficulty are the second parts of the expression for Π_a. If one writes it as a tensor as described above we find:

$$(\text{second part of } \mathbb{T}_a) = \begin{bmatrix} 2iB_{aBD'}z^{D'}\bar{z}^{C} & 0 \\ 0 & -2iB_{aC'D}\bar{z}^{D}z^{B'} \end{bmatrix}$$

It is now easily seen that this is indeed trace free as is required of the Chern connection.

Remarks:

The twistor hypersurface C.R. manifold has degenerate curvature. It is three times algebraically special along the vector fields $\partial/\partial z_A$, in the sense that any two insertions of this vector field and/or its complex conjugate into the curvature yields zero. In particular the S tensor has only two independant real components ($N^a N^b B_{ab}$ and $N^a \varepsilon^{bcde} N_d T_e \nabla_b \pi_{ac}$), whereas in general, for a five dimensional C.R. manifold, the S tensor has five real components.

This algebraic speciality corresponds to the foliation of PN by holomorphic curves, the spin spaces at each point of the hypersurface. This becomes a foliation by holomorphic two surfaces on N. (For a description of holomorphic curves in Lorentzian C.R. manifolds see R. Bryant [6].)

Characterisation of twistor C.R. manifolds.

The algebraic speciality is stronger than that required for the existence of the curves, which only requires double algebraic speciality. (See also R. Penrose [22] on this point.)

The twistor C.R. manifolds are more special than general five dimensional C.R. manifolds that are foliated by holomorphic CP^1 s. This follows from the fact that the antiholomorphic vector field transverse to the CP^1's can be restricted so as to depend quadratically on holomorphic homogenous coordinates on each CP^1 in the twistor case (see §1.5). This is clearly not possible in general.

The triple algebraic speciality (as defined above) is, in fact, sufficient for a five dimensional C.R. manifold to be a hypersurface twistor C.R. manifold:

Firstly it guarantees the existence of holomorphic curves. Secondly it guarantees that the curvature vanishes when restricted to any given holomorphic curve, so the first order neighbourhood of a curve is equivalent to that in the flat case, which in turn implies the quadratic condition on the holomorphic vectors transverse to the curves. (See §1.5 theorem 1 for a description of this quadratic condition, and its sufficiency for a C.R. manifold to be of the twistor type.) This proves the proposition:

Proposition: A five dimensional C.R. manifold that is foliated by holomorphic CP^1s is a twistor C.R. manifold iff the curvature form vanishes when restricted to each holomorphic CP^1.

Note that this is slightly stronger than triple degeneracy of the S tensor, one is also requiring that the P,Q,R and V tensors, be special. This may, however, be implied by use of the Bianchi identities.

Some more canonical structures:

There are further canonical structures on a twistor C.R. manifold, an origin twistor $X^{\alpha\beta}$ and an infinity twistor $I^{\alpha\beta}$ at each point of N. These are defined by the existence of canonically defined two forms on N:

$$z_A\, z_B\, d^2x^{A'B'} = X_{\alpha\beta} DZ^\alpha \wedge DZ^\beta \quad \text{and} \quad I_{\alpha\beta} DZ^\alpha \wedge DZ^\beta = dz^{A'} \wedge dz_A$$

As usual $X_{\alpha\beta}$ is conformally invariant whereas $I_{\alpha\beta}$ is not. Note that $X^{\alpha\beta}$ defines the holomorphic curves. Expressed as a two form as above, it is the annihilator of them, or, expressed as a bivector:

$$X^{\alpha\beta} \leftrightarrow (\partial/\partial z_{A'}) \wedge (\partial/\partial z^{A'})$$

it defines the distribution generating the holomorphic curves.

(This can be used to understand the theory of holomorphic curves. For the bivector to be purely holomorphic we require $Z^{\alpha}X_{\alpha\beta} = 0$. For simplicity of the bivector we require $X^{\alpha\beta}X_{\alpha\beta} = 0$. Integrability requires:

$$DX_{\alpha\beta} \wedge DZ^{\alpha} \wedge DZ^{\beta} = \alpha \wedge X_{\alpha\beta} DZ^{\alpha} \wedge DZ^{\beta}$$

for some one form α. Derivatives of this equation then provide conditions on the curvature, the first being $X_{\alpha\epsilon}X^{\beta\epsilon}\Pi^{\alpha\ \gamma}_{\ \beta\ \delta}X^{\delta\xi}X_{\gamma\eta} = 0$.)

The triple algebraic speciality condition as stated above becomes:

$$\Pi^{\alpha\ \gamma}_{\ \beta\ \delta}X_{\gamma\xi}X^{\delta\epsilon} = 0$$

This provides algebraic equations on $X^{\alpha\beta}$ which in general will have a unique solution.

This allows us to compare the pullback of the local twistor connection to N with the Chern connection on N. We still have the same expressions for $DX^{\alpha\beta}$ and, as noted above, $DDX^{\alpha\beta} = 0$. However $DI^{\alpha\beta}$ no longer vanishes as a consequence of the vacuum field equations. Indeed:

$$DI_{\alpha\beta} = \begin{bmatrix} 0 & i\pi_{A}^{\ B'} \\ -i\pi_{B}^{\ A'} & 0 \end{bmatrix}$$

So that $DI_{\alpha\beta} = 0$ implies that the hypersurface is embedable in flat space-time. $DI_{\alpha\beta}$ contains the information of all the restriction of the curvature from the ambient space-time.

We have the following essentially obvious proposition.

Proposition: The curvature of the Chern connection vanishes iff the hypersurface together with its initial data is embeddable in conformally flat space-time (modulo topological considerations).

Proof: If the curvature vanishes then PN is the "flat" hyperquadric PM. The holomorphic $\mathbb{CP}^1$'s in PN provide a minkowski space M into which the hypersurface is naturally embedded together with its conformal initial data

by virtue of the correspondance between points of the hypersurface and holomorphic $\mathbb{CP}^1$'s in PN. The converse is trivial.□

The constraint equations:

The subject of the constraint equations and the evolution will be treated in some detail in §4. I will only here derive the formulae that I will use later in the thesis.

The expression for the connection can be put into a somewhat more pleasing form. This will be of use later. Define:

$$\theta^A = i z_{A'} dx^{AA'}, \quad \bar{\theta}^{A'} = -i \bar{z}_A dx^{AA'}$$

Then we can write: $\pi_{ab} dx^b = \eta_{aB} \theta^B + \bar{\eta}_{aB'} \bar{\theta}^{B'}$

where: $$\eta_{A'AB} = u^{-1} (\Xi_{A\ ABC} + T_{AA'} G_{0\circ BC} - (1/3) G_{0\circ} T_{A\ (B}{}^{\circ}{}_{C)A}) \bar{z}^C$$

Here, as usual, $u = (T^{AA'} z_{A'} \bar{z}_A)$. The curvature quantities are as defined in §1.2, except:

$$\Xi_{A'BCD} = T_{A'}{}^{A} \Xi_{ABCD}$$

In particular it is worth noting that:

$$\eta_{A\ \ A}^{\ A} = (T_{A'}{}^{A} G_{0\circ AC} - (1/2) G_{0\circ} T_{A'C}) \bar{z}^C$$

The vanishing of this quantity is the constraint equations. In §4 I will show that this is indeed the appropriate twistorial expression of the constraint equations.

§2.4 Applications to algebraically special space-times.

Since my work on this subject has not yet reached fruition, I will just mention the preliminary results without much proof.-

Let M be a space-time with a geodesic shear free congruence determined by a spinor field $o^{A'}$. Let PS be the factor space of the congruence, ie the space of null geodesics in the congruence. The condition that the congruence is shear free is equivalent to the condition that the part of the twistor distribution on the spin bundle tangent to $z^{A} = o^{A'}$, descends to PS. This gives PS the structure of a C.R. manifold. The C.R. structure on PS is nondegenerate iff the congruence is twisting.

From now on assume PS is nondegenerate as a C.R. manifold, ie assume that the twist of the congruence is everywhere non vanishing.

Structures on PS arising from its C.R. structure:

We have the usual array of structures on PS; the tautological C.R. line bundle $S \to PS$, is given by factoring out $S_{A'}M|_{z \cdot o = 0}$ by the null geodesic spray. Υ, θ and $\omega = d\theta$ all descend to S. The holomorphic volume form, however, doesn't have such a nice space-time representative.

The question naturally arises as to what structures are required on S in order to rebuild M. Clearly it is necessary to impose field equations on M in order to cut down the functional freedom of M to at most functions of three variables.

This is easily answered by identifying the various terms in the standard reductions of the vacuum field equations for algebraically special space-times. The answer is contained in the following proposition:

Proposition: Let M be a space-time satisfying the vacuum field equations and: $o^{A'} o^{B'} \Psi_{A'B'C'D'} = 0$

Then M can be rebuilt from S if on S we have a holomorphic 1-form ι homogenous of degree 2, $d\iota \wedge \iota = 0$.

Proof: The proof is not very direct. It involves fixing a holomorphic section of S over PS such that when ι is pulled back to PS it is closed. Then coordinates $(u,\zeta,\bar{\zeta})$ are chosen such that, (confusing θ and ι with their pullbacks to PS via the holomorphic section of S):

$\iota = d\zeta$, $\theta = du + z d\zeta + \bar{z} d\bar{\zeta}$ where $z \equiv z(u,\zeta,\bar{\zeta})$.

This data together with the coordinate freedom then corresponds to that resulting from the reduction of the field equations as in Debney, Kerr, and Schild [11]. The notation used here, however, follows P. Sommers, [23].

The space-time is then reconstructed as PS$\times\mathbb{R}$, with the metric being given in terms of various derivatives of z.□

The field equations:

Define the vector fields Δ and δ by:

$\Delta \lrcorner \theta = 1$, $\Delta \lrcorner \iota = \Delta \lrcorner \bar{\iota} = 0$, and: $\delta \lrcorner \iota = 1$, $\delta \lrcorner \bar{\iota} = \delta \lrcorner \theta = 0$.

The field equations are then equivalent to:

$$\Delta\delta\bar{\delta}\bar{z} = (\Delta\delta z)(\Delta\bar{\delta}\bar{z})$$

and $$\delta\bar{\delta}\bar{z} - \bar{\delta}\bar{\delta}z = 0$$

If one assumes $o^{A'} \Psi_{A'B'C'D'} = 0$, ie type N, the field equations reduce to: $\delta\Delta\bar{\delta}\bar{z} = 0$, and $\delta\delta\bar{\delta}\bar{z} - \bar{\delta}\bar{\delta}z = 0$.

Ψ_4 is then nonzero if: $\Delta\Delta\delta z \neq 0$.

Remarks:

If one restricts $z^{A'} dz_{A'}$ to $z_{A'} o^{A'} = 0$, it then descends to S, when

$o^{A'}o^{B'}o^{C'}\Psi_{A'B'C'D'} = 0$, and $o^{A'}o^{B'}\bar{o}^{A}\Phi_{A'B'AB} = 0$. This gives ι on S. We will see later in §4 that $\iota = z^{A'}dz_{A'}$ on $S_{A'}M$ plays an essential role in the formulation of the vacuum field equations in terms of hypersurface twistors.

If one wants to incorporate nonzero Ψ_2, then one must include more data on PS.

This leads to some interesting questions:

1) Algebraically special vacuum space-times depend on free functions of two variables, whereas 3-dimensional C.R. manifolds depend on free functions of three variables. Is it possible to charachterise those C.R. 3-manifolds that arise from from vacuum algebraically special space-times, or at least vacuum type N space-times?

2) Can one make a simplifying ansatz on the structure of the C.R. manifold in order to simplify the field equations sufficiently to either solve them or prove, for instance, the existence of an asymptotically simple type N vacuum space-time? A holomorphic construction for solutions generalising those in the nontwisting case would be particularly nice, although possibly unlikely.

3) Can one generalise the result from linearised theory that, based on a given C.R. manifold in PN, there exists solutions to the type N field equations depending on a free holomorphic function of two variables?

In order to attack problem 1, it is necessary to calculate the Cartan (Chern) connection of PS in terms of space-time quantities. I have as yet been unable to do this although it appears a tractable problem. One can calculate it in terms of the z function used above, but the result is sufficiently messy that it is impossible to interpret.

E. Cartan has made an exhaustive study of 3-dimensional C.R. manifolds

[8]. In particular E. Cartan has classified all the homogenous three dimensional C.R. manifolds. (He also derived the three dimensional version of the "Chern" connection.) These provide suitable ansatze for the structure of the C.R. manifold. I have studied the case of the hypersphere in some detail. In particular the Hauser solutions are based on the hypersphere, and one may hope that there are more.

Checking that various ansatze have solutions is a simple application of Cartan-Kahler theory [9], so it should be possible to see how many solutions there are based on the hypersphere.

Unfortunately the ansatz is difficult to insert into the above framework, and I have only been able to do so by making further assumptions. That is essentially that ι has the same form as it does in the flat case (this is satisfied in the case of the Hauser solutions). This yields one complex and one real equation on one real function. However R. Bryant has shown that these equations have only at most one nontrivial solution. This is presumably the Hauser solution.

What is required is a reduction of the field equations into which one can insert the ansatz of the structure of the C.R. manifold independantly of knowledge of ι. This would also give a handle on problem three above.

It would also be nice to relate these considerations to the more general results in §4.

§3 Linearised general relativity.

In this chapter I will be concerned with linearised general relativity in the form of a spin two field with conformal weight zero on Minkowski space. This is the appropriate weight for active linearised gravity rather than an ordinary passive spin two field.

First I will derive the deformation in the complex structure of ambitwistor space, $\mathcal{A}$, the space of complex null geodesics in $\mathbb{M}$, corresponding to a general variation of the metric. This will be given in terms of a hamiltonian in $H^1(\mathcal{A},\mathcal{O}(1,1))$. I will then derive the form of the hamiltonian corresponding to variations in the metric satisfying the linearised field equations. (I will also present hamiltonians that give rise to linearised metrics satisfying the Bach equations and which give the space of charged null geodesics with respect to some electromagnetic field). Evidence is provided that the procedure will generalise to nonlinear theory for the case of the Bach equations. The vacuum equations don't appear so amenable.

These ideas will then be used to give a linearised analysis of the structure and evolution of hypersurface twistor spaces. First I will perform an analysis for flat hypersurfaces using Cech cohomology. Then I will generalise this to extrinsically curved hypersurfaces using Dolbeault cohomology.

