# Branching interval partition diffusions, the on/off Brownian snake, and random walks on Coxeter interchange graphs




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## 0.2 Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this dissertation are original.

The work presented in Chapter 3 titled *The on/off Brownian snake* is a collaboration between the author and his collaborator Dave Jacobi which is planned to be published as a pre-print soon.

The work presented in Chapter 4 titled *Random walks on Coxeter interchange graphs* is a collaboration between the author and his collaborators Brett Kolesnik, Rivka Mitchell, Tomasz Przybyłowski and resulted in a pre-print submitted to the Electronic Journal of Probability, see https://arxiv.org/abs/2401.17210.

# Abstract

This thesis investigates three new stochastic processes recently introduced to the literature. In Chapter 2 we introduce and study branching interval partition diffusions in their natural generality. We let interval widths evolve independently according to a general real-valued diffusion subject only to conditions that ensure finite lifetimes of intervals and allow the continuous generation of new intervals. The latter is governed by the Pitman-Yor excursion measure of the real-valued diffusion and an associated spectrally positive Lévy process to order both these excursions and their start times. This generalises previous work by Forman, Pal, Rizzolo and Winkel on the self-similar case and gives rise to a new class of general Markovian homogeneous Crump-Mode-Jagers-type branching processes with characteristics varying diffusively during their lifetimes.

In Chapter 3 we define what we call an on/off Brownian snake. We use this to construct on/off super Brownian motion recently introduced to the literature by Blath and Jacobi and which is a measure-valued branching process with a dormant state and an active state. Our construction mirrors the construction of super Brownian motion from the Brownian snake by Le Gall. We use the on/off Brownian snake to obtain results concerning the range of on/off super Brownian motion.

In Chapter 4 we study Coxeter tournaments. A tournament is an orientation of a graph. Vertices are players and edges are games, directed away from the winner. Kannan, Tetali and Vempala and McShine showed that tournaments with given score sequence can be rapidly sampled, via simple random walks on the interchange graphs of Brualdi and Li. These graphs are generated by the cyclically directed triangle, in the sense that traversing an edge corresponds to the reversal of such a triangle in a tournament. We study Coxeter tournaments on Zaslavsky's signed graphs. These tournaments involve collaborative and solitaire games, as well as the usual competitive games. The interchange graphs are richer in complexity, as a variety of other generators are involved. We prove rapid mixing by an intricate application of Bubley and Dyer's method of path coupling, using a delicate re-weighting of the graph metric. Geometric connections with the Coxeter permutahedra introduced by Ardila, Castillo, Eur and Postnikov are discussed.

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# Chapter 1

# Introduction

There is an extensive and varied collection of Markov processes in the literature today. It is often useful to establish results about their distributions or to obtain convergence theorems and study scaling limits of such processes. These results have real world applications in instances where we study phenomena with time-scales that are too long or sizes that are too large to simulate. For instance, populations can be modelled as branching processes that have measure-valued scaling limits as the population sizes tend to infinity. Also, Markov processes run for long time scales can reach an equilibrium stationary distribution and understanding the nature of these distributions is often an active interest.

Science and mathematics constantly inspire new probabilistic models and this thesis takes three recent creations and investigates their properties. Chapter 2 concerns interval partition diffusions that were first introduced in 2020, [34]; Chapter 3 looks at a measure-valued process known as on/off super Brownian motion introduced in 2023, [14]; Chapter 4 is based on Coxeter tournaments which again were first introduced in 2023, [58]. These new models are exciting avenues of research and have potential for further exploration.

## 1.1 Branching interval partition diffusions

### 1.1.1 Background

Branching processes are used in mathematics to model populations of individuals. The branching processes we study are related to Crump–Mode–Jagers processes, see Jagers [51]. These are branching processes where each individual is born from a single parent and over the course of the individual's lifetime it gives birth to offspring independently of other individuals with rate given by the birth rate. In our setting we have individuals represented by intervals in an interval partition, and new individuals born correspond to new intervals generated between existing intervals.

Interval partitions are collections of disjoint open intervals of $[0,\infty)$ and can be found in several different areas of mathematics, see e.g. [44, 45, 85]. In our setting the lengths of intervals evolve diffusively during the lifetime of the individuals they represent who then die when the length hits zero. We give the formal definition of interval partitions.

**Definition 1.1.1** (Interval Partitions)**.** For $M > 0$, an *interval partition of length M* is a set $\beta$ of intervals of $[0,M]$ that are open and disjoint, and which cover $[0,M]$ up to a Lebesgue-null set. We write $\|\beta\|$ for the length $M$. An *interval partition* is an interval partition of length $M$ for some $M$. We define $\mathscr{I}_H$ to be the space of interval partitions.

Interval partitions can be constructed by concatenating ordered collections of intervals, e.g. the concatenation of intervals $(0,b)$ and $(0,c)$ would be given by $\{(0,b),(b,b+c)\}$ where we have shifted the interval $(0,c)$ by $b$. More generally we can concatenate interval partitions $\beta$ and $\gamma$ by shifting the all intervals of $\gamma$ by $\|\beta\|$ and we denote the resulting interval partition by $\beta \star \gamma$ in this case. We use the same terminology as in [34]: we refer to stochastic processes which take values in $\mathscr{I}_H$ as *interval partition evolutions* or more simply as *IP-evolutions*. We refer to IP-evolutions that are continuous and have the Markov property with respect to some filtered probability space as *interval partition diffusions* or *IP-diffusions*. The class of interval partitions, $\mathscr{I}_H$, along with a suitable metric from [35], form a complete and separable metric space on which we can define our IP-diffusions and we state this metric in the next section.

Self-similar IP-diffusions are studied by Forman, Pal, Rizzolo and Winkel in [33, 34, 37] also [39], with Shi, and are used in the monograph by the same authors, [38], to solve the Aldous conjecture about a diffusion on a space of continuum trees, Aldous [3]. We follow their technique for the construction of IP-diffusions that uses Lévy processes and excursions of diffusions. The work in Chapter 2 is the first piece of work on general non-self-similar IP-diffusions.

It is shown in [60, 61] that spectrally positive Lévy processes are the contour functions of splitting trees, also with infinite birth rates. We use such Lévy processes to position the individuals in our IP-diffusions. We make this rigorous in the next section and we call this Lévy process the scaffolding for its role in positioning. It is not the case that the infinite birth rate corresponds to there being infinitely many individuals at all times. In the case where the Lévy process is of bounded variation this leads to interval partition processes where there are only finitely many individuals alive at any given fixed point in time a.s. and this is the case in Theorem 2.1.7 which we also state in the next section. When the Lévy process is of infinite variation there are infinitely many individuals alive at any fixed point in time a.s., see Theorem 2.1.6.

We lift the necessary definitions and theorems from Section 2.1 to give a framework for our main results.

### 1.1.2 Framework of main results

We define one-dimensional diffusions with the framework given by Revuz and Yor [88, Chapter VII.3 and VII.4]. In our setting we consider a Markov process $\mathbf{Y} = (\mathbf{Y}_t, t \geq 0)$ on a state space $E$ and for $x \in E$ we define $(\Omega, \mathscr{F}, \mathbb{P}_x)$ to be the probability space for $\mathbf{Y}$ with an initial value $\mathbf{Y}_0 = x$ and we denote the expectation operator by $\mathbb{E}_x$. We assume $\mathbf{Y}$ has the following properties:

1. the state space $E$ is of the form $E = [0, c]$ where $c$ is finite or of the form $E = [0, c)$ where $c$ is either finite or infinite;

2. the paths of $\mathbf{Y}$ are continuous on $E$ and that $\mathbf{Y}$ has the strong Markov property;

3. for all $x, y \in (0, c)$ we assume $\mathbb{P}_x(T_y < \infty) > 0$ where $T_y := \inf\{t : \mathbf{Y}_t = y\}$.

Certain diffusions can be expressed as solutions to stochastic differential equations as follows. Specifically, there exist locally bounded Borel-measurable functions $\mu : (0, c) \to \mathbb{R}$, $\sigma : (0, c) \to (0, \infty)$ and a Brownian motion $\mathbf{B}$ such that under $\mathbb{P}_x$, $\mathbf{Y}$ satisfies the stochastic differential equation

$$d\mathbf{Y}_t = \mu(\mathbf{Y}_t)dt + \sigma(\mathbf{Y}_t)d\mathbf{B}_t; \quad \mathbf{Y}_0 = x, \tag{1.1}$$

for all $x \in (0, c)$. More details on diffusions of this type can be found in e.g. [88, Chapter VII.2].

We now state assumptions on the diffusions that we consider in this paper.

*Assumption* A. A diffusion $\mathbf{Y}$ on state space $E$ satisfies *Assumption A* if:

1. We have that $\mathbb{P}_x(T_0 < \infty) > 0$ for some $x \in (0, c)$ (equivalently for all $x \in (0, c)$).

2. For all $0 < y < x < c$ we have that $\mathbb{E}_x(T_y)$ is finite.

3. There exist locally bounded Borel-measurable functions $\mu : (0, c) \to \mathbb{R}$, $\sigma : (0, c) \to (0, \infty)$ and a Brownian motion $\mathbf{B}$ such that $\mathbf{Y}$ satisfies the stochastic differential equation in (1.1).

4. The functions $\mu$ and $\sigma^2$ defined in $A3$ are continuous on $(0, b]$ for some $b \in E \backslash \{0\}$.

We label these conditions $A1$, $A2$, $A3$, $A4$.

The functions $(\mu(x), x \in (0, c))$ and $(\sigma^2(x), x \in (0, c))$ are known as the infinitesimal drift and the infinitesimal variance of the diffusion. We note that for the most general class of diffusions do not satisfy $A3$ but in this case a diffusion can still be characterised by a function and a measure that are known as their scale function and speed measure, see [88, Chapter VII.3]. In particular, the scale function is a function $s : E \to [0, \infty)$ such that for any $a$, $w$, $x$ in $E$ with $0 \leq a < x < w \leq c$ we have

$$\mathbb{P}_x(T_w < T_a) = \frac{s(x) - s(a)}{s(w) - s(a)}.$$

This function is unique up to normalising constants in the sense that for $c'$, $c'' > 0$ if $s$ is a scale function for a diffusion then the function $\tilde{s} : E \to [0,\infty)$ given by $\tilde{s}(x) := c's(x) + c''$ satisfies the equation above and so is also a scale function for the diffusion. We take $c'' = 0$ in our setting. The speed measure can similarly be defined from the diffusion but will not be needed in the following. We believe that it is possible to create IP-evolutions from this general class of diffusions, and we note Theorem 2.1.4 which gives results in this direction. We further note that $A1$ and $A2$ have equivalent formulations in terms of the scale function and speed measure that make them easy to determine for a given diffusion and we state these in our version of Assumption A in Section 2.1.

We will now assume $A1$, $A2$, $A3$ for the remainder of this introduction for simplicity as these assumptions all feature in our main results. In this setting the scale function $s$ function is differentiable with derivative

$$s'(x) = \exp\Big( - \int_{z=b}^{x} \frac{2\mu(z)}{\sigma^2(z)} dz \Big)$$

where $b \in (0,c)$ determines the normalising constant $c'$ for $s$. Furthermore the diffusion has a Pitman-Yor excursion measure as defined in [84] and we state the definition below. The definition uses two other diffusions: the 0-diffusion which is our original diffusion stopped at 0, i.e. $(\mathbf{Y}_{t\wedge T_0}, t \geq 0)$; the $\uparrow$-diffusion has infinitesimal drift $\mu^{\uparrow}$ and infinitesimal variance $(\sigma^2)^{\uparrow}$ defined in terms of our original diffusion as

$$\mu^{\uparrow}(x) = \mu(x) + \frac{s'(x)}{s(x)}\sigma^2(x); \quad (\sigma^2)^{\uparrow}(x) = \sigma^2(x).$$

**Definition 1.1.2** (Pitman-Yor excursion measure $\nu$)**.** We define the $\sigma$-finite excursion measure $\nu$ on the space $\mathscr{E}$ of positive continuous excursions by the following conditions:

1. $\nu(f \equiv 0) = \nu(f(0) \neq 0) = 0$;
2. for all $y \in E\backslash\{0\}$ we have $\nu\left(T_y(f) < \infty\right) = 1/s(y)$ where $s$ is the scale function for the diffusion;
3. for all $y \in E\backslash\{0\}$, the conditional distribution $\nu\left(\cdot | T_y(f) < \infty\right)$ is the law of the following $\mathbf{V} = (\mathbf{V}_t)_{t\in[0,\infty)}$. The process $(\mathbf{V}_t)_{t\in[0,T_y)}$ has the law of an $\uparrow$-diffusion started at 0 and run until it hits $y$, and $\left(\mathbf{V}_{T_y+t}\right)_{t\in[0,\infty)}$ has the law of an independent 0-diffusion started at $y$.

Like the scale function, this measure is unique up to a normalising constant in the sense that if $\nu$ is an excursion measure and $k > 0$ then $k\nu$ is also an excursion measure (the normalising constant $c'$ for $s$ determines the normalising constant for $\nu$).

Our first main result, Theorem 2.1.3, concerns this excursion measure. For an excursion $f \in \mathscr{E}$, let $\zeta(f)$ denote the lifetime of $f$, i.e. $\zeta(f) := \sup(\{t : f(t) > 0\})$. In the theorem we give sufficient conditions for

$$\int_{x=0}^{\infty} (x^2 \wedge x) \nu(\zeta \in dx) < \infty.$$

Under this condition we can construct recurrent Lévy processes from the lifetimes of excursions in the following way. let $\mathbf{N}'$ be a PRM with intensity $\mathrm{Leb} \otimes \nu$ on $[0,\infty) \times \mathscr{E}$. We define the recurrent SPLP $\mathbf{X} = (\mathbf{X}_t, t \geq 0)$ by the almost sure limits

$$\mathbf{X}_t := \lim_{\varepsilon \downarrow 0} \left( -t \int_{x=\varepsilon}^{\infty} x \nu(\zeta \in dx) + \sum_{\substack{(s,f) \text{ atom of } \mathbf{N}' \\ 0<s<t}} \zeta(f) \mathbb{1}_{\zeta(f)>\varepsilon} \right). \tag{1.2}$$

The PRM $\mathbf{N}'$ takes values in a certain class of $\sigma$-finite point measures on $[0,\infty) \times \mathscr{E}$ which we denote by $\mathscr{N}$, and the SPLP $\mathbf{X}$ is a càdlàg process and we denote the set of càdlàg paths by $\mathscr{D}$. Specifically, the class $\mathscr{N}$ consists of all the point measures $N$ for which the limit in (1.2) exists for all $t > 0$ when replacing $\mathbf{N}'$ by $N$. The Lévy measure for $\mathbf{X}$ is $\nu(\zeta \in \cdot)$ which we refer to as the lifetime under the Pitman-Yor excursion measure. More generally, for a suitable point measure $N \in \mathscr{N}$, let $\xi(N)$ be the càdlàg function, $(X_t, t \geq 0)$, obtained when replacing $\mathbf{N}'$ by $N$ in (1.2). We call $\mathbf{X}$ scaffolding and we use it to construct interval partitions in the following way.

**Definition 1.1.3** (Skewer function)**.** Let $I \subseteq J$ be countable sets, $N := \sum_{i \in I} \delta_{(r_i,f_i)}$, $N' := \sum_{i \in J} \delta_{(r_i,f_i)} \in \mathscr{N}$, and let $X = \xi(N')$. We define the aggregate health process $((M^y_{N,X})(t), t \in [0,\infty))$ by

$$M^y_{N,X}(t) := \sum_{\substack{(r,f) \text{ atom of } N: \\ 0 \leq r < t}} f(y - X_{r-}).$$

We also define the total health process at $t \in [0,\infty)$ as $(M^y_{N,X}(t), y \in \mathbb{R})$. When $M^y_{N,X}(\infty) < \infty$ we define the skewer for $(N,X)$ by

$$\mathrm{skewer}(y,N,X) := \{(M^y_{N,X}(t-), M^y_{N,X}(t)) : t \in [0,\infty), M^y_{N,X}(t-) < M^y_{N,X}(t)\}.$$

Note that in this case $\mathrm{skewer}(y,N,X)$ is an interval partition of length $M^y_{N,X}(\infty)$. We define $\mathrm{skewer}(y,N,X)$ to be $\emptyset$, the empty interval partition, if $M^y_{N,X}(\infty)$ is infinite. Finally we define $\overline{\mathrm{skewer}(N,X)} := (\mathrm{skewer}(y,N,X), y \geq 0)$.

In this setting we call a Poisson random measure $\mathbf{N}$ spindles and the pair $(\mathbf{N},\mathbf{X})$ a scaffolding-and-spindles pair, see Figure 1.1. We give the aggregate health process and total health process these names as we interpret the lengths of the intervals as health levels for the individuals they represent. The idea is that intervals of greater length take longer on average to hit 0 and so those individuals can be considered to be healthier. In Theorem 2.1.4 we give necessary and sufficient

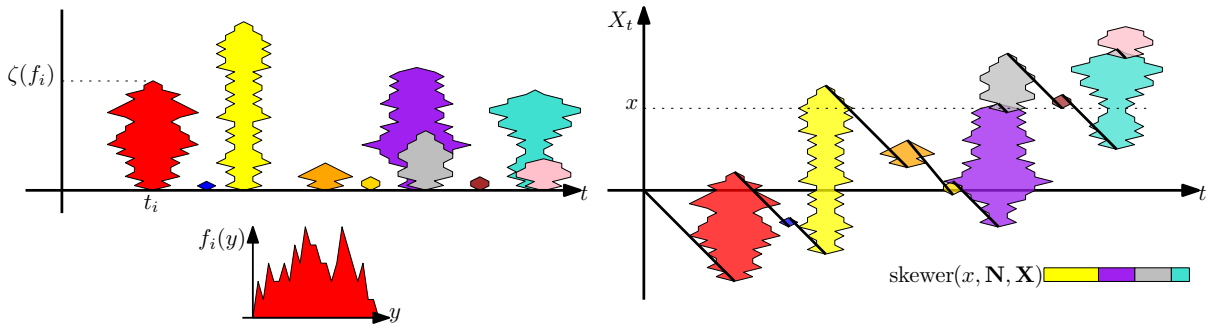



Figure 1.1: Left: A PRM of health excursions represented with time vertically moving up page and reflected to produce spindle-like shapes, with the excursion of a single spindle underneath. Right: the scaffolding process (black) constructed from the PRM, along with the interval partition generated by the skewer process at a fixed level $x$.

conditions for the summability of the aggregate health process at level $y = 0$ which means the skewer process at that level generates an interval partition.

We now consider further restrictions on the class of diffusions under which we prove the existence of IP-diffusions. We state the complete and separable metric space that we use from [35].

**Definition 1.1.4** (Interval partition metric space $(\mathscr{I}_H, d'_H)$)**.** Recall $\mathscr{I}_H$, the set of interval partitions from above. To define $d'_H$, we first define the notion of a correspondence between two interval partitions, and distortion of a correspondence. We use the standard notation of $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

1. A *correspondence* between two interval partitions $\beta$ and $\gamma$ is a tuple $(\beta, \gamma, (U_i, V_i)_{i\in[N]})$ involving two collections of intervals $U_1, \ldots, U_N$ from $\beta$ and $V_1, \ldots, V_N$ from $\gamma$, for some $N \in \mathbb{N}$, such that each is ordered with respect to the left-to-right ordering of their interval partitions.

2. The *distortion* of a correspondence $(\beta, \gamma, (U_i, V_i)_{i\in[N]})$ is defined to be

$$\mathrm{dis}(\beta, \gamma, (U_i, V_i)_{i\in[N]}) := \max \begin{Bmatrix} \sum_{i=1}^N |\mathrm{Leb}(U_i) - \mathrm{Leb}(V_i)| + \|\beta\| - \sum_{i=1}^N \mathrm{Leb}(U_i), \\ \sum_{i=1}^N |\mathrm{Leb}(U_i) - \mathrm{Leb}(V_i)| + \|\gamma\| - \sum_{i=1}^N \mathrm{Leb}(V_j) \end{Bmatrix}.$$

3. The distance $d'_H(\beta, \gamma)$ between interval partitions $\beta$ and $\gamma$ is defined to be the infimum of distortions over the set of all possible correspondences:

$$d'_H(\beta, \gamma) := \inf_{N\geq 0, (U_i, V_i)_{i\in[N]}} \mathrm{dis}(\beta, \gamma, (U_i, V_i)_{i\in[N]}).$$

We are nearly in a position where we can state our main results. We impose different restrictions on the diffusions we consider and we do this by considering spatial transformations

$g : E \to g(E)$. We consider strictly increasing and continuous transformations $g$ with $g(0) = 0$. We define $g(c) = \lim_{x \uparrow c} g(x) \leq \infty$ and therefore $g(E) = [0, g(c))$ or $g(E) = [0, g(c)]$.

Let $\mathbf{Y} = (\mathbf{Y}_t, t \geq 0)$ be a diffusion in state space $E$ that satisfies the conditions given so far and let $\mathbf{Z} = (\mathbf{Z}_t, t \geq 0)$ be the process defined by $\mathbf{Z}_t = g(\mathbf{Y}_t)$ for all $t \geq 0$. Then $\mathbf{Z}$ is a diffusion on $g(E)$ that satisfies $A1$ and $A2$. We refer to $\mathbf{Y}$ as the untransformed diffusion throughout the paper to avoid confusion between $\mathbf{Y}$ and $\mathbf{Z}$.

We now use transformations in the form given above to consider a class of diffusions under which we can prove the existence of IP-diffusions. These diffusions are branching processes in the sense that all intervals evolve independently and as mentioned previously new intervals are generated between existing intervals. The evolution of the individual intervals follows the law of the one-dimensional diffusion that we use to generate excursions in the PRM.

*Assumption B.* A diffusion $\mathbf{Z}$ satisfies *Assumption B* if it is of the form $\mathbf{Z} = g(\mathbf{Y})$, where $\mathbf{Y}$ is a diffusion that satisfies $A1$, $A2$, $A3$ of Assumption A with state space $E \subseteq [0, \infty)$ of the form $E = [0, c]$ or $E = [0, c)$ for some $c \leq \infty$ and furthermore is such that:

1. the Pitman-Yor excursion measure $\nu_{\mathbf{Y}}$ (equivalently $\nu_{\mathbf{Z}}$) satisfies
$$\int_{x=0}^{\infty} (x^2 \wedge x) \nu_{\mathbf{Y}}(\zeta \in dx) < \infty = \int_{x=0}^{\infty} x \nu_{\mathbf{Y}}(\zeta \in dx);$$
2. $\sigma^2_{\mathbf{Y}}(x) = 4x$;

there exists $\varepsilon_0 \in (0, c)$ such that

3. for some $0 < \alpha^- \leq \alpha^+ < 1$ we have
$$-2\alpha^+ \leq \mu_{\mathbf{Y}}(x) \leq -2\alpha^- \text{ for all } x \in (0, \varepsilon_0]$$

(where the subscript $\mathbf{Y}$ in $\mu$ and $\sigma^2$ denotes that they are the infinitesimal drift and variance for $\mathbf{Y}$), and $g : E \to g(E)$ is a strictly increasing continuous function with $g(0) = 0$, $g(E) \subseteq [0, \infty)$ and

4. there exists $q_0 > \alpha^+$ such that the transformation $g$ is $q$-Hölder continuous on $[0, \varepsilon_0)$ for all $q < q_0$, i.e. for all $q < q_0$ there exists $C_q \geq 0$ such that for all $x$, $y \in [0, \varepsilon_0)$ we have $|g(x) - g(y)| \leq C_q |x - y|^q$.

Finally, we say $\mathbf{Z} = g(\mathbf{Y})$ satisfies Strong Assumption B if as well as the above we have that

5. the transformation $g$ satisfies $0 < \liminf_{x \to 0} \frac{g(x)}{x} \leq \infty$ or $0 \leq \limsup_{x \to 0} \frac{g(x)}{x} < \infty$.

We note that while $B1$ appears cryptic that if the untransformed diffusion satisfies $A4$, as well as $A1$, $A2$, $A3$, $B2$, $B3$, then it also satisfies $B1$ by our results on Pitman-Yor excursion measures given in Theorem 2.1.3.

Under the conditions given in Assumption B we can generate IP-diffusions.

**Theorem 1.1.5** (Theorem 2.1.6)**.** *Consider a diffusion* $\mathbf{Z}$ *that satisfies Assumption B and let* $\nu_{\mathbf{Z}}$ *be its Pitman-Yor excursion measure. Let* $(\mathbf{N},\mathbf{X})$ *be a scaffolding-and-spindles pair where* $\mathbf{N}$ *has intensity Leb* $\otimes \nu_{\mathbf{Z}}$ *stopped at an a.s. finite random variable* $T$*. Then there exists a modification* $(\gamma^y, y \geq 0)$ *of* $\overline{skewer}(\mathbf{N},\mathbf{X})$ *which is a continuous* IP-evolution *on* $(\mathscr{I}_H, d'_H)$*. Furthermore the process* $\mathbf{X}$ *has a local time that is continuous in time and level. Let* $\tau^0$ *be the right-continuous inverse of the local time of* $\mathbf{X}$ *at level* 0*. Then for* $T = \tau^0(s)$*, for some* $s > 0$*,* $(\gamma^y, y \geq 0)$ *is a simple Markov* IP-evolution *in an appropriate filtered probability space and if furthermore* $\mathbf{Z}$ *satisfies Strong Assumption B then it is an* IP-diffusion*.*

We prove Theorem 2.1.6 in Section 2.4.6. The IP-diffusion of Theorem 2.1.6, $(\gamma^y, y \geq 0)$, takes initial states $\gamma^0$ in a large class of interval partitions, see Corollary 2.4.22, and the IP-diffusion can be started from an even larger class of interval partitions, see Theorem 2.4.35. We refer to $(\gamma^y, y \geq 0)$ as a *branching IP-diffusion* because the IP-diffusions from Theorem 2.4.35 have the following branching property: for IP-diffusions $(\gamma_1^y, y \geq 0)$, $(\gamma_2^y, y \geq 0)$ with initial states $\gamma_1$, $\gamma_2$, the IP-diffusion with initial state $\gamma_1 \star \gamma_2$ (the concatenation of interval partitions $\gamma_1$ and $\gamma_2$) has the same distribution as $(\gamma_1^y \star \gamma_2^y, y \geq 0)$.

We give another set of assumptions in Assumption C which give a disjoint family of diffusions to Assumption B.

*Assumption* C. A diffusion $\mathbf{Z}$ satisfies *Assumption C* if it is of the form $\mathbf{Z} = g(\mathbf{Y})$, where $\mathbf{Y}$ is a diffusion that satisfies *A*1, *A*2, *A*3 of Assumption A with state space $E \subseteq [0,\infty)$ of the form $E = [0,c]$ or $E = [0,c)$ for some $c \leq \infty$ and furthermore is such that:

1. $\sigma^2_{\mathbf{Y}}(x) = 4x$;

there exists $\varepsilon_1 \in (0,c)$ such that

2. for some $0 < \beta^- \leq \beta^+ < 1$ we have

$$2\beta^- \leq \mu_{\mathbf{Y}}(x) \leq 2\beta^+ \text{ for all } x \in (0,\varepsilon_1];$$

3. and $g : E \to g(E)$ is a strictly increasing continuous function with $g(0) = 0$, $g(E) \subseteq [0,\infty)$ and

$$\int_{y=0}^{\varepsilon_1} \frac{g(y^{1/(1-\beta^-)})}{y^2} dy < \infty.$$

**Theorem 1.1.6** (Theorem 2.1.7)**.** *Consider a diffusion* $\mathbf{Z}$ *under Assumption C with excursion measure* $\nu$ *and let* $(\mathbf{N},\mathbf{X})$ *be scaffolding-and-spindles pair where* $\mathbf{N}$ *has intensity Leb* $\otimes \nu$ *stopped at an a.s. finite valued random variable* $T$*. Then* $\overline{skewer}(\mathbf{N},\mathbf{X})$ *is a continuous* IP-evolution *on* $(\mathscr{I}_H, d'_H)$*. Furthermore we have that* $\mathbf{X}$ *is a Lévy process of bounded variation and when* $T$ *denotes the* $n^{th}$ *hitting time for* $\mathbf{X}$ *of level* 0 *for some* $n \in \mathbb{N}$ *then* $\overline{skewer}(\mathbf{N},\mathbf{X})$ *is a simple-Markov* IP-evolution *in an appropriate filtered probability space.*

### 1.1.3 Further results and outlook

These theorems give in general non-self-similar IP-diffusions and open up many interesting further directions for research. We list these below.

Under Theorems 2.1.6 and 2.1.7 we do not explicitly state the initial conditions of the IP-diffusion. In both cases the initial state is random and has the law of the skewer process when the scaffolding is suitably stopped. It is reasonable to ask about IP-diffusions with a fixed initial condition. Under Assumption C we only consider initial states given by interval partitions that consist of only finitely many interval blocks. The same initial states work in Assumption B but we extend this to a larger class of interval partitions in Theorem 2.4.35 for which we can construct IP-evolutions that are continuous on $[0,\infty)$. Also in Theorem 2.4.35 we consider a larger class of initial states and for which we can show continuity on $(0,\infty)$. An even larger class of interval partitions is considered in Proposition 2.3.12 on which we show that we can construct scaffolding and spindles with the fixed initial condition for the skewer at level 0 given by the interval partitions in the class. Further investigation into these areas could find the largest possible classes of diffusions for each of these cases.

1. Can we identify a natural state space in which the diffusions evolve such that the process can start continuously from anywhere in the state space?

Under Assumption C we only show the existence of continuous simple Markov IP-evolutions. Under strong Assumption B we require a technical condition *B*5 that may not be necessary. In both instances we struggle to prove the strong Markov property due to our previous point that we do not have IP-evolutions with arbitrary initial conditions.

2. Can we relax the conditions under Strong Assumption B and also prove the strong Markov property under Assumption C?

Another area of note is that the Assumptions B and C do not cover all possible diffusions. We do not construct IP-diffusions with untransformed diffusions which have infinitesimal drift $\mu(x) \approx 0$ for $x$ near 0. This could be future research and we explore such untransformed diffusions in Example 2.1.12.

3. Can we show the existence of IP-diffusions with untransformed diffusions that have $\mu(x) \approx 0$ for $x$ near 0 (i.e. diffusions between Assumptions B and C)?

Finally we believe that our research should extend to time-inhomogeneous IP-diffusions. Imagine for example a branching process for individuals over a summer and winter seasonal variation. Formally we would have the diffusions be time-dependent, i.e. in the scaffolding-and-spindles construction, and this would result in a different excursion law for each level but the scaffolding in this setting would still be Markovian a Lévy-type process, see e.g. Jacob and Schilling [50], instead of a Lévy process.

4. Can we show the existence of time-inhomogeneous IP-diffusions?

## 1.2 The on/off Brownian snake

### 1.2.1 Background

Superprocesses are measure-valued branching processes, see e.g. Etheridge [32]. Measure-valued processes have long been used to study scaling limits for populations that live and move in some spatial domain and continue to feature in new research, see e.g. [24, 26, 81]. Recently, research has begun to explore the effect of dormancy on the long term behaviour of these systems. In the literature this is done through the incorporation of "seed banks" in the model correspond to states where the particle is dormant and does not move or die or reproduce, see e.g. [13, 69]. Our focus is one such process known as on/off super Brownian motion (ooSBM) which was recently defined by Blath and Jacobi [14]. The process ooSBM is analogous to a process without dormancy known as super Brownian motion (SBM). SBM is one of the first measure-valued processes to be studied, see [29, 94]. In Chapter 3 we use Brownian snakes to study ooSBM.

Brownian snakes have been used to study superprocesses, see e.g. [11, 67]. We give a brief overview of the snakes that we use. Let $\xi$ be a Markov process on $[0,\infty)$ with two independent spatial components $B$, $S$ where

1. $B$ is a Brownian motion on $\mathbb{R}^d$ for some $d \in \mathbb{N}$;
2. $S$ is a compound Poisson subordinator on $[0,\infty)$ with unit drift and jump intensity $\pi(ds) = c\tilde{c}e^{-\tilde{c}s}ds$.

We consider Brownian snakes with spatial motion given by $\xi$ and we give some properties of such Brownian snakes. We define a killed path in $E := \mathbb{R}^d \times [0,\infty)$ to be a càdlàg function $w : [0,\zeta) \to E$, where we call $\zeta = \zeta(w) \in (0,\infty)$ the lifetime of $w$. For $x \in E$ we let $\mathscr{W}_x$ denote the set of all the killed paths $w$ such that $w(0) = x$ and we let $x$ denote a path of length 0 started from $x$. A Brownian snake $X = (X_t, t \geq 0)$ started from $x$ is a $\mathscr{W}_x$-valued Markov process and three key properties of $(X_t, t \geq 0)$ are:

1. for each fixed $t$ the process $X_t : [0, \zeta(X_t)) \to E$ has the distribution of $\xi$ started from $x$ and with lifetime $\zeta(X_t)$;
2. the process $(\zeta(X_t), t \geq 0)$, known as the contour process of $(X_t, t \geq 0)$, is a reflecting Brownian motion on $[0,\infty)$ started from 0;
3. for $t, t' > 0$ we have $X_t(s) = X_{t'}(s)$ for all $s \in [0, \min(\{\zeta(X_t), \zeta(X_{t'})\}))$.

The process $(X_t, t \geq 0)$ is continuous and strong Markov with respect to an appropriate complete and separable metric that we define in Section 3.2. Heuristically, we can think of the dynamics of $(X_t, t \geq 0)$ as follows. As $\zeta(X_t)$ decreases the path of $X_t$ can be thought of as being erased and as $\zeta(X_t)$ increases the paths $X_t$ can be thought of as being extended with an independent increment according to the dynamics of $\xi$.

In Chapter 3 we use the Brownian snake described above to construct a coupling of ooSBM and SBM on which their ranges coincide. Heuristically, the range of a measure-valued process is the set of all spatial locations ever visited by the process over all points in time and our result shows us that the range of ooSBM is equal in distribution to the range of SBM.

In the next section we lift the key definitions and results from Section 3.1 to give an overview of the project and the main result.

### 1.2.2 Framework of main results

We first define branching Brownian motion (BBM) and on/off branching Brownian motion (ooBBM) with critical binary branching.

**Definition 1.2.1** (Branching Brownian motion (BBM) and on/off branching Brownian motion (ooBBM))**.** *Branching Brownian motion* is a discrete particle system that evolves according to the following dynamics:

1. each particle independently moves according to the law of a Brownian motion in $\mathbb{R}^d$ during its lifetime;

2. a particle splits into two particles at the same spatial location with rate $\gamma/2$ for some $\gamma \geq 0$;

3. a particle dies with rate $\gamma/2$.

*On/off branching Brownian motion* is a discrete particle system where particles take values in $\mathbb{R}^d \times \{0,1\}$. The first component is the spatial position of the particle and the second component denotes whether a particle is active (taking value 1) or dormant (0). We furthermore have the following.

1. Each "active" particle independently has the spatial motion of a Brownian motion in $\mathbb{R}^d$. It also has splitting rate $\gamma/2$ and death rate $\gamma/2$ for some $\gamma \geq 0$ as before. Finally an "active" particle switches to a "dormant" particle with rate $c$ for some $c > 0$.

2. Each "dormant" particle does not move or split or die. Each particle stays in the same spatial position and switches to an "active" particle with rate $\tilde{c}$ for some $\tilde{c} > 0$.

We define $BBM(\gamma)$ to be a BBM with parameter $\gamma$ and $ooBBM(\gamma, c, \tilde{c})$ to be an ooBBM with parameter $\gamma$, dormancy rate $c$ and wake-up rate $\tilde{c}$.

BBM takes values in the space of finite measures on $\mathbb{R}^d$, $\mathscr{M}_F(\mathbb{R}^d)$, and ooBBM takes values in the space of pair of finite measures on $\mathbb{R}^d$, $\mathscr{M}_F(\mathbb{R}^d)^2$, where the first component is unit Dirac masses of dormant particles and the second component is the active particles.

SBM and ooSBM are both measure-valued processes. We define the Prokhorov metric for finite measures next and we state some simple properties after following [28, Section A.2.5].

For a metric space $(E, d_E)$ we define the *Prokhorov metric* $\rho$ for a pair of finite measures on $(E, d_E)$, given by $P, Q \in \mathscr{M}_F(E)$, as

$$\rho(P,Q) = \inf\left(\left\{\varepsilon > 0 : P(F) \le Q(F^\varepsilon) + \varepsilon; \quad Q(F) \le P(F^\varepsilon) + \varepsilon \quad \text{for all } F \in \mathscr{C}\right\}\right)$$

where $\mathscr{C}$ is the collection of closed subsets of $E$ and $F^\varepsilon := \{x \in E : \inf(\{d_E(x,y) : y \in F\}) \le \varepsilon\}$. The Prokhorov metric metrizes weak convergence and is complete and separable if $(E, d_E)$ is complete and separable.

SBM and ooSBM are scaling limits of BBM and ooBBM. To formulate this precisely, we denote by $\mathbb{D}([0,\infty), E)$ the space of measure-valued càdlàg paths $f : [0,\infty) \to E$ where $(E, d_E)$ is a complete and separable metric space, equipped with the Skorokhod topology, see e.g. [12, Section 16].

**Proposition 1.2.2** (The existence of super Brownian motion (SBM), see e.g. [32])**.** *Let $\mu \in \mathscr{M}_F(\mathbb{R}^d)$. Then there exists a measure-valued process $(\mathscr{X}_s)_{s\ge 0}$ that takes values in $\mathscr{M}_F(\mathbb{R}^d)$ with $\mathscr{X}_0 = \mu$ which is the scaling limit of BBMs in the following sense. Let $\tilde{\mathscr{Z}}^\varepsilon = (\tilde{\mathscr{Z}}^\varepsilon_s)_{s\ge 0}$ be $BBM(\gamma/\varepsilon)$ with $\tilde{\mathscr{Z}}^\varepsilon_0$ distributed as a Poisson random measure with intensity $\mu/\varepsilon$. Then we have*

$$(\varepsilon \tilde{\mathscr{Z}}^\varepsilon_s)_{s\ge 0} \to (\mathscr{X}_s)_{s\ge 0}$$

*as $\varepsilon$ goes to 0, in distribution on $\mathbb{D}([0,\infty), \mathscr{M}_F(\mathbb{R}^d))$. This process $(\mathscr{X}_s)_{s\ge 0}$ is called $SBM(\gamma)$ started at $\mu$.*

**Proposition 1.2.3** (The existence of on/off super Brownian motion (ooSBM), Blath and Jacobi [14])**.** *Let $(\mu_{\mathrm{d}}, \mu_{\mathrm{a}}) \in \mathscr{M}_F(\mathbb{R}^d)^2$. Then there exists a measure-valued Markov process $((\mathscr{Y}^{(\mathrm{d})}_s, \mathscr{Y}^{(\mathrm{a})}_s), s \ge 0)$ that takes values in $\mathscr{M}_F(\mathbb{R}^d)^2$ with $(\mathscr{Y}^{(\mathrm{d})}_0, \mathscr{Y}^{(\mathrm{a})}_0) = (\mu_{\mathrm{d}}, \mu_{\mathrm{a}})$ which is the scaling limit of ooBBM in the following sense. For $\varepsilon > 0$, let $((\mathscr{Z}^{(\varepsilon,\mathrm{d})}_s, \mathscr{Z}^{(\varepsilon,\mathrm{a})}_s))_{s\ge 0}$ be $ooBBM(\gamma/\varepsilon, c, \tilde{c})$ with $\mathscr{Z}^\varepsilon_0$ distributed as a Poisson random measure with intensity $(\mu_{\mathrm{d}}/\varepsilon, \mu_{\mathrm{a}}/\varepsilon)$. Then we have*

$$((\varepsilon \mathscr{Z}^{(\varepsilon,\mathrm{d})}_s, \varepsilon \mathscr{Z}^{(\varepsilon,\mathrm{a})}_s))_{s\ge 0} \to ((\mathscr{Y}^{(\mathrm{d})}_s, \mathscr{Y}^{(\mathrm{a})}_s))_{s\ge 0}$$

*as $\varepsilon$ goes to 0 in distribution, on $\mathbb{D}([0,\infty), \mathscr{M}_F(\mathbb{R}^d)^2)$. The process $(\mathscr{Y}_s)_{s\ge 0}$ is called $ooSBM(\gamma, c, \tilde{c})$ started at $(\mu_{\mathrm{d}}, \mu_{\mathrm{a}})$.*

We define the support of a measure noted in Cohn [25, Section 7.4] and the range of a measure-valued process, see e.g. Etheridge [32, Definition 6.7].

**Definition 1.2.4** (The support of a measure and the range of a measure-valued process)**.** Let $\mu \in \mathscr{M}_F(\mathbb{R}^d)$. Then $\mathbb{R}^d$ has a largest open subset $U_\mu$ such that $\mu(U) = 0$ – this is the union of all open subsets $U'$ such that $\mu(U') = 0$. The *support* of $\mu$ is defined to be $\text{supp}(\mu) := \mathbb{R}^d \backslash U_\mu$.

For a measure-valued process $\mathscr{X} = (\mathscr{X}_s)_{s\geq 0}$ we define the *range* of $\mathscr{X}$ to be $\mathscr{R}(\mathscr{X}) = \overline{\bigcup_{s>0} \text{supp}(\mathscr{X}_s)}$, that is the closure of the union of supports over all positive times.

We state our main theorem here.

**Theorem 1.2.5** (Theorem 3.1.8, the equality of ranges for SBM and ooSBM)**.** *Let $\mu_\text{d}$, $\mu_\text{a} \in \mathscr{M}_F(\mathbb{R}^d)$ be finite measures and let $\gamma$, $c$, $\tilde{c} > 0$. Let $\mathscr{X}$ be an SBM$(\gamma)$ with initial measure $\mu_\text{d} + \mu_\text{a}$ and let $\mathscr{Y} = (\mathscr{Y}^{(\text{d})}, \mathscr{Y}^{(\text{a})})$ be an ooSBM$(\gamma, c, \tilde{c})$ with initial measure $(\mu_\text{d}, \mu_\text{a})$. Then we have that $\mathscr{R}(\mathscr{X})$ is equal in distribution to $\mathscr{R}(\mathscr{Y}^{(\text{d})}) \cup \mathscr{R}(\mathscr{Y}^{(\text{a})})$.*

*Moreover there exists a coupling for $\mathscr{X}$ and $\mathscr{Y}$ such that $\mathscr{R}(\mathscr{X}) = \mathscr{R}(\mathscr{Y}^{(\text{d})}) \cup \mathscr{R}(\mathscr{Y}^{(\text{a})})$ a.s.*

### 1.2.3 Further results and outlook

Our key result concerning Brownian snakes is given in Theorem 3.4.6. It is an identification of the finite dimensional distributions of ooSBM with arbitrary initial condition by a Poisson random measure of Brownian snakes. We give a summary of the result here.

**Theorem 1.2.6** (Theorem 3.4.6)**.** *Let $\mu_\text{d}$, $\mu_\text{a}$ be finite measures on $\mathbb{R}^d$ and let $c$, $\tilde{c} > 0$. There exists a Poisson random measure of Brownian snakes $\sum_{i\in I} \delta_{x^i}$ from which we can construct a process $(\mathscr{Z}^{(\text{d})}, \mathscr{Z}^{(\text{a})}) = (\mathscr{Z}^{(\text{d})}_s, \mathscr{Z}^{(\text{a})}_s)_{s\geq 0}$ which has the finite-dimensional distributions of ooSBM$(4, c, \tilde{c})$ started from $(\mu_\text{d}, \mu_\text{a})$.*

Our abridged version of the theorem stated here is rather cryptic as we do not specify the Poisson random measure that we consider or how to construct $(\mathscr{Z}^{(\text{d})}, \mathscr{Z}^{(\text{a})})$. We also note, for any $\gamma > 0$, that the result can be generalised to ooSBM$(\gamma, c, \tilde{c})$. The Poisson random measure we use is defined in Definition 3.2.7 and the full version of the theorem is given in Section 3.4. The ideas follow the construction of SBM from a Poisson random measure of Brownian snakes in [67, Theorem 4.4.2]. Informally, we use the component $S$ of the spatial motion $\xi = (B, S)$ to insert dormancy into the motion of the particles.

The superprocess ooSBM is a scaling limit of discrete particle systems where the branching is binary. More generally, there exist superprocesses that are the scaling limit of non-binary branching processes. These processes have been constructed with what are known as Lévy snakes, see [67, Chapter 5], which has a contour process with distribution derived from a general

Lévy process rather than from Brownian motion. In [67, Chapter 5] Lévy snakes are used to construct superprocesses that are the scaling limit of discrete particle systems with non-binary branching. It is plausible that our construction of ooSBM could be generalised to scaling limits of discrete particle systems with dormancy.

1. Can we use Lévy snakes to construct superprocesses that are the scaling limits of discrete branching processes with dormancy and with non-binary branching?

Another possible research direction would be to uncover more properties of ooSBM. Our snake representation in the theorem stated above could provide easy access to tackling these questions. For instance, it is stated in [14, Corollary 3.3] that the total mass of ooSBM is positive for all time a.s. but it is also stated in [14, Corollary 3.8] that the total mass of the active component hits zero with positive probability. We believe that the active component hits zero infinitely often a.s. Another area of ooSBM to study is the support of the process. Unlike the ranges of ooSBM and SBM, there does not seem to be a natural coupling between the support of ooSBM at a given time and the support of SBM at a given time. However, as the Brownian snake is used to identify properties of the support of SBM, see [67, Theorem 4.5.2], it is plausible that our representation of ooSBM could play a role in uncovering properties about the support of ooSBM.

2. Can we use our construction of ooSBM from Brownian snakes to uncover more properties about the superprocess?

## 1.3 Random walks on Coxeter interchange graphs

### 1.3.1 Background

For a graph $G = (V, E)$ a lazy simple random walk on the graph is a Markov chain $(X_m)_{m\geq 0}$ such that for $i$, $j \in V$ we have $\mathbb{P}(X_{m+1} = j | X_m = i)$ is equal to $1/2$ if $i = j$ and equal to $1/2d_i$ if $\{i, j\}$ is an edge in $E$ where $d_i := |\{k \in V : \{i,k\} \in E\}|$ denotes the degree of $i$. This random walk is aperiodic and for a connected graph $G$ the random walk is irreducible and so it converges in distribution to its unique stationary $\pi$ on $V$, i.e. for all $i, j \in V$,

$$\mathbb{P}(X_m = j | X_0 = i) \to \pi(y) \text{ as } m \to \infty.$$

We note that $\pi(j)$ is also the asymptotic proportion of time spent at $j \in V$. We call a graph regular if for some $d \geq 0$ we have $d_i = d$ for all $i \in V$ and we note that for a connected and regular graph the stationary distribution for a lazy simple random walk on the graph is the uniform distribution.

We study lazy simple random walks on interchange graphs of Coxeter tournaments with specified score sequences. These graphs are connected, Theorem 5, and they have also been

shown to be regular in recent work by Kolesnik, Mitchell, Przybyłowski [57]. The work we present in Chapter 4 is the next work in the series and is a collaboration with these authors. The rest of this background section has the following structure:

1. we introduce (classical) tournaments and the notion of a score sequence $\mathbf{s}$ for a tournament;
2. we define an interchange graph for tournaments with a specified score sequence;
3. we state the existing result by Kannan, Tetali and Vempala, [53], and McShine, [72], which concerns rapid-mixing of lazy simple random walks on the interchange graphs of tournaments with specified score sequences.

Our work in Chapter 4 generalises the above results to what we call Coxeter tournaments as introduced by Kolesnik and Sanchez [58]. We define Coxeter tournaments and give our main results next in Section 1.3.2.

A tournament is an oriented graph $\mathscr{T} = ([n], E)$. Each vertex $i \in [n]$ is considered to be a player and each edge is a game between that pair of players. An edge between vertices $i$ and $j$ and directed towards $j$, which we denote by $i \to j$, represents a game between players $i$ and $j$ that is won by player $i$ and lost by player $j$. We add exactly one directed edge $e \in E$ between $i$ and $j$ for every pair of players $i$, $j$. Tournaments have been studied in the literature and go as far back as the 1950s, see [64] which looks at hierarchies within animal societies (for example, the pecking order for a group of chickens).

Let $\mathscr{T} = ([n], E)$ be a tournament. We define a score sequence $\mathbf{s}$ for $\mathscr{T}$ in the following way. We define the number of "wins" for player $i$, $d_i^{\mathrm{w}}$, and the number of "losses" for player $i$, $d_i^{\mathrm{l}}$, in the following way. We have

$$d_i^{\mathrm{w}} := |\{j \in V : (i \to j) \in E\}| \quad d_i^{\mathrm{l}} := |\{j \in V : (j \to i) \in E\}|.$$

Note that $d_i^{\mathrm{w}} + d_i^{\mathrm{l}} = n - 1$. We now define the score sequence for $\mathscr{T}$ as

$$\mathbf{s} = (s_1, \ldots, s_n), \quad s_i := \frac{1}{2}(d_i^{\mathrm{w}} - d_i^{\mathrm{l}}) \quad \text{for } 1 \leq i \leq n.$$

We let $\mathrm{Score}(n)$ denote the set of all possible score sequences for a tournament with vertex set $[n]$. We can now define the interchange graph with a specified score sequence of tournaments. These are graphs with undirected edges where vertices represent tournaments. In particular, for a specified score sequence $\mathbf{s}$, the vertices of the interchange graph $\mathrm{IntGr}(n, \mathbf{s})$ are tournaments $\mathscr{T}$ with vertex set $V := [n]$ and with score sequence $\mathbf{s}$. We define the vertex set $\mathrm{Tour}(n, \mathbf{s})$. Two tournaments $\mathscr{T} = ([n], E)$, $\mathscr{T}' = ([n], E')$ have an edge between them if and only if the following conditions are met. There exist $i$, $j$, $k \in [n]$ distinct such that

- we have $(i \to j)$, $(j \to k)$, $(k \to i) \in \mathscr{T}$;

- we have $(i \to k)$, $(k \to j)$, $(j \to i) \in \mathscr{T}'$;
- all other edges of $\mathscr{T}$ and $\mathscr{T}'$ are identical, i.e. if $x \notin \{i,j,k\}$ or $y \notin \{i,j,k\}$ then $(x \to y) \in \mathscr{T}$ if and only if $(x \to y) \in \mathscr{T}'$.

In other words, two tournaments with vertex set $[n]$ and score sequence $\mathbf{s}$ are adjacent in $\mathrm{IntGr}(n,\mathbf{s})$ if and only if you can "flip" a triangle in one to get to the other. The set of possible score sequences can be seen in Landau [64] which establishes the set of possible "win sequences" $(d_1^{\mathrm{w}},\ldots,d_n^{\mathrm{w}})$.

We now describe the result for the rapid mixing for a lazy simple random walk $(X_t, t \geq 0)$ on an interchange graph with a specified score sequence, $\mathrm{IntGr}(n,\mathbf{s})$. For $\pi$ the stationary distribution, we define

$$\tau(m) = \max_{x \in \mathrm{Tour}(n,\mathbf{s})} \frac{1}{2} \sum_{y \in \mathrm{Tour}(n,\mathbf{s})} |\mathbb{P}(X_m = y | X_0 = x) - \pi(y)|.$$

It is a simple operation to determine that this is non-increasing in $m$. The *mixing time* is defined as

$$t_{\mathrm{mix}}(\varepsilon) = \inf\{m \geqslant 0 : \tau(m) \leqslant \varepsilon\}.$$

Then $(X_m, m \geq 0)$ is said to be *rapidly mixing* if $t_{\mathrm{mix}} \coloneqq t_{\mathrm{mix}}(1/4)$ is bounded by a polynomial in $m$. For $\varepsilon < 1/4$ we have $t_{\mathrm{mix}}(\varepsilon) \leq \lceil \log_2(\varepsilon^{-1}) \rceil t_{\mathrm{mix}}(1/4)$ and so it suffices to find asymptotic bounds for $t_{\mathrm{mix}}$.

**Theorem 1.3.1** (Rapid-mixing for lazy simple random walk on tournaments with specified score sequences, [53], [72])**.** *Fix any* $\mathbf{s} \in \mathrm{Score}(n)$. *Then a lazy simple random walk* $(\mathscr{T}_m : m \geqslant 0)$ *on* $\mathrm{IntGr}(n,\mathbf{s})$ *mixes in time* $t_{\mathrm{mix}} = O(n^3 \log n)$ *as* $n \to \infty$.

This result was shown for certain score sequences in [53] before later being shown for all possible score sequences in [72]. In applications we note that random walks on regular connected graphs are rapidly mixing can be used to generate approximate uniform random samples over a set in a process known as Markov Chain Monte Carlo, see [95, Chapter 3]. Therefore this result gives a method to sample an approximately uniform random tournament with a specified score sequence.

The set-up above can naturally be generalised to include different types of games. For instance, a collaborative game would consist of a game where either both players win or both players lose and a solitaire game would be a game with only a single player who either wins or loses. Coxeter tournaments are generalisations of this type and include collaborative games and two types of solitaire games; we give a precise formulation in Section 1.3.2. Coxeter tournaments are related to other areas of mathematics outside of probability and we refer to [58] which introduces Coxeter tournaments and explains these related fields.

In the next section we define precisely Coxeter tournaments which are the focus of our study. As we have done for the previous chapters, we lift the necessary definitions and results from Chapter 4 to give a framework for our main results.

### 1.3.2 Framework of main results

We define the Coxeter tournaments $A_{n-1}$, $B_n$, $C_n$, $D_n$, $n \in \mathbb{N}$. The tournaments we study have links with root systems that use the same notation $A_{n-1}$, $B_n$, $C_n$, $D_n$ and we discuss this link further in Chapter 4. The case $A_{n-1}$ is the simplest and has already been established as tournaments in an alternative equivalent formulation in the previous section. We state it here again along with the generalisations $B_n$, $C_n$, $D_n$ and this hopefully makes it clear exactly what we mean in the following.

A *signed graph* $\mathscr{S}$ on $[n] = \{1, 2, \ldots, n\}$ has a set of signed edges $E(\mathscr{S})$. The four possible types of edges are:

1. *negative edges* $e_{ij}^-$ between two vertices $i$ and $j$,
2. *positive edges* $e_{ij}^+$ between two vertices $i$ and $j$,
3. *half edges* $e_i^h$ with only one vertex $i$, and
4. *loops* $e_i^\ell$ at a vertex $i$.

A *Coxeter tournament* $\mathscr{T}$ on a signed graph $\mathscr{S}$ is an orientation of $\mathscr{S}$. More formally, $\mathscr{T} = (w_e : e \in E(\mathscr{S}))$, with all $w_e \in \{0, 1\}$. We think of each $e \in E(\mathscr{S})$ as a *game*, and $w_e$ as indicating its outcome. The *score sequence* is given by, cf. (4.2),

$$\mathbf{s}(\mathscr{T}) = \sum_{e \in E(\mathscr{S})} (w_e - 1/2)\mathbf{e},$$

where $\mathbf{e}$ is the vector corresponding to the signed edge $e$, given by $\mathbf{e}_{ij}^\pm = \mathbf{e}_i \pm \mathbf{e}_j$, $\mathbf{e}_i^h = \mathbf{e}_i$ and $\mathbf{e}_i^\ell = 2\mathbf{e}_i$. In other words:

1. negative edges $e_{ij}^-$ are *competitive games* in which one of $i, j$ wins and the other loses a $1/2$ point,
2. positive edges $e_{ij}^+$ are *collaborative games* in which $i, j$ both win or lose a $1/2$ point,
3. half edges $e_i^h$ are *(half edge) solitaire games* in which $i$ wins or loses a $1/2$ point, and
4. loops $e_i^\ell$ are *(loop) solitaire games* in which $i$ wins or loses 1 point.

If $\mathbf{s}(\mathscr{T}) = \mathbf{0}_n$ we say that $\mathscr{T}$ is *neutral*. The *complete* $\Phi$*-graph* $\mathscr{K}_\Phi$ contains all possible negative edges for type $\Phi = A_{n-1}$. In the other types $\Phi = B_n$, $C_n$ and $D_n$, the *complete* $\Phi$*-graph* $\mathscr{K}_\Phi$ contains all possible negative and positive edges $e_{ij}^{\pm}$. These are all the edges for $D_n$. In type $B_n$ it also contains all possible half edges $e_i^h$. In type $C_n$ it also contains all possible loops $e_i^\ell$. We call $\mathscr{S}$ a $\Phi$*-graph* if $\mathscr{S} \subset \mathscr{K}_\Phi$. A $\Phi$*-tournament* on $\mathscr{S}$ is an orientation of $\mathscr{S}$. When $\mathscr{S}$ is $\mathscr{K}_\Phi$, we will simply call it a $\Phi$*-tournament*. We next define the Coxeter interchange graph $\mathrm{IntGr}(n, \mathbf{s})$ for a given score sequence.

The sets $\mathrm{Tour}(n, \mathbf{s})$ are generated by the cyclic triangle $\Delta_c$. In the previous section we define explicitly what it means to "flip" a triangle in a tournament to go from one tournament to an adjacent one in an interchange graph. In the Coxeter setting, the idea is the same except for the fact that there are now additional types of edge in the tournament that correspond to the additional types of games.

In all types $B_n$, $C_n$ and $D_n$, in addition to $\Delta_c$, we also require a *balanced triangle* $\Delta_b$. In type $B_n$, there are also three *neutral pairs* $\Omega_1$, $\Omega_2$ and $\Omega_3$. On the other hand, in type $C_n$, there are also two *neutral clovers* $\Theta_1$ and $\Theta_2$. See Figures 4.5–4.7. In the figures below, we draw:

1. competitive games as edges directed away from their winner,
2. collaborative games as solid/dotted lines if won/lost,
3. half edge solitaire games as half edges directed away/toward from their (only) endpoint if won/lost, and
4. loop solitaire games as solid/dotted loops if won/lost.

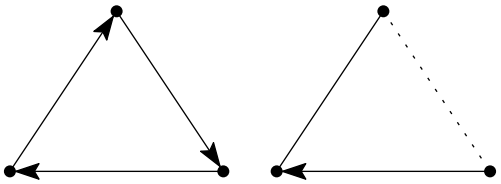


Figure 1.2: The cyclic and balanced triangles $\Delta_c$ and $\Delta_b$ are generators in all types $B_n$, $C_n$ and $D_n$.



Figure 1.3: The neutral pairs $\Omega_1$, $\Omega_2$ and $\Omega_3$ are additional generators in type $B_n$.

The *reversal* $\mathscr{T}^*$ of a Coxeter tournament $\mathscr{T} = (w_e : e \in E(\mathscr{S}))$ on $\mathscr{S}$ is obtained by reversing the outcome of all games in $\mathscr{T}$. That is, $\mathscr{T}^* = (w_e^* : e \in E(\mathscr{S}))$, where $w_e^* = 1 - w_e$. For

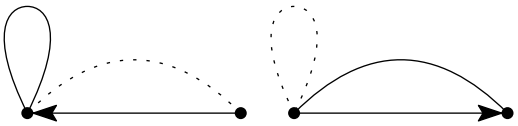


Figure 1.4: The neutral clovers $\Theta_1$ and $\Theta_2$ are additional generators in type $C_n$.

tournaments $\mathscr{X}$, $\mathscr{T}$ we write $\mathscr{X} \subset \mathscr{T}$ if all the edge set of $\mathscr{X}$ is a subset of the edge set of $\mathscr{T}$ and if the corresponding $w_e$ values are the same across both tournaments. If $\mathscr{X} \subset \mathscr{T}$, we let $\mathscr{T} * \mathscr{X}$ denote the Coxeter tournament obtained from $\mathscr{T}$ by reversing the outcome of all games in $\mathscr{X}$. In particular, $\mathscr{T}^* = \mathscr{T} * \mathscr{T}$.

The *Coxeter interchange graph* $\mathrm{IntGr}(\Phi, \mathbf{s})$ is a multigraph and has a vertex $v(\mathscr{T})$ for each $\mathscr{T} \in \mathrm{Tour}(\Phi, \mathbf{s})$, where $\mathrm{Tour}(\Phi, \mathbf{s})$ is the set of all Coxeter tournaments on $\mathscr{K}_\Phi$ with score sequence $\mathbf{s}$. Vertices $v(\mathscr{T}_1), v(\mathscr{T}_2)$ are neighbours if $\mathscr{T}_2 = \mathscr{T}_1 * \mathscr{G}$, for some copy $\mathscr{G} \subset \mathscr{T}_1$ of a type $\Phi$ generator. If $\mathscr{G}$ is a neutral clover, we add a double edge, and otherwise we add a single edge. Therefore the double edges feature only in the interchange graphs $\mathrm{IntGr}(C_n, \mathbf{s})$. We give an example of an interchange graph in this figure from the chapter and we also state our main results. In the following, for $\Phi = B_n$, $C_n$ or $D_n$, let $\mathrm{Score}(\Phi)$ be the set of all possible score

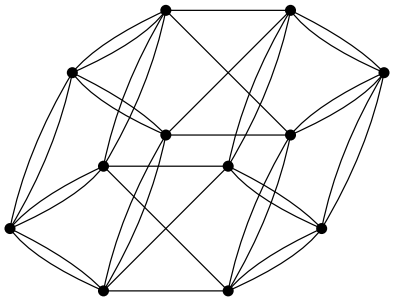


Figure 1.5: The "snare drum" interchange graph, arising in $C_3$, consists of two crystals joined by double edges.

sequences on $\mathscr{K}_\Phi$.

**Theorem 1.3.2** (Theorem 5)**.** *Let $\Phi = B_n$, $C_n$ or $D_n$. Fix any $\mathbf{s} \in \mathrm{Score}(\Phi)$. The Coxeter interchange graph $\mathrm{IntGr}(\Phi, \mathbf{s})$ is connected and its diameter $D = O(n^2)$.*

**Theorem 1.3.3** (Theorem 4)**.** *Let $\Phi = B_n$, $C_n$ or $D_n$. Fix any $\mathbf{s} \in \mathrm{Score}(\Phi)$. Then lazy simple random walk $(\mathscr{T}_k : k \geqslant 0)$ on $\mathrm{IntGr}(\Phi, \mathbf{s})$ mixes in time $t_{\mathrm{mix}} = O(n^3 \log n)$ if $\Phi = B_n$ or $D_n$, and in time $t_{\mathrm{mix}} = O(n^4 \log n)$ if $\Phi = C_n$.*

To summarise, our project looks at graphs such as the one in Figure 1.5 and determines how quickly a random walk reaches stationarity on such a graph. This graph was illustrated above for a specified score sequence in $C_3$. As $n$ increases the graphs of $C_n$ grow in size but the structure of the graphs have similarities to the one shown here.

### 1.3.3 Further results and outlook

These results open up several interesting research questions. Firstly, we address the different mixing time upper bound in the $C_n$ case. For the $A_{n-1}$, $B_n$ and $D_n$ the mixing time is $O(n^3 \log n)$ whereas in the $C_n$ case it is of order $O(n^4 \log n)$. It is not clear for any of the cases whether these upper asymptotic bounds are optimal and we give some insight for the difference between $C_n$ and the other two cases $B_n$ and $D_n$ here.

To show the mixing times result we use a technique called path-coupling. We state the result here which we restate later in Theorem 3.

**Theorem 1.3.4** (Path coupling, [17])**.** *Consider a Markov chain $(X_m)_{m\geq 0}$ on the vertices of a connected graph $G = (V,E)$. Let $\delta$ denote the graph distance, which assigns weight $\delta(u,v) = 1$ to edges $\{u,v\} \in E$. Let $w$ be a re-weighting, assigning some $w(u,v) \geqslant 1$ to $\{u,v\} \in E$. Let $w(x,y)$ be the re-weighted distance between $x,y$ (that is, the minimal total weight of edges along a path between them). Let*

$$D_w = \max_{x,y\in V} w(x,y)$$

*denote the re-weighted diameter of $G$. Suppose that, for some $\alpha > 0$, for each $\{x',x''\} \in E$ there is a coupling $(X_1',X_1'')$ with $(X_0',X_0'') = (x',x'')$ so that*

$$\mathbb{E}[w(X_1',X_1'')] \leqslant (1-\alpha)w(x',x'').$$

*Then $(X_m)_{m\geq 0}$ mixes in time $t_{\text{mix}} = O(\alpha^{-1}\log D_w)$ (i.e. there exists a constant $C > 0$ such that for any graph with a one-step path-coupling as given as above we have that $t_{\text{mix}} \leq C\alpha^{-1}\log D_w$).*

We apply this result with $w = \delta$ for the cases $B_n$ and $D_n$ but for the case $C_n$ we could not find a contractive one-step with the metric $w = \delta$. Instead we use a metric $w$ that assigns a slightly longer length to the single edges (and keeps the length of an edge which is part of a double edge as 1). This does lead to a one-step contractive path-coupling but ultimately the different metric is what leads to the worse asymptotic upper bound. We make precise exactly what the situation is for this case below.

We define interchange networks and then extended interchange networks which are subgraphs of the interchange graphs. In Chapter 4 we show that finding path-couplings on these overlapping subgraphs leads to a path-coupling on the whole graph in the manner given in the theorem above.

**Definition 1.3.5.** For $\mathscr{T}_1, \mathscr{T}_2 \in \text{Tour}(\Phi,\mathbf{s})$ at distance two in $\text{IntGr}(\Phi,\mathbf{s})$, we define the *interchange network* $N(\mathscr{T}_1,\mathscr{T}_2)$ to be the union over all paths of length two between $\mathscr{T}_1$ and $\mathscr{T}_2$.

We call a pair of vertices in $N(\mathscr{T}_1,\mathscr{T}_2)$ with distance two from each other an "antipodal" pair.

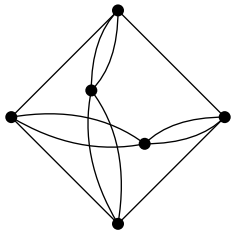


Figure 1.6: The crystal extended interchange network.

**Definition 1.3.6.** We define the *extended interchange network* $\hat{N}(\mathscr{T}_1, \mathscr{T}_2)$ to be the union of $N(\mathscr{T}_1', \mathscr{T}_2')$ over all "antipodal" pairs $\mathscr{T}_1', \mathscr{T}_2'$ in $N(\mathscr{T}_1, \mathscr{T}_2)$ at distance two.

The example of an extended interchange network that is relevant here is called the crystal, as shown in Figure 4.25 here. It is this type of extended interchange network with which we had trouble with path-coupling with the metric $w = \delta$. The path-coupling result we were able to achieve uses another parameter $\gamma$ to define the new metric $w$ where we define $\gamma$ as follows. We define the crystal degree of a double edge to be the number of crystals that contain the double edge and we define $\gamma = \gamma(C_n, \mathbf{s})$ to be the maximal crystal degree over all double edges. This quantity appears abstract here without further details of our proof technique but it suffices to note that we show, in Lemma 4.5.8, that $\gamma \leq \min\{2n, d\}$ where $d$ is the degree of the interchange graph $\mathrm{IntGr}(C_n, \mathbf{s})$. We note from [57] that $d = O(n^3)$ in the $B_n$, $C_n$, $D_n$ cases.

**Theorem 1.3.7** (Theorem 14)**.** *Let $\Phi = C_n$. Fix any $\mathbf{s} \in \mathrm{Score}(\Phi)$. Then lazy simple random walk $(\mathscr{T}_m : m \geqslant 0)$ on the Coxeter interchange graph $\mathrm{IntGr}(C_n, \mathbf{s})$ is rapidly mixing. If there are no crystals in $\mathrm{IntGr}(C_n, \mathbf{s})$ (when $\gamma = 0$) then $t_{\mathrm{mix}} = O(d \log n)$. Otherwise, we have $t_{\mathrm{mix}} = O(\gamma d \log n)$.*

As $d = O(n^3)$ and $\gamma \leq \min\{d, 2n\}$, we have that the mixing time has $O(n^4 \log n)$ as an upper bound as stated in Theorem 4. Ultimately, the question is whether the presence of these crystals truly does act as a bottleneck to the mixing of the Markov chain or whether this is an artefact of our proof technique and that in fact the mixing rates in the $C_n$ case is comparable to the mixing rates in the $A_{n-1}$, $B_n$ and $D_n$ cases. Future research in this direction could improve the upper bound in all cases and may find a common asymptotic upper bound for the $A_{n-1}$, $B_n$, $C_n$, $D_n$ cases.

1. Are these mixing times bounds given in Theorem 4 optimal, particularly in the $C_n$ case?

Another possible direction is to explore random tournaments and random Coxeter tournaments. In this new setting, instead of an oriented edge between players $i$ and $j$, we have a value $p_{ij} = 1 - p_{ji} \in [0, 1]$ which we interpret as the probability that player $i$ would beat player $j$ in a game. The score sequence would be interpreted as the relative strength of each player. This generalisation again has links to other areas of mathematics outside of probability but we leave

it to [58] to provide an overview to these other areas. The collection of possible score sequences for random tournaments, known in this setting as mean score sequences, was established in Moon [75]. The corresponding result for score sequences for random Coxeter tournaments was established recently in [58].

We identify two natural directions to proceed. We consider random tournaments only in the following but we note that our ideas could generalise to the Coxeter setting. Let $0 < \varepsilon \leq 1$ and consider a random tournament. Let $i, j, k \in [n]$ with $p_{ij}$, $p_{jk}$ and $p_{ki} \geq \varepsilon$. Then we could "shift" the triangle $i \to j \to k \to i$ by $\varepsilon$ with new probabilities

$$\tilde{p}_{ij} \coloneqq p_{ij} - \varepsilon; \quad \tilde{p}_{jk} \coloneqq p_{jk} - \varepsilon; \quad \tilde{p}_{ki} \coloneqq p_{ki} - \varepsilon$$

and keep all other probabilities fixed. Note that for $\varepsilon = 1$ this corresponds to flipping the triangle $i \to j \to k \to i$. Our first approach would be to then consider a random walk where at each time step we pick a triangle for which we can do an "$\varepsilon$-shift" and then to perform the shift. It may be possible to rescale time of the random walk along with the size of $\varepsilon$ to obtain a diffusive limit for the random walk keeping $n$ fixed.

The other direction would be to construct a random walk as follows. As previously we identify a triangle at each time step with three distinct vertices $i$, $j$, $k$ but now we define a random variable $S \sim \text{Uniform}[0, \min\{p_{ij}, p_{jk}, p_{ki}\}]$ and perform an "$S$-shift" on the triangle $i \to j \to k \to i$. This random walk would be on the space of random tournaments with a specified score sequence and the mixing times for this walk could be studied. To summarise we have possible future research questions given by the following.

2. Do there exist diffusions or random walks on the space of random tournaments with specified score sequences? Do these generalise to the Coxeter setting? How quickly do they converge to stationarity?

Other possible research questions are given in Section 4.7.

# Chapter 2

# Branching Interval Partition Diffusions

This chapter was uploaded to ArXiv, [18], https://arxiv.org/abs/2402.08548. It was also submitted to the Latin American Journal of Probability and Mathematical Statistics (ALEA) in March 2024. Apart from minor edits, and incorporating the abstract and bibliography into the overall abstract and bibliography of this dissertation, it is presented here in the state in which it was submitted.

## 2.1 Introduction

Continuous-time branching processes with finite birth rates and independent identically distributed (iid) lifetimes of individuals are known as Crump–Mode–Jagers (CMJ) processes, as studied in Jagers [51, Chapter 6]. For these processes individuals are born from single parents, and during their lives they can in turn give birth to more individuals until they die after some iid positive random times after their births; those with constant birth rates are known as homogeneous and those for which all births produce exactly one offspring are known as binary. Specifically, CMJ processes are the population size processes and so take values in $\mathbb{N}$. In the homogeneous binary cases individuals produce single offspring after iid exponential random times and this repeats until their deaths (or never occurs if they die before their first reproduction event). We call the difference between birth and death the *lifetime* of the individual and the interval of time between the birth and death the *life* of the individual. To produce Markovian homogeneous binary CMJ processes we require either the lifetimes to be infinite or for the lifetimes to be exponentially distributed.

We study homogeneous binary branching processes with infinite birth rates where individuals are marked with non-negative characteristics that continuously vary during their lives, and we construct continuous Markovian interval-partition-valued processes by the method shown in Forman, Pal, Rizzolo, and Winkel [34]. Interval partitions are ordered collections of intervals with summable lengths; we define these below following Aldous [2, Section 17] and Pitman [83, Chapter 4].

**Definition 2.1.1** (Interval partitions)**.** For $M > 0$, an *interval partition of length M* is a set $\beta$ of intervals of $[0,M]$ that are open and disjoint, and which cover $[0,M]$ up to a Lebesgue-null set. We write $\|\beta\|$ for the length $M$. An *interval partition* is an interval partition of length $M$ for some $M$. We define $\mathscr{I}_H$ to be the space of interval partitions.

Interval partitions can be constructed by concatenating ordered collections of intervals, e.g. the concatenation of intervals $(0,b)$ and $(0,c)$ would be given by $\{(0,b),(b,b+c)\}$ where we have shifted the interval $(0,c)$ by $b$. More generally we can concatenate interval partitions $\beta$ and $\gamma$ by shifting the all intervals of $\gamma$ by $\|\beta\|$ and we denote the resulting interval partition by $\beta \star \gamma$ in this case.

In our model individuals do not live forever and the characteristics take the value 0 at the births and deaths and are positive during the rest of the lives. We construct interval partitions in Definition 2.1.2 which are concatenations of intervals with lengths given by these characteristics. The characteristics themselves, as functions of the ages of the individuals they mark, are continuous non-negative excursions away from 0 of length given by the lifetime of the individual they mark. For this reason we refer to a characteristic at a point in time as the *level of health*, or just as the *health*, for the individual it marks at that point in time, and we refer to the continuous excursion of marks over an individual's life as its *health excursion*.

Lambert [60] and Lambert and Uribe Bravo [61] study spectrally positive Lévy processes (SPLPs) that are the contour functions of classes of splitting trees and are known as jumping chronological contour processes (JCCPs). For any $x \geq 0$ the jumps across level $x$ represent the individuals alive at that point in time. We construct interval partitions from the health excursions and the SPLP by the method from [34]. Definition 2.1.2 formally describes this construction and Figure 2.1 uses terminology from [34] which we explain in Section 2.1.1. The result is that at every point in time an interval partition is produced with a one-to-one correspondence of intervals and individuals alive at that point in time, and each interval has a length given by its corresponding individual's health at that point in time. These interval partitions therefore, as functions of the time for the branching process, form an interval-partition-valued process. This paper continues the investigation into these processes. We use the same terminology as in [34]: we refer to stochastic processes which take values in $\mathscr{I}_H$ as *interval partition evolutions* or more simply as *IP-evolutions*. We refer to IP-evolutions that are continuous and have the Markov property with respect to some filtered probability space as *interval partition diffusions* or *IP-diffusions*. The construction can be considered to give rise to branching processes as new intervals are generated between existing intervals and we discuss this further after Theorem 2.1.6. We generalise the class of IP-diffusions that have this branching property beyond the self-similar cases that have already been established in [34]. We use a complete and separable metric on interval partitions $d'_H$, from Forman et al. [35], which we state in Definition 2.1.5.

When we consider the health excursions rather than just the lifetimes of the individuals, the birth rate is a measure on the space of continuous non-negative excursions away from 0, and it is this rate that we are referring to when we say the birth rate is constant. We set the birth rate to be the Pitman-Yor excursion measure of a non-negative diffusion, from Pitman and Yor [84], and we explore the range of possible non-negative diffusions for which this is applicable. The Pitman-Yor excursion measure is more general than the standard Itô excursion measure as it includes cases where 0 is an exit boundary of the diffusion whereas the Itô excursion measure only includes cases where 0 is a regular boundary point of the diffusion (we give the relevant details in Section 2.1.2). Under certain conditions for summability of the health levels, that we specify in Theorem 2.1.4, this leads to IP-evolutions where the interval lengths all evolve independently with the same continuous Markovian dynamics given by the specified non-negative diffusion. It is these dynamics on the interval lengths which make possible the continuity and Markov property for the IP-evolutions. This is also why we choose excursions away from 0, as this preserves continuity at the points of birth and death of individuals; when an individual is born a new interval grows continuously out of 0 in the IP-evolution, with the parent's interval on the left, and it dies when this health level returns to 0.

Excursion measures of diffusions are not finite but $\sigma$-finite, and this is why the processes that we study have an infinite birth rate. However, most contribute small health levels and have small lifetimes in the sense that, for any positive threshold $\varepsilon > 0$ there is a finite rate of births with lifetimes or maximum health levels greater than $\varepsilon$. The continuity and the Markov property of the IP-evolutions do not follow immediately from the continuity and Markov property of the one-dimensional diffusions, indeed, it is not even the case that we can produce an IP-evolution from the Pitman-Yor excursion measure of any diffusion. This is because we may not be able to construct the SPLP "scaffolding" or, even if we can, the health levels may not be summable. In Section 2.1.2, Theorem 2.1.3 gives conditions for the construction of the SPLP, Theorem 2.1.4 gives conditions for the summability of health levels, and Theorems 2.1.6 and 2.1.7 give conditions for the existence of continuous Markovian IP-evolutions.

### 2.1.1 Construction of the SPLP "scaffolding" and the "skewer process"

Let $\Lambda$ be a Lévy measure on $(0,\infty)$ for a recurrent Lévy process, i.e. a measure such that $\int_{x=0}^{\infty}(x^2 \wedge x)\Lambda(dx)$ is finite, and let $\mathbf{J}$ be a Poisson random measure (PRM) with intensity $\mathrm{Leb} \otimes \Lambda$ on $[0,\infty) \times [0,\infty)$ (where "Leb" is the Lebesgue measure). If $\int_{x=0}^{\infty} x\Lambda(dx)$ is finite then we can construct a recurrent SPLP of bounded variation

$$\mathbf{X}_t := -t \int_{x=0}^{\infty} x\Lambda(dx) + \sum_{\substack{(s,j) \text{ atom of } \mathbf{J}: \\ 0<s<t}} j.$$

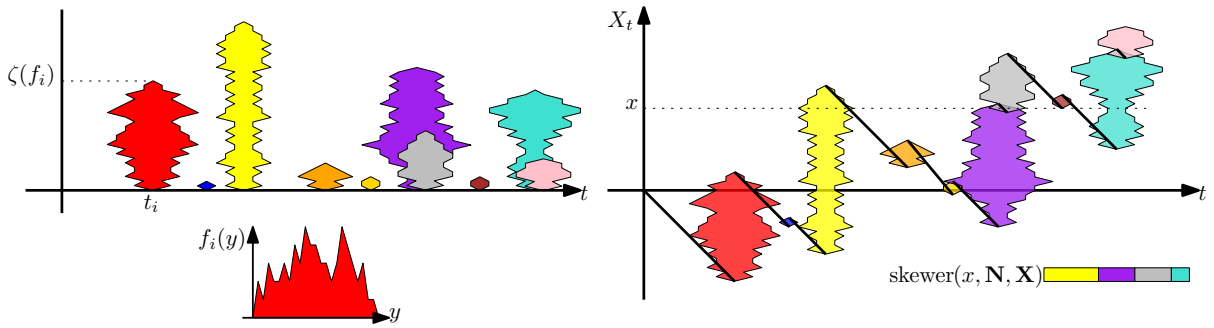



Figure 2.1: Left: A PRM of health excursions represented with time vertically moving up page and reflected to produce spindle-like shapes, with the excursion of a single spindle underneath. Right: the scaffolding process (black) constructed from the PRM, along with the interval partition generated by the skewer process at a fixed level $x$.

