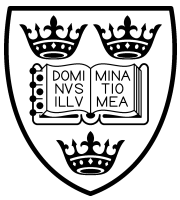
Sampling the Sensitivity of Climate Models



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"When you have eliminated the impossible,
whatever remains, however improbable,
must be the truth."

Sherlock Holmes

"Mi illumino di immenso."

"Mattina", di G. Ungaretti

This thesis is dedicated to

Ivan, who taught me to walk in the sun,
Katerina, because we'll never give up,
Daria, "life is one, so play cool".
Will, because it was funnily strange
to be shot by a banana,
Monica, because next time we are going to use proper shoes,
climbing Hertford College.
Davide e Andrea: because you always let me feel home,
and sometimes you don't need neither to speak,
a joke or a box of charms, it is enough !
Richard, for the sweet smile behind a chocolate,
Kate, because I know you are there and care about me,
Gareth, because reality is different,
and Tym, because I love to be madly in trouble !

For the dodgy nights we spent together,
in the Comlab or making pasta or cycling around shouting songs.
For the problems we shared, the hopes, the joys and the delusions,
for the long chats in front of a cup of tea,
for all the things we learned one from each other
that are not written in any book,
for all the true people we meet every day.

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Abstract

The sensitivity of climate models in particular, and nonlinear models in general, is a topic of great interest. Quantifying this sensitivity by brute force numerical exploration is often computationally untractable due to the large number of parameters involved, and it is widely hoped that a small number of "effective parameters" (linear combinations of the physical parameters) may dominate the model response and hence ease the task of sensitivity analysis.

A new method for identifying this "effective dimension" of the model response is introduced and applied to two simple models. General issues of identifying the response are discussed. No low dimensional response is found in the more complex model considered.

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Introduction

Modern climatological models typically contain a large number of adjustable parameters, whose values are often based on empirical estimates or on measurements. In order to assess the robustness and significativity of the results obtained in climatological simulations, it is of paramount importance to measure the sensitivity of models to pertubations in the parameter values used.

Performing the sensitivity analysis of a climate model often requires a large computational effort, especially if in the model has a large number of parameters. A basic question of climatologists is thus to investigate whether it is possible to find methods which allow for neglecting some parameters when performing the sensitivity analysis of a climate model, thereby avoiding a lot of computational work.

The aim of this project is to develop and to test a new method to investigate the effective dimensionality of the model response and, eventually, to identify a reduced set of linear combinations of parameters which is sufficient to explore the model response.

The background ideas behind the method presented can be summarised as follows: we consider a generic climate model and construct a map which associates a climatological value $\langle \vec{x} \rangle$ with each parameter choice. Choosing a range of parameters and applying this map, we obtain a set of points in the state space, whose dimension generically is equal to the dimension of the parameter space. If this set is an hypersurface of a smaller dimension, this could imply that in the model analysed there are some parameters that depend on others, and that there exist therefore some relations between the parameters, which describe an hypersurface in the parameter space. Then, to perform a sensitivity analysis, it would not be necessary to perturb all the parameters in the initial range: we would only need to perturb the parameters corresponding to the hypersurface. If *all* these hypotheses hold, the aim of climatologists would be, then, to find this hypersurface.

In this work we are going to analyse whether this is the case for two models: the classical Lorenz '63 model and a 36 dimensional version of the Lorenz '95 model [Lor63, Lor95].

This dissertation is organised in three chapters. In the first chapter we introduce the basic concepts and assumptions we consider in the treatment, and then we expound in more detail the method to be investigated and the aim of our analysis. In the second chapter we perform the analysis on the Lorenz '63 model, presenting the methodologies used, and finally, in the third chapter, we perform the same analysis on the Lorenz '95 model. In the conclusions we summarise the results obtained.

Chapter 1

The basic ideas

In this chapter we introduce the quantities and concepts with which we will deal with in this work: we are going to give some definitions, present some general properties and state the assumptions made in the treatment.

In the first section we introduce the concept of climate model in general and then we describe which kind of climate models we will consider in our treatment. In the second section we define what is meant by climatological values, expound some of their properties and we explain how they will be computed in our analysis. In the third section we introduce the concept of sensitivity of a climate model in general, we define then the macroscopic climate sensitivity and finally we introduce the concept of sensitivity diagram, showing some of its properties.

With the knowledge of these concepts, in the last section, we expound the background ideas of this work, explaining the reasons and the aims of this project.

1.1 Climate models

Climate models are an attempt to simulate the climate system. For this reason it is useful, before describing the climate models themselves, to spend few words introducing the concept of climate system.

Many definitions of climate system have been attempted: most of them present it as the " ... totality of the atmosphere, hydrosphere, cryosphere, biosphere and geosphere and their interactions ... ". Usually the idea of "interaction" between these environmental elements is particularly emphasised, so that also the physical laws governing the system, the internal forcings, feedback mechanisms, etc. are included in the definition itself.

In our treatment we are going to consider the climate system not from a detailed physical point of view, but from a more general and mathematical point of view: we will regard it as a dynamical system, i.e. a system whose future status evolves from its present status according to precise physical laws or mathematical equations.

The climate system is a dynamical system in transient balance, "transient" because of the presence of external forcing affecting the climate of the planet, but usually it is possible to consider it as a closed dynamical system, because the external forcings

are varying very slowly.
(For further details see [MHS97]).

The climate models we are going to consider in this work consist of systems of deterministic equations, hence the system is not affected by random phenomena and the dynamics of the physical system modelled is uniquely determined by the system itself.

We assume that our models consist of a finite system of ordinary differential equations (ODEs): this choice is not so restrictive as it could seem, in fact, usually, any closed dynamical system of finite mass can be treated as a finite collection of point masses and so the equations governing their motion are a finite set of ordinary differential equations. It can happen that this set of equations is so highly intractable that the collections of point masses must be approximated by a continuous distribution of mass: continuous quantities are so introduced in the model and then the governing laws are expressed by a set of partial differential equations: these equations are rarely solvable analytically (especially if they are nonlinear) and solving them with numerical procedures implies a substitution of the continuous quantities by a finite set of quantities as functions of time (for example the values of the continuous variables on the points of a grid), so that the governing laws become a finite set of ordinary differential equations again. (See [Lor63]).

Generically our climate model can then be written as

$$\frac{d\vec{x}}{dt} = f(\vec{x}; \vec{\mu}) \ , \tag{1.1}$$

where $\vec{x} \in \mathbb{R}^m$ is a vector containing the variables, i.e. the physical quantities of the modelled dynamical system, $\vec{\mu} \in \mathbb{R}^p$ is a vector containing the parameters of the model and $f : \mathbb{R}^{m+p} \to \mathbb{R}^m$ is a function expressing the relations existing between variables, parameters and the derivatives of the variables with respect to time t.

Given a fixed set of parameters $\vec{\mu}$, the solution of the system (1.1) is given by a trajectory $\vec{x}(t)$: it describes the future state of the climate and, since the system is deterministic, it exists and is uniquely determined by the initial condition $\vec{x}_0$, describing the present state of the climate.

The system (1.1) is an autonomous system. We observe, in fact, that it is always possible, formally, to write a system of ordinary differential equations as an autonomous system of ordinary differential equations: if the time appears explicitly on the right hand side of the equation, we could substitute it with a new variable and then we could add to the equations of the system the trivial differential equation which states that the derivative of the new variable with respect to the time is equal to one.

Since our study concerns climate, we expect that our dynamical system behaves in a interesting way from a climatological point of view, hence we expect that the solution of our system does not diverge as the time evolves, but reaches a "climatologically well defined status" (which we will define properly in the next section). In our treatment we consider then only models whose solutions remain bounded in a finite region of the state space as the time increases.

Climate models often exhibit chaotic behaviour: for a deterministic system with bounded solutions this means that for an open set of initial conditions the trajectories $\vec{x}(t)$, solutions of our system, do not set to fixed points or to periodic or quasi-periodic orbits as the time goes to infinity, but they have an aperiodic long term behaviour. In this case the solutions exhibit sensitive dependence on initial conditions, i.e. they are unstable with respect to modifications of small amplitude so that two trajectories starting nearby separate exponentially fast (mathematically this phenomenon [1] occurs for system with positive Lyapunov exponents).

In our work we are going to consider models that exhibit chaotic behaviour. This assumption implies that the system (1.1) is nonlinear (and hence that f is a nonlinear function), since nonlinearity is a necessary condition for deterministic chaos. Further the dimension of the state space must be at least equal to three[2], in fact the Poincaré-Bendixon theorem implies that deterministic chaos cannot occur in a two-dimensional autonomous system of ODEs (and deterministic chaos cannot occur in a one-dimensional continuous dynamical system).
(For further details see [Mal00, Lor63, Str94, MHS97]).

In our treatment we are going to consider two models introduced by E.N. Lorenz in the years 1963 and 1995 (see [Lor63], [Lor95]).

We will describe these models more precisely in the next chapters, in this context we just want to summarise their general properties: both these models consist of a finite system of deterministic autonomous ordinary differential equations, their solutions lie in a bounded region of the state space, they are governed by nonlinear evolution laws and both exhibit chaotic behaviour for certain choices of the parameters. The former is a three-dimensional system, and so its solutions can be easily visualised as trajectories in a three-dimensional space; the latter is a N-dimensional system, where N is a finite number greater than three which can be chosen arbitrarily.

The Lorenz '63 is one of the most simple but complete chaotic models: it is for these reasons that we choose to analyse this easily visualizable "toy model" before investigating the more complicated Lorenz '95 model.

1.2 Climatological values

In this section we introduce the main quantities we are dealing with in this work: the climatological values.

Since our study concerns climate, our interest is focused on quantities which describe the behaviour of the physical system averaged over a long time period. Then, given a model described by the system (1.1), rather than in its solution $\vec{x}(t)$, we are interested in the mean of the solution on a large time interval. Qualitatively this

[1] The same phenomenon occurs in most meteorological models, because of the high nonlinearity of their processes. This is the reason why, because of the inability to specify accurately the initial conditions, forecasts diverge from the real weather, or at least from other forecasts, over a period of few days.

[2] Climate models typically have dimensions greater than 10^4, so $m > 3$ is not a constraint.

quantity, which we refer to as climatological values, can be thought as the climate state reached by our dynamical system in a climatological time period.

We define the analytical climatological value as

$$\langle \vec{x} \rangle_\infty := \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \vec{x}(t)\ dt\ . \tag{1.2}$$

Analogously we define as analytic climatological values each component of the vector $\langle \vec{x} \rangle_\infty$. (See [LAH00]).

These quantities are well defined since we have assumed that the solution $\vec{x}(t)$ of our system (which exists and is uniquely determined given an initial condition, since the system is deterministic) remains circumscribed in a bounded region of the state space: for this reason the trajectory $\vec{x}(t)$, as the time goes to infinity, either settles to a fixed point in the state space, or starts turning on a periodic orbit, or approaches a chaotic attractor, hence the analytic climatological value results is either the fixed point, or the baricentre of the periodic orbit or of the attractor.

Moreover we observe that the analytic climatological value not only depends on the system itself, since we are considering a deterministic system, but also on the initial condition: the trajectory $\vec{x}(t)$ could end up to different fixed points or periodic orbits or even different attractors, depending on which basin of attraction the initial condition lies in.

Since in our treatment we are going to use a numerical approach, we cannot use the analytical definition of climatological value, hence we introduce the analogous concept of numerical climatological value

$$\langle \vec{x} \rangle := \frac{\Delta t}{\tau_{\text{end}} - \tau_{\text{trans}}} \sum_{t_n = \tau_{\text{trans}}}^{\tau_{\text{end}}} \vec{x}(t_n)\ , \tag{1.3}$$

where $\vec{x}(t_n)$ is the numerical solution of our nonlinear system evaluated at the time t_n, Δt is the time step of the numerical method used, τ_{trans} is called the transient time and is the time from which we start to consider the solution to compute its mean, and τ_{end} is the length of the time interval in which we perform our numerical integration.

As before, we define as numerical climatological value each component of the vector $\langle \vec{x} \rangle$ and, as before, these quantities are well defined, assuming we are developing our numerical integration with a stable algorithm.

The analytical climatological value is not equal to the numerical climatological value, but a good approximation of it can be obtained following the procedure described below, and for this reason, in our work we will simply use the term climatological value, overlapping sometimes the two concepts.

To evaluate $\langle \vec{x} \rangle$ we have to choose the transient time τ_{trans} so that the solution $\vec{x}(t_n)$ for $t_n \geq \tau_{\text{trans}}$ is "near" its final climate state: in this way we cut the transient solution $\vec{x}(t_n)$ for $t_n < \tau_{\text{trans}}$ arising from the choice of a particular initial condition, which otherwise would introduce an error in evaluating the averaged behaviour of the system (figure 1.1).

