



# Rough Analysis and Stochastic Partial Differential Equations

Ruhong JIN  
St Catherine's College  
University of Oxford



A thesis submitted for the degree of  
DOCTOR OF PHILOSOPHY  
Supervisor: Prof. Rama Cont  
Trinity 2024

# Acknowledgements

The completion of this doctoral dissertation would not have been possible without the support and guidance of many individuals. I would like to take this opportunity to express my deepest gratitude to all those who have contributed to my journey.

First and foremost, I would like to thank my supervisor, Prof. Rama Cont, for his unwavering support, insightful guidance, and continuous encouragement throughout my research. Your expertise, patience, and dedication have been invaluable, and I am deeply grateful for the opportunity to learn from you.

I would also like to extend my sincere thanks to my collaborator, Prof. Nicolas Perkowski. Your collaboration and input have significantly enriched my research, and I am fortunate to have had the opportunity to work with you. Your feedback and suggestions have been instrumental in shaping this dissertation.

Additionally, I would like to acknowledge the contributions of several other individuals who have assisted me with discussions and provided valuable insights during my research. My heartfelt thanks go to Lukas Johannes Gräfner, Toyomu Matsuda, Sarah-Jean Meyer and Huanyu Yang for their thought-provoking discussions and helpful suggestions. Your input has been crucial in refining my ideas and improving the quality of my work.

I am profoundly grateful to my examiners, Prof. Massimiliano Gubinelli and Dr. Tommaso Rosati, for taking the time to review my work and provide constructive feedback. Your critical insights and recommendations have greatly improved the quality of this dissertation.

I would like to acknowledge the financial support from Mathematical Institute, University of Oxford, which made this research possible. Your generous funding has allowed me to focus on my studies and achieve my academic goals without financial burden. I am truly appreciative of your support.

Finally, I would like to express my heartfelt gratitude to my family. To my parents, who have always believed in me and encouraged me to pursue my dreams, your love and support have been my greatest source of strength. To my girl friend, Jiayi Li,

thank you for your patience, understanding, and unwavering support throughout this journey. Your constant encouragement and belief in me have been a source of inspiration.

This dissertation is dedicated to all of you. Thank you for being an integral part of this journey.

# Abstract

Rough analysis involves the study of systems governed by (partial) differential equations driven by irregular ('rough') signals. This thesis explores advancements in pathwise Itô formula with arbitrary rough signal, singular stochastic partial differential equations (SPDEs) and its applications.

In the first part of the research, with Prof. Rama Cont, we extend the Itô calculus to paths with finite  $p$ -variation, where  $p$  is any positive real number. This generalization builds upon previous work that handled integer values of  $p$ , introducing a fractional Itô calculus in a purely pathwise setting. The new framework includes the local Caputo fractional derivative and provides a comprehensive Itô formula applicable to a broader class of paths.

The second part of the research delves into singular SPDEs, focusing on applications in superprocess and fluid dynamics. In collaboration with Prof. Nicolas Perkowski, we investigate the compact support property of rough super Brownian motion and extend the model to include branching random walks with infinite variance off-spring distributions. Furthermore, we address fractional stochastic Landau-Lifshitz Navier-Stokes equations, demonstrating existence and uniqueness of energy solutions, and exploring the large-scale behavior of the systems in the super critical cases.

# Contents

<b>1</b>	<b>Overview</b>	<b>1</b>
<b>2</b>	<b>Fractional Ito-Föllmer calculus</b>	<b>6</b>
2.1	Introduction	6
2.2	Preliminaries	8
2.2.1	Pathwise calculus for paths with finite p-th variation	8
2.2.2	Fractional derivatives and fractional Taylor expansions	9
2.2.3	Local fractional derivative	11
2.3	Fractional Itô Calculus	15
2.3.1	Change of variable formula	15
2.3.2	Extension to path-dependent functionals and $\phi$ -variation	31
2.4	An isometry formula for the pathwise integral	36
2.5	Multi-dimensional extensions	39
<b>3</b>	<b>Compact Support property of Rough Super Brownian Motion</b>	<b>43</b>
3.1	Introduction	43
3.2	Notations	45
3.2.1	Notations on the regularity	45
3.2.2	Notations on weights	48
3.3	Rough Super Brownian Motion	51
3.4	Interior estimates	56
3.5	Proof of the main theorem	67
<b>4</b>	<b>A Generalized Rough Super Brownian Motion</b>	<b>73</b>
4.1	Introduction	73
4.2	Notation	75
4.2.1	Notation on regularity	75
4.2.2	Notation on operators	78
4.2.3	Other notations	79