Now suppose that $\Lambda$ instead satisfies the condition $\int_{x=0}^{\infty}(x^2 \wedge x)\Lambda(dx) < \infty = \int_{x=0}^{\infty} x\Lambda(dx)$. In this case, for any $\varepsilon > 0$, we can construct a recurrent SPLP of bounded variation using the measure $\Lambda_\varepsilon(dx) := \mathbb{1}_{x>\varepsilon}\Lambda(dx)$ as we did above and take the limit $\varepsilon \downarrow 0$ is to obtain a recurrent SPLP of unbounded variation, see (2.1). For our applications the Lévy measure $\Lambda$ we will use is of the form $\nu(\zeta \in \cdot)$ where $\nu$ is the Pitman-Yor excursion measure for our specified diffusion (a measure on $\mathscr{E}$, the space of continuous positive excursions) and $\zeta$ is a function, $\zeta : \mathscr{E} \to [0,\infty)$, which gives the length of excursions which we call the lifetimes.

More formally, continuous positive excursions are functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(y) = 0$ for all $y \leq 0$, and there exists some $x > 0$ such that $f(y) = 0$ for all $y > x$, and finally for $y \in (0,x)$ that $f(y) > 0$ and $f$ is continuous on $[0,x]$; the lifetime $\zeta$ of $f$ is then $\zeta(f) = x$. We use the lifetimes as the jump sizes for the SPLP. Suppose $\int_{x=0}^{\infty}(x^2 \wedge x)\nu(\zeta \in dx) < \infty$, and let $\mathbf{N}'$ be a PRM with intensity Leb $\otimes\, \nu$ on $[0,\infty) \times \mathscr{E}$. We define the recurrent SPLP $\mathbf{X} = (\mathbf{X}_t, t \geq 0)$ by the almost sure limits

$$\mathbf{X}_t := \lim_{\varepsilon \downarrow 0} \left( -t \int_{x=\varepsilon}^{\infty} x\nu(\zeta \in dx) + \sum_{\substack{(s,f) \text{ atom of } \mathbf{N}' \\ 0<s<t}} \zeta(f)\mathbb{1}_{\zeta(f)>\varepsilon} \right). \tag{2.1}$$

The PRM $\mathbf{N}'$ takes values in a certain class of $\sigma$-finite point measures on $[0,\infty) \times \mathscr{E}$ which we denote by $\mathscr{N}$, and the SPLP $\mathbf{X}$ is a càdlàg process and we denote the set of càdlàg paths by $\mathscr{D}$. Specifically, the class $\mathscr{N}$ consists of all the point measures $N$ for which the limit in (2.1) exists for all $t > 0$ when replacing $\mathbf{N}'$ by $N$. The Lévy measure for $\mathbf{X}$ is $\nu(\zeta \in \cdot)$ which we refer to as the lifetime under the Pitman-Yor excursion measure. More generally, for a suitable point measure $N \in \mathscr{N}$, let $\xi(N)$ be the càdlàg function, $(X_t, t \geq 0)$, obtained when replacing $\mathbf{N}'$ by $N$ in (2.1).

We call $\mathbf{X}$ the scaffolding and $\mathbf{N}'$ the point measure of spindles. The reason for these names can be seen in the simplified version $(N,X)$ in Figure 2.1. In the SPLP, there is a one-to-one correspondence between individuals and jumps. Moreover, the jump heights are equal to the

individuals' lifetimes, and levels over which each jump crosses are the times for which the corresponding individual is alive in our model. Therefore, in the figure, the health excursion can be placed vertically in each jump. The excursions themselves look like spindles as we have width equal to health and they are symmetric for better visual representation on the diagram. We next define the skewer function.

**Definition 2.1.2** (Skewer function)**.** Let $J$ be a countable set and let $N' := \sum_{i\in J}\delta_{(r_i,f_i)} \in \mathscr{N}$. Let $I \subseteq J$ be a subset and let $N := \sum_{i\in I}\delta_{(r_i,f_i)}$. In this setting we write $N \leq N'$ to denote the fact that $N'$ has all the atoms of $N$. Let $X = \xi(N')$. We define the *aggregate health process* $((M^y_{N,X})(t), t \in [0,\infty))$ by

$$M^y_{N,X}(t) := \sum_{\substack{(r,f) \text{ atom of } N:\\ 0\leq r<t}} f(y - X_{r-}).$$

We also define the *total health process* at $t \in [0,\infty)$ as $(M^y_{N,X}(t), y \in \mathbb{R})$. When $M^y_{N,X}(\infty) < \infty$ we define the skewer for $(N,X)$ by

$$\text{skewer}(y,N,X) := \{(M^y_{N,X}(t-), M^y_{N,X}(t)) : t \in [0,\infty), M^y_{N,X}(t-) < M^y_{N,X}(t)\}.$$

Note that in this case skewer$(y,N,X)$ is an interval partition of length $M^y_{N,X}(\infty)$. We define skewer$(y,N,X)$ to be $\emptyset$, the empty interval partition, if $M^y_{N,X}(\infty)$ is infinite. For random measures $\mathbf{N} \leq \mathbf{N}'$ and scaffolding $\mathbf{X} = \xi(\mathbf{N}')$ we define the scaffolding-and-spindles construction of $(\mathbf{N},\mathbf{X})$ to be the IP-evolution given by $\overline{\text{skewer}}(\mathbf{N},\mathbf{X}) := (\text{skewer}(y,\mathbf{N},\mathbf{X}), y \geq 0)$. Unless stated otherwise, we take $\mathbf{N}'$ to be a PRM and we take

$$\mathbf{N} = \sum_{\substack{(r,f) \text{ atom of } \mathbf{N}':\\ 0\leq r\leq T}} \delta_{(r,f)}$$

where $T$ is a.s. positive finite random variable and then in this case we say $\mathbf{N}$ is *stopped at* $T$, we refer to $\mathbf{N}$ as a *stopped PRM* and we refer to $(\mathbf{N},\mathbf{X})$ as a *stopped scaffolding-and-spindles pair*.

### 2.1.2 Main results

We define one-dimensional diffusions with the framework given by Revuz and Yor [88, Chapter VII.3]. In our setting we consider a Markov process $\mathbf{Y}$ whose state space $E$ is of the form $E = [0,c]$ where $c$ is finite or of the form $E = [0,c)$ where $c$ is either finite or infinity. We assume that the paths of $\mathbf{Y}$ are continuous on $E$ and that $\mathbf{Y}$ has the strong Markov property. We further assume that our diffusion is regular on $(0,c)$, which means that for all points $x$, $y \in (0,c)$ the probability of a diffusion reaching point $y$ having started from point $x$ is positive, i.e. $\mathbb{P}_x(T_y < \infty) > 0$ where $T_y$ is the hitting time of the point $y$ by the diffusion. With these

assumptions, there exists a strictly increasing function $s : E \to [0,\infty)$ such that for any $a$, $w$, $x$ in $E$ with $0 \leq a < x < w \leq c$ we have

$$\mathbb{P}_x(T_w < T_a) = \frac{s(x) - s(a)}{s(w) - s(a)}, \tag{2.2}$$

where the function $s$ is called a scale function of $\mathbf{Y}$. The scale function is unique up to a normalising constant and we take $s(0) = 0$. We also define for $v \in (a,w)$

$$G_{(a,w)}(x,v) = \frac{s(x) - s(a)}{s(w) - s(a)} (s(w) - s(v)) \mathbb{1}_{x<v<w} + \frac{s(w) - s(x)}{s(w) - s(a)} (s(v) - s(a)) \mathbb{1}_{a<v<x} \tag{2.3}$$

which we call the Green's function for the diffusion on $(a,w)$. Then there exists a unique measure $M$ on $(0,c)$, which is known as the speed measure, such that for any $0 \leq a < x < w \leq c$ we have

$$\mathbb{E}_x(T_a \wedge T_w) = \int_{v=a}^{w} G_{(a,w)}(x,v) M(dv). \tag{2.4}$$

Certain diffusions can be expressed as solutions to stochastic differential equations as follows. Specifically, there exist locally bounded Borel-measurable functions $\mu : (0,c) \to \mathbb{R}$, $\sigma : (0,c) \to (0,\infty)$ and a Brownian motion $\mathbf{B}$ such that under $\mathbb{P}_x$, $\mathbf{Y}$ satisfies the stochastic differential equation

$$d\mathbf{Y}_t = \mu(\mathbf{Y}_t)dt + \sigma(\mathbf{Y}_t)d\mathbf{B}_t; \quad \mathbf{Y}_0 = x, \tag{2.5}$$

for all $x \in (0,c)$. More details on diffusions of this type can be found in e.g. [88, Chapter VII.2].

We now state assumptions on the diffusions that we consider in this paper.

*Assumption* A. A diffusion $\mathbf{Y}$ on state space $E$ satisfies *Assumption A* if:

1. We have that $\mathbb{P}_x(T_0 < \infty) > 0$ for some $x \in (0,c)$ which is equivalent to $\int_{y=0}^{x} s(y)M(dy) < \infty$ (equivalently for all $x \in (0,c)$ in both cases).

2. For all $0 < y < x < c$ we have that $\mathbb{E}_x(T_y)$ is finite. This is satisfied if and only if $M(E\backslash[0,x])$ is finite for all $x \in (0,c)$.

3. There exist locally bounded Borel-measurable functions $\mu : (0,c) \to \mathbb{R}$, $\sigma : (0,c) \to (0,\infty)$ and a Brownian motion $\mathbf{B}$ such that $\mathbf{Y}$ satisfies the stochastic differential equation in (2.5).

4. The functions $\mu$ and $\sigma$ defined in $A3$ are continuous on $(0,b]$ for some $b \in E\backslash\{0\}$.

We label these conditions $A1$, $A2$, $A3$, $A4$.

We use the representation of stochastic differential equations later in Section 2.4.2 but for now we simply note that for a process $Y$ that under Assumption A we have

$$\mu(x) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}_x(Y_h - x); \quad \sigma^2(x) := (\sigma(x))^2 = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}_x((Y_h - x)^2) \quad \text{for } x \in (0,b)$$

where $b$ is the constant in $A4$. The functions $\mu$ and $\sigma^2$ are referred to as the infinitesimal drift and the infinitesimal variance of the diffusion. Heuristically, for $x \in (0,c)$, the infinitesimal drift $\mu(x)$ describes the average direction of a diffusion at position $x$ will move in and the infinitesimal variance $\sigma^2(x)$ describes the roughness of the path of a diffusion position $x$.

The first condition $A1$ is equivalent to 0 being an attainable boundary point for the diffusion, i.e. a regular or exit boundary point according to the Feller boundary point classification, see [54, Table 15.6.1]. The boundaries of the diffusion are the points 0 and $c$. In words, a regular boundary point is one where you can restart the diffusion from the boundary once the diffusion first hits the boundary, i.e. $\mathbb{P}_0(T_x < \infty) > 0$; a typical example is the boundary point 0 for a Brownian motion in $[0,\infty)$ that is reflected at 0. An exit boundary point is where this behaviour does not occur, and instead, you have that the diffusion starting in the interior of the state space can reach the point 0 with positive probability, but the diffusion cannot reenter the interior after hitting 0. We note that the boundary point $c$ is regular exactly when $s(c) := \lim_{x \uparrow c} s(x)$ is finite equivalently when the state space $E$ is equal to $[0,c]$ and not $[0,c)$. In later sections this case sometimes needs special consideration, and we may identify it by any of the equivalent conditions just stated depending on which is most relevant to the section. Furthermore, as some of the measures that we use in this paper are not necessarily atom-free we take care when we define the limits of integration. We write $\int_{w=a}^{b}$ for an integral over $(a,b)$ and we write $\int_{w \in (a,b]}$ to integrate over $(a,b]$ (and then similarly to integrate over $[a,b)$ or $[a,b]$).

We now state our first results which use the hypotheses in Assumption A.

**Theorem 2.1.3** (Lévy measure)**.** *Consider a diffusion that satisfies $A1$, $A2$ of Assumption A. Then it has a Pitman-Yor excursion measure $\nu$ and we have that*

$$\int_{x=0}^{\infty} x\nu(\zeta \in dx) < \infty \tag{2.6}$$

*if and only if*

$$M(E) < \infty, \tag{2.7}$$

*which is equivalent to 0 being a regular boundary point for the diffusion. In this setting, the lifetime under $\nu$ is a Lévy measure for a recurrent SPLP of bounded variation.*

*On the other hand, if we have that: the diffusion satisfies all of Assumption A; the condition in (2.7) does not hold (i.e. 0 is an exit boundary of the diffusion); and we have*

$$\int_{v=0}^{b} \int_{y=0}^{v} s(y)M(dy)M(dv) < \infty \text{ where } b \in (0,c) \text{ is as in } A4 \tag{2.8}$$

*then we have*

$$\int_{x=0}^{\infty} (x^2 \wedge x)\nu(\zeta \in dx) < \infty = \int_{x=0}^{\infty} x\nu(\zeta \in dx). \tag{2.9}$$

*In this setting, the lifetime under $\nu$ is a Lévy measure for a recurrent SPLP of unbounded variation.*

Theorem 2.1.3 is proven in Section 2.3.1. We believe that, under $A1$, $A2$, the condition in (2.8) is necessary as well as sufficient for the lifetime under $\nu$ to be a Lévy measure for a recurrent process (i.e. we believe that $A3$ and $A4$ are not necessary), but proving this is beyond the scope of this paper. The Lévy measure condition of Theorem 2.1.3 means that we can construct a scaffolding-and-spindles pair $(\mathbf{N},\mathbf{X})$. Later, in Proposition 2.3.12, we define and construct $(\mathbf{N}_\beta,\mathbf{X}_\beta)$, for $\beta \in \mathscr{I}_H$. This is a scaffolding-and-spindles pair for which we prove that $\overline{\text{skewer}}(\mathbf{N}_\beta,\mathbf{X}_\beta)$ an IP-evolution such that $\text{skewer}(0,\mathbf{N}_\beta,\mathbf{X}_\beta) = \beta$ a.s.

Throughout the rest of the paper we consider diffusions along with spatial transformations $g : E \to g(E)$ that are strictly increasing and continuous with $g(0) = 0$. We define $g(c) = \lim_{x\uparrow c} g(x) \leq \infty$ and therefore $g(E) = [0,g(c))$ or $g(E) = [0,g(c)]$. Let $\mathbf{Y} = (\mathbf{Y}_t, t \geq 0)$ be a diffusion in state space $E$ and let $\mathbf{Z} = (\mathbf{Z}_t, t \geq 0)$ be the process defined by $\mathbf{Z}_t = g(\mathbf{Y}_t)$ for all $t \geq 0$. Then $\mathbf{Z}$ is a diffusion and has scale function $s_{\mathbf{Z}}(x) = s_{\mathbf{Y}}(g^{-1}(x))$ where $s_{\mathbf{Y}}$ is the scale function of $\mathbf{Y}$ and speed measure $m_{\mathbf{Z}}$ which is the image of the speed measure for $\mathbf{Y}$, $m_{\mathbf{Y}}$, under $g$, see e.g. [88, Exercise VII.3.18]. Therefore we can note that $\mathbf{Y}$ satisfies $A1$ (respectively $A2$) if and only if $\mathbf{Z}$ satisfies $A1$ (respectively $A2$). Furthermore we see that under $A1$, $\mathbf{Y}$ and $\mathbf{Z}$ have Pitman-Yor excursion measures $\nu_{\mathbf{Y}}$ and $\nu_{\mathbf{Z}}$, see [84, Section 3], and as the spatial transformation $g$ does not affect the lifetime $\zeta$ of an excursion we have $\nu_{\mathbf{Z}}(\zeta \in \cdot) = \nu_{\mathbf{Y}}(\zeta \in \cdot)$.

We refer to $\mathbf{Y}$ as the untransformed diffusion throughout the paper to avoid confusion between $\mathbf{Y}$ and $\mathbf{Z}$. This technique of using spatial transformations on untransformed diffusions to construct IP-diffusions was first used in [34], and we give the details of their setting in Example 2.1.8.

**Theorem 2.1.4** (Aggregate health summability)**.** *In the setting described above, consider a diffusion* $\mathbf{Z}$ *constructed from an untransformed diffusion* $\mathbf{Y}$ *along with a spatial transformation* $g$*. Suppose that* $\nu = \nu_{\mathbf{Z}}$ *(equivalently* $\nu = \nu_{\mathbf{Y}}$*) satisfies* (2.6) *or* (2.9) *and let* $(\mathbf{N},\mathbf{X})$ *be scaffolding-and-spindles pair where* $\mathbf{N}$ *has intensity Leb* $\otimes \nu_{\mathbf{Z}}$ *and stopped at an a.s. finite random variable* $T$*.*

*If* $\nu = \nu_{\mathbf{Z}}$ *satisfies* (2.6)*, then the aggregate health process* $M^0_{\mathbf{N},\mathbf{X}}(T)$ *is finite a.s. If* $\nu = \nu_{\mathbf{Z}}$ *satisfies* (2.9)*, then* $M^0_{\mathbf{N},\mathbf{X}}(T)$ *is finite a.s. if and only if*

$$\int_{y=0}^{b} g(y) M_{\mathbf{Y}}(dy) < \infty \text{ for some (equivalently all) } b \in (0,c), \tag{2.10}$$

*or equivalently that* $\int_{z=0}^{b} z M_{\mathbf{Z}}(dz)$ *is finite for some (or all)* $b \in (0,g(c))$*. Furthermore in this case we have that* $M^0_{\mathbf{N},\mathbf{X}}(T)$ *is infinite a.s. when* (2.10) *does not hold.*

Theorem 2.1.4 is proven in Section 2.3.2.

Our next results involve continuity of IP-evolutions and so we now give the definition of the complete and separable metric space of interval partitions $(\mathscr{I}_H, d'_H)$ that we use from Forman et al. [35].

**Definition 2.1.5** (Interval partition metric space $(\mathscr{I}_H, d'_H)$)**.** Recall $\mathscr{I}_H$, the set of interval partitions, from Definition 2.1.1. To define $d'_H$, we first define the notion of a correspondence between two interval partitions, and the distortion of a correspondence. We use the standard notation of $[n] = \{1, \dots, n\}$ for $n \in \mathbb{N}$.

1. A *correspondence* between two interval partitions $\beta$ and $\gamma$ is a tuple $(\beta, \gamma, (U_i, V_i)_{i\in[N]})$ involving two collections of intervals $U_1, \dots, U_N$ from $\beta$ and $V_1, \dots, V_N$ from $\gamma$, for some $N \in \mathbb{N}$, such that each is ordered with respect to the left-to-right ordering of their interval partitions.

2. The *distortion* of a correspondence $(\beta, \gamma, (U_i, V_i)_{i\in[N]})$ is defined to be

$$\operatorname{dis}(\beta, \gamma, (U_i, V_i)_{i\in[N]}) := \max \left\{ \begin{matrix} \sum_{i=1}^N |\operatorname{Leb}(U_i) - \operatorname{Leb}(V_i)| + \|\beta\| - \sum_{i=1}^N \operatorname{Leb}(U_i), \\ \sum_{i=1}^N |\operatorname{Leb}(U_i) - \operatorname{Leb}(V_i)| + \|\gamma\| - \sum_{i=1}^N \operatorname{Leb}(V_j) \end{matrix} \right\}.$$

3. The distance $d'_H(\beta, \gamma)$ between interval partitions $\beta$ and $\gamma$ is defined to be the infimum of distortions over the set of all possible correspondences:

$$d'_H(\beta, \gamma) := \inf_{N\geq 0, (U_i, V_i)_{i\in[N]}} \operatorname{dis}(\beta, \gamma, (U_i, V_i)_{i\in[N]}).$$

We use this metric in our main results below. Our main results below in Theorems 2.1.6 and 2.1.7 show the existence of IP-diffusions. Now we give a set of assumptions for one-dimensional diffusions in Assumption B.

*Assumption* B. A diffusion $\mathbf{Z}$ satisfies *Assumption B* if it is of the form $\mathbf{Z} = g(\mathbf{Y})$, where $\mathbf{Y}$ is a diffusion that satisfies $A1$, $A2$, $A3$ of Assumption A with state space $E \subseteq [0, \infty)$ of the form $E = [0, c]$ or $E = [0, c)$ for some $c \leq \infty$ and furthermore is such that:

1. the Pitman-Yor excursion measure $\nu_{\mathbf{Y}}$ satisfies (2.9) where $\nu = \nu_{\mathbf{Y}}$ (equivalently $\nu = \nu_{\mathbf{Z}}$);

2. $\sigma^2_{\mathbf{Y}}(x) = 4x$;

there exists $\varepsilon_0 \in (0, c)$ such that

3. for some $0 < \alpha^- \leq \alpha^+ < 1$ we have

$$-2\alpha^+ \leq \mu_{\mathbf{Y}}(x) \leq -2\alpha^- \text{ for all } x \in (0, \varepsilon_0]$$

(where the subscript $\mathbf{Y}$ in $\mu$ and $\sigma^2$ denotes that they are the infinitesimal drift and variance for $\mathbf{Y}$), and $g : E \to g(E)$ is a strictly increasing continuous function with $g(0) = 0$, $g(E) \subseteq [0, \infty)$ and

4. there exists $q_0 > \alpha^+$ such that the transformation $g$ is $q$-Hölder continuous on $[0, \varepsilon_0)$ for all $q < q_0$, i.e. for all $q < q_0$ there exists $C_q \geq 0$ such that for all $x$, $y \in [0, \varepsilon_0)$ we have $|g(x) - g(y)| \leq C_q |x - y|^q$.

Finally, we say $\mathbf{Z} = g(\mathbf{Y})$ satisfies Strong Assumption B if as well as the above we have that

5. the transformation $g$ satisfies $0 < \liminf_{x \to 0} \frac{g(x)}{x} \leq \infty$ or $0 \leq \limsup_{x \to 0} \frac{g(x)}{x} < \infty$.

We believe that $B1$ follows from $B2$ and $B3$ but we could not show this. We do show however in Proposition 2.4.1 that if the untransformed diffusion additionally satisfies $A4$ then we have that $B2$ and $B3$ implies $B1$. We give intuition behind the nature of hypothesis $B5$ with an example of a transformation $g$ that does not satisfy the hypothesis in Example 2.4.28. The hypothesis $B1$ ensures the scaffolding process is a SPLP of unbounded variation. The reason behind our fixed choice of $\sigma_{\mathbf{Y}}^2$ in $B2$ is that we compare our untransformed diffusions to squared Bessel processes with infinitesimal variance $4x$ and infinitesimal drift $-2\alpha^-$ or $-2\alpha^+$. We see in Example 2.1.8 that squared Bessel processes of this type featured previously as the untransformed diffusions for the IP-diffusions in [34]. This comparison with squared Bessel processes is used in a Kolmogorov–Čentsov continuity argument for the IP-evolution in Section 2.4.4 along with $B3$ and $B4$. The condition $B2$ should not be seen as restrictive as we can apply suitable transformations $g$ to consider diffusions with different infinitesimal variances $\sigma_{\mathbf{Z}}^2$ and we discuss this in Section 2.1.3.

The hypothesis $B5$ is used in Section 2.4.6 to show the continuity of several related functions on interval partition space. These continuity results lead to a proof of the Strong Markov property based on a continuity in the initial state proof, adapting the arguments used to show the Strong Markov property for self-similar IP-diffusions in [34, Section 6.4].

**Theorem 2.1.6** (Branching IP-diffusions when 0 is an exit point)**.** *Consider a diffusion* $\mathbf{Z}$ *that satisfies Assumption B and let* $\nu_{\mathbf{Z}}$ *be its Pitman-Yor excursion measure. Let* $(\mathbf{N}, \mathbf{X})$ *be a scaffolding-and-spindles pair where* $\mathbf{N}$ *has intensity Leb* $\otimes \nu_{\mathbf{Z}}$ *stopped at an a.s. finite valued random variable* $T$*. Then there exists a modification* $(\gamma^y, y \geq 0)$ *of* $\overline{skewer}(\mathbf{N}, \mathbf{X})$ *which is a continuous* IP-evolution *on* $(\mathscr{I}_H, d'_H)$*. Furthermore the process* $\mathbf{X}$ *has a local time that is continuous in time and level. Let* $\tau^0$ *be the right-continuous inverse of the local time of* $\mathbf{X}$ *at level* 0*. Then for* $T = \tau^0(s)$*, for some* $s > 0$*,* $(\gamma^y, y \geq 0)$ *is a simple Markov* IP-evolution *in an appropriate filtered probability space and if furthermore* $\mathbf{Z}$ *satisfies Strong Assumption B then it is an* IP-diffusion*.*

We prove Theorem 2.1.6 in Section 2.4.6. In Theorem 2.4.35 we show that the IP-diffusion $(\gamma^y, y \geq 0)$ of Theorem 2.1.6 can be started from a large class of initial states. We refer to $(\gamma^y, y \geq 0)$ as a *branching IP-diffusion* because the IP-diffusion has the following branching property. For independent copies of the IP-diffusion $(\gamma_1^y, y \geq 0)$, $(\gamma_2^y, y \geq 0)$ with initial states $\gamma_1$,

$\gamma_2$, the process $(\gamma_1^y \star \gamma_2^y, y \geq 0)$ (i.e. for each $y \geq 0$ we take the concatenation of interval partitions $\gamma_1^y$ and $\gamma_2^y$) is the IP-diffusion with initial state $\gamma_1 \star \gamma_2$.

We now state Assumption C which considers a disjoint set of diffusions to Assumption B.

*Assumption* C. A diffusion $\mathbf{Z}$ satisfies *Assumption C* if it is of the form $\mathbf{Z} = g(\mathbf{Y})$, where $\mathbf{Y}$ is a diffusion that satisfies $A1$, $A2$, $A3$ of Assumption A with state space $E \subseteq [0,\infty)$ of the form $E = [0,c]$ or $E = [0,c)$ for some $c \leq \infty$ and furthermore is such that:

1. $\sigma^2_{\mathbf{Y}}(x) = 4x$;

there exists $\varepsilon_1 \in (0,c)$ such that

2. for some $0 < \beta^- \leq \beta^+ < 1$ we have

$$2\beta^- \leq \mu_{\mathbf{Y}}(x) \leq 2\beta^+ \text{ for all } x \in (0,\varepsilon_1];$$

3. and $g : E \to g(E)$ is a strictly increasing continuous function with $g(0) = 0$, $g(E) \subseteq [0,\infty)$ and

$$\int_{y=0}^{\varepsilon_1} \frac{g(y^{1/(1-\beta^-)})}{y^2} dy < \infty.$$

The hypotheses $C1$, $C2$ and $C3$ are used together to prove continuity of IP-evolutions in Section 2.4.5. The hypothesis $C1$ plays the corresponding role of $B2$ in Assumption B and the generality of the transformation $g$ is what allows us to consider diffusions with different infinitesimal variances, $\sigma^2_{\mathbf{Z}}$. We discuss this in Section 2.1.3.

Theorem 2.1.7 is our second main result.

**Theorem 2.1.7** (Branching IP-evolutions when 0 is a regular point)**.** *Consider a diffusion* $\mathbf{Z}$ *under Assumption C with excursion measure* $\nu$ *and let* $(\mathbf{N},\mathbf{X})$ *be scaffolding-and-spindles pair where* $\mathbf{N}$ *has intensity Leb* $\otimes \nu$ *stopped at an a.s. finite valued random variable* $T$*. Then* $\overline{skewer}(\mathbf{N},\mathbf{X})$ *is a continuous* IP-evolution *on* $(\mathscr{I}_H, d'_H)$*. Furthermore we have that* $\mathbf{X}$ *is a Lévy process of bounded variation and when* $T$ *denotes the* $n^{th}$ *hitting time for* $\mathbf{X}$ *of level* 0 *for some* $n \in \mathbb{N}$ *then* $\overline{skewer}(\mathbf{N},\mathbf{X})$ *is a simple-Markov* IP-evolution *in an appropriate filtered probability space.*

We prove Theorem 2.1.7 in Section 2.4.5. We believe in this setting that $\overline{\text{skewer}}(\mathbf{N},\mathbf{X})$ is an IP-diffusion but showing this is beyond the scope of the paper. We do not believe that Theorems 2.1.6 and 2.1.7 cover all the one-dimensional diffusions from which it is possible to construct continuous Markovian IP-evolutions by the construction of scaffolding and spindles described in Section 2.1.1, although we do believe that all such diffusions must satisfy either (2.7) or both (2.8) and (2.10).

As in Assumption B the untransformed diffusions in Assumption C can bounded between two squared Bessel processes processes with different dimensions. As squared Bessel processes with dimension $\delta \in \mathbb{R}$ satisfy $A1$ exactly when $\delta \in (-2,2)$ we believe our ranges for the drift in $B3$ where $\mu_{\mathbf{Y}}(x)$ takes values in a closed subset of $(-2,0)$ for $x$ near 0 and in $C2$ where $\mu_{\mathbf{Y}}(x)$ takes values in a closed subset of $(0,2)$ for $x$ near 0 should encompass a large class of the diffusions for which it is possible to generate branching IP-diffusions. Indeed, this only leaves boundary cases open where $\mu_{\mathbf{Y}}$ is not bounded away from $-2$, 0, 2 as well as cases where the transformations $g$ do not satisfy the conditions needed in $B4$, $B5$, $C3$.

We give examples in the next section of diffusions which satisfy the hypotheses in Assumptions A, B and C.

### 2.1.3 Examples

In this section we list a few examples of diffusions that satisfy the hypotheses in the previous section. The diffusions in our selected examples satisfy $A3$ and we begin with some consequences.

Consider a diffusion that satisfies $A1$, $A2$ and $A3$. For any $b \in (0,c)$, we can define a scale function $s$ with derivative $s'$ and its speed measure $M$ with density $m$ by, for $x \in (0,c)$,

$$s'(x) = \exp\left(-\int_{z=b}^{x} \frac{2\mu(z)}{\sigma^2(z)} dz\right); \quad m(x) = \frac{2}{\sigma^2(x)s'(x)} = \frac{2}{\sigma^2(x)} \exp\left(\int_{z=b}^{x} \frac{2\mu(z)}{\sigma^2(z)} dz\right); \tag{2.11}$$

$s(x) = \int_{y=0}^{x} s'(y)dy$; $M((l,r)) = \int_{y=l}^{r} m(y)dy$, see e.g. [88, Exercise VII.3.20]. We were free to choose the lower limit $b$ here and in Section 2.4.1 we take $b$ to be $\varepsilon_0$ when under Assumption B and to be $\varepsilon_1$ when under Assumption C. Different values in the interior only change $s'$, $s$, $m$ by the same constant scale factor and such changes do not affect any of the conditions in our results.

If $g$ is a twice differentiable function with continuous second derivative, and if $\mu_{\mathbf{Y}}$ and $\sigma^2_{\mathbf{Y}}$ are continuous on $(0,c)$, then we have, for $x \in (0,c)$, that

$$\mu_{\mathbf{Z}}(g(x)) = \frac{1}{2}\sigma^2_{\mathbf{Y}}(x)g''(x) + \mu_{\mathbf{Y}}(x)g'(x); \quad \sigma^2_{\mathbf{Z}}(g(x)) = \sigma^2_{\mathbf{Y}}(x)(g'(x))^2, \tag{2.12}$$

and we further have that

$$s_{\mathbf{Z}}(g(x)) = s_{\mathbf{Y}}(x); \quad s'_{\mathbf{Z}}(g(x)) = \frac{1}{g'(x)} s'_{\mathbf{Y}}(x); \quad m_{\mathbf{Z}}(g(x)) = \frac{1}{g'(x)} m_{\mathbf{Y}}(x),$$

see e.g. Karlin and Taylor [54, Theorem 15.2.1]. Similarly, the results in Theorem 2.1.3 hold for the diffusion $\mathbf{Z}$ if and only if they hold for the diffusion $\mathbf{Y}$. With the equations given in (2.12) we give a heuristic method that can be used to check if certain diffusions satisfy Assumption B or C:

1. First you take a diffusion under Assumption A with infinitesimal drift $\mu_{\mathbf{Z}}$ and infinitesimal variance $\sigma^2_{\mathbf{Z}}$.

2. Then you should find a solution $g$ to the equation $\sigma^2_{\mathbf{Z}}(g(x)) = 4x(g'(x))^2$ with $g(0) = 0$.

3. With this solution $g$ you can then find a solution $\mu_{\mathbf{Y}}(x)$ to the equation

$$\mu_{\mathbf{Z}}(g(x)) = 2xg''(x) + \mu_{\mathbf{Y}}(x)g'(x).$$

4. The original diffusion satisfies (strong) Assumption B if $\mu_{\mathbf{Y}}$ satisfies $B3$ and if $g$ satisfies $B4$ (and $B5$). On the other hand the original diffusion satisfies Assumption C if $\mu_{\mathbf{Y}}$ satisfies $C2$ and if $g$ satisfies $C3$.

We used this heuristic method for the processes in Example 2.1.9 and we verified that they satisfy Assumption C in Proposition 2.1.10.

The first example concerns self-similar diffusions. Our main reference in this work, [34], determined all the self-similar diffusions for which it is possible to use the scaffolding and spindles construction for IP-diffusions and we state their results here.

**Example 2.1.8** (Self-similar diffusions)**.** *This family of IP-diffusions has already been established in [34, Theorems 1.3, 1.4]. Let $k > 0$, $0 < \alpha < 1$, $\alpha < q < \infty$ and consider an untransformed diffusion $\mathbf{Y}$ on $[0,\infty)$ and a transformation $g$ with*

$$\mu_{\mathbf{Y}}(y) = -2\alpha; \quad \sigma^2_{\mathbf{Y}}(y) = 4y; \quad g(y) = ky^q.$$

*Here $\mathbf{Y}$ is a squared Bessel process with dimension $-2\alpha$. Let $\mathbf{Z}$ be the diffusion $\mathbf{Z} = g(\mathbf{Y})$. Then $\mathbf{Z}$ satisfies Assumption B and can be used to generate IP-diffusions. These are all the self-similar diffusions for which the construction of IP-diffusions that we use in this paper is possible, see [63, Theorem 5.1] for a parametrisation of self-similar Markov processes.*

The second example concerns Wright–Fisher diffusions, which are well-known across the literature; we quote the relevant information about them from [54, Example 15.6.8]. The Wright–Fisher model is a gene frequency model with mutation parameters that describes a population with two different types, say type 1 and type 2. The diffusion measures the proportion of the population which is type 1, and therefore takes values in $[0,1]$. There are two parameters, $\gamma_1$, $\gamma_2 \geq 0$ which determine the rate of mutation from one type to another (i.e. the parameter $\gamma_1$ determines the rates of mutation from type 1 to type 2). They have infinitesimal drift and infinitesimal variance given by

$$\mu_{\mathbf{Z}}(z) = \gamma_2(1-z) - \gamma_1 z, \quad \sigma^2_{\mathbf{Z}}(z) = z(1-z). \tag{2.13}$$

**Example 2.1.9** (Wright–Fisher gene frequency model involving mutation parameters)**.** *The Wright–Fisher diffusion with mutation parameters $\gamma_1 \geq 0$ and $0 < \gamma_2 < 1/2$ for IP-diffusions can be used to generate continuous simple Markov IP-evolutions.*

*We prove the existence of IP-evolutions in this case in Proposition 2.1.10 below.*

**Proposition 2.1.10.** *The Wright–Fisher diffusions as defined in Example 2.1.9 satisfy Assumption C and therefore can be used to generate continuous simple Markov IP-evolutions.*

*Proof.* Let $\mathbf{Y}$ be defined the diffusion that satisfies $A3$ on $[0,\pi^2]$ with

$$\mu_{\mathbf{Y}}(y) = 1 + \sqrt{y}\Big(2\gamma_2 \cot\Big(\frac{\sqrt{y}}{2}\Big) - \cot(\sqrt{y}) - 2\gamma_1 \tan(\sqrt{y})\Big); \quad \sigma^2_{\mathbf{Y}}(y) = 4y; \quad y \in (0,\pi^2).$$

With the transformation $g(y) = \sin^2\left(\sqrt{y}/2\right)$ we have the Wright–Fisher diffusion as described above; this can be checked with the formulae in (2.12). We now verify the diffusion satisfies Assumption C and therefore, by Theorem 2.1.7, there exist continuous simple Markov IP-evolutions. It is equivalent to check that $\mathbf{Z}$ satisfies $A1$, $A2$ rather than check that diffusion $\mathbf{Y}$ does. The diffusion $\mathbf{Z}$ has scale function and speed measure with the following asymptotic properties:

$$\begin{aligned} s'(z) &= O(z^{-2\gamma_2}), & m(z) &= O(z^{2\gamma_2-1}) & &\text{as } z \to 0;\\ s'(z) &= O((1-z)^{-2\gamma_1}), & m(z) &= O((1-z)^{1-2\gamma_1}) & &\text{as } z \to 1. \end{aligned}$$

Therefore

$$\int_{z=0}^{b} s(z)m(z)dz = O\Big(\int_{z=0}^{b} z^{1-2\gamma_2} z^{2\gamma_2-1} dz\Big) = O(b) \quad \text{as } b \to 0.$$

Similarly, the condition $\int_{z=b}^{1} m_{\mathbf{Z}}(z)dz < \infty$ can be checked.

The drift satisfies $\mu_{\mathbf{Y}}(y) = 4\gamma_2 + o(1)$ for $y$ near 0 and therefore there exists $\varepsilon > 0$ such that $0 < 2\gamma_2 \leq \mu_{\mathbf{Y}}(y) \leq 2\gamma_2 + 1 < 2$ for $y \in (0,\varepsilon)$. Furthermore, $g(y) = O(y)$ near 0 and so

$$\int_{y=0}^{b} \frac{g(y^{1/(1-\gamma_2)})}{y^2} dy = O\Big(\int_{y=0}^{b} y^{-2+1/(1-\gamma_2)} dy\Big) = O(b^{\gamma_2/(1-\gamma_2)}) \quad \text{as } b \to 0$$

and this is again finite. ■

We believe that Wright–Fisher diffusions with $\gamma_2 = 0$ should also lead to continuous Markov IP-evolutions, but our methods were not powerful enough to prove this. On the other hand, the requirement that $\gamma_1 > 0$ is necessary here to satisfy $A2$.

The next example concerns Cox-Ingersoll-Ross processes, Cox, Ingersoll and Ross [27]. This family of diffusions is an alternative gene frequency model with two mutation parameters in a scaling where the mutation likelihood is larger in magnitude in the scaling limit, and in this context they are known as Laguerre diffusions after the Laguerre orthogonal polynomials involved, see [54, Example 15.2.F(d)] for details. They also appears as a population growth model in the same reference, see [54, Example 15.13.C]. Finally, they are also referred to in [27, Equation (17)] and have financial applications.

**Example 2.1.11** (Cox-Ingersoll-Ross processes/Laguerre Diffusions)**.** *For $a > 0$ and $b$, $c \in \mathbb{R}$ the infinitesimal drift and infinitesimal variance is given by*

$$\mu_{\mathbf{Z}}(z) = bx + c; \quad \sigma^2_{\mathbf{Z}}(z) = az.$$

*The diffusion can be constructed with the transformation $g(y) = (a/4)y$ and the untransformed diffusion with $\mu_{\mathbf{Y}}(y) = by + 4c/a$ and $\sigma^2_{\mathbf{Y}}(y) = 4y$. This satisfies Assumption C when $0 < 2c < a$ and $b < 0$ and therefore in these cases, by Theorem 2.1.7, can be used to generate continuous simple Markov IP-evolutions.*

This final example is something of a middle case between Assumptions B and C for which we have some results but no proof for the existence of continuous Markov branching IP-evolutions.

**Example 2.1.12.** *Consider diffusions $\mathbf{Z}$ of the form $\mathbf{Z} = g(\mathbf{Y})$, where $\mathbf{Y}$ is a diffusion that satisfies A1, A2, A3 with state space $E \subseteq [0,\infty)$ of the form $E = [0,c]$ or $E = [0,c)$ for some $c \leq \infty$ and furthermore is such that:*

1. *$\sigma^2_{\mathbf{Y}}(x) = 4x$;*

*there exists $\varepsilon_2 \in (0,c)$ such that*

2. *$\mu_{\mathbf{Y}}(x) = 0$ for all $x \in (0,\varepsilon_2]$;*

*(where the subscript $\mathbf{Y}$ in $\mu$ and $\sigma^2$ denote that they are the infinitesimal drift and variance for $\mathbf{Y}$), and $g : E \to g(E)$ is a strictly increasing continuous function with $g(0) = 0$, $g(E) \subseteq [0,\infty)$ and*

3. *there exists $q_0' > 0$ such that the transformation $g$ is $q$-Hölder continuous on $[0,\varepsilon_0)$ for all $q < q_0'$, i.e. for all $q < q_0'$ there exists $C_q' \geq 0$ such that for all $x$, $y \in [0,\varepsilon_2)$ we have $|g(x) - g(y)| \leq C_q'|x-y|^q$.*

*Note that $\mathbf{Y}$ satisfies A4 with $b = \varepsilon_2$. We define a scale function $s_{\mathbf{Y}}$ and a speed measure $m_{\mathbf{Y}}$ by setting $b = \varepsilon_2$ in* (2.11) *so that*

$$s'_{\mathbf{Y}}(x) = 1, \quad s_{\mathbf{Y}}(x) = x, \quad m_{\mathbf{Y}}(x) = \frac{1}{2x} \quad \text{for } x \in (0,\varepsilon_2). \tag{2.14}$$

*Condition* (2.7) *is not satisfied: $M_{\mathbf{Y}}((0,\varepsilon_2)) = \infty$; furthermore we have that* (2.8) *is satisfied as*

$$\int_{v=0}^{\varepsilon_2}\int_{y=0}^{v} s_{\mathbf{Y}}(y)m_{\mathbf{Y}}(y)dy m_{\mathbf{Y}}(v)dv = \int_{v=0}^{\varepsilon_2}\int_{y=0}^{v}\frac{1}{2}dy\frac{1}{2v}dv = \frac{\varepsilon_2}{4}.$$

*Therefore the Pitman-Yor excursion measure satisfies* (2.9) *by Theorem 2.1.3. Furthermore, for $q < q_0'$*

$$\int_{v=0}^{\varepsilon_2} g(v)m_{\mathbf{Y}}(v)dv \leq \int_{v=0}^{\varepsilon_2}\frac{C_q'}{2}v^{q-1}dv = \frac{C_q'}{2q} < \infty$$

*and so $M^0_{\mathbf{N},\mathbf{X}}(T)$ is finite for a stopped scaffolding-and-spindles pair $(\mathbf{N},\mathbf{X})$ by Theorem 2.1.4.*

Further examples that satisfy Assumption A but not Assumptions B or C that could in theory generate branching IP-evolutions include reflected Brownian motion with or without a constant drift on $E = [0, c]$ for some finite $c$; reflected Brownian motion on $E = [0, \infty)$ with a constant negative drift; reflected Ornstein-Uhlenbeck process on $E = [0, \infty)$. All these processes satisfy (2.6) (as 0 is a regular boundary point for these processes) and so we have aggregate health summability by Theorem 2.1.4.

### 2.1.4 Related work

Interval partitions can be found in several different areas of probability. As noted in [34, Introduction], interval partitions are the ranges of subordinators [85], the complements of the zero-set of a Brownian bridges [44, Example 3], the products of stick-breaking schemes [44, Example 2], as well as the limits of compositions of natural numbers [45]. Furthermore, in our setting the class of interval partitions, $\mathscr{I}_H$ along with the metric $d'_H$ in Definition 2.1.5 serve as a useful complete and separable metric space to explore branching processes with diffusively evolving characteristics and to produce continuous Markovian processes. That said, we feel that these diffusions on the space of interval partitions should be taken as objects of interest in their own right.

Using only the lifetimes of the individuals, and not the health excursions, the branching processes we study give rise to splitting trees, which can be found in the literature in e.g. Geiger and Kersting [42]. A splitting tree is a rooted planar tree that represents the type of branching processes that we study. It is a rooted tree where branch points represent births, leaves represent deaths, and the edges of varying length represent the times between birth or death events. Starting with the ancestor at the root, one can trace the path along the edge until a branch or boundary point. In the case of a boundary point, the ancestor had no offspring and the tree is a single edge of length given by the ancestor's lifetime. In the case of a branch point, the left-most edge denotes the ancestor and the other edge denotes its first offspring. Continuing to follow the left-most edges at each branch point gives the collection of edges that represent the ancestor, and the sum of lengths is the lifetime. All the other edges at the branch points denote the offspring of the ancestor. Repeating this process for all descendants would return all the information about the branching process. The collection of individuals alive in the branching process at time $t$ is therefore in bijection with the collection of points of distance $t$ from the root.

The skewer process draws similarities to the Ray-Knight theorems, see Revuz and Yor [88, Theorems XI.2.1,XI.2.3] for a reference, which show that the local time process, indexed by level, of certain stopped Brownian motions are squared Bessel processes. In our setting we have interval partitions instead of local times, although, we find in Lemma 2.3.16 that the local time at any fixed level is a function of the interval partition at that level a.s. In Forman et al. [33] they show a stronger result for the self-similar cases: this same function applied to the interval

partitions at all levels simultaneously gives the local times at those levels a.s. We still believe that this holds in non-self-similar cases but our methods were not powerful enough to prove it.

A motivation for our primary source, [34], and related papers by the same authors, [33,37,39], is to generate interval-partition-valued diffusions to explore diffusively evolving masses of the sub-trees of a spine of a continuum random tree. This is part of their monograph, [38], to solve the Aldous conjecture, Aldous [3], showing the existence of a diffusion on the space of continuum trees with the Brownian Continuum Random tree as the stationary distribution.

### 2.1.5 Structure of this paper

Section 2.2 explores the relevant literature we use to prove our results. In Section 2.2.1 we look at the general theory of SPLPs with reference Bertoin [9], and state key results about excursions of SPLPs from Pardo et al. [80]. In Section 2.2.2 we give a summary of the Pitman-Yor excursion measure from [84] as well as the Williams disintegration for the Pitman-Yor excursion measure in Theorem 2.2.7, and we give an overview of the theory of disintegration. The disintegration theory also allows us to consider conditional measures such as $\nu(\cdot|\zeta = t)$, for $t > 0$, which denote the probability distribution of a health excursion for an individual with a lifetime equal to $t$. In Section 2.2.3 we explore Green's functions as these along with the Williams decomposition are useful for calculations for all our main results.

Then in Section 2.3 we prove results for diffusions under Assumption A. In Section 2.3.1 we prove Theorem 2.1.3 which is necessary to have the SPLP "scaffolding". Then, in Section 2.3.2 we prove Theorem 2.1.4 which determines that the levels of health are summable at a fixed point in time, and this is necessary to produce interval partitions. Section 2.3.3 proves necessary disintegration results for the excursion measure of the SPLP. In Section 2.3.4 we restrict cases in Assumption A to find the possible initial states for an IP-evolution.

In Section 2.4.1 we consider the parameters for the untransformed diffusion under Assumption B, which we use in Section 2.4.2 where we show conditions for the recurrent SPLP of unbounded variation to have local time that is jointly continuous in time and space. This is required in our Kolmogorov–Čentsov continuity criterion argument in Section 2.4.4 along with some further calculations in Section 2.4.3. We do not use a Kolmogorov–Čentsov argument for diffusions under Assumption C as we do not have an equivalent to the continuity of local time result that was given in Section 2.4.2 that only holds for SPLPs of unbounded variation. We prove Theorems 2.1.6 (without the strong Markov property) and 2.1.7 in Section 2.4.5. We then show the strong Markov property for Theorem 2.1.6 under Strong Assumption B in Section 2.4.6 by showing continuity in the initial state.

## 2.2 Preliminaries

In this section we give an overview of the results we use from the existing literature. We explore the excursions of a spectrally one-sided Lévy process away from level 0 as described in Bertoin [9] and Pardo, Pérez and Rivero [80]. We also use the excursion measure of a one-dimensional diffusion from Pitman and Yor [84]; in particular we use the Williams decomposition [84, Description (3.3)] which describes the Pitman-Yor excursion measure. We require a notion of a conditional expectation for $\sigma$-finite measures, and for this we use the concept of a disintegration from Chang and Pollard [21]. Finally, we look at Revuz and Yor [88] for the Green's function for real-valued diffusions and use a result from Karlin and Taylor [54, Section 15] which applies to such diffusions under Assumption *A*3.

We define path spaces and various functions on these spaces.

**Definition 2.2.1.** We define the following metric spaces of functions:

1. Let $\mathscr{D}$ be the Skorokhod space of one-dimensional càdlàg functions $g : \mathbb{R} \to \mathbb{R}$. Let $\mathscr{C}$ be the metric space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ under the uniform metric.

2. Let $X : \mathscr{D} \to \mathscr{D}$ be the canonical process on $\mathscr{D}$, and let $Y$ be its restriction to $\mathscr{C}$, given by

$$X_t(\omega) = \omega(t), \quad \omega \in \mathscr{D}; \qquad Y_t(\omega) = \omega(t), \quad \omega \in \mathscr{C}.$$

   We have defined two processes here, $X$ and $Y$ for clarity later on.

3. Let $\mathscr{D}_{\text{exc}} \subset \mathscr{D}$ be the subspace of $\mathscr{D}$ consisting of càdlàg excursions away from 0, i.e.

$$\mathscr{D}_{\text{exc}} := \left\{ g \in \mathscr{D} \middle| \exists z \in (0,\infty) : g|_{(-\infty,0]\cup[z,\infty)} = 0, g(x) > 0 \text{ for all } x \in (0,z) \right\}.$$

   Let $\mathscr{E} \subset \mathscr{C}$ of be the subspace of $\mathscr{D}_{exc}$ of continuous positive excursions from 0, given by

$$\mathscr{E} = \left\{ f \in \mathscr{C} \middle| \exists z \in (0,\infty) : f|_{(-\infty,0]\cup[z,\infty)} = 0; f \text{ positive, continuous on } [0,z] \right\} = \mathscr{D}_{\text{exc}} \cap \mathscr{C}.$$

**Definition 2.2.2.** We define functions on the spaces given in Definition 2.2.1:

1. For $g \in \mathscr{D}_{\text{exc}}$, we define $\zeta(g) := \sup\{s \geq 0 : g(s) > 0\}$, and we call $\zeta(g)$ the *lifetime* of $g$.

2. For $g \in \mathscr{D}_{\text{exc}}$, we define $A(g) := \sup\{g(s), 0 \leq s \leq \zeta(g)\}$, and we call $A(g)$ the *amplitude* of $g$.

3. For $g \in \mathscr{D}_{\text{exc}}$, we define $R(g) := \inf\{s > 0 : g(s) = A(g) \text{ or } g(s-) = A(g)\}$.

4. For $h \in \mathscr{D}$, we define $T_0^+(h) := \inf\{t > 0 : h(t) \geq 0\}$. We call $T_0^+(h)$ the crossing time of 0 for $h$, and we call $h(T_0^+(h)-)$ and $h(T_0^+(h))$ the *undershoot* of $h$ and the *overshoot* of $h$ respectively.

5. For $h \in \mathscr{D}$, and $E \subset \mathbb{R}$ measurable, we define $T_E(h) := \inf\{t > 0, h(t) \in E\} \in [0,\infty]$. For $y \in \mathbb{R}$, we define $T_y(h) := T_{\{y\}}(h)$ and we call $T_y(h)$ the *hitting time of* $y$ for $h$.

Finally, we formally define the notion of concatenating interval partitions from [34, Section 2.2].

**Definition 2.2.3.** Let $(\beta_a)_{a\in\mathscr{A}}$ denote a family of interval partitions indexed by a totally ordered set $(A, \preceq)$. Let $S(a-) := \sum_{b\prec a}\|\beta_b\|$ for $a \in \mathscr{A}$. If $S(a-) < \infty$ for every $a \in \mathscr{A}$, then we define the concatenation

$$\bigstar_{a\in\mathscr{A}}\beta_a := \{(x+S(a-), y+S(a-)) : a \in \mathscr{A}, (x,y) \in \beta_a\}$$

and otherwise we define the concatenation as the empty interval partition $\emptyset$. When $\mathscr{A} = \{a_1, a_2\}$, $a_1 \prec a_2$ we denote this by $\beta_{a_1} \star \beta_{a_2}$. If $\sum_{a\in\mathscr{A}}\|\beta_a\| < \infty$, we call this collection $(\beta_a)_{a\in\mathscr{A}}$ summable.

### 2.2.1 Itô excursion theory for recurrent SPLPs of bounded and of unbounded variation

As discussed in Section 2.1.1, the scaffolding $\mathbf{X}$ is a recurrent SPLP either of bounded variation or of unbounded variation but with no Gaussian component. We discuss the excursions of the SPLPs in both cases.

Firstly, let $\mathbf{X}$ be a recurrent SPLP of bounded variation. For $x \in \mathbb{R}$, let $\mathbb{P}_x$ be the distribution on $\mathscr{D}$ of $\mathbf{X}$ started at $\mathbf{X}_0 = x$, and let $\mathbb{E}_x$ denote the expectation operator. For all $y \in \mathbb{R}$, the level $y$ is irregular ($\mathbb{P}_y(T_y(X) = 0) = 0$), Bertoin [9, Corollary VII.5]. For any fixed level $y \in \mathbb{R}$ the number of returns to $y$ after some time $t > 0$ is finite a.s., [60, Lemma 4.2], but the set of levels for which the number of returns to a level is infinite is dense in the range of the SPLP a.s., [60, Theorem 5.3].

Now instead consider a recurrent SPLP $\mathbf{X}$ of unbounded variation and with no Gaussian component. For all $y \in \mathbb{R}$, the level $y$ is regular ($\mathbb{P}_y(T_y(X) = 0) = 1$) and instantaneous ($\mathbb{P}_y(T_{\mathbb{R}\setminus y}(X) = 0) = 1$) for $\mathbf{X}$, [9, Chapter IV.1-4]. Therefore we can define the local time $\ell^y$ for $\mathbf{X}$ at level $y$. The local time at all levels $y \in \mathbb{R}$ is an occupation density for the Lévy process such that for any bounded and measurable function $h$ and any $t > 0$ we have

$$\int_{y=-\infty}^{\infty} h(y)\ell^y(t)dy = \int_{s=0}^{t} h(\mathbf{X}_s)ds \quad \text{a.s.}$$

The local time at level $y$ is a continuous increasing function $\ell^y : [0,\infty) \to [0,\infty)$, and so we can define a right inverse $\tau^y$. Define $\tau^y : [0,\infty) \to [0,\infty)$ as, for $s > 0$, $\tau^y(s) = \inf\{t > 0 : \ell^y(t) > s\}$. Note that $\tau^0$ is a subordinator, [9, Theorem IV.8]. Then, following [9, Equation IV.6], we can define the excursion process $e := (e_s, s \geq 0 : \tau^0(s) - \tau^0(s-) > 0)$ for $\mathbf{X}$,

$$e_s = \left(\mathbf{X}_{\tau^0(s-)+r}, 0 \leq r < \tau^0(s) - \tau^0(s-)\right), \tag{2.15}$$

where each $e_s \in \mathscr{D}_{\text{exc}}$. There are only countably many points $s$ such that $\tau^0(s) - \tau^0(s-) > 0$ a.s. From [9, Theorem IV.10] we note that $e$ is a Poisson point process.

Define $\mathbf{S} = (\mathbf{S}_t)_{t\in[0,\infty)}$, where, for $t > 0$, $\mathbf{S}_t := \sup_{0\le s\le t} \mathbf{X}_s$, i.e. $\mathbf{S}$ is the running supremum of $\mathbf{X}$. The process $\mathbf{S} - \mathbf{X}$ is non-negative for all $t > 0$ and only hits the level 0 at the points where $\mathbf{X}$ attains new suprema. Note that the level 0 is also regular and instantaneous for $\mathbf{S} - \mathbf{X}$ and we can define $\overline{e}$ to be the excursion process for $\mathbf{S} - \mathbf{X}$ in the same way that we defined $e$ for $\mathbf{X}$. Let $n$ be the characteristic measure for $e$; i.e. the excursion measure for $\mathbf{X}$, and let $\overline{n}$ the characteristic measure for $\overline{e}$; i.e. the excursion measure for $\mathbf{S} - \mathbf{X}$. The Laplace exponent, $\psi : [0,\infty) \to \mathbb{R}$, for $\mathbf{X}$ is formally defined to be the function that satisfies

$$\mathbb{E}_0(e^{-\lambda X_t}) = e^{t\psi(\lambda)},$$

and in our setting is equal to

$$\psi(\lambda) = \int_{x=0}^{\infty} \left(e^{-\lambda x} - 1 + \lambda x\right) \Lambda(dx), \tag{2.16}$$

see [9, Chapter VII.1].

As the recurrent SPLP $\mathbf{X}$ has no Gaussian component we have that, up to a null set, all excursions under $n$ have the following structure a.s.: initially the excursion takes values in $(-\infty, 0)$, before entering $(0,\infty)$ by a jump across 0, and then taking values in $(0,\infty)$ until hitting 0, [80, Theorem 3].

We define a relation between the excursion measure $n$ of $\mathbf{X}$ and the excursion measure $\overline{n}$ for $\mathbf{S} - \mathbf{X}$, with the scale function $\hat{W}$ for $\hat{\mathbf{X}} := -\mathbf{X}$, defined in [9, Section VII.2]. The scale function $\hat{W} : [0,\infty) \to [0,\infty)$ is a continuous increasing function which relates to the Laplace exponent for $\hat{\mathbf{X}}$ in the following way: for all $\lambda > 0$,

$$\int_{x=0}^{\infty} e^{-\lambda x} \hat{W}(x) dx = \frac{1}{\psi(\lambda)}.$$

Let $\hat{\mathbb{P}}_x$ be the probability measure for the law of the dual process $\hat{\mathbf{X}}$ started at $\hat{\mathbf{X}}(0) = x$. The scale function $\hat{W}$ for $\hat{\mathbf{X}}$ solves the two-sided exit problem for $\hat{\mathbf{X}}$, [9, Theorem VII.8]: for $x, y > 0$, the probability that $\hat{\mathbf{X}}$ makes its first exit from $[-x, y]$ at $y$ is

$$\hat{\mathbb{P}}_0\Big(\inf_{0\le s\le T_y}(X_s) \ge -x\Big) = \frac{\hat{W}(x)}{\hat{W}(x+y)}.$$

In the following, we use notation such as

$$\overline{n}(g(t) \in A, \zeta > t) := \overline{n}(\{g \in \mathscr{D}_{\text{exc}} : g(t) \in A, \zeta(g) > t\}),$$

where $A$ is a measurable set of $\mathbb{R}$ in this setting. To relate $n$ to $\overline{n}$, we use an instance of a general result for recurrent SPLPs from [80, Theorem 3 (iii)]: there exists $k_1 > 0$ such that for all $y$,

$z > 0$

$$n\left(-g\left(T_0^+-\right) \in dy, g\left(T_0^+\right) \in dz, T_0^+ > t\right) = k_1 \int_{x=0}^{\infty} \left(\hat{W}(x) - \hat{W}(x-y)\right) \overline{n}(g(t) \in dx, \zeta > t) \Lambda(dz+y)dy, \quad (2.17)$$

where $\hat{W}(t) := 0$ for $t < 0$.

For $x > 0$, when we consider events up of $\hat{\mathbf{X}}$ up to a finite fixed time $t$ (i.e. any event $A$ such that $A \in \mathscr{F}_t$, the natural filtration for the process $\mathbf{X}$), $\hat{\mathbb{P}}_x^{\uparrow}$ is the distribution on $\mathscr{D}$ of $\hat{\mathbf{X}}$ when started from $x$ and conditioned to be positive in $(0,t)$, which can be found in [9, Equation VII.6]:

$$\hat{\mathbb{P}}_x^{\uparrow}(X_t \in dy) := \frac{\hat{W}(y)}{\hat{W}(x)} \hat{\mathbb{P}}_x(X_t \in dy, T_{(-\infty,0)}(X) > t).$$

Furthermore, [9, Proposition VII.14] states that the probability measure for $\mathbf{X}$ conditioned to stay positive when started at level 0, $\mathbb{P}_0^{\uparrow}$, can be defined by the limit $\lim_{x \downarrow 0} \mathbb{P}_x^{\uparrow}$. Then, from [9, Proposition VII.15], we have that for some $k_2 > 0$ and for any $x, t > 0$,

$$\overline{n}(g(t) \in dx, \zeta(g) > t) = \frac{k_2}{\hat{W}(x)} \hat{\mathbb{P}}_0^{\uparrow}(X_t \in dx). \quad (2.18)$$

### 2.2.2 Pitman-Yor excursion theory for one-dimensional diffusions and disintegration theory

Consider a diffusion that satisfies Assumption $A1$. We construct an excursion measure following Pitman and Yor [84, Section 3]. This is equivalent to the Itô excursion measure when 0 is a regular boundary point, but is more general as it also applies in the case where 0 is an exit boundary point.

Before we define the excursion measure, we need to define two more diffusions – the 0-diffusion and the $\uparrow$-diffusion. For $x > 0$, the 0-diffusion started at $x$ is simply the original diffusion started at $x$ which we stop when it hits level 0. For $x > 0$, the $\uparrow$-diffusion started at the level $x$ has law given by the limit, as $y \uparrow \infty$, of the conditional probability measure for the original diffusion (again started at $x$) conditioned on $\{T_y < T_0\}$; details of this can be found in [84, Section 3]. We can define the $\uparrow$-diffusion starting at 0 and in this case the process never returns to level 0. The scale function $s^{\uparrow}$ and the speed measure $M^{\uparrow}$ of the $\uparrow$-diffusion can be determined in terms of $s$ and $M$:

$$s^{\uparrow}(x) = \frac{-1}{s(x)}; \quad M^{\uparrow}(dx) = s(x)^2 M(dx). \quad (2.19)$$

The scale function $s^{\uparrow}$ result follows from the defining characteristic of a scale function in (2.2) and some simple calculations. The speed measure result is noted in the literature in e.g. Salminen et al. [89]. With the $\uparrow$-diffusion and the 0-diffusion we can define the Pitman-Yor excursion measure $\nu$.

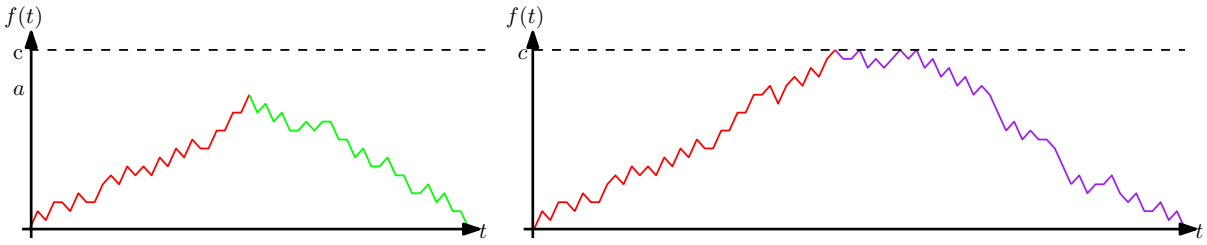



Figure 2.2: Left: an excursion with amplitude $a \in (0,c)$, constructed from an $\uparrow$-diffusion (red) and a time-reversed $\uparrow$-diffusion (green) both started at 0 and stopped at $a$. Right: an excursion with amplitude $c$, constructed from an $\uparrow$-diffusion stopped at $c$ (red) and a 0-diffusion started at $c$ and stopped at 0 (purple).

**Definition 2.2.4** (Pitman-Yor excursion measure $\nu$)**.** We define the $\sigma$-finite excursion measure $\nu$ on the space of positive continuous excursions $\mathscr{E}$ by the following conditions:

1. $\nu(f \equiv 0) = \nu(f(0) \neq 0) = 0$;
2. for all $y \in E\backslash\{0\}$ we have $\nu\left(T_y(f) < \infty\right) = 1/s(y)$;
3. for all $y \in E\backslash\{0\}$, the conditional distribution $\nu\left(\cdot | T_y(f) < \infty\right)$ is the law of the following $\mathbf{V} = (\mathbf{V}_t)_{t\in[0,\infty)}$. The process $(\mathbf{V}_t)_{t\in[0,T_y)}$ has the law of an $\uparrow$-diffusion started at 0 and run until it hits $y$, and $\left(\mathbf{V}_{T_y+t}\right)_{t\in[0,\infty)}$ has the law of an independent 0-diffusion started at $y$.

The interpretation for $\nu$ is that the excursions come into $E\backslash\{0\}$ from the level 0 according to the $\uparrow$-diffusion and then evolve in $E\backslash\{0\}$ according to the 0-diffusion. Like the scale function, this measure is unique up to a normalising constant in the sense that if $\nu$ is an excursion measure and $k > 0$ then $k\nu$ is also an excursion measure (the normalising constant for $s$ determines the normalising constant for $\nu$).

We give an overview of the results from Chang and Pollard [21] on the theory of disintegration, which will allow us to condition over infinite dimensional measures such as the Pitman-Yor excursion measure $\nu$.

**Definition 2.2.5** (Disintegration)**.** Let $\lambda$ be a $\sigma$-finite measure on a metric space $\mathscr{X}$ and let $S : \mathscr{X} \to \mathbb{R}$ be a measurable map. Then $\lambda$ has a disintegration $\{\lambda(\cdot|S=t)\}_{t\in[0,\infty)}$ with respect to $S$, known as an $S$-disintegration, if

1. $\lambda(\cdot|S=t)$ is a probability measure on $\mathscr{X}$ concentrated on $\{S=t\}$. In other words, $\lambda(\{S\neq t\}|S=t) = 0$ for $\lambda(S\in\cdot)$-almost all $t$;
2. for any $F : \mathscr{X} \to [0,\infty)$, $t \mapsto \int_{g\in\mathscr{X}} F(g)\lambda(dg|S=t)$ is measurable and
3. $\int_{g\in\mathscr{X}} F(g)\lambda(dg) = \int_{t\in\mathbb{R}} \int_{g\in\mathscr{X}} F(g)\lambda(dg|S=t)\lambda(S\in dt)$.

The next theorem states sufficient conditions for a function $S : \mathscr{X} \to \mathbb{R}$ on a complete and separable space to disintegrate a measure $\lambda$.

**Theorem 2.2.6** (Existence Theorem for a disintegration for $\lambda$)**.** *Suppose $(\mathscr{X}, \Sigma(\mathscr{X}), \lambda)$ is a complete and separable space and that $S$ is a measurable map. Then $\lambda$ has an $S$-disintegration. Furthermore, the measures $\lambda(\cdot|S=t)$ are uniquely determined up to an almost sure equality: that is to say that if $\lambda_t$ is another $S$-disintegration then $\lambda(S^{-1}(t \in [0,\infty] : \lambda_t \neq \lambda(\cdot|S=t))) = 0$.*

Definition 2.2.5 and Theorem 2.2.6 are given in greater generality in [21, Definition 1] and [21, Theorem 1] respectively – what we refer to as an $S$-disintegration they refer to as a $(S, S\lambda)$-disintegration. The Williams decomposition in Theorem 2.2.7 satisfies the conditions of Definition 2.2.5 and so is an $A$-disintegration of $\nu$. Furthermore, by Theorem 2.2.6 there exists an $\zeta$-disintegration of $\nu$.

We abuse notation and write $s(dw)$, for $w \in E\backslash\{0\}$, to denote the Lebesgue-Stieltjes measure on $E\backslash\{0\}$ as defined from the continuous increasing scale function $s$. Formally, $s(dw)$ is the Borel measure $\phi(dw)$ for which we have for any $0 \leq x \leq y < c$ that $\phi((x,y)) = s(y) - s(x)$. We remark that under Assumption $A3$ we have that $s(dw) = s'(w)dw$ but that $s$ is not in general differentiable.

The following description of the Williams decomposition of $\nu$ depends on whether $s(c) < \infty$. If $s(c) < \infty$ then the level $c$ is a regular boundary point for the diffusion, and therefore the diffusion can hit $c$ in finite time. In this case the process reflects off the point $c$ and the Williams decomposition of the excursion measure therefore includes a case where the amplitude $A$ is equal to $c$ which can occur with positive probability. The graph on the right in Figure 2.2 shows an excursion that hits level $c$ in this setting.

**Theorem 2.2.7** (Williams decomposition of $\nu$)**.** *Consider a diffusion that satisfies $A1$. Then its Pitman-Yor excursion measure $\nu$ is the unique measure that satisfies the following:*

1. *$\nu(A \in dw) = s(dw)/s(w)^2$ for $w \in (0,c)$. If $s(c)$ is finite, we further have that $\nu(A = c) = 1/s(c)$.*

2. *For $w \in (0,c)$, a process $\mathbf{V}$ under $\nu(\cdot|A=w)$ has an a.s. unique time at which it attains its amplitude, given by $R$, so that $\mathbf{V}_R = w$. Furthermore, the processes*

$$(\mathbf{V}_t, 0 \leq t \leq R) \text{ and } \left(\mathbf{V}_{\zeta - t}, 0 \leq t \leq \zeta - R\right)$$

   *are independent $\uparrow$-diffusions each started at the level $0$ and then stopped at time $T_w$. See the graph on the left in Figure 2.2.*

3. *In the case where $s(c)$ is finite the process $\mathbf{V}$ under $\nu(\cdot|A=c)$ is defined and evolves as the following. The process $(\mathbf{V}_t)_{t\in[0,T_c)}$ has the law of an $\uparrow$-diffusion started at $0$ run until it hits $c$, and $(\mathbf{V}_{T_c+t})_{t\in[0,\infty)}$ has the law of an independent $0$-diffusion started at $c$. See the graph on the right in Figure 2.2.*

Theorem 2.2.7 is proven in [84, Section 3] in slightly less generality, but the generalisation in our version follows from this.

### 2.2.3 Green's functions of diffusions

In this section we explore the properties of Green's functions of diffusions that we use in this paper. For a probability measure $\mathbb{Q}$ we let $\mathbb{E}_{\mathbb{Q}}$ denote the expectation operator under $\mathbb{Q}$. For a diffusion on a state space $E$ and for $x \in E$ let $\mathbb{Q}_x$ denote the probability measure of the diffusion started at $x$. This first result is from [88, Corollary VII.3.8].

**Theorem 2.2.8.** *Consider a diffusion on a state space $E \subseteq [0,\infty)$ which is of the form $E = [0,c)$ or $E = [0,c]$ for some $c \leq \infty$. Let $a$, $w \in E$ with $a < w$, and let $F : \mathbb{R} \to \mathbb{R}$ be a locally bounded Borel function. We have for $x \in (a,w)$ that*

$$\mathbb{E}_{\mathbb{Q}_x}\left(\int_{s=0}^{T_a \wedge T_w} F(Y_s)ds\right) = \int_{v=a}^{w} F(v)G_{(a,w)}(x,v)M(dv)$$

*where $G_{(a,w)}$ is the Green's function for the diffusion on $(a,w)$ as in* (2.3)*.*

The next result is from [88, Proposition VII.3.10] and includes some immediate consequences that apply to our setting.

**Lemma 2.2.9.** *Consider a diffusion which satisfies A*1 *and has speed measure $M$ such that $M((0,b)) < \infty$ for some $b \in E$ and let $F : \mathbb{R} \to [0,\infty)$ be a locally bounded Borel function. Then, for $w \in E$, we have*

$$\mathbb{E}_{\mathbb{Q}_0}\left(\int_{s=0}^{T_w} F(Y_s)ds\right) = \int_{v=0}^{w} F(v)(s(w)-s(v))M(dv).$$

*In particular, if a diffusion $\mathbf{Y}$ satisfies A*1*, A*2 *and $s(c)$ is finite then $c$ is a regular boundary point for $\mathbf{Y}$ and we can define the diffusion $\hat{\mathbf{Y}} := c - \mathbf{Y}$. Also, using "hats" for anything associated with $\hat{\mathbf{Y}}$, we have that $\hat{\mathbf{Y}}$ satisfies A*1*, $M_{\hat{\mathbf{Y}}}((0,b)) < \infty$ for some $b \in E$ and*

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_c}\left(\int_{s=0}^{T_0} F(Y_s)ds\right) &= \mathbb{E}_{\hat{\mathbb{Q}}_0}\left(\int_{s=0}^{T_c} F(c-Y_s)ds\right)\\ &= \int_{v=0}^{c} F(c-v)(s_{\hat{\mathbf{Y}}}(c)-s_{\hat{\mathbf{Y}}}(v))M_{\hat{\mathbf{Y}}}(dv) = \int_{v=0}^{c} F(v)s_{\mathbf{Y}}(v)M_{\mathbf{Y}}(dv).\end{aligned}$$

We now look at results about $\uparrow$-diffusions. For $x \in E$, we define $\mathbb{Q}_x^{\uparrow}$ to be the distribution of the $\uparrow$-diffusion started from $x$. This final lemma uses results from Karlin and Taylor [54, Section 15.3].

**Lemma 2.2.10.** *Consider a diffusion that satisfies A*1*, A*3 *and has infinitesimal drift $\mu$ and infinitesimal variance $\sigma^2$ that are continuous on $E$. Then we can define the $\uparrow$-diffusion in this setting and we have that the infinitesimal drift $\mu^{\uparrow}$ and infinitesimal variance $(\sigma^2)^{\uparrow}$ for the $\uparrow$-diffusion are the continuous functions*

$$\mu^{\uparrow}(x) = \mu(x) + \frac{s'(x)}{s(x)}\sigma^2(x); \quad (\sigma^2)^{\uparrow}(x) = \sigma^2(x). \tag{2.20}$$

*Furthermore, for $a > 0$, we define functions $U_{(a,w)}$ and $V_{(a,w)}$ from $U$ and $V$ in [54, Equation 15.3.37]: for $x \in (a,w)$ let*

$$U_{(a,w)}(x) := \mathbb{E}_{\mathbb{Q}_x^\uparrow}\Big((T_a \wedge T_w)^2\Big), \quad V_{(a,w)}(x) := 2\mathbb{E}_{\mathbb{Q}_x^\uparrow}(T_a \wedge T_w). \tag{2.21}$$

*Then $U_{(a,w)}(\cdot)$, $V_{(a,w)}(\cdot)$ satisfy:*

$$\frac{1}{2}(\sigma^2)^\uparrow(x)U''_{(a,w)}(x) + \mu^\uparrow(x)U'_{(a,w)}(x) + V_{(a,w)}(x) = 0. \tag{2.22}$$

*Furthermore, there exists a unique solution $(U_{(a,w)}(x))_{x\in(a,w)}$ to (2.22) with boundary conditions $U_{(a,w)}(a) = 0$ and $U_{(a,w)}(w) = 0$, which is given by*

$$U_{(a,w)}(x) = k_{(a,w)}(s^\uparrow(w) - s^\uparrow(x)) + \int_{y=x}^{w}\int_{z=a}^{y}(s^\uparrow)'(y)V_{(a,w)}(z)m^\uparrow(z)dydz \text{ for } x \in (a,w),$$

*where*

$$k_{(a,w)} = \frac{-1}{s^\uparrow(w) - s^\uparrow(a)}\int_{y=a}^{w}\int_{z=a}^{y}(s^\uparrow)'(y)V_{(a,w)}(z)m^\uparrow(z)dzdy.$$

We prove the solution satisfies the differential equation (2.22) in Lemma 2.5.2 in the Appendix. The formulae for $\mu^\uparrow$ and $(\sigma^2)^\uparrow$, stated in (2.20), can be found in the literature, e.g. Pinsky [82, p. 371]. In (2.4) we have $V_{(a,w)}(x)$ in terms of the speed measure and the Green's function for the diffusion on $(a,w)$ which is given in (2.3). We can then make substitutions for $s^\uparrow$ and $m^\uparrow$ using (2.19). We take monotonic limits as $a$ decreases to 0 and then as $x$ decreases to 0 to obtain

$$\mathbb{E}_{\mathbb{Q}_0^\uparrow}(T_w) = \int_{v=0}^{w}(s^\uparrow(w) - s^\uparrow(v))M^\uparrow(dv) = \int_{v=0}^{w}\Big(\frac{1}{s(v)} - \frac{1}{s(w)}\Big)s(v)^2M(dv) \text{ for } w \in E, \tag{2.23}$$

$$\mathbb{E}_{\mathbb{Q}_0^\uparrow}(T_w^2) = 2\int_{y=0}^{w}\int_{z=0}^{y}\int_{v=z}^{w}\left(\frac{1}{s(v)} - \frac{1}{s(w)}\right)s(v)^2m(v)dvs(z)^2m(z)dz\frac{s'(y)}{s(y)^2}dy \tag{2.24}$$

for $w \in E$. This ultimately follows as 0 is not a regular or exit boundary point for the $\uparrow$-diffusion and so we have $\lim_{a\downarrow 0}\mathbb{E}_{\mathbb{Q}_x^\uparrow}(T_a \wedge T_w) = \mathbb{E}_{\mathbb{Q}_x^\uparrow}(T_w)$ and $\lim_{x\downarrow 0}\mathbb{E}_{\mathbb{Q}_x^\uparrow}(T_w) = \mathbb{E}_{\mathbb{Q}_0^\uparrow}(T_w)$ and similarly for $\mathbb{E}_{\mathbb{Q}_0^\uparrow}(T_w^2)$.

## 2.3 Scaffolding and spindles – Assumption A

In this section we give results for the interval partition produced at a fixed level $y \in \mathbb{R}$ by our scaffolding-and-spindles construction.

### 2.3.1 Proof of Theorem 2.1.3

We use Green's functions to determine the Lévy measure condition in Theorem 2.1.3. We prove that with the conditions of Theorem 2.1.3 the lifetime measure under the Pitman-Yor excursion measure is the jump measure of a recurrent SPLP, and the conclusion of the theorem follows. Our diffusion takes values state space $E \subseteq [0,\infty)$ of the form $E = [0,c)$ or $E = [0,c]$ where $c \leq \infty$. For $b \in (0,c)$ we can split the excursions $f \in \mathscr{E}$ of the diffusion based on whether or not they have amplitude $A(f)$ greater than $b$:

$$\begin{aligned}\int_{x=0}^{\infty} (x^2 \wedge x)\nu(\zeta \in dx) &= \int_{f\in\mathscr{E}} \left(\zeta(f)^2 \wedge \zeta(f)\right)\nu(df) \\ &\leq \int_{f\in\mathscr{E}} \zeta(f)^2 \mathbb{1}_{A(f)\leq b}\nu(df) + \int_{f\in\mathscr{E}} \zeta(f)\mathbb{1}_{A(f)>b}\nu(df). \quad (2.25)\end{aligned}$$

**Lemma 2.3.1.** *Consider a diffusion that satisfies A*1 *and A*2 *with Pitman-Yor excursion measure* $\nu$. *Then for* $b \in (0,c)$ *we have that*

$$\int_{f\in\mathscr{E}} \zeta(f)\mathbb{1}_{A(f)>b}\nu(df) < \infty.$$

*Proof.* The disintegration described in the Williams decomposition in Theorem 2.2.7 gives the excursion conditional on $A = w$ as the concatenation of two $\uparrow$-diffusions, the second one being time-reversed so that the concatenated pair form a non-negative excursion of amplitude $w$, see the left graph in Figure 2.2. Therefore for $w \in (0,c)$ the expectation of the lifetime under $\nu(\cdot|A=w)$ is given by

$$\int_{f\in\mathscr{E}} \zeta(f)\nu(df|A=w) = 2\int_{f\in\mathscr{E}} R(f)\nu(df|A=w) = 2\mathbb{E}_{\mathbb{Q}_0^{\uparrow}}(T_w).$$

We have calculated the expectation of $T_w$ under $\mathbb{Q}_0^{\uparrow}$ in (2.23), and so

$$\int_{f\in\mathscr{E}} \zeta(f)\nu(df|A=w) = 4\int_{v=0}^{w} \left(\frac{1}{s(v)} - \frac{1}{s(w)}\right) s(v)^2 M(dv).$$

We integrate over the amplitude $A = w \in (b,c)$ and take the upper bound of $1/s(v)$ for $1/s(v) - 1/s(w)$ to obtain

$$\begin{aligned}\int_{f\in\mathscr{E}} \zeta(f)\mathbb{1}_{A(f)\in(b,c)}\nu(df) &= 4\int_{w=b}^{c}\int_{v=0}^{w} \frac{s(v)^2}{s(w)^2}\left(\frac{1}{s(v)} - \frac{1}{s(w)}\right) M(dv)s(dw) \\ &\leq 4\int_{w=b}^{c}\int_{v=0}^{w} s(v)M(dv)\frac{1}{s(w)^2}s(dw).\end{aligned} \quad (2.26)$$

Finally, we split the integral for $v \in [0,b]$ and for $v \in (b,w)$ to show the claim.

$$\int_{w=b}^{c}\int_{v=0}^{b} s(v)M(dv)\frac{1}{s(w)^2}s(dw) \leq \frac{1}{s(b)}\int_{v=0}^{b} s(v)M(dv),$$

which is finite by $A1$ and

$$\int_{w=b}^{c}\int_{v=b}^{w} s(v)M(dv)\frac{1}{s(w)^2}s(dw) = \int_{v=b}^{c}\int_{w=v}^{c}\frac{1}{s(w)^2}s(dw)s(v)M(dv) \leq M((b,c))$$

which is finite by $A2$.