Further, to have a significant evaluation of $\langle \vec{x} \rangle$, we have to choose the length of the integration time interval, τ_{end}, so that the time interval $[\tau_{\text{trans}}, \tau_{\text{end}}]$ is long enough

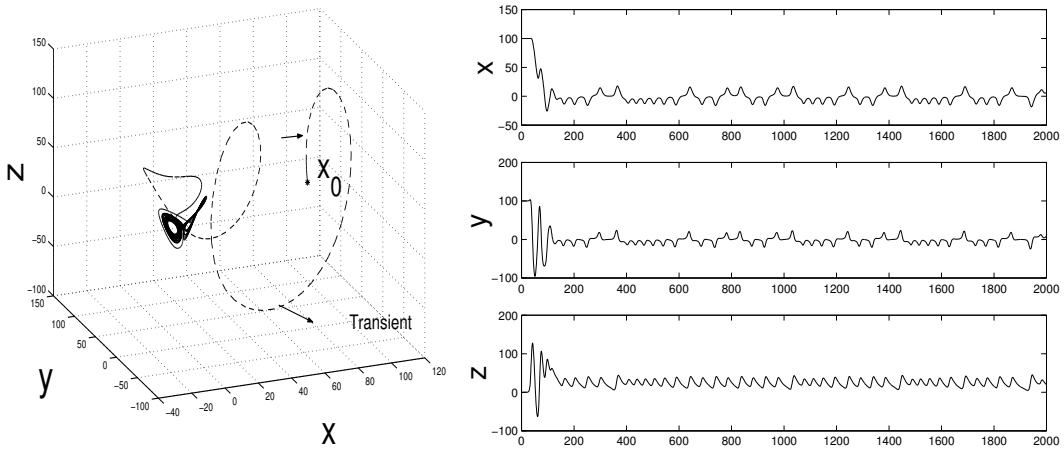


Figure 1.1: Solution of the Lorenz system with the parameters set as $r = 28, \sigma = 10, b = 8/3$ and with the initial condition $(x_0, y_0, z_0) = (100, 100, 0)$. In three dimensions (left) the solution, after a transient behaviour (dashed line), approaches the Lorenz attractor and the climatological value would be the baricentre of the attractor.

to produce a good sample of values of the solution $\vec{x}(t_n)$ for $t_n \in [\tau_{\text{trans}}, \tau_{\text{end}}]$, in order to evaluate a significative mean (figure 1.1).

The optimal choice of the values of τ_{trans} and τ_{end} can vary significantly depending on the system considered, on its parameters, on the initial conditions, on the numerical method and resources used, etc. In this work these values are assigned in order to obtain an acceptable fixed sampling error. In this way the integration time-scales are adjusted to the particular model (parameters and even initial conditions) as required. In this context we approximate the sampling error as the absolute value of the difference between the mean of the solution $\vec{x}(t_n)$ evaluated on the first and on the second half of the time interval $[\tau_{\text{trans}}, \tau_{\text{end}}]$.

1.3 Sensitivity of climate models

In this section we introduce one of the focal concepts involved our analysis: the sensitivity of a climate model.

Qualitatively the sensitivity of a climate model is the response of the model to the variations of its parameters. Modellers perform the sensitivity analysis of their modelled climate systems to examine the behaviour of the models with respect to the parameters, with the aim of assessing the robustness of the results obtained against parameter changes.

Referring to the system (1.1), we define as macroscopic climate sensitivity the quantity $\Delta \langle x_i \rangle_\infty / \Delta \mu_j$, which represents the slope of the analytical climatological value $\langle x_i \rangle_\infty$ with respect to a finite perturbation of the parameter μ_j.

For long time integrations the numerically evaluated macroscopic climate sensitivity $\Delta \langle x_i \rangle / \Delta \mu_j$ converges to $\Delta \langle x_i \rangle_\infty / \Delta \mu_j$ (see [LAH00]). We could hence study the macroscopic sensitivity of the climate model (1.1) sketching for each variable $x_i,\ i : 1 \dots m$ and each parameter $\mu_j,\ j : 1 \dots p$, the diagram of the climatological

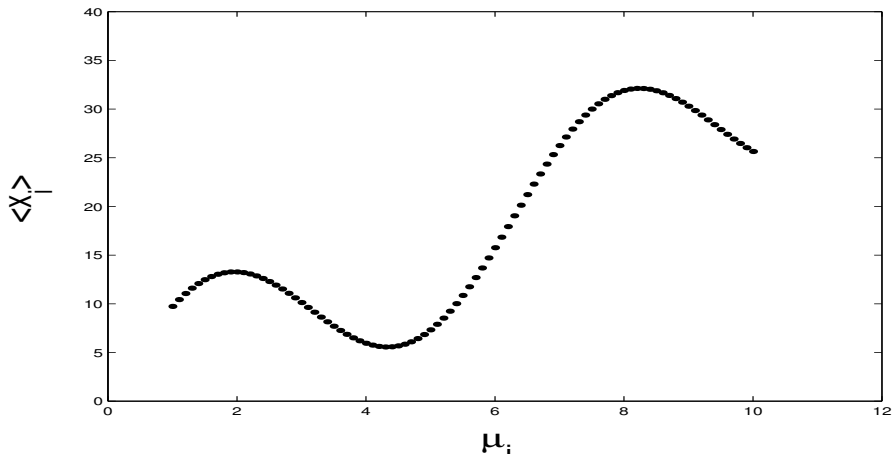


Figure 1.2: Example of sensitivity diagram of the generic climatological value $\langle x_i \rangle$ against the parameter μ_j.

values $\langle x_i \rangle$ against the parameter μ_j (figure 1.2).

We observe that in general it is not guaranteed that, sketching a sensitivity diagram, we obtain a function: we could have, for example, a multi-valued map. In fact for each parameter μ_j the corresponding climatological value $\langle x_i \rangle$ is computed running the model with that particular choice of the parameter μ_j (and keeping the other parameters fixed), finding the solution $\vec{x}(t)$ and then evaluating the climatological value $\langle x_i \rangle$ itself. If for the same choice of parameters we integrate the system using initial conditions which lie in the basins of attraction of different fixed points, periodic orbits or chaotic attractors, we will generically obtain different climatological values.

This implies also that in some cases the macroscopic sensitivity could even not be well defined.

We observe, further, that since the climatological values are either fixed points, or the baricentres of periodic orbits or chaotic attractors, then the sensitivity diagram can be reinterpreted as the generalisation of the bifurcation diagrams commonly used to perform the stability analysis of nonlinear dynamical systems, in which the baricentres of periodic orbits and chaotic attractors are added to the stable equilibria.

1.4 Proposed work

In this section we are going to present the basic ideas of the new method for investigating the effective dimensionality of the response of a generic model. In the next chapters we are going to apply these ideas to the Lorenz '63 and Lorenz '95 models.

We consider the generic climate model (1.1). We construct, then, the map

$$\begin{array}{rccc} \mathcal{M}: & \mathbb{R}^p & \longrightarrow & \mathbb{R}^m \\ & \vec{\mu} & \longmapsto & \langle \vec{x} \rangle \end{array} , \tag{1.4}$$

which associates a point in the state space with each point of the space of the parameters: for each choice of the parameters $\vec{\mu}$, the solution of the system $\vec{x}(t)$ is computed,

and then the climatological value $\langle \vec{x} \rangle$ is evaluated (figure 1.3).

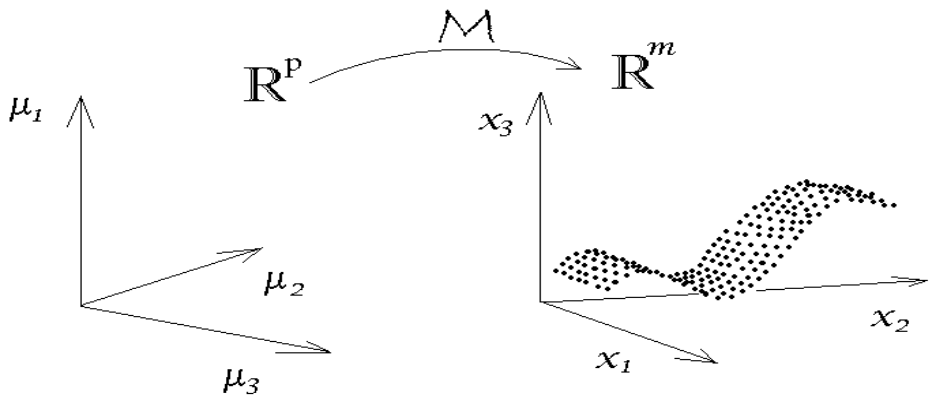


Figure 1.3: Map between the parameter space and the state space: at each choice of the parameters $\vec{\mu}$ is associated the climatological value $\langle \vec{x} \rangle$.

Choosing a range of parameters and applying this map, we obtain a "cloud" of points in the state space $\mathbb{R}^m$, whose dimension generically would be equal to p.

Let's suppose for a moment that we are in the happy situation in which the cloud is an hypersurface in $\mathbb{R}^m$ whose effective dimension is $q < p$: this could imply that in the model analysed there are only q parameters which are effectively independent one from each other, and that there exist $p - q$ relations between the parameters of the model itself. Then, if we could find these relations, it would be redundant to perturb all the parameters performing a sensitivity analysis: we would need to perturb only the parameters lying on an hypersurface of dimension q, determined by the relations, in the space of the parameters $\mathbb{R}^p$.

In this work we are going to analyse this hypothesis.

Therefore, performing our analysis on our models, first of all we are going to generate the cloud and investigate its nature. Then we are going to find its effective dimensionality using the SVD algorithm. The further step would be to implement a method to find the q-dimensional hypersurface in the parameter space, provided it exists.

Chapter 2

Lorenz '63

In this chapter we apply our analysis to the Lorenz '63 model.

In the first section we introduce the Lorenz '63 system: we explain its origin and importance, its physical meaning and finally we recall briefly the main properties we are interested in. In the second section we perform the sensitivity analysis of the Lorenz '63 system in order to understand in more details its behaviour before proceeding with our analysis: we focus initially on the sensitivity analysis for one parameter, finding numerically the solutions of the system, giving explicit examples of climatological values and explaining with particular attention the meaning and properties of the sensitivity diagram; then we sketch the sensitivity diagrams for all the parameters and summarise their characteristics; finally, we comment these results and we introduce a new assumption, in order to proceed with our work. In the third section we perform the analysis described in section 1.4: we find the Lorenz '63 cloud, we show, applying the SVD algorithm, that it is approximately a one-dimensional hypersurface in $\mathbb{R}^3$ and finally we find the one-dimensional hypersurface in the parameter space which is the anti-image of the cloud mapped by $\mathcal{M}$.

2.1 The Lorenz '63 system

In 1963 Lorenz published a paper introducing a dynamical system which later became famous in the literature as the Lorenz System.

This system is largely studied since it is, without being trivial, one of the more simple deterministic systems which exhibits chaotic behaviour, and it is hence used as a prototype to analyse some properties of deterministic chaos.

Initially it was introduced by Lorenz as a very simple model of the dynamics of the atmosphere, in order to show that a correct long-range weather forecast is impossible, unless the initial conditions are known exactly: in fact since the system is chaotic it is sensitively dependent to the initial conditions, hence even solutions starting from close initial conditions will be far apart after some time, then, because the inevitable errors in the weather observations, a prediction of a sufficiently distant future is usually incorrect.

The Lorenz system describes the two-dimensional convection occurring in a layer of fluid of uniform depth when the temperature difference between the upper and the lower surfaces is maintained at a constant value. The model for this physical system was previously studied by Rayleigh and Saltzman: they found its governing equations, the conditions on the parameters for which convection occurs, and then, using Fourier series expansions, they derived a finite set of ordinary differential equations, numerically solvable. Lorenz, following the same iter but using truncated Fourier series, found the nondimensionalised system

$$\begin{cases} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \quad . \end{cases} \tag{2.1}$$

The quantities $x = x(t)$, $y = y(t)$ and $z = z(t)$ describe the dynamics of the system; they are respectively proportional to the intensity of the convective motion, to the temperature difference between the ascending and descending streams, and to the distortion of the vertical temperature profile from linearity. They depend on time and the dot denotes their time derivative. The constants σ, r and b are parameters depending on the physical characteristics of the system; more precisely, σ is the Prandtl number, $r = R_a/R_c$ and $b = 4/(1 + a^2)$, where R_a is the Rayleigh number, R_c the critical value of R_a above which the modelled convective motion occurs, and a is the parameter which determines this critical value (precisely $R_c = \pi^4(1 + a^2)^3/a^2$). (For further details see [Lor63]).

The Lorenz system consists of three nonlinear ordinary differential equations.

The system is invariant under the transformation $(x, y, z) \to (-x, -y, z)$, i.e. under the reflection in the z-axis. This implies that if there exists a solution of the system of the form $(x(t), y(t), z(t))$, then either it is symmetric with respect to the z-axis, or the system admits another solution of the form $(-x(t), -y(t), z(t))$.

The fixed points of the system are found by setting the left hand side of the equations (2.1) equal to zero and solving the system obtained: they are the origin $(0, 0, 0)$ and the points $C^{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$, when $r > 1$. In $r = 1$ there is a super-critical pitchfork bifurcation.