(Note that throughout this chapter $\mathcal{A}$ will be confused with the total space of the $\mathcal{O}(-1,-1)$ line bundle over $\mathcal{A}$. Only on the total space of this line bundle can we do any symplectic geometry as ω does not descend to $\mathcal{A}$. Sections of $\mathcal{O}(n,n)$ are good hamiltonians since when pulled back to this line bundles they are just functions. In order to use functions of different homogeneity as hamiltonians we will have to work on a

neighbourhood of $Z^{\alpha}W_{\alpha}$ in $\mathbb{T}\times\mathbb{T}^{*}$. Another notational confusion I will indulge in is that by $\mathcal{A}$ or $\mathbb{T}$ I will in general mean suitable regions thereof. That is let U be some stein neighbourhood in $\mathbb{CM}$, then by $\mathbb{T}$, I will mean $Z^{\alpha} \in \mathbb{T}$ such that the corresponding α-plane intersects U.)

§3.1: Deformations of ambitwistor space.

§3.1.1: Introducing the hamiltonians:

Deformations of ambitwistor space, $\mathcal{A}$, are given by $H^1(\mathcal{A},\Theta)$, where Θ is the sheaf of sections of the tangent bundle. Obstructions to exponentiating such first order deformations are given by $H^2(\mathcal{A},\Theta)$. M.G. Eastwood has shown that $H^2(\mathcal{A},\Theta)$ vanishes so that we do not need to worry about the obstructions. I will however show that the obstructions to the deformations I am considering vanish.

Let $\mathcal{U} = \{U_i\}$ be some Stein cover of $\mathcal{A}$, and let $X_{ij} \in H^1(\mathcal{A},\Theta)$ be a Cech representative of some deformation with respect to this cover. By the second theorem in §1.5, we see that such a deformation will correspond to some variation h_{ab} in the conformal structure on $\mathbb{M}$ and a variation in the conformal connection which will in general include conformal torsion. Since we wish to rule out conformal torsion we must require that the deformation X_{ij} preserves the symplectic potential θ, and therefore the symplectic form, $\omega = d\theta$.

The structure of a symplectic form together with a symplectic potential is locally trivial by Darboux's theorem. So if X_{ij} deforms $(\mathcal{A},\theta,\omega)$ to $(\mathcal{A}',\theta',\omega')$ to first order, we can find a vector field X_i on each U_i of $\mathcal{U}$ which takes θ' to θ and ω' to ω, and so we can adjust X_{ij} by a coboundary, $X_i - X_j$, to ensure $\pounds_{X_{ij}}\theta = 0 = \pounds_{X_{ij}}\omega$. Such X_{ij} will then be the most general deformations of $\mathcal{A}$ which lead to variations in the conformal structure of $\mathbb{M}$ with no variation in the conformal torsion.

The equation $\pounds_{X_{ij}}\omega = 0$ implies that X_{ij} is a hamiltonian vector field:

$X_{ij} = X_{H_{ij}}$ where $X_{H_{ij}} \lrcorner \omega + dH_{ij} = 0$ for some hamiltonian H_{ij}.

$\pounds_{X_{ij}}\theta = 0$ then implies that H_{ij} has homogeneity degree (1,1), so that

$H_{ij} \in H^1(\mathcal{A}, \mathcal{O}(1,1))$.

This follows form the fact that $\theta = \Upsilon \lrcorner \omega$ so that $\pounds_V \theta = 0 = \pounds_V \omega$ implies $[V,\Upsilon] = 0$ so V has homogeneity degree 0. This in turn implies, since ω has homogeneity (1,1), the hamiltonian of V must also have homogeneity (1,1).)

So $H^1(\mathcal{A}, \mathcal{O}(1,1))$ should be isomorphic to first order variations in the conformal structure of $\mathbb{CM}$.

Lemma: We have the following isomorphism:

$H^1(\mathcal{A}, \mathcal{O}(1,1)) \cong \{\text{Trace free } h_{ab}\}/\{\text{Trace free part of } \nabla_{(a}\xi_{b)}\}$.

Proof: We can evaluate H_{ij} on $\mathbb{CM}$ directly as follows. Consider the relative deRham sequence on $PS_A' \otimes PS_A$ relative to the foliation by lifts of null geodesics:

$$0 \to \mathcal{O}_{\mathcal{A}}(1,1) \xrightarrow{p^*} \mathcal{O}_{PS_A' \otimes PS_A}(1,1) \xrightarrow{z^{A'} w^A \nabla_{AA'}} \mathcal{O}_{PS_A' \otimes PS_A}(2,2) \to 0$$

Here p is the projection $p: PS_A' \otimes PS_A \to \mathbb{CM}$, and the second map is differentiation along the null geodesic spray. This leads to a long exact sequence two segments of which are of interest:

1) $.. \to H^0(PS_A' \otimes PS_A, \mathcal{O}(1,1)) \to H^0(PS_A' \otimes PS_A, \mathcal{O}(2,2)) \to H^1(\mathcal{A}, \mathcal{O}(1,1)) \to 0$

2) $.. \to H^1(PS_A' \otimes PS_A, \mathcal{O}(2,2)) \to H^2(\mathcal{A}, \mathcal{O}(1,1)) \to H^2(PS_A' \otimes PS_A, \mathcal{O}(1,1)) \to ..$

The first follows from $H^1(PS_A' \otimes PS_A) = 0$. The first sequence provides the isomorphism: $H^1(\mathcal{A}, \mathcal{O}(1,1)) \cong \{\text{T.f } h_{ab}\}/\{\text{T.f. } \nabla_{(a}\xi_{b)}\}$

Where t.f. in front of a two index tensor means "the trace free part of". So we see that $H^1(\mathcal{A}, \mathcal{O}(1,1))$ consists of variations of the conformal structure modulo those obtained by Lie dragging the existing one along a vector field. This is just as expected from the above argument. Note that h_{ab} has the appropriate conformal weight for a variation in the conformal

structure.□

Remark: The first and last terms in the second sequence vanish, so therefore does the middle one. So $H^2(\mathcal{A},\mathcal{O}(1,1))$ vanishes. These are the obstructions to the deformations we are considering.

It remains to show that the deformation creates the same variation in the conformal structure as that given be the above correspondence. We can do this in much the same way as one does with the linearised nonlinear graviton.

First note that the above correspondance is given by the following procedure:

Take $H_{ij} \in H^1(\mathcal{A},\mathcal{O}(1,1))$. Pull it back to $PS_A \otimes PS_A$. On $PS_A \otimes PS_A$ $H^1(\mathcal{O}(1,1))$ is trivial so that we can split p^*H_{ij}:

$$p^*H_{ij} = H_i - H_j$$

Since p^*H_{ij} is pulled back from $\mathcal{A}$, $z^{A'}\bar{z}^{A}\nabla_{AA'}(p^*H_{ij}) = 0$, so that:

$$z^{A'}w^{A}\nabla_{AA'}H_i = z^{A'}w^{A}\nabla_{AA'}H_j = h$$

Therefore h is global on the spin bundles and homogenous of degree (2,2), so $h = z^{A'}z^{B'}w^{A}w^{B}h_{A'B'AB}$. Note that only the trace free part of h_{ab} is thus determined. The gauge freedom is in $H_i \to H_i + G$, where G is a homogenous degree (1,1) global function on the spin bundles. So $G = z^{A}w^{A}\xi_a$ and under this gauge change $t.f.(h_{ab}) \to t.f.(h_{ab} + \nabla_{(a}\xi_{b)})$, as required.

Performing the deformation.

To calculate the variation in the metric caused by the deformation we assume that the deformation only deforms the tangent spaces of the "celestial quadrics" to first order, not the quadrics themselves. We restrict the deforming vector field to a quadric, where it then splits:

$$X_{H_{ij}} = X_{H_i} - X_{H_j}$$

We then see where it takes the sections of the normal bundle of the quadric which correspond to null vectors at the corresponding point in $\mathbb{M}$, that is those which vanish at some point. This can be articulated as follows.

Lift the deforming vector field to $PS_A, @PS_A$ so that it is Lie dragged along the null geodesic spray. On $PS_A, @PS_A$, $X_{H_{ij}}$ splits:

$$X_{H_{ij}} = X_{H_i} - X_{H_j},$$

Since $VH_i \neq 0$, we must regard the $O(1,1)$ bundle on $PS_A, @PS_A$ as being embedded in $T^*\mathbb{M}$ as $p_a p^a = 0$, and take the hamiltonian vector fields of H_i w.r.t. the ambient symplectic structure. We will only require the derivatives of H_i along $p^2 = 0$, so there is no problem. We can now represent V, the null geodesic spray, by the hamiltonian vector field of p^2.

The variation in the null geodesic spray, V, is then just given by:

$$\begin{aligned} \delta V &= [X_{H_i}, V] = [X_{H_j}, V] \\ &= [X_{p^2}, X_{H_i}] = X_{[p^2, H_i]} = X_{VH_i} = X_h \end{aligned}$$

Here I have used the fact that the commutator of two hamiltonian vector fields is the hamiltonian vector field of the Poisson bracket of the hamiltonians. This, in its turn is equal to the derivative of one of the hamiltonians along the vector field of the other.

So we see that the hamiltonian of the null geodesic spray, $g_{ab}p^a p^b$, varies by $h_{ab}p^a p^b$. So h_{ab} is indeed the variation of the conformal structure.

Remark: The above result goes through identically in curved space

§3.1.2: Field equations.

So far the results are not so interesting as we have not yet tried to solve any field equations.

Proposition: Hamiltonians in $H^1(\mathcal{A},\mathcal{O}(1,1))$ that give rise to variations in the conformal structure conformal to variations in the metric satisfying the linearised vacuum field equations are those of the form:

$$H = (Z^{\alpha} I_{\alpha\beta} \partial/\partial W_{\beta}) \tilde{g}(W) + (W_{\alpha} I^{\alpha\beta} \partial/\partial Z^{\beta}) g(Z)$$

$$= (\pi^{A'} \partial/\partial \tilde{\omega}^{A}) \tilde{g}(W) + (\tilde{\pi}^{A} \partial/\partial \omega^{A}) g(Z)$$

Where $g(Z) \in H^1(\mathbb{PT},\mathcal{O}(2))$, and in the above formulae is assumed to be pulled back to $\mathcal{A}$. Similarly for $\tilde{g}(W)$.

Proof: When the evaluation procedure above is applied to the A.S.D. part of the hamiltonian, it becomes just the ordinary evaluation of $(W_{\alpha} I^{\alpha\beta} \partial/\partial Z^{\beta}) g(Z)$ on $\mathbb{M}$ as a potential for the for the corresponding linearised Weyl spinor as in Ward [1]. The major differance is that because of the extra derivative we find that we are evaluating the second potential, h^{+}_{ab}, for the corresponding Weyl spinor, and further, because of the factor of $I_{\alpha\beta}$, we find it has the appropriate conformal weight for linearised gravity.□

Remark: Although this form of the hamiltonian was originally motivated from Penrose's scattering theory expressions in [19], one can derive it more succintly as follows:

The general linearised A.S.D. vacuum solution corresponds to an infinitesimal deformation of a region of $\mathbb{PT}$ given by:

$$X = (\partial/\partial \omega_{A}) g(Z) \partial/\partial \omega^{A} = I^{\alpha\beta} (\partial/\partial Z^{\alpha}) g(Z) \partial/\partial Z^{\beta}$$

This induces a deformation of $\mathcal{A}$. The total space of the $\mathcal{O}(1,1)$ bundle on $\mathcal{A}$ is the cotangent bundle of $\mathbb{PT}$. The corresponding deforming vector

field on T^*PT is obtained by Lie dragging the 1-forms, ie the Lie lift of X above. So the lift $\tilde{X}$ of X is determined by $\pounds_{\tilde{X}}\theta = 0$. That is:

$$\tilde{X} \lrcorner \omega + d(\tilde{X} \lrcorner \theta) = 0$$

So $\tilde{X}$ is a hamiltonian vector field with hamiltonian $\tilde{X} \lrcorner \theta$ which depends only on X, and equals: $\tilde{X} \lrcorner \theta = (W_\alpha I^{\alpha\beta} \partial/\partial Z^\beta) g(Z)$

On $\mathcal{A}$, since we are working with linearised theory, we can add the S.D. contributions to the A.S.D. contribution to get the general expression.

Generalisations to different fields and equations: I will just present the results without too much proof as the proofs are analagous to those above.

The Bach equations:

The Bach equations on a space-time are:

$$B_{ab} = (\nabla_{(A}{}^{D'} \nabla_{B)}{}^{D'} + \Phi_{AB}{}^{C'D'}) \Psi_{A\ B'C\ D'} = 0$$

B_{ab} is the Bach tensor, and it is automatically trace free, divergence free, conformally invariant, and is equal to the corresponding quantity constructed from Ψ_{ABCD}. The last property implies that it automatically vanishes on a conformally S.D. or A.S.D. backround.

Lemma: Deformations giving rise to variations of the conformal structure on $\mathbb{M}$ satisfying the Bach equations are given by hamiltonians of the form: $H = Z^\alpha \tilde{g}_\alpha(W) + W_\alpha g^\alpha(Z)$

Where $g^\alpha(Z)$ is the pullback from twistor space of an element of $H^1(PT, \mathcal{O}^\alpha(1))$. The S.D. part is similarly defined.

Proof: As in the remark above, clearly $W_\alpha g^\alpha(Z)$ is the hamiltonian for the Lie lift to T^*PT of $g^\alpha \partial/\partial Z^\alpha$ on PT. This is clearly the most general deformation leading to a conformally A.S.D. linearised metric.

To prove the lemma we must therefore show that a linearised metric

satisfying the linearised Bach equations can be split into a conformally S.D. and a conformally A.S.D. part. To do this we can use an exact sequence (see also Baston and Mason [TN20]). It is the generalised deRham sequence on $\mathbb{M}$ obtained by tensoring through the ordinary deRham sequence with the local twistor bundle, (t.f. $\mathbb{T}^{\alpha}{}_{\beta}$), ie twistors $E^{\alpha}{}_{\beta}$ with $E^{\alpha}{}_{\alpha} = 0$. The sequence is:

$$0 \to \mathbb{C}^{\alpha}{}_{\beta} \to \Omega^{0\,\alpha}{}_{\beta} \to \Omega^{1\,\alpha}{}_{\beta} \to \Omega^{2\,\alpha}{}_{\beta} \to \Omega^{3\,\alpha}{}_{\beta} \to \Omega^{4\,\alpha}{}_{\beta} \to 0$$

Where the maps are all given by D, the local twistor covariant exterior derivative (except the first two maps which are just injections).

The sequence is easily seen to be exact by observing that, in a covariantly constant frame, the sequence decouples into 15 copies of the ordinary deRham sequence.