4.3	Model Formulation . . . . .	79
4.4	On variants of the Parabolic Anderson Model . . . . .	83
4.4.1	Discrete case . . . . .	84
4.4.2	Continuous case . . . . .	89
4.5	Existence as limit of particle systems . . . . .	93
4.5.1	For the case $\varkappa < \frac{2}{1+\beta}$ . . . . .	93
4.5.2	For the general $\varkappa > \frac{2}{1+\beta}$ . . . . .	100
4.6	Martingale problem . . . . .	101
4.7	Properties of rough Super Brownian Motion . . . . .	106
<b>5</b>	<b>Fractional stochastic Landau-Lifshitz Navier-Stokes equations in dimension <math>d \geq 3</math>: Existence and (non-)triviality</b> . . . . .	<b>110</b>
5.1	Introduction . . . . .	110
5.2	Notation and preliminaries . . . . .	113
5.2.1	Fourier Analysis . . . . .	113
5.2.2	Gaussian Analysis . . . . .	114
5.3	Infinitesimal generator of the truncated equation . . . . .	116
5.4	Existence and uniqueness of energy solutions for $\theta > \frac{d}{2}$ . . . . .	126
5.5	Solution on the whole space for $\theta \leq 1$ and $N < \infty$ . . . . .	130
5.6	Large-scale behavior for $\theta \leq 1$ . . . . .	131
5.6.1	Limiting behaviour . . . . .	131
5.6.2	Implication for fluctuating hydrodynamics . . . . .	136
<b>A</b>	<b>Appendix of Chapter 2</b> . . . . .	<b>139</b>
A.1	Properties of Fractional Derivatives . . . . .	139
A.2	Proof of Corollary 2.2.14 . . . . .	141
<b>B</b>	<b>Appendix of Chapter 3</b> . . . . .	<b>142</b>
B.1	Proofs of technical lemmas on interior estimates . . . . .	142
<b>C</b>	<b>Appendix of Chapter 4</b> . . . . .	<b>151</b>
C.1	Integer-valued random measures . . . . .	151
C.2	Canonical decomposition of a semi-martingale and Itô's formula . . . . .	153
C.3	Lemmas for Branching random walk . . . . .	154
C.4	Auxiliary lemmas . . . . .	155
	<b>Bibliography</b> . . . . .	<b>158</b>

# Chapter 1

## Overview

Rough analysis may be defined as the analysis of systems governed by (partial) differential equations driven by irregular ('rough') signals. It originated from the discovery of the Rough path theory by Terry Lyons [70, 69], where he realized that, to make sense of controlled differential equations, we need to lift the paths of the driving process to certain graded objects. This approach was further developed in [46, 47] using a *local* viewpoint by Massimiliano Gubinelli where he constructed integrals with respect to reference rough path by introducing the idea of *controlled path*. This idea was then generalized to the world of distributions to build solutions for certain sub-critical singular Stochastic Partial Differential Equations (SPDEs) by Hairer [54] for his regularity structure theory and Gubinelli-Imkeller-Perkowski [49] for paracontrolled distributions.

These theories try to provide structures or building blocks to build desired integrals or products. An early attempt in this direction was Hans Föllmer's [38] pathwise approach to Itô calculus, where he used increments along partitions instead of signatures. This was further developed and extended by Cont and Perkowski [24] to rougher paths with finite  $p$ -th order variation when  $p$  is an integer. In the first part of my doctoral research I extended these results to the case where  $p$  is not an integer [23] and developed a 'fractional Ito calculus' in a purely pathwise setting.

