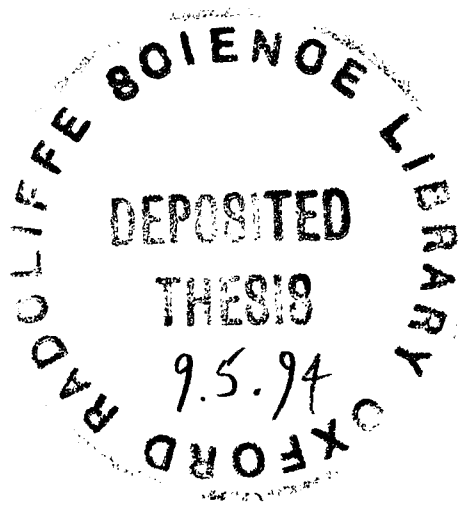
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16th July 1993

Intersection Topologies

by

Mark R. Jones



Thesis submitted in support of an application for leave to supplicate for the degree of Doctor of Philosophy.

To my parents

Acknowledgements

"y_i"

(Old joke, Brilliant Oxford Topology Youth Movement)

My warmest thanks go to Dr. Mike Reed for all of his help and encouragement, as well as the occasional dalliance with blackmail, during my three years of postgraduate study. Warmest thanks also to all of the members of the Analytic Topology group for their support, suggestions and patience without them, this piece of work could never have been completed. Special thanks to Paul for always putting up with new ideas over *innumerable* coffee-breaks, tea-breaks, laughing-breaks and just plain breaks, and to Ian for some of the most contrived attempts at making me irate that I have ever heard. I would also like to thank Dr. John Roe for initially firing my enthusiasm for becoming a mathematician.

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Abstract

Intersection Topologies

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D.Phil. Thesis
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Given two topologies, $\mathcal{T}_1$ and $\mathcal{T}_2$ on the same set X, the *intersection topology* with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ is the topology with basis

$$\{U_1 \cap U_2 : U_1 \in \mathcal{T}_1,\ U_2 \in \mathcal{T}_2\}$$

Equivalently, $\mathcal{T}$ is the join of $\mathcal{T}_1$ and $\mathcal{T}_2$ in the lattice of topologies on the set X. This thesis is concerned with analysing some particular classes of intersection topologies, and also with making some more general remarks about the technique.

Reed was the first to study intersection topologies in these terms, and he made an extensive investigation of intersection topologies on a subset of the reals of cardinality $\aleph_1$, where the topologies under consideration are the inherited real-line topology and the topology induced by an ω_1-type ordering of the set. We consider the same underlying set, and describe the properties of the intersection topology with respect to the inherited Sorgenfrey line topology and an ω_1-type order topology, demonstrating that, whilst most of the properties possessed by Reed's class are shared by ours, the two classes are strictly disjoint.

A useful characterisation of the intersection topology is as the diagonal of the product of the two topologies under consideration. We use this to prove some general properties about intersection topologies, and also to show that the intersection topology with respect to a first countable, hereditarily separable space and an ω_1-type order topology can never be locally compact.

Results about the real line and Sorgenfrey line intersections with ω_1 use various properties of the two lines. We demonstrate that most of the basic properties of the intersection topology require only the hereditary separability of $\mathbb{R}$, and give examples to show that 'hereditary' is essential here. We also show that results about normality, ω_1-compactness and the property of being perfect, all of which are set-theoretic in the classes of real-ω_1 and Sorgenfrey-ω_1 intersection topologies, can be shown to generalise to the class of intersection topologies with respect to separable generalised ordered spaces and ω_1.

Contents

1 Introduction **5**

1.1 Definitions and notation 8

1.2 The study of the lattice of topologies 13

1.3 Pol's space as an intersection topology 16

2 Sorgenfrey-ω_1 intersection topologies **24**

2.1 On intersection topologies w.r.t. the real line and ω_1 25

2.2 Carrying Reed's results over to Sorgenfrey-ω_1 intersection topologies 28

2.3 Carrying Kunen's results over to Sorgenfrey-ω_1 intersection topologies 40

2.4 A difference between Sorgenfrey-ω_1 and real-ω_1 intersection topologies 46

3 The intersection topology as the diagonal of the product **49**

3.1 A new characterisation 49

3.2 Bešlagić's Dowker product constructions 56

4 Hereditarily separable-ω_1 and hereditarily Lindelöf-ω_1 intersection topologies **58**

4.1 Some generalities 59

4.2 Hereditarily separable-ω_1 intersection topologies 60

4.3 Hereditarily Lindelöf-ω_1 intersection topologies 63

4.4 Dropping hereditary separability 64

4.5 The bow-tie space 69

5 The intersection topology w.r.t. a separable GO-space and the countable ordinals **72**

5.1 Separable GO-spaces and almost G_δ-diagonals 73

5.2 Applications to the intersection topology 80

Chapter 1

Introduction

"He's out," said Pooh sadly. "That's what it is. He's not in. I shall have to go on a fast Thinking Walk by myself. Bother!"

(*The House at Pooh Corner*, A. A. Milne)

In 1936, Garret Birkhoff published 'On the combination of topologies' [Bf], a paper in which he observes that the family of all topologies on a set can be ordered by letting $\mathcal{T}_1 < \mathcal{T}_2$ iff $\mathcal{T}_1 \subset \mathcal{T}_2$. He notes that this ordering gives the family of such topologies a lattice structure - for any two topologies, $\mathcal{T}_1$ and $\mathcal{T}_2$, on a set, there is a topology $\mathcal{T}_1 \wedge \mathcal{T}_2$, the *meet* of $\mathcal{T}_1$ and $\mathcal{T}_2$, which is the greatest topology contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ (and in fact $\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2$), and there is a topology $\mathcal{T}_1 \vee \mathcal{T}_2$, the *join* of $\mathcal{T}_1$ and $\mathcal{T}_2$, which is the least topology containing both $\mathcal{T}_1$ and $\mathcal{T}_2$. The lattice has a greatest element the discrete topology, where every set is open and a least element - the indiscrete topology, where only the whole set and the null set are open. Moreover, the lattice is complete, that is to say there is a greatest topology contained in each element of an arbitrary family of topologies, and there is a least topology containing each element of an arbitrary family of topologies.

In spite of its fundamental importance as a concept in general topology, little attention has been paid to the study of this lattice and the effect of lattice operations on topologies. There has been some work carried out on *complementation* in the lattice (by Berri, Steiner, Anderson, Watson and others), and Meyer, Neumann-Lara and Wilson have considered joins of orderable topologies. We briefly review this work in what follows.

It seems intuitive that topological properties should behave very poorly with respect to

the lattice meet operation, for there seems no reason why, even if two topologies on the same set have very similar properties, that the open sets themselves should coincide (and thus be elements of the meet). Indeed, of all the separation axioms, only T_1 is preserved by the meet operation. This is easy to see: if $\{\mathcal{T}_\lambda : \lambda \in \Lambda\}$ is a family of T_1-topologies on a set X and $x \in X$, then the set $\{x\}$ is closed w.r.t. each $\mathcal{T}_\lambda$. But then $\{x\}$ is closed w.r.t. $\bigwedge_{\lambda\in\Lambda}\mathcal{T}_\lambda$ for all $x \in X$, and the meet is therefore T_1. Because the operation performs so badly from a topological viewpoint, we shall refer to the meet only sparingly. In this thesis, we concentrate instead on the join operation in terms of its effect on the lattice of topologies on a set, which is much better-behaved. The work was motivated by a paper of Reed's [R_1], in which he makes the following definition:

Definition 1.1 *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies defined on the set X, then the* intersection topology, *$\mathcal{T}$, with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ is the topology on X with basis*

$$\{U_1 \cap U_2 : U_1 \in \mathcal{T}_1,\ U_2 \in \mathcal{T}_2\}$$

Evidently, the definition of the intersection topology coincides with that of the join of the two topologies (for the join *must* contain all sets of the form $\{U_1 \cap U_2 : U_1 \in \mathcal{T}_1,\ U_2 \in \mathcal{T}_2\}$). Both definitions are useful, and we shall refer to the 'join' or the 'intersection topology' interchangeably throughout the thesis.

Reed observes that although there had been no general study of its properties, there had been several notable applications of the intersection topology in the literature, for example by Pol to construct a normal, locally metrizable, nonparacompact space in [Po] (which we analyse in Section 1.3), by Pol and Pol to construct a hereditarily normal, strongly zero-dimensional space with a subspace of positive dimension in [PP] and by Reed himself to produce results concerning Q-sets [R_2], S- and L-spaces [R_3] and normality conditions [R_4]. Bešlagić [B_1, B_2, B_3] too uses ideas related to the intersection topology (discussed in Section 3.2). Reed rectifies this omission in [R_1] with some general remarks about the technique and an extensive investigation of a particular class of intersection topologies the class $\mathcal{C}$ such that $(X, \mathcal{T}) \in \mathcal{C}$ if and only if $X = \{x_\alpha : \alpha < \omega_1\} \subseteq \mathbb{R}$ and $\mathcal{T}$ is the intersection topology with respect to the inherited real-line topology $\mathcal{T}_{\mathbb{R}}$ on X and the order topology $\mathcal{T}_{\omega_1}$ on X of type ω_1. The construction here echoes that of the non-regular S- and L-spaces of Hajnal and Juhász [HJ]. They take an uncountable subset of the reals, well-order it in type ω_1, and then refine the usual topology on the

set by declaring all initial segments w.r.t. the order open. This is then a hereditarily separable, non-Lindelöf space. If all initial segments are declared closed instead, we obtain a hereditarily Lindelöf, non-separable space. In both cases, the resulting spaces are Hausdorff but not regular. They are a simple kind of intersection topology too, this time w.r.t. the real-line topology and a non-T_1 topology generated by initial segments (or final segments) as basic open sets.

In addition, Reed demonstrates that the class $\mathcal{C}$ contains a collectionwise Hausdorff, non-normal, first countable, T_3 space (in fact, the first submetrizable example to be locally separable or to have cardinality $\aleph_1$), and uses this example to resolve a question about his 'Moore space machine' [R_5, R_6]. In a more recent paper [CR], Reed and Cook prove that if the Moore space machine is applied to a member of the class $\mathcal{C}$ then, under $\mathbf{MA} + \neg\mathbf{CH}$, the resulting space is a normal Moore space whose square is not normal.

This particular class $\mathcal{C}$ of intersection topologies has since been further investigated by Gruenhage in [Gr], Kunen in [Ku] and van Douwen in [vD], and we review these results in what follows. Reed points out in [R_1] that all of the results contained in that paper about the class $\mathcal{C}$ remain valid for *any* separable metric topology intersected with an ω_1-type order topology. It is from this viewpoint that Kunen and van Douwen establish further results about $\mathcal{C}$.

Our aim is twofold. Much of the emphasis of the thesis is concerned with intersection topologies w.r.t the countable ordinals and some other topology, thus generalising the results of Reed, Kunen, van Douwen and Gruenhage to larger classes of spaces, and determining which properties of separable metric spaces are critical in their results about the class $\mathcal{C}$. Our other interest is in making more general remarks about the join of two topologies, and considering which properties of the starting topologies will result in which properties of the join.

The remainder of this introductory chapter consists of three parts. The first contains some basic definitions and an outline of the notation used in the thesis. Then we review the study of the lattice of topologies in the literature. And lastly, motivated by Reed's observation that Pol's space may be re-interpreted as an intersection topology, we analyse this space from this perspective.

Chapter 2 is a self-contained study of what we will call the class $\mathcal{S}$, where $(X, \mathcal{T}) \in \mathcal{S}$

if and only if $X = \{x_\alpha : \alpha < \omega_1\} \subseteq \mathbb{R}$ and $\mathcal{T}$ is the intersection topology with respect to the inherited *Sorgenfrey* line topology $\mathcal{T}_{\mathbb{S}}$ on X and the order topology $\mathcal{T}_{\omega_1}$ on X of type ω_1. The Sorgenfrey line is not metrizable, of course, and the absence of a countable base makes many of the results proved about $\mathcal{C}$ difficult to carry over to $\mathcal{S}$. But we do eventually succeed in showing that $\mathcal{C}$ and $\mathcal{S}$ share virtually all of the properties which Reed, Kunen, van Douwen and Gruenhage show that $\mathcal{C}$ possesses, whilst also demonstrating that the two classes are strictly disjoint.

Chapter 3 is concerned with some more general results, following from the interesting characterisation of the join of two spaces as the diagonal of their product. In Chapter 4 we consider two well-behaved classes of spaces - hereditarily separable and hereditarily Lindelöf spaces (the real line and the Sorgenfrey line belong to both classes) and show which results about their intersection topologies with ω_1 generalise from $\mathcal{C}$ and $\mathcal{S}$.

One set of results which is absent from the generalisations in Chapter 4 is that from Kunen's paper, concerning normality, 'perfectness' and ω_1-compactness of intersection topologies with ω_1. In Chapter 5, we give a partial generalisation to the class of separable GO-spaces, where it is shown that Kunen's results *do* hold.

1.1 Definitions and notation

Anything not defined here can be found in [E] or [KV].

Elementary definitions

A set is *clopen* if it is both closed and open. A space is *zero-dimensional* if it has a basis of clopen sets. A *G_δ-set* is a countable intersection of open sets. An *F_σ-set* is a countable union of closed sets. A space is *perfect* if every closed set is a G_δ. A point X is *isolated* if the set $\{x\}$ is open. A set $K \subseteq X$ is said to be *discrete* if no point of K is a limit point of K; equivalently, if the relative topology on K is discrete. A collection $\mathcal{H}$ of sets is said to be *discrete* if, for each $x \in X$, there exists an open set U_x containing x such that $|\{H \in \mathcal{H} : U_x \cap H\} \neq \emptyset| \leq 1$. A space is *$\omega_1$-compact* if every set of cardinality $\aleph_1$ has a limit point; equivalently, if there are no uncountable closed discrete subspaces. A space X is *scattered* if every nonempty subspace $Y \subseteq X$ has an isolated point w.r.t. the

subspace topology. If P is a topological property, then a space X is said to be *hereditarily* P if every subspace of X has property P.

Cardinal functions

The *weight* of X, $w(X)$, is the least cardinality of a basis for X. (If $w(X) = \aleph_0$ then X is *second countable*.) The *density* of X, $d(X)$, is the least cardinality of a dense subset of X. (If $d(X) = \aleph_0$ then X is *separable*.) The *character* of $x \in X$ is the least cardinality of a neighbourhood base at that point, and the *character* of X, $\chi(X)$, is the supremum of the characters of all of the points of X. (If $\chi(X) = \aleph_0$ then X is *first countable*.) X is *countably tight* if, for all $A \subseteq X$ and $x \in \overline{A}$, there exists countable $B \subseteq A$ such that $x \in \overline{B}$. The *cellularity* of X, $c(X)$ is the supremum of the cardinalities of families of pairwise disjoint nonempty open subsets of X. (If $c(X) = \aleph_0$ then X has the *countable chain condition*.)

Generalised metric and separation properties

A sequence $(\mathcal{G}_n)$ of open covers of a space X is a *development* if, for each $x \in X$, the set $\{\mathrm{St}(x, \mathcal{G}_n) : n \in \omega\}$ is a local base at x. A *developable* space is a space which has a development. A *Moore space* is a regular developable space. A space X has a G_δ-*diagonal* if the diagonal of $X \times X$ is a G_δ; equivalently, X has a sequence $(\mathcal{U}_n)$ of open covers such that $\bigcap_{n\in\omega} \mathrm{St}(x, \mathcal{U}_n) = \{x\}$ for all $x \in X$. A space $(X, \mathcal{T})$ is *submetrizable* if there exists a metrizable topology $\mathcal{M} \subseteq \mathcal{T}$. A related notion is *cometrizability*. A space $(X, \mathcal{T})$ is cometrizable if $\mathcal{T}$ has a separable metrizable subtopology $\mathcal{M}$ such that if $x \in U \in \mathcal{T}$, then there exists $V \in \mathcal{T}$ such that $x \in V \subseteq \overline{V}^{\mathcal{M}} \subseteq U$.

A space X is *semi-stratifiable* if there is a function G which assigns to each $n \in \omega$ and each closed set $H \subseteq X$, an open set $G(n, H)$ containing H such that $H = \bigcap_n G(n, H)$, and if $H \subseteq K$, then $G(n, H) \subseteq G(n, K)$. If, moreover, $H = \bigcap_n \overline{G(n, H)}$, then X is *stratifiable*. A space is *monotonically normal* if to each pair (H, K) of disjoint closed subsets of X, one can assign an open set $D(H, K)$ such that, firstly, $H \subseteq D(H, K) \subseteq \overline{D(H, K)} \subseteq X \setminus K$, and, secondly, if $H \subseteq H'$ and $K \supseteq K'$ then $D(H, K) \subseteq D(H', K')$.

A space X is said to have a *base of countable order* if there is a basis $\mathcal{B}$ for X such that

if $x \in B_n \in \mathcal{B}$ and (B_n) is strictly decreasing, then $\{B_n : n \in \omega\}$ is a local base at x. It has a *W_δ-diagonal* if there is a sequence of bases $(\mathcal{B}_n)$ for X such that if $x \in B_n \in \mathcal{B}_n$ and (B_n) is decreasing, then $\{x\} = \bigcap_n B_n$. A collection $\mathcal{F}$ of subsets of a space X is a *k-network* if, whenever K is a compact subset of an open set U, there exists a finite $\mathcal{F}' \subseteq \mathcal{F}$ such that $K \subseteq \bigcup \mathcal{F}' \subseteq U$. A regular space with a σ-locally finite (countable) k-network is called an *$\aleph$-space* (*$\aleph_0$-space*). X is a *P-space* if $X \times M$ is normal for all metric spaces M.

A space X is *(strongly) collectionwise Hausdorff* if every closed discrete set D can be *(strongly) screened*, that is, there exists a pairwise disjoint (discrete) collection $\mathcal{V}$ of open sets covering D such that, for each $V \in \mathcal{V}$, $|V \cap D| = 1$. A space X is *collectionwise normal* if every discrete collection of closed sets can be screened by a pairwise disjoint collection of open sets. A space is *pseudonormal* if two disjoint closed sets, one of which is countable, can be separated by disjoint open sets. A space is *completely regular* if, for nonempty C closed in X and $x \notin C$, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(C) = \{1\}$.

LOTS and GO-spaces

If X is a linearly ordered set, then we can define a topology on X by taking as a sub-basis all sets $(\leftarrow, x) = \{y : y < x\}$ and $(x, \rightarrow) = \{y : y > x\}$ for $x \in X$. This is called the *linear order topology*, and X thus equipped is a *linearly ordered topological space*, which we shall call a *LOTS*. A subset U of a LOTS is *convex* if, for all $z \in X$ such that $x < z < y$ where $x, y \in U$, we have $z \in U$.

A space X is called a *GO-space* (generalised ordered space) if it is homeomorphic to a subspace of a LOTS; equivalently, if there is a linear order on X such that the topology includes the topology generated by the linear order and also has a basis of sets which are convex with respect to the linear order.

Let X be a GO-space, and suppose (A, B) is an ordered pair of disjoint open sets such that $X = A \cup B$ and $a < b$ whenever $a \in A$, $b \in B$. (A, B) is then a *gap* if A has no maximal point and B has no minimal point. (A, B) is a *pseudogap* if both A and B are nonempty, and either (i) A has no maximal point, but B has a minimal point, or (ii) A has a maximal point, but B has no minimal point.

In a LOTS or a GO-space, we write $(\leftarrow, x]$, $[x, \rightarrow)$, $(x, y]$ for $\{z : z \leq x\}$, $\{z : z \geq x\}$ and $\{z : x < z \leq y\}$ respectively. $[x, y]$, (x, y) and $[x, y)$ are defined similarly.

Covering properties

Suppose $\mathcal{U}$ is a cover of a space X. $\mathcal{V}$ *refines* $\mathcal{U}$ if $\mathcal{V}$ is a cover of X and, for all $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U$. $\mathcal{V}$ *star-refines* $\mathcal{U}$ if $\mathcal{V}$ is a cover of X and, for all $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $\mathrm{St}(V, \mathcal{V}) \subseteq U$. An open cover $\{V_\alpha : \alpha < \kappa\}$ is a *shrinking* of the family $\{U_\alpha : \alpha < \kappa\}$ if, for all $\alpha < \kappa$, $\overline{V}_\alpha \subseteq U_\alpha$.

A collection $\mathcal{U}$ of subsets of X is *locally finite (countable)* if each point has a neighbourhood meeting only finitely many (only countably many) members of $\mathcal{U}$, and is *point finite (countable)* if each point is a member of only finitely many (only countably many) members of $\mathcal{U}$. For each of these properties P (and for P 'disjoint' or 'discrete'), a collection $\mathcal{U}$ has property σ-P if $\mathcal{U}$ can be expressed as a countable union of subcollections, each with property P.

A space is *(countably) paracompact* if every (countable) open cover has a locally finite open refinement, *(countably) metacompact* if every (countable) open cover has a point finite open refinement, *metaLindelöf* if every open cover has a point countable open refinement, *screenable* if every open cover has a σ-disjoint open refinement and *subparacompact* if every open cover has a σ-discrete closed refinement.

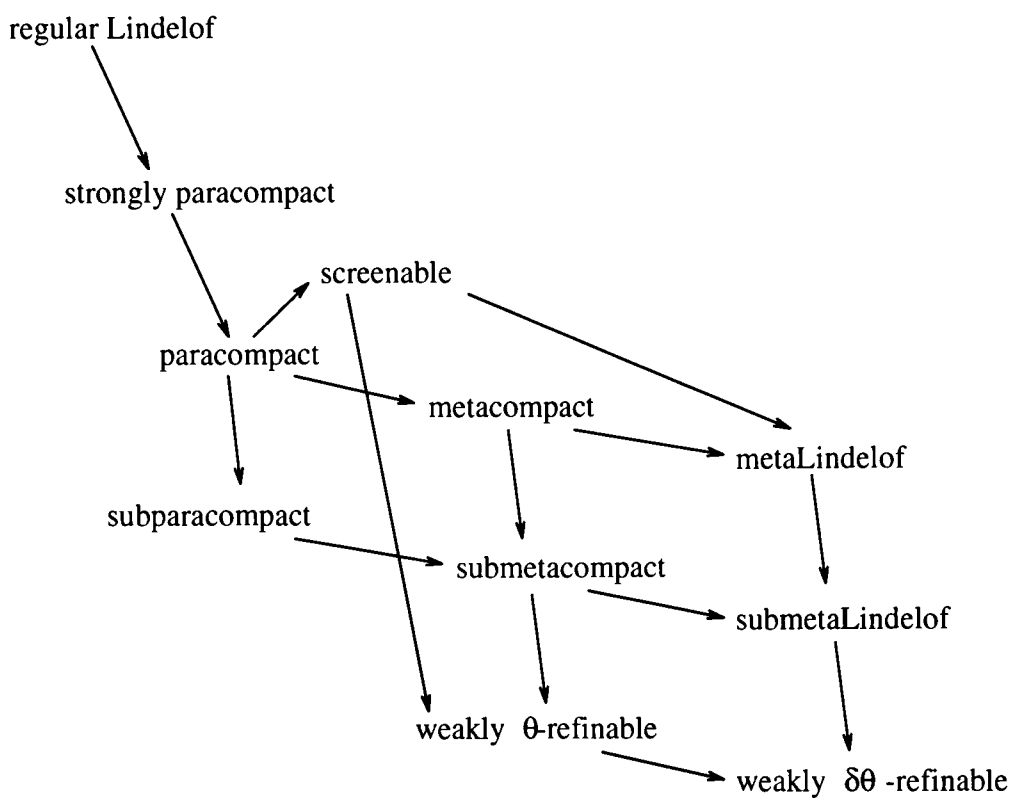
X is *weakly θ-refinable* if for any open cover $\mathcal{U}$, there is an open refinement $\mathcal{V} = \bigcup_n \mathcal{V}_n$ such that if $x \in X$, then there is some $n \in \omega$ such that $x \in \bigcup \mathcal{V}_n$ and x meets only finitely many members of $\mathcal{V}_n$. If this condition is strengthened to require that each $\mathcal{V}_n$ covers, then X is *submetacompact* or *θ-refinable*.

X is *weakly $\delta\theta$-refinable* if for any open cover $\mathcal{U}$, there is an open refinement $\mathcal{V} = \bigcup_n \mathcal{V}_n$ such that if $x \in X$, then there is some $n \in \omega$ such that $x \in \bigcup \mathcal{V}_n$ and x meets only countably many members of $\mathcal{V}_n$. If this condition is strengthened to require that each $\mathcal{V}_n$ covers, then X is *submetaLindelöf* or *$\delta\theta$-refinable*.

A space X is *realcompact* if it can be embedded as a closed subspace of the Tychonoff product of copies of the real line. X is *sequentially compact* if each countably infinite set has an infinite subset that converges somewhere. A completely regular space is *Čech-*

complete if it is a G_δ-set in its Stone-Čech compactification; equivalently, there exists a sequence $(\mathcal{U}_n)$ of open covers such that, whenever $\{M_n : n \in \omega\}$ are nonempty closed sets for $n \in \omega$ with $M_{n+1} \subseteq M_n$ and $M_n \subseteq U_n$ for some $U_n \in \mathcal{U}_n$, the intersection of the M_ns is nonempty. A space is *shrinking* if every open cover has a shrinking.

We illustrate the relationships between some of these properties in the following diagram:



Set theory

We write ω for the first infinite ordinal, and ω_1 for the first uncountable ordinal. $\aleph_0$ and $\aleph_1$ are their respective cardinalities. We write $\mathfrak{c}$ for the cardinality of the continuum $2^{\aleph_0}$.

Let μ be a limit ordinal. A subset $C \subseteq \mu$ is a *closed unbounded* or *club* set if it is unbounded in μ and is closed w.r.t. the usual order topology. A subset $S \subseteq \mu$ is a *stationary* set if it meets every club set in μ. There are two lemmas about stationary sets in ω_1 which we use throughout the thesis. They are:

Lemma 1.2 (Ulam) (COUNTABLE DECOMPOSITION LEMMA) *If $H = \bigcup_{i<\omega} H_i$ is stationary in ω_1, then there is some $i < \omega$ for which H_i is stationary.*

Lemma 1.3 (Fodor) (PRESSING DOWN LEMMA) *If H is stationary in ω_1 , and if*

$f : H \to \omega_1$ such that $f(\alpha) < \alpha$ in ω_1 for each $\alpha \in H$, then there is some $\beta < \omega_1$ such that $\{\alpha \in H : f(\alpha) = \beta\}$ is stationary in ω_1.