Projecting out primary parts we find:

$$\{c.k.v's\} \to \{\xi_b\} \to \{t.f.h_{ab}\} \to \begin{matrix}\{\Psi_{ABCD}\}\\ \oplus \\ \{\Psi_{A'B'C'D'}\}\end{matrix} \to \{B_{ab}\} \to \{V_a\} \to 0$$

Here c.k.v. stands for conformal killing vector. The first map is just injection. The second is $\xi_a \to$ t.f.$\nabla_{(a}\xi_{b)}$. The third is the map from a variation in the metric to the corresponding variation in the Weyl curvature. The fourth map is:

$$(\Psi_{ABCD}, \Psi_{A'B'C'D'}) \to \nabla_{A'}{}^{C}\nabla_{B'}{}^{D}\Psi_{ABCD} - \nabla_{A}{}^{C'}\nabla_{B}{}^{D'}\Psi_{A'B'C'D}$$

that is, the difference between the Bach tensor constructed from the A.S.D. Weyl spinor, and that from the S.D. Weyl spinor. The last map is just the divergence.

So if the Bach tensor of a variation in the metric, h_{ab}, vanishes, and if Ψ_{ABCD} and $\Psi_{A'B'C'D'}$ are the corresponding variations in the Weyl spinors, $(\Psi_{ABCD}, 0)$ is mapped to 0 by the third map and so it is in the image of the second. So we can find a variation in the metric, h^{+}_{ab} with

Weyl spinors $(\Psi_{ABCD},0)$, and similarly one with Weyl spinors $(0,\Psi_{A'B'C'D'})$, h^-_{ab}. So h_{ab} is gauge equivalent to $h^+_{ab} + h^-_{ab}$.□

Electromagnetism:

The other interesting field equations are those of electromagnetism, (since I am only considering linearised theory, Yang-Mills theory would decouple into several copies of electromagnetism). There are two ways of approaching this, the first is to consider charged null geodesics, and the second is to consider null geodesics with a charged phase (ie an ordinary null geodesic together with a spinor aligned along it, $\pi_{A'}$ parallely propagated according to $\pi^{A'}\pi^{A}(\nabla_{AA'} + iA_{AA'})\pi_{A'} = 0$). The first is realised to first order with the hamiltonian:

$H = f(Z) + \tilde{f}(W)$, where $f(Z)$ is pulled back from $H^1(\mathbb{PT},\mathcal{O})$, similarly for $\tilde{f}(W)$. An interesting feature of this is that if $\tilde{f} = 0$, and we regard the deformation as occurring on $T^*\mathbb{PT}$, then we can exponentiate the deformation in such a way that the deformation shifts the zero section of $T^*\mathbb{PT}$ over $\mathbb{PT}$ by $df(Z)$ on overlaps. This is the Penrose transform of the charged symplectic structure on $T^*\mathbb{M}$, where the zero section of $T^*\mathbb{M}$ is regarded as being shifted by the potential for the electromagnetic field.

The second approach, considering null geodesics with charged phase, is realised with the following hamiltonian: $H = Z\cdot W(f(Z) + \tilde{f}(W))$

This twists the $\mathcal{O}(1,-1)$ bundle on $\mathbb{A}$. Note that although this doesn't deform the underlying complex manifold on $\mathbb{A}$, if we extend the deformation to $\mathbb{PT}\times\mathbb{PT}^*$, it does deform the underlying manifold. The significance of this is unclear.

Generalisation to nonlinear theories: Clearly all the above deformations can be exponentiated when either the S.D. or the A.S.D. parts

of the hamiltonians vanish, as they then reduce to linearised versions of the standard "leg break" nonlinear constructions. However one would like to be able to do better than this.

The Bach equations:

The kunneth formula shows that these hamiltonians are precisely those which restrict to $\mathcal{A}$ from $\mathbb{T}\times\mathbb{T}^*$. The Kunneth formula on $\mathbb{T}\times\mathbb{T}^*$ gives:

$$H^1(\mathbb{T}\times\mathbb{T}^*,\mathcal{O}(1,1)) = H^0(\mathbb{T},\mathcal{O}(1))\otimes H^1(\mathbb{T}^*,\mathcal{O}(1)) \oplus H^1(\mathbb{T},\mathcal{O}(1))\otimes H^0(\mathbb{T}^*,\mathcal{O}(1))$$

It can be seen that the right hand side consists of hamiltonians of the form presented above for the Bach equations. Conversely R. Baston has calculated the obstruction to extending an element of $H^1(\mathcal{A},\mathcal{O}(1,1))$ off $\mathcal{A}$ onto $\mathbb{T}\times\mathbb{T}^*$. The obstruction to 5^{th} order is precisely the Bach tensor. The above argument shows that if this vanishes, then it will extend to $\mathbb{T}\times\mathbb{T}^*$.

However, there are obstructions to exponentiating such an infinitesimal deformation which lie in $H^2(\ ,\mathcal{O})$; if the vector fields used are Hamiltonian we need only look at the obstruction in $H^2(\ ,\mathcal{O}(1,1))$; the map from $H^1(\ ,\mathcal{O}(1,1))$ to $H^2(\ ,\mathcal{O}(1,1))$ is given by Poisson bracket of the Hamiltonians on triple overlaps.

Although these vanish on $\mathcal{A}$, they do not on $\mathbb{T}\times\mathbb{T}^*$. However $H^2(\ ,\mathcal{O}(1,1))$ vanishes on the first five formal neighbourhoods of $\mathcal{A}$ in $\mathbb{T}\times\mathbb{T}^*$. So one may conjecture that in the full theory the existence of the first four formal neighbourhoods of A are guaranteed in general and the existence of the fifth characterises the vanishing of the Bach tensor. It seems that an extension to all orders constrains the space-time to be either S.D. or A.S.D.. However an extension to sixth order seems to imply that the space-time be conformal to Einstein. For more details see Baston and Mason in TN21.

The vacuum equations:

I will now restrict attention to the gravitational case.

To deform $\mathcal{A}$ to higher order we will need to know how to deform one curved space of null geodesics correponding to a manifold M conformal to vacuum to another, infinitesimally. A variation, h_{ab}, in the metric on a curved manifold takes an Einstein metric (ie one with vanishing trace free Ricci curvature) to another if the variation in Φ_{ab} is zero.

$$\delta\Phi_{ab} = \nabla_{(A|}{}^{C'}\nabla_{(A'}{}^{C}h_{B')|B)CC'} + \nabla_{(A'|}{}^{C}\nabla_{(A}{}^{C'}h_{B)|B')CC'} = 0$$

We can express this as a condition on $h = z^{A'}z^{B'}w^{A}w^{B}h_{A'B'AB}$ by introducing the operators:

$$X = z^{A'}\nabla_{AA'}(\partial/\partial w_A), \text{ and } \tilde{X} = w^{A}\nabla_{AA'}(\partial/\partial z_{A'}).$$

Then the above equation becomes: $(X\tilde{X} + \tilde{X}X)h = 0$.

This will only yield a condition on the corresponding hamiltonian, H, if $[V, X\tilde{X} + \tilde{X}X] = 0$, when commuted as operators. (V is as usual the null geodesic spray.) However $[V,X] = z^{A'}z^{B'}\tilde{R}_{A'B'}$, and $[V, X\tilde{X} + \tilde{X}X]$ gives rise to several terms involving curvature and derivatives of curvature.

So there is little hope of being able to identify the appropriate hamiltonians in the curved case with just a local condition on H on $\mathcal{A}$.

(Note that in the flat case the operator X descend to $\mathcal{A}$ to become:

$X = -i(2+\tilde{Y})\tilde{\pi}^{A}\partial/\partial\omega^{\hat{A}} = -i(2+\tilde{Y})Z^{\alpha}I_{\alpha\beta}\partial/\partial W_{\beta}$)

Essentially the problem is that the major feature which distinguishes the hamiltonians which solve the field equations from the rest in linearised theory is the existence of the twistor and dual twistor polarizations of $\mathcal{A}$. These just do not survive on a space of null geodesics that corresponds to a conformally curved space-time. However, they do survive if we are allowed to choose a space-like hypersurface. Much of the rest of this thesis will be concerned with applying the above ideas to

hypersurface twistors.

Encoding the conformal factor:

There are two gaps in the above discussion. Firstly we would like to characterise the hamiltonians which solve the field equations geometrically, and secondly we would like to know how to encode the conformal factor.

In the nonlinear graviton construction these two gaps are filled simultaneously with the introduction of the form $\iota = \pi^{A'} d\pi_{A'} = I_{\alpha\beta} dZ^{\alpha} \wedge dZ^{\beta}$, and the Poisson bivector, $I^{\alpha\beta} \partial/\partial Z^{\alpha} \wedge \partial/\partial Z^{\beta}$. They simultaneously provide the conformal factor and determine which are allowable deformations, the allowable ones being those which preserve the two structures. If we pull the two structures back to $\mathbb{T}\times\mathbb{T}^*$, we can convert the Poisson bivector into a two form using the symplectic structure. This two form is $I^{\alpha\beta} dW_{\alpha} \wedge dW_{\beta} = d\tilde{\iota}$, where $\tilde{\iota} = I^{\alpha\beta} W_{\alpha} \wedge dW_{\beta}$.

To find how the conformal factors vary under the deformation, we can Lie drag the two forms along the deforming vector field. We have the following result:

Lemma: If X is the deforming vector field corresponding to a variation in the metric that satisfies the linearised vacuum equations, then:

$\pounds_X I_{\alpha\beta} dZ^{\alpha} \wedge dZ^{\beta} = \pounds_X \iota = 0 \mod dW_{\alpha}$ and

$\pounds_X I^{\alpha\beta} dW_{\alpha} \wedge dW_{\beta} = \pounds_X \tilde{\iota} = 0 \mod dZ^{\alpha}$.

Proof: Straightforward calculation.□

The converse is probably true but I have not proved it.

So to first order we still have $\iota \mod dW_{\alpha}$, and $\tilde{\iota} \mod dZ^{\alpha}$. These will provide the conformal factors for the infinitesimally deformed space-time.

Of interest also is the actual variation in ι and $\tilde{\iota}$. These are given

by: $\mathfrak{L}_{\chi} \imath = (\mathfrak{L}_{\xi})^2 dg(W)$ where $\tilde{\xi} = Z^{\alpha} I_{\alpha\beta} \partial/\partial W_{\beta}$. Similarly for $\tilde{\imath}$.

Further remarks: The variation of $\tilde{\imath}$ can be thought of as a 1-form valued cohomology class on $\mathcal{A}$ in $H^1(\mathcal{A}, \Omega^1(2,0))$ encoding the information of the self dual part of the field. Similarly the variation of $\tilde{\imath}$ encodes the informaion of the A.S.D. part of the field.

Clearly $\imath$ and $\tilde{\imath}$ do not survive for finite deformations unless the deformation is either entirely S.D. or A.S.D..

To first order, when the deformation is purely A.S.D., $\imath$ is entirely preserved, but $\tilde{\imath}$ picks up the one form valued cohomology part. This is reflected in the fully nonlinear case; let $P\mathcal{T}$ be a deformed twistor space; then $w^A dw_A$ and $dw^A \wedge dw_A$ descend from the spin bundles to $T^*P\mathcal{T}$ modulo forms pulled back from $P\mathcal{T}$ (when the trace free Ricci tensor vanishes). On $T^*P\mathcal{T}$, $dw^A \wedge dw_A$ is closed modulo forms pulled back from $P\mathcal{T}$. When $dw^A \wedge dw_A$ is converted to a bivector using the symplectic structure on $T^*P\mathcal{T}$, it is well defined as a bivector modulo vectors up the fibres of $T^*P\mathcal{T}$, and pushes down to $P\mathcal{T}$ to become the familiar Poisson bivector.

We will see that $\imath$ and $\tilde{\imath}$ do survive on the spin bundles and hypersurface twistor spaces. They are then neither holomorphic forms, elements of $H^1(\mathcal{A}, \Omega^1(2,0))$, nor, $H^1(\mathcal{A}, \Omega^1(0,2))$ respectively. We will see, however, that it is still possible to articulate the field equations and even determine the evolution of the hypersurface twistor spaces with them.

§3.2: Hypersurface twistors in linearised theory, the Cech approach.

§3.2: The analysis using Cech cohomology:

In this section I will derive the infinitesimal deformation of the hypersurface twistor spaces corresponding to flat hypersurfaces in $\mathbb{M}$. I will also derive an expression for the evolution which will generalise, not only to non flat hypersurfaces but also to curved space. However for these generalisations we will require special Dolbeault representatives for the various quantities involved. These will be introduced in §3.3.

Flat space-like hypersurfaces: Consider a foliation of $\mathbb{M}$ by space-like hypersurfaces, the level surfaces of some linear time function, t. The hypersurface twistors, $P\tau_t$, determined by the hypersurface at time t, H_t, are just the null geodesics in the complexification of H_t, $\mathbb{C}H_t$. Therefore for each t we can identify $P\tau_t$ as a 3-dimensional submanifold of $\mathcal{A}$. As t varies, $P\tau_t$ varies as a submanifold of $\mathcal{A}$. The basic idea, roughly, is to restrict the deforming vector field on $\mathcal{A}$ to each $P\tau_t$, and thus obtain the infinitesimal deformation of each $P\tau_t$. The time derivative of this deformation will yield the evolution.

We need to locate each $P\tau_t$ in $\mathcal{A}$. $P\tau_t$ can be located as the fixed points of the involution on $\mathcal{A}$ determined by reflecting each null geodesic in the hypersurface H_t. This reflection takes an α-plane into a β-plane and vice-versa. Explicitly:

Choose an origin in $\mathbb{M}$, and a timelike vector $T^{AA'}$, such that $T_aT^a = 2$ and $t = x^{AA'}T_{AA'}$. Null vectors of the form $\pi^{A'}\alpha^A = N^a$ when reflected in T^a become $T^aN^bT_b - N^a = (T^{AB'}\pi_{B'})(T^{A'B}\alpha_B)$. So the α-plane with spinor $\pi_{A'}$ through $x \in H_t$, gets reflected into the β-plane with spinor $T_A{}^{B'}\pi_{B'}$.

We find that: $(\omega^A, \pi_{A'}) \to (T_A{}^{B'}\pi_{B'}, -\omega^BT_B{}^{A'} + it\pi^{A'})$

This is more neatly expressed in terms of CP^3 quantities. The

hypersurface H_t is determined as the intersection with $\mathbb{M}$ of the hyperplane, $H_{t\alpha\beta}X^{\alpha\beta} = 0$ in $\mathbb{CP}^3$, where $H_{t\alpha\beta}$ is a vector in $\mathbb{C}^4$ that is real and normalised to have length 4, so that $H_{t\alpha\dot{\gamma}}H_t^{\dot{\gamma}\beta} = \delta^{\beta}_{\ \alpha}$. Reflection of a twistor in this hyperplane is then just: $Z^{\alpha} \to H_{t\alpha\beta}Z^{\beta}$

Similarly: $W_{\alpha} \to H_t^{\alpha\beta}W_{\beta}$

The relation $H_{t\alpha\beta}H_t^{\beta\dot{\gamma}} = \delta_{\alpha}^{\ \dot{\gamma}}$ then implies that this is indeed an involution. (The above relations are proved with much twistor algebra.)

