

Optimal Decisions in Finance: Passport Options and the Bonus Problem

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Abstract

The object of this thesis is the study of some new financial models. The common feature is that they all involve optimal decisions. Some of the decisions take the form of a control and we enter the theory of stochastic optimal control and of Hamilton-Jacobi-Bellman (HJB) equations. Other decisions are “binary” and we deal with the theory of optimal stopping and free boundary problems.

Throughout the thesis we will prefer a heuristic and intuitive approach to a too technical one which could hide the underlying ideas.

In the first part we introduce the reader to option pricing, HJB equations and free boundary problems, and we review briefly the use of these mathematical tools in finance.

The second part of the thesis deals with passport options. The pricing of these exotic options involves stochastic optimal control and free boundary problems. Finally, in the last part we study the end-of-the-year bonus for traders: how to optimally reward a trader?

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Part I

Option Pricing and Optimal Decisions

This section is a broad introduction to the thesis.

In the first chapter we are going to talk about option pricing. We are going to go through the Black-Scholes model in depth. We are going to underline his weaknesses and his strengths. We are going to try to justify why we are going to work under the Black-Scholes assumptions.

The second chapter deals with the mathematics of optimal decisions. Two types of decisions are going to be studied. The first one is the choice between two different states and the decision maker has to figure out which is best. Usually the decision maker has the right to switch states only once. The underlying mathematics are optimal stopping and free boundary problems. The second one allows the decision maker to choose between a large range of states. He usually is allowed to change states as often as he likes. The underlying mathematics are stochastic optimal control and HJB equations.

Finally the third chapter gives three typical examples of models involving optimal decisions in finance. We are going to talk about asset allocation in continuous time, option pricing in the presence of transaction costs and on exercising an American option at a non-optimal time.

As we have mentioned earlier, the first part of the thesis is a broad introduction. The second part (Passport options) is based on Ahn et al. (1999) and Penaud et al. (1999). The third part (Bonus) is based on Ahn et al. (2000). I have performed all numerical computations in the thesis apart from figures 6.1, 6.2 and 6.3 which were performed by Hyungsok Ahn.

Chapter 1

Option Pricing

1.1. Introduction

The idea of options is certainly not new. Ancient Romans, Grecians, and Phoenicians traded options against outgoing cargoes from their local seaports. When used in relation to financial instruments, options are generally defined as a “contract between two parties in which one party has the right but not the obligation to do something, usually to buy or sell some underlying asset”. Call options are contracts giving the right to buy something, while put options entitle the holder to sell something.

As the option gives its holder a right, it must have financial value. The evaluation of the fair option price is the first task of option pricing. Note that the option holder cannot lose more than the amount of money he has spent for buying the option, whereas the writer could lose a lot of money indeed. The two fundamental questions that arise are

- What is the price of the option?
- What is the risk taken by the writer?

In fact these two questions are very linked indeed: the price of an option minimizes the

risk taken by the writer¹. In the Black-Scholes world, the risk actually goes down to zero.

But let's first take a quick look at the history of option pricing. Louis Bachelier anticipated much of what has become standard fare in financial theory: random walk of financial market prices, Brownian motion and martingales (before Einstein and Wiener!). His PhD thesis (*Théorie de la spéculation*, 1900) marks the birth of both the continuous time mathematics of stochastic processes and the continuous time economics of option pricing. It was unfortunately not appreciated by his contemporaries (his PhD did not receive a good grade despite the fact that the president of the committee was Henri Poincaré). Unknown for half a century, his work was rediscovered in the 1950's by Samuelson. Option pricing took a new start and finally in 1973 Black and Scholes derived what is now the reference model of option pricing.

Let's give a more precise definition of a European call. A European call option with strike price E , maturity T on an underlying asset of price S gives the right to its owner to buy the underlying at time T for the price E . Of course, the holder of such an option will only exercise his option if the price of the underlying at time T is greater than E . A European call option can therefore be seen as a contract that gives $\max(S - E, 0)$ at time T to his holder.

A European *put* gives its holder the right to *sell* the asset for E at time T . It can be seen as a contract that gives $\max(E - S, 0)$ at time T to its holder.

European calls and puts are the most basic options. Note that all sorts of features can be added to the contract. In particular, one very common feature is the right for the holder to exercise the option before time T . The option is then said to be *American*. These options are actually traded in the market and their prices are driven by supply and demand. However, some other options are less common and are only traded over the counter (OTC). For traded options there is a price given by the market and a price given by the option pricing model. A trader would probably buy an option if its market value

¹We are being vague on the definition of risk. It will be made clearer later.

is below the theoretical price. However these situations do not arise for OTC options as there is no market price. The institution relies then on its financial engineers to find the best price and the best risk management strategies.

Remark: If one knows the price of a European call, one can deduce the price of the corresponding put with same strike and same maturity thanks to the put-call parity formula

$$\text{call} - \text{put} = S - e^{-r(T-t)}E \quad (1.1)$$

where r is the risk-free interest rate and t is the present time. The above formula is very powerful as it is model-independent. In particular, this formula tells us that a rise in the call price implies a rise in the put price (for t and S fixed).

So one should be very careful about comments of the type “Option prices are driven by supply and demand. If the market thinks that the underlying is going to go up, they will buy calls and sell puts. Call prices are therefore going to go up and put prices are going to go down”. Of course this would violate put-call parity.

1.2. The Black-Scholes Model

1.2.1. The Black-Scholes assumptions

There are a lot of assumptions in the Black-Scholes model. Some of them can be relaxed easily within the Black-Scholes framework of option pricing. Others would require a different approach to the problem. We are going to discuss the different assumptions one by one.

- *The underlying follows a lognormal random walk with constant drift and volatility*

In other words, returns are independent and identically distributed random variables. The money won (or lost) by the agent is proportional to the money he has put in his “bet”. The randomness of the outcome does not depend on the past, does not depend on time nor on the value of the underlying (dS/S does not depend on S). And finally, the randomness involved is Gaussian. It is indeed the obvious first choice in a continuous model involving randomness.

The lognormal random walk assumption is without any doubt the most important assumptions in the model. The stochastic differential equation (SDE) for the price of the underlying is:

$$dS = \mu S dt + \sigma S dW \quad (1.2)$$

where μ is the drift of the underlying, σ its volatility, and dW is a Gaussian random variable with mean zero and variance dt . The solution to the above stochastic differential equation is

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}. \quad (1.3)$$

Only one parameter (σ) characterizes the randomness involved in the price process of the underlying. In reality, price processes seem to be more sophisticated and need more parameters to describe them (Bouchaud and Potters (2000)). Indeed, statistical analysis of stock price processes reveals that most of them are not lognormal. Jumps occur in stock price processes (see Merton (1976)) while the lognormal random walk is a continuous process. Estimation of the volatility at different time scales gives different values for σ (Bouchaud and Potters (2000)), which is again a contradiction with the lognormal random walk hypothesis. And the historical pdf seems to have fatter tails and have a higher peak in the middle (see Bouchaud and Potters (2000)).

Different approaches are available to tackle these problems. One way is to “improve” the process modelling the stock prices. One could make the volatility stochastic in (1.2) (see Hull and White (1987) for example) or one could move out of the Gaussian world

and model the changes in price returns by more complex distributions (see Bouchaud and Potters (2000)). The Black-Scholes framework is still available for stochastic volatility models but it is not possible to replicate the payoff with the underlying any more. Perfect replication would require hedging with the underlying *and* another option. However models involving probability distributions which are not Gaussian require a completely different approach. Indeed, one cannot use Itô's lemma any more and one has to tackle the option pricing problem from a global (integral) point of view (see Bouchaud and Potters (2000)). Again it is not possible to replicate the payoff perfectly. When one uses these models (stochastic volatility and non-Gaussian distribution), one needs to evaluate the parameters from historical data. But one has to be aware of market prices of traded instruments as well!

An alternative approach is to study the volatility surface. There is a 1-1 correspondence between the volatility and the option price. This 1-1 correspondence is guaranteed for vanilla options. But it is not necessarily true for exotic options. For a given option, one can therefore find the volatility which corresponds to the option price in the Black-Scholes world. It is called the implied volatility. Practitioners do not talk about option prices, they talk about implied volatilities instead. To the question "What is the price of a call maturity 6 months strike 100?", they would answer "20%". The interesting phenomenon is that the implied volatility is not constant across expiry dates and strike prices. The main assumption of the model is to say that the volatility is constant and we end up with a non-constant implied volatility. Practitioners draw the volatility surface (i.e. the implied volatility as a function of strike and time to maturity) and develop theory for its understanding. In some sense, it is somewhat strange as the volatility surface is just the image of the market prices through some bijection (the Black-Scholes operator). The thing is that traders have gained a certain feeling about the meaning of implied volatility throughout the years. This is why they go further and further in this direction. See Black (1989) for intuitive tricks of modifying Black-Scholes.

At this point it is worth mentioning the model of Avellaneda et. al (1995) and Lyons

(1995). It allows the volatility to be anything in a prescribed interval. The option is priced by taking the worst volatility. Unfortunately, this often leads to an unrealistic option price. It can be improved via optimal static hedging though.

In this section we have been vague about the nature of the underlying. Roughly speaking stock price processes in emerging markets seem to be quite far away from the lognormal random walk while the random walk assumption is not too bad in FX markets. However, in energy markets (Electricity, Oil, Natural gas), no one has not found yet a satisfactory way of modelling the price processes.

- *The interest rate r is constant*

In this thesis we are going to assume that r is constant. This assumption is fine for options which have a short time life (typically less than one year), so there is not much error produced by assuming that r is constant. We could even assume that the interest rate is a known function of time without having to change the framework (see Wilmott (2000)).

- *There are no dividends*

Most companies pay dividends to their shareholders. If we assume that the amount and dates of the dividends are predictable, it is easy to incorporate them in the model (see Wilmott (2000)). For unpredictable dividends, we refer to Bakstein and Wilmott (1999).

- *Trading takes place continuously*

This means that it is possible to buy and sell the underlying continuously. It is obviously an idealisation of the real world. For option pricing when the hedger re-hedges his position at fixed time intervals δt , the hedge is not perfect and a small error occurs at each time step (see Wilmott (2000)). Hedging discretely would be perfect if the underlying followed a binomial process. But if we assume that the underlying follows a continuous process, discrete hedging will give a small hedging error at each time step.

- *There are no transaction costs*

Market participants have to pay transaction costs when they buy or sell assets. When we suppose the existence of transaction costs, we have to relax the hypothesis of continuous trading at the same time otherwise the total cost would be infinite. There are two types of transaction costs models. The first type says that the writer rehedges every δt whatever happens. The result is a nonlinear PDE (see Wilmott (2000)). The second type is more interesting as it is going to be the first example of the application of stochastic control in finance: all the time, the hedger asks himself whether it is optimal or not to re-hedge his position (Hodges and Neuberger (1989), Davis et al. (1993)). He makes his decision according to some utility maximization criteria: he tries to maximize his profit according to his personal attitude towards risk. We are going to go through such a model in the third chapter.

- *There is no arbitrage*

As a consequence,

All risk-less portfolios must earn the same return

As a consequence, if we build a risk-less portfolio out of risky instruments, this portfolio must earn the same return as the risk-free interest rate. As we are going to see in the next section, the random walks of an option and of its underlying are completely correlated: it is possible to build a risk-less portfolio out of them. By writing that this portfolio must earn as much money as if we had put the money in the bank, we obtain the pricing equation.

Remark: Throughout the thesis we are going to work under the Black-Scholes assumptions in order to concentrate on the new issues.

1.2.2. The Black-Scholes equation

We first are going to explain briefly Itô's lemma (see Bjork (1998) for a rigorous definition). If f is a function of t and W , where t is time and W is a Brownian motion (dW is a random variable with mean zero and variance dt), then

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial W}dW + \frac{1}{2} \frac{\partial^2 f}{\partial W^2}dt. \quad (1.4)$$

This is because $dW^2 \rightarrow dt$ as $dt \rightarrow 0$. One way of understanding that is to see how the Brownian motion can be obtained from a discrete process. The discrete process $W(t + \delta t) - W(t) = \sqrt{\delta t}$ with probability 1/2 and $W(t + \delta t) - W(t) = -\sqrt{\delta t}$ with probability 1/2 tends to W_t as $\delta t \rightarrow 0$ (see Feller (1966)). As $\delta W^2 = \delta t$, it is not surprising to have the continuous equivalent of this property as well. Applying Itô's lemma to the option price $V(t, S)$ gives

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}dt \quad (1.5)$$

Now the writer builds a portfolio $\Pi = -V + \Delta S$. The change in his portfolio in a small time step is

$$d\Pi = -\frac{\partial V}{\partial t}dt + (\Delta - \frac{\partial V}{\partial S})dS - \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}dt. \quad (1.6)$$

The writer can make the change in his portfolio deterministic by choosing $\Delta = \frac{\partial V}{\partial S}$. His portfolio is then risk-less and thus must earn the same return as the interest rate:

$$d\Pi = r\Pi dt. \quad (1.7)$$

After division by dt in the above equation, we obtain the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (1.8)$$

This parabolic partial differential equation (PDE) is solved together with the final condition

$$V(T, S) = \max(S - E, 0). \quad (1.9)$$

Note that the option price does not depend on the drift of the underlying (we have got rid of dS by delta hedging).

The analysis would remain exactly the same for an option with a payoff $f(S)$. The price of an option which pays out $f(S)$ at time T can be found by solving the Black-Scholes PDE with the final condition $V(T, S) = f(S)$. For options whose payoffs depend on other variables, one has to incorporate the new variables in the analysis.

Remark: One could see Δ -hedging as replicating the payoff. To see that we are going to have a look at the replicating portfolio P of the hedger. At time t its value is the option price and it evolves in the following way:

$$dP = rPdt + \Delta(dS - rSdt) \quad (1.10)$$

so that $P(T, S) = f(S)$. In the above equation, ΔdS is the change in the portfolio due to the change in the underlying while $r(P - \Delta S)dt$ is the effect of the interest rate on the cash part of the portfolio. From this point of view the hedger replicates the payoff by trading the underlying. And the option price is the cash needed in order to make the replication possible. But as we mentioned earlier perfect replication is not always possible. For example if we move out of the Black-Scholes world and assume that the underlying follows a truncated Lévy process (see Bouchaud and Potters (2000)) it is not possible to replicate the payoff exactly. There is no Δ which makes the change in the portfolio completely deterministic. The pricing model has to take the remaining randomness into account. A market in which perfect replication is possible is said to be *complete*.

1.2.3. The Martingale approach

There are three other mathematical frameworks for option pricing:

- The binomial approach (see Cox et al. (1979)).
- The optimal investment approach (see Rogers (1998)).
- The martingale approach. The martingale approach to option pricing has its origins in the seminal papers of Harrison and Kreps (1979) and Harrison and Pliska (1981).

Academics tend to prefer the martingale approach, but the PDE approach is just as popular in the industry.

The solution of the Black-Scholes PDE can be solved using a stochastic representation formula à la Feynman-Kac (see Bjork (1998)):

$$V(t, S) = e^{-r(T-t)} \hat{E}[f(S)] \quad (1.11)$$

where the S process is defined by the dynamics

$$\begin{aligned} dS &= rSdt + \sigma SdW \\ S(t) &= S. \end{aligned} \quad (1.12)$$

These dynamics define the risk-neutral (or martingale) measure. Under this measure the underlying follows a risk-neutral random walk with drift r and volatility σ . The martingale leads to such an expression for the option price. It does not need to find the PDE first though. In this thesis we do not need to know the techniques behind the martingale approach and we refer to Baxter and Rennie (1996).

Chapter 2

Decision Making in Finance

2.1. Early Exercise

Although European options allow the holder to exercise the option at expiry only, American options allow him to exercise it at any time up to expiry. The pricing of American options is therefore different as the holder has extra rights. There are two ways of tackling this problem: via the martingale approach or via the PDE approach. The martingale approach is going to involve the theory of optimal stopping as the PDE approach is going to involve the theory of free boundary problems. American options are therefore another example of the nice connection between these two theories.

We find the price of an American option by making sure that the writer can replicate at least the payoff whatever the holder does with his exercise right. Indeed, if there was an exercise strategy that would make it impossible for the writer to replicate the payoff, there would be a clear arbitrage opportunity. And we want to find the smallest price that guarantees that replicating is possible.

In fact, there exists an exercise strategy which corresponds to the option price. It is represented by a curve $S^*(t)$ which separate the (t, S) plane into a region in which it

is optimal not to exercise and a region in which it is optimal to exercise. What do we mean by optimal? We mean that it is the strategy which leads to the minimum initial wealth needed by the writer for replicating the payoff. So the writer will make a profit if the holder does not follow the optimal strategy. This idea is developed in the third chapter (“On trading American options”). We now ask two questions:

- What is the optimal strategy?
- What is the connection with PDE’s?

The price of an American put option with the martingale approach is

$$V(t, S) = \sup_{t \leq \tau \leq T} E[e^{-r\tau} \max(E - S_\tau)] \quad (2.1)$$

where E is the expectation under the risk neutral measure (see Bjork (1998)) and τ is the exercise time. It is an optimal stopping problem. The price of an American put option via the PDE approach can be obtained as follows: the option price has to be greater than the payoff all the time:

$$V(t, S) \geq \max(E - S, 0). \quad (2.2)$$

Indeed, if it was not the case, there would be an arbitrage opportunity. By buying the option and exercising instantaneously, it would be possible to make an instantaneous risk-less profit. Financially speaking, two situations can arise:

- The option price is strictly greater than the payoff. In this case it is optimal for the holder to keep the option and the Black-Scholes PDE is satisfied.
- The option price is equal to the payoff. In this case it is optimal for the holder to exercise the option and the Black-Scholes PDE is violated.

Intuitively, one can understand the pricing of an American option as follows: Given $V(T, S)$, we are looking for the maximum of $V(0, S)$ over all Markovian trading strategies. At each point (t, S) , we ask ourselves: “What would make the option more expen-

sive? Exercise or continue?”

- If the Black-Scholes PDE gives a price which is more expensive than the payoff, then continue (and the Black-Scholes PDE is satisfied).
- If the Black-Scholes PDE gives a price which is cheaper than the payoff, then exercise (the Black-Scholes PDE is then violated and $V(t, S) = V(T, S)$). In this case the Black-Scholes equality turns into an inequality ($<$) because the price is corrected upwards. The mathematical formulation is called the linear complementary problem:

$$\begin{cases} \mathcal{L}_{BS}V = \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \leq 0 \\ V(t, S) \geq \max(E - S, 0) \\ (\mathcal{L}_{BS}V) \cdot \max(E - S, 0) = 0 \end{cases}$$

For the theoretical aspects of these problems such as regularity conditions we refer to Friedman (1988).

A numerical example Let's consider the following American put: $T = 0.5$, $\sigma = 0.2$, $r = 0.06$, $E = 100$.

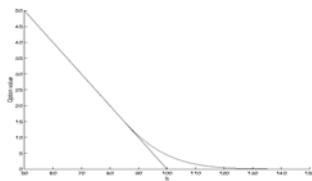


Figure 2.1. Price of the American put

We have solved the free boundary problem by using the explicit finite difference method (FDM) (see Wilmott (2000)). At each grid point we check which gives the

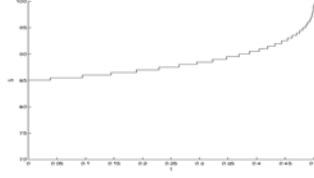


Figure 2.2. Optimal free boundary $S^*(t)$.

highest option price, the PDE scheme or the payoff. In Figure 2.2, the optimal free boundary is defined to be the “first” price for which the payoff is greater than the outcome of the PDE scheme. The steps look of the free boundary is a consequence of our discretization procedure in our numerical scheme. Note that $S^*(t)$ divides the plane between the continuation region (above) and the exercise region (below).