The stability analysis of the equilibria is realized by studying the sign of the real part of the eigenvalues of the Jacobian of the system evaluated at the equilibria themselves. For the origin the characteristic equation is

$$(b + \lambda)[\lambda^2 + (\sigma + 1)\lambda - \sigma(r - 1)] = 0 \tag{2.2}$$

It has three roots with negative real part when $r < 1$. When $r > 1$ it has a positive root. The origin is therefore a stable equilibrium for $r < 1$ and an unstable one for $r > 1$. For the equilibria $C^{\pm}$ the characteristic equation is

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r - 1) = 0 \tag{2.3}$$

A more elaborate analysis shows that the three roots have negative real part for values of r immediately above 1, and that in a certain range of r there are two

complex conjugate roots whose real part changes sign when r reaches the critical value $r_c = \sigma(\sigma + b + 3)/(\sigma - b - 1)$. The two fixed points $C^{\pm}$ are hence unstable equilibria when $r > r_c > 1$ and stable equilibria in the other cases. In $r = r_c > 1$ there is a sub-critical Hopf bifurcation.

We summarise this stability analysis with a bifurcation diagram, in which we sketch the x-coordinate of the fixed points as r varies, assuming to keep σ and b fixed (figure 2.1).
(Further details can be found in [Str94, chap.9], [Dra92, chap.8], [Lor63]).

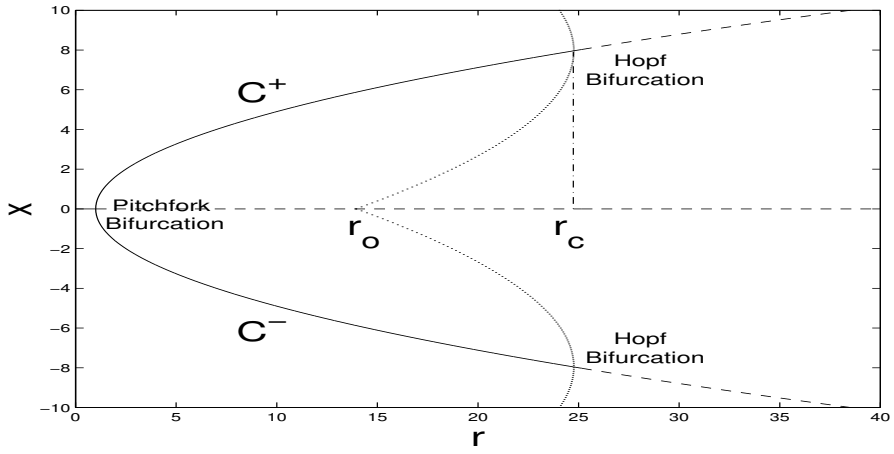


Figure 2.1: Bifurcation diagram of the Lorenz system. The diagram is the plot of the x-coordinate of the fixed points against the parameter r, with $\sigma = 10$ and $b = 8/3$ fixed. The continuous lines denote stable equilibria, the dashed lines denote unstable equilibria; the dotted lines denote the positions of the unstable limit cycles of the sub-critical Hopf bifurcations occurring for $r = r_c = \sigma(\sigma + b + 3)/(\sigma - b - 1) \simeq 24.74$.

2.2 Sensitivity analysis

In order to understand better the sensitivity analysis of the Lorenz system, we focus, in all the first part of this section, on the sensitivity of the system against the parameter r, fixing the other two parameters at $\sigma = 10$ and $b = 8/3$, so that we can refer easily to the results showed in the previous section and, in particular, to the bifurcation diagram 2.1.

When $0 < r < 1$ we found that the system has only a fixed point, stable, in the origin. As the time develops, hence, the generic solution $\vec{x}(t)$ falls into the origin, which is so the unique climatological value for this choice of the parameters. (Figure 2.2).

When $1 < r < r_c$ the system has an unstable equilibrium in the origin and two stable equilibria $C^{\pm}$. The solutions $\vec{x}(t)$, depending on the initial conditions, fall in one of these two stable equilibria, and so the points $C^{\pm}$ are climatological values. We observe that for the same choice of parameters there are two different climatological values. It is interesting to notice that for $1 < r < r_o$ the trajectories simply spin around the stable equilibrium until falling into it, but for $r_o < r < r_c$ the trajectories

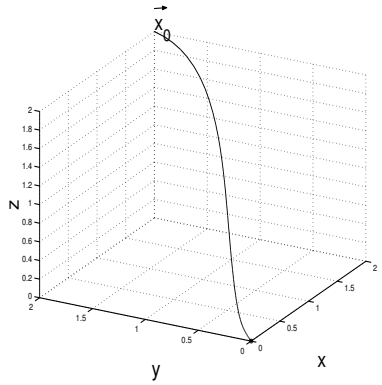


Figure 2.2: Solution of the Lorenz system for $r = 1/2$, $\sigma = 10$ and $b = 8/3$, integrating from the initial condition $(2, 2, 2)$. The solution falls into the origin, unique stable equilibrium and hence climatological value.

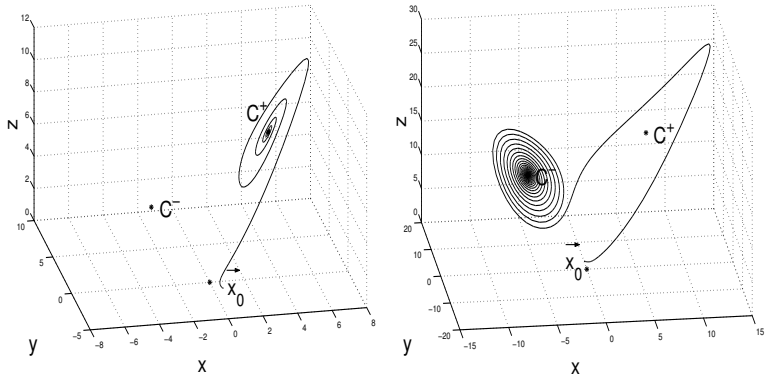


Figure 2.3: Solutions of the Lorenz system respectively for $r = 8$ (left) and $r = 18$ (right), fixing the other parameters as $\sigma = 10$, $b = 8/3$ and integrating from random initial condition in the cube $[-1, 1]^3$. The solutions, depending on the initial condition, fall in one of the two stable equilibria $C^{\pm}$, which are then climatological values. We observe that for this choice of the parameters $r_o \simeq 13.93$ and $r_c \simeq 24.74$: the solution for $r_o < r = 18 < r_c$ turns around a stable equilibrium before spinning around and falling into the other one, because of the presence of the unstable limit cycles of the sub-critical Hopf bifurcation.

turn around a stable equilibrium and then spin around and fall into the other one: this is due to the presence of the unstable limit cycles of the sub-critical Hopf bifurcation. In particular when $r = r_o$ there is an homoclinic trajectory starting and ending in the origin. (Figure 2.3).

When $r > r_c$ the system has only unstable fixed points: the solution $\vec{x}(t)$ starts "rotating" with chaotic behaviour approaching the Lorenz attractor as the time goes to infinity, and so the correspondent climatological value is the baricentre of the attractor. (Figure 2.4).

We observe that for some values of r below r_c some trajectories, found integrating the system from slightly different initial conditions but with the same choice of parameters, fall either on the attractor or in one of the fixed points $C^{\pm}$. In fact there exists a range of r where the Lorenz attractor coexists with the stable equilibria $C^{\pm}$ and in that range the climatological values could be either the baricentre of the attractor or

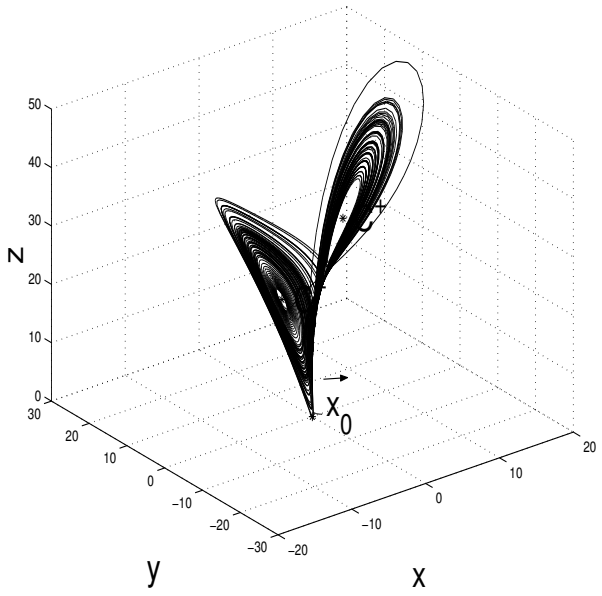


Figure 2.4: Solution of the Lorenz system for $\sigma = 10$, $b = 8/3$ and $r = 28 > r_c \simeq 24.74$, integrating from a random initial condition in the cube $[-1, 1]^3$. The solution falls on the Lorenz attractor as the time goes to infinity and so the climatological value is the baricentre of the attractor.

the stable equilibria. This implies that we could have multiple climatological values, even of different nature, for the same choice of parameters. (Figure 2.5).

We observe further that for some values of $r > r_c$ the trajectory $\vec{x}(t)$ settles on a periodic orbit, and so the climatological value that results is the baricentre of the periodic orbit. Moreover, because of the symmetry of the system, if the orbit is not symmetric respect to the z-axis, there exists at least another solution, which can be found integrating from a different initial condition, which ends up to the periodic orbit symmetric to the previous one. This implies again that we could have multiple climatological values for the same choice of parameters. (Figure 2.6).

Studying further the behaviour of the solutions of the Lorenz system would lead us into a intricate "jungle" of attractors, limit cycles, bifurcations, etc. This analysis is beyond the scope of this work, and hence we are not going to treat it. In any case, we mentioned these examples to illustrate that in general the solutions $\vec{x}(t)$ fall either in one of the fixed points $C^{\pm}$, or on the Lorenz attractor, or end up in a periodic orbit, and that the corresponding climatological values are, then, either the fixed points themselves, or the baricentres of the attractor or of the periodic orbit. We observe, further, that since we noticed there can be coexistence of fixed points, fixed points and attractor or periodic orbits, we can have multiple climatological values for the same choice of parameters.
(For further details see [Str94, chap.9], [Dra92, chap.8], [Lor63]).

The resulting sensitivity diagram of the x-coordinate of the climatological values against r, is shown in figure 2.7.

We observe that this diagram corresponds to the bifurcation diagram 2.1, where

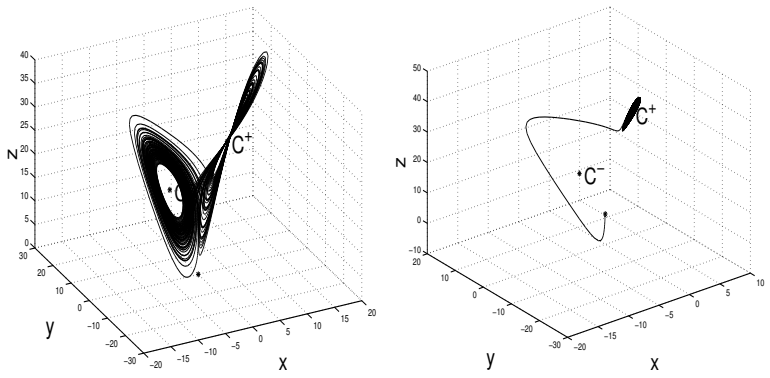


Figure 2.5: Solutions of the Lorenz system for $r = 24.1 < r_c$, $\sigma = 10$, $b = 8/3$, integrating from two different random initial conditions in the cube $[-1, 1]^3$. The solution falls on the Lorenz attractor for one initial condition and on a fixed point for an other initial condition, hence the climatological values are different in the two cases.

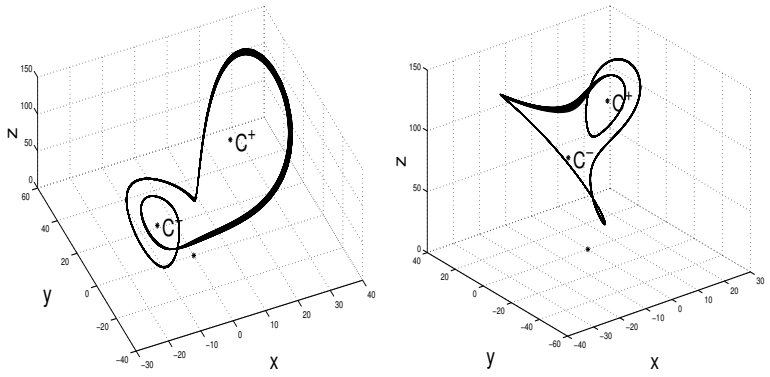


Figure 2.6: Solutions of the Lorenz system for $r \simeq 99.6 > r_c$, $\sigma = 10$, $b = 8/3$, integrating from two different random initial conditions in the cube $[-1, 1]^3$. The solutions fall on two different periodic orbits and so the climatological values result to be the two different baricentres of these orbits.