With a choice of the origin and infinity we find:

$$H_{t\alpha\beta} = \begin{bmatrix} 0 & T_A^{\ B'} \\ -T^{A'}_{\ \ B} & t\varepsilon^{A'B'} \end{bmatrix}$$

This will be referred to as the time twistor, it determines the hypersurface at time t. $H_{t\alpha\beta}$ is skew.

In particular note that $H_{t\alpha\beta} = H_{0\alpha\beta} + tI_{\alpha\beta}$.

So the involution on ambitwistor space is just:

$$(Z^{\alpha}, W_{\beta}) \to (H_t^{\alpha\dot{\gamma}}W_{\dot{\gamma}}, H_{t\beta\dot{\gamma}}Z^{\dot{\gamma}})$$

In particular $P\mathbb{T}_t$ is a section of $\mathbb{A}$ over $P\mathbb{T}$ where $\mathbb{A}$ is regarded as the cotangent bundle of $P\mathbb{T}$. The section is given by:

$$Z^{\alpha} \to (Z^{\alpha}, H_{t\beta\dot{\gamma}}Z^{\dot{\gamma}})$$

To find the deformation of $P\mathbb{T}_t$, we must restrict the deforming vector field on $\mathbb{A}$ to $P\mathbb{T}_t$, and then project it along the fibres of the projection $\mathbb{A} \to P\mathbb{T}$ to obtain a vector field tangent to $P\mathbb{T}_t$. So with hamiltonian $H = (\xi g(Z) + \tilde{\xi}\tilde{g}(W))$, where $\xi = I^{\alpha\beta}W_{\alpha}\partial/\partial Z^{\beta}$ we will obtain the deforming vector field:

$$X_t = I^{\alpha\beta}(\partial g/\partial Z^{\alpha})\partial/\partial Z^{\beta} + (\partial(\tilde{\xi}\tilde{g})/\partial W^{\alpha})(\partial/\partial Z^{\alpha})\big|_{W_{\dot{\gamma}} = H_{t\dot{\gamma}\delta}Z^{\delta}}$$

As can be seen, the A.S.D. part of the deformation is just the same as that for the nonlinear graviton, this part remains constant with time. The S.D. part varies with time through its dependance on $H_{t\alpha\beta}$. The S.D.

part is just about as general as one could imagine and does not appear to preserve any interesting local structure. This makes it appear very difficult to characterise hypersurface twistor spaces corresponding to initial data sets conformal to one satisfying the constraint equations by means of any intrinsic holomorphic structure.

The evolution: The evolution is somewhat neater. This is obtained by differentiating the above expression with respect to time. This yields:

$$E_t = \partial/\partial t\ X_t = (\partial/\partial W_\alpha)(\tilde{\xi}^2\tilde{g}(W))\partial/\partial Z^\alpha \Big|_{W_\delta = H_{t\gamma\delta}Z^\delta}$$

Where, as above $\tilde{\xi} = Z^\alpha I_{\alpha\beta}\partial/\partial W_\beta$.

It can be seen that the effect of the time derivative is to take the derivative of $\tilde{g}$ along $\tilde{\xi}$. This follows frome the fact that:

$H_{t\alpha\beta} = H_{0\alpha\beta} + tI_{\alpha\beta}$, so: $[\partial/\partial t\ W_\alpha] = \tilde{\xi}\ W_\alpha$ when $W_\alpha = H_{t\alpha\beta}Z^\beta$.

This expression can be regarded as being obtained by taking the hamiltonian vector field of an evolution hamiltonian $H_E = \tilde{\xi}^2\tilde{g}(W)$ on $\mathbb{T}\times\mathbb{T}^*$, and then restricting this expression to $P\tau_t$, ie $W_\alpha = H_{t\alpha\beta}Z^\beta$.

Remarks: 1) If the S.D. part of the field vanishes, there is no evolution, and all the $P\tau_t$ are the same, and are just the linearised nonlinear graviton.

2) Since the A.S.D. part of the deformation doesn't evolve, it doesn't affect any of the results in linearised theory. So in the following two sections I will miss it out.

3) The time derivative was taken holding the Z^α constant, so the $P\tau_t$'s are identified along the flat α-planes. Since we are working with Cech cohomology, the information of how the hypersurfaces are identified is lost. This is not just lost in the coboundary freedom, but also in the mechanics of the deformation procedure in Cech theory. In the nonlinear

theory we have a particular identification of the $P\tau_t$'s, at least on N. To describe this we will need a preferred Dolbeault representative.

4) It is possible to extend this analysis to shear free null hypersurfaces as well, but these are too special to be of general interest. The hypersurface twistor spaces project out the A.S.D. part of the field.

Another analysis of interest is a power series expansion of the googly part of the deformation as one comes in from $\mathcal{I}$. The result is suggestive, but difficult to interpret. An interesting feature of this calculation is that the power series "forgets" the information of which hypersurfaces are used to take the limit to $\mathcal{I}$. It may be hoped that this could give some insight into a linearised googly deformation.

§3.3 Some comments on Dolbeaut cohomology.

The basic reference for Dolbeaut cohomology in twistor theory is Woodhouse [28]. In this section I will derive some extensions of these results that are used later in the thesis. In particular I will need Dolbeult representatives for derivatives of elements of $H^1(\mathbb{T},\mathcal{O}(2))$. I will however prove results that I use from Woodhouse in order to make this section relatively self contained. (For what its worth, the presentation is new.)

The Cech Dolbeault isomorphism:

The basic Cech Dolbeault isomorphism for H^1's arises as follows; take a Cech representive on a complex manifold, M, α_{ij}, relative to some cover $\mathcal{U} = \{U_i\}$ of M. Then we can split α_{ij}, $\alpha_{ij} = \alpha_i - \alpha_j$, where α_i are smooth functions on each U_i. Since α_{ij} was holomorphic, $\bar{\partial}\alpha_i = \bar{\partial}\alpha_j = \alpha$, say. α is then the Dolbeault representative corresponding to α_{ij}.

(Note that this necessitates using both holomorphic and anti-holomorphic coordinates on M. On $\mathbb{A}$ we will have to use $(Z^\alpha, \bar{Z}^{\bar{\alpha}}, W_\alpha, \bar{W}_{\bar{\alpha}})$. If we are representing $\mathbb{A}$ as the spin bundles restricted to a hypersurface, we will have to use four spin bundles, S_A, $\bar{S}_{\bar{A}}$, $S_{A'}$ and $\bar{S}_{\bar{A}'}$. Only on a lorentzian slice can we identify $\bar{S}_{\bar{A}}$ with $S_{A'}$, etc. The nontrivial aspect of the use of Dolbeaut theory is to find distinguished representatives, and use reality structures in such a way as to be able to minimize these irritations.)

Representatives on the spin bundle:

For the purposes of this discussion I will restrict attention to cohomology classes on $\mathbb{PN}$ of positive homogeneity. $\mathbb{N}$ will be identified with the primed spin bundle restricted to a general space-like hypersurface

in $\mathbb{M}$, $S_{A'}|_H$. The easiest example is that of a homogeneity degree zero twistor function, $f_{ij}(Z)$. Because the cohomology class is trivial on $\mathbb{CP}^1$'s, we can split $f_{ij}(Z)$ on each fibre of the spin bundle over H:

$$f_{ij}(Z) = f_i(x, z_{A'}) - f_j(x, z_{B'})$$

The corresponding Dolbeault representative is $f = \bar{\partial} f_i$. Now the antiholomorphic tangent space is just the part of $D = \{z^{A'} \nabla_{AA'}, \partial/\partial \bar{z}_A\}$ tangent to H. $\bar{\partial} f_i$ is determined by the knowledge of each derivative of f_i along an element of D tangent to H. By construction $\partial/\partial \bar{z}_A f_i = 0$, and from Ward [1,page 46] we have $z^{A'} \nabla_{AA'} f_i = z^{A'} A_{AA'}$, where $A_{AA'}$ is the A.S.D. Maxwell potential corresponding to f_{ij}.

So the corresponding Dolbeault class can be represented by $A_a dx^a$. It can be checked that the freedom in choosing different Dolbeault representatives that are trivial on each fibre is in 1-1 correspondence with the gauge freedom in A_a.

Note that as a Dolbeault form, $A_a dx^a$ is only defined when contracted into an antiholomorphic vector, and so one can write it equivalently, as in Woodhouse §6, as:

$$u^{-1} A_{AA'} z^{A'} dx^{AB'} \hat{z}_{B'} \quad \text{or even:} \quad u^{-2} (A_{AA'} \overset{\vee}{z}{}^A z^A)(z_{B'} \overset{\vee}{z}{}_B dx^{BB'})$$

Where $u = (\hat{z}^{A'} z_{A'}) = (T^{AA'} z_{A'} \bar{z}_A)$ and $\overset{\vee}{z}_A = T_A{}^{A'} z_{A'}$.

Higher homogeneity cohomology classes are treated analogously. The neutrino field ν_A is slightly different. It doesn t require a Hertz potential, and its Dolbeaut representative is just:

$$u^{-1} \nu_A \hat{z}_A dx^{AA'}.$$

The proof again is just as above. Note that in this case the restricted gauge freedom is trivial. It would have to consist of $\bar{\partial}$ of global functions on $\mathbb{M}$, homogenous of degree -1, and holomorphic in $z_{A'}$.

For the purpose of the linearised analysis of hypersurface twistor

theory we will need the Dolbeault representatives for homogeneity degree 2 twistor functions and their derivatives. So consider $g_{ij}(Z) \in H^1(PT, O(2))$, and the corresponding field, Ψ_{ABCD}, first potential, $\chi_{A'ABC}$, second potential, h_{ab}, and third potential, $\phi_{AA'B'C'}$. (I shall also restrict the gauge by requiring $\nabla_A^{A'} \phi_{BA'B'C'} = 0$ and requiring that all the potentials are maximally symmetric and trace free. It is possible to perform the following calculations without such gauge fixing, but then one has to be more careful with the positioning of indices etc.)

From Woodhouse, (or the argument used above) we have that the Dolbeault representative of g that is trivial on each fibre is:

$$z^{B'} z^{C'} \phi_{AA'B'C'} dx^{AA'}$$

Again the gauge freedom in $\phi_{AA'B'C}$ corresponds to the freedom in different choices of Dolbeault representative.

Dolbeault representatives for derivatives of cohomology classes.

We will also need Dolbeault representatives for $(\partial/\partial\omega^A)g(Z)$ and higher derivatives. The result is:

$$(\partial/\partial\omega^A)g(Z) \text{ corresponds to } z^{A'} h_{AA'b} dx^b$$

Similarly $(\partial/\partial\omega^A)(\partial/\partial\omega^B)g(Z) \quad \leftrightarrow \quad \chi_{cAB} dx^c$

These are most easily calculated using helicity lowering operators.

Let $X^\alpha = (\omega^A, \pi_{A'})$. Then the cohomology class $(X^\alpha \partial/\partial Z^\alpha)g(Z)$ corresponds to the spin 3 field $\phi_{ABC} = \Psi_{ABCD}\omega^D$.

A second potential for ϕ_{ABC} is:

$$\eta_{aB'} = h_{aBB'}\omega^B - i\phi_{aB'C'}\pi^{C'}$$

So the corresponding Dolbeault representative is $\eta_{aB'} z^{B'} dx^a$. Projecting out the secondary part with the infinity twistor yields the expression above. Similarly if the cohomology class $(X^\alpha \partial/\partial Z^\alpha)^2 g(Z)$

corresponds to the spin 1 field $\phi_{AB} = \Psi_{ABCD}\omega^C\omega^D$, then the corresponding potential is:

$$A_a = \chi_{aBC}\omega^B\omega^C - 2ih_{aBB'}\omega^B\pi^{B'} - 2\phi_{aB'C'}\pi^{B'}\pi^{C'}$$

Remark: This expression is taken from Woodhouse §10, it can be checked directly, or derived from the generalised deRham sequence using the local twistor D on the local twistor bundles $\mathbb{T}_{(\alpha\beta\ldots\delta)}$ on space-time.

Note that if we set $X^\alpha = Z^\alpha = (0, z_{A'})$ we obtain the Dolbeault representative for $(Z^\alpha\partial/\partial Z^\alpha)^2 g(Z) = 4g(Z)$, as is necessary for consistency.

An interesting feature of this approach is that $A_a dx^a$ is a Dolbeault representative for the second derivative of a positive homogeneity twistor function on $P\mathbb{T}$, whereas in Woodhouse it was regarded as being derived from $F = d(A_a dx^a)$, where F was regarded as the push down from $P\mathbb{T}^*$ of $(X^\alpha W_\alpha)^2 f(W)\,(\varepsilon^{\beta\gamma\delta\epsilon} W_\beta dW_\gamma \wedge dW_\delta \wedge dW_\epsilon)$, where $f(W)$ is the homogeneity -6 function on $P\mathbb{T}^*$ corresponding to Ψ_{ABCD}. It should be possible to build this procedure into a straightforward treatment of the twistor transform.

So the corresponding Dolbeault representative is $A_a dx^a$. Projecting out the tertiary part with the infinity twistor yields the expression above.

Finally we will also need the third derivative of $g(Z)$. $(X^\alpha\partial/\partial Z^\alpha)^3 g(Z)$ correspond to the neutrino field $v_A = \Psi_{ABCD}\omega^B\omega^C\omega^D$. Then the corresponding Dolbeault representative is $u^{-1} v_A \hat{z}_{A'} dx^{AA'}$. It can be seen that the primary, secondary, and tertiary parts vanish, leaving:

$$(\partial/\partial\omega^A)(\partial/\partial\omega^B)(\partial/\partial\omega^C)g(Z) \leftrightarrow (z^{A'}\hat{z}_{A'})^{-1}\Psi_{ABCD}\hat{z}_{D'}dx^{DD'}$$

Remarks:

Although these forms were regarded as being defined on the hypersurface C.R. manifold, which was thought of as being identified with $\mathbb{M}$ in $\mathbb{T}$, they are in fact obviously defined on the whole spin bundle. The

only reference to the hypersurface is in the operation $z_{A'} \to \hat{z}_{A'}$.

Field equations:

The pullback to N of two representatives for the same cohomology class on two different hypersurfaces differ by a holomorphic form. This can be seen from the fact that for, for instance, $A_{\bar{a}}dx^a$, $d(A_{\bar{a}}dx^a)$ vanishes when restricted to the twistor distribution $D = \{z^{A'}\nabla_{AA'}, \partial/\partial z_{A'}\}$. So the Lie drag of $A_a dx^a$ along the null geodesic spray is therefore $\bar{z}^A \phi_{AB} z_{B'} dx^{BB'}$, which vanishes when regarded as a Dolbeault form.

