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DYNAMICALLY STABLE SETS IN INFINITE STRATEGY SPACES

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Dynamically Stable Sets in Infinite Strategy Spaces*

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Abstract

Evolutionary game theory has largely focused on finite games. Dynamic stability is harder to attain in infinite strategy spaces; Bomze (*Monatshefte für Mathematik* **110**, 1990, 189–206) and Oechssler and Riedel (*Economic Theory* **17**, 2001, 141–162) provide conditions for the stability of rest points under the replicator dynamics. Here, conditions are given for the stability of *sets* of strategies under this process. *Journal of Economic Literature* Classification Number: C70, C72.

Key Words: replicator dynamics; evolutionary stability; continuous strategy spaces; stable sets.

1 Introduction

Taylor and Jonker's (1978) replicator dynamics are the canonical dynamic model of evolutionary game theory. Their properties in finite games and connections with static equilibrium concepts are by now well understood (Hofbauer and Sigmund 1998, Weibull 1995). Moreover, their importance as a special case of the more general payoff-monotonic and payoff-positive dynamics, and as an approximation of certain learning models (Börgers and Sarin 1997, Schlag 1998, Hopkins 2002, Benaïm and Weibull 2003), is well recognized. But their behavior in games with infinite strategy sets has only recently been investigated (Bomze 1990, 1991, Seymour 2000, Oechssler and Riedel 2001, 2002, Eshel and Sansone 2003, Cressman 2005, Cressman, Hofbauer, and Riedel 2005), due to the attendant technical difficulties.

Chief amongst these is the task of providing a notion of "distance" between populations as compelling as the natural Euclidean distance of the familiar finite-dimensional setting.

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The leading contenders for this title are the variational (or strong) topology and the weak topology on the (infinite-dimensional) vector space of signed measures on the strategy space. This contest is unlikely to produce a clear winner, since arguments in favor of both topologies can be offered (Oechssler and Riedel 2001, 2002). In this paper, the variational topology is adopted, in part because of the wider class of payoff functions that it allows, as discussed in the next section.

This conceptual thicket is not without analytical fruit; evolutionarily stable strategies are no longer necessarily asymptotically stable in infinite strategy spaces, suggesting some ESS to be more stable than others as a finite strategy grid becomes arbitrarily fine. Under the variational topology, however, Bomze's (1990) "strong uninvadability" refinement does imply a weakened form of asymptotic stability, and indeed a homogeneous, "uninvadable" population delivers the undiluted version (Oechssler and Riedel 2001). In this paper, a setwise generalization of strong uninvadability—"strong immutability"—is defined for *faces* of strategies and shown to give asymptotic stability under the replicator dynamics. The weaker notion of a "strongly immovable face" is also defined, and shown to imply Lyapunov stability. Finally, in case sets more general than faces are called upon, Bomze's result is shown to extend to "strongly uninvadable sets."

Interest in setwise stability concepts is firmly established in finite evolutionary settings (Thomas 1985d, Gilboa and Matsui 1991, Swinkels 1992, Ritzberger and Weibull 1995, Balkenborg and Schlag 2001), and in game theory more generally (Kalai and Samet 1984, Kohlberg and Mertens 1986, Basu and Weibull 1991). Indeed, pointwise solution concepts have certain defects which setwise concepts need not share (Kohlberg and Mertens 1986, Dufwenberg, Norde, Reijnierse, and Tijs 2001). The stability concepts developed in this paper are of particular use in models with equivalent outcomes, including certain public-project mechanisms, the evolution of preferences and mixed-strategy models. To illustrate their utility, several examples of applications are given at the end of the paper.

2 Dynamics

Consider a symmetric two-player game with arbitrary (Borel) strategy sets $S \subset \mathbb{R}^n$, and a bounded, Borel-measurable payoff function $f : S \times S \to \mathbb{R}$. This game is played by random pairwise matchings amongst a population of players programmed to pure strategies. The play of the population as a whole is described by a probability measure P on the measure space $(S, \mathscr{B})$, where $\mathscr{B}$ denotes the Borel σ-algebra on S. The *support* of a population P, $\mathrm{supp}(P)$, is the unique (relatively) closed subset of S such that the measure of its complement is 0 but every open set that intersects it has positive measure.[1] Δ denotes the simplex of all populations P on $(S, \mathscr{B})$; Δ_A the set of all populations P with $\text{supp}(P) \subseteq A$, $A \in \mathscr{B}$—i.e. the *face* of a particular Borel set A. For given $A \in \mathscr{B}$, $A^c := S \backslash A$ is the complement of A.

The distribution of the players' strategies evolves in continuous time according to Taylor and Jonker's (1978) replicator dynamics, adapted for the infinite strategy space (Bomze 1990, Oechssler and Riedel 2001). In particular, the analysis takes place on the Banach space $\mathscr{M}^e(S, \mathscr{B})$ of signed measures endowed with the *supremum* or *variational norm*,

$$\|\mu\| = \sup_f \left| \int f \, d\mu \right|,$$

where the sup is taken over all measurable functions $f : S \to \mathbb{R}$ bounded by 1, $\sup_{s \in S} |f(s)| \leq 1$.[2] For the present purposes, the variational norm's specialization to probability measures,

$$\|P - Q\| = 2 \sup_{A \in \mathscr{B}} |P(A) - Q(A)|,$$

(see Shiryaev 1995, p. 360) is particularly useful.

The average payoff of population P against population Q is

$$E(P, Q) = \int_S \int_S f(x, y) Q(dy) P(dx). \tag{1}$$

The replicator dynamics then, as usual, increase the frequency of strategies that are successful relative to the prevailing average fitness:

$$Q'(t)(A) = \int_A \sigma(x, Q(t)) Q(t)(dx), \qquad Q(0) = P, \tag{2}$$

for all $A \in \mathscr{B}$, where

$$\begin{aligned} \sigma(x, Q) :=& \int_S f(x, y) Q(dy) - \int_S \int_S f(x, y) Q(dy) Q(dx) \\ =& E(\delta_x, Q) - E(Q, Q) \end{aligned} \tag{3}$$

is the success of strategy x if the population is Q, δ_x denoting a Dirac measure with mass 1 on x. Note that the usual finite-strategy-set replicator dynamics constitute a special case of this framework.

The use of the variational norm revolves around the question of what is meant by the

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[1] Such a set exists by virtue of P's Borel structure (Royden 1988).

[2] ν is a *signed measure* on $(S, \mathscr{B})$ if there are two finite measures μ_1 and μ_2 such that for all sets $A \in \mathscr{B}$, $\nu(A) = \mu_1(A) - \mu_2(A)$.

"distance" between populations in the infinite-strategy-space setting. The variational norm captures the most common evolutionary sense of distance, in that a mutated population $(1-\eta)P+\eta Q$ is close to the population P in the variational topology for small η (Oechssler and Riedel 2001). However, the variational norm is also quite a strong measure of distance, so that, for example, two distinct homogeneous populations δ_x and δ_y, $x \neq y$, always have maximal distance from one another in the variational topology, no matter how close are their strategies x and y in the natural metric on S (Oechssler and Riedel 2002).