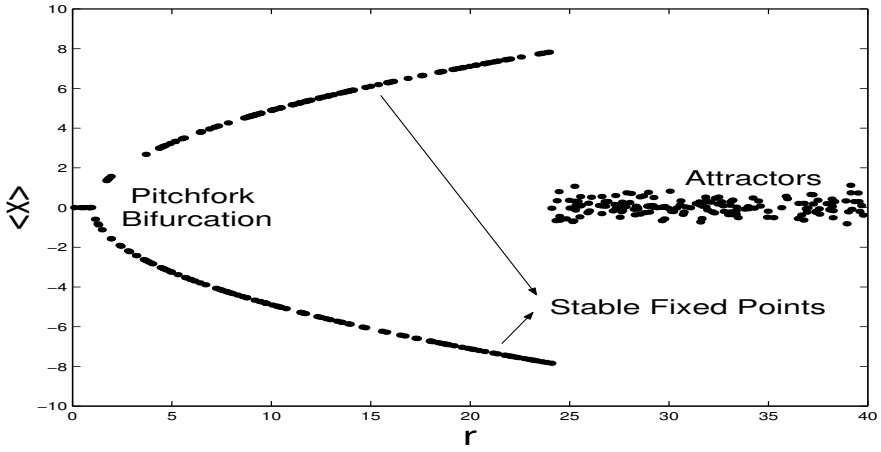


Figure 2.7: Sensitivity diagram of the climatological value $\langle x \rangle$ against the parameter r obtained from the Lorenz system, fixing the other parameters as $\sigma = 10$, $b = 8/3$ and integrating from random initial condition in the cube $[-1, 1]^3$. We observe that the sensitivity diagram corresponds to the classical bifurcation diagram (figure 2.1) with the stable equilibria and, added, the x-coordinate of the baricentres of the attractors.

the x-position of the baricentres of the attractors are added to the stable equilibria.

For the values of r immediately below r_c we can observe the coexistence of an attractor and fixed points: in fact in our diagram we notice, in this small range of r, the presence of both the corresponding climatological values.

The climatological values $\langle x \rangle$ we found for the chaotic behaviour are approximately zero: there is some sampling error in their evaluation but they should be exactly zero, in fact they are the x-coordinates of the baricentre of the attractor that, because of the symmetry of the system lies on the z-axis; the sampling error arises since we cannot integrate till infinity, evaluating so the analytical climatological value. In any case, increasing the length of the integration time interval we observed a shrinking of the sampling error.

We observe that, again because of the symmetry of the system and because of the symmetry of the solutions, sketching the sensitivity diagram of the climatological value $\langle y \rangle$ against r, we would obtain the same diagram.

Performing the sensitivity analysis of the Lorenz system for the parameters r, σ and b we obtain qualitatively similar results.

The solutions of the Lorenz system either spiral into fixed points, or on the attractor or settle on periodic orbits, and the corresponding climatological values result to be either the fixed points or the baricentres of the attractor or of the periodic orbits.

Because of the presence of the two fixed points $C^{\pm}$ the sensitivity diagram appears like a double-valued function of the parameters; in a small range near the value of the parameter where chaos starts occurring, there is coexistence of attractors and fixed points; moreover, there is, for some choices of the parameters, coexistence of periodic orbits. Therefore there is more than one climatological value for the same choice of parameters and the sensitivity diagram is a multi-valued map.

As we have just noticed previously, because of the symmetric behaviour of the sys-

tem, when chaotic behaviour occurs the x-coordinate of the baricentre of the attractor is zero, and hence the climatological value $\langle x \rangle$ is zero too.

Again, because of symmetry reasons, the sensitivity diagram of the climatological value $\langle x \rangle$ results to be equal to the sensitivity diagram of the climatological value $\langle y \rangle$.

The sensitivity diagrams we obtain are plotted in the figure 2.8.

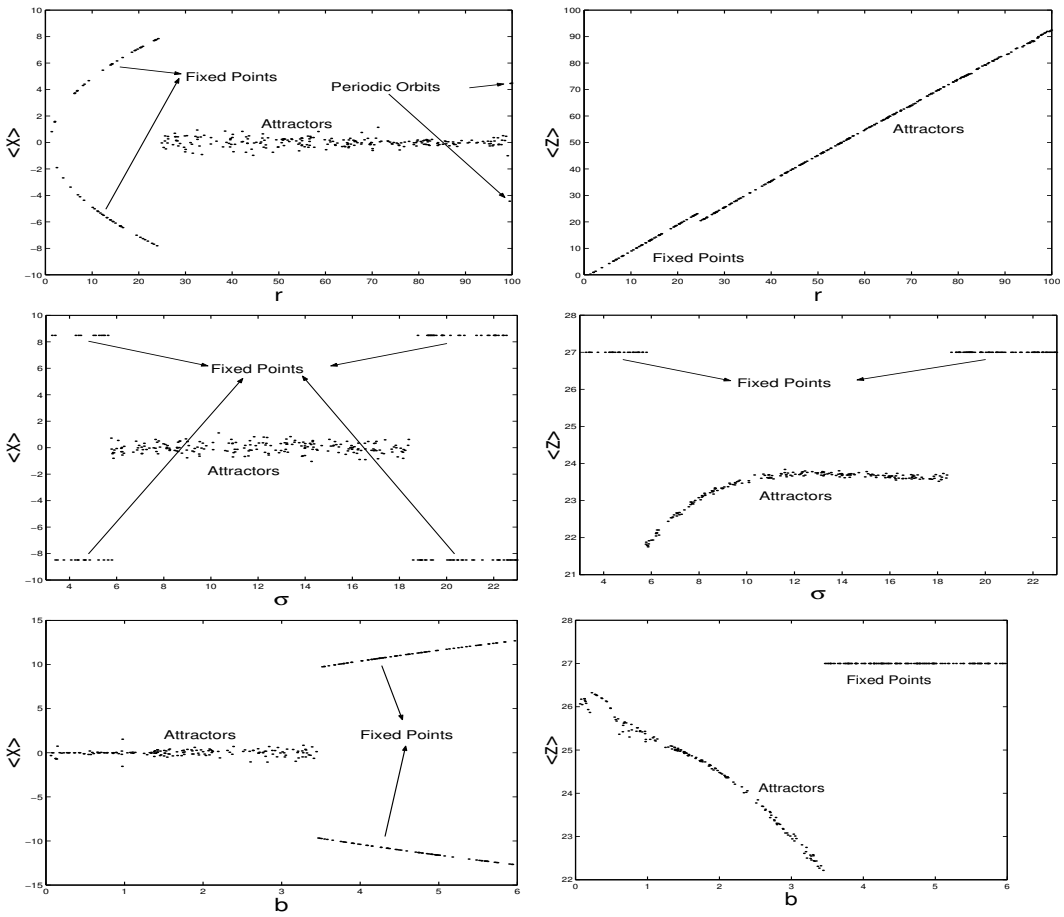


Figure 2.8: Sensitivity diagrams of the climatological values $\langle x \rangle$ and $\langle z \rangle$ against the parameters r, σ and b obtained from the Lorenz system integrating from random initial condition in the cube $[-1, 1]^3$. The parameters not varying are fixed at $r = 28$, $\sigma = 10$ and $b = 8/3$. The sensitivity diagram of the climatological value $\langle y \rangle$ is not sketched, because it is equal to the one of $\langle x \rangle$. As we indicate, the climatological values are either stable equilibria or baricentres of the attractors or of periodic orbits. The sensitivity diagrams appear as multi-valued maps. The climatological values $\langle x \rangle$ (and hence $\langle y \rangle$) corresponding to the chaotic solutions are approximately equal to zero.

In view of the future work, we would like to point out the following observations on our sensitivity analysis.

As we have just noticed, in the Lorenz '63 system we could have more than one climatological value for the same choice of parameters. This implies that the sensitivity analysis leads to a multi-valued map and hence from this it follows that the cloud we are going to find performing our analysis could not be an hypersurface. To avoid this "problem" we will consider, in our work, only the solutions of the system

ending on the Lorenz attractor, which is unique for each choice of the parameters. So, performing the sensitivity analysis, we obtain approximately a function and the cloud, as we will see, results to be approximately an hypersurface. We should add, hence, to our assumptions that we are going to restrict our analysis to the chaotic behaviour of climate systems, in order to avoid the presence of multi-valued maps and their consequent effects. We have to observe that this approach does not apply successfully to generic dynamical systems, in fact there exist dynamical systems where there coexist more that one attractor for the same choice of parameters (see [Mal00, Cas00]), so that bifurcations could occur even if only the chaotic cases are considered. In any case, it is generally believed that climate models do not have multiple attractors, hence we are quite confident in the utility of this restriction.

From the sensitivity diagrams above we can deduce approximately the range in which we should consider our parameters in order to generate the chaotic cloud. In any case this should be done with a certain carefulness since, taking the Cartesian product of the three ranges found, could lead to choices of parameters for which chaotic solutions does not occur. This could happen either because there exists a choice of parameters for which there is coexistence of an attractor and of fixed points (or periodic orbits), or, more simply, because the sensitivity diagrams we are considering are effectively the response of the system along three orthogonal lines in the parameter space passing through the point $r = 28$, $\sigma = 10$ and $b = 8/3$, so that for different values of these parameters the chaotic intervals could be different. In generating the chaotic cloud we will look for and neglect the points generated by those solutions which are not chaotic.

We have just observed that the climatological values $\langle x \rangle$ and $\langle y \rangle$, when chaotic behaviour occurs, are always equal to zero. Further we notice from the sensitivity diagrams that the climatological value $\langle z \rangle$ varies as a function of the parameters. We expect then that the chaotic cloud would be approximately a segment along the z-axis and that its vertical coordinate is a function of the three parameters r, σ and b.

2.3 The Lorenz '63 cloud

In this section we perform the analysis described in section 1.4 on the Lorenz '63 system: we find and analyse the Lorenz '63 cloud, we show, applying the SVD algorithm, that it is approximately a one-dimensional hypersurface in $\mathbb{R}^3$, and finally we find the one-dimensional hypersurface in the parameter space which is the anti-image of the cloud mapped by $\mathcal{M}$.

To generate the cloud associated to the Lorenz '63 system we construct the map

$$\begin{array}{rccc} \mathcal{M}: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (\sigma, b, r) & \longmapsto & (\langle x \rangle, \langle y \rangle, \langle z \rangle) \ , \end{array} \tag{2.4}$$

which associates with each point (σ, b, r) of the parameter space the point $(\langle x \rangle, \langle y \rangle, \langle z \rangle)$ in the state space, found integrating the Lorenz '63 system for that particular choice of the parameters and then evaluating from the solution the climatological value.

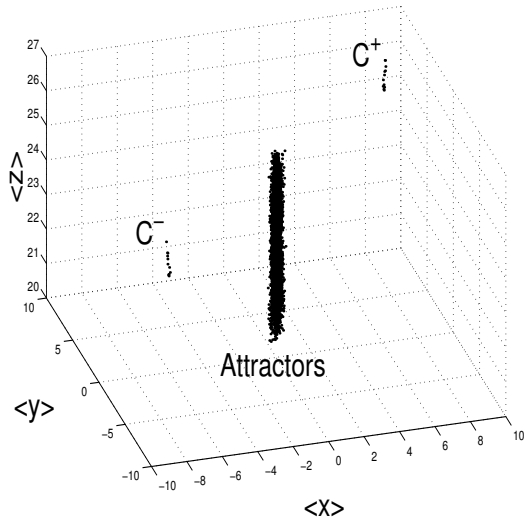


Figure 2.9: Cloud obtained applying the map (2.4) to the Lorenz '63 system taking the parameters in the range $(\sigma, b, r) \in [7, 11] \times [7/3, 3] \times [26, 30]$. The long central vertical cloud of points is formed by climatological values obtained from chaotic solutions; the small two clouds of points on the two sides are formed by climatological values corresponding to fixed points.

The Lorenz '63 cloud we obtained is plotted in figure 2.9.

We evaluated each image point numerically, using a fourth-order Runge-Kutta method, with time step $\Delta t = 0.001$. Each solution is evaluated starting the integration from a random initial condition in the cube $[-1, 1]^3$. The transient is cut at the computational time $\tau_{\text{trans}} = 100$ and the integration is performed until the computational time $\tau_{\text{end}} = 1500$: the choices of these values are done using the methodology described in section 1.2. The climatological values are then evaluated as the mean of $x(t_n)$, $y(t_n)$ and $z(t_n)$ on the interval $[\tau_{\text{trans}}, \tau_{\text{end}}]$ and the component of the sampling error occurring in their evaluation are computed as the absolute value of the difference between the correspondent climatological values evaluated on the first and on the second half of that interval.

The map is applied to 5000 points of the parameter space, chosen randomly in the range $(\sigma, b, r) \in [7, 11] \times [7/3, 3] \times [26, 30]$.

The range of the parameters has been chosen considering the sensitivity diagrams 2.8, and deducing from them the intervals which could be reliable to give only chaotic solutions. Actually, in this range there are also some points (σ, b, r) of the parameter space for which $r < r_c = \sigma(\sigma + b + 3)/(\sigma - b - 1)$: in these cases, we find the two equilibria $C^{\pm}$ as climatological values. The cloud, because of the presence of these "disturbing" points, is not even an hypersurface! Further, because of their presence, the distribution of the climatological values forming the cloud is not Gaussian (figure 2.10), therefore, applying the SVD algorithm to this cloud, we would obtain less effective and even misleading results! To bypass these problems we consider only the climatological values obtained from chaotic solutions and we neglect the other solutions, eliminating the corresponding points from the cloud: so, we perform our analysis only on the chaotic cloud.