ZFC stands for Zermelo-Frankel set theory with the Axiom of Choice. We use the following consistent strengthenings of **ZFC** in this thesis. **CH** is the *Continuum Hypothesis*, that $\mathfrak{c} = \aleph_1$. $\mathbf{MA}_\kappa$ is the statement: 'Whenever $\langle \mathbb{P}, \leq \rangle$ is a nonempty c.c.c. partial order and $\mathcal{D}$ is a family of $\leq \kappa$ dense subsets of $\mathbb{P}$, then there is a filter G in $\mathbb{P}$ such that $\forall D \in \mathcal{D}$, $G \cap D \neq \emptyset$.' **MA** is then the statement: '$\forall \kappa < 2^\omega$, $\mathbf{MA}_\kappa$ holds.' **SH**, the *Suslin Hypothesis* is the statement: 'There are no Suslin lines', i.e. every c.c.c. LOTS is separable. Finally, $\diamondsuit$ is the statement: 'There are sets $A_\alpha \subseteq \alpha$ for $\alpha < \omega_1$ such that, for all $A \subseteq \omega_1$, the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary.'

Notation

As usual, $\mathbb{R}$ denotes the set of reals and $\mathbb{Q}$ denotes the set of rationals. An ω_1-type order topology on a set X will always be denoted by $\mathcal{T}_{\omega_1}$. If we refer to $(X, \mathcal{T}_{\omega_1})$ then implicitly it will be assumed that X has been equipped with some ω_1-type order as $X = \{x_\alpha : \alpha < \omega_1\}$. So we will call a subset of X stationary or club if it has the corresponding property with respect to the ω_1-type order, and sometimes we use the shorthand $(\alpha, \beta]$ to denote the set $\{x_\theta : \alpha < \theta \leq \beta\}$, and so on. For $x, y \in X$ we say $x < y$ in ω_1 if $\beta < \alpha$, where $x = x_\beta$ and $y = x_\alpha$. We shall occasionally use the convention that, for $x \in X$, $\alpha_x \in \omega_1$ is such that $x = x_{\alpha_x}$.

1.2 The study of the lattice of topologies

As mentioned in the introduction, there has been little work carried out on the study of the lattice of topologies. In this section we review this work.

Complementation in the lattice

Topologies $\mathcal{T}$ and $\mathcal{U}$ on the same set are said to be *complementary* iff their meet is the indiscrete topology and their join is the discrete topology. In lattice terminology, $\mathcal{T} \wedge \mathcal{U} = 0$ and $\mathcal{T} \vee \mathcal{U} = 1$. Note that both the discrete topology and the indiscrete

topology are uniquely complemented, so call any topology other than these two a *proper* topology.

In the 1960s, a considerable amount of work was carried out on the study of complementation. Two questions in particular were considered. When can a topology be complemented? And how many complementary topologies can it then have? In 1966, Steiner [Ste_1] showed that the lattice of all topologies on any set is complemented, that is *every* topology has a complementary topology. In 1968, Schnare [Sc_1] showed that in fact any proper topology has infintely many complements, and a year later [Sc_2] established that on an infinite set of cardinality κ, the number of complements of a proper topology is between κ and 2^{2^κ}. He also exhibited topologies with exactly κ, 2^κ and 2^{2^κ} complements, so that under **GCH**, there are exactly three possible values for the number of complements, and they are all attained.

Watson 'rediscovered' the subject in 1989, and establishes in [Wt_1] that the number of distinct complements of a topology on a set of cardinality $\aleph_n$ is either 1 or $\aleph_n^\kappa$, where $0 < \kappa \leq \aleph_n$ or $2^{\aleph_n + 2^{\omega_i}}$, where $0 \leq i \leq n$. For a general cardinal, κ, the set of possible numbers of complements of topologies on a set of cardinality κ remains unknown.

A slightly different notion is that of T_1-*complement.* Here we consider the lattice of T_1 topologies only, a sublattice of the full lattice of topologies. (Note that, of the separation axioms, it is only sensible to consider T_1-complements because no other separation axiom is preserved by meets and thus no other class of T_i-spaces forms a lattice.) The 0 in this lattice is not the indiscrete topology, of course, because this topology fails to be T_1. Instead, the 0 is the cofinite topology, and then the T_1-complement is defined in the obvious way. This lattice is not complemented, as shown in [Ste_2]. Steiner and Steiner [SS] show that of any pair of T_1-complements on a countable set, at least one fails to be Hausdorff, and Anderson and Stewart [AS] show that of any pair of T_1-complements on an arbitrary set, at least one fails to be first countable Hausdorff. Again, after a lull in the study of the subject, Watson established in [Wt_2] that there *is* a $T_{3\frac{1}{2}}$ topology (on a set of size $(2^\omega)^+$) which is in fact its own T_1-complement. In 1971, Anderson [An] had also shown that there are at least κ mutually T_1-complementary topologies on a set of cardinality κ, and when Watson re-approached the subject, he and Steprāns established that there do not exist uncountably many mutually T_1-complementary topologies on ω [SW]. They also show that it is consistent that there do not exist $\aleph_2$ many mutually

T_1-complementary topologies on ω_1, but that under **CH**, there exist $2^{\aleph_1}$ many mutually T_1-complementary topologies on ω_1.

In [Wt_3], Watson poses the following general questions about the subject:

- Which topology on a set of size n has the largest number of complements?
- What is the maximum number of pairwise complementary T_0-topologies on a set of size n? (The answer is known to lie asymptotically between $\frac{n}{100 \log n}$ and $0 \cdot 486n$.)
- Can every lattice be homomorphically embedded in the lattice of topologies on some set?

JO-topologies

A *JO-topology* on a set X is a topology which can be expressed as the lattice join of linearly ordered topologies. For a cardinal κ, a JO-topology, τ, is said to be JO[κ] if it can be expressed as the join of κ orderable topologies.

JO-topologies have been studied in [Me] and [MNW], where the relationship between JO-topologies and GO-topologies (subspaces of linearly ordered topologies) is explored. Evidently JO[1]-topologies are simply linearly ordered topologies. There exist examples of topologies which are JO[2] but not GO, and it is not difficult to show that every GO-topology is a JO-topology. What is still unknown is whether every GO-topology is JO[2] (or even JO[n] for some $n < \omega$).

The Sorgenfrey line is known to be JO[2] (as proved in [Me]), and the Michael line is at most JO[3]. In [MNW], it is shown that X is JO[2] if X is embeddable in a linearly ordered space D, where D is either metrizable and locally separable, or connected with countable cellularity. More generally, if $(X, \leq, \tau)$ is a GO-space and κ is the number of pseudogaps of $(X, \leq, \tau)$, then (X, τ) is JO[κ].

The intersection topologies w.r.t. the real line and the countable ordinals in [R_1] are JO[2]-topologies, of course, since both $\mathbb{R}$ and ω_1 (with their usual topologies) are LOTS, and many of the intersection topologies we consider in this thesis will be JO. In particular, the Sorgenfrey-ω_1 intersection topologies considered in Chapter 2 are JO[3]-topologies, because as mentioned earlier, the Sorgenfrey line is JO[2].

Joins of rational topologies

In [JK], Jayanthan and Kannan take a countable set and analyse the join of a collection of topologies on the set, all of which are homeomorphic to the rationals. Having observed that the join of any two so-called rational topologies is a metrizable non-compact topology, they demonstrate that the converse also holds - any non-compact metrizable topology on a countable set arises as the join of two rational topologies. They also show that corresponding results hold for a join of arbitrarily many rational topologies any such join is Tychonoff non-compact, and any Tychonoff non-compact topology on a countable set arises as the join of rational topologies.

They also observe that, by similar methods, one can prove that every completely metrizable, zero-dimensional, separable space of cardinality $\mathfrak{c}$ is a join of two irrational topologies. Here, the converse does not hold, however, as the join of two irrational topologies need not be *completely* metrizable. The question of how to characterise the joins of two, or even arbitrarily many, irrational topologies remains open.

1.3 Pol's space as an intersection topology

In [Po], Pol constructed an example of a topological space that settled many open questions in topology. Pol's space was the first to be:

- perfectly normal and locally metrizable, yet not metrizable
- collectionwise normal with a G_δ-diagonal and not paracompact
- a perfect T_3-space but not subparacompact.

Pol's construction has since been employed and modified to construct a hereditarily normal, strongly zero-dimensional space with a subspace of positive dimension (by Pol and Pol [PP]) and various other examples in the dimension theory of perfectly normal spaces (by Gruenhage and Pol [GP]).

As Reed observes in [R_1], the construction employed in [Po] may be re-interpreted as essentially the intersection topology with respect to the countable ordinals with the order topology and the Baire space of weight ω_1. Indeed, Pol's space provided the initial

impetus for a more general study of intersection topologies w.r.t. a metrizable space and ω_1, and in particular motivated Reed's study of the class $\mathcal{C}$, using the most familiar metric space of all. In this section, we take Pol's example and analyse it as an intersection topology.

Pol's construction technique

In [Po], Pol takes a metric space of uncountable weight and refines the topology to yield the required space.

Let X be a set and ρ a metric on X such that the weight of (X, ρ), $w(X, \rho) = \aleph_1$. Suppose that, for $\xi < \omega_1$, we have constructed sets X_ξ satisfying the following conditions:

1. $X_1 \subset X_2 \subset \ldots \subset X_\xi \subset \ldots \subset X$,
2. $\overline{X_\xi}^\rho = X_\xi$,
3. $w(X_\xi, \rho) \leq \aleph$,
4. $X = \bigcup_{\xi<\omega_1} X_\xi$ and, for ξ a limit ordinal, $X_\xi = \overline{\bigcup_{\alpha<\xi} X_\alpha}^\rho$.

Such sets can be obtained, for example, by taking $X_\xi = \overline{\{x_\alpha : \alpha < \xi\}}^\rho$ for a set $\{x_\alpha : \alpha < \omega_1\}$ dense in the space (X, ρ).

The required topology on the set X is that generated by the basic open sets $U \cap X_\xi$, where U is open with respect to ρ and $\xi < \omega_1$.

Pol goes on to analyse this modified topology. He shows that the clopen sets X_ξ are homeomorphic to subspaces of the products $(X_\xi, \rho) \times W(\xi + 1)$ (where $W(\alpha)$ is the set α equipped with the order topology), whence each X_ξ is metrizable, so that X is locally metrizable. X is further shown to be perfect and collectionwise normal, and fails to be paracompact provided that the metric ρ is sufficiently complicated. The precise condition is formulated in terms of the function $\chi : X \to \omega_1$, defined by

$$\chi(x) = \min\{\xi : x \in X_\xi\} \text{ for } x \in X.$$

If $\chi(X)$ is stationary in ω_1 then X is not paracompact. Pol observes that this condition depends on the metric ρ alone; namely, it is equivalent to the property that the met-

ric space (X, ρ) cannot be expressed as the union of countably many locally separable subspaces.

Pol's space as an intersection topology

It is fairly evident from the construction that some notions of intersection topologies can be applied. We already have a topology, ρ, on X. If we let τ be the topology generated by basic open sets $\{X_\xi : \xi < \omega_1\}$, then the Pol space topology is simply the intersection topology w.r.t. ρ and τ. Unfortunately, in general the topology τ fails to be particularly interesting. Since all of the basic open sets are nested, it even fails to be T_1. In general this is the best we can do, and essentially the same is true of the constructions in [PP] and [GP].

However, in some cases, it turns out that the topology τ interacts with the metric topology ρ in a particularly straightforward way. In the case we consider here, we take our metric space (X, ρ) to be a particular subspace of $B(\omega_1)$, the Baire space of weight ω_1. We apply Pol's technique to the space, note that there is an obvious ω_1-type order on this subspace, and show that all open sets w.r.t. the topology induced by this order are open in the resultant Pol's space. Then this example is actually the intersection topology w.r.t. our subspace of $B(\omega_1)$ and the countable ordinals.

First we consider a topology $\mathcal{T}$ defined on the set X of countable limit ordinals and demonstrate that $(X, \mathcal{T})$ can be embedded in $B(\omega_1)$.

The topology $\mathcal{T}$ is defined as follows. For each countable limit ordinal, α, we fix an increasing sequence $\{\alpha_n : n \in \omega\}$ of countable ordinals cofinal with α. The basis for $\mathcal{T}$ is then the collection $\{V_{\alpha,n} : \alpha \in X, n \in \omega\}$ where

$$V_{\alpha,n} = \{\beta \in X : \alpha_m = \beta_m \ \ \forall m \leq n\}$$

Lemma 1.4 *$(X, \mathcal{T})$ is homeomorphic to a subspace of $B(\omega_1)$, the Baire space of weight ω_1.*

Proof For a cardinal $\alpha \geq \aleph_0$, the *Baire space*, $B(\alpha)$, of weight α is a metrizable topology on the set α^ω (see [E, Example 4.2.12]). The metric, ρ, compatible with the topology is

defined thus for $(\xi_i)_{i\in\omega}$, $(\eta_i)_{i\in\omega}$ in α^ω:

$$\rho((\xi_i)_{i\in\omega},(\eta_i)_{i\in\omega}) = \begin{cases} 1 & \text{if } \xi_1 \neq \eta_1 \\ 0 & \text{if } \xi_i = \eta_i \text{ for all } i \in \omega \\ 1/k & \text{if } \xi_i = \eta_i \text{ for all } i < k \text{ but } \xi_k \neq \eta_k \end{cases}$$

Define $f : X \to B(\omega_1)$ by $f(\alpha) = (\alpha_i)_{i\in\omega}$ (where $(\alpha_i)_{i\in\omega}$ is the fixed sequence cofinal with α). f is evidently injective. ($\alpha_n = \beta_n \ \forall n \Rightarrow \lim \alpha_n = \lim \beta_n \Rightarrow \alpha = \beta$).

If U is open in $B(\omega_1)$ and $\alpha \in f^{-1}(U)$, then there exists $k \in \omega$ such that $B_{1/k}(f(\alpha)) \subseteq U$. But then if $f(\beta) \in B_{1/k}(f(\alpha))$, then $\alpha_i = \beta_i$ for all $i < k$, whence $\beta \in V_{\alpha,k-1}$. Hence $\alpha \in V_{\alpha,k-1} \subseteq f^{-1}(U)$, whence $f^{-1}(U)$ is open in X. Thus f is continuous.

Is $g \stackrel{\text{def}}{=} f^{-1}|_{\text{im f}}$ continuous? Note that $g(\alpha_1,\alpha_2,\ldots) = \lim \alpha_n$. If U is open in X and $(\alpha_1,\alpha_2,\ldots) \in g^{-1}(U)$, then $\alpha = \lim \alpha_n \in U$, and since U is open, there exists $N \in \omega$ such that $V_{\alpha,N} \subseteq U$. But if $(\beta_1,\beta_2,\ldots) \in \text{im}f$ with $\rho((\alpha_i),(\beta_i)) < 1/(N+1)$, then $\alpha_i = \beta_i$ for all $i \leq N$, so that $\beta \in V_{\alpha,N} \subseteq U$. Hence $B_{1/(N+1)}((\alpha_1,\alpha_2,\ldots)) \subseteq g^{-1}(U)$ and g is continuous. Therefore $(X,\mathcal{T})$ is homeomorphic to a subspace of $B(\omega_1)$, consisting of those sequences cofinal with the limit ordinals used in the definition of $\mathcal{T}$.

□

Corollary 1.5 *$(X,\mathcal{T})$ is metrizable.*

Lemma 1.6 *$(X,\mathcal{T})$ has weight $\aleph_1$.*

Proof If $(X,\mathcal{T})$ were second countable, then, since it is metrizable, it would be separable. But let $\{\alpha^n : n \in \omega\}$ be a countable subset of $(X,\mathcal{T})$. Let $\beta > \sup\ \alpha^n$. Then there exists $N \in \omega$ such that, if $m \geq N$, then $\beta_m > \sup\ \alpha^n$ (since $\{\beta_m : m \in \omega\}$ is cofinal with β), i.e., $\beta_m > \alpha^n$ for all $n \in \omega, m \geq N$. But then if $n \in \omega$, $\alpha^n_N \neq \beta_N$ (since $\alpha^n_N < \alpha_N$ for all $n \in \omega$), whence $\alpha^n \notin V_{\beta,N}$ for any $n \in \omega$. Hence $\{\alpha^n : n \in \omega\}$ is not dense in $(X,\mathcal{T})$ and $(X,\mathcal{T})$ cannot be separable. Since $|X| = \aleph_1$, we must therefore have $w(X,\mathcal{T}) = \aleph_1$.

□

Having shown that $(X,\mathcal{T})$ is metrizable (say by a metric ρ) and of uncountable weight, we can certainly apply Pol's technique to the space. The countable limit ordinals have order type ω_1 when the order they inherit from ω_1 is preserved, so we may write

$$X = \{x_\alpha : \alpha < \omega_1\}$$

When referring to the sequence of ordinals cofinal with points of X, we use the notation $\{x_{\alpha_n} : n \in \omega\}$ for the sequence cofinal with x_α. For $\xi < \omega_1$, we define $X_\xi = \overline{\{x_\alpha : \alpha < \xi\}}^\rho$. Clearly the sets X_ξ satisfy conditions (1)-(4) of Pol's technique, and the topology on X generated by the sets $U \cap X_\xi$, where U is ρ-open and $\xi < \omega_1$, is duly an example of Pol's space.

Our aim now is to show that that sets which are open with respect to the ω_1-type order topology on X are actually open in the Pol space topology. This follows from the way in which the order on X and the topology ρ interact:

Lemma 1.7 *The subspace $\{x_\alpha : \xi < \alpha < \omega_1\}$ is open in (X, ρ) for all $\xi < \omega_1$.*

Proof Let $\alpha > \xi$ in ω_1. Note that this implies that $x_\alpha > x_\xi$ in X.
Since the sequence $\{x_{\alpha_n} : n \in \omega\}$ is cofinal with x_α in ω_1, there exists some $N \in \omega$ such that, if $n \geq N$, then $x_{\alpha_n} > x_\xi$.
Hence, if $\beta \leq \xi$, then $x_{\alpha_N} > x_\xi \geq x_\beta \geq x_{\beta_N}$ and therefore $x_{\alpha_N} \neq x_{\beta_N}$.
But this implies that $x_\beta \notin V_{x_\alpha, N}$ for all $\beta \leq \xi$,whence

$$x_\alpha \in V_{x_\alpha, N} \subseteq \{x_\gamma : \xi < \gamma < \omega_1\}$$

demonstrating that the required set is open.

□

Lemma 1.8 *$X_{\xi+1} = \{x_\alpha : \alpha \leq \xi\}$ for all $\xi < \omega_1$.*

Proof $X_{\xi+1} = \overline{\{x_\alpha : \alpha < \xi + 1\}}^\rho = \overline{\{x_\alpha : \alpha \leq \xi\}}^\rho$.
Since, by Lemma 1.7 ,$\{x_\alpha : \xi < \alpha < \omega_1\}$ is ρ-open, its complement, $\{x_\alpha : \alpha \leq \xi\}$, is ρ-closed, whence

$$X_{\xi+1} = \overline{\{x_\alpha : \alpha \leq \xi\}}^\rho = \{x_\alpha : \alpha \leq \xi\}$$

□

We are now in a position to prove

Theorem 1.9 *The example of Pol's space generated above is precisely the intersection topology with respect to the set X equipped with the ω_1-order topology and the topology on X induced by the metric ρ, which is itself homeomorphic to the subspace of the Baire space, $B(\omega_1)$, of weight ω_1.*

Proof That (X, ρ) is homeomorphic to a subspace of $B(\omega_1)$ was shown in Lemma 1.4. A basic open set in the intersection topology is of the form $\{x_\gamma : \alpha < \gamma \leq \beta\} \cap U$, where $\alpha < \beta < \omega_1$ and U is ρ-open, and

$$\begin{aligned} \{x_\gamma : \alpha < \gamma \leq \beta\} \cap U &= \{x_\gamma : \gamma \leq \beta\} \cap \{x_\gamma : \alpha < \gamma < \omega_1\} \cap U \\ &= X_{\beta+1} \cap [\{x_\gamma : \alpha < \gamma < \omega_1\} \cap U] \qquad \text{(by Lemma 1.8)} \end{aligned}$$

which is open in the Pol space topology (since the term in square brackets is ρ-open by Lemma 1.7).

Conversely, a basic open set in the Pol space topology is of the form $U \cap X_\xi$, where U is ρ-open and $\xi < \omega_1$. Now $X_\xi = \overline{\{x_\alpha : \alpha < \xi\}}^\rho$ is equal to either $\{x_\alpha : \alpha < \xi\}$ or $\{x_\alpha : \alpha \leq \xi\}$, since $\{x_\alpha : \xi < \alpha < \omega_1\}$ is ρ-open by Lemma 1.7. In either case, X_ξ is open in the order topology on X, whence $U \cap X_\xi$ is open in the intersection topology. We conclude that the two topologies are identical.

□

Note that here we have an intersection topology w.r.t. a metrizable space and the countable ordinals which *is* perfectly normal, a property not allowed in the joins of separable metrizable spaces and ω_1, as shown by Kunen (Theorem 2.5). Hence separability of the spaces in [Ku] is essential.

We show now that this example of Pol's space is indeed a perfectly normal, locally metrizable space which fails to be paracompact. *All* examples generated by the technique are perfectly normal and locally metrizable, so all we need to show is the following:

Theorem 1.10 *The example of Pol's space generated by (X, ρ) is not paracompact.*

Proof In [Po], Pol observed that the space X is non-paracompact if (X, ρ) cannot be expressed as the union of countably many locally separable subspaces. We show that (X, ρ) cannot be expressed thus.

Suppose it could. Let $Y \subseteq (X, \rho)$ be a locally separable subspace. Therefore, each $\alpha \in Y$ has a separable open neighbourhood. Since (X, ρ) is metric, this implies that each $\alpha \in Y$ has a separable *basic* open neighbourhood, i.e., for some $m_\alpha \in \omega$, $V_{\alpha,m_\alpha} \cap Y$ is separable. By precisely the same argument as that used in Lemma 1.6, this indicates that $V_{\alpha,m_\alpha} \cap Y$ is countable.

Now, we are assuming that $X = \bigcup_{n\in\omega} Y_n$, where each Y_n is locally separable. X, the set of limit ordinals, is stationary in ω_1, whence the Countable Decomposition Lemma gives us that Y_n is stationary in ω_1 for some $n < \omega$.

Since Y_n is locally separable, we have, for all $\alpha \in Y_n$, $m_\alpha \in \omega$ such that $V_{\alpha,m_\alpha} \cap Y$ is countable. Hence

$$Y_n = \bigcup_{m\in\omega} \{\alpha \in Y_n : m_\alpha = m\}$$

Therefore, for some $M \in \omega$, since Y_n is stationary, $\{\alpha \in Y_n : m_\alpha = M\}$ is stationary in ω_1. Call this set Z.

By the Pressing Down Lemma, there exists a stationary subset $Z_1 \subseteq Z$ and $n_1 \in \omega_1$ such that $\alpha_1 = n_1$ for all $\alpha \in Z_1$. But then there exists a stationary subset $Z_2 \subseteq Z_1$ and $n_2 \in \omega_1$ such that $\alpha_2 = n_2$ for all $\alpha \in Z_2$. By induction, we construct a sequence $Z_M \subseteq Z_{M-1} \subseteq \ldots \subseteq Z_1 \subseteq Z$ and $n_M \in \omega_1$ such that $\alpha_M = n_M$ for all $\alpha \in Z_M$.

But then we have Z_M a stationary subset of $Z = \{\alpha \in Y_n : m_\alpha = M\}$ such that, for all $\alpha \in Z_M$,

$$\alpha_1 = n_1,\ \alpha_2 = n_2, \ldots,\ \alpha_M = n_M$$

Now, let $\beta \in Z_M$. Then $m_\beta = M$, and, for all $\alpha \in Z_M$,

$$\alpha_1 = \beta_1 = n_1,\ \alpha_2 = \beta_2 = n_2, \ldots,\ \alpha_M = \beta_M = n_M$$

whence $\alpha \in V_{\beta,M} = V_{\beta,m_\beta}$ so that $Z_M \subseteq V_{\beta,m_\beta}$ and $Z_M = Z_M \cap Y_n \subseteq V_{\beta,m_\beta} \cap Y_n$, which is a contradiction since Z_M is uncountable (a stationary subset of ω_1) and $V_{\beta,m_\beta} \cap Y_n$ is countable.

□

Evidently only some of the properties of the metric space (X, ρ) are used in the above analysis. Given any metrizable space (X, ρ) of cardinality and weight $\aleph_1$, we can apply Pol's technique to obtain a perfectly normal, locally metrizable space, and as Pol observes, this space will fail to be paracompact if (X, ρ) cannot be expressed as the union of countably many locally separable subspaces. To carry out this construction, we require an ω_1-type order on X, and some condition on this order and its relation to the topology induced by ρ is needed if the Pol space topology and intersection topology (with respect to the order topology induced by this order and the metrizable topology) are to coincide. A suitable condition is to insist that the order on $X = \{x_\alpha : \alpha < \omega_1\}$ is chosen so that

$\{x_\alpha : \alpha \leq \xi\}$ is ρ-closed for all $\xi < \omega_1$. Then Lemmas 1.7 and 1.8 follow easily and the two topologies are indeed identical.

In the next chapter, we review results about the class $\mathcal{C}$ of intersection topologies w.r.t. the real line and the countable ordinals. $\mathcal{C}$ provides simple (though set-theoretic) analogues to Pol's construction. In particular, under **CH**, $\mathcal{C}$ contains a collectionwise normal space which has a G_δ-diagonal and which is not paracompact - it is also the first such example to have cardinality $\aleph_1$ and to be ω_1-compact. Moreover, under **MA** + $\neg$**CH**, *every* member of $\mathcal{C}$ is a perfect T_3 space which fails to be subparacompact.

Chapter 2

Sorgenfrey-ω_1 intersection topologies

THE OBJECT OF EDUCATION IS TO TEACH
US TO LOVE WHAT IS BEAUTIFUL.

(*The Republic*, Plato)

This chapter is an in-depth study of the class of intersection topologies w.r.t. the Sorgenfrey line and the countable ordinals. We shall take subsets of $\mathbb{R}$ of cardinality ω_1, equip them with an ω_1-type order and consider the intersection topology w.r.t. the inherited Sorgenfrey line topology and the order topology resulting from the ω_1-type order. This gives us a class of intersection topologies analogous to Reed's class $\mathcal{C}$ of intersection topologies w.r.t. the real line and the countable ordinals. For much of this chapter, we will be concerned with comparing $\mathcal{C}$ with what we shall call the class $\mathcal{S}$ of Sorgenfrey-ω_1 intersection topologies. It turns out that the two classes share many of the properties which Reed, Kunen, van Douwen and Gruenhage demonstrated were possessed by $\mathcal{C}$.