The condition that $d(A_{\bar{a}}dx^a)$ vanishes when restricted to the twistor distribution D is the condition that is equivalent to the field equations on A_a, since the restriction of $d(A_{\bar{a}}dx^a)$ to D is precisely the self dual part of the field obtained from A_a.

Constraint equations, spin 1: We find that $A_{\bar{a}}dx^a$ is always $\bar{\partial}$ closed when pulled back to N from $S_A|_{\bar{H}}$ for all $A_{\bar{a}}dx^a$. This is because the antiholomorphic tangent space is:

$$\bar{H} = \{\bar{z}^{\dot{A}} z^{A'} \nabla_{AA'}, \partial/\partial \bar{z}_{\dot{A}}\}$$

Since $A_{\bar{a}}dx^a$ doesn't depend on $\bar{z}_{\dot{A}}$, $dA_{\bar{a}}dx^a$ is of the form $F_{\bar{a}\bar{b}}dx^a \wedge dx^b$, which necessarily vanishes when restricted to $\bar{H}$, as $\bar{H}$ only contains one horizontal vector. $A_{\bar{a}}dx^a|_{\bar{H}}$ only defines the 3-vector part of $A_{\bar{a}}$, $\underline{a}$. This reflects the fact that there are no constraints for this prescription of initial data.

Constraint equations, spin 2: For the same reason as in the remark above, the Dolbeault representative for the second derivative of g(Z) is automatically $\bar{\partial}$ closed. However there are nontrivial constraints on the specification of $(h_{\bar{a}\bar{b}}, \chi_{\bar{a}BC})$ as initial data for a spin 2 field. (The $\bar{a}$ index is assumed to be projected into the hypersurface.)

The constraints appear in the condition that a given representative $g_{\alpha\beta} \in H^1(PT, O_{(\alpha\beta)})$ is actually a second derivative. This condition can be written: $\partial_{[\alpha} g_{\beta]\delta} = 0$.

In the case of the representative $g_{AB} = dx^C \gamma_{cAB}$, the constraint equations on γ_{cAB} are equivalent to:

$$(\partial/\partial\omega^A) g_{BC} - (\partial/\partial\omega^B) g_{AC} = 0$$

<u>Compatibility with the "$\vee$" operation:</u> One of the reasons why these representatives will be of particular use is that they are well behaved under the map from twistor space to dual twistor space given by reflecting the twistor in the chosen hypersurface. Call this map "$\vee$".

If $(x, z_{A'}) \in \mathbb{N}$:

$$\vee[(x, z_{A'})] = (X, w_A = \overset{\vee}{z}_A) \in \mathbb{N}^*$$

where, as before, $\overset{\vee}{z}_A = T_A{}^{A'} z_{A'}$. This is the appropriate generalisation to non flat hypersurfaces of the map from twistor space to dual twistor space used in the last section:

$$Z^\alpha \to H_{\alpha\beta} Z^\beta = \overset{\vee}{Z}_\alpha$$

So if we have some cohomology class $\alpha \in H^1(\mathbb{N}, O(n))$, where $n \geq -1$, the above type of representative will be of the form:

$$\alpha = \alpha_a(x, z, \bar{z}) dx^a, \text{ where } \alpha_{AA'} z^A \text{ is holomorphic in } z_{A'}.$$

Pulling this back to $\mathbb{N}^*$ via $\vee$, we will obtain the form $\vee^*\alpha$ given by:

$\vee^*(\alpha) = \alpha_a(x, w_A T^A{}_{A'}, \bar{w}_A T^{A'}{}_A) dx^a$ such that $(\vee^*\alpha)_{AA'} T^A{}_A w_A$ is holomorphic in w_A. This is the condition that $\vee^*(\alpha)$ is $\bar{\partial}$ closed on $\mathbb{N}^*$.

However the restricted gauge freedom only matches appropriately when $n = 0$ or -1. This will nevertheless be good enough for our purposes, since these are the only cases that will arise. Note that for $n = -1$, the restricted gauge freedom is trivial anyway.

§3.4: Hypersurface twistors in linearised theory, the Dolbeault approach.

In this section I will show how the formulae of §3.2 work for general hypersurfaces in $\mathbb{M}$ if one uses the specially chosen Dolbeault representatives represented in §3.3.

I shall just work with a linearised self dual field. The contribution of a linearised anti self dual field is exactly that of a linearised nonlinear graviton and does not evolve. Since I am working with linearised theory there is no interaction between the S.D. part and the A.S.D. part and so there is no interest in considering the A.S.D. part.

So consider a variation of the metric h_{ab} on $\mathbb{M}$, such that the variation in the connection is $\delta\gamma_{aB'C'} = \nabla_{B(B'}h_{C')}{}^{B}{}_{a}$, and the variation in the unprimed spin connection is zero. Let the variation in the S.D. Weyl spinor be $\Psi_{A'B'C'D'}$, and the corresponding homogeneity degree 2 dual twistor function be g(W).

Consider a foliation of $\mathbb{M}$ by space-like surfaces, H_t, given by the level surfaces of some time function t which is constrained for convenience such that $(\nabla_a t)(\nabla^a t) = 2$, so that $T_a = \nabla_a t$ satisfies $T_a T^a = 2$.

We will as usual identify $\mathbb{M}$ with $S_A\,|_H$.

The variation of canonical structures on the spin bundle:

The various forms and vector fields on S_A, vary according to (see §1.6):

$$\delta\nabla_{AA'} = h_{AA'}{}^{b}\nabla_b + \gamma_{aB'}{}^{C'}z_{C'}\partial/\partial z_{B'}, \qquad \delta(\partial/\partial z_{A'}) = 0 \text{ and } \delta(\partial/\partial\bar{z}_A) = 0$$

$$\delta dx^{AA'} = -h^{AA'}{}_{b}dx^{b}, \qquad \delta dz_{A'} = -\gamma_{bA'}{}^{C'}z_{C'}dx^{b} \text{ and } \delta d\bar{z}_A = 0$$

The twistorial structures, $\theta = z_{A'}\bar{z}_A dx^{AA'}$, $\iota = z^{A'}dz_{A'}$, $\bar{\iota}$, and $\Upsilon = z_{A'}\partial/\partial z_{A'}$, and $\bar{\Upsilon}$ vary as follows.

$$\delta\theta = z_{A'}\bar{z}_A h^{AA'}{}_{b}dx^{b}, \quad \delta\iota = dx^{b}\gamma_{b}{}^{A'}{}_{C'}z_{A'}z^{C'}, \quad \delta\bar{\iota} = 0 \text{ and } \delta\Upsilon = \delta\bar{\Upsilon} = 0.$$

From §3.3 we can recognise $\delta\theta$ and $\delta\imath$ as dolbeault representatives for derivatives of g(W) as follows:

$$\delta\theta \leftrightarrow (\pi^{A'}\partial/\partial\tilde{\omega}^{A'})g(W)|_{W=\bar{Z}}, \text{ and } \delta\imath \leftrightarrow (\pi^{A'}\partial/\partial\tilde{\omega}^{A'})^2 g(W)|_{W=\bar{Z}}.$$

Note that, in the flat case, $z_{A'}$ is the pullback of $\pi_{A'}$ to the spinbundle. This notation is used, (following Woodhouse [28]) because $\partial/\partial z_{A'} \neq \partial/\partial\pi_{A'}$.

Remarks:

The expression for $\delta\theta$ can be understood in terms of the deformation using the hamiltonian $H_{ij} = (\pi^{A'}\partial/\partial\tilde{\omega}^{A'})g_{ij}(W)$. The indices i and j refer to some cover $\mathcal{U} = \{U_i\}$ of $\mathcal{A}$. We can identify $\mathcal{A}$ with $\mathcal{S}_A, \mathcal{O}\mathcal{S}_A$ restricted to the complexification of the hypersurface.

In order to produce the deformation one splits $H_{ij} = H_i - H_j$, and then Lie drags each U_i along the integral curves of the hamiltonian vector field of each H_i. Under this operation $\theta \to \theta + \delta\theta$, where $\delta\theta = \mathfrak{L}_{X_{H_i}}\theta = \bar{\partial}H_i$. $\delta\theta$ matches up correctly on overlaps since: $\mathfrak{L}_{X_{H_{ij}}}\theta = 0$.

The expression for $\delta\imath$ is of course just the Dolbeault representative of the corresponding Cech expression presented in §3.2. Note in particular that the corresponding Cech expression is the evolution hamiltonian as presented in §3.2. So $\delta\imath$ can be regarded as a Dolbeault repreesentative for the evolution hamiltonian.

The variation in the C.R. structure.

In order to avoid confusion between B for boundary, and spinor indices, I will write $\bar{\partial}$ for the $\bar{\partial}_B$ operator. $\bar{\partial}$ can be written:

$$\bar{\partial} = u^{-2}\hat{z}_{B'}\bar{z}_B dx^{BB'}\check{z}_A z_{A'}\nabla_{AA'} + d\bar{z}_A\partial/\partial\bar{z}_A$$

Where $u = z_{A'}\bar{z}_A T^{AA'}$, $\hat{z}_{A'} = T_{A'}{}^A z_A$ and $\check{z}_A = T_A{}^{A'} z_{A'}$.

Its variation is an element of $H^1_B(\mathbb{N},T'\mathbb{N})$. Here $T'\mathbb{N}$ denotes the sheaf of sections of $\mathbb{C}T\mathbb{N}/\bar{H}$, where $\bar{H}$ is the antiholomorphic tangent space on $\mathbb{N}$. $T'\mathbb{N}$ is a C.R. vector bundle with $\bar{\partial}$ operator as given in §1.3.

The variation of $\bar{\partial}$ is given as follows:

$$\delta\bar{\partial} = dx^a z_{B'}\{u^{-1}h_a{}^{BB'}\hat{z}^{C'}\nabla_{BC'} + \delta_a{}^{B'}{}_{C'}\partial/\partial z_{C'}\}$$

The form part of the above expression is only defined when contracted into an antiholomorphic vector field, and the vector part is understood to be projected along the null geodesic spray so as to be tangent to the hypersurface.

We wish to show that this is indeed the Dolbeault representative of the Cech formula obtained in §3.2. The corresponding Cech representative is:

$$(\partial/\partial W_\alpha)(\pi^{A'}\partial/\partial\tilde{\omega}^{A'})g(W)\partial/\partial Z^\alpha|_{W_\alpha} = H_{t\alpha\beta}Z^\beta$$

The Dolbeault representative for $(\partial/\partial W_\alpha)(\partial/\partial\tilde{\omega}^B)g(W)$ is:

$$(idx^c h_{cB'}{}^A,\ dx^c v_{cB'A'}) \in H^1(\mathbb{N}^*,\ T^\alpha\otimes S_{B'})$$

This form is, as usual, only defined when contracted into a vector in the antiholomorphic tangent space on $\mathbb{N}^*$. Mapping it to an element of $H^1(\mathbb{N},T^\alpha\otimes S_{B'})$ by setting $W_\alpha = H_{t\alpha\beta}Z^\beta$ doesnt change the above form except that it is now only defined when contracted into a vector in $\bar{H}$ on $\mathbb{N}$.

Given a constant local twistor $X^\alpha = (\omega^A,\pi_{A'})$, we can represent the vector field $X^\alpha\partial/\partial Z^\alpha$ on $S_A|_\mu$ by:

$$X^\alpha\partial/\partial Z^\alpha \sim iu^{-1}\omega^{A}\hat{z}^{A'}\nabla_{AA'} + \pi_{A'}\partial/\partial z_A$$

Where the $\sim$ denotes the fact that they are not quite equal but differ by an antiholomorphic vector field which is irrelevant to our situation. (This is proved using the fact that, on functions pulled back from $\mathbb{N}$, $\nabla_{AA'}$

acts as: $\nabla_{AA'} = \imath z_{A'}\partial/\partial\omega^{A} - i\bar{z}_{A}\partial/\partial\bar{\omega}^{A}$ and $\partial/\partial z_{A'} = ix^{AA'}\partial/\partial\omega^{A} + \partial/\partial\pi_{A'})$

So finally we obtain that:

$$(\partial/\partial W_{\alpha})(\pi^{B'}\partial/\partial\tilde{\omega}^{B'})g(W)\partial/\partial Z^{\alpha}\Big|_{W_{\alpha} = H_{t\alpha\beta}Z^{\beta}}$$

has the following Dolbeault representative:

$$dx^{c}{}_{z}{}^{B'}(u^{-1}h_{cB'}{}^{A}{}_{\hat{z}}{}^{\hat{A}'}\nabla_{AA'} + \gamma_{cB'A'}\partial/\partial z_{A'})$$

as required.

The evolution of the C.R. structure:

As before the expression for the evolution is somewhat simpler. The actual evolution is obtained by Lie dragging the $\bar{\partial}$ operator along the null geodesic spray. This is most easily done by calculating the result in curved space and then restricting to linearised theory, where the expression is the same. We find:

$$(d/dt)\bar{\partial} = u^{-2}dx^{EE'}\bar{z}_{E}\hat{z}_{E'}z^{A'}z^{B'}z^{C'}\Psi_{A'B'C'D}\partial/\partial z_{D'}$$

Where as usual $u = (z_{A'}\bar{z}_{A}T^{AA'})$. According to §3.2 this should be the Dolbeault representative of:

$$(\partial/\partial W_{\alpha})(\pi^{\hat{A}}\partial/\partial\tilde{\omega}^{\hat{A}'})g(W)\Big|_{W_{\alpha}=\overset{\vee}{Z}_{\alpha}}\quad(\partial/\partial Z^{\alpha})$$

From §3.3 we found that the Dolbeault representative of the third derivative of $g(W)$ is:

$$(\partial/\partial\tilde{\omega}^{A'})(\partial/\partial\tilde{\omega}^{B'})(\partial/\partial\tilde{\omega}^{C'})g(W) \leftrightarrow \tilde{u}^{-1}\Psi_{A'B'C'D'}\hat{w}_{D}dx^{DD'}$$

Where $\tilde{u} = (w_{A}\bar{w}_{A}T^{AA'})$. All the other third derivatives vanished. Setting $W^{\alpha} = H_{t\alpha\beta}Z^{\beta}$ we obtain the representative:

$$u^{-1}\Psi_{A'B'C'D'}\bar{z}_{D}dx^{d}$$

So we finally find:

$$(\partial/\partial W_{\alpha})(\pi^{B'}\partial/\partial\tilde{\omega}^{B'})^{2}g(W)\Big|_{W=\overset{\vee}{Z}}(\partial/\partial Z^{\alpha})$$

corresponds to:

$$u^{-1}dx^{AA'}\bar{z}_{A}\Psi_{A'B'C'D'}z^{B'}z^{C'}\partial/\partial z_{D'}$$

This, as a Dolbeault form, agrees with the actual expression for the evolution given above.