2.2. Utility

2.2.1. Introduction

Question: How do we make a decision when the outcome is uncertain?

The mathematical tools which allow us to quantify the different attitudes towards reward and risk are the utility functions and stochastic control.

In order to make a decision one has to define two things:

- An attitude towards reward and risk (risk-loving, risk-averse, objective).
- A criterion (an expected profit to maximize, a risk to minimize...)

Once these two questions have been answered, the problem of making the decision is

set up.

The standard way of measuring the attitude towards reward and risk is the use of utility functions. It is the object of the following section. Once this function is defined the natural criterion is to maximize the expectation of the utility function. Indeed, the basic utility theorem states that the expectation of the utility provides the objective index for comparing the desirability of rewards by a risk sensitive decision maker.

Another method of obtaining a “best” decision follows the principle of minimization of regret (Savage (1951)). The idea is to minimize the difference between the absolute best payoff and the payoff corresponding to our decision.

2.2.2. Utility functions

A utility function U assigns a subjective value of money to its objective value. As an investor cannot think that more money is worth less, utility functions are increasing. We are going to assume that they are continuous too. Here is a summary of the connection between the curvature of a utility function and the attitude of the investor towards risk:

<i>Risk Attitude</i>	Lover	Averse	Neutral
<i>Curvature</i>	$\frac{\partial^2 U}{\partial x^2} > 0$	$\frac{\partial^2 U}{\partial x^2} < 0$	$\frac{\partial^2 U}{\partial x^2} = 0$

Indeed, combining Taylor series leads to

$$\frac{1}{2} \left(U(x + \epsilon) + U(x - \epsilon) \right) = U(x) + \frac{\epsilon^2}{2} U''(x) + \dots \quad (2.3)$$

The left hand side is the expected utility of taking part in a game which pays out $x + \epsilon$ with probability $1/2$ and $x - \epsilon$ with probability $1/2$. The first term in the right-hand side is the utility of not taking part in the game and keeping the initial wealth of x . One can see that positive convexity corresponds to an investor who likes risk and concavity corresponds to an investor who is risk-averse.

For more theory and use of utility functions in finance we refer to Merton (1990) and Tapiero (1998).

2.2.3. Utility functions in practice

Utility functions are tools which make it easy to compare which is best between different risk/rewards situations. The desirability of a risky-reward situation can be measured by the expected utility

$$E[U(x)] = \int U(x)P(x)dx \quad (2.4)$$

where x is the set of rewards which occur with probability P . The expected utility is therefore a tool for the decision maker: typically he would have to choose between different situations with different probabilities P 's. If he knows his utility function he could choose which situation is the most suited to his attitude towards reward and risk. The best situation would be the one that maximizes his expected utility.

Remark: One might think that decision making should therefore be a straightforward process. But we have assumed that the decision maker knows the utility function. This leads to another question: how does he find out his utility function? It is the same old problem... It is not easy to quantify things in finance. But it is always useful to see the different results given by different utility functions.

2.3. Stochastic Optimal Control

2.3.1. Introduction

In the second part of our DPhil thesis, we are going to price passport options. In the third part, we are going to study the end-of-the-year bonus for traders. Both problems (when translated in mathematical terms) are stochastic control problems. We will explain the modelling in the second and in the third part. In this section we are going to study stochastic optimal control problems from a mathematical point of view.

Stochastic optimal control found its first application in automatic control (see Kamil and Chui (1996) for a brief history and introduction to automatic control)

Intuitively: A stochastic process generates a probability density function (pdf) at time T (maturity). If the stochastic process is controlled, we have a control on the pdf at maturity. So if we are trying to maximize the expectation of some function of the random variable at maturity (the expectation), maybe there is a control process that would do the trick.

- What is this control process?
- What is the corresponding expectation?

The two fundamental ideas we are going to use in order to answer these questions are the following:

- One can write a (controlled) PDE for the evolution of the expectation.
- The control process can be found by local optimization.

More formally: In this section we are going to follow Bjork (1998). Let $\mu(t, x, q)$ and $\sigma(t, x, q)$ be the drift and the volatility of the following controlled stochastic differential equation:

$$\begin{aligned} dX_t &= \mu dt + \sigma dW_t \\ X_0 &= x_0 \end{aligned} \tag{2.5}$$

In the above, $\mu : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^{n \times k}$, where n is the dimension of the Brownian motion W_t and k is the dimension of the control process q_t . We suppose that the value q_t of the control process is only allowed to depend on past observed values of the state process X_t , i.e. there is a deterministic function g , $g : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^k$ such that

$$q_t = g(t, X_t) \tag{2.6}$$

and g is called a feedback control law.

In most cases q also has to satisfy some control constraints, i.e. $q_t \in Q$, where Q is included in $\mathbb{R}^k$. A control law is said to be *admissible* if

- i) $q(t, x) \in Q \ \forall t > 0, \forall x \in \mathbb{R}^n$
- ii) For every initial point (t, x) the SDE

$$\begin{aligned} dX_s &= \mu(q(s, X_s))ds + \sigma(q(s, X_s))dW_s \\ X_t &= x \end{aligned} \tag{2.7}$$

has a unique solution. The class of admissible control laws is denoted by $\mathcal{Q}$.

Let $F : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}$ and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$. In finance F will often be the utility function and ϕ the bequest function. We define the value function $J_0 : \mathcal{Q} \rightarrow \mathbb{R}$ by

$$J_0(q) = E \left[\int_0^T F(t, X_t^q, q_t) dt + \phi(X_T^q) \right], \tag{2.8}$$

where we have written X_t^q (not X_t) to stress the dependence on q . The stochastic control problem is to maximize $J_0(q)$ over all $q \in \mathcal{Q}$, and we define the optimal value J_0^* by

$$J_0^* = \sup_{q \in \mathcal{Q}} J_0(q). \quad (2.9)$$

If there exists an admissible control law q^* such that

$$J_0(q^*) = J_0^* \quad (2.10)$$

we say that q^* is an optimal control law.

Given an optimal control problem, two natural questions arise:

- i) Does there exist an optimal control law?
- ii) Given that an optimal control law exists, how do we find it?

To answer the first question, we refer to the literature (Fleming and Soner (1992) for example). In the next section, we are going to see how we can answer to the second question by solving a PDE.

2.3.2. Bellman's optimality principle of dynamic programming

“An optimal policy has the property that, whatever the initial conditions are, the remaining decision must constitute an optimal policy with regard to the state resulting from the first decision.” (Bellman (1957), p83).

The idea can be captured very easily with a very simple example which illustrates the basic idea.

Suppose we are interested in finding the shortest auto route between Los Angeles and New York City. Further suppose that we know that the shortest route between LA and New York goes through Chicago. Then Bellman's optimality principle states

the obvious fact that in this case, the Chicago to New York leg of the shortest journey from LA to New York will be identical to the shortest auto route between Chicago and New York. Why is this obvious observation useful? Because it can result in a lot of computational savings in finding the best path from LA to New York: if we find the best path from LA to Chicago, then we only need to add on the shortest auto distance between Chicago and New York, if we already know the answer to that. Note that this example assumes that the shortest path does not depend on time.

2.3.3. The Hamilton-Jacobi-Bellman equation

Following Bjork (1998), we assume that there exists an optimal control law q^* and that the optimal value function (called V from now on) is smooth enough.

Sketch of the derivation of the HJB equation: For (t, x) fixed, we define the following control law:

$$\tilde{q}(s, y) = \begin{cases} q(s, y), & \text{if } s \in (t, t + h) \\ q^*(s, y), & \text{if } s \in (t + h, T) \end{cases}$$

where $h > 0$ and $t + h < T$. By comparing the two expectations resulting from $\tilde{q}$ and q^* , we get an inequality. By letting $h \rightarrow 0$, we obtain a variational inequality. It turns into the PDE for $q = q^*$.

Proof: The value function J is defined by

$$J(t, x, q) = E\left[\int_t^T F(t, X_t, q_t)dt + \phi(X_T)\right]. \quad (2.11)$$

The optimal value function V is defined by

$$V(t, x) = \sup_{q \in \mathcal{Q}} J(t, x, q). \quad (2.12)$$

Assuming that q^* exists, we have

$$J(t, x, q^*) = V(t, x) \quad (2.13)$$

$$J(t, x, \tilde{q}) = E^{t,x} \left[\int_t^{t+h} F(x, X_s^q, q_s) ds \right] + E^{t,x} \left[V(t+h, X_{t+h}^q) \right] \quad (2.14)$$

So

$$V(t, x) \geq E^{t,x} \left[\int_t^{t+h} F(x, X_s^q, q_s) ds + V(t+h, X_{t+h}^q) \right] \quad (2.15)$$

Let's use Itô's lemma on $V(t+h, X_{t+h}^q)$: the stochastic integral $\int_t^{t+h} \nabla_x V(s, X_s) \sigma dW$ vanishes and $V(t, x)$ cancels out:

$$E^{t,x} \left[\int_t^{t+h} [F(s, X_s^q, q_s) + \frac{\partial V}{\partial t}(s, X_s^q) + \mathcal{A}^q V(s, X_s^q)] ds \right] \leq 0 \quad (2.16)$$

where $\mathcal{A}^q = \sum_{i=1}^n \mu_i^q(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n C_{i,j}^q(t, x) \frac{\partial^2}{\partial x_i \partial x_j}$ and $C^q(t, x) = \sigma(t, x, q) \sigma(t, x, q)'$.

Now we divide everything by h and we go to the limit as $h \rightarrow 0$. This leads to

$$F(t, x, q) + \frac{\partial V}{\partial t}(t, x) + \mathcal{A}^q V(t, x) \leq 0 \quad (2.17)$$

This holds for all choices of $q \in \mathcal{Q}$, and there is equality if and only if $q = q^*$. So

$$\frac{\partial V}{\partial t} + \sup_{q \in \mathcal{Q}} [F + \mathcal{A}^q V] = 0. \quad (2.18)$$

Theorem: V satisfies the HJB equation

$$\begin{cases} \frac{\partial V}{\partial t} + \sup_{q \in \mathcal{Q}} \{F(t, x, q) + \mathcal{A}^q V(t, x)\} = 0 \\ V(T, x) = \phi(x) \quad \forall x \in R^n \end{cases}$$

And $\forall (t, x) \in [0, T] \times R^n$, the supremum is attained by $q = q^*(t, x)$.

The above theorem states that if V is the optimal value function and if q^* is the optimal control, then V satisfies the HJB equation and q^* realises the supremum in the HJB equation. But what about the other way around?

Verification theorem: Suppose that two functions $H(t, x)$ and $g(t, x)$ are such that

- H solves the HJB equation.
- g is an admissible control law.
- For each (t, x) the supremum in the HJB equation is attained by $q = g$.

Then

- The optimal value function of the control problem is H .
- g is an optimal control law.

Proof (1st step): Choose a control law $q \in \mathcal{Q}$ and a point (t, x) . Let's insert the process X_s^q in the function H :

$$\begin{aligned} H(T, X_T^q) &= H(t, x) + \int_t^T \left(\frac{\partial H}{\partial t}(s, X_s^q) + \mathcal{A}^q(s, X_s^q) \right) ds + \\ &\quad + \int_t^T \nabla_x H(s, X_s^q) \sigma^q(s, X_s^q) dW_s \end{aligned} \quad (2.19)$$

As H solves the HJB equation,

$$\frac{\partial H}{\partial t}(t, x) + F(t, x, q) + \mathcal{A}^q H(t, x) \leq 0 \quad \forall q \in \mathcal{Q} \quad (2.20)$$

$$H(t, x) \geq \int_t^T F^q(s, X_s^q) ds + \phi(X_T^q) + \int_t^T \nabla_x H(s, X_s^q) \sigma^q dW_s \quad (2.21)$$

Taking expectations,

$$H(t, x) \geq J(t, x, q) \quad (2.22)$$

And finally

$$H(t, x) \geq \sup_{q \in \mathcal{Q}} J(t, x, q) = V(t, x). \quad (2.23)$$

2nd step Choose $q = g$, so

$$\frac{\partial H}{\partial t} + F^g + \mathcal{A}^g H = 0 \quad (2.24)$$

and

$$H(t, x) = E^{t,x} \left[\int_t^T F^g(s, X_s^g) ds + \phi(X_T^g) \right] = J(t, x, g). \quad (2.25)$$

But

$$V(t, x) \geq J(t, x, g), \quad (2.26)$$

so

$$H(t, x) \geq V(t, x) \geq J(t, x, q) = V(t, x) \quad (2.27)$$

and finally

$$H(t, x) = V(t, x) = J(t, x, q). \quad (2.28)$$

This proves that $H = V$ and that q is the optimal control law.

Remark: In general HJB equations are not easy to solve. But the HJB equations we are going to encounter are not too bad. In the worst case, the term to maximize will be a quadratic function in the control parameter. It is therefore easy to find the optimal control.

Literature: HJB equations were originally used in finance for asset allocation problems (see Merton (1990)). Stochastic control problems arise in option pricing models in the presence of transaction costs as well. Indeed the hedger has to make the decision whether to re-hedge or not according to his utility (see Davis et al. (1993)).

Part II

Passport Options

Chapter 3

Introduction

Market participants trade risky assets for better returns, at least on average. As long as the risk remains significant, however, investment in these assets does not guarantee a positive payoff and traders are prone to become a victim of their own strategies. Nowadays, a variety of financial instruments allow traders to generate non-linear returns and hence enable them to avoid extreme financial loss to some extent. An example of such an instrument is the passport or perfect trader option.

3.1. The contract

The passport option is a call option on the balance of a trading account. The trading account is equal to zero when the contract is sold. It goes up when the noteholder is good and it goes down when he is bad. The option holder trades an asset (or several assets) and takes the net profit of all the trades he made before maturity. The issuer is liable for the net loss. This option certainly makes the fund management straightforward and safer as long as fund managers are willing to pay the up-front premium for the passport option. Note that the writer has to know permanently the current position

of the noteholder. If the writer does not know what the noteholder is doing, the issuer cannot hedge himself. From now on we assume that the noteholder does not actually trade anything: he just tells his orders to the issuer. The issuer has to deal with it.

The first version of the passport option was launched by Bankers Trust. The option holder trades one asset with the position limit specified in the contract and with an immunity from the net loss he may incur in trading.

More precisely, let's call π the value of the trading account, q the number of underlyings held by the noteholder (the position of the noteholder), r the risk-free interest rate and T the maturity of the option. At time $t = 0$, no trading has been performed within the contract yet and $\pi = 0$. At time $t > 0$, the trading account evolves in the following way:

$$d\pi = r(\pi dt - qSdt) + qdS. \quad (3.1)$$

Let's take a closer look at the above equation. The term $\pi - qS$ represents the holding in cash of the trading account. The rate at which the π bit of the cash evolves does not have to be the same as the risk-free interest rate, but we are going to make this choice in order to keep things simple so that we can focus on other extensions of the passport option. In the "literature section" we give references for when the "risk-free rate" of the contract differs from the risk-free interest rate. More explicitly, the equation for the evolution of the trading account would be

$$d\pi = \lambda\pi dt - rqSdt + qdS \quad (3.2)$$

where λ is specified in the contract. But let's come back to (3.1). The second term in (3.1) is $-qrSdt$. It represents the cost of borrowing qS . We assume that the cost of borrowing is equal to the risk-free interest rate. So $r(\pi - qS)dt$ is the return of the cash part of the trading account. And finally the last term in (3.1) is qdS . It represents the effect of the change in the underlying price.

The last feature which has to be specified in the contract is the position limit L . The noteholder is not allowed to go long or short beyond this position limit:

$$|q| \leq L. \quad (3.3)$$

At maturity the issuer has to give $\max(\pi, 0)$ to the noteholder.

3.2. Pricing

The puzzle in the price valuation is that the option holder's trading strategy is not known *a priori*. As in the American option valuation, we surmount the uncertainty of the option holder's strategy by maximising the price over all admissible trading strategies. As a result the price of the option under the complete market assumption (see Harrison and Pliska (1981), for example) is given by

$$\sup_{|q| \leq L} E \left[e^{-rT} \max(\pi_T(q), 0) \right] \quad (3.4)$$

where $q = \{q(t), 0 \leq t < T\}$ is the position of the option holder on the underlying asset, π_T is the balance of the trading account at the maturity T , r is the risk-free interest rate, L is the position limit ($|q| \leq L$), and E is the expectation under the risk-neutral measure, also known as the equivalent martingale measure which evaluates the value of the replicating portfolio as an expectation. Hyer, Lipton-Lifschitz, and Pugachevsky (1997) describe the price (3.4) as a solution of the HJB equation of the corresponding control problem. Provided that the underlying asset pays no dividend and that the growth rate of the cash balance in the account specified in the contract

coincides with the risk-free rate (the so called symmetric case), the equation for the option price $V(t, S, \pi)$ becomes:

$$-\frac{\partial V}{\partial t} = r\left(\pi \frac{\partial V}{\partial \pi} + S \frac{\partial V}{\partial S} - V\right) + \frac{\sigma^2 S^2}{2} \sup_{|q| \leq L} \left(q^2 \frac{\partial^2 V}{\partial \pi^2} + 2q \frac{\partial^2 V}{\partial \pi \partial S} + \frac{\partial^2 V}{\partial S^2} \right) \quad (3.5)$$

with terminal data $\max(\pi, 0)$. The first term in the right side indicates that the return of the replicating portfolio coincides with the risk-free rate. The balance of the trading account inherits the movement of the underlying asset except that its volatility is amplified (or condensed) by the amount of assets the customer holds. Thus the meaning of the last term in (3.5) is that the risk-less portfolio of the issuer must absorb the curvature effect (i.e., gamma) in the worst case.

We are not going to go through the details of the analysis because it is a particular case of the next chapter (multi-dimensional passport options). But we are going to state the results and give an intuitive explanation for them. We can assume $L = 1$ without loss of generality because

$$V_{L=L^*}(t, S, \pi) = V_{L=1}(t, L^* S, \pi). \quad (3.6)$$

Indeed the payoff is unchanged (it depends on π only) and the PDE is unchanged as well.

It is possible to reduce the dimension of the problem by introducing the variable $z = \frac{\pi}{S}$: the option price is

$$V(t, S, \pi) = S\phi(t, z) \quad (3.7)$$

and this new function ϕ is a solution of

$$-\frac{\partial \phi}{\partial t} = \frac{\sigma^2}{2} \max_{|q| \leq 1} \left[(z - q)^2 \frac{\partial^2 \phi}{\partial z^2} \right] \quad (3.8)$$

together with the final condition $\phi(T, z) = \max(z, 0)$. By arguing that ϕ is always convex in z (see Hyer et al. 1997), the equation simplifies to

$$-\frac{\partial \phi}{\partial t} = \frac{\sigma^2}{2} \max_{|q| \leq 1} \left[(z - q)^2 \right] \frac{\partial^2 \phi}{\partial z^2} \quad (3.9)$$

and the optimal strategy is

$$q^* = \begin{cases} 1 & \text{if } \pi < 0 \\ -1 & \text{if } \pi > 0. \end{cases}$$

The PDE for ϕ is therefore

$$-\frac{\partial \phi}{\partial t} = \frac{\sigma^2}{2} (1 + |z|)^2 \frac{\partial^2 \phi}{\partial z^2}. \quad (3.10)$$

We are going to give an intuitive explanation of these results in the next section.