The chapter is divided into four sections. In the first we review the study of the class $\mathcal{C}$. The second considers Reed's results from [R_1] and demonstrates that, in the main, they carry over to the Sorgenfrey case. Then we consider Kunen's results from [Ku] and demonstrate that they too carry over to $\mathcal{S}$. Only in the last section do we show that the two classes - $\mathcal{C}$ and $\mathcal{S}$ are indeed different. In fact we show that they are completely disjoint - if $C \in \mathcal{C}$ and $S \in \mathcal{S}$, then C and S are *not* homeomorphic.

2.1 On intersection topologies w.r.t. the real line and ω_1

As Reed points out in [R_1], all of the results he proves about intersection topologies w.r.t. the real line and the countable ordinals also hold if the real line is replaced by an *arbitrary* separable metrizable space. In this section, we will only explicitly state results about the former, but it should be borne in mind that this generalisation holds throughout.

Fix a set $X \subseteq \mathbb{R}$ of size $\aleph_1$. Let $\mathcal{T}_{\mathbb{R}}$ be the inherited real-line topology on X and let $\mathfrak{T}$ be the class of ω_1-type order topologies on X. Then, using lattice notation, if $\mathcal{T}_{\omega_1} \in \mathfrak{T}$, $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$ is the intersection topology with respect to $\mathcal{T}_{\mathbb{R}}$ and $\mathcal{T}_{\omega_1}$. Reed's class $\mathcal{C}$ from [R_1] is the class of all such intersection topologies.

Reed established numerous properties shared by all members of $\mathcal{C}$:

Theorem 2.1 (Reed) *If $(X, \mathcal{T}) \in \mathcal{C}$, then $(X, \mathcal{T})$ is a completely regular, submetrizable, collectionwise Hausdorff, pseudonormal, countably metacompact space with a base of countable order and a G_δ-diagonal. However, $(X, \mathcal{T})$ is neither submetacompact, metaLindelöf, cometrizable nor locally compact.*

The remaining results in [R_1], along with those established in [Gr], [Ku] and [vD], are all concerned with normality, ω_1-compactness and 'perfectness' in the class $\mathcal{C}$. It transpires that these properties are closely related and that they are dependent on the model of set theory used.

Results true in **ZFC** include the following:

Theorem 2.2 (Reed) *If $(X, \mathcal{T}) \in \mathcal{C}$, then the following are equivalent:*

1. *X is strongly collectionwise Hausdorff.*
2. *X is countably paracompact.*
3. *X is normal.*
4. *X is collectionwise normal.*

Theorem 2.3 (Reed, Kunen) *If $(X, \mathcal{T}) \in \mathcal{C}$ is ω_1-compact, then $(X, \mathcal{T})$ is normal.*

Theorem 2.4 (Reed) *There exists $(X, \mathcal{T}) \in \mathcal{C}$ which is not normal.*

Theorem 2.5 (Kunen) *$(X, \mathcal{T}) \in \mathcal{C}$ is not perfectly normal.*

Under **CH**, the following results hold:

Theorem 2.6 (Gruenhage) **(CH)** *$(X, \mathcal{T}) \in \mathcal{C}$ is not perfect.*

Theorem 2.7 (Kunen) **(CH)** *For all $(X, \mathcal{T}) \in \mathcal{C}$, there exists a closed unbounded subset D of X such that D is normal.*

But under **MA** + ¬**CH**, the situation with regard to normality and perfectness is virtually reversed:

Theorem 2.8 (Reed) **(MA + ¬CH)** *$(X, \mathcal{T}) \in \mathcal{C}$ is perfect.*

Theorem 2.9 (Gruenhage) **(MA)** *$(X, \mathcal{T}) \in \mathcal{C}$ is not normal.*

Kunen also established that an 'intermediate' model of set theory, with regard to normality and perfectness of these topologies, exists:

Theorem 2.10 (Kunen) *There exists a model of set theory in which some members of $\mathcal{C}$ are perfect and others are normal.*

Van Douwen developed the following useful characterisation of ω_1-compactness in the class $\mathcal{C}$:

Lemma 2.11 (van Douwen) *$(X, \mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}) \in \mathcal{C}$ is ω_1-compact if and only if every uncountable $\mathcal{T}_{\mathbb{R}}$-closed subset of X is $\mathcal{T}_{\omega_1}$-stationary.*

This lemma allows van Douwen to analyse the class $\mathcal{C}$ with regard to ω_1-compactness in a variety of models of set theory. For example, he shows:

- Under **CH**, given *any* subset $X \subseteq \mathbb{R}$ of cardinality ω_1, it is possible to order X in such a way as to identify all of the uncountable $\mathcal{T}_{\mathbb{R}}$-closed sets with stationary sets in ω_1. Hence, there is some $\mathcal{T}_{\omega_1} \in \mathfrak{T}$ such that $(X, \mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1})$ is ω_1-compact.

- Under **MA**+$\neg$**CH**, no $(X, \mathcal{T}) \in \mathcal{C}$ is ω_1-compact. (This also follows from Theorems 2.3 and 2.9.)

- Recall that a *Lusin space*, $(X, \mathcal{T})$, is an uncountable space such that each uncountable subspace of X is dense in some open subspace of $(X, \mathcal{T})$. Van Douwen shows that if $(X, \mathcal{T}_{\mathbb{R}})$ is Lusin, then there exists $\mathcal{T}_{\omega_1} \in \mathfrak{T}$ such that $(X, \mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1})$ is ω_1-compact. Since $\neg$**CH** is also consistent with the existence of Lusin sets, it is consistent with $\neg$**CH** that there is some ω_1-compact $(X, \mathcal{T}) \in \mathcal{C}$.

- If we identify an uncountable $\mathcal{T}_{\mathbb{R}}$-closed subset of X with a nonstationary subset of ω_1, then Lemma 2.11 tells us that $(X, \mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1})$ will not be ω_1-compact, so it is always possible to order X in this way.

In [R_1], Reed constructs an example of a collectionwise normal element of $\mathcal{C}$ under **CH** using a Lusin set. Van Douwen makes it apparent that the link between Lusin sets and normality in $\mathcal{C}$ is crucial:

Theorem 2.12 (van Douwen) *Let $X \subseteq \mathbb{R}$ be such that every nonempty $\mathcal{T}_{\mathbb{R}}$-open subset of X is uncountable. Then, the following are equivalent:*

1. *$(X, \mathcal{T}_{\mathbb{R}})$ is Lusin.*
2. *For all $\mathcal{T}_{\omega_1} \in \mathfrak{T}$, $(X, \mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1})$ is normal.*

This allows him to prove (in a somewhat different way from that used by Reed in Theorem 2.4) that there is always a non-normal element of $\mathcal{C}$ (since there are always sets which fail to be Lusin). Kunen proved a slightly less convenient characterisation of perfectness in the class $\mathcal{C}$:

Lemma 2.13 (Kunen) *If $(X, \mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}) \in \mathcal{C}$, then the following are equivalent:*

1. *$(X, \mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1})$ is perfect.*
2. *Every club set is a G_δ-set w.r.t. $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$.*
3. *Every club set contains a club set which is a G_δ w.r.t. $\mathcal{T}_{\mathbb{R}}$.*

In [vD], van Douwen asks if there is some characterisation, in the spirit of Lemmas 2.11 and 2.13, of when elements of $\mathcal{C}$ will be normal. As yet, no such characterisation has been established.

2.2 Carrying Reed's results over to Sorgenfrey-ω_1 intersection topologies

The *Sorgenfrey line* is the real line $\mathbb{R}$ with the topology generated by the half-open intervals $\{(a, b] : a, b \in \mathbb{R}\}$. Clearly it is finer than $\mathbb{R}$ with the usual topology, and it fails to be metrizable. Proofs in $[R_1]$ tend to rely quite heavily on the existence of a countable base in separable metrizable spaces, and its absence in the Sorgenfrey line forces some substantial changes in the proofs of analogous results. However, the Sorgenfrey line is hereditarily separable, a 'countability' property that will prove crucial, and of course it has other useful properties like first countability and normality. We will eventually show that virtually all of the results which Reed proves about $\mathcal{C}$ also hold in the class $\mathcal{S}$.

Our objective in this section is to see how the results of $[R_1]$ and [Ku] carry over from the class $\mathcal{C}$ to what we shall call the class $\mathcal{S}$, where $(X, \mathcal{T}) \in \mathcal{S}$ if and only if $X = \{x_\alpha : \alpha < \omega_1\} \subseteq \mathbb{R}$ and $\mathcal{T}$ is the intersection topology with respect to the inherited *Sorgenfrey* line topology $\mathcal{T}_{\mathbb{S}}$ on X and the order topology $\mathcal{T}_{\omega_1}$ on X of type ω_1.

We consider five topologies on a set $X \subseteq \mathbb{R}$ of cardinality $\aleph_1$ ordered as $\{x_\alpha : \alpha < \omega_1\}$. These are

- the inherited real-line topology, $\mathcal{T}_{\mathbb{R}}$
- the order topology with respect to the ω_1-type order, $\mathcal{T}_{\omega_1}$
- the inherited Sorgenfrey line topology, $\mathcal{T}_{\mathbb{S}}$
- the intersection topology with respect to $\mathcal{T}_{\mathbb{R}}$ and $\mathcal{T}_{\omega_1}$, which we denote by $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$
- the intersection topology with respect to $\mathcal{T}_{\mathbb{S}}$ and $\mathcal{T}_{\omega_1}$, which we denote by $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$

If $(X, \mathcal{T}) \in \mathcal{S}$, $\mathcal{B}_{\mathbb{R}} = \{B_i : i < \omega\}$ will denote a countable basis for $(X, \mathcal{T}_{\mathbb{R}})$. For each limit ordinal α, $\{\alpha_i\}_{i<\omega}$ will denote a fixed sequence converging to α in ω_1 such that, for all $i < \omega$, $\alpha_i < \alpha_{i+1} < \alpha$. Finally, if $x = x_\alpha \in X$, $\mathcal{G}_{\omega_1}(x)$ will denote a local base at x in

$(X, \mathcal{T}_{\omega_1})$, where $\mathcal{G}_{\omega_1}(x) = \{\{x\}\}$ if α is a nonlimit ordinal and $\mathcal{G}_{\omega_1}(x) = \{G_i(x) : i < \omega\}$, where $G_i(x) = \{x_\beta : \alpha_i < \beta \leq \alpha\}$, if α is a limit ordinal.

Lemma 2.14 *If $(X, \mathcal{T}) \in \mathcal{S}$, then $(X, \mathcal{T})$ is a completely regular, first countable, locally countable, submetrizable (hence with a G_δ-diagonal) space.*

Proof Evidently, if spaces $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are completely regular, then so is their intersection topology. The same holds for first countability. Local countability of $(X, \mathcal{T}_{\omega_1})$ implies local countability of $(X, \mathcal{T})$ (since $\mathcal{T}_{\omega_1} \subseteq \mathcal{T}$) and submetrizability of $(X, \mathcal{T}_{\mathbb{S}})$ (since $\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{T}_{\mathbb{S}}$) implies submetrizability of $(X, \mathcal{T})$ (since $\mathcal{T}_{\mathbb{S}} \subseteq \mathcal{T}$).

□

Theorem 2.15 *Let $X = \{x_\alpha : \alpha < \omega_1\}$ and let $K \subseteq X$. Let $\mathcal{F}$ be the class of first countable, T_3 topologies on X which are finer than the ω_1-type order topology, $\mathcal{T}_{\omega_1}$. If K is a metrizable subspace of $(X, \mathcal{T}_{\omega_1})$ then $K \subseteq U$, where U is open and σ-discrete w.r.t. $\mathcal{T}_{\omega_1}$, and metrizable as a subspace of $(X, \mathcal{T}_F)$ for* all *$F \in \mathcal{F}$.*

Proof Let K be such a subspace. Then by Lemma 3 of [R_1], K is nonstationary. Let C be a club disjoint from K, let $\alpha_0 = \inf C$, and for all $\alpha \in C$, let $\alpha_+ = \inf\{\beta > \alpha : \beta \in C\} \in C$. Then if $U = [0, \alpha_0) \cup \bigcup_{\alpha \geq \alpha_0}(\alpha, \alpha_+)$, we have $K \subseteq U$.
Now let $\mathcal{T}_F \in \mathcal{F}$. Then each (α, α_+) is a countable subspace of $(X, \mathcal{T}_F)$, hence is also first countable and T_3, and therefore metrizable. Each (α, α_+) is also $\mathcal{T}_{\omega_1}$-open, and hence $\mathcal{T}_F$-open.
Therefore U is a disjoint union of $\mathcal{T}_F$-open metrizable subspaces of $(X, \mathcal{T}_F)$, hence is itself a metrizable subspace of $(X, \mathcal{T}_F)$. Moreover, U is $\mathcal{T}_{\omega_1}$-open and nonstationary, whence, again by Lemma 3 of [R_1], is σ-discrete w.r.t. $\mathcal{T}_{\omega_1}$.

□

This theorem gives us a useful characterisation of nonstationary subsets of X in terms of the intersection topology on X:

Corollary 2.16 *If $(X, \mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}) \in \mathcal{S}$ and $K \subseteq X$, then the following are equivalent:*

1. *K is σ-discrete w.r.t. $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$.*

2. *K is σ-discrete w.r.t. $\mathcal{T}_{\omega_1}$.*

3. *K is not stationary.*

4. *K is a metrizable subspace of $(X, \mathcal{T}_{\omega_1})$.*

5. *K is a metrizable subspace of $(X, \mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1})$.*

6. *$K \subseteq U$, where U is an open, σ-discrete, metrizable subspace of $(X, \mathcal{T}_{\omega_1})$, and of $(X, \mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1})$, and of $(X, \mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1})$.*

Proof (1) $\Rightarrow$ (3): Suppose $K = \bigcup_{i<\omega} K_i$, where for each $i < \omega$, K_i is discrete in $(X, \mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1})$ and K is stationary in $(X, \mathcal{T}_{\omega_1})$.

Then there is some $m < \omega$ such that K_m is stationary in $(X, \mathcal{T}_{\omega_1})$.

Since K_m is discrete, we have, for all $x \in K_m$, a $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-open $U(x) = G(x) \cap B(x)$ such that $U(x) \cap K_m = \{x\}$ where $G(x) \in \mathcal{G}_{\omega_1}(x)$ and $x \in B(x) \in \mathcal{T}_{\mathbb{S}}$.

By the Pressing Down Lemma, there exists $\gamma \in \omega_1$ and $L \subseteq K_m$ such that L is stationary in $(X, \mathcal{T}_{\omega_1})$ and, for all $x \in L$, $G(x) = \{x_\beta : \gamma < \beta \leq \alpha_x\}$. Since $(X, \mathcal{T}_{\mathbb{S}})$ is hereditarily separable, L has a countable subset M which is dense in L with respect to $(X, \mathcal{T}_{\mathbb{S}})$.

Choose $z \in L$ such that $\alpha_z > \sup\{\alpha_x : x \in M\}$ (whence $z \notin M$).

We have $U(z) \cap (K_m \setminus \{z\}) = \emptyset$, and therefore, since $L \subseteq K_m$

$$[\{x_\beta : \gamma < \beta \leq \alpha_z\} \cap B(z)] \cap (L \setminus \{z\}) = \emptyset \tag{2.1}$$

Since $B(z)$ is open in $(X, \mathcal{T}_{\mathbb{S}})$, there is some $m \in M \cap B(z)$, but also

$$m \in M \subseteq \{x_\beta : \gamma < \beta \leq \alpha_z\}$$

and since $m \in M$, $z \notin M$, we must have $m \in L \setminus \{z\}$, contradicting equation (2.1).

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): Proved in [vDL].

(2) $\Rightarrow$ (1): $\mathcal{T}_{\omega_1} \subseteq \mathcal{T}$, so that a discrete subspace of $(X, \mathcal{T}_{\omega_1})$ is also a discrete subspace of $(X, \mathcal{T})$.

(4) $\Rightarrow$ (6) follows straightforwardly from the previous theorem because both the $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$ and $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$ intersection topologies are finer than the ordinal topology and are first countable and T_3.

(6) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (1): Suppose Y is a metrizable subspace of $(X, \mathcal{T})$. Then Y is metaLindelöf. Each first countable, locally countable, metaLindelöf space is the disjoint union of a collection of countable clopen subsets. Hence Y is σ-discrete.

□

Lemma 2.17 *If* $(X, \mathcal{T}) \in \mathcal{S}$, *then* $(X, \mathcal{T})$ *is pseudonormal.*

Proof Let H, K be disjoint closed sets in $(X, \mathcal{T}) \in \mathcal{S}$ where K is countable. Clearly $(X, \mathcal{T}) = U \cup V$, where $U \cap V = \emptyset$, U and V are both open, U is countable, and $K \subseteq U$. Since each countable, regular, first countable space is metrizable, K and $U \cap H$ can be separated by open subsets of U. Hence K and H can be separated by open sets in $(X, \mathcal{T})$.

□

Lemma 2.18 *If* $(X, \mathcal{T}) \in \mathcal{S}$, *then* $(X, \mathcal{T})$ *has a base of countable order.*

Proof From Lemma 2.14, $(X, \mathcal{T})$ is locally countable and first countable, hence locally metrizable. Each locally metrizable space has a base of countable order [WW].

□

Recall that ω_1 has virtually no covering properties apart from countable ones. It is countably compact and yet not compact, so it cannot be weakly $\delta\theta$-refinable or even *pure* (in the sense of Arhangelsk'iĭ [Ar]), since both properties are enough to make countably compact spaces compact. Intersection topologies w.r.t. the reals and ω_1 *can* be pure, since any space with a G_δ-diagonal is pure, but Reed shows that no member of $\mathcal{C}$ can be submetacompact or metaLindelöf, both very weak covering conditions. We demonstrate a strengthening of this for $\mathcal{S}$ in the next lemma, and the proof works identically for a member of $\mathcal{C}$. Recall that weak $\delta\theta$-refinability is implied by both submetacompactness and metaLindelöfness.

Lemma 2.19 *If* $(X, \mathcal{T}) \in \mathcal{S}$, *then* $(X, \mathcal{T})$ *is not weakly* $\delta\theta$*-refinable.*

Proof Let $\mathcal{U}$ be the $\mathcal{T}$-open cover $\{\{x_\beta : \beta < \alpha\} : \alpha < \omega_1\}$, and suppose $(X, \mathcal{T})$ weakly $\delta\theta$-refinable, so that there is an open refinement $\mathcal{G} = \bigcup_{n \in \omega} \mathcal{G}_n$ of $\mathcal{U}$ where, for all $x \in X$, there exists $n_x \in \omega$ such that $1 \leq \mathrm{ord}(x, \mathcal{G}_{n_x}) \leq \aleph$.

By the Countable Decomposition Lemma, there exist stationary $S \subseteq X$ and $N \in \omega$ such that $1 \leq \mathrm{ord}(x, \mathcal{G}_N) \leq \aleph$ for all $x \in S$. Note that if $G \in \mathcal{G}$, then, since $\mathcal{G}$ refines $\mathcal{U}$,

$G \subseteq \{x_\beta : \beta < \alpha\}$ for some $\alpha < \omega_1$. But then every element of $\mathcal{G}$ is countable, and so if $\text{ord}(x, \mathcal{G}_N) \leq \aleph$, then $\text{St}(x, \mathcal{G}_N)$ is countable. We show that this leads to a contradiction.

For all $x_\alpha \in S$, there exist $f(\alpha) < \alpha$ and $U_\alpha \in \mathcal{T}_{\mathbb{S}}$ such that $\alpha \in U_\alpha$ and

$$\{x_\beta : f(\alpha) < \beta \leq \alpha\} \cap U_\alpha \subseteq \text{St}(x_\alpha, \mathcal{G}_N)$$

The Pressing Down Lemma then implies that there exist stationary $T \subseteq S$ and $\gamma \in \omega_1$ such that $f(\alpha) = \gamma$ for all $\alpha \in T$. $(X, \mathcal{T}_{\mathbb{S}})$ is hereditarily separable, so let K be a countable dense subset of T and, for all $k \in K$, let $\{B_n(k) : n \in \omega\}$ be a local base at k. Since U_α is open in T, we have, for all $\alpha \in T$, $n_\alpha \in \omega$ and $k_\alpha \in K$ such that $B_{n_\alpha}(k_\alpha) \subseteq U_\alpha$. But then the Countable Decomposition Lemma implies that there exist Z stationary in T, $n \in \omega$ and $k \in K$ such that $n_\alpha = n$ and $k_\alpha = k$ for all $\alpha \in Z$, that is

$$\{x_\beta : \gamma < \beta \leq \alpha\} \cap B_n(k) \subseteq \text{St}(x_\alpha, \mathcal{G}_N)$$

Now $k \in T$, so suppose $k = x_\xi$. If $\alpha \in Z$ such that $\alpha > \xi$, then

$$x_\xi \in \{x_\beta : \gamma < \beta \leq \alpha\} \cap B_n(k) \subseteq \text{St}(x_\alpha, \mathcal{G}_N)$$

and therefore $x_\alpha \in \text{St}(x_\xi, \mathcal{G}_N)$. But then $Z \cap \{x_\beta : \beta > \xi\} \subseteq \text{St}(k, \mathcal{G}_N)$ so that $\text{St}(k, \mathcal{G}_N)$ is uncountable, with $k \in S$, a contradiction.

□

The next lemma is proved in considerably greater generality in Chapter 3 (See Theorem 3.12) but we retain the original proof for the Sorgenfrey case here.

Lemma 2.20 *If $(X, \mathcal{T}) \in \mathcal{S}$, then $(X, \mathcal{T})$ is not locally compact.*

We need the preliminary result:

Lemma 2.21 *If X is an uncountable subset of $\mathbb{R}$, then there exists $Y \subseteq X$ such that each point of Y is an uncountable limit point of Y from both the left and the right in $\mathcal{T}_{\mathbb{R}}$ and $|X \setminus Y| \leq \omega$.*

Proof Let Y be the set of points of X which are uncountable limit points of X from both the left and the right in $\mathcal{T}_{\mathbb{R}}$. Let $x \in X \setminus Y$. Then there exists $y_x \in \mathbb{R}$ such that either $(y_x, x] \cap X$ is countable, or $[x, y_x) \cap X$ is countable. Suppose $X \setminus Y$ is uncountable.

Then, without loss of generality, there is a subset $Z \subseteq X \setminus Y$ such that, for all $x \in Z$, $(y_x, x] \cap X$ is countable. Now $\{(y_x, x] : x \in Z\}$ is a cover of Z, and each $(y_x, x]$ is open in $(X, \mathcal{T}_{\mathbb{S}})$, the Sorgenfrey line, which we know to be hereditarily Lindelöf. Therefore, there exists a countable subcover $\{(y_{x_i}, x_i] : i \in \omega\}$ for some $\{x_i\}_{i\in\omega} \subseteq Z$, whence $Z \subseteq \bigcup_{i\in\omega}(y_{x_i}, x_i] \cap X$, implying that some $(y_{x_i}, x_i] \cap X$ is uncountable, a contradiction. Thus $X \setminus Y$ is countable.

Now if $y \in Y$ and $x < y$ then, since y is an uncountable limit point of X from the left in $\mathcal{T}_{\mathbb{R}}$, $|(x, y] \cap X| \geq \omega_1$, and therefore

$$|(x, y] \cap Y| = |((x, y] \cap X) \setminus (X \setminus Y)| \geq \omega_1$$

since $|X \setminus Y| \leq \omega$.

Thus, y is an uncountable limit point of Y from the left in $\mathcal{T}_{\mathbb{R}}$ and, similarly, from the right.

□

Proof of Lemma 2.20 Let Y be a subset of X with the properties listed in Lemma 2.21, and let K be a countably dense subset of Y with respect to $\mathcal{T}_{\mathbb{S}}$.

Suppose $(X, \mathcal{T})$ is locally compact.

Then, for each $x \in Y$ such that α_x is a limit ordinal in ω_1, there exists $k_1(x)$, $k_2(x) \in K$ and $y_x \in X$ such that $k_1(x) < k_2(x) < x$ in $\mathcal{T}_{\mathbb{S}}$ and:

1. $G(x) = \{z \in X : y_x < z < x\}$ in $\mathcal{T}_{\omega_1}$.
2. $B(x) = (k_1(x), x]$ in $(X, \mathcal{T}_{\mathbb{S}})$.
3. $\overline{U(x)}$ is compact in $(X, \mathcal{T})$, where $U(x) = G(x) \cap B(x)$.
4. $G(x) \cap (k_1(x), k_2(x)) = \emptyset$.

Condition (4) follows because if no such $U(x)$ and $k_2(x)$ existed, then each neighbourhood of x in $(X, \mathcal{T}_{\mathbb{S}})$ would contain a sequence converging to $k_1(x)$ from the right in $\mathcal{T}_{\mathbb{R}}$, and hence this sequence would have a limit point in neither $(X, \mathcal{T}_{\mathbb{S}})$ nor $(X, \mathcal{T})$, contradicting compactness of $\overline{U(x)}$.

Applying the Pressing Down Lemma and Countable Decomposition Lemma, it follows that there exists a stationary set H in ω_1, $y \in X$ and $k_1, k_2 \in K$ such that, for all $x \in H$, $y_x = y$, $k_1(x) = k_1$ and $k_2(x) = k_2$.

Since $k_1 \in Y$, $(k_1, k_2) \cap Y$ is uncountable, hence there exists $x \in H$ such that x is a limit point of $(k_1, k_2) \cap Y$ in ω_1. But then there exists $x' \in Y$ such that

$$x' \in G(x) \cap (k_1(x), k_2(x))$$

which is a contradiction.

□

Lemma 2.22 *If $(X, \mathcal{T}) \in \mathcal{S}$ then $(X, \mathcal{T})$ is collectionwise Hausdorff.*

Proof Let H be a closed discrete subset of $(X, \mathcal{T})$. By Corollary 2.16, there is an open, metrizable subspace of $(X, \mathcal{T})$ containing H. Since each metrizable space is collectionwise Hausdorff, H can be screened in U, and thus in $(X, \mathcal{T})$.

□

Our last lemma, showing that each member of $\mathcal{S}$ is countably metacompact, requires the following result, demonstrating that, given a $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-open set, there is a corresponding $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$-open set, contained in the first, such that the difference between the two is nonstationary. Thus, in some sense, the two intersection topologies are 'close'. Note that this is the first lemma requiring any more than the first countability, regularity, hereditary separability and hereditary 'Lindelöfness' of the Sorgenfrey line.