Remarks:

Gauge freedom:

The two cases where we have used the map ν to pullback Dolbeault forms, were on homogeneity degree 0 and -1 forms. These are the two cases where the restricted cohomology freedom is compatible with the map. This is essential for the procedure to make sense.

The restricted cohomology gauge freedom for the expression for the evolution is actually trivial. It would need to consist of global homogeneity degree -1 functions holomorphic in $z_{A'}$. However there is still a large amount of unrestricted cohomology freedom. The actual expression for the evolution is necessarily unique as the identification between two successive hypersurface twistor spaces is fixed. It is notable that the two different prescriptions give the same preferred representative.

It might be argued that this derivation of the evolution is not twistorial in the sense that to produce the preferred representative we need to use the $\mathbb{CP}^1$s corresponding to points of the hypersurface.

However we can motivate this choice of representative in a more "twistorial" fashion by regarding the knowledge of $\iota + \delta\iota$ as a fixed one form as "twistorial". This is more appropriate in curved space where ι has no residual gauge freedom. However $\delta\iota$ is a preferred Dolbeault representative of the evolution hamiltonian, and deriving the evolution from $\iota + \delta\iota$ as evolution hamiltonian gives the appropriate unique representative for the evolution.

The deformed infinity twistor:

The infinity twistor is the main structure I will use in the following chapter to generalise the results of this section to the full nonlinear theory. In order to motivate the calculations there it is useful to know how it varies as a result of the deformation.

The deformation rotates the infinity twistor. If we deform with the hamiltonian vector field of $Z^{\alpha}I_{\alpha\beta}(\partial/\partial W_{\beta})g(W)$ we can determine the rotation of $I_{\alpha\beta}$ under the deformation by examining the action of the deformation on $I_{\alpha\beta}dZ^{\alpha}\wedge dZ^{\beta}$.

Represent the cohomology class $g(W)$ by $g_{ij}(W)$ with respect to some open cover $\mathcal{U} = \{U_i\}$. In order to perform the deformation we split $g_{ij}(W)$ into $g_i(W,\bar{W}) - g_j(W,\bar{W})$ and Lie drag each U_i along the hamiltonian flows of each $\tilde{\xi}g_i(W,\bar{W})$. (Where $\tilde{\xi} = Z^{\alpha}I_{\alpha\beta}\partial/\partial W_{\beta}$.)

Since the Lie drag of $I_{\alpha\beta}dZ^{\alpha}\wedge dZ^{\beta}$ along the hamiltonian flows of $\tilde{\xi}g_{ij}(W)$ is zero mod dW_{α}, it will survive on the infinitesimally deformed space (mod dW_{α}) to become: $I_{\alpha\beta}dZ^{\alpha}\wedge dZ^{\beta} + \delta(I_{\alpha\beta}dZ^{\alpha}\wedge dZ^{\beta})$ where $\delta(I_{\alpha\beta}dZ^{\alpha}\wedge dZ^{\beta})$ is the Lie drag of $I_{\alpha\beta}dZ^{\alpha}\wedge dZ^{\beta}$ along the hamiltonian flows of $\tilde{\xi}g_i(W,\bar{W})$. We have after a short calculation:

$$\delta(I_{\alpha\beta}dZ^{\alpha}\wedge dZ^{\beta}) = (\partial/\partial\bar{W}_{\bar{\alpha}})(\partial/\partial W_{\beta})\tilde{\xi}g(W,\bar{W})I_{\beta\bar{\gamma}}dZ^{\bar{\gamma}}\wedge d\bar{W}_{\bar{\alpha}}$$

We are only interested in the result on $\mathbb{M}$, where $W = \bar{Z}$ and $\bar{W} = Z$, so setting $\partial/\partial\bar{Z}_{\alpha} = \bar{\partial}^{\alpha}$, and $\partial/\partial Z^{\alpha} = \partial_{\alpha}$, we find:

$$\delta(I_{\alpha\beta}dZ^{\alpha}\wedge dZ^{\beta}) = \bar{\xi}\bar{\partial}^{\alpha}\partial_{\beta}g(\bar{Z},Z)I_{\alpha\bar{\gamma}}dZ^{\bar{\gamma}}\wedge dZ^{\beta}.$$

So we see that we can regard $I_{\alpha\beta}$ as having been rotated as follows:

$$I_{\alpha\beta} \to I_{\alpha\beta} + G_{[\alpha}{}^{\bar{\gamma}}I_{\beta]\bar{\gamma}} \quad \text{where} \quad G_{\bar{\alpha}}{}^{\beta} = \bar{\xi}\bar{\partial}^{\beta}\partial_{\alpha}g(\bar{Z},Z)$$

§4: The structure and evolution of hypersurface twistor spaces in curved space-time.

In the first part of this chapter I will outline and discuss the various twistorial objects that live on hypersurface twistor spaces. The constraint equations are characterised by use of the infinity twistor. The $*$ map, which reflects a twistor in the hypersurface to obtain a dual twistor, is also discussed and shown to be equivalent, at least for our purposes, to the local information of a "time" twistor, $H_{\alpha\beta}$. This generalises the skew twistor used in §3.2. Essentially it contains the information of the time function used to foliate the space-time with hypersurfaces. It can be used to characterise the minimal surface condition. The second part of this chapter is be concerned with calculating the evolution of the C.R. structure using these structures. This is all motivated by close analogy with the results from linearised theory in the last chapter.

§4.1: Structure:

To start with I will be concerned with structures on N the space of scaled null geodesics in a Lorentzian space-time, M, relative to a space-like hypersurface H. I will only be interested in structures which have some claim to being twistorial, in the sense that they do not have built into them the information of the location of the holomorphic curves corresponding to points of the hypersurface.

As usual identify N with $S_{A'}|_H$. As mentioned in §1.4 we have the following conformally invariant structures on N:

$\theta = z_{A'}\bar{z}_A dx^{AA'}$, $\omega = d\theta$, $Y = z_{A'}\partial/\partial z_{A'}$, and $\bar{Y}$.

These structures are all kinematical in the sense that any two spaces endowed with such structure are locally the same.

Also conformally invariant is the holomorphic volume form:

$v = z^{A'} dz_{A'}.z_{B'}.z_{C'}.d^2x^{B'C'}$, and dv.

These determine the C.R. structure. These structures are manifestly twistorial in the sense that they can be derived from the C.R. structure on PN (see §2.3).

In order to break conformal invariance we shall require knowledge of the infinity twistor $I_{\alpha\beta}$, where the twistor indices refer to those used in the Chern connection in §2.3. With the infinity twistor we can construct various forms and vector fields:

$\tau = I_{\alpha\beta} DZ^\alpha \wedge DZ^\beta = dz^{A'} \wedge dz_{A'}$, $\iota = I_{\alpha\beta} Z^\alpha DZ^\beta = z^{A'} dz_{A'} = Y \lrcorner \tau$.

$\xi = I^{\alpha\beta}\bar{Z}_\alpha \partial/\partial Z^\beta = u^{-1}\bar{z}^{\dot{A}}\bar{z}^A \nabla_{AA'}$, where $u = z_{A'}\bar{z}_A T^{AA'}$

note $\iota = \bar{\xi} \lrcorner \omega$.

We can also construct indexed quantities: $dz^{A'} = I_{\alpha\beta} DZ^\beta$

The primed index corresponds to the form taking values in $I_{\alpha\beta}\mathbf{T}^\beta$ which is equal to $S^{A'}$ (see §2.3).

The other structure we required in order to calculate the evolution was the map ν: $N^* \to N$ which corresponded to reflecting the dual twistor in the hypersurface. If $(x,w_A) \in N^*$, then $\nu(x,w_A) = (x,T_A{}^{A'}z_{A'}) \in N$ is the appropriate map.

Interpretation and discussion:

Unlike the case of local twistors or the nonlinear graviton, $DI_{\alpha\beta} \neq 0$. So, in particular, $d\tau \neq 0$, and $[\xi,\bar{\xi}] \neq 0$. Indeed in the last chapter it was shown that, at least in linearised theory, ι should be thought of as a Dolbeault representative for the second derivative of the homogeneity degree 2 dual twistor function corresponding to the S.D. part of the Weyl curvature. In particular we had:

$$\delta\iota \leftrightarrow (\pi^{A'}\partial/\partial\tilde{\omega}^{A'})^2 g^{\bar{\alpha}}(W,\bar{W})d\bar{W}_{\bar{\alpha}}$$

Where now $g^{\bar{\alpha}}(W,\bar{W})d\bar{W}_{\bar{\alpha}}$ is the Dolbeault form on $\mathbb{T}^*$ corresponding to $\Psi_{A'B'C'D'}$. Since we are working on N, we must restrict this form to $W = Z$:

$$\delta\iota \leftrightarrow (\pi^{A'}\partial/\partial\bar{\omega}^{A'})^2 g_{\alpha}(\bar{Z},Z)dZ^{\alpha}$$

If we now assume analyticity and complexify N to obtain $\mathbb{C}N$, ι will become the analogue of: $(\pi^{A'}\partial/\partial\tilde{\omega}^{A})^2 g_{\alpha}(W,Z)dZ^{\alpha}$ (*)

So we see that we cannot distinguish between the dependance of ι on Z^{α} and $\bar{W}_{\bar{\alpha}}$. (In linearised theory we were only able to do so because of the availability of a considerable amount of background structure which we do not posess in curved space-time.)

In particular we cannot check whether ι is $\bar{\partial}$ closed so we cannot argue that ι should have the status of a Dolbeault form. Nevertheless we can proceed on the basis that it is a curved space analogue of (*). We shall see that this interpretation works very well.

We can similarly regard $dz^{A'} = I_{\alpha\beta}DZ^{\beta}$ as a curved space analogue of:

$$(\partial/\partial\tilde{\omega}^{A'})(\pi^{B'}\partial/\partial\tilde{\omega}^{B'})g_{\gamma}(W,Z)dZ^{\gamma}.$$

Finally we can regard $I_{\alpha\beta}$ as an exponentiated version of:

$$I_{\alpha\beta} + I_{\gamma[\alpha}\tilde{\xi}\tilde{\partial}^{\gamma}\partial_{\beta]}g_i(W,Z)$$

Where $g_i(W,Z)$ is as defined in the last section of the last chapter. Note that $Z^{\alpha}\partial_{\alpha}g_i(W,Z) = 0$ since g is homogenous of degree zero in Z.

$I_{\alpha\beta}$ is clearly the most basic of the structures outlined above, all the others can be derived from it. Knowledge of $I_{\alpha\beta}$ is equivalent to knowledge of τ.

Note that the gauge freedom of these quantities has disappeared. This is because the gauge freedom in the linearised cased corresponded to infinitesimal coordinate changes. In the nonlinear case this is replaced by the full coordinate freedom.

The full field equations:

In §3.3 it was pointed out that the full field equations for a field with Dolbeault representative defined on the spin bundle are equivalent to the condition that the Lie drag of the representative along the null geodesic spray vanishes as a Dolbeault form. Since ι is a dual twistor cohomology class, $\pounds_V\iota$ should vanish when restricted to the dual twistor antiholomorphic distribution:

$$\bar{D} = \{\bar{z}^{A}\nabla_{AA'}, \partial/\partial z_{A}\}$$

As usual V denotes the null geodesic spray, $V = z^{A'}\bar{z}^{A}\nabla_{AA'}$.

A straightforward calculation yields:

$$\pounds_V z^{A'}dz_{A'} = \Psi_{A'B'C'D'}z^{A'}z^{B}z^{C'}\bar{z}_{D}dx^{DD'} + \Phi_{A'B'CD}z^{A'}z^{B'}\bar{z}^{C}z_{D'}dx^{DD'}$$

The $\Psi_{A'B'C'D'}$ term can be seen to vanish when restricted to $\bar{D}$, whereas the Φ_{ab} term doesn't. The vanishing of $\pounds_V\iota$ when restricted to $\bar{D}$ is equivalent to $\Phi_{ab} = 0$, that is the Einstein equations. (A cosmological constant is allowed with this formulation.)

This can be regarded as asserting that ι and τ remain constant as

holomorphic forms,if we bear in mind that the notion of holomorphicity is evolving. (This is equivalent to the old observation that $\nabla_{A'}{}^{A}\Psi_{ABCD} = 0$ is a linear equation, except for the fact that Ψ_{ABCD} determines how $\nabla_{AA'}$ varies.)

A formula for the derivative of the infinity twistor:

In order to proceed we will need an expression analogous to the Dolbeault equivalent in linearised theory of:

$$(\partial/\partial W_{\alpha})(\partial/\partial\bar{\omega}^{B'})(\pi^{C'}\partial/\partial\bar{\omega}^{C'})g(W) = g_{\alpha B'} \text{ say.}$$

This is necessary for two reasons, firstly to obtain the expression for the evolution, and secondly in order to express the constraint equations.

We can obtain the curved space analogue of this derivative by taking the derivative of the infinity twistor using the Chern connection.

We can write: $$DI_{\alpha\beta} = I_{\alpha\beta;\gamma}DZ^{\gamma} + I_{\alpha\beta}{}^{;\gamma}D\bar{Z}_{\gamma}$$

Then $I_{\alpha\beta}{}^{;\gamma}$ is the curved space analogue of $(\partial/\partial W_{\gamma})I_{\alpha\beta}$.

Note that we have the usual ambiguity in the definition of $I_{\alpha\beta}{}^{;\gamma}$ arising from the fact that $Z^{\alpha}D\bar{Z}_{\alpha} + \bar{Z}_{\alpha}DZ^{\alpha} = 0$. This ambiguity can be fixed by choosing the $I_{\alpha\beta}{}^{;\gamma}$ in the equivalence class such that $D(I_{\alpha\beta}{}^{;\gamma}D\bar{Z}_{\gamma})$ is trace free on its two form indices. (There is no obstruction to this choice since $D^2I_{\alpha\beta}$ is necessarily trace free on its two form indices, since the curvature is.) This is a general phenomena with the Chern connection.