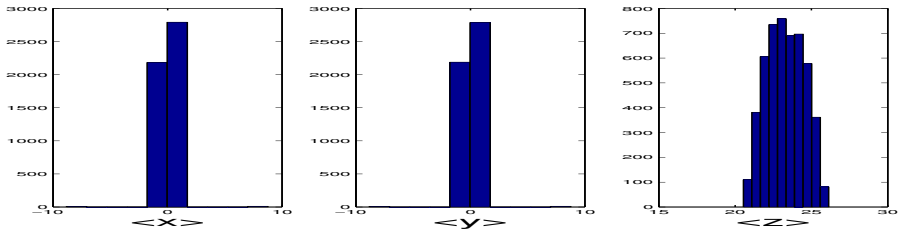


Figure 2.10: Histograms of the coordinates of the climatological values forming the Lorenz '63 cloud sketched in the figure 2.9. The presence of the equilibria $C^{\pm}$ affects the distribution of the x and y-coordinates so that there are two small peaks on the sides of the correspondent diagrams and the values are not Gaussian distributed.

The particular values within this range of the parameters are chosen also on the basis that we want to obtain a cloud of points "significant" to perform the SVD algorithm obtaining useful information about the effective dimension of the cloud itself:

- For the procedure to work, the range has to be large enough in order to obtain a sufficiently large sample of points which show the effective dimension of the cloud.

  If the range is too small we could obtain a too small cloud of points, which, potentially, lies inside a bigger and differently shaped cloud of points. It would be, considering the sampling error occurring in the evaluations of the points of the cloud, like obtaining a "point" inside the "hypersurface". If this case occurs, applying the SVD algorithm to this too small cloud, we would find that all the singular values have the same size, since the cloud would not be distinguishable from a "ball", because of the inevitable presence of sampling error. We could so erroneously conclude that the dimension of this small cloud is equal to the dimension of the state space (but the "hypersurface" could have a smaller effective dimension)!

  We can sort out rigorously if this is the case, comparing the size of the cloud, represented by the singular values in the SVD algorithm, with the projections of the sampling error of the climatological values along the singular vectors themselves: if they are approximately the same we are probably dealing with a 0-dimensional object.

  For the Lorenz '63 system the range chosen is clearly large enough, in fact we can see even from figure 2.9, that there is a predominant direction, along the z-axis, in which the chaotic cloud spreads.

- Moreover, for the method to work, the range of the parameters has to be not too large, otherwise we could obtain an excessively large cloud, which could be an hypersurface which bends and flips over.

  In that case the distributions of the climatological values would not be Gaussian and applying the SVD algorithm we would obtain misleading results.

Again in our case we are not in this situation, in fact we can see, again from figure 2.9, that the chaotic cloud we obtain is approximately a straight segment along the z-axis.

To apply the SVD algorithm to our data we organise the coordinates of the climatological values composing the chaotic cloud in a matrix $X \in \mathbb{R}^{n\times 3}$ as follows

$$X = \begin{bmatrix} \langle x_1 \rangle & \langle y_1 \rangle & \langle z_1 \rangle \\ \vdots & \vdots & \vdots \\ \langle x_n \rangle & \langle y_n \rangle & \langle z_n \rangle \end{bmatrix} . \tag{2.5}$$

Then we translate the origin of the variable space to the baricentre of the cloud, obtaining the matrix $X_{\text{cent}} = X - X_{\text{mean}}$, where the matrix X_{mean} is simply the matrix composed by all lines containing the coordinates of the baricentre itself. Applying the SVD to the centred matrix X_{cent} we obtain $X_{\text{cent}} = U\Sigma V^T$, where $\Sigma \simeq \text{diag}\,(\,85.0003, 19.8700, 0.0426\,)$ and the first two columns of V are

$$\vec{v}_1 \simeq \begin{bmatrix} 0.0035 \\ 0.0035 \\ -1.0000 \end{bmatrix} , \qquad \vec{v}_2 \simeq \begin{bmatrix} -0.7071 \\ -0.7071 \\ -0.0049 \end{bmatrix} . \tag{2.6}$$

The predominance of the first singular value shows that the cloud spreads prevalently along the direction of the first singular vector v_1, i.e. in the vertical direction. The second singular value and singular vector show that there is a secondary spreading direction parallel to the diagonal of the xy-plane: as we have previously explained, this spreading is due to the sampling error and will decrease as we increase the integration time. The third singular value is very small if compared to the previous two, so that we can neglect the spreading along the third orthogonal singular vector. The significativity of this analysis is confirmed comparing the size of the singular values, which represents the dimension of the cloud, with the projections of the sampling error along the singular vectors.

As we expected according to figure 2.9, from the SVD analysis we conclude that the Lorenz '63 chaotic cloud is approximately a one-dimensional hypersurface in the state space $\mathbb{R}^3$, spreading along the vertical direction.

From the results we have just obtained, we observe that the Lorenz '63 cloud is generated by perturbing three parameters, but that its effective dimension is approximately equal to one. Then, following the hypothesis presented in section 1.4, we suppose that there is only one effective independent parameter and that there exist two relations between the three parameters. These relations describe a curve in the parameter space that is mapped by the map (2.4) again in the Lorenz '63 cloud. Therefore, to perform the sensitivity analysis of the Lorenz '63 system, it would be redundant to perturb the parameters in the three-dimensional range we chose in the beginning, and it would be enough to consider the points of the parameter space lying on this curve: our aim is then to find this curve.

In order to simplify our treatment, we approximate the map (2.4) with the local linearization, on the range of parameters and variables we are analysing, of the map which associates at each point of the parameter space the corresponding point of the Lorenz '63 cloud projected along the direction of the relevant singular vectors. To find this approximation we consider the data we have collected until now, i.e. the set of parameters (σ_i, b_i, r_i), for $i : 1, \dots, n$, which are mapped by the map (2.4) into the points of the Lorenz '63 cloud $(\langle x_i \rangle, \langle y_i \rangle, \langle z_i \rangle)$, for $i : 1, \dots, n$. We build, so, the linear map

$$\begin{array}{rccl} M: & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (\sigma_i, b_i, r_i) & \longmapsto & m_\sigma \sigma_i + m_b b_i + m_r r_i + m_0 \ , \end{array} \tag{2.7}$$

whose coefficients $(m_\sigma, m_b, m_r, m_0)$ are given by the solution of the system

$$\begin{bmatrix} \sigma_1 & b_1 & r_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_n & b_n & r_n & 1 \end{bmatrix} \begin{bmatrix} m_\sigma \\ m_b \\ m_r \\ m_0 \end{bmatrix} = \begin{bmatrix} \langle x_1 \rangle & \langle y_1 \rangle & \langle z_1 \rangle \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \langle x_n \rangle & \langle y_n \rangle & \langle z_n \rangle \end{bmatrix} \begin{bmatrix} \vdots \\ \vec{v}_1 \\ \vdots \end{bmatrix} . \tag{2.8}$$

We compute this solution performing an SVD of the matrix of the parameters, and solving, then, the system

$$U \Sigma V^T \ \vec{m}_1 = X \ \vec{v}_1 = \vec{b} \quad \Rightarrow \quad \vec{m}_1 = V \Sigma^{-1} U^T \ \vec{b} \ , \tag{2.9}$$

where the matrix Σ^{-1} is equal to the inverse of Σ, $\mathrm{diag}\,(\dots, 1/\sigma_j, \dots)$. If the matrix Σ is singular, Σ^{-1} is still equal to $\mathrm{diag}\,(\dots, 1/\sigma_j, \dots)$, but when the singular value σ_j is small, the entry $1/\sigma_j$ is replaced by zero. The solution $\vec{m}_1$ so found is the best least-square solution of the system (2.8), and so the map M is the "best least-square" approximation of our map (see [PTVF92, chap. 2.6]).

For the Lorenz '63 cloud considered we obtain that the coefficients of the map M are approximately

$$\vec{m}_1 = \begin{bmatrix} m_\sigma \\ m_b \\ m_r \\ m_0 \end{bmatrix} \simeq \begin{bmatrix} -0.2547 \\ 1.8585 \\ -0.9819 \\ 1.5529 \end{bmatrix} , \tag{2.10}$$

and that the map associates with each point (σ_i, b_i, r_i) of the parameter space approximately the value $-\langle z_i \rangle$.

Given a fixed image of this map, each point of the plane

$$m_\sigma \sigma + m_b b + m_r r + m_0 + \langle z_i \rangle = 0 \tag{2.11}$$

of the space of the parameters is mapped into the same point of the cloud. Because of the generality of the climatological value $\langle z_i \rangle$, we find a set of parallel planes in the space of the parameters and each of them is mapped in one unique point of the cloud. It is enough, then, to consider only one point from each of these planes, to

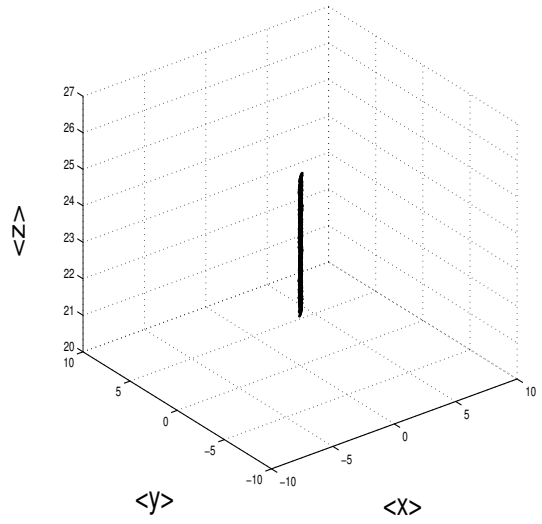


Figure 2.11: Cloud obtained applying the map (2.4) to the Lorenz '63 system taking the parameters on the line of the parameter space defined by (2.13).

generate the cloud. Therefore, to perform, eventually, a sensitivity analysis, we can consider only the points of the parameter space lying on the line cutting these planes: this line is the curve we aim. We can choose, for example the line perpendicular to the set of planes, given by the parametric equations

$$\begin{cases} \sigma &= m_\sigma\, s + \sigma_0 \\ b &= m_b\, s + b_0 \\ r &= m_r\, s + r_0 \quad , \end{cases} \tag{2.12}$$

where $s \in [s_0, s_n]$ is the arc-length parameter and (σ_0, b_0, r_0) is a point through which the line passes. These values have to be chosen appropriately, so that the curve is perpendicular to all the planes which cover the range of parameters considered initially.

In the Lorenz '63 case analysed we consider the line

$$\begin{cases} \sigma &= 4\, s + 7 \\ b &= 2/3\, s + 7/3 \\ r &= 4\, s + 26 \qquad\qquad\qquad s \in [0, 1] \ , \end{cases} \tag{2.13}$$

which is the diagonal of the range chosen at the beginning.

As expected, applying the map (2.4) and evaluating the climatological values with the same computational procedures as before but, this time, taking the images of 2000 points of the parameter space chosen randomly on the line defined by (2.13), approximately we obtain again the Lorenz '63 chaotic cloud (figure 2.11).

The previous procedure can be generalised to the general case of a climate model of the form (1.1), in which we consider p parameters and we find, by the SVD algorithm, that the cloud is an hypersurface of effective dimension q, with $1 < q < p$. In this

case we should solve the q systems

$$R\vec{m}_j = X\vec{v}_j \qquad\qquad j : 1, \ldots, q \ , \tag{2.14}$$

as we did for the system (2.8), in order to find the coefficients defining the linear map

$$M = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{m_1} & \vdots & \vec{m_q} \\ \vdots & \vdots & \vdots \end{bmatrix} \tag{2.15}$$

which best approximates the map associating each point in the parameter space with the projections of the corresponding point of the cloud along the relevant singular vectors. Proceeding as before, we obtain q hyperplanes, defined by the $\vec{m}_j$, in the space of the parameters $\mathbb{R}^p$, whose intersection is mapped into a unique point of the cloud: so the orthogonal space to this intersection give us the q-dimensional hypersurface that we should consider to perturb the parameters to perform a sensitivity analysis.

Summarising, in this section, performing the analysis described in section 1.4, we found that the response of the Lorenz '63 model to a sensitivity analysis is an effectively one-dimensional hypersurface in $\mathbb{R}^3$. Thus we assume that there is only one effective independent parameter and we found the one-dimensional hypersurface in the parameter space which is the anti-image of the Lorenz '63 cloud.

Chapter 3

Lorenz '95

In this chapter we apply our analysis to the Lorenz '95 model.

In the first section we introduce the Lorenz '95 system, the particular setting we assume for it, and the different cases of forcings we are going to consider developing our study, in order to outline the properties and understand the responses of the system. In the second section we apply the analysis described in section 1.4 to these different forcings: in all cases we considered we obtain that the system has an n-dimensional response to an n-dimensional forcing: this implies there is no method which might reduce the dimensionality of the sensitivity analysis.

3.1 The Lorenz '95 system

In 1995 Lorenz, during his opening speech on the occasion of a meeting on predictability at the ECMWF centre, introduced a new climate model.

The model was introduced to have a simple mathematical system which described the behaviour of the atmosphere and, in particular, of the error in the forecast compared with the knowledge of the leading Lyapunov exponent, in order to obtain relevant information about the long-range predictability of the system itself. The system is more interesting from the mathematical point of view, rather than for its reality in describing physical processes, but in any case it can be used as a very simple model of the dynamics of the climate system.

The system is a one-dimensional atmospheric model.