Theorem 2.23 *Let U be a $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-open set. Then if $\tilde{U} = \mathrm{Int}_{\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}} U$, $U \setminus \tilde{U}$ is nonstationary in ω_1.*

Proof Suppose $U \setminus \tilde{U}$ is stationary in ω_1.
Since U is open, for each $x = x_\alpha \in U \setminus \tilde{U}$, there exist $f(\alpha) < \alpha$ in ω_1 and $y_x < x$ in $\mathbb{R}$ such that

$$x \in (y_x, x] \cap \{x_\theta : f(\alpha) < \theta \leq \alpha\} \subseteq U$$

By the Pressing Down Lemma and countability of $\mathbb{Q}$ there exist $\gamma \in \omega_1$, $r \in \mathbb{Q}$ and Z stationary in ω_1 such that $f(\alpha) = \gamma$ and $r \in (y_x, x]$ for all $x = x_\alpha \in Z$, i.e.

$$x \in (r, x] \cap \{x_\theta : \gamma < \theta \leq \alpha\} \subseteq U \tag{2.2}$$

Now, let $x_\alpha, x_\beta \in Z$ with $x_\alpha < x_\beta$ in $\mathbb{R}$. Since $x_\alpha \in Z$ but $x_\alpha \notin \tilde{U}$, we must have

$$(r, x_\beta) \cap \{x_\theta : \gamma < \theta \leq \alpha\} \not\subseteq U$$

But by (2.2), $(r, x_\alpha] \cap \{x_\theta : \gamma < \theta \leq \alpha\} \subseteq U$ so that

$$(x_\alpha, x_\beta) \cap \{x_\theta : \gamma < \theta \leq \alpha\} \not\subseteq U \tag{2.3}$$

Also $x_\beta \in Z$ so that $(r, x_\beta] \cap \{x_\theta : \gamma < \theta \leq \beta\} \subseteq U$, and since $r < x_\alpha < x_\beta$, we have $(x_\alpha, x_\beta) \subseteq (r, x_\beta]$. Hence

$$(x_\alpha, x_\beta) \cap \{x_\theta : \gamma < \theta \leq \beta\} \subseteq U \tag{2.4}$$

Comparing (2.3) and (2.4) shows that $\alpha > \beta$, i.e. $x_\alpha < x_\beta$ in Z iff $\alpha > \beta$ in ω_1. But then we have an ω_1-type order on an uncountable subset of $\mathbb{R}$ reversing the usual order on $\mathbb{R}$, which is impossible.

□

Lemma 2.24 *If $(X, \mathcal{T}) \in \mathcal{S}$ then $(X, \mathcal{T})$ is countably metacompact.*

Proof Let $\{U_n : n \in \omega\}$ be a cover of X by $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-open sets. For all $n \in \omega$, let $V_n = \mathrm{Int}_{\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}} U_n$, a $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$-open set.

By Theorem 2.23, $U_n \setminus V_n$ is nonstationary for all n. Let $V = \bigcup_{n \in \omega} V_n$. Then $X \setminus V = \bigcup_{n \in \omega}(U_n \setminus V)$. Since $V_n \subseteq V$, $U_n \setminus V \subseteq U_n \setminus V_n$, so that

$$X \setminus V \subseteq \bigcup\nolimits_{n \in \omega}(U_n \setminus V_n)$$

a countable union of nonstationary sets, hence nonstationary.

By Corollary 2.16, $X \setminus V \subseteq W$, open, σ-discrete and metrizable, both as a subspace of $(X, \mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1})$ and of $(X, \mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1})$. Note that $X \setminus W \subseteq \bigcup_{n \in \omega} V_n$.

Now. $(X, \mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1})$ is countably metacompact (by Lemma 6 of [R$_1$]), and $\{V_n : n \in \omega\} \cup \{W\}$ is an open cover of X by $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$-open sets, so has a point-finite open refinement $\mathcal{V}$, covering X, consisting of $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$-, and hence $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-open sets.

Moreover, W is metrizable as a subspace of $(X, \mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1})$, and is therefore countably metacompact w.r.t. $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$. Since $\{U_n : n \in \omega\}$ covers W, it has a point-finite refinement $\mathcal{W}$, covering W, consisting of $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-open sets, which, without loss of generality, are all contained in W (which is open).

If we let $\mathcal{V}' = \{A \in \mathcal{V} : A \cap W \neq \emptyset\}$, then each element of $\mathcal{V}'$ must be contained in some V_n, hence in some U_n, and $\mathcal{V}'$ covers $X \setminus W$. $\mathcal{W}$ covers W, and also refines $\{U_n : n \in \omega\}$. And since $\mathcal{V}$ and $\mathcal{W}$ are both point-finite covers of X, $\mathcal{V}' \cup \mathcal{W}$ is point-finite.

Then $\mathcal{V}' \cup \mathcal{W}$ is the desired point-finite refinement of $\{U_n : n \in \omega\}$.

□

Theorem 2.25 *If $(X, \mathcal{T}) \in \mathcal{S}$, then $(X, \mathcal{T})$ is a completely regular, pseudonormal, collectionwise Hausdorff, countably metacompact, submetrizable space with a base of countable order and a G_δ-diagonal. However, $(X, \mathcal{T})$ is neither weakly $\delta\theta$-refinable nor locally compact.*

This, our analogue of Theorem 2.1, considers all of the properties of Reed's paper apart from cometrizability. The proof of the result about this property in [R_1] relies heavily on the existence of a countable base in $(X, \mathcal{T}_{\mathbb{R}})$, and as yet we have failed to generalise it.

Theorem 2.8 generalises to $\mathcal{S}$.

Theorem 2.26 (**MA+¬CH**) *If $(X, \mathcal{T}) \in \mathcal{S}$, then $(X, \mathcal{T})$ is perfect.*

Proof Under **MA**, each separable metrizable space of cardinality $< \mathfrak{c}$ has the property that each of its subsets is a G_δ-set [Sh, Ta]. Hence it follows that each subset of X is a G_δ-set with respect to $(X, \mathcal{T}_{\mathbb{R}})$ and thus, since $\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{T}_{\mathbb{S}}$, each subset of X is a G_δ-set with respect to $(X, \mathcal{T}_{\mathbb{S}})$. Immediately, this implies that $(X, \mathcal{T})$ is perfect.

□

Two examples

Reed goes on, in [R_1], to consider normality in the class $\mathcal{C}$, and describes the construction of two examples – one not normal, the other collectionwise normal (under **CH**) – to illustrate how normality depends on the particular ω_1-type order chosen on X and the model of set theory used. We demonstrate here how both of these examples can be carried over to the Sorgenfrey case.

Example 2.27 *There exists $(X, \mathcal{T}) \in \mathcal{S}$ such that $(X, \mathcal{T})$ is not normal.*

Proof Most of the proof follows that of Example 1 in [R_1], but some modifications are required towards the end.

Let C be a Cantor set in $\mathbb{R}$, $M \subseteq C$ and $|M| = \aleph_1$. Let $Y \subseteq \mathbb{R} \setminus C$ be such that $|Y| = \aleph_1$ and Y is uncountable in each open interval in $\mathbb{R}$. Define $X = Y \cup M$. Let

$\mathcal{B}$ denote the countable basis $\{(a,b) \cap X : a,b \in \mathbb{Q}\}$ for $(X, \mathcal{T}_{\mathbb{R}})$. For all $x \in X$, let $\{B_i(x) : i < \omega\}$ denote a nonincreasing local base at x in $(X, \mathcal{T}_{\mathbb{S}})$ such that, for all $i < \omega$, $x \in X$, $B_i(x) = (z, x]$, where $z \in \mathbb{Q}$. Enumerate $\mathbb{Q}$ as $\{z_i : i < \omega\}$. Define, for all $x \in M$, $i < \omega$ $S_i(x)$, a countably dense subset of $B_i(x)$ such that:

1. $S_i(x) \subseteq Y$,

2. if $i \neq j$, then $S_i(x) \cap S_j(y) = \emptyset$ for all $x, y \in M$, and

3. if $x \neq y$, then $S_i(x) \cap S_j(y) = \emptyset$ for all $i, j < \omega$.

[To do this, enumerate M as $\{m_\alpha : \alpha < \omega_1\}$. Define $S_i(x_0)$ for all $i < \omega$, then $S_i(x_1)$ for all $i < \omega$, and so on. When constructing $S_i(x_\alpha)$ for $\alpha < \omega_1$, only countably many points have been chosen from $B_i(x_\alpha)$ already, and we know by construction that Y is uncountable in $B_i(x_\alpha)$.] Let $H = M \cup \bigcup\{S_i(x) : i < \omega, x \in M\}$.

The ω_1-type order on X is exactly that in [R_1]. We let L_1 be the set of ordinals in ω_1 which are limits of nonlimit ordinals but not limits of limit ordinals, and let L_2 be the set of ordinals in ω_1 which are limits of L_1 but which are not limits of limits of L_1. L_2 is discrete in ω_1, hence, by (2)$\Rightarrow$(6) of Corollary 2.16, L_2 is contained in an open metrizable subset of ω_1 with the order topology. Hence, for each $\alpha \in L_2$, let $G(\alpha)$ denote a basic open set in ω_1 containing α such that, if $\alpha \neq \beta$, then $G(\alpha) \cap G(\beta) = \emptyset$. Furthermore, for each $\alpha \in L_2$, let $\{\alpha_i : i < \omega\} \subseteq G(\alpha)$ be such that $\{\alpha_i\}_{i<\omega}$ converges to α in ω_1, and, for each $i < \omega$, $\alpha_i \in L_1$. Now, for each $\alpha \in L_2$ and $0 < i < \omega$, let

$$G_i(\alpha) = \{\beta \in G(\alpha) : \alpha_{i-1} < \beta < \alpha_i \text{ in } \omega_1\}$$

and let $G_0(\alpha) = \{\beta \in G(\alpha) : \beta \leq \alpha_0\}$. Our ω_1-type order on X is defined as follows. Identify the points of M and the ordinals of L_2 in a 1-1 fashion, i.e., $M = \{x_\alpha : \alpha \in L_2\}$. Then, for each $x_\alpha \in M$ and $i < \omega$, identify the points of $S_i(x_\alpha)$ and $G_i(\alpha)$ in 1-1 fashion. Finally, identify the points of $X \setminus H$ and $\omega_1 \setminus \bigcup\{G(\alpha) : \alpha \in L_2\}$ in a 1-1 fashion such that, for each $B \in \mathcal{B}$, some subset of B is identified with a stationary set in ω_1. Let $(X, \mathcal{T}_{\omega_1})$ denote X with the topology induced by this ω_1-type ordering.

Consider $(X, \mathcal{T})$, the intersection topology with respect to $(X, \mathcal{T}_{\mathbb{S}})$ and $(X, \mathcal{T}_{\omega_1})$. M and $X \setminus H$ are obviously disjoint, and they are also both closed. ($X \setminus H$ is closed in $(X, \mathcal{T}_{\omega_1})$ because it is identified with $\omega_1 \setminus \bigcup\{G(\alpha) : \alpha \in L_2\}$, which is closed in ω_1. M is closed

in $(X, \mathcal{T}_{\mathbb{R}})$ because any limit points it has in $\mathbb{R}$ are in C, hence *not* in X (which is $Y \cup M$). But any set closed in $(X, \mathcal{T}_{\mathbb{R}})$ is closed in $(X, \mathcal{T}_{\mathbb{S}})$.) We show that they cannot be separated by open sets in $(X, \mathcal{T})$. Suppose there exist open V, W in $(X, \mathcal{T})$ such that $M \subseteq V$, $X \setminus H \subseteq W$ with $V \cap W = \emptyset$. For each $x = x_\alpha \in M$ and $i < \omega$, let $U_i(x) = \bigcup_{j \leq i} \{x_\beta : \beta \in G_j(\alpha)\}$. Since V is open, for all $x \in M$, there is some $i_x < \omega$ such that $x \in U_{i_x}(x) \cap (z_{i_x}, x] \subseteq V$. Hence, for some $i < \omega$, and some $z \in \mathbb{Q}$, $i_x = i$ and $z_{i_x} = z$ for uncountably many $x \in M$. Moreover, there is some $r \in \mathbb{Q}^+$ such that $x \geq z + r$ for uncountably many of these x. Denote by M' the set of all such x, i.e. for all $x \in M'$,

$$i_x = i \quad z_{i_x} = z \quad x \geq z + r$$

By construction, there exists a stationary subset of $X \setminus H$ with respect to $(X, \mathcal{T}_{\omega_1})$ contained in $(z, z + r)$. Thus, there is some $q \in (X \setminus H) \cap (z, z + r)$ such that q is a limit point of M' in $(X, \mathcal{T}_{\omega_1})$ (since M' is uncountable). If $G(q)$ is an open set in $(X, \mathcal{T}_{\omega_1})$ containing q, then $S_i(x) \subseteq G(q)$ for some $x \in M'$, where $S_i(x)$ is dense in $(z, x]$ with respect to $(X, \mathcal{T}_{\mathbb{R}})$. [To see this, observe that $G(q)$ is open in $(X, \mathcal{T}_{\omega_1})$ and contains q, where there exists a sequence in M' converging to q in $G(q)$. Now, $G(q)$ contains a basic open set $\{x_\beta : \alpha < \beta \leq \alpha_q\}$ for some $\alpha \in \omega_1$, so we may choose $x, y \in \{x_\beta : \alpha < \beta \leq \alpha_q\} \cap M'$ such that $x < y$ in ω_1. Now M' is a subset of M, which is identified with L_2, so we have basic open sets in ω_1, $G(\alpha_x)$ and $G(\alpha_y)$ such that $G(\alpha_x) \cap G(\alpha_y) = \emptyset$. Then

$$G(\alpha_x) \subseteq \{\beta \in \omega_1 : \alpha_y < \beta \leq \alpha_q\} \subseteq \{\beta \in \omega_1 : \alpha < \beta \leq \alpha_q\} \subseteq \{\alpha_z : z \in G(q)\}$$

whence $S_i(x) \subseteq G(q)$ (since $S_i(x)$ is identified with $G_i(\alpha_x)$) and therefore $S_i(x)$ is dense in $B_i(x) = (z, x]$ with respect to $(X, \mathcal{T}_{\mathbb{S}})$.]

Now we have $q \in X \setminus H$ is a limit point of V with respect to $(X, \mathcal{T}_{\omega_1})$ (since $M' \subseteq V$). We show now that it is actually a limit point of V with respect to $(X, \mathcal{T})$. Let $G(q) \cap B(q)$ be an open set in $(X, \mathcal{T})$ containing q, where $G(q)$ is open in $(X, \mathcal{T}_{\omega_1})$ and $B(q)$ is open in $(X, \mathcal{T}_{\mathbb{S}})$. $S_i(x) \subseteq G(q)$ for some $x \in M'$, and $S_i(x)$ is dense in $(z, x]$ with respect to $(X, \mathcal{T}_{\mathbb{S}})$. For all $x \in M'$, $x \in U_i(x) \cap (z, x] \subseteq V$. Recall $S_i(x) \subseteq U_i(x)$. Since $S_i(x)$ is dense in $(z, x]$, there exists $y \in S_i(x)$ such that $y \in S_i(x) \cap (z, x] \cap B(q)$ with $y \neq q$. (Remember $q \in (z, z + r) \subseteq (z, x]$, whence $(z, x] \cap B(q)$ is an open interval in $(z, x]$.) So we have

$$y \in S_i(x) \cap (z, x] \subseteq U_i(x) \cap (z, x] \subseteq V$$

and

$$y \in S_i(x) \cap B(q) \subseteq G(q) \cap B(q)$$

i.e. an arbitrary $\mathcal{T}$-open neighbourhood of q contains a point of V, implying $q \in X \setminus H$ is a limit point of V with respect to $(X, \mathcal{T})$, so that M and $X \setminus H$ cannot be separated by open sets in $(X, \mathcal{T})$.

□

Example 2.28 (**CH**) *There exists* $(X, \mathcal{T}) \in \mathcal{S}$ *which is collectionwise normal, but not perfect.*

Proof Under **CH**, there exists a Lusin set $X \subseteq \mathbb{R}$ [L, Ma]. A *Lusin set* is an uncountable subset X of $\mathbb{R}$ such that each uncountable subset of X is dense in some interval. Reed points out in [R_1] that, in fact, for each uncountable subset Y of X, there exists an open interval in $\mathbb{R}$ such that Y is uncountable in each subinterval.

Let $\mathcal{B}$ denote a countable basis for $(X, \mathcal{T}_{\mathbb{R}})$, and let $(X, \mathcal{T}_{\omega_1})$ be such that X is stationary in each element of $\mathcal{B}$ with respect to $(X, \mathcal{T}_{\omega_1})$. We show that $(X, \mathcal{T})$ is ω_1-compact. Suppose not, and let K be an uncountable, closed, discrete subset of $(X, \mathcal{T})$. Then there exists $B_K \in \mathcal{B}$ such that K is uncountable in each element of $\mathcal{B}$ contained in B_K. However, there exists $Y \subseteq B_K$ such that Y is stationary with respect to $(X, \mathcal{T}_{\omega_1})$ and $Y \cap K = \emptyset$.

Since K is closed in $(X, \mathcal{T}_{\mathbb{S}})$, we have, for each $x \in Y$, $G(x)$ open in $(X, \mathcal{T}_{\omega_1})$ and $y_x < x$ (in $\mathbb{R}$) with $y_x \in \mathbb{Q}$ such that

$$x \in G(x) \cap (y_x, x] \quad (y_x, x] \subseteq B_K \quad (G(x) \cap (y_x, x]) \cap K = \emptyset$$

Thus, for some $y \in \mathbb{Q}$, $y_x = y$ for all x in a stationary subset $Y' \subseteq Y$ of $(X, \mathcal{T}_{\omega_1})$ (by the Countable Decomposition Lemma). Again by the Countable Decomposition Lemma, there exists $z < y$ such that $x > z$ for all x in a stationary subset $Y'' \subseteq Y'$ of $(X, \mathcal{T}_{\omega_1})$. Now there is some $B \in \mathcal{B}$ such that $B \subseteq (y, z) \subseteq B_K$, whence K is uncountable in B. Therefore, there exists $q \in Y''$ such that q is a limit point of $B \cap K$ with respect to $(X, \mathcal{T}_{\omega_1})$, whence $G(q) \cap (B \cap K) \neq \emptyset$. *But* $q \in Y''$, and therefore $(G(q) \cap (y, q]) \cap K = \emptyset$ and $B \subseteq (y, z) \subseteq (y, q]$, which is a contradiction.

We now apply Theorem 2.31, which tells us that in the class $\mathcal{S}$, ω_1-compactness implies normality. Therefore $(X, \mathcal{T})$ is normal. Now, suppose we have a discrete collection of

subsets of X. By picking one point from each subset, we obtain a closed discrete set. Since X is ω_1-compact, this closed discrete set must be countable, and thus any discrete collection of subsets of X must be countable. But in a normal space, any countable, discrete collection of closed sets can be separated by disjoint open sets, and therefore X is collectionwise normal.

$(X, \mathcal{T})$ is not perfect, however, since in a perfect space each discrete set is the union of countably many closed, discrete subsets. However, there is an uncountable subset Y of X such that Y is identified with the nonlimit ordinals in ω_1, and hence Y is discrete in $(X, \mathcal{T})$. If Y were the union of countably many closed, discrete subsets, then Y would be countable since $(X, \mathcal{T})$ is ω_1-compact.

□

Note that this last example cannot possess a stronger normality property such as monotone normality monotonically normal spaces with a G_δ-diagonal are paracompact [Pa], and no member of $\mathcal{S}$ can be paracompact.

Here we have considered virtually all of the properties in Reed's original paper and shown that they carry over to the class $\mathcal{S}$. There are a couple of omissions. Reed manages to show that no member of $\mathcal{C}$ was cometrizable, and as yet we have failed to see how to adapt the proof to the Sorgenfrey case it seems to rely heavily on the existence of a countable base for $\mathcal{T}_{\mathbb{R}}$. More serious, perhaps, is the absence of any analogue of Theorem 2.2, showing that strong collectionwise Hausdorffness, countable paracompactness, normality and collectionwise normality all coincide in $\mathcal{C}$. Again, the existence of a countable base is used throughout the proof of this theorem in [R_1].

2.3 Carrying Kunen's results over to Sorgenfrey-ω_1 intersection topologies

Theorem 2.23 indicates that the $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$ and $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$ topologies are 'close'. Kunen [Ku] has a corresponding theorem saying that the $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$ and ordinary real-line topologies are 'close'. It is with some intuitive satisfaction, then, that we see the following result holds, the analogue of Lemma 1 of [Ku], saying that the $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$ and real-line topologies are 'close':

Lemma 2.29 *If U is $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-open, then there is an $\mathcal{T}_{\mathbb{R}}$-open V such that $U \setminus V$ is nonstationary and $V \setminus U$ is countable.*
If H is $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-closed, then there is an $\mathcal{T}_{\mathbb{R}}$-closed K such that $K \setminus H$ is nonstationary and $H \setminus K$ is countable.

Proof Obviously the second statement follows from the first.
Let U be $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-open. By Theorem 2.23, there exists $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$-open $\tilde{U}$ such that $\tilde{U} \subseteq U$ and $U \setminus \tilde{U}$ is nonstationary in ω_1. By Lemma 1 of [Ku], there exists $\mathcal{T}_{\mathbb{R}}$-open V such that $\tilde{U} \setminus V$ is nonstationary and $V \setminus \tilde{U}$ is countable. Now $U \setminus V \subseteq (\tilde{U} \setminus V) \cup (U \setminus \tilde{U})$, a union of two nonstationary sets, and hence nonstationary. Moreover, $V \setminus U \subseteq V \setminus \tilde{U}$ and is therefore countable.

□

With this result we have been able to carry over most of the results in [Ku] from the $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$ intersection topology to the $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$ intersection topology, using virtually identical proofs to those in [Ku]. Particularly useful is the following lemma:

Lemma 2.30 *The following are equivalent:*

1. *$(X, \mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1})$ is ω_1-compact.*
2. *$(X, \mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1})$ is ω_1-compact.*
3. *All nonstationary $\mathcal{T}_{\mathbb{R}}$-closed sets are countable.*
4. *All nonstationary $\mathcal{T}_{\mathbb{S}}$-closed sets are countable.*
5. *All nonstationary $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$-closed sets are countable.*
6. *All nonstationary $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-closed sets are countable.*

Proof (1)⇔(3)⇔(5) was proved in [Ku].
(2)⇒(6): Suppose there exists an uncountable nonstationary closed set H. Then there exists a club set C disjoint from H. But then we can let K be an uncountable subset of H such that any two distinct elements of K have an element of C between them. But then K is closed discrete, contradicting (2).
(6)⇒(2): Let $C \subseteq X$ closed discrete in $(X, \mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1})$. By Corollary 2.16, C is not

stationary in ω_1, whence by (6), C is countable.

(4)⇒(3) is trivial. Any $\mathcal{T}_{\mathbb{R}}$-closed set is $\mathcal{T}_{\mathbb{S}}$-closed.

(6)⇒(4) is trivial.

(3)⇒(6): Suppose (3) holds. Let H be a nonstationary $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-closed set. Then by the second part of Lemma 2.29, there exists $\mathcal{T}_{\mathbb{R}}$-closed K such that $K \setminus H$ is nonstationary and $H \setminus K$ is countable. Now $K \subseteq H \cup (K \setminus H)$, so that K must be nonstationary, hence countable by (3). But $H \subseteq K \cup (H \setminus K)$, so that H must be countable too.

□

We now demonstrate that Kunen's results about the class $\mathcal{C}$ also hold true for $\mathcal{S}$. The proofs which follow are usually modifications of the proofs in [Ku].

Theorem 2.31 *If $(X, \mathcal{T}) \in \mathcal{S}$ is ω_1-compact, then $(X, \mathcal{T})$ is normal.*

Proof Let H_1 and H_2 be $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-closed and disjoint in X. By Lemma 2.29, let K_1 and K_2 be $\mathcal{T}_{\mathbb{R}}$-closed with $K_1 \setminus H_1$ and $K_2 \setminus H_2$ nonstationary, and $H_1 \setminus K_1$ and $H_2 \setminus K_2$ countable. Since H_1 and H_2 are disjoint, $K_1 \cap K_2 \subseteq (K_1 \setminus H_1) \cup (K_2 \setminus H_2)$ and hence is nonstationary. Therefore, by Lemma 2.30, $K_1 \cap K_2$ is countable.

Now let $\alpha < \omega_1$ be such that all elements of $H_1 \setminus K_1$, $H_2 \setminus K_2$ and $K_1 \cap K_2$ are below α in ω_1. Then X is partitioned into clopen sets $[0, \alpha]$ and $[\alpha + 1, \omega_1)$. On $[0, \alpha]$, H_1 and H_2 can be separated because countable regular spaces are normal. On $[\alpha + 1, \omega_1)$, H_1 and H_2 are contained in disjoint $\mathcal{T}_{\mathbb{R}}$-closed sets $K_1 \cap [\alpha + 1, \omega_1)$ and $K_2 \cap [\alpha + 1, \omega_1)$, which can be separated because the Sorgenfrey line is hereditarily normal.

□

The converse to this theorem is not true, but what can be said is

Theorem 2.32 *If $(X, \mathcal{T}) \in \mathcal{S}$ is normal, then there is a club C which is ω_1-compact in its relative topology.*

Proof $\mathcal{T}_{\mathbb{R}}$ is second countable, so there is a club F which misses all nonstationary basic open sets, and thus *all* nonstationary $\mathcal{T}_{\mathbb{R}}$-open sets. Since F, when equipped with the subspace topology, is also a Sorgenfrey-ω_1 intersection topology, we may assume without loss of generality that $F = X$, i.e. all nonempty $\mathcal{T}_{\mathbb{R}}$-open sets are stationary, so that every club is $\mathcal{T}_{\mathbb{R}}$-dense in X.

Let P be the set of limit ordinals β such that, for some $\alpha < \beta$, $(\alpha, \beta]$ is not $\mathcal{T}_{\mathbb{R}}$-dense in X. Suppose (for a contradiction) that P is stationary. Then, for all $\beta \in P$, there exists $f(\beta) < \beta$ such that $\overline{(f(\beta), \beta]}^{\mathbb{R}} \neq X$, so there exists $n_\beta \in \omega$ such that $(f(\beta), \beta] \cap B_{n_\beta} = \emptyset$. Using the Pressing Down Lemma and the Countable Decomposition Lemma, we find that there exist stationary Q, $n \in \omega$ and $\gamma \in \omega_1$ such that $f(\beta) = \gamma$ and $n_\beta = n$ for all $\beta \in Q$. But then $(\gamma, \omega_1) \cap B_n = \emptyset$ so that $\overline{(\gamma, \omega_1)}^{\mathbb{R}} \neq X$, contradicting our assumption that all clubs are $\mathcal{T}_{\mathbb{R}}$-dense. So P is non-stationary.