The final part we need to consider are excursions that hit level $c$, and this occurs when $s(c)$ is finite (equivalently when $c$ is a regular boundary point of the diffusion). In this case, as stated in the Williams decomposition in Theorem 2.2.7, we have that conditional on $\{A = c\}$ the excursion follows a probability distribution that consists of an $\uparrow$-diffusion started at 0 and stopped at $T_c$, the hitting time of $c$, followed by an independent 0-diffusion started at $c$, i.e. a diffusion started at $c$ and stopped at $T_0$.

We calculate the contributions. The first term is given for $w = c$ in (2.23) and the second contribution is given in Lemma 2.2.9. The combination is

$$\begin{aligned}\int_{f\in\mathscr{E}}\zeta(f)\nu(df,A=c) &= \nu(A=c)\Big(\mathbb{E}_{\mathbb{Q}_0^{\uparrow}}(T_c)+\mathbb{E}_c(T_0)\Big)\\ &= \frac{1}{s(c)}\int_{y\in(0,c]}\Big(2-\frac{s(y)}{s(c)}\Big)s(y)M(dy).\end{aligned} \tag{2.27}$$

This is less than $(2/s(c))\int_{y=0}^{c} s(y)M(dy)$. The result follows as $\int_{y=0}^{b} s(y)M(dy)$ is finite by $A1$ and we have that $\int_{y=b}^{c} s(y)M(dy)$ is finite because $s(c)$ is finite and as $M((b,c))$ is finite by $A2$. ■

**Lemma 2.3.2.** *Consider a diffusion that satisfies Assumption A and let $\nu$ be the Pitman-Yor measure. Let $b \in (0,c)$ be as in A4. Then we have*

$$\int_{f\in\mathscr{E}}\zeta(f)^2\mathbb{1}_{A(f)\leq b}\nu(df) \leq 8\int_{v=0}^{b}\int_{z=0}^{v} s(z)m(z)dzm(v)dv. \tag{2.28}$$

*Proof.* As we assume $A3$ in this lemma we can write $M(dv) = m(v)dv$ and $s(dw) = s'(w)dw$, i.e. $M$ has a density $m$ with respect to the Lebesgue measure and $s$ is differentiable. As in the previous lemma, we use from Theorem 2.2.7 that, for $w \in (0,b)$, $R$ and $\zeta - R$ are independent and identically distributed under $\nu(\cdot|A=w)$ with the distribution of $T_w$ under $\mathbb{Q}_0^{\uparrow}$. By expanding $\zeta^2 = (R+(\zeta-R))^2$ we obtain the upper bound

$$\begin{aligned}\int_{f\in\mathscr{E}}\big(R(f)^2+2R(f)\big(\zeta(f)-R(f)\big)+(\zeta(f)-R(f))^2\big)\nu(df|A=w)&\\ \leq 4\int_{f\in\mathscr{E}}R(f)^2\nu(df|A=w).&\end{aligned}$$

To find an upper bound for the right-hand-side we consider a different diffusion $\tilde{\mathbf{Z}}$ with infinitesimal drift $\tilde{\mu}$ and infinitesimal drift $\tilde{\sigma}^2$ given by

$$\tilde{\mu}(w) := \mu(w),\quad \tilde{\sigma}^2(w) := \sigma^2(w) \text{ for } w\in(0,b);\qquad \tilde{\mu}(w) := \mu(b),\quad \tilde{\sigma}^2(w) := \sigma^2(b) \text{ for } w \geq b.$$

Let $\tilde{s}$, $\tilde{m}$ and $\tilde{\nu}$ be the scale function, speed measure density and Pitman-Yor excursion measure for $\tilde{\mathbf{Z}}$. Then $\tilde{\mathbf{Z}}$ is a diffusion that satisfies $A1$ and $A3$ and $\tilde{\mu}$ and $\tilde{\sigma}^2$ are continuous on $E$ and so we can apply Lemma 2.2.10 and, substituting the expectation for $T_w^2$, we obtain

$$\int_{f\in\mathscr{E}} R(f)^2\tilde{\nu}(df|A=w) = 2\int_{y=0}^{w}\int_{z=0}^{y}\int_{v=z}^{w}\left(\frac{1}{\tilde{s}(v)}-\frac{1}{\tilde{s}(w)}\right)\tilde{s}(v)^2\tilde{m}(v)dv\tilde{s}(z)^2\tilde{m}(z)dz\frac{\tilde{s}'(y)}{\tilde{s}(y)^2}dy$$

for $w\in(0,b)$. However as, for $w\in(0,b)$, we have $\tilde{s}(w)=s(w)$, $\tilde{m}(w)=m(w)$ and also $\nu(\cdot|A=w)=\tilde{\nu}(\cdot|A=w)$ by the Williams decomposition in Theorem 2.2.7, we can substitute the terms in the equation above. We then integrate over $A=w\in(0,b)$ and take the same upper bound of $1/s(v)$ for $1/s(v)-1/s(w)$ as in the previous lemma to get

$$\int_{w=0}^{b}\int_{f\in\mathscr{E}}\zeta(f)^2\nu(df|A=w)dw \leq 8\int_{w=0}^{b}\int_{y=0}^{w}\int_{z=0}^{y}\int_{v=z}^{w}s(v)m(v)dvs(z)^2m(z)dz\frac{s'(y)}{s(y)^2}dy\frac{s'(w)}{s(w)^2}dw$$

for $w\in(0,b)$. We swap the order of integration and then extend the range of the now inner $y$ and $w$ integrals to upper bound the integral above by

$$\int_{z=0}^{b}\int_{v=z}^{b}\int_{w=v}^{c}\int_{y=z}^{c}\frac{s'(y)}{s(y)^2}dy\frac{s'(w)}{s(w)^2}dws(v)m(v)dvs(z)^2m(z)dz \leq \int_{z=0}^{b}\int_{v=z}^{b}m(v)dvs(z)m(z)dz,$$

where we again use the same upper bound of $1/s(v)$ for $1/s(v)-1/s(w)$ and also in $z$. Finally we swap the order of integration of $z$ and $v$ to give the upper bound in the claim. ■

*Proof of Theorem 2.1.3.* Consider a diffusion under $A1$ and $A2$. Then as in (2.26), but now without the amplitude constraint of $A(f)>b$, we note that

$$\int_{f\in\mathscr{E}}\zeta(f)\nu(df) = 4\int_{w=0}^{c}\int_{v=0}^{w}\frac{s(v)^2}{s(w)^2}\left(\frac{1}{s(v)}-\frac{1}{s(w)}\right)M(dv)s(dw)$$

which we can rearrange to give

$$4\int_{v=0}^{c}\int_{w=v}^{c}\Big(\frac{s(v)}{s(w)^2}-\frac{s(v)^2}{s(w)^3}\Big)s(dw)M(dv).$$

We split into two cases. When $s(c)$ is infinite the above term is equal to $2M((0,c))$ and therefore we have the Pitman-Yor excursion measure satisfies (2.6) if and only if (2.7) holds. When $s(c)$ is finite we have the above integral is equal to

$$2M((0,c)) - \frac{4}{s(c)}\int_{v=0}^{c}s(v)M(dv) + \frac{2}{s(c)^2}\int_{v=0}^{c}s(v)^2M(dv).$$

To complete the proof in this case it suffices to show that $\int_{v=0}^{c}s(v)M(dv)$ is finite. We have $\int_{v=0}^{b}s(v)M(dv)$ is finite by $A1$ and furthermore we have that $\int_{v=b}^{c}s(v)M(dv)$ is finite as $M((b,c))$ and $s(c)$ are both finite.

For the second setting described in Theorem 2.1.3 (where the diffusion satisfies all of Assumption A) the result follows immediately from Lemmas 2.3.1 and 2.3.2. ■

### 2.3.2 Proof of Theorem 2.1.4

Throughout this section we consider a diffusion that satisfies $A1$ and $A2$ and has a Pitman-Yor excursion measure $\nu$ that satisfies (2.9), which we recall means that for a scaffolding-and-spindles pair in this setting, the scaffolding $\mathbf{X}$ is an SPLP of unbounded variation with no Gaussian component (we deal with the bounded variation case in the final proof of Theorem 2.1.4 as the end of this section). With this in mind, we begin with a lemma that applies to such Lévy processes.

**Lemma 2.3.3** (Excursion undershoot-overshoot measure)**.** *Let $\mathbf{X}$ be a recurrent SPLP of unbounded variation with Lévy measure $\Lambda$ and with no Gaussian component. Let $n$ be the excursion measure for the law of $\mathbf{X}$. For some constant $k_0 > 0$, the undershoot and overshoot joint measure under the excursion measure $n$ is given by the following measure on $(0,\infty)^2$*

$$n(-g(T_0^+-) \in dy, g(T_0^+) \in dz) = k_0 \Lambda(dz+y)dy. \tag{2.29}$$

*Proof.* We substitute the expression for the excursion measure of $\mathbf{S}-\mathbf{X}$ from (2.18) into (2.17) to give

$$\begin{aligned} &n\left(-g(T_0^+-) \in dy, g(T_0^+) \in dz, T_0^+ > t\right) \\ &\qquad = k_1 \left( \int_{x=0}^{\infty} \left(\hat{W}(x) - \hat{W}(x-y)\right) \frac{k_2}{\hat{W}(x)} \hat{\mathbb{P}}^{\uparrow}(X_t \in dx) \right) \Lambda(dz+y)dy. \end{aligned}$$

For $x > 0$ and $y > 0$, $\left(\hat{W}(x) - \hat{W}(x-y)\right)/\hat{W}(x) \leq 1$. Also, $\lim_{x \downarrow 0} \left(\hat{W}(x) - \hat{W}(x-y)\right)/\hat{W}(x) = 1$ by monotonicity. Finally, for $X$ with law $\mathbb{P}^{\uparrow}$, we have that $X_t \to 0$ as $t \downarrow 0$ a.s. Also we have $n(T_0^+ = 0) = 0$, see [80, Theorem 3], and so

$$\begin{aligned} n\left(-g(T_0^+-) \in dy, g(T_0^+) \in dz\right) &= \lim_{t \downarrow 0} n\left(-g(T_0^+-) \in dy, g(T_0^+) \in dz, T_0^+ > t\right) \\ &= k_1 k_2 \left( \lim_{t \downarrow 0} \int_{x=0}^{\infty} \frac{\hat{W}(x) - \hat{W}(x-y)}{\hat{W}(x)} \hat{\mathbb{P}}^{\uparrow}(X_t \in dx) \right) \Lambda(dz+y)dy \\ &= k_0 \Lambda(dz+y)dy \text{ where } k_0 := k_1 k_2, \end{aligned}$$

and where the limit of measures is a vague limit. ∎

**Definition 2.3.4.** For a complete and separable metric space $S$, let $\mathscr{N}(S)$ be the set of counting measures on $S$ that are boundedly finite. We equip $\mathscr{N}(S)$ with the $\sigma$-algebra $\Sigma(\mathscr{N}(S))$ generated by the evaluation maps $N \mapsto N(B)$ for Borel subsets $B \subseteq S$. Let $N \in \mathscr{N}([0,\infty) \times \mathscr{E})$ and define

1. $\text{len}(N) := \inf\{t > 0 : N([t,\infty) \times \mathscr{E}) = 0\} \in [0,\infty]$;
2. $N|_{[a,b]}$ is $N$ restricted to the interval $[a,b]$: $N|_{[a,b]} := \sum_{t \in [a,b], (t,f_t) \text{ atom of } N} \delta_{(t,f_t)}$.

3. $N|^{\leftarrow}_{[a,b]}$ is $N$ with the time index shifted back by $a$: $N|^{\leftarrow}_{[a,b]} := \sum_{t\in[a,b],(t,f_t)\text{ atom of }N} \delta_{(t-a,f_t)}$;

4. for an atom $(t,f_t)$ of $N$ we define, for $y \in (\xi_N(t-),\xi_N(t))$ (recall (2.1) and paragraph below the equation for definition of $\xi_N$), the decomposition of the spindle $f_t$ between the component of $f_t$ that falls below level $y$, denoted as $\check{f}_t \in \mathscr{E}$, and the component of $f_t$ that is above level $y$, denoted as $\hat{f}_t \in \mathscr{E}$:

$$\check{f}^y_t(z) := f_t(z)\mathbb{1}_{z\in[0,y-\xi_N(t-))}, \qquad \hat{f}^y_t(z) := f_t(y-\xi_N(t-)+z)\mathbb{1}_{z\in[0,\xi_N(t)-y]},$$

and we write $\check{f}_t := \check{f}^0_t$ and $\hat{f}_t := \hat{f}^0_t$;

5. we define a subspace of $\mathscr{N}([0,\infty)\times\mathscr{E})$. Let $\mathscr{N}^{\mathrm{sp}}_{\mathrm{fin}}$ be the set of counting measures with the additional properties:

   (a) $N(\{t\}\times\mathscr{E}) \le 1$ for every $t \in [0,\infty)$,

   (b) $N([0,t]\times\{f\in\mathscr{E} : \zeta(f) > z\}) < \infty$ for every $t, z > 0$,

   (c) $\mathrm{len}(N) < \infty$ and limit $\xi_N(t)$ exists for each $t \in [0,\mathrm{len}(N)]$;

6. we define the subspace $\mathscr{N}^{\mathrm{sp}} \subset \mathscr{N}$ by saying that $N \in \mathscr{N}^{\mathrm{sp}}$ if and only if $N|_{[0,t]} \in \mathscr{N}^{\mathrm{sp}}_{\mathrm{fin}}$ for each $t \ge 0$.

With the above definitions, we can define the following sets that contain measures associated with càdlàg excursions away from 0. We define

$$\mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} := \{N \in \mathscr{N}^{\mathrm{sp}}_{\mathrm{fin}} : \xi(N) \in \mathscr{D}_{\mathrm{exc}}\},$$

$$\mathscr{N}^{\mathrm{sp}}_{+\mathrm{cld}} := \{N \in \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} : \inf_t \xi_N(t) = 0\}, \qquad \mathscr{N}^{\mathrm{sp}}_{-\mathrm{cld}} := \{N \in \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} : \sup_t \xi_N(t) = 0\},$$

to classify the possible excursions of $\mathbf{X}$ marked with the points of $\mathbf{N}$. We split $N \in \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} \setminus (\mathscr{N}^{\mathrm{sp}}_{-\mathrm{cld}} \cup \mathscr{N}^{\mathrm{sp}}_{+\mathrm{cld}})$ into what we call its clade component $N^+ \in \mathscr{N}^{\mathrm{sp}}_{+\mathrm{cld}}$ and its anti-clade component $N^- \in \mathscr{N}^{\mathrm{sp}}_{-\mathrm{cld}}$ by

$$N^- := N|_{[0,T_0^+)} + \delta(T_0^+, \check{f}^0_{T_0^+}), \qquad N^+ := \delta(0,\hat{f}^0_{T_0^+}) + N|^{\leftarrow}_{(T_0^+,\infty)}, \tag{2.30}$$

(and we define $(N^-,N^+)$ to be $(N,0)$ for $N \in \mathscr{N}^{\mathrm{sp}}_{-\mathrm{cld}}$ and similarly for $N \in \mathscr{N}^{\mathrm{sp}}_{+\mathrm{cld}}$). The terminology is inherited from our main reference [34]. The component $N^+$ represents all the descendents of an individual, and this is therefore a clade of the total population. The term anti-clade was then coined due to it being the inverse to the clade. Finally, we use the term bi-clade for the combination of the clade and the anti-clade. We also refer to elements of $\mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}$, $\mathscr{N}^{\mathrm{sp}}_{+\mathrm{cld}}$ and $\mathscr{N}^{\mathrm{sp}}_{-\mathrm{cld}}$ as bi-clades, clades and anti-clades respectively.

In Definition 2.3.5 we define $V^y$ to be the set of all complete excursions about level $y$ and we define $V^y_0$ to be $V^y$ along with the possibly incomplete first and last excursions in the following way from [34, Definition A.1].

**Definition 2.3.5.** For a càdlàg function $g \in \mathscr{D}$ and $y \in \mathbb{R}$ we define

$$V^y(g) := \left\{ [a,b] \subset [0,\mathrm{len}(g)] \,\middle|\, \begin{array}{c} a<b<\infty; g(t-) \neq y \neq g(t) \text{ for } t \in (a,b); \\ g(a-)=y \text{ or } g(a)=y; g(b-)=y \text{ or } g(b)=y \end{array} \right\}.$$

The set $V_0^y(g) \supset V^y(g)$ then includes any incomplete first or last excursions. To define this rigorously, let

$$T^y(g) := \inf(\{t \in [0,\mathrm{len}(g)] : g(t)=y \text{ or } g(t-)=y\} \cup \{\mathrm{len}(g)\}),$$
$$T^y_*(g) := \sup(\{t \in [0,\mathrm{len}(g)] : g(t)=y \text{ or } g(t-)=y\} \cup \{0\}).$$

Then, for $y \neq 0$, we include $[0,T^y(g)] \cap [0,\infty)$ in $V_0^y(g)$, even if $T^y(g)$=0. Also, if $T^y_*(g) < \mathrm{len}(g)$ or $g(\mathrm{len}(g)) \neq y$, then we include $[T^y_*(g),\mathrm{len}(g)] \cap [0,\infty)$ in $V_0^y(g)$. For $N \in \mathscr{N}^{\mathrm{sp}}$ we define $V^y(N) := V^y(\xi(N))$ and similarly for $V_0^y$.

Finally, we need to take care with possible non-degenerate excursions and so we carefully define intervals $I_N^y(a,b)$ to be one of $[a,b]$, $(a,b]$, $[a,b)$, $(a,b)$ as follows: we exclude the point $a$ iff $a<b$ and $g(a-)<y=g(a)$; we exclude $b$ iff both $a<b$ and $g(b-)=y<g(b)$.

With this definition we can consider the measure on excursions of the process $\mathbf{X} := \xi(\mathbf{N})$ where the jumps are marked by the spindles that generate them. For $N \in \mathscr{N}^{\mathrm{sp}}_{\mathrm{fin}}$, let $V^y(N)$ for $y \in \mathbb{R}$, $N \in \mathscr{N}^{\mathrm{sp}}$ be the set of intervals of complete excursions of $\mathbf{X}$, so that $[a,b] \in V^y(N)$ means that the path $(\xi_N(a+t), 0 \le t \le b-a)$ is an excursion about level $y$ – we define this rigorously in Proposition 2.5.4 in the Appendix. As $\mathbf{X}$ is of unbounded variation we have that, for each level $y \in \mathbb{R}$, there are no degenerate excursions of $\mathbf{X}$ a.s. and $\mathbf{X}$ has local times. Furthermore, the local times are injective a.s. for each $y$ in the sense that for $[a,b]$, $[c,d] \in V^y$, $\ell^y(a) \neq \ell^y(c)$ unless $[a,b]=[c,d]$; this is shown in Millar [74] which summarises whether Lévy processes either first jump across or first hit fixed levels. While it holds at any fixed level a.s., across all levels simultaneously this would be false. We define $F^0$ for $N \in \mathscr{N}^{\mathrm{sp}}$ and $\mathbf{F}^0$ for $\mathbf{N}$ by

$$F^0(N) := \sum_{[a,b] \in V^0(N)} \delta_{\ell^0(a), N|^{\leftarrow}_{[a,b]}}, \qquad \mathbf{F}^0 := F^0(\mathbf{N}). \tag{2.31}$$

We note that $\mathbf{F}^0$ is a Poisson random measure where each element is the collection of spindles that generate an excursion about level 0 for the process $\xi_N$. We define the measure $\nu_{\mathrm{cld}}$ on $\mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}$ by

$$\nu_{\mathrm{cld}}(A) := \mathbf{E}(\mathbf{F}^0([0,1] \times A)) \text{ for a measurable set } A \subset \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}.$$

The bi-clades here will have a unique jump that crosses zero, that is to say that the set of all bi-clades with no unique crossing jump has measure 0 under $\nu_{\mathrm{cld}}$, and the unique jump will have associated with it an excursion of our diffusion. Therefore we can discuss the crossing width $m^0$ of the unique crossing jump:

$$m^0(N) := \sum_{(r,f_r) \text{ atom of } N} \max\{f_r((-\xi_N(r-))-), f_r(-\xi_N(r-))\}; \;\; N \in \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}. \tag{2.32}$$

The reason that we define $m^0$ in this way is that it makes sense for $N \in \mathscr{N}^{\mathrm{sp}}_{-\mathrm{cld}} \cup \mathscr{N}^{\mathrm{sp}}_{+\mathrm{cld}}$. Indeed, we have

$$m^0(N) = f_{T_0^+(\xi_N)}(-\xi_N(T_0^+(\xi_N)-)) \text{ for } N \in \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} \backslash (\mathscr{N}^{\mathrm{sp}}_{-\mathrm{cld}} \cup \mathscr{N}^{\mathrm{sp}}_{+\mathrm{cld}}).$$

This second formulation for the crossing width is more intuitive. We use the following notation, similar in style to that used in Section 2.2.1: take as an example, for a measurable set $A$ of $[0,\infty) \times (-\infty, 0] \times [0,\infty)$,

$$\begin{aligned}&\nu_{\mathrm{cld}}((m^0, \xi_N(T_0^+-), \xi_N(T_0^+)) \in A) \\ &\qquad := \nu_{\mathrm{cld}}(\{N \in \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} : (m^0(N), \xi_N(T_0^+(\xi_N)-), \xi_N(T_0^+(\xi_N))) \in A\}).\end{aligned}$$

**Proposition 2.3.6.** *Consider a diffusion that satisfies A*1 *and A*2 *with Pitman-Yor excursion measure* $\nu$ *that satisfies* (2.9)*. Let* $(\mathbf{N}, \mathbf{X})$ *be a scaffolding-and-spindles pair for this diffusion. Then, in the setting described above, we have that*

$$\sum_{(t,N') \text{ atom of } F^0(\mathbf{N})} \delta_{t, m^0(N')}$$

*is a Poisson random measure with intensity* $\nu_{cld}(m^0 \in dx) = k_0 M(dx)$ *where* $k_0$ *is the constant in Lemma 2.3.3 for the SPLP* $\mathbf{X}$*.*

*Proof.* The fact that this measure is a Poisson random measure with intensity $\nu_{\mathrm{cld}}(m^0 \in dx)$ follows from the fact that $\mathbf{F}^0$ is a Poisson random measure with intensity $\nu_{\mathrm{cld}}$. It remains to show $\nu_{\mathrm{cld}}(m^0 \in dx) = k_0 M(dx)$. We rearrange the result in Lemma 2.3.3 to obtain

$$\nu_{\mathrm{cld}}(-\xi_N(T_0^+-) \in dv, \xi_N(T_0^+) - \xi_N(T_0^+-) \in du) = k_0 dv \nu(\zeta \in du) \text{ for } k_0 > 0, \text{ for } 0 < v < u,$$

where $k$ is the constant stated in Lemma 2.3.3. Recall that $\Lambda$ is the jump measure of $\mathbf{X}$ and therefore $\nu(\zeta \in \cdot) = \Lambda(\cdot)$ in this context. Let $H$ be a positive measurable function $H : [0,\infty) \to [0,\infty)$. We claim that

$$\nu_{\mathrm{cld}}\big(H(m^0)\big) = \int_{N \in \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}} H(m^0(N)) \nu_{\mathrm{cld}}(dN) = k_0 \int_{y \in E} H(y) M(dy).$$

First note that

$$\begin{aligned}&\int_{N \in \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}} H(m^0(N)) \nu_{\mathrm{cld}}(dN) \\ &\qquad = \int_{u=0}^{\infty} \int_{v=0}^{u} \int_{x \in E} H(x) \nu_{\mathrm{cld}}(m^0 \in dx, -\xi_N(T_0^+-) \in dv, \xi_N(T_0^+) - \xi_N(T_0^+-) \in du)\end{aligned}$$

and in the next step we disintegrate $\nu_{\mathrm{cld}}(m^0 \in \cdot)$ with respect to $(-\xi_N(T_0^+-), \xi_N(T_0^+))$. This is possible as the bi-clade measure $\nu_{\mathrm{cld}}$ can be obtained from marking the jumps of an excursion of

a càdlàg process under the excursion measure of the Lévy process $\mathbf{X}$, [34, Proposition 4.9]. We obtain

$$k_0 \int_{u=0}^{\infty} \int_{v=0}^{u} \int_{x\in E} H(x) \nu_{\mathrm{cld}}(m^0 \in dx | -\xi_N(T_0^+-) = v, \xi_N(T_0^+) - \xi_N(T_0^+-) = u) dv \nu(\zeta \in du)$$
$$= k_0 \int_{u=0}^{\infty} \int_{v=0}^{u} \int_{f\in\mathscr{E}} H(f(v)) \nu(df | \zeta = u) dv \nu(\zeta \in du),$$

by (2.32) and the fact that $\nu_{\mathrm{cld}}(\mathscr{N}^{\mathrm{sp}}_{-\mathrm{cld}} \cup \mathscr{N}^{\mathrm{sp}}_{+\mathrm{cld}}) = 0$. We can rewrite this as

$$k_0 \int_{u=0}^{\infty} \int_{f\in\mathscr{E}} \int_{v=0}^{\zeta(f)} H(f(v)) dv \nu(df | \zeta = u) \nu(\zeta \in du),$$

where $f$ represents the excursion associated with the unique crossing jump and $\nu$ is the Pitman-Yor excursion measure of our diffusion. We can express this in terms of an integral against the Pitman-Yor excursion measure as

$$k_0 \int_{f\in\mathscr{E}} \Big( \int_{v=0}^{\zeta(f)} H(f(v)) dv \Big) \nu(df).$$

From Theorem 2.2.7 we see this disintegrates with respect to the amplitude $A$ to give

$$k_0 \int_{w\in E\setminus\{0\}} \int_{f\in\mathscr{E}} \Big( \int_{v=0}^{\zeta(f)} H(f(v)) dv \Big) \nu(df | A = w) \nu(A \in dw). \tag{2.33}$$

As shown in the left graph of Figure 2.2, for $w \in (0,c)$ the Williams decomposition result gives a disintegration with the conditional distribution $\nu(\cdot | A = w)$ being the concatenation of two $\uparrow$-diffusions both stopped at $T_w$. Therefore for the $w \in (0,c)$ in (2.33) we have that

$$\int_{w=0}^{c} \int_{f\in\mathscr{E}} \Big( \int_{v=0}^{\zeta(f)} H(f(v)) dv \Big) \nu(df | A = w) \nu(A \in dw) = \int_{w=0}^{c} 2\mathbb{E}_{\mathbb{Q}_0^{\uparrow}} \Big( \int_{t=0}^{T_w} H(Y_t) dt \Big) \frac{s(dw)}{s(w)^2}.$$

which is equal to

$$\int_{w=0}^{c} \int_{y=0}^{w} 2H(y)(s^{\uparrow}(w) - s^{\uparrow}(y)) M^{\uparrow}(dy) \frac{s(dw)}{s(w)^2}$$

by Theorem 2.2.8 (where we have taken monotonic limits $a \downarrow 0$ and $x \downarrow 0$ as we did in Lemma 2.2.9). With the equations for $s^{\uparrow}$ and $M^{\uparrow}$ measure from (2.19), we substitute these terms and swap the order of integration to obtain

$$2 \int_{y=0}^{c} \int_{w=y}^{c} \left( \frac{1}{s(y)} - \frac{1}{s(w)} \right) \frac{1}{s(w)^2} s(dw) H(y) s(y)^2 M(dy).$$

We split into cases depending on whether or not $s(c)$ is finite. If $s(c)$ is infinite then $E = [0,c)$ and the above integral is equal to $\int_{y\in E} H(y) M(dy)$. When $s(c)$ is finite we instead have that $E = [0,c]$ and the above integral is equal to

$$\int_{y\in E} \Big( 1 - 2\frac{s(y)}{s(c)} + \frac{s(y)^2}{s(c)^2} \Big) H(y) M(dy).$$

Furthermore when $s(c)$ is finite we have that the measure $\nu(A \in \cdot)$ has an atom at $c$ and so there is a positive contribution to the integral in (2.33) at $w = c$. We can calculate this from the Williams decomposition in Theorem 2.2.7 to obtain

$$\nu(A = c)\Big(\mathbb{E}_{\mathbb{Q}_0^\uparrow}\Big(\int_{y=0}^{T_c} H(Y_s)ds\Big) + \mathbb{E}_{\mathbb{Q}_c}\Big(\int_{s=0}^{T_0} H(Y_s)ds\Big)\Big).$$

We have that $\nu(A = c)$ is equal to $1/s(c)$ and

$$\mathbb{E}_{\mathbb{Q}_0^\uparrow}\Big(\int_{y=0}^{T_c} H(Y_s)ds\Big) + \mathbb{E}_{\mathbb{Q}_c}\Big(\int_{s=0}^{T_0} H(Y_s)ds\Big)$$
$$= \int_{y\in E} H(y)\Big(\frac{1}{s(y)} - \frac{1}{s(c)}\Big)s(y)^2 M(dy) + \int_{y\in E} H(y)s(y)M(dy)$$

which equals $\int_{y\in E} H(y)\Big(2s(y) - \frac{s(y)^2}{s(c)}\Big)M(dy)$. Therefore we have (2.33) is equal to $\int_{y\in E} H(y)M(dy)$ in both cases as required. ■

With Proposition 2.3.6, we can prove Theorem 2.1.4.

*Proof of Theorem 2.1.4.* Let $(\mathbf{N},\mathbf{X})$ be a scaffolding-and-spindles pair stopped at an a.s. finite $T$ for a diffusion $\mathbf{Z}$. In the setting where $\nu_{\mathbf{Z}}$ satisfies (2.6) for $\nu = \nu_{\mathbf{Z}}$, the scaffolding $\mathbf{X}$ is an SPLP of bounded variation. Therefore the set $\{r \in [0,t] : \mathbf{X}_{r-} < 0 \leq \mathbf{X}_r\}$ is finite a.s., i.e. there are only finitely many crossing widths to sum; therefore, the aggregate health $M^0_{\mathbf{N},\mathbf{X}}(T)$ at level 0 is finite as it is a finite sum of positive values.

In the setting where $\nu_{\mathbf{Z}}$ satisfies (2.9) for $\nu = \nu_{\mathbf{Z}}$ we can apply Proposition 2.3.6. This proposition states that, for fixed $t > 0$, the aggregate fitness process $M^0_{\mathbf{N},\mathbf{X}}(t)$ is a sum over a Poisson random measure with intensity $\mathrm{Leb} \otimes M_{\mathbf{Z}}$ on $[0,t] \times E\backslash\{0\}$. Let $b := 1 \wedge c/2 \in (0,c)$. We deal with the sum of crossing widths less than $b$ and the crossing widths of size greater than $b$ separately. The crossing widths of size less than $b$ form a Poisson random measure with intensity $\mathrm{Leb} \otimes M_{\mathbf{Z}}$ on $[0,t] \times (0,b)$. Therefore, by Campbell's Theorem, see [56, Section 3.2], these are summable if and only if

$$\int_{[0,t]\times(0,b)} z dt M_{\mathbf{Z}}(dz) = t\int_{z=0}^{b} z M_{\mathbf{Z}}(dz) < \infty$$

and this is finite by the Assumption in the theorem. The crossing widths of size greater than $b$ are from a Poisson random measure with intensity $\mathrm{Leb} \otimes M_{\mathbf{Z}}$ on $[0,t] \times [b,c]$. Note that

$$\int_{[0,t]\times[b,c]} dt M_{\mathbf{Z}}(dz) = t M_{\mathbf{Z}}([b,c])$$

and this is finite by $A2$ and therefore there are only finitely many such crossings so the sum is clearly finite. The result then follows for the aggregate fitness process evaluated at the a.s. finite time $T$.

To see the result in terms of an untransformed diffusion $\mathbf{Y}$ with transformation $g$, apply Proposition 2.3.6 to diffusion $\mathbf{Y}$ to obtain that the crossing widths in this case are a Poisson random measure with intensity $\mathrm{Leb} \otimes M_{\mathbf{Y}}$. We apply the transformation $g$ to these crossing widths and this is equal in distribution to the crossing widths from the diffusion $\mathbf{Z}$. The condition for summability of the small crossing widths then becomes the condition in (2.10). ■

### 2.3.3 Disintegration with respect to the crossing width

In this section we disintegrate the bi-clade measure $\nu_{\mathrm{cld}}$ with respect to the crossing width $m^0$. As in the previous section, we assume that our diffusion satisfies $A1$ and $A2$ and that the Pitman-Yor excursion measure $\nu$ satisfies (2.9). We define some operators that time-reverses excursions and point measures.

**Definition 2.3.7.** Let $f \in \mathscr{E}$ and $N \in \mathscr{N}^{\mathrm{sp}}_{\mathrm{fin}}$. Then

1. We define the time-reversal for $f$ by

$$\mathscr{R}_{\mathrm{spdl}}(f) := (f((\zeta(f)-y)-), y \in \mathbb{R}), \tag{2.34}$$

   following the definition given in [34, Equation (2.16)].

2. We also define the time-reversal for $N$ by

$$\mathscr{R}_{\mathrm{cld}}(N) := \sum_{(t,f) \text{ atom of } N} \delta(\mathrm{len}(N)-t, \mathscr{R}_{\mathrm{spdl}}(f)), \tag{2.35}$$

   following the definition given in [34, Equation (4.19)].

The reversibility of the excursions under the Pitman-Yor excursion measure $\nu$, see e.g. the Williams decomposition in [84, Section 3], along with the independence between excursions away from 0 for the Lévy process $\mathbf{X}$, means that

$$\nu_{\mathrm{cld}}(A) = \nu_{\mathrm{cld}}(\mathscr{R}_{\mathrm{cld}}(A)) \text{ for a measurable } A \subset \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}. \tag{2.36}$$

The next result is a generalisation of [34, Lemma 4.13], and it proves conditional independence between the clade and the anti-clade given the crossing width.

**Lemma 2.3.8** (Mid-spindle Markov property)**.** *Consider a diffusion that satisfies $A1$ and $A2$ with Pitman-Yor excursion measure $\nu$ that satisfies* (2.9)*, and let $(\mathbf{N}, \mathbf{X})$ be a scaffolding-and-spindles construction for this diffusion. Let $T$ be a stopping time either of the form $T^{\geq y}$ for some $y > 0$ or of the form*

$$T^y_{n,j} := \inf\left\{ t > 0 \,\middle|\, \int_{[0,t]\times\mathscr{E}} \mathbb{1}_{f(y-\mathbf{X}(s-))>1/n} d\mathbf{N}(s,f) \geq j \right\} \text{ for } y \in \mathbb{R}, j, n \in \mathbb{N}, \tag{2.37}$$

*and let $f_T$ denote the spindle of $\mathbf{N}$ at this stopping time $T$. Then, conditionally given $f_T(y-\mathbf{X}(T-)) = x > 1/n$, we have that*

$$(\mathbf{N}|_{[0,T)}, \check{f}_T^y) \text{ is independent of } (\mathbf{N}|^{\leftarrow}_{(T,\infty)}, \hat{f}_T^y)$$

*(see Definition 2.3.4 for $\check{f}_T^y$ and $\hat{f}_T^y$). Under this conditional law, $\mathbf{N}|^{\leftarrow}_{(T,\infty)}$ is a Poisson random measure with intensity measure Leb $\otimes\, \nu$ independent of $\hat{f}_T$, which is a 0-diffusion started at $x$.*

*Proof.* The proof given in [34, Lemma 4.13] applies in our setting as we have the following conditions: the Pitman-Yor description of an excursion that hits level $1/n$ is given by an $\uparrow$-diffusion started at 0 and stopped at $1/n$, followed by an independent 0-diffusion started at $1/n$; the Poisson random measure $\mathbf{N}$ has a strong Markov property; the scaffolding process $\mathbf{X}$ is of unbounded variation and so we have $T^{\geq y} = \inf_{n\geq 1} T^y_{n,1}$. ■

For $x > 0$, let $\mathbb{Q}^0_x$ be the probability measure on $\mathscr{D}$ for the 0-diffusion started from $x$.

**Lemma 2.3.9.** *In the setting of Lemma 2.3.8, for $n > 1$ and $x > 1/n$ we have that*

$$\begin{aligned}
&\mathbf{P}\Big(\hat{f}_{T^0_{n,1}} \in A, \check{f}_{T^0_{n,1}} \in B, \mathbf{N}|_{(\tau^0(\ell^0(T^0_{n,1})-),T^0_{n,1})} \in C, \mathbf{N}|^{\leftarrow}_{(T^0_{n,1},\tau^0(\ell^0(T^0_{n,1})))} \in D \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big) \\
&= \int_{f\in A} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f)})} \in D) \mathbb{Q}^0_x(df) \int_{f'\in \mathscr{R}_{spdl}(B)} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f')})} \in \mathscr{R}_{cld}(C)) \mathbb{Q}^0_x(df').
\end{aligned}$$

*Proof.* First note by the conditional independence property given in Lemma 2.3.8 we have, for $x > 0$, that

$$\mathbf{P}\Big(\hat{f}_{T^0_{n,1}} \in A, \check{f}_{T^0_{n,1}} \in B \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big) = \mathbb{Q}^0_x(A)\mathbf{P}\Big(\check{f}_{T^0_{n,1}} \in B \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big).$$

Let $\lambda > 0$ and let $e_\lambda$ be an exponential random variable with rate $\lambda$. The excursions of $\mathbf{X}$ away from 0 are indexed by its local time. Therefore $\mathbf{N}|_{[0,\tau^0(e_\lambda)]}$ consists of a collection of complete bi-clades. Therefore, by reversibility of the clade measure, noted in (2.36), we have

$$\mathbf{N}|_{[0,\tau^0(e_\lambda)]} \overset{d}{=} \mathscr{R}_{cld}\Big(\mathbf{N}|_{[0,\tau^0(e_\lambda)]}\Big). \tag{2.38}$$

Now, on the event $\{T^0_{n,1} < \tau^0(e_\lambda)\}$ there exists $j \in \mathbb{N}$ such that

$$T^0_{n,1}(\mathbf{N}|_{[0,e_\lambda]}) = \tau^0(e_\lambda) - T^0_{n,j}(\mathscr{R}_{\mathrm{cld}}(\mathbf{N}|_{[0,e_\lambda]})). \tag{2.39}$$

Furthermore, we have that

$$\begin{aligned}
&\mathbf{P}\Big(\check{f}_{T^0_{n,1}} \in B \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big) \\
&\qquad = \lim_{\lambda\downarrow 0} \sum_{j=1}^{\infty} \mathbf{P}\Big(\check{f}_{T^0_{n,1}} \in B, T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1} \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big)
\end{aligned}$$

as $\lim_{\lambda\downarrow 0}\mathbf{P}(T^0_{n,1} < \tau^0(e_\lambda)) = 1$. We consider each term in the sum. We apply the time reversal and the mid-spindle Markov property (Lemma 2.3.8) to obtain

$$\mathbf{P}\Big(\check{f}_{T^0_{n,1}} \in B \Big| T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1}, f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big) \\ = \mathbf{P}\Big(\hat{f}_{T^0_{n,j}} \in \mathscr{R}_{\mathrm{spdl}}(B) \Big| f_{T^0_{n,j}}(-\mathbf{X}(T^0_{n,j}-)) = x\Big)$$

where the conditioning $T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1}$ was able to be removed due to the conditional independence given $f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x$ from the mid-spindle Markov property. Let $p_\lambda = \mathbf{P}(\tau^0(e_\lambda) > T^0_{n,1})$. Then

$$\mathbf{P}\Big(T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1} \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big) = \mathbf{P}\Big(T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1}\Big) = p_\lambda^j(1-p_\lambda).$$

Pulling all this together we obtain

$$\lim_{\lambda\downarrow 0}\mathbf{P}\Big(\check{f}_{T^0_{n,1}} \in B, T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1} \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big) \\ = \lim_{\lambda\downarrow 0}\sum_{j=1}^{\infty} p_\lambda^j(1-p_\lambda)\mathbb{Q}^0_x(\mathscr{R}_{\mathrm{spdl}}(B)),$$

which equals $\mathbb{Q}^0_x(\mathscr{R}_{\mathrm{spdl}}(B))$ because $p_\lambda \to 1$ as $\lambda \to 0$. The generalisation to the statement in the proposition follows from the Lemma 2.3.8; the strong Markov property of $\mathbf{N}$ at $T^0_{n,1}$ leads to the independence of the Poisson random measures $\mathbf{N}|_{[0,T^0_{n,1})}$ and $\mathbf{N}|^{\leftarrow}_{(T^0_{n,1},\infty)}$. ■

We now describe the regular conditional distribution of $\nu_{\mathrm{cld}}$ disintegrated with respect to $m^0$. Let $x > 0$ denote the crossing width. Let $\hat{f}$ and $\check{f}$ be independent 0-diffusions started at $x$. Let $\check{\mathbf{N}}$ and $\hat{\mathbf{N}}$ be independent Poisson random measures with intensity $\mathrm{Leb} \otimes \nu$.

We define the idea of concatenating point measures in the analogous way to how we defined concatenating interval partitions in Definition 2.2.3. Let $N_i \in \mathscr{N}^{\mathrm{sp}}_{\mathrm{fin}}$ be a collection of point measures where $i \in I$ is an ordered index set. We define $N := \bigstar_{i\in I} N_i$ to be the concatenation with respect to the ordering of $I$ in the following way. On the condition that for all $i \in I$ we have $S(j-) := \sum_{j\in I: j<i}\mathrm{len}(N_j) < \infty$ we take $N$ to be the point measure with length $\sum_{i\in I}\mathrm{len}(N_i) \le \infty$ such that $N|^{\leftarrow}_{(S(i-),S(i-)+\mathrm{len}(N_i))} = N_i$. If the condition does not hold we take $N = \emptyset$. When $I = \{1,2,\ldots,n\}$ is a finite set we write $N_1 \star \cdots \star N_n$ for $\bigstar_{i\in I} N_i$.

We define a function to stick spindles together: for $f_1$, $f_2$ excursions we define $f_{1\to 2}$ by

$$f_{1\to 2}(y) := \begin{cases} f_1(\zeta(f_1) - y) & \text{for} \quad 0 < y < \zeta(f_1) \\ f_2(y - \zeta(f_1)) & \text{for} \quad \zeta(f_1) \le y \le \zeta(f_1) + \zeta(f_2) \\ 0 & \text{otherwise.} \end{cases}$$

Define the map for paths $f_1$, $f_2$ and for $N_1, N_2 \in \mathscr{N}^{\mathrm{sp}}$

$$H(f_1, f_2, N_1, N_2) := \mathscr{R}_{\mathrm{cld}}(N_1|_{[0,T^{-\zeta(f_1)})}) \star \delta_{(0,f_{1\to 2})} \star N_2|_{[0,T^{-\zeta(f_2)})},$$

and define the measure $\mathbf{M}$ by

$$\mathbf{M} := H(\check{f}, \hat{f}, \check{\mathbf{N}}, \hat{\mathbf{N}}),$$

and let $\nu_x$ be the distribution of $\mathbf{M}$ on $\mathscr{N}^{\mathrm{sp}}_{\mathrm{cld}}$. We note from the independence in the construction that for $x > 0$ and $G_1 \subseteq \mathscr{N}^{\mathrm{sp}}_{+\mathrm{cld}}$, $G_2 \subseteq \mathscr{N}^{\mathrm{sp}}_{-\mathrm{cld}}$ that

$$\nu_x(N^+ \in G_1, N^- \in G_2) = \nu_x(N^+ \in G_1)\nu_x(N^- \in G_2).$$

Furthermore, let $\hat{f}$ be a 0-diffusion started at $x$, independent of $\mathbf{N}$. Let $\hat{T}^0 := \inf\{t > 0 : \xi_{\mathbf{N}}(t) = -\zeta(\hat{f})\}$. Then

$$\overline{N}^+_x := \delta(0, \hat{f}) + \mathbf{N}|_{[0,\hat{T}^0)}$$

has the law $\nu_x(N^+ \in \cdot)$, and $\mathscr{R}_{\mathrm{cld}}(\overline{N}^+_x)$ has the law $\nu_x(N^- \in \cdot)$.

**Proposition 2.3.10.** *Consider a diffusion that satisfies A*1 *and A*2 *with Pitman-Yor excursion measure* $\nu$ *that satisfies* (2.9)*. Then* $\nu_x$*,* $x \in E$*, is a disintegration of* $\nu_{cld}$ *with respect to* $m^0$*.*

*Proof.* This proof concerns kernels as defined in Kallenberg [52, Chapter I.3]. Let $\mu_x$ be the distribution of $(\check{f}, \hat{f}, \check{\mathbf{N}}, \hat{\mathbf{N}})$. We first show that $x \mapsto \mu_x$ is a kernel. First note that $x \mapsto \mathbf{P}(\mathbf{N} \in \cdot)$ is a kernel and we also have that $x \mapsto \mathbb{Q}^0_x$ is a kernel. The first follows as the image is independent of $x$ and the second follows as the measure $\mathbb{Q}^0_x$ is continuous in the weak sense and is therefore measurable. It therefore follows that $x \mapsto \mu_x$ is a kernel as we have that $\mu_x = \mathbb{Q}^0_x \otimes \mathbb{Q}^0_x \otimes \mathbf{P}(\mathbf{N} \in \cdot) \otimes \mathbf{P}(\mathbf{N} \in \cdot)$, see [52, Lemma I.3.3]. The result then follows as $H$ is a measurable map and $\nu_x$ is $\mu_x(H^{-1}(\cdot))$, and therefore the map $x \mapsto \nu_x$ is measurable and therefore is a kernel, see [52, Lemma I.3.1]. Therefore the map on $\mathbb{R}_+ \to \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}$ given by $x \mapsto \nu_x$ is a kernel because the map $H$ from $(\check{f}, \hat{f}, \check{\mathbf{N}}, \hat{\mathbf{N}})$ to $\mathbf{M}$ is measurable.

We now verify that $(\nu_x, x > 0)$ do integrate to the bi-clade measure $\nu_{\mathrm{cld}}$. Let $A$, $B \subset \mathscr{E}$ measurable and let $C$, $D \subset \mathscr{N}^{\mathrm{sp}}_{\mathrm{fin}}$ measurable. By definition of $\nu_{\mathrm{cld}}$ we have that $\nu_{\mathrm{cld}}(H(A \times B \times C \times D) \cap \{m^0 > 1/n\})$ equals

$$\begin{aligned}
&\nu_{\mathrm{cld}}(\hat{f}_{T^+_0} \in A, \check{f}_{T^+_0} \in B, N^-|_{[0,\mathrm{len}(N^-))} \in C, N^+|_{[0,\mathrm{len}(N^+))} \in D, m^0 > 1/n) \\
&\quad = k_0 M(E \backslash [0, 1/n]) \mathbf{P}(\hat{f}_{T^0_{n,1}} \in A, \check{f}_{T^0_{n,1}} \in B, \mathbf{N}|_{[\tau^0(\ell^0(T^0_{n,1})-), T^0_{n,1})} \in C, \mathbf{N}|^{\leftarrow}_{(T^0_{n,1}, \tau^0(\ell^0(T^0_{n,1})))} \in D).
\end{aligned}$$

Let $B' = \mathscr{R}_{\mathrm{spdl}}(B)$ and $C' = \mathscr{R}_{\mathrm{cld}}(C)$. Then with the result from Lemma 2.3.9 along with the crossing width intensity measure from Proposition 2.3.6 this is equal to

$$\int_{x \in E \backslash [0,1/n]} \int_{f \in B'} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f)})} \in C') \mathbb{Q}^0_x(df) \int_{f' \in A} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f')})} \in D) \mathbb{Q}^0_x(df') k_0 M(dx).$$

We take the limit as $n$ tends to infinity to obtain

$$\begin{aligned}
&\nu_{\mathrm{cld}}(H(A \times B \times C \times D)) = \nu_{\mathrm{cld}}(\hat{f}_{T^+_0} \in A, \check{f}_{T^+_0} \in B, N^-|_{[0,\mathrm{len}(N^-))} \in C, N^+|_{[0,\mathrm{len}(N^+))} \in D) \\
&= \int_{x \in E \backslash \{0\}} \int_{f \in B'} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f)})} \in C') \mathbb{Q}^0_x(df) \int_{f' \in A} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f'))})} \in D) \mathbb{Q}^0_x(df') 2k_0 M(dx).
\end{aligned}$$

By the definition of $\nu_x$ and $\mu_x$ this is equal to

$$\int_{x\in E\setminus\{0\}} \mu_x(A\times B\times C\times D)\nu_{\mathrm{cld}}(m^0\in dx) = \int_{x\in E\setminus\{0\}} \nu_x(H(A\times B\times C\times D))\nu_{\mathrm{cld}}(m^0\in dx).$$

The generality of $A, B\in\mathscr{E}$, and $C, D\in\mathscr{N}_{\mathrm{fin}}^{\mathrm{sp}}$ is sufficient here to prove the proposition by the monotone class theorem. ■

We also note from the proof of the proposition above that $\mu_x$, $x\in E\setminus\{0\}$, is in fact a disintegration of $\nu_{\mathrm{cld}}(\check{f}_{T_0^+}\in\cdot, \hat{f}_{T_0^+}\in\cdot, N^-|_{[0,\mathrm{len}(N^-))}\in\cdot, N^+|_{[0,\mathrm{len}(N^+))}\in\cdot)$.

### 2.3.4 Lévy process scaffolding from fixed initial interval partition

In this section we give the construction of scaffolding-and-spindles pairs started from certain interval partitions $\beta\in\mathscr{I}_H$, which we denote as $(\mathbf{N}_\beta, \mathbf{X}_\beta)$. We give the construction of $\mathbf{N}_\beta$ in Definition 2.3.11 and we prove that this construction is possible under certain conditions in Proposition 2.3.12. We define random measures $\mathbf{N}_\beta$ on the space $\mathscr{N}_{\mathrm{fin}}^{\mathrm{sp}}$.

**Definition 2.3.11.** Let $\mathscr{I}_H(E) := \{\gamma\in\mathscr{I}_H : \mathrm{Leb}(U)\in E\ \forall U\in\gamma\}$ and let $\beta\in\mathscr{I}_H(E)$. If $\beta=\varnothing$ then define $\mathbf{N}_\beta$ to be the zero measure. Otherwise, for each interval $U$ of $\beta$ we construct each $\mathbf{N}_U$ independently in the following way. First let $f_U$ have law $\mathbb{Q}^0_{\mathrm{Leb}(U)}$ and let $\mathbf{N}$ be a Poisson random measure with intensity $\mathrm{Leb}\otimes\nu$. Let $T := \inf\{t>0 : \xi_{\mathbf{N}}(t) = -\zeta(f_U)\}$ and let $\mathbf{N}_U := \delta(0,f_U) + \mathbf{N}|_{[0,T]}$.

Under the condition that $\sum_{U\in\beta}\mathrm{len}(\mathbf{N}_U)$ is finite, let $\mathbf{N}_\beta := \bigstar_{U\in\beta}\mathbf{N}_U$ (and otherwise take $\mathbf{N}_\beta$ to be the zero measure). Let $\mathbf{X}_\beta := \xi(\mathbf{N}_\beta)$ be the corresponding scaffolding constructed from $\mathbf{N}_\beta$. We write $\mathbf{P}_\beta$ to denote the law of $\mathbf{N}_\beta$ on $\mathscr{N}_{\mathrm{fin}}^{\mathrm{sp}}$.

In all the results that follow we consider intervals partitions $\beta\in\mathscr{I}_H(E)$. We give sufficient conditions on an interval partition $\beta$ for concatenating clades $(\mathbf{N}_U, U\in\beta)\subset\mathscr{N}_{+\mathrm{cld}}^{\mathrm{sp}}$ as in Definition 2.3.11.

**Proposition 2.3.12.** *Consider a diffusion that satisfies A1, A2 with a Pitman-Yor excursion measure $\nu$ that satisfies (2.9). Further suppose that*

$$\limsup_{x\downarrow 0}\frac{1}{x}\Big(\int_{v\in E} s(v\wedge x)M(dv)\Big) =: k_3 < \infty. \tag{2.40}$$

*Then, in the setting of Definition 2.3.11, for any interval partition $\beta\in\mathscr{I}_H(E)$, we have that $\sum_{U\in\beta} len(\mathbf{N}_U)$ is finite a.s. and hence $(\mathbf{N}_\beta, \mathbf{X}_\beta)$ is a non-trival scaffolding-and-spindles pair (unless $\beta=\emptyset$).*

*Proof.* We calculate the expected lifetime of an interval with initial width $x$, for $x \in E$, and we define the function $r(x) \coloneqq \mathbb{E}_{\mathbb{Q}^0_x}(\zeta)$ to be this expectation. We first assume that $s(c)$ is finite and we calculate $\mathbb{E}_{\mathbb{Q}^0_x}(T_0 \wedge T_c)$. For $0 < a < x < w < c$ we use $\mathbb{E}_{\mathbb{Q}^0_x}(T_a \wedge T_w)$ from (2.4) and we can take the monotonic limits of $a \downarrow 0$ and $w \uparrow c$ to find

$$\mathbb{E}_{\mathbb{Q}_x}(T_0 \wedge T_c) = \int_{v \in E} \Big( \frac{s(x)}{s(c)} \Big( s(c) - s(v) \Big) \mathbb{1}_{x<v} + \frac{s(v)}{s(c)} \Big( s(c) - s(x) \Big) \mathbb{1}_{v \leq x} \Big) M(dv).$$

To account for the path from $c$ to 0 on the event that $T_c < T_0$ we have

$$\mathbb{E}_{\mathbb{Q}_x} \Big( (T_0 - T_c) \mathbb{1}_{T_c < T_0} \Big) = \mathbb{Q}_x(T_c < T_0) \mathbb{E}_{\mathbb{Q}_c}(T_0).$$

Firstly we have $\mathbb{Q}_x(T_c < T_0) = s(x)/s(c)$ (see (2.2)). The second term can be calculated from Lemma 2.2.9 to give

$$\mathbb{E}_{\mathbb{Q}_x} \Big( (T_0 - T_c) \mathbb{1}_{T_c < T_0} \Big) = \frac{s(x)}{s(c)} \int_{v \in E} s(v) M(dv).$$

We add the two contributions to obtain

$$r(x) = \int_{v \in E} s(v \wedge x) M(dv) \tag{2.41}$$

In the case where $s(c)$ is infinite we use $\mathbb{E}_{\mathbb{Q}^0_x}(T_a \wedge T_w)$ from (2.4) and we can take the monotonic limits of $a \downarrow 0$ and $w \uparrow c$ and this equals $\mathbb{E}_{\mathbb{Q}^0_x}(T_0)$ and this limit is as in (2.41).

Under the assumption given in (2.40), there exists $\varepsilon' > 0$ such that for all $x < \varepsilon'$ we have that $r(x) \leq (1 + k_3)x$. Therefore, for an interval partition $\beta \in \mathscr{I}_H(E)$ we have

$$\mathbb{E}\Bigg( \sum_{\substack{U \in \beta \\ \mathrm{Leb}(U) < \varepsilon'}} \zeta(f_U) \Bigg) = \sum_{\substack{U \in \beta \\ \mathrm{Leb}(U) < \varepsilon'}} r(\mathrm{Leb}(U)) \leq \sum_{\substack{U \in \beta \\ \mathrm{Leb}(U) < \varepsilon'}} (1 + k_3) \mathrm{Leb}(U) = (1 + k_3) \|\beta\| < \infty. \tag{2.42}$$

Therefore we have that $\sum_{U \in \beta \mathrm{Leb}(U) < \varepsilon'} \zeta(f_U) < \infty$ a.s. and as there are only finitely many intervals $U$ of the interval partition $\beta$ that have length greater than $\varepsilon'$ we have that $\sum_{U \in \beta} \zeta(f_U) < \infty$ a.s. From here the result follows; to see this consider a scaffolding-and-spindles pair $(\mathbf{N}, \mathbf{X})$ stopped at $T_{-\sum_{U \in \beta} \zeta(f_U)}$, the hitting time of $-\sum_{U \in \beta} \zeta(f_U)$ by $\mathbf{X}$. Let $\preceq$ denote the total order of the intervals in the interval partition $\beta$ and, for $U \in \beta$, define $S_U, W_U$ by

$$S_U \coloneqq T_{-\sum_{V \in \beta : V \prec U} \zeta(f_V)}; \quad W_U = T_{-\sum_{V \in \beta : V \preceq U} \zeta(f_V)}.$$

Finally let $\mathbf{N}_U \coloneqq \delta_{(0, f_U)} + \mathbf{N}|^{\leftarrow}_{[S_U, W_U]}$ and $\mathbf{N}_\beta = \bigstar_{U \in \beta} \mathbf{N}_U$. Then $\sum_{U \in \beta} \mathrm{len}(\mathbf{N}_U)$ is equal to $T_{-\sum_{U \in \beta} \zeta(f_U)}$ which is finite a.s. and by the Strong Markov property for the Poisson random measure $\mathbf{N}$ and we have that $\mathbf{N}_\beta$ has the distribution given in Definition 2.3.11. ∎

We show in Corollary 2.3.13 that (2.40) is satisfied for a large class of diffusions and in Remark 2.3.14 we highlight a class of diffusions that satisfy $A1$, $A2$ and (2.9) but that do not satisfy (2.40).

**Corollary 2.3.13.** *Consider a diffusion that satisfies $A1$, $A2$, $A3$ with Pitman-Yor excursion measure that satisfies* (2.9) *and suppose that there exists $k_4 < 0$ such that*

$$\limsup_{y \downarrow 0} \mu(y) = k_4 < 0.$$

*Then the diffusion satisfies* (2.40) *and therefore for any interval partition $\beta \in \mathscr{I}_H(E)$ the construction of $(\mathbf{N}_\beta, \mathbf{X}_\beta)$ as in Definition 2.3.11 exists and len$(\mathbf{N}_\beta)$ is finite a.s. Furthermore we have that the function r, defined as $r(x) = \mathbb{E}_{\mathbb{Q}_x^0}(\zeta)$ in the proof of Proposition 2.3.12, has a continuous derivative $r'$ on $(0,c)$ and $\limsup_{x\to 0} r'(x) < \infty$.*

*Proof.* As our diffusion satisfies $A3$ we can use the equations for the scale function derivative and speed measure density from (2.11). This means that we can calculate $r'(x)$ and note that the condition in (2.40) is satisfied if $\limsup_{x\to 0} r'(x)$ is finite because this is greater than or equal to $\limsup_{x\to 0} r(x)/x$. We obtain

$$r'(x) = s'(x) \int_{v=x}^{c} m(v)dv$$

and this function is continuous on $(0,c)$. By the condition given in the corollary, there exists $\varepsilon > 0$ such that for $y < \varepsilon$ we have $\mu(y) < k_4/2$. Now, for $x < \varepsilon$, we have that

$$s'(x) \int_{v=x}^{\varepsilon} m(v)dv = \int_{v=x}^{\varepsilon} \frac{2}{\sigma^2(v)} \exp\Big( \int_{z=x}^{v} \frac{2\mu(z)}{\sigma^2(z)} dz \Big) dv \leq \int_{v=x}^{\varepsilon} \frac{2}{\sigma^2(v)} \exp\Big( \int_{z=x}^{v} \frac{k_4}{\sigma^2(z)} dz \Big) dv$$

and therefore

$$\begin{aligned} \limsup_{x\downarrow 0} s'(x) \int_{v=x}^{\varepsilon} m(v)dv &\leq \limsup_{x\downarrow 0} \int_{v=x}^{\varepsilon} \frac{2}{\sigma^2(v)} \exp\Big( \int_{z=x}^{v} \frac{k_4}{\sigma^2(z)} dz \Big) dv \\ &= \limsup_{x\downarrow 0} \frac{-2}{k_4} \Big( 1 - \exp\Big( \int_{z=x}^{\varepsilon} \frac{k_4}{\sigma^2(z)} dz \Big) \Big) \leq \frac{-2}{k_4} < \infty. \end{aligned} \tag{2.43}$$

It remains to show that $\limsup_{x\downarrow 0} s'(x) \int_{v=\varepsilon}^{c} m(v)dv$ is finite. We have that $\int_{v=\varepsilon}^{c} m(v)dv$ is finite by $A2$. Also as $\mu$ is negative locally at 0 we have that the limit $\lim_{x\downarrow 0} s'(x)$ exists and equals

$$s'(0) := \exp\left( \int_{z=0}^{b} \frac{2\mu(z)}{\sigma^2(z)} dz \right)$$

where $b \in (0,c)$ is from (2.11) as the function $s'(x)$ is monotonic decreasing for $x \in (0,\varepsilon)$. Therefore we have an upper bound for $\limsup_{x\to 0} r(x)/x$ and for $\limsup_{x\to 0} r'(x)$ of $-2/k_4 + s'(0) \int_{v=\varepsilon}^{c} m(v)dv$ and from this we see that the diffusion satisfies (2.40) by Proposition 2.3.12 and therefore a construction of $(\mathbf{N}_\beta, \mathbf{X}_\beta)$ exists. ∎

We note that the untransformed diffusions under Assumption B satisfy the condition in Corollary 2.3.13 and so satisfy (2.40). This does not extend to all diffusions under Assumption B as (2.40) is not invariant under spatial transformations $g$ (unlike our previous conditions with the speed measure and scale function in (2.7) and (2.8)).

*Remark* 2.3.14. The untransformed diffusions in Example 2.1.12 satisfy $A1$, $A2$ and (2.9) but do not satisfy (2.40): using (2.14) for the density $m$ of the speed measure $M$ and for the scale function $s$ we have, for $x \in (0, \varepsilon_2)$,

$$\begin{aligned}\frac{1}{x}\Big(\int_{v\in E} s(v\wedge x)m(v)dv\Big) &= \frac{1}{x}\Big(\int_{v=0}^{x}\frac{1}{2}dv + \int_{v=x}^{\varepsilon_2} x\frac{1}{2v}dv + xM((\varepsilon_2,c])\Big)\\ &= \frac{-\log(x)+1+\log(\varepsilon_2)}{2} + M((\varepsilon_2,c])\end{aligned}$$

and this goes to infinity as $x$ goes to zero.

The purpose of Proposition 2.3.12 and Corollary 2.3.13 is to give a framework from which we can develop a simple Markov property for the IP-evolutions for Theorem 2.1.6. For the remainder of propositions and lemmas in this section we assume (2.40) and so that for any interval partition $\beta \in \mathscr{I}_H(E)$ we have a non-trivial scaffolding-and-spindles pair $(\mathbf{N}_\beta, \mathbf{X}_\beta)$. We now define several functions that will allow us to construct suitable filtrations in space and in time, and then prove the simple Markov property in the setting of the Poisson random measure $\mathbf{N}$ on the probability space $(\Omega, \mathscr{A}, \mathbf{P})$. We omit the exact proofs, rather, we give references to results from [34] and explain how they generalise to our setting. First we define $M$-diversity which is crucial for the arguments for the Markov property.

**Definition 2.3.15.** Let $\beta$ be an interval partition and $t \geq 0$. Then we define the $M$-diversity of $\beta$ at $t$ as the following limit when it exists:

$$\mathscr{D}^M_\beta(t) := \lim_{x\to 0}\frac{1}{M(E\backslash[0,x])}\Big|\Big\{(a,b)\in\beta : |b-a| > x, b \leq t\Big\}\Big| \in [0,\infty), \tag{2.44}$$

and if the limit does not exist then we say that $\beta$ at $t$ does not have an $M$-diversity. Furthermore, for an interval $U = (a,b) \in \beta$ we define the limit $\mathscr{D}^M_\beta(U) := \mathscr{D}^M_\beta(b)$, when this limit exists. Finally, we also define the limsup-$M$-diversity by

$$\overline{\mathscr{D}}^M_\beta(t) := \limsup_{x\to 0}\frac{1}{M(E\backslash[0,x])}\Big|\Big\{(a,b)\in\beta : |b-a| > x, b \leq t\Big\}\Big| \in [0,\infty], \tag{2.45}$$

and the limit always exists for any interval partition $\beta$, and for an interval $U \in \beta$ we define $\overline{\mathscr{D}}^M_\beta(U) := \overline{\mathscr{D}}^M_\beta(b)$. Let $\mathscr{I}_M \subset \overline{\mathscr{I}}_M$ be

$$\mathscr{I}_M := \{\beta \in \mathscr{I}_H : \forall t > 0, \mathscr{D}^M_\beta(t) \text{ exists}\}; \quad \overline{\mathscr{I}}_M := \{\beta \in \mathscr{I}_H : \forall t > 0, \overline{\mathscr{D}}^M_\beta(t) < \infty\}, \tag{2.46}$$

and let $\mathscr{I}_M(E) = \mathscr{I}_H(E) \cap \mathscr{I}_M$ for a measurable subset $E \subseteq [0,\infty)$ and similarly for $\overline{\mathscr{I}}_M(E)$. We substitute the $M$ for $m$ and write $\mathscr{D}^m_\beta$ for $\mathscr{D}^M_\beta$ and similarly for the other terms when the speed measure $M$ has a density $m$, and we call replace the $M$- for $m$- in their names.

The $M$-diversity defined here is a generalisation of the notion of $\alpha$-diversity used in [34]. In their work the $\alpha$-diversity features in a metric $d_\alpha$ that they consider. While we only use the metric $d'_H$ in this paper (a metric that does not have diversity in its definition) we use the $M$-diversity of interval partitions to determine the Markov property of the IP-evolutions. Specifically, the $M$-diversity is used as an index label to order the components $\mathbf{N}_U$ of $\mathbf{N}_\beta$ to define another random measure in (2.50), and we use this random measure to show a conditional independence result in Proposition 2.3.19.

**Lemma 2.3.16.** *Consider a diffusion that satisfies A*1 *and A*2 *of Assumption A with speed measure and scale function that satisfies* (2.40)*, and for which the Pitman-Yor excursion measure* $\nu$ *satisfies* (2.9)*. Let* $(\mathbf{N},\mathbf{X})$ *be a stopped scaffolding-and-spindles pair for this diffusion with an a.s. finite T, defined on* $(\Omega,\mathscr{A},\mathbf{P})$*. Then, for* $y\in\mathbb{R}$ *and* $t>0$*,* $\beta^y := skewer(y,\mathbf{N},\mathbf{X})$ *has an M-diversity at t a.s. We further have for all intervals* $U\in\beta^y$ *simultaneously that* $\mathscr{D}^M_{\beta^y}(U)=k_0\ell^y(U)$ *a.s.*

*Proof.* As $\nu$ satisfies (2.9) it does not satisfy (2.6) and so, by the first part of Theorem 2.1.3, we have that $M(E)=\infty$. We first assume that $T=\tau^y(a)$ for some $a>0$ where $\tau^y$ is the inverse local time at level $y$, see Section 2.2.1, and prove the case for general $T$ after. Let $x\in E\backslash\{0\}$. By considering the Poisson random measure given in Proposition 2.3.6 we see that, for $u\le a$,

$$\sum_{(t,N') \text{ atom of } F^0(\mathbf{N}|^{\leftarrow}_{[T_y,\tau^y(u)]})} \mathbb{1}_{\{m^0(N')>x\}} \tag{2.47}$$

is a Poisson random variable with mean $k_0uM(E\backslash[0,x])$ (where $T_y$ is the first hitting time of level $y$, see Definition 2.2.2, and $F^0$ is defined in (2.31)) as the sum is the number of atoms in an interval of the Poisson random measure over the interval $E\backslash[0,x]$. This sum is finite as the mean of the Poisson random variable is finite by $A2$ of Assumption A. Furthermore, for $0<x'<x$, we have

$$\sum_{(t,N') \text{ atom of } F^0(\mathbf{N}|^{\leftarrow}_{[T_y,\tau^y(u)]})} \mathbb{1}_{\{x'<m^0(N')\le x\}}$$

is a Poisson random variable with finite mean $k_0uM((x',x])$ independent of (2.47) as this sum is the number of atoms over the interval $(x',x]$ of the Poisson random measure in Proposition 2.3.6. Finally

$$\sum_{(t,N') \text{ atom of } F^0(\mathbf{N}|^{\leftarrow}_{[0,T_y]})} \mathbb{1}_{\{m^0(N')>0\}}$$

is at most 1 (and so will not impact the diversity limit). As we similarly have independence for the Poisson random variables over disjoint intervals $(x',x]$, and as we have already noted

$M(E) = \infty$, then by the strong law of large numbers $\mathscr{D}^M_{\beta^y}(\tau^y(u))$ exists and equals $k_0 u$ for all $0 \le u \le a$. Now, as for all $0 < u \le a$ we have

$$\sum_{(t,N') \text{ atom of } F^0(\mathbf{N}|^{\leftarrow}_{[\tau^y(u-),\tau^y(u)]})} \mathbb{1}_{\{m^0(N')>0\}}$$

is at most one a.s. we have that $\mathscr{D}^M_{\beta^y}(t)$ exists for all $t \le \tau^y(a)$ where $\mathscr{D}^M_{\beta^y}(t) = \mathscr{D}^M_{\beta^y}(\tau^y(\ell^y(t)))$ which is equal to $k_0 \ell^y(t)$ a.s.

Finally, we note for the case of general finite $T$ that $\mathbf{P}(\bigcup_{a>0}\{T < \tau^y(a)\}) = 1$ and so the result applies for general $T$. ■

**Lemma 2.3.17.** *Consider a diffusion that satisfies A*1 *and A*2 *of Assumption A with speed measure and scale function that satisfies* (2.40)*, and for which the Pitman-Yor excursion measure* $\nu$ *satisfies* (2.9)*. Let* $\beta \in \mathscr{I}_H(E)$ *and let* $(\mathbf{N}_\beta, \mathbf{X}_\beta)$ *be a scaffolding-and-spindles pair started from* $\beta$ *for this diffusion as in Definition 2.3.11.*

*Then for* $y > 0$ *fixed, the local times of the excursions about level y are distinct a.s. in the sense that for* $[a,b]$, $[c,d] \in V^y(\mathbf{N}_\beta)$, $\ell^y(a) \ne \ell^y(c)$ *unless* $[a,b] = [c,d]$*, where* $\ell$ *denotes the local time of* $\xi(\mathbf{N}_\beta)$ *and where* $V^y$ *is defined in Definition 2.3.5.*

*Furthermore, the crossing widths are summable, and the M-diversities exist for all* $U \in \beta^y :=$ *skewer*$(y, \mathbf{N}_\beta, \mathbf{X}_\beta)$ *and are distinct in the sense that for* $U \ne V \in \beta^y$ *we have that* $\mathscr{D}^M_{\beta^y}(U) \ne \mathscr{D}^M_{\beta^y}(V)$.

*Proof.* First note that $\xi_{\mathbf{N}_U}$ is a Lévy process of unbounded variation started from a random level $\zeta(f_U)$, where $f_U$ is the incomplete spindle with initial value $\mathrm{Leb}(U)$, so that $\xi_{\mathbf{N}_U}(0) \sim \mathbb{Q}^0_{\mathrm{Leb}(U)}(\zeta \in \cdot)$. For $y > 0$ at most finitely many of the $\xi_{\mathbf{N}_U}$ from $\mathbf{N}_\beta$ intersect level $y$. To see this, recall from the proof of Proposition 2.3.12 where we note that the process $\mathbf{X}_\beta$ with the jumps for the incomplete spindles $(\zeta(f_U), U \in \beta)$ removed produces a Lévy process $\mathbf{X}$ started at 0 and stopped at hitting time of $-\sum_{U\in\beta}\zeta(f_U)$. For every $\xi_{\mathbf{N}_U}$ that intersects level $y$ we either have that $\zeta(f_U)$ has length of at least $y/2$ or we have an upcrossing of $\mathbf{X}$ of size $y/2$. As there are finitely many such $\zeta(f_U)$ and finitely many such upcrossings we have our assertion. From this, the local times claim follows from Millar [74]; see Proposition 2.5.4 in the Appendix for a full description.

The summability of the crossing widths at level $y$ follows from Theorem 2.1.4 along with the fact that only finitely many of the $\mathbf{N}_U$ intersect level $y$. The $M$-diversities claim follows from the previous two results of this proposition along with Lemma 2.3.16. ■

We now define cutoff functions which are used to define the filtrations from [34]. The idea is to have a filtration indexed by level $y \in \mathbb{R}$ that can be used to define a Markov property for the skewer process.

$$\sigma^y_N(t) := \mathrm{Leb}\{u \le t : \xi_N(u) \le y\},$$

$$\begin{aligned}\text{cutoff}^{\leq y}_{\xi(N)}(s) &:= \xi_N(\sup\{t \geq 0 : \sigma^y_N(t) \leq s\}) - \min\{y,0\},\\ \text{cutoff}^{\geq y}_{\xi(N)}(s) &:= \xi_N(\sup\{t \geq 0 : t - \sigma^y_N(t) \leq s\}) - \max\{y,0\},\end{aligned} \tag{2.48}$$

$$\begin{aligned}\text{cutoff}^{\leq y}_N &:= \sum_{(t,f)\in N} \mathbb{1}_{\xi_N(t-)<y<\xi_N(t)}\delta(\sigma^y_N(t), \check{f}^y_t) + \mathbb{1}_{\xi_N(t)\leq y}\delta(\sigma^y_N(t), f_t),\\ \text{cutoff}^{\geq y}_N &:= \sum_{(t,f)\in N} \mathbb{1}_{\xi_N(t-)<y<\xi_N(t)}\delta(t-\sigma^y_N(t), \check{f}^y_t) + \mathbb{1}_{\xi_N(t-)\geq y}\delta(t-\sigma^y_N(t), f_t),\end{aligned} \tag{2.49}$$

**Definition 2.3.18** (Filtrations)**.** We define a filtration indexed by level $y \geq 0$, $(\overline{\mathscr{F}^y}, y \geq 0)$, and a filtration, indexed by time, $(\overline{\mathscr{F}_t}, t \geq 0)$.

1. We define $(\mathscr{F}^y, y \geq 0)$ to be the least right-continuous filtration under which the maps $N \mapsto \text{cutoff}^{\leq y}_{\xi(N)}$ and $N \mapsto \text{cutoff}^{\leq y}_N$ are $\mathscr{F}^y$-measurable for $y \geq 0$.
2. We define $(\mathscr{F}_t, t \geq 0)$ to be the least right-continuous filtration on $\mathscr{N}^{\text{sp}}$ under which $N \mapsto N|_{[0,t]}$ is $\mathscr{F}_t$-measurable for $t \geq 0$.

We define $(\overline{\mathscr{F}}_t, t \geq 0)$ and $(\overline{\mathscr{F}}^y, y \geq 0)$ to be the $\mathbf{P}$-completions of $(\mathscr{F}_t, t \geq 0)$ and $(\mathscr{F}^y, y \geq 0)$ respectively on $\mathscr{N}^{\text{sp}}_{\text{fin}}$, where $(\Omega, \mathscr{A}, \mathbf{P})$ is the probability space on which the PRM $\mathbf{N}$ is defined.

For $\beta \in \mathscr{I}_H(E)$ recall $\mathbf{N}_\beta$, the $\mathbf{N}_U$ for $U \in \beta$ and the $\mathbf{P}_\beta$ probability measure from Definition 2.3.11. We define the collection of excursion intervals about a fixed level $y \geq 0$ for $N \in \mathscr{N}^{\text{sp}}$ by

$$F^{\geq y}_0(N) = \sum_{[a,b]\in V^y_0(N)} \delta\Big(\ell^y_N(a), (N|^{\leftarrow}_{[a,b]})^+\Big).$$

By a slight abuse of notation with the above definition for $\beta \in \mathscr{I}_M(E)$ we define for a measure $\mathbf{N}_\beta$ as in Definition 2.3.11

$$F^{\geq 0}_0(\mathbf{N}_\beta) := \sum_{U\in\beta} \delta(k_0^{-1}\mathscr{D}^M_\beta(U), \mathbf{N}_U), \tag{2.50}$$

and we set $F^{\geq 0}_0(\mathbf{N}_\beta)$ to be the zero measure when $\beta \in \mathscr{I}_H(E)\backslash\mathscr{I}_M(E)$. We state results from [34, Section 5] that generalise to our setting.

**Proposition 2.3.19.** *Consider a diffusion that satisfies A*1 *and A*2 *of Assumption A with speed measure and scale function that satisfies* (2.40)*, and for which the Pitman-Yor excursion measure* $\nu$ *satisfies* (2.9)*. Let* $(\mathbf{N}, \mathbf{X})$ *be a scaffolding-and-spindles pair where* $\mathbf{N}$ *has intensity Leb* $\otimes \nu$ *stopped at* $T$*, an a.s. finite time* $T$ *with the following properties:*

1. $S^0 := \ell^0(T)$ *is measurable in* $\overline{\mathscr{F}}^0$*;*
2. $\mathbf{X} < 0$ *on the time interval* $(\tau^0(S^0-), T)$*.*

*Then for each* $y \geq 0$*, the measure* $F^{\geq y}_0(\mathbf{N})$ *is conditionally independent of* $\overline{\mathscr{F}}^y$ *given* $\beta^y :=$ *skewer*$(\mathbf{N}, \mathbf{X}, y)$*, with the regular conditional distribution* $\mathbf{P}_{\beta^y}(F^{\geq 0}_0 \in \cdot)$*.*

*Proof.* The proof of [34, Proposition 5.6] applies in our setting as we have the following:

1. The scaffolding $\mathbf{X}$ is a Lévy process of unbounded variation and therefore we can decompose $\mathbf{N}$ into biclades at every fixed level $y$ of $\mathbf{X}$: $(\mathbf{N}|_I^{\leftarrow}, I \in V_0^y(\mathbf{N}))$.

2. We have a mid-spindle Markov property (Lemma 2.3.8).

3. We have a disintegration for the bi-clade excursion measure $\nu_{\text{cld}}$ with respect to the crossing width $m^0$ (Proposition 2.3.10).

4. Local times of the excursions at a fixed positive level $y > 0$ are equal to the $M$-diversities of the interval partitions a.s. (Lemma 2.3.16). These are measurable functions of the skewer function at that level.

■

This result is sufficient to prove the simple Markov property with respect to the filtration generated by the skewer process. For a probability measure $\mu$ on the space of interval partitions $\mathscr{I}_H(E)$, we define the probability measure $\mathbf{P}_\mu$ on the space $\mathscr{N}_{\text{fin}}^{\text{sp}}$ by $\mathbf{P}_\mu(\cdot) := \int \mathbf{P}_\beta(\cdot)\mu(d\beta)$.