This problem can be resolved by adopting the topology of weak convergence, but at the cost of quite stringent requirements for stability, and a narrower class of payoff functions giving well-defined dynamics.[3] In particular, the variational topology has the advantage of not requiring continuity of f, which is particularly pertinent for the setwise stability concepts considered here, since they will frequently arise in cases with discontinuous payoff functions. Moreover, in these cases, players will have more or less blunt incentives not to drift over the payoff discontinuity points in the manner critiqued by the weak-topology approach. It can also be argued that δ_x and δ_y, $x \neq y$, should not be considered "close" to one another in any case, irrespective of the distance between x and y, since transition from one to the other requires a high degree of coordinated mutation amongst the members of the population. Why should it be any easier for the whole population simultaneously to mutate to one strategy rather than another?

With these arguments in mind, the variational topology is adopted here, and it will play an important role in the analysis. Oechssler and Riedel (2001) prove that, in this setting, the replicator dynamics are well defined if the payoff function f is bounded—an assumption that is maintained here.

3 Stability

It is well known that symmetric Nash equilibria are stationary under the replicator dynamics. And whilst there are stationary states of the replicator dynamics that are not symmetric Nash equilibria, no such state satisfies even the weak requirement of Lyapunov stability.[4] The set of symmetric Nash equilibria in a game can of course be refined using Lyapunov and asymptotic stability, but these criteria can be somewhat cumbersome to check in individual applications. With finite strategy sets, the static criteria of evolutionary and neutral stability (Maynard Smith 1974, 1982) are helpful in this regard. In the notation of Section 2, a population P is

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[3] An alternative resolution—though one which restricts mutations themselves to be local in the natural metric—is to adopt the "maximal shift topology" investigated by Eshel and Sansone (2003).

[4] This result, due to Bomze (1986) for finite strategy spaces, readily carries over to infinite strategy spaces if f is bounded (so that the bilinear functional $E(P, Q)$ is continuous in the variational topology).

an *evolutionarily stable state* (ESS) if, for every "mutation" Q, there is an *invasion barrier* $\varepsilon(Q) > 0$ such that, for all $0 < \eta \leq \varepsilon(Q)$,

$$E(P, (1-\eta)P + \eta Q) > E(Q, (1-\eta)P + \eta Q). \tag{4}$$

If the inequality in (4) is weak, then $\varepsilon(Q)$ is called a *neutrality barrier* and P is a *neutrally stable state* (NSS). In the finite case with pairwise interactions, every ESS is asymptotically stable in the replicator dynamics (Taylor and Jonker 1978, Hofbauer, Schuster, and Sigmund 1979), and every NSS is Lyapunov stable (Thomas 1985d, Bomze and Weibull 1995).[5] However, in infinite strategy spaces, Oechssler and Riedel (2001) show that the former result no longer holds, so that a stronger concept than ESS is necessary.

Vickers and Cannings (1987) call a population *uninvadable* if it has a uniform invasion barrier—that is, an $\varepsilon > 0$ such that (4) holds for all Q and all $0 < \eta \leq \varepsilon$. The corresponding case with a uniform neutrality barrier is termed an *unbeatable* population by Bomze and Weibull (1995). Clearly uninvadability implies evolutionary stability whilst unbeatability implies neutral stability. If we allow mutations amongst the *whole population* to populations that are "nearby" in the variational topology, even stronger stability concepts are required. Bomze (1990) calls a population P *strongly uninvadable* if there is a barrier $\varepsilon > 0$ such that for all populations $R \neq P$ with $\|R - P\| \leq \varepsilon$, we have

$$E(P, R) > E(R, R). \tag{5}$$

If the inequality in (5) is weak, then P is called *strongly unbeatable* (Bomze and Weibull 1995). Strong uninvadability implies uninvadability, and if the equilibrium population state is homogeneous, then the criteria coincide. Oechssler and Riedel's (2001) Theorem 3 shows that, in infinite strategy spaces, every uninvadable, homogeneous population is Lyapunov stable, and—if f is continuous—"weakly attracting" (see Definition 2 below). But of course, such a population may not exist; and in many such cases, a setwise stability concept will be useful.

Bernhard Thomas was the first to extend evolutionary stability to sets of strategies. A set of populations $\Pi \subseteq \Delta$ is called an *evolutionarily stable (ES) set* (Thomas 1985d) if it is nonempty and closed, and each $P \in \Pi$ has some neighborhood U such that $E(P, R) \geq E(R, R)$ for all $R \in U$, with strict inequality if $R \notin \Pi$.[6] As Balkenborg and Schlag (2001) point out, this setwise stability criterion in fact extends the notion of uninvadability—rather than

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[5] For a review of results in finite strategy spaces, see Weibull (1995). For the complications arising outside the case of pairwise interactions—and the resulting implications of different notions of evolutionary stability—see Bomze and Weibull (1995).

[6] See also Bomze (1998), as well as Thomas (1985b,c) and Cressman, Garay, and Varga (2003) for applications to population genetics. Alternative setwise stability criteria include Swinkels' (1992) "equilibrium evolutionarily stable sets" and Gilboa and Matsui's (1991) "cyclically stable sets."

evolutionary stability—to sets of populations. They also argue that a more appropriate setwise extension of uninvadability has the specified properties hold for some ε-neighborhood of any $P \in \Pi$, where the barrier $\varepsilon > 0$ is independent of the particular population P (as well as R).[7] Clearly a population that belongs to such a set is neutrally stable, but the set as a whole has the properties of uninvadability with respect to populations not in the set. Correspondingly, if ε-closeness is defined on the variational topology, we require the following concept.

Definition 1 *A set of populations* $\Pi \subseteq \Delta$ *is a* strongly uninvadable set *if and only if it is nonempty and there is a barrier* $\varepsilon > 0$ *such that each* $P \in \Pi$ *has* $E(P, R) \geq E(R, R)$ *for all* R *with* $\|R - P\| \leq \varepsilon$*, with strict inequality if* $R \notin \Pi$*. If the inequality remains weak for* $R \notin \Pi$*, then* Π *is a* strongly unbeatable set*.*

The following result—proved in the Appendix—unsurprisingly holds for elements of such sets.

Proposition 1 *If* P *belongs to a strongly unbeatable set* $\Pi \subseteq \Delta$*, then* P *is a Nash equilibrium.*

To investigate the dynamic stability properties of such sets, it is first necessary to extend the standard concepts (e.g. Oechssler and Riedel's (2001) Definition 6) to a setwise framework. Given a set of populations $\Pi \subseteq \Delta$, let $d(Q, \Pi) := \inf_{P \in \Pi} \|Q - P\|$.

Definition 2 *Let* $\Pi^* \subseteq \Delta$ *be a closed set of populations. Then:*

- Π^* *is* Lyapunov stable *if for all* $\varepsilon > 0$ *there exists* $\eta > 0$ *such that* $d(Q(0), \Pi^*) < \eta \Rightarrow d(Q(t), \Pi^*) < \varepsilon$ *for all* $t > 0$*;*
- Π^* *is* weakly attracting *if there exists* $\varepsilon > 0$ *such that* $d(Q(0), \Pi^*) < \varepsilon \Rightarrow Q(t) \to \Pi^*$ *weakly (i.e. the Prohorov distance between* $Q(t)$ *and* Π^* *vanishes).*

The latter concept is one possible formulation of asymptotic stability for an infinite strategy space. In finite strategy spaces, every ES set is asymptotically stable (Weibull 1995). Since an ES set has the properties of uninvadability, one might conjecture that a similar result would hold for strongly uninvadable sets in the infinite case. This is shown to be true for faces Δ_A, $A \in \mathscr{B}$. The restriction to faces is necessary for the same reason as Oechssler and Riedel's (2001) requirement of homogeneity for single populations; namely, so that the measure on A can be considered without regard to its distribution. For more general sets, we will demonstrate a slightly weaker form of stability.