3.3. Intuitive Explanation

3.3.1. Why is there no interest rate in the equation?

Let's look at the evolution of $d\pi$ under the risk-neutral measure:

$$\begin{aligned} d\pi &= r\pi dt + \Delta(dS - rSdt) \\ &= r\pi dt + \Delta(rSdt + \sigma SdW - rSdt) \\ &= \Delta\sigma SdW + r\pi dt \end{aligned} \quad (3.11)$$

where Δ is the number of assets held by the writer. The $r\pi dt$ term cancels out with the discount factor e^{-rt} of (3.4).

To summarize, the risk-free interest rate does not appear in the pricing equation for

two reasons:

- i) the cost of borrowing cancels out with the risk-neutral drift of dS .
- ii) the risk-free rate of the trading account cancels out with the discount factor in the option price.

3.3.2. The optimal strategy

Why $q^* = 1$ when $\pi < 0$ and $q^* = -1$ when $\pi > 0$?

I have given a few talks on passport options and every time I was asked the following question: “Surely the noteholder would be better long if μ is greater than r ”. The thing is that when pricing the option we are not trying to maximize the noteholder’s expected return! (This is actually the subject of one of the following chapters). We are in the risk-neutral world and we are looking for the strategy which maximizes the option price. And of course, if μ is large, the noteholder will probably go long, and he will probably make more money than if he had followed the optimal strategy. But q^* is still the strategy that maximizes the option price. In section 6-3, we are going to see that the passport option can be seen as an at-the-money call¹ with the right to switch this contract at any time into an in-the-money put if the underlying goes up (i.e. the trading account is positive) or into an out-of-the-money put if the underlying goes down (i.e. the trading account is negative). So when $\pi > 0$ the noteholder can choose between an in-the-money call ($q = 1$) and an in-the-money put ($q = -1$), The in-the-money put being more expensive, the optimal strategy is then $q^* = -1$. Similar arguments lead to $q^* = 1$ when $\pi < 0$.

¹By “at-the-money” we actually mean “strike $e^{r(T-t)}S_t$ ”.

3.3.3. Similarity solution

The option price V is a function of three variables: t, S and π . But this function of three variables has some symmetry. On the lines $\pi/S = \text{const}$ the option price is proportional to S . This nice property can be used to reduce the dimensionality of the pricing equation. This saves a lot of computation time when it comes to solve the PDE via a FDM scheme.

3.4. Literature Review

Hyer et al. study as well the case in which the underlying asset pays dividends and the growth rate of the trading account does not coincide with the risk-free rate. They provide numerical results for this more general setting, as does Nagayama (1999). Andersen et. al. (1998) study the case when the cost of borrowing is not part of the trading account and the asset pays dividends. Shreve and Vecer (2000) study the problem in the more general case when $q \in [\alpha, \beta]$. They study deeply the vacation call ($q \in [0, 1]$) and the vacation put ($q \in [-1, 0]$). Henderson and Hobson (2000) and Delbaen and Yor independently developed a new way of computing the value of the passport option in the symmetric case. Their argument is based on local time and Skorohod's lemma. Henderson and Hobson show as well that in the symmetric case the optimal trading strategy is unchanged if we allow the volatility to be a function of t and S , as long as it is an increasing function of the stock price. Delbaen and Yor study the case with zero interest rate via hitting time for a Bessel process. Passport options with stochastic volatility are studied by Henderson and Hobson (1999). In particular they find that the optimal strategy is the same as in the constant volatility model.

3.5. Outline of this Part

Various types of passport options are traded in the market, depending on the number of underlying assets and on exotic trading constraints. We establish the pricing equations for such options using stochastic control and optimal stopping times. In constructing a replicating strategy, the issuer must tune his position with the trading strategy performed by the option holder, not the one that maximises the price. Remember that the noteholder's current strategy is known to the issuer. If the issuer did not know the strategy he would not be able to hedge. What happens if the issuer hedges with the optimal strategy (not the strategy actually chosen by the noteholder)? Well, the issuer does not replicate the payoff at all! Think of the case where the optimal strategy is to go long and the noteholder chooses to go short.

In chapter 4, we describe the pricing equation for the multi-asset passport option in terms of the HJB equation. Only passport options on one asset have been studied so far. Chapter 4 is therefore a generalization of results known in one dimension to N dimensions. We show that the price maximising trading strategy has its value in the boundary of the position limit. Also we reduce the number of variables using the similarity solution. This reduces the burden of severe numerical computation.

In chapter 5, we investigate several exotic passport options. Due to the complexity of the notations and the numerical procedures we restrict our treatment to the single asset passport option. In particular, we consider two exotic trading constraints: (i) a restriction on the number of trades; (ii) a restriction on the time between trades. We describe the price maximising strategy as a sequence of optimal stopping times. The resulting pricing equations become multiple layers of free boundary partial differential equations (PDE). Only Andersen et. al. and Delbaen and Yor have been looking at discrete constraints: their model allows the noteholder to change q only at times which are fixed in advance. We provide as well a unified methodology for resolving further complications caused by imposing a penalty at each trade.

In chapter 6, we investigate the utility of the two positions, short and long, in trading passport options. We consider two models, one with perfectly specified drift and the other with imperfect information. The main result of this chapter is that the issuer makes a profit if the noteholder does not follow the optimal strategy. Indeed, for a given strategy there corresponds an option price. So if the noteholder follows a strategy which is different from the price maximizing strategy, the issuer will make a profit equal to the difference between the two option prices. There is no literature so far on the gains that the issuer and the buyer could make. In the final section we discuss a robust method for hedging passport options.

Chapter 4

Multi-dimensional Passport Options

Let $S = (S^{(1)}, \dots, S^{(n)})'$ be a set of tradeable assets. We assume that $S^{(i)}$ follows a lognormal random walk:

$$dS^{(i)} = \mu_i S^{(i)} dt + \sigma_i S^{(i)} dW^{(i)}$$

where $W = (W^{(1)}, \dots, W^{(n)})$ is a standard Brownian motion with correlation matrix $[\rho_{ij}]_{n \times n}$. The option holder maintains $q^{(i)}$ shares of the i -th asset, and the balance on the trading account π evolves as

$$d\pi = r \left(\pi - \sum_{i=1}^n q^{(i)} S^{(i)} \right) dt + \sum_{i=1}^n q^{(i)} dS^{(i)} \quad (4.1)$$

where r is the risk-free interest rate. The first term appearing in (4.1) describes how the cash balance (i.e., deposit/withdraw) in the trading account is compounded. The payoff of the passport option holder at the maturity is $\max(\pi_T, 0)$.

As in the one-dimensional passport option we can assume $L = 1$ because

$$V_{L=L^*}(t, S_1, \dots, S_n, \pi) = V_{L=1}(t, L^* S_1, \dots, L^* S_n, \pi). \quad (4.2)$$

4.1. Description of the Pricing Equation

In what follows we will use rather abstract notations: ∇ and ∇^2 are the gradient (the first derivative) and the Hessian¹ (the second derivative), respectively. First we state the equation, and then we will try to justify it financially.

We designate X to be the vector of size $n + 1$ consisting of π and S . For the time being, we restrict the trading strategy q to be a Markov policy (also known as a feedback control), meaning that $q(t)$ is a function of t and X_t . We define the value function of the problem as follows:

$$V(t, X) = \sup_{|q| \leq 1} E \left[e^{-r(T-t)} \max(\pi, 0) \right] \quad (4.3)$$

where the supremum is taken over all feasible Markov policies with position limit $L = 1$. By q we mean $\sup |q_i|$. The following result is the generalization to n dimensions of the pricing equation of Hyer et. al.:

Let $X = (\pi, S_1, \dots, S_n)'$. The value function $V(t, X)$ defined in (4.3) satisfies the following HJB equation:

$$-\frac{\partial V}{\partial t} = r \left(\langle \nabla V, X \rangle - V \right) + \frac{1}{2} \cdot \sup_{|q| \leq 1} \left\{ \langle q, Cq \rangle \frac{\partial^2 V}{\partial \pi^2} + 2 \langle \nabla_0^2 V, Cq \rangle + \langle S, DS \rangle \right\}, \quad (4.4)$$

$$V(T, x) = \max(\pi, 0),$$

where $C = [\rho_{ij} \sigma_i \sigma_j S_i S_j]_{n \times n}$, $D = [\rho_{ij} \sigma_i \sigma_j \nabla_{ij}^2 V]_{n \times n}$, and $\nabla_0^2 V = (\nabla_{01}^2 V, \dots, \nabla_{0n}^2 V)'$ and $\langle, \rangle$ is the scalar product.

If the contract permits the option holder to trade only one asset ($n = 1$), the HJB equation (4.4) is reduced to (3.5). Obtaining the above result is a standard procedure in stochastic control and one can check that a unique viscosity solution to (4.4) exists. In addition, the existence of a classical solution to (4.4) provides a justification for

¹The Hessian of V is then a $(n + 1) \times (n + 1)$ matrix consisting of entries $\nabla_{ij}^2 V$ at the (i, j) position.

restricting the class of trading strategies to that of Markov policies, which is known as the “verification theorem” in stochastic control theory. Discussing this in detail is beyond the scope of the thesis, and we refer to Fleming and Soner (1992). The implication of the theorem is that the value function of the control problem will not be increased by a non-Markovian policy as long as the status at the moment is the same. Thus, as long as the issuer of the passport option knows the current prices of the underlying assets and the balance of the trading account of his customer, it is not important how his customer ends up with the current balance.

Here we sketch an informal derivation of the HJB equation (4.4), and argue that it is the pricing equation for our problem. Under the complete market assumption, the issuer of the option can construct a risk-free portfolio using the underlying assets. Hence we consider the following portfolio:

$$\Pi = \langle \Delta, S \rangle - V^q = \sum_{i=1}^n \Delta^{(i)} S^{(i)} - V^q \quad (4.5)$$

where V^q is the value of the option corresponding to the known trading strategy q . As in the derivation of the Black-Scholes PDE it is going to be possible to construct a riskless portfolio. Let’s determine the correct Δ to make (4.5) risk-less. Due to the Markovian nature, the value V^q of the option will depend upon the prices of $S^{(i)}$, $i = 1, \dots, n$, and the balance of the trading account π at the moment. Applying Itô’s formula to V^q yields:

$$dV^q = \langle \nabla V^q, dX \rangle + \frac{\partial V^q}{\partial t} dt + \frac{1}{2} \nabla^2 V^q (dX, dX) \quad (4.6)$$

where $\nabla^2 V^q(,)$ is the bilinear form² defined by the Hessian of V . Recall that X consists of π and S . Thus we ascertain that the only random factor in the growth of Π at the moment is contained in

$$\langle \Delta, dS \rangle - \langle \nabla V^q, dX \rangle = \sum_{i=1}^n (\Delta^{(i)} - \nabla_i V^q - q^{(i)} \frac{\partial V^q}{\partial \pi}) dS^{(i)} - r \left(\pi - \langle q, S \rangle \right) \frac{\partial V^q}{\partial \pi} dt$$

where $\nabla_i V^q$ is the i th row of ∇V^q . In order to eliminate the risk caused by the random growth factor dS , we must choose $\Delta = (\nabla_i V^q + q^{(i)} \frac{\partial V^q}{\partial \pi})'_{i=1, \dots, n}$. Now the risk-less portfolio has to grow as the risk-free rate r :

$$-r \left(\pi - \langle q, S \rangle \right) \frac{\partial V^q}{\partial \pi} dt - \frac{\partial V^q}{\partial t} dt - \frac{1}{2} \nabla^2 V^q(dX, dX) = r \Pi dt$$

which is equivalent to

$$-\frac{\partial V^q}{\partial t} dt = r \left(\langle \nabla V^q, X \rangle - V^q \right) dt + \frac{1}{2} \nabla^2 V^q(dX, dX). \quad (4.7)$$

which is the pricing equation for V^q .

But this was for when we knew in advance the trading strategy of the noteholder! What is the option price when the noteholder is allowed to trade as he wishes? Remember that the option price can be expressed as the discounted expected payoff under the risk-neutral measure:

$$V^q(t, X) = e^{-r(T-t)} E[\max(\pi, 0)]. \quad (4.8)$$

The option price $V(t, X)$ when the noteholder can trade as he wants is

$$V(t, X) = \sup_{|q| \leq 1} V^q(t, X) \quad (4.9)$$

²We have $\nabla^2 V^q(dX, dX) = \langle \nabla^2 V dX, dX \rangle$.

which leads to (4.3). For this stochastic control problem, Bellman's principle is

$$V(t, X) = e^{-r\delta t} \sup_{|q| \leq 1} E \left[V(t + \delta t, X + \delta X) \right] \quad (4.10)$$

and the HJB equation for the price of the passport option is (4.4).

Note that the delta is given by

$$\Delta = \frac{\partial V}{\partial S} + q \frac{\partial V}{\partial \pi} \quad (4.11)$$

and it is therefore necessary for the hedger to be aware of the current trading position of the noteholder.

4.2. Properties of the Pricing Equation

We will discover several properties of the pricing equation (4.4). First, the equation (4.4) does not depend upon the risk-free rate r . Second, we construct a similarity solution in such a way that the number of space variables in the equation can be reduced. This reduces the computation time and error in solving the equation numerically. Finally, we will discuss the behaviour of q^* the price maximising trading strategy.

4.2.1. No interest rate in the equation

Inspecting the terms in the pricing equation reveals that the value function V is homogeneous of degree 1 in space variables:

$$V(t, \lambda X) = \lambda V(t, X) \text{ for each scalar } \lambda. \quad (4.12)$$

Indeed, the transformation $X \rightarrow \lambda X$ leaves the PDE unchanged and multiplies the payoff by λ . And the PDE is linear in V . Differentiating this identity with respect to λ and evaluating it at $\lambda = 1$, we obtain the following:

The solution V of the HJB equation (4.4) satisfies $\langle \nabla V, X \rangle = V$.

As a result, the price of the passport option doesn't depend upon the risk-free rate. As we mentioned earlier in the section, the growth rate of the cash balance in the trading account is the value specified in the contract. If it differs from the risk-free rate r then the price of the option depends upon the growth rate as well as the risk-free rate. Also if the terminal payoff is not $\max(\pi, 0)$, the result may fail.

4.2.2. Similarity solution

Next we consider a diffeomorphism ξ on the state space $\mathbb{R} \times (\mathbb{R}^+)^n$ which satisfies the following properties: for all $\lambda \in \mathbb{R}$,

$$\begin{aligned}\xi_i(\lambda X) &= \xi_i(X), & i = 0, \dots, n-1, \\ \xi_n(\lambda X) &= \lambda \xi_n(X),\end{aligned}$$

for each X in $\mathbb{R} \times (\mathbb{R}^+)^n$. Define u via $V(t, X) = V(t, \xi^{-1} \circ \xi(X)) = u(t, \xi(X))$. Then we must have

$$\lambda u(t, \xi(x)) = u(t, \xi(\lambda X)) = u(t, \xi_0(X), \dots, \xi_{n-1}(X), \lambda \xi_n(X)).$$

Therefore we conclude that $u(t, \xi) = \xi_n \phi(t, \xi_0, \dots, \xi_{n-1})$ for some function ϕ . For example, suppose we choose a diffeomorphism ξ on $\mathbb{R} \times (\mathbb{R}^+)^n$ defined by

$$\xi(x) = \left(\frac{X_0}{X_n}, \dots, \frac{X_{n-1}}{X_n}, X_n \right).$$

Then the value function V can be rewritten as $V(t, X) = X_n \phi(t, z, y)$ where $z = \pi/X_n$ and y is a vector of size $n - 1$ with its element $y_i = X_i/X_n, i = 1, \dots, n - 1$. And one can then obtain the HJB equation satisfied by ϕ :

$$-\frac{\partial \phi}{\partial t} = \frac{1}{2} \cdot \sup_{|q| \leq 1} \left\{ \langle q, \hat{C}q \rangle \frac{\partial^2 \phi}{\partial z^2} + 2 \langle \nabla_0^2 \phi, \Lambda q \rangle + \langle y, \hat{D}y \rangle \right\}, \quad (4.13)$$

$$\phi(T, z, y) = \max(z, 0),$$

where $\hat{D} = [(\sigma_n^2 - \rho_{in}\sigma_i\sigma_n - \rho_{jn}\sigma_j\sigma_n + \rho_{ij}\sigma_i\sigma_j)\nabla_{ij}\phi]_{(n-1) \times (n-1)}$, $\nabla_0^2 \phi = (\nabla_{01}^2 \phi, \dots, \nabla_{0(n-1)}^2 \phi)'$ and where

$$\begin{aligned} \langle q, \hat{C}q \rangle &= \sigma_n^2(z - q_n)^2 - 2\sigma_n(z - q_n) \sum_{i=1}^{n-1} \rho_{in}\sigma_i y_i q_i + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \rho_{ij}\sigma_i\sigma_j y_i y_j q_i q_j, \\ (\Lambda q)_i &= \sigma_n^2(z - q_n)y_i + \sum_{j=1}^{n-1} \sigma_j (\rho_{ij}\sigma_i - \rho_{nj}\sigma_n) y_i y_j q_j + \rho_{ni}\sigma_n\sigma_i y_i q_n. \end{aligned}$$

When $n = 1$, y is nullified and all the sums vanish. Thus the value function for the single-asset passport option can be written as $V(t, S, \pi) = S\phi(t, \pi/S)$ where $\phi(t, z)$ satisfies

$$-\frac{\partial \phi}{\partial t} = \frac{\sigma^2}{2} \cdot \sup_{|q| \leq 1} (z - q)^2 \frac{\partial^2 \phi}{\partial z^2} \quad \left(\text{or} = \frac{\sigma^2}{2} (1 + |z|)^2 \frac{\partial^2 \phi}{\partial z^2} \right) \quad (4.14)$$

with terminal data $\max(z, 0)$.

4.2.3. The optimal strategy

Next we discuss the behaviour of q^* the price maximising trading strategy. Since the terminal data $\max(\pi, 0)$ is convex in π , $\frac{\partial^2 V}{\partial \pi^2}$ stays positive as long as the solution of (4.4) does not explode. Also note that C is a positive definite matrix. Therefore the

supremum in the HJB equation (4.4) is always attained on the boundary of the feasible set. Of course this will fail when the terminal data is not convex.

The price maximising trading strategy q^* satisfies $|q_t^*| = 1$ for all t almost surely.