**Proposition 2.3.20** (Simple Markov property of the skewer process under $\mathbf{P}_\mu$)**.** *Consider a diffusion that satisfies A*1 *and A*2 *of Assumption A and either with speed measure and scale function that satisfies* (2.40) *and for which the Pitman-Yor excursion measure* $\nu$ *satisfies* (2.9)*, or with Pitman-Yor excursion measure that satisfies* (2.6)*. Let* $\mu$ *be a probability measure on* $\mathscr{I}_H(E)$*. Let* $z > 0$ *and* $0 \le y_1 < \cdots < y_n$*. Let* $\eta : \mathscr{N}_{fin}^{sp} \to [0,\infty)$ *be* $\overline{\mathscr{F}}^z$ *measurable, and let* $f : (\mathscr{I}_H(E))^n \to [0,\infty)$ *be measurable. Then*

$$\mathbf{P}_\mu[\eta f(skewer(z+y_j,\cdot,\xi(\cdot)), j \in [n])] = \int \eta(N)\mathbf{P}_{skewer(z,N)}[f(skewer(y_j,\cdot,\xi(\cdot)), j \in [n])]\mathbf{P}_\mu(dN).$$

*Proof.* We split the proof into two cases. If the Pitman-Yor excursion measure satisfies (2.6), the SPLP $\mathbf{X}$ is of bounded variation a.s. and therefore there are only finitely many crossings at level $y$ a.s. Therefore the simple Markov property follows in this case as the concatenation of the simple Markov properties for each incomplete spindle and its SPLP extension added to it.

For the other case, the proof follows from the previous results and it uses the same argument as given in [34, Corollary 6.7] with the following conditions:

1. For an interval partition $\beta \in \mathscr{I}_H(E)$, and fixed $y > 0$, the local times of the excursions of $\xi(\mathbf{N}_\beta)$ about level $y$ are distinct a.s. (Lemma 2.3.17).

2. We have conditional independence of $F_0^{\geq y}(\mathbf{N})$ and $\overline{\mathscr{F}}^y$ given skewer$(\mathbf{N}, \xi(\mathbf{N}), y)$, for certain stopping times $T$ (Proposition 2.3.19). In particular, for an interval partition $\beta \in \mathscr{I}_H(E)$, and for $U \in \beta$, it is shown in [34, Proposition 5.10] that $F_0^{\geq y}(\mathbf{N}_U)$ is conditionally independent of $\overline{\mathscr{F}}^y$ given skewer$(y, \mathbf{N}_U, \xi_{\mathbf{N}_U})$. As only finitely many of the $\mathbf{N}_U$, $U \in \beta$, will contribute to the skewer process at levels above $y$, the result follows.

■

## 2.4 Continuity and Markov property – Assumptions B and C

In this section we prove Theorems 2.1.6 and 2.1.7. The majority of the section is dedicated to the proof of Theorem 2.1.6 which is completed in Section 2.4.6. The proof of Theorem 2.1.7 is simpler and is completed at the end of Section 2.4.4.

### 2.4.1 Scale function and speed measure under Assumptions B' and C'

We now define a modification of Assumption B which we will use to prove intermediate results in this paper.

*Assumption* B'. A diffusion $\mathbf{Y}$ with state space $E \subseteq [0,\infty)$ of the form $E = [0,c]$ or $E = [0,c)$ for some $c \leq \infty$ satisfies *Assumption B'* if it satisfies $A1$, $A2$, $A3$, $B1$, $B2$, $B3$.

Note that this is Assumption B in the cases where the transformation $g$ is the identity map. We define Assumption C' in the same way for Assumption C.

*Assumption* C'. A diffusion $\mathbf{Y}$ with state space $E \subseteq [0,\infty)$ of the form $E = [0,c]$ or $E = [0,c)$ for some $c \leq \infty$ satisfies *Assumption C'* if it satisfies $A1$, $A2$, $A3$, $C1$, $C2$.

We use the definition of the derivative $s'$ of a scale function $s$ from (2.11) and we take $b$ in (2.11) to be $\varepsilon_0$ when under Assumption B' and to be $\varepsilon_1$ when under Assumption C'. We obtain upper and lower bounds for the speed measure and scale function from the corresponding quantities in the squared Bessel case given above. We have under Assumption B', $x \in (0, \varepsilon_0)$

$$\varepsilon_0^{-\alpha^+} x^{\alpha^+} = \exp\left(\int_{z=x}^{\varepsilon_0} \frac{-\alpha^+}{z} dz\right) \leq s'(x) = \exp\left(\int_{z=x}^{\varepsilon_0} \frac{\mu(z)}{2z} dz\right) \leq \varepsilon_0^{\alpha^-} x^{\alpha^-} \tag{2.51}$$

and under Assumption C', $x \in (0, \varepsilon_1)$

$$\varepsilon_1^{\beta^-} x^{-\beta^-} = \exp\left(\int_{z=x}^{\varepsilon_1} \frac{\beta^-}{z} dz\right) \leq s'(x) = \exp\left(\int_{z=x}^{\varepsilon_1} \frac{\mu(z)}{2z} dz\right) \leq \varepsilon_1^{\beta^+} x^{-\beta^+}. \tag{2.52}$$

For the density of the speed measure this results in under Assumption B', $x \in (0, \varepsilon_0)$

$$\frac{1}{2}\varepsilon_0^{\alpha^-} x^{-1-\alpha^-} \leq m(x) \leq \frac{1}{2}\varepsilon_0^{\alpha^+} x^{-1-\alpha^+} \tag{2.53}$$

and under Assumption C', $x \in (0, \varepsilon_1)$

$$\frac{1}{2}\varepsilon_1^{-\beta^+} x^{-1+\beta^+} \leq m(x) \leq \frac{1}{2}\varepsilon_1^{-\beta^-} x^{-1+\beta^-}. \tag{2.54}$$

For $s(x)$ we have under Assumption B'

$$\frac{\varepsilon_0^{-\alpha^+}}{1+\alpha^+} x^{1+\alpha^+} = \varepsilon_0^{-\alpha^+} \int_{y=0}^{x} y^{\alpha^+} dy \leq s(x) \leq \varepsilon_0^{-\alpha^-} \int_{y=0}^{x} y^{\alpha^-} dy = \frac{\varepsilon_0^{-\alpha^-}}{1+\alpha^-} x^{1+\alpha^-} \tag{2.55}$$

for $x \in (0, \varepsilon_0)$. It immediately follows that diffusions under Assumption C have scaffolding of bounded variation because 0 is a regular point for these diffusions. As for diffusions under Assumption B, if we further assume $A4$ then $B1$ is immediate and is not required as a condition. We include this in the result below for completeness.

**Proposition 2.4.1.** *Consider a diffusion* $\mathbf{Z}$ *that satisfies Assumption B and let* $(\mathbf{N}, \mathbf{X})$ *be a scaffolding-and-spindles pair for this diffusion which is stopped at some a.s. finite random time* $T$*. Then we have the aggregate health process* $M^0_{\mathbf{N},\mathbf{X}}(T)$ *is finite a.s.*

*Furthermore, if instead we have a diffusion that satisfies all of Assumption A and B*2*, B*3 *then the diffusion also satisfies B*1.

*Proof.* For the first result we consider an untransformed diffusion $\mathbf{Y}$ along with a transformation $g$ which gives rise to our diffusion $\mathbf{Z}$ and we note $\mathbf{Y}$ satisfies Assumption B' so we can use the bound in (2.53) for its speed measure $m_{\mathbf{Y}}$. Let $q \in (\alpha^+, q_0)$ and let $C_q$ be the Hölder constant for $g$ as in from Assumption $B4$. Then we have

$$\int_{y=0}^{\varepsilon_0} g(y) m_{\mathbf{Y}}(y) dy \leq \frac{1}{2} C_q \varepsilon_0^{\alpha^+} \int_{y=0}^{\varepsilon_0} y^{-1-\alpha^++q} dy = \frac{C_q \varepsilon_0^{\alpha^+}}{2(q-\alpha^+)} < \infty.$$

For the second result we begin by noting that on $(0, \varepsilon_0)$ the scale function for $\mathbf{Y}$, $s_{\mathbf{Y}}$, is such that

$$s'_{\mathbf{Y}}(x) = \exp\left( \int_{z=x}^{\varepsilon_0} \frac{\mu_{\mathbf{Y}}(z)}{2z} dz \right),$$

and so as $\mu_{\mathbf{Y}}$ is negative on $(0, \varepsilon_0)$ by $B3$ we have $s'_{\mathbf{Y}}$ is increasing on $(0, \varepsilon_0)$, and so $s_{\mathbf{Y}}(x) \leq x s'_{\mathbf{Y}}(x)$ for $x \in (0, \varepsilon_0)$. Recall from (2.11) that $m_{\mathbf{Y}}(x) = 2/(\sigma^2_{\mathbf{Y}}(x) s'_{\mathbf{Y}}(x))$. Then, as $\sigma^2_{\mathbf{Y}}(x) = 4x$ by $B2$ we have

$$\int_{x=0}^{\varepsilon_0} m_{\mathbf{Y}}(x) \int_{z=0}^{x} s_{\mathbf{Y}}(z) m_{\mathbf{Y}}(z) dz dx \leq \frac{1}{2} \int_{x=0}^{\varepsilon_0} m_{\mathbf{Y}}(x) \int_{z=0}^{x} dz dx$$

which by the bounds for $m_{\mathbf{Y}}$ given in (2.53) (that follow from $B2$ and $B3$) is less than or equal to

$$\frac{\varepsilon_0^{\alpha^+}}{4} \int_{x=0}^{\varepsilon_0} x^{-\alpha^+} dx$$

which is finite, and so the condition in (2.8) is satisfied. Also, by the lower bound for $m_{\mathbf{Y}}(x)$ given in (2.53) we have

$$M_{\mathbf{Y}}((0,\varepsilon_0)) \geq \frac{\varepsilon_0^{\alpha^-}}{2} \int_{x=0}^{\varepsilon_0} x^{-1-\alpha^-} dx = \infty,$$

and so the condition in (2.7) is not satisfied and therefore by Theorem 2.1.3 we have that the Pitman-Yor excursion measure for $\mathbf{Y}$ (equivalently for $\mathbf{Z}$) satisfies (2.9) as required. ■

### 2.4.2 Continuity of local time

In our proof of Theorem 2.1.6 we use the continuity in space and time of the local time of the scaffolding process $\mathbf{X}$. To show continuity of the local time, we first prove some technical lemmas and we use them to show a necessary and sufficient condition for the continuity of local time on Laplace exponents of SPLPs with infinite variation, from Lambert and Simatos [62, Lemma 2.2]. We note that necessary and sufficient conditions for the space-time continuity of the local time of Lévy processes more generally had been determined by Barlow [8] but the conditions in [62] are more accessible for us to verify in our setting. To begin, we define the measure of the overshoot (or equivalently of the undershoot) $\chi$ from the Lévy measure $\nu(\zeta \in \cdot)$ as

$$\chi(\cdot) := \nu_{\mathrm{cld}}(\zeta(\hat{f}_{T_0^+}) \in \cdot) = \nu_{\mathrm{cld}}(\zeta(\check{f}_{T_0^+}) \in \cdot),$$

where the second equality is an immediate consequence of (2.36). We also have from Lemma 2.3.3 that

$$\chi(dz) = \nu_{\mathrm{cld}}(\zeta(\check{f}_{T_0^+}) \in dz) = \int_{y=0}^{\infty} \nu(\zeta \in dz+y)dy = \nu(\zeta \in (z,\infty))dz.$$

In the next lemma we also use the stochastic integral equation representation for the diffusion based on the stochastic differential equation representation given in (2.5). Let $\mathbb{Q}_x$ denote the probability measure for the diffusion under Assumption B'. For $\alpha \in (0,1)$, let ${}^{-2\alpha}\mathbb{Q}_u(\cdot)$ denote the law of a diffusion with infinitesimal drift $\mu(x) = -2\alpha$ and infinitesimal variance $\sigma^2(x) = 4x$ (and we again use a superscript 0 to denote a 0-diffusion and a superscript $\uparrow$ to denote an $\uparrow$-diffusion as before).

**Lemma 2.4.2.** *Consider a diffusion that satisfies Assumption B'. Let $\nu$ be its Pitman-Yor excursion measure and let $\chi$ be defined from $\nu$ as above. Then there exist $\varepsilon_2 > 0$ and $k_5 > 0$ such that*

$$\chi((x,\infty)) \geq k_5 x^{-\alpha^-} \text{ for all } x \in (0,\varepsilon_2).$$

*Proof.* First note that we have

$$\chi((x,\infty)) = \nu_{\mathrm{cld}}(\zeta(\hat{f}_{T_0^+}) > x) = k_0 \int_{u=0}^{\infty} m(u)\mathbb{Q}_u^0(\zeta > x)du, \tag{2.56}$$

where $k_0$ is the constant given in Lemma 2.3.3. We can bound the density of the speed measure $m$ below by (2.53). Let $u \in (0,1)$, and let $B$ be a standard Brownian motion on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Define the stochastic processes $Y$, $Y^+$, $Y^-$ on this space by the stochastic integral equations,

$$Y_t = u + \int_{s=0}^{t} \mu(Y_s)ds + 2\int_{s=0}^{t} \sqrt{|Y_s|}dB_s; \;\; Y_t^{\pm} = u - 2\alpha^{\pm}t + 2\int_{s=0}^{t} \sqrt{|Y_s^{\pm}|}dB_s, \tag{2.57}$$

until they hit 0 when they are stopped. We have that

$$\mathbb{P}(Y \in \cdot) = \mathbb{Q}_u^0(\cdot); \;\; \mathbb{P}(Y^+ \in \cdot) = {}^{-2\alpha^+}\mathbb{Q}_u^0(\cdot); \;\; \mathbb{P}(Y^- \in \cdot) = {}^{-2\alpha^-}\mathbb{Q}_u^0(\cdot),$$

and furthermore that

$$Y_t^+ \leq Y_t \leq Y_t^- \text{ for } t \in (0, T_0(Y^+) \wedge T_{\varepsilon_0}(Y^-)) \;\; \mathbb{P}\text{-a.s.},$$

where the $\varepsilon_0$ is as in Assumption B'. Therefore, by using these inequalities along with the marginal distributions of $Y$, $Y^+$, $Y^-$ under $\mathbb{P}$, we have that

$$\begin{aligned}\mathbb{Q}_u^0(\zeta > x) = \mathbb{P}(T_0(Y) > x) &\geq \mathbb{P}\Big(T_0(Y^+) > x, \sup_{0 \leq t \leq T_0(Y^-)} \{Y_t^-\} < \varepsilon_0\Big) \\ &\geq {}^{-2\alpha^+}\mathbb{Q}_u^0(\zeta > x) - {}^{-2\alpha^-}\mathbb{Q}_u^0(A \geq \varepsilon_0).\end{aligned}$$

Göing-Jaeschke and Yor [46, Equation (13)] state $\zeta$ under ${}^{-2\alpha}\mathbb{Q}_u^0$ has distribution InverseGamma$(1+\alpha, u/2)$, i.e.

$${}^{-2\alpha}\mathbb{Q}_u^0(\zeta \in dx) = \frac{1}{\Gamma(1+\alpha)}\Big(\frac{u}{2}\Big)^{1+\alpha} x^{-2-\alpha} \exp\Big(-\frac{u}{2x}\Big). \tag{2.58}$$

By the two-sided exit problem and (2.55) we have ${}^{-2\alpha^-}\mathbb{Q}_u^0(A \geq \varepsilon_0) = (u/\varepsilon_0)^{1+\alpha^-}$. For $x < \varepsilon_0/4$, we now state an upper bound for $\chi((x,\infty))$ by using the upper bound for $\mathbb{Q}_u^0(\zeta > x)$ given above along with the upper bound for $m$ given in (2.53). We also upper bound $\mathbb{Q}_u^0(\zeta > x)$ by $\mathbb{Q}_u^0(\zeta \in (x, \varepsilon_0))$ and obtain

$$\begin{aligned}&\chi((x,\infty)) \\ &\geq \int_{u=0}^{\varepsilon_0} \frac{k_0 \varepsilon_0^{\alpha^-}}{2} u^{-1-\alpha^-} \Big( \int_{y=x}^{\varepsilon_0/2} \Big( \Big( \frac{1}{\Gamma(1+\alpha^+)} \frac{u^{1+\alpha^+}}{y^{2+\alpha^+}} \exp\Big(-\frac{u}{2y}\Big) dy - \Big(\frac{u}{\varepsilon_0}\Big)^{1+\alpha^-} \Big) \wedge 0 \Big) \Big) du\end{aligned}$$

which is greater than or equal to

$$\begin{aligned}&\int_{u=0}^{\varepsilon_0} \frac{k_0 \varepsilon_0^{\alpha^-}}{2} u^{-1-\alpha^-} \int_{y=x}^{\varepsilon_0/2} \frac{1}{\Gamma(1+\alpha^+)} \frac{u^{1+\alpha^+}}{y^{2+\alpha^+}} \exp\Big(-\frac{u}{2y}\Big) dy du \\ &\qquad - \int_{u=0}^{\varepsilon_0} \frac{k_0 \varepsilon_0^{\alpha^-}}{2} u^{-1-\alpha^-} \int_{y=x}^{\varepsilon_0/2} \Big(\frac{u}{\varepsilon_0}\Big)^{1+\alpha^-} dy du.\end{aligned}$$

We consider the two double integrals separately. For the first double integral we may swap the integrals to find and restrict $u \in (0,\varepsilon_0)$ to $u \in (0,2y)$ as $y \in (x,\varepsilon_0/2)$. We get a lower bound of

$$\frac{k_0\varepsilon_0^{\alpha^-}}{2\Gamma(1+\alpha^+)}\int_{y=x}^{\varepsilon_0/2} y^{-2-\alpha^+}\int_{u=0}^{2y} u^{\alpha^+-\alpha^-}\exp\Big(-\frac{u}{2y}\Big)dudy$$

and we can bound $\exp(-u/2y)$ from below by $1/e$ to get a lower bound of

$$\frac{k_0\varepsilon_0^{\alpha^-}}{2\Gamma(1+\alpha^+)e}\int_{y=x}^{\varepsilon_0/2} y^{-2-\alpha^+}\int_{u=0}^{2y} u^{\alpha^+-\alpha^-}dudy = \frac{2^{\alpha^+-\alpha^-}k_0\varepsilon_0^{\alpha^-}}{\Gamma(1+\alpha^+)e(1-\alpha^++\alpha^-)}\int_{y=x}^{\varepsilon_0/2} y^{-1-\alpha^-}dy$$

which is equal to

$$\frac{2^{\alpha^+-\alpha^-}k_0\varepsilon_0^{\alpha^-}}{\Gamma(1+\alpha^+)e(1-\alpha^++\alpha^-)\alpha^-}\Big(x^{-\alpha^-}-\Big(\frac{\varepsilon_0}{2}\Big)^{-\alpha^-}\Big).$$

For the second double integral we see the upper bound given by

$$\int_{u=0}^{\varepsilon_0}\frac{k_0\varepsilon_0^{\alpha^-}}{2}u^{-1-\alpha^-}\int_{y=0}^{\varepsilon_0/2}\Big(\frac{u}{\varepsilon_0}\Big)^{1+\alpha^-}dydu = \frac{k_0}{2\varepsilon_0}\int_{u=0}^{\varepsilon_0}\int_{y=0}^{\varepsilon_0/2}dydu = \frac{k_0\varepsilon_0}{4}.$$

Therefore the claim in the lemma is satisfied for some $k_5$ and $\varepsilon_2$. ■

**Lemma 2.4.3.** *In the same setting as in Lemma 2.4.2, we have that, for some positive constant $k_6$,*

$$\chi((x,\infty)) \le k_6(x^{-\alpha^+}+1) \text{ for all } x \in (0,\infty),$$

*Proof.* We use the formulation of $\chi((x,\infty))$ as given in (2.56), as well as the coupled processes $Y$, $Y^+$ and $Y^-$ as in Lemma 2.4.2. We note that

$$\begin{aligned}\mathbb{Q}_u^0(\zeta > x) &= \mathbb{P}(T_0(Y) > x)\\ &= \mathbb{P}\Big(T_0(Y) > x, \sup_{0\le t\le T_0(Y^-)}\{Y_t^-\} \ge \varepsilon_0\Big) + \mathbb{P}\Big(T_0(Y) > x, \sup_{0\le t\le T_0(Y^-)}\{Y_t^-\} < \varepsilon_0\Big)\\ &\le \mathbb{P}(A(Y^-) \ge \varepsilon_0) + \mathbb{P}(\zeta(Y^-) > x) = {}^{-2\alpha^-}\mathbb{Q}_u^0(A \ge \varepsilon_0) + {}^{-2\alpha^-}\mathbb{Q}_u^0(\zeta > x),\end{aligned}$$

where the $\varepsilon_0$ is as in Assumption B'. Therefore we have that $\chi((x,\infty))$ is less than or equal to

$$\frac{k_0\varepsilon_0^{\alpha^+}}{2}\int_{u=0}^{\varepsilon_0} u^{-1-\alpha^+}\Big({}^{-2\alpha^-}\mathbb{Q}_u^0(A \ge \varepsilon_0) + {}^{-2\alpha^-}\mathbb{Q}_u^0(\zeta > x)\Big)du + k_0\int_{u=\varepsilon_0}^{c}\mathbb{Q}_u^0(\zeta > x)m(u)du$$

and we bound this above by

$$\frac{k_0\varepsilon_0^{\alpha^+}}{2}\int_{u=0}^{\varepsilon_0} u^{-1-\alpha^+}\Big(u^{1+\alpha^-} + \int_{y=x}^{\infty}\frac{1}{\Gamma(1+\alpha^-)}\frac{u^{1+\alpha^-}}{y^{2+\alpha^-}}\exp\Big(-\frac{u}{2y}\Big)dy\Big)du + \frac{k_0}{2}M(E\backslash[0,\varepsilon_0]).$$

We deal with the two parts of the integral separately. We have

$$\begin{aligned}&\frac{k_0\varepsilon_0^{\alpha^+}}{2}\int_{u=0}^{\varepsilon_0} u^{-1-\alpha^+}\int_{y=x}^{\infty}\frac{1}{\Gamma(1+\alpha^-)}\frac{u^{1+\alpha^-}}{y^{2+\alpha^-}}\exp\Big(-\frac{u}{2y}\Big)dydu\\ &\qquad\le \frac{k_0\varepsilon_0^{\alpha^+}}{2\Gamma(1+\alpha^-)}\int_{y=x}^{\infty} y^{-2-\alpha^-}\int_{u=0}^{\varepsilon_0} u^{\alpha^--\alpha^+}dudy\end{aligned}$$

which equals

$$\frac{k_0\varepsilon_0^{1+\alpha^-}}{2\Gamma(1+\alpha^-)(1+\alpha^--\alpha^+)}\int_{y=x}^{\infty} y^{-1-\alpha^+}dy = \frac{k_0\varepsilon_0^{1+\alpha^-}}{2\Gamma(1+\alpha^-)(1+\alpha^--\alpha^+)\alpha^+}x^{-\alpha^+}.$$

For the second part of the integral we have that

$$\frac{k_0\varepsilon_0^{\alpha^+}}{2}\int_{u=0}^{\varepsilon_0} u^{\alpha^--\alpha^+}du = \frac{k_0\varepsilon_0^{1+\alpha^-}}{2(1+\alpha^--\alpha^+)},$$

and so

$$\chi((x,\infty)) \leq \frac{k_0\varepsilon_0^{1+\alpha^-}}{2\Gamma(1+\alpha^-)(1+\alpha^--\alpha^+)\alpha^+}x^{-\alpha^+} + \frac{k_0\varepsilon_0^{1+\alpha^-}}{2(1+\alpha^--\alpha^+)} + \frac{k_0}{2}M(E\backslash[0,\varepsilon_0])$$

and the result follows. ■

**Lemma 2.4.4.** *Consider a diffusion that satisfies Assumption B', and let $\psi$ be the Laplace exponent for the scaffolding process* $\mathbf{X}$*, see* (2.16)*. Then we have for $\lambda \in (0,\infty)$ that*

$$\psi(\lambda) = \int_{x=0}^{\infty} \lambda^2 e^{-\lambda x}\chi((x,\infty))dx. \tag{2.59}$$

*Proof.* This follows by integration-by-parts applied twice to $\psi(\lambda) = \int_{x=0}^{\infty}(e^{-\lambda x} - 1 + \lambda x)\Lambda(dx)$ (where we recall that $\Lambda(\cdot) = \nu(\zeta \in \cdot)$) and we require the upper bound for $\chi((x,\infty))$ in Lemma 2.4.3 to do this. ■

We now use a necessary and sufficient condition from Lambert [62] to determine that the local time is a.s. jointly continuous in time and space.

**Proposition 2.4.5.** *Consider a diffusion* $\mathbf{Z}$ *under Assumption B and let* $(\mathbf{N},\mathbf{X})$ *be a scaffolding-and-spindles construction. Then the local time $(\ell_t^u : t,u \in \mathbb{R})$ is jointly continuous a.s.*

*Proof.* Without loss of generality we can assume our diffusion satisfies Assumption B'. This is because, as discussed in the introduction after Theorem 2.1.3, the lifetime under the excursion measure is not affected by spatial transformations, i.e. $\nu_{\mathbf{Z}}(\zeta \in \cdot) = \nu_{\mathbf{Y}}(\zeta \in \cdot)$ where $\mathbf{Y}$ is an untransformed transformation that satisfies Assumption B'.

By Lambert [62, Lemma 2.2], as $\mathbf{X}$ is a SPLP that satisfies (2.9), a necessary and sufficient condition for the local time to be jointly continuous a.s. is

$$\int_c^{\infty} \frac{d\lambda}{\psi(\lambda)\sqrt{\log(\lambda)}} < \infty$$

for any (equivalently for all) $c > 0$. Now, with the lower bound for $\chi((x,\infty))$ from Lemma 2.4.2 we have that

$$\psi(\lambda) \geq k_5\int_{x=0}^{\delta} \lambda^2 e^{-\lambda x}x^{-\alpha^-}dx.$$

We let $\lambda > 1/\delta$ and consider the Taylor expansion of $e^{-\lambda x}$ to obtain that

$$\psi(\lambda) \geq k_5 \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{n+2}}{n!} \int_{x=0}^{1/\lambda} x^{n-\alpha^-}\, dx = k_5 \Big( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1-\alpha^-)} \Big) \lambda^{1+\alpha^-}$$

and therefore

$$\int_{\lambda=1/\delta}^{\infty} \frac{d\lambda}{\psi(\lambda)\sqrt{\log(\lambda)}} \leq \Big( k_5 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1-\alpha^-)} \Big)^{-1} \int_{\lambda=1/\delta}^{\infty} \frac{d\lambda}{\lambda^{1+\alpha^-}\sqrt{\log(\lambda)}} < \infty.$$

■

### 2.4.3 Some further calculations

In this section we give upper bounds of certain probabilities and expectations for diffusions with a state space $E = [0,\infty)$ and we use these results in our proof of Theorem 2.1.6. In Section 2.4.3.1 we prove results for a general such diffusions conditioned to not hit 0 and we reference the literature on quasi-stationary distributions, Méléard and Villemonais [73]. In Section 2.4.3.2 we consider squared Bessel cases of negative dimension.

#### 2.4.3.1 Expectation for a diffusion conditioned not to hit level 0

In this section we calculate conditional moments for the deviation of a diffusion with state space $E \subseteq [0,\infty)$. As before, for $s > 0$, $\mathbb{Q}_s$ denotes the law of the diffusion started from $s$ and $\mathbb{Q}_s^0$ denotes the law of the diffusion started from $s$ and stopped at 0.

**Lemma 2.4.6.** *Consider a diffusion that satisfies A*1 *of Assumption A. Then for r, $s > 0$, $0 < a < c$ and $0 < t \leq r$ we have*

$$\mathbb{Q}_s^0(Y_t \in E\backslash[0,a)) \leq \mathbb{Q}_s^0(Y_t \in E\backslash[0,a) | \zeta > r).$$

*Proof.* Let $t \in (0,r)$. We first note that

$$b := \mathbb{Q}_s^0(\zeta > r | Y_t \in [0,a)) \leq \mathbb{Q}_a^0(\zeta > r-t) \leq \mathbb{Q}_s^0(\zeta > r | Y_t \in E\backslash[0,a)) =: c,$$

by the Markov property and almost-sure continuity of the sample paths. Let $p = \mathbb{Q}_s^0(Y_t \in E\backslash[0,a))$. We then have that

$$\mathbb{Q}_s^0(Y_t \in E\backslash[0,a))\mathbb{Q}_s^0(\zeta > r) = p(cp + b(1-p)) \leq cp = \mathbb{Q}_s^0(Y_t \in E\backslash[0,a), \zeta > r)$$

which implies that

$$\mathbb{Q}_s^0(Y_t \in E\backslash[0,a)) \leq \mathbb{Q}_s^0(Y_t \in E\backslash[0,a) | \zeta > r).$$

Taking $t \uparrow r$ we have by a.s. continuity of the sample paths that

$$\mathbb{Q}_s^0(Y_r \in E\backslash[0,a)) \leq \mathbb{Q}_s^0(Y_r \in E\backslash[0,a) | \zeta > r).$$

■

**Lemma 2.4.7.** *In the same setting as Lemma 2.4.6, for r, s, $u > 0$ and $0 < a < c$ we have*

$$\mathbb{Q}_s^0(Y_r \in E\backslash[0,a) | \zeta > r) \leq \mathbb{Q}_s^0(Y_r \in E\backslash[0,a) | \zeta > r+u).$$

*Proof.* In a very similar fashion to the previous lemma we first note that

$$b_u := \mathbb{Q}_s^0(\zeta > r+u | Y_r \in (0,a)) \leq \mathbb{Q}_a^0(\zeta > u) \leq \mathbb{Q}_s^0(\zeta > r+u | Y_r \in E\backslash[0,a)) =: c_u,$$

and let $p := \mathbb{Q}_s^0(Y_r \in (a,\infty)) \leq \mathbb{Q}_s^0(\zeta > y) =: q$. Then we have that

$$\mathbb{Q}_s^0(\zeta > r+u, Y_r \in E\backslash[0,a)) = c_u p.$$

Furthermore note that

$$\mathbb{Q}_s^0(\zeta > r+u | \zeta > r) = c_u \frac{p}{q} + b_u \frac{q-p}{q} \leq c_u, \quad \mathbb{Q}_s^0(Y_r \in E\backslash[0,a) | \zeta > r) = \frac{p}{q},$$

and therefore

$$\mathbb{Q}_s^0(Y_r \in E\backslash[0,a), \zeta > r+u | \zeta > r) = \frac{pc}{q} \geq \mathbb{Q}_s^0(Y_r \in E\backslash[0,a) | \zeta > r) \mathbb{Q}_s^0(\zeta > r+u | \zeta > r),$$

and the claim follows. ∎

This final lemma, Lemma 2.4.9, uses the following convergence result for the $\uparrow$-diffusion from [73, Theorem 27].

**Lemma 2.4.8.** *In the same setting as the previous lemmas, for r, s, $a > 0$ we have*

$$\lim_{u\to\infty} \mathbb{Q}_s^0(Y_r \in E\backslash[0,a) | \zeta > r+u) = \mathbb{Q}_s^{\uparrow}(Y_r \in E\backslash[0,a)).$$

**Lemma 2.4.9.** *Let $F : [0,\infty) \to \mathbb{R}$ be an absolutely continuous increasing function. Then, in the same setting as the previous lemmas, for r, $s > 0$ we have*

$$\mathbb{E}_{\mathbb{Q}_s^0}(F(Y_r)) \leq \mathbb{E}_{\mathbb{Q}_s^0}(F(Y_r) | \zeta > r) \leq \mathbb{E}_{\mathbb{Q}_s^{\uparrow}}(F(Y_r)).$$

*Proof.* We first note $F$ has derivative $F'$ Lebesgue-almost everywhere, see e.g. Kallenberg [52, Theorem 2.15]. We therefore have that

$$\mathbb{E}_{\mathbb{Q}_s^0}(F(Y_r)) = \int_{z=0}^{\infty} \mathbb{Q}_s^0(Y_r \in E\backslash[0,z)) F'(z) dz,$$

and similarly for $\mathbb{E}_{\mathbb{Q}_s^0}(F(Y_r) | \zeta > r)$ and $\mathbb{E}_{\mathbb{Q}_s^{\uparrow}}(F(Y_r))$. By Lemmas 2.4.6, 2.4.7 and 2.4.8 we have that for any $a \in (0,\infty)$

$$\begin{aligned}\mathbb{Q}_s^0(Y_r \in E\backslash[0,a)) &\leq \mathbb{Q}_s^0(Y_r \in E\backslash[0,a) | \zeta > r) \\ &\leq \lim_{u\to\infty} \mathbb{Q}_s^0(Y_r \in E\backslash[0,a) | \zeta > r+u) = \mathbb{Q}_s^{\uparrow}(Y_r \in E\backslash[0,a)),\end{aligned}$$

and the result follows. ∎

#### 2.4.3.2 Certain calculations concerning squared Bessel processes with negative dimension

In this section we work with squared Bessel processes with dimension $-2\alpha$, where $\alpha \in (0,1)$. We make a slight abuse of notation and write ${}^{-2\alpha}\mathbb{Q}_s^0$ in place of $\mathbb{E}_{-2\alpha\mathbb{Q}_s^0}$ for readability.

**Lemma 2.4.10.** *Let $r$, $q > 0$, $0 < \alpha, s < 1$ and $n \in \mathbb{N}$. Then we have*

$$ {}^{-2\alpha}\mathbb{Q}_s^0(|Y_r - s|^{qn} | \zeta > r) \leq {}^{-2\alpha}\mathbb{Q}_s^0(|Y_r - s|^{qn}) + {}^{4+2\alpha}\mathbb{Q}_s^0(|Y_r - s|^{qn}). $$

*Proof.* We note that the functions

$$ |z-s|^{qn}\mathbb{1}_{z\geq s}; \quad -|z-s|^{qn}\mathbb{1}_{z\leq s}, $$

are absolutely continuous increasing functions that are Lebesgue differentiable almost everywhere. Therefore, by Lemma 2.4.9 we have that

$$ \begin{aligned} {}^{-2\alpha}\mathbb{Q}_s^0(|Y_r - s|^{qn} | \zeta > r) &\leq {}^{-2\alpha}\mathbb{Q}_s^{\uparrow}(|Y_r - s|^{qn}\mathbb{1}_{z\geq s} | \zeta > r) - {}^{-2\alpha}\mathbb{Q}_s^0(-|Y_r - s|^{qn}\mathbb{1}_{z\leq s} | \zeta > r) \\ &\leq {}^{4+2\alpha}\mathbb{Q}_s^0(|Y_r - s|^{qn}) + {}^{-2\alpha}\mathbb{Q}_s^0(|Y_r - s|^{qn}). \end{aligned} $$

■

We establish upper and lower bounds for the probability that the lifetime $\zeta$ is greater than $r$ under ${}^{-2\alpha}\mathbb{Q}_s^0$.

**Lemma 2.4.11.** *For $r$, $s > 0$, $0 < \alpha < 1$, we have*

$$ {}^{-2\alpha}\mathbb{Q}_s^0(\zeta \leq r) \leq \frac{\Gamma(1+\alpha+v)2^v}{\Gamma(1+\alpha)}\frac{r^v}{s^v}; \quad {}^{-2\alpha}\mathbb{Q}_s^0(\zeta > r) \leq \frac{\Gamma(1+\alpha-v)}{\Gamma(1+\alpha)2^v}\frac{s^v}{r^v} \tag{2.60} $$

*for $v > 0$ and for $0 < v < 1+\alpha$ respectively.*

*Proof.* For the first inequality let $v > 0$ and apply Markov's inequality and (2.58) to see that

$$ \begin{aligned} {}^{-2\alpha}\mathbb{Q}_s^0(\zeta \leq r) &= {}^{-2\alpha}\mathbb{Q}_s^0\left(\frac{1}{\zeta^v} \geq \frac{1}{r^v}\right) \\ &\leq {}^{-2\alpha}\mathbb{Q}_s^0\left(\frac{1}{\zeta^v}\right) r^v = r^v \int_{x=0}^{\infty} \frac{s^{1+\alpha}}{2^{1+\alpha}\Gamma(1+\alpha)} \frac{1}{x^{2+\alpha+v}} \exp\left(-\frac{s}{2x}\right) dx \end{aligned} $$

and we integrate out the probability density function of an InverseGamma$(1+\alpha+v, s/2)$ to obtain

$$ \frac{\Gamma(1+\alpha+v)2^v}{\Gamma(1+\alpha)}\frac{r^v}{s^v}\int_{x=0}^{\infty} \frac{s^{1+\alpha+v}}{2^{1+\alpha+v}\Gamma(1+\alpha+v)} \frac{1}{x^{2+\alpha+v}} \exp\left(-\frac{s}{2x}\right) dx = \frac{\Gamma(1+\alpha+v)2^v}{\Gamma(1+\alpha)}\frac{r^v}{s^v}. $$

For the second inequality let $0 < v < 1+\alpha$, we use the same ideas:

$$ \begin{aligned} {}^{-2\alpha}\mathbb{Q}_s^0(\zeta > r) &\leq \frac{1}{r^v} {}^{-2\alpha}\mathbb{Q}_s^0(\zeta^v) \\ &= \frac{1}{r^v}\int_{x=0}^{\infty} \frac{s^{1+\alpha}}{2^{1+\alpha}\Gamma(1+\alpha)} \frac{1}{x^{2+\alpha-v}} \exp\left(-\frac{s}{2x}\right) dx = \frac{\Gamma(1+\alpha-v)}{\Gamma(1+\alpha)2^v}\frac{s^v}{r^v}, \end{aligned} $$

where we have integrated out the probability distribution function of an InverseGamma$(1+\alpha-v, s/2)$. ■

We also establish a further lemma with some useful probability bounds for squared Bessel processes.

**Lemma 2.4.12.** *Let $\gamma \in (-\infty, 0) \cup [2, \infty)$, and $d > 0$. Then there exists a constant $C = C(\gamma, d)$ such that for any $r$, $s > 0$, we have*

$$ {}^{\gamma}\mathbb{Q}_s^0(|Y_r - s|^d) \leq C r^{d/2}(r^{d/2} \vee s^{d/2}). $$

*Proof.* We use [33, Equation (19)] to note that for $d \geq 2$ and $\gamma \in \mathbb{R}$ that we have

$$ \begin{aligned} \left({}^{\gamma}\mathbb{Q}_s(|Y_r - s|^d)\right)^{(1/d)} &\leq |\gamma| r + 2\sqrt{(d-1)(|s| r + r^2(|\gamma| + (d-2)))} \\ &\leq \left(|\gamma| + 2\sqrt{(d-1)(1 + (|\gamma| + (d-2)))}\right) r^{1/2}\left(r^{1/2} \vee s^{1/2}\right) \end{aligned} \tag{2.61} $$

Squared Bessel processes with negative dimension never return to the positive half-line after the first hitting time of 0 a.s. Also, squared Bessel processes with dimension greater than or equal to 2 that start positive never hit 0 a.s. Therefore, for $\gamma \in (-\infty, 0) \cup [2, \infty)$ we have that

$$ \left({}^{\gamma}\mathbb{Q}_s^0(|Y_r - s|^d)\right)^{(1/d)} \leq \left(|\gamma| + 2\sqrt{(d-1)(1 + (|\gamma| + (d-2)))}\right) r^{1/2}\left(r^{1/2} \vee s^{1/2}\right) $$

for $d \geq 2, \gamma \in (-\infty, 0) \cup [2, \infty)$ by (2.61). Now let $d < 2$. Then $4/d > 2$ and so we can apply Jensen's inequality to obtain that

$$ {}^{\gamma}\mathbb{Q}_s^0(|Y_r - s|^d) \leq ({}^{\gamma}\mathbb{Q}_s^0(|Y_r - s|^4))^{d/4} \leq C(\gamma, 4)^{d/4} r^{d/2}\left(r^{d/2} \vee s^{d/2}\right) $$

for $0 < d < 2, \gamma \in (-\infty, 0) \cup [2, \infty)$. ∎

**Lemma 2.4.13.** *Let $n \in \mathbb{N}$, $0 < \alpha, \gamma < 1$, $0 < \kappa < 1 - \alpha$, and $\alpha + \kappa < q < 1$. Then there exists a constant $C = C(n, q, \alpha, \gamma, \kappa) > 0$ such that for $0 < r \leq 1$ we have*

$$ \int_{s=0}^{1} {}^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn}) s^{-1-\alpha} ds \leq C r^{n((q-\alpha-\kappa)\wedge q/2)}. $$

*Proof.* We can apply Lemma 2.4.12 as

$$ {}^{-2\gamma}\mathbb{Q}_s^0(|Y_r^q - s^q|^n) \leq {}^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn}) \text{ for } 0 < q < 1, n \geq 1. $$

We split the integral. We first consider the expectation on the event $\{\zeta \leq r\}$ and use the upper bound for ${}^{-2\gamma}\mathbb{Q}_s^0(\zeta \leq r)$ in (2.60) with $v = qn - \alpha - \kappa$:

$$ \begin{aligned} &\int_{s=0}^{1} {}^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn}\, \mathbb{1}_{\zeta \leq r}) s^{-1-\alpha} ds \\ &\qquad \leq \frac{\Gamma(1 + \gamma + qn - \alpha - \kappa) 2^{qn-\alpha-\kappa}}{\Gamma(1+\alpha)} \int_{s=0}^{1} s^{-1-\alpha} s^{qn} \frac{r^{qn-\alpha-\kappa}}{s^{qn-\alpha-\kappa}} ds \\ &\qquad = \frac{\Gamma(1 + \gamma + qn - \alpha - \kappa) 2^{qn-\alpha-\kappa}}{\Gamma(1+\alpha)\kappa} r^{qn-\alpha-\kappa}. \end{aligned} $$

We now consider the integral on the event $\{\zeta > r\}$ on the range $0 < s < r$. By Lemmas 2.4.10 and 2.4.12 we have that

$$^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn}|\zeta > r) \leq (C(-2\gamma, qn) + C(4+2\gamma, qn))r^{qn/2}(s^{qn/2} \vee r^{qn/2}),$$

where $C(-2\gamma, qn)$ and $C(4+2\gamma, qn)$ are from Lemma 2.4.12. Let
$K = (C(-2\gamma, qn) + C(4+2\gamma, qn))$. We use (2.60) with $v = \alpha + \kappa$ to bound above $^{-2\gamma}\mathbb{Q}_s^0(\zeta > r)$:

$$\begin{aligned}\int_{s=0}^{r\wedge 1} {}^{-2\gamma}\mathbb{Q}_s^0[|Y_r - s|^{qn}\mathbb{1}_{\zeta>r}]s^{-1-\alpha}ds &\leq \frac{K\Gamma(1+\gamma-\alpha-\kappa)}{\Gamma(1+\gamma)2^{\alpha+\kappa}}\int_{s=0}^{r\wedge 1}\frac{s^{\alpha+\kappa}}{r^{\alpha+\kappa}}s^{-1-\alpha}r^{qn}ds \\ &\leq \frac{K\Gamma(1+\gamma-\alpha-\kappa)}{\Gamma(1+\gamma)2^{\alpha+\kappa}}r^{qn-\alpha-\kappa}\int_{s=0}^{1}s^{-1+\kappa}ds \\ &= \frac{K\Gamma(1+\gamma-\alpha-\kappa)}{\Gamma(1+\gamma)2^{\alpha+\kappa}\kappa}r^{qn-\alpha}.\end{aligned}$$

We note that

$$r^{qn-\alpha-\kappa} \leq r^{n((q-\alpha-\kappa)\wedge q/2)} \text{ for } 0 < r < 1.$$

Finally, we consider the integral on the event $\{\zeta > r\}$ over the range $r \wedge 1 \leq s \leq 1$. We split this calculation into two parts to deal with the cases $q/2 \geq q-\alpha-\kappa$ and $q/2 < q-\alpha-\kappa$ separately. In the case where $q/2 \geq q-\alpha-\kappa$ we use the bound in (2.60) for $^{-2\gamma}\mathbb{Q}_s^0(\zeta > r)$ with $0 < v = \alpha+\kappa-q/2 < 1+\alpha$:

$$\begin{aligned}&\int_{s=r\wedge 1}^{1} {}^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn}\ \mathbb{1}_{\zeta>r})s^{-1-\alpha}ds \\ &\qquad\leq \frac{K\Gamma(1+\gamma+q/2-\alpha-\kappa)}{\Gamma(1+\alpha)2^{\alpha+\kappa-q/2}}\int_{s=r\wedge 1}^{1}\frac{s^{\alpha+\kappa-q/2}}{r^{\alpha+\kappa-q/2}}s^{-1-\alpha}r^{qn/2}s^{qn/2}ds \\ &\qquad\leq \frac{K\Gamma(1+\gamma+q/2-\alpha-\kappa)}{\Gamma(1+\alpha)2^{\alpha+\kappa-q/2}}r^{qn/2+q/2-\alpha-\kappa}\int_{s=0}^{1}s^{-1+\kappa}ds \\ &\qquad\leq \frac{K\Gamma(1+\gamma+q/2-\alpha-\kappa)}{\Gamma(1+\alpha)2^{\alpha+\kappa-q/2}\kappa}r^{n(q-\alpha-\kappa)} \text{ for } q/2 \geq q-\alpha-\kappa.\end{aligned}$$

For the case where $q/2 < q-\alpha-\kappa$ we do not use (2.60) to bound $^{-2\gamma}\mathbb{Q}_s^0(\zeta > r)$ (we instead use the trivial bound of 1):

$$\begin{aligned}\int_{s=r\wedge 1}^{1} {}^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn}\mathbb{1}_{\zeta>r})s^{-1-\alpha}ds &\leq K\int_{s=r\wedge 1}^{1}s^{-1-\alpha}r^{qn/2}s^{qn/2}ds \\ &= K\int_{s=r\wedge 1}^{1}s^{-1-\alpha+qn/2}dsr^{qn/2} \\ &\leq Kr^{qn/2}\int_{s=0}^{1}s^{-1+\kappa}ds = \frac{K}{\kappa}r^{qn/2}.\end{aligned}$$

■

### 2.4.4 Continuity of the skewer process

In this section we prove the continuity results for Theorems 2.1.6 and 2.1.7. We begin with some definitions. Consider a diffusion under Assumption B' with Pitman-Yor excursion measure $\nu$, and a transformation $g : E \to g(E)$ (where $E = [0,c)$ or $E = [0,c]$ for some $c \leq \infty$) that satisfies the conditions stated in Assumption B. Without loss of generality we can take the $\varepsilon_0$ in Assumption B to be less than 1 and we do this here. As before, let $\mathbf{N}$ be a stopped PRM with intensity $\mathrm{Leb} \otimes \nu$, where the stopping is at an a.s. finite random variable $T$. Finally, for $M$, $\varepsilon_0 > 0$ we define $\mathbf{N}_{M,\varepsilon_0}$ by

$$\mathbf{N}_{M,\varepsilon_0} := \mathbb{1}_{\sup_{u\in\mathbb{R}} \ell^u(T)<M} \sum_{\substack{(t,f) \text{ atom of } \mathbf{N}: \\ A(f)<\varepsilon_0}} \delta_{(t,f)}. \tag{2.62}$$

**Lemma 2.4.14.** *Let $x$, $y \in \mathbb{R}$, $z \in \{x,y\}$, and let $k \in \mathbb{N}$. Let $\pi_k$ denote the set of compositions of $k$, i.e. $\pi_k := \{(n_1,\ldots,n_a) | a, n_i \in \mathbb{N}\backslash\{0\}, \sum_i n_i = k\}$. Then we have that*

$$\begin{aligned}
&\mathbf{E}\left(\left(\sum_{\substack{(t,f) \text{ atom of } \mathbf{N}_{M,\varepsilon_0}: \\ \mathbf{X}_{t-}<z<\mathbf{X}_t, \ell^z(t)>0}} \left|g\big(f(y-\mathbf{X}_{t-})\big) - g\big(f(x-\mathbf{X}_{t-})\big)\right|\right)^k\right) \\
&\leq C_q^k \sum_{\pi_k} (k_0 M)^a \prod_{i=1}^{a} \int_{s=0}^{\varepsilon_0} m(s)\Big({}^{-2\alpha^+}\mathbb{Q}_s^0(|Y_{|x-y|} - s|^{qn_i}) + {}^{-2\alpha^-}\mathbb{Q}_s^0(|Y_{|x-y|} - s|^{qn_i})\Big) ds,
\end{aligned} \tag{2.63}$$

*where $C_q$ is the Hölder constant for $g$ as in Assumption B and $k_0$ is as in Proposition 2.3.6.*

*Proof.* The bound is trivial for $x = y$ so assume $x \neq y$. First recall from Assumption B that there exists $C_q > 0$ such that $|g(a) - g(b)| \leq C_q |a-b|^q$ for $0 < a, b < \varepsilon_0$. We can bound the term on the left-hand-side of (2.63) by

$$C_q^k \mathbf{E}\left(\left(\sum_{\substack{(t,f) \text{ atom of } \mathbf{N}: A(f)<\varepsilon_0 \\ 0<\ell^z(t)<M, \mathbf{X}_{t-}<z<\mathbf{X}_t}} \left|f(y-\mathbf{X}_{t-}) - f(x-\mathbf{X}_{t-})\right|^q\right)^k\right). \tag{2.64}$$

As $T^z$ is a stopping time for $\mathbf{X}$, we have by Lemma 2.3.9 that the process $\xi(\mathbf{N}|^{\leftarrow}_{[T^z,\infty)})$ can be split into excursions about level 0, and all these excursions a.s. include a single jump across level 0. We have a.s. a one-to-one correspondence between crossing across a level and excursions about the level. Therefore the sum in (2.64) less than or equal to a sum indexed by the Poisson point process of crossing widths at level $z$ and is summing functionals of the clades or anti-clades associated with each of these crossing widths. Specifically, let $\mathbf{N}'$ be the unstopped Poisson random measure, then

$$\mathbf{J} := \sum_{\substack{t:0<\ell^z(t)<M, \\ A(f)<\varepsilon_0}} \delta_{(\ell^z(t), \mathbf{N}'|^{\leftarrow}_{(\ell^z(t)-,\ell^z(t))})}$$

is distributed as a Poisson random measure with intensity

$$\mathrm{Leb}|_{[0,M]} \otimes k_0 \int_{s=0}^{\varepsilon_0} m(s) \nu_{\mathrm{cld}}(\mathbb{1}_{A(f_{T_0^+})<\varepsilon_0} N \in \cdot | f_{T_0^+}(-\xi_N(T_0^+-)) = s) ds,$$

where $k_0$ is as in Proposition 2.3.6 and $\varepsilon_0$ is as in Assumption B. Let $w \in \{x,y\}\backslash\{z\}$. Then, we define

$$F(N) := |f_{T_0^+}(w-z-\xi_N(T_0^+-)) - f_{T_0^+}(-\xi_N(T_0^+-))|^q \text{ for } N \in \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}$$

and therefore the expectation in (2.64) is bounded above by

$$\mathbf{E}\left(\left(\sum_{\substack{(s,N) \text{ atom} \\ \text{of } \mathbf{J}}} F(N)\right)^k\right) = \sum_{\pi_k} \mathbf{E}\left(\sum_{\substack{(s_1,N_1),\ldots,(s_a,N_a) \\ \text{distinct atoms of } \mathbf{J}}} \prod_{i=1}^{a} F(N_i)^{n_i}\right).$$

We apply Campbell's formula for Poisson random measures, see Kingman [56, Equation (3.14)], to obtain, for a composition $(n_1,\ldots,n_a)$ of $k$, that the expectation in the right-hand-side of the above equation equals

$$(k_0 M)^a \prod_{i=1}^{a} \int_{s=0}^{\varepsilon_0} ds m(s) \nu_{\mathrm{cld}}\Big(|f_{T_0^+}(w-z-\xi_N(T_0^+-)) - f_{T_0^+}(-\xi_N(T_0^+-))|^{qn_i} \mathbb{1}_{A(f_{T_0^+})<\varepsilon_0} \Big| f_{T_0^+}(-\xi_N(T_0^+-)) = s\Big).$$

Now, let $\tilde{\mathbf{Y}}$ be a diffusion with infinitesimal parameters

$$\sigma^2_{\tilde{\mathbf{Y}}}(x) = 4x; \quad \mu_{\tilde{\mathbf{Y}}}(x) = \begin{cases} \mu(x) \text{ for } x \in (0,\varepsilon_0]; \\ \mu(\varepsilon_0) \text{ for } x > \varepsilon_0, \end{cases} \tag{2.65}$$

and let $\tilde{\nu}_{\mathrm{cld}}$ be the bi-clade excursion measure for $\tilde{\mathbf{Y}}$. Then for any measurable set $B \subseteq \mathscr{C}$ of continuous paths we have for $C := \{N \in \mathscr{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} : f_{T_0^+} \in B\}$ that

$$\nu_{\mathrm{cld}}\Big(\{A(f_{T_0^+}) < \varepsilon_0\} \cap C \Big| f_{T_0^+}(-\xi_N(T_0^+-)) = s\Big) = \tilde{\nu}_{\mathrm{cld}}\Big(\{A(f_{T_0^+}) < \varepsilon_0\} \cap C \Big| f_{T_0^+}(-\xi_N(T_0^+-))) = s\Big)$$

because the two excursion measures differ only on excursions with amplitude greater than $\varepsilon_0$. Indeed, you can construct a Poisson random measure $\mathbf{N}$ with bi-clade measure $\nu_{\mathrm{cld}}$ by first constructing a Poisson random measure $\tilde{\mathbf{N}}$ with bi-clade measure $\tilde{\nu}_{\mathrm{cld}}$, removing the finitely many bi-clades with spindles with amplitude greater than $\varepsilon_0$ (up to a finite time $T$) and adding finitely many bi-clades with spindles with amplitude greater than $\varepsilon_0$ since such bi-clades have finite intensity under the bi-clade measures.

In Proposition 2.3.10 we established that the bi-clade measure about a fixed level disintegrates with respect to the crossing width. Let $n \in \mathbb{N}$. The terms in the sum in (2.64) are the functionals

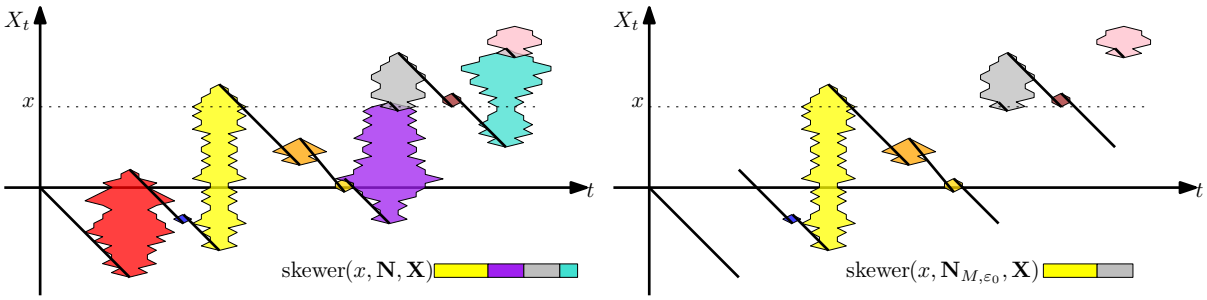



Figure 2.3: Left: The skewer process at level $x$ constructed from the PRM $\mathbf{N}$ and scaffolding $\mathbf{X}$. Right: The skewer process at level $x$ constructed from the reduced PRM $\mathbf{N}_{M,\varepsilon_0}$ from (2.62) and scaffolding $\mathbf{X}$.

of the clade (if $z = \min\{x,y\}$) or the anti-clade (if $z > \min\{x,y\}$) of excursions about level $z$. Therefore we have an upper bound given by

$$\tilde{\nu}_{\text{cld}}\Big(|f_{T_0^+}(w-z-\xi_N(T_0^+-)) - f_{T_0^+}(-\xi_N(T_0^+-))|^{qn}\Big| f_{T_0^+}(-\xi_N(T_0^+-))) = s\Big) \\ = \tilde{\mathbb{Q}}_s^0(|Y_{|x-y|} - s|^{qn}),$$

and as $-2\alpha^+ \le \mu_{\tilde{\mathbf{Y}}}(x) \le -2\alpha^-$ we have, through a coupling argument, that

$$\tilde{\mathbb{Q}}_s^0(|Y_{|x-y|} - s|^{qn}) \le {}^{-2\alpha^+}\mathbb{Q}_s^0(|Y_{|x-y|} - s|^{qn}) + {}^{-2\alpha^-}\mathbb{Q}_s^0(|Y_{|x-y|} - s|^{qn}), \tag{2.66}$$

which gives the bound stated in this lemma. ■

The next lemma deals with the first spindle to cross level $z > 0$. In this case, we do not have a complete excursion about level $z$, and moreover the distribution of the crossing width is not clear. However, it is sufficient to know that some probability density function $p_z, z \in [0,\infty)$ gives the distribution for the crossing width where the anti-clade has infimum less than or equal to $-z$, i.e. the distribution of

$$\nu_{\text{cld}}(f_{T_0^+}(-\xi_N(T_0^+-))) \in \cdot \,|\, \inf \xi_N \le -z).$$

**Lemma 2.4.15.** *In the setting of Lemma 2.4.14 we have that*

$$\mathbf{E}\left(\left(\sum_{\substack{(t,f)\ atom\ of\ \mathbf{N}:\\ t<T, \ell^z(t)=0,\\ A(f)<\varepsilon_0, \mathbf{X}_{t-}<z<\mathbf{X}_t}} \Big|g\big(f(y-\mathbf{X}_{t-})\big) - g\big(f(x-\mathbf{X}_{t-})\big)\Big|\right)^k\right) \\ \le C_q^k \int_{s=0}^{\varepsilon_0} p_z(s)\Big({}^{-2\alpha^+}\mathbb{Q}_s^0(|Y_{|x-y|} - s|^{qk}) + {}^{-2\alpha^-}\mathbb{Q}_s^0(|Y_{|x-y|} - s|^{qk})\Big)ds \tag{2.67}$$

*where $p_z$ is the probability density function on $[0,\infty)$ that gives the distribution for $\nu_{cld}(m^0 \in \cdot|\zeta^+(N^-) > z)$ (i.e. the distribution for the crossing width under $\nu_{cld}$ conditional on the anti-clade having depth at least $-z$) and where $C_q$ is the Hölder constant given in Assumption B.*

*Proof.* We use the same method as in the previous lemma. The sum on the left-hand-side consists of the first spindle to cross level $z$ a.s. if it exists and has amplitude less than $\varepsilon_0$. We can then apply Lemma 2.3.8 with stopping time $T^{\geq z}$. This tells us that the spindle above level $z$ evolves diffusively. We compare this to the diffusion $\tilde{\mathbf{Y}}$ to upper bound this expectation in the same way as in Lemma 2.4.14. ■

**Theorem 2.4.16.** *Consider a diffusion that satisfies Assumption B. Let* $(\mathbf{N},\mathbf{X})$ *be a scaffolding-and-spindles pair for this diffusion stopped at an a.s. finite random variable* $T$ *on some probability space* $(\Omega,\mathscr{A},\mathbf{P})$. *Then there exists a continuous IP-evolution that is a modification of* $(skewer(y,\mathbf{N},\mathbf{X}), y\in\mathbb{R})$ *which takes values on* $(\mathscr{I}_H(g(E)), d'_H)$.

*Proof.* First assume $g(E)$ is closed and $x, y \geq 0$. Let $\kappa > 0$ be small enough so that $\alpha^+ + \kappa < q$. Let $\mathbf{N}_{M,g(\varepsilon_0)}$ be as in (2.62) but now with $(\mathbf{N},\mathbf{X})$ a scaffolding-and-spindles pair for our diffusion under Assumption B. We first show that there exists $A \geq 0$ such that

$$\mathbf{E}\left(\left(\sum_{\substack{(t,f)\text{ atom of }\mathbf{N}_{M,\varepsilon_0}:\\ \mathbf{X}_{t-}<x<\mathbf{X}_t\\ \text{or }\mathbf{X}_{t-}<y<\mathbf{X}_t}}\left|f(y-\mathbf{X}_{t-})-f(x-\mathbf{X}_{t-})\right|\right)^k\right)\leq A|x-y|^{k(q/2\wedge(q-\alpha^+-\kappa))}. \tag{2.68}$$

Note we do not have the $g$ transformations applied to the spindles $f$ in the equation above as these spindles are from the Poisson random measure for the diffusion under Assumption B and not the untransformed diffusion under Assumption B' as in the previous lemmas. First we upper bound sum in the equation above by

$$\left(\sum_{\substack{(t,f)\text{ atom of}\\ \mathbf{N}_{M,\varepsilon_0}:t<T,\\ \mathbf{X}_{t-}<x<\mathbf{X}_t,\\ \ell^x(t)=0}}+\sum_{\substack{(t,f)\text{ atom of}\\ \mathbf{N}_{M,\varepsilon_0}:t<T,\\ \mathbf{X}_{t-}<y<\mathbf{X}_t,\\ \ell^y(t)=0}}+\sum_{\substack{(t,f)\text{ atom of}\\ \mathbf{N}_{M,\varepsilon_0}:t<T,\\ \mathbf{X}_{t-}<x<\mathbf{X}_t,\\ \ell^x(t)>0}}+\sum_{\substack{(t,f)\text{ atom of}\\ \mathbf{N}_{M,\varepsilon_0}:t<T,\\ \mathbf{X}_{t-}<y<\mathbf{X}_t,\\ \ell^y(t)>0}}\right)\left|f(y-\mathbf{X}_{t-})-f(x-\mathbf{X}_{t-})\right|.$$

Now note that for any $a, b, c, d \geq 0$ we have that $(a+b+c+d)^k \leq 4^{k-1}(a^k+b^k+c^k+d^k)$ and therefore we can deal with each of the sums in the term above individually. We apply Lemmas 2.4.13 and 2.4.14 and the upper bound for $m$ given in (2.53) to obtain an upper bound for the third sum in the equation above:

$$\mathbf{E}\left(\left(\sum_{\substack{(t,f)\text{ atom of }\mathbf{N}_{M,\varepsilon_0}:\\ t<T,\mathbf{X}_{t-}<x<\mathbf{X}_t,\ell^x(t)>0}}\left|f(y-\mathbf{X}_{t-})-f(x-\mathbf{X}_{t-})\right|\right)^k\right)\leq A'|x-y|^{k(q/2\wedge(q-\alpha^+-\kappa))},$$

for some $A' \geq 0$ and similarly for the fourth sum. We deal with the first two sums in the same way, using Lemma 2.4.15 and therefore obtain the upper bound given in (2.68). We now consider the $\mathbf{P}$-random times

$$S_0 := 0; \quad S_n := \inf\{t > S_{n-1} : (t,f)\text{ atom of }\mathbf{N},\ \ A(f)\geq\varepsilon_0\}\text{ for } n\geq 1. \tag{2.69}$$

As $\mathbf{N}$ is a stopped PRM stopped at random time $T$ there exists $P \in \mathbb{N}$ such that $S_{P-1} \leq T < S_P$. For $1 \leq n \leq P$ we define the random measures

$$\mathbf{N}_{M,\varepsilon_0,n} \coloneqq \sum_{\substack{(t,f) \text{ atom of } \mathbf{N}_{M,\varepsilon_0}: \\ t \in [0,T] \cap (S_n, S_{n+1})}} \delta_{(t,f)} \text{ for } n \geq 0$$

so that $\mathbf{N}_{M,\varepsilon_0} = \sum_{n=0}^{P} \mathbf{N}_{M,\varepsilon_0,n}$. Let $n \in \mathbb{N}$. Consider the correspondence between $\text{skewer}(x, \mathbf{N}_{M,\varepsilon_0,n}, \mathbf{X})$ and $\text{skewer}(y, \mathbf{N}_{M,\varepsilon_0,n}, \mathbf{X})$ given by pairing all the intervals generated by spindles that cross both levels $x$ and $y$. This is a correspondence as we only have finitely many spindles cross level $x$ and level $y$ $\mathbf{P}$-a.s and has distortion

$$\sum_{\substack{(t,f) \text{ atom} \\ \text{of } \mathbf{N}_{M,\varepsilon_0,n}: \\ \mathbf{X}_{t-} < x \leq y < \mathbf{X}_t}} \left| f(y - \mathbf{X}_{t-}) - f(x - \mathbf{X}_{t-}) \right| + \max \Big\{ \sum_{\substack{(t,f) \text{ atom} \\ \text{of } \mathbf{N}_{M,\varepsilon_0,n}: \\ x \leq \mathbf{X}_{t-} < y < \mathbf{X}_t}} f(y - \mathbf{X}_{t-}), \sum_{\substack{(t,f) \text{ atom} \\ \text{of } \mathbf{N}_{M,\varepsilon_0}: \\ \mathbf{X}_{t-} < x < \mathbf{X}_t \leq y}} f(x - \mathbf{X}_{t-}) \Big\}.$$

This sum is less than or equal to the sum in (2.68). Therefore we have $\overline{\text{skewer}}(\mathbf{N}_{M,\varepsilon_0,n}, \mathbf{X})$ satisfies the Kolmogorov-Čentsov continuity criterion and so there exists a continuous modification $(\gamma^y_{M,n}, y \geq 0)$. We can take $M$ to infinity as by Proposition 2.4.5 we have that $\sup_{u \in \mathbb{R}} \ell^u(T)$ is finite $\mathbf{P}$-a.s. and so we have a continuous modification $(\gamma^y_n, y \geq 0)$ of $\overline{\text{skewer}}(\mathbf{N}|_{(S_n, S_{n+1})}, \mathbf{X})$.

From here we can construct a continuous modification of $\overline{\text{skewer}}(\mathbf{N}, \mathbf{X})$. For $n \in \mathbb{N}$ let $f_n$ be the $n^{\text{th}}$ spindle of $\mathbf{N}$ with amplitude greater than or equal to $\varepsilon_0$, i.e. so that $\delta_{(S_n, f_n)}$ is an atom of $\mathbf{N}$. Then we define the IP-evolution $(\gamma^y, y \geq 0)$ by

$$\gamma^y \coloneqq \bigstar_{n=1}^{P} \left( \gamma^y_n \star \{(0, f_n(y - \mathbf{X}_{S_n-}))\} \right), \tag{2.70}$$

where $\star$ and $\bigstar$ are the concatenate operations from Definition 2.2.3. Note there are only finitely many spindles with amplitude greater than or equal to $\varepsilon_0$ on $[0,T]$ $\mathbf{P}$-a.s. Therefore we have that $(\gamma^y, y \geq 0)$ is the concatenation of finitely many $d'_H$-continuous IP-evolutions (the IP-evolutions $(\{(0, f_n(y - \mathbf{X}_{S_n}))\}, y \geq 0)$ are clearly $d'_H$-continuous) and therefore $(\gamma^y, y \geq 0)$ is $d'_H$-continuous. Finally we note that concatenating modifications gives modifications and so $(\gamma^y, y \geq 0)$ is a modification of $\overline{\text{skewer}}(\mathbf{N}, \mathbf{X})$ on $(\mathscr{I}_H, d'_H)$. The subspace $(\mathscr{I}_H(g(E)), d'_H)$ is closed and so this modification exists on the appropriate subspace. To finish the proof assume $g(E)$ is not closed. This only occurs if $g(c)$ is finite and $g(E) = [0, g(c))$. In this case however we still obtain our result as we do not take a modification of the diffusions with amplitude greater than $\varepsilon_0$ and as each of these diffusions with amplitude greater than $\varepsilon_0$ do not hit $g(c)$ then $(\gamma^y, y \geq 0)$ does take values on the subspace $(\mathscr{I}_H(g(E)), d'_H)$.

Finally, assume $x, y \leq 0$. Then this case follows ultimately by (2.36). If $x \leq 0 \leq y$ then we have the result by the previous two cases and the triangle inequality. ■

We conclude this section with the analogous result for diffusions under Assumption C.

**Proposition 2.4.17.** *Let* $\mathbf{Z}$ *be a diffusion that satisfies Assumption C, and let* $(\mathbf{N},\mathbf{X})$ *be a scaffolding-and-spindles pair stopped at an a.s. finite stopping time. Then* $\overline{skewer}(\mathbf{N},\mathbf{X})$ *is a.s. continuous and takes values on* $(\mathscr{I}_H(g(E)),d'_H)$.

*Proof.* Let $\mathbf{Y}$ be an untransformed diffusion that satisfies Assumption C' and $g$ be a transformation that satisfies $C3$ of Assumption C such that $\mathbf{Z}=g(\mathbf{Y})$.

By the bounds in (2.54) we have that $M((0,\varepsilon_0))$ is finite. Therefore by Theorem 2.1.3 the Pitman-Yor excursion measure for $\mathbf{Y}$ (equivalently $\mathbf{Z}$) satisfies (2.6). For the diffusion $\mathbf{Z}$ let $(\mathbf{N},\mathbf{X})$ be a scaffolding-and-spindles pair stopped at an a.s. finite stopping time.

To show continuity of the IP-evolution we first show that the sum of the amplitudes of all of the spindles is finite. From Theorem 2.2.7 we know that the intensity of spindles for unstopped $\mathbf{N}$ with amplitude greater than $x$ is $1/s(x)$. Therefore, the amplitudes of the spindles form a Poisson point process with intensity $s'(x)/s(x)^2$ which, when $\mathbf{N}$ is stopped at an a.s. finite random time, is summable a.s. With the substitution $y=s(x)$ we have that it is equal to

$$\int_{x=0}^{b} g(x)\frac{s'(x)}{s(x)^2}dx=\int_{y=0}^{s(b)}\frac{g(s^{-1}(y))}{y^2}dy<\infty \tag{2.71}$$

for some (equivalently all) $b\in(0,c)$ (we have included the integral obtained with the substitution $y=s(x)$ here as we use it below). By the bound in (2.52) on $s'$ we have

$$\frac{\varepsilon_1^{\beta^-}}{1-\beta^-}x^{1-\beta^-}\leq s(x) \text{ for } x\in(0,\varepsilon_1).$$

Therefore for $y=\left(\varepsilon_1^{-\beta^-}(1-\beta^-)\right)^{1/(1-\beta^-)}x^{1/(1-\beta^-)}$ we have

$$x=\frac{\varepsilon_1^{\beta^-}}{1-\beta^-}y^{(1-\beta^-)}\leq s(y) \text{ for } y\in(0,\varepsilon_1).$$

This means that

$$s^{-1}(x)\leq\left(\varepsilon_1^{-\beta^-}(1-\beta^-)\right)^{1/(1-\beta^-)}x^{1/(1-\beta^-)} \qquad \text{for } x<s(\varepsilon_1)$$

and so the second integral in (2.71) is less than or equal to

$$\int_{y=0}^{s(b)}\frac{g\left(\left(\varepsilon_1^{-\beta^-}(1-\beta^-)\right)^{1/(1-\beta^-)}y^{1/(1-\beta^-)}\right)}{y^2}dy \qquad \text{provided that } b<\varepsilon_1.$$

To put this into the form of $C3$ requires a linear transformation $z=ay$ for $a=\left(\varepsilon_1^{-\beta^-}(1-\beta^-)\right)$. Therefore, the integral in (2.71) is finite by $C3$ and the summability of the amplitudes follows.

From here we can show that $\overline{\text{skewer}}(\mathbf{N},\mathbf{X})$ is continuous a.s. with respect to the metric $d'_H$. Let $I$ be an index set for the spindles of $\mathbf{N}$, $\varepsilon > 0$ and $C = \sum_{i\in I} A(f_i) < \infty$ be the sum of amplitudes Let $(t_1,f_1),\dots,(t_K,f_K)$ be atoms of $\mathbf{N}$ such that $\sum_{i=1}^{K} A(f_i) > C - \varepsilon/2$, and let $\tilde{\mathbf{N}}$ be the Poisson random measure $\mathbf{N}$ with the atoms $(t_1,f_1),\dots,(t_K,f_K)$ removed. The spindles $f_i$ are continuous functions with finite lifetimes so are uniformly continuous. Let $\delta > 0$ such that for any $1 \le i \le K$ and any $s$, $t$ with $|s-t| < \delta$ we have $|f_i(s) - f_i(t)| < \varepsilon/2K$. Let $x$, $y \in \mathbb{R}$ with $|x-y| < \delta$. Then, for the interval partitions skewer$(x,\mathbf{N},\mathbf{X})$ and skewer$(y,\mathbf{N},\mathbf{X})$ we can define the correspondence of size at most $K$ that matches every interval generated by spindle $f_i$ in skewer$(x,\mathbf{N},\mathbf{X})$ to the corresponding interval in skewer$(y,\mathbf{N},\mathbf{X})$. The remaining intervals in skewer$(x,\mathbf{N},\mathbf{X})$ or skewer$(y,\mathbf{N},\mathbf{X})$ have sum less than $\varepsilon/2$ and therefore the distortion is equal to

$$\sum_{j=1}^{K} |f_i(x-\mathbf{X}_{t_i-}) - f_i(y-\mathbf{X}_{t_i-})| + \max\left\{ \sum_{(t,f)\text{ atom of }\tilde{\mathbf{N}}} f(x-\mathbf{X}_{t-}), \sum_{(t,f)\text{ atom of }\tilde{\mathbf{N}}} f(y-\mathbf{X}_{t-}) \right\} \le K\frac{\varepsilon}{2K} + \frac{\varepsilon}{2}$$

and therefore $d'_H(\text{skewer}(x,\mathbf{N},\mathbf{X}),\text{skewer}(y,\mathbf{N},\mathbf{X})) \le \varepsilon$. As in the previous result the process takes values on the subspace $(\mathscr{I}_H(g(E)),d'_H)$ as required. ■

### 2.4.5 Further continuity results and proof of Theorem 2.1.7

For a measurable map $g : E \to g(E)$ (where as before $E = [0,c)$ or $E = [0,c]$ for some $c \le \infty$) we define $g^* : \mathscr{I}_H(E) \to g^*(\mathscr{I}_H(E))$ in the following way: for an interval partition $\beta = (U_i, i \in I)$ such that $\sum_{i\in I} g(\text{Leb}(U_i))$ is finite we define $g^*(\beta)$ to be the interval partition $\gamma := (V_i, i \in I)$ where we have $\text{Leb}(V_i) = g(\text{Leb}(U_i))$ and the order of the intervals in $\beta$ and $\gamma$ is consistent; when $\sum_{i\in I} g(\text{Leb}(U_i))$ is infinite we take $g^*(\beta) := \emptyset$, the empty interval partition. Note that, $g^*(\mathscr{I}_H(E)) \subseteq \mathscr{I}_H(g(E))$. Also, as $g$ is an injection, $g^*$ restricted to $\{\beta \in \mathscr{I}_H(E) : \sum_{U\in\beta} g(\text{Leb}(U)) < \infty\}$ has an inverse $(g^*)^{-1}$ and for any $\gamma \in g^*(\mathscr{I}_H(E))$ we have that $(g^*)^{-1}(\gamma) = (g^{-1})^*(\gamma)$. We also note that $g^*$ is a measurable function. To see this, first define the functions $(g_n, n \in \mathbb{N})$ by $g_n(x) = g(x)$ if $x > 1/n$ and $g_n(x) = x$ otherwise and observe that $g^*$ is the limit of the sequence of the functions $(g_n^*, n \in \mathbb{N})$. The measurability of $g^*$ then follows by the fact that for all $n \in \mathbb{N}$ the functions $g_n^*$ are continuous. The proof of continuity of the functions $(g_n^*, n \in \mathbb{N})$ follows from the continuity of $g$ and that for any interval partition $\beta$ the interval partition differs from its image $g_n^*(\beta)$ on only finitely many interval widths (see also Lemma 2.4.29).

The next step is to develop a simple Markov property. In Corollary 2.3.13 we have that for an interval partition $\beta \in \mathscr{I}_H(E)$ and a diffusion that satisfies Assumption B', we can construct a scaffolding-and-spindles pair $(\mathbf{N}_\beta,\mathbf{X}_\beta)$ as in Definition 2.3.11. We state a corollary here for diffusions under Assumption B.

**Corollary 2.4.18.** *Consider a diffusion* $\mathbf{Z}$ *that satisfies Assumption B and let* $\gamma \in g^*(\mathscr{I}_H(E))$. *Consider a scaffolding-and-spindles pair started from* $\gamma$, *denoted as* $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ *as in Definition 2.3.11. Then we have that* $len(\mathbf{N}_\gamma) = \sum_{U\in\gamma} len(\mathbf{N}_U)$ *is finite a.s.*

*Proof.* Let $\mathbf{Y}$ be the untransformed diffusion that satisfies Assumption B', let $\gamma \in g^*(\mathscr{I}_H(E))$ and then let $\beta \in \mathscr{I}_H(E)$ such that $g^*(\beta) = \gamma$. As mentioned above, by Corollary 2.3.13 we can construct a scaffolding-and-spindles pair $(\mathbf{N}^{\mathbf{Y}}_\beta, \mathbf{X}^{\mathbf{Y}}_\beta)$ for $\mathbf{Y}$ started from $\beta$ and $\text{len}(\mathbf{N}_\beta)$ is finite a.s. Therefore we can define a scaffolding-and-spindles pair for diffusion $\mathbf{Z}$ and started from $\gamma$ by

$$\mathbf{N}^{\mathbf{Z}}_\gamma = \sum_{(t,f) \text{ atom of } \mathbf{N}^{\mathbf{Y}}_\beta} \delta_{(t,g(f))},$$

where for a function $(f(x), x \in \mathbb{R})$ we write $g(f)$ to be the function $(g(f(x)), x \in \mathbb{R})$. Then define $\mathbf{X}^{\mathbf{Z}}_\gamma = \xi(\mathbf{N}^{\mathbf{Z}}_\gamma)$ which is equal to $\mathbf{X}^{\mathbf{Y}}_\beta$ as the spatial transformation of the spindles does not change the scaffolding. ■

We now state a continuity result for an IP-evolution with a fixed initial state. For $N \in \mathscr{N}^{\text{sp}}_{\text{fin}}$, let $\zeta^+(N) := \sup_{t\in\mathbb{R}} \xi_N(t)$.

**Lemma 2.4.19.** *Consider a diffusion* $\mathbf{Z}$ *that satisfies Assumption B and let* $\gamma \in g^*(\mathscr{I}_H(E))$. *Let* $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ *be a scaffolding-and-spindles pair* $\gamma$ *as in Definition 2.3.11. Then there exists a modification of* $\overline{skewer}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ *that we call* $(\gamma^y, y \geq 0)$ *that is continuous for* $y \in (0,\infty)$. *Furthermore, for* $y \in (0,\infty)$ *fixed, we have* $\gamma^y \in g^*(\mathscr{I}_H(E))$ *a.s.*

*Proof.* Let $n \in \mathbb{N}$ and $\gamma \in g^*(\mathscr{I}_H(E))$. From Corollary 2.4.18 we can construct $(\mathbf{N}_U, U \in \gamma)$ as in Definition 2.3.11. Recall that $\mathbf{N}_U := \delta_{(0,f_U)} + \mathbf{N}|_{[0,T^{-\zeta(f_U)}]}$ where $\mathbf{N}$ is a Poisson random measure. The process $(\text{skewer}(y, \mathbf{N}|_{[0,T^{-\zeta(f_U)}]}, \xi(\mathbf{N}|_{[0,T^{-\zeta(f_U)}]})), y \in \mathbb{R})$ has a continuous modification by Theorem 2.4.16 and it follows that $\overline{\text{skewer}}(\mathbf{N}_U, \mathbf{X}_U)$ has a continuous modification. Also, from the fact that $\sum_{U\in\gamma} \text{len}(\mathbf{N}_U)$ is finite a.s., from Corollary 2.4.18, we have that for only finitely many $U \in \gamma$ that $\zeta^+(\mathbf{N}_U) > 1/n$. Therefore we can conclude that there exists a continuous modification of $\overline{\text{skewer}}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ on $(1/n, \infty)$. As this is true for all $n \in \mathbb{N}$, we have a continuous modification on $(0,\infty)$.