Given $A \in \mathscr{B}$, and a population R with $R(A) \in (0, 1)$, let R_A be the probability measure given by $R_A(B) := R(A \cap B)/R(A)$, $\forall B \in \mathscr{B}$.[8] Consider the following weakening of strong uninvadability for faces.

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[7] This is their "uniform evolutionarily stable set," which also drops Thomas' requirement that the set be closed.

[8] The $R(A) \in (0, 1)$ proviso is required for existence of the measures R_A and R_{A^c}.

Definition 3 *Δ_A is a* strongly immutable face *if and only if it is nonempty and there is a barrier $\varepsilon \in (0,1)$ such that $E(R_A, R) > E(R, R)$ for all R with $d(R, \Delta_A) \in (0, \varepsilon]$.*

Note that the usual weak inequality proviso for $R \in \Delta_A$ is superfluous here, since $R_A \equiv R$ if $R \in \Delta_A$.

Proposition 2 *If Δ_A is strongly uninvadable, then it is strongly immutable.*

If A is the singleton $\{x\}$—so that $\Delta_A = R_A = \delta_x$—then the converse clearly also applies, so that δ_x is strongly uninvadable if and only if it is strongly immutable. But in general Δ_A can be strongly immutable and not strongly uninvadable. Note in particular that Proposition 1 does not extend to strongly immutable faces; not all elements of a strongly immutable Δ_A are necessarily Nash equilibria.

The concept of a strongly immutable face allows the following generalization of Oechssler and Riedel's (2001) Theorem 3.

Theorem 1 *If Π^* is a strongly immutable face, then:*

- *it is Lyapunov stable;*
- *if additionally f is continuous, Π^* is weakly attracting.*

Remark The importance of strong immutability—rather than strong uninvadability—derives from the possibility of mutation amongst the *whole population* within the bounds prescribed by the variational topology. This is unimportant in the singleton case $A = \{x\}$, where a mutated population R has $R_A \equiv \delta_x$; when dealing with nonsingleton sets, by contrast, a mutation away from a given population $P \in \Delta_A$ can give $R_A \neq P$. So whilst a strongly uninvadable face is certainly Lyapunov stable and (for continuous f) weakly attracting, strong immutability provides a weaker sufficient condition.

An alternative characterization of a strongly immutable face is available, in terms of the following two constituents.

Definition 4 *A* strictly fitter mutation *away from Δ_A is a population R, with $R(A) \in (0,1)$, for which $E(R_{A^c}, R_A) > E(R_A, R_A)$. A* secret-handshake mutation *away from Δ_A is a population R, with $R(A) \in (0,1)$, for which $E(R_{A^c}, R_A) = E(R_A, R_A)$ and $E(R_{A^c}, R_{A^c}) \geq E(R_A, R_{A^c})$.*

Roughly speaking, a strictly fitter mutation away from Δ_A is a population whose strategies outside of A do better against its strategies in A than they themselves do. A secret-handshake mutation away from Δ_A is a population R whose strategies outside of A: (a) do equally well against R's strategies in A as do the latter; and (b) do at least as well against themselves as R's strategies in A do against them. Secret handshakes (Robson 1990) are a familiar evolutionary phenomenon; the secret-handshake mutations defined here are somewhat different, but transpose the essence of the concept to the present setting.

Proposition 3 *There are neither any strictly fitter mutations nor secret-handshake mutations away from a nonempty* Δ_A *if and only if it is strongly immutable.*

This result parallels the equivalence between Thomas' (1985d) notion of evolutionary stability for single populations and the concept of uninvadability (Bomze and Weibull 1995), and is often useful for applications. Note that, whilst not all elements of a strongly immutable face need be Nash, the absence of strictly fitter mutations does accord the set as a whole a Nash-like property. This feature is shared by faces that are "closed under better replies" (Ritzberger and Weibull 1995), all of which are strongly immutable (by Proposition 3) and many of which contain non-Nash elements.

In cases where a strongly immutable face fails to exist, the following natural weakening of the concept may be useful.

Definition 5 Δ_A *is a* strongly immovable face *if and only if it is nonempty and there is a barrier* $\varepsilon \in (0,1)$ *such that* $E(R_A, R) \geq E(R, R)$ *for all* R *with* $d(R, \Delta_A) \in (0, \varepsilon]$.

Proposition 4 *If* Δ_A *is strongly unbeatable, then it is strongly immovable.*

Theorem 2 *If* Π^* *is a strongly immovable face, then it is Lyapunov stable.*

The proofs of these results are obvious variants on their counterparts for strong immutability. A useful equivalent characterization of strong immovability is again available. Define a *firm-handshake mutation* away from Δ_A to be a population R, with $R(A) \in (0,1)$, for which $E(R_{A^c}, R_A) = E(R_A, R_A)$ and $E(R_{A^c}, R_{A^c}) > E(R_A, R_{A^c})$.

Proposition 5 *There are neither any strictly fitter mutations nor firm-handshake mutations away from a nonempty* Δ_A *if and only if it is strongly immovable.*

This result parallels the equivalence between Thomas' (1985d) notion of weak evolutionary stability for single populations and the concept of unbeatability (Bomze and Weibull 1995).

Stable sets in evolutionary models are often not faces though, and hence it is desirable to consider the stability of more general sets. Here we can demonstrate a weaker form of stability for strongly uninvadable sets by exploiting Bomze's (1990, 1991) decreasing "cross entropy" results. Given a population P absolutely continuous with respect to another population Q, $P \ll Q$, the *cross entropy* or Kullback–Leibler (1951) distance of Q with respect to P is

$$K_{Q:P} = \int \log\left(\frac{dP}{dQ}\right) dP = \int \frac{dP}{dQ} \log\left(\frac{dP}{dQ}\right) dQ,$$

where dP/dQ is a Radon–Nikodym derivative of P with respect to Q. $K_{Q:P}$ is nonnegative and vanishes if and only if $Q = P$; moreover, $\|Q - P\|^2 \leq K_{Q:P}$ by Bomze's (1991) Lemma 3. The set

$$\mathscr{U}_{\Pi} = \{Q \in \Delta : (\exists P \in \Pi : P \ll Q \text{ and } K_{Q:P} < \delta)\}$$

can thus be considered a "vicinity" of Π in some sense, though not strictly speaking a "neighborhood;" $K_{Q:P}$ is asymmetric in Q and P, and the sets $\mathscr{U}_P$, $\delta > 0$, do not define a topology (Bomze 1991). It also clearly captures a stronger notion of "closeness" than does the variational topology.

Using this apparatus, we have the following extension of Bomze's (1990) Theorem 2.

Theorem 3 *If* $\Pi^* \subseteq \Delta$ *is a strongly uninvadable set, then every trajectory* $(Q(t))_{t \geq 0}$ *with* $Q(0) \in \mathscr{U}_{\Pi^*}$ *converges weakly to* Π^* *as* $t \to \infty$*, provided* $\delta > 0$ *is small enough.*

The useful properties of the Kullback–Leibler distance compensate here for the reduced structure on the sets.