Hence, there is a club C of limit ordinals disjoint from P. We demonstrate that C is ω_1-compact. By Lemma 2.30, it is enough to show that if $H \subset C$ is closed non-stationary, then H is countable. Fix some club $D \subseteq C$ such that $D \cap H = \emptyset$, and, using normality, let U be open such that $D \subseteq U$ and $\overline{U} \cap H = \emptyset$. By Lemma 2.29, let V be $\mathcal{T}_{\mathbb{R}}$-open such that $U \setminus V$ is nonstationary and $V \setminus U$ is countable. We now show that H can contain no elements above $\sup(V \setminus U)$. Suppose γ were such an element. Since $\gamma \in H$ and $H \cap \overline{U} = \emptyset$, there exists $B \in \mathcal{T}_{\mathbb{S}}$ with $(\alpha, \gamma) \cap B$ disjoint from U. We may assume $\alpha > \sup(V \setminus U)$, so that $(\alpha, \gamma] \cap B$ is also disjoint from V. Since V contains a club, $\overline{V}^{\mathbb{R}} = X$. V is also $\mathcal{T}_{\mathbb{R}}$-open, so since $\gamma \in C$ (and thus not in P), $(\alpha, \gamma] \cap V$ is $\mathcal{T}_{\mathbb{R}}$-dense, contradicting $(\alpha, \gamma] \cap V \cap B = \emptyset$.

□

Theorem 2.33 **(CH)** *For all $(X, \mathcal{T}) \in \mathcal{S}$, there exists a closed unbounded subset D of X such that D is normal.*

Proof By **CH**, let $\{C_\alpha : \alpha < \omega_1\}$ be the nonstationary $\mathcal{T}_{\mathbb{R}}$-closed subsets of X. For all $\alpha < \omega_1$, there is a club D_α such that $C_\alpha \cap D_\alpha = \emptyset$. Let

$$D = \{\gamma : \forall \beta < \gamma (\gamma \in D_\beta)\}$$

D is the diagonal intersection of club sets and is therefore itself club. Moreover, if $\gamma \in D \cap C_\alpha$, then for all $\beta < \gamma$, $\gamma \notin C_\beta$. But $\gamma \in C_\alpha$, whence $\alpha \geq \gamma$, i.e. $D \cap C_\alpha \subseteq \alpha + 1$, and is therefore countable.

So D has countable intersection with each nonstationary $\mathcal{T}_{\mathbb{R}}$-closed subset of X. Therefore, any nonstationary $\mathcal{T}_{\mathbb{R}}$-closed subset of D is countable, hence ω_1-compact by Lemma 2.30 (since the subspace topology on D, a club, as induced by X is itself a Sorgenfrey-ω_1 intersection topology), hence normal by Theorem 2.31.

□

Theorem 2.34 *No $(X, \mathcal{T}) \in \mathcal{S}$ can be both perfect and normal.*

Proof Let $(X, \mathcal{T}) \in \mathcal{S}$ be normal.

By Theorem 2.32, there is some club $C \subseteq X$ which is ω_1-compact in its relative topology. Let D be a club in C. Suppose D is a G_δ. Then

$$X \setminus D = \bigcup_{n \in \omega} Y_n$$

where each Y_n is closed. Since D is club, $X \setminus D$ is nonstationary, and therefore each Y_n must be nonstationary. But, by Lemma 2.30, each Y_n is countable, and hence $X \setminus D$ must be too. Therefore D is co-countable. *But* there exist clubs which fail to be co-countable, and C cannot, therefore be perfect, whence X cannot be perfect.

□

Lemma 2.35 *If G is a G_δ-set w.r.t. $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$, then there is a G_δ-set w.r.t. $\mathcal{T}_{\mathbb{R}}$, H, such that $G \setminus H$ is nonstationary and $H \subseteq G$.*

Proof $G = \bigcap_{n \in \omega} U_n$ where each U_n is $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$-open. By Lemma 2.29, there is a $\mathcal{T}_{\mathbb{R}}$-open V_n such that $U_n \setminus V_n$ is nonstationary and $V_n \setminus U_n$ is countable. Let $H' = \bigcap_{n \in \omega} V_n$, where H' is a G_δ w.r.t. $\mathcal{T}_{\mathbb{R}}$.

$$G \setminus H' = \bigcup_{m \in \omega} (\bigcap_{n \in \omega} U_n \setminus V_m) \subseteq \bigcup_{m \in \omega} (U_m \setminus V_m)$$

which is nonstationary.

Let $H = H' \cap G$. Then $G \setminus H = G \setminus H'$ is nonstationary, and certainly $H \subseteq G$. Moreover

$$H = H' \cap G = H' \cap (X \setminus (H' \setminus G))$$

Now $H' \setminus G \subseteq \bigcup_{m \in \omega} (V_m \setminus U_m)$ which is countable, and therefore $X \setminus (H' \setminus G))$ is co-countable, hence a G_δ, and thus H is the intersection of two G_δs w.r.t. $\mathcal{T}_{\mathbb{R}}$, and is therefore itself a G_δ w.r.t. $\mathcal{T}_{\mathbb{R}}$.

□

Theorem 2.36 *If $(X, \mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}) \in \mathcal{S}$, then the following are equivalent:*

1. *$(X, \mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1})$ is perfect.*
2. *Every club set is a G_δ-set w.r.t. $\mathcal{T}_{\mathbb{S}} \vee \mathcal{T}_{\omega_1}$.*

3. *Every club set contains a club set which is a G_δ-set w.r.t. $\mathcal{T}_{\mathbb{R}}$.*

Proof (1)⇒(2) is trivial.

(2)⇒(3): Suppose all clubs are G_δ-sets. Let G be a club, whence by the previous lemma, there is a G_δ w.r.t. $\mathcal{T}_{\mathbb{R}}$, H_1, such that $G \setminus H_1$ is nonstationary and $H_1 \subseteq G$. G is club, and $G \setminus H_1$ is nonstationary, so there exists club $G_1 \subseteq H_1$.

Inductively construct a sequence $\{G_n\}$ of clubs and a sequence $\{H_n\}$ of G_δs w.r.t. $\mathcal{T}_{\mathbb{R}}$ such that $H_{n+1} \subseteq G_n \subseteq H_n$ for all n.

Let $D = \bigcap_{n\in\omega} G_n$, which is club. Since $H_{n+1} \subseteq G_n \subseteq H_n$ for all n, $\bigcap_{n\in\omega} G_n = \bigcap_{n\in\omega} H_n$, whence D is also a G_δ w.r.t. $\mathcal{T}_{\mathbb{R}}$.

(3)⇒(1): Assume (3) and let U be open. We show it's F_σ.

Let V be as in Lemma 2.29. V is an F_σ, so we are finished if we can show that $U \setminus V$ is an F_σ and $V \setminus U$ is a G_δ. $V \setminus U$ is a G_δ (using the ordinal topology).

Let $Y = U \setminus V$. Since Y is nonstationary, (3) implies that there exists a club D with D a G_δ and $D \cap Y = \emptyset$. Say $D = \bigcap_{m\in\omega} W_m$, where each W_m is open. Set $Y = \bigcup_{n\in\omega} Y_n$, where each Y_n is separated by the points of D, so that in the ordinal topology it can only have points of D as limit points. Then Y is the union of sets of the form $Y_n \setminus W_m$, each of which is closed and discrete in the intersection topology. Hence Y is an F_σ.

□

Theorem 2.37 (**CH**) $(X, \mathcal{T}) \in \mathcal{S}$ *is not perfect.*

Proof Suppose $(X, \mathcal{T})$ perfect. Then, by Theorem 2.33, there is a club $D \subseteq X$ such that D is normal. But D is then perfectly normal and, since it is a club, also a Sorgenfrey-ω_1 intersection topology. But this contradicts Theorem 2.34.

□

Theorem 2.38 (**MA** + ¬**CH**) $(X, \mathcal{T}) \in \mathcal{S}$ *is not normal.*

Proof This follows directly from Theorems 2.26 and 2.34.

□

It is not without some concern that one reaches this stage of the chapter! As yet, there has been *no* result to differentiate $\mathcal{C}$ and $\mathcal{S}$. Could it be that given $C \in \mathcal{C}$, there exists $S \in \mathcal{S}$ such that C and S are homeomorphic? And vice versa? Fortunately, this is not the case, as we prove in the next section.

2.4 A difference between Sorgenfrey-ω_1 and real-ω_1 intersection topologies

Theorem 2.39 *Let X be an intersection topology w.r.t. $\mathcal{T}_{\mathbb{R}}$ and $\mathcal{T}_{\omega_1}$, and let Y be the Sorgenfrey line. Then $X \times Y$ is collectionwise Hausdorff.*

Proof The idea of the proof is as follows. We take an arbitrary closed discrete subset, K, of $X \times Y$ and show that its projection onto X must be nonstationary. A theorem of Reed's tells us that we can find an open metrizable subset, U, of X containing this projection. We separate the points of K in $U \times Y$, and because $U \times Y$ is open in $X \times Y$, we have separated the points of K in $X \times Y$ as well.

So, let K be a closed discrete subset of $X \times Y$.
First we show that $\pi_1(K)$ (i.e. the projection of K onto X) is nonstationary. Suppose $\pi_1(K)$ were stationary.
For all $\mathbf{x} = (x, y) \in K$, there exists $\mathcal{T}_{\mathbb{R}} \vee \mathcal{T}_{\omega_1}$-open $U_{\mathbf{x}}$ and $r_{\mathbf{x}} \in \mathbb{Q}$ with $r_{\mathbf{x}} < y$ such that

$$\{U_{\mathbf{x}} \times (r_{\mathbf{x}}, y]\} \cap K = \{\mathbf{x}\}$$

where $U_{\mathbf{x}} = B_{n_{\mathbf{x}}} \cap \{x_\theta : f(\alpha) < \theta \leq \alpha\}$ with $B_{n_{\mathbf{x}}} \in \mathcal{B}_{\mathbb{R}}$ and $x = x_\alpha$.
Since $\pi_1(K)$ is stationary, we have, by the Pressing Down Lemma, stationary $L \subseteq K$ and $\gamma \in \omega_1$ such that $f(\alpha) = \gamma$ for all $\alpha \in L$. Now, L is uncountable, so for some $r \in \mathbb{Q}$, $n \in \omega$, $L_{r,n} = \{\mathbf{x} \in L : r_{\mathbf{x}} = r, n_{\mathbf{x}} = n\}$ is uncountable. For such r, n, let $M = L_{r,n}$. Then for all $\mathbf{x} \in M$,

$$\{U_{\mathbf{x}} \times (r, y]\} \cap K = \{\mathbf{x}\}$$

where $U_{\mathbf{x}} = B_n \cap \{x_\theta : \gamma < \theta \leq \alpha\}$ (and $x = x_\alpha$).

Claim: $\pi_2(M)$ is uncountable.

Proof: Suppose not. Then, since M is uncountable, there exists $z \in Y$ such that $y = z$ for uncountably many $\mathbf{x} = (x, y) \in M$. Let $M' = \{\mathbf{x} \in M : y = z\}$.

For all such $\mathbf{x}$,

$$\{U_{\mathbf{x}} \times (r, z]\} \cap K = \{\mathbf{x}\}$$

(with $U_{\mathbf{x}}$ as above).

Let $\mathbf{x}_1, \mathbf{x}_2 \in M'$ where $\mathbf{x}_i = (x_{\alpha_i}, z)$ and, without loss of generality, $x_{\alpha_1} > x_{\alpha_2}$ in

$\mathbb{R}$. Then $\mathbf{x}_2 \notin U_{\mathbf{x}_1} \times (r, z]$. Therefore, $x_{\alpha_2} \notin U_{\mathbf{x}_1} = B_n \cap \{x_\theta : \gamma < \theta \leq \alpha_1\}$.

But $x_2 \in U_{\mathbf{x}_2} = B_n \cap \{x_\theta : \gamma < \theta \leq \alpha_2\}$ and therefore $\alpha_2 > \alpha_1$.

Hence we have an ω_1-type order on an uncountable subset, $\pi_1(M')$, of $\mathbb{R}$, reversing the usual order on $\mathbb{R}$, a contradiction. Hence $\pi_2(M)$ is uncountable, completing the proof of the claim.

Now, suppose $y_1 < y_2$ in $\pi_2(M)$.

Then, there exist $\mathbf{x}_1, \mathbf{x}_2 \in M$ such that $\mathbf{x}_i = (x_i, y_i)$ for $i = 1, 2$. We have

$$\{U_{\mathbf{x}_2} \times (r, y_2]\} \cap K = \{\mathbf{x}_2\}$$

Now $r < y_1 < y_2$, so $y_1 \in (r, y_2]$ and $\mathbf{x}_1 \in K$ but $\mathbf{x}_1 \neq \mathbf{x}_2$. Therefore

$$x_1 \notin U_{\mathbf{x}_2} = B_n \cap \{x_\theta : \gamma < \theta \leq \alpha_2\}$$

where $x_i = x_{\alpha_i}$.

Now,

$$x_1 \in U_{\mathbf{x}_1} = B_n \cap \{x_\theta : \gamma < \theta \leq \alpha_1\}$$

so $x_1 \in B_n$. Hence we must have

$$x_1 \notin \{x_\theta : \gamma < \theta \leq \alpha_2\}$$

But $\alpha_1 > \gamma$ and so $\alpha_1 > \alpha_2$.

Hence, in $\pi_2(M)$, $y_1 < y_2$ iff $\alpha_1 > \alpha_2$. But $\pi_2(M)$ is uncountable, and we therefore have an ω_1-type order on an uncountable subset of $\mathbb{R}$, reversing the usual order, a contradiction.

Therefore $\pi_1(K)$ is nonstationary.

By Lemma 3 of [R_1], $\pi_1(K) \subseteq U$, an open, metrizable subset of X.

Consider $U \times Y$. We have K a closed discrete subset of $U \times Y$. U is metrizable and Y is the Sorgenfrey line, hence paracompact and perfectly normal. But a theorem of Michael's [Mi] says that the product of a paracompact, perfectly normal space and a metrizable space is paracompact, i.e. $U \times Y$ is paracompact. But every paracompact space is collectionwise Hausdorff, and the set K can therefore be screened by open sets in $U \times Y$ and thus, since $U \times Y$ is open in $X \times Y$, by open sets in $X \times Y$.

Therefore, $X \times Y$ is collectionwise Hausdorff.

□

Theorem 2.40 *Let X be an intersection topology w.r.t. $\mathcal{T}_{\mathbb{S}}$ and $\mathcal{T}_{\omega_1}$ and let Y be the Sorgenfrey line. Then $X \times Y$ is not collectionwise Hausdorff.*

Proof Let $\nabla = \{(x, -x) : x \in X\}$, the backwards diagonal in $X \times Y$ (where we consider $X, Y \subseteq \mathbb{R}$ here). ∇ is evidently closed discrete in $X \times Y$ because basic open neighbourhoods of points on ∇ are 'below' ∇. Suppose the points of ∇ can be separated by disjoint open sets in $X \times Y$.

Let $x = x_\alpha \in X$. Let the open set round $(x, -x)$ be U_x, where

$$U_x = ((y_x, x] \cap \{x_\theta : f(\alpha) < \theta \leq \alpha\}) \times (z_x, -x]$$

and $y_x < x$, $z_x < -x$, $f(\alpha) < \alpha$. By the Pressing Down Lemma and Countable Decomposition Lemma, there exists Z stationary in X such that, for some $r, r' \in \mathbb{Q}$, $\gamma \in \omega_1$, we have $r \in (y_x, x]$, $r' \in (z_x, -x]$ and $f(\alpha) = \gamma$ for all $x = x_\alpha \in Z$. For all such x, let

$$V_x = ((r, x] \cap \{x_\theta : \gamma < \theta \leq \alpha\}) \times (r', -x]$$

Since all U_x's are disjoint, we also have $V_{x_\alpha} \cap V_{x_\beta} = \emptyset$ for all $x_\alpha, x_\beta \in Z$ with $\alpha \neq \beta$. *But*, if $x_\alpha > x_\beta$ in $\mathbb{R}$, where $x_\alpha, x_\beta \in Z$, then

$$\begin{aligned} V_{x_\alpha} \cap V_{x_\beta} &= ((r, x_\alpha] \cap (r, x_\beta] \cap \{x_\theta : \gamma < \theta \leq \alpha\} \cap \{x_\theta : \gamma < \theta \leq \beta\}) \times \\ &\qquad ((r', -x_\alpha] \cap (r', -x_\beta]) \\ &= ((r, x_\beta] \cap \{x_\theta : \gamma < \theta \leq \min\{\alpha, \beta\}\}) \times (r', -x_\alpha] \end{aligned}$$

But $(r', -x_\alpha] \neq \emptyset$, and therefore, since $V_{x_\alpha} \cap V_{x_\beta} = \emptyset$, we have

$$(r, x_\beta] \cap \{x_\theta : \gamma < \theta \leq \min\{\alpha, \beta\}\} = \emptyset$$

Thus, $x_\beta \notin \{x_\theta : \gamma < \theta \leq \min\{\alpha, \beta\}\}$, whence $\alpha < \beta$. But then we have an ω_1-type order on an uncountable subset, Z, of $\mathbb{R}$, reversing the usual real-line order, which is a contradiction.

□

This last pair of theorems gives us the immediate result:

Corollary 2.41 *If $C \in \mathcal{C}$ and $S \in \mathcal{S}$, then C and S are not homeomorphic.*

Chapter 3

The intersection topology as the diagonal of the product

MATHEMATICS, RIGHTLY VIEWED, POSSESSES NOT ONLY TRUTH, BUT ALSO SUPREME BEAUTY - A BEAUTY COLD AND AUSTERE, LIKE THAT OF SCULPTURE.

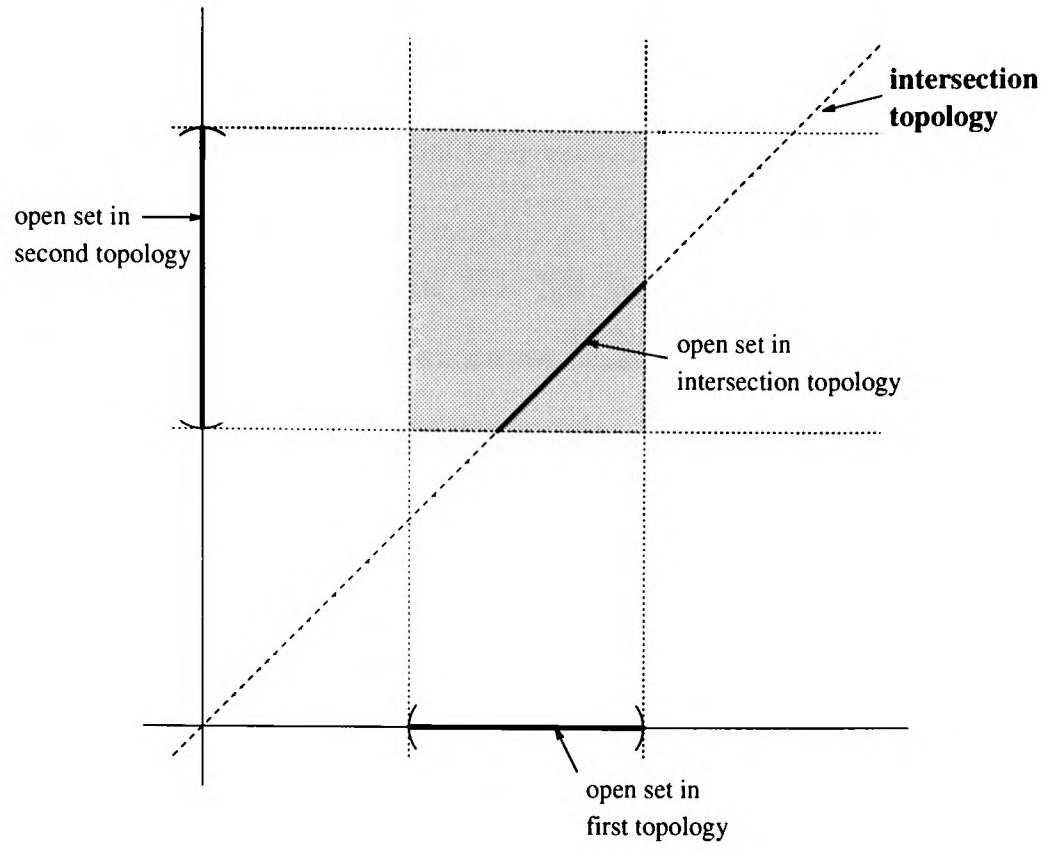
(*Mysticism and Logic*, Bertrand Russell)

In this chapter, we analyse one of the most useful characterisations of the intersection topology, namely as the diagonal of the product of the two topological spaces under consideration. This gives us some new machinery to deal with the intersection topology, and we give several examples of results that follow using this technique. From the point of view of this thesis, the most useful result is that the intersection topology w.r.t. a first countable, hereditarily separable, Hausdorff space space and ω_1 is *never* locally compact, generalising Reed's result about the intersection of ω_1 with a separable metric space.

3.1 A new characterisation

Formally, we have

Theorem 3.1 *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on some set X, then the intersection topology $\mathcal{T}_1 \vee \mathcal{T}_2$ on X is homeomorphic to the diagonal of the product $(X, \mathcal{T}_1) \times (X, \mathcal{T}_2)$, equipped with the subspace topology.*



Proof This is fairly straightforward, as indicated by the diagram above. Let Δ be the diagonal of $X \times X$. Define $f : X \to \Delta \subseteq X \times X$ by $f(x) = (x, x)$. Evidently, this map is a bijection from the space X to the diagonal of the product. Now, a basic open set in $(X, \mathcal{T}_1 \vee \mathcal{T}_2)$ is of the form $U_1 \cap U_2$ where $U_1 \in \mathcal{T}_1$, $U_2 \in \mathcal{T}_2$. Under f, this is mapped to $\{(x, x) : x \in U_1 \cap U_2\} = \{(x, y) : x \in U_1,\ y \in U_2,\ x = y\} = (U_1 \times U_2) \cap \Delta$, which is open in the subspace topology on Δ. And conversely, a basic open set in the topology on Δ is of the form $(U_1 \times U_2) \cap \Delta$, and the inverse image of this set under f is $U_1 \cap U_2$, which is open in $\mathcal{T}_1 \vee \mathcal{T}_2$. Hence f is a homeomorphism.

□

Now suppose we have a property P, shared by $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ which is preserved by finite products and is also hereditary. Then, $(X, \mathcal{T}_1) \times (X, \mathcal{T}_2)$ will have property P, and thus so will Δ (equipped with the subspace topology). But we have just shown that Δ is homeomorphic to $(X, \mathcal{T}_1 \vee \mathcal{T}_2)$, and hence the intersection topology has property P as well.

Corollary 3.2 *Suppose $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ share any of the properties P listed below. Then the intersection topology $(X, \mathcal{T}_1 \vee \mathcal{T}_2)$ also has property P.*
P is any one of: T_0, T_1, T_2, T_3, $T_{3\frac{1}{2}}$, metrizability, character $< \kappa$, weight $< \kappa$, zero-dimensionality, semi-stratifiability, stratifiability, hereditary disconnectedness, BCO, W_δ-diagonal, $\aleph$-space, $\aleph_0$-space.

Proof All of the properties P listed are finitely productive and hereditary, so that by the above observations the results follow immediately.

□

The separation property missing from this list is, of course, normality, which fails to be preserved by taking joins. As an example, take the set $[0,1] \times [0,1]$ and let $\mathcal{T}_{\mathbb{R}}$ and $\mathcal{T}_{\mathbb{S}}$ be the real-line and Sorgenfrey line topologies on $[0,1]$ respectively. If we then put $\mathcal{T}_1 = \mathcal{T}_{\mathbb{R}} \times \mathcal{T}_{\mathbb{S}}$ and $\mathcal{T}_2 = \mathcal{T}_{\mathbb{S}} \times \mathcal{T}_{\mathbb{R}}$, then both $\mathcal{T}_1$ and $\mathcal{T}_2$ are normal, yet it is easy to see that $\mathcal{T}_1 \vee \mathcal{T}_2 = \mathcal{T}_{\mathbb{S}} \times \mathcal{T}_{\mathbb{S}}$, and, as is well-known, the square of the Sorgenfrey line is not normal.

The range of topological properties preserved by these two operations taking finite products and subspaces is relatively small. If we insist that the property, P, we consider is finitely productive but only *closed* hereditary, then the intersection topology will have property P provided we add an extra condition to the relationship between $\mathcal{T}_1$ and $\mathcal{T}_2$:

Definition 3.3 *If $\mathcal{T}$ and $\mathcal{U}$ are topologies on a set X, then they are said to be* pairwise Hausdorff *if, for any $x \neq y$ in X, there exist $\mathcal{T}$-open U and $\mathcal{U}$-open V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

It appears that this property was first considered by Weston [We] in 1957, where he defines what it means for two topologies on the same set to be *consistent* his definition coincides precisely with pairwise Hausdorffness. The terminology we use is taken from the theory of bitopological spaces Kelly [Ke] defined pairwise Hausdorffness in bitopological spaces in 1963, along with notions of higher separation axioms in this class of spaces.

Lemma 3.4 *The diagonal, Δ, is closed in $(X, \mathcal{T}_1) \times (X, \mathcal{T}_2)$ if and only if $\mathcal{T}_1$ and $\mathcal{T}_2$ are pairwise Hausdorff.*

Proof is trivial. □

Note that if the lattice *meet* of the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ is Hausdorff, then the diagonal is automatically closed and the topologies are pairwise Hausdorff.

Similar reasoning to that used above then gives us

Corollary 3.5 *Suppose $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ share any of the properties P listed below, and that $\mathcal{T}_1$ and $\mathcal{T}_2$ are pairwise Hausdorff. Then the intersection topology $(X, \mathcal{T}_1 \vee \mathcal{T}_2)$*

also has property P.

P is any one of: compactness, local compactness, sequential compactness, Čech-completeness, realcompactness.

Proof All of the properties P listed are finitely productive and closed hereditary, so the results follow immediately.