It remains to show that for $y \in (0,\infty)$ fixed, we have $\gamma^y \in g^*(\mathscr{I}_H(E))$ a.s. Let $y > 0$. For each $U \in \gamma$ we have that $\mathbf{N}_U$ is constructed by an initial incomplete spindle $f_U$ and then a stopped Poisson random measure $\mathbf{N}$ with stopping time $T^{-\zeta(f_U)}$. We claim that
$\text{skewer}(y, \mathbf{N}_U, \mathbf{X}_U) \in g^*(\mathscr{I}_H(E))$ a.s. By the spatial homogeneity of the scaffolding-and-spindles pair and the independence of $f_U$ and $\mathbf{N}$ it is sufficient to show that $\text{skewer}(0, \mathbf{N}, \mathbf{X}) \in g^*(\mathscr{I}_H(E))$ a.s. We have that $\text{skewer}(0, \mathbf{N}, \mathbf{X})$ is in $\mathscr{I}_H(E)$ by the first part of Proposition 2.4.1 and we can conclude that it is in fact in $g^*(\mathscr{I}_H(E))$ by again using the first part of Proposition 2.4.1 with the identity transformation $g(x) = x$ (this tells us the untransformed spindles produce an interval

partition and therefore the transformed spindles produce an interval partition in $g^*(\mathscr{I}_H(E))$ a.s.).

We concatenate the interval partitions skewer$(y, \mathbf{N}_U, \mathbf{X}_U)$ for the finitely many of the $U \in \gamma$ with $\zeta^+(\mathbf{N}_U) > 1/n$ to form skewer$(y, \mathbf{N}_\gamma, \mathbf{X}_\gamma)$ and conclude that skewer$(y, \mathbf{N}_\gamma, \mathbf{X}_\gamma) \in g^*(\mathscr{I}_H(E))$ a.s. and therefore the modification $\gamma^y \in g^*(\mathscr{I}_H(E))$ a.s. ■

We define the filtration $(\mathscr{F}^y_{\mathscr{J}}, y \geq 0)$ to be the right-continuous filtration generated by $(\gamma^y, y \in (0,\infty))$ on $\Omega := \mathscr{C}((0,\infty), \mathscr{I}_H)$, the class of all paths in $\mathscr{I}_H$ continuous on $(0,\infty)$. We define a filtered probability space $(\Omega, \mathscr{F}_{\mathscr{J}}, (\mathscr{F}^y_{\mathscr{J}}, y \geq 0), \mathbb{P}_\gamma)$ where $\mathscr{F}_{\mathscr{J}} := \bigcup_{y \geq 0} \mathscr{F}^y_{\mathscr{J}}$, and $\mathbb{P}_\gamma(\cdot)$ is the probability measure

$$\mathbb{P}_\gamma(\cdot) = \mathbf{P}((\gamma^y, y \geq 0) \in \cdot) \tag{2.72}$$

where $(\gamma^y, y \geq 0)$ is the modification of $\overline{\text{skewer}}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ that is continuous on $(0,\infty)$ that is given in the lemma above.

**Corollary 2.4.20** (Simple Markov property under Assumption B)**.** *Consider a diffusion under Assumption B. Let $\mu$ be a probability distribution on interval partitions that is supported on $g^*(\mathscr{I}_H(E))$, and define the probability measure $\mathbb{P}_\mu(\cdot) = \int_{\gamma \in g^*(\mathscr{I}_H(E))} \mathbb{P}_\gamma(\cdot)\mu(d\gamma)$. For any $y \in (0,\infty)$ let $\theta_y$ be the shift operator, and let $\eta, f : \mathscr{C}((0,\infty), \mathscr{I}_H) \to [0,\infty)$ be measurable where $\eta$ is measurable with respect to $\mathscr{F}^y_{\mathscr{J}}$. Then we have that*

$$\mathbb{P}_\mu\big[\eta f \circ \theta_y\big] = \mathbb{P}_\mu\big[\eta \mathbb{P}_{\gamma^y}[f]\big].$$

*Proof.* This proof uses the same arguments as the analogous result [34, Corollary 6.13]; specifically, by Lemma 2.4.19 we have that the sample paths a.s. are in $\Omega$ and the simple Markov property for the skewer process on a larger probability space is given in Proposition 2.3.20. ■

We now show the analogous result for diffusions under Assumption C and prove Theorem 2.1.7. We let $\mathscr{I}_{\text{fin}} \subset \mathscr{I}_H$ be the interval partitions that consist of finitely many intervals and $\mathscr{I}_{\text{fin}}(g(E)) = \mathscr{I}_{\text{fin}} \cap \mathscr{I}_H(g(E))$. For any interval partition $\gamma \in \mathscr{I}_{\text{fin}}(g(E))$ we define $\mathbb{P}_\gamma$ as before and note that the scaffolding-and-spindles pair $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ can always be constructed for such initial interval partitions $\gamma$. Let $\Omega_0 := \mathscr{C}([0,\infty), \mathscr{I}_H)$ be the space of continuous paths on $\mathscr{I}_H$ indexed by $[0,\infty)$.

*Proof of Theorem 2.1.7.* Let $\mathbf{Z}$ be a diffusion under Assumption C, and let $\gamma \in \mathscr{I}_{\text{fin}}(g(E))$. Let $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ be a scaffolding-and-spindles pair started at $\gamma$ as defined in Definition 2.3.11. For $y \geq 0$ let $\gamma^y = \text{skewer}(y, \mathbf{N}_\gamma, \mathbf{X}_\gamma)$. Then $(\gamma^y, y \geq 0)$ is continuous by the same reasoning as in Proposition 2.4.17; for each $y$ we have only concatenated finitely many interval partitions.

In this setting let $(\mathscr{F}^y_{\mathscr{J}}, y \geq 0)$ be the right-continuous filtration generated by $(\gamma^y, y \geq 0)$ on the smaller space $\Omega_0$, rather than on $\Omega$ as we had done previously. The simple Markov property for the skewer process from Proposition 2.3.20 applies in this setting as for fixed $y > 0$ the

interval partition $\gamma^y \in \mathscr{I}_{\text{fin}}(g(E))$ a.s. To see this, note that for each $U \in \gamma$ the SPLP $\mathbf{X}_U$ is of bounded variation and so only crosses a fixed level finitely many times a.s., [9, Corollary VII.5], and therefore $\gamma^y$ consists of finitely many intervals as it is the concatenation of finitely many interval partitions that each consist of finitely many intervals. To conclude the proof for the setting described in Theorem 2.1.7 we note that, for $n \in \mathbb{N}$, if we stop the PRM $\mathbf{N}$ of a scaffolding-and-spindles pair $(\mathbf{N},\mathbf{X})$ at the $n^{\text{th}}$ hitting time of 0 then $\text{skewer}(0,\mathbf{N},\mathbf{X}) \in \mathscr{I}_{\text{fin}}(g(E))$ a.s. ■

We used $\Omega_0 \subset \Omega$ under Assumption C whereas under Assumption B we were required to use $\Omega$ as we did not have a proof of continuity at time 0. However, if we restrict the class of initial interval partitions then we can use $\Omega_0$ to state the simple Markov property under Assumption B, and we do this in the following subsection. This subsection is not necessary to prove Theorem 2.1.6 but we include it to show the class of initial interval partitions for which we have a proof of continuity at time 0.

#### 2.4.5.1 Continuity at time 0 under Assumption B

To prove the results of this section we define certain classes of interval partitions. For any $q > 0$ and any class of interval partitions $\mathscr{I} \subseteq \mathscr{I}_H$ we define

$$\mathscr{I}^{(q)} := \{\beta = (U_i, i \in I) \in \mathscr{I} : \sum_{i \in I} \text{Leb}(U_i)^q < \infty\}.$$

Note that for $q \geq 1$ we have that $\mathscr{I}^{(q)} = \mathscr{I}$. We now state a corollary of Theorem 2.4.16.

**Lemma 2.4.21.** *Consider a diffusion that satisfies Assumption B' with a stopped scaffolding-and-spindles pair* $(\mathbf{N},\mathbf{X})$ *on some probability space* $(\Omega,\mathscr{A},\mathbf{P})$*, and let* $q > \alpha^+$*. Then, for* $(\text{skewer}(y,\mathbf{N},\mathbf{X}), y \geq 0)$*, there exists a modification* $(\beta^y, y \geq 0)$ *which is continuous on* $(\mathscr{I}_H^{(q)}(E), d'_H)$*.*

*Proof.* We first note that for $q \in (0,1)$ we can apply Theorem 2.4.16 with a diffusion $\mathbf{Z}$ constructed from an untransformed diffusion $\mathbf{Y}$ and transformation $h(x) = x^q$, to produce the continuous IP-evolution $(\phi^y, y \geq 0)$. Also, note that the map $(h^{-1})^*$ is continuous on the space of interval partitions $\mathscr{I}_H$ (this ultimately follows because the map $h^{-1}$ is Lipschitz on $[0,1]$, see also Lemma 2.4.29). Therefore $((h^{-1})^*(\phi^y), y \geq 0)$ is $\mathbf{P}$-a.s. continuous in $\mathscr{I}_H$.

Now, $(\phi^y), y \geq 0)$ is a modification of $h^*(\overline{\text{skewer}}(\mathbf{N},\mathbf{X}))$ and therefore $((h^{-1})^*(\phi^y), y \geq 0)$ is a modification of $\overline{\text{skewer}}(\mathbf{N},\mathbf{X})$. Also, again by Theorem 2.4.16, there exists a continuous process $(\beta^y, y \geq 0)$ which is a modification of $\overline{\text{skewer}}(\mathbf{N},\mathbf{X})$ and so is also a modification of $((h^{-1})^*(\phi^y), y \geq 0)$. As both $(\beta^y, y \geq 0)$ and $((h^{-1})^*(\phi^y), y \geq 0)$ are both continuous processes they are therefore indistinguishable from one another, and therefore $(\beta^y, y \geq 0)$ is a continuous process in $\mathscr{I}_H^{(q)}(E)$ a.s. ■

As in the previous sections, we consider a diffusion $\mathbf{Z}$ under Assumption B, which is equal in distribution to $g(\mathbf{Y})$ where $\mathbf{Y}$ is a diffusion under Assumption B' and $g : E \to g(E)$ is a continuous bijection as in Assumption B. Let $q_0$, $\alpha^+$, $\alpha^-$ be the associated constants and let $q \in (\alpha^+, q_0)$. Finally, let $m_{\mathbf{Y}}$ and $m_{\mathbf{Z}}$ be the speed measure densities of $\mathbf{Y}$ and $\mathbf{Z}$. We now state a corollary of Lemma 2.3.16 and Lemma 2.4.21.

**Corollary 2.4.22.** *Consider a stopped scaffolding-and-spindles pair $(\mathbf{N}, \mathbf{X})$ for the diffusion $\mathbf{Z}$ which satisfies Assumption B, and let $(\gamma^y, y \geq 0)$ be a continuous modification of $\overline{skewer}(\mathbf{N}, \mathbf{X})$. Then for any fixed $y \geq 0$ we have that*

$$\gamma^y \in g^*(\mathscr{I}^{(q)}_{m_{\mathbf{Y}}}(E)) = \left\{\gamma = (V_i, i \in I) \in g^*(\mathscr{I}_{m_{\mathbf{Y}}}(E)) : \sum_{i \in I} g^{-1}(Leb(V_i))^q < \infty\right\} \ \mathbf{P} - a.s.$$

*Proof.* Let $\tilde{\mathbf{N}}$ be the Poisson random measure of untransformed spindles. The $m_{\mathbf{Y}}$-diversity of $\overline{\text{skewer}}(\tilde{\mathbf{N}}, \mathbf{X})$ at any fixed level has already been established in Lemma 2.3.16. This shows that $\text{skewer}(y, \mathbf{N}, \mathbf{X}) \in g^*(\mathscr{I}_{m_{\mathbf{Y}}}(E))$ a.s. and so $\gamma^y \in g^*(\mathscr{I}_{m_{\mathbf{Y}}}(E))$.

It remains to show that $\gamma^y \in g^*(\mathscr{I}^{(q)}_H(E))$. To do this we note that the process $(\gamma^y, y \geq 0)$ is a modification of $g^*(\overline{\text{skewer}}(\tilde{\mathbf{N}}, \mathbf{X}))$. Therefore $((g^{-1})^*(\gamma^y), y \geq 0)$ is a modification of $\overline{\text{skewer}}(\tilde{\mathbf{N}}, \mathbf{X})$. By Lemma 2.4.21 $\overline{\text{skewer}}(\tilde{\mathbf{N}}, \mathbf{X})$ has a continuous modification that takes values in $\mathscr{I}^{(q)}_H(E)$, and the result follows as for fixed $y \geq 0$ we have that $(g^{-1})^*(\gamma^y) = \text{skewer}(y, \tilde{\mathbf{N}}, \mathbf{X})$ a.s. ■

The following lemma concerns the initial spindles of the IP-evolution, i.e. the spindles that are not equal to 0 at time 0.

**Lemma 2.4.23.** *Consider a diffusion that satisfies Assumption B'. Let $(a_i, i \in I)$ be a potentially infinite sequence of positive numbers such that $\sum_{i \in I} a_i^{q \wedge 1} < \infty$. Consider a family of independent diffusions $(f_i, i \in I)$ distributed as independent diffusions started from $(a_i, i \in I)$ stopped at 0 on some common probability space $(\Omega, \mathscr{A}, \mathbf{P})$, i.e. $f_i \sim \mathbb{Q}^0_{a_i}$. Then we have that $\sum_{i \in I} A(f_i)^q < \infty$ $\mathbf{P}$-a.s.*

*Proof.* By the coupling given in (2.57) we have for $0 < a < y < \varepsilon_0$ that $\mathbb{Q}^0_a(A > y) \leq {}^{-2\alpha^-}\mathbb{Q}^0_a(A > y)$ which in turn is equal to $(a/y)^{1+\alpha^-}$ (by (2.2) and as $x^{1+\alpha^-}$ for $x \in [0, \infty)$ is a scale function for a squared Bessel process with dimension $-2\alpha^-$). Therefore we have

$$\sum_{i \in I : a_i < \varepsilon_0} \mathbf{P}(A(f_i) > \varepsilon_0) \leq \varepsilon_0^{-1-\alpha^-} \sum_{i \in I} a_i^{1+\alpha^-} < \infty,$$

and so by the Borel-Cantelli lemma and by the fact that only finitely many $a_i$ are such that $a_i \geq \varepsilon_0$, we have that a.s. only finitely many of the incomplete spindles $f_i$ will have amplitude greater than $\varepsilon_0$.

Furthermore, again with the upper bound for $\mathbb{Q}^0_a(A > y)$ stated above, we have

$$\mathbf{E}(A(f_i)\mathbb{1}_{A(f_i)<\varepsilon_0}) = \int_{y=0}^{\varepsilon_0} \mathbf{P}(y \le A(f_i) < \varepsilon) dy.$$

This is less than or equal to

$$\int_{y=0}^{\varepsilon_0} \mathbb{Q}^0_{a_i}(A > y) dy = \int_{y=0}^{\varepsilon_0} \Big( (a_i/y)^{1+\alpha^-} \wedge 1 \Big) dy = \Big( \frac{1}{\alpha^-} + 1 \Big) a_i - \frac{a_i^{1+\alpha^-}}{\alpha^- \varepsilon_0^{\alpha^-}} \le \Big( \frac{1}{\alpha^-} + 1 \Big) a_i.$$

Therefore, when $q < 1$, we have that

$$\mathbf{E}(A(f_i)^q \mathbb{1}_{A(f_i)<\varepsilon_0}) \le \Big( \mathbf{E}(A(f_i)\mathbb{1}_{A(f_i)<\varepsilon_0}) \Big)^q \le C_q \Big( \frac{1}{\alpha^-} + 1 \Big)^q a_i^q$$

where $C_q$ is the Hölder constant from Assumption B. Therefore we have that $\sum_{i \in I} \mathbf{E}(A(f_i)^q : A(f_i) < \varepsilon_0)$ is finite, and so $\sum_{i \in I : A(f_i) < \varepsilon_0} A(f_i)^q$ is finite $\mathbf{P}$-a.s. ■

In the next lemma we give a result that compares certain interval partitions with skewer$(0, \mathbf{N}, \mathbf{X})$ for a particular stopped scaffolding-and-spindles pair $(\mathbf{N}, \mathbf{X})$, which we then use in a coupling argument in the proposition after to prove continuity of a IP-evolutions with certain initial interval partitions.

**Lemma 2.4.24** (Interval partition domination)**.** *Consider a diffusion* $\mathbf{Z}$ *that satisfies Assumption B and let* $\gamma \in \overline{\mathscr{I}}_{m_\mathbf{Z}}(g(E))$*. Recall the notation* $\mathscr{D}$ *and* $\overline{\mathscr{D}}$ *from Definition 2.3.15 and let* $(\mathbf{N}, \mathbf{X})$ *be a scaffolding-and-spindles pair stopped at* $\tau^0(k_0^{-1}(\overline{\mathscr{D}}^{m_\mathbf{Z}}_\gamma(\infty) + 1))$ *(where* $\tau^0$ *is the inverse local time of* $\mathbf{X}$ *at level* $0$ *and* $k_0$ *is the constant from Lemma 2.3.3), and let* $\tilde{\gamma} := skewer(0, \mathbf{N}, \mathbf{X})$*. Firstly we have that*

$$\mathscr{D}^{m_\mathbf{Z}}_{\tilde{\gamma}}(\infty) = \overline{\mathscr{D}}^{m_\mathbf{Z}}_\gamma(\infty) + 1 \ a.s.$$

*Also, with positive probability we have that there exists a matching between* $\gamma$ *and* $\tilde{\gamma}$ *such that every interval of* $\gamma$ *is matched with a larger interval of* $\tilde{\gamma}$ *(but that does not necessarily respect the total-order of the intervals). We say that* $\tilde{\gamma}$ *dominates* $\gamma$ *by matching in this case. Finally we have that* $g^*(\overline{\mathscr{I}}_{m_\mathbf{Y}}(E)) = \overline{\mathscr{I}}_{m_\mathbf{Z}}(g(E))$ *and also* $g^*(\mathscr{I}_{m_\mathbf{Y}}(E)) = \mathscr{I}_{m_\mathbf{Z}}(g(E))$*.*

*Proof.* The diversity of $\tilde{\gamma}$ follows from Lemma 2.3.16. The second part of this lemma generalises [34, Lemma 6.8] to interval partitions with finite $m_\mathbf{Z}$-diversity and the proof is the same. It requires the aggregate health summability under Assumption B which was shown in Proposition 2.4.1. The result further generalises to any interval partition with a finite limsup-$m_\mathbf{Z}$-diversity as intervals can be added to an interval partition with finite limsup-$m_\mathbf{Z}$-diversity to produce an interval partition with a finite $m_\mathbf{Z}$-diversity and, therefore, the domination result carries over.

For the final part first note that if a pair of interval partitions $\hat{\beta}$, $\hat{\gamma} \neq \emptyset$ satisfy $g^*(\hat{\beta}) = \hat{\gamma}$ then $\hat{\beta} \in \mathscr{I}_{m_\mathbf{Y}}(E)$ if and only if $\hat{\gamma} \in \mathscr{I}_{m_\mathbf{Z}}(g(E))$ and also $\mathscr{D}^{m_\mathbf{Y}}_{\hat{\beta}}(\infty) = \mathscr{D}^{m_\mathbf{Z}}_{\hat{\gamma}}(\infty)$ (and similarly for

limsup diversities). Therefore for $\hat{\gamma} \in \overline{\mathscr{I}}_{m_{\mathbf{Z}}}(g(E))$, to show $\hat{\gamma} \in g^*(\overline{\mathscr{I}}_{m_{\mathbf{Y}}}(E))$ it suffices to show that $\sum_{V\in\hat{\gamma}} g^{-1}(\mathrm{Leb}(V))$ is finite.

Let $\hat{\gamma} \in \overline{\mathscr{I}}_{m_{\mathbf{Z}}}(g(E))$ and for the diffusion $\mathbf{Z}$ let $(\mathbf{N},\mathbf{X})$ be a scaffolding-and-spindles pair stopped at $\tau^0(k_0^{-1}(\overline{\mathscr{D}}_{\gamma}^{m_{\mathbf{Z}}}(\infty)+1))$. Let $\tilde{\gamma} := \mathrm{skewer}(0,\mathbf{N},\mathbf{X})$. As untransformed diffusions also satisfy Assumption B (as the identity transformation $g(x)=x$ satisfies the hypotheses in Assumption B) we also have the aggregate health summability result for the untransformed diffusions by Proposition 2.4.1. Therefore we have $\tilde{\beta} := (g^{-1})^*(\tilde{\gamma})$ is also a non-trivial interval partition (as well as $\tilde{\gamma}$) because $\tilde{\beta}$ is equal in distribution to $\mathrm{skewer}(0,\tilde{\mathbf{N}},\tilde{\mathbf{X}})$ for a stopped scaffolding-and-spindles pair $(\tilde{\mathbf{N}},\tilde{\mathbf{X}})$ for the untransformed diffusion. Furthermore as we have with positive probability that $\tilde{\gamma}$ dominates $\hat{\gamma}$ we can take a realisation of $\tilde{\gamma}$ and $\tilde{\beta}$ where $\tilde{\gamma}$ dominates $\hat{\gamma}$ and we find that

$$\sum_{V\in\hat{\gamma}} g^{-1}(\mathrm{Leb}(V)) \leq \sum_{V\in\tilde{\gamma}} g^{-1}(\mathrm{Leb}(V)) = \sum_{U\in\tilde{\beta}} \mathrm{Leb}(U) = \|\tilde{\beta}\| < \infty$$

and therefore we have that $\hat{\gamma} \in g^*(\overline{\mathscr{I}}_{m_{\mathbf{Y}}}(E))$ as required.

Therefore we can conclude that $g^*(\overline{\mathscr{I}}_{m_{\mathbf{Y}}}(E)) = \overline{\mathscr{I}}_{m_{\mathbf{Z}}}(g(E))$ as it is immediate that $g^*(\overline{\mathscr{I}}_{m_{\mathbf{Y}}}(E)) \subseteq \overline{\mathscr{I}}_{m_{\mathbf{Z}}}(g(E))$. The other case $g^*(\mathscr{I}_{m_{\mathbf{Y}}}(E)) = \mathscr{I}_{m_{\mathbf{Z}}}(g(E))$ follows by the same reasoning. ■

**Proposition 2.4.25.** *Let $\gamma = (V_i, i\in I) \in g^*(\overline{\mathscr{I}}^{(q)}_{m_{\mathbf{Y}}}(E))$ and let $(\mathbf{N}_\gamma,\mathbf{X}_\gamma)$ be a scaffolding-and-spindles pair for $\gamma$ as in Definition 2.3.11. Then there exists a modification of $\overline{\mathrm{skewer}}(\mathbf{N}_\gamma,\mathbf{X}_\gamma)$ that is continuous on $[0,\infty)$.*

*Proof.* As in the proof of Theorem 2.4.16 first assume that $g(E)$ is closed. Let $\gamma \in g^*(\overline{\mathscr{I}}_{m_{\mathbf{Y}}}(E)) = \overline{\mathscr{I}}_{m_{\mathbf{Z}}}(g(E))$ and let $\tilde{\gamma}$ be a random interval partition such that $\mathscr{D}_{\tilde{\gamma}}^{m_{\mathbf{Z}}}(\infty) = \overline{\mathscr{D}}_{\gamma}^{m_{\mathbf{Z}}}(\infty)+1$ and that dominates $\gamma$ as in Lemma 2.4.24. We describe a coupling for a scaffolding-and-spindles pair $(\mathbf{N}_\gamma,\mathbf{X}_\gamma)$ and another stopped scaffolding-and-spindles pair $(\mathbf{N},\mathbf{X})$.

Let $(\tilde{V}_i, i\in\tilde{I})$ be the intervals of $\tilde{\gamma}$ and let $I\subset\tilde{I}$ so that $(V_i,i\in I)$ denotes the matching intervals of $\gamma$. For each $i\in I$, we consider an independent Brownian motion $(B^i_y, y\geq 0)$ and we define $f_{\tilde{V}_i}$ and $f_{V_i}$ to be strong solutions to the stochastic differential equations

$$\begin{aligned} df_{\tilde{V}_i}(y) &= \mathrm{Leb}(\tilde{V}_i) + \mu_{\mathbf{Z}}(f_{\tilde{V}_i}(y))dy + \sigma_{\mathbf{Z}}(f_{\tilde{V}_i}(y))dB^i_y; \\ df_{V_i}(y) &= \mathrm{Leb}(V_i) + \mu_{\mathbf{Z}}(f_{V_i}(y))dy + \sigma_{\mathbf{Z}}(f_{V_i}(y))dB^i_y, \end{aligned}$$

and we stop $f_{\tilde{V}_i}$ and $f_{V_i}$ when they hit 0. With this construction we have that $(f_{\tilde{V}_i}, i\in I)$ are independent diffusions with the same dynamics and with initial states $(\tilde{V}_i, i\in I)$, and similarly for $(f_{V_i}, i\in I)$. We also have for each $i\in I$ that $\zeta(f_{V_i}) \leq \zeta(f_{\tilde{V}_i})$ and for each $y>0$ that $f_{V_i}(y) \leq f_{\tilde{V}_i}(y)$. We then define a pair of point measures $(\mathbf{N},\mathbf{N}_\gamma)$. For $z>0$, let $\mathbb{Q}^0_z$ denote the law of the diffusion

$\mathbf{Z}$ started from $z$, and let $\nu$ be the Pitman-Yor excursion measure of $\mathbf{Z}$. Consider independent Poisson random measures $(\mathbf{N}_i, i \in I)$ with intensity $\mathrm{Leb} \otimes \nu$ and scaffolding $\xi(\mathbf{N}_i) = \mathbf{X}_i$ for $i \in I$; we use this to define

$$\mathbf{N}_{\tilde{V}_i} := \delta(0, f_{\tilde{V}_i}) + \mathbf{N}_i|_{[0,\tilde{T}_i]} \quad \text{where} \quad \tilde{T}_i := \inf\{t > 0 : \mathbf{X}_i(t) = -\zeta(f_{\tilde{V}_i})\}.$$

We further define $\mathbf{N}_\gamma := \bigstar_{i\in I} \mathbf{N}_{V_i}$ where $\mathbf{N}_{V_i}$ is given by

$$\mathbf{N}_{V_i} := \delta(0, f_{V_i}) + \mathbf{N}_i|^{\leftarrow}_{(T_i,\tilde{T}_i]} \quad \text{with} \quad T_i := \inf\{t \geq 0 : \mathbf{X}_i(t) = \zeta(f_{V_i}) - \zeta(f_{\tilde{V}_i})\} \tag{2.73}$$

and $\tilde{T}_i$ is as stated above. Let $T = \sum_{i\in I} T_i$ and note $T$ is a $\mathbf{P}$-a.s. finite random variable. Finally let $\mathbf{N} := \bigstar_{i\in I} \mathbf{N}_{\tilde{V}_i}$ and $\mathbf{N}_\gamma := \bigstar_{i\in I} \mathbf{N}_{V_i}$.

The rest of the proof uses the same ideas as the proof of Theorem 2.4.16. Let $q$, $\kappa > 0$ such that $\alpha^+ + \kappa < q < q_0$ where $\alpha^+$, $q_0$ are as in Assumption B and let $\rho := (q - \alpha^+ - \kappa) \wedge q/2$. We recall $\mathbf{N}_{M,\varepsilon_0}$ from (2.62) and define $\mathbf{N}_{\gamma,M,\varepsilon_0}$:

$$\mathbf{N}_{M,\varepsilon_0} := \mathbb{1}_{\sup_{u\in\mathbb{R}} \ell^u(T) < M} \sum_{\substack{(t,f) \text{ atom of} \\ \mathbf{N}: A(f) < \varepsilon_0}} \delta_{(t,f)}; \quad \mathbf{N}_{\gamma,M,\varepsilon_0} := \mathbb{1}_{\sup_{u\in\mathbb{R}} \ell^u(T) < M} \sum_{\substack{(t,f) \text{ atom of } \mathbf{N}_\gamma: \\ A(f) < \varepsilon_0, f(0) = 0}} \delta_{(t,f)}$$

where $\ell^u$ for $u \in \mathbb{R}$ denotes the local time of $\xi(\mathbf{N})$. We note that the set of atoms of $\mathbf{N}_{\gamma,M,\varepsilon_0}$ is a subset of the set of atoms of $\mathbf{N}_{M,\varepsilon_0}$ and so we can use the bound in (2.68) as $\mathbf{N}$ is equal in distribution to the concatenation of the clade components of a Poisson random measure stopped at $\tau^0(k_0^{-1}\overline{\mathscr{D}}_\gamma^{m_\mathbf{Z}}(\infty) + 1)$, see (2.30). The constant in front of the bound in (2.68) will change from $A$ to $A/p$ where $p := \mathbf{P}(\tilde{\gamma} \text{ dominates } \gamma) > 0$ and the definition of domination is given in Lemma 2.4.24 (note that the fact that the domination of $\gamma$ by $\tilde{\gamma}$ does not respect the order of the intervals in the interval partition does not impact the upper bound for the sum).

At this point we need to incorporate the initial spindles and the spindles with amplitude greater than $\varepsilon_0$. In the proof of Theorem 2.4.16, we only had the spindles with amplitude greater than $\varepsilon_0$ to add in, and as there were only finitely many of these we could concatenate finitely many continuous IP-evolutions to finish the proof. In this setting however as we have potentially infinitely many initial spindles we need to take care to incorporate these.

We adapt the proof of the Kolmogorov-Čentsov continuity criterion from [88, Theorem 1.2.1]. We define

$$D_m := \{i2^{-m} : i \in \mathbb{N} \cap [0, 2^m)\} \text{ for } m \in \mathbb{N}; \quad D := \bigcup_{m\in\mathbb{N}} D_m$$

and

$$M_\rho := \sup_{x,y\in D, x\neq y} |x-y|^{-\rho} \left\{ \sum_{\substack{(t,f) \text{ atom of } \mathbf{N}_{M,\varepsilon_0}:, \\ \mathbf{X}_{t-} < x < \mathbf{X}_t \text{ or} \\ \mathbf{X}_{t-} < y < \mathbf{X}_t}} \left| g\big(f(y - \mathbf{X}_{t-})\big) - g\big(f(x - \mathbf{X}_{t-})\big) \right| \right\}.$$

Then it follows from the proof in [88, Theorem 1.2.1] that $M_\rho$ is finite $\mathbf{P}$-a.s. as we have shown the bound in (2.68) with $\mathbf{N}_{\gamma,M,\varepsilon_0}$ in place of $\mathbf{N}_{M,\varepsilon_0}$. Let $\mathbf{N}_{\gamma,M} \coloneqq \mathbb{1}_{\sup_{u\in\mathbb{R}} \ell^u(T)<M}\mathbf{N}_\gamma$. From this point we show that $\overline{\text{skewer}}(\mathbf{N}_{\gamma,M},\mathbf{X}_\gamma)$ is $d'_H$-continuous on $D$. Let $\varepsilon>0$, let $K\in\mathbb{N}$, $i_1,\dots,i_K\in I$ such that $\sum_{i\in I:i\neq i_1,\dots,i_K} A(f_{V_i})<\varepsilon/4$, and let $K'\in\mathbb{N}$ and $(t_1,f_1),\dots,(t_{K'},f_{K'})$ be the atoms of $\mathbf{N}_\gamma$ with amplitude greater than or equal to $\varepsilon_0$. There exists $\delta_1>0$ such that for $x,y\in D$ with $|x-y|<\delta_1$ we have

$$\sum_{j=1}^{K}\left|g(f_{V_j}(y))-g(f_{V_j}(x))\right|<\frac{\varepsilon}{4};\quad \sum_{j=1}^{K'}\left|g\big(f_j(y-\mathbf{X}_{t_j-})\big)-g\big(f_j(x-\mathbf{X}_{t_j-})\big)\right|<\frac{\varepsilon}{4}.$$

Furthermore, for $x,y\in D$ with $|x-y|<\delta_2\coloneqq(\varepsilon/4M_\rho)^{1/\rho}$ we have

$$\sum_{\substack{(t,f)\text{ atom of }\mathbf{N}_{M,\varepsilon_0}:,\\ \mathbf{X}_{t-}<x<\mathbf{X}_t\text{ or}\\ \mathbf{X}_{t-}<y<\mathbf{X}_t}}\left|g\big(f(y-\mathbf{X}_{t-})\big)-g\big(f(x-\mathbf{X}_{t-})\big)\right|<\frac{\varepsilon}{4}$$

Let $\delta=\min\{\delta_1,\delta_2\}$. Then for all $x,y\in D$ with $|x-y|<\delta$ the sum of $|g(f(y-\mathbf{X}_{t-}))-g(f(x-\mathbf{X}_{t-}))|$ over all atoms $(t,f)$ in $\mathbf{N}_{\gamma,M}$ is less than $\varepsilon$ $\mathbf{P}$-a.s. This is an upper bound for the distortion between $\text{skewer}(x,\mathbf{N}_{\gamma,M},\mathbf{X}_\gamma)$ and $\text{skewer}(y,\mathbf{N}_{\gamma,M},\mathbf{X}_\gamma)$ for the correspondence that matches intervals with each other if they both are generated by the same spindle. This is a correspondence as we have only finitely many spindles with lifetime greater than $|x-y|$; it is the same correspondence used in Theorem 2.4.16. We can take $M$ to infinity at this stage as by Proposition 2.4.5 the local time of $\mathbf{X}$ is continuous and therefore bounded. Therefore we have that $\overline{\text{skewer}}(\mathbf{N}_\gamma,\mathbf{X}_\gamma)$ is continuous on $D$ $\mathbf{P}$-a.s. To finish, we can define for $y\in[0,\infty)$

$$\gamma^y\coloneqq\lim_{\substack{s\to y\\ s\in D}}\text{skewer}(s,\mathbf{N}_\gamma,\mathbf{X}_\gamma)$$

and $(\gamma^y,y\geq 0)$ is a continuous modification of $\overline{\text{skewer}}(\mathbf{N}_\gamma,\mathbf{X}_\gamma)$. This modification is in the appropriate subspace $(\mathscr{I}_H(g(E)),d'_H)$ as this subspace is closed. Now assume $g(E)$ is not closed. To complete the argument in this case we avoid taking modifications of IP-evolutions with intervals great than $\varepsilon_0$. As in Theorem 2.4.16 there are only finitely many spindles with amplitude greater than $\varepsilon_0$ and so we can consider stopping times in (2.69) with $\mathbf{N}_\gamma$ in place of $\mathbf{N}$. Denoting these stopping times by $(S^\gamma_n,n\in\mathbb{N})$ we can obtain finitely many skewer processes $(\overline{\text{skewer}}(\mathbf{N}_\gamma|_{(S^\gamma_n,S^\gamma_{n+1})}),n\leq P)$ for some $P\in\mathbb{N}$ and concatenate these with the skewer processes generated by the individual spindles with amplitude larger than $\varepsilon_0$ as in (2.70). ■

*Remark* 2.4.26. The previous proposition shows path continuity at 0 for certain initial states but not for all possibles ones. This is a partial response to the claim just after [34, Corollary 6.19] that states "We believe that $d'_H$-path-continuity extends to time 0" which is in reference to their IP-evolutions generated from self-similar diffusions. As their self-similar diffusions satisfy our Assumption B we have confirmed that this belief is correct for at least some initial states.

**Proposition 2.4.27** (Simple Markov property under Assumption B on $\Omega_0$)**.** *Consider a diffusion under Assumption B. Let $\gamma \in g^*(\overline{\mathscr{I}}^{(q)}_{m_{\mathbf{Y}}}(E))$ and let $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ be a scaffolding-and-spindles pair with initial state skewer$(0, \mathbf{N}_\gamma, \mathbf{X}_\gamma) = \gamma$ a.s. as in Definition 2.3.11. Let $(\gamma^y, y \geq 0)$ be a modification of $\overline{\text{skewer}}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ that is continuous on $[0,\infty)$ which exists by Proposition 2.4.25. Recall the filtration $(\mathscr{F}^y_{\mathscr{J}}, y \geq 0)$ from Corollary 2.4.20. We define $(\Omega_0, \mathscr{F}, (\mathscr{F}^y_{\mathscr{J}}, y \geq 0), \mathbb{P}_\gamma)$, a filtered probability space, where $\Omega_0$ is the set of all paths over $\mathscr{I}_H$ that are continuous on $[0,\infty)$ and with initial states in $g^*(\overline{\mathscr{I}}^{(q)}_{m_{\mathbf{Y}}})$, $\mathscr{F} := \bigcup_{y\geq 0} \mathscr{F}^y_{\mathscr{J}}$, and also $\mathbb{P}_\gamma(\cdot) := \mathbf{P}((\gamma^y, y \geq 0) \in \cdot)$. Finally, let $\mu$ be a probability distribution on interval partitions that is supported on $g^*(\overline{\mathscr{I}}^{(q)}_{m_{\mathbf{Y}}}(E))$, and define the probability measure $\mathbb{P}_\mu(\cdot) = \int_{\gamma \in g^*(\mathscr{I}_H(E))} \mathbb{P}_\gamma(\cdot)\mu(d\gamma)$. Then $(\gamma^y, y \geq 0)$ is a simple Markov process on $(\Omega_0, \mathscr{F}, (\mathscr{F}^y_{\mathscr{J}}, y \geq 0), \mathbb{P}_\mu)$.*

*Proof.* This proof uses the same arguments as the analogous result [34, Corollary 6.13]. In particular, we have continuity of the sample paths from Proposition 2.4.25 with initial states in $g^*(\overline{\mathscr{I}}^{(q)}_{m_{\mathbf{Y}}}(E))$ as well as the simple Markov property for the skewer process given in Proposition 2.3.20. Furthermore, at each level we have, for $y \in (0,\infty)$, that the skewer$(y, \mathbf{N}_\gamma, \mathbf{X}_\gamma) \in g^*(\mathscr{I}^{(q)}_{m_{\mathbf{Y}}}(E)) \subset g^*(\overline{\mathscr{I}}^{(q)}_{m_{\mathbf{Y}}}(E))$ a.s. by Corollary 2.4.22. Therefore, for any fixed $y \in (0,\infty)$, the continuous modification takes values in $g^*(\overline{\mathscr{I}}^{(q)}_{m_{\mathbf{Y}}}(E))$ a.s. as required. ■

## 2.4.6 Strong Markov property and proof of Theorem 2.1.6

In this section we assume that the diffusion satisfies Strong Assumption B, that is Assumption B along with the assumption that either $\liminf_{x\to 0} g(x)/x > 0$ or $\limsup_{x\to 0} g(x)/x < \infty$. Note that the identity transformation $g(x) = x$ satisfies these conditions, and therefore Strong Assumption B is more restrictive than Assumption B but more general than Assumption B'. Before we begin the proof we give a class of transformations $g$ that do not satisfy $B5$ of Strong Assumption B to show that this really is a restriction.

**Example 2.4.28** (Transformation $g$ that does not satisfy $B5$)**.** *Consider a function $g : [0, 1/32] \to [0, 1/32]$ such that for all $n \geq 3$ we have:*

$$g(2^{-2^n}) = 2^{-2^n}; \;\; g(2^{-2^n-1}) = 2^{-2^n-n}; \;\; g(2^{-2^{n+1}+1}) = 2^{-2^{n+1}+n},$$

*and where for all $n \geq 3$, $x \in [2^{-2^{n+1}}, 2^{-2^{n+1}+1}]$, $[2^{-2^{n+1}+1}, 2^{-2^n-1}]$, $[2^{-2^n-1}, 2^{-2^n}]$ that $g$ is smooth and strictly increasing and $g(0) = 0$. Then $g$ is a smooth continuous strictly increasing bijection and moreover*

$$\liminf_{x\downarrow 0} \frac{g(x)}{x} \leq \lim_{n\to\infty} \frac{g(2^{-2^n-1})}{2^{-2^n-1}} = \lim_{n\to\infty} 2^{-n+1} = 0;$$

$$\limsup_{x\downarrow 0} \frac{g(x)}{x} \geq \lim_{n\to\infty} \frac{g(2^{-2^{n+1}+1})}{2^{-2^{n+1}+1}} = \lim_{n\to\infty} 2^{n-1} = \infty.$$

We state key results about the transformations $g^*$ and $(g^{-1})^*$ which apply under Strong Assumption B.

**Lemma 2.4.29.** *Let $g : E \to g(E)$ be a strictly increasing continuous function with $g(0) = 0$ where $E \subseteq [0,\infty)$ is of the form $E = [0,c)$ or $E = [0,c]$ for some $c > 0$ and $g(E) \subseteq [0,\infty)$. If $\limsup_{x\to 0} g(x)/x < \infty$ then the map $g^* : \mathscr{I}_H(E) \to \mathscr{I}_H(g(E))$ is injective and continuous. If instead $\liminf_{x\to 0} g(x)/x > 0$ then the map $(g^{-1})^* : \mathscr{I}_H(g(E)) \to \mathscr{I}_H(E)$ is injective and continuous.*

*Proof.* First suppose that $\limsup_{x\to 0} g(x)/x < \infty$. Let $\varepsilon > 0$, let $\beta \in \mathscr{I}_H(E)$ and let $(a_j)_{j\in\mathbb{N}}$ be the interval lengths of $\beta$ in a non-decreasing order and let $(U_j, j \in \mathbb{N})$ be the corresponding intervals of $\beta$. If $g(c)$ is finite let $D = (a_1 + g(c))/2$ and if $g(c)$ is infinite let $D = a_1 + 1$. Let $C \coloneqq \sup_{x\in(0,D]} g(x)/x$. We note that $\sum_{j\in\mathbb{N}} g(a_j) \le C \sum_{j\in\mathbb{N}} a_j$ which is finite so $g^*(\beta)$ is a non-trivial interval partition. The fact that $g^*$ is injective then follows immediately from $g$ being injective.

Let $K \in \mathbb{N}$ be large enough so that $\sum_{j>K} a_j < \varepsilon/4C$. As $g$ is continuous, then $g$ is uniformly continuous on $[0,D]$, and therefore let $\tilde{\varepsilon} > 0$ be small enough so that for $x$, $y \in [0,D]$ with $|x-y| < \tilde{\varepsilon}$ we have $|g(x) - g(y)| < \varepsilon/2K$.

Let $\delta \coloneqq \frac{1}{2}\min\{\tilde{\varepsilon}, a_K/2, \varepsilon/4C, D - a_1\}$. Let $\gamma \in \mathscr{I}_H(E)$ such that $d'_H(\beta,\gamma) < \delta$. Consider a correspondence $(\tilde{U}_j, \tilde{V}_j)_{1\le j\le \tilde{K}}$ from $\beta$ to $\gamma$ with distortion less than $\delta$. Since $\delta < a_K$, we must have that $\tilde{K} \ge K$ and $\{U_j\}_{1\le j\le K} \subseteq \{\tilde{U}_j\}_{1\le j\le \tilde{K}}$. Let $(V_j)_{1\le j\le K}$ denote the intervals of $\gamma$ paired in the correspondence with the respective $U_j$, and let $b_j \coloneqq \mathrm{Leb}(V_j)$.

Consider the correspondence of $g^*(\beta)$ and $g^*(\gamma)$ that uses the corresponding interval pairs to $(U_j, V_j)_{1\le j\le K}$. This has distortion

$$\sum_{j=1}^{K} |g(a_j) - g(b_j)| + \max\Big\{ \|g^*(\beta)\| - \sum_{j=1}^{K} g(a_j), \|g^*(\gamma)\| - \sum_{j=1}^{K} g(b_j) \Big\}.$$

First, let $1 \le j \le K$. Then $|a_j - b_j| < \tilde{\varepsilon}$ and therefore we have $|g(a_j) - g(b_j)| < \varepsilon/2K$. Also note that $\|\beta\| - \sum_{j=1}^{K} a_j < \varepsilon/4C$, and therefore we have

$$\|\gamma\| - \sum_{j=1}^{K} b_j \le d'_H(\beta,\gamma) + \|\beta\| - \sum_{j=1}^{K} a_j < \frac{\varepsilon}{2C}.$$

From this we can conclude that

$$\max\Big\{ \|g^*(\beta)\| - \sum_{j=1}^{K} g(a_j), \|g^*(\gamma)\| - \sum_{j=1}^{K} g(b_j) \Big\} < \frac{\varepsilon}{2}$$

and the first result follows. To prove the second result, we show that if $\liminf_{x\to 0} g(x)/x > 0$ then

$$\limsup_{x\to 0} \frac{g^{-1}(x)}{x} \le \Big( \liminf_{y\to 0} \frac{g(y)}{y} \Big)^{-1} < \infty$$

and the result then follows as previously. To see this let $(x_n, n \geq 1)$ be a positive sequence that converges to 0. Then $(g^{-1}(x_n), n \geq 1)$ is also a sequence that converges to 0. Therefore we have that

$$\liminf_{n\to\infty} \frac{x_n}{g^{-1}(x_n)} = \liminf_{n\to\infty} \frac{g(g^{-1}(x_n))}{g^{-1}(x_n)} \geq \liminf_{y\to 0} \frac{g(y)}{y} > 0.$$

Therefore $\limsup_{n\to\infty} g^{-1}(x_n)/x_n \leq 1/(\liminf_{y\to 0} g(y)/y) < \infty$ and the second result follows in the same way as the first result. ■

Lemma 2.4.29 states that under the condition $\liminf_{x\to 0} g(x)/x > 0$ we have the equality of sets $g^*(\mathscr{I}_H(E))$ and $\mathscr{I}_H(g(E))$. We recall a similar result in Lemma 2.4.24 which applies to smaller spaces of interval partitions where interval partitions have diversities; in that setting, under Assumption B, we have the equality of $g^*(\mathscr{I}_{m_{\mathbf{Y}}}(E))$ and $\mathscr{I}_{m_{\mathbf{Z}}}(g(E))$.

*Remark* 2.4.30. In fact the conditions in Lemma 2.4.29 are necessary as well as sufficient in the sense that if $\limsup_{x\to 0} g(x)/x = \infty$ then the map $g^*$ is not injective and if $\liminf_{x\to 0} g(x)/x = 0$ then the map $(g^{-1})^*$ is not injective (they both map certain interval partitions to $\emptyset$ as the transformed intervals are not summable).

We now give certain bounds for the diffusion under Strong Assumption B which we will use to prove continuity in the initial state in Proposition 2.4.34.

**Lemma 2.4.31.** *Consider a diffusion* $\mathbf{Z}$ *that satisfies Strong Assumption B with untransformed diffusion* $\mathbf{Y}$ *and spatial transformation* $g$ *such that* $\liminf_{x\to 0} g(x)/x > 0$*, and let* $z > 0$ *and let* $\varepsilon > 0$*. Then there exists* $k_7 = k_7(\varepsilon, z) > 0$ *such that for any interval partition* $\gamma \in \mathscr{I}_H(g(E))$ *such that* $\|\gamma\| < k_7$ *we have that*

$$\mathbf{P}(\zeta^+(\mathbf{X}_\gamma) < z) > 1 - \varepsilon,$$

*where* $\mathbf{X}_\gamma$ *is constructed as in Definition 2.3.11 and* $\zeta^+(X) := \sup_{t\in[0,len(X)]} X(t)$ *for* $X \in \mathscr{D}$.

*Proof.* Let $\varepsilon, z > 0$. The injectivity of $(g^{-1})^*$ from Lemma 2.4.29 leads to the surjectivity of $g^*$ and so $g^*(\mathscr{I}_H(E)) = \mathscr{I}_H(g(E))$. Also we note that the untransformed diffusion satisfies Corollary 2.3.13. Therefore we can create a scaffolding-and-spindles pair $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ with an initial interval partition $\gamma \in \mathscr{I}_H(g(E))$ as in Definition 2.3.11 where we construct the scaffolding by adding $\zeta(f_{V_i})$ jumps for each interval $V_i$ of the initial interval partition $\gamma$. We will compare this to an SPLP with a single initial jump $P = \sum_{U\in\gamma} \zeta(f_U)$ at the beginning of the process which we run until it hits level 0 and we will denote this process as $\mathbf{X}_P$. By a natural coupling we have $\mathbf{X}_P(t) \geq \mathbf{X}_\gamma(t)$ for all $t \geq 0$ and therefore the event $\{\zeta^+(\mathbf{X}_\gamma) > z\}$ is a subset of by $\{\zeta^+(\mathbf{X}_P) > z\}$. We construct $\mathbf{X}_P$ by considering a Lévy process started at $\sum_{U\in\beta} \zeta(f_U)$ such that $\mathbf{X}_P - \sum_{U\in\beta} \zeta(f_U)$ has the same distribution as an SPLP $\mathbf{X}$ with Lévy measure $\nu(\zeta \in \cdot)$.

Let $W$ be the scale function for the Lévy process $\mathbf{X}$. This function is continuous and increasing with $W(0) = 0$, [9, Theorem VII.8], and therefore there exists $x > 0$ such that

$W(x)/W(z) \leq \varepsilon/2$. Also note that for the empty interval partition $\emptyset$ we have $d'_H(\gamma,\emptyset) = \|\gamma\|$. Recall that we can apply Proposition 2.3.12 to the untransformed diffusion by Corollary 2.3.13. By Lemma 2.4.29 $(g^{-1})^*$ is continuous at $\emptyset$ and therefore there exists $k_7 > 0$ such that for all $\gamma \in \mathscr{I}_H(g(E))$ with $\|\gamma\| < k_7$ we have $\|(g^{-1})^*(\gamma)\| < \varepsilon x/2(1+k_3)$ where $k_3$ is constant from Proposition 2.3.12. Recall $\varepsilon'$ from the same proposition and take $k_7 < g(\varepsilon')$. Then, by (2.42), we therefore have that

$$\mathbf{E}\Big(\sum_{i\in I}\zeta(f_{V_i})\Big) \leq (1+k_3)\sum_{i\in I} g^{-1}(\mathrm{Leb}(V_i)) = (1+k_3)\|(g^{-1})^*(\gamma)\|.$$

Therefore, for $\gamma \in \mathscr{I}_H(g(E))$ such that $||\gamma|| < k_7$, we have that

$$\mathbf{P}\big(\zeta^+(\mathbf{X}_\gamma) > z\big) \leq \mathbf{P}\big(\zeta^+(\mathbf{X}_P) > z\big) \leq \mathbf{P}\Big(\sum_{i\in I}\zeta(f_{V_i}) > x\Big) + \frac{W(x)}{W(z)} \leq \varepsilon.$$

■

In the following lemma, for $x \in g(E)$, let $\mathbb{Q}^0_x(\cdot)$ denote the law of the diffusion under Assumption B.

**Lemma 2.4.32.** *Consider a diffusion under Assumption B, let $D \in g(E)$ and let $d_0$, $d_1 > 0$. Then there exists $d_2 > 0$ such that for all $a$, $b \in [0,D]$ with $a < b < a + d_2$ we have*

$$\mathbb{Q}^0_b(T_a < d_1) > 1 - d_0.$$

*Proof.* This result uses the Markov inequality. We first note that

$$\mathbb{Q}^0_b(T_a \geq d_1) \leq \frac{\mathbb{E}_{\mathbb{Q}^0_b}(T_a)}{d_1}$$

and recall from Corollary 2.3.13 that the function $r(x) := \mathbb{E}_{\mathbb{Q}^0_{g^{-1}(x)}}(\zeta)$ has a derivative that is bounded above near 0 (we have the transformation $g^{-1}$ here as the corollary we use only applies to untransformed diffusions). Therefore, setting $C := \sup_{x\in(0,g^{-1}(D))} r'(x)$ we have for all $0 \leq a < b \leq D$

$$\mathbb{E}_{\mathbb{Q}^0_b}(T_a) = r(g^{-1}(b)) - r(g^{-1}(a)) \leq C|g^{-1}(b) - g^{-1}(a)|.$$

Now, $g^{-1}$ is uniformly continuous on $[0,D]$ and so there exists $d_2 > 0$ such that for $a$, $b \in [0,D]$ with $b > a$ and $b - a < d_2$ we have $g^{-1}(b) - g^{-1}(a) < d_0 d_1/C$ and the result follows. ■

**Lemma 2.4.33** (Continuity in the initial state for $\overline{\text{skewer}}(\mathbf{N}_U, \mathbf{X}_U)$)**.** *Let $d_3$, $d_4$, $z > 0$, $D \in g(E)$ and let $a \in [0,D]$. Then there exists $d_5 > 0$ such that for $b \in [0,D]$ with $|a-b| < d_5$ we have the following. Let $U = (0,a)$ and $V = (0,b)$. Then, on some probability space $(\Omega, \mathscr{A}, \mathbf{P})$, there exists a coupled pair of scaffolding and spindles $(\mathbf{N}_U, \mathbf{X}_U)$, $(\mathbf{N}_V, \mathbf{X}_V)$ started from $U$ and $V$ as in Definition 2.3.11 on which we have*

$$\mathbf{P}(d'_H(\mathit{skewer}(z, \mathbf{N}_U, \mathbf{X}_U), \mathit{skewer}(z, \mathbf{N}_V, \mathbf{X}_V)) > d_3) < d_4.$$

*Proof.* Let $d_3$, $d_4$, $z > 0$ and let $a \leq D \in g(E)$. Let $d_6 > 0$ be small enough that $W(z-2d_6)/W(z) > 1 - d_4/5$ where $W$ is the scale function for the SPLP from Lemma 2.4.31. Let $d_7 = d_2/2$ where $d_2$ is the constant obtained from Lemma 2.4.32 when applied with $d_0 = d_4/5$ and $d_1 = d_6$. Let $f$ be a 0-diffusion started at $(a+d_7) \wedge D$ and note that

$$\mathbf{P}\Big(G_0 := \Big\{T_{(a-d_7)\vee 0}(f) < d_6\Big\}\Big) > 1 - \frac{d_4}{5}$$

where $T_{(a-d_7)\vee 0}(f)$ is the hitting time of $(a-d_7)\vee 0$ by $f$. The diffusion $f$ has a finite lifetime and so is uniformly continuous and therefore there exists $d_8 > 0$ such that

$$\mathbf{P}\Big(G_1 := \Big\{|f(x) - f(y)| < \frac{d_3}{2} \;\; \forall x, y \in \mathbb{R} : |x-y| < d_8\Big\}\Big) > 1 - \frac{d_4}{5}.$$

Now consider an independent Poisson random measure $\mathbf{N}$ with scaffolding $\mathbf{X}$ and which is stopped at $T_{-\zeta(f)}$. First note that

$$\mathbf{P}\Big(G_2 := \Big\{\sup_{t \in [T_{-\zeta(f)+2d_6}, T_{-\zeta(f)}]} \mathbf{X}_t \leq z - \zeta(f)\Big\}\Big) \geq \frac{W(z-2d_6)}{W(z)} > 1 - \frac{d_4}{5}$$

by the independence of $f$ and $\mathbf{N}$ and the strong Markov property of $\mathbf{X}$ and where we follow convention that the supremum over the empty set is $-\infty$. Furthermore note that $(\text{skewer}(y, \mathbf{N}, \mathbf{X}), y \in \mathbb{R})$ has a uniformly continuous modification $(\gamma^y, y \in \mathbb{R})$ by Theorem 2.4.16. Let $d_9 > 0$ be small enough so that

$$\mathbf{P}\Big(G_3 := \Big\{d'_H(\gamma^x, \gamma^y) < d_3/2 \text{ for all } x, y \in \mathbb{R} : |x-y| < d_9\Big\}\Big) > 1 - \frac{d_4}{5}.$$

Now, let $d_{10} = d_2/2$ where $d_2$ is now the constant obtained from Lemma 2.4.32 when applied with $d_1 = d_8 \wedge d_9$ and $d_0 = d_4/5$ and let $d_5 := d_7 \wedge d_{10}$. Then we have

$$\mathbf{P}\Big(G_4 := \Big\{T_{(a-d_5)\vee 0}(f) - T_{(a+d_5)\wedge D}(f) < d_8 \wedge d_9\Big\}\Big) > 1 - \frac{d_4}{5}.$$

Now let $b \in [0, D]$ with $|a-b| < d_5$. Let $f_U(\cdot) := f(T_a + \cdot)$ and let $f_V(\cdot) := f(T_b + \cdot)$. We define

$$\tilde{\mathbf{N}}_U := \mathbf{N}|_{[0, T_{-\zeta(f_U)}(\mathbf{N})]}; \;\; \mathbf{N}_U := \delta_{(0, f_U)} + \tilde{\mathbf{N}}_U; \;\; \tilde{\mathbf{N}}_V := \mathbf{N}|_{[0, T_{-\zeta(f_V)}(\mathbf{N})]}; \;\; \mathbf{N}_V := \delta_{(0, f_V)} + \tilde{\mathbf{N}}_V.$$

Note that

$$\mathbf{P}\Big(G_5 := \Big\{\gamma^{z-\zeta(f_U)} = \text{skewer}(z - \zeta(f_U), \mathbf{N}, \mathbf{X}), \;\; \gamma^{z-\zeta(f_V)} = \text{skewer}(z - \zeta(f_V), \mathbf{N}, \mathbf{X})\Big\}\Big) = 1$$

by the spatial homogeneity of the scaffolding-and-spindles pair and the independence of $f_U$ and $\mathbf{N}$ and of $f_V$ and $\mathbf{N}$. Therefore we have

$$\text{skewer}(z, \tilde{\mathbf{N}}_U, \mathbf{X}_U) = \gamma^{z-\zeta(f_U)}, \;\; \text{skewer}(z, \tilde{\mathbf{N}}_V, \mathbf{X}_V) = \gamma^{z-\zeta(f_V)} \text{ on } G_0 \cap G_2 \cap G_5.$$

Furthermore we have that

$$|f_U(z) - f_V(z)| < d_3/2 \text{ on } G_1 \cap G_4; \quad \text{and} \quad d'_H(\gamma^{z-\zeta(f_U)}, \gamma^{z-\zeta(f_V)}) < d_3/2 \text{ on } G_3 \cap G_4.$$

Therefore, as

$$\begin{aligned} &d'_H(\text{skewer}(z, \mathbf{N}_U, \mathbf{X}_U), \text{skewer}(z, \mathbf{N}_V, \mathbf{X}_V)) \\ &\qquad \leq |f_U(z) - f_V(z)| + d'_H(\text{skewer}(z, \tilde{\mathbf{N}}_U, \mathbf{X}_U), \text{skewer}(z, \tilde{\mathbf{N}}_V, \mathbf{X}_V)) \end{aligned}$$

then it follows that

$$d'_H(\text{skewer}(z, \mathbf{N}_U, \mathbf{X}_U), \text{skewer}(z, \mathbf{N}_V, \mathbf{X}_V)) < d_3 \text{ on } G := G_0 \cap G_1 \cap G_2 \cap G_3 \cap G_4 \cap G_5,$$

and as $\mathbf{P}(G) > 1 - d_4$ the result follows. ■

**Proposition 2.4.34** (Continuity in the initial state)**.** *Consider a diffusion under Strong Assumption B for which* $\liminf_{x\to 0} g(x)/x > 0$. *For* $\gamma \in g^*(I_H(E))$ *recall* $\mathbb{P}_\gamma(\cdot) := \mathbf{P}((\gamma^y, y \geq 0) \in \cdot)$ *from* (2.72)*, where* $(\gamma^y, y \geq 0)$ *is a modification of* $\overline{skewer}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ *that is continuous for* $y \in (0,\infty)$. *Let* $\mathbb{E}_\gamma$ *be the expectation operator for* $\mathbb{P}_\gamma$. *Let* $z > 0$ *and let* $f : \mathscr{I}_H \to [0,\infty)$ *be a bounded and continuous function. Then the map* $\gamma \mapsto \mathbb{E}_\gamma(f(\gamma^z))$ *is continuous on* $(\mathscr{I}_H(g(E)), d'_H)$.

*Proof.* This was shown for self-similar diffusions in [34, Proposition 6.15]. We adapt the proof to generalise to our setting.

As in Lemma 2.4.31 we are under the assumption $\liminf_{x\to 0} g(x)/x > 0$ and so $g^*(\mathscr{I}_H(E)) = \mathscr{I}_H(g(E))$. Let $\gamma \in \mathscr{I}_H(g(E))$, $\varepsilon, z > 0$, and let $(\Omega, \mathscr{F}_\mathscr{I}, \mathbb{P}_\gamma)$ be the probability space as defined in Section 2.4.5, where $\Omega := \mathscr{C}((0,\infty), \mathscr{I}_H)$, $\mathbb{P}_\gamma$ is the probability measure for $(\gamma^y, y \geq 0)$ and $\mathscr{F}_\mathscr{I}$ is the natural filtration for $(\gamma^y, y \geq 0)$. As $f$ is continuous at $\gamma^z$ we can define a $(0,\infty)$-valued random variable $A_\varepsilon(\gamma^z)$ on $(\Omega, \mathscr{F}_\mathscr{I}, \mathbb{P}_\gamma)$ for which we have that for $\beta \in \mathscr{I}_H(g(E))$ we have $d'_H(\gamma^z, \beta) < A_\varepsilon(\gamma^z)$ implies that $|f(\gamma^z) - f(\beta)| < \varepsilon/3$. Let $M := \sup_{\beta \in \mathscr{I}_H(g(E))} f(\beta)$. Then there is $\tilde{\varepsilon} > 0$ such that $\mathbb{P}_\gamma(A_\varepsilon(\gamma^z) < \tilde{\varepsilon}) < \varepsilon/3M$. We take $\hat{\gamma} \in \mathscr{I}_H(g(E))$ and consider a probability space $(\tilde{\Omega}, \mathscr{A}, \mathbf{P})$ for a pair of processes $(\gamma^y, y \geq 0)$ and $(\hat{\gamma}^y, y \geq 0)$ which have marginal distributions $\mathbb{P}_\gamma$ and $\mathbb{P}_{\hat{\gamma}}$ respectively; we will later specify an explicit coupling. By Lemma 2.4.29 note that $g^*$ is surjective so as the untransformed diffusion satisfies Corollary 2.3.13 it is possible to construct IP-evolutions $(\gamma^y, y \geq 0)$ and $(\hat{\gamma}^y, y \geq 0)$ with initial states $\gamma$ and $\hat{\gamma}$. We first note that

$$\left|\mathbb{E}_\gamma(f(\gamma^z)) - \mathbb{E}_{\hat{\gamma}}(f(\hat{\gamma}^z))\right| \leq \mathbf{E}\Big(|f(\gamma^z) - f(\hat{\gamma}^z)|\Big).$$

Then we can further bound the expectation above by

$$\mathbf{E}\Big(|f(\gamma^z) - f(\hat{\gamma}^z)|\Big(\mathbb{1}_{d'_H(\gamma^z,\hat{\gamma}^z)<\tilde{\varepsilon}, A_\varepsilon(\gamma^z)\geq\tilde{\varepsilon}} + \mathbb{1}_{d'_H(\gamma^z,\hat{\gamma}^z)\geq\tilde{\varepsilon}} + \mathbb{1}_{A_\varepsilon(\gamma^z)<\tilde{\varepsilon}}\Big)\Big).$$

The first of the three terms is less than or equal to $\varepsilon/3$; the second is less than or equal to $M\mathbf{P}(d'_H(\gamma^z,\hat{\gamma}^z)\geq\tilde{\varepsilon})$; and the third is less than or equal to $\varepsilon/3$. Therefore it suffices to show that there is some $\delta>0$ such that for $\hat{\gamma}\in\mathscr{I}_H(g(E))$ with $d'_H(\gamma,\hat{\gamma})<\delta$ we have a coupled pair of IP-evolutions $(\gamma^y,y\geq 0)$ and $(\hat{\gamma}^y,y\geq 0)$ starting from these two initial states, with

$$\mathbf{P}(d'_H(\gamma^z,\hat{\gamma}^z)>\tilde{\varepsilon})<\frac{\varepsilon}{3M}. \tag{2.74}$$

To this end, let $U_1, U_2, \ldots$ denote the blocks of $\gamma$, listed in non-increasing order by mass and let $a_j := \mathrm{Leb}(U_j)$.

Let $k_7(\varepsilon/9M,z)$ be the constant from Lemma 2.4.31 and let $K$ be large enough that $\sum_{j>K}a_j<\frac{1}{2}k_7$. If $a_1<g(c)$ then take $D\in g(E)$ such that $a_1<D<g(c)$ and if $a_1=g(c)$ then take $D=a_1=g(c)$. For $1\leq j\leq K$ let $\delta_j=d_5$ as in Lemma 2.4.33 with $d_3=\tilde{\varepsilon}/K$, $d_4=\varepsilon/9MK$, $D$ as above and $a=a_j$. Now let

$$\delta<\begin{cases}\min\{\delta_1,\ldots,\delta_K,a_K,\frac{k_7}{2},D-a_1\} \text{ if } a_1<D<g(c),\\ \min\{\delta_1,\ldots,\delta_K,a_K,\frac{k_7}{2}\} \text{ if instead } a_1=D=g(c).\end{cases}$$

Let $\hat{\gamma}\in\mathscr{I}_H(g(E))$ with $d'_H(\gamma,\hat{\gamma})<\delta$. By definition of $d'_H$, there exists a correspondence $(\tilde{U}_j,\tilde{V}_j)_{1\leq j\leq\tilde{K}}$ from $\gamma$ to $\hat{\gamma}$ with distortion less than $\delta$. Since $\delta<a_K$, we get $\tilde{K}\geq K$ and $\{U_j\}_{1\leq j\leq K}\subset\{\tilde{U}_j\}_{1\leq j\leq\tilde{K}}$. Let $(V_j)_{1\leq j\leq K}$ denote the terms paired with the respective $U_j$ in the correspondence. For $1\leq j\leq K$, let $b_j:=\mathrm{Leb}(V_j)$. Let $(V_j,j>K)$ be the remaining intervals of $\hat{\gamma}$. Note that necessarily we have $|a_j-b_j|<\delta$ for $1\leq j\leq K$. Consider coupled scaffolding and spindles pairs $(\mathbf{N}_{U_j},\mathbf{X}_{U_j})$, $(\mathbf{N}_{V_j},\mathbf{X}_{V_j})$ independently for each $1\leq j\leq K$ as in Lemma 2.4.33. For $j>K$ we consider independent scaffolding and spindles $(\mathbf{N}_{U_j},\mathbf{X}_{U_j})$ and $(\mathbf{N}_{V_j},\mathbf{X}_{V_j})$. We construct coupled scaffolding and spindles $(\mathbf{N}_\gamma,\mathbf{X}_\gamma)$ and $(\mathbf{N}_{\hat{\gamma}},\mathbf{X}_{\hat{\gamma}})$ constructed from coupled scaffolding and spindles $(\mathbf{N}_{U_j},\mathbf{X}_{U_j})$ and $(\mathbf{N}_{V_j},\mathbf{X}_{V_j})$ for $j\in\mathbb{N}$ as in Definition 2.3.11. Let $(\gamma^y,y\geq 0)$ and $(\hat{\gamma}^y,y\geq 0)$ be the modifications of $\overline{\mathrm{skewer}}(\mathbf{N}_\gamma,\mathbf{X}_\gamma)$ and $\overline{\mathrm{skewer}}(\mathbf{N}_{\hat{\gamma}},\mathbf{X}_{\hat{\gamma}})$ that are continuous on $(0,\infty)$ which are shown to exist in Lemma 2.4.19. First we make the obvious note that

$$\mathbf{P}\Big(F_0:=\Big\{\gamma^z=\mathrm{skewer}(z,\mathbf{N}_\gamma,\mathbf{X}_\gamma);\ \ \hat{\gamma}^z=\mathrm{skewer}(z,\mathbf{N}_{\hat{\gamma}},\mathbf{X}_{\hat{\gamma}})\Big\}\Big)=1$$

as $(\gamma^y,y\geq 0)$ and $(\hat{\gamma}^y,y\geq 0)$ are modifcations of the skewer processes. Also, for $1\leq j\leq K$, we have $|a_j-b_j|<\delta_j$ and so

$$\mathbf{P}\Big(F_1^j:=\Big\{d'_H(\mathrm{skewer}(z,\mathbf{N}_{U_j},\mathbf{X}_{U_j}),\mathrm{skewer}(z,\mathbf{N}_{V_j},\mathbf{X}_{V_j}))<\frac{\tilde{\varepsilon}}{K}\Big\}\Big)>1-\frac{\varepsilon}{9MK} \text{ for } 1\leq j\leq K.$$

Furthermore we have

$$\mathbf{P}\Big(F_2:=\Big\{\forall j>K,\zeta^+(\mathbf{N}_{U_j})<z\Big\}\Big)\geq 1-\frac{\varepsilon}{9M} \text{ by Lemma 2.4.31.}$$

Also we have that

$$||\hat{\gamma}||-\sum_{j=1}^K b_j\leq d'_H(\gamma,\hat{\gamma})+||\gamma||-\sum_{j=1}^K a_j<\delta+\frac{1}{2}k_7<k_7.$$

Therefore

$$\mathbf{P}\Big(F_3 := \Big\{\zeta^+(\mathbf{N}_{V_j}) < z, \forall j > K\Big\}\Big) \geq 1 - \frac{\varepsilon}{9M} \text{ again by Lemma 2.4.31.}$$

Pulling all these together we conclude

$$\mathbf{P}\Big(F := F_0 \cap \Big(\bigcap_{j=1}^{K} F_1^j\Big) \cap F_2 \cap F_3\Big) \geq 1 - \frac{\varepsilon}{3M},$$

and we now show that $d'_H(\gamma^z, \hat{\gamma}^z) < \tilde{\varepsilon}$ on $F$ to finish the proof. First we note that on $F_2$ $\gamma^z$ consists only of intervals from spindles from $\mathbf{N}_{U_j}$ for $1 \leq j \leq K$ and similarly for $\hat{\gamma}^z$ on $F_3$ and therefore

$$d'_H(\gamma^z, \hat{\gamma}^z) \leq \sum_{j=1}^{K} d'_H(\text{skewer}(z, \mathbf{N}_{U_j}, \mathbf{X}_{U_j}),\ \text{skewer}(z, \mathbf{N}_{V_j}, \mathbf{X}_{V_j})) \text{ on } F_0 \cap F_2 \cap F_3,$$

and so $d'_H(\gamma^z, \hat{\gamma}^z) \leq \tilde{\varepsilon}$ on $F$ and the result follows. ■

**Theorem 2.4.35** (Strong Markov property)**.** *Consider a diffusion under Strong Assumption B and then let $\gamma \in g^*(\mathscr{I}_H(E))$. Let $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ be a scaffolding-and-spindles pair started from $\gamma$ as in Definition 2.3.11 and let $(\gamma^y, y \geq 0)$ be a modification that is continuous on $(0,\infty)$ as in Lemma 2.4.19. Recall the appropriate filtered probability space a filtered probability space $(\Omega, \mathscr{F}_\mathscr{I}, (\mathscr{F}^y_\mathscr{I}, y \geq 0), \mathbb{P}_\gamma)$ as used in Corollary 2.4.20. Then we have that $(\gamma^y, y \geq 0)$ is a strong Markov process on this filtered probability space.*

*Furthermore, in the case where $\gamma \in g^*(\overline{\mathscr{I}}^{(q)}_{m_\mathbf{Y}}(E))$ we have that $(\gamma^y, y \geq 0)$ is continuous at 0 and so $(\gamma^y, y \geq 0)$ is an IP-diffusion.*

*Proof.* We first consider a diffusion that satisfies Strong Assumption B and we begin with the case where $\liminf_{x\to 0} g(x)/x > 0$. As in [34, Proposition 6.17] we extend the result in Proposition 2.4.34 to the claim of continuity of the process

$$\gamma \mapsto \mathbb{E}_\gamma\Big(\prod_{i=1}^{m} f_i(\gamma^{y_i})\Big),$$

where $f_1, \ldots, f_m : \mathscr{I}_H \to [0,\infty)$ are bounded and continuous. This can be shown by induction on $m$ using the simple Markov property from Corollary 2.4.20. Then we show the strong Markov property from this by the method given by Kallenberg [52, Theorem 17.17] where the continuity in the initial state stated above then works in place of the Feller property that is used in the result in [34, Proposition 6.17].

Next we consider the case where $\limsup_{x\to 0} g(x)/x < \infty$. We first note that Proposition 2.4.34 applies to the identity transformation $g(x) = x$. Also we have the map
$g^* : \mathscr{I}_H(E) \to g^*(\mathscr{I}_H(E))$ is continuous and bijective by Lemma 2.4.29 which means that the Strong Markov property in the untransformed case leads to the strong Markov property in the transformed case.

Finally for the case where $\gamma \in g^*(\overline{\mathscr{I}}^{(q)}_{m_{\mathbf{Y}}}(E))$ we use Proposition 2.4.27 which states that $(\gamma^y, y \geq 0)$ is continuous on $[0,\infty)$ as required. ■

We can now prove Theorem 2.1.6.

*Proof of Theorem 2.1.6.* First consider the case where we have a diffusion under Assumption B and we have a scaffolding-and-spindles pair $(\mathbf{N},\mathbf{X})$ for this diffusion stopped at an a.s. finite random time $T$. Then there exists a continuous modification of $\overline{\text{skewer}}(\mathbf{N},\mathbf{X})$ by Theorem 2.4.16 and the furthermore $\mathbf{X}$ has a local time that is continuous in time and space by Proposition 2.4.5.

Now, instead consider an untransformed diffusion under Assumption B' and let $s > 0$. Let $(\tilde{\mathbf{N}},\tilde{\mathbf{X}})$ be a scaffolding-and-spindles pair for this untransformed diffusion that is stopped at $\tau^0(s)$. Then by Corollary 2.3.13 the diffusion satisfies (2.40). Therefore, as it also satisfies $B1$ we can apply Proposition 2.3.19 as $\tau^0(s)$ satisfies the conditions of the proposition. This gives a regular conditional distribution of the clade component for level 0 given $\beta^0 := \text{skewer}(0,\tilde{\mathbf{N}},\tilde{\mathbf{X}})$. By Proposition 2.3.20 we have a simple Markov property for $\overline{\text{skewer}}(\tilde{\mathbf{N}},\tilde{\mathbf{X}})$ on the probability space $(\Omega,\mathscr{A},\mathbf{P})$ for $\tilde{\mathbf{N}}$.

We have aggregate health summability at level 0 of the transformed diffusion by Proposition 2.4.1 and Theorem 2.1.4 and so we also have the simple Markov property in the transformed case. Moreover, under Strong Assumption B, by Theorem 2.4.35 we have a strong Markov property for a modification of the transformed skewer process on the space of paths continuous on $(0,\infty)$ which we denote as $(\Omega,\mathscr{F},\mathbb{P}_\mu)$ where $\mu$ is the probability distribution on $g^*(\mathscr{I}_H(E))$ of $g^*(\beta^0)$. ■

# 2.5 Appendix

**Lemma 2.5.1.** *Let $\nu$ be a $\sigma$-finite measure on $\mathscr{E}$. Then, for any measurable map $S : \mathscr{E} \to [0,\infty]$, there exists a S-disintegration $(\nu_t)_{t\in[0,\infty]}$. This disintegration is uniquely determined up to an almost sure equivalence, i.e. if $(\nu^*_t)_{t\in[0,\infty]}$ is another S-disintegration of $\nu$ then $\nu(S^{-1}(t \in [0,\infty] : \nu_t \neq \nu^*_t)) = 0$.*

Lemma 2.5.2 restates Lemma 2.2.10. The Green's function for the $\uparrow$-diffusion is given by

$$G^{\uparrow}_{(a,w)}(x,v) = 2\frac{s^{\uparrow}(x)-s^{\uparrow}(a)}{s^{\uparrow}(w)-s^{\uparrow}(a)}\Big(s^{\uparrow}(w)-s^{\uparrow}(v)\Big)m^{\uparrow}(v)\mathbb{1}_{x<v<w} + 2\frac{s^{\uparrow}(w)-s^{\uparrow}(x)}{s^{\uparrow}(w)-s^{\uparrow}(a)}\Big(s^{\uparrow}(v)-s^{\uparrow}(a)\Big)m^{\uparrow}(v)\mathbb{1}_{a<v<x},$$

Recall the functions $U_{(a,w)}$ and $V_{(a,w)}$ from (2.21):

$$U_{(a,w)}(x) = \mathbb{E}_{\mathbb{Q}^{\uparrow}_x}\big(\big(T_a \wedge T_w\big)^2\big), \quad V_{(a,w)}(x) = 2\mathbb{E}_{\mathbb{Q}^{\uparrow}_x}\big(T_a \wedge T_w\big).$$

We have from Section 2.2.3 that for $0 < a < w < c$ that $U_{(a,w)}$ is the solution to the following differential equation, from (2.22), in $x \in (a,w)$

$$\frac{1}{2}\sigma^2(x)U''_{(a,w)}(x) + \mu^{\uparrow}(x)U'_{(a,w)}(x) + V_{(a,w)}(x) = 0,$$

with boundary conditions $U_{(a,w)}(a) = U_{(a,w)}(w) = 0$.

**Lemma 2.5.2** (Proof of Lemma 2.2.10)**.** *We claim that the solution to the differential equation in* (2.22)*, with boundary conditions given by $U(a) = U(w) = 0$, is the following:*

$$U_{(a,w)}(x) := \mathbb{E}_{\mathbb{Q}^{\uparrow}_x}\left(\left(T_a \wedge T_w\right)^2\right) = k(a,w)(s^{\uparrow}(w) - s^{\uparrow}(x)) + \int_{y=x}^{w} (s^{\uparrow})'(y) \int_{z=a}^{y} \frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)} dz dy,$$

*where $k(a,w)$ is a constant given by*

$$k(a,w) = \frac{-1}{s^{\uparrow}(w) - s^{\uparrow}(a)} \int_{y=a}^{w} (s^{\uparrow})'(y) \int_{z=a}^{y} \frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)} dz dy,$$

*and where $V_{(a,w)}(x)$ is*

$$V_{(a,w)}(x) = 2\mathbb{E}_{\mathbb{Q}^{\uparrow}_x}\left(T_a \wedge T_w\right) = 2\int_{v=0}^{w} G^{\uparrow}_{(a,w)}(x,v)dv,$$

*where $G^{\uparrow}$ is the Green's function for the $\uparrow$-diffusion.*

*Proof.* We first note that the $\uparrow$-diffusion satisfies $A3$ and that the infinitesimal drift $\mu^{\uparrow}$ and infinitesimal variance $(\sigma^2)^{\uparrow}$ are continuous functions on $(0,c)$. Therefore the differential equation in (2.22), from [54, Equation 15.3.37], has solution given in (2.21).

We now check that the boundary conditions are satisfied:

$$U_{(a,w)}(w) = k(a,w)(s^{\uparrow}(w) - s^{\uparrow}(w)) + \int_{y=w}^{w} (s^{\uparrow})'(y) \int_{z=a}^{y} \frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)} dz dy = 0,$$

$$\begin{aligned} U_{(a,w)}(a) =& \frac{-1}{s^{\uparrow}(w) - s^{\uparrow}(a)} \int_{y=a}^{w} (s^{\uparrow})'(y) \int_{z=a}^{y} \frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)} dz dy (s^{\uparrow}(w) - s^{\uparrow}(a)) \\ &+ \int_{y=a}^{w} (s^{\uparrow})'(y) \int_{z=a}^{y} \frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)} dz dy = 0. \end{aligned}$$

Now we show that the differential equation itself is solved by this solution. Taking a derivative with respect to $x$ gives

$$U'_{(a,w)}(x) = -k(a,w)(s^{\uparrow})'(x) - (s^{\uparrow})'(x) \int_{z=0}^{x} \frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)} dz.$$

Then multiplying by $1/(s^{\uparrow})'(x) = \exp\left(\int_{t=b}^{x} \frac{2\mu^{\uparrow}(t)}{\sigma^2(t)} dt\right)$ gives

$$\exp\left(\int_{t=1}^{x} \frac{2\mu^{\uparrow}(t)}{\sigma^2(t)} dt\right) U'_{(a,w)}(x) = \frac{1}{(s^{\uparrow})'(x)} U'_{(a,w)}(x) = -k(a,w) - \int_{z=0}^{x} \frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)} dz.$$

Finally taking another a derivative with respect to $x$ gives

$$\exp\left(\int_{t=b}^{x}\frac{2\mu^{\uparrow}(t)}{\sigma^2(t)}dt\right)\left(U''_{(a,w)}(x)+\frac{2\mu^{\uparrow}(x)}{\sigma^2(x)}U'_{(a,w)}(x)\right)=\frac{-2V_{(a,w)}(x)}{(s^{\uparrow})'(x)\sigma^2(x)},$$

and multiplying by $(s^{\uparrow})'(x)=\exp\left(\int_{t=b}^{x}\frac{-2\mu^{\uparrow}(t)}{\sigma^2(t)}dt\right)$, adding $\frac{2V_{(a,w)}(x)}{\sigma^2(x)}$, then multiplying by $\frac{1}{2}\sigma^2(x)$ gives the differential equation in the form of (2.22). Therefore the result follows by standard results for the uniqueness of solutions to differential equations. ■

This final result generalises [36, Lemma 8] to our setting. The other relevant results from this supplementary paper apply without any modification.