4.3. Numerical Example : Dual Passport (n=2)

The HJB equation for V is

$$\begin{aligned}
-\frac{\partial V}{\partial t} = & r(S_1 \frac{\partial V}{\partial S_1} + S_2 \frac{\partial V}{\partial S_2} + \pi \frac{\partial V}{\partial \pi} - V) + \\
& + \frac{\sigma_1 S_1^2}{2} \frac{\partial^2 V}{\partial S_1^2} + \frac{\sigma_2 S_2^2}{2} \frac{\partial^2 V}{\partial S_2^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \\
& + \max_{|q_1| \leq 1, |q_2| \leq 1} \left[q_1^2 \frac{\sigma_1 S_1^2}{2} \frac{\partial^2 V}{\partial \pi^2} + q_1 (\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial \pi \partial S_1} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial \pi \partial S_2}) + \right. \\
& + q_2^2 \frac{\sigma_2 S_2^2}{2} \frac{\partial^2 V}{\partial \pi^2} + q_2 (\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial \pi \partial S_2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial \pi \partial S_1}) + \\
& \left. + \rho q_1 q_2 \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial \pi^2} \right]. \tag{4.15}
\end{aligned}$$

And the HJB equation for ϕ is

$$\begin{aligned}
-\frac{\partial V}{\partial t} = & \max_{|q_1| \leq 1, |q_2| \leq 1} \left[(\sigma_2^2 (z - q_2)^2 - 2\rho \sigma_1 \sigma_2 y_1 q_1 + \sigma_1^2 y_1^2 q_1^2) \phi_z z + \right. \\
& + (\sigma_2^2 (z - q_2) y + q_1 \sigma_1 (\sigma_1 - \rho \sigma_2) y^2 + q_2 \rho \sigma_1 \sigma_2 y) \phi_{yz} + \\
& \left. + y^2 (\sigma_2^2 - 2\rho \sigma_1 \sigma_2 + \sigma_1^2) \phi_{yy} \right] \tag{4.16}
\end{aligned}$$

In the following numerical example, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$ and $\rho = 0.5$ (ρ is the correlation coefficient). We have solved the above equation via an explicit FDM scheme.

Figure 4.1 is a graph of the price of the 6 months dual passport option with constraint $q_1 q_2 = 0$: i.e., the option holder may use his passport to enter into different territories,

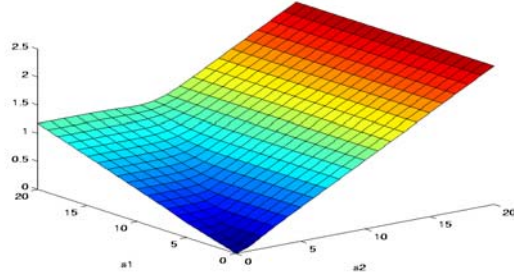


Figure 4.1. Price of dual passport option with constraints $q^{(1)}q^{(2)} = 0$.

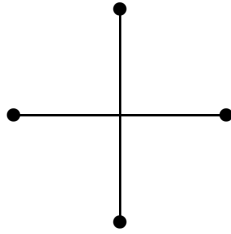


Figure 4.2. Candidates for the optimal strategy with the constraint $q_1q_2 = 0$.

but given any time the physical presence has to be in either one of them. Intuitively this constraint reduces the effect of correlation by not allowing the two assets to coalesce in the trading account. Thus pricing and hedging are somewhat less sensitive to the possible misspecification of the correlation. In this example, the volatility of S_2 dominates that of S_1 . Thus, when the price of S_1 is greater than that of S_2 , the price the option is robust to the change of prices in S_1 . The optimal strategy is one of the following four: $(q_1, q_2) = (1, 0), (-1, 0), (0, 1), (0, -1)$.

Figure 4.4 is the case when there is no constraint. Obviously the contract without the constraint $q^{(1)}q^{(2)} = 0$ is more expensive than the one with the constraint. The optimal strategy is one of the following four: $(q_1, q_2) = (1, 1), (1, -1), (-1, 1), (-1, -1)$. Indeed in the expression to maximize over q_1 and q_2 the coefficients of both q_1^2 and q_2^2 are positive. The option price is then much more sensitive to the correlation coefficient. If $\rho = 1$ the expected value of the payoff will be very high with $q_1 = q_2 = 1$.

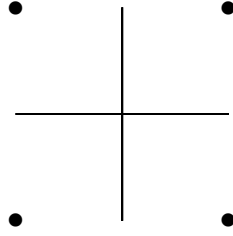


Figure 4.3. Candidates for the optimal strategy when there is no constraint

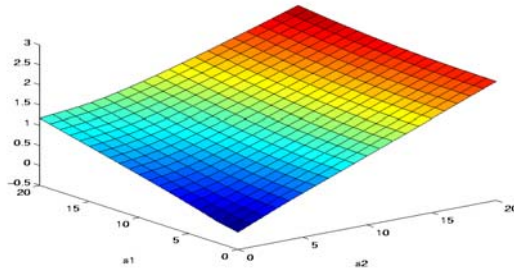


Figure 4.4. Price of dual passport option without constraints.

Chapter 5

Exotic Passport Options

In this section, we discuss the various exotic features that we can add to the original vanilla passport option. We will confine the scope of our discussion to the single-asset passport option. We are going to provide numerical examples for most of the contracts. We solve all the PDE's using the explicit FDM.

5.1. Maximum number of trades

In this passport option the noteholder is not allowed to change his q more than N times, with N specified in the contract. It is clear that the price maximising strategy has its value (i.e., the amount of holding) equal to one of the limits ± 1 . Then the price maximising strategy can be identified by a sequence of optimal stopping times. We will describe the option price with multiple layers of free boundary PDE's. We refer to Van Moerbeke (1976) for the justification of the equivalence of the optimal stopping problems and free boundary PDE's.

5.1.1. One trade only

When the option holder is permitted to trade only once, an analytical approach is available. We demonstrate this case first. We will start by arguing that the value of the option is maximised when the option holder trades at time 0. Suppose that the option holder buys one share of the underlying at time $\tau \in [0, T)$. Then the balance of his trading account at the maturity becomes $\pi_T = S_T - S_\tau e^{r(T-\tau)}$, and hence the value of $\max(\pi_T, 0)$ at time τ is the same as that of the European call with strike $S_\tau e^{r(T-\tau)}$:

$$S_\tau \left(N\left(\frac{1}{2}\sigma\sqrt{T-\tau}\right) - N\left(-\frac{1}{2}\sigma\sqrt{T-\tau}\right) \right) = S_\tau g(T-\tau) \quad (5.1)$$

where N is the distribution function of the standard normal random variable. Equation (5.1) defines g . If the option holder shorts the underlying at time $\tau \in [0, T)$, the balance ends up $\pi_T = S_\tau e^{r(T-\tau)} - S_T$ at the maturity. In this case, the value of $\max(\pi_T, 0)$ at time τ coincides with that of the European put with strike $S_\tau e^{r(T-\tau)}$, which is identical to (5.1) by the put-call parity. Note that $t \rightarrow g(t)$ is a decreasing function. Therefore

$$E\left[e^{-r\tau} S_\tau g(T-\tau)\right] \leq E\left[e^{-r\tau} S_\tau\right] g(T) = S_0 g(T). \quad (5.2)$$

The last equality is obtained from the fact that $e^{-rt} S_t$ is a martingale under the risk neutral measure and by the optional sampling theorem (see Karatzas and Shreve (1988), for example). As a result, the stopping time that maximises the option value must be identically 0, and the price of the option is $Sg(T)$. This coincides with that of the European call with strike Se^{rT} , or equivalently the European put with the same strike.

The hedging strategy is straightforward in this case. At time 0, the issuer buys $g(T)$ shares of the underlying asset and stays with the position until the option holder demands a trade. This costs the issuer exactly the price of the option he wrote. Suppose the option holder wants to trade at time τ . The issuer cashes $g(T-\tau)$ shares of his assets, and pays for a call or put at strike $S_\tau e^{r(T-\tau)}$, depending on whether the option holder

wants to be long or short in the underlying asset. From then on, the vanilla option takes care of the rest. Therefore when τ is greater than zero, the issuer gains $g(T) - g(T - \tau)$ shares of the underlying which he may cash in anytime. A risk neutral choice is to cash $-g'(T - t)dt$ shares of the underlying assets during $[T - t, T - t + dt]$ until the customer demands a trade.

5.1.2. N trades only

Our goal is to describe the price maximising value function when only a finite number of trades are permitted. Note that our model is different from the ones of Andersen et. al. and Delbaen and Yor. Indeed their model allows the strategy q to be modified at N discrete times only $(t_0, t_1, \dots, t_{N-1})$ where $t_0 = 0$ and $t_N = T$.

As we discovered earlier in the case when only one trade is permitted, the worst case for the issuer is when his customer demands a trade at the very moment he signs the contract. So for the time being, we will assume that one trade is made at time 0.

We consider $V^{(n+)}$ and $V^{(n-)}$, where n is the number of trades still to be made and $+/-$ indicates the current position, long or short. Then the price of the option is the maximum of $V^{(n+)}(0, S, 0)$ and $V^{(n-)}(0, S, 0)$, if $n + 1$ trades are allowed, because we assumed one trade is made at time 0. These functions are homogeneous of degree 1 in space variables, and hence $\pi \frac{\partial V^{(n\pm)}}{\partial \pi} + S \frac{\partial V^{(n\pm)}}{\partial S} = V^{(n\pm)}$ regardless of n . If the option holder is not allowed to trade any more ($n = 0$), the value functions evolve without an obstacle:

$$\begin{aligned} \mathcal{L}^+ V^{(0+)} &= \frac{\partial V^{(0+)}}{\partial t} + \frac{\sigma^2 S^2}{2} \left(\frac{\partial^2 V^{(0+)}}{\partial S^2} + 2 \frac{\partial^2 V^{(0+)}}{\partial S \partial \pi} + \frac{\partial^2 V^{(0+)}}{\partial \pi^2} \right) = 0, \\ \mathcal{L}^- V^{(0-)} &= \frac{\partial V^{(0-)}}{\partial t} + \frac{\sigma^2 S^2}{2} \left(\frac{\partial^2 V^{(0-)}}{\partial S^2} - 2 \frac{\partial^2 V^{(0-)}}{\partial S \partial \pi} + \frac{\partial^2 V^{(0-)}}{\partial \pi^2} \right) = 0, \end{aligned}$$

with terminal data $V^{(0\pm)}(T, S, \pi) = \max(\pi, 0)$. Now we investigate $V^{(n+)}$ for $n > 0$. Suppose that $\Pi = \Delta S - V^{(n+)}$ is the risk-less portfolio for the issuer. As we described

in Section 2, the issuer must choose $\Delta = \frac{\partial V^{(n+)}}{\partial S} + q \frac{\partial V^{(n+)}}{\partial \pi}$ where q is the actual trading strategy performed by his customer. Since the risk-less portfolio must grow at least at the risk-free rate r ,

$$\mathcal{L}^+ V^{(n+)} = \frac{\partial V^{(n+)}}{\partial t} + \frac{\sigma^2 S^2}{2} \left(\frac{\partial^2 V^{(n+)}}{\partial \pi^2} + 2 \frac{\partial^2 V^{(n+)}}{\partial S \partial \pi} + \frac{\partial^2 V^{(n+)}}{\partial S^2} \right) \leq 0. \quad (5.3)$$

The equality must hold at least in one case to avoid arbitrage, and when it does, it is the worst case for the issuer. Suppose that $\mathcal{L}^+ V^{(n+)}$ is strictly less than 0 in a situation. This means that the option holder staying with long position is no longer the worst case for the issuer. Thus the value $V^{(n+)}$ must coincide with the residual value $V^{((n-1)-)}$. On the other hand, if $V^{(n+)}$ exceeds $V^{(n-1)-}$, the trade demanded by the option holder does not provoke the worst case for the issuer. Thus $\mathcal{L}^+ V^{(n+)}$ vanishes. Combining these, we obtain the linear complementary problem:

$$\begin{cases} \mathcal{L}^+ V^{(n+)} \leq 0 \\ V^{(n+)} \geq V^{((n-1)-)} \\ \mathcal{L}^+ V^{(n+)} \cdot (V^{(n+)} - V^{((n-1)-)}) = 0 \end{cases}$$

with the terminal condition $V^{(n+)}(T, S, \pi) = \max(\pi, 0)$. Similarly, $V^{(n-)}$ satisfies the following:

$$\begin{cases} \mathcal{L}^- V^{(n-)} \leq 0 \\ V^{(n-)} \geq V^{(n-1)+} \\ \mathcal{L}^- V^{(n-)} \cdot (V^{(n-)} - V^{(n-1)+}) = 0 \end{cases}$$

with terminal data $V^{(n-)}(T, S, \pi) = \max(\pi, 0)$. If the contract specifies a fixed amount of penalty, say p , on each trade the option holder engages, then we replace the free boundary conditions in the above sets of equations by

$$V^{(n+)} \geq V^{((n-1)-)} + p \quad \text{and} \quad V^{(n-)} \geq V^{((n-1)+)} + p.$$

Now we discuss how the issuer hedges the option. Suppose that n trades are allowed so that the maximum of $V^{((n-1)+)}$ and $V^{((n-1)-)}$ is the price. The price is proportional to that of the underlying asset, and hence we may write it as $Sg_n(T)$. When we construct $V^{(k\pm)}$, $k \geq 0$, we assumed that the option holder trades at time 0 while he may not have to. If the option holder indeed trades at time 0, the issuer follows $V^{(k\pm)}$, $k = n-1, \dots, 0$, tracking his customer's position (+/- as long/short) and the number of trades to be made. For example, if the customer holds 0.7 shares of the underlying and if he is allowed to engage 3 more trades, the issuer holds $\Delta = \frac{\partial V^{(3+)}}{\partial S} + 0.7 \frac{\partial V^{(3+)}}{\partial \pi}$.

Next, suppose that the option holder is not sure about the market direction at time 0 and he waits until time $\tau > 0$. In this case, the issuer buys $g_n(T)$ shares of the underlying asset at time 0. Since $S_\tau g_n(T - \tau)$ is sufficient for the issuer to hedge the option from the moment (i.e., τ) his customer makes the first trade and since g_n is decreasing, the issuer gains $g_n(T) - g_n(T - \tau)$ shares of the underlying asset. As before the issuer may cash in $g_n(T) - g_n(T - \tau)$ shares gradually until his customer decides to make the first trade.

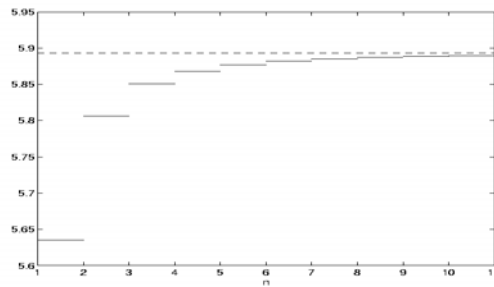


Figure 5.1. Limited number of trades.

See the end of the next section for comments of the above picture.

5.2. Passport Option with a Clock

Suppose that the contract states that the option holder is allowed to trade only after a specified time, say ω , has elapsed since the last trade. We introduce the idea of a clock which keeps track of time since the last trade. The clock is reset to zero immediately after each time the option holder trades, and it keeps ticking until its hand reaches ω , where it remains until the next trade. To model this, we introduce an additional time variable θ :

$$\theta(t) = \begin{cases} t - \tau_i, & \text{if } \tau_i \leq t < \tau_i + \omega, \\ \omega, & \text{if } \tau_i + \omega \leq t < \tau_{i+1} \end{cases}$$

where τ_i and τ_{i+1} are adjacent trading times. The option holder is allowed to trade only when the clock is dormant, i.e. $\theta = \omega$.

We will describe the value of the option using price maximising value functions, $V^{(+)}$ and $V^{(-)}$. The status described by $V^{(+)}(t, S, \pi, \theta)$, for $\theta \in (0, \omega)$ is that the option holder is currently long in asset, but not allowed to trade. $V^{(+)}(t, S, \pi, \omega)$ is for the case when the option holder is currently long in asset and he is allowed to trade which puts him in a short position in asset. $V^{(+)}(t, S, \pi, 0)$ describes the very moment the option holder puts himself in a long position in asset. $V^{(-)}$ describes the opposite case. As before these functions are homogeneous of degree 1 in space variables, and hence $\pi \frac{\partial V^{\pm}}{\partial \pi} + S \frac{\partial V^{\pm}}{\partial S} = V^{\pm}$.

The evolution of the price maximising value functions $V^{(\pm)}$ will depend on the status of the clock, active or dormant. First we consider the case when the clock is active ($\theta < \omega$). The option holder is not allowed to trade and all he can do is to watch the

clock ticking anxiously. The price maximising value functions evolve naturally (i.e., without any obstacles) as the clock ticks:

$$\begin{aligned}\frac{\partial V^{(+)}}{\partial t} + \frac{\partial V^{(+)}}{\partial \theta} + \frac{\sigma^2 S^2}{2} \left(\frac{\partial^2 V^{(+)}}{\partial S^2} + 2 \frac{\partial^2 V^{(+)}}{\partial \pi \partial S} + \frac{\partial^2 V^{(+)}}{\partial \pi^2} \right) &= 0, \\ \frac{\partial V^{(-)}}{\partial t} + \frac{\partial V^{(-)}}{\partial \theta} + \frac{\sigma^2 S^2}{2} \left(\frac{\partial^2 V^{(-)}}{\partial S^2} + 2 \frac{\partial^2 V^{(-)}}{\partial \pi \partial S} + \frac{\partial^2 V^{(-)}}{\partial \pi^2} \right) &= 0,\end{aligned}\quad (5.4)$$

Next, we suppose that the clock is dormant ($\theta = \omega$), and hence the option holder is allowed trade. Then the value functions evolve with several rules. First, the price maximising value functions must stay above the residual values:

$$V^{(+)}(t, S, \pi, \omega) \geq V^{(-)}(t, S, \pi, 0) \quad \text{and} \quad V^{(-)}(t, S, \pi, \omega) \geq V^{(+)}(t, S, \pi, 0). \quad (5.5)$$

Second, the risk-less portfolio for the issuer grows at least at the risk free rate:

$$\mathcal{L}^+ V^{(+)} = \frac{\partial V^{(+)}}{\partial t} + \frac{\sigma^2 S^2}{2} \left(\frac{\partial^2 V^{(+)}}{\partial S^2} + 2 \frac{\partial^2 V^{(+)}}{\partial \pi \partial S} + \frac{\partial^2 V^{(+)}}{\partial \pi^2} \right) \leq 0, \quad (5.6)$$

$$\mathcal{L}^- V^{(-)} = \frac{\partial V^{(-)}}{\partial t} + \frac{\sigma^2 S^2}{2} \left(\frac{\partial^2 V^{(-)}}{\partial S^2} + 2 \frac{\partial^2 V^{(-)}}{\partial \pi \partial S} + \frac{\partial^2 V^{(-)}}{\partial \pi^2} \right) \leq 0, \quad (5.7)$$

where each function is evaluated at (t, S, π, ω) . Here we dropped $V_\theta^{(\pm)}$ as the clock is no longer ticking. The third rule combines the first two in the following sense. $\mathcal{L}^+ V^{(+)} < 0$ indicates that trading demand does not provoke the worst case. This happens only when $V^{(+)}(t, S, \pi, \omega) = V^{(-)}(t, S, \pi, 0)$. On the other hand, if $V^{(+)}(t, S, \pi, \omega)$ exceeds $V^{(-)}(t, S, \pi, 0)$, waiting does provoke the worst case. In summary,

$$\mathcal{L}^+ V^{(+)}(t, \pi, S, \omega) \cdot \left(V^{(+)}(t, S, \pi, \omega) - V^{(-)}(t, S, \pi, 0) \right) = 0. \quad (5.8)$$

In the opposite case, we have

$$\mathcal{L}^- V^{(-)}(t, S, \pi, \omega) \cdot \left(V^{(-)}(t, S, \pi, \omega) - V^{(+)}(t, S, \pi, 0) \right) = 0. \quad (5.9)$$

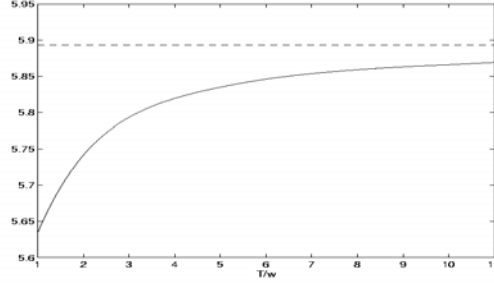


Figure 5.2. Price of a passport option with a clock.