It consists of the set of N equations

$$\dot{x}_k = -x_{k-2}x_{k-1} + x_{k-1}x_{k+1} - x_k + F_k \qquad \text{for } k : 1, \dots, N \ , \tag{3.1}$$

with the assumption that $x_i = x_{i+N}$, for $i \in \mathbb{Z}$.

The N variables x_k may be thought of as the values of some atmospheric quantity in N sectors of a latitude circle, and may therefore imagined as describing the behaviour of the system in N points, equally spaced, distributed along a circle. They depend on time and the dot denotes time derivatives.

The linear terms on the right hand side of the equations simulate the internal dissipation, and the quadratic terms simulate the advection occurring in the system.

Further, the constant terms F_k represent the external forcings which affect the evolution of the atmosphere.
(For further details see [Lor95]).

The system is governed by a finite number of nonlinear ordinary differential equations: in our work we assume this number to be $N = 36$, which seems to be a geophysically interesting choice, since in that way we describe the dynamics of the atmosphere each ten degrees longitude.

When the parameters F_k are all equal to a unique value F, if F assumes small values all the solutions decay to the steady solution $x_1 = \cdots = x_N = F$. If F is somewhat larger most solutions are periodic. If F is still larger chaos occurs. As for the Lorenz '63 system, we are interested in solutions which exhibit chaotic behaviour, hence we are going to consider this last case (figure 3.1).

In our work, in order to understand better the behaviour of the system, we consider, performing our analysis, three different choice of forcings:

1. Initially we consider all the F_k equal to the same value F.

2. We consider then all the F_k having different random values.

3. Finally we consider

   (a) $F_k = A\cos(2 \cdot 2\pi/N \cdot k + \varphi) + F, \quad$ for $k : 1, \ldots, N$

   (b) $F_k = A\cos(3 \cdot 2\pi/N \cdot k + \varphi) + F. \quad$ for $k : 1, \ldots, N$

   Here each forcing F_k is the value, in the correspondent grid point $k \cdot 2\pi/N$, of a global sinusoidal external forcing with, respectively, wave-number equal to 2 and 3.

The first case is studied to perform our analysis first on a simple case, in which only one parameter is perturbed, in order to control if the range in which we chose to perturb the parameters F_k is appropriate. We expect in that case to obtain, performing the analysis described in section 1.4, a one-dimensional cloud.

The second case is the general case: we perturb the N parameters and we study the response on the cloud obtained. We should have, in general, a N-dimensional cloud and we want to investigate if its effective dimension is smaller.

The third case is analysed because, since a wave-like propagation in the solution of the system was observed (figure 3.2, see [Han98]), it is possible that the climatological solutions also exhibit a wave structure. Hence we want to investigate the effects on the cloud of different wave-number global forcings and see if there are some relations with the response of the system itself.

The key idea behind this choice is that we would like to think about a global forcing which is equal to the linear combination of different wave-number sinusoidal curves, i.e. it is expressed like a Fourier series, so that

$$F_k = \sum_i a_i \cos(i \cdot 2\pi/N \cdot k) + \sum_i b_i \sin(i \cdot 2\pi/N \cdot k) + a_0 \ . \tag{3.2}$$

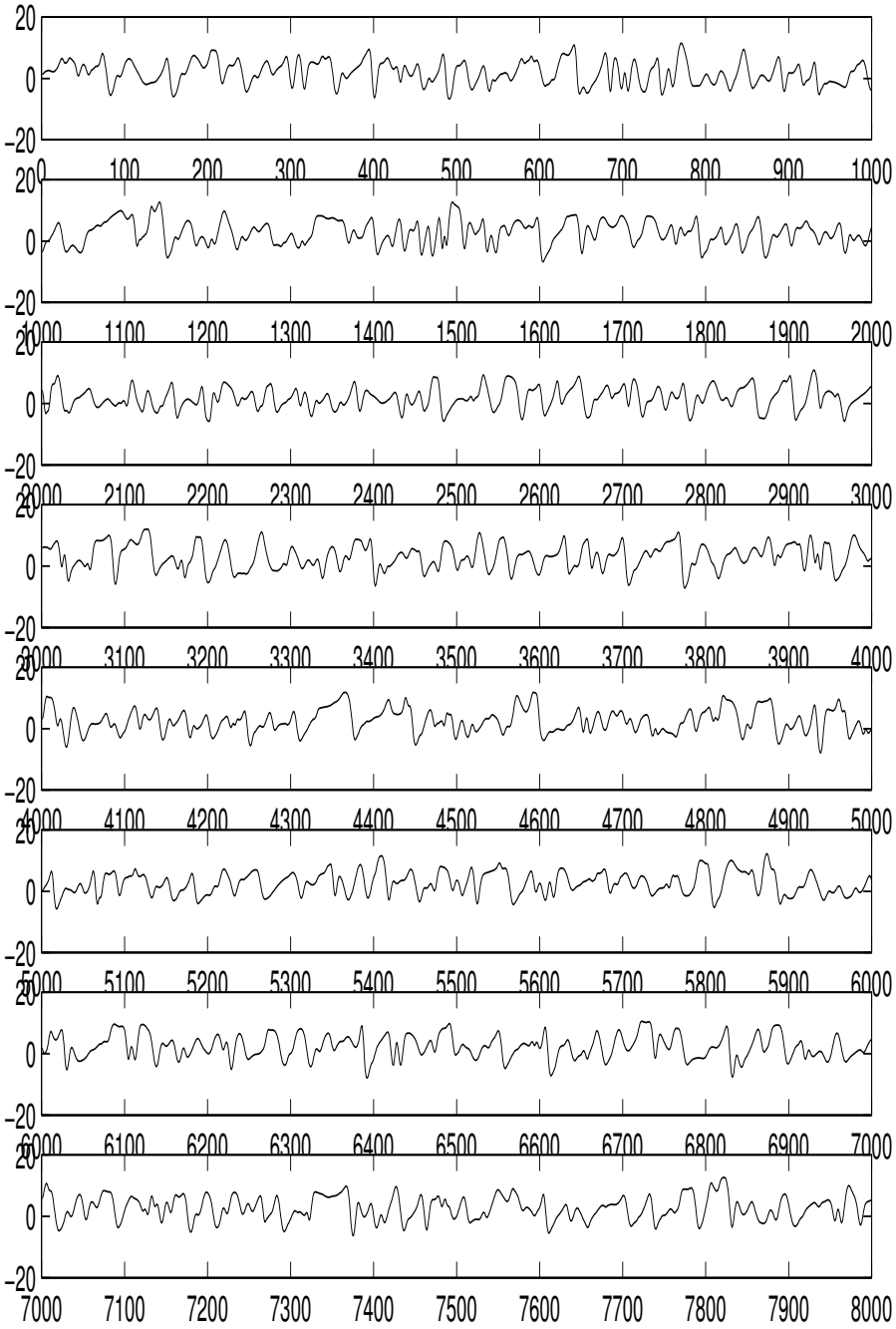


Figure 3.1: Diagram of the variable $x_k(t)$ against the time, obtained integrating the Lorenz '95 system with constant forcing $F_k = F = 8$ and evaluating the solutions from the random initial conditions $x_{k0} \in [-1, 1]$, for $k : 1, \ldots, N$. We notice that the solution exhibits chaotic behaviour.

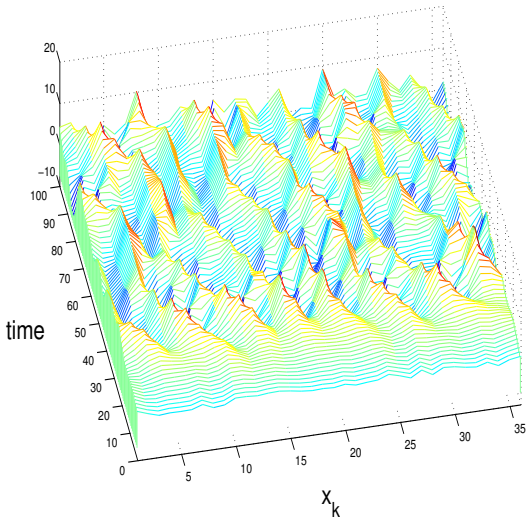


Figure 3.2: Solution of the Lorenz '95 system with constant forcing $F_k = F = 8$ evaluated from the N random initial condition $x_{k0} \in [-1, 1]$: we notice a wave-like propagation in the solution.

We would like then to consider our climatological values as the values on a circular discrete grid of a periodic function, which can also be expressed as a Fourier series. We would like to investigate if there are some particular wave-numbers for which the response of our system is stronger. We are thinking essentially to perform our analysis on the Fourier coefficients of the global forcing and of the climatological values forming the cloud.

We observe then that, reinterpreting the first and the second case under the light of this analysis, the first case is equivalent to assume that the only non-null Fourier coefficient of the global forcing is $a_0 = F$, and the second case is the general case in which all the Fourier coefficients are non-null.

3.2 The Lorenz '95 cloud

In this section we perform the analysis described in section 1.4 on the Lorenz '95 system for the three choices of forcing previously described.

For all the the three cases, we initially generate the cloud associated with the Lorenz '95 system by the map

$$\begin{array}{rccc} \mathcal{M}: & \mathbb{R}^N & \longrightarrow & \mathbb{R}^N \\ & \vec{F} & \longmapsto & \langle \vec{x} \rangle \end{array} , \tag{3.3}$$

which associates with each point $\vec{F} = (F_1, \ldots, F_N)$ of the parameter space the point $\langle \vec{x} \rangle = (\langle x_1 \rangle, \ldots, \langle x_N \rangle)$ in the state space, found integrating the Lorenz '95 system for that particular choice of the parameters and then evaluating from the solution the climatological value.

For each of the three cases we evaluated each image point numerically, using a fourth-order Runge-Kutta method, with time step $\Delta t = 0.005$. Each solution is

evaluated starting the integration from a random initial condition in the hypercube $[-1,1]^N$. The transient is cut at the computational time $\tau_{\text{trans}} = 500$ and the integration is performed until the computational time $\tau_{\text{end}} = 5500$: the choices of these values are made keeping in account the observations done in section 1.2. The climatological values are then evaluated as the mean of $x_k(t_n)$ on the interval $[\tau_{\text{trans}}, \tau_{\text{end}}]$ and the component of the sampling error occurring in their evaluation are computed as the absolute value of the difference between the correspondent climatological values evaluated on the first and on the second half of that interval.

In all the three cases we perform the map on $n = 3000$ points, and we save in a matrix $X \in \mathbb{R}^{3000\times(3\times N)}$ the forcings, the climatological values and the component of the sampling error, organising the data as follows

$$X = \begin{bmatrix} F_1^{(1)} & \dots & F_N^{(1)} & \langle x_1\rangle^{(1)} & \dots & \langle x_N\rangle^{(1)} & e_1^{(1)} & \dots & e_N^{(1)} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ F_1^{(n)} & \dots & F_N^{(n)} & \langle x_1\rangle^{(n)} & \dots & \langle x_N\rangle^{(n)} & e_1^{(n)} & \dots & e_N^{(n)} \end{bmatrix} . \tag{3.4}$$

First forcing

In the first case we set all the F_k equal to the same value F, and to generate the cloud we consider random values of F in the interval $[6, 10]$.

Performing the SVD algorithm on the matrix of the climatological values, we obtain one singular value definitely predominant on the others

$$\sigma_1 \simeq 53.1202 , \qquad \sigma_i \simeq O(1) \quad \text{for } i : 2, \dots, N . \tag{3.5}$$

The significativity of this analysis is confirmed comparing the size of the singular values, which represents the dimension of the cloud, with the projections of the sampling error along the singular vectors.

This result implies, as we were expecting, that the cloud we obtained spreads prevalently in one direction and so that it is approximately a one-dimensional hypersurface in $\mathbb{R}^N$, in fact we are varying only one parameter in generating it.

This result, also, shows us that the range chosen to perturb F is not too small, so that, generating the cloud, we are not incurring in the case we find a "point" inside the "hypersurface" (see page 19, for the Lorenz '63 case), in fact we obtain a one-dimensional object and not a N-dimensional object. Moreover the range is not too large, a fact that could affect our SVD analysis, since the cloud could be a bent hypersurface which flips over (see page 19, for the Lorenz '63 case), otherwise we would obtain more than one relevant singular value. For this range of the parameters we obtain, hence, a significant sample of points of the Lorenz '95 cloud (see page 19, for the Lorenz '63 case).

The singular vector associated with the predominant singular value is

$$v_1 \simeq \begin{bmatrix} 0.1667 \\ \vdots \\ 0.1667 \end{bmatrix} , \tag{3.6}$$

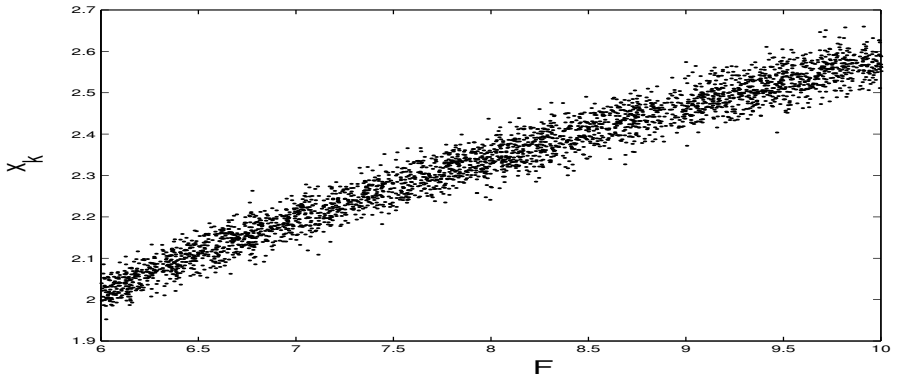


Figure 3.3: Response of the generic climatological value $\langle x_k \rangle$, against the forcing F, for the first case of the Lorenz '95 cloud analysed. The diagram shows that as the forcing increases the climatological values increase too, apparently approximately linearly.

hence the cloud is oriented along the diagonal of the first quadrant of $\mathbb{R}^N$.