**Lemma 2.5.3.** *For fixed $y\in\mathbb{R}$, the map $N\mapsto skewer(y,N,\xi_N)$ is measurable on the set $\mathscr{N}^{sp}_{fin}$. Recall that we map $(y,N,\xi_N)$ to the empty interval partition $\emptyset$ in the event that the aggregate mass is infinite; i.e. when $M^{y}_{N,\xi(N)}(\infty)=\infty$, see Definition 2.1.2.*

*Proof.* The proof [36, Lemma 8] that the map restricted to the finite aggregate mass summability case is measurable generalises to our setting because the $\sigma$-algebra of $(\mathscr{I}_H,d'_H)$ is the same as $(\mathscr{I}_H,d_H)$ ( [35, Theorem 2.3(b)]), where for $\beta$ and $\gamma\in\mathscr{I}_H$ we define $d_H(\beta,\gamma)$ to be the Hausdorff distance between $[0,\|\beta\|]\backslash\beta$ and $[0,\|\gamma\|]\backslash\gamma$. The extension to the map to include the case where the aggregate mass is infinite follows as the subset $\mathscr{N}^{\mathrm{sp}}_{\mathrm{fin}}$ for which the aggregate mass at level $y$ is finite is measurable. ■

## 2.5.1 Excursion intervals

This section follows [34, A Excursion intervals] very closely. All their measure theoretic results hold in our setting, and we state the results and state why they generalise to our case.

**Proposition 2.5.4.** *Let $\mathbf{X}$ be an SPLP with jump measure that satisfies* (2.9)*. For $y\in\mathbb{R}$ recall the concept of the sets of excursion intervals $V^{y}(\mathbf{X})$ and $V^{y}_{0}(\mathbf{X})$ from Definition 2.3.5. Let $N\in\mathscr{N}^{sp}$ and $y\in\mathbb{R}$. Then we have the following a.s.:*

1. $V^{y}(\mathbf{X})=\Big\{[a,b]\subset[0,\infty)|a<b;\mathbf{X}_{a-}=y=\mathbf{X}_b;\ and\ \mathbf{X}_t\neq 0\ for\ t\in(a,b)\Big\}$.
2. *For $I,J\in V^{y}_{0}(\mathbf{X})$, $I\neq J$, the set $I\cap J$ is either empty or a single shared endpoint.*
3. *If two interval $[a,b]$, $[b,c]\in V^{y}_{0}(\mathbf{X})$ share an endpoint $b$ then $\mathbf{X}$ does not have a jump at $b$.*
4. *For every $t\notin\bigcup_{I\in V^{y}_{0}(\mathbf{X})}I$, we have that $\mathbf{X}_{t-}=\mathbf{X}_t=y$.*
5. $Leb\Big([0,\infty)\backslash\bigcup_{I\in V^{y}_{0}(\mathbf{X})}I\Big)=0$.
6. *There are no degenerate excursions of $\mathbf{X}$ about level $y$, that is to say that for all excursions we have $0<T^{+}_{0}<\zeta$ a.s.*

7. *Local times* $(\ell^y(t), t \geq 0)$ *of* $\mathbf{X}$ *exist and are injective with respect to the excursion intervals in the sense that, for* $[a,b]$, $[c,d] \in V_0^y$, $\ell^y(a) \neq \ell^y(c)$ *unless* $[a,b] = [c,d]$.

8. *If* $y > 0$, *we have* $T^y > T^{\geq y} := \inf\{t \geq 0 : \mathbf{X}_t \geq y\}$.

9. *For* $[a,b] \in V^y(N)$, *the process* $N|^{\leftarrow}_{I^y_N(a,b)}$ *is a bi-clade. Furthermore, the set* $\{N|^{\leftarrow}_{I^y_N(a,b)} : [a,b] \in V_0^y(N)\}$ *partitions the spindles of* $N$.

*Proof.* These results are given in the self-similar cases in [34, Proposition A.2-4]. The proofs in [34, Proposition A.2] immediately generalise. As for the results of Proposition [34, Proposition A.3], they rely on the exit properties of Lévy processes characterised by Millar [74] which applies to our general setting. The final part, from [34, Proposition A.4], is a result of the careful constructions of the intervals $I^y_N(a,b)$ and also applies in our general setting. ■

We conclude this section with definitions, specifically concerning incomplete excursions, to formally define the cutoff processes stated in (2.48) and (2.49). We first define $T^{\geq y}$ and $T^{\geq y}_*$ in the same way as we have defined $T^y$ and $T^y_*$ and then with these define $N^{\leq y}_{\text{first}}$, $N^{\leq y}_{\text{last}}$, $N^{\geq y}_{\text{first}}$, $N^{\geq y}_{\text{last}}$:

$$T^{\geq y} := \inf(\{t \in [0, \text{len}(N)] : \xi_N(t) \geq y\} \cup \{\text{len}(N)\});$$
$$T^{\geq y}_* := \sup(\{t \in [0, \text{len}(N)] : \xi_N(t-) \leq y\} \cup \{0\});$$

$$N^{\leq y}_{\text{first}} = \left(N|_{[0,T^{\geq y})} + \mathbb{1}_{\xi_N(T^{\geq y}-)<y} \boldsymbol{\delta}_{(T^{\geq y}, \check{f}^y_{T^{\geq y}})}\right) \mathbb{1}_{T^{\geq y} \neq T^{\geq y}_*};$$
$$N^{\geq y}_{\text{last}} = N|^{\leftarrow}_{(T^{\geq y}_*, \text{len}(N)]} + \mathbb{1}_{\xi_N(T^{\geq y}_*-)>y} \boldsymbol{\delta}_{(0, \hat{f}^y_{T^{\geq y}_*})};$$
$$N^{\leq y}_{\text{last}} = N|^{\leftarrow}_{[T^y_*, T^{\geq y}_*)} + \mathbb{1}_{y \neq \xi_N(\text{len}(N)); \xi_N(T^{\geq y}_*-)<(y \wedge \xi_N(T^{\geq y}_*))} \boldsymbol{\delta}_{(T^{\geq y}_* - T^y_*, \check{f}^y_{T^{\geq y}_*})};$$
$$N^{\geq y}_{\text{first}} = \left(N|^{\leftarrow}_{(T^{\geq y}, T^y]} + \mathbb{1}_{y \neq 0; \xi_N(T^{\geq y}-)>y \vee 0} \boldsymbol{\delta}_{(0, \hat{f}^y_{T^{\geq y}})}\right) \mathbb{1}_{T^{\geq y} \neq T^{\geq y}_*}.$$

# Chapter 3

# The on/off Brownian snake and range coupling for on/off super-Brownian motion

The work presented here is a collaboration between the author and his collaborator Dave Jacobi and is planned to be published as a pre-print soon.

## 3.1 Introduction

Superprocesses are measure-valued branching processes, see e.g. Etheridge [32]. We consider two superprocesses: super Brownian motion (SBM) and on/off super Brownian motion (ooSBM). These superprocesses are scaling limits of branching Brownian motion (BBM), see e.g. Etheridge [32, Chapter 1], and of on/off branching Brownian motion (ooBBM), [14, Definition 1.1]. We first define BBM and ooBBM with critical binary branching.

**Definition 3.1.1** (Branching Brownian motion (BBM) and on/off branching Brownian motion (ooBBM))**.** *Branching Brownian motion* is a discrete particle system that evolves according to the following dynamics:

1. each particle independently moves according to the law of a Brownian motion in $\mathbb{R}^d$ during its lifetime;
2. a particle splits into two particles at the same spatial location with rate $\gamma/2$ for some $\gamma \geq 0$;
3. a particle dies with rate $\gamma/2$.

*On/off branching Brownian motion* is a discrete particle system where particles take values in $\mathbb{R}^d \times \{0,1\}$. The first component is the spatial position of the particle and the second component denotes whether a particle is active (taking value 1) or dormant (0). We furthermore have the following.

1. Each "active" particle independently has the spatial motion of a Brownian motion in $\mathbb{R}^d$. It also has splitting rate $\gamma/2$ and death rate $\gamma/2$ for some $\gamma \geq 0$ as before. Finally an "active" particle switches to a "dormant" particle with rate $c$ for some $c > 0$.

2. Each "dormant" particle does not move or split or die. Each particle stays in the same spatial position and switches to an "active" particle with rate $\tilde{c}$ for some $\tilde{c} > 0$.

We define *BBM*$(\gamma)$ to be a BBM with parameter $\gamma$ and *ooBBM*$(\gamma, c, \tilde{c})$ to be an ooBBM with parameter $\gamma$, dormancy rate $c$ and wake-up rate $\tilde{c}$.

BBM takes values in the space of finite measures on $\mathbb{R}^d$, $\mathscr{M}_F(\mathbb{R}^d)$, and ooBBM takes values in the space of pair of finite measures on $\mathbb{R}^d$, $\mathscr{M}_F(\mathbb{R}^d)^2$, where the first component is unit Dirac masses of dormant particles and the second component is the active particles.

We have included the degenerate case $\gamma = 0$ in the definition above as this is a useful notion later (in Lemma 3.2.3). Note that BBM$(0)$ and with a single initial particle is simply a Brownian motion. We refer to ooBBM$(0, c, \tilde{c})$ as *on/off Brownian motion* with dormancy rate $c$ and wake-up rate $\tilde{c}$, (ooBM$(c, \tilde{c})$).

We now define super Brownian motion (SBM) and on/off super Brownian motion (ooSBM). SBM is well studied in the literature as one of the original measure-valued branching processes. It is also known as the Dawson-Watanabe process after its introduction by Watanabe [94] and its use in stochastic evolution equations by Dawson [29]. The process ooSBM has been introduced more recently in Blath and Jacobi [14] and follows current interest in the effects of dormancy in population biology, see e.g. [13, 69]. The effect of dormancy in ooSBM impacts the properties of the superprocess. For example, their research shows in [14, Theorem 1.10] that ooSBM does not die out in finite time, unlike SBM. Our research concerns the range of ooSBM, where heuristically the range of a superprocess is the set of spatial locations ever visited by the process over its lifetime, see Definition 3.1.5 below. In Theorem 3.1.8 we show a coupling between the range of ooSBM and the range of SBM which confirms the prediction given in [14, Remark 1.13].

SBM and ooSBM are both measure-valued processes. We define the Prokhorov metric for finite measures next and we state some simple properties after following [28, Section A.2.5].

For a metric space $(E, d_E)$ we define the *Prokhorov metric* $\rho$ for a pair of finite measures on $(E, d_E)$, given by $P, Q \in \mathscr{M}_F(E)$, as

$$\rho(P,Q) = \inf\left(\left\{\varepsilon > 0 : P(F) \leq Q(F^\varepsilon) + \varepsilon;\ \ Q(F) \leq P(F^\varepsilon) + \varepsilon \quad \text{for all } F \in \mathscr{C}\right\}\right)$$

where $\mathscr{C}$ is the collection of closed subsets of $E$ and $F^\varepsilon := \{x \in E : \inf(\{d_E(x,y) : y \in F\}) \leq \varepsilon\}$. The Prokhorov metric metrizes weak convergence and yields a complete and separable metric space if $(E, d_E)$ is complete and separable.

SBM and ooSBM are scaling limits of BBM and ooBBM. To formulate this precisely, we denote by $\mathbb{D}([0,\infty),E)$ the space of measure-valued càdlàg paths $f:[0,\infty)\to E$ where $(E,d_E)$ is a complete and separable metric space, equipped with the Skorokhod topology and with the Skorokhod metric that we denote by $d^{\mathrm{Sko}}$, see e.g. [12, Section 16].

**Proposition 3.1.2** (The existence of super Brownian motion (SBM), see e.g. [32])**.** *Let $\mu\in\mathscr{M}_F(\mathbb{R}^d)$. Then there exists a measure-valued process $(\mathscr{X}_s)_{s\geq 0}$ that takes values in $\mathscr{M}_F(\mathbb{R}^d)$ with $\mathscr{X}_0=\mu$ which is the scaling limit of BBMs in the following sense. Let $\tilde{\mathscr{Z}}^{\varepsilon}=(\tilde{\mathscr{Z}}^{\varepsilon}_s)_{s\geq 0}$ be BBM$(\gamma/\varepsilon)$ with $\tilde{\mathscr{Z}}^{\varepsilon}_0$ distributed as a Poisson random measure with intensity $\mu/\varepsilon$. Then we have*

$$(\varepsilon\tilde{\mathscr{Z}}^{\varepsilon}_s)_{s\geq 0}\to(\mathscr{X}_s)_{s\geq 0}$$

*as $\varepsilon$ goes to 0, in distribution on $\mathbb{D}([0,\infty),\mathscr{M}_F(\mathbb{R}^d))$. This process $(\mathscr{X}_s)_{s\geq 0}$ is called SBM$(\gamma)$ started at $\mu$.*

**Proposition 3.1.3** (The existence of on/off super Brownian motion (ooSBM), Blath and Jacobi [14])**.** *Let $(\mu_{\mathrm{d}},\mu_{\mathrm{a}})\in\mathscr{M}_F(\mathbb{R}^d)^2$. Then there exists a measure-valued Markov process $((\mathscr{Y}^{(\mathrm{d})}_s,\mathscr{Y}^{(\mathrm{a})}_s),s\geq 0)$ that takes values in $\mathscr{M}_F(\mathbb{R}^d)^2$ with $(\mathscr{Y}^{(\mathrm{d})}_0,\mathscr{Y}^{(\mathrm{a})}_0)=(\mu_{\mathrm{d}},\mu_{\mathrm{a}})$ which is the scaling limit of ooBBM in the following sense. For $\varepsilon>0$, let $((\mathscr{Z}^{(\varepsilon,\mathrm{d})}_s,\mathscr{Z}^{(\varepsilon,\mathrm{a})}_s))_{s\geq 0}$ be ooBBM$(\gamma/\varepsilon,c,\tilde{c})$ with $\mathscr{Z}^{\varepsilon}_0$ distributed as a Poisson random measure with intensity $(\mu_{\mathrm{d}}/\varepsilon,\mu_{\mathrm{a}}/\varepsilon)$. Then we have*

$$((\varepsilon\mathscr{Z}^{(\varepsilon,\mathrm{d})}_s,\varepsilon\mathscr{Z}^{(\varepsilon,\mathrm{a})}_s))_{s\geq 0}\to((\mathscr{Y}^{(\mathrm{d})}_s,\mathscr{Y}^{(\mathrm{a})}_s))_{s\geq 0}$$

*as $\varepsilon$ goes to 0, in distribution on $\mathbb{D}([0,\infty),\mathscr{M}_F(\mathbb{R}^d)^2)$. The process $(\mathscr{Y}_s)_{s\geq 0}$ is called ooSBM$(\gamma,c,\tilde{c})$ started at $(\mu_{\mathrm{d}},\mu_{\mathrm{a}})$.*

The superprocess ooSBM can also be characterised through its evolution equations, see [14, Theorem 1.3]. We make a remark about the distributional results in Proposition 3.1.3.

*Remark* 3.1.4. In [14], ooSBM and ooBBM are actually represented as processes in $\mathscr{M}_F(\mathbb{R}^d\times\{0,1\})$ but it will be more convenient for us to represent them in $\mathscr{M}_F(\mathbb{R}^d)^2$ where the two components are the restrictions of the measure to the dormant and active parts. As the map from $\mathscr{M}_F(\mathbb{R}^d\times\{0,1\})$ to $\mathscr{M}_F(\mathbb{R}^d)^2$ is a homeomorphism, all results can be easily transferred.

We define the support of a measure noted in Cohn [25, Section 7.4] and the range of a measure-valued process, see e.g. Etheridge [32, Definition 6.7].

**Definition 3.1.5** (The support of a measure and the range of a measure-valued process)**.** Let $\mu\in\mathscr{M}_F(\mathbb{R}^d)$. Then $\mathbb{R}^d$ has a largest open subset $U_\mu$ such that $\mu(U)=0$ – this is the union of all open subsets $U'$ such that $\mu(U')=0$. The *support* of $\mu$ is defined to be $\mathrm{supp}(\mu):=\mathbb{R}^d\backslash U_\mu$.

For a measure-valued càdlàg process $\mathscr{X}=(\mathscr{X}_s)_{s\geq 0}$ we define the *range* of $\mathscr{X}$ to be $\mathscr{R}(\mathscr{X})=\overline{\bigcup_{s>0}\mathrm{supp}(\mathscr{X}_s)}$, that is the closure of the union of supports over all positive times.

We note that $\mathrm{supp}(\mu)$ is the smallest closed set whose complement has measure zero under $\mu$. Furthermore we note that a point $x \in \mathbb{R}^d$ is in $\mathrm{supp}(\mu)$ if and only if every open neighbourhood of $x$ has positive measure under $\mu$. These sets given by the range and the support of a superprocess may be hard to understand. We follow [32, Chapter 6] to define the Hausdorff dimension of the range and the support.

**Definition 3.1.6** (The Hausdorff dimension)**.** Let $A \subseteq \mathbb{R}^d$ be a measurable set. For $\delta > 0$ we call a $\delta$-cover any collection of open sets $U_j, j \in J$ such that

$$A \subseteq \bigcup_{j \in J} U_j; \quad \sup_{j \in J} \mathrm{Leb}(U_j) < \delta.$$

We define $\mathscr{H}^s_\delta(A) = \inf \sum_{j \in J} |U_j|^s$ where the infimum is taken over all $\delta$-covers and we define $\mathscr{H}^s(A) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(A)$. The *Hausdorff dimension* of $A$, which we denote $\dim(A)$, is defined to be the unique number such that

$$\mathscr{H}^s(A) = \infty \text{ for all } 0 \leq s < \dim(A); \quad \text{and} \quad \mathscr{H}^s(A) = 0 \text{ for all } \dim(A) < s < \infty.$$

We state well-known results about SBM, see e.g. [32, Theorem 6.15] and see also [67, Theorem 4.5.2].

**Proposition 3.1.7** (The dimension of the support and the range of SBM)**.** *Let $(\mathscr{X}_s, s \geq 0)$ be SBM. Then we have that for $y > 0$, conditional on $\mathscr{X}_y$ being non-zero, that*

$$dim(supp(\mathscr{X}_y)) = 2 \wedge d \quad a.s.$$

*Furthermore, for the SBM $\mathscr{X}$, we have that*

$$dim(\mathscr{R}(\mathscr{X})) = 4 \wedge d \quad a.s.$$

We state our main theorem here.

**Theorem 3.1.8** (The equality of ranges for SBM and ooSBM)**.** *Let $\mu_{\mathrm{d}}$, $\mu_{\mathrm{a}} \in \mathscr{M}_F(\mathbb{R}^d)$ be finite measures and let $\gamma$, $c$, $\tilde{c} > 0$. Let $\mathscr{X}$ be an SBM$(\gamma)$ with initial measure $\mu_{\mathrm{d}} + \mu_{\mathrm{a}}$ and let $\mathscr{Y} = (\mathscr{Y}^{(\mathrm{d})}, \mathscr{Y}^{(\mathrm{a})})$ be an ooSBM$(\gamma, c, \tilde{c})$ with initial measure $(\mu_{\mathrm{d}}, \mu_{\mathrm{a}})$. Then we have that $\mathscr{R}(\mathscr{X})$ is equal in distribution to $\mathscr{R}(\mathscr{Y}^{(\mathrm{d})}) \cup \mathscr{R}(\mathscr{Y}^{(\mathrm{a})})$.*

*Moreover there exists a coupling for $\mathscr{X}$ and $\mathscr{Y}$ such that $\mathscr{R}(\mathscr{X}) = \mathscr{R}(\mathscr{Y}^{(\mathrm{d})}) \cup \mathscr{R}(\mathscr{Y}^{(\mathrm{a})})$ a.s.*

*In particular we have that $dim(\mathscr{R}(\mathscr{Y}^{(\mathrm{d})}) \cup \mathscr{R}(\mathscr{Y}^{(\mathrm{a})})) = 4 \wedge d$ a.s.*

The dimension of the range follows immediately from the range coupling and Proposition 3.1.7. The range operator $\mathscr{R}$ as defined in Definition 3.1.5 is not continuous on the space of measure-valued càdlàg paths. Therefore, we cannot infer our range coupling result in Theorem

3.1.8 directly from coupling the ranges of BBM and ooBBM and using the scaling limits in Propositions 3.1.2 and 3.1.3.

Our proof technique utilises the Brownian snake with discontinuous spatial motion as defined in Bertoin, Le Gall and Le Jan [11]. In particular we study a Brownian snake with spatial motion given by a Brownian component and a second independent component given by a compound Poisson process with unit drift. This has similarities with the Poisson snake introduced in Abraham and Serlet [1] which is a Brownian snake with spatial motion given by a Poisson process. We also note a related problem in Delmas and Dhersin [30] where they used the Brownian snakes along with time-changes to show the existence of superprocesses related to SBM that they refer to as SBM with interactions. In their work they apply a random time-change to each of the paths of the path-valued Brownian snake. Their work does not directly apply to our setting as they require the time change $\phi$ to satisfy $\phi'(s) > 0$ for all $s > 0$ and all paths and we would require $\phi'(s) = 0$ during the dormancy periods of paths and so we instead use the literature on Brownian snakes with discontinuous spatial motion mentioned above.

### 3.1.1 Structure of the chapter

In Section 3.2 first we define the Brownian snake, see Theorem 3.2.1. We also define an age process in Definition 3.2.4 for the snake that tracks how long each particle has lived as the sum of its active time and dormant time. We then define an on/off Brownian snake in Definition 3.2.7.

In Section 3.3 we define the *h-erased contour function* for an excursion and use this to construct BBM and ooBBM from the same on/off Brownian snake in Proposition 3.3.7.

In Section 3.4, Lemma 3.4.1 we give an alternative characterisation of the ooBBM and BBM constructed in Proposition 3.3.7. We use this in Theorem 3.4.6, along with the notion of an exit local time given in Proposition 3.4.3, to construct a measure-valued process with the finite-dimensional distributions of ooSBM. Then, in Section 3.5 we show that a continuous modification of this measure-valued process has the same range as an SBM. From this, we can prove Theorem 3.1.8.

## 3.2 Construction

We recall the set-up given in [11, Section 4] for a Brownian snake with discontinuous spatial motion. Let $\xi$ be a càdlàg Borel right Markov process with values in a complete and separable metric space $(E, d_E)$ and let the process $\xi$ be defined on the Skorokhod space $\mathbb{D}([0,\infty), E)$. In our setting $E = \mathbb{R}^d \times [0,\infty)$ and $d_E$ is the Euclidean metric. Furthermore our spatial motion $\xi$ is of the form $(B,S)$ where $B$ and $S$ are independent, $B$ is a $d$-dimensional Brownian motion and $S$

is a compound Poisson subordinator with unit drift and jump intensity $\pi(ds) = c\tilde{c}e^{-\tilde{c}s}ds$. For $x \in \mathbb{R}^d \times [0,\infty)$ let $P_x^\xi$ the distribution on $\mathbb{D}([0,\infty), \mathbb{R}^d \times [0,\infty))$ of $\xi$ with initial value $x$.

We define a killed path in $E$ to be a càdlàg function $w : [0,\zeta) \to E$, where we call $\zeta = \zeta(w) \in (0,\infty)$ the lifetime of $w$. Furthermore, for all $x \in E$, we call $x$ a killed path with lifetime 0. Let $\mathscr{W}$ be the set of all killed paths and for $x \in E$ let $\mathscr{W}_x$ denote the set of all the killed paths $w$ such that $w(0) = x$.

We define, for $w$, $w' \in \mathscr{W}$

$$d(w,w') = d_E(w(0),w'(0)) + |\zeta(w) - \zeta(w')| + \int_{y=0}^{\zeta(w)\wedge\zeta(w')} (d^{\mathrm{Sko}}(w|_{[0,y]}, w'|_{[0,y]}) \wedge 1)dy, \quad (3.1)$$

where $w|_{[0,y]}$ denotes the restriction of $w$ to $[0,y]$. This defines a metric $d$ on $\mathscr{W}$, and $(\mathscr{W},d)$ is a complete and separable metric space. However we note that a killed path does not necessarily have a left limit at $\zeta(w)$.

Consider $w \in \mathscr{W}_x$ with lifetime $\zeta(w) > 0$, i.e. $w = (w(y), 0 \le y < \zeta(w))$. If $0 \le a < \zeta(w)$, and $b \ge a$, we let $Q_{a,b}(w,dw')$ be the unique probability measure on $\mathscr{W}_x$ such that

1. $\zeta(w') = b$, $Q_{a,b}(w,dw')$ a.s.
2. $w'(y) = w(y)$, for all $y \in [0,a]$, $Q_{a,b}(w,dw')$ a.s.
3. the law under $Q_{a,b}(w,dw')$ of $(w'(a+y), 0 \le y < b-a)$ is the law of $(\xi_y, 0 \le y < b-a)$ under $P^\xi_{w(a)}$.

When the left limit $w(a-)$ exists we can extend this definition to the case $a = \zeta(w)$ and in this instance we modify the third point so that the law of $(w'(a+y), 0 \le y < b-a)$ is the law of $(\xi_y, 0 \le y < b-a)$ under $P^\xi_{w(a-)}$.

Note that $Q_{0,0}(w,dw') = \delta_x(dw')$ and $Q_{0,b}(w,dw')$ is the law of $(\xi_y, 0 \le y < b)$ under $P_x^\xi$ which we denote by $P_x^{\xi,b}(dw')$. We also set $Q_{0,b}(x,dw') = P_x^{\xi,b}(dw')$.

Let $R = (R_t, t \ge 0)$ be a reflecting Brownian motion on $[0,\infty)$ and for $r \ge 0$ let $P_r^R$ denote the law of $R$ with $R_0 = r$. We denote by $\theta_t^r(dadb)$ the joint distribution of $(\inf(\{R_{t'} : t' \in [0,t]\}), R_t)$ under $P_r^R$:

$$\theta_t^r(dadb) = \frac{2(r+b-2a)}{\sqrt{2\pi t^3}} \exp -\Big(\frac{(r+b-2a)^2}{2t}\Big) \mathbb{1}_{(0<a<r\wedge b)} dadb + \frac{\sqrt{2}}{\sqrt{\pi t}} \exp -\Big(\frac{(r+b)^2}{2t}\Big) \mathbb{1}_{(0<b)} \delta_0(da)db.$$

For $r > 0$ we define a set $\mathscr{G}_r \subset \mathscr{C}([0,\infty),[0,\infty))$. We let $g : [0,\infty) \to [0,\infty) \in \mathscr{G}_r$ be the functions with $g_0 = r$ and the following properties:

1. for some $t > 0$ we have $\inf(\{g_{t'}, 0 \le t' \le t\}) < g_0$;

2. for all $\alpha \in (0,1/2)$ we have that $g$ has Hölder continuous paths with exponent $\alpha$.

**Theorem 3.2.1** (Bivariate Brownian snake)**.** *In the setting given above, let $x \in E = \mathbb{R}^d \times [0,\infty)$ and let $w \in \mathscr{W}_x$. Then there exists a continuous time-homogeneous strong Markov $\mathscr{W}_x$-valued process $X = (X_t, t \geq 0)$ with $X_0 = w$ with the following properties:*

1. *the transition kernels are given by the formula*

$$K_t(w', dw'') = \int_{[0,\infty)^2} \theta_t^{\zeta(w')}(dadb) Q_{a,b}(w', dw'').$$

*Let $\mathbb{P}_w$ denote the distribution for this snake on $\mathscr{C}([0,\infty), \mathscr{W}_x)$.*

2. *Furthermore the process $R = (R_t, t \geq 0)$ where $R_t := \zeta(X_t)$, which we call the* contour process *for the snake, is a reflecting Brownian motion on $[0,\infty)$ (started from $\zeta(w)$);*

3. *Finally let $g \in \mathscr{G}_{\zeta(w)}$. Then the conditional distribution of $(X_t, t \geq 0)$ given the contour process $R = g$ is that of a continuous time-inhomogeneous Markov process. Let $w' \in \mathscr{W}_x$, suppose that either $w'(\zeta(w')-)$ exists or $\inf(\{g_u, t \leq u \leq t'\}) < g_t$ for some $t' > t$. Then we can explicitly state the transition kernel as*

$$K^g_{t,t'}(w', dw'') = Q_{\inf(\{g_u, t \leq u \leq t'\}), g_{t'}}(w', dw''),$$

*and this collection of time-inhomogeneous Markov processes form a family of regular conditional distributions for the strong Markov process defined above.*

*Proof.* The first two points are stated in [11, Proposition 5]. Their proof quotes the proof of Le Gall [66, Theorem 1.1]. They explain how the proof in [66] generalises to their more general setting in [11] and as the third point is an intermediate result in the proof of [66] we can conclude this additional result by the same logic. In particular, in [66], the proof first considers a fixed contour process, in our case $g \in \mathscr{G}_r$, then constructs a probability measure for the snake by stating the marginals for a fixed number of time points $t_1 < \cdots < t_n$ using the kernels $K^g_{t_i,t_{i+1}}$ and then applies the Kolmogorov-Čentsov theorem. This applies and suffices in our setting, as $K^g_{t,t'}(w', dw'')$-a.e. $w''$ has a left limit at $\zeta(w'')$ and $P^R_r(\mathscr{G}_r) = 1$. ∎

We can also start the process from $x$, the path of length 0 and we write $\mathbb{P}_x$ for this distribution and $\mathbb{E}_x$ for this expectation operator.

In our setting for $X_t = (B_t, S_t)$ we cannot exclude the case where $\lim_{y \to \zeta(X_t)-} S_t(y) = \infty$ and so we must take care when we deal with this component of the snake.

It is noted in [11, Section 4] that $\mathbb{P}_w$-a.s. for every $t < t'$, the killed paths $X_t$, $X_{t'}$ coincide for $r < m(t,t') := \inf(\{\zeta(X_{r'}) : r' \in [t,t']\})$ and that they also coincide at $r = m(t,t')$ provided that $m(t,t') < \zeta(X_t) \wedge \zeta(X_{t'})$. We use this property throughout and we refer to this as the *snake property*.

In the setting of Le Gall [67] the spatial motion is continuous and he works with stopped paths $\mathscr{W}'$ where $w \in \mathscr{W}'$ is a continuous map $w : [0, \zeta] \to E' := \mathbb{R}^d$, where $\zeta = \zeta(w) < \infty$ as before. There is a complete and separable metric space $(\mathscr{W}', d')$ with metric $d'$ defined by

$$d'(w, w') = |\zeta(w) - \zeta(w')| + \sup\big(\{d_{E'}(w(y \wedge \zeta(w)), w'(y \wedge \zeta(w')), y \geq 0\}\big).$$

which is a stronger metric than $d$ on $\mathscr{W}'$ obtained by replacing $E$ and $\mathscr{W}$ by $E'$ and $\mathscr{W}'$ in (3.1). We give a brief discussion about new setting in Remark 3.2.2 below.

*Remark* 3.2.2 (Head of the snake). In the setting of Le Gall [67] the spatial motion is continuous. In this setting there exists a Brownian snake $(W_t)_{t\geq 0}$, with distribution $\mathbb{P}'_{w'}$ on $\mathscr{C}([0,\infty), \mathscr{W}')$, that is continuous with respect to the metric $d'$ and has a continuous *head of the snake* process $(\hat{W}_t = W_t(\zeta(W_t)))_{t\geq 0}$ $\mathbb{P}'_{w'}$-a.s.

We can also define a snake $(V_t)_{t\geq 0}$ with spatial motion given by a subordinator $S$ with the same contour process that is continuous with respect to the metric $d$ on a common probability space. We can show that the path-valued process with both spatial components $(W_t, V_t)_{t\geq 0}$ is continuous with respect to the metric $d$ on the product space. By a suitable coupling with the bivariate snake $X$ given in Theorem 3.2.1 we have that $(W, V)$ is a modification of $X$ and both processes are path-continuous with respect to the metric $d$.

We can conclude, for all $w \in \mathscr{W}$ whose left limit at $\zeta(w)$ exists, that $\mathbb{P}_w$-a.s. the Brownian component $(B_t, t \geq 0)$ of the snake $X$ has a head $\hat{B}_t = B_t(\zeta(B_t)-)$ for all $t \geq 0$ and that $(\hat{B}_t, t \geq 0)$ is continuous. This follows as two continuous processes that are modifications of one another are indistinguishable, see e.g. Revuz-Yor [88, Definition I.1.7].

For $x \in E$, $w \in \mathscr{W}_x$, let $\mathbb{E}_w$ denote the expectation operator for the Brownian snake started at $w$. The transition kernels for the strong Markov process above are such that for every $0 \leq t' \leq t$ and for every non-negative measurable functional $\Phi$ on $\mathscr{W}_x$, we have

$$\mathbb{E}_w(\Phi(X_t)|(X_r, r \leq t')) = \int K_{t-t'}(X_{t'}, dw)\Phi(w), \quad \mathbb{P}_w - \text{a.s.}$$

where $\mathbb{E}_w$ is the expectation operator for $X$ under $\mathbb{P}_w$.

For $x \in E$, $w \in \mathscr{W}_x$ and $g \in \mathscr{G}_{\zeta(w)}$, we define $\mathbb{P}_{w|g}$ to be the probability measure for the time-inhomogeneous Markov process with contour process $g$ on the space of measure-valued functions. Let $\mathbb{E}_{w|g}$ be the expectation operator. Then for any bounded continuous $\mathscr{W}_x$-valued function $\Phi$ we have that

$$\mathbb{E}_{w|g}(\Phi(X_t)|(X_r, r \leq t')) = \int K^g_{t',t}(X_{t'}, dw')\Phi(w'), \quad \mathbb{P}_{w|g} - \text{a.s.}$$

We explore more properties of the bivariate Brownian snake in Section 3.4, in particular the notion of an exit local time.

We give a summary of the excursion measure $n$ for reflecting Brownian motion. We define the space $\mathscr{E} \subset \mathscr{C}([0,\infty),[0,\infty))$ of excursions as continuous functions on $[0,\infty)$ that take positive values on some interval $(0,\sigma)$ and the value 0 at 0 and on $[\sigma,\infty)$ for some $\sigma \in (0,\infty)$. For an excursion $f = (f_t, t \geq 0)$ and for $y > 0$ we define $T_y(f) := \inf(\{t \in [0,\infty) : f_t = y\})$. The excursion measure $n$ for $R$ (reflecting Brownian motion on $[0,\infty)$) is the unique sigma finite measure on $\mathscr{E}$ with the following properties, see e.g. [88, Chapter XII].

1. For all $y \in (0,\infty)$ we have $n(T_y(f) < \infty) = 1/2y$.

2. For all $y \in (0,\infty)$, the conditional distribution $n(\cdot | T_y(f) < \infty)$ is the distribution of the following process $Z = (Z_t)_{t\geq 0}$. We have that $(Z_t)_{0\leq t\leq T_y(Z)}$ has the law of a Bessel process of dimension 3 started from 0, and let us denote this by $P_0^{\mathrm{Bes}(3)}$. Then the process $(Z_{T_y(Z)+t})_{t\geq 0}$ is independent of $(Z_t)_{0\leq t\leq T_y(Z)}$ and has the law of a Brownian motion on $[0,\infty)$ started at $y$ and absorbed when it hits 0. The Bessel process of dimension 3 is a diffusion with infinitesimal drift $\mu(y) = 1/y$ and infinitesimal variance $v(y) = 1$, see e.g. Revuz-Yor [88, Chapter XI].

Note we have for $x \in E$ and $w \in \mathscr{W}_x$ that $\mathbb{P}_w(\cdot) = \int_g P^R_{\zeta(w)}(dg)\mathbb{P}_{w|g}(\cdot)$. Furthermore, for $x \in E$, we can define an excursion measure for the bivariate snake denoted by $\mathbb{N}_x$ and given by $\mathbb{N}_x(\cdot) = \int_f n(df)\mathbb{P}_{x|f}(\cdot)$.

We now return to the spatial process $\xi$ itself and prove a lemma that we use later.

**Lemma 3.2.3.** *Let $b \in \mathbb{R}^d$ and consider a process $\xi = (\xi_y, y \geq 0)$ with distribution $P^{\xi}_{(b,0)}$. Let $B$ and $S$ be the components of $\xi$, so that $\xi = (B,S)$. Let*

$$H^{(s)}(\xi) := \inf\{y \in [0,\infty) : S(y) \geq s\}$$

*and let $A^{(s)}(\xi) := 1$ if $H^{(s+\varepsilon)}(\xi) > H^{(s)}(\xi)$ for all $\varepsilon > 0$ and $A^{(s)}(\xi) := 0$ otherwise. Then $(B(H^{(s)}(\xi)), A^{(s)}(\xi))_{s\geq 0}$ is an ooBM$(c,\tilde{c})$ (i.e., an ooBBM$(0,c,\tilde{c})$) started from $(b,1) \in \mathbb{R}^d \times \{0,1\}$, i.e. a single particle that does not die or split and that switches between dormant and active states.*

*Proof.* Let $C = (C_s, s \geq 0)$ be given by $C_s := B(H^{(s)}(\xi))$ and let $\tilde{A} = (\tilde{A}_s, s \geq 0)$ be given by $\tilde{A}_s := A^{(s)}(\xi)$. We show that $(C,\tilde{A})$ has the correct spatial motion and dormancy dynamics. Let $T := \inf(\{y : S(y-) \neq S(y)\})$. The time $T$ represents the first discontinuity point of $S$ (and $\xi$). We have that

$$C_s = B_s, \quad \tilde{A}_s = 1, \quad 0 \leq s < T.$$

Let $T' := S(T) - S(T-)$. Then we have that

$$C_s = B_T, \quad \tilde{A}_s = 0, \quad T \leq s < T + T'.$$

Now by construction we have that $T \sim \text{Exp}(c)$ and $T' \sim \text{Exp}(\tilde{c})$ which gives the correct dormancy dynamics. The spatial motion is also correct as $C$ evolves with the dynamics of a Brownian motion when $\tilde{A} = 1$ and $C$ does not move when $\tilde{A} = 0$. The result follows by applying the same argument to each subsequent discontinuity point of $S$. ■

We now define what we call the *age process* for a snake.

**Definition 3.2.4** (Age process)**.** Fix $s > 0$, let $x \in E$ and $w \in \mathscr{W}_x$ and consider a bivariate snake $X = (X_t, t \geq 0)$ started from $w$. For $t \geq 0$ let $B_t$ and $S_t$ be the Brownian and subordinator components of $X_t$, so that $X_t = (B_t, S_t)$. Let $R = (R_t, t \geq 0)$ be the contour process for $X$ given by, for $t \geq 0$, $R_t := \zeta(X_t)$. For $t \geq 0$ we define

$$H_t^X(s) := \inf(\{y \in [0, R_t) : S_t(y) \geq s\} \cup \{R_t\})$$

and we note $H_t^X(s) = H^{(s)} \circ X_t$. We say that $t$ is active at (time) $s$ if $H_t^X(s+\varepsilon) > H_t^X(s)$ for all $\varepsilon > 0$ and $t$ is dormant at (time) $s$ otherwise. We define a function $A_t^X : [0,\infty) \to \{0,1\}$ which is $A_t^X(s) = 1$ if $t$ is active at $s$ and $A_t^X(s) = 0$ otherwise (note again that $A_t^X(s) = A^{(s)} \circ X_t$). Finally we define what we call the residual process $(\text{Res}_t^X(s))_{t \geq 0}$ at $s$ by $\text{Res}_t^X(s) := R_t - H_t^X(s)$. We drop the superscript $X$ notation and write $H_t := H_t^X$, $A_t := A_t^X$, $\text{Res}_t := \text{Res}_t^X$ when the snake $X$ is clear from context.

We now state some properties of the age process.

**Lemma 3.2.5.** *Let $s > 0$, let $x \in E$, let $w \in \mathscr{W}_x$. For $X$ a snake under $\mathbb{P}_w$ (respectively $\mathbb{N}_x$) we have that $(H_t^X(s))_{t \geq 0}$, $(Res_t^X(s), t \geq 0)$ are both continuous $\mathbb{P}_w$-a.s. (respectively $\mathbb{N}_x$-a.e.). Let $R = (R_t, t \geq 0)$ be the contour process defined by $R_t := \zeta(X_t)$. Define $V_s := \{t \geq 0 : Res_t^X(s) > 0\}$. We have that $V_s$ is an open set $\mathbb{P}_w$-a.s. ($\mathbb{N}_x$-a.e.). Also, for the open set $V_s = \cup_{j \in J_s}(a_j, b_j)$, where the sets $(a_j, b_j)$ are disjoint open intervals over some countable index set $J_s$, we have for each $j \in J_s$ the function $t \mapsto H_t^X(s)$ is constant on $[a_j, b_j]$ $\mathbb{P}_w$-a.s. ($\mathbb{N}_x$-a.e.).*

*Proof.* Let $t > 0$. We consider two cases. First assume $H_t^X(s) < R_t$ and let $\delta := R_t - H_t^X(s)$. Then, by continuity of the contour process $(R_{t'}, t' \geq 0)$ there exists $\delta' > 0$ such that for $t' \in [t - \delta', t + \delta']$ we have $R_{t'} > H_t^X(s) + \delta/2$. In this case, by the snake property of $X$, we have that $t' \mapsto H_{t'}^X(s)$ is constant on $[t - \delta', t + \delta']$ and so $t' \mapsto H_{t'}^X(s)$ is continuous at $t$. It also shows us that $V_s$ is open.

Before we show continuity for the case where $H_t^X(s) = R_t$, we show the final part of the lemma. Let $(a_j, b_j)$ be an open interval of $V_s$. Let $(c_j, d_j)$ be the largest open interval with $(a_j, b_j) \subseteq (c_j, d_j)$ and for which constancy extends, i.e. such that $t \mapsto H_t^X(s)$ is constant on $(c_j, d_j)$. It follows by the same arguments as in the previous paragraph that $(c_j, d_j) = (a_j, b_j)$. Moreover, we have that $X_{a_j}|_{[0, R_{a_j})} = X_t|_{[0, R_{a_j})}$ for $t \in (a_j, b_j)$ by the snake property and so we can conclude that the constancy extends to the boundary point $a_j$. By the same logic we have that the constancy extends to the other boundary point $b_j$.

Now we assume that $H_t^X(s) = R_t$ and we show continuity at $t$ in this case. We show left- and right-continuity separately. Let $(t_n, n \in \mathbb{N})$ be a non-decreasing sequence that converges to $t$, i.e. $t_n \uparrow t$. If $t_n \in V_s$ there exists $j_n \in J_s$ such that $t_n \in ((a_{j_n}, b_{j_n}))$, $t' \mapsto H_{t'}^X(s)$ is constant on $[a_{j_n}, b_{j_n}]$ and $a_n$, $b_n \notin V_s$. Define the sequence $t_n'$ by $t_n' = b_{j_n}$ for $t_n \in V_s$ and $t_n' = t_n$ otherwise. Note that this for this new sequence we have $t_n' \uparrow t$. Therefore we have

$$\lim_{n\to\infty} H_{t_n}^X(s) = \lim_{n\to\infty} H_{t_n'}^X(s) = \lim_{n\to\infty} R_{t_n'} = R_t = H_t^X(s).$$

This shows left-continuity at $t$. The same argument holds for right-continuity where we use $a_n$ in place of $b_n$ to define the sequence $(t_n', n \geq 1)$. ■

We give a remark about the age process.

*Remark* 3.2.6. The age process can be interpreted as a stopping line for the snake, see e.g. Chauvin [22]. Furthermore there are similarities to the notion of a frost for fragmentation processes as defined in Bertoin [10].

At this point we briefly mention a possible confusion due to notation where an element of $\mathbb{R}^d \times \{0,1\}$ denotes the position of a particle of an ooBBM as defined in Definition 3.1.1 and an element of $E = \mathbb{R}^d \times [0,\infty)$ denotes the starting point of a bivariate snake as in Theorem 3.2.1. The latter notation is used in the next definition.

**Definition 3.2.7** (The on/off Brownian snake)**.** Consider finite measures $\mu_{\mathrm{d}}$ and $\mu_{\mathrm{a}}$ on $\mathbb{R}^d$. Let $(\Omega, \mathscr{F}, \mathbb{P}_{(\mu_{\mathrm{d}},\mu_{\mathrm{a}})})$ be a probability space, let $\mathbb{E}_{(\mu_{\mathrm{d}},\mu_{\mathrm{a}})}$ be the expectation operator, and suppose there exists on this space a Poisson random measure $\sum_{i\in I} \delta_{f^i}$ on the space of excursions in $\mathbb{R}_+$ with intensity measure $(\|\mu_{\mathrm{d}}\| + \|\mu_{\mathrm{a}}\|)n(\cdot)$ (where $n$ is the Itô excursion measure for Brownian motion) for which we can mark the excursions in the following way. For $i \in I$:

1. with probability $\|\mu_{\mathrm{d}}\|/(\|\mu_{\mathrm{d}}\| + \|\mu_{\mathrm{a}}\|)$ we mark $f^i$ with $(b_i, s_i)$ where $b_i$ is sampled from the normalised probability distribution $\tilde{\mu}_d := \mu_{\mathrm{d}}/\|\mu_{\mathrm{d}}\|$ and $s_i$ is sampled independently from the $\mathrm{Exp}(\tilde{c})$ distribution (the exponential distribution with density $\tilde{c}e^{-\tilde{c}z}$, $z > 0$);

2. with probability $\|\mu_{\mathrm{a}}\|/(\|\mu_{\mathrm{d}}\| + \|\mu_{\mathrm{a}}\|)$ we mark $f^i$ with $(b_i, s_i)$ where $b_i$ is sampled from the normalised probability distribution $\tilde{\mu}_a := \mu_{\mathrm{a}}/\|\mu_{\mathrm{a}}\|$ and $s_i = 0$.

We then mark the pair $((b_i, s_i), f^i)$ with a snake $x^i$ where the snake is sampled from the probability distribution $\mathbb{P}_{(b_i,s_i)|f^i}(\cdot)$. Let

$$\sum_{i\in I} \delta_{((b_i,s_i),f^i,x^i)} \tag{3.2}$$

be the Poisson random measure we obtain. We call this *the on/off Brownian snake*.

We have abused terminology somewhat here as what we have defined as an on/off Brownian snake is not a path-valued process that evolves in time (as is the usual setting for snakes) but instead is a Poisson random measure of path-valued processes that evolve in time.

## 3.3 Identification

In this section we construct both BBM and ooBBM from the same on/off Brownian snake defined in Definition 3.2.7. Recall the stopping time $T_h(f)$ for $h > 0$ for an excursion function $f \in \mathscr{E}$ from the previous section. We extend this definition to a $\mathscr{W}$-valued $X$ and define $T_h(X) := \inf(\{t \in [0,\infty) : \zeta(X_t) = h\})$. We also extend the definition of $\sigma$ and write $\sigma(X)$ for the lifetime of the contour process $(\zeta(X_t), t \geq 0)$ in the same way. We give a decomposition for the distribution $\mathbb{N}_x(\cdot | T_h < \infty)$ on $\mathscr{C}([0,\infty), \mathscr{W})$.

**Proposition 3.3.1.** *Let $x \in E$. The snake $X$ under the probability measure $\mathbb{N}_x(\cdot | T_h < \infty)$ can be expressed as the concatenation of the following two snakes.*

1. *$(\zeta(X_t), 0 \leq t \leq T_h(X))$ has the distribution $P_0^{Bes(3)}$. Conditional on $(\zeta(X_t), 0 \leq t \leq T_h(X))$ being given by $g := (g_t)_{0 \leq t \leq T_h(g)}$, the process $(X_t, 0 \leq t \leq T_h(X))$ is a time-inhomogeneous snake with distribution $\mathbb{P}_{x|g}(\cdot)$.*

2. *Conditional on $X_{T_h(X)} = w$ for an $E$-valued killed path $w$, the processes $(X_t, 0 \leq t \leq T_h(X))$ and $(X_{T_h(X)+t}, t \geq 0)$ are independent and the latter has law $\mathbb{P}_w(\cdot)$.*

*Proof.* The proof follows almost immediately from Theorem 3.2.1 and the representation of the excursion measure $n$ for Brownian motion given at the end of the previous section. The second part follows by the Markov property of the time-inhomogeneous snake $X$ conditionally given the contour process and of the decomposition $\mathbb{P}_w(\cdot) = \int_g P^R_{\zeta(w)}(dg)\mathbb{P}_{w|g}(\cdot)$. ■

*Remark* 3.3.2. The "stopped Bessel snake" given by $(X_t, 0 \leq t \leq T_h(f))$ under $\mathbb{N}_x(\cdot | T_h < \infty)$ in the proposition above is also a strong Markov process and the proof follows the same logic as for the Brownian snake. However we do not require this here. We also note that we can represent the distribution of the complete snake process $(X_t, t \geq 0)$ under $\mathbb{N}_x(\cdot | T_h < \infty)$ as $\mathbb{N}_x(\cdot | T_h < \infty) = \int_f \mathbb{P}_{x|f}(\cdot) n(df | T_h < \infty)$.

We now recall the construction of $h$-erased contour processes from Le Gall [65] with vertices that form an alternating exponential random walk, see also Neveu and Pitman [78]. We use these processes to construct BBM and ooBBM. For an excursion $f = (f_t, t \geq 0) \in \mathscr{E}$ with $T_h(f) < \infty$ we define the following. Let $h > 0$, let $\alpha_0^{(h)} = 0$ and for at most finitely many $p' \geq 0$, following the construction set out in [65, Section 4], we can define

$$\gamma^{(h)}_{2p'} = \beta^{(h)}_{p'} = \inf\left\{t > \alpha^{(h)}_{p'} : f_t \geq \inf\{f_{t'} : \alpha^{(h)}_{p'} \leq t' \leq t\} + h\right\}, \quad Z^{(h)}_{2p'} = f_{\beta^{(h)}_{p'}} - h,$$

$$\gamma^{(h)}_{2p'+1} = \alpha^{(h)}_{p'+1} = \inf\left\{t > \beta^{(h)}_{p'} : f_t \leq \sup\{f_{t'} : \beta^{(h)}_{p'} \leq t' \leq t\} - h\right\}, \quad Z^{(h)}_{2p'+1} = f_{\alpha^{(h)}_{p'+1}}.$$

These stopping times define an alternating walk $(Z^{(h)}_p)_{0 \leq p \leq M^{(h)}}$ where $M^{(h)}$ is the largest $p \geq 0$ for which $\gamma^{(h)}_p < \infty$. Let $\tilde{M}^{(h)} := \lfloor M^{(h)}/2 \rfloor$. Let $Z^{(h)}_{M^{(h)}+1} := 0$ and $\gamma^{(h)}_{M^{(h)}+1} := \sigma(f)$. We define

$f^{(h)} : [0,\infty) \to [0,\infty)$ to be the *h-erased contour function* with local extrema given by the points $(\gamma_p^{(h)})_{0\le p\le M^{(h)}+1}$, i.e.

$$f^{(h)}(t) := \begin{cases} 0, & 0 \le t < \gamma_0^{(h)} \quad \text{or} \quad t \ge \gamma_{M^{(h)}+1}^{(h)} \\ Z_p^{(h)} + \frac{(t-\gamma_p^{(h)})(Z_{p+1}^{(h)} - Z_p^{(h)})}{(\gamma_{p+1}^{(h)} - \gamma_p^{(h)})}, & \gamma_p^{(h)} \le t < \gamma_{p+1}^{(h)}, 0 \le p \le M^{(h)}. \end{cases}$$

We recall the construction of a tree from a contour function, see e.g. Le Gall [67, Chapter 2].

**Definition 3.3.3** (Tree from contour function)**.** We define a shifted excursion to be a continuous function $g : [0,\infty) \to [0,\infty)$ such that there exists $r \ge 0$ such that $g(t) = 0$ for $t \in [0,r]$ and the function $\tilde{g} : [0,\infty) \to [0,\infty)$ is an excursion where we have $\tilde{g}(t) := g(r+t)$ for $t \ge 0$. We define $\sigma(g) := r + \sigma(\tilde{g})$.

We define a tree from a shifted excursion $g : [0,\infty) \to [0,\infty)$ as the following:

1. Each point $t \in [0,\infty)$ corresponds to a vertex of the tree at height $g(t)$;

2. for $t$, $t' \in [0,\infty)$, $t < t'$, the vertex $t$ is an ancestor of the vertex $t'$ if and only if $g(t) = \inf(\{g(r) : r \in [t,t']\})$. More generally, the quantity $\inf(\{g(r) : r \in [t,t']\})$ is the height of the most recent common ancestor to $t$ and $t'$ and we call any $r$ for which this infimum is attained the most recent common ancestor of $t$ and $t'$;

3. the distance between the vertices $t$ and $t'$ is defined to be $\tilde{d}(t,t') = g(t) + g(t') - 2\inf(\{g(r) : r \in [t,t']\})$ and we identify $t$ and $t'$ (we write $t \sim t'$) if $\tilde{d}(t,t') = 0$.

The tree coded by $g$ is the quotient set $[0,\infty)/\sim$ with the distance $\tilde{d}$ and the genealogical relation defined in (2). We call the equivalence class of level 0 the root and we note that the root is an ancestor of all the vertices.

Finally we note that $[0,\infty)/\sim$ is equivalent to $[0,\sigma(g)]/\sim$. We use this second formulation throughout as it highlights the shifted excursion $g$ from which the tree is constructed.

We note that $f^{(h)}$ defined above is a shifted excursion and we consider the tree $[0,\sigma(f^{(h)})]/\sim$ constructed from it as in Definition 3.3.3. Such trees are known as the *h-erased trees of* $f$, see Neveu [79]. The tree $[0,\sigma(f^{(h)})]/\sim$ consists of leaves at distance $(Z_{2p'+1}^{(h)})_{0\le p'\le \tilde{M}^{(h)}}$ from the root and branch points at distance $(Z_{2p'}^{(h)})_{1\le p'\le \tilde{M}^{(h)}}$ from the root. We can split the rooted tree into its edges $(e_p^{(h)})_{1\le p\le M^{(h)}}$ which have as endpoints the vertices $(\gamma_p^{(h)})_{0\le p\le M^{(h)}}$ of the tree where for each $p$ the upper end of the edge $e_p^{(h)}$ has height $Z_p^{(h)}$. Furthermore, the edge $e_p^{(h)}$ is isometric to the interval $[Y_p^{(h)}, Z_p^{(h)}]$ where $Y_p^{(h)}$ is the height of the most recent common ancestor of $\gamma_p^{(h)}$ among the $\gamma_{2p'}^{(h)}$. See Figure 3.1 for an illustration.

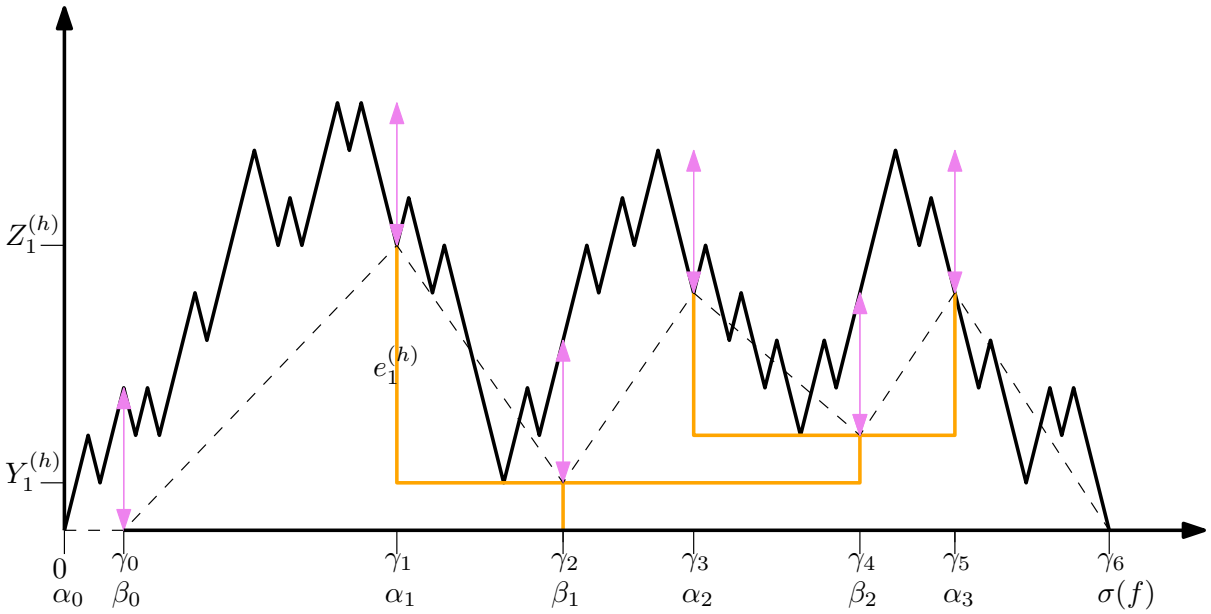



Figure 3.1: In black bold we have the excursion $f$ and in black dashed we have the contour function $f^{(h)}$. In orange we have the tree $[0,\sigma(f^{(h)})]/\sim$. We have labelled one edge $e_1^{(h)}$ of $[0,\sigma(f^{(h)})]/\sim$ and the corresponding values $Y_1^{(h)}$ and $Z_1^{(h)}$. The purple arrowed lines indicate the differences of $h$ that determine the times $(\gamma_p^{(h)})_{0\le p\le M^{(h)}}$. In this case $M^{(h)}=5$, the number of vertical edges of the tree.

*Remark* 3.3.4. The tree in Definition 3.3.3 could also be encoded as a genealogical tree with edge lengths and Ulam-Harris indexing of vertices, see e.g. [65], but we code it as above to keep indexing notation consistent with the bivariate snake.

We define a distribution for an alternating random walk that we refer to below. Fix $h>0$ as above and let $(E_j^{(h)})_{j\in\mathbb{N}}$ be a family of independent Exponential random variables where $E_j^{(h)}\sim\mathrm{Exp}(1/h)$. Let $C_k^{(h)}:=\sum_{j=1}^{k}(-1)^{j+1}E_j^{(h)}$ and let $K^{(h)}:=\inf\{n\ge 1: C_k^{(h)}<0\}$. Finally, let $(\tilde{C}_k^{(h)})_{1\le k\le K^{(h)}}$ be defined by $\tilde{C}_k^{(h)}:=C_k^{(h)}$ for $1\le k<K^{(h)}$, $\tilde{C}_k^{(h)}:=0$ for $k=K^{(h)}$. We denote the distribution of $(\tilde{C}_k^{(h)})_{1\le k\le K^{(h)}}$ by $Q^{(h)}$.

Let us now assume that $f$ is an excursion under the probability distribution $n(\cdot|T_1<\infty)$. Under this assumption Le Gall shows in the proof of [65, Theorem 6] that the process $(Z_p^{(1)})_{1\le p\le M^{(1)}+1}$ has distribution $Q^{(1)}$ as defined above. Le Gall also proves in [65, Theorem 3] that the rooted tree given by $[0,\sigma(f^{(1)})]/\sim$ is the genealogical tree of a critical binary branching process with iid $\mathrm{Exp}(2)$ lifetimes.

*Remark* 3.3.5. Everything stated above generalises to any $h>0$ in the sense that $[0,\sigma(f^{(h)})]/\sim$ is the genealogical tree of a critical binary branching process with $\mathrm{Exp}(2/h)$ lifetimes under $n(\cdot|T_h<\infty)$.

Let $h>0$ and now assume that $f\in\mathscr{E}$ is a random excursion with probability distribution $n(\cdot|T_h<\infty)$. We make some observations on the structure of the tree. The root of the tree

is at $\gamma_0^{(h)}$,the first edge ends at $\gamma_{q^*}^{(h)}$ for the $1 \le q^* \le M^{(h)}$ for which $Z_{q^*}^{(h)}$ is minimal, $q^* := \text{argmin}(\{Z_p^{(h)} : 1 \le p \le M^{(h)}\})$. With probability $1/2$ we have $\gamma_{q^*}^{(h)} = \alpha_1^{(h)}$ in which case the tree consists of a single edge or $\gamma_{q^*}^{(h)} = \beta_r^{(h)}$ for some $r$. In the second case we have the that the ancestor has two subtrees. It is shown in [65, Lemma 4] that $Z_{q^*}^{(h)} \sim \text{Exp}(2/h)$ and that, conditional on $\gamma_{q^*}^{(h)}$ having descendants, we have that the processes

$$(Z_p^{(h)} - Z_{q^*}^{(h)})_{1 \le p \le q^*}; \quad ((Z_{q^*+p}^{(h)} - Z_{q^*}^{(h)})^+)_{1 \le p \le M^{(h)}+1-q^*}$$

are independent both with law $Q^{(h)}$. In the second case here we take the positive part denoted by $(\cdot)^+$ for notational convenience so that for the final point $(Z_{M^{(h)}+1}^{(h)} - Z_{q^*}^{(h)})^+ = 0$ as we require. The process then repeats recursively for both the subtrees.

We also note from the proof of [65, Theorem 6] for all $1 \le p \le M^{(h)}$ that $Z_p^{(h)} \le f_{\gamma_p^{(h)}}$. This is because for $Z_{2p'+1}^{(h)} = f_{\gamma_{2p'+1}^{(h)}}$ and $Z_{2p'}^{(h)} = f_{\gamma_{2p'}^{(h)}} - h$ for $0 \le p' \le \tilde{M}^{(h)}$.

Furthermore, for $0 \le p' \le \tilde{M}^{(h)} - 1$, Le Gall states that $Z_{2p'}^{(h)} = \inf(\{f_t : \alpha_{p'}^{(h)} \le t \le \beta_{p'}^{(h)}\})$ and we can further note that $Z_{2p'}^{(h)} \le \inf(\{f_t : \beta_{p'}^{(h)} \le t \le \alpha_{p'+1}^{(h)}\})$ because

$$f_{\beta_{p'}^{(h)}} \le \sup\{f_{t'} : \beta_{p'}^{(h)} \le t' \le t\} \quad \text{for all } t > \beta_{p'}^{(h)} \quad \text{and so} \quad \alpha_{p'+1}^{(h)} \le \inf\{t' > \beta_{p'}^{(h)} : f_{t'} = f_{\beta_{p'}^{(h)}} - h\}$$

by definition of the points $(\alpha_{p'}^{(h)}, 1 \le p' \le \tilde{M}^{(h)} + 1)$. Therefore

$$Z_{2p'}^{(h)} = \inf(\{f_t : \alpha_{p'}^{(h)} \le t \le \alpha_{p'+1}^{(h)}\}) \quad \text{for all } 0 \le p' \le \tilde{M}^{(h)} - 1 \tag{3.3}$$

which also verifies for us that the tree does indeed lie beneath the excursion, as illustrated in Figure 3.1.

During the lifetime of a particle the splitting and dying rates do not affect the spatial motion. We can use the Lemma 3.2.3 in the next proposition to verify that each particle individually has a spatial motion that follows the correct distribution. In this proposition we have a snake $X$ and we condition on its contour process to $f$. We consider, for $h > 0$, the process $f^{(h)}$ and the tree $[0, \sigma(f^{(h)})]/\sim$ and we use the edges of this tree represented as the intervals $[Y_p^{(h)}, Z_p^{(h)})$ along with the time changes given by the function $H^{(s)}$ to define a particle system. We use the $\mathbb{R}^d$ component of the snake to give spatial locations to these particles in (3.4). All this together gives ooBBM.

**Proposition 3.3.6.** *Let $x = (b, 0) \in E = \mathbb{R}^d \times [0, \infty)$, $h > 0$ and let $X = (W, V)$ be a snake with distribution $\mathbb{N}_x(\cdot | T_h < \infty)$. Then the process $(\tilde{\mathscr{Z}}_s^{(h,\mathrm{d})}, \tilde{\mathscr{Z}}_s^{(h,\mathrm{a})})_{s \ge 0}$ defined by*

$$(\tilde{\mathscr{Z}}_s^{(h,\mathrm{d})}, \tilde{\mathscr{Z}}_s^{(h,\mathrm{a})}) := \Big( \sum_{\substack{p: A^X_{\gamma_p^{(h)}}(s) = 0, \\ H^X_{\gamma_p^{(h)}}(s) \in [Y_p^{(h)}, Z_p^{(h)})}} \delta_{W_{\gamma_p^{(h)}}(H^X_{\gamma_p^{(h)}}(s))}, \sum_{\substack{p: A^X_{\gamma_p^{(h)}}(s) = 1 \\ H^X_{\gamma_p^{(h)}}(s) \in [Y_p^{(h)}, Z_p^{(h)})}} \delta_{W_{\gamma_p^{(h)}}(H^X_{\gamma_p^{(h)}}(s))} \Big) \tag{3.4}$$

*is an ooBBM$(2/h, c, \tilde{c})$ and with initial state $(0, \delta_b)$, i.e. starting from a single active particle at spatial location b.*

*Proof.* We drop the superscripts $X$ for this proof as we only deal with a single snake $X$ throughout. We consider the $h$-erased tree of its contour process. On the event where this tree consists of a single edge $e_1^{(h)}$ the argument is simple.

As noted in Remark 3.3.5, this edge has an Exp$(h/2)$-distributed length. As $\mathbb{N}_x(\cdot) = \int_f \mathbb{P}_{x|f}(\cdot) n(df)$ and hence $\mathbb{N}_x(\cdot | T_h < \infty) = \int_f \mathbb{P}_{x|f}(\cdot) n(df | T_h < \infty)$, we obtain from the third part of Theorem 3.2.1 that $X_{\gamma_1^{(h)}}$ is distributed as $(\xi(y), 0 \le y < E^{(2/h)})$, where $\xi \sim P^{\xi}_{(b,0)}$ and $E^{(2/h)} \sim \text{Exp}(2/h)$ are independent. Hence $(\mathscr{Z}_s^{(h,\mathrm{d})}, \mathscr{Z}_s^{(h,\mathrm{a})})_{s \ge 0}$ consists of a single particle that starts at position $b$ in the active state and we have the spatial motion and dormancy of the particle by Lemma 3.2.3 have the required dynamics of a single particle in an ooBBM.

Note that the points $\gamma_p^{(h)}$ for $1 \le p \le M^{(h)}$ are non-random under $\mathbb{P}_{x|f}$. Furthermore the left limits $X_{\gamma_p^{(h)}}(Z_p^{(h)} -)$ exist for $1 \le p \le M^{(h)}$ (this is immediately apparent for $p$ even as $Z_p^{(h)} < f_{\gamma_p^{(h)}}$ and also follows for $p$ odd where $Z_p^{(h)} = f_{\gamma_p^{(h)}}$). Therefore we can condition on the contour process of $X$ and use the snake property at the points $(\gamma_p^{(h)})_{1 \le p \le M^{(h)}}$ up to heights $(Z_p^{(h)})_{1 \le p \le M^{(h)}}$. Condition on $X$ to have contour process $f$. For the given genealogical tree $[0, \sigma(f^{(h)})]/\sim$ we show the particle system has the correct spatial motion and dormancy dynamics.

We show the joint distribution for each pair of particles. We do this below and we use the snake property of $X$ and the most recent common ancestor of the pair in the genealogical tree. The argument generalises for the joint distribution of all particles by repeating this argument by looking at all the most recent common ancestors for all subsets of particles and repeating the steps below. The spatial motion and dormancy dynamics for each individual particle follows from Lemma 3.2.3.

Let $k$, $j \in \mathbb{N}$ such that $1 \le k < j \le M^{(h)}$. Consider the particles denoted by the vertices $\gamma_k^{(h)}$ and $\gamma_j^{(h)}$ in $[0, \sigma(f^{(h)})]/\sim$. The spatial motion and dormancy dynamics of the particle indexed by $\gamma_k^{(h)}$ is given by

$$(W_{\gamma_k^{(h)}}(H_{\gamma_k^{(h)}}(s)), A_{\gamma_k^{(h)}}(s))_{s'_k \le s < s''_k}$$

where

$$s'_k := \sup\left(\left\{s : H_{\gamma_k^{(h)}}(s) = Y^{(h)}_{\gamma_k^{(h)}}\right\}\right); \quad s''_k := \inf\left(\left\{s : Z_{\gamma_k^{(h)}} = H_{\gamma_k^{(h)}}(s) = Z^{(h)}_{\gamma_k^{(h)}}\right\}\right).$$

More generally for $1 \le p \le M^{(h)}$ define $s'_p$ in the same we defined $s'_k$. We note that the particle $\gamma_k^{(h)}$ has the correct dynamics over its lifetime by Lemma 3.2.3. We get a same result for the dynamics of the particle at $\gamma_j$. It remains to verify that the initial states and the joint law of the spatial motions of the particles follow the correct distribution.

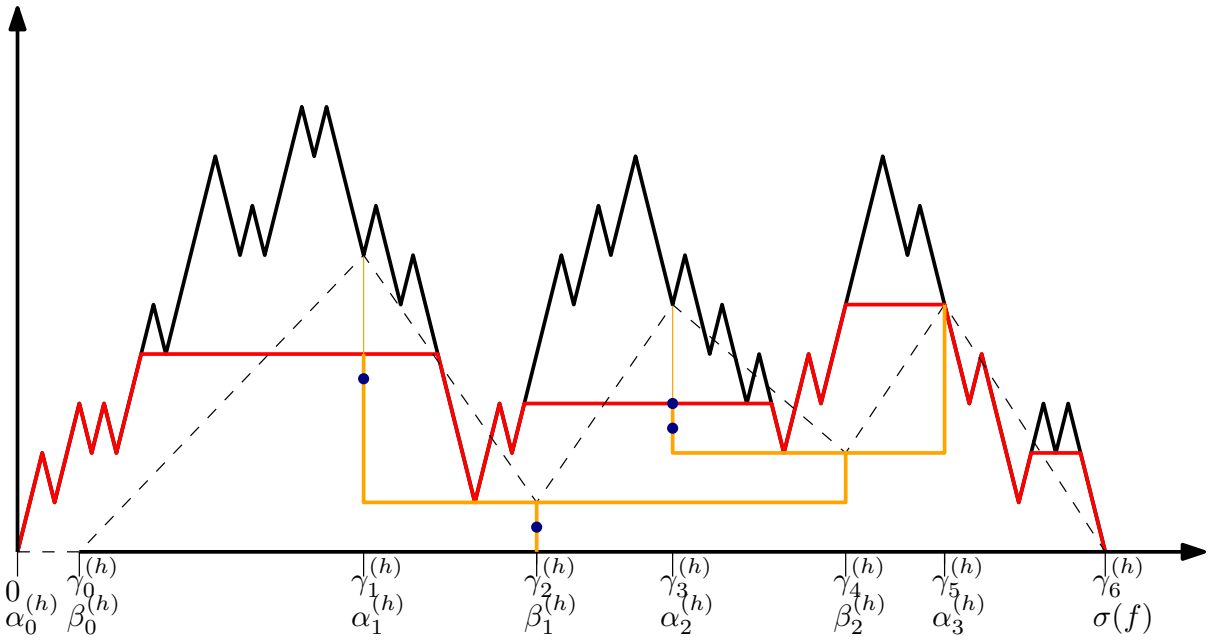



Figure 3.2: In black bold we have the excursion $f$ for the same excursion as in Figure 3.1. In black dashed we have the contour process $f^{(h)}$. In red we have the age process $(H_t(s))_{0\le s\le\sigma(f)}$ and in orange we have the tree $[0,\sigma(f^{(h)})]/\sim$ which is bold up to where it is stopped. The dark blue dots represent jumps for paths of the subordinator component of the snake that determines the age process.

We first note that $\inf(\{f_t : t \in [\gamma_k^{(h)}, \gamma_j^{(h)}]\}) = Z_{\gamma_{2l}}^{(h)}$ for some $k \le 2l \le j$, $\gamma_k^{(h)} \le \gamma_{2l}^{(h)} \le \gamma_j^{(h)}$ (by (3.3), note it is possible that $\gamma_{2l}^{(h)} = \gamma_k^{(h)}$ or $\gamma_{2l}^{(h)} = \gamma_j^{(h)}$). We consider the snake at $\gamma_k^{(h)}$ and $\gamma_j^{(h)}$. By the snake property we have that

$$(X_{\gamma_k^{(h)}}(y))_{0\le y<Z_{2l}^{(h)}} = (X_{\gamma_j^{(h)}}(y))_{0\le y<Z_{2l}^{(h)}}$$

and therefore

$$(W_{\gamma_k^{(h)}}(H_{\gamma_k^{(h)}}(s)), A_{\gamma_k^{(h)}}(s))_{0\le s<s'_{2l}} = (W_{\gamma_k^{(h)}}(H_{\gamma_j^{(h)}}(s)), A_{\gamma_j^{(h)}}(s))_{0\le s<s'_{2l}}$$

Furthermore we have $(X_{\gamma_k^{(h)}}(y))_{Z_{2l}^{(h)}<y<Z_k^{(h)}}$ is conditionally independent of $(X_{\gamma_j^{(h)}}(y))_{Z_{2l}^{(h)}<y<Z_j^{(h)}}$ given $X_{\gamma_{2l}^{(h)}}$ and that therefore $(W_{\gamma_k^{(h)}}(H_{\gamma_k^{(h)}}(s)), A_{\gamma_k^{(h)}}(s))_{s'_{2l}<s\le s'_k}$ is conditionally independent of $(W_{\gamma_j^{(h)}}(H_{\gamma_j^{(h)}}(s)), A_{\gamma_j^{(h)}}(s))_{s'_{2l}<s\le s'_j}$.

Finally we note that the contour of $X$ has distribution $n(\cdot|T_h<\infty)$ and is such that $(Z_p^{(h)}, 1 \le p \le M^{(h)}+1)$ has distribution $Q^{(h)}$ and under this distribution we have particles of the tree split and die during their active states with rate $1/h$ each which are the correct rates for ooBBM. ■

We now consider the Poisson random measure that we call the on/off Brownian snake $\sum_{i\in I}\delta_{((b_i,s_i),f^i,x^i)}$ which we defined in Definition 3.2.7. For $i \in I$, and for $s, t \ge 0$, let $H_t^i(s) := H_t^{x^i}(s)$ and $A_t^i(s) := A_t^{x^i}(s)$. Also let $Y_{p,i}^{(h)}$, $Z_{p,i}^{(h)}$, $\gamma_{p,i}^{(h)}$ be analogous to $Y_p^{(h)}$, $Z_p^{(h)}$, $\gamma_p^{(h)}$ for the

excursion $f^i$. Let $x^i := (w^i, v^i)$ where $w^i$ denotes the Brownian spatial component of the snake and $v^i$ denotes the subordinator component of the snake. Note also for $i \in I$, $s < s_i$ and $0 \le t \le \sigma(f^i)$ we have $A^i_t(s) = 0$. We can now give our main result of the section.

**Proposition 3.3.7.** *Recall the on/off Brownian snake from Definition 3.2.7 with index set I. We define the following measure-valued processes for $i \in I$ and for $s \ge 0$:*

$$(\mathscr{Z}^{(h,\mathrm{d})}_{s,i}, \mathscr{Z}^{(h,\mathrm{a})}_{s,i}) := \Big( \sum_{\substack{p: A^i_{\gamma^{(h)}_{p,i}}(s)=0, \\ H^i_{\gamma^{(h)}_{p,i}}(s) \in [Y^{(h)}_{p,i}, Z^{(h)}_{p,i})}} \delta_{w^i_{\gamma^{(h)}_{p,i}}(H^i_{\gamma^{(h)}_{p,i}}(s))}, \sum_{\substack{p: A^i_{\gamma^{(h)}_{p,i}}(s)=1, \\ H^i_{\gamma^{(h)}_{p}}(s) \in [Y^{(h)}_{p,i}, Z^{(h)}_{p,i})}} \delta_{w^i_{\gamma^{(h)}_{p,i}}(H^i_{\gamma^{(h)}_{p,i}}(s))} \Big).$$

*Then we have that $((\mathscr{Z}^{(h,\mathrm{d})}_s, \mathscr{Z}^{(h,\mathrm{a})}_s))_{s\ge 0}$ where $(\mathscr{Z}^{(h,\mathrm{d})}_s, \mathscr{Z}^{(h,\mathrm{a})}_s) := (\sum_{i\in I} \mathscr{Z}^{(h,\mathrm{d})}_{s,i}, \sum_{i\in I} \mathscr{Z}^{(h,\mathrm{a})}_{s,i})$ for $s \ge 0$ is an ooBBM$(2/h, c, \tilde{c})$ starting from a Poisson random measure with intensity $(\mu_\mathrm{d}/2h, \mu_\mathrm{a}/2h)$.*

*Furthermore the process $(\mathscr{Z}^{(h)}_y)_{y\ge 0}$, where for $y \ge 0$ we define*

$$\mathscr{Z}^{(h)}_y := \sum_{i\in I} \sum_{p: y \in [Y^{(h)}_{p,i}, Z^{(h)}_{p,i})} \delta_{w^i_{\gamma^{(h)}_{p,i}}(y)},$$

*is a BBM$(2/h)$ starting from a Poisson random measure with intensity $(\mu_\mathrm{d} + \mu_\mathrm{a})/2h$.*

*Proof.* We first show that $\mathscr{Z}^{(h)}_0$ and $(\mathscr{Z}^{(h,\mathrm{d})}_0, \mathscr{Z}^{(h,\mathrm{a})}_0)$ have the correct distributions.
Let $\mu' := \mu_a + \mu_d$ and let $\tilde{\mu}' := \mu'/\|\mu'\|$. The random measure $\mathscr{Z}^{(h)}_0$ is then a Poisson random measure with intensity

$$n(T_h < \infty)\mu'(\cdot) = \mu'(\cdot)/2h$$

and $(\mathscr{Z}^{(h,\mathrm{a})}_0, \mathscr{Z}^{(h,\mathrm{d})}_0)$ consists of particles under a Poisson random measure with intensity $(\mu_d/2h, \mu_a/2h)$ according to Definition 3.2.7. For the ooBBM case we have by Proposition 3.3.6 for each initial active particle the subsequent behaviour then has the correct distribution and the particles with a dormant initial state switch to an active particle at the correct rate since the $s_i$ are sampled from Exp$(\tilde{c})$. Finally, note that particles evolve independently under the Poisson random measure. The case for BBM is simpler. ■

## 3.4 Local time measure

In this section we use the notion of downcrossings for local times, see e.g. Mörters and Peres [77]. For $R$ a reflecting Brownian motion and for $0 \le a < b$ we can define $\tau_0 := 0$, and for $n \ge 1$

$$\sigma_n := \inf(\{t > \tau_{n-1} : R(t) = b\}), \qquad \tau_n := \inf(\{t > \sigma_n : R(t) = a\}),$$

and $R^{(j)} : [0, \tau_n - \sigma_n] \to [0, \infty)$ given by $R^{(j)}(t') := R(\sigma_n + t')$ is called the $j^{\text{th}}$ downcrossing of $[a,b]$. We also refer to the points $(\tau_n, j \in \mathbb{N})$ as the downcrossings of $[a,b]$. We also refer

to these as $(b-a)$-downcrossings to level $a$. We also define $\tilde{R}^{(j)} : [0, \sigma_n - \tau_{n-1}] \to [0,\infty)$ by $\tilde{R}^{(j)}(t') := R(\tau_{n-1} + t')$ as the $j^{\text{th}}$ upcrossing of $[a,b]$. In this section we are interested in the downcrossings of $[0,h]$, for $h > 0$, of the residual process $(\text{Res}_t^X(s))_{t\geq 0}$ for $s \geq 0$, of snakes $X$.