□

Unfortunately, this result is less useful to us than we may have hoped. As the following proposition indicates, only a limited class of topologies are pairwise Hausdorff w.r.t. an ω_1-type order topology.

Proposition 3.6 *Suppose $|X| = \omega_1$ and that $\mathcal{T}_{\omega_1}$ is an ω_1-type order topology on X. If $\mathcal{T}$ is a first countable topology on X, and $\mathcal{T}$ and $\mathcal{T}_{\omega_1}$ are pairwise Hausdorff, then $\mathcal{T}$ must be locally countable.*

Proof Order X as $\{x_\alpha : \alpha < \omega_1\}$, and let $\alpha \neq \beta$ in ω_1. Then there exist $\mathcal{T}$-open U and $\mathcal{T}_{\omega_1}$-open V such that $x_\alpha \in U$, $x_\beta \in V$ and U and V are disjoint.
Fix α and let $B_1 \supseteq B_2 \supseteq \ldots$ be a local base at x_α w.r.t. $\mathcal{T}$. For all $\beta > \alpha$, there exist $f(\beta) < \beta$ such that $x_\beta \in \{x_\theta : f(\beta) < \theta \leq \beta\} \subseteq V$ and $n_\beta \in \omega$ such that $B_{n_\beta} \subseteq U$. Then

$$B_{n_\beta} \cap \{x_\theta : f(\beta) < \theta \leq \beta\} = \emptyset$$

Let $n \in \omega$ such that $n_\beta = n$ for all β in some stationary set. Then $B_n \cap \{x_\theta : f(\beta) < \theta \leq \beta\} = \emptyset$ for all such β. Then by the Pressing Down Lemma, there is a stationary set, Z in ω_1 and some $\gamma \in \omega_1$ such that $f(\beta) = \gamma$ for all $\beta \in Z$. But then we must have $B_n \cap \{x_\theta : \theta > \gamma\} = \emptyset$, whence $B_n \subseteq \{x_\theta : \theta \leq \gamma\}$, i.e. B_n is countable.
Therefore we have that an arbitrary point of X has a countable $\mathcal{T}$-neighbourhood, and hence $(X, \mathcal{T})$ is locally countable.

□

Corollary 3.7 *If $(X, \mathcal{T})$ is first countable and Lindelöf with $|X| = \omega_1$, then $\mathcal{T}$ and $\mathcal{T}_{\omega_1}$ cannot be pairwise Hausdorff.*

Proof is trivial, as an uncountable Lindelöf space cannot be locally countable. □

For completeness, we include some other examples of this characterisation of the intersection topology.

Proposition 3.8 *If $(X, \mathcal{T}_1)$ is perfectly normal with any of the properties P listed below, and $(X, \mathcal{T}_2)$ is a σ-compact metric space, then the intersection topology $\mathcal{T}_1 \vee \mathcal{T}_2$ also has property P.*
P is any one of: Lindelöf, paracompact, screenable, metacompact, subparacompact, metaLindelöf, submetacompact, submetaLindelöf, weakly θ-refinable, weakly $\delta\theta$-refinable.

Proof Let P be any of the properties listed in the statement of the proposition. We combine the following theorems with the above characterisation of the intersection topology:

- [Mo] If X is perfectly normal, and Y is metrizable, then $X \times Y$ is perfectly normal.
- [Bu, Thm. 6.1] If X is regular with property P and Y is regular and σ-compact, then $X \times Y$ has property P.
- [Bu, Thm. 7.1] If X is a perfect space with property P, then P is hereditary in X.

□

Proposition 3.9 *Suppose $(X, \mathcal{T})$ is normal, countably paracompact, ω_1-compact and countably tight. Then if $\mathcal{T}$ and $\mathcal{T}_{\omega_1}$ are pairwise Hausdorff, their intersection topology is normal.*

Proof follows from the result of [GNP] that if $(X, \mathcal{T})$ has the properties as listed, then $X \times \omega_1$ is normal. □

Proposition 3.10 *Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are GO-space topologies on X with G_δ-diagonals. Suppose further that $(X, \mathcal{T}_1)$ is perfect. Then $(X, \mathcal{T}_1 \vee \mathcal{T}_2)$ is subparacompact.*

Proof This follows directly from the result of [Al] that if X and Y are GO-spaces with G_δ-diagonals and X is perfect, then $X \times Y$ is hereditarily subparacompact.
□

Proposition 3.11 **(CH)** *Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are Lindelöf GO-space topologies on X, both with G_δ-diagonals. Then $(X, \mathcal{T}_1 \vee \mathcal{T}_2)$ is subparacompact.*

Proof Again, this follows directly from the result of [Al] that, under **CH**, if X and Y are Lindelöf GO-spaces with G_δ-diagonals then $X \times Y$ is hereditarily subparacompact. □

Our final main result in this section is the most important in terms of its relationship to the rest of the thesis. It generalises the result in [R_1] that the intersection topology w.r.t. a separable metric space and the countable ordinals is never locally compact. It transpires that the only properties of the separable metric space actually necessary here are first countability, Hausdorffness and hereditary separability, and the proof of the result uses the characterisation of the intersection topology as the diagonal of the product. Recall that in Corollaries 3.2 and 3.5 we considered topological properties which are productive and hereditary, either with respect to all subspaces or with respect to all closed subspaces. Local compactness is closed hereditary, but it is also hereditary w.r.t. *open* subspaces, and, indeed any subspace of a locally compact space which is the intersection of an open set and a closed set will be locally compact. Moreover, a converse holds here a subspace of a Hausdorff space is locally compact only if it is the intersection of a closed set and an open set. It is this result which allows us to show that in the following theorem, the diagonal cannot be locally compact.

Theorem 3.12 *If $(X, \mathcal{T})$ is a first countable, Hausdorff, hereditarily separable space and $\mathcal{T}_{\omega_1}$ is an ω_1-type order topology on X, then $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is* not *locally compact.*

Lemma 3.13 [E, Thm. 3.3.9] *If X is Hausdorff and $Y \subseteq X$ is locally compact, then Y can be expressed in the form $F \cap V$, where F is closed in X and V is open in X.*

Proof of theorem For convenience, by X we shall mean the topological space $(X, \mathcal{T})$, and by ω_1 the topological space $(X, \mathcal{T}_{\omega_1})$.
Suppose $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is locally compact. Then by Lemma 3.13, $\triangle = F \cap V$, where F is closed in $X \times \omega_1$ and V is open in $X \times \omega_1$. Evidently, without loss of generality, we may assume that $F = \overline{\triangle}$.

The diagonal, $\triangle \subseteq V$, which is open, and therefore, for all $\alpha < \omega_1$, $(x_\alpha, x_\alpha) \in V$. Hence, there exist a $\mathcal{T}$-open U_α containing x_α and $f(\alpha) < \alpha$ such that

$$(x_\alpha, x_\alpha) \in U_\alpha \times (f(\alpha), \alpha] \subseteq V$$

By the Pressing Down Lemma, there exists Z stationary in ω_1 and $\gamma < \omega_1$ such that $f(\alpha) = \gamma$ for all $\alpha \in Z$, i.e.

$$U_\alpha \times (\gamma, \alpha] \subseteq V$$

for all $\alpha \in Z$, so that $(x_\alpha, x_\beta) \in V$ for all $\gamma < \beta \leq \alpha$ such that $\alpha \in Z$.

$(X, \mathcal{T})$ is hereditarily separable, so let Y be a countable dense subset of (γ, ω_1) with respect to $\mathcal{T}$. Let $\alpha > \sup Y$ such that $\alpha \in Z$.
Let $\{B_n : n \in \omega\}$ be a decreasing local base at x_α (with respect to $\mathcal{T}$) and for all $n \in \omega$, let

$$y_n \in (B_n \cap (\gamma, \omega_1)) \cap Y$$

The set $\{y_n : n \in \omega\}$ must be infinite (since $(X, \mathcal{T})$ is Hausdorff). It therefore has an accumulation point with respect to ω_1, say β. Then, if $\xi < \beta$, we have that $(\xi, \beta]$ contains infinitely many of the y_n's. Therefore

$$B_n \cap (\xi, \beta] \neq \emptyset \quad \forall n \in \omega, \forall \xi < \beta$$

so that

$$(B_n \times (\xi, \beta]) \cap \triangle \neq \emptyset$$

and hence $(x_\alpha, x_\beta) \in \overline{\triangle}$.
Note that, since β is an accumulation point of Y, $\gamma < \inf Y \leq \beta \leq \sup Y < \alpha$. Then

$$(x_\alpha, x_\beta) \in \{x_\alpha\} \times (\gamma, \alpha] \subseteq V$$

and $(x_\alpha, x_\beta) \in V \cap \overline{\triangle} = \triangle$. But this is a contradiction, because $\beta < \alpha$.

□

Corollary 3.14 *If $(X, \mathcal{T})$ is first countable, Hausdorff, locally compact and hereditarily separable with $|X| = \omega_1$, then $\mathcal{T}$ and $\mathcal{T}_{\omega_1}$ cannot be pairwise Hausdorff.*

Proof If they were, Corollary 3.5 would imply that $\mathcal{T} \vee \mathcal{T}_{\omega_1}$ was locally compact, since both $\mathcal{T}$ and $\mathcal{T}_{\omega_1}$ are.

□

So having already ruled out the possibility of pairwise Hausdorffness of a first countable, Lindelöf topology and ω_1 in Corollary 3.7, there is also no way that a first countable,

locally compact S-space (such as the Kunen line or the Ostaszewski line) and ω_1 can be pairwise Hausdorff either.

Glancing at the proof of Theorem 3.12, it is apparent that in the only place in which hereditary separability is used, it is actually sufficient to assume only that all *co-countable* subspaces of $(X, \mathcal{T})$ are separable. This means that, for example, the following holds:

Corollary 3.15 *The intersection topology w.r.t. the tangent disc space and the countable ordinals is never locally compact.*

Proof Although the tangent disc space is not hereditarily separable, all co-countable subsets of it are.

□

3.2 Bešlagić's Dowker product constructions

A *Dowker space* is a normal space whose Cartesian product with the unit interval $[0, 1]$ is not normal. In 1951, Dowker [D] showed that $X \times [0, 1]$ is normal if and only if X is countably paracompact, so a Dowker space is a normal space which fails to be countably paracompact. In 1955, Rudin [Ru_1] showed that the existence of a Suslin line implied the existence of a Dowker space, so when the existence of a Suslin line was shown to be independent of **ZFC** (by Solovay and Tennenbaum [ST] in 1971), the existence of a Dowker space was shown to at least be consistent. In 1971, Rudin [Ru_2] constructed a Dowker space without any additional set-theoretic assumptions, and it remains the only known 'real' example. However, all of its cardinal functions are large, so the search for a 'nicer' Dowker space one which is first countable, separable or with cardinality at most $\mathfrak{c}$ continues.

In his thesis [Sta], Starbird asks if a product $X \times Y$ can be Dowker without X or Y being Dowker. Wage announced in [Wg] in 1978 that **CH** implies that the answer to Starbird's question is yes, but he never published a proof. Since then, Bešlagić has produced three examples of so-called *Dowker products*. From our point of view, they are interesting because in each case, two topologies, $\mathcal{T}_0$ and $\mathcal{T}_1$ are constructed on a set X and the resulting topological spaces $X_i = (X, \mathcal{T}_i)$ are considered. X_0 and X_1 are shown to be countably paracompact and normal, but $X_0 \times X_1$ is shown to be Dowker by demonstrating

that $X_0 \times X_1$ is normal, yet the diagonal of $X_0 \times X_1$ is a closed subspace of $X_0 \times X_1$ and *not* countably paracompact. (In each case, the diagonal is shown to be closed in $X_0 \times X_1$ by demonstrating that there exists a Hausdorff topology $\mathcal{T}_2 \subseteq \mathcal{T}_0 \cap \mathcal{T}_1$.) Then $X_0 \times X_1$ cannot be countably paracompact either. Because the diagonal is homeomorphic to the join of X_0 and X_1, in each case we have corresponding results about Dowker intersection topologies. We review Bešlagić's results here and consider their intersection topology analogues.

- [B_1] ($\diamondsuit$) There exist topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ on $\omega_1 \times \omega$ such that $X_0 = (\omega_1 \times \omega, \mathcal{T}_0)$ and $X_1 = (\omega_1 \times \omega, \mathcal{T}_1)$ are normal, first countable, countably paracompact, locally compact, shrinking P-spaces, $X_0 \times X_1$ is collectionwise normal and countably paracompact, and yet the diagonal of $X_0 \times X_1$ is closed and Dowker.

- [B_2] (**CH**) Let L be a Lusin subset of $\mathbb{R}$. There exist topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ on L such that $X_0 = (L, \mathcal{T}_0)$ and $X_1 = (L, \mathcal{T}_1)$ are normal and countably paracompact, $X_0 \times X_1$ is normal and countably paracompact, and yet the diagonal of $X_0 \times X_1$ is closed and Dowker.

- [B_3] ($\diamondsuit$) There exist topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ on ω_1 such that $X_0 = (\omega_1, \mathcal{T}_0)$ and $X_1 = (\omega_1, \mathcal{T}_1)$ are perfectly normal (and thus countably paracompact), $X_0 \times X_1$ is normal and countably paracompact, and yet the diagonal of $X_0 \times X_1$ is closed and Dowker.

This gives us the following examples of Dowker intersection topologies:

Example 3.16 *There exist topologies $\mathcal{T}_0$ and $\mathcal{T}_1$*

1. *on $\omega_1 \times \omega$ (under $\diamondsuit$) which are normal, first countable, countably paracompact, locally compact, shrinking P-spaces, or*

2. *on a Lusin subset of $\mathbb{R}$ (under* **CH***) which are normal and countably paracompact, or*

3. *on ω_1 (under $\diamondsuit$) which are perfectly normal,*

such that in all cases the intersection topology $\mathcal{T}_0 \vee \mathcal{T}_1$ is Dowker.

Chapter 4

Hereditarily separable-ω_1 and hereditarily Lindelöf-ω_1 intersection topologies

HUMAN SPEECH IS LIKE A CRACKED KETTLE ON WHICH WE TAP CRUDE RHYTHMS FOR BEARS TO DANCE TO, WHILE WE LONG TO MAKE MUSIC THAT WILL MELT THE STARS.

(*Madame Bovary*, Gustave Flaubert)

In this chapter, we consider how the results of Chapter 2 can be generalised to larger classes of spaces the class of hereditarily separable spaces and the class of hereditarily Lindelöf spaces. These classes are very closely related, but it transpires that, although the majority of the results about Sorgenfrey-ω_1 intersection topologies carry over to the hereditarily separable case, the hereditarily Lindelöf case is more subtle, depending on the model of set theory used. We also demonstrate how further generalisations, to the class of separable spaces, are very limited, and illustrate this with two examples. Finally, we look at the bow-tie space, an easily comprehensible example of a hereditarily separable, hereditarily Lindelöf space, and explain how the results proved in Section 2.3 about Sorgenfrey-ω_1 intersection topologies can be carried over to bow-tie space intersections with ω_1. Firstly, though, we make some preliminary, more general comments about intersection topologies with the countable ordinals.

4.1 Some generalities

There are some properties, possessed by ω_1, which *any* intersection topology with a regular, first countable space automatically inherits:

Theorem 4.1 *Let $(X, \mathcal{T})$ be T_3 and first countable. Then $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ will be completely regular, first countable, zero-dimensional, locally countable, pseudonormal and scattered with a base of countable order. Moreover, it cannot be separable or Lindelöf, or have the countable chain condition.*

Proof First countability follows from Corollary 3.2. Pseudonormality, local countability and a base of countable order follow from proofs identical to those of Lemmas 2.14, 2.17, and 2.18. Any refinement of ω_1 is scattered and evidently fails to be Lindelöf or separable, or have the countable chain condition. $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is regular by Corollary 3.2 again, and the equivalence of regularity, complete regularity and zero-dimensionality follows in the class of all bijective preimages of ω_1 by [Go, Prop. 2.2.6].

□

Note that Section 4.4 contains an example of an intersection topology w.r.t. a first countable T_3 space and ω_1 which is discrete and metrizable, so 'non-Lindelöf' is the best we can do here. And, of course, Theorem 2.9 demonstrates that, consistently at least, such intersection topologies need not be normal. So complete regularity and pseudonormality are also the best we can do.

An easy result in the general case is

Proposition 4.2 *If $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is countably compact, then so is $(X, \mathcal{T})$.*

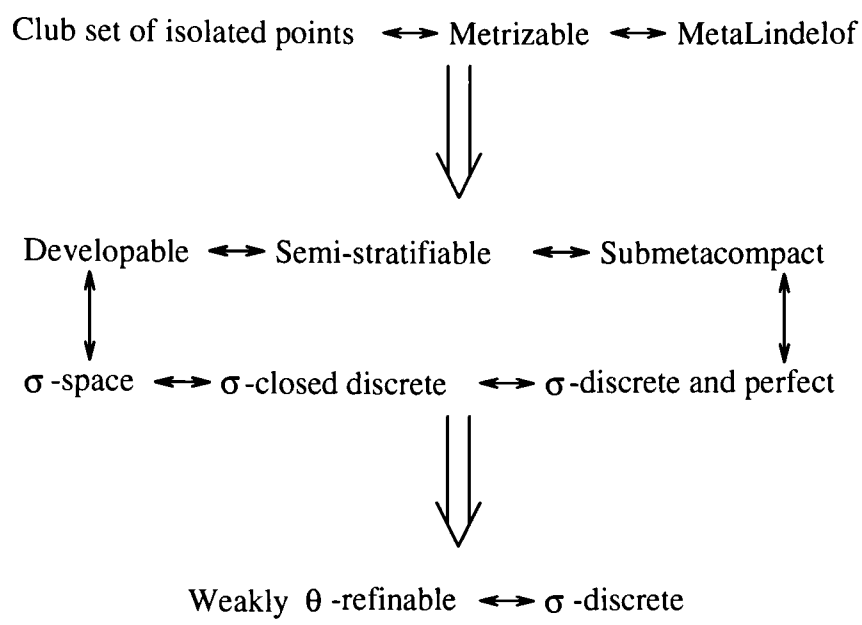
Proof Any countable $\mathcal{T}$-open cover of X is a $\mathcal{T} \vee \mathcal{T}_{\omega_1}$-open cover, and therefore has a finite subcover.

□

Of course, the converse does not hold. Any metrizable space has a G_δ-diagonal. So if $(X, \mathcal{T})$ is countably compact, separable and metrizable, then $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ has a G_δ-diagonal. But a countably compact space with a G_δ-diagonal is compact [C], and therefore $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ cannot be countably compact.

The last proposition in this section deals with covering properties in this class of intersection topologies.

Proposition 4.3 *In the class of intersection topologies w.r.t. a first countable, T_3 space and the countable ordinals, we have the following implications:*



Proof All of the implications follow trivially from results about σ-scattered spaces in [N], and results about bijective preimages of ω_1, as analysed in [Go, Props. 2.3.2, 2.3.3]. □

4.2 Hereditarily separable-ω_1 intersection topologies

Hereditary separability turns out to be *the* crucial property of the Sorgenfrey line in determining the properties of Sorgenfrey-ω_1 intersection topologies. Observe that the proof of Corollary 2.16 requires only hereditary separability (and first countability and regularity) of $(X, \mathcal{T}_{\mathbb{S}})$, and thus we have

Proposition 4.4 *If $(X, \mathcal{T})$ is a first countable, T_3, hereditarily separable space and $K \subseteq X$, then the following are equivalent:*

1. *K is σ-discrete w.r.t. $\mathcal{T} \vee \mathcal{T}_{\omega_1}$.*
2. *K is σ-discrete w.r.t. $\mathcal{T}_{\omega_1}$.*
3. *K is not stationary.*
4. *K is a metrizable subspace of $(X, \mathcal{T}_{\omega_1})$.*

5. *K is a metrizable subspace of $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$.*

6. *$K \subseteq U$, where U is an open, σ-discrete, metrizable subspace of $(X, \mathcal{T}_{\omega_1})$, and of $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$.*

This was the vital result required to carry over Reed's results to the class $\mathcal{S}$, and we can use it here too to show that, in addition to the properties described in Theorem 4.1, we also have,

Theorem 4.5 *Let $(X, \mathcal{T})$ be first countable, T_3 and hereditarily separable. Then $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is collectionwise Hausdorff and fails to be weakly $\delta\theta$-refinable or locally compact.*

Proof Collectionwise Hausdorffness and the failure of weakly $\delta\theta$-refinability follow from proofs identical to those of Lemmas 2.19 and 2.22. $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is not locally compact by Theorem 3.12.

□

One property missing here is the property of having a G_δ-diagonal. Real-ω_1 and Sorgenfrey-ω_1 intersection topologies have G_δ-diagonals trivially, as both the real line and the Sorgenfrey line have G_δ-diagonals, and any refinement of a space with a G_δ-diagonal has one too. In general, though, a first countable, T_3, hereditarily separable space need not have a G_δ-diagonal, and in this case, the intersection topology need not have one either, as the following example indicates.

Example 4.6 *There is a subspace of the double arrow space which, when intersected with ω_1, fails to have a G_δ-diagonal.*

Recall that the double arrow space is the set $[0,1] \times \{0,1\}$ equipped with the lexicographic order topology, a hereditarily separable, hereditarily Lindelöf LOTS without a G_δ-diagonal (though it only 'just' fails to, as we show in Lemma 5.1).

Proof Let A be a subspace of $[0,1]$ of cardinality $\aleph_1$, dense in $[0,1]$ and let $X = A \times \{0,1\}$. Put $X_0 = A \times \{0\}$ and $X_1 = A \times \{1\}$. Order X in order type ω_1 as $\{\mathbf{x}_\alpha : \alpha < \omega_1\}$ such that every Euclidean-open subset of X is stationary in ω_1. Let $\mathcal{T}$ and $\mathcal{T}_{\omega_1}$ be the inherited double arrow space topology and ω_1-type order topology respectively.

A basis at (x,n) in the topology, $\mathcal{T}$ is of the form $\{B_k(x,n) : k \in \omega\}$ where

$$B_k(x,n) = \begin{cases} ((x-1/k, x] \times \{0\}) \cup ((x-1/k, x) \times \{1\}) & \text{if } n = 0 \\ ((x, x+1/k) \times \{0\}) \cup ([x, x+1/k) \times \{1\}) & \text{if } n = 1 \end{cases}$$

Suppose $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ has a G_δ-diagonal sequence, $\{\mathcal{U}_n : n \in \omega\}$. We may assume that each $\mathcal{U}_n$ consists of basic open sets, so that, without loss of generality, $\mathcal{U}_n = \{U_n^{\mathbf{x}} : \mathbf{x} \in X\}$, where

$$U_n^{\mathbf{x}_\alpha} = (f_n(\alpha), \alpha] \cap B_{k(\mathbf{x}_\alpha, n)}(\mathbf{x})$$

(and $(\xi, \eta]$ is shorthand for $\{\mathbf{x}_\alpha : \xi < \alpha \leq \eta\}$ as usual).

Since $\{\mathcal{U}_n : n \in \omega\}$ is a G_δ-diagonal sequence, $(x,1) \notin \bigcap_{n\in\omega} \mathrm{St}((x,0), \mathcal{U}_n)$ for all $x \in A$. Then, for all $x \in A$, there exists n_x such that $(x,1) \notin \mathrm{St}((x,0), \mathcal{U}_{n_x})$. By the Countable Decomposition Lemma, there exists stationary $Z' \subseteq A$ and $N \in \omega$ such that $(x,1) \notin \mathrm{St}((x,0), \mathcal{U}_N)$ for all $x \in Z'$. Apply Lemma 2.21 to find $Z \subseteq Z'$ such that $|Z' \setminus Z| \leq \omega$ and each point of Z is an uncountable limit point of Z, w.r.t. $\mathcal{T}_{\mathbb{R}}$ from both the left and the right. Then Z too is stationary.

Let Q be a countable $\mathcal{T}$-dense subset of Z. Let $x \in Z$. Since x is an uncountable limit point of Z from the left, there exists $q_x \in Q \cap (B_{k((x,0),N)}(x,0) \cap Z)$. Then since q_x is an uncountable limit point of Z from the right, there exists $r_x \in Q \cap ((q_x, x) \cap Z)$. By the Countable Decomposition Lemma and the Pressing Down Lemma, there exists stationary $W \subseteq Z$ such that $q_x = q$, $r_x = r$ and $f_N(\alpha) = \gamma$ for all $x \in W$, where $(x,0) = \mathbf{x}_\alpha$, $q, r \in Q$ and $\gamma \in \omega_1$, so that

$$((q,r) \times \{0,1\}) \cap (\gamma, \alpha] \subseteq U_N^{\mathbf{x}_\alpha}$$

for all $\mathbf{x}_\alpha = (x,0)$ such that $x \in W$.

Now, since Z is uncountable, there exists $a \in Z \cap (q,r)$ such that if $(a,0) = \mathbf{x}_\xi$ and $(a,1) = \mathbf{x}_\eta$ then $\xi, \eta > \gamma$. W is also uncountable so there is some $x \in W$ such that if $(x,0) = \mathbf{x}_\alpha$ then $\alpha > \max\{\xi, \eta\}$. (So we have $\gamma < \xi, \eta < \alpha$.) Then

$$(a,0) \in ((q,r) \times \{0,1\}) \cap (\gamma, \alpha] \subseteq U_N^{\mathbf{x}_\alpha}$$

so that $U_N^{\mathbf{x}_\alpha} \subseteq \mathrm{St}((a,0), \mathcal{U}_N)$. But we also have $(a,1) \in U_N^{\mathbf{x}_\alpha}$ so that $(a,1) \in \mathrm{St}((a,0), \mathcal{U}_N)$, which is a contradiction.

□

4.3 Hereditarily Lindelöf-ω_1 intersection topologies

Here we consider an arbitrary first countable, regular, hereditarily Lindelöf space and determine the properties of intersection topologies with the countable ordinals. Again, we are generalising results from the Sorgenfrey line case, as, of course, the Sorgenfrey line is hereditarily Lindelöf.

The distinction between hereditary separability and hereditary 'Lindelöfness' in regular spaces is a very subtle one. An *S-space* is a first countable, T_3, hereditarily separable space which fails to be hereditarily Lindelöf, and an *L-space* is a first countable, T_3, hereditarily Lindelöf space which fails to be hereditarily separable. Study of these spaces has long been a set-theoretic topologist's favourite. In 1935, Kurepa [Kr] showed that a Suslin line is an L-space, and Jech proved the consistency of a Suslin line, hence of an L-space, in 1967. In 1972, Rudin [Ru_3] constructed an S-space from a Suslin line. S- and L-spaces have also been shown to be consistent with **CH**, $\diamondsuit$ and $\mathbf{MA} + \neg\mathbf{CH}$. In 1983, Todorčević [To] showed the consistency of 'there are no S-spaces', but the question of whether there is *always* an L-space remains open.