4 Applications

Equivalent outcomes The setwise stability concepts explored above will most obviously arise in games with sets of payoff-equivalent outcomes. The simplest setting in which such equivalent outcomes will exist is that where the payoff function is independent of the opponent's action, $f(x, y) = g(x)$, and has a set A of maximizers. In this case,

$$\begin{aligned} E(R_A, R) &= \int_S \int_S g(x) R(dy) R_A(dx) \\ &= \int_S g(x) R_A(dx) \\ &> \int_S g(x) R(dx) = E(R, R) \end{aligned}$$

for all R with $d(R, \Delta_A) \in (0, 1)$; i.e. Δ_A is strongly immutable. Therefore, Δ_A is Lyapunov stable and, for continuous g, weakly attracting by Theorem 1. Note, moreover, that there is no $x \in A$ such that δ_x is uninvadable, and indeed no proper subset of Δ_A that can be weakly attracting.

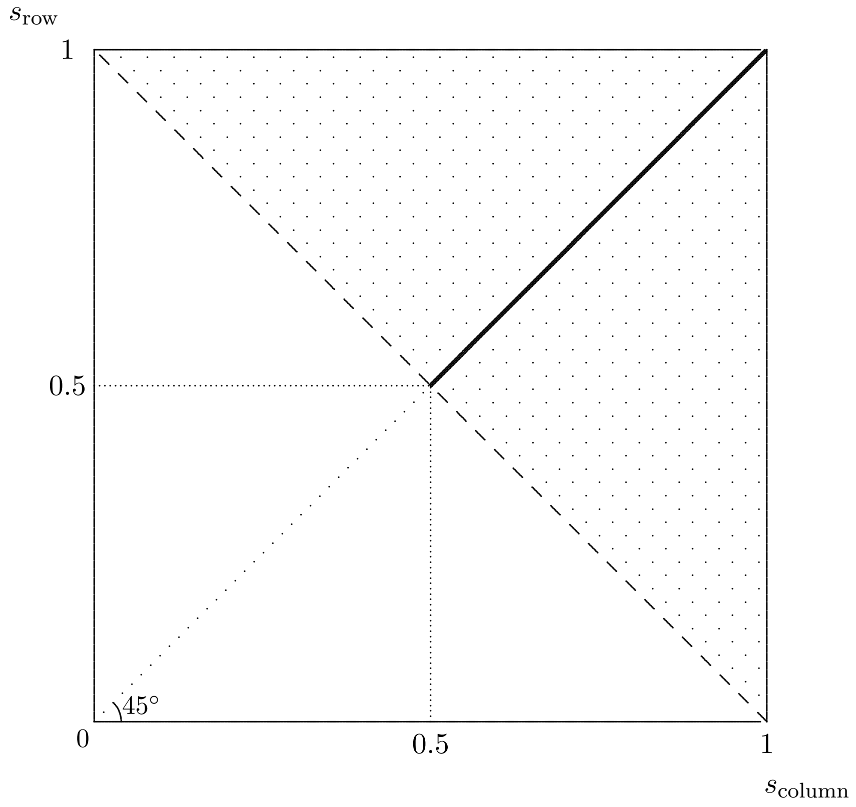

Figure 1: Public-project game

Public projects Such sets of equivalent outcomes are also of interest in numerous economic applications. Consider a public-project game where two players receive a fixed, positive payoff v from a public good, which is provided if and only if the sum of their reported payoffs is sufficiently high, with the cost $c < 2v$ then shared equally between them.[9] More formally, each player i's strategy set is the closed unit interval $[0,1] \subset \mathbb{R}$, from which he selects his payoff report s_i; the good is provided iff $s_{\text{row}} + s_{\text{column}} \geq 1$. The Nash equilibria of this game occupy the shaded region of Figure 1. The symmetric equilibria—i.e. the face $\Delta_{[0.5,1]}$ of all measures on the intersection of the equilibrium set with the 45°-line—are of course all stationary under the replicator dynamics, but none are uninvadable. $\Delta_{[0.5,1]}$ is, however, strongly immutable and hence Lyapunov stable.

To see this note that, for any population R with $R([0.5,1]) \in (0,1)$,

$$E(R_{[0.5,1]}, R_{[0.5,1]}) = v - c/2 \geq E(R_{[0,0.5)}, R_{[0.5,1]}),$$

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[9] This game is a special case of Mas-Colell, Whinston, and Green's (1995) Example 23.C.1 of Groves–Clarke mechanisms applied to public projects. We do not, however, consider the issues surrounding transfers to individuals and efficiency.

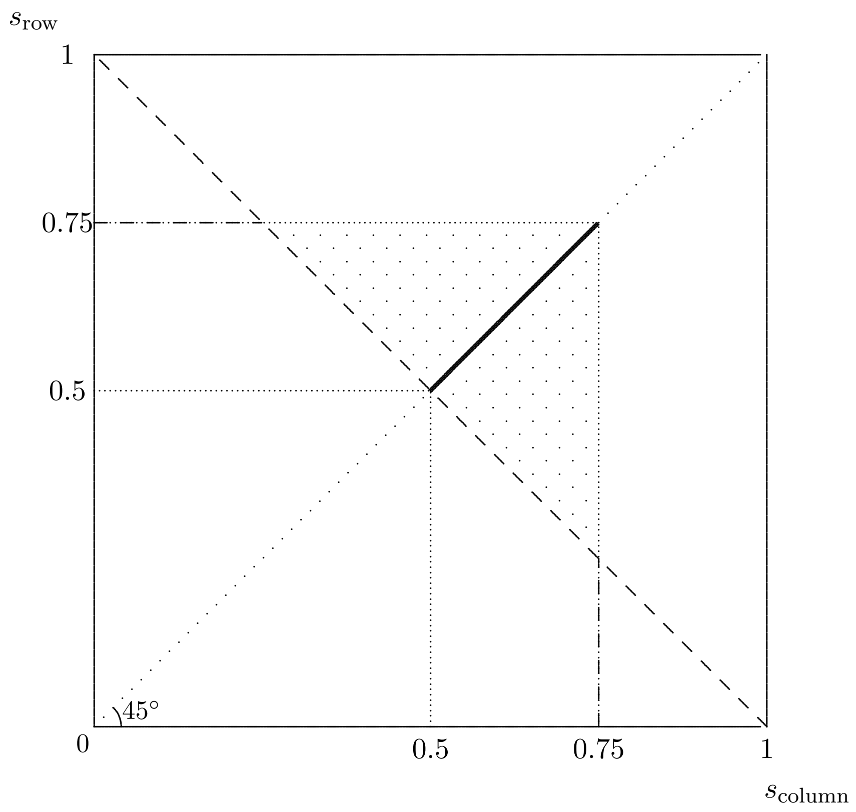

Figure 2: Modified public-project game

whilst $E(R_{[0.5,1]}, R_{[0.5,1]}) = E(R_{[0,0.5)}, R_{[0.5,1]})$ implies that

$$E(R_{[0,0.5)}, R_{[0.5,1]}) = E(R_{[0.5,1]}, R_{[0,0.5)}) = v - c/2 > 0 = E(R_{[0,0.5)}, R_{[0,0.5)}).$$

Hence, there are neither any strictly fitter nor secret-handshake mutations away from $\Delta_{[0.5,1]}$, which is thus strongly immutable by Proposition 3 and Lyapunov stable by Theorem 1.