For the rest of the section we work with the Poisson random measure that we called the on/off Brownian snake in Definition 3.2.7. We recall from (3.2) that this is the Poisson random measure

$$\sum_{i\in I} \delta_{((b_i,s_i),f^i,x^i)}.$$

For $i \in I$ we write $x^i = (w^i, v^i)$ for the Brownian component $w^i$ and the subordinator component $v^i$ of the snake $x^i$. We also write $\hat{w}^i$ for $w^i(\zeta(x^i)-)$ (recall Remark 3.2.2) and we also $\hat{v}^i$ for $v^i(\zeta(x^i)-)$ when this exists. Also we set $y_i := (b_i, s_i)$. We define this Poisson random measure on a probability space $(\Omega, \mathscr{F}, \mathbb{P}_{(\mu_\text{d},\mu_\text{a})})$.

For $i \in I$ we define $\text{Res}_t^i(s) := \text{Res}_t^{x^i}(s) = f_t^i - H_t^i(s)$. For $h > 0$, consider the finitely many stopping times $(S^{h,\text{a}}_{s,n,i})_{1\leq n\leq M_i^{h,\text{a}}}$, $(S^{h,\text{d}}_{s,n,i})_{1\leq n\leq M_i^{h,\text{d}}}$, $(T^{h,\text{a}}_{s,n,i})_{0\leq n\leq M_i^{h,\text{a}}}$ and $(T^{h,\text{d}}_{s,n,i})_{0\leq n\leq M_i^{h,\text{d}}}$ for the snake given by $T^{h,\text{a}}_{s,0,i} = T^{h,\text{d}}_{s,0,i} = 0$ and for finitely many $n$ we have

$$S^{h,\text{a}}_{s,n,i} := \inf\{t > T^{h,\text{a}}_{s,n-1,i} | \text{Res}_t^i(s) > h, A_t^i(s) = 1\}; \quad T^{h,\text{a}}_{s,n,i} := \inf\{t > S^{h,\text{a}}_{s,n,i} | \text{Res}_t^i(s) = 0\},$$

$$S^{h,\text{d}}_{s,n,i} := \inf\{t > T^{h,\text{d}}_{s,n-1,i} | \text{Res}_t^i(s) > h, A_t^i(s) = 0\}; \quad T^{h,\text{d}}_{s,n,i} := \inf\{t > S^{h,\text{d}}_{s,n,i} | \text{Res}_t^i(s) = 0\},$$

where $M_i^{h,\text{d}} \in \mathbb{N}$ (respectively $M_i^{h,\text{a}}$) is the largest number $n$ such that $S^{h,\text{d}}_{s,n,i}$ (respectively $S^{h,\text{a}}_{s,n,i}$) is finite. We have the following identification of the of the ooBBM defined in the previous section. We show these stopping times in Figure 3.3.

**Lemma 3.4.1.** *Recall the Poisson random measure from Definition 3.2.7 and its index set $I$. Let $i \in I$ and let $h > 0$. Recall $(\mathscr{Z}^{(h,\text{d})}_{s,i}, \mathscr{Z}^{(h,\text{a})}_{s,i})$ from Proposition 3.3.7. Then we have*

$$(\mathscr{Z}^{(h,\text{d})}_{s,i}, \mathscr{Z}^{(h,\text{a})}_{s,i}) = \Big( \sum_{n=1}^{M_i^{h,\text{d}}} \delta_{w^i_{S^{h,\text{d}}_{s,n,i}}(H^i_{S^{h,\text{d}}_{s,n,i}}(s))}, \sum_{n=1}^{M_i^{h,\text{a}}} \delta_{w^i_{S^{h,\text{a}}_{s,n,i}}(H^i_{S^{h,\text{a}}_{s,n,i}}(s))} \Big).$$

*Proof.* First we consider the case $T_h(f^i) = \infty$. In this case both sides are equal to the zero measure and the claim of the lemma is trivially satisfied. For the remainder of the proof we work consider the case of $i \in I$ such that $T_h(f^i) < \infty$.

We first show that each up-/downcrossing pair $S^{h,\text{d}}_{s,n,i}$, $T^{h,\text{d}}_{s,n,i}$ (respectively $S^{h,\text{a}}_{s,n,i}$, $T^{h,\text{a}}_{s,n,i}$) corresponds to a point $\gamma^{(h)}_{p,i}$ with $H^i_{\gamma^{(h)}_{p,i}}(s) \in [Y^{(h)}_{p,i}, Z^{(h)}_{p,i})$ and $\gamma^{(h)}_{p,i}$ dormant (respectively active) at $s$ $\mathbb{P}_{(\mu_\text{d},\mu_\text{a})}$-a.s. Similarly for each such $\gamma^{(h)}_{p,i}$ there is a unique pair $S^{h,\text{d}}_{s,n,i}$, $T^{h,\text{d}}_{s,n,i}$ (respectively $S^{h,\text{a}}_{s,n,i}$, $T^{h,\text{a}}_{s,n,i}$) $\mathbb{P}_{(\mu_\text{d},\mu_\text{a})}$-a.s. We then show that they give the same Dirac masses in the sum and the lemma follows. We add Figure 3.3 for clarity which shows the correspondence.

First note that $\alpha^{(h)}_{p',i}$ is a $h$-downcrossing of $f^i$ to level $Z^{(h)}_{2p'-1,i}$ and furthermore note that on $[\alpha^{(h)}_{p',i}, \beta^{(h)}_{p',i})$ there are no $h$-upcrossings to any level by definition of the time $\beta^{(h)}_{p',i}$ being the

first time there is an increase of $h$ from the running infimum after time $\alpha^{(h)}_{p',i}$. Also we can see that $Z^{(h)}_{2p',i} = f^i_{\beta^{(h)}_{p',i}} - h = \inf(\{f^i_t : \alpha^{(h)}_{p',i} \le t \le \beta^{(h)}_{p',i}\})$ as there is exactly one $h$-downcrossing time to level $y$ in $[\alpha^{(h)}_{p',i}, \beta^{(h)}_{p',i}]$ for all $p'$ with $y \in [Z^{(h)}_{2p',i}, Z^{(h)}_{2p'-1,i}]$. Therefore we have a one-to-one correspondence except at the unique $s$ for which $H^i_{\gamma^{(h)}_{2p'-1,i}}(s) = Z^{(h)}_{2p'-1,i}$ because in this instance we count a downcrossing at $t$ even though the particle is deemed to have died at that point (this occurs for example for the pair $S^{h,\mathrm{a}}_{s,2,i}$, $T^{h,\mathrm{a}}_{s,2,i}$ in Figure 3.3). But as the $Z^{(h)}_{2p'-1,i}$ levels are continuously distributed this does not occur for fixed $s$ $\mathbb{P}_{(\mu_\mathrm{d},\mu_\mathrm{a})}$-a.s. this does not affect our argument.

Furthermore we have that $H^i_t(s)$ is constant for $t \in [S^{h,\mathrm{a}}_{s,n,i}, T^{h,\mathrm{a}}_{s,n,i})$ by Lemma 3.2.5 and that therefore $\inf(\{f^i_t : t \in [S^{h,\mathrm{a}}_{s,n,i}, T^{h,\mathrm{a}}_{s,n,i}]\}) = f^i_{T^{h,\mathrm{a}}_{s,n,i}}$. There exists a unique $\gamma^{(h)}_{p,i}$ for which $H^i_{\gamma^{(h)}_{p,i}}(s) \in [Y^{(h)}_{p,i}, Z^{(h)}_{p,i})$ (note $\gamma^{(h)}_{p,i}$ is not necessarily $\alpha^{(h)}_{p',i}$). Finally, we have by the snake property at points $S^{h,\mathrm{a}}_{s,n,i} - T_h(f^i)$, $\gamma^{(h)}_{p,i} - T_h(f^i)$, $T^{h,\mathrm{a}}_{s,n,i} - T_h(f^i)$ for the strong Markov process $(x^i_{T_h(f^i)+t}, t \ge 0)$, see Proposition 3.3.1, the equality of $w^i_{\gamma^{(h)}_{p,i}}(H^i_{\gamma^{(h)}_{p,i}}(s))$ and $w^i_{S^{h,\mathrm{a}}_{s,n,i}}(H^i_{S^{h,\mathrm{a}}_{s,n,i}}(s))$ and $A^i_{\gamma^{(h)}_{p,i}}(H^i_{\gamma^{(h)}_{p,i}}(s)) = 1$ and also as required.

The same argument follows for the dormant case. ■

In Proposition 3.4.3 we give results about the residual process. This is an instance of a more general result from [11] that we state here.

**Lemma 3.4.2** (Residual lifetime process, Section 4 of [11])**.** *Let $x \in E$ and let $w \in \mathcal{W}_x$. Consider a snake $X$ under $\mathbb{P}_w$ (respectively $\mathbb{N}_x$). Let $D$ be an open subset of $E$, let $x \in D$ and let $F = E \backslash D$. For $v \in \mathcal{W}_x$, set $\tau(v) := \inf(\{t \ge 0, v(t) \notin D\})$. Assume that $P^\xi_x(\tau < \infty) > 0$. Then define*

$$U_t := (\zeta(X_t) - \tau(X_t))^+ \qquad \eta_r := \inf\left(\left\{t, \int_0^t \mathbb{1}_{\{\tau(X_u) < \zeta(X_u)\}} du > r\right\}\right),$$

*and then the process $(U_{\eta_r}, r \ge 0)$ is a reflecting Brownian motion $\mathbb{P}_w$-a.s., (respectively is a stopped reflecting Brownian motion $\mathbb{N}_x$-a.e.). Furthermore, let $\ell(\cdot)$ denote the local time for $(U_{\eta_r}, r \ge 0)$ with the standard normalisation in the sense that the local time is the limit of the number of $h$-downcrossings multiplied by $h$ as $h$ tends to $0$. Then under $\mathbb{N}_x$ we have that*

$$\mathbb{N}_x(\ell(\infty)) = P^\xi_x(\tau(\xi) < \infty). \tag{3.5}$$

*Proof.* This result is given in [11] at the start of Section 4 prior to the introduction of exit measures. In particular, (3.5) is an instance of [11, Equation (35)]. ■

**Proposition 3.4.3.** *Fix $s > 0$ and let $x = (b, s') \in E$ with $s' < s$. We define the time changes*

$$\eta(s,u,i) := \inf\left\{t' \ge 0 : \int_{r=0}^{t'} \mathbb{1}_{\{Res_r(s) > 0\}} dr \ge u\right\};$$

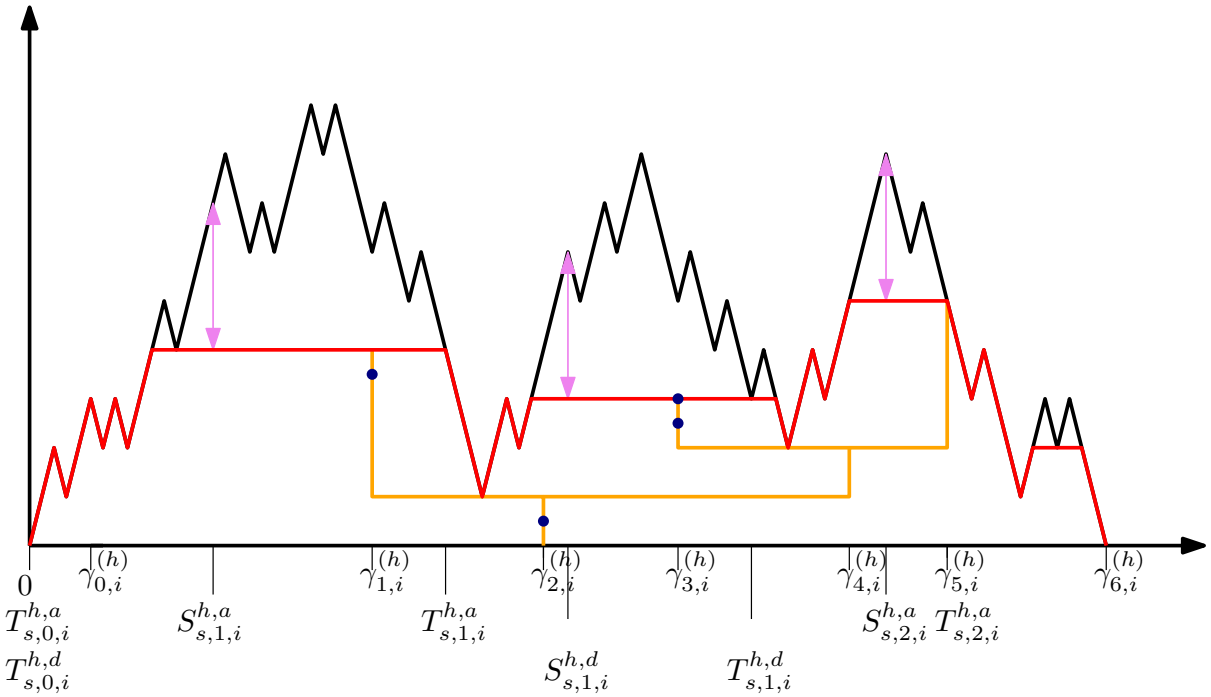



Figure 3.3: For $i \in I$ we show the downcrossing stopping times $(S^{h,\mathrm{d}}_{s,n,i}, T^{h,\mathrm{d}}_{s,n,i})_{1\le n\le M_i^{h,\mathrm{d}}}$, $(S^{h,\mathrm{a}}_{s,n,i}, T^{h,\mathrm{a}}_{s,n,i})_{1\le n\le M_i^{h,\mathrm{a}}}$. In black bold we have $f^i$ which is taken to be the same excursion $f$ as in Figures 3.1 and 3.2 for convenience. The tree given in orange is the tree for the $h$-erased contour process of $f^i$ and this tree is stopped at the age process $(H^i_t(s))_{0\le s\le\sigma(f)}$ given in red. The dark blue dots represent jumps for paths of the subordinator components $v^i$ of the snake. The purple arrowed lines here indicate the differences of $h$ that determine the downcrossing stopping times.

$$\eta(s,u,i,\mathrm{a}) := \inf\Big\{t' \geq 0 : \int_{r=0}^{t'} \mathbb{1}_{\{Res_r(s)>0, A_r^i(s)=1\}} dr \geq u\Big\};$$

$$\eta(s,u,i,\mathrm{d}) := \inf\Big\{t' \geq 0 : \int_{r=0}^{t'} \mathbb{1}_{\{Res_r(s)>0, A_r^i(s)=0\}} dr \geq u\Big\}.$$

*Then the maps $u \mapsto Res_{\eta(s,u,i)}(s)$, $u \mapsto Res_{\eta(s,u,i,\mathrm{a})}(s)$ and $u \mapsto Res_{\eta(s,u,i,\mathrm{d})}(s)$ are stopped reflecting Brownian motions $\mathbb{N}_x$-a.e. Furthermore we have that the local times of $(Res_{\eta(s,u,i)}(s))_{u\geq 0}$, $(Res_{\eta(s,u,i,\mathrm{a})}(s))_{u\geq 0}$ and $(Res_{\eta(s,u,i,\mathrm{d})}(s))_{u\geq 0}$ at level 0, denoted by $\tilde{\ell}_{s,i}$, $\tilde{\ell}^{\mathrm{a}}_{s,i}$, $\tilde{\ell}^{\mathrm{d}}_{s,i}$ and normalised as in the previous lemma, are such that*

$$\mathbb{N}_x\Big(\tilde{\ell}_{s,i}(\infty)\Big) = \mathbb{N}_x\Big(\tilde{\ell}^{\mathrm{a}}_{s,i}(\infty)\Big) + \mathbb{N}_x\Big(\tilde{\ell}^{\mathrm{d}}_{s,i}(\infty)\Big) = 1.$$

*Proof.* Consider the open set $D = \mathbb{R}^d \times [0,s)$ in the setting of Lemma 3.4.2. In this setting the process $(U_t, t \geq 0)$ is exactly the process given by $(\mathrm{Res}_{\eta(s,u,i)}(s))_{u\geq 0}$.

Now instead consider the set $D = \mathbb{R}^d \times [0,\infty)\backslash\{s\}$ in the setting of the previous lemma. In this instance the process $(U_t, t \geq 0)$ is exactly the process given by $(\mathrm{Res}_{\eta(s,u,i,\mathrm{a})}(s))_{u\geq 0}$.

The final case follows as

$$\mathrm{Res}_t(s) = \mathrm{Res}_t(s)\mathbb{1}_{A_t(s)=0} + \mathrm{Res}_t(s)\mathbb{1}_{A_t(s)=1}$$

and so the process $(\mathrm{Res}_{\eta(s,u,i,\mathrm{d})}(s))_{u\geq 0}$ is a reflecting Brownian motion as it is defined by first taking the process $(\mathrm{Res}_{\eta(s,u,i)}(s))_{u\geq 0}$ and removing excursions that correspond to $(\mathrm{Res}_{\eta(s,u,i,\mathrm{a})}(s))_{u\geq 0}$. For the final part we use (3.5) and note that

$$\mathbb{N}_x\Big(\tilde{\ell}_{s,i}(\infty)\Big) = P_x^{\xi}\Big(\inf(\{y : S(y) \geq s\}) < \infty\Big) = 1 > P_x^{\xi}\Big(\inf(\{y : S(y) = s\}) < \infty\Big) = \mathbb{N}_x\Big(\tilde{\ell}^{\mathrm{a}}_{s,i}(\infty)\Big)$$

where $S$ denotes the subordinator component of $\xi$. Furthermore we have that

$$\mathbb{N}_x\Big(\tilde{\ell}^{\mathrm{d}}_{s,i}(\infty)\Big) = 1 - P_x^{\xi}(\inf(\{y : S(y) = s\}) < \infty) = P_x^{\xi}(\inf(\{y : S(y) = s\}) = \infty) \in (0,1),$$

where we note that the final probability is in $(0,1)$ as the subordinator can either cross or hit level $s$, see e.g. [9, Theorem III.5]. ∎

We later apply this result above to the snakes in the Poisson random measure given in Definition 3.2.7. Before that though we briefly make a remark on particles that start dormant and do not wake up before time $s$.

*Remark* 3.4.4. Fix $s \geq 0$. In the setting of the Definition 3.2.7, for $i \in I$, we have that for $s_i > s$ so that $H_t^i(s) = 0$, $A_t^i(s) = 0$ and $\mathrm{Res}_t^i(s) = f_t^i$ for all $t \in [0, \sigma(f^i)]$. In this instance the stopping times $S^{h,\mathrm{d}}_{s,n,i}$, $T^{h,\mathrm{d}}_{s,n,i}$ are then simply the up- and down-crossing times of $[0,h]$ for $f^i$. As $f^i$ is an excursion, this is one such downcrossing if $T_h(f^i) < \infty$ and a zero downcrossings otherwise.

The case where $s_i = s$ does not affect our analysis as for fixed $s$ we have $s \neq s_i$ for all $i \in I$ $\mathbb{P}_{(\mu_{\mathrm{d}},\mu_{\mathrm{a}})}$-a.s.

We will define the representation of ooSBM in Theorem 3.4.6 but first we state a technical lemma that we will require in the proof of the theorem.

**Lemma 3.4.5.** *Consider a (possibly infinite) collection of independent reflecting Brownian motions $(R^j, j \in J)$ stopped at the local times at zero $(C_j, j \in J)$ on some common probability space $(\Omega, \mathscr{F}, P)$. Let $C = \sum_{j\in J} C_j$ and suppose $C$ is finite. For $j \in J$ let $D_h^j$ be the number of downcrossings from level $h$ to level $0$ for $R^j$. Then we have that*

$$\lim_{h\downarrow 0} h \sum_{j\in J} D_h^j = C \quad P\text{-a.s.}$$

*Proof.* We first note that for $h > 0$, $(D_h^j, j \in J)$ form a family of independent Poisson random variables with $D_h^j \sim \text{Poisson}(C_j/2h)$. We recall that the sum of independent Poisson random variables is again a Poisson random variable and so $\sum_{j\in J} D_h^j \sim \text{Poisson}(C/2h)$. The result then follows by the strong law of large numbers for Poisson processes. ■

**Theorem 3.4.6.** *Fix $s > 0$. For $t \geq 0$, $s_i > s$ let $Res_t^{i,\mathrm{d}}(s) := Res_t^i(s)\mathbb{1}_{A_t^i(s)=0}$ and $Res_t^{i,\mathrm{a}}(s) := Res_t^i(s)\mathbb{1}_{A_t^i(s)=1}$. Firstly, there exist local time processes $\ell_{s,i}^{\mathrm{a}}(\cdot)$ and $\ell_{s,i}^{\mathrm{d}}(\cdot)$ at level $0$ for $(Res_t^{i,\mathrm{d}}(s))_{t\geq 0}$ and $(Res_t^{i,\mathrm{a}}(s))_{t\geq 0}$.*

*Secondly, we have*

$$(2h\mathscr{Z}_s^{(h,\mathrm{d})}, 2h\mathscr{Z}_s^{(h,\mathrm{a})}) \xrightarrow[h\to 0]{a.s.} (\mathscr{Z}_s^{(\mathrm{d})}, \mathscr{Z}_s^{(\mathrm{a})}),$$

*on the space of $\mathscr{M}_F(\mathbb{R}^d)^2$. The limits $((\mathscr{Z}_s^{(\mathrm{d})}, \mathscr{Z}_s^{(\mathrm{a})}))_{s\geq 0}$ have the finite-dimensional distributions of ooSBM$(4, c, \tilde{c})$ started at $(\mu_{\mathrm{d}}, \mu_{\mathrm{a}})$.*

*Finally for any continuous bounded function $\Phi : \mathbb{R}^d \to [0,\infty)$ we have that*

$$\int_x \Phi(x)\mathscr{Z}_s^{(\mathrm{d})}(dx) = e^{-\tilde{c}s}\int_x \Phi(x)\mu_{\mathrm{d}}(dx) + \sum_{i\in I: s_i<s}\int_{t=0}^{\sigma(f^i)} \Phi(\hat{w}_t^i)\ell_{s,i}^d(dt)$$

*and*

$$\int_x \Phi(x)\mathscr{Z}_s^{(\mathrm{a})}(dx) = \sum_{i\in I: s_i<s}\int_{t=0}^{\sigma(f^i)} \Phi(\hat{w}_t^i)\ell_{s,i}^a(dt).$$

*Proof.* For the first part of the theorem suppose $i \in I$ and $s_i < s$. The processes $(\mathrm{Res}_t^{i,\mathrm{d}}(s))_{t\geq 0}$, $(\mathrm{Res}_t^{i,\mathrm{a}}(s))_{t\geq 0}$ are simply the processes $(\mathrm{Res}^i_{\eta(s,t',i,\mathrm{a})})_{t'\geq 0}$, $(\mathrm{Res}^i_{\eta(s,t',i,\mathrm{d})})_{t'\geq 0}$ from Proposition 3.4.3 with a time change and so by Proposition 3.4.3 we have the existence of the local time processes $\ell_{s,i}^{\mathrm{d}}(\cdot)$ and $\ell_{s,i}^{\mathrm{a}}(\cdot)$.

Now we show the second part of the theorem and first we consider $i \in I$ with $s_i > s$. We have

$$\Big(2h\sum_{i\in I: s_i>s}\sum_{n=1}^{M_i^{h,\mathrm{d}}} \delta_{S_{s,n,i}^{h,\mathrm{a}}}, 2h\sum_{i\in I: s_i>s}\sum_{n=1}^{M_i^{h,\mathrm{a}}} \delta_{S_{s,n,i}^{h,\mathrm{a}}}\Big) = \Big(2h\sum_{i\in I: s_i>s, T_h(f^i)<\infty} \delta_{T_h(f^i)}, 0\Big)$$

which means that

$$\Big(2h\sum_{i\in I:s_i>s}\mathscr{Z}^{(h,\mathrm{d})}_{s,i},2h\sum_{i\in I:s_i>s}\mathscr{Z}^{(h,\mathrm{d})}_{s,i}\Big)=\Big(2h\sum_{i\in I:s_i>s,T_h(f^i)<\infty}\delta_{b_i},0\Big)$$

and this is a Poisson random measure with intensity $(e^{-\tilde{c}s}\mu_{\mathrm{d}}(db)/2h,0)$ rescaled by $2h$ and this converges to the measure $(\int_b e^{-\tilde{c}s}\mu_{\mathrm{d}}(db),0)$ as $h$ tends to zero $\mathbb{P}_{(\mu_{\mathrm{d}},\mu_{\mathrm{a}})}$-a.s. by the strong law of large number for Poisson processes, see Dudley [31, Theorem 11.4.1].

Now we return to $i\in I$ with $s_i<s$ (note as discussed in Remark 3.4.4 there are no $i\in I$ with $s_i=s$ $\mathbb{P}_{(\mu_{\mathrm{d}},\mu_{\mathrm{a}})}$-a.s.). The points $(S^{h,\mathrm{a}}_{s,n,i})_{1\le n\le M^{h,\mathrm{a}}_i}$ are the left-endpoints of the $h$-downcrossings to level 0 for the process $(\mathrm{Res}^i_{\eta(s,\mathrm{a},\mathrm{t},\mathrm{i})})_{t\ge0}$ and so we have that

$$\Big(2h\sum_{n=1}^{M^{h,\mathrm{d}}_i}\delta_{S^{h,\mathrm{d}}_{s,n,i}}(\cdot),2h\sum_{n=1}^{M^{h,\mathrm{a}}_i}\delta_{S^{h,\mathrm{a}}_{s,n,i}}(\cdot)\Big)\xrightarrow[a.s.]{}(\ell^{\mathrm{d}}_{s,i}(\cdot),\ell^{\mathrm{a}}_{s,i}(\cdot)) \tag{3.6}$$

on the space $\mathscr{M}_F([0,\infty)^2)$, see [77, Chapter 6]. Strictly that reference applies to reflecting Brownian motion but the convergence is not affected by the time changes we have here. Let $\ell_{s,i}:=\ell^{\mathrm{d}}_{s,i}+\ell^{\mathrm{a}}_{s,i}$ and for $x\in E$ let $\ell_s$ denote the corresponding measure under $\mathbb{N}_x$. Then we have by the compensation formula for Poisson random measures that

$$\mathbb{E}_{(\mu_{\mathrm{d}},\mu_{\mathrm{a}})}\Big[\sum_{i\in I:s_i<s}\ell_{s,i}(\infty)\Big]=\int_{b\in\mathbb{R}^d}\mathbb{N}_{(b,0)}(\ell_s(\infty))\mu_{\mathrm{a}}(db)+\int_{b\in\mathbb{R}^d}\int_{q=0}^{s}\mathbb{N}_{(b,q)}(\ell_s(\infty))\nu_s(dq)\mu_{\mathrm{d}}(db)$$

where $\nu_s(dq):=\mathbb{1}_{0\le q\le s}\tilde{c}e^{-\tilde{c}q}$. We note here that for $b\in\mathbb{R}^d$, $0<q<s$ that $\mathbb{N}_{(b,q)}(\ell_s(\infty)\in\cdot)=\mathbb{N}_{(b,0)}(\ell_{s-q}(\infty)\in\cdot)$. By the second part of Proposition 3.4.3 this is equal to $\|\mu_{\mathrm{a}}\|+(1-e^{-\tilde{c}s})\|\mu_{\mathrm{d}}\|$ which is finite and therefore $\sum_{i\in I,s_i<s}\ell_{s,i}(\infty)<\infty$ $\mathbb{P}_{(\mu_{\mathrm{d}},\mu_{\mathrm{a}})}$-a.s.

Now let $\varepsilon>0$ and let $M:=\sup(\{\Phi(b),b\in\mathbb{R}^d\})<\infty$. Let $N\in\mathbb{N}$ and $i_1,\dots,i_N\in I$ such that $\sum_{i\in I:i\neq i_k\forall k\in[N]}\ell^{\mathrm{a}}_{s,i}(\infty)<\varepsilon/M$. By Lemma 3.4.5 the downcrossings of the remaining contributions (conditional of the local times at level 0) are such that

$$\lim_{h\to0}2h\sum_{i\in I:i\neq i_k\forall k\in[N],s_i<s}(M^{h,\mathrm{d}}_i+M^{h,\mathrm{a}}_i)<\varepsilon/M. \tag{3.7}$$

We now consider

$$\begin{aligned}\liminf_{h\to0}\sum_{i\in I}\int_{t=0}^{\sigma(f^i)}\Phi(w^i_t(H^i_t(s)))2h\sum_{n=1}^{M^{h,\mathrm{a}}_i}\delta_{S^{h,\mathrm{a}}_{s,n,i}}(dt)\\ \ge\sum_{k=1}^{N}\liminf_{h\to0}\int_{t=0}^{\sigma(f^i)}\Phi(w^i_t(H^i_t(s)))2h\sum_{n=1}^{M^{h,\mathrm{a}}_i}\delta_{S^{h,\mathrm{a}}_{s,n,i}}(dt)\ge\int_{b\in\mathbb{R}^d}\Phi(b)\mathscr{Z}^{(\mathrm{a})}_s(db)-\varepsilon\end{aligned}$$

where the last inequality is by (3.7) where we have the total mass of all the contributions we have not included has size at most $\varepsilon/M$. The convergence is due to the fact in the limit as $h$ tends

to 0 we are evaluating at the head of the Brownian component of the snake, $\hat{w}^i = w^i(\zeta(x^i)-)$, $\ell$-a.e. and we recall that the head of the snake is continuous as noted in Remark 3.2.2. Similarly

$$\limsup_{h\to 0}\sum_{i\in I}\int_{t=0}^{\sigma(f^i)}\Phi(w_t^i(H_t^i(s)))2h\sum_{n=1}^{M_i^{h,\mathrm{a}}}\delta_{S_{s,n,i}^{h,\mathrm{a}}}(dt)$$
$$\leq \sum_{k=1}^{N}\limsup_{h\to 0}\int_{t=0}^{\sigma(f^i)}\Phi(w_t^i(H_t^i(s)))2h\sum_{n=1}^{M_i^{h,\mathrm{a}}}\delta_{S_{s,n,i}^{h,\mathrm{a}}}(dt)+\varepsilon \leq \int_{b\in\mathbb{R}^d}\Phi(b)\mathscr{Z}_s^{(\mathrm{a})}(db)+\varepsilon.$$

As $\varepsilon$ is arbitrary we have the convergence for the active component at time $s$. The argument for the dormant component is the same except there is the contribution from the $i \in I$ with $s_i > s$ to include but this is shown to converge as $h \to 0$ in the first part of this proof.

To show the convergence in finite dimensional distributions recall from Proposition 3.1.3 that ooSBM is the limit of of ooBBM. Consider any $(s_1,\ldots,s_l)$. Then we have the a.s convergence holds jointly for $s_1$, …, $s_l$ and by Proposition 3.3.7 these finite dimensional distributions converge to the finite dimensional distributions of ooSBM.

The final part of the theorem follows from the previous parts, namely (3.6). ■

We finish this section with a note that everything above applies to give a representation of SBM.

*Remark* 3.4.7. If we take the switching rate from active to dormant as $c = 0$ in everything we have done above then we obtain a limit which has the finite dimensional distributions of SBM. This has already been done in Le Jan [68] and our result above can be thought as a generalisation to the case of ooSBM.

## 3.5 Representation of on/off super Brownian motion

We state a general result about random variables with common finite dimensional distributions which can be shown via standard proof techniques.

**Proposition 3.5.1.** *Let $(E',d')$ be a complete and separable metric space and $(\Omega,\mathscr{F},\mathbb{P})$ a probability space on which we have random variables $Q_t : \Omega \to E'$, $t \geq 0$. Suppose further that there is another probability space $(\Omega',\mathscr{F}',\mathbb{P}')$ on which random variables $Q'_t : \Omega' \to E'$ form a path-continuous process with the same finite-dimensional distributions as $Q = (Q_t, t \geq 0)$. Then $(Q_t, t \geq 0)$ has a modification $(\tilde{Q}_t, t \geq 0)$ with continuous paths, i.e. for each fixed $t \geq 0$ we have $\mathbb{P}(\tilde{Q}_t = Q_t) = 1$.*

The idea behind the proof is that uniform continuity on $[0,T]\cap\mathbb{Q}$, for $T > 0$, is $\mathscr{B}(E')^{[0,\infty)\cap\mathbb{Q}}$-measurable and from this we can define a continuous modification of $(Q_t, t \geq 0)$.

We can now prove Theorem 3.1.8.

*Proof of Theorem 3.1.8.* We work on the probability space $(\Omega, \mathscr{F}, \mathbb{P}_{(\mu_{\mathrm{d}},\mu_{\mathrm{a}})})$ of the on/off Brownian snake defined in Definition 3.2.7. We work with an SBM $\mathscr{X} = (\mathscr{X}_y, y \geq 0)$ constructed from the snake via the standard methods of [67, Theorem 4.4.2] given as follows. There exists a process given by $\tilde{\mathscr{X}} := (\tilde{\mathscr{X}}_y, y \geq 0)$ which is the measure-valued process for which we have for any continuous and bounded function $\Phi : \mathbb{R}^d \to [0,\infty)$ that

$$\int_x \Phi(x)\tilde{\mathscr{X}}_y(dx) = \sum_{i\in I}\int_{t=0}^{\sigma(f^i)} \Phi(\hat{w}^i_t)\ell'_{y,i}(dt) \quad y > 0,$$

where $\ell'_{y,i}(\cdot)$ is the local time for $f^i$ at level $y$ and we set $\tilde{\mathscr{X}}_0 := \mu_{\mathrm{d}} + \mu_{\mathrm{a}}$. There exists a continuous modification for $\tilde{\mathscr{X}}$ and we take $\mathscr{X} = (\mathscr{X}_y, y \geq 0)$ to be this modification.

By Proposition 3.5.1 we can consider a continuous modification of the process $(\mathscr{Z}^{(\mathrm{d})}_s, \mathscr{Z}^{(\mathrm{a})}_s)_{s\geq 0}$ established in Theorem 3.4.6 and we denote this process by $\mathscr{Y} = ((\mathscr{Y}^{(\mathrm{d})}_s, \mathscr{Y}^{(\mathrm{a})}_s), s \geq 0)$. We note here that the construction in Theorem 3.4.6 had the finite dimensional distributions of ooSBM$(4, c, \tilde{c})$ but this argument can be generalised to any fixed $\gamma > 0$ for ooSBM$(\gamma, c, \tilde{c})$.

By the continuity of $\mathscr{Y}$ we have that a.s.

$$\mathscr{R}(\mathscr{Y}) = \overline{\bigcup_{s\in\mathbb{Q},s\geq 0} \mathrm{supp}(\mathscr{Y}^{(\mathrm{d})}_s)} \cup \overline{\bigcup_{s\in\mathbb{Q},s\geq 0} \mathrm{supp}(\mathscr{Y}^{(\mathrm{a})}_s)} = \overline{\bigcup_{s\in\mathbb{Q},s\geq 0} \mathrm{supp}(\mathscr{Z}^{(\mathrm{d})}_s)} \cup \overline{\bigcup_{s\in\mathbb{Q},s\geq 0} \mathrm{supp}(\mathscr{Z}^{(\mathrm{a})}_s)}.$$

We further note that a.s.

$$\mathrm{supp}(\mathscr{Z}^{(\mathrm{a})}_{s,i}) = \overline{\{\hat{w}^i_{T^{h_m,\mathrm{a}}_{s,n,i}} : h_m = 1/m, m \in \mathbb{N}, 1 \leq n \leq M^{h_m,\mathrm{a}}_i\}}$$

by the construction of $\mathscr{Z}^{(\mathrm{a})}_{s,i}$ and we also have

$$\mathrm{supp}(\mathscr{Z}^{(\mathrm{d})}_{s,i}) = \overline{\mathrm{supp}(\mu_{\mathrm{d}}) \cup \{\hat{w}^i_{T^{h_m,\mathrm{d}}_{s,n,i}} : h_m = 1/m, m \in \mathbb{N}, 1 \leq n \leq M^{h_m,\mathrm{d}}_i\}}$$

by the final part of Theorem 3.4.6 (where the addition of $\mu_{\mathrm{d}}$ accounts for the fact that there is always a proportion of the initial dormant mass that has not woken up before time $s$). We have $\hat{w}^i_{T^{h_m,\mathrm{d}}_{s,n,i}}, \hat{w}^i_{T^{h_m,\mathrm{a}}_{s,n,i}} \in \overline{\{\hat{w}^i_t | t \in (0, \sigma(f^i))\}}$ and by continuity of the snake at time zero we have $\mathrm{supp}(\mu_{\mathrm{d}}) \subseteq \mathrm{supp}(\mathscr{X}_0)$ and so we have $\mathscr{R}(\mathscr{Y}) \subseteq \mathscr{R}(\mathscr{X}) = \overline{\bigcup_{i\in I}\{\hat{w}^i_t | t \in (0, \sigma(f^i))\}}$ a.s. This can be seen as the support of local time measures is precisely the zero-set, see [88, Proposition VI.2.5]. By the continuity of the head of the snakes this is equal to

$$\overline{\bigcup_{i\in I}\{\hat{w}^i_t | t \in \mathbb{Q} \cap (0, \sigma(f^i))\}}$$

Fix $t \in \mathbb{Q} \cap (0, \sigma(f^i))$. Then $\mathbb{P}_{(\mu_{\mathrm{d}},\mu_{\mathrm{a}})}$-a.s. there exists $N \in \mathbb{N}$ such that $f^i_t > 2^{-N}$ and such that there is no dormancy jump for $v^i$ in the interval $[f^i_t - 2^{-N}, f^i_t)$, i.e. we have $(v^i(f^i_t - 2^{-N} + r), 0 \leq$

$r < 2^{-N}$) is continuous. Therefore there exist dyadic levels $t'_n$ approximating $\hat{v}^i_t$ (the head of the subordinator part of the snake which exists here in this case $\mathbb{P}_{(\mu_\mathrm{d},\mu_\mathrm{a})}$-a.s.) such that there are downcrossing times at these levels that converge to $t$. To see this, let $s'_n := 2^{-n}\lfloor 2^n H^i_t(\hat{v}^i_t)\rfloor$ and then for $n$ large enough there exists $m \in \mathbb{N}$ such that $|T^{2^{-n},\mathrm{d}}_{s'_n,m,i} - t| \leq 2^{-n}$. Hence by the continuity of the Brownian snake we have a.s. that $\hat{w}^i_t \in \mathscr{R}(\mathscr{Y})$ for all $t \in \mathbb{Q} \cap (0, \sigma(f^i))$ and all $i \in I$ and so we have $\mathscr{R}(\mathscr{X}) \subseteq \mathscr{R}(\mathscr{Y})$ a.s. as required. ■

# Chapter 4

# Random Walks On Coxeter Interchange Graphs

The work presented here is a collaboration between the author and his collaborators Brett Kolesnik, Rivka Mitchell, Tomasz Przybyłowski and resulted in a pre-print submitted to the Electronic Journal of Probability, see [19], https://arxiv.org/abs/2401.17210. We have made changes based on feedback from the reviewers for the journal, and we have incorporated the abstract and bibliography into the overall abstract and bibliography of this dissertation. Otherwise, it is presented here in the state in which it was submitted.

## 4.1 Introduction

A tournament is an orientation of a graph. We think of vertices as players and edges as games, the orientation of which indicates the winner. Tournaments are related to the geometry of the permutahedron $\Pi_{n-1}$, which is a classical polytope in discrete convex geometry. See, e.g., Stanley [93], Ziegler [100], and Kolesnik and Sanchez [59].

Classical combinatorics is related to the root system of type $A_n$. Coxeter combinatorics is concerned with extensions to the other roots systems of types $B_n$, $C_n$ and $D_n$ (and sometimes also the finite, exceptional types $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$). For example, works by Galashin, Hopkins, McConville and Postnikov [40, 41] have investigated Coxeter versions of the chip-firing game (the sandpile model).

Recently, Kolesnik and Sanchez [58] introduced the Coxeter analogue of graph tournaments, which are associated with orientations of signed graphs, as in Zaslavsky [99], and the Coxeter permutahedra $\Pi_\Phi$, recently introduced by Ardila, Castillo, Eur and Postnikov [5]. Coxeter tournaments involve collaborative and solitaire games, as well as the usual competitive games in graph tournaments.

In this work, we show (see Theorem 4 below) that random walks rapidly mix on the sets of Coxeter tournaments with given score sequence, that is, on the fibres of the Coxeter permutahedra

$\Pi_\Phi$. Informally, this means that the walk is close to uniform in the fibre after a short amount of time, yielding an efficient way to sample from this set of interest.

Many combinatorial properties of these structures remain mysterious. The purpose of this work is to explore the associated Coxeter interchange graphs, which encode their combinatorics, via random walks. These graphs, introduced by Kolesnik, Mitchell and Przybyłowski [57], generalize the interchange graphs introduced by Brualdi and Li [16]. Rapid mixing in the classical setting was established by Kannan, Tetali and Vempala [53] and McShine [72]. We recover these results by our general strategy.

Let us emphasize that even the classical interchange graphs appear to be difficult to describe in general. Indeed, Brualdi and Li [16, p. 151] state that they have "a rich and fascinating combinatorial structure and that much remains to be determined." Even counting the number of vertices is of "considerable interest and considerable difficulty" [16, p. 143].

Beginning with Spencer [92], and subsequent works by McKay [70], McKay and Wang [71], and Isaev, Iyer and McKay [49], asymptotic estimates for the number of vertices in the interchange graphs have been found only for fibres of points near the centre of $\Pi_{n-1}$. In the other extreme, Chen, Chang and Wang [23] showed that, for certain points near the boundary of $\Pi_{n-1}$, the interchange graph is the classical hypercube.

The Coxeter interchange graphs are richer still. Therefore, in broad terms, we show in this work that random walks rapidly mix on a wide and intricate class of graphs.

Rapid mixing can sometimes be used to approximately count sets of interest. Roughly speaking, this is because the uniform measure $\pi$ on a set $S$ is related to the size of the set $\pi = 1/|S|$. See, e.g., Sinclair [91] for more details. The current work might serve as a first step towards developing efficient approximate counting schemes for the fibres of the Coxeter permutahedra $\Pi_\Phi$.

As this work touches on a variety of subjects (combinatorics, geometry, algebra and probability), some preliminaries are required before we can state our results precisely. In Section 4.1.1, we discuss the literature related to tournaments and the standard permutahedron $\Pi_{n-1}$ (of type $A_{n-1}$). Our results are discussed informally in Sections 4.1.2 and 4.1.3. Further background on tournaments, root systems, signed graphs and combinatorial geometry is in Section 4.2 and in the previous works in this series [57–59]. Our main result is stated formally in Section 4.3. See Sections 4.4–4.6 for the proofs. Finally, a number of open problems and future directions are listed in Section 4.7.

We hope that this work will serve as an invitation to step into the Coxeter "worlds" (of types $B_n$, $C_n$ and $D_n$). We believe that Coxeter combinatorics is fertile ground, where algebraists, combinatorialists, geometers and probabilists can open new lines of fruitful communication. In particular, many problems in discrete probability likely have a Coxeter analogue, waiting to be discovered.

### 4.1.1 Context

A *tournament* is an orientation of the complete graph $K_n$, encoded as some $T = (w_{ij} : i > j)$ with all $w_{ij} \in \{0,1\}$. Each edge $\{i,j\}$ in $K_n$ is oriented as $i \to j$ if $w_{ij} = 1$ or $i \leftarrow j$ if $w_{ij} = 0$. We think of each edge as a game, directed away from the winner. The *win sequence*

$$\mathbf{w}(T) = \sum_{i>j} [w_{ij}\mathbf{e}_i + (1 - w_{ij})\mathbf{e}_j]$$

lists the total number of wins by each player, where $\mathbf{e}_i \in \mathbb{Z}^n$ are the standard basis vectors. We let $\mathbf{w}_n = (0, 1, \ldots, n-1)$ denote the *standard win sequence,* corresponding to the *transitive* (acyclic) tournament in which $w_{ij} = 1$ for all $i > j$. In a sense, $\mathbf{w}_n$ and its permutations are as "spread out" as possible.

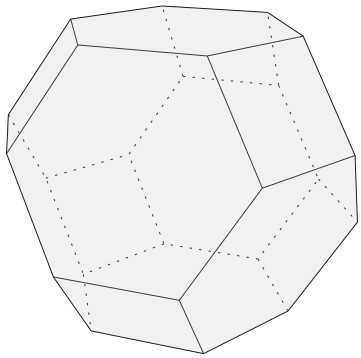


Figure 4.1: The permutahedron $\Pi_3 \subset \mathbb{R}^4$, projected into $\mathbb{R}^3$. Its 24 vertices correspond to the permutations of the standard win sequence $\mathbf{w}_4 = (0,1,2,3)$.

Results by Rado [87] and Landau [64] imply that the set $\mathrm{Win}(n)$ of all win sequences is precisely the set of lattice points in the permutahedron $\Pi_{n-1}$, that is, $\mathrm{Win}(n) = \mathbb{Z}^n \cap \Pi_{n-1}$. We recall that $\Pi_{n-1}$ is a classical polytope in discrete geometry (see, e.g., Ziegler [100]), obtained as the convex hull of $\mathbf{w}_n$ and its permutations, see Figure 4.1. By Stanley [93], win sequences are in bijection with spanning forests $F \subset K_n$. (The volume of $\Pi_{n-1}$ is the number of spanning trees $T \subset K_n$.) See Postnikov [86] for generalizations.

It is convenient to make a linear shift

$$\Pi'_{n-1} = \Pi_{n-1} - \frac{n-1}{2}\mathbf{1}_n, \tag{4.1}$$

where $\mathbf{1}_n = (1, \ldots, 1) \in \mathbb{Z}^n$. Note that this re-centers the polytope at the origin $\mathbf{0}_n = (0, \ldots, 0) \in \mathbb{Z}^n$. The *score sequence*

$$\mathbf{s}(T) = \mathbf{w}(T) - \frac{n-1}{2}\mathbf{1}_n,$$

associated with the win sequence $\mathbf{w}(T)$ of a tournament $T$, is given by

$$\mathbf{s}(T) = \sum_{i>j} (w_{ij} - 1/2)(\mathbf{e}_i - \mathbf{e}_j). \tag{4.2}$$

This shift corresponds to awarding a $\pm 1/2$ point for each win/loss. We let $\mathrm{Score}(n)$ denote the set of all possible score sequences.

Although the set $\mathrm{Score}(n)$ has a simple, geometric description, the set $\mathrm{Tour}(n,\mathbf{s})$, of tournaments with given score sequence $\mathbf{s}$, appears to be quite combinatorially complex.

Kannan, Tetali and Vempala [53] investigated simple random walk as a way of sampling from $\mathrm{Tour}(n,\mathbf{s})$. However, rapid mixing was proved only for $\mathbf{s}$ sufficiently close to $\mathbf{0}_n$. McShine [72] established rapid mixing in time $O(n^3 \log n)$, for *all* $\mathbf{s} \in \mathrm{Score}(n)$, by an elegant application of Bubley and Dyer's [17] method of path coupling, which was relatively new at the time. See Section 4.2.3 below for an overview. We note that Sarkar [90] has shown that mixing takes $\Omega(n^3)$ for some sequences.

More specifically, in [53, 72], the random walks are on the *interchange graphs* $\mathrm{IntGr}(n,\mathbf{s})$ introduced by Brualdi and Li [16]. Any two tournaments with the same score sequence have the same number of copies of the *cyclic triangle* $\Delta_c$ (see Figure 4.5). Moreover, if some $T$ contains a copy $\Delta$ of $\Delta_c$, then the tournament $T * \Delta$, obtained by reversing the orientation of all edges in $\Delta$, has the same score sequence as $T$. These observations are the key to exploring $\mathrm{Tour}(n,\mathbf{s})$. The graph $\mathrm{IntGr}(n,\mathbf{s})$ has a vertex $v(T)$ for each $T \in \mathrm{Tour}(n,\mathbf{s})$ and two $v(T_1), v(T_2)$ are neighbours if $T_2 = T_1 * \Delta$ for some copy $\Delta \subset T_1$ of $\Delta_c$. It can be shown that $\mathrm{IntGr}(n,\mathbf{s})$ is connected. In this sense, $\Delta_c$ *generates* the set $\mathrm{Tour}(n,\mathbf{s})$.

The permutahedron $\Pi_{n-1}$ is related with the standard root system of type $A_{n-1}$, as the symmetric group $S_n$ is the Weyl group of type $A_{n-1}$. We refer to, e.g., the standard text by Humphreys [48] for background on root systems. We recall that Killing [55] and Cartan [20] classified all (irreducible, crystallographic) root systems (up to isomorphism) as the infinite families $A_{n-1}$, $B_n$, $C_n$ and $D_n$ and the finite exceptional types $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$.

Coxeter permutahedra $\Pi_\Phi$, recently studied by Ardila, Castillo, Eur and Postnikov [5], are obtained by replacing the role of $S_n$ in the definition of $\Pi_{n-1}$ with the Weyl group $W_\Phi$ of a root system $\Phi$. See, e.g., Figure 4.2.

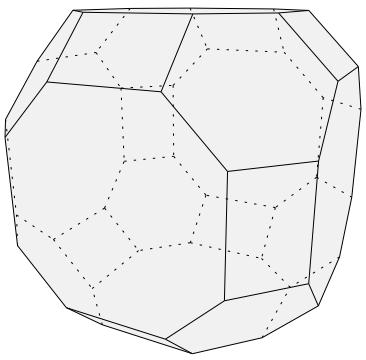


Figure 4.2: The Coxeter permutahedron of type $C_3$.

The previous works in this series [57, 58] studied the connection between the polytopes $\Pi_\Phi$ and *Coxeter tournaments,* which are related to orientations of signed graphs, as developed by

Zaslavsky [97–99]. As mentioned above, these tournaments involve collaborative and solitaire games, as well as the usual competitive games in classical graph tournaments.

### 4.1.2 Purpose

In this work (see Theorem 4) we show that simple random walks mix rapidly on the *Coxeter interchange graphs* $\mathrm{IntGr}(\Phi,\mathbf{s})$. These graphs encode the combinatorics of the sets $\mathrm{Tour}(\Phi,\mathbf{s})$, of Coxeter tournaments with a given score sequence $\mathbf{s}$, and give structural information about the fibres of the Coxeter permutahedra $\Pi_\Phi$. We focus on the non-standard types $\Phi = B_n$, $C_n$ and $D_n$.

We also show (see Theorem 5) that all Coxeter interchange graphs are connected and we bound their diameter. In constructing our random walk couplings, we uncover various other fine, structural properties of the graphs $\mathrm{IntGr}(\Phi,\mathbf{s})$, and hence the sets $\mathrm{Tour}(\Phi,\mathbf{s})$, which might be of independent (algebraic, geometric, etc.) interest.

### 4.1.3 Discusssion

Path coupling is a powerful method for establishing rapid mixing (see Section 4.2.3). As already mentioned, path coupling was used in [72]. We will also use this method, however, the application in the Coxeter setting is significantly more delicate.

As discussed above, the interchange graphs in type $A_{n-1}$ are generated by a single neutral tournament, namely, the cyclic triangle $\Delta_c$. On the other hand, in the Coxeter setting, there are a number of other generators which play a role (see Figures 4.5–4.7). A fascinating interplay arises, as these generators can interact in a variety of interesting ways. As such, the Coxeter interchange graphs are much richer in complexity. Likewise, the analysis of random walks on these structures is more involved.

The types increase in difficulty in order $A_{n-1}$, $D_n$, $B_n$, $C_n$. Type $C_n$ is especially challenging, due to the presence of loops in some of the generators, which we call *clovers* (see Figure 4.7). These generators correspond to double edges in an interchange graph. In particular, a special type of structure, which we call a *crystal* (see Figure 4.25), can appear in type $C_n$ interchange graphs. The crystal arises in $C_3$ when $\mathbf{s} = (2,1,1)$. Crystals can also be found as subgraphs in larger type $C_n$ interchange graphs, for instance, in the *snare drum* in Figure 4.3, when $\mathbf{s} = (-1,0,1)$. In other examples, such as the *tambourine* in Figure 4.4, when $\mathbf{s} = (0,0,0)$ is the centre of the polytope, there are no crystals, but interesting structure nonetheless.

The presence of crystals in type $C_n$ interchange graphs leads to two main issues. The first is in extending certain natural couplings on various small subgraphs (the *extended networks* discussed in Sections 4.5.1 and 4.5.2) to a unified coupling on the entire interchange graph. To overcome this difficulty, we will prove a number of detailed combinatorial properties of the Coxeter interchange graphs. We classify the types of subgraphs (see Figures 4.14, 4.17

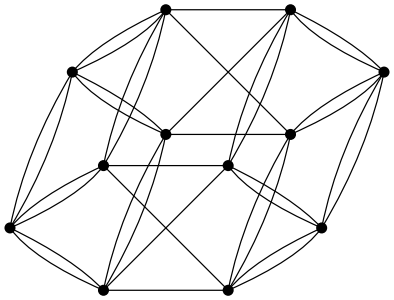


Figure 4.3: The snare drum interchange graph $\mathrm{IntGr}(C_3, \mathbf{s})$, when $\mathbf{s} = (-1, 0, 1)$, is the Cartesian product of a double edge and the crystal (see Figure 4.25).

and 4.25), which together form the full graph, and study the ways in which they can intersect. For example, one crucial property (see Theorem 4.5.7) is that any two crystals can share at most one single edge. Without this property, it seems that a path coupling argument would not be possible.

The second issue caused by crystals is in obtaining a "contractive" (see Section 4.2.3) coupling. In applying path coupling, we will need to re-weight the graph metric in a specific way, which accounts for the occurrence of crystals. Loosely speaking, the choice of weights is related to the fact that, in our random walk couplings, crystals work like "switches," that convert single edges to double edges, and vice versa.

The overall coupling used to establish rapid mixing in the Coxeter setting is quite elaborate. See, e.g., Figures 4.28–4.30 below. The classical type $A_{n-1}$ result [53, 72] is a special case of the argument depicted in Figure 4.27.

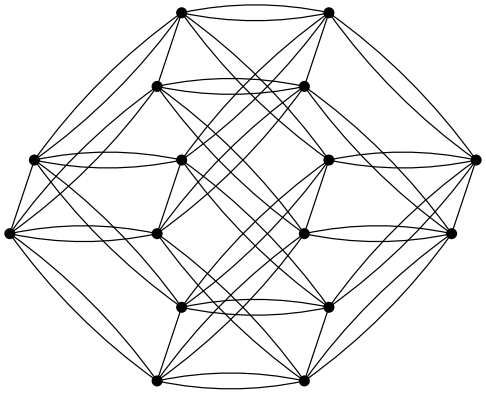


Figure 4.4: The tambourine interchange graph $\mathrm{IntGr}(C_3, \mathbf{s})$, when $\mathbf{s} = (0, 0, 0)$ is the centre of the type $C_3$ permutahedron. This graph is the Cartesian product of a single edge and the cube of double edges.

## 4.2 Background

We refer to [5], Humphreys [48], Zaslavsky [97–99], and the previous works in this series [57,58] for a detailed background on root systems, signed graphs and their connections to discrete geometry. In this section, we will only recall what is used in the current work.

### 4.2.1 Coxeter tournaments

A *signed graph* $\mathscr{S}$ on $[n] = \{1,2,\ldots,n\}$ has a set of signed edges $E(\mathscr{S})$. The four possible types of edges are:

- *negative edges* $e_{ij}^-$ between two vertices $i$ and $j$,
- *positive edges* $e_{ij}^+$ between two vertices $i$ and $j$,
- *half edges* $e_i^h$ with only one vertex $i$, and
- *loops* $e_i^\ell$ at a vertex $i$.

We note that classical graphs $G$ correspond to signed graphs $\mathscr{S}$ with only negative edges.

In this work, we focus on the *complete signed graphs* $\mathscr{K}_\Phi$ of types $\Phi = B_n$, $C_n$ and $D_n$. These signed graphs contain all possible negative and positive edges $e_{ij}^\pm$. In type $B_n$ (resp. $C_n$), all possible half edges $e_i^h$ (resp. loops $e_i^\ell$) are also included. We call a signed graph $\mathscr{S}$ a $\Phi$-*graph* if $\mathscr{S} \subset \mathscr{K}_\Phi$.

Most of the results in the literature on classical (type $A_{n-1}$) graph tournaments restricts to the case that $G$ is the complete graph $K_n$. We note that the signed graph $\mathscr{K}_{A_{n-1}}$ with all possible negative edges (and no other types of signed edges) corresponds to the classical complete graph $K_n$.

A *Coxeter tournament* $\mathscr{T}$ on a signed graph $\mathscr{S}$ is an orientation of $\mathscr{S}$. When $\mathscr{S}$ is unspecified, our default assumption will be that $\mathscr{S} = \mathscr{K}_\Phi$. More formally, $\mathscr{T} = (w_e : e \in E(\mathscr{S}))$, with all $w_e \in \{0,1\}$. We think of each $e \in E(\mathscr{S})$ as a *game*, and $w_e$ as indicating its outcome. (We think of $E(\mathscr{S})$ as having a natural ordering, so that $(w_e : e \in E(\mathscr{S}))$ holds all information about the orientation of $\mathscr{S}$ under $\mathscr{T}$. However, we could, somewhat pedantically, instead write $\mathscr{T} = \{(e, w_e) : e \in E(\mathscr{S})\}$.)

The *score sequence* is given by, cf. (4.2),

$$\mathbf{s}(\mathscr{T}) = \sum_{e \in E(\mathscr{S})} (w_e - 1/2)\mathbf{e},$$

where $\mathbf{e}$ is the vector corresponding to the signed edge $e$, given by $\mathbf{e}_{ij}^\pm = \mathbf{e}_i \pm \mathbf{e}_j$, $\mathbf{e}_i^h = \mathbf{e}_i$ and $\mathbf{e}_i^\ell = 2\mathbf{e}_i$. In other words:

- negative edges $e_{ij}^-$ are *competitive games* in which one of $i, j$ wins and the other loses a $1/2$ point,
- positive edges $e_{ij}^+$ are *collaborative games* in which $i, j$ both win or lose a $1/2$ point,
- half edges $e_i^h$ are *(half edge) solitaire games* in which $i$ wins or loses a $1/2$ point, and
- loops $e_i^\ell$ are *(loop) solitaire games* in which $i$ wins or loses 1 point.

If $\mathbf{s}(\mathscr{T}) = \mathbf{0}_n$ we say that $\mathscr{T}$ is *neutral*.

We let $\mathbf{s}_\Phi$ denote the *standard score sequence* corresponding to the Coxeter tournament in which all $w_e = 1$. We note that, in some contexts, $\mathbf{s}_\Phi$ is called the *Weyl vector*. It is also the sum of the *fundamental weights* of the root system $\Phi$. See, e.g., [47, 48] for more details.

As discussed in [5], the *Coxeter $\Phi$-permutahedron* $\Pi_\Phi$ is the convex hull of the orbit of $\mathbf{s}_\Phi$ under the Weyl group $W_\Phi$ of type $\Phi$. Thus $\mathbf{s}_\Phi$ is a distinguished vertex of $\Pi_\Phi$. Note that the symmetric group $S_n$ is the Weyl group of standard type $\Phi = A_{n-1}$ and the Weyl vector is $\mathbf{s}_n = \mathbf{w}_n - \frac{n-1}{2}\mathbf{1}_n$, so $\Pi'_{n-1}$ (see (4.1) above) is the $\Phi$-*permutahedron* of standard type $\Phi = A_{n-1}$.

In [58], we showed that $\Pi_\Phi$ is precisely the set of all possible *mean* score sequences of *random* Coxeter tournaments, thereby establishing a Coxeter analogue of a classical result of Moon [75]. The next work in this series [57] focused on deterministic Coxeter tournaments. The set $\mathrm{Score}(\Phi)$ of all score sequences of Coxeter tournaments was classified, generalizing the classical result of Landau [64] discussed above.

The precise characterization of $\mathrm{Score}(\Phi)$ is somewhat technical, involving a certain weak sub-majorization condition and additional parity conditions in types $C_n$ and $D_n$. The proof is constructive, in that it shows how to build a Coxeter tournament with any given score sequence. See [57, Theorem 4] for more details.

### 4.2.2 Interchange graphs

The set $\mathrm{Tour}(\Phi, \mathbf{s})$ of all Coxeter tournaments on $\mathscr{K}_\Phi$ with given score sequence $\mathbf{s}$ was also investigated in [57]. Coxeter analogues of the interchange graphs $\mathrm{IntGr}(n, \mathbf{s})$ discussed above were introduced. Recall that the sets $\mathrm{Tour}(n, \mathbf{s})$ are generated by the cyclic triangle $\Delta_c$. In the Coxeter setting, there are additional generators.

In all types $B_n$, $C_n$ and $D_n$, in addition to $\Delta_c$, we also require a *balanced triangle* $\Delta_b$. In type $B_n$, there are also three *neutral pairs* $\Omega_1$, $\Omega_2$ and $\Omega_3$. On the other hand, in type $C_n$, there are also two *neutral clovers* $\Theta_1$ and $\Theta_2$. See Figures 4.5–4.7. In figures depicting Coxeter tournaments, we will draw:

- competitive games as edges directed away from their winner,
- collaborative games as solid/dotted lines if won/lost,

- half edge solitaire games as half edges directed away/toward from their (only) endpoint if won/lost, and
- loop solitaire games as solid/dotted loops if won/lost.

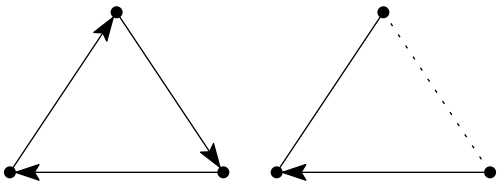


Figure 4.5: The cyclic and balanced triangles $\Delta_c$ and $\Delta_b$ are generators in all types $B_n$, $C_n$ and $D_n$.



Figure 4.6: The neutral pairs $\Omega_1$, $\Omega_2$ and $\Omega_3$ are additional generators in type $B_n$.

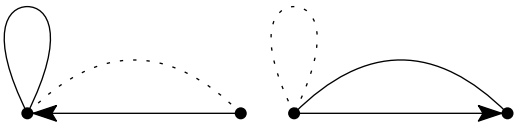


Figure 4.7: The neutral clovers $\Theta_1$ and $\Theta_2$ are additional generators in type $C_n$.

The *reversal* $\mathscr{T}^*$ of a Coxeter tournament $\mathscr{T} = (w_e : e \in E(\mathscr{S}))$ on $\mathscr{S}$ is obtained by reversing the outcome of all games in $\mathscr{T}$. That is, $\mathscr{T}^* = (w_e^* : e \in E(\mathscr{S}))$, where $w_e^* = 1 - w_e$. If $\mathscr{X} \subset \mathscr{T}$, we let $\mathscr{T} * \mathscr{X}$ denote the Coxeter tournament obtained from $\mathscr{T}$ by reversing the outcome of all games in $\mathscr{X}$. In particular, $\mathscr{T}^* = \mathscr{T} * \mathscr{T}$.

The *Coxeter interchange graph* $\mathrm{IntGr}(\Phi, \mathbf{s})$ has a vertex $v(\mathscr{T})$ for each $\mathscr{T} \in \mathrm{Tour}(\Phi, \mathbf{s})$. Vertices $v(\mathscr{T}_1), v(\mathscr{T}_2)$ are neighbours if $\mathscr{T}_2 = \mathscr{T}_1 * \mathscr{G}$, for some copy $\mathscr{G} \subset \mathscr{T}_1$ of a type $\Phi$ generator. If $\mathscr{G}$ is a neutral clover, we add a double edge, and otherwise we add a single edge.

For instance, the "snare drum," in Figure 4.3 above, is $\mathrm{IntGr}(C_3, \mathbf{s})$ when $\mathbf{s} = (-1, 0, 1)$.

The decision to represent clovers as double edges might seem arbitrary at first sight, however, there is a good reason. As it turns out, rather miraculously, this adjustment makes the type $C_n$ interchange graphs degree regular. Furthermore, the degree of $\mathrm{IntGr}(\Phi, \mathbf{s})$ is related to distances in $\Pi_\Phi$ in the following way. Let $\|\mathbf{x}\|^2 = \sum_i x_i^2$ denote the squared length of $\mathbf{x} \in \mathbb{R}^n$.

Recall that $\mathrm{Score}(\Phi)$ is the set of all possible score sequences of Coxeter tournaments on $\mathscr{K}_\Phi$. As discussed above, this set is classified in [57].

**Theorem 1** ([57]). *Let $\Phi = B_n$, $C_n$ or $D_n$. Fix any $\mathbf{s} \in \mathrm{Score}(\Phi)$. Then the Coxeter interchange graph $\mathrm{IntGr}(\Phi, \mathbf{s})$ is regular, with degree given by*

$$d(\Phi, \mathbf{s}) = \frac{\|\mathbf{s}_\Phi\|^2 - \|\mathbf{s}\|^2}{2},$$

*where $\mathbf{s}_\Phi$ is the standard score sequence.*

In particular, $d(\Phi, \mathbf{s}) = O(n^3)$.

We observe that, as $\mathbf{s}$ moves closer to the center $\mathbf{0}_n$ of the polytope $\Pi_\Phi$, the degree $d(\Phi, \mathbf{s})$ of $\mathrm{IntGr}(\Phi, \mathbf{s})$ increases. This is in line with the intuition that Coxeter tournaments with $\mathbf{s}$ closer to $\mathbf{0}_n$ (i.e., closer to being neutral) should contain more copies of the (neutral) generators.

Let us note that $\mathbf{s}$ with $\|\mathbf{s}\|^2 = \|\mathbf{s}_\Phi\|^2$ are precisely the vertices of $\Pi_\Phi$. For such $\mathbf{s}$, we have $d(\Phi, \mathbf{s}) = 0$, in line with the fact there is a unique Coxeter tournaments with score sequence $\mathbf{s}$. Indeed, such a tournament is transitive, in the sense that it contains no copy of a neutral generator, and so its interchange graph is single isolated vertex.

In [57], we observed that such a result also holds for graph tournaments, in relation to the standard permutahedron, yielding a geometric interpretation of the classical result (see, e.g., Moon [76]) that any two tournaments with the same win/score sequence have the same number of cyclic triangles.

In closing, let us emphasize the the neutral generators in Figures 4.5–4.7 were identified in [57]. Theorem 1, proved therein, identifies the degree of the interchange graphs. However, in the current work, we will show (see Theorem 5 below) that the interchange graphs are connected. It is this result that justifies calling these structures "generators," in the sense that the entire space $\mathrm{Tour}(\Phi, \mathbf{s})$ is obtained by iteratively reversing copies of these specific neutral structures.

### 4.2.3 Path coupling

We recall that an aperiodic, irreducible discrete-time Markov chain $(X_n)$ on a finite state space $\Omega$ has a unique *equilibrium* $\pi$ on $S$ such that, for all $x, y \in \Omega$,

$$p_n(x, y) = \mathbb{P}(X_n = y | X_0 = x) \to \pi(y),$$

as $n \to \infty$. We note that $\pi(y)$ is the asymptotic proportion of time spent at state $y \in \Omega$. The maximal total variation distance from $\pi$ by time $n$,

$$\tau(n) = \max_{x \in \Omega} \frac{1}{2} \sum_{y \in S} |p_n(x, y) - \pi(y)|,$$

is non-increasing. The *mixing time* is defined as

$$t_{\mathrm{mix}} = \inf\{n \geqslant 0 : \tau(n) \leqslant 1/4\}.$$

A Markov chain is said to be *rapidly mixing* if $t_{\text{mix}}$ is bounded by a polynomial in $\log|\Omega|$.

Path coupling was introduced by Bubley and Dyer [17]. See, e.g., Aldous and Fill [4, Sec. 12.1.12] or Levin, Peres and Wilmer [95, Sec. 14.2] for reformulations of the original result that are closer in appearance to that of the following.

Consider a connected graph $G = (V,E)$. The *graph distance* $\delta(x,y)$ is the minimal number of edges in a path between $x$ and $y$. The *diameter* is $D = \max_{x,y\in V} \delta(x,y)$ is the maximal length of such a path in $G$.

**Definition 2.** We say that $G = (V,E)$ is *weighted* by $w$ if each edge $\{u,v\} \in E$ is assigned some *weight* $w(u,v) \geqslant 1$. The *weighted distance* $w(x,y)$ is the minimal total weight path between $x$ and $y$. Likewise, $D_w = \max_{x,y\in V} w(x,y)$ is the *weighted diameter*.

The usual graph distance $\delta$ corresponds to the $w$ for which $w(u,v) = 1$ for all $\{u,v\} \in E$.

**Theorem 3** (Path coupling, [17])**.** *Consider a Markov chain $(X_n)$ on a connected graph $G = (V,E)$, weighted by $w$. Suppose that, for some $\alpha > 0$, for each $\{x',x''\} \in E$ there is a coupling $(X_1',X_1'')$ with $(X_0',X_0'') = (x',x'')$ so that*

$$\mathbb{E}[w(X_1',X_1'')] \leqslant (1-\alpha)w(x',x'').$$

*Then $(X_n)$ mixes in time $t_{\text{mix}} = O(\alpha^{-1}\log D_w)$.*

Often this result is applied with $w = \delta$, and, indeed, this will suffice for us in types $B_n$ and $D_n$. In this case, path coupling has the intuitive interpretation that if the chain is "contractive" in expectation, then it is rapidly mixing.

On the other hand, in the more complicated type $C_n$, we will select a careful re-weighting $w$ that takes into account some of the more intricate features in the interchange graphs of this type.

Finally, note that the type $C_n$ interchange graphs are, in fact, *multi*graphs. Specifically, some pairs of vertices (corresponding to clover reversals) are joined by double edges, as in Figures 4.3 and 4.4 above. This is for technical convenience, as it makes the graph regular, and thereby our coupling procedure easier to explain. We note that Theorem 3 still applies, since a Markov chain $(Y_n)$ on a multigraph $M$ with some double edges is equivalent to the Markov chain $(X_n)$ on the graph $G$, obtained by collapsing each double edge in $M$ into a single edge, and combining the two associated edge crossing probabilities.

## 4.3 Main result

Our main result shows that random walks rapidly mix on the Coxeter interchange graphs.

**Theorem 4.** *Let $\Phi = B_n$, $C_n$ or $D_n$. Fix any $\mathbf{s} \in \mathrm{Score}(\Phi)$. Then lazy simple random walk $(\mathscr{T}_n : n \geqslant 0)$ on $\mathrm{IntGr}(\Phi, \mathbf{s})$ mixes in time $t_{\mathrm{mix}} = O(n^3 \log n)$ if $\Phi = B_n$ or $D_n$, and in time $t_{\mathrm{mix}} = O(n^4 \log n)$ if $\Phi = C_n$.*

Rapid mixing in type $A_{n-1}$, proved in [53, 72], follows as a special case of our proof of this result in type $D_n$.

We note that the classification of $\mathrm{Score}(\Phi)$ in [58] is constructive, which allows us to initialize the random walk in the first place.

In fact, we will prove sharper bounds (see Theorems 13 and 14 below). In types $B_n$ and $D_n$, we will show that $t_{\mathrm{mix}} = O(d \log n)$, where $d$ is the degree (see Theorem 1 above) of the interchange graph $\mathrm{IntGr}(\Phi, \mathbf{s})$. The result above follows, since $d = O(n^3)$. In type $C_n$, we will show that $t_{\mathrm{mix}} = O(\gamma d \log n)$, where $\gamma$ is a certain quantity (see Theorem 4.5.8) satisfying $\gamma \leqslant \min\{d, 2n\}$. We call $\gamma$ the *maximal crystal degree* of the interchange graph. Roughly speaking, it is maximal number of crystals, all containing the same double edge. This quantity is related to the re-weighting $w$ that we will use in applying Theorem 3 in type $C_n$. We note that re-weighting arguments have been used before, e.g., in the work of Wilson [96].

## 4.4 Connectivity

Before proving Theorem 4, we will first establish the following combinatorial result, giving further structural information (beyond its regularity, given by Theorem 1) about the Coxeter interchange graphs.

**Theorem 5.** *Let $\Phi = B_n$, $C_n$ or $D_n$. Fix any $\mathbf{s} \in \mathrm{Score}(\Phi)$. The Coxeter interchange graph $\mathrm{IntGr}(\Phi, \mathbf{s})$ is connected and its diameter $D = O(n^2)$.*

This result is a corollary of Theorem 4.4.3 (the "reversing lemma") proved at the end of this section. A number of preliminaries are required. First, in the next subsection, we will find a way of encoding Coxeter tournaments as special types of directed graphs.

### 4.4.1 Z-frames

Oriented signed graphs were studied by Zaslavsky [99]. From this point of view (see [99, Fig. 1]), each oriented signed edge in an oriented signed graph $\mathscr{S}$ is the union of at most two directed half edges. We modify this idea, by adding a named endpoint to each half-edge, which we call a *match*. This will allow us to prove certain structural facts using graph theory techniques. We call such a structure a *Z-frame*.

**Definition 6.** A *Z-frame* $\mathbf{Z}$ is a directed, bipartite multigraph on disjoint sets of *players* $V$ and *matches* $M$, such that every match has degree 1 or 2.

This concept is fairly general, and not all Z-frames correspond to a Coxeter tournament $\mathscr{T}$ on some $\mathscr{S} \subset \mathscr{K}_\Phi$. However, each such $\mathscr{T}$ has a unique representation as a Z-frame $\mathbf{Z}(\mathscr{T})$. We think of $\mathbf{Z}(\mathscr{T})$ as revealing the "inner directed graph structure" of $\mathscr{T}$. Players in $\mathbf{Z}(\mathscr{T})$ correspond to vertices in $\mathscr{T}$. Recall that each game in $\mathscr{T}$ corresponds to an oriented signed edge. Each such game is associated with a match in $\mathbf{Z}(\mathscr{T})$.

We will think of edges directed away/toward players $v \in V$ as positively/negatively charged. (That being said, positive/negative edges in a Z-frame should not be confused with positive/negative edges in a Coxeter tournament.)

We say that $\mathbf{Z}$ is *neutral* if all players $v \in V$ have net zero charge, i.e., $\deg^+(v) - \deg^-(v) = 0$, where $\deg^\pm(v)$ is the number of positive/negative edges incident to $v$. We put $\deg(v) = \deg^+(v) + \deg^-(v)$. Note that, if $\mathbf{Z}$ is *neutral* then $\deg(v)$ is even.

**Definition 7.** Let $\mathscr{T}$ be a Coxeter tournament on a signed graph $\mathscr{S}$ on $[n]$. Let $\mathbf{Z}(\mathscr{T})$ be the Z-frame on $V = [n]$ and $M = \{m_e : e \in \mathscr{S}\}$, with the following directed edges:

- If a competitive game $e^-_{ij} \in \mathscr{S}$ between players $i$ and $j$ is won (resp. lost) by $i$ in $\mathscr{T}$, we include two directed edges $i \to m^-_{ij} \to j$ (resp. $i \leftarrow m^-_{ij} \leftarrow j$).
- If a collaborative game $e^+_{ij} \in \mathscr{S}$ between players $i$ and $j$ is won (resp. lost) in $\mathscr{T}$, we include two directed edges $i \to m^+_{ij} \leftarrow j$ (resp. $i \leftarrow m^+_{ij} \to j$).
- If a half edge solitaire game $e^h_i \in \mathscr{S}$ by player $i$ is won (resp. lost) in $\mathscr{T}$, we include one directed edge $i \to m^h_i$ (resp. $i \leftarrow m^h_i$).
- If a loop solitaire game $e^\ell_i \in \mathscr{S}$ by player $i$ is won (resp. lost) in $\mathscr{T}$, we include two directed edges $i \rightrightarrows m^\ell_i$ (resp. $i \leftleftarrows m^\ell_i$).

See Figure 4.8 for an example of a Coxeter tournament $\mathscr{T}$ and its corresponding Z-frame $\mathbf{Z}(\mathscr{T})$.

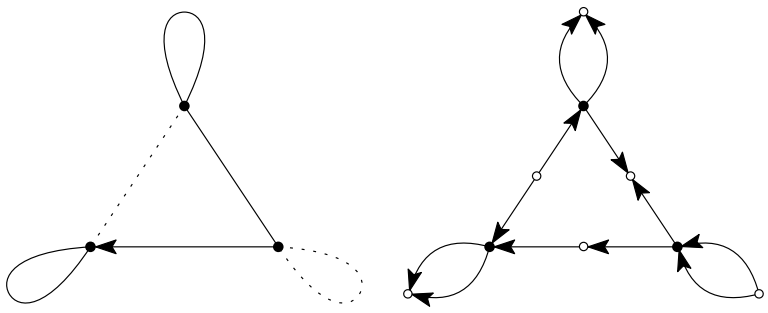


Figure 4.8: A $C_3$-tournament $\mathscr{T}$ and its Z-frame $\mathbf{Z}(\mathscr{T})$. Players/matches are drawn as black/white dots.

The reason for the above terminology is that directed edges in $\mathbf{Z}(\mathscr{T})$ with a positive/negative charge correspond to positive/negative contributions to the score sequence $\mathbf{s}(\mathscr{T})$. Note that,

in competitive/collaborative games, the two charges are opposing/aligned (regardless of the outcome of the game). See Table 4.1.

Table 4.1: Signed edges $e$ in a signed graph $\mathscr{S}$ can be oriented in one of two ways $w_e \in \{0,1\}$ by a Coxeter tournament $\mathscr{T}$. The contribution to the score sequence $\mathbf{s}(\mathscr{T})$ and representation in the Z-frame $\mathbf{Z}(\mathscr{T})$ are given below.

| $e$ | $w_e$ | $\mathbf{s}$ | $\mathbf{Z}$ |
|---|---|---|---|
| $e_{ij}^-$ | 1 | $+\mathbf{e}_{ij}^-/2 = +\mathbf{e}_i/2 - \mathbf{e}_j/2$ | $i \to m_{ij}^- \to j$ |
| | 0 | $-\mathbf{e}_{ij}^-/2 = -\mathbf{e}_i/2 + \mathbf{e}_j/2$ | $i \leftarrow m_{ij}^- \leftarrow j$ |
| $e_{ij}^+$ | 1 | $+\mathbf{e}_{ij}^+/2 = +\mathbf{e}_i/2 + \mathbf{e}_j/2$ | $i \to m_{ij}^+ \leftarrow j$ |
| | 0 | $-\mathbf{e}_{ij}^+/2 = -\mathbf{e}_i/2 - \mathbf{e}_j/2$ | $i \leftarrow m_{ij}^+ \to j$ |
| $e_i^h$ | 1 | $+\mathbf{e}_i^h/2 = +\mathbf{e}_i/2$ | $i \to m_i^h$ |
| | 0 | $-\mathbf{e}_i^h/2 = -\mathbf{e}_i/2$ | $i \leftarrow m_i^h$ |
| $e_i^\ell$ | 1 | $+\mathbf{e}_i^\ell/2 = +\mathbf{e}_i$ | $i \rightrightarrows m_i^\ell$ |
| | 0 | $-\mathbf{e}_i^\ell/2 = -\mathbf{e}_i$ | $i \leftleftarrows m_i^\ell$ |

### 4.4.2 Decomposing Z-frames

Our aim is to decompose neutral Coxeter tournaments into irreducible neutral parts. In this section, we will do this for Z-frames in general.