The relevance of this to our work is as follows. Szentmiklóssy [Sz] shows that, under $\mathbf{MA} + \neg\mathbf{CH}$, there are *no* first countable L-spaces, so that every first countable, T_3, hereditarily Lindelöf space is hereditarily separable. So trivially, all of our results follow in this model:

Theorem 4.7 ($\mathbf{MA} + \neg\mathbf{CH}$) *Let $(X, \mathcal{T})$ be first countable, T_3 and hereditarily Lindelöf. Then $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is collectionwise Hausdorff and fails to be weakly $\delta\theta$-refinable or locally compact.*

It turns out, though, that first countable L-spaces are consistent with **ZFC**. Such spaces of cardinality $\aleph_1$ have been constructed from **CH** (in [vDTW]) and from a Suslin line (in [Pn]). And every L-space contains a subspace which is *left-separated in type ω_1*, that is, a subspace which can be ordered in type ω_1 in such a way that every initial segment is closed. This provides us with the following example.

Example 4.8 (**CH** *or* $\neg$**SH**) *Let $(X, \mathcal{T})$ be a first countable L-space which is left-separated in type ω_1, and let $X = \{x_\alpha : \alpha < \omega_1\}$ be the order with respect to which*

$\mathcal{T}$ is left-separated. Then if $\mathcal{T}_{\omega_1}$ is the order topology induced by this ordering, the intersection topology $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is discrete.

Proof $(X, \mathcal{T})$ is left-separated. Therefore $\{x_\beta : \beta < \alpha\}$ is closed for all $\alpha < \omega_1$ (w.r.t. $\mathcal{T}$), whence $\{x_\beta : \beta \geq \alpha\}$ is open for all $\alpha < \omega_1$. *But*, $\{x_\beta : \beta \leq \alpha\}$ is open w.r.t. $\mathcal{T}_{\omega_1}$, so $\{x_\beta : \beta \geq \alpha\} \cap \{x_\beta : \beta \leq \alpha\} = \{x_\alpha\}$ is open in $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ for all $\alpha < \omega_1$.

□

Hence we see that in some models, it is possible to have quite different properties in the hereditarily Lindelöf case recall that in the hereditarily separable case, metrizability and local compactness were ruled out in intersection topologies with ω_1, and these are evidently properties which our example above has trivially. Note also that when $(X, \mathcal{T})$ is a Suslin line, it cannot have a G_δ-diagonal (for if it did, then it would be a LOTS with a G_δ-diagonal, and such spaces are metrizable) and yet the intersection topology $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ trivially does.

One final property we can include here is realcompactness.

Proposition 4.9 *Let $(X, \mathcal{T})$ be first countable, T_3 and hereditarily Lindelöf. Then $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is realcompact.*

Proof Gillman and Jerison prove in [GJ, Cor. 8.18] that any refinement of a first countable, T_3, Lindelöf space is realcompact.

□

4.4 Dropping hereditary separability

By now, it should be apparent that most of Reed's results about the intersection topology w.r.t. a separable metric space and ω_1 carry over to the first countable, T_3, hereditarily separable case. What happens, though, if we consider a space which is first countable, T_3 and separable though not hereditarily? Such a space is the *tangent disc space*, also called the *Niemytski plane* or *Moore plane*. This has as underlying set the closed upper half-plane, $\mathbb{R} \times \{y \in \mathbb{R} : y \geq 0\}$. Basic neighbourhoods for points above the x-axis are the usual Euclidean neighbourhoods, while basic neighbourhoods for a point $(x, 0)$ on

the x-axis consist of the point $(x,0)$ together with the interior of a disc in the upper half-plane that is tangent to the x-axis at $(x,0)$. The tangent disc space is a separable Moore space, with rational points in the half-plane a countable dense subset, but the x-axis is a closed discrete subset of size $\mathfrak{c}$, and it cannot, therefore, be hereditarily separable or metrizable. We look at two constructions based on the tangent disc space, chosen to illustrate that the intersection topology can possess much stronger topological properties as soon as $(X,\mathcal{T})$ is no longer hereditarily separable.

Construction I

Take a subspace, X, of the tangent disc space of cardinality $\aleph_1$ such that

- X contains an uncountable dense subset of the x-axis.
- X contains uncountably many points from every Euclidean open set in the half-plane.

and give it the subspace topology, $\mathcal{T}$. Let Y be the intersection of X with the x-axis. Note then that Y is discrete, and therefore non-separable.

We may now consider the intersection topology of $\mathcal{T}$ and some ω_1-type order topology on X. Identify Y with the limit ordinals in ω_1, and $X \setminus Y$ with the nonlimits. Note that Y is discrete in $\mathcal{T}$ and hence also in $\mathcal{T} \vee \mathcal{T}_{\omega_1}$. *But* it is also stationary. This is our first indication of behaviour different from that in the hereditarily separable case recall (Proposition 4.4) that all σ-discrete subspaces of $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ are nonstationary for hereditarily separable $\mathcal{T}$.

We do know that $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is $T_{3\frac{1}{2}}$, first countable, locally countable and pseudonormal with a base of countable order, as these properties follow for all first countable, T_3 topologies, $\mathcal{T}$ (Theorem 4.1). By Corollary 3.15, this intersection topology cannot be locally compact. It is evidently submetrizable, because it refines the usual Euclidean topology on X. For our next observation, we need two results of Nyikos [N]:

Lemma 4.10 (Nyikos) *Let X be σ-scattered. Then X is σ-closed discrete iff X is submetacompact with a G_δ-diagonal.*

Lemma 4.11 (Nyikos) *A σ-scattered space is developable iff it is first countable and submetacompact.*

Now, as has been observed earlier, all ω_1 intersection topologies are scattered (because they are refinements of scattered ω_1). The above two lemmas therefore allow us to prove:

Proposition 4.12 $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ *is a Moore space.*

Proof Let $X_0 = (\mathbb{R} \times [1, \infty)) \cap X$ and, for $n \geq 1$, let $X_n = (\mathbb{R} \times [\frac{1}{n+1}, \frac{1}{n}]) \cap X$. Evidently, $X = Y \cup \bigcup_{n=0}^{\infty} X_n$. Y is closed w.r.t. $\mathcal{T}_{\omega_1}$ and discrete w.r.t. $\mathcal{T}$, hence closed discrete w.r.t. $\mathcal{T} \vee \mathcal{T}_{\omega_1}$. Each $X_n \subseteq Y \setminus X$, which is discrete, hence they are all discrete too. Moreover they are all closed w.r.t. $\mathcal{T}$, and therefore $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is σ-closed discrete. Then by Lemma 4.10, X is submetacompact. By our observations above, X is first countable, and hence by Lemma 4.11, X is developable. But we know X is T_3, and hence a Moore space.

□

Thus we see that this intersection topology can have much stronger properties than are allowed in the hereditarily separable case. $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is not, however, metrizable, as the next proposition makes clear in a fairly drastic manner.

Proposition 4.13 $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ *is not metaLindelöf.*

Proof Y is closed discrete in $(X, \mathcal{T})$. If $x_\alpha \in Y$, then let U_α be a $\mathcal{T}$-open set such that $U_\alpha \cap Y = x_\alpha$. If $x_\alpha \notin Y$, then let U_α be a $\mathcal{T}$-open set such that $U_\alpha \cap Y = \emptyset$. Let $\mathcal{V}$ be an open refinement, in the topology $\mathcal{T} \vee \mathcal{T}_{\omega_1}$, of the cover $\{U_\alpha : \alpha \in \omega_1\}$. We show that $\mathcal{V}$ cannot be point-countable.

For each $x_\alpha \in Y$, let V_α be a basic open set in $\mathcal{T} \vee \mathcal{T}_{\omega_1}$ contained in some element of $\mathcal{V}$ such that $x_\alpha \in V_\alpha$. Note that each V_α must be contained in a different element of $\mathcal{V}$, because no two elements of Y can be in the same element of some $W \in \mathcal{V}$. Now

$$V_\alpha = (f(\alpha), \alpha] \cap G_n(\alpha)$$

where $f(\alpha) < \alpha$ in ω_1 and $G_n(\alpha) = \{x_\alpha\} \cup B_{1/n}((x_\alpha, 1/n))$, a basic open set in the tangent disc topology.

For all α, there exists $\mathbf{q}_\alpha \in \mathbb{Q}^2$ and $k_\alpha \in \omega$ such that $B_{1/k_\alpha}(\mathbf{q}_\alpha) \subseteq G_n(\alpha)$. Now Y is stationary, and so we can apply the Pressing Down Lemma and Countable Decomposition Lemma to obtain a stationary $Z \subseteq Y$, $\gamma \in \omega_1$, $k \in \omega$ and $\mathbf{q} \in \mathbb{Q}^2$ such that $k_\alpha = k$, $\mathbf{q}_\alpha = \mathbf{q}$ and $f(\alpha) = \gamma$ for all $\alpha \in Z$. Hence

$$(\gamma, \alpha] \cap B_{1/k}(\mathbf{q}) \subseteq V_\alpha$$

for all $\alpha \in Z$.

Choose $x_\theta \in B_{1/k}(\mathbf{q})$ such that $\theta > \gamma$. (Recall that we insisted X be uncountable in every $\mathcal{T}$-open set.) Then, since Z is unbounded, x_θ must belong to uncountably many V_α's, and hence to uncountably many members of $\mathcal{V}$.

Hence, no open refinement of $\{U_\alpha : \alpha \in \omega_1\}$ is point countable, and therefore $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ cannot be metaLindelöf.

□

By modifying the space a little, we can obtain even stronger properties.

Construction II

Consider the tangent disc space with only rational points above the x-axis and again a dense subset of the x-axis of cardinality $\aleph_1$. Note that the space is no longer hereditarily separable with respect to co-countable subsets, and thus Corollary 3.15 no longer applies. We *could* now have local compactness of the intersection topology.

We re-order X in type ω_1 as follows. Partition $X \setminus Y$ into positive and non-positive parts (left and right of the y-axis), and identify $X \setminus Y$ with $2\omega + 1$ in such a way that $(1, 1)$ is identified with the point ω, $(-1, 1)$ is identified with the point 2ω, the rest of the positive part of $X \setminus Y$ is identified with $(\omega, 2\omega)$, and the rest of the non-positive part is identified with $[0, \omega)$. Then *all* points above the axis are isolated in the intersection topology. Identify Y with the remainder of ω_1 in an arbitrary way. $\mathcal{T}$-neighbourhoods of $x_\alpha \in Y$ reach up into $X \setminus Y$, but miss Y, and $\mathcal{T}_{\omega_1}$-neighbourhoods of the form $(2\omega, \alpha]$ separate y from the rest of X. So, in the intersection topology, each point of Y is isolated.

Hence, $\mathcal{T} \vee \mathcal{T}_{\omega_1}$ is actually the discrete topology on X. In particular, it is locally compact and metrizable, further properties ruled out in the hereditarily separable case. Note also

that this example makes clear that there is no way Kunen's results can be extended this widely perfect normality is ruled out for separable metrizable-ω_1 intersections, yet evidently the above example *is* perfectly normal.

In [R_1], Reed observes that under **CH**, the intersection topology w.r.t. the Michael line and the countable ordinals can be metrizable, so that the join of a nonmetrizable, T_3 topology and ω_1 can be metrizable. Construction II improves on his example - the space X is nonmetrizable and T_3 and yet its intersection topology with ω_1 is metrizable, and this is true without employing **CH**.

It is fairly evident that this example was carefully constructed so that the intersection topology turned out to be discrete. However *regardless* of the way in which we order X, the intersection topology will always be metrizable. To prove this, we need a result from Good's thesis [Go]. Good studies the class $\mathcal{W}$ of all bijective preimages of ω_1, evidently containing as a subclass all intersection topologies w.r.t. $\mathcal{T}_{\omega_1}$ and some other topology. Consider his next result:

Lemma 4.14 (Good) *If $X \in \mathcal{W}$ is first countable and T_3, then it is metrizable iff it has a club set of isolated points.*

We can now prove

Proposition 4.15 *Regardless of the way in which X is ordered in type ω_1, the resulting intersection topology is metrizable.*

Proof $X \setminus Y$ is countable, so we can choose $\alpha > \sup(X \setminus Y)$. Consider $\beta > \alpha$, and then we must have $x_\beta \in Y$. If U is a basic $\mathcal{T}$ neighbourhood of x_β, then it is a tangent disc, so that $U \cap Y = \{x_\beta\}$, that is $U \subseteq \{x_\beta\} \cup (X \setminus Y) \subseteq \{x_\beta\} \cup \{x_\gamma : \gamma < \alpha\}$. Hence $U \cap (\alpha, \beta] = \{x_\beta\}$. So we have each point of (α, ω_1) isolated in the intersection topology. But this is a club set, and $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is first countable and T_3, so that by Lemma 4.14, the intersection topology is metrizable.

□

These examples should make clear how critical the *heredity* of separability of $(X, \mathcal{T})$ is in determining the properties of the intersection topology. If $(X, \mathcal{T})$ were *hereditarily* separable, then we could not even have $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ weakly $\delta\theta$-refinable.

4.5 The bow-tie space

In the next chapter, we will show that Kunen's results about the class $\mathcal{C}$ also hold in the class of intersection topologies w.r.t. separable GO-spaces and ω_1. It seems appropriate here to describe another example where Kunen's results also hold. The *bow-tie* or *butterfly space* of McAuley [McA] is a space with countable network (and hence is hereditarily separable and hereditarily Lindelöf) which fails to be developable. Its underlying set is $\mathbb{R}^2$. Basic neighbourhoods of points not on the x-axis are the usual Euclidean neighbourhoods, while those on the x-axis resemble a bow-tie. Basic neighbourhoods at $\mathbf{x} = (x, 0)$ are of the form

$$U_n(\mathbf{x}) = \{\mathbf{x}\} \cup \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{y} - \mathbf{x}| < 1/n, |\mathbf{y} - (x, \pm n)| > n\}$$

Evidently this space, $(X, \mathcal{T}_{\mathbb{B}})$, is submetrizable, with the ordinary Euclidean topology as a subtopology. In the next chapter, the rôle which such a subtopology plays in helping to prove Kunen's results about the intersection topology will become more apparent recall that Kunen's results hold for an arbitrary separable metrizable space, and thus in particular for the Euclidean topology on $\mathbb{R}^2$. For now, we content ourselves with proving only the following proposition, and assuring the reader that analogues of Kunen's results then follow easily (by a method identical to that in Sections 2.3 and 5.2). This proposition essentially says that the intersection topology w.r.t. the bow-tie topology and ω_1 is 'close' to the underlying intersection topology w.r.t. the Euclidean topology and ω_1.

Proposition 4.16 *If U is $\mathcal{T}_{\mathbb{B}} \vee \mathcal{T}_{\omega_1}$-open and $\tilde{U} = \mathrm{Int}_{\mathcal{T}_{\mathbb{R}^2} \vee \mathcal{T}_{\omega_1}} U$, then $U \setminus \tilde{U}$ is nonstationary.*

Proof Let Y be the x-axis in $\mathbb{R}^2$, and let the ω_1-type order on X be $\{\mathbf{x}_\alpha : \alpha < \omega_1\}$. Suppose $U \setminus \tilde{U}$ is stationary. Then either

1. $(U \setminus \tilde{U}) \cap Y$ is stationary, or
2. $(U \setminus \tilde{U}) \setminus Y$ is stationary.

Suppose (1) holds. For $x < y$ in $\mathbb{R}$, let

$$G_y(x) = \{(x, 0)\} \cup \{\mathbf{z} \in \mathbb{R}^2 : |\mathbf{z} - (x, 0)| < y - x, |\mathbf{z} - (x, \pm\frac{1}{y - x})| > \frac{1}{y - x}, z_2 > x\}$$

where $\mathbf{z} = (z_1, z_2)$. $G_y(x)$ is the 'right bow' of a bow-tie centred at $(x, 0)$ and with its rightmost point at $(y, 0)$, so that each basic open neighbourhood of $(x, 0)$ contains such a set.

For all $(x, 0) = \mathbf{x}_\alpha \in (U \setminus \widetilde{U}) \cap Y$, there exist $f(\alpha) < \alpha$ and $y_x \in \mathbb{Q}$ such that

$$G_{y_x}(x) \cap (f(\alpha), \alpha] \subseteq U$$

Applying the Pressing Down Lemma and Countable Decomposition Lemma, we find Z stationary in ω_1, $y \in \mathbb{Q}$ and $\gamma \in \omega_1$ such that $f(\alpha) = \gamma$ and $y_x = y$ for all $(x, 0) \in Z$, that is

$$G_y(x) \cap (\gamma, \alpha] \subseteq U$$

for all $(x, 0) = \mathbf{x}_\alpha \in Z$.

Now, let $\mathbf{x}_\alpha = (x_\alpha, 0), \mathbf{x}_\beta = (x_\beta, 0) \in Z$ with $x_\alpha < x_\beta$ in $\mathbb{R}$. Then $G_y(x_\alpha) \setminus \{x_\alpha\}$ is a $\mathcal{T}_{\mathbb{R}^2}$-open set round x_β, and $x_\beta \notin \widetilde{U}$, so that

$$(G_y(x_\alpha) \setminus \{x_\alpha\}) \cap (\gamma, \beta] \not\subseteq U$$

whence

$$G_y(x_\alpha) \cap (\gamma, \beta] \not\subseteq U$$

But $x_\alpha \in Z$, so that

$$G_y(x_\alpha) \cap (\gamma, \alpha] \subseteq U$$

These together imply that $\alpha < \beta$, so that $x_\alpha < x_\beta$ in $\mathbb{R}$ iff $\alpha < \beta$. But this is a contradiction, because there is no ω_1-type order on an uncountable subset of $\mathbb{R}$ preserving the usual order.

Suppose (2) holds. Well, the bow-tie topology coincides with the ordinary Euclidean topology on $(U \setminus \widetilde{U}) \setminus Y$, so that $\mathrm{Int}_{\mathcal{T}_{\mathbb{R}^2} \vee \mathcal{T}_{\omega_1}}(U \setminus Y) = U \setminus Y$, whence $U \setminus Y = \widetilde{U} \setminus Y$ and therefore $(U \setminus \widetilde{U}) \setminus Y = \emptyset$, a contradiction.

□

This is a very specialised result, of course, and the proof appears to rely very much on the particular structure of the bow-tie space. Some kind of generalisation seems feasible. Among the properties of the bow-tie space is stratifiability, a well-behaved generalised metric property. All stratifiable spaces are submetrizable, and it is the author's conjecture that an analogous result about the relationship between the intersection topologies

associated to a separable stratifiable space and its corresponding metric subtopology must exist. If this is the case, then it would be more or less the strongest result obtainable, for the spaces in Section 4.4 are separable and semi-stratifiable, and it was demonstrated there that Kunen's results fail to hold in so general a setting.

Chapter 5

The intersection topology w.r.t. a separable GO-space and the countable ordinals

THEN FELT I LIKE SOME WATCHER OF THE SKIES
WHEN A NEW PLANET SWIMS INTO HIS KEN;
OR LIKE STOUT CORTEZ WHEN WITH EAGLE EYES
HE STAR'D AT THE PACIFIC – AND ALL HIS MEN
LOOK'D AT EACH OTHER WITH A WILD SURMISE –
SILENT, UPON A PEAK IN DARIEN.

(*On first looking into Chapman's 'Homer'*, John Keats)

In the proof that the results of Kunen about normality, perfectness and ω_1-compactness apply to the intersection topology w.r.t the Sorgenfrey line and ω_1, it becomes apparent that there are a few properties which are crucial. First is the submetrizability of the Sorgenfrey line, and, moreover, particularly 'close' submetrizability, in the sense that basic open sets in the Sorgenfrey line and those in the real line differ by only one point (the 'end-point' of the half-open Sorgenfrey interval). Second is the orderability of the Sorgenfrey line. And lastly we use its separability.

Generalising these results to the join of an *arbitrary* separable GO-space and ω_1 seemed reasonable, but the problem that immediately becomes apparent is that in the general case there is no obvious metrizable topology to employ. Indeed, separable GO-spaces are not in general submetrizable: the Alexandroff double arrow space (the space $[0,1]\times\{0,1\}$ equipped with the lexicographic order topology) is a compact and separable LOTS but does not have a countable base, so is not submetrizable.

However, there are some well-behaved topologies on the set $[0,1] \times \{0,1\}$. First is the ordinary Euclidean topology (inherited by $[0,1] \times \{0,1\}$ as a subspace of $\mathbb{R}^2$), but the problem here is that the topology is too coarse an approximation Euclidean-open sets like $[0,1] \times \{0\}$ have empty interior w.r.t. the double arrow topology and it even fails to be a subtopology of the double arrow space. To remedy this, we consider instead the rather odd (not even T_0!) subtopology

$$\mathcal{T} = \{U \times \{0,1\} : U \in \mathcal{T}_{\mathbb{R}}\}$$

where $\mathcal{T}_{\mathbb{R}}$ is the Euclidean topology on $[0,1]$. Evidently this topology is 'close' to the double arrow topology in the same way that the Sorgenfrey line is 'close' to the real line. And although it fails to be metrizable, it *is* second countable, and this turns out to be enough.

It is surprising to discover that a similar process can be applied to an arbitrary separable GO-space. In [Br], Brod proves that every separable GO-space is the union of two submetrizable subspaces but she fails to analyse the subtopologies any more thoroughly. In this chapter, we show that any separable GO-space, $(X, \mathcal{T})$, has a second countable subtopology $\mathcal{M}$ with the property that if $\mathcal{T}_{\omega_1}$ is an ω_1-type order topology on X, and if $U \in \mathcal{T} \vee \mathcal{T}_{\omega_1}$ with $\tilde{U} = \mathrm{Int}_{\mathcal{M} \vee \mathcal{T}_{\omega_1}} U$, then $U \setminus \tilde{U}$ is nonstationary. This result allows us to carry Kunen's results over into this more general setting.

5.1 Separable GO-spaces and almost G_δ-diagonals

The first section shows that although not all separable GO-spaces have a G_δ-diagonal (again, the double arrow space is a counterexample), they always possess a sequence with a slightly weaker property. We then use this property to construct the subtopology $\mathcal{M}$.

Lemma 5.1 *If $(X, \mathcal{T})$ is a separable GO-space, then there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that*

1. *$\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for all n, and*
2. *if $x \in X$, then $|\bigcap_{n \in \omega} \mathrm{St}(x, \mathcal{U}_n)| \leq 2$.*

Proof Let D be a countable dense subset of X, and let I be the set of 'one point intervals' (i.e. those points x for which $\{x\} = (a, b)$ for some $a, b \in X$). Enumerate D as $\{d_n : n \in \omega\}$ and I as $\{i_n : n \in \omega\}$.

Let $n \in \omega$. Order the finite set $\{d_m : m \leq 2n\}$ by $<$ as $a_1 < a_2 < \ldots < a_{2n}$. Put

$$\mathcal{V}_n = \{(\leftarrow, a_2), (a_1, a_3), (a_2, a_4), \ldots, (a_{2n-2}, a_{2n}), (a_{2n-1}, \rightarrow)\}$$

Order $\{i_m : m \leq n\}$ as $b_1 < b_2 < \ldots < b_n$, and put

$$\mathcal{W}_n = \{(\leftarrow, b_1), \{b_1\}, (b_1, b_2), \{b_2\}, (b_2, b_3), \ldots, (b_{n-1}, b_n), \{b_n\}, (b_n, \rightarrow)\}$$

Let $\mathcal{U}_n$ be the 'intersection' of $\mathcal{V}_n$ and $\mathcal{W}_n$ in the obvious way, so that $\mathcal{U}_n$ is the coarsest cover refining both $\mathcal{V}_n$ and $\mathcal{W}_n$. Clearly $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$.

Now, suppose, for some $x, y \in X$, $(x, y) \neq \emptyset$. Then, since D is dense, there exists $d \in (x, y) \cap D$. There are two cases:

- Suppose $(x, y) = \{d\}$. Then $d \in I$ and therefore $d = i_n$ for some n. Hence $\mathrm{St}(x, \mathcal{U}_n) \subseteq (\leftarrow, d)$ and $\mathrm{St}(y, \mathcal{U}_n) \subseteq (d, \rightarrow)$, and so $x \notin \mathrm{St}(y, \mathcal{U}_n)$ and $y \notin \mathrm{St}(x, \mathcal{U}_n)$.
- Otherwise, without loss of generality, $(x, d) \neq \emptyset$. Then there is some $c \in (x, d) \cap D$. For some n, $c, d \in \{d_m : m \leq 2n\}$. Hence $\mathrm{St}(x, \mathcal{U}_n) \subseteq (\leftarrow, d)$ and $\mathrm{St}(y, \mathcal{U}_n) \subseteq (c, \rightarrow)$, and so $x \notin \mathrm{St}(y, \mathcal{U}_n)$ and $y \notin \mathrm{St}(x, \mathcal{U}_n)$.

So, if $x < y$ and $x \in \bigcap_{n \in \omega} \mathrm{St}(y, \mathcal{U}_n)$, then $(x, y) = \emptyset$.

Suppose, for a contradiction, that $\bigcap_{n \in \omega} \mathrm{St}(x, \mathcal{U}_n)$ contains three different points x, y, z. Either

- $x < y < z$, but then $(x, z) \neq \emptyset$, so that by the above $z \notin \bigcap_{n \in \omega} \mathrm{St}(x, \mathcal{U}_n)$, which is a contradiction, or
- $y < x < z$, but then $z, y \in \mathrm{St}(x, \mathcal{U}_n) \forall n$, so that $(y, x) = (x, z) = \emptyset \Rightarrow (y, z) = \{x\} \Rightarrow x \in I \Rightarrow \mathrm{St}(x, \mathcal{U}_N) = \{x\}$ for some N, contradicting $z, y \in \bigcap_{n \in \omega} \mathrm{St}(x, \mathcal{U}_n)$, or
- $y < z < x$, which is the same as the first case.

Hence, we must have $|\bigcap_{n\in\omega}\mathrm{St}(x,\mathcal{U}_n)| \leq 2$ for all $x \in X$.