Suppose now that the public-good provider only observes a noisy signal $\tilde{s}_i = s_i + E_i$ of each player's reported payoff, with E_i a continuous random variable on $(-\psi, \psi)$, $\psi \in (0, 0.5)$. In this case, with the good provided iff $\tilde{s}_{\text{row}} + \tilde{s}_{\text{column}} \geq 1$, the good is provided with probability 1 iff $s_{\text{row}} + s_{\text{column}} \geq 1 + 2\psi$. Also, since each E_i is continuous, the payoff function f is now continuous, so that the second condition of Theorem 1 is met. Hence, $\Delta_{[0.5+\psi,1]}$ is both Lyapunov stable and weakly attracting.

Consider now the same game (without noise), but where, in the case $s_{\text{row}} + s_{\text{column}} < 1$, a reported payoff exceeding 0.75 by one of the players leads the public-good provider still to provide the good, but then to levy the *whole* cost $c > v$ on that player. This reduces

(*a*)

	L	R
U	2 2	0 0
D	0 0	1 1

(*b*)

	L	R
U	$2 + x_c$ $2 + x_r$	0 x_r
D	x_c 0	1 1

Figure 3: (a) Coordination in fitnesses (b) Payoffs given biases

the Nash-equilibrium set to the shaded area in Figure 2, with the symmetric subset now $\Delta_{[0.5,0.75]}$. Moreover, strong immutability fails for $\Delta_{[0.5,0.75]}$, given that any $R \notin \Delta_{[0.5,0.75]}$ with $\text{supp}(R) \subseteq [0.5, 1]$ and $R([0.5, 0.75]) \in (0, 1)$ serves as a secret-handshake mutation. However, there are no *firm*-handshake mutations away from $\Delta_{[0.5,0.75]}$, which is thus strongly *immovable* by Proposition 5, and hence Lyapunov stable by Theorem 2.

Preference evolution Sets of equivalent outcomes could of course arise in many other games, and in particular are quite natural (indeed, seemingly unavoidable) in models of preference evolution (see, e.g., Samuelson 2001, Dekel, Ely, and Yilankaya 2007, Heifetz, Shannon, and Spiegel 2007). In such models, players play rationally for given preferences, but those preferences are free and subject to evolutionary selection according to their success in an underlying game of biological fitness. Suppose, for instance, that a population of players is repeatedly matched to play the pure coordination game in Figure 3(a), where the numbers represent *fitnesses.* Each player $i \in \{r, c\}$ then also has an additive *bias* $x_i \in (-5, 5)$ in favor of one strategy or the other, so that the *payoffs* to each possible outcome are as depicted in Figure 3(b). Suppose that this game is common knowledge, so that in particular both fitnesses and biases are observable by the players. However, the players choose their strategies in order to maximize payoffs, not fitnesses, and thus play a Nash equilibrium of the payoff game in Figure 3(b); if there is more than one such equilibrium, each of the possible equilibria is assumed to be played with strictly positive probability. The preference biases $\{x_r, x_c\}$ are then shaped by the replicator dynamics (2) with $S = (-5, 5)$ and $f(x, y)$ determined by the fitnesses in Figure 3(a).

The "game" on which the replicator dynamics act in this case is depicted in Figure 4, the Nash equilibria occupying the shaded regions where coordination is enforced by a dominant strategy for at least one of the players. The face $\Delta_{(1,5)}$ consists of all symmetric equilibria

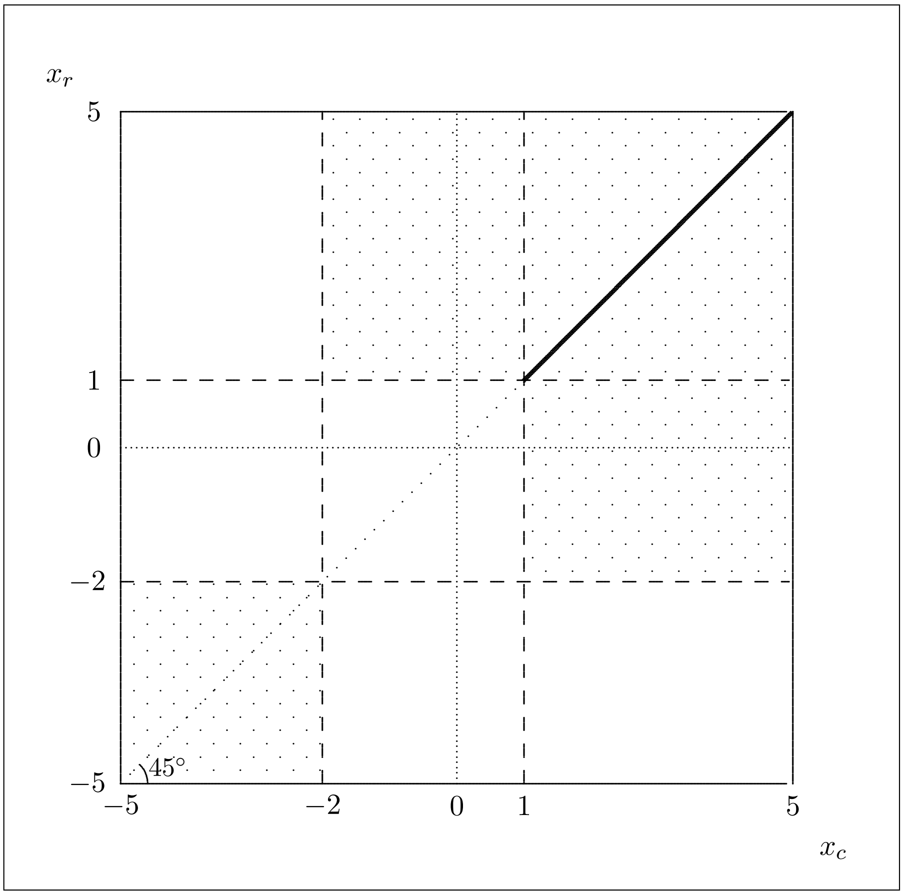

Figure 4: Stable biases

sustaining coordination on the Pareto-dominant (fitness) equilibrium $\{U, L\}$ through dominant strategies for both players. Clearly there can be no strictly fitter mutations away from this face, whilst $E(R_{(1,5)}, R_{(1,5)}) = E(R_{(-5,1]}, R_{(1,5)})$ implies that

$$E(R_{(-5,1]}, R_{(1,5)}) = E(R_{(1,5)}, R_{(-5,1]}) = 2 > E(R_{(-5,1]}, R_{(-5,1]})$$

for any R with $R((1,5)) \in (0,1)$, precluding secret-handshake mutations as well. Therefore, $\Delta_{(1,5)}$ is strongly immutable by Proposition 3 and Lyapunov stable by Theorem 1. Furthermore, if the probabilities assigned to multiple equilibria in the payoff game vary continuously with x_r and x_c, then f is continuous, and $\Delta_{[1,5)}$ becomes weakly attracting. By contrast, the face $\Delta_{(-5,-2)}$ of symmetric equilibria supporting coordination on the Pareto-*dominated* equilibrium $\{D, R\}$ is vulnerable to secret-handshake mutations R with support divided between $(-5,-2)$ and $[-2,1)$, for which

$$E(R_{[-2,1)}, R_{[-2,1)}) > 1 = E(R_{(-5,-2)}, R_{[-2,1)}) = E(R_{[-2,1)}, R_{(-5,-2)}) = E(R_{(-5,-2)}, R_{(-5,-2)}).$$

This provides a dynamic example then of the key result from the preference-evolution literature that stability implies efficiency under observable preferences (Dekel, Ely, and Yilankaya 2006).