From §2.3 we find:

$$DI_{\alpha\beta} = \begin{bmatrix} 0 & i\pi_{B}{}^{A'} \\ -i\pi_{A}{}^{B'} & 0 \end{bmatrix}$$

Where $\pi_{a} = \pi_{ab}dx^{b} = \eta_{a\beta}DZ^{\beta} + \bar{\eta}_{a}{}^{\beta}D\bar{Z}_{\beta}$

and $\eta_a{}^{\beta} = (0, \bar{\eta}_{aB'})$ where:

$$\bar{\eta}_{AA'B'} = \tilde{u}^{-1}(\Xi_{AA'B'C'} + T_{AA'}G_{0}{}_{B'C'} - (1/3)G_{00}T_{A(B'}\varepsilon_{C')A'})z^{C'}$$

So we obtain:

$$I_{\alpha\beta;\gamma} = \begin{bmatrix} 0 & i\bar{\eta}_{B}{}^{A'\gamma} \\ -i\bar{\eta}_{A}{}^{B'\gamma} & 0 \end{bmatrix}$$

Constraint equations:

In §3.3 it was pointed out that the constraint equations on the potentials for a spin two field were equivalent to the following condition on g_{AB}: $\varepsilon^{AB}(\partial/\partial\omega^{A})g_{BC} = 0$

Where $g_{AB} = dx^{C}\gamma_{cAB}$, is the Dolbeault representative for the second derivative of the corresponding homogeneity degree 2 twistor function.

We can duplicate this here. It was remarked above that $dz_{A} = I_{\alpha\beta}DZ^{\beta}$, should be regarded as the generalisation from linearised theory of the Dolbeault representative for $I_{\alpha\beta}\tilde{\partial}^{\beta}\tilde{\xi}g(W)$ where as before:

$\tilde{\partial}^{\alpha} = \partial/\partial W_{\alpha}$, and $\tilde{\xi} = I_{\alpha\beta}Z^{\alpha}\tilde{\partial}^{\beta}$.

So the above expression of the constraint equations becomes:

$$\tilde{\partial}^{\alpha}(I_{\alpha\beta}\tilde{\partial}^{\beta}\tilde{\xi}g(W)) = 0$$

In the curved case we have:

$$D(I_{\alpha\beta}DZ^{\beta}) = (DI_{\alpha\beta})DZ^{\beta}, \text{ since } D^{2}Z^{\alpha} = 0$$
$$= (I_{\alpha\beta;\gamma})DZ^{\gamma}\wedge DZ^{\beta} + (I_{\alpha\beta}{}^{;\gamma})D\bar{Z}_{\gamma}\wedge DZ^{\beta}$$

So clearly the constraint equations should be equivalent to:

$I_{\alpha\beta}{}^{;\beta} = 0$

Using the above expression for $I_{\alpha\beta}{}^{;\gamma}$ we find:

$$I_{\alpha\beta}{}^{;\beta} = (0, \bar{\eta}_{AB'}{}^{B'})$$

As remarked in §2.3 this is precisely the constraints:

$$\bar{\eta}_{AB'}{}^{B'} = (G_{0}{}_{A'B'}T^{B'}{}_{A} - (1/2)G_{00}T_{AA'})z^{A'}$$

Using the fact that $I_{\alpha\beta}$ is real we can write this condition equivalently in the following form:

$$I_{[\alpha\beta;\gamma]} = 0$$

A characterisation of the constraints in terms of the curvature of the Chern connection:

We can use this to obtain the conditions on the curvature of the Chern connection that the constraints be satisfied. However, to do this we must take a derivative of the above expression, so that we obtain an expression for the derivative of the constraints. These only imply constraints when the hypersurface is asymptotically flat or compact, and then they allow a cosmological constant.

Let $DI_{\alpha\beta}{}^{;\gamma} = I_{\alpha\beta}{}^{;\gamma\delta}D\bar{Z}_{\delta} + I_{\alpha\beta}{}^{;\gamma}{}_{\delta}DZ^{\delta}$

Then $D^2I_{\alpha\beta} = (I_{\alpha\beta}{}^{;\gamma}{}_{\delta} - I_{\alpha\beta;\delta}{}^{\gamma})D\bar{Z}_{\gamma}DZ^{\delta}$

However $D^2I_{\alpha\beta} = I_{\gamma[\alpha}\Pi^{\gamma}{}_{\beta]}{}^{\delta}{}_{\epsilon}D\bar{Z}_{\delta}DZ^{\epsilon}$. So we see that the derivative of the constraints becomes: $I_{\gamma[\alpha}\Pi^{\gamma}{}_{\beta}{}^{\delta}{}_{\delta]} = I_{\alpha\gamma}\Pi^{\alpha}{}_{[\beta}{}^{\gamma}{}_{\delta]} = 0$ (*)

It should be possible to regard this as an algebraic condition determining $I_{\alpha\beta}$, along with all the other conditions that $I_{\alpha\beta}$ must satisfy. That is $I_{\alpha\beta}$ should be real, simple, $I_{\alpha\beta}X^{\alpha\beta} = 2$, $Z^{\alpha}I_{\alpha\beta} \neq 0$. (Note that $X^{\alpha\beta}$ is determined by the curvature by $D^2X^{\alpha\beta} = 0$.) Then, with either compactness or asymptotic flatness we should hopefully be able to deduce that $I_{\alpha\beta}{}^{;\beta} = 0$ follows from (*). Various details remain to be checked.

If the above is true then the existence of an appropriate solution to (*) is a characterisation of those twistor C.R. manifolds corresponding to initial data sets satisfying the constraints.

Note that if we do allow a cosmological constant we can allow $I_{\alpha\beta}$ to become nonsimple in order to ensure $I_{\alpha\beta}{}^{;\beta} = 0$.

§4.2: The evolution:

I shall now attempt to derive the expression for the evolution of the C.R. structure following the formula discovered for linearised theory.

The formula in linearised theory was:

$$(d/dt)\bar{\delta} = \{\tilde{\delta}^{\alpha}(\tilde{\xi}^{\bar{a}})g^{\bar{\alpha}}(W,\bar{W})d\bar{W}_{\bar{\alpha}}\}\big|_{W=\overset{\vee}{Z}}\,\delta_{\alpha}$$

This is where a problem arises in formulating the analogy with linearised theory. As mentioned above, we at best know the analogue of:

$$I_{\alpha\beta} + I_{\gamma[\alpha}\tilde{\xi}\tilde{\delta}^{\gamma}g_{\beta]}(W,Z)$$

It is now impossible to distinguish $\bar{W}$ from Z in this expression so that when we restrict to $W = \overset{\vee}{Z}$, we do not know whether any particular Z in the expression should be left alone or to be set equal to $\bar{\overset{\vee}{Z}}$. We can however write an expression for the evolution motivated from the above by making a specific well defined choice of which Z's should remain fixed, and which should be set equal to $\bar{\overset{\vee}{Z}}$.

To motivate the expression consider taking the above linearised formula for $I_{\alpha\beta}$, contracting Z^{α} into it and Lie dragging it along $\partial/\partial W^{\alpha}$. We then obtain the expression:

$$\mathcal{E}' = \{\bar{\delta}^{\alpha}\bar{\xi}^{\bar{a}}g_{\beta}(\bar{Z},Z)dZ^{\beta}\}\delta_{\alpha}$$

(I have here restricted this expression to $\mathcal{N}$, ie $W = \bar{Z}$.)

If we could have distinguished between Z and $\bar{W}$, we would now obtain the expression for the evolution by restricting the expression to $W = \overset{\vee}{Z}$. Instead we can define a holomorphic vector valued antiholomorphic one form $\mathcal{E}$ by defining:

$$\mathcal{E} = \{\bar{\delta}^{\alpha}\bar{\xi}^{\bar{a}}g_{\beta}(\bar{Z},Z)(\vee^{*}d\bar{W}^{\beta})\}\delta_{\alpha}$$

Here $\vee^{*}d\bar{W}^{\alpha}$ is the pullback of $d\bar{W}^{\alpha}$ to $\mathcal{N}$ from $\mathcal{N}^{*}$ with the map $\vee: \mathcal{N} \to \mathcal{N}^{*}$. In curved space this is given by taking the expression for $D\bar{W}^{\alpha}$ in terms of (x,w_{A}), and setting $w_{A} = T_{A}{}^{A'}z_{A'}$. This yields:

$$\vee^{*}(D\bar{W}^{\alpha}) = (\hat{z}_{A'}dx^{AA'},d(\hat{z}_{A'}))$$

The derivation of this expression for $(\vee^{*}D\bar{W}^{\alpha})$ uses the triviality of

the Chern connection along the holomorphic $\mathbb{CP}^1$'s in order to identify $\mathbf{T}^\alpha$ at $(x,z_{A'})$ and $\vee(x,z_{A'})$. This requires extra structure over and above knowledge of $\vee$.

Rather than use the map $\vee$, we can instead perform the map in the tangent space using a two index "time" twistor $H_{\alpha\beta}$, and put $H^{\alpha\beta}D\bar{Z}_\beta$ instead of $(\vee^* D\bar{W}^\alpha)$. Certainly knowledge of $\vee$ and the triviality of the Chern connection along the holomorphic $\mathbb{CP}^1$'s imply knowledge of $H_{\alpha\beta}$. $H_{\alpha\beta}$ is given by:

$$H_{\alpha\beta} = \begin{bmatrix} K_{AB} & iT_A{}^{B'} \\ -iT^{A'}{}_B & 0 \end{bmatrix}$$

Where $K_{AB} = u^{-1}K_{AA'BB'}\hat{z}^{A'}z^{A'}$, and K_{ab} is the extrinsic curvature.

The analogue of $\mathfrak{E}$ in curved space is now:

$$\mathfrak{E} = I_{\alpha\beta}{}^{;\gamma}Z^\alpha(H^{\beta\delta}D\bar{Z}_\delta)\partial/\partial Z^\gamma$$

Where now the indices are those associated to the Chern connection.

(Note that $\mathbf{T}^\alpha$ was identified with $TN/\bar{H}$, so by $A^\alpha\partial/\partial Z^\alpha$ I mean the element of $TN/\bar{H}$ corresponding to A^α.)

We have the following result:

Proposition: When the constraints are satisfied, $\mathfrak{E}$, as defined above, provides the evolution of the $\bar{\partial}$ operator in accordance with the Einstein vacuum equations.

Proof:

First I will calculate the actual evolution of the $\bar{\partial}$ operator in general. We have:

$$\bar{\partial} = u^{-2}dx^{AA'}\hat{z}_{A'}\bar{z}_A z^{\vee B}z^{B'}\nabla_{BB'} + d\bar{z}_A\partial/\partial\bar{z}_A$$

In order to calculate the rate of change of $\bar{\partial}$ from one hypersurface

to the next we will Lie drag it along $u^{-1}V$, where V is the null geodesic spray, $V = z^{A'}\bar{z}^{A}\nabla_{AA'}$. A short calculation yields:

$$(d/dt)\bar{\partial} = \pounds_{V/u}\ \bar{\partial}_B = u^{-2}dx^{AA'}\hat{z}_{A'}\bar{z}_A\Psi_{A'B'C'D'}z^{A'}z^{B'}z^{C'}\partial/\partial z_{D'}$$

Where I have ignored all the terms with antiholomorphic vectors, and one forms that annihilate antiholomorphic vectors. The only term that contributes is the holomorphic part of the commutator of V and $\overset{\vee}{z}^{A}z^{A'}\nabla_{AA'}$. This just yields the Weyl spinor term above.

We now come to calculating $\mathcal{E}$.

The terms in the expression for $\mathcal{E}$ are already calculated and yield:

$$I_{\alpha\beta}{}^{;\overset{\vee}{\gamma}}Z^{\alpha}(\overset{*}{\vee}D\bar{W}^{\alpha}) = z^{A'}\bar{\eta}_{A'A}{}^{\overset{\vee}{\gamma}}\hat{z}_{B'}dx^{AB'}$$

We have $\bar{\eta}_a{}^{\beta} = (0,\bar{\eta}_{aB'})$ so as a vector field $\bar{\eta}_a{}^{\overset{\vee}{\gamma}}$ corresponds to:

$$\bar{\eta}_a{}^{\beta}\partial/\partial Z^{\beta} = \bar{\eta}_{aB}\ \partial/\partial z_{B'}$$

Furthermore when the constraints are satisfied:

$$\bar{\eta}_{AA'B'} = u^{-1}\Xi_{AA'B'C'}z^{C'}$$

Putting all this together we obtain:

$$\mathcal{E} = u^{-1}dx^{AA'}\hat{z}_{A'}\Xi_{AB'C'D'}z^{B'}z^{C'}\partial/\partial z_{D'}$$

If we set $(d/dt)\bar{\partial} = \mathcal{E}$ we effectively force $\Psi_{A'B'C'D'} = \Xi_{A'B'C'D'}$, but $\Psi_{A'B'C'D'} - \Xi_{A'B'C'D'} = \Phi_{A\ B'C'D'}$, where Φ is the trace free part of the projection of the Ricci tensor into the hypersurface. The vacuum evolution equations are precisely $\Phi_{A'B'C'D'} = 0$.□

Remarks:

If one regards knowledge of $I_{\alpha\beta}$ and $H_{\alpha\beta}$ as allowable twistorial information, the above derivation of the evolution and constraints are purely twistorial in the sense that the information of the space-time points was not used. (The information of the location of the holomorphic curves corresponding to space-time points is essentially encoded in the

knowledge of $\chi^{\alpha\beta}$, which was not used in any of the above formulae.)

It is possible to give a derivation of the expression for the evolution in a more geometrical vein, that is, without using the Chern connection. The basic idea is to work on the direct sum of the primed and unprimed spin bundles restricted to the (real) hypersurface, which is to be regarded as a convenient subset of A. Then use $\imath$ as the evolution hamiltonian, convert it to a holomorphic vector valued holomorphic one form using a Poisson bivector. (This is in fact well defined.) The problem arises that $\imath$ has homogeneity degree (2,0) so that we cannot use the symplectic structure on A to do this, since it is only well defined on hamiltonians homogenous of degree (n,n). So one has to cook a Poisson bivector mimicing $(\partial/\partial Z^{\alpha}) \wedge (\partial/\partial W_{\alpha})$ from flat space. If we define:

$$\partial/\partial\tilde{\omega}^{A'} = \tilde{u}^{-1} w^{A} \nabla_{AA'} - \tilde{u}^{-2} \bar{w}_{A'} w^{B} \bar{w}^{B'} \nabla_{BB'}$$

The Poisson bivector is then defined to be:

$$(\partial/\partial\omega^{A}) \wedge (\partial/\partial w_{A}) + (\partial/\partial\tilde{\omega}^{A'}) \wedge (\partial/\partial z_{A'})$$

Since we are only interested in the holomorphic vector field part of the answer, we obtain:

$$\mathcal{E}' = \pounds_{(\partial/\partial\tilde{\omega}^{A'})} \imath \, (\partial/\partial z_{A'})$$

restricting this to $w_{A} = T_{A}^{\ A'} z_{A'}$ then yields the correst answer for the evolution as given above.

The only difficulty with this approach is that in flat space-time the Poisson bivector above doesn't quite agree with $(\partial/\partial Z^{\alpha}) \wedge (\partial/\partial W_{\alpha})$. Nevertheless I feel that it should be possible to make sense of this approach if it is possible to work on an analogue of a neighbourhood of $\mathbb{A}$ in $\mathbb{T}\times\mathbb{T}^{*}$, and thus use the correct Poisson bivector.