□

We will refer to this sequence frequently in what follows, so we make the

Definition 5.2 *An* almost G_δ-diagonal sequence *on a space X is a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that*

1. *$\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for all n, and*
2. *if $x \in X$, then $|\bigcap_{n\in\omega}\mathrm{St}(x,\mathcal{U}_n)| \leq 2$.*

Now, GO-spaces with G_δ-diagonal sequences are submetrizable, and what the almost G_δ-diagonal sequence allows us to do is split the space into two subspaces, both of which have G_δ-diagonals. Applied to the double arrow space with a little care, these two subspaces are precisely the two 'arrows'.

Lemma 5.3 *Let $\{\mathcal{U}_n : n \in \omega\}$ be an almost G_δ-diagonal sequence for the separable GO-space X. The equivalence relation $\sim$ on X defined by*

$$x \sim y \Leftrightarrow \bigcap \mathrm{St}(x,\mathcal{U}_n) = \bigcap \mathrm{St}(y,\mathcal{U}_n)$$

decomposes X into equivalence classes of size at most 2, with each element of D in a different equivalence class.

Proof Suppose $y \sim x$, so that $\bigcap \mathrm{St}(y,\mathcal{U}_n) = \bigcap \mathrm{St}(x,\mathcal{U}_n)$, where $x \neq y$. Then $y \in \bigcap \mathrm{St}(y,\mathcal{U}_n) = \bigcap \mathrm{St}(x,\mathcal{U}_n)$. Conversely, suppose $y \in \bigcap \mathrm{St}(x,\mathcal{U}_n)$, where $y \neq x$. Since $|\bigcap \mathrm{St}(x,\mathcal{U}_n)| \leq 2$, we must have $\bigcap \mathrm{St}(x,\mathcal{U}_n) = \{x,y\}$. But then $y \in \mathrm{St}(x,\mathcal{U}_n) \forall n$, whence $x \in \mathrm{St}(y,\mathcal{U}_n) \forall n$, so that $x \in \bigcap \mathrm{St}(y,\mathcal{U}_n)$. But again, $|\bigcap \mathrm{St}(y,\mathcal{U}_n)| \leq 2$, so that $\bigcap \mathrm{St}(y,\mathcal{U}_n) = \{x,y\}$. Hence $\bigcap \mathrm{St}(y,\mathcal{U}_n) = \bigcap \mathrm{St}(x,\mathcal{U}_n)$. We conclude that $x \sim y$ iff $y \in \bigcap \mathrm{St}(x,\mathcal{U}_n)$.

Hence, the equivalence class of x is simply $\bigcap \mathrm{St}(x,\mathcal{U}_n)$, which, by Lemma 5.1, has size at most 2.

Now, let $c, d \in D$ with $c \neq d$. For some N, we have $c, d \in \{d_m : m \leq 2N\}$. Suppose, without loss of generality, that $c < d$. Then $\mathrm{St}(c, \mathcal{U}_N) \subseteq (\leftarrow, d)$, so that $d \notin \mathrm{St}(c, \mathcal{U}_N)$ whence $d \notin \bigcap \mathrm{St}(c, \mathcal{U}_n)$, and therefore $d \not\sim c$.
Hence c and d belong to different equivalence classes.

□

Now look at the set of equivalence classes.
Pick one point from each class, an element of D if possible, and call this set of points X_0. Each of the remaining points of X is contained in an equivalence class with a point of X_0. Call this set of remaining points X_1. Then X is the disjoint union of X_0 and X_1, and $D \subseteq X_0$ (for, if an equivalence class contains two points, then by the previous lemma, only one of them can be in D).
To each element $y \in X_1$, we can associate an element $x \in X_0$ for which $x \sim y$. We write $y = x'$. If $A \subseteq X_0$, write $A' = \{y \in X_1 : y \sim x \text{ for some } x \in A\}$.
Observe that, as shown in the proof of Lemma 5.1, if x' exists then there are no points of X between x and x' (i.e. $(x, x') = \emptyset$ if $x < x'$ or $(x', x) = \emptyset$ if $x' < x$). Note also that $x < y$ in X_0 iff $x' < y'$ in X_1 (provided that both of these exist).
X_0 and X_1 are the two 'arrows' of the space. The reason for decomposing X in this manner becomes apparent in the next lemma. Recall that every GO-space has an underlying LOTS subtopology generated by the linear order on the space [Lu].

Lemma 5.4 *Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be the subspace topologies induced by $\mathcal{T}$ on X_0 and X_1 respectively. Then*

- *$(X_0, \mathcal{T}_0)$ is submetrizable, with the underlying LOTS $(X_0, \mathcal{M}_0)$ as a metrizable subtopology.*
- *$\mathcal{M}_1 = \{U' : U \in \mathcal{M}_0\}$ is a topology on X_1, and is a metrizable subtopology of $\mathcal{T}_1$.*
- *If $U \in \mathcal{M}_0$ then $U \cup U' \in \mathcal{T}$.*

Proof Let $\mathcal{M}_0$ be the LOTS topology associated with $(X_0, \mathcal{T}_0)$. Note that $\mathcal{M}_0 \subseteq \mathcal{T}_0$.

We show first that $(X_0, \mathcal{M}_0)$ is separable metric.
Let $\{\mathcal{U}_n : n \in \omega\}$ be an almost G_δ-diagonal sequence for X. Let $\mathcal{U}_n^0 = \{U \cap X_0 : U \in \mathcal{U}_n\}$. Note that since $D \cup I \subseteq X_0$, each $\mathcal{U}_n^0$ is actually a cover of X_0 by sets open in the LOTS

topology $\mathcal{M}_0$.

It is easy to verify that for all $x \in X_0$, $\mathrm{St}(x, \mathcal{U}_n^0) = \mathrm{St}(x, \mathcal{U}_n) \cap X_0$. Moreover, $\bigcap \mathrm{St}(x, \mathcal{U}_n^0) = \bigcap (\mathrm{St}(x, \mathcal{U}_n) \cap X_0) = X_0 \cap \bigcap \mathrm{St}(x, \mathcal{U}_n) = \{x\}$ (for, if $x \sim y \neq x$, then $y \in X_1$).

Hence $\mathcal{U}_n^0$ is a G_δ-diagonal sequence for $(X_0, \mathcal{M}_0)$. But a LOTS topology with a G_δ-diagonal is metrizable, so $(X_0, \mathcal{M}_0)$ is metrizable.

Moreover, $(X_0, \mathcal{M}_0)$ is separable because $\mathcal{M}_0$ is contained in separable $\mathcal{T}_0$.

Now, let $\mathcal{M}_1 = \{U' : U \in \mathcal{M}_0\}$. We show that $(X_1, \mathcal{M}_1)$ is a separable metric GO-space, and is a subtopology of $(X_1, \mathcal{T}_1)$.

1. $\mathcal{M}_1$ is a topology on X_1 (using $U' \cap V' = (U \cap V)'$ and $\bigcup U'_\lambda = (\bigcup U_\lambda)'$).

2. We show that $(X_1, \mathcal{M}_1)$ is a GO-space, by demonstrating that $\mathcal{M}_1$ refines the LOTS topology on X_1 and has a base of convex sets. Let $a', b' \in X_1$ with $a' < b'$.

   Claim: $(a', b') \cap X_1 = ((a, b) \cap X_0)'$

   Let $c \in (a, b) \cap X_0$ and suppose c' exists.

   $c \in (a, b) \cap X_0 \Leftrightarrow a < c < b \Leftrightarrow a' < c' < b' \Leftrightarrow c' \in (a', b') \cap X_1$

   Hence $(a', b') \cap X_1 \in \mathcal{M}_1$, since $(a, b) \cap X_0$ is $\mathcal{M}_0$-open, and so $\mathcal{M}_1$ refines the LOTS topology on X_1. Moreover, $(X_0, \mathcal{M}_0)$ has a base, $\mathcal{B}$, of convex sets, and obviously $\{B' : B \in \mathcal{B}\}$ is then a base of convex sets for $(X_1, \mathcal{M}_1)$.

3. $(X_0, \mathcal{M}_0)$ is separable metric, and therefore has a countable base. So $(X_1, \mathcal{M}_1)$ has a countable base too. But $(X_1, \mathcal{M}_1)$ is also a GO-space, hence T_3, and is therefore separable metric.

4. Let $U \in \mathcal{M}_0$.

   Claim: $U \cup U' \in \mathcal{T}$

   Method: For all $x \in U \cup U'$, we show that $\exists c < x < d$ (with $c, d \in X$) such that $(c, d) \subseteq U \cup U'$. Since $\mathcal{T}$ is GO, it contains the LOTS topology, whence $(c, d) \in \mathcal{T}$.

   - Suppose $x \in U$. Then, since $\mathcal{M}_0$ is the LOTS topology on X_0, there exist $a, b \in X_0$ such that $x \in (a, b) \cap X_0 \subseteq U$. There are now three cases:

     Case (i) $(a, x) \neq \emptyset, (x, b) \neq \emptyset$. Then there exists $c \in (a, x) \cap D, d \in (x, b) \cap D$. Clearly, $(c, d) \cap X_0 \subseteq (a, b) \cap X_0 \in \mathcal{M}_0$, but what about $(c, d) \cap X_1$? Well,

take $y' \in (c,d) \cap X_1$. Then $y' \sim y$, where $y \in [c,d]$ (because we know $(y,y') = \emptyset$). Then $y \in (a,b) \cap X_0$, whence $y' \in ((a,b) \cap X_0)'$. And therefore $(c,d) \cap X_1 \subseteq ((a,b) \cap X_0)'$. Hence

$$x \in (c,d) \subseteq ((a,b) \cap X_0) \cup ((a,b) \cap X_0)' \subseteq U \cup U'$$

Case (ii) $(a,x) = (x,b) = \emptyset$. Then $(a,b) = \{x\}$, whence

$$x \in (a,b) \subseteq U \subseteq U \cup U'$$

Case (iii) $(a,x) = \emptyset, (x,b) \neq \emptyset$. Let $d \in (x,b) \cap D$. We have $x \in (a,d)$, so $(a,d) \cap X_0 \subseteq (a,b) \cap X_0 \subseteq U$. What about $(a,d) \cap X_1$? Well, if $y' \in (a,d) \cap X_1$, then $y' \sim y \in X_0$. Since $(a,x) = \emptyset$, we must have $y' \in [x,d)$, but $y' \neq x$ because $y' \in X_1$ and $x \in X_0$. Since $(y,y') = \emptyset$, we must have $y \in [x,d] \subseteq (a,b)$, and therefore $y \in ((a,b) \cap X_0)'$. So we have

$$x \in (a,d) \subseteq ((a,b) \cap X_0) \cup ((a,b) \cap X_0)' \subseteq U \cup U'$$

The case $(a,x) \neq \emptyset, (x,b) = \emptyset$ works in exactly the same way.

- Suppose $x' \in U'$. Then $x' \sim x \in U$, and there exist $a, b \in X_0$ such that $x \in (a,b) \cap X_0 \subseteq U$. Above we determined $c, d \in X_0$ such that

$$x \in (c,d) \subseteq ((a,b) \cap X_0) \cup ((a,b) \cap X_0)' \subseteq U \cup U'$$

Since $x' \in X_1, x' \neq c, x' \neq d$ and $(x,x') = \emptyset$, we have

$$x' \in (c,d) \subseteq U \cup U'$$

as required.

Hence we have $U \cup U' \in \mathcal{T}$.

5. Note, then, that $U' = (U \cup U') \cap X_1 \in \mathcal{T}_1$, so that $\mathcal{M}_1$ is a subtopology of $\mathcal{T}_1$.

□

We can now define the second countable subtopology on X. This is completely analogous to the process for the double arrow space described earlier: a basic open set in the second countable subtopology on X will consist of the points of a basic open set in the metrizable subtopology on X_0 ($U \times \{0\}$ in the double arrow space) together with the points in X_1 which are $\sim$-equivalent to the points of this open set ($U \times \{1\}$ in the double arrow space).

Lemma 5.5 *Let $\mathcal{M}$ be the topology on X defined by $\mathcal{M} = \{U \cup U' : U \in \mathcal{M}_0\}$. Then $(X, \mathcal{M})$ is a second countable subtopology of $\mathcal{T}$.*

Proof That $\mathcal{M}$ *is* a topology is fairly evident.
$\mathcal{M}_0$ is separable metrizable, hence has a countable base $\mathcal{B}$. But then $\{B \cup B' : B \in \mathcal{B}\}$ is a countable base for $(X, \mathcal{M})$. Finally, $\mathcal{M} \subseteq \mathcal{T}$ by part 4 of the previous lemma.

□

The next lemma demonstrates how well-behaved this new topology is.

Lemma 5.6 *If $a, b \in X_0$, with $x \in (a, b) \cap X_0$, then there is some $M \in \mathcal{M}$ such that $x \in M \subseteq (a, b)$.*
If $a', b' \in X_1$, with $x' \in (a', b') \cap X_1$, then there is some $M \in \mathcal{M}$ such that $x' \in M \subseteq (a', b')$.

Proof

- Let $a, b \in X_0$ with $x \in (a, b) \cap X_0$.
  Let c be an arbitrary element of $(a, b) \cap X_0$. If c' exists, then $(c, c') = \emptyset$, and therefore $c' \in [a, b]$. But $a, b \in X_0$ and $c' \in X_1$, so $c' \neq a, c' \neq b$, i.e. $c' \in (a, b)$. Hence, $((a, b) \cap X_0)' \subseteq (a, b)$, and therefore
  $$x \in ((a, b) \cap X_0) \cup ((a, b) \cap X_0)' \subseteq (a, b)$$
  Put $M = ((a, b) \cap X_0) \cup ((a, b) \cap X_0)'$. Then $M \in \mathcal{M}$ and $x \in M \subseteq (a, b)$.

- Let $a', b' \in X_1$ with $x' \in (a', b') \cap X_1 = ((a, b) \cap X_0)'$.
  Then $x \in (a, b) \cap X_0$ and therefore, by the above, $x \in ((a, b) \cap X_0) \cup ((a, b) \cap X_0)' \subseteq (a, b)$. Now, if $c \in (a, b) \cap X_0$, then, because $(a, a') = \emptyset$ and $(b, b') = \emptyset$ (and $c \neq a', c \neq b'$), we have $c \in (a', b')$. *And* $((a, b) \cap X_0)' = (a', b') \cap X_1$, so $x' \in M \subseteq (a', b')$, where $M = ((a, b) \cap X_0) \cup ((a, b) \cap X_0)' \in \mathcal{M}$.

□

The reason for constructing this subtopology will become clear in the next section, where we derive results about the intersection topology with ω_1.

5.2 Applications to the intersection topology

Having constructed the subtopology $\mathcal{M}$, we can now use it (in much the same way as we used $\mathcal{T}_{\mathbb{R}}$ as a subtopology of $\mathcal{T}_{\mathbb{S}}$) to derive results about the intersection topology with ω_1.

Lemma 5.7 *If U is $\mathcal{T} \vee \mathcal{T}_{\omega_1}$-open and $\tilde{U} = \mathrm{Int}_{\mathcal{M} \vee \mathcal{T}_{\omega_1}} U$, then $U \setminus \tilde{U}$ is nonstationary.*

Proof Let U be $\mathcal{T} \vee \mathcal{T}_{\omega_1}$-open, and suppose $U \setminus \tilde{U}$ is stationary, for a contradiction. Assume, without loss of generality, that $(U \setminus \tilde{U}) \cap X_0$ is stationary, and call this set W. Let $\mathcal{B}$ be a base of convex sets for $\mathcal{T}$.

For all $x = x_\alpha \in W$, there exists $f(\alpha) < \alpha$ in ω_1 and $B(x) \in \mathcal{B}$ such that

$$x \in B(x) \cap (f(\alpha), \alpha] \subseteq U$$

$(X, \mathcal{T})$ is separable GO, hence hereditarily separable, so let Q be a countable dense subset of $(X_0, \mathcal{T}_0)$. Since $x \in B(x) \cap X_0$, $B(x) \cap X_0 \neq \emptyset$ and is open in $(X_0, \mathcal{T}_0)$, we have that there is some $q_x \in B(x) \cap Q$.

By the Pressing Down Lemma and Countable Decomposition Lemma, there exist $q \in Q, \gamma \in \omega_1$ and Z' stationary in ω_1 such that $f(\alpha) = \gamma$ and $q_x = q$ for all $x = x_\alpha \in Z'$, i.e.

$$B(x) \cap (\gamma, \alpha] \subseteq U \tag{5.1}$$

where $q \in B(x)$.

Without loss of generality, there exists $Z \subseteq Z'$ such that Z is stationary and $q < x$ for all $x \in Z$. Since $\mathcal{B}$ is a base of *convex* sets,

$$[q, x] \subseteq B(x) \quad \forall x \in Z \tag{5.2}$$

Let $x_\alpha, x_\beta \in Z$ with $x_\alpha < x_\beta$. Since $q, x_\alpha, x_\beta \in X_0$, we have, by Lemma 5.6, that there exists $M \in \mathcal{M}$ such that

$$x_\alpha \in M \subseteq (q, x_\beta) \tag{5.3}$$

Therefore, since $x_\alpha \notin \tilde{U}$, $M \cap (\gamma, \alpha] \not\subseteq U$ and so, by (5.3),

$$(q, x_\beta) \cap (\gamma, \alpha] \not\subseteq U$$

By (5.1) and (5.2)

$$[q, x_\alpha] \cap (\gamma, \alpha] \subseteq U$$

and therefore

$$(x_\alpha, x_\beta) \cap (\gamma, \alpha] \not\subseteq U \tag{5.4}$$

But, $x_\beta \in Z$ and so (5.1),(5.2) imply that

$$[q, x_\beta] \cap (\gamma, \beta] \subseteq U$$

and therefore

$$(x_\alpha, x_\beta) \cap (\gamma, \beta] \subseteq U \tag{5.5}$$

Equations (5.4) and (5.5) imply that $\alpha > \beta$. Hence $x_\alpha < x_\beta$ iff $\alpha > \beta$.
Let $\{\alpha_\lambda : \lambda \in \omega_1\}$ be a strictly increasing sequence of countable ordinals such that, for all $\lambda \in \omega_1$, $(\alpha_\lambda, \alpha_{\lambda+1}) \neq \emptyset$. Then, for all λ, $x_{\alpha_{\lambda+1}} < x_{\alpha_\lambda}$ and, since there is some $\beta_\lambda \in (\alpha_\lambda, \alpha_{\lambda+1})$, $x_{\beta_\lambda} \in (x_{\alpha_{\lambda+1}}, x_{\alpha_\lambda})$ so that $(x_{\alpha_{\lambda+1}}, x_{\alpha_\lambda}) \neq \emptyset$. But then $\{(x_{\alpha_{\lambda+1}}, x_{\alpha_\lambda}) : \lambda \in \omega_1\}$ is a disjoint collection of uncountably many nonempty open subsets of $(X, \mathcal{T})$, contradicting separability of $(X, \mathcal{T})$.

□

Here we have an analogue of Theorem 2.23 for an arbitrary separable GO-space. It can be used to derive generalisations of Kunen's results in precisely the same way we did in Section 2.3. We illustrate this with some examples and then state the remaining results with guidance as to their proofs. The next lemma uses the second countability of $\mathcal{M}$ to prove that the $\mathcal{M}$ and $\mathcal{M} \vee \mathcal{T}_{\omega_1}$ topologies are 'close'.

Lemma 5.8 *If U is $\mathcal{M} \vee \mathcal{T}_{\omega_1}$-open, then there is an $\mathcal{M}$-open V such that $U \setminus V$ is nonstationary and $V \setminus U$ is countable.*
If H is $\mathcal{M} \vee \mathcal{T}_{\omega_1}$-closed, then there is an $\mathcal{M}$-closed K such that $K \setminus H$ is nonstationary and $H \setminus K$ is countable.

Proof Obviously the second result follows from the first.
Let $\{B_i : i \in \omega\}$ be a base for second countable $\mathcal{M}$. Partition U into disjoint sets $\{Q_i : i \in \omega\}$ such that for all $x_\alpha \in Q_i$, there is some $f(\alpha) < \alpha$ such that $x_\alpha \in (f(\alpha), \alpha] \cap B_i \subseteq U$. By the Pressing Down Lemma, $B_i \setminus U$ is countable whenever Q_i is stationary, so let $V = \bigcup\{B_i : Q_i \text{ is stationary}\}$. Then $U \setminus V$ is contained in $\bigcup\{Q_i : Q_i \text{ is nonstationary}\}$.

□

Lemma 5.7 shows that the $\mathcal{T} \vee \mathcal{T}_{\omega_1}$ and $\mathcal{M} \vee \mathcal{T}_{\omega_1}$ topologies are close, and the last lemma showed that the $\mathcal{M} \vee \mathcal{T}_{\omega_1}$ and $\mathcal{M}$ topologies are close. The next lemma is therefore no surprise:

Lemma 5.9 *If U is $\mathcal{T} \vee \mathcal{T}_{\omega_1}$-open, then there is an $\mathcal{M}$-open V such that $U \setminus V$ is nonstationary and $V \setminus U$ is countable.*
If H is $\mathcal{T} \vee \mathcal{T}_{\omega_1}$-closed, then there is an $\mathcal{M}$-closed K such that $K \setminus H$ is nonstationary and $H \setminus K$ is countable.

Proof Again, the second statement obviously follows from the first.
Let U be $\mathcal{T} \vee \mathcal{T}_{\omega_1}$-open. By Lemma 5.7, there exists $\tilde{U} \subseteq U$ such that $\tilde{U} \in \mathcal{M} \vee \mathcal{T}_{\omega_1}$ and $U \setminus \tilde{U}$ is nonstationary. But by Lemma 5.8, there exists an $\mathcal{M}$-open V such that $\tilde{U} \setminus V$ is nonstationary and $V \setminus \tilde{U}$ is countable. Now,

$$U \setminus V \subseteq (\tilde{U} \setminus V) \cup (U \setminus \tilde{U})$$

which is a union of two nonstationary sets, and hence nonstationary. Moreover $V \setminus U \subseteq V \setminus \tilde{U}$ and is therefore countable.

□

Lemma 5.10 *The following are equivalent:*

1. *$(X, \mathcal{M} \vee \mathcal{T}_{\omega_1})$ is ω_1-compact.*
2. *$(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is ω_1-compact.*
3. *All nonstationary $\mathcal{M}$-closed sets are countable.*
4. *All nonstationary $\mathcal{T}$-closed sets are countable.*
5. *All nonstationary $\mathcal{M} \vee \mathcal{T}_{\omega_1}$-closed sets are countable.*
6. *All nonstationary $\mathcal{T} \vee \mathcal{T}_{\omega_1}$-closed sets are countable.*

Proof (1)⇔(3)⇔(5) was proved in [Ku].
(2)⇒(6): Suppose there exists an uncountable nonstationary closed set H. Then there exists a club set C disjoint from H. But then we can let K be an uncountable subset of H such that any two distinct elements of K have an element of C between them. But then K is closed discrete, contradicting (2).
(6)⇒(2): Let $C \subseteq X$ closed discrete in $(X, \mathcal{M} \vee \mathcal{T}_{\omega_1})$. By Proposition 4.4, C is not stationary in ω_1, whence by (6), C is countable.
(4)⇒(3) is trivial. Any $\mathcal{M}$-closed set is $\mathcal{T}$-closed.

(6)⇒(4) is trivial.

(3)⇒(6): Suppose (3) holds. Let H be a nonstationary $\mathcal{T} \vee \mathcal{T}_{\omega_1}$-closed set. Then by the second part of Lemma 5.9, there exists $\mathcal{M}$-closed K such that $K \setminus H$ is nonstationary and $H \setminus K$ is countable. Now $K \subseteq H \cup (K \setminus H)$, so that K must be nonstationary, hence countable by (3). But $H \subseteq K \cup (H \setminus K)$, so that H must be countable too.

□

Theorem 5.11 *If $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is ω_1-compact, then $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is normal.*

Proof Let H_1 and H_2 be closed and disjoint. By Lemma 5.9, let K_1 and K_2 be $\mathcal{M}$-closed with $K_1 \setminus H_1$ and $K_2 \setminus H_2$ nonstationary, $H_1 \setminus K_1$ and $H_2 \setminus K_2$ countable. Then $K_1 \cap K_2$ is nonstationary and hence countable by Lemma 5.10, (2)⇒(3).
Let $\alpha < \omega_1$ be such that all elements of $H_1 \setminus K_1$, $H_2 \setminus K_2$ and $K_1 \cap K_2$ are below α. Then X is partitioned into $\mathcal{T}_{\omega_1}$-clopen sets $[0, \alpha]$ and $[\alpha + 1, \omega_1)$. Provided we can separate H_1 and H_2 on each of these, we're done.

- On $[0, \alpha]$, H_1 and H_2 can be separated because countable regular spaces are normal.
- On $[\alpha + 1, \omega_1)$, H_1 and H_2 can be separated because they are contained in disjoint $\mathcal{M}$-closed sets K_1 and K_2, which can be separated in $\mathcal{T}$ because $\mathcal{M} \subseteq \mathcal{T}$ and $\mathcal{T}$, being a GO-space topology, is hereditarily normal.

□

The remaining results of this section are direct generalisations of Theorems 2.33 to 2.38. The proofs generalise easily, by replacing $\mathcal{T}_{\mathbb{R}}$ by $\mathcal{M}$ throughout, and $\mathcal{T}_{\mathbb{S}}$ by $\mathcal{T}$ throughout.

Theorem 5.12 **(CH)** *There exists a closed unbounded subset D of $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ such that D is normal.*

Theorem 5.13 *$(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ is not perfectly normal.*

Lemma 5.14 *If G is a G_δ-set w.r.t. $\mathcal{T} \vee \mathcal{T}_{\omega_1}$, then there is a G_δ-set w.r.t. $\mathcal{M}$, H, such that $G \setminus H$ is nonstationary and $H \subseteq G$.*

Theorem 5.15 *The following are equivalent for $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$:*

1. $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ *is perfect.*

2. *Every club set is a* G_δ*-set w.r.t.* $\mathcal{T} \vee \mathcal{T}_{\omega_1}$.

3. *Every club set contains a club set which is a* G_δ*-set w.r.t.* $\mathcal{M}$.

Theorem 5.16 (CH) $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ *is not perfect.*

Theorem 5.17 (MA + ¬CH) $(X, \mathcal{T} \vee \mathcal{T}_{\omega_1})$ *is not normal.*

If these results can be generalised to some larger class of spaces, then it is unclear to the author how to do so. The proofs of Lemma 5.7 and of Proposition 4.16 both rely on some kind of order structure in the space, either directly, as in the case of all separable GO-spaces, or indirectly, as with the bow-tie space, where the real-line is embedded. Finding some kind of comparable order structure in, say, an arbitrary separable stratifiable or even arbitrary separable monotonically normal space would appear to be a difficult problem.

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