Mixed strategies Typically, players are restricted to pure strategies in the replicator dynamics, and mixed strategies are viewed as descriptions of play at the population level. But an alternative approach is to allow individual players to adopt mixed strategies, which can then be replicated just as readily as pure strategies (Zeeman 1981, Hines 1982, Akin 1982, Thomas 1985a, Bomze 1990, Cressman 1990, 1992b, Weissing 1991). This has the advantage of delivering the converse implication between evolutionary and asymptotic stability, but the model sits uneasily at best with the usual finite setting. By contrast, as Oechssler and Riedel (2001) point out, this approach fits quite straightforwardly into the present framework. If T is a finite set of pure strategies, of cardinality n, then let $\Sigma(T) := \{x \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=1}^n x_i = 1\}$—the $(n-1)$-dimensional simplex—be the infinite strategy set S from which our population of players can choose. With a payoff matrix $U : T \times T \to \mathbb{R}$, we have

$$f(x, y) = xUy = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j U_{ij},$$

and hence

$$\begin{aligned} E(P, Q) &= \int_{\Sigma(T)} \int_{\Sigma(T)} xUyQ(dy)P(dx) \\ &= \int_{\Sigma(T)} xU \int_{\Sigma(T)} yQ(dy)P(dx) \\ &= \mu(P)U\mu(Q), \end{aligned} \tag{6}$$

where $\mu(P) := \int_{\Sigma(T)} xP(dx)$ is the mean mixed strategy in population P.

Consider in particular the symmetric 3×3 game with payoff matrix

$$U = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

The symmetric Nash equilibria of this game employ the strategies $\{x \in \Sigma(T) : x_1 = x_2\}$, as illustrated in Figure 5. These equilibrium outcomes are not payoff-equivalent, but for any symmetric equilibrium x and any $y \in \Sigma(T)$, $f(x, y) - f(y, y) = (y_1 - y_2)^2$, so that $f(x, y) \geq f(y, y)$ for all $y \in \Sigma(T)$, with equality only for symmetric equilibrium y. This example appears in the usual (finite) pure-strategy setting in Weibull (1995) (and also in Cressman 1992b). There, the set of all populations with equal numbers playing the pure

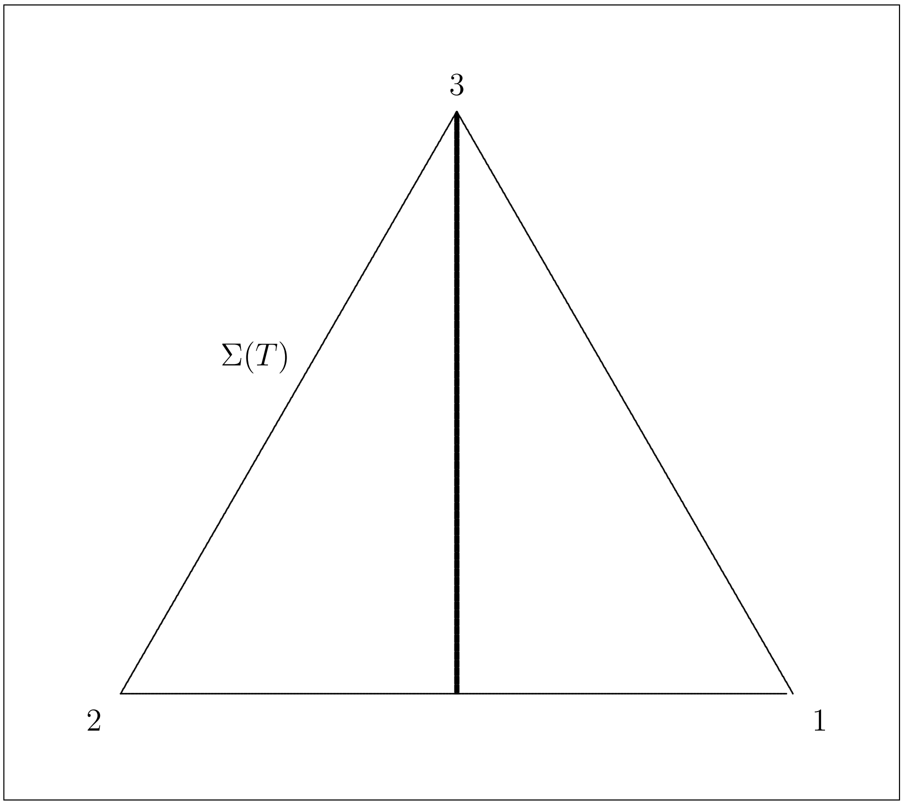

Figure 5: Stable mixed strategies

strategies s_1 and s_2 is an evolutionarily stable set. Once players are allowed to play mixed strategies, however, this set is strongly unbeatable but not strongly uninvadable; moreover, it is not a face, so that Theorems 1 and 2 will not apply. Nonetheless, it can be shown using (6) that $E(P,Q) - E(Q,Q) = (\mu_1(Q) - \mu_2(Q))^2$ if $\mu_1(P) = \mu_2(P)$, so that firm-handshake mutations away from the face $\Delta_{\{x\in\Sigma(T):x_1=x_2\}}$ are impossible $(E(R_{\{x\in\Sigma(T):x_1=x_2\}}, R_{\{x\in\Sigma(T):x_1\neq x_2\}}) \geq E(R_{\{x\in\Sigma(T):x_1\neq x_2\}}, R_{\{x\in\Sigma(T):x_1\neq x_2\}})$ for all R with $R(\{x \in \Sigma(T) : x_1 = x_2\}) \in (0,1))$. Since strictly fitter mutations are also clearly impossible ($\mu(P)$ being a symmetric equilibrium for all $P \in \Delta_{\{x\in\Sigma(T):x_1=x_2\}}$), $\Delta_{\{x\in\Sigma(T):x_1=x_2\}}$ is strongly immovable by Proposition 5 and hence Lyapunov stable by Theorem 2.

More generally, given that a mixture of mixed strategies yields just another mixed strategy, there will clearly be sets of populations that are behaviorally equivalent in evolutionary mixed-strategy games with random pairwise matching. The dynamic process will find such sets indistinguishable, so that setwise stability concepts become key. Faces, however, are inadequate to capture the behavioral equivalencies associated with mixed-strategy models; since the population mean mixed strategy determines population payoffs and thus stability, we will be interested in sets of populations sharing the same mean, which will not in general be faces. This is clear even from the 3 × 3 example above; $\Delta_{\{x\in\Sigma(T):x_1=x_2\}}$ is strongly immovable but *not* strongly immutable (and not weakly attracting), since any $R \notin \Delta_{\{x\in\Sigma(T):x_1=x_2\}}$ with $R(\{x \in \Sigma(T) : x_1 = x_2\}) \in (0,1)$ and $\mu_1(R) = \mu_2(R)$ serves as a secret-handshake mutation. This is where Theorem 3 becomes useful, since the set $\{P \in \Delta : \mu_1(P) = \mu_2(P)\}$ is strongly uninvadable ($E(P,Q) - E(Q,Q) = (\mu_1(Q) - \mu_2(Q))^2 > 0$ if $\mu_1(Q) \neq \mu_2(Q)$), and hence satisfies Theorem 3's weakening of asymptotic stability. Moreover, Theorem 3 should have power quite generally in such mixed-strategy models.