The conditions (5.5) to (5.9) define a system of linear complementary problem. As in the case without clock, the price is the maximum of $V^{(+)}(0, S, 0, 0)$ and $V^{(-)}(0, S, 0, 0)$, and hedging is a matter of tracking the position of the option holder. If there is a fixed penalty p on each trade, we replace the free boundary conditions in (5.5) by

$$V^{(+)}(t, S, \pi, \omega) \geq V^{(-)}(t, S, \pi, 0) + p \quad \text{and} \quad V^{(-)}(t, S, \pi, \omega) \geq V^{(+)}(t, S, \pi, 0) + p.$$

Numerical Results

The price of the single-asset passport option is proportional to the price of the underlying asset, even with discrete trading constraints. Let $g(n)$ be the proportional constant when n trades are permitted (i.e., the maximum of $V^{(n-1)+}$ and $V^{(n-1)-}$) and $g_c(\omega)$ for the clock. Figure 5.1 shows the value of $100 \cdot g$, thus the price of the option when the underlying asset price is 100, for $n = 1, \dots, 10$. The dotted line is the price of the passport option without trading constraints, i.e., $n = \infty$. The volatility is 20%

and the maturity is six months. Figure 5.2 (b) is the price of the passport option with clock, $100 \cdot g_c$. Parameters are the same as (a). We have chosen the variable T/ω for the horizontal axis. Thus the number of permitted trades is the greatest integer not more than the variable T/ω . Again the dotted line is the case when there is no clock, i.e., $\omega = 0$. Note that the option with clock is less expensive. This is because it has more restriction even with the same number of trades are allowed. In both cases we have used the explicit FDM.

Also it is noteworthy that the price of the option with 10 trades is already close to that of the unconstrained passport option. In practice, however, it is customary that the contract compels the option holder to refrain from frequent trades. In order to maintain a delta-neutral position, the issuer of the passport option needs to trade the underlying asset at least as often as his customer. When the customer trades very often, the issuer is burdened with excessive transaction costs.

5.3. Chooser Passport Options

At a time $t_1 \in [0, T]$ specified in the contract, the noteholder has to choose between the two following payoffs at time T : $\max(\pi, 0)$ and $\max(-\pi, 0)$.

So the noteholder should have his trading account as far away as possible from zero at time t_1 : if π is positive he will choose $\max(\pi, 0)$ and if it is negative he will go for $\max(-\pi, 0)$. According to the choice of payoff he will have made at t_1 , what he must do after t_1 is clear. But what he has got to do before t_1 is not that obvious: should he try to increase π or should he try to decrease it? Looking at this from a different point of view, the chooser passport option would be useful to the noteholder who thinks there is going to be a large price movement between $t = 0$ and $t = t_1$, but does not know in which direction: he thinks his trading account will be large in absolute value at t_1 and would choose at that time the appropriate payoff.

To price this contract we first need to establish the “Put-Call parity” for Passport Options.

Let C be price of the ‘Vanilla’ passport option which expires at T with payoff $\max(\pi, 0)$ and P the price of the Vanilla passport option which expires at T with payoff $\max(-\pi, 0)$.

So the “Put-Call parity” for passport options is

$$P - C + \pi = 0,$$

which is valid at any time $t \in [0, T]$.

At time t_1 , the price of the chooser passport option is

$$V(t_1, S, \pi) = \max(C(t_1, S, \pi), P(t_1, S, \pi)).$$

Using “Put-Call parity” in the above equation, we get

$$V(t_1, S, \pi) = C(t_1, S, \pi) + \max(0, -\pi)$$

and we can find $V(S, 0, 0) = S\phi(0, 0)$ by solving the following problem:

$$\left\{ \begin{array}{l} \Phi(T, z) = \max(z, 0) \\ -\frac{\partial \Phi}{\partial t} = \frac{\sigma^2}{2}(1 + |z|)^2 \frac{\partial^2 \Phi}{\partial z^2} \text{ for } t_1 \leq t \leq T \\ \phi(t_1, z) = \Phi(t_1) + \max(0, -z) \\ -\frac{\partial \phi}{\partial t} = \frac{\sigma^2}{2}(1 + |z|)^2 \frac{\partial^2 \phi}{\partial z^2} \text{ for } 0 \leq t \leq t_1. \end{array} \right.$$

Figure 5.3 gives an example of a chooser passport option. For $t_1 = 0$ the option is a standard vanilla passport option. For $t_2=0.5$ the option is a passport option with payoff $|\pi|$. It is therefore equal to twice the price of the vanilla passport option.

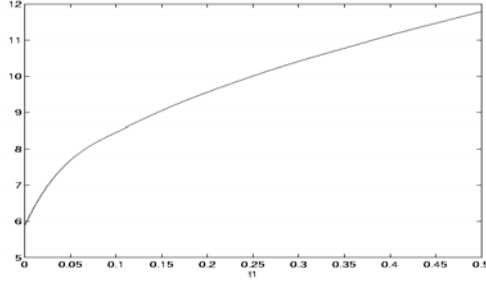


Figure 5.3. Chooser passport option.

5.4. Barrier Passport Options

The barrier passport option is inspired by barrier options: if the trading account reaches a certain positive number (say B) which is specified in the contract, the barrier passport option automatically expires and the noteholder pockets B . This option is cheaper than the Vanilla one, and can be useful to the noteholder who cannot see the point of keeping a passport option when the trading account is large.

This option is priced as usual barrier options: we add the boundary condition

$$V(t, S, B) = B.$$

Again the option price does not depend on the interest rate and the pricing equation becomes

$$-\frac{\partial V}{\partial t} = \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \frac{\sigma^2 S^2}{2} \left[\frac{\partial^2 V}{\partial \pi^2} + 2 \left| \frac{\partial^2 V}{\partial \pi \partial S} \right| \right]$$

which has to be solved in the domain $(t, S, \pi) \in [0, +\infty) \times (-\infty, B] \times [0, T]$ with $V(T) = \max(\pi, 0)$ and $V(t, S, B) = B$. Note that no similarity solution is available as we need to know explicitly the value of S in order to apply the boundary condition. If the barrier depended on π/S only we would have had a similarity solution. Figure 2 gives an example of a barrier passport option. The four lines which are not straight are (from below) for $B = 7$, $B = 8$, $B = 9$ and $B = 10$.

Note that the value of the option on the barrier could be a function of time (for example). We could have had a boundary condition of the type $V(t, S, -B) = 0$ as well to make the option cheaper. It is important to notice that if the boundary conditions make the payoff non-convex (e.g. $V(t, S, B) = 0$), the maximum does not have to be attained on $|q| = 1$.

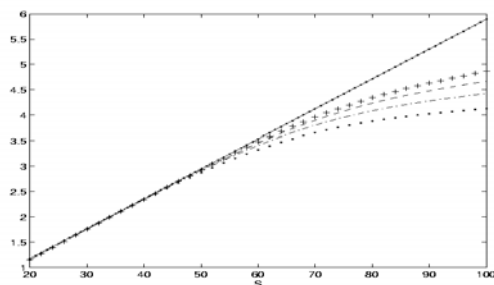


Figure 5.4. Barrier passport option.

5.5. Smooth Passport Options

As the Vanilla passport option specifies that $|q| \leq L$, the smooth trading passport option has a constraint on $\frac{\partial q}{\partial t}$ instead: $|q_t| \leq L'$. This means that the noteholder cannot change his position from $q = 1$ to $q = -1$ (for example) instantaneously. Indeed, there is a maximum speed (namely L') at which q can be changed by the noteholder. If L' and T are not too large, this option will be cheaper than the Vanilla one and maybe more attractive to the noteholder who does intend to trade smoothly.

The option price depends on q and the HJB equation for V becomes¹

$$-\frac{\partial V}{\partial t} = \frac{\sigma^2 S^2}{2} \left(\frac{\partial^2 V}{\partial S^2} + q^2 \frac{\partial^2 V}{\partial \pi^2} + 2q \frac{\partial^2 V}{\partial \pi \partial S} \right) + \max_{|\frac{\partial q}{\partial t}| \leq L'} \left[\frac{\partial V}{\partial q} \frac{\partial q}{\partial t} \right],$$

with the usual final condition $V(T, S, \pi, q) = \max(\pi, 0)$. Indeed, it is now q_t that the noteholder can choose and it is therefore among all possible (Markovian) $\frac{\partial q}{\partial t}$'s that the maximisation takes place.

Using $V(t, S, \pi, q) = S\phi(t, z, q)$, where $z = \pi/S$, the HJB equation satisfied by the similarity solution ϕ is

$$-\frac{\partial \phi}{\partial t} = \frac{\sigma^2}{2} (z - q)^2 \frac{\partial^2 \phi}{\partial z^2} + L' \left| \frac{\partial \phi}{\partial q} \right|,$$

with $\phi(T) = \max(z, 0)$ as usual.

5.6. The Piecewise Smooth Trader Passport Option

In the smooth trader passport option, the constraint on the slope of q might be too restrictive in certain circumstances: imagine the trader suddenly changes his mind about the direction of the asset. He would be stuck in a position he does not like at all. As a remedy to this rather annoying feature, we introduce an extra clause in the contract

¹ $V(t, \lambda S, \lambda \pi, q) = \lambda V(t, S, \pi, q)$ and the option price does not depend on r ,

which allows the noteholder a certain number of jumps (say N) in $q(t)$. In order to avoid large jumps, we reintroduce the position limit L : $|q| \leq L$.

We call V^n the Option price when n jumps are allowed; V^0 is just the Pure Smooth Trading Passport Option described in the first part of the section.

For each t^*, S^*, π^*, q^* , the Option price $V^n(t^*, S^*, \pi^*, q^*)$ should be larger than $V^{n-1}(t^*, S^*, \pi^*, q)$ for all q 's in $[-L, L]$. Indeed, if the noteholder decides to exercise his right to have a jump in q^* at t^* , then $q^*(t^* + dt)$ could be any q allowed in the contract.

In terms of the similarity solution ϕ , the linear complementary problem is

$$\begin{cases} \mathcal{L}\phi^n \leq 0 \\ \phi^n \geq \max_{|q| \leq L} \{\phi^{n-1}\} \\ \mathcal{L}\phi^n \cdot (\phi^n - \max_{|q| \leq L} \{\phi^{n-1}\}) = 0 \end{cases}$$

with $\phi(T) = \max(z, 0)$ and where $\mathcal{L}$ is such that

$$\mathcal{L}\phi = \begin{cases} \frac{\partial \phi}{\partial t} + \frac{\sigma^2}{2}(z - q)^2 \frac{\partial^2 \phi}{\partial z^2} + \max_{|q_t| \leq L'} \left[\frac{\partial \phi}{\partial q} q_t \right] & \text{if } |q| < L \\ \frac{\partial \phi}{\partial t} + \frac{\sigma^2}{2}(z - q)^2 \frac{\partial^2 \phi}{\partial z^2} + \max_{0 \leq q_t \leq L'} \left[\frac{\partial \phi}{\partial q} q_t \right] & \text{if } q = -L \\ \frac{\partial \phi}{\partial t} + \frac{\sigma^2}{2}(z - q)^2 \frac{\partial^2 \phi}{\partial z^2} + \max_{-L' \leq q_t \leq 0} \left[\frac{\partial \phi}{\partial q} q_t \right] & \text{if } q = L \end{cases}$$

Financially speaking: either it is optimal for the noteholder to exercise his right to have a jump, in which case $\phi^n = \max_{|q| \leq L} \{\phi^{n-1}\}$ and as a consequence $\mathcal{L}\phi^n < 0$, or it is not optimal for the noteholder to exercise his right to have a jump and the “smooth trader passport option equation” is valid together with the constraint $\phi^n > \max_{|q| \leq L} \{\phi^{n-1}\}$. We are going to meet more linear complementary in the next sections: we only justify their second inequalities because the first inequality and the equality can be “deduced” from it (because all these passport options obey the same rules).

A typical price of a piecewise smooth trader passport option is given in Table 5.1.

Table 5.1
Values of various Passport Options, $T = 0.5$, $\sigma = 0.2$, $S = 100$.

<i>Type</i>	Vanilla	Barrier	Chooser	P.Smooth	Reset	D.Stake	M.Potion
				$L = 1$			
<i>Parameters</i>		$B = 15$	$t_1 = 0.05$	$L' = 4$	$n = 1$	$L = 0.1$	
				$n = 1$			
<i>Price</i>	5.8931	5.3299	7.6944	5.6245	7.6348	7.4370	7.8470

5.7. The Reset Passport Option

This option allows the noteholder to erase, say N times, all that he has done so far; in other words, he can come back N times to his initial trading account which is zero.

If, during the life of the option, the trading account is negative and quite far from zero, the noteholder would think it would be a waste of time to carry on trading: but what if he thinks he would make money in future trading? The reset passport option is a remedy to this dilemma: whenever he wants, the noteholder can set his trading account back to zero, and therefore hope to have a positive trading account at expiry. As it gives an extra right to the noteholder, this option is more expensive than the Vanilla one.

For each t, S and π , the option price $V^n(t, S, \pi)$ should be larger than $V^{n-1}(t, S, 0)$, the price of the option if the noteholder decides to reset his trading account (n means that the noteholder is allowed to reset his trading account n times). The linear complementary problem satisfied by the similarity solution is, for $n \geq 1$,

$$\begin{cases} \mathcal{L}\phi^n \leq 0 \\ \phi^n(t, z) \geq \phi^{n-1}(t, 0) \\ \mathcal{L}\phi^n \cdot (\phi^n(t, z) - \phi^{n-1}(t, 0)) = 0 \end{cases}$$

with the usual final condition $\phi(T, z) = \max(z, 0)$.

In the above formulation, ϕ^0 is the similarity solution of the Vanilla passport option and $\mathcal{L}$ is such that

$$\mathcal{L}\phi = \frac{\partial\phi}{\partial t} + \frac{\sigma^2}{2}(1 + |z|)^2 \frac{\partial^2\phi}{\partial z^2}.$$

Figure 5.5 gives an example of a reset passport option.

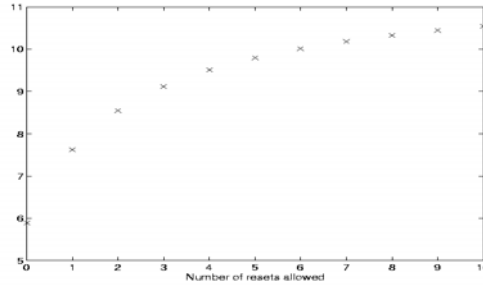


Figure 5.5. Price of a Reset passport option.

5.8. The Double Stake Passport Option

This passport option allows the noteholder to change his position limit from 1 to 2 during a period of time specified in the contract (say L).

Imagine that at a time during the life of the option the noteholder, for some reason, is pretty sure which direction the asset is going to go. Then he can decide to enter the double stake period and can therefore make more profit if he is right (but more ‘loss’ if he is wrong).

If $S\phi(t, z)$ is the price of the double stake passport option, it must be larger than $S\Phi(t, z, 0)$, the price of the option which specifies that the position limit is 2 from time t to time $t + L$. The linear complementary problem for ϕ is therefore

$$\begin{cases} \mathcal{L}\phi \leq 0 \\ \phi(t, z) \geq \Phi(t, z, 0) \\ \mathcal{L}\phi \cdot (\phi(t, z) - \Phi(t, z, 0)) = 0 \end{cases}$$

where $\mathcal{L}$ is as in the previous section and $\phi(T, z) = \max(z, 0)$. But we need to find Φ as well; between t and $t + L$, a clock τ is ticking and $L = 2$:

$$\begin{cases} -\frac{\partial \Phi}{\partial t} - \frac{\partial \Phi}{\partial \tau} = \frac{\sigma^2}{2}(2 + |z|)^2 \frac{\partial^2 \Phi}{\partial z^2} & \text{if } 0 \leq \tau < L \\ -\frac{\partial \Phi}{\partial t} = \frac{\sigma^2}{2}(1 + |z|)^2 \frac{\partial^2 \Phi}{\partial z^2} & \text{if } \tau = L \\ \Phi(T, z) = \max(z, 0). \end{cases}$$

Figure 5.6 gives an example of a double stake passport option.

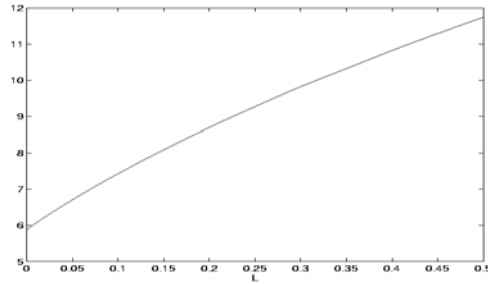


Figure 5.6. Double stake passport option.

5.9. The Magic Potion Passport Option

This passport option allows the noteholder to make disappear part of the history of his trading account; more explicitly, he can decide at any time (called t_1) to drink the magic potion. At any time $t_2 > t_1$, he can make magic happen: what happened to the trading account between t_1 and t_2 disappears, thus

$$\pi(t_2) = \max(\pi(t_1), \pi(t_2)).$$

If $\pi(t_1)$ is quite large and the noteholder wants to take more risks without losing what he has already gained, then he should drink the Magic Potion in order to be able to come back to $\pi(t_1)$ whenever he wants. The linear complementary problem satisfied by the similarity solution ϕ is

$$\left\{ \begin{array}{l} \mathcal{L}\phi \leq 0 \\ \mathcal{L}\Phi \leq 0 \\ \mathcal{L}\psi = 0 \\ \phi(t, z^*) \geq \Phi(t, z^*; z^*) \\ \Phi(t, z; z^*) \geq \psi(t, z^*) \\ \mathcal{L}\phi \cdot (\phi(t, z^*) - \Phi(t, z^*; z^*)) = 0 \\ \mathcal{L}\Phi \cdot (\Phi(t, z; z^*) - \psi(t, z^*) = 0) \end{array} \right.$$

where $\mathcal{L}$ is as in the previous section and ϕ , Φ , and ψ obey the usual final condition.

In the above formulation, $S\phi(t, z)$ is the price of the magic passport option; it is greater than $S\Phi(t, z; z)$, the option price if the noteholder decides to drink the magic potion. And $S\Phi(t, z; z^*)$ is itself greater than $S\psi(t, z^*)$, the option price at the time the noteholder decides to make magic happen (z^* being the value of z at the time the noteholder drank the magic potion).

Table I gives typical values of various passport options.

5.10. The Switch Passport Option

This passport option allows the noteholder to trade with two assets (α) and (β); however he cannot hold positions in both assets at the same time. Moreover, he is allowed to switch from a position in one asset to a position in the other asset at most N times (N is specified in the contract).

We note S_α and S_β the price of (α) and (β) respectively, σ_α and σ_β their volatilities, q_α and q_β the positions of the noteholder, and ρ the correlation coefficient between the two assets.