Recall that a *trail* is a walk in which no edge is visited twice.

**Definition 8.** We call a trail $\mathbf{T}$ of (directed) edges in a Z-frame *closed* if it starts and ends at the same vertex, and otherwise we call it *open*. A trail is *neutral* if the two consecutive edges at each player $v \in V$ have opposite charges. The *length* $\ell$ of a trail is its number of matches.

Note that the edges along a trail in a $Z$-frame do not repeat, and are connected to each other in alternation by a player/match. Also note that neutral open trails start and end at distinct final matches (the left and rightmost matches along the trail). See Figure 4.9.

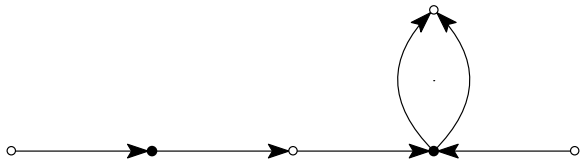


Figure 4.9: An open neutral trail of length $\ell = 4$.

**Lemma 4.4.1.** *Any neutral Z-frame* $\mathbf{Z}$ *can be decomposed into an edge-disjoint union* $\mathbf{Z} = \bigcup_i \mathbf{T}_i$ *of closed neutral trails and open neutral trails, such that no two open trails have a common final match.*

*Proof.* Let $v \in V$. Since $\mathbf{Z}$ is neutral, $\deg_+(v) = \deg_-(v)$. Therefore we may pair each edge directed away from $v$ with an edge directed toward $v$. Note that any such pair is a neutral trail. We let $\mathscr{P}_v$ denote the set of all such pairs. Then $\bigcup_{v \in V} \mathscr{P}_v$ is a decomposition of $\mathbf{Z}$ into an edge-disjoint union of neutral trails. Select a decomposition $\mathbf{Z} = \bigcup_i \mathbf{T}_i$ with the minimal number of neutral trails. Open trails in this decomposition cannot have a common final match, as otherwise they could be concatenated to form a single longer trail, contradicting minimality. ■

Let $\mathbf{Z}$ be a neutral Z-frame. We say that $\mathbf{Z}$ is *reducible* if it contains a non-empty neutral $\mathbf{Z}' \subsetneq \mathbf{Z}$. Otherwise, $\mathbf{Z}$ is *irreducible*.

Each vertex $v$ in a neutral trail has even degree, since $\deg^+(v) = \deg^-(v)$ and $\deg(v) = \deg^+(v) + \deg^-(v)$. As discussed, vertices $v$ in a neutral $\mathbf{Z}$ have even $\deg(v)$. Our next result observes that if $\mathbf{Z}$ is irreducible, then all $\deg(v) \leqslant 4$.

**Lemma 4.4.2.** *Let $\mathbf{Z}$ be an irreducible neutral Z-frame. Then $\mathbf{Z}$ is a neutral trail and $\deg(v) \in \{0,2,4\}$ for all $v \in V$.*

*Proof.* By Theorem 4.4.1, $\mathbf{Z}$ is a neutral trail, and so all $\deg(v)$ are even. If $v$ is an isolated vertex that plays no solitaire games, then $\deg(v) = 0$. Otherwise, if $\mathbf{Z}$ is non-trivial, we will argue that $\deg(v) \in \{2,4\}$ are the only possibilities. To see this, start at any $v$ along the trail, and then follow the trail. Consider the charges of the edges incident to $v$, in the order in which they are visited by the trail. If the first and second charges are opposing, then the trail is complete with $\deg(v) = 2$. Otherwise, suppose they are both positive (the other case is symmetric). Since $\mathbf{Z}$ is neutral, the 3rd charge is negative. Finally, consider the 4th charge. If it were positive, then the sub-trail between the third negative edge incident to $v$ and the forth positive edge incident to $v$ would be neutral. Therefore, the 4th charge is negative, and so the trail is complete with $\deg(v) = 4$. ■

### 4.4.3 Reversing Coxeter tournaments

Applying the results of the previous section, we obtain the following result for Z-frames $\mathbf{Z}(\mathscr{T})$ of Coxeter tournaments $\mathscr{T}$.

**Lemma 9.** *Let $\Phi = B_n$, $C_n$ or $D_n$. Let $\mathbf{Z}(\mathscr{T})$ be the Z-frame of a neutral Coxeter tournament $\mathscr{T}$ on a signed $\Phi$-graph $\mathscr{S} \subset \mathscr{K}_\Phi$. Consider a decomposition $\mathbf{Z}(\mathscr{T}) = \bigcup_i \mathbf{T}_i$ into neutral trails given by Theorem 4.4.1. If $\Phi = C_n$ or $D_n$ then all trails $\mathbf{T}_i$ are closed. If $\Phi = B_n$ then possibly some $\mathbf{T}_i$ are open.*

*Proof.* This result follows by noting that if $\mathscr{T}$ is of type $C_n$ or $D_n$ then all matches in $\mathbf{Z}(\mathscr{T})$ are degree 2. Therefore, there are no open neutral trails in the decomposition, so each $\mathbf{T}_i$ is closed. On the other hand, in type $B_n$ it is possible to have open and closed trails, since in this case possibly some matches are degree 1. ■

Note that a tournament $\mathscr{T}$ on a signed graph $\mathscr{S}$ is neutral if and only if its Z-frame $\mathbf{Z}(\mathscr{T})$ is neutral. Naturally, we call such a $\mathscr{T}$ *irreducible* if $\mathbf{Z}(\mathscr{T})$ is irreducible, and *reducible* otherwise.

**Lemma 10.** *Suppose that $\mathscr{S}$ is a $D_n$-graph in which all players $v \in V$ have degree four. Then any neutral tournament $\mathscr{T}$ on $\mathscr{S}$ is reducible.*

*Proof.* The proof is by contradiction. Suppose that $\mathscr{T}$ on $\mathscr{S}$ is neutral and that $\mathbf{Z}(\mathscr{T})$ is irreducible. By Theorem 4.4.1, $\mathbf{Z}(\mathscr{T})$ is in fact a single neutral trail, which for convenience we will denote by $\mathbf{T}$. Following the consecutive edges of $\mathbf{T}$, we can find a closed trail $\mathbf{T}' \subset \mathbf{T}$ which:

- starts and ends at some player $v$,
- visits no player $u \neq v$ more than once along the way,
- and is neutral everywhere except at $v$.

Let us assume that the two directed edges in $\mathbf{T}'$ incident to $v$ are positive, since the other case is symmetric.

Consider the extension $\mathbf{T}' \cup \mathbf{T}''$ of $\mathbf{T}'$, as it departs $v$ via some negatively charged directed edge, until it eventually returns to some player $u \in \mathbf{T}'$ for the first time. Note that $u \neq v$, as else, since $\mathbf{T}$ is irreducible, it would follow that $\mathbf{T} = \mathbf{T}' \cup \mathbf{T}''$, and then that there are degree 2 vertices in $\mathbf{T}$ along $\mathbf{T}'$. Since $\mathbf{T}'$ is neutral at $u$, the edges in $\mathbf{T}'$ incident to $u$ have opposing charges. Let $\mathbf{T}' = \mathbf{T}'_+ \cup \mathbf{T}'_-$, where $\mathbf{T}'_\pm \subset \mathbf{T}'$ is the trail between $u$ and $v$ which includes the positive/negative edge in $\mathbf{T}'$ incident to $u$, as in Figure 4.10. To conclude, consider the charge of the directed edge in $\mathbf{T}''$ along which $u$ is revisited. To obtain the required contradiction, note that if this charge is positive (resp. negative) then $\mathbf{T}'_- \cup \mathbf{T}''$ (resp. $\mathbf{T}'_+ \cup \mathbf{T}''$) is a neutral trail. ■

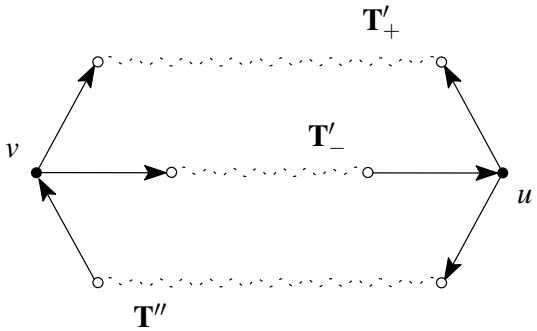



Figure 4.10: In this instance, the directed edge in $\mathbf{T}''$ along which $u$ is revisited is positive, so $\mathbf{T}'_- \cup \mathbf{T}''$ is a neutral trail. If it were negative, then $\mathbf{T}'_+ \cup \mathbf{T}''$ would be neutral. Curved dotted lines represent the continuation of a trail.

Finally, we turn our attention to the main result of this section.

**Lemma 4.4.3** (Reversing lemma)**.** *Let $\Phi = B_n$, $C_n$ or $D_n$. Let $\mathscr{T}$ be a tournament on the complete signed graph $\mathscr{K}_\Phi$, and let $\mathscr{I} \subset \mathscr{T}$ be a neutral sub-tournament. If $\mathscr{I}$ has $\ell \geqslant 3$ games then $\mathscr{I}$ can be reversed in a series of at most $\ell - 2$ type $\Phi$ generator reversals.*

The assumption that $\mathscr{T}$ is on $\mathscr{K}_\Phi$ is crucial, since not all of the games in the generators used to reverse $\mathscr{I}$ will be in $\mathscr{I}$ itself. However, after the series of reversals, only the games in $\mathscr{I}$ will have been reoriented, i.e., all games in $\mathscr{T} \setminus \mathscr{I}$ will be restored to their initial orientations.

*Proof.* Without loss of generality we may assume that $\mathscr{I}$ is irreducible. In this case, by Theorem 4.4.1, the Z-frame $\mathbf{Z}(\mathscr{I})$ is a single neutral trail. For simplicity, we will speak of $\mathscr{I}$ as a tournament and trail interchangeably.

We first address the simplest case of open neutral trails, which appears only in type $B_n$. We proceed by induction on the length $\ell$. The smallest open trails, with $\ell = 3$, are the neutral pairs $\Omega_i$ (as in Figure 4.6) themselves, which are clearly reversible in a single reversal. For longer open neutral trails $\mathscr{I}$, consider any (half edge) solitaire game $h$ in $\mathscr{T}$ played by some $u$ in $\mathscr{I}$, which is not an endpoint of the trail. Using $h$, we can first reverse either the part of the trail that is to the "left" of $u$ or else the part which is to the "right." We can do this using the inductive hypothesis, as one of the parts is neutral and both have length smaller than $\ell$. Then, using the reversal of this game $h^*$, we can reverse the other part of the trail in turn. See Figure 4.11 for an example.

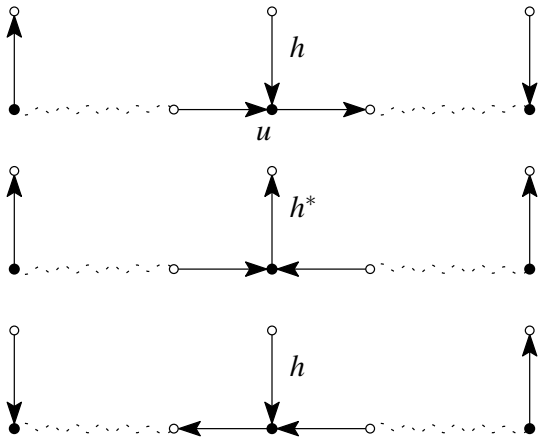



Figure 4.11: In type $B_n$: Reversing an open neutral trail of length $\ell > 3$, using a (half edge) solitaire game $h$ played by some $u$ in the "middle" of the trail. In this example, from top to bottom, we first reverse the "right" side of $\mathscr{I}$ using $h$, and then the "left" side using $h^*$.

Next, we turn to the case of closed neutral trails. Recall that closed trails do not have solitaire half edge games, so from this point on we assume that $\Phi = C_n$ or $D_n$. If $\mathscr{I}$ is a generator, then the result clearly holds. If it is not, then let $k \geqslant 3$ be the number of vertices in $\mathscr{I}$. If $k = 3$, one can verify directly that the result holds. See Figure 4.12.

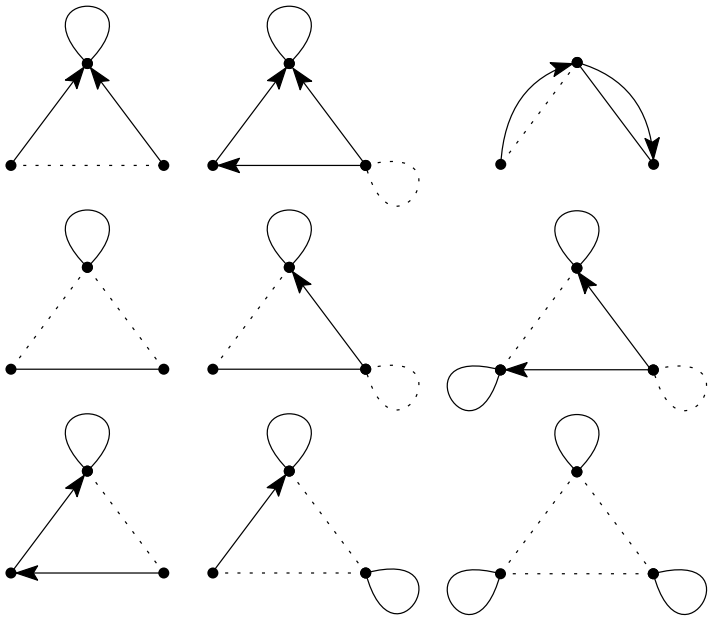


Figure 4.12: Up to symmetry, it suffices to consider the above neutral tournaments on three vertices. Note that each of these tournaments with $\ell$ games can be reversed in $\ell - 2$ steps.

For $k \geqslant 4$, we aim to find a pair of vertices $i$, $j$ in $\mathscr{I}$ which do not play a game with each other in $\mathscr{I}$. Once we find such a pair, the argument is similar to the case of open trails above; the difference being that, in the case of closed trails, we will either first reverse the "top/bottom" (instead of the "left/right") of the trail, and then the other side in turn, as in Figure 4.13.

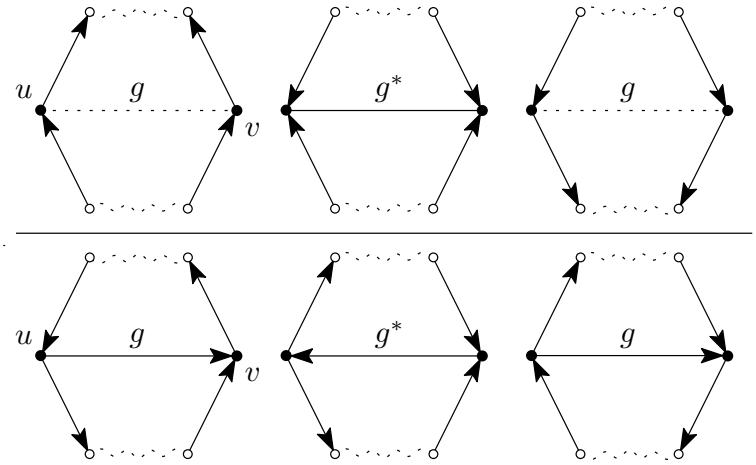



Figure 4.13: Reversing $\mathscr{I}$ when its Z-frame is a closed neutral trail. *Above:* If the charges of the edges incident to $u$ and $v$ are "aligned," we use the collaborative game $g$ between $u, v$ to reverse $\mathscr{I}$. *Below:* Otherwise, if they are "unaligned," we use the competitive game between $u$ and $v$.

To this end, suppose that every pair of vertices in $\mathscr{I}$ plays at least one game with each other. As $k \geqslant 4$, this implies that the degree of every player in $\mathscr{I}$ is at least 3. Moreover, if some $v$ in $\mathscr{I}$ plays a loop game, then $\deg(v) \geqslant 5$. However, as $\mathscr{I}$ is neutral and irreducible, this would

contradict Theorem 4.4.2. Thus, $\mathscr{I}$ is a 4-regular $D_n$-graph. But this is also impossible, by Lemma 10. Therefore, for all $k \geqslant 4$, there exists a pair $i, j$ of players which do not play a game with each other, and this concludes the proof. ■

Finally, using the reversing lemma, we will prove the main result of this section.

*Proof of Theorem 5.* Let $\mathscr{T}, \mathscr{T}' \in \mathrm{Tour}(\Phi, \mathbf{s})$. The distance between $\mathscr{T}$ and $\mathscr{T}'$ is the smallest number of the generator reversals which transforms $\mathscr{T}$ into $\mathscr{T}'$. The games in the difference $\mathscr{D} = \mathscr{T} \setminus \mathscr{T}'$ are precisely those which need to be reversed. Since $\mathbf{s}(\mathscr{T}) = \mathbf{s}(\mathscr{T}')$, it follows that $\mathscr{D}$ is neutral. Therefore, by Theorem 4.4.3, there is a path from $v(\mathscr{T})$ to $v(\mathscr{T}')$ in $\mathrm{IntGr}(\Phi, \mathbf{s})$ of length $O(n^2)$. Hence $\mathrm{IntGr}(\Phi, \mathbf{s})$ is connected and its diameter $D = O(n^2)$. ■

## 4.5 Interchange networks

In this section, for ease of exposition, we will speak of Coxeter tournaments $\mathscr{T}$ and their corresponding vertices $v(\mathscr{T})$ in IntGr interchangeably.

**Definition 11.** For $\mathscr{T}_1, \mathscr{T}_2 \in \mathrm{Tour}(\Phi, \mathbf{s})$ at distance two in $\mathrm{IntGr}(\Phi, \mathbf{s})$, we define the *interchange network* $N(\mathscr{T}_1, \mathscr{T}_2)$ to be the union over all paths of length two between $\mathscr{T}_1, \mathscr{T}_2$.

Note that each path of length two between such $\mathscr{T}_1, \mathscr{T}_2$ corresponds to a way of reversing the *difference* $\mathscr{D} = \mathscr{T}_1 \setminus \mathscr{T}_2$ between $\mathscr{T}_1$ and $\mathscr{T}_2$.

### 4.5.1 Classifying networks

In this section, we will classify the possibilities for $(N, \mathscr{D})$. As we will see, this is the key to applying path coupling (Theorem 3 above) in Section 4.6 below. In the classical case of type $A_{n-1}$ there is only one possibility for $N$ (a "single diamond"), and this is the reason why such a simple contractive coupling (as in Figure 4.27 below) is possible. As it turns out, this continues to hold in types $B_n$ and $D_n$, but the underlying reasons are more complicated. Type $C_n$, on the other hand, is significantly more complex, as then the structure of $N$ can take various other forms.

It can be seen that any two distinct generators $\mathscr{G}_1 \neq \mathscr{G}_2$ are either *disjoint* $\mathscr{G}_1 \cap \mathscr{G}_2 = \emptyset$ or else have exactly one game $g$ in common $\mathscr{G}_1 \cap \mathscr{G}_2 = \{g\}$. In this case, we say that $\mathscr{G}_1, \mathscr{G}_2$ are *adjacent*.

Note that if a path of length two from $\mathscr{T}_1$ to $\mathscr{T}_2$ passes through some $\mathscr{T}_{12}$, then there are two generators $\mathscr{G}_1, \mathscr{G}_2 \subset \mathscr{T}_{12}$ such that $\mathscr{T}_i = \mathscr{T}_{12} * \mathscr{G}_i$, for $i \in \{1, 2\}$. In this way, every such path of length two from $\mathscr{T}_1$ to $\mathscr{T}_2$ is determined by a *midpoint* $\mathscr{T}_{12}$ and a pair of generators $\mathscr{G}_1, \mathscr{G}_2 \subset \mathscr{T}_{12}$.

There are three possible networks when $\mathscr{G}_1, \mathscr{G}_2$ are disjoint. We call these the *single*, *double* and *quadruple diamonds*. See Figure 4.14. Recall that double edges in IntGr correspond to

neutral clover reversals. All other types of reversals (neutral triangles and pairs) are represented as single edges.

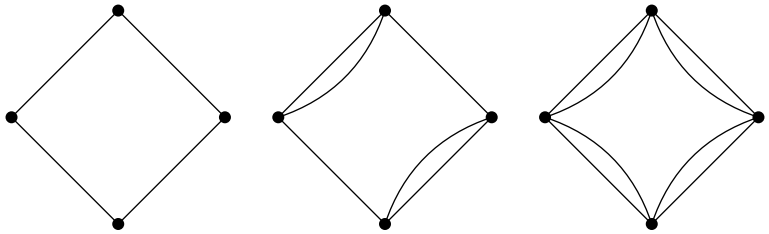


Figure 4.14: *Left to right:* The single, double and quadruple diamond interchange networks.

The following result classifies the types of networks $N(\mathscr{T}_1, \mathscr{T}_2)$ when $\mathscr{G}_1, \mathscr{G}_2$ are disjoint.

**Lemma 4.5.1.** *Suppose that there is a path of length two between $\mathscr{T}_1, \mathscr{T}_2$ in* $\mathrm{IntGr}(\Phi, \mathbf{s})$ *that passes through midpoint $\mathscr{T}_{12}$, with associated generators $\mathscr{G}_i \subset \mathscr{T}_{12}$ such that $\mathscr{T}_i = \mathscr{T}_{12} * \mathscr{G}_i$. Suppose that $\mathscr{G}_1, \mathscr{G}_2$ are disjoint. Then if exactly zero, one or two of the $\mathscr{G}_i$ are clovers then the network $N(\mathscr{T}_1, \mathscr{T}_2)$ is a single, double or quadruple diamond, respectively.*

*Proof.* Clearly, there are exactly two paths from $\mathscr{T}_1$ to $\mathscr{T}_2$. These paths correspond to reversing the disjoint generators $\mathscr{G}_1^*, \mathscr{G}_2 \subset \mathscr{T}_1$ in series, in one of the two possible orders. See Figure 4.15. ■

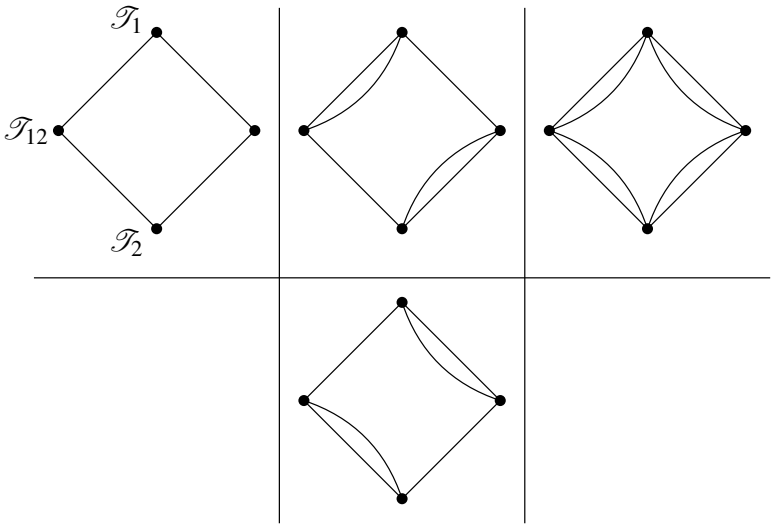



Figure 4.15: *Left to right:* $N(\mathscr{T}_1, \mathscr{T}_2)$ when exactly zero, one or two of the disjoint $\mathscr{G}_1, \mathscr{G}_2$ is a clover (single, double and quadruple diamonds). In each cell, the leftmost vertices are $\mathscr{T}_1$, $\mathscr{T}_{12}$ and $\mathscr{T}_2$, as in the first cell.

The case that $\mathscr{G}_1, \mathscr{G}_2$ are adjacent is more involved. It is useful to note that, in this case, the difference $\mathscr{D} = \mathscr{T}_1 \setminus \mathscr{T}_2$ is a neutral tournament with exactly four games on either three or four vertices. Even so, there are a number of cases to consider, and the key to a concise argument is

grouping symmetric cases together. For a Coxeter tournament $\mathscr{T}$, we define its *projection graph* $\pi(\mathscr{T})$ to be the graph obtained by changing each:

- oriented negative/positive edge (i.e., competitive/collaborative game) into an undirected edge,
- oriented half edge (i.e., half edge solitaire game) into an undirected half edge,
- oriented loop (i.e., loop solitaire game) into an undirected loop.

There are a number of ways that two generators $\mathscr{G}_1, \mathscr{G}_2$ can be adjacent. However, there are only four possibilities for their projected difference $\pi(\mathscr{D})$. We call these the *square*, *tent*, *fork* and *hanger*, as in Figure 4.16. The following observation is essentially self-evident, and can be verified by an elementary case analysis. We omit the proof.

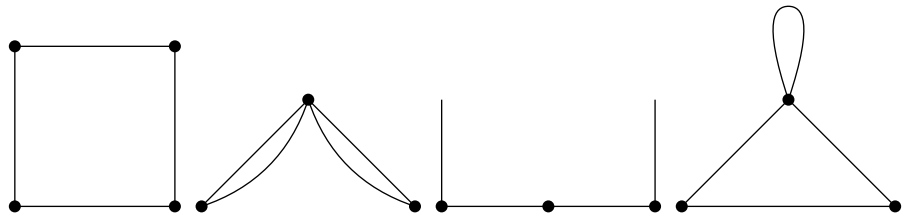


Figure 4.16: *Left to right:* The square, tent, fork, and hanger.

**Lemma 4.5.2.** *Assume the same set up as Theorem 4.5.1, except that instead $\mathscr{G}_1, \mathscr{G}_2$ are adjacent. Let $\mathscr{D} = \mathscr{T}_1 \setminus \mathscr{T}_2$. Then:*

1. *If $\mathscr{G}_1, \mathscr{G}_2$ are neutral triangles on four/three vertices then $\pi(\mathscr{D})$ is a square/tent.*
2. *If $\mathscr{G}_1, \mathscr{G}_2$ are neutral clovers then $\pi(\mathscr{D})$ is a tent.*
3. *If $\mathscr{G}_1$ is a neutral pair and $\mathscr{G}_2$ is a neutral pair or triangle then $\pi(\mathscr{D})$ is a fork.*
4. *If $\mathscr{G}_1$ is a neutral clover and $\mathscr{G}_2$ is a neutral triangle then $\pi(\mathscr{D})$ is a hanger.*

In addition to the single and double diamond networks in Figure 4.14, there are two additional networks that can occur when $\mathscr{G}_1, \mathscr{G}_2$ are adjacent. We call these the *split* and *heavy diamonds*, see Figure 4.17.

The following result classifies the types of networks $N(\mathscr{T}_1, \mathscr{T}_2)$ when $\mathscr{G}_1, \mathscr{G}_2$ are adjacent.

**Lemma 4.5.3.** *Assume the same set up as Theorem 4.5.2 (with $\mathscr{G}_1, \mathscr{G}_2$ are adjacent). Let $\mathscr{D} = \mathscr{T}_1 \setminus \mathscr{T}_2$ and $N = N(\mathscr{T}_1, \mathscr{T}_2)$. Then:*

1. *If $\Phi = B_n$ or $D_n$, then $N$ is a single diamond.*

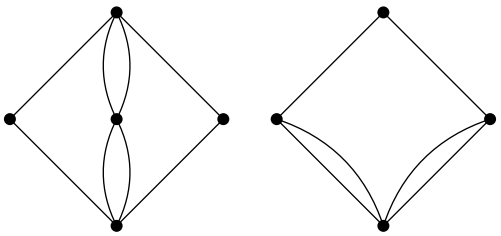


Figure 4.17: *Left to right:* The split and heavy diamond interchange networks.

2. *If $\Phi = C_n$ and $\pi(\mathscr{D})$ is a square, then $N$ is a single diamond.*

3. *If $\Phi = C_n$ and $\pi(\mathscr{D})$ is a tent, then $N$ is a split diamond.*

4. *If $\Phi = C_n$ and $\pi(\mathscr{D})$ is a hanger, then $N$ is a double or heavy diamond.*

*Proof.* **Case 1a.** We start with the simplest case that $\Phi = B_n$ or $D_n$ and $\mathscr{G}_1, \mathscr{G}_2$ are adjacent neutral triangles on four vertices, so that $\pi(\mathscr{D})$ is a square. Note that, to reverse $\mathscr{D}$ in two steps, we must reverse exactly two edges in $\mathscr{D}$ in each step. As such, no neutral pairs will be involved in reversing $\mathscr{D}$ in two steps. There are exactly two ways to reverse $\mathscr{D}$. For each pair of "antipodal" vertices in $\mathscr{D}$, consider the two games played between the pair. Exactly one of the two games $g$ allows us to reverse the games in $\mathscr{D}$ on one "side" of $g$. Then, in turn, we can use $g^*$ to reverse the other two games in $\mathscr{D}$. See Figure 4.18 for all the possible cases of $\mathscr{D}$.

**Case 1b.** Suppose that $\Phi = B_n$ or $D_n$ and $\mathscr{G}_1, \mathscr{G}_2$ are adjacent neutral triangles on three vertices, so that $\pi(\mathscr{D})$ is a tent. In this case, both of the (competitive and collaborative) games between the "base" vertices lead to a way of reversing $\mathscr{D}$. See Figure 4.19.

**Case 1c.** Suppose that $\Phi = B_n$ and that one of $\mathscr{G}_1, \mathscr{G}_2$ is a neutral pair and the other is an adjacent neutral pair or triangle, so that $\pi(\mathscr{D})$ is a fork. Then the half edge game played by the "middle" vertex and exactly one of the games between the "base" vertices lead to ways of reversing $\mathscr{D}$. See Figure 4.20.

By Cases 1a–c, statement (1) follows, that is, in types $B_n$ and $D_n$ the network $N$ is always a single diamond, as in Figure 4.21.

**Case 2.** In type $C_n$, the case that $\pi(\mathscr{D})$ is a square follows by the same argument as in types $B_n$ and $D_n$. Indeed, recall that any reversal of $\mathscr{D}$ in two steps will involve reversing exactly two games in $\mathscr{D}$ in each step. Therefore, no clovers will be involved in such a reversal of $\mathscr{D}$, and so once again $N$ is a single diamond, yielding statement (2).

**Case 3a.** Suppose that $\Phi = C_n$ and that $\pi(\mathscr{D})$ is a tent formed by two adjacent neutral triangles on three vertices. Then by Case 1b, $N$ contains a single diamond. However, using the loop game $\ell$ played by the "middle" vertex, we obtain an additional path of length two between $\mathscr{T}_1, \mathscr{T}_2$. We can use $\ell$ to reverse the two games on one "side" of the tent. Then, in turn, we can use $\ell^*$ to reverse the other two games. See Figure 4.22. Hence $N$ is a split diamond in this case.

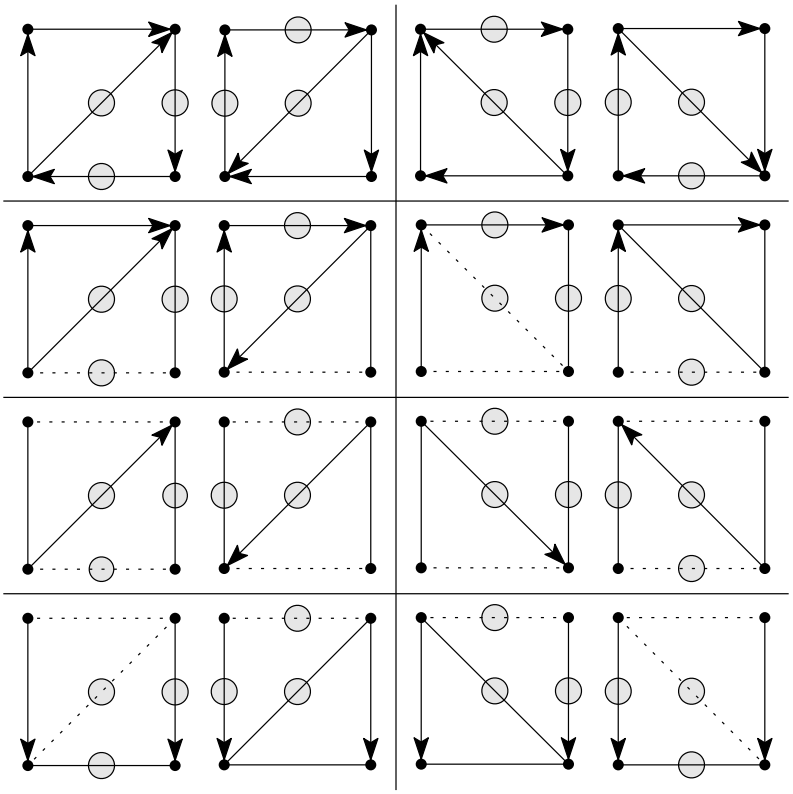


Figure 4.18: Reversing when $\pi(\mathscr{D})$ is a square in types $B_n$, $C_n$ or $D_n$. Each row corresponds to one of the four possible configurations of $\mathscr{D}$. In each cell, the shaded circles indicate how to reverse $\mathscr{D}$ in two steps, using one of the games between an "antipodal" pair of players along $\mathscr{D}$. Note that each row has exactly two cells, as there is always exactly two ways to reverse $\mathscr{D}$ in two steps. The "middle" game (not in the square itself) is reversed twice, so returned to its original orientation.

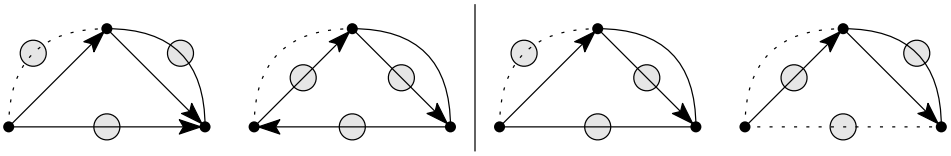


Figure 4.19: Reversing when $\pi(\mathscr{D})$ is a tent formed by two neutral triangles in types $B_n$ and $D_n$. This figure has only one row, as there is only one possibility for $\mathscr{D}$.

**Case 3b.** Suppose that $\Phi = C_n$ and that $\pi(\mathscr{D})$ is a tent formed by two adjacent neutral clovers. Then $\mathscr{D}$ is on three vertices, and so by Case 1b, we find that $N$ is a split diamond, once again.

By Cases 3a–b, statement (3) follows. The difference between Cases 3a and 3b are depicted in the first column of Figure 4.23.

**Case 4.** Finally, suppose that $\Phi = C_n$ and that $\pi(\mathscr{D})$ is a hanger. We will argue that this case corresponds to the second and third columns in Figure 4.23.

Note that, in this case, exactly one of $\mathscr{G}_1, \mathscr{G}_2$ is a neutral clover and the other is an adjacent neutral triangle. Suppose that the loop game $\ell$ in $\mathscr{D}$ is played by vertex $x$ and that the other two

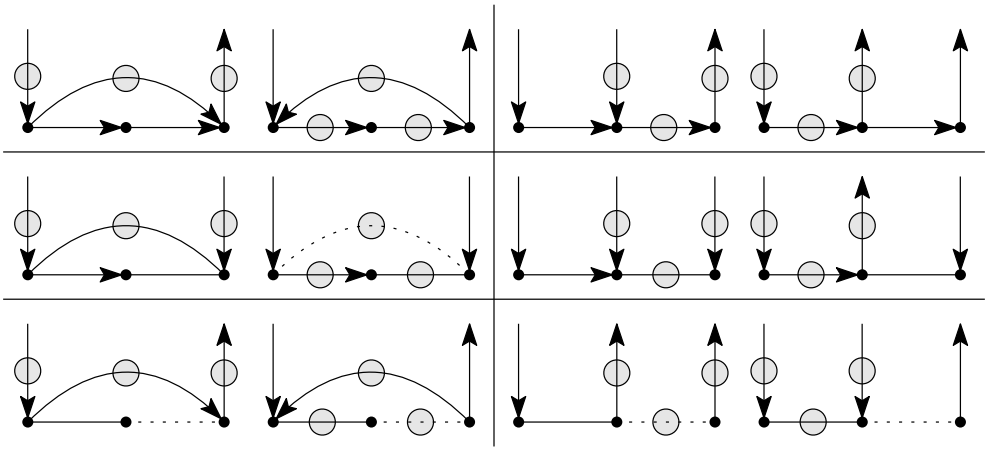


Figure 4.20: Reversing when $\pi(\mathscr{D})$ is a fork in type $B_n$. Each row corresponds to a possible configuration for $\mathscr{D}$.

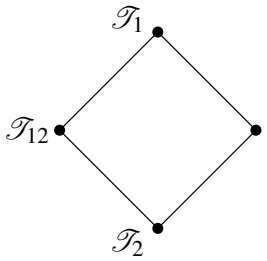



Figure 4.21: When $\mathscr{G}_1, \mathscr{G}_2$ are adjacent, the only possible $N(\mathscr{T}_1, \mathscr{T}_2)$ in types $B_n$ and $D_n$ is a diamond.

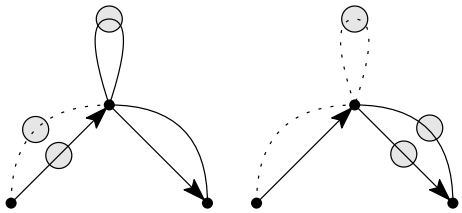


Figure 4.22: The additional way of reversing when $\pi(\mathscr{D})$ is a tent in type $C_n$.

vertices in $\mathscr{D}$ are $y, z$. Note that any way of reversing $\mathscr{D}$ in two steps will involve reversing $\ell$ exactly once, and so each path of length two from $\mathscr{T}_1$ to $\mathscr{T}_2$ will contain exactly one double edge.

The four cases in the second and third columns in Figure 4.23 can be seen by considering the other games played between $x, y$ and $x, z$ that are not in $\mathscr{D}$. Depending on their outcomes, each such game either creates a clover with loop $\ell$ at $x$ or else forms a neutral triangle together with the two "opposite" games in $\mathscr{D}$. After this clover/triangle is reversed, the triangle/clover, which was not initially, becomes present. See Figure 4.24.

The proof is complete. ■

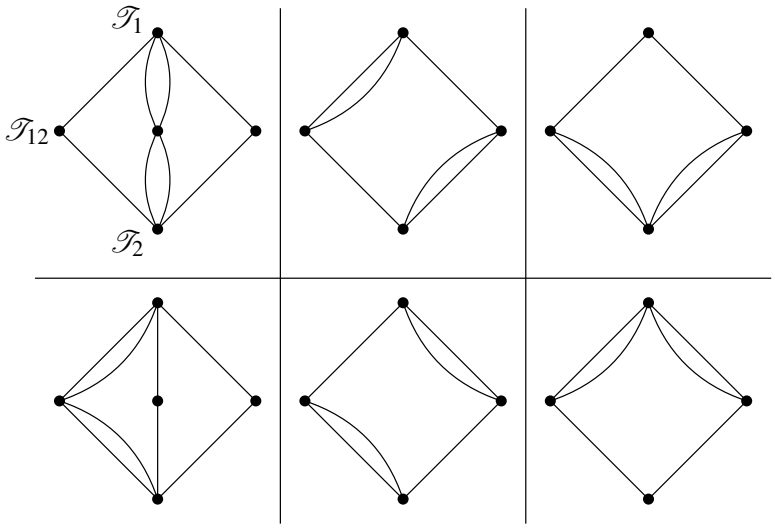



Figure 4.23: The additional (other than the single diamond) possible $N(\mathscr{T}_1, \mathscr{T}_2)$ in type $C_n$, when $\mathscr{G}_1, \mathscr{G}_2$ are adjacent. *From left to right:* Split, double and heavy diamonds. In each cell, the leftmost vertices are $\mathscr{T}_1$, $\mathscr{T}_{12}$ and $\mathscr{T}_2$, as in the first cell.

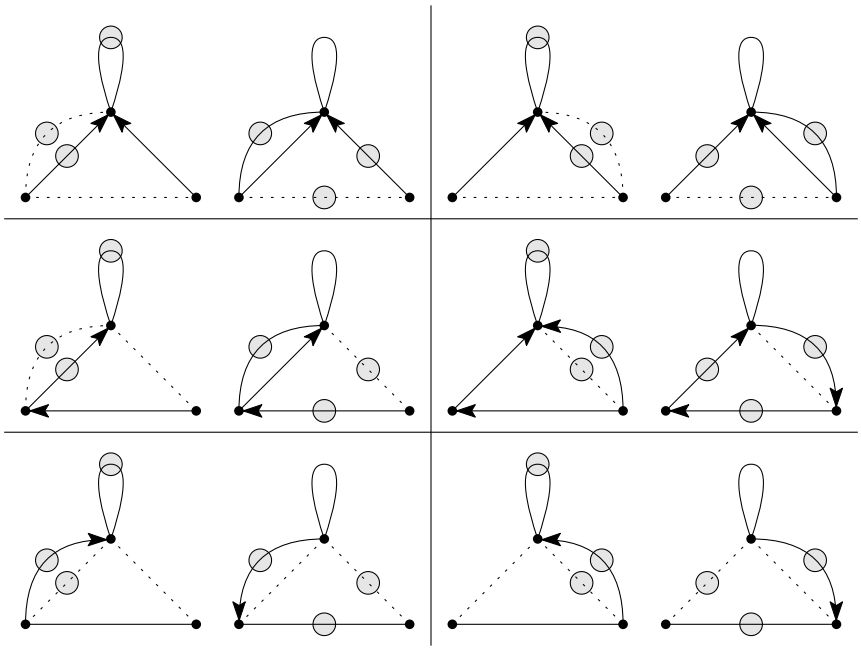


Figure 4.24: Reversing when $\pi(\mathscr{D})$ is a hanger in type $C_n$. Each row corresponds to a possible configuration for $\mathscr{D}$.

### 4.5.2 Extended networks

Theorems 4.5.1 and 4.5.3 above classify the types of interchange networks $N(\mathscr{T}_1, \mathscr{T}_2)$. Recall that such a network contains all paths of length two between $\mathscr{T}_1, \mathscr{T}_2$.

**Definition 12.** We define the *extended interchange network* $\hat{N}(\mathscr{T}_1, \mathscr{T}_2)$ to be the union of $N(\mathscr{T}_1', \mathscr{T}_2')$ over all "antipodal" pairs $\mathscr{T}_1', \mathscr{T}_2'$ in $N(\mathscr{T}_1, \mathscr{T}_2)$ at distance two.

Single, double and quadruple diamonds are "stable," in the sense that $\hat{N} = N$. In contrast,

split and heavy diamond networks extend to a type of structure, which we call a *crystal*. See Figure 4.25.

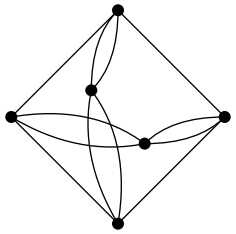


Figure 4.25: The crystal extended interchange network.

*Remark* 4.5.4. All of the interchange networks that we have described, except the single diamond, can be found in Figure 4.3 above. This demonstrates how interchange networks can overlap and mesh together to form the interchange graph of a given score sequence.

**Lemma 4.5.5.** *Suppose that $\mathscr{T}_1, \mathscr{T}_2$ are at distance two in* $\mathrm{IntGr}(\Phi, \mathbf{s})$. *Let $N = N(\mathscr{T}_1, \mathscr{T}_2)$ and $\hat{N} = \hat{N}(\mathscr{T}_1, \mathscr{T}_2)$.*

1. *If $N$ is a single, double or quadruple diamond, then the extended interchange network $\hat{N} = N$.*

2. *Otherwise, if $N$ is a split or heavy diamond, then the extended interchange network $\hat{N}$ is a crystal.*

*Proof.* Statement (1) is clear, and can be seen by inspection. On the other hand, statement (2) follows by repeated application of Theorem 4.5.3, considering the various antipodal pairs in $N$.

**Case 1.** If $N$ is a split diamond, as in the first column of Figure 4.23, then consider $\mathscr{T}_1', \mathscr{T}_2'$ in $N$ that are incident to only single edges in $N$. By Theorem 4.5.3, it follows that $N(\mathscr{T}_1', \mathscr{T}_2')$ is a split diamond, and therefore $\hat{N}$ is a crystal.

**Case 2.** If $N$ is a heavy diamond, as in the third column of Figure 4.23, then consider $\mathscr{T}_1', \mathscr{T}_2'$ in $N$, each of which incident to exactly one single edge and one double edge in $N$. By Theorem 4.5.3, it follows that $N(\mathscr{T}_1', \mathscr{T}_2')$ is a split diamond, and therefore there is a path of length two between them consisting of two single edges in $\hat{N}$. Then, applying Theorem 4.5.3, once again, but this time to the midpoint $\mathscr{T}_{12}'$ along this path and $\mathscr{T}_3$ in $N$ that is incident to two single edges in $N$, we find that $\hat{N}$ is a crystal, as claimed. ■

Recall that two generators are either disjoint or have exactly one game in common. A similar property holds for extended networks.

**Lemma 4.5.6.** *Any two distinct extended networks $\hat{N} \neq \hat{N}'$ are either edge-disjoint or have exactly one single or double edge in common.*

*Proof.* By Theorem 4.5.5 there are only four types of extended networks. By an elementary case analysis, it can be seen that any two antipodal (distance two) vertices $\mathscr{T}_1, \mathscr{T}_2$ in an extended network $\hat{N}$ give rise to the same extended network $\hat{N}$. That is, $\hat{N} = \hat{N}(\mathscr{T}_1, \mathscr{T}_2)$, for any such $\mathscr{T}_1, \mathscr{T}_2$. From this observation the result follows, since if two extended networks $\hat{N}, \hat{N}'$ share at least three vertices, then they necessarily have at least one antipodal pair of vertices in common. ■

### 4.5.3 Properties of crystals

In this section, we obtain two key properties of crystals, which will play a crucial role in the type $C_n$ couplings discussed in Section 4.6.2 below.

First, we note that crystals cannot share a single edge. We will use this, together with Theorem 4.5.6, to extend natural couplings on networks to a coupling on the full interchange graph.

**Lemma 4.5.7.** *Suppose that $\hat{N} \neq \hat{N}'$ are distinct crystals in an interchange network* $\mathrm{IntGr}(C_n, \mathbf{s})$. *Then $\hat{N}$, $\hat{N}'$ are either edge-disjoint or share a double edge. That is, no such $\hat{N} \neq \hat{N}'$ share a single edge.*

*Proof.* By the proof of Theorems 4.5.3 and 4.5.5, it can be seen that each crystal is associated with three players. Each double edge in the crystal corresponds to reversing a neutral clover involving two of them, and each single edge corresponds to reversing a neutral triangle involving all three.

By Theorem 4.5.6, it suffices to show that two crystals $\hat{N} \neq \hat{N}'$ cannot share a single edge. To see this, simply note that otherwise both tournaments joined by this single edge would contain a tournament on three players with three neutral triangles, which is impossible. See Figure 4.26. ■

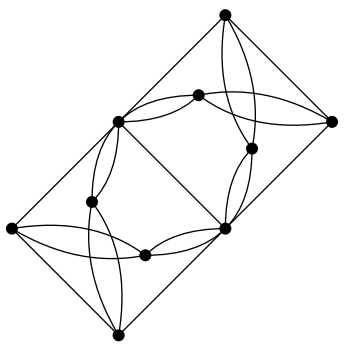


Figure 4.26: It is impossible for two (distinct) crystals to share a single edge, as depicted above, since there are at most two neutral triangles on any given three players in a tournament. However, each of the two "middle" vertices in this figure are incident to three single edges.

The previous result shows that single edges can be in at most one crystal. Double edges, on the other hand, can be in more than one. The following result gives an upper bound on this number, which is related to the re-weighting $w$ of the graph metric, discussed in Section 4.6.2, under which our coupling will be contractive.

Recall that $d = d(C_n, \mathbf{s})$ is the degree of $\mathrm{IntGr}(C_n, \mathbf{s})$.

**Lemma 4.5.8.** *Any given double edge in an interchange network* $\mathrm{IntGr}(C_n, \mathbf{s})$ *is contained in at most* $\min\{d, 2n\}$ *crystals.*

*Proof.* Consider a double edge between some $\mathscr{T}_1, \mathscr{T}_2$. By Theorem 4.5.6, each crystal containing it corresponds to an additional double edge or two single edges incident to $\mathscr{T}_1$. It follows that there are at most $(d-2)/2$ such crystals.

The second bound is somewhat more complicated. Recall, as noted in the proof of Theorem 4.5.7, that each crystal is associated with three players. Suppose that a double edge between some $\mathscr{T}_1, \mathscr{T}_2$ is associated with a neutral clover involving players $i, j$. We claim that, for any other player $k$, there are at most two crystals associated with $i, j, k$. To see this, observe that, if there were three, then one of $\mathscr{T}_1, \mathscr{T}_2$ would be incident to four single edges in these crystals. However, this would imply that in one of $\mathscr{T}_1, \mathscr{T}_2$ there are four neutral triangles on $i, j, k$, which is impossible. Indeed, as noted in the proof of Theorem 4.5.7, there can be at most two. Therefore, there are at most $2(n-2)$ crystals containing any given double edge. (In fact, the upper bound $n-2$ can be proved, but involves a more careful analysis.) ■

# 4.6 Rapid mixing

Using the results of the previous section, we show that simple random walk on any given $\mathrm{IntGr}(\Phi, \mathbf{s})$ is rapidly mixing. The idea is to first define couplings on extended networks $\hat{N}$. We then argue that these couplings are compatible, and extend to a full coupling.

In types $B_n$ and $D_n$, rapid mixing then follows by Theorem 3, using the standard weighting $w = \delta$ given by the graph distance $\delta$ in $\mathrm{IntGr}(\Phi, \mathbf{s})$. In type $C_n$, we will need to select a special re-weighting $w \neq \delta$, accounting for the presence of crystals in the interchange graphs of this type.

For a tournament $\mathscr{T} \in \mathrm{Tour}(\Phi, \mathbf{s})$, we let $\mathscr{E}(\mathscr{T})$ denote the set of edges in $\mathrm{IntGr}(\Phi, \mathbf{s})$ incident to $v(\mathscr{T})$.

## 4.6.1 Coupling in $B_n$ and $D_n$

We begin with the simplest cases of types $B_n$ and $D_n$. These types are the most straightforward, since then all networks $N(\mathscr{T}_1, \mathscr{T}_2)$ are single diamonds, and no re-weighting of the graph metric is necessary.

**Theorem 13.** *Let $\Phi = B_n$ or $D_n$. Fix any $\mathbf{s} \in \mathrm{Score}(\Phi)$. Then lazy simple random walk $(\mathscr{T}_n : n \geqslant 0)$ on the Coxeter interchange graph $\mathrm{IntGr}(\Phi, \mathbf{s})$ is rapidly mixing in time $t_{\mathrm{mix}} = O(d \log n)$.*

*Proof.* Let $\Phi = B_n$ or $D_n$. Consider two copies of lazy simple random walk $(\mathscr{T}'_n)$ and $(\mathscr{T}''_n)$ on $\mathrm{IntGr}(\Phi, \mathbf{s})$, started from neighbouring $\mathscr{T}'_0, \mathscr{T}''_0 \in \mathrm{Tour}(\Phi, \mathbf{s})$. Then $\mathscr{T}''_0 = \mathscr{T}'_0 * \mathscr{G}$ for some type $\Phi$ generator $\mathscr{G} \subset \mathscr{T}'_0$. In this sense, the random walks start at distance 1.

We will construct a contractive coupling of $\mathscr{T}'_1, \mathscr{T}''_1$, such that the expected distance between $\mathscr{T}'_1, \mathscr{T}''_1$ is strictly less than 1, for every choice of $\mathscr{T}'_0, \mathscr{T}''_0$. In fact, in this coupling, the $\mathscr{T}'_1, \mathscr{T}''_1$ will coincide with probability $1/d$, and otherwise remain at distance 1.

More specifically, we couple $\mathscr{T}'_1, \mathscr{T}''_1$ using the natural bijection $\psi$ from $\mathscr{E}(\mathscr{T}'_0)$ to $\mathscr{E}(\mathscr{T}''_0)$, which fixes the edge $\{\mathscr{T}'_0, \mathscr{T}''_0\}$ and pairs "opposite" edges in each single diamond containing $\{\mathscr{T}'_0, \mathscr{T}''_0\}$. By Theorems 4.5.1 and 4.5.3, for each type $\Phi$ generator $\mathscr{G}' \neq \mathscr{G} \subset \mathscr{T}'_0$, the network $N(\mathscr{T}''_0, \mathscr{T}'_0 * \mathscr{G}')$ is a single diamond. As such, there are exactly two paths of length two from $\mathscr{T}''_0$ to $\mathscr{T}'_0 * \mathscr{G}'$. One such path passes through $\mathscr{T}'_0$. We let

$$\psi(\{\mathscr{T}'_0, \mathscr{T}'_0 * \mathscr{G}'\}) = \{\mathscr{T}''_0, \mathscr{T}''_0 * \mathscr{G}''\} \tag{4.3}$$

be the first edge along the other such path. By Theorem 1 and Theorems 4.5.5 and 4.5.6, $\psi$ is a bijection from $\mathscr{E}(\mathscr{T}'_0)$ to $\mathscr{E}(\mathscr{T}''_0)$.

Finally, we define the coupling of $\mathscr{T}'_1, \mathscr{T}''_1$ as follows. Let $\{\mathscr{T}'_0, \mathscr{T}'_0 * \mathscr{G}'\}$ be a uniformly random edge in $\mathscr{E}(\mathscr{T}'_0)$ and $r_0$ a Bernoulli$(1/2)$ random variable (i.e., a fair "coin flip"). If $\mathscr{G}' = \mathscr{G}$, we put $\mathscr{T}'_1 = \mathscr{T}''_1 = \mathscr{T}'_0$ if $r_0 = 0$ and $\mathscr{T}'_1 = \mathscr{T}''_1 = \mathscr{T}''_0$ if $r_0 = 1$. In this case, the coupling contracts. On the other hand, if $\mathscr{G}' \neq \mathscr{G}$ we put $\mathscr{T}'_1 = \mathscr{T}'_0$ and $\mathscr{T}''_1 = \mathscr{T}''_0$ if $r_0 = 0$, and $\mathscr{T}'_1 = \mathscr{T}'_0 * \mathscr{G}'$ and $\mathscr{T}''_1 = \mathscr{T}''_0 * \mathscr{G}''$ if $r_0 = 1$, where $\mathscr{G}''$ is given by the bijection $\psi$ in (4.3). See Figure 4.27.

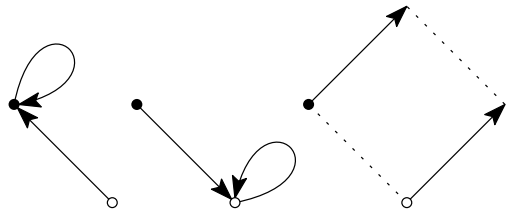


Figure 4.27: A contractive coupling in $B_n$ and $D_n$. The black and white vertices represent the starting positions $\mathscr{T}'_0, \mathscr{T}''_0$. In the configuration at right, corresponding to $\mathscr{G}' \neq \mathscr{G}$, the loops have been omitted, since in this case the walks are either both lazy or not.

In this coupling, $\mathscr{T}'_1 = \mathscr{T}''_1$ with probability $1/d$. Otherwise, they remain at distance 1. By Theorem 5, the diameter of $\mathrm{IntGr}(\Phi, \mathbf{s})$ is $D = O(n^2)$. Therefore, by Theorem 3, the mixing time is bounded by $O(d \log n)$. ∎

*Remark* 4.6.1. Rapid mixing for classical (type $A_{n-1}$) tournaments follows as a special case of the argument above.

### 4.6.2 Coupling in $C_n$

Finally, we investigate mixing in type $C_n$.

Recall that, in types $B_n$ and $D_n$, the coupling was determined by an edge pairing, given by a bijection $\psi$ from $\mathscr{E}(\mathscr{T}_0')$ to $\mathscr{E}(\mathscr{T}_0'')$, where $\mathscr{T}_0', \mathscr{T}_0''$ are neighbouring tournaments in $\mathrm{IntGr}(\Phi, \mathbf{s})$. The coupling in type $C_n$ is also determined by such a $\psi$, however, since there are a number of different networks in type $C_n$, the pairing is more involved.

The fact (see Theorem 4.5.7) that distinct crystals cannot share a single edge is crucial. Otherwise it would not be possible to extend couplings on extended networks to a full coupling. Roughly speaking, this is because (see Case 1b in the proof of Theorem 14 below) single edges $\{\mathscr{T}_0', \mathscr{T}_0''\}$ in a crystal will need to be paired with one of the edges in a double edge of the same crystal. As such, if there were two crystals with the same single edge, a bijective pairing would not be possible.

Furthermore, there is an additional complication in type $C_n$. As it turns out, the pairing $\psi$ does not lead to a contractive coupling, with respect to the graph distance in $\mathrm{IntGr}(C_n, \mathbf{s})$. The problem concerns the case that the initial starting positions $\mathscr{T}_0', \mathscr{T}_0''$ are joined by a single edge in a crystal. In this case, the natural coupling is only "neutral" (i.e., with $\alpha = 0$ in Theorem 3), rather than contractive.

There are (at least) three ways to overcome this difficulty, leading to increasingly better bounds on the mixing time. The first way is to apply Bordewich and Dyer's [15] path coupling without contraction, leading to an upper bound $O(dn^4) = O(n^7)$. It is also possible to apply path coupling at time $t = 2$, since at this point the coupling (with respect to the usual graph metric) becomes contractive. In doing so, the key is to observe that, in the problematic case that $\mathscr{T}_0', \mathscr{T}_0''$ are joined by a single edge in a crystal, if $\mathscr{T}_1'$ stays within the crystal then the pairing $\psi$ (described below) selects a $\mathscr{T}_1''$ in the crystal such that $\mathscr{T}_1', \mathscr{T}_1''$ are now joined by a double edge. This argument leads to an upper bound of $O(d^2 \log n) = O(n^6 \log n)$.

We will present a third strategy, by re-weighting the metric, which yields a better bound.

Recall (see Theorem 4.5.7) that single edges in $\mathrm{IntGr}(C_n, \mathbf{s})$ are contained in at most one crystal. Double edges, on the other hand, can be contained in more than one. We define the *crystal degree* of a double edge to be the number of crystals containing it. We let $\gamma = \gamma(C_n, \mathbf{s})$ denote the *maximal crystal degree,* over all double edges. By Theorem 4.5.8, we have that $\gamma \leqslant \min\{d, 2n\}$, where $d = d(C_n, \mathbf{s})$ is the degree of $\mathrm{IntGr}(C_n, \mathbf{s})$.

We will assume throughout that $\gamma > 0$. Indeed, if $\gamma = 0$, then there are no crystals in the interchange graph. In this case, a straightforward modification of the proof of Theorem 13 (using the standard graph metric) shows that $t_{\mathrm{mix}} = O(d \log n)$.

We will prove the following result, using the weighting $w$ that puts $w = 1$ on each edge in a double edge and $w = 1 + 1/\gamma$ on each single edge.

To be clear, this choice of $w$ re-weights the graph distance between neighbouring vertices joined by single edges, but not those joined by double edges. Specifically, if $u, v$ are joined by a single edge then $w(u,v) = 1 + 1/\gamma$, and if $u, v$ are joined by a double edge then each edge is given weight 1, and so the weighted distance between $u, v$ remains $w(u,v) = 1$ (see Definition 2).

**Theorem 14.** *Let $\Phi = C_n$. Fix any $\mathbf{s} \in \mathrm{Score}(\Phi)$. Then lazy simple random walk $(\mathscr{T}_n : n \geqslant 0)$ on the Coxeter interchange graph $\mathrm{IntGr}(C_n, \mathbf{s})$ is rapidly mixing. If there are no crystals in $\mathrm{IntGr}(C_n, \mathbf{s})$ (when $\gamma = 0$) then $t_{\mathrm{mix}} = O(d \log n)$. Otherwise, we have $t_{\mathrm{mix}} = O(\gamma d \log n)$.*

In particular, this result implies that $t_{\mathrm{mix}} = O(n^4 \log n)$.

*Proof.* As discussed, let us assume that $\gamma > 0$, as otherwise a simple adaptation of the general reasoning in types $B_n$ and $D_n$ (the proof of Theorem 13) shows that $t_{\mathrm{mix}} = O(d \log n)$.

Consider two copies of lazy simple random walk $(\mathscr{T}'_n)$ and $(\mathscr{T}''_n)$ on $\mathrm{IntGr}(C_n, \mathbf{s})$, started from neighbours $\mathscr{T}'_0, \mathscr{T}''_0 \in \mathrm{Tour}(C_n, \mathbf{s})$. Let $\mathscr{G}$ be such that $\mathscr{T}''_0 = \mathscr{T}'_0 * \mathscr{G}$.

The first step is to obtain an edge pairing $\psi$, which will associate a $\mathscr{T}''_1$ to each possible $\mathscr{T}'_1$. Then we will show that this coupling is contractive, with respect to the re-weighting $w = 1 + 1/\gamma$ on single edges and $w = 1$ on each edge in double edges. The key in this regard will be the classification of extended networks, established in Section 4.5.

By Theorem 4.5.7, there are three cases to consider:

1. $\mathscr{G} = \Delta$ is a neutral triangle, and the edge $\{\mathscr{T}'_0, \mathscr{T}''_0\}$ is in
   (a) no crystal,
   (b) exactly one crystal.
2. $\mathscr{G} = \Theta$ is a neutral clover, and the double edge between $\mathscr{T}'_0, \mathscr{T}''_0$ is in $\gamma' \leqslant \gamma$ crystals.

In these cases, we will construct couplings with the following properties:

- In Case 1a, either $\mathscr{T}'_1 = \mathscr{T}''_1$, or else $\mathscr{T}'_1, \mathscr{T}''_1$ are again joined by a single edge.
- In Case 1b, either $\mathscr{T}'_1, \mathscr{T}''_1$ are joined by a double edge in the crystal, or else $\mathscr{T}'_1, \mathscr{T}''_1$ are joined by a single edge.
- In Case 2, either $\mathscr{T}'_1, \mathscr{T}''_1$ are joined by a single edge in some crystal containing the double edge between $\mathscr{T}'_0, \mathscr{T}''_0$, or else $\mathscr{T}'_1, \mathscr{T}''_1$ are joined by a double edge.

Note that, under these couplings, crystal networks work like "switches," in that they move single edges to double edges, and vice versa. Also note that, it is Cases 1b and 2 in which the choice of $w$ is crucial. We put weight $w = 1 + 1/\gamma$ on single edges so that, as we will see, the couplings in these cases are contractive.

**Case 1a.** Suppose that $\mathscr{G} = \Delta$ is a neutral triangle, and that the single edge $\{\mathscr{T}_0', \mathscr{T}_0''\}$ is not contained in a crystal. Then, by Theorems 4.5.1 and 4.5.3, all extended networks $\hat{N}$ containing $\{\mathscr{T}_0', \mathscr{T}_0''\}$ are single and double diamonds. As such, it is only slightly more complicated to construct a contractive coupling in this case, than it was in types $B_n$ and $D_n$ above. We proceed as depicted in Figure 4.28 (cf. Figure 4.27).

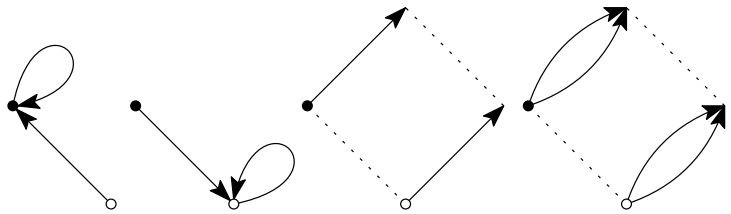


Figure 4.28: *Case 1a:* A contractive coupling, when the starting positions (black and white vertices) are joined by a single edge, which is not in a crystal.

Once again (as in the proof of Theorem 13), using Theorem 1 and Theorems 4.5.5 and 4.5.6, we find a bijection $\psi$ from from $\mathscr{E}(\mathscr{T}_0')$ to $\mathscr{E}(\mathscr{T}_0'')$ that fixes the edge $\{\mathscr{T}_0', \mathscr{T}_0''\}$ and pairs "opposite" edges in single and double diamonds containing $\{\mathscr{T}_0', \mathscr{T}_0''\}$. and so correspond to the same generator.

For each edge $\{\mathscr{T}_0', \mathscr{T}_0' * \mathscr{G}'\}$ in a single diamond $\hat{N}$ containing $\{\mathscr{T}_0', \mathscr{T}_0''\}$, we let

$$\psi(\{\mathscr{T}_0', \mathscr{T}_0' * \mathscr{G}'\}) = \{\mathscr{T}_0'', \mathscr{T}_0'' * \mathscr{G}''\} \tag{4.4}$$

be the "opposite" edge in $\hat{N}$.

Likewise, for each double edge, consisting of two copies $\{\mathscr{T}_0', \mathscr{T}_0' * \mathscr{G}'\}^{(i)}$, with $i \in \{1,2\}$, of the same edge in a double diamond network $\hat{N}$ containing $\{\mathscr{T}_0', \mathscr{T}_0''\}$, we let

$$\psi(\{\mathscr{T}_0', \mathscr{T}_0' * \mathscr{G}'\}^{(i)}) = \{\mathscr{T}_0'', \mathscr{T}_0'' * \mathscr{G}''\}^{(i)}, \tag{4.5}$$

with $i \in \{1,2\}$, be the "opposite" edges in $\hat{N}$.

We couple $\mathscr{T}_1', \mathscr{T}_1''$ as follows. Let $\{\mathscr{T}_0', \mathscr{T}_0' * \mathscr{G}'\}$ be a uniformly random edge in $\mathscr{E}(\mathscr{T}_0')$ and $r_0$ a Bernoulli$(1/2)$. (Note that, this is a uniformly random edge, not generator. Indeed, neutral clovers $\mathscr{G}' = \Theta$ corresponding to double edges are twice as likely to be selected as neutral triangles $\mathscr{G}' = \Delta$.) If $\mathscr{G}' = \mathscr{G}$, we put $\mathscr{T}_1' = \mathscr{T}_1'' = \mathscr{T}_0'$ if $r_0 = 0$ and $\mathscr{T}_1' = \mathscr{T}_1'' = \mathscr{T}_0''$ if $r_0 = 1$. On the other hand, if $\mathscr{G}' \neq \mathscr{G}$, we put $\mathscr{T}_1' = \mathscr{T}_0'$ and $\mathscr{T}_1'' = \mathscr{T}_0''$ if $r_0 = 0$ and $\mathscr{T}_1' = \mathscr{T}_0' * \mathscr{G}'$ and $\mathscr{T}_1'' = \mathscr{T}_0'' * \mathscr{G}''$ if $r_0 = 1$, where $\mathscr{G}''$ is given by the bijection $\psi$, defined in (4.4) or (4.5) above.

Note that $\mathscr{T}_1' = \mathscr{T}_1''$ with probability $1/d$. Otherwise, $\mathscr{T}_1', \mathscr{T}_1''$ are again joined by a single edge. As such

$$\mathbb{E}[w(\mathscr{T}_1', \mathscr{T}_1'')] = (1 - 1/d) w(\mathscr{T}_0', \mathscr{T}_0''). \tag{4.6}$$

**Case 1b.** Suppose that $\mathscr{G} = \Delta$ is a neutral triangle, and that the single edge $\{\mathscr{T}_0', \mathscr{T}_0''\}$ is contained in *exactly* one crystal. Once again, by Theorem 4.5.6, all single and double diamonds

and the one crystal containing $\{\mathscr{T}_0', \mathscr{T}_0''\}$ are otherwise edge-disjoint. In this case, the bijection can no longer fix $\{\mathscr{T}_0', \mathscr{T}_0''\}$, as in Case 1a. Rather, we will need to use this edge in a non-trivial way in order to define the edge pairing within the crystal.

The bijection $\psi$, in this case, is defined in the same way as in Case 1a for the edges in each single and double diamond. On the other hand, for the edges in the crystal, we define $\psi$ as indicated in Figure 4.29. That is, the two single edges in the crystal incident to $\mathscr{T}_0'$ (one of which is $\{\mathscr{T}_0', \mathscr{T}_0''\}$) are paired with the double edges in the crystal incident to $\mathscr{T}_0''$, and vice versa. By Theorem 1, $\psi$ is a bijection from $\mathscr{E}(\mathscr{T}_0')$ to $\mathscr{E}(\mathscr{T}_0'')$. We stress here that $\psi$ pairs edges, not generators, and it is critical, in this case, that there is only one crystal containing $\{\mathscr{T}_0', \mathscr{T}_0''\}$ (since it can only be paired once).

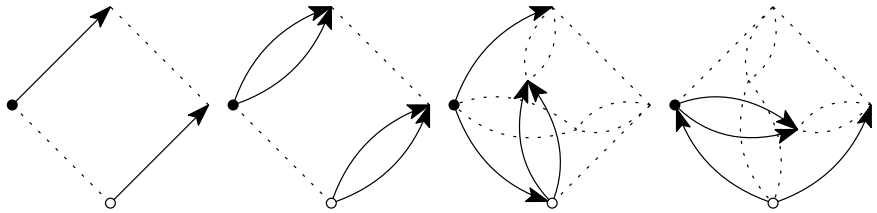


Figure 4.29: *Case 1b:* A neutral coupling, when the starting positions (black and white vertices) are joined by a single edge, which is in exactly one crystal. Note that, if the walks stay in the crystal, they move to a pair joined by a double edge.

To couple $\mathscr{T}_1', \mathscr{T}_1''$, we let $\{\mathscr{T}_0', \mathscr{T}_0' * \mathscr{G}'\}$ be a uniformly random edge in $\mathscr{E}(\mathscr{T}_0')$ and $r_0$ a Bernoulli$(1/2)$. If $r_0 = 0$, we put $\mathscr{T}_1' = \mathscr{T}_0'$ and $\mathscr{T}_1'' = \mathscr{T}_0''$. If $r_0 = 1$, we put $\mathscr{T}_1' = \mathscr{T}_0' * \mathscr{G}'$ and $\mathscr{T}_1'' = \mathscr{T}_0'' * \mathscr{G}''$, where $\mathscr{G}''$ is given by the bijection $\psi$.

In this case, with probability $2/d$ the pair $\mathscr{T}_1', \mathscr{T}_1''$ remains in the crystal, but are now joined by a double edge. Otherwise, $\mathscr{T}_1', \mathscr{T}_1''$ are again joined by a single edge. Therefore,

$$\begin{aligned} \mathbb{E}[w(\mathscr{T}_1', \mathscr{T}_1'')] &= (1-2/d)(1+1/\gamma) + 2/d \\ &= \left[1 - \frac{2}{d(1+\gamma)}\right] w(\mathscr{T}_0', \mathscr{T}_0''), \end{aligned} \tag{4.7}$$

since $w(\mathscr{T}_0', \mathscr{T}_0'') = 1 + 1/\gamma$.

**Case 2.** Finally, suppose that $\mathscr{G} = \Theta$ is a neutral clover. Suppose that $\{\mathscr{T}_0', \mathscr{T}_0''\}$ is contained in $\gamma' \leqslant \gamma$ crystals.

By Theorems 4.5.1 and 4.5.3, all extended networks $\hat{N}$ containing $\{\mathscr{T}_0', \mathscr{T}_0''\}$ are double and quadruple diamonds and crystals. As in the previous cases, we define $\psi$ in this case by pairing "opposite" edges in the double and quadruple diamonds. In this case, $\psi$ fixes the two edges in the double edge between $\mathscr{T}_0', \mathscr{T}_0''$. Note that if $\{\mathscr{T}_0', \mathscr{T}_0''\}$ is in a crystal, then one of $\mathscr{T}_0', \mathscr{T}_0''$ is incident to two single edges in the crystal and the other is incident to a double edge $\neq \{\mathscr{T}_0', \mathscr{T}_0''\}$ in the crystal. We define $\psi$ on each such crystal by pairing these edges, as

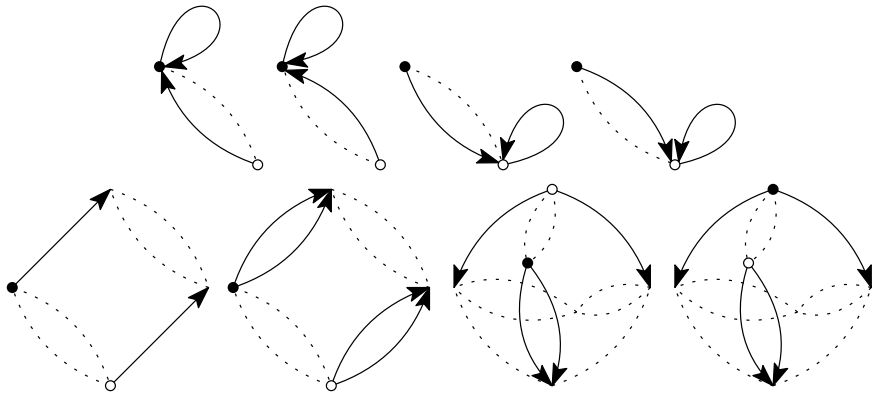


Figure 4.30: *Case 2:* A contractive coupling, when the starting positions (black and white vertices) are joined by a double edge.

indicated in Figure 4.30. Once again, applying Theorem 1, we see that $\psi$ is a bijection from $\mathscr{E}(\mathscr{T}_0')$ to $\mathscr{E}(\mathscr{T}_0'')$.

To couple $\mathscr{T}_1', \mathscr{T}_1''$, we let $\{\mathscr{T}_0', \mathscr{T}_0' * \mathscr{G}'\}$ be a uniformly random edge in $\mathscr{E}(\mathscr{T}_0')$ and $r_0$ a Bernoulli$(1/2)$. If $\mathscr{G}' = \mathscr{G}$, we put $\mathscr{T}_1' = \mathscr{T}_1'' = \mathscr{T}_0'$ if $r_0 = 0$ and $\mathscr{T}_1' = \mathscr{T}_1'' = \mathscr{T}_0''$ if $r_0 = 1$. Otherwise, if $\mathscr{G}' \neq \mathscr{G}$, we put $\mathscr{T}_1' = \mathscr{T}_0'$ and $\mathscr{T}_1'' = \mathscr{T}_0''$ if $r_0 = 0$ and $\mathscr{T}_1' = \mathscr{T}_0' * \mathscr{G}'$ and $\mathscr{T}_1'' = \mathscr{T}_0'' * \mathscr{G}''$ if $r_0 = 1$, where $\mathscr{G}''$ is given by the bijection $\psi$.

In this case, $\mathscr{T}_1' = \mathscr{T}_1''$ with probability $2/d$. With probability $\gamma'/d$, the pair $\mathscr{T}_1', \mathscr{T}_1''$ move within one of the $\gamma'$ crystals containing $\{\mathscr{T}_0', \mathscr{T}_0''\}$, and are then joined by a single edge. Otherwise, with probability $1-(2+\gamma')/d$, $\mathscr{T}_1', \mathscr{T}_1''$ remain joined by a double edge. Therefore,

$$\begin{aligned}\mathbb{E}[w(\mathscr{T}_1', \mathscr{T}_1'')] &= \frac{\gamma'}{d}(1+1/\gamma) + 1 - \frac{2+\gamma'}{d} \\ &= 1 - \frac{2\gamma - \gamma'}{d\gamma} \\ &\leqslant (1-1/d) w(\mathscr{T}_0', \mathscr{T}_0''), \end{aligned} \tag{4.8}$$

since $w(\mathscr{T}_0', \mathscr{T}_0'') = 1$.

By (4.6)–(4.8), we may apply Theorem 3 with $\alpha = O(1/\gamma d)$. Note that, by Theorem 5, it follows that $D_w = O((1+1/\gamma)D) = O(n^2)$. We conclude that the mixing time is bounded by $O(\gamma d \log n)$, as claimed. ■

## 4.7 Future directions

We conclude with a list possibilities for future study.

1. It remains open to find lower bounds for the mixing time, and to determine whether our bounds are sharp. In types $A_{n-1}$, $B_n$ and $D_n$, we might conjecture so, at least up to logarithmic factors.

Recall that, in these types, we have shown that $t_{\text{mix}} = O(d \log n)$, for *any* score sequence, where $d$ is the degree of the interchange graph. In type $A_{n-1}$, Sarkar [90] has shown that $t_{\text{mix}} = \Omega(n^3)$ for a special class of score sequences with $d = \Theta(n^3)$ and a "bottleneck" that is simple to analyze. A similar argument also works in the other types $B_n$, $C_n$ and $D_n$. Perhaps at least $t_{\text{mix}} = \Omega(d)$ can be shown to hold in general.

As discussed, in type $A_n$, Chen, Chang and Wang [23] have shown that the interchange graph is the hypercube, for some very specific score sequences. This shows, at least in some cases, that the bound $t_{\text{mix}} = O(d \log n)$ is sharp, *with* the logarithmic factor.

In type $C_n$, on the other hand, we have used a re-weighting of the metric to show that $t_{\text{mix}} = O(\gamma d \log n)$, where $\gamma$ is the maximal crystal degree. Perhaps other techniques can lead to an improvement. However, we think that crystals in type $C_n$ are a genuine obstacle, so it might be surprising if, in fact, $t_{\text{mix}} = O(d \log n)$ also in this type.

2. Recall that Theorem 5 shows that the interchange graphs are connected with diameter $D = O(n^2)$. It might be of theoretical interest to find a precise formula for $D$, or at least good bounds, as a function of $\mathbf{s}$. Note that Theorem 1 above (proved in [57]) gives such a formula for the degree $d$. We also note that in [16] some results are proved about the diameter $D$ in type $A_{n-1}$.

3. Recall that each edge in the interchange graph corresponds to a generator reversal. Generators are the smallest neutral structures. It might be interesting to consider a generalization, in which neutral structures up to a given size can be reversed in a single step, and to quantify the decrease in the mixing time.

   Related to this, Gioan [43] has studied cycle and cocycle reversing systems, and these have been generalized by Backman [6, 7]. One might pursue Coxeter analogues of these results.

4. A *graphical zonotope* is a polytope obtained as a Minkowski sum of line segments, where the sum is indexed by the edges of the graph (see, e.g., Ziegler [100]). The permutahedron

$$\Pi_{n-1} = \mathbf{w}_n + \sum_{1 \leqslant i < j \leqslant n} [\mathbf{0}_n, \mathbf{e}_j - \mathbf{e}_i]$$

   is the graphical zonotope of the complete graph $K_n$. Likewise, the Coxeter versions $\Pi_\Phi$ are obtained as sums indexed by the edges in the complete signed graphs $\mathscr{K}_\Phi$. It could be interesting to study random walks on the fibres of other graphical zonotopes.

   That being said, our current arguments take full advantage of the symmetry of $\mathscr{K}_\Phi$. Once some edges become unavailable, it is more challenging (or even impossible) to show connectivity (and bound the diameter) of the interchange graph, and to devise a path coupling (which we have accomplished, via a non-trivial edge pairing argument).

5. Rapid mixing can be a starting point for approximate counting. It would be interesting if our result could help with counting the number of vertices in interchange graphs, for a general score sequence. As already discussed, these have been approximated (see [49, 70, 71, 92]) only in type $A_{n-1}$ and when $\mathbf{s}$ is close to the centre of $\Pi_{n-1}$.

6. In this work, we have studied random walks on interchange graphs associated with score sequences. However, one could also, quite naturally, try to study random walks on the set of score sequences itself. In type $A_{n-1}$, all lattice points are score sequences. In types $B_n$, $C_n$ and $D_n$ the score sequences are more complicated sets of points, characterized in the previous work in this series [57].

7. Finally, we recall that the interchange graphs in Figures 4.3 and 4.4 are Cartesian products. Also recall that, in type $A_n$, some interchange graphs are the hypercube [23], which are a simple example of a product graph. It might be enlightening to investigate the product structure of interchange graphs more generally.

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