5 Conclusion

The concept of an evolutionarily stable set is useful when a game with a finite strategy space has a set of strategies that are not ESS's individually, but which behave like ESS's with respect to strategies outside the set. In games with infinite strategy spaces, however, the ES set—like other notions of evolutionary stability—requires reformulation in order to reproduce the dynamic stability properties of the finite-space concept. In this paper, we have introduced the appropriate extensions of ES sets to the infinite setting with the variational topology, and proved their stability under the replicator dynamics. The new concepts are of most obvious use in games with equivalent outcomes, of which there are many in economics. Evolutionary mixed-strategy models, for example, are but one member of the class of "population games," where a summary statistic determines the payoff of a population (Bomze 1991), and for which sets of strategies yielding the same value of the summary statistic will be of interest.

Equivalencies—and thus setwise stability concepts—also arise naturally in extensive-form games (Cressman 1992a), through unreached information sets. Infinitely repeated games constitute a particularly natural application, given their infinite strategy spaces and sets of equivalent Nash equilibria. In particular, consider the Nash equilibrium of the infinitely repeated Prisoner's Dilemma where both players defect in every subgame. This equilibrium belongs to a large set of payoff-equivalent Nash equilibria with the same perpetual defection along the equilibrium path but different behavior off it. The evolutionary process studied here will find these Nash equilibria indistinguishable, suggesting the application of setwise stability criteria. Of course, such an approach would ignore the extensive-form structure of the game, and one can say a good deal more by taking this seriously (see, e.g., Cressman 2003). But as in the case of mixed-strategy models, setwise stability concepts can address the problem of "spurious" duplication of strategies that dogs pointwise stability concepts in the standard normal-form analysis.

Appendix

Proof of Proposition 1. Suppose otherwise. Then there exists some $Q \in \Delta$ such that $E(Q,P) > E(P,P)$. Letting $R = (1-\eta)P + \eta Q$, clearly $\|P - R\| \leq 2\eta$. Equation (1) then gives

$$\begin{aligned} E(P,R) - E(R,R) &= \int_S \int_S f(x,y)((1-\eta)P + \eta Q)(dy)P(dx) \\ &\quad - \int_S \int_S f(x,y)((1-\eta)P + \eta Q)(dy)((1-\eta)P + \eta Q)(dx) \\ &= \eta\left(E(P,P) - E(Q,P)\right) \\ &\quad - \eta^2\left(E(P,P) - E(Q,P) + E(Q,Q) - E(P,Q)\right), \end{aligned}$$

which is negative for η sufficiently small, in violation of the strong unbeatability of Π. ■

Proof of Proposition 2. Given $B \in \mathscr{B}$, consider the decomposition $B = B_A \cup B_{A^c}$, where $B_A := B \cap A$ and $B_{A^c} := B \cap A^c$. Clearly

$$\begin{aligned} |R(B) - R_A(B)| &= |R(B_A) + R(B_{A^c}) - R_A(B_A) - R_A(B_{A^c})| \\ &= |(R(B_{A^c}) - R_A(B_{A^c})) - (R_A(B_A) - R(B_A))| \\ &\leq \max\{|R(B_{A^c}) - R_A(B_{A^c})|, |R_A(B_A) - R(B_A)|\}, \end{aligned}$$

since both $(R(B_{A^c}) - R_A(B_{A^c}))$ and $(R_A(B_A) - R(B_A))$ are positive. Moreover,

$$\begin{aligned} |R(B_{A^c}) - R_A(B_{A^c})| &\leq |R(B_{A^c}) - R_A(B_{A^c}) + R(B^c \cap A^c) - R_A(B^c \cap A^c)| \\ &= |R(A^c) - R_A(A^c)|, \end{aligned}$$

whilst

$$\begin{aligned} |R_A(B_A) - R(B_A)| &\leq |R_A(B_A) - R(B_A) + R_A(B^c \cap A) - R(B^c \cap A)| \\ &= |R_A(A) - R(A)|, \end{aligned}$$

so that $\sup_{B \in \mathscr{B}} |R(B) - R_A(B)| = |R(A^c) - R_A(A^c)| = |R(A) - R_A(A)| = R(A^c)$.

Now, for any R with $d(R, \Delta_A) \in (0, \varepsilon]$,

$$\begin{aligned} d(R, R_A) &= 2 \sup_{B \in \mathscr{B}} |R(B) - R_A(B)| \\ &= 2\, R(A^c) \\ &= 2\, |R(A^c) - P(A^c)|,\ \forall P \in \Delta_A \\ &\leq d(R, \Delta_A). \end{aligned}$$

But of course $R_A \in \Delta_A$ by definition, and hence $d(R, R_A) \geq d(R, \Delta_A)$. It follows that $d(R, R_A) = d(R, \Delta_A) \leq \varepsilon$. Thus, $E(R_A, R) > E(R, R)$ by strong uninvadability of Δ_A. ■

Proof of Theorem 1. By definition of Δ_A, $Q(A) < 1$ for all $Q \in \Delta \backslash \Delta_A$, so that $|Q(A) - P(A)| > 0$ for all $P \in \Delta_A$ and hence $d(Q, \Delta_A) > 0$. Letting $\xi(Q) = d(Q, \Delta_A)/2$, it is clear that the ball $B_{\xi(Q)}(Q) \subset \Delta \backslash \Delta_A$ for all $Q \in \Delta \backslash \Delta_A$. Hence, $\Delta \backslash \Delta_A$ is open in the variational topology, and Δ_A is closed.

If $Q(0) \in \Delta_A$, then $Q'(t)(A) = 0$, $\forall t > 0$, trivially. So consider $Q(0) \notin \Delta_A$ such that $d(Q(0), \Delta_A) \leq \varepsilon$, where ε is Δ_A's strong immutability barrier. From (1), (2) and (3),

$$\begin{aligned} Q'(t)(A) &= \int_A \left(E(\delta_x, Q(t)) - E(Q(t), Q(t))\right) Q(t)(dx) \\ &= \int_A E(\delta_x, Q(t)) Q(t)(dx) - \int_A E(Q(t), Q(t)) Q(t)(dx) \\ &= \int_A \int_S f(x, y) Q(t)(dy) Q(t)(dx) - E(Q(t), Q(t)) \int_A Q(t)(dx) \\ &= Q(t)(A) \int_S \int_S f(x, y) Q(t)(dy) Q_A(t)(dx) - Q(t)(A) E(Q(t), Q(t)) \\ &= Q(t)(A) \left(E(Q_A(t), Q(t)) - E(Q(t), Q(t))\right), \end{aligned}$$

which is strictly positive if $d(Q(t), \Delta_A) \in (0, \varepsilon]$ by strong immutability of Δ_A. Note that $Q_A(0) \in \Delta_A$ and, from the proof of Proposition 2, that $d(Q(0), Q_A(0)) = d(Q(0), \Delta_A) \leq \varepsilon$. $Q(t)(A)$ is thus strictly increasing, and hence $d(Q(t), \Delta_A) < \varepsilon$ for all $t > 0$.