Let's define $\mathcal{L}^\alpha$ and $\mathcal{L}^\beta$ as follows:

$$\begin{aligned} \mathcal{L}^\alpha \phi = & \frac{\partial \phi}{\partial t} + (\sigma_\beta^2 z^2 - 2q_\alpha \rho \sigma_\alpha \sigma_\beta y z + q_\alpha^2 \sigma_\alpha^2 y^2) \frac{\partial^2 \phi}{\partial z^2} + \\ & + (\sigma_\beta^2 y z + q_\alpha \sigma_\alpha (\sigma_\alpha - \rho \sigma_\beta) y^2) \frac{\partial^2 \phi}{\partial y \partial z} + y^2 (\sigma_\beta^2 - 2\rho \sigma_\alpha \sigma_\beta + \sigma_\alpha^2) \frac{\partial^2 \phi}{\partial y^2} \end{aligned} \quad (5.10)$$

and

$$\begin{aligned} \mathcal{L}^\beta \phi = & \frac{\partial \phi}{\partial t} - 2q_\alpha \rho \sigma_\alpha \sigma_\beta y z \frac{\partial^2 \phi}{\partial z^2} + \\ & + (\sigma_\beta^2 (z - q_\beta) y + q_\beta \rho \sigma_\alpha \sigma_\beta y) \frac{\partial^2 \phi}{\partial y \partial z} + y^2 (\sigma_\beta^2 - 2\rho \sigma_\alpha \sigma_\beta + \sigma_\alpha^2) \frac{\partial^2 \phi}{\partial y^2} \end{aligned} \quad (5.11)$$

Note that $\mathcal{L}^\alpha$ is the operator that we get when we force $q_\beta = 0$ in the pricing PDE for dual passport option (see chapter 4), and $\mathcal{L}^\beta$ is the one we get when we force $q_\alpha = 0$. In chapter 4, it is shown that the price V of a dual passport option admits a similarity solution ϕ , and $V(t, S_\alpha, S_\beta, \pi) = S_\beta \phi(t, z, y)$, where $z = \frac{\pi}{S_\beta}$ and $y = \frac{S_\alpha}{S_\beta}$.

If $N = 0$, it is quite straightforward to price the switch passport option: we can find ϕ^0 in the following way:

- 1) solve $\mathcal{L}^\alpha \phi = 0$ to get $\phi^{\alpha,0}$
- 2) solve $\mathcal{L}^\beta \phi = 0$ to get $\phi^{\beta,0}$
- 3) $\phi^0 = \max(\phi^{\alpha,0}, \phi^{\beta,0})$

If $N \geq 1$, the mathematical structure of the problem is similar to the one developed in section 6-1 for the discrete trading passport option. Hence, for $n = 1 \cdots N$, we solve the following complementary problem:

$$\left\{ \begin{array}{l} \mathcal{L}^\alpha \phi^{\alpha,n} \leq 0 \\ \phi^{\alpha,n} \geq \phi^{\beta,n-1} \\ \mathcal{L}^\alpha \phi^{\alpha,n} \cdot (\phi^{\alpha,n} - \phi^{\beta,n-1}) = 0 \\ \mathcal{L}^\beta \phi^{\beta,n} \leq 0 \\ \phi^{\beta,n} \geq \phi^{\alpha,n-1} \\ \mathcal{L}^\beta \phi^{\beta,n} \cdot (\phi^{\beta,n} - \phi^{\alpha,n-1}) = 0 \\ \phi^n = \max(\phi^{\alpha,n}, \phi^{\beta,n}). \end{array} \right.$$

As the vanilla passport option is an insurance against trading loss, some exotic versions can interest investors with various views on the market and with various risk attitudes. The price of the financial products we have defined can be obtained by solving HJB equations or free boundary problems.

Chapter 6

Trading Passport Options

In this chapter we come back to the Vanilla passport option. we are going to perform hedging simulations. We are going to examine how the option holder utilizes his option and how much the issuer gains by selling the option. The investor who owns a passport option may construct his trading strategy to maximise his utility, predicting the market movement. When the physical trend of the market differs from the risk-neutral drift significantly, the option holder will benefit as long as he has a correct view on μ and is not unlucky. At the same time, the issuer will gain from the difference between the price maximising trading strategy and the trading strategy performed by his customer. We will focus on the case for the single asset passport option with the position limit $L = 1$.

Finally we are going to see a more robust hedging method.

The three numerical examples of this section have been performed by H. Ahn.

6.1. Perfectly Specified Drift Model

Modelling the gain by investors may be fictitious when one assumes a perfectly specified drift μ . However, the presumption allows us to evaluate the gain by selling passport

options to a transcendental investor who trades ideally. Therefore, if the gain turns out to be significant in this case, then the result remains persuasive.

First we assume that the option holder finds his strategy by solving the value of the maximum expected utility of the payoff:

$$u(t, S, \pi) = \sup_{|q| \leq 1} E \left[e^{-r(T-t)} U \left(\max(\pi_T(q), 0) \right) \right] \quad (6.1)$$

where E is the expectation under the physical measure and U is the option holder's utility function which is increasing in its argument. One can check that u satisfies the following HJB equation. The same equation arises in Ahn et. al. (2000). The function u represents the expected end-of-the-year bonus of a trader and q is the strategy he should follow in order to maximize his expected bonus.

$$-\frac{\partial u}{\partial t} = r\pi \frac{\partial u}{\partial \pi} + \mu S \frac{\partial u}{\partial S} - ru + \sup_{|q| \leq 1} \left(qS(\mu - r) \frac{\partial u}{\partial \pi} + \frac{1}{2} \sigma^2 S^2 \left\{ q^2 \frac{\partial^2 u}{\partial \pi^2} + 2q \frac{\partial^2 u}{\partial \pi \partial S} + \frac{\partial^2 u}{\partial S^2} \right\} \right), \quad (6.2)$$

$$u(T, \pi, S) = U \left(\max(\pi, 0) \right)$$

where μ is the physical drift of the underlying asset. If U is non-convex (in fact, most of the popular utility functions are concave because of risk aversion), $\frac{\partial^2 U}{\partial \pi^2}$ need not be positive, and consequently, the utility maximising strategy can have its value anywhere inside the position limit. But an investor with a passport option in his hands should not be afraid of risk. Buying a passport option is an act of risk aversion. But once one is insured it makes sense to take more risk! If $U(x) = x$, $\frac{\partial^2 u}{\partial \pi^2}$ stays positive, as in the case the value function of the option, and hence the utility maximising strategy has its value either ± 1 . The interpretation of the linear utility is that the option holder maximises expected return, which can be transformed from (6.1) by scaling it with the price of the option. A motivation for studying such utility is that the investor's portfolio is already insured by the passport option he owns and that it is tractable. In this case u is

homogeneous of degree 1 in space variables, and hence it has a similarity solution of the form $u(t, \pi, S) = Sh(t, \pi/S)$. Furthermore $h(t, z)$ satisfies the following HJB equation:

$$-\frac{\partial h}{\partial t} = (\mu - r)(h - z \frac{\partial h}{\partial z}) + \sup_{|q| \leq 1} \left(\frac{1}{2} \sigma^2 (z - q)^2 \frac{\partial^2 h}{\partial z^2} + q(\mu - r) \frac{\partial h}{\partial z} \right) \quad (6.3)$$

with the terminal data $\max(z, 0)$. From this, we obtain the option holder's trading strategy:

$$q = \text{sign} \left(\frac{\mu - r}{\sigma^2} \cdot \frac{\frac{\partial h}{\partial z}}{\frac{\partial^2 h}{\partial z^2}} - z \right). \quad (6.4)$$

When μ coincides with the risk-free rate r , (6.3) agrees with the price maximising value function for the option (4.14), and q in (6.4) coincides with the price maximising strategy q^* . If μ differs from r , then the option holder's choice will be different from the price maximising strategy.

Next we discuss the issuer's hedging strategy, Δ . In chapter 5, we explained that the hedging strategy must be in tune with the actual trading strategy performed by the option holder, and that it is given by $\Delta = \frac{\partial V}{\partial S} + q \frac{\partial V}{\partial \pi}$ where V solves the HJB equation (3.5). Then the profit of the issuer becomes:

$$P = V(0, 0, S_0) + \int_0^T e^{-rt} \Delta (dS - rSdt) - e^{-rT} V(T, \pi_T, S_T). \quad (6.5)$$

The first term is the price of the option he collects in cash, the second is the result of the delta hedging, and the third is the present value of the potential liability. Applying Itô's formula to V yields:

$$\begin{aligned} P &= - \int_0^T e^{-rt} \cdot \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \left\{ \frac{\partial^2 V}{\partial S^2} + 2q \frac{\partial^2 V}{\partial \pi \partial S} + q^2 \frac{\partial^2 V}{\partial \pi^2} \right\} \right) (t, \pi_t, S_t) dt \\ &= \frac{1}{2} \sigma^2 \int_0^T e^{-rt} S_t^2 \cdot \left((q^{*2} - q^2) \frac{\partial^2 V}{\partial \pi^2} + 2(q^* - q) \frac{\partial^2 V}{\partial \pi \partial S} \right) (t, \pi_t, S_t) dt \end{aligned} \quad (6.6)$$

where q^* is the price maximising strategy and q is the strategy performed by the option holder. Here we have exploited $\pi \frac{\partial V}{\partial \pi} + S \frac{\partial V}{\partial S} - V = 0$ as well as (3.5):

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} = -\frac{\sigma^2 S^2}{2} \left(q^{*2} \frac{\partial^2 V}{\partial \pi^2} + 2q^* \frac{\partial^2 V}{\partial \pi \partial S} \right) \quad (6.7)$$

Recall that $V(t, \pi, S)$ has a similarity solution $S\phi(t, \pi/S)$ where ϕ is defined in (4.14). In particular, we have

$$\frac{\partial^2 V}{\partial \pi \partial S} = -\frac{\pi}{S^2} \frac{\partial^2 \phi}{\partial z^2} \quad \text{and} \quad \frac{\partial^2 V}{\partial \pi^2} = \frac{1}{S} \frac{\partial^2 \phi}{\partial z^2}.$$

Also recall that $q^* = -\text{sign}(z)$. Thus we have a further reduction in the integrand of (6.6):

$$S^2 \cdot \left((q^{*2} - q^2) \frac{\partial^2 V}{\partial \pi^2} + 2(q_* - q) \frac{\partial^2 V}{\partial \pi \partial S} \right) = \frac{\partial^2 \phi}{\partial z^2} \left(t, \frac{\pi}{S} \right) \cdot \left(2|\pi| + 2q\pi + (1 - q)^2 S \right). \quad (6.8)$$

Now suppose that the option holder finds his strategy by maximising the expected return. Then, as we computed earlier in (6.4), the option holder's strategy q depends on π and S only through the ratio π/S and has its value either ± 1 . Hence the last term in (6.8) drops out, and the profit of the issuer becomes:

$$P = \sigma^2 \int_0^T e^{-rt} S_t \cdot \left(|z_t| + q(t, z_t) z_t \right) \frac{\partial^2 \phi}{\partial z^2} (t, z_t) dt \quad (6.9)$$

where z is the ratio π/S . To obtain the expected profit $E[P]$ of the issuer, we define

$$g(t, \pi, S) = \sigma^2 E^{t, \pi, S} \left[\int_t^T e^{-r\tau} S_\tau \cdot \left(|Z_\tau| + q(\tau, Z_\tau) Z_\tau \right) \frac{\partial^2 \phi}{\partial z^2} (\tau, Z_\tau) d\tau \right].$$

Again we observe that g has a similarity solution of the form $g(t, \pi, S) = S\psi(t, \pi/S)$ and that $\psi(t, z)$ satisfies the following PDE:

$$-\frac{\partial \psi}{\partial t} = (\mu - r)(q - z) \frac{\partial \psi}{\partial z} + \mu \psi + \frac{1}{2} \sigma^2 (q - z)^2 \frac{\partial^2 \psi}{\partial z^2} + \sigma^2 e^{-rt} \left(|z| + zq(t, z) \right) \frac{\partial^2 \phi}{\partial z^2}.$$

subject to $\psi(T, z) = 0$. To solve this equation, we need to obtain ϕ from (4.14) and q from (6.3).

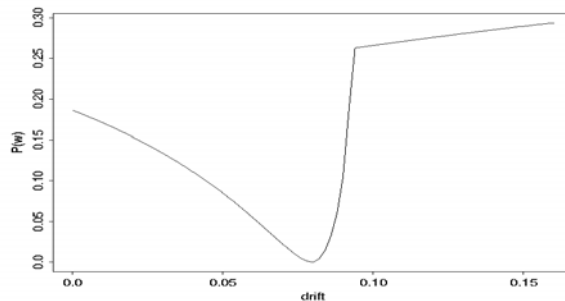


Figure 6.1. Issuer's expected gain versus the drift of the underlying asset.

Figure 4 shows the expected gain by the issuer as a function of μ , the physical drift, that is $\psi(0, 0)$ against μ . The asset volatility is 20% annum and the maturity of the option is 6 months. We calculate the profit $100 \cdot \psi$ at 81 different values of the physical drift from zero to 16%. When the drift coincides with the risk-free rate $r = .08$ (i.e., 8% annum), the gain vanishes. The issuer gains more as the gap between the drift and the risk-free rate become larger as the price maximizing optimal boundary and the expected utility optimal boundary become more and more different.

6.2. Imperfect Information Model

When the physical drift is positive, the price of the asset increases in the long run. However, on a short time period, the change in the price is mainly due to the volatility.

Thus, even if the drift is positive, the price may fall in a short period. We use the term “market direction” for the direction of the price in a short period to distinguish it from the drift.

We investigate how an investor benefits from buying a passport option in the environment where he must guess the market direction from imperfect information. Thus the strategy performed by the option holder will be different from the one that maximises the price and the issuer gains as long as he hedges well. In the following, we describe our simulation model. Suppose that the price of the asset is a lognormal random walk with physical drift μ and volatility σ . Then the probability that the price at time τ is greater than the price at time 0 is given by

$$P(S_\tau > S_0) = N\left(\sqrt{\tau}\left(\frac{\mu}{\sigma} - \frac{1}{2}\sigma\right)\right) \quad (6.10)$$

where N is the distribution function for the standard normal random variable. For example, if $\sigma = 0.2$ (i.e., 20% annum) and $\tau = 0.05$ (18 days, roughly), then $\mu_+ = 0.2466$ yields 60% chance of rise and $\mu_- = -0.2066$ yields 60% chance of fall. Thus, if the investor guesses a rising market when $\mu_+ = 0.2466$ (or a falling market when $\mu_- = -0.2066$), then he will be correct only 60% of the time. In general, we may choose $\mu_+(p)$ and $\mu_-(p)$ so that the probability of a correct guess becomes p . As we alternate μ_+ and μ_- , the investor guesses the market direction correctly only $100 \cdot p\%$ of the times.

Figure 5 shows the mean profit made by the issuer (the thin line) and the mean profit made by the option holder (the broken line), as a function of the guessing probability. We alternate μ_+ and μ_- 10 times during the life of the option which we set at six months. As before, the volatility is 20% annum and the initial asset price \$100. The curves in the picture are in fact the present values of the mean profits (i.e., it is already discounted). Thus the issuer beats the risk-free rate regardless of the correct guessing probability. If the option holder guesses correctly 53.8% of the time, then he beats the risk-free rate as well. Figure 6 shows a scatter-plot of 5,000 simulations when the option holder guess

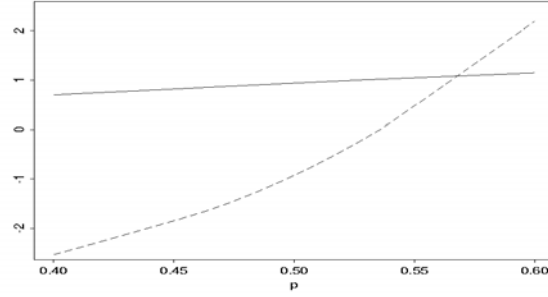


Figure 6.2. Profits and correct guessing probabilities.

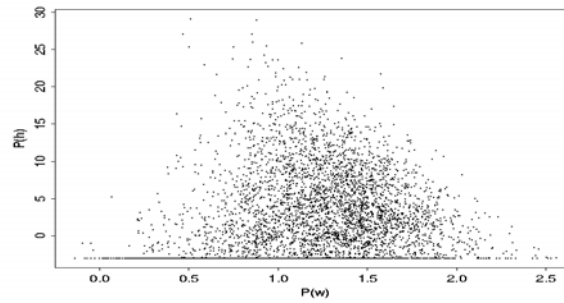


Figure 6.3. Profits of writer (w) and holder (h).

correctly 60% of the time. The horizontal axis is for the issuer (writer) and the vertical is for the option holder. The price of the option is 2.94. The mean profit is 1.14 for

the issuer and 2.16 for the option holder. Thus the mean return for the option holder is 72%. The few points at which the issuer loses money are the result of discrete hedging in simulation (100 rebalancings) and of errors in numerical computations.

6.3. A Robust Hedging Strategy

The risk management of a passport option requires constant rebalancing of the writer's hedge. Even if the noteholder does not change his trading strategy, the delta is going to change and the issuer will have to rebalance his portfolio. Moreover, if the noteholder decides to change dramatically his position, the issuer will have to change his position dramatically too in order to remain delta-hedged. Indeed a discontinuity in q would imply a discontinuity in Δ .

Question: Is there a more static and robust way to hedge a passport option?

In fact, a passport option is like a call with strike $E_0 = e^{r(T)}S_0$, with the right to change it at any time t into a put with strike $E_1 = 2S_t - S_0e^{r(T-t)}$. And then the right to change the put into a call with strike $E_2 = 2S - E_1$. And so on...

So the issuer only needs to buy the appropriate vanilla calls and puts to hedge the option! The noteholder will have to rebalance his position much less often. Indeed he will have to rebalance it only at the times at which the noteholder decides to change his strategy.

The only drawback with this hedging strategy is the eventual lack of liquidity of option markets. In particular, if the noteholder makes a lot of money, the issuer will have to purchase deep in-the-money options. These options are not as liquid as the at-the-money options and this has to be taken into account in the pricing of the option. Note that FX markets are very liquid and are an ideal market for the risk management

of passport options.

Remark: And with this way of seeing things, one understands better the optimal trading strategy q^* . When $\pi > 0$, the in-the-money put is more expensive than the in-the-money call.

Part III

Bonus

Chapter 7

Position of the Problem

Traders are compensated by bonuses, in addition to their basic salary. The most simple compensation structure is to have no structure at all: the bank waits until the end of the year, and then gives large bonuses to the traders it wants to keep. But a bank which adopts such a policy would have no control on the traders' trading strategies. On the other hand, if a bonus structure is designed in advance, the bank could assume that the trader would follow the trading strategy which maximizes his expected bonus. The bank would therefore have some control on the trader's trading behaviour.

Of course, the story is not quite as simple as this. Traders have limits imposed on their available capital and on the gearing and type of instruments they are allowed to trade. Nevertheless little is known about how to optimally reward a trader and there is a need to develop quantitative methods to understand the consequences of a bonus structure.

In this final part we are going to build a framework for the study of this problem and explore a variety of possible compensation structures.

We assume that the trader is allowed to trade only one risky asset; the price of the risky asset S follows the lognormal random walk

$$dS = \mu S dt + \sigma S dX \quad (7.1)$$

where μ is the drift of the asset, σ its volatility, and dX is a Gaussian random variable with mean zero and variance dt .

The trader can put his money in a risk-free asset as well; his trading account π is zero at $t = 0$, and it satisfies the following stochastic differential equation

$$d\pi = r\pi dt + q(dS - rS dt), \quad (7.2)$$

where r is the risk-free interest rate and q is the position of the trader, i.e. the number of risky assets that he holds. He will have some restriction such as $|q| \leq L$, L is the position limit.