We observed further that larger forcings F correspond to larger climatological values $\langle x_k \rangle$, for each $k : 1, \ldots, N$, so that it seems that as the parameter grows the climatological values all grow as well (figure 3.3).

Second forcing

In the second case we generate the cloud choosing, at each integration, for each F_k a random different value in the interval $[6, 10]$.

Performing the SVD algorithm on the matrix of climatological values obtained, we do not obtain any dominant singular value: they all decrease homogeneously from ~ 34.4695 to ~ 6.5375.

This implies, that the cloud spreads approximately homogeneously in all the N directions in the state space, and hence it seems to be approximately an N-dimensional object.

We can verify that we are actually sampling points from a effectively N-dimensional cloud, checking that the range chosen to perturb the F_k permits to obtain a significant sample of points composing the Lorenz '95 cloud:

- The range is large enough so that the cloud obtained is distinguishable from a "point" inside the "hypersurface" (see page 19, for the Lorenz '63 case).

  This follows from the analysis of the sampling errors occurring in the evaluation of the climatological values: in fact, we evaluated the mean sampling error $\vec{\bar{e}} = (\bar{e}_1, \ldots, \bar{e}_k)$, computing each component $\bar{e}_k$ as the mean of the sampling errors $e_k^{(i)}$ over all the integrations. We projected this error along the directions of the principal components of the cloud, taking its scalar product with the singular vectors v_k, for $k : 1, \ldots, N$. We found that these projections are small if compared with the dimensions of the cloud, described by the singular values (which are the standard deviations of the climatological values projected onto each principal axis).

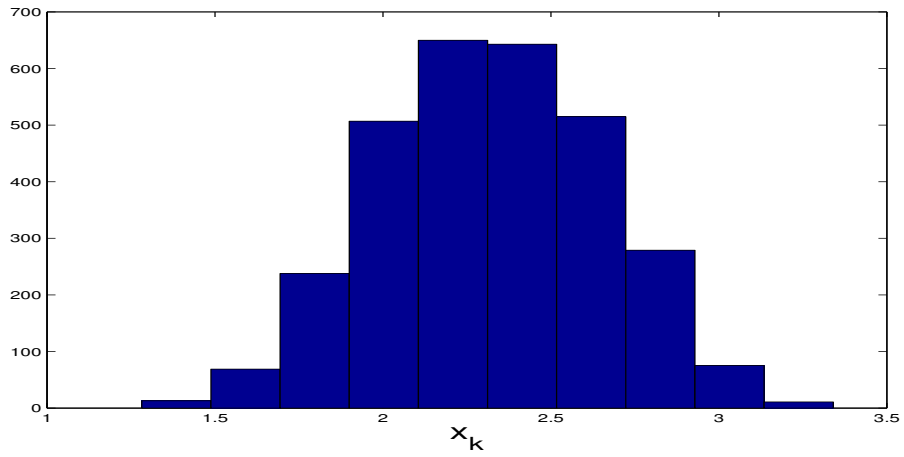


Figure 3.4: Histogram of the generic climatological value $\langle x_k \rangle$, k-th coordinate of the points forming the Lorenz '95 cloud, for the general case analysed (F_k chosen randomly in the interval $[6, 8]$ for $k : 1, \ldots, N$, at each integration). The distribution is approximately Gaussian.

Hence we are not dealing with a 0-dimensional object, because in that case the sampling error projections should be comparable with the singular values.

- Further, the range is not too large, a fact that could affect our SVD analysis, since the cloud could be a bent hypersurface which flips over (see page 19, for the Lorenz '63 case). In fact, the distributions of the climatological values found are approximately Gaussian (figure 3.4).

This result contrasts with the hypothesis made in section 1.4, in fact, in this case of the Lorenz '95 system, perturbing N parameters we obtain an effectively N-dimensional cloud. Therefore we cannot assume that there are dependent and independent parameters in performing a sensitivity analysis: consequently we cannot apply to the Lorenz '95 model the method suggested to reduce the computational work in performing a sensitivity analysis.

Third forcing

In the third case to generate a Lorenz '95 cloud we consider at each integration the global external forcing as a sinusoidal curves with random phase φ, chosen in the interval $[0, 2\pi]$, and random amplitude A, chosen in the interval $[-2, 2]$, oscillating around the value $F = 8$, with wave-number equal respectively at 2 and 3. Each F_k is, then, the value of this curve at the k-th grid point $k \cdot 2\pi/N$, for $k : 1, \ldots, N$, and we consider the two cases

(a) $F_k = A\cos(2 \cdot 2\pi/N \cdot k + \varphi) + F, \quad$ for $k : 1, \ldots, N$,

(b) $F_k = A\cos(3 \cdot 2\pi/N \cdot k + \varphi) + F, \quad$ for $k : 1, \ldots, N$.

Performing the SVD algorithm on the matrix of the climatological values obtained, we obtain two dominant singular values

$$\sigma_1 \simeq 22.7401 \ , \quad \sigma_2 \simeq 21.6502 \ , \qquad \sigma_i \simeq O(1) \ \text{ for } i : 3, \ldots, N \ , \tag{3.7}$$

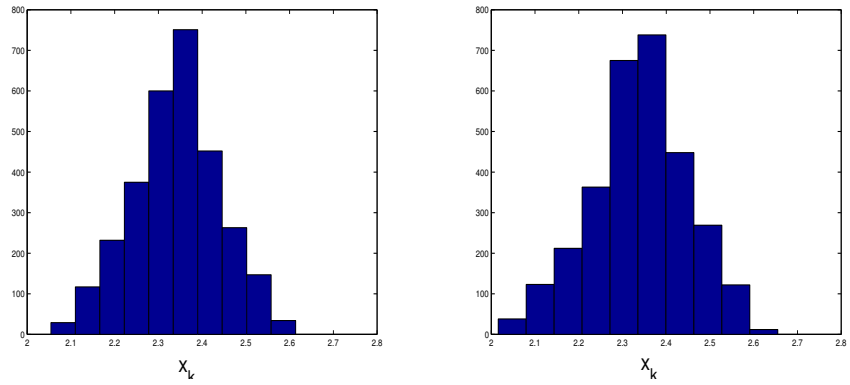


Figure 3.5: Histograms of the generic climatological values $\langle x_k \rangle$, k-th coordinate of the points forming the Lorenz '95 cloud, for the third case analysed, i.e. considering at each integration of the Lorenz '95 system a sinusoidal global forcing with wave-number equal respectively at 2 (left) and 3 (right), and setting each F_k as the value of this curve at the k-th grid point $k \cdot 2\pi/N$, for $k : 1, \ldots, N$. The distributions are approximately Gaussian.

for the case in which the wave-number is equal to 2, and

$$\sigma_1 \simeq 25.7857 \ , \quad \sigma_2 \simeq 24.5331 \ , \qquad \sigma_i \simeq O(1) \ \text{ for } i : 3, \ldots, N \ , \tag{3.8}$$

for the case in which the wave-number is equal to 3.

The significativity of this analysis is confirmed comparing the size of the singular values, which represents the dimension of the cloud, with the projections of the sampling error along the singular vectors.

This implies, as we were expecting, that the cloud we obtain spreads prevalently in two direction and so that it is approximately a two-dimensional hypersurface in $\mathbb{R}^N$, in fact we are varying two parameters in generating it, the phase φ and the amplitude A of the global forcing.

This result, also, shows us that the range chosen to perturb F is not too small, so that, generating the cloud, we are not incurring in the case we find a "point" inside the "hypersurface" (see page 19, for the Lorenz '63 case), in fact we obtain a two-dimensional object and not a N-dimensional object. Moreover the range is neither too large, fact that could affect our SVD analysis, since the cloud could be a bended hypersurface which flips over (see page 19, for the Lorenz '63 case), in fact, the distributions of the climatological values found are approximately Gaussian (figure 3.5).

A plot of the singular vectors v_1 and v_2 corresponding to the dominant singular values shows a sinusoidal behaviour with respectively wave-number equal to 2 or 3, for the two cases considered (figure 3.6). Further we notice that, for each integration of the Lorenz '95 system, considering a sinusoidal global forcing $\vec{F} = (F_1, \ldots, F_N)$ and the climatological values $\langle \vec{x} \rangle = (\langle x_1 \rangle, \ldots, \langle x_N \rangle)$ obtained applying the map 3.3 to that choice of parameters, and plotting these two sets, we obtain the same sinusoidal shape with the same wave number (figure 3.7). The response of the system to a sinusoidal global forcing with wave number equal to 2 and 3 is again approximately sinusoidal and with the same wave-number.

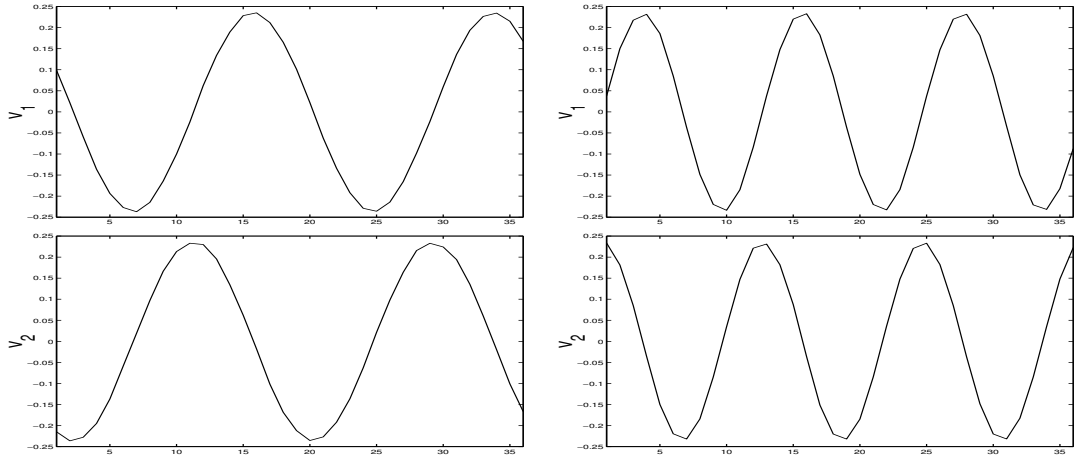


Figure 3.6: Plots of the component of the singular vectors v_1 and v_2 correspondent to the dominant singular values obtained from the SVD of the climatological values matrix in the the third case analysed, i.e. considering at each integration of the Lorenz '95 system a sinusoidal global forcing with wave-number equal respectively at 2 (left) and 3 (right), and setting each F_k as the value of this curve at the k-th grid point $k \cdot 2\pi/N$, for $k : 1, \ldots, N$. The response of the system is again a sinusoidal curve with wave-number equal respectively at 2 and 3.

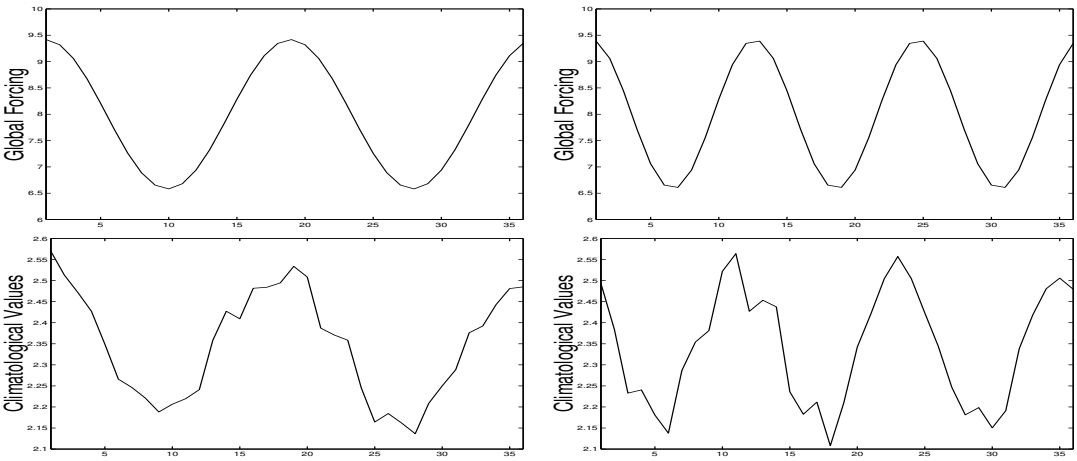


Figure 3.7: Plots of the global forcing $\vec{F} = (F_1, \ldots, F_N)$ and the corresponding climatological values $\langle \vec{x} \rangle = (\langle x_1 \rangle, \ldots, \langle x_N \rangle)$ obtained applying the map 3.3. The response of the Lorenz '95 system to a sinusoidal global forcing with wave-number equal to 2 (left) and 3 (right) is approximately again sinusoidal, with the same phase and with wave-number respectively equal to 2 and 3.