Note also that the constraints would also translate into the more reasonable form, on the real slice of $\mathbb{T}\times\mathbb{T}^{*}$: $\partial\tau = 0$.

That is that τ should be holomorphically closed.

Such a formulation would be more attractive than the above.

Using different lapse functions:

The information of the lapse is introduced in the twistor $H_{\alpha\beta}$. In the above the lapse was set equal to one. We can however change this by using a different $H_{\alpha\beta}$.

$H_{\alpha\beta}$ is the curved space analogue of the twistor which defines the hypersurface in linearised theory. We can effectively define $H_{t\alpha\beta}$ by:

$$X^{\alpha\beta}H_{t\alpha\beta} = x^0 - t,$$

where x^0 is some time coordinate, and t is the constant value this time coordinate takes at the hypersurface.

So the hypersurface at time t is defined by $X^{\alpha\beta}H_{t\alpha\beta} = 0$. This is appropriate for any choice of time function, x^0, but we must then use an $H_{t\alpha\beta}$ that depends on x. On $x^0 = t$ we will then have $DX^{\alpha\beta}H_{t\alpha\beta} = 0$ so that:

$$H_{t\alpha\beta} = \begin{bmatrix} K_{AB} & i\nabla_A{}^{B'}x^0 \\ -i\nabla^{A'}{}_B x^0 & (x^0 - t)\varepsilon^{A'B'} \end{bmatrix}$$

where K_{AB} has yet to be determined. The above derivation suggests that K_{AB} should be: $K_{AB} = u^{-1}nK_{AA'BB'}z^{A'}\hat{z}^{B'}$

Where $n = T^a\nabla_a x^0$ is the lapse.

This yields a non skew $H_{t\alpha\beta}$. For the above derivation we only need the skew part.

Clearly we can chose any such "time" twistor in curved space, and different choices of x^0 will give rise to different lapse functions. (We must of course restrict x^0 so that it is constant on the hypersurface we are considering.) Of particular interest in questions concerning evolving initial data is being able to use a maximal slicing of space-time. It is

easily seen that we have the following lemma:

Lemma: The slicing is maximal iff:

$$H_{t\alpha\beta}I^{\alpha\beta} = 0.$$

Proof From the above we see that this equation is just $K = 0$, where K is the trace of the extrinsic curvature. This is just the maximal slicing condition.□

The initial value problem:

It would be nice to extend the above results to a resolution of the initial value problem for hypersurface twistors. Apart from the C.R. structure itself we require knowledge of $I_{\alpha\beta}$ and $H_{\alpha\beta}$ in order to evolve C.R. structure. We must then however know how to evolve $I_{\alpha\beta}$ and $H_{\alpha\beta}$.

A well defined way to evolve $I_{\alpha\beta}$ and $H_{\alpha\beta}$ is to Lie drag the forms $I_{\alpha\beta}DZ^{\alpha}\wedge DZ^{\beta}$ and $H_{\alpha\beta}DZ^{\alpha}\wedge DZ^{\beta}$ along the null geodesic spray.

As mentioned above, $I_{\alpha\beta}DZ^{\alpha}\wedge DZ^{\beta}$ only picks up antiholomorphic parts, when evolved in vacuum which can be unambiguously discarded. Unfortunately the evolution of $H_{\alpha\beta}$ is not so well behaved. For flat hypersurfaces in linearised theory we had $(d/dt)H_{t\alpha\beta} = I_{\alpha\beta}$. However for extrinsically curved hypersurfaces we have correction terms involving the extrinsic curvature. I have not yet seen how to incorporate these extra terms in a natural way. Nevertheless they are of course calculable from the initial data as specified above.

Further development:

In order to have a cast iron initial value theory for twistor C.R. manifolds we must understand the evolution of $H_{\alpha\beta}$. In particular it would be nice to have some twistorial understanding of how to propagate a maximal slicing of space-time.

Another approach to this problem might be to try to derive $H_{\alpha\beta}$ from the C.R. structure. It is not clear, however, that one should expect a twistor C.R. manifold to be able to tell the time. It certainly cannot in flat space. However in linearised theory it might be able to tell the time by measuring the amount of deformation by the S.D. Weyl curvature compared to the A.S.D. Weyl curvature. In particular when the constraints are satisfied we have that $DI^{\alpha\beta}H_{\alpha\beta} = 0$. However $X^{\alpha\beta}H_{\alpha\beta} = 0$ is also required in order to get the correct result for the evolution and we do not wish to have to know $X^{\alpha\beta}$. This is equivalent to knowing the location of the holomorphic curves corresponding to events on the hypersurface.

Another irritation with the above is that one of the primary motivations for examining the evolution of hypersurface twistors is to produce symplectic transformations of twistor space corresponding to unitary transformations of the quantum theory.

Although the above expression was produced from an evolution hamiltonian, we then performed the $\sim$ map to provide the final expression. Since the evolution preserves the symplectic form, it is trivial that the evolution can be expressed in a hamilonian fashion directly. It would be nice to find an elegant expression for this hamiltonian

Another development of interest would be to generalise the above to finite evolutions. To do this it would probably be more appropriate to work with a Cech framework, since nonlinear phenomena are then more easy to describe.

One would assume analyticity and embed the C.R. manifold in the hypersurface twistor space. The condition $I_{[\alpha\beta;\delta]}$ is probably necessary and sufficient to be able to extend $I_{\alpha\beta}DZ^{\alpha}\wedge DZ^{\beta}$ off N. Furthermore one

could assume that one has solved the Monge Ampere equations on a neighbourhood of N in the twistor space and thus obtain a Kahler connection which would restrict to N to be the Chern connection. So one has a lot of structure to work with. Especially attractive is the prospect of being able to evolve for a finite time using a generating function obtained by integrating the evolution hamiltonian, referred to above, along its integral curves.

Another interesting problem would be turn some of the above ideas, into a holomorphic construction for solutions of the constraint equations. The conditions $I_{[\alpha\beta;\gamma]} = 0$ are very geometric. However, the existence of a solution to this equation, along with all the other necessary relations that $I_{\alpha\beta}$ must satisfy, provides conditions on the curvature of the Chern connection which may be difficult to satisfy holomorphically.

Another generalisation would be to include some conformally invariant matter fields, e.g. electromagnetism. This should be straightforward to study in linearised theory, and as we have seen, the results tend to generalise to curved space-time in a relatively straightforward fashion.

§5 Some comments on superficial twistors and the canonical formalism.

In this chapter I will show that the Sparling three-form can be interpreted as the hamiltonian of general relativity. I will then show how the angular momentum twistor associated to a two surface in space-time can be interpreted as a matrix of hamiltonians generating deformations of some spanning three-surface of the two surface. Further problems will then be discussed.

§5.1: The Sparling three form:

Define:

$$\Lambda = iz_{A'}d\bar{z}_{A'}dx^{AA'}$$

$$\Gamma = idz_{A'}\wedge d\bar{z}_{A'}dx^{AA'}$$

In [25] G.A.J. Sparling pointed out that:

$$d\Lambda = \Gamma - z^{A'}\bar{z}^{A}G_{AA'b}d^3x^b$$

Where G_{ab} is the Einstein tensor. So, in particular, $d\Gamma = 0$ iff the vacuum equations are satisfied.

The usual expression for the gravitational hamiltonian is (modulo boundary terms):

$$H = \int_\Sigma n^a G_{ab} d^3x^b \qquad (*)$$

Where n^a is the lapse shift vector and Σ is some space-like hypersurface on which the hamiltonian is being evaluated. So if n^a is taken to be null, $n^a = \alpha^A\alpha^{A'}$ we can write:

$$H = \int_\Sigma (\Gamma - d\Lambda)|_{z^{A'}=\alpha^{A'}}$$

Casting off the boundary term we find:

$$H = \int_\Sigma \Gamma|_{z^{A'}=\alpha^{A'}} \qquad (**)$$

The only nontrivial point about showing that this is the gravitational hamiltonian is arguing that the expression involving Γ alone has the appropriate boundary terms.

Regge and Teitelboim have provided the condition for a hamiltonian

agreeing with (*) modulo boudary terms to be the correct hamiltonian of general relativity. This condition is essentially that the hamiltonian should give the appropriate equations of motion "quasi-locally".

This is articulated as follows:

Let (p^{ab},q_{ab}) be coordinates on the gravitational phase space represented in terms of tensor fields on Σ where Σ is some 3-surface with boundary. Σ is regarded as a finite three surface, so that the initial data (p^{ab},q_{ab}) has no particular fall off conditions near $\partial\Sigma$. Let (r^{ab},s_{ab}) be a variation of (p^{ab},q_{ab}), then for a general choice of H one will have:

$$\delta H(r^{ab},s_{ab}) = \int_{\Sigma} A^{ab}s_{ab} + B_{ab}r^{ab}\, d^3x + \oint_{\partial\Sigma} C^{ab}s_{ab} + D_{ab}r^{ab}\, d^2x$$

Only if the surface terms vanish and there are no derivatives of (r^{ab},s_{ab}) in the integrand can we argue that:

$\delta H/\delta s_{ab} = A^{ab}$ and $\delta H/\delta r^{ab} = B_{ab}$ and impose hamiltons equations:

$$dp^{ab}/dt = -A^{ab} \text{ and } dq_{ab}/dt = B_{ab}$$

If this were not the case the formally correct expressions for $\delta H/\delta s_{ab}$ would be nonlocal in the p's and q's.

We have the following lemma:

Lemma: The hamiltonian in (**) has the appropriate boundary terms.

Proof: To see this consider the extended phase space consisting of spin coefficients and solder forms: $(\gamma_{\underline{aB}\,\underline{C}\,\underline{C}'},\bar{\gamma}_{\underline{aBC}},dx^{A\underline{A}'})$.

Clearly the variation of the hamiltonian (**) is ultralocal in these coordinates, and no problem with boundary terms can arise. However we now have to face the fact that we now have more hamiltonians and constraints. The new constraints are merely that the connection constructed out of the spin coefficients should be torsion free, and they generate frame rotations

and so do not give rise to any observables of interest. The variation of the new hamiltonians can easily be written so that they are ultralocal in the variation.□

Remark:

The above observation was made by several people simultaneously after various talks by Abhay Ashtekar, see [2,25].

G.A.J. Sparling originally motivated the form Λ by suggesting that $\Lambda+\bar{\Lambda}$ should be regarded as the natural skew part of the tensor of which the Fefferman metric was the symmetric part. This was an inspired mistake since, as noted in §2.3, the natural skew part is just the symplectic form on the spin bundle, $\omega = \Lambda - \bar{\Lambda}$.

The total hamiltonian:

The total hamiltonian of general relativity together with matter fields is: $$H_{tot} = \int_\Sigma (G_{ab} + 8\pi T_{ab})n^a \, d^3x^b$$

Where T_{ab} is the stress energy tensor for the matter fields. Note that it vanishes when the field equations hold. We have seen above that the G_{ab} term should be supplemented by a boundary term, $d\Lambda$. This yields, for null lapse-shift vector as above, in the presence of the field equations:

$$H_{tot} = \int_\Sigma d\Lambda \,|_{(z^{A'}=\alpha^{A'})} = \int_{\partial\Sigma} \Lambda \,|_{(z^{A'}=\alpha^{A'})}.$$

So the total hamiltonian is given as a surface integral when the field equations are satisfied. This as one expects. Note that Λ is the Witten-Nester form as used by Witten to prove the positive energy theorem (see [2,25]).

The angular momentum twistor:

The angular momentum twistor associated to a two surface, σ, is meant to

describe the total linear and angular momentum of the gravitational field and sources on a 3-surface spanning σ.

In symplectic geometry there is a direct correspondance between observables and canonical transformations of the phase space. The canonical transformation corresponding to an observable is generated by the hamiltonian vector field of the observable.

Observables such as linear and angular momentum correspond to canonical transformations that arise from Lie dragging the fields along killing vector fields.

In curved space there are no killing vectors in general, however all we need in the above expression is the notion of a killing vector at the two surface, σ. The restriction of Killing s equation to σ has too many solutions to be useful in this respect. However a two surface twistor provides a notion of a Killing vector at σ.

$$Z^{\alpha} = (\omega^{A}, \pi_{A'}) \leftrightarrow k^{AA'} = \omega^{A}\pi^{A'}$$

This is provided by the kinematical sequence, [18]. Setting $\omega^{A} = \alpha^{A}$ and $\pi^{A'} = \alpha^{A'}$ in the above formula for the hamiltonian yields:

$$H_{tot} = \oint_{\sigma} \pi_{A'} d\omega_{A} \wedge dx^{AA'} = \oint_{\sigma} \pi_{A'}\pi_{B'} d^{2}x^{A'B'}$$
$$= A_{\alpha\beta} Z^{\alpha} Z^{\beta}$$

The first step follows by use of the twistor equation: $\nabla_{AA'} \omega^{B} = \varepsilon_{A}{}^{B} \pi_{A'}$ and the second by use of Tod's formula for the angular momentum twistor.

Remarks:

Unfortunately the above results do not take us very far. They are more an indication that there is work to be done. I will just mention some of the ideas I feel should be pursued.

Quasi-local mass.

Some insight may be obtained into the difficulties of the quasi-local

mass construction by formulating a hypersurface twistorial interpretation. This is easily done using the detailed analogies between linearised theory and curved space-time hypersurface twistors. This may give some insight into natural adjustments of either the formula, or its interpretation, which would enable the construction to make better sense in the contorted case.

Canonical formalism:

I have made some attempts to reformulate the canonical formalism in terms of hypersurface twistors. In particular I have found a formula for the symplectic form on the gravitational phase space in terms of variations of the C.R. structure. I have not presented it here as I am still not sure of its naturality, and because it will probably be superseded with an expression using the indexed cohomology classes presnted in §3.3.

A problem with using the above expressions for the hamiltonian are that the forms Λ and Γ are not twistorial, they do not descend to twistor space in the flat case. My hope is that it may be possible to regard them as primary parts of some twistorial structure. For instance:

$$\Lambda = X_{\alpha\beta} I^{\beta\dot{\gamma}} D\bar{Z}_{\dot{\gamma}\wedge} DZ^{\alpha}$$

is the primary part of $I^{\beta\dot{\gamma}} D\bar{Z}_{\dot{\gamma}\wedge} DZ^{\alpha}$.

One of the motivations for having a twistorial version of the canonical formalism is to make contact with standard approaches to the quantisation of gravity. In particular ideas of A. Ashtekar [3] seem particularly suitable for a twistorial interpretation. (However, whether it is a suitable approach to the quantisation of gravity is open to a certain amount of doubt.)

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