Now, Oechssler and Riedel's (2001) Theorem 6 implies that

$$Q(t)(A) = \int_A \exp\left(\int_0^t \sigma(x, Q(s))\, ds\right) Q(0)(dx),$$

for all $A \in \mathscr{B}$. Convergence of $Q(t)(A)$ then necessitates that

$$\int_0^\infty \sigma(x, Q(s))\, ds < \infty$$

for $Q(0)$-almost every x. Together with uniform continuity of the map $t \mapsto \sigma(x, Q(t))$ (established in the proof of Oechssler and Riedel's (2001) Theorem 3), this yields a vanishing fitness differential $\sigma(x, Q(t))$ for $Q(0)$-almost every x. Hence, if $P \in \Delta_A$ is absolutely continuous with respect to $Q(0)$, denoted $P \ll Q(0)$, then—since f is assumed to be bounded—

$$\begin{aligned}\lim_{t\to\infty} E(P, Q(t)) - E(Q(t), Q(t)) &= \lim_{t\to\infty} \int_S E(\delta_x, Q(t))P(dx) - E(Q(t), Q(t)) \\ &= \int_S \lim_{t\to\infty} \sigma(x, Q(t))P(dx) = 0\end{aligned}$$

by Lebesgue's dominated convergence theorem.

The set Δ of all populations is compact in the weak topology, and with f continuous, the expected payoff difference $E(P, Q) - E(Q, Q)$ is continuous in Q with respect to the weak topology. Letting R be a weak accumulation point of the trajectory $(Q(t))_{t\geq 0}$, we have that

$$0 = \lim_{t\to\infty}[E(P, Q(t)) - E(Q(t), Q(t))] = E(P, R) - E(R, R),$$

for any $P \in \Delta_A$ such that $P \ll Q(0)$. Since we have already established that Δ_A is Lyapunov stable, R must be within ε of Δ_A, and hence of R_A. And given that $R_A \in \Delta_A$ is absolutely continuous with respect to R, and hence with respect to $Q(0)$, $E(R_A, R) = E(R, R)$ and $R \in \Delta_A$ by strong immutability of Δ_A. ■

Proof of Proposition 3. For any R with $d(R, \Delta_A) \in (0, \varepsilon]$, the fact that $R = R(A)R_A + R(A^c)R_{A^c}$ implies that

$$R_A - R = R(A^c)(R_A - R_{A^c}),$$

and hence—by bilinearity of E—that

$$\begin{aligned}E(R_A - R, R) &= R(A^c)E(R_A - R_{A^c}, R) \\ &= R(A^c)E(R_A - R_{A^c}, R(A)R_A + R(A^c)R_{A^c}) \\ &= R(A)R(A^c)E(R_A - R_{A^c}, R_A) + R(A^c)^2E(R_A - R_{A^c}, R_{A^c}).\end{aligned}$$

This is strictly positive if and only if

$$R(A)(E(R_A - R_{A^c}, R_A)) > (1 - R(A))(E(R_{A^c} - R_A, R_{A^c})). \tag{7}$$

"⇒": In the absence of strictly fitter mutations, $E(R_A - R_{A^c}, R_A) \geq 0$. If this inequality holds strictly, then (7) must hold for ε sufficiently small, since $R(A) \to 1$ as $\varepsilon \to 0$. Moreover, in the absence of secret-handshake mutations, $E(R_A - R_{A^c}, R_A) = 0$ implies

$E(R_{A^c} - R_A, R_{A^c}) < 0$, so that (7) must again hold. Δ_A is thus a strongly immutable face.

"$\Leftarrow$": Suppose otherwise. Then Δ_A is strongly immutable and there exists either a strictly fitter mutation or a secret-handshake mutation away from Δ_A. By strong immutability of Δ_A, (7) must hold for all R with $d(R, \Delta_A) \in (0, \varepsilon]$. But if $R = R(A)R_A + R(A^c)R_{A^c}$ is a strictly fitter mutation, then so is $(1-\eta)R_A + \eta R_{A^c}$ for all $0 < \eta < 1$; and since $E(R_A - R_{A^c}, R_A) < 0$, it follows that (7) is violated for η sufficiently small, a contradiction. Similarly, if R is a secret-handshake mutation, then so is $(1-\eta)R_A + \eta R_{A^c}$ for all $0 < \eta < 1$; and since $E(R_A - R_{A^c}, R_A) = 0$ and $E(R_{A^c} - R_A, R_{A^c}) \geq 0$, (7) is again violated, giving a contradiction for η sufficiently small. Thus, neither a strictly fitter nor a secret-handshake mutation away from a strongly immutable Δ_A can exist. ∎

Proof of Theorem 3. Let $\delta = \varepsilon^2$, where ε is the strong uninvadability barrier of Π^*. Then, for any $Q(0) \in \mathscr{U}_{\Pi^*} \backslash \Pi^*$, there exists a $P \in \Pi^*$ such that $P \ll Q(0)$, $K_{Q(0):P} < \delta$ and hence $\|Q(0) - P\| < \varepsilon$, with $E(P, Q(0)) > E(Q(0), Q(0))$ following by strong uninvadability of Π^*. By Bomze's (1991) Lemma 2(a), $P \ll Q(t)$ for all $t \geq 0$, so that $K_{Q(t):P}$ is well defined and nonnegative, and from his Lemma 2(b),

$$\begin{aligned} K_{Q(t):P} &= K_{Q(0):P} - \int_S \int_0^t \sigma(x, Q(s))\, ds\, P(dx) \\ &= K_{Q(0):P} - \int_0^t \int_S \sigma(x, Q(s))\, P(dx)\, ds \\ &= K_{Q(0):P} - \int_0^t E(P, Q(s)) - E(Q(s), Q(s))\, ds, \end{aligned} \tag{8}$$

where the second line exploits Fubini's theorem. Given uniform continuity of the map $t \mapsto \sigma(x, Q(t))$ (established in the proof of Oechssler and Riedel's (2001) Theorem 3), which in turn yields continuity of $t \mapsto E(P, Q(t)) - E(Q(t), Q(t))$, the time derivative of $K_{Q(t):P}$ at time t exists and equals $E(Q(t), Q(t)) - E(P, Q(t))$. This is nonpositive for $\|Q(t) - P\| \leq \varepsilon$ by strong uninvadability of Π^*, so that $0 \leq K_{Q(t):P} \leq K_{Q(0):P} < \delta$ and $\|Q(t) - P\| < \varepsilon$ for all $t \geq 0$.

Therefore, the integrand on the right-hand side of (8) is nonnegative (and uniformly continuous in $s \geq 0$), and since (8) is bounded below by 0, it follows that

$$E(P, Q(t)) - E(Q(t), Q(t)) \to 0 \quad \text{as } t \to \infty.$$

Letting R be a weak accumulation point of the trajectory $(Q(t))_{t \geq 0}$, we know from above that $\|Q(t) - P\| < \varepsilon$ for all $t \geq 0$, which implies that $\|R - P\| \leq \varepsilon$. Therefore, $R \in \Pi^*$ by strong uninvadability of Π^*. ∎

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