The sinusoidal response obtained in the last forcing case considered, motivates to develop further our analysis on the Fourier coefficients [1] of the global forcing and of the climatological values forming the cloud.

We consider the components of each point of the Lorenz '95 cloud to be the values of a periodic function living on a circular discrete grid. This function can be expressed as a Fourier series. We find the set of the Fourier coefficients associated with the climatological values, and we organise the data found in a matrix, and then we perform the SVD algorithm on this set.

Applying the SVD to the Fourier coefficients matrix generated in the third case analysed, we obtain two dominant singular values

$$\sigma_1 \simeq 5.3599 \ , \quad \sigma_2 \simeq 5.1030 \ , \qquad \sigma_i \simeq O(0.1) \quad \text{for } i : 3, \dots, N \ , \tag{3.9}$$

for the case in which the wave-number is equal to 2, and

$$\sigma_1 \simeq 6.0778 \ , \quad \sigma_2 \simeq 5.7825 \ , \qquad \sigma_i \simeq O(0.1) \quad \text{for } i : 3, \dots, N \ , \tag{3.10}$$

for the case in which the wave-number is equal to 3.

This implies, as we were expecting, that we have a two dimensional response, in fact we are varying two parameters, the phase φ and the amplitude A of the global forcing.

The singular vectors v_1 and v_2 corresponding to the dominant singular values have components all of the order of $O(0.001)$. The only exception are the two components corresponding to the Fourier coefficients a_2 and b_2, in the case of a global forcing with wave-number equal to 2, and the two components corresponding to the Fourier coefficients a_3 and b_3, in the case of a global forcing with wave-number equal to 3. In these two cases both components are of order $O(0.1)$. This definitely implies that the response of the system to a sinusoidal global forcing with wave number equal at 2 and 3 is again sinusoidal and with the same wave number.

This case suggests that there is not a "favoured" wave-number response of the system, but it seems to reflect the forcing wave-number.

In fact, applying the SVD of the Fourier coefficients matrix generated in the second case analysed, which we remind can be reinterpreted as the general case in which we consider a global forcing expressed as a Fourier series with all non-null Fourier coefficients, again we do not obtain any dominant singular value: they all decrease homogeneously from ~ 8.1245 to ~ 1.1269. This implies that there is not a particular wave-number for which the response of the Lorenz '95 system is stronger.

Applying the SVD to the Fourier coefficients matrix generated in the first case analysed, which is analogous to the general case in which we consider a global forcing which can be expressed as a Fourier series but with all null Fourier coefficients except than $a_0 = F$, we obtain one singular value definitely predominant on the others

$$\sigma_1 \simeq 8.8534 \ , \qquad \sigma_i \simeq O(0.1) \quad \text{for } i : 2, \dots, N \ . \tag{3.11}$$

[1] We observe that in our case, since we have a grid with N points, the only significant Fourier coefficients are a_0, a_i for $i : 1, \dots, N/2$ and b_i for $i : 1, \dots, N/2 - 1$, where the a_i are the coefficients of the $\cos(ix)$ and the b_i are the coefficients of the $\sin(ix)$.

This implies, as we were expecting, that we have a one effective dimensional response, as we are varying only one parameter $F = a_0$.

The singular vector v_1 corresponding to the predominant singular value has components all of the order of $O(0.001)$ except the component corresponding to the Fourier coefficient a_0, which is of order $O(1)$. This implies that the response of the system to a continuous global forcing with wave-number 0 has again a wave-number equal to 0, and further it confirms that perturbing the parameter $F = a_0$ affects the mean value of the climatological values, as we previously found (figure 3.3).

Summarising in all the cases explored we found that the Lorenz '95 model has n-dimensional response to a n-dimensional forcing, even developing the analysis in Fourier space. Therefore we cannot assume that there are dependent and independent parameters in performing a sensitivity analysis and we cannot apply to the Lorenz '95 model the method suggested to reduce computational efforts in performing a sensitivity analysis.

Conclusions

A new method for estimating the number of independent degrees of freedom in climate model sensitivity has been presented and explored. This method has been applied to the Lorenz '63 and to Lorenz '95 model, obtaining two completely different results.

Lorenz '63 behaves as expected when formulating the hypothesis on which our method is based. In fact we find that the Lorenz '63 chaotic cloud, generated perturbing three parameters, is effectively a one-dimensional hypersurface in $\mathbb{R}^3$. We assume then that there is a dependence between the parameters, and we find the one-dimensional hypersurface in the parameter space which is the anti-image of the Lorenz '63 chaotic cloud. Perturbing so only the parameters lying on this hypersurface we should be able to perform a sensitivity analysis with a reduced computational effort.

The Lorenz '95 system behaves rather differently: in all the cases analysed, we obtain that the system has an n-dimensional response to an n-dimensional forcing, so that we cannot assume that there are dependent and independent parameters. So the method suggested cannot be applied to this system.

These results let us conclude that the method investigated works, but that in general a lower effective dimensional response cannot always be found.

Time constraints limited our analysis to the these simple systems, where symmetry constraints simplify the problem. The test we have developed is generally applicable and we hope to apply it to more complex models, specifically model 2 of Lorenz '95 [Lor95], in the future.

Appendix A

SVD algorithm

The SVD algorithm is a standard method to find the best-fitting set of orthogonal axes approximating a set of data in an Euclidean space.

We give here a brief description of this method, explaining its meaning and its properties. (See [MH87, chap. 2], [PTVF92, chap 2.6, 14, 15]).

We consider a set of n objects, or data, or points, each one identifiable by m attributes, or coordinates, in an Euclidean space, which, for simplicity, we identify with $\mathbb{R}^m$.

We organise the data into the matrix $X \in \mathbb{R}^{n\times m}$ as follows

$$X = \begin{bmatrix} x_1^{(1)} & \dots & x_m^{(1)} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ x_1^{(n)} & \dots & x_m^{(n)} \end{bmatrix} \tag{A.1}$$

We search for the set of orthogonal axes which are the best fit of our points in the Euclidean space $\mathbb{R}^m$, in order to replace our initial set of data with these m axes.

We observe that if the set of data is not centred into the origin of $\mathbb{R}^m$, then the principal axis would indicate just the direction of the position of our set, and the others would be orthogonal to it, not giving us any useful information about the shape and orientation of the set (figure A.1), that is what we aim. Therefore we need to translate the set of data, locating its baricentre into the origin.

We organise the "new" centred data into the matrix $X_{\text{cent}} \in \mathbb{R}^{n\times m}$,

$$X_{\text{cent}} = \begin{bmatrix} x_1^{(1)} - \bar{x}_1 & \dots & x_m^{(1)} - \bar{x}_m \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ x_1^{(n)} - \bar{x}_1 & \dots & x_m^{(n)} - \bar{x}_m \end{bmatrix}, \tag{A.2}$$

where the quantities $\bar{x}_j$, for $j : 1 \dots m$, are the jth-coordinate of the baricentre of our data, and are evaluated as the the means of the coordinates $x_j^{(i)}$, for $i : 1 \dots n$.

We wish now to find the unit vector $\vec{v} \in \mathbb{R}^m$ which gives the direction of the axis which best fits our centred data. The axis is the best-fitting one when the sum of the distances between the points and the axis is minimal. This condition is equal to require that the sum of the square of the projection of the points along the axis is maximal. This quantity is given by

$$(X_{\text{cent}}\vec{v})^T(X_{\text{cent}}\vec{v}) \ , \tag{A.3}$$

and it is the variance of the points when projected onto this axis. In order to find the unit vector $\vec{v}$, therefore, we need to maximise the Lagrangian

$$L = \vec{v}^T X_{\text{cent}}^T X_{\text{cent}}\vec{v} - \lambda(\vec{v}^T\vec{v} - 1) \ , \tag{A.4}$$

where λ is a Lagrangian multiplier. The maximum is reached if the derivative of the Lagrangian is equal to zero: differentiating we obtain

$$2X_{\text{cent}}^T X_{\text{cent}}\vec{v} - 2\lambda\vec{v} = 0 \quad \Rightarrow \quad X_{\text{cent}}^T X_{\text{cent}}\vec{v} = \lambda\vec{v} \ . \tag{A.5}$$

From this equation we can already observe that the principal axis is determined by the eigenvector $\vec{v_j}$ of the matrix $X_{\text{cent}}^T X_{\text{cent}}$ correspondent to the eigenvalue λ_j for which the quantity $\lambda_j/\|\vec{v_j}\|$ is maximised.

We consider now the SVD of the matrix X_{cent}

$$X_{\text{cent}} = U\Sigma V^T = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{u_1} & \vdots & \vec{u_m} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{bmatrix} \begin{bmatrix} \dots & \vec{v_1} & \dots \\ \dots & \dots & \dots \\ \dots & \vec{v_m} & \dots \end{bmatrix} \ . \tag{A.6}$$

Plugging this decomposition into (A.5) we obtain

$$V\Sigma^2 V^T\vec{v} = \lambda\vec{v} \ . \tag{A.7}$$

The solutions of this equation are the singular vectors $\vec{v_1}, \dots, \vec{v_m}$ and the correspondent eigenvalues $\lambda_1, \dots, \lambda_m = \sigma_1^2, \dots, \sigma_m^2$: the directions of the best-fitting orthogonal axes are hence given by these vectors and the principal one is the one corresponding to the largest eigenvalue. The eigenvalues represent the figure of merit for each axis and, more precisely, they are the variances of the set of points projected onto each axis; the singular values are hence the standard deviations.

We have shown that, given a set of points organised in a matrix X, it is sufficient to perform the SVD of the associated centred matrix X_{cent} to obtain the best-fitting set of orthogonal axis of our data. Further the singular values, giving the standard deviations of the set of points projected onto each axis, show us which are the predominant directions along which our set of points spreads. If the distribution of our data is Gaussian, the SVD algorithm gives an idea of the shape that our set of data assumes into the Euclidean space: the less Gaussian is the distribution, the less precise will be this image, and the SVD algorithm will give just the set of the best-fitting axis for the data (figure A.1).

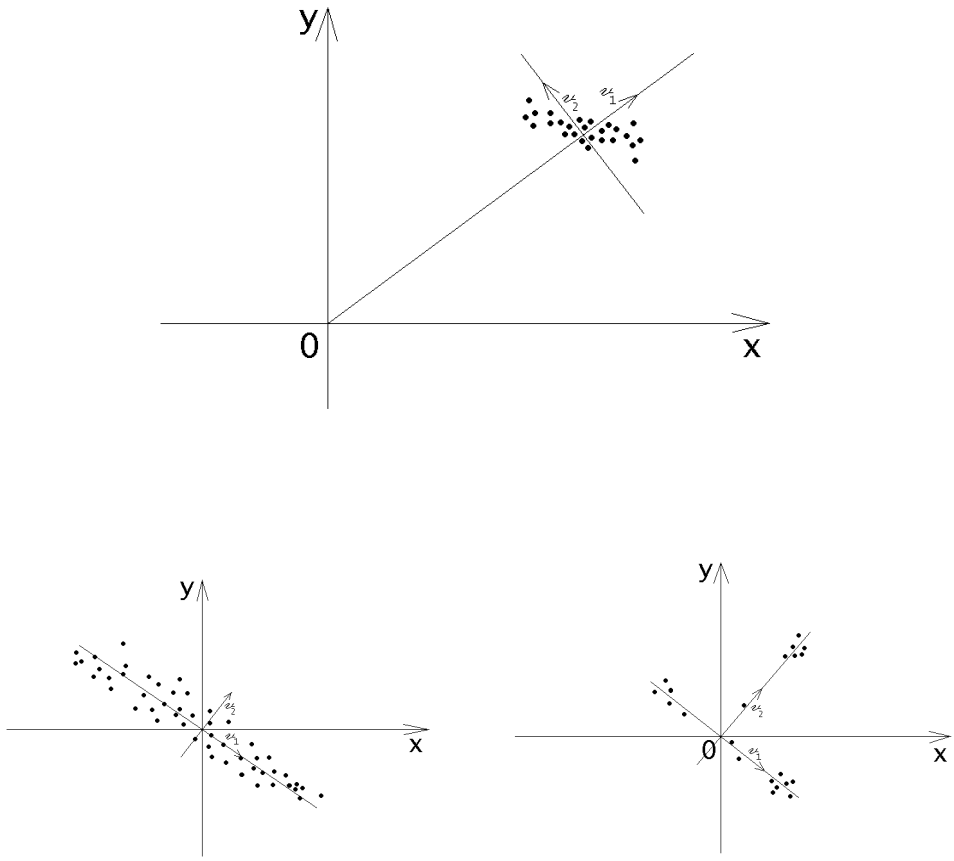


Figure A.1: Best-fitting set of orthogonal axis for a set of points in $\mathbb{R}^2$. When the set of points is not centred into the origin (top) the principal axis indicates the direction in which the data are located. When the points are centred into the origin, if they have a Gaussian distribution (bottom left) the SVD algorithm indicates the principal components along which the set spreads, giving an idea of its shape, otherwise (bottom right) it gives just the set of the best-fitting axes.

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