We introduce the function V , the expected bonus of the trader. It depends on t, S, π and eventually other variables if the bonus depends on other variables. We assume that the trader will choose the trading strategy which maximizes the expected bonus: the choice of optimal strategy becomes a stochastic control problem whose solution, a second-order non-linear PDE in two (or more) state variables, characterises both the optimal strategy and the expected bonus. In most cases we can reduce the dimension of the problem. Once we have found the optimal trading strategy, we perform Monte-Carlo simulations of optimal trading and deduce the histograms of the trader's profit and of the bank's profit. For a given bonus structure the bank can therefore quantify what profit both the trader and the bank are going to make. It can therefore choose the bonus structure which fits best with its risk preferences.

In the first chapter we concentrate on different fixtures we can put into the bonus structure: we start from the simple bonus structure where the bank pays out a percentage of the trader's profit. As this type of compensation does not have much control on the

risks the trader could take, there is a need to turn to bonus structures which depend on the risks taken by the trader over the year: we incorporate the Sharpe ratio of the trader's trading account in the bonus structure. To make our model more realistic we put the trader's skill into the equation as well. Finally we look at the consequences of adding a feature in the contract which fires the trader if his trading account goes down to some prescribed value. In the next chapter we concentrate on the horizon problem: the bank has a long-term horizon whereas the trader might have short term plans! We will try to see how the bank can make the trader's horizon match with the bank's horizon by making the bonus structure path-dependent.

Chapter 8

Various Bonus Structures

8.1. Simple bonus structure

In this section, we are going to study the following bonus structure: at the end of the year, the trader gets a percentage of his profit

$$\text{bonus} = \lambda \max(\pi_T, 0) \tag{8.1}$$

where λ is the percentage fixed in advance, and $T = 1$. This bonus structure is basic and we are going to explain our mathematical framework in details.

The trader's bonus can be seen as a call option on his trading account: if the trading account is positive at the end of the year, then he chooses to exercise his option and gets his bonus. On the other hand, if he has performed badly and has lost money, then nothing happens: he does not get any bonus. Let's call $V(t, S, \pi)$ the expected value of the bonus. Note that V does not depend on the paths $S(t)$ and $q(\cdot)$ because equations (7.1) and (7.2) show that the trading account π is Markovian in the state variables S, π and q ; since the bonus structure does not depend on the paths of S and π , neither may V and the optimal strategy q^* (see Fleming and Soner 1992).

Let $V^q(t, S, \pi)$ denote the value corresponding to the trading strategy $q(\cdot)$, where q does not need to be optimal. The expected bonus is given by

$$V^q(t, S, \pi) = E[\exp^{-r(T-t)} \lambda \max(\pi_T, 0)] \quad (8.2)$$

where the expectation is taken under the physical measure. Now we use the fact that the trader chooses the trading strategy which maximizes his expected bonus

$$V(t, S, \pi) = \max_{|q| \leq L} V^q(t, S, \pi), \quad (8.3)$$

where V is the maximal expected bonus. So the stochastic control problem is

$$V(t, S, \pi) = \max_{|q| \leq L} E[\exp^{-r(T-t)} \lambda \max(\pi_T, 0)] \quad (8.4)$$

As for passport options we can derive a HJB equation for the problem. The only difference is that the expectation is taken under the physical measure¹

$$-\frac{\partial V}{\partial t} = \mu S \frac{\partial V}{\partial S} + r\pi V_\pi - rV + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \max_{|q| \leq L} [q(S(\mu - r) \frac{\partial V}{\partial \pi} + \sigma^2 S^2 \frac{\partial^2 V}{\partial \pi \partial S}) + q^2 \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial \pi^2}] \quad (8.5)$$

with final condition

$$V(T, S, \pi) = \lambda \max(\pi, 0). \quad (8.6)$$

Note that μ and r come into the PDE. Again, this is because the expectation is taken under the physical measure.

As for passport options we assume $L = 1$. We do not lose generality because $V_{L=L^*}(t, S, \pi) = V_{L=1}(t, L^*S, \pi)$.

¹For passport options it was the risk-neutral measure because of delta hedging.

Now, the HJB equation can be simplified thanks to the symmetry of the problem; indeed, if we introduce the variable $z = \frac{\pi}{S}$, then $V(t, S, \pi) = S\phi(t, z)$, where ϕ satisfies

$$-\frac{\partial \phi}{\partial t} = (\mu - r)\phi - (\mu - r)z \frac{\partial \phi}{\partial z} + \max_{|q| \leq 1} [q(\mu - r) \frac{\partial \phi}{\partial z} + \frac{\sigma^2}{2}(z - q)^2 \frac{\partial^2 \phi}{\partial z^2}] \quad (8.7)$$

with the final condition $\phi(T, z) = \lambda \max(z, 0)$.

Note that similar equations arise in the pricing of passport options (see Ahn et. al. 2000) and part 2.

Numerical results

We use an explicit finite difference scheme to solve the PDE (see Wilmott 2000 for explanation of these methods in finance). For each grid point, we find the optimal strategy q^* by checking the three candidates: $q_1 = -1$, $q_2 = 1$, and q_3 , which is the value for which the q -derivative of the term to maximize vanishes. Once the optimal strategy is found, we perform standard simulations of the asset price; we simulate the corresponding optimal trading strategy and we obtain histograms of the trader's bonus and of the bank's profit. We assume that the trader trades approximately once a day.

In the following numerical examples, $T - t = 1$, $S = 50$, $\mu = 0.1$, $\sigma = 0.2$, $r = 0.0$, $\lambda = 0.1$ and δt , the time between two consecutive trades, is 0.004. So we look at the trader's and bank's profit in the case where the trader is allowed to trade a stock whose value is 50, drift is 10%, volatility is 20%, interest rate is zero, and the trader will get 10% of his eventual profit at the end of the year. We perform 100000 Monte-Carlo simulations of the trader following the strategy q^* .

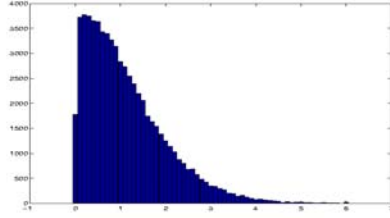
The optimal strategy q^* is always equal to one: in our example, the buy and hold strategy is optimal.

In figure 8.1a, we plot the histogram of the profit made by the trader: the probability of the trader receiving no bonus at the end of the year is 0.3459. His expected bonus is 0.7344 and the standard deviation of his bonus is 0.8938.

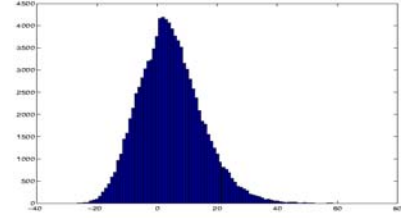
In figure 8.1b, we plot the histogram of the profit made by the bank. Its expected profit is 4.4436 and the standard deviation of its profit is 10.4638.

Figure 8.1c is the histogram of $\pi(1)$, the trading account at the end of the year. It is the sum of the bank's profit and of the trader's profit. As the optimal strategy is equal to one and $r = 0$, we have $\pi(1) = S(1) - S(0)$. This explains why the distribution of $\pi(1)$ is lognormal.

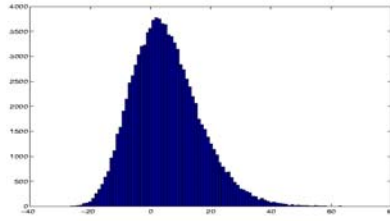
Finally, in figure 8.1d, we plot the mean (+) and the standard deviation (o) of the bank's profit as a function of λ . The greater λ , the more the bank has to pay out at the end of the year.



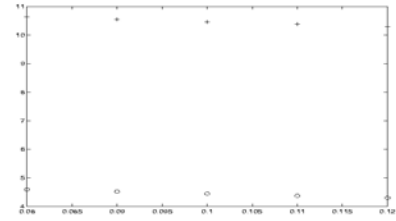
(a) Trader's bonus.



(b) Bank's profit.



(c) Distribution of $\pi(1)$.



(d) Bank's profit as a function of λ ,
mean: + and standard deviation: o.

Figure 8.1. Numerical results for the simple bonus structure.

8.2. Sharpe ratio

Things get more interesting and perhaps more sensible, if the bonus also depends on the realised Sharpe ratio. Indeed, the bank can now reward the trader according to the profit he has made and to the risk he has taken to obtain this profit. It is clear that for the bank, the ideal trader would make a large profit by taking very little risk. So we introduce the new variable I , the variance of the trading account. It evolves according to

$$dI = q^2 \sigma^2 S^2 dt \quad (8.8)$$

so

$$I = \int_0^t q^2 \sigma^2 S^2 dt. \quad (8.9)$$

At time $t = 0$, $\pi = I = 0$ and the trader begins to trade the underlying asset. At the end of the year, the bank gives the trader a bonus depending on the profit made, $\pi(T)$ and the Sharpe ratio

$$\frac{\pi}{\sqrt{I}}. \quad (8.10)$$

We are ignoring the risk-free interest rate that should be in the Sharpe ratio.

Again,

$$\max_{|q| \leq 1} (dV - rV dt) = 0. \quad (8.11)$$

The equation for $V(t, S, \pi, I)$ is therefore

$$\begin{aligned} & \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + r\pi \frac{\partial V}{\partial \pi} - rV + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \\ & + \max_{|q| \leq 1} \left[q(S(\mu - r)) \frac{\partial V}{\partial \pi} + \sigma^2 S^2 \frac{\partial^2 V}{\partial \pi \partial S} + q^2 \left(\frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial \pi^2} + \sigma^2 S^2 \frac{\partial V}{\partial I} \right) \right] = 0 \end{aligned} \quad (8.12)$$

with final condition

$$V(T, S, \pi, I) = \max(\pi, 0) P\left(\frac{\pi}{\sqrt{I}}\right). \quad (8.13)$$

A suitable form for the function P would be monotonically increasing from zero to a constant $c < 1$.

Again, a similarity solution is available: if we introduce the variables $z_1 = \frac{\pi}{S}$ and $z_2 = \frac{I}{S^2}$, then

$$V(t, S, \pi, I) = S\phi(t, z_1, z_2), \quad (8.14)$$

where ϕ satisfies

$$\frac{\partial \phi}{\partial t} + (\mu - r)\phi - (\mu - r)z_1 \frac{\partial \phi}{\partial z_1} + \max_{|q| \leq 1} \left[q(\mu - r) \frac{\partial \phi}{\partial z_1} + q^2 \sigma^2 \frac{\partial \phi}{\partial z_2} + \frac{\sigma^2}{2} (z_1 - q)^2 \frac{\partial^2 \phi}{\partial z_1^2} \right] = 0 \quad (8.15)$$

with final condition

$$\phi(T, z_1, z_2) = \max(z_1, 0) P\left(\frac{z_1}{\sqrt{z_2}}\right) \quad (8.16)$$

Numerical results

In the following example, we take the same assumptions as in the previous section. The only difference lies in the bonus payoff: we choose $P(x) = 0.2 \frac{x^2}{x^2 + 2}$. This function is a realistic example of a monotonically increasing function from 0 to 0.2.

In figure 8.2a, we plot the bonus structure as a function of π and $\frac{\pi}{\sqrt{I}}$.

In figure 8.2b, we plot ϕ as a function of z_1 and z_2 .

In figure 8.2c, we plot the optimal trading strategy q^* as a function of z_1 and z_2 . When z_1 is negative, the trading account is negative and the optimal strategy is 1: the trader has to take risks if he wants to maximize his expected bonus. On the other hand, the optimal strategy is no longer 1 when the trading account is positive: this reflects the fact that the bonus structure depends on the Sharpe ratio.

In figure 8.3a,b,c and d, we plot simulations of S, q^*, π and $\frac{\pi}{\sqrt{I}}$ respectively. The trader is quite lucky in the first 6 months and his Sharpe ratio is 2.5, which is very good. At this point, if the Sharpe ratio increases, the expected bonus would not increase by much. And if the Sharpe ratio decreases, the expected bonus would be comparatively

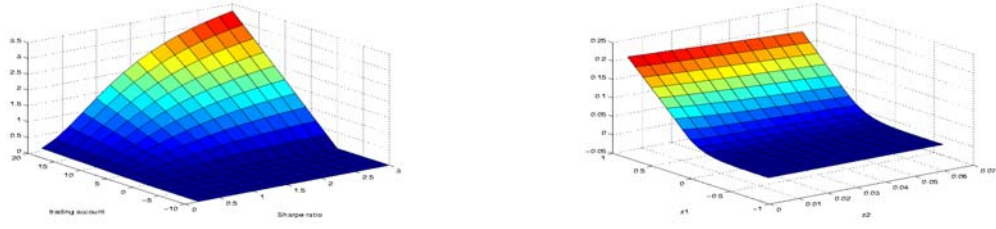
more affected. This is why it is not worth taking too much risk. The reason for that is the choice of function P .

In figure 8.4a, we plot the histogram of the trader's bonus. The probability of the trader receiving no bonus at the end of the year is 0.3461. His expected bonus is 0.7113 and the standard deviation of the bonus is 1.1577.

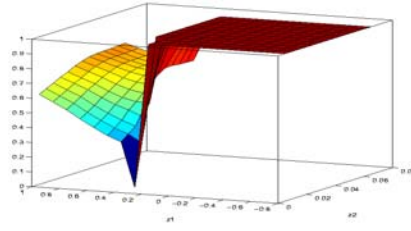
In figure 8.4b, we plot the histogram of the bank's profit. Its mean is 4.3784 and its standard deviation is 9.6277.

Figure 8.4c is the histogram of $\pi(1)$. This time the optimal strategy is not always equal to one and the distribution does not have to be lognormal any more. This histogram need to be compared to the one of figure 8.1c, where the bonus structure does not depend on the Sharpe ratio. The histogram of figure 8.4c has a peak near 20 and its tail is smaller. This reflects the fact that if the trader starts the year well and manages to put his trading account above some critical value (which can be found from 8.2.c), then q^* will become lower and the trading will stabilize around that critical value.

Finally, in figure 8.4d, we plot the histogram of the Sharpe ratio. Its mean is 0.4399 and its standard deviation is 1.0611.

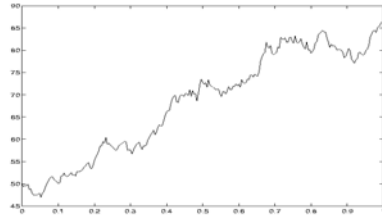


(a) Trader's bonus as a function of π and $\frac{\pi}{\sqrt{I}}$ (b) Value of ϕ as a function of z_1 and z_2 .

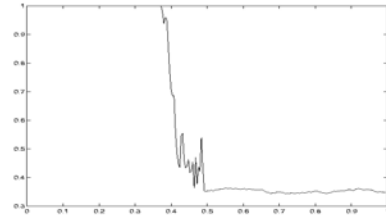


(c) The trading strategy q at time $t = T - 1$ which maximises the expected bonus.

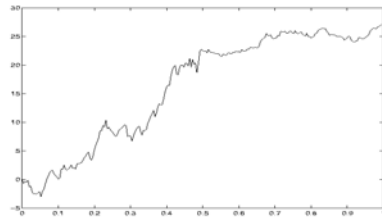
Figure 8.2. Bonus structure with the Sharpe ratio and subsequent values of ϕ and q^* .



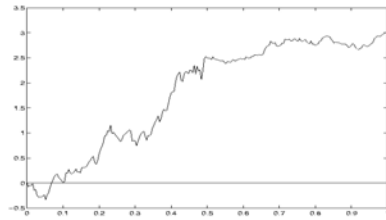
(a) Simulation of the asset price.



(b) Simulation of the optimal strategy.

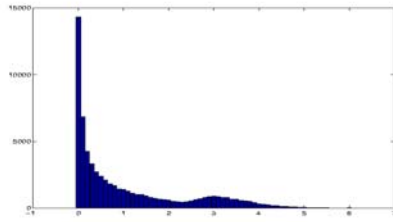


(c) Simulation of the trading account.

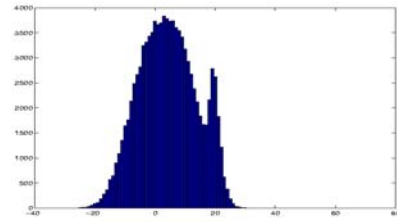


(d) Simulation of the Sharpe ratio.

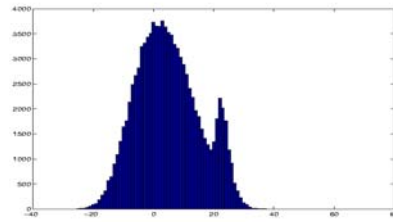
Figure 8.3. Simulation of S and subsequent values of q^* , π and $\frac{\pi}{\sqrt{I}}$.



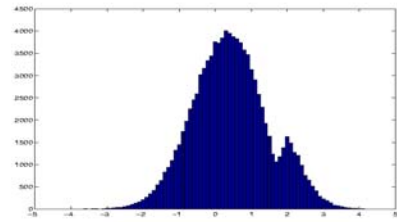
(a) Histogram of the trader's bonus.



(b) Histogram of the bank's profit.



(c) Histogram of $\pi(T)$.



(d) Histogram of the Sharpe ratio.

Figure 8.4. Various histograms when the Sharpe ratio is in the bonus structure.

8.3. Skill

There is little point in rewarding traders, or even hiring them, if they do not possess some skill. Often ‘skill’ is quantified by the Sharpe ratio. Here we want to suggest something a little more complex, but more realistic. We will model a possible way in which traders act, incorporating a skill factor that quantifies how much correct information they receive.

We are going to assume that our trader gets important and correct information about the direction of the market a fraction p of the time. If the information is that the market will rise, he buys to the position limit, if the information is that the market will fall he sells to the same limit. The remaining $1 - p$ he trades according to the optimal strategy.

Let $p = \alpha\sqrt{dt}$ be the probability that the trader gets told the outcome. Note that p needs to scale with $\sqrt{dt}$ otherwise $E[d\pi]$ would not scale with dt or p would not have any influence on $E[d\pi]$. If he gets told the outcome he trades up to the position limit. We get

$$\phi_i(n^-, z_1, z_2) = \begin{cases} \phi_{i+1}(n^+, 0, 0) + z_1 P_i(\frac{z_1}{\sqrt{z_2}}) & \text{if } z_1 \text{ is positive} \\ \phi_1(n^+, 0, 0) & \text{if } z_1 \text{ is negative} \end{cases}$$

The trading account evolves on the following way:

$$d\pi = \begin{cases} r\pi dt + q(dS - rSdt) & \text{with probability } 1-p \\ r\pi dt + C|dS - rSdt| & \text{with probability } p \end{cases}$$

This leads to

$$\frac{E[d\pi]}{dt} = r\pi - qrS + q\mu S + \alpha C\sigma S \sqrt{\frac{2}{\pi}} \quad (8.17)$$

The PDE for V is therefore

$$\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + (r\pi + \alpha\sigma S\omega) \frac{\partial V}{\partial \pi} - rV + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} +$$

$$+ \max_{|q| \leq 1} \left[q(S(\mu - r) \frac{\partial V}{\partial \pi} + \sigma^2 S^2 \frac{\partial^2 V}{\partial \pi \partial S}) + q^2 \left(\frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial \pi^2} + \sigma^2 S^2 \frac{\partial V}{\partial I} \right) \right] = 0 \quad (8.18)$$

where $\omega = \sqrt{\frac{2}{\pi}} = 0.7978846 \dots$, with the 'other' $\pi = 3.1415926 \dots$

The similarity solution ϕ satisfies

$$-\frac{\partial \phi}{\partial t} = (\mu - r)\phi - (\mu - r)z_1 \frac{\partial \phi}{\partial z_1} + \alpha \sigma \omega \frac{\partial \phi}{\partial z_1} + \max_{|q| \leq 1} \left[q(\mu - r) \frac{\partial \phi}{\partial z_1} + q^2 \sigma^2 \frac{\partial \phi}{\partial z_2} + \frac{\sigma^2}{2} (z_1 - q)^2 \frac{\partial^2 \phi}{\partial z_1^2} \right], \quad (8.19)$$

with the usual final condition.

Numerical results

Given α , we can calculate the means and standard deviations of the profits made by the bank and the trader via Monte-Carlo simulations. We perform 100000 MC simulations. In the following examples, $\alpha = 0.5$ and $\delta t = 0.004$. So $p = 0.03162 \dots$

At each time step, there is a probability of p for the trader to know the direction of the next price change.

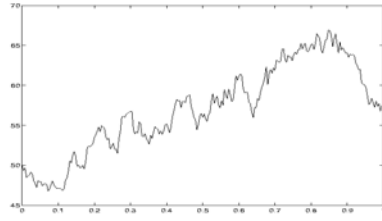
In figure 8.5a, b, c and d, we plot simulations of S, q^*, π and $\frac{\pi}{\sqrt{I}}$ respectively. Note that q^* jumps to -1 when the trader gets told that the asset price is going to go down.

In figure 8.6a, we plot the histogram of the trader's bonus. The probability of the trader receiving no bonus is 0.2133. The expected profit of the trader is 1.1311 and the standard deviation of his profit is 1.3986.

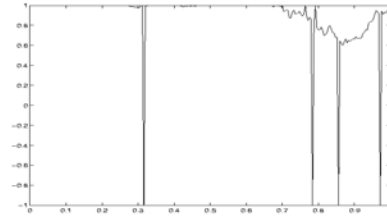
In figure 8.6b, we plot the histogram of the bank's profit. The mean is 7.7908 and the standard deviation is 9.0146. Note that both histograms in Figure 8.6 are compressed compared to the histograms of Figure 8.4 (trader with no skill). The effect of the skill is to make the process sometimes deterministic; it is therefore not surprising to obtain histograms which are less spread out.

In figure 8.7a, we plot the mean (+) and the standard deviation (o) of the trader's profit as a function of the skill factor α . Both the mean and the standard deviation increase with α .

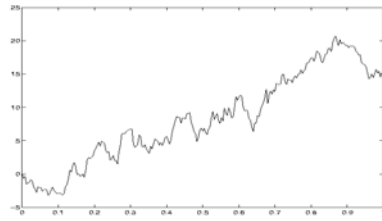
In figure 8.7b, we plot the mean (+) and the standard deviation (o) of the bank's



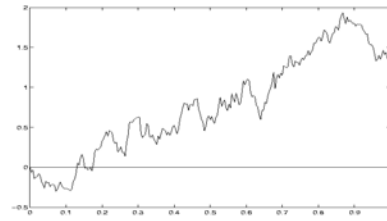
(a) Simulation of the asset price.



(b) Simulation of the optimal strategy.



(c) Simulation of the trading account.

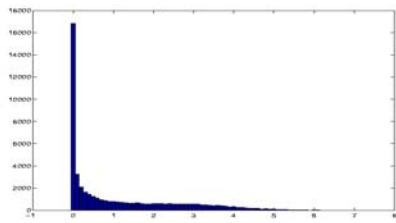


(d) Simulation of the Sharpe ratio.

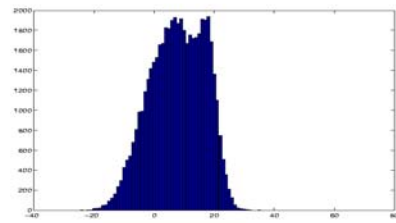
Figure 8.5. Simulation of S and subsequent values of q^* , π and $\frac{\pi}{\sqrt{I}}$ when the trader's skill factor is put in the model.

profit as a function of the skill factor α . As α increases, the mean increases and the standard deviation decreases. This explains why the shape of the histogram of figure 8.6b is more concentrated than the one of figure 8.4b.

A skilled trader is good both for the trader's bonus and for the bank's profit.

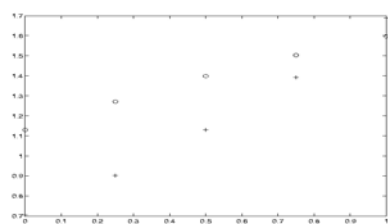


(a) Histogram of the trader's bonus.

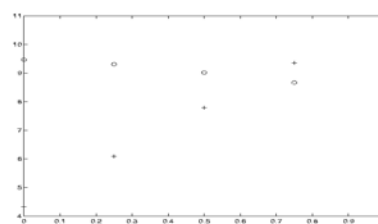


(b) Histogram of the bank's profit.

Figure 8.6. Histograms when the trader's skill factor is added in the model.



(a) Mean (+) and standard deviation (o) of the trader's profit as a function of the skill factor.



(b) Mean (+) and standard deviation (o) of the bank's profit as a function of the skill factor.

Figure 8.7. Trader's bonus and bank's profit as a function of the trader's skill factor.

8.4. Firing

If π gets negative, the optimal strategy for the trader becomes $q^* = 1$: the trader might be tempted to take very large risks in the hope of bringing π back to a positive value. The bank does not want traders to take large risks; its policy should therefore discourage this type of behavior. For example, the bank could fire the trader if its trading account reaches a given negative value. This condition would discourage the trader to take large risks; indeed, if its trading account gets dangerously close to the red zone after a series of bad luck, he would get out of this zone using his skill, not his luck.

The PDE is the same, but we now have the new boundary condition

$$V(t, S, F, I) = 0 \tag{8.20}$$

where F is the negative value of the trading account for which the trader gets fired. This boundary condition actually means that if the trading account ever reaches F , then the trader does not get any bonus at the end of the year. So we suppose that these two perspectives are just as bad as each other from the trader's point of view. If one thinks that the prospect of no bonus is not as bad as the prospect of getting fired, one can replace the boundary condition by $V(t, S, F, I) = G$, with G a suitable negative function.

Note that no similarity solution is available and computation time is therefore much longer.

Numerical results

In the following numerical examples, we use the same inputs as in the previous section apart from δt : we use $\delta t = 0.008$.

Figure 8.8 shows the optimal strategy q^* as a function of π and I ($S = 50$ and $F = -9$). Note that when the trading account is negative, the optimal strategy is not 1 anymore: this should discourage the trader from taking large risks.

In figure 8.9a, we plot the mean (+) and the standard deviation (σ) of the trader's profit as a function of F . The trader would rather have no firing boundary condition!

In figure 8.9b, we plot the mean and the standard deviation of the bank's profit as a function of F . The firing boundary condition has a bad effect on the bank's expected profit. But it can be seen as an insurance against traders losing a lot of money.

In figure 8.10, we look at the probability for the trader to get fired for different values of F .

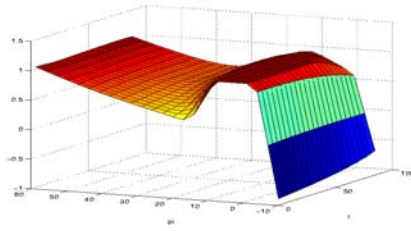
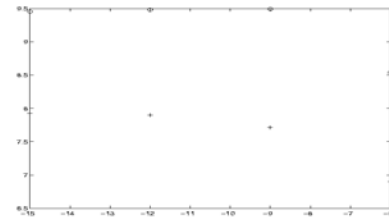
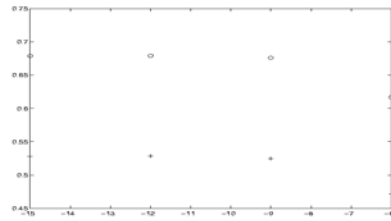


Figure 8.8. The optimal strategy q^* as a function of π and I in the 'firing' model.



(a) Mean (+) and standard deviation (o) of the trader's profit as a function of F . (b) Mean (+) and standard deviation (o) of the bank's profit as a function of F .

Figure 8.9. Trader's bonus and bank's profit as a function of F , the value of the trading for which the trader gets fired.

F	-15	-12	-9	-6
<i>Probability</i>	0.013	0.0358	0.0845	0.1362

Figure 8.10. Probability of getting fired for different F 's.

Another issue in the bonus problem is that the bank has a long-term horizon in its investments whereas the trader might have a shorter horizon. Indeed, the trader is not going to work for the bank forever and could therefore trade according to a shorter time horizon. So there is a need for the bank to choose a bonus structure which takes this problem into account: a bonus structure which would push the trader to have a longer term horizon.

8.5. High-water mark

One way to ensure that the trader's fate is matched with the bank's is to include a "high-water mark" in the model. These types of features are used in hedge funds managers' compensation contracts. This simply means that the bank keeps track of the accumulated trading account of the trader. For the trader to have a bonus, his accumulated profit (since time $t = 0$) must be positive, regardless how well he has done over the year. Moreover, the bonus is calculated on the above water profit only. For example, if the trader brings his trading account from -1 to 3 over a year, his above water profit will be 3 , not 4 . These two conditions ensure that the trader's main goal is to keep the accumulated profit above water.

In this section we drop the Sharpe ratio to keep things simple and the bonus structure is

$$\text{bonus} = \lambda \min(\max(\pi_N - e^r \pi_{N-1}, 0), \max(\pi_N, 0)). \quad (8.21)$$

So the usual PDE is satisfied between bonus times and at each time $n = 1, \dots, N - 1$ (where N is the time horizon of the bank) there is a jump condition:

$$V(n^-, S, \pi; \pi_{n-1}) = V(n^+, S, 0; e^r \pi_{n-1} + \pi) + \lambda \min(\max(\pi - e^r \pi_{n-1}, 0), \max(\pi, 0)) \quad (8.22)$$

In the above equation, the parameter is the accumulated trading account. As usual a similarity solution is available: $V(t, S, \pi; \pi_n) = S\phi(t, z; z_n)$ with $z = \frac{\pi}{S}$ and $z_n = \frac{\pi_n}{S}$.

In terms of the similarity solution, the jump conditions are

$$\phi(n^-, z; z_{n-1}) = \phi(n^+, 0; e^r z_{n-1} + z) + \lambda \min(\max(z - e^r z_{n-1}, 0), \max(z, 0)) \quad (8.23)$$

and the final condition is

$$\phi(N, z; z_{N-1}) = \lambda \min(\max(z - e^r z_{N-1}, 0), \max(z, 0)). \quad (8.24)$$

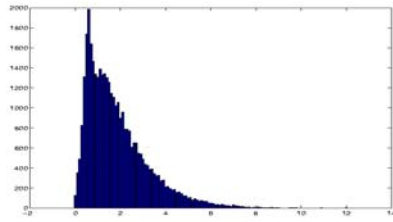
Numerical results

In the following numerical example, we look at a two years period ($N = 2$).

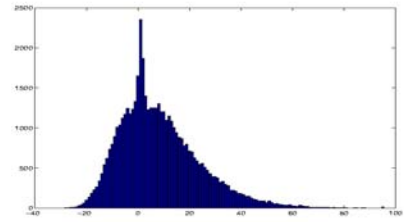
In figure 8.11a, we plot the histogram of the profit made by the trader. The mean is 1.3569 and the standard deviation is 1.1559.

In figure 8.11b, we plot the histogram of the profit made by the bank. The mean is 14.3830 and the standard deviation is 15.0506.

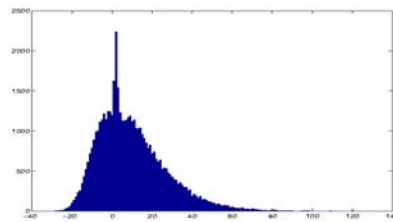
In figure 8.11c, we plot the histogram of $\pi(2)$. If the model had had no high-water mark, the distribution of the trading account would have been the sum of two independent and identically distributed lognormal distributions, which is clearly not the case.



(a) Histogram of the trader's bonus.



(b) Histogram of the bank's profit.



(c) Histogram of $\pi(2)$.

Figure 8.11. Histograms in the high-water mark model.

8.6. Another model

Another way to attract traders with a long-term horizon is to improve the bonus structure of good traders. More explicitly, the bank changes the function P to a greater function from one year to the next one if the trader performs well. In his first year the trader's bonus is $\max(\pi, 0)P_1(\frac{\pi}{\sqrt{I}})$. In his second year, his bonus structure is the same if he has not made money in his first year. However it is $\max(\pi, 0)P_2(\frac{\pi}{\sqrt{I}})$ if he has made money in his first year, with $P_2 \geq P_1$. Similarly, for the third year he could move from P_1 to P_2 , or from P_2 to P_3 , with $P_3 \geq P_2$. However, if he loses money in his second year, he goes back down to P_1 . We are going to write the equations for this problem in order to find the optimal strategy q^* . Looking at the trader's histogram profit and its own histogram of profit, the bank could then decide which P_n 's fit the best with its risk preferences.

As the expected bonus clearly depends on which bonus structure is 'activated', this makes the problem slightly more complicated. Indeed, we have now N final conditions (N is the time horizon of the bank):

$$V_i(N, S, \pi, I) = \max(\pi, 0)P_i(\frac{\pi}{\sqrt{I}}), \quad (8.25)$$

for $i = 1, \dots, N$. The parameter i tells which bonus structure is activated. For example, if $i = N$, this means that the trader has made money every single year and his bonus at time N will be $\max(\pi, 0)P_N(\frac{\pi}{\sqrt{I}})$. At each bonus time there will be jump conditions. At time n there will be n jump conditions :

$$V_i(n^-, S, \pi, I) = \begin{cases} V_{i+1}(n^+, S, 0, 0) + \pi P_i(\frac{\pi}{\sqrt{I}}) & \text{if } \pi \text{ is positive} \\ V_1(n^+, S, 0, 0) & \text{if } \pi \text{ is negative} \end{cases}$$

for $i = 1, \dots, n$. There are only n jump conditions at time n because the trader cannot do better than making money every year...

The usual HJB equation is satisfied between the bonus times and our stochastic control problem is well-defined. Note as well that the dimension of the problem can be reduced via similarity solutions. The jump conditions are then

$$\phi_i(n^-, z_1, z_2) = \begin{cases} \phi_{i+1}(n^+, 0, 0) + z_1 P_i(\frac{z_1}{\sqrt{z_2}}) & \text{if } z_1 \text{ is positive} \\ \phi_1(n^+, 0, 0) & \text{if } z_1 \text{ is negative} \end{cases}$$

The final conditions are

$$\phi_i(N, z_1, z_2) = \max(z_1, 0) P_i(\frac{z_1}{\sqrt{z_2}}), \quad (8.26)$$

Numerical results

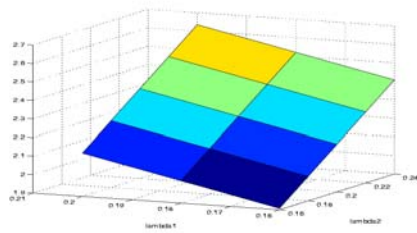
In the following numerical example, we take $N = 2$ and $P_i(x) = \lambda_i \frac{x^2}{x^2+2}$.

In figure 8.12a, we plot the trader's expected profit as a function of λ_1 and λ_2 .

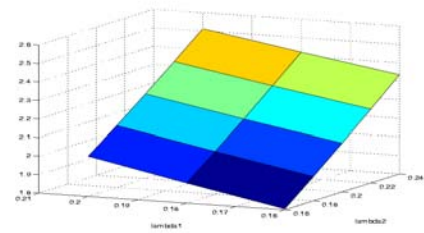
In figure 8.12b, we plot the standard deviation of the trader's profit as a function of λ_1 and λ_2 .

In figure 8.12c, we plot the bank's expected profit as a function of λ_1 and λ_2 .

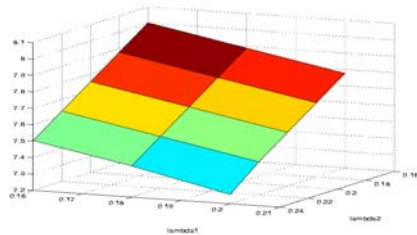
In figure 8.12d, we plot the standard deviation of the bank's profit as a function of λ_1 and λ_2 .



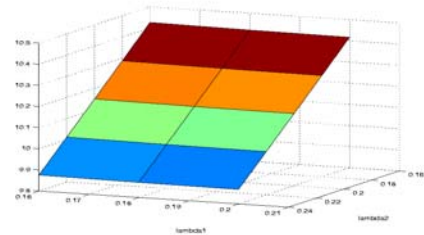
(a) The trader's expected profit as a function of λ_1 and λ_2 .



(b) The standard deviation of the trader's profit as a function of λ_1 and λ_2 .



(c) The bank's expected profit as a function of λ_1 and λ_2 .



(d) The standard deviation of the bank's profit as a function of λ_1 and λ_2 .

Figure 8.12. Trader's bonus and bank's profit as a function of λ_1 and λ_2 .

The end-of-the-year bonus is a source of motivation for the trader throughout the year and a way for the bank to keep its good traders. However a badly designed bonus structure could have undesired effects on the trader's trading behaviour. Indeed, the trader could be tempted to take large risks or he could trade with a short-term view. For example, the simple bonus structure which pays out a percentage of the profit encourages the trader to be long the asset up to the position limit. There is therefore a need to develop quantitative methods so that the bank can adopt a bonus policy which would fit with its risk preferences.

In this section we have built a mathematical framework for the study of this problem. We have studied various bonus structures of practical interest. We found the trader's optimal trading strategy by solving HJB equations. In our numerical examples, we have looked at the effect of the bonus on the optimal strategy, on the distribution of the trader's profit and on the distribution of the bank's profit. We have found that the bonus depending on the Sharpe ratio stops the optimal strategy from being equal to the position limit in the case where the trading account is positive. Our solution for tackling the case where the trading account is negative is to add a feature in the contract which says that the trader gets fired if his trading account falls to a prescribed value. Our mathematical framework allows us to model the trader's skill as well. The skill is quantified by a skill factor that represents how much correct information the trader receives. Its effect is to add an advection term in the HJB equation. The solution that we have proposed in order to encourage the trader to have a long-term horizon is to make the bonus structure path-dependent. We have studied two such models.

Bonus compensation of traders is a very important issue: there are numerous examples of traders taking a lot of risk in the hope of a higher bonus, and losing a lot of money. Yet no quantitative methods had been developed to tackle this problem. In this section, we have built a mathematical framework for the study of the bonus problem and we have determined the features that a good bonus structure should depend on.

Conclusion

In this thesis we have abundantly developed models involving HJB equations and free boundary problems. In finance a large number of situations arise in which the investor has to make decisions of the type

- “I have the right to do this. Shall I do it now or shall wait until later?”.
- “I have to choose a control parameter² over a large range of possible choices. Which one shall I choose?”

In our thesis we have studied such problems. We have been able to translate the problems in mathematical terms. Indeed, problems of the first type correspond to free boundary problems as problem of the second type are correspond to HJB equations.

Most of the literature on stochastic optimal control requires advanced technical knowledge. We have preferred an intuitive approach. By using repeatedly the key ideas behind the theory we hope to have helped democratising these mathematical tools.

²typically a number of assets to hold.

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