In the second part of my doctoral research, I have explored applications of rough analysis to the study of singular SPDEs, in collaboration with Prof. Nicolas Perkowski, with a focus on the application of Parabolic Anderson Model (PAM) to the Branching Random Walk in Random Environment (BRWRE) as well as a probabilistic approach to solving singular SPDEs related to fluid dynamics.

## Pathwise Itô Calculus

The Itô formula states that for any  $C^2$  function  $f$  and a semi-martingale  $S : [0, T] \rightarrow \mathbb{R}$ , one can obtain a decomposition

$$f(S(t)) - f(S(0)) = \int_0^t f'(S(u))du + \frac{1}{2} \int_0^t f''(S(u))d[S](u).$$

Here in the formula, only the quadratic part  $[S]$  is determined in the stochastic sense and Föllmer pointed out that the path-wise definition of this term leads to a path-wise Itô calculus, which has been used for model-free formulation in continuous-time finance. It is also related to rough path theory and can be constructed using a canonical rough path associated with  $S$  (see Remark 2.2.4 and [24]).

Comparing with the rough path theory where one can build theories on very general path, the purpose of the research is trying to investigate the possibility of generalizing the above formula to more general paths.

In collaboration with Rama Cont, we derive a general Itô formula for paths with finite  $p$ th-variation, where  $p \geq 1$  may be any real number, extending previous results obtained in [24] in the integer case. The Itô remainder term is shown to be zero in general when  $p$  is not an integer but sometimes can also have non-zero remainder term. We use a local version of the Caputo fractional derivative.

## Singular SPDEs

Singular SPDEs arise in many physical systems, for example in random surface growth which is modeled via the Kardar-Parisi-Zhang (KPZ) equation:

$$\partial_t h = \Delta h + |\nabla h|^2 - \infty + \xi, \tag{1.0.1}$$

and the intermittency model for PAM

$$\partial_t u = \Delta u + \eta u - \infty u, \tag{1.0.2}$$

where  $\xi$  is the space time white noise and  $\eta$  is a spatial white noise. The 'singular' term before the SPDEs means in general, we need to do some renormalization for the equation. For example the infinity term in equations (1.0.1) and (1.0.2).

The difficulty of solving the singular SPDEs lies in the difficulty in defining products of distributions. It is well-known that in order to define a product of two distribution, the sum of their Hölder exponents should be positive. However, it is very common in the singular SPDEs that the sum of their Hölder exponents is strictly smaller than 0. So we have to specify the meaning of the products that appear in the equation. It

is the same in the SDE case since the Hölder regularity of  $\frac{dB_t}{dt}$  is only  $-\frac{1}{2}-$ , where  $B_t$  is the Brownian motion. The integrand of the Itô integral can be viewed as time integral of products of  $\frac{dB_t}{dt}$  and some functional of  $B_t$ , which is of  $\frac{1}{2}-$  regularity. In fact, the Itô integral actually simply gives a meaning of this product and it leads to the well-known Itô integral.

Four different approaches have been proposed to solve such singular SPDEs:

- The regularity structure theory introduced by Martin Hairer [54], where he builds the local approximation of solutions using tree-like objects.
- The paracontrolled distribution method introduced by Gubinelli, Imkeller and Perkowski [49], where they used the bony product to decompose the products and build the required data for the equation.
- Energy solution or the probability construction from (cylindrical) martingale problem introduced by Gubinelli and Jara [50]. It is then further developed in [51, 45] to a general situation. The main idea is to build the infinitesimal generator of the singular SPDEs in the  $L^2$  space using Malliavin Calculus.
- The renormalization method introduced by Kupiainen [62] and the flow equation method introduced by Duch [28], where they build the effective equations or effective force on different scales. This method doesn't need to define the products of two distributions as the probabilistic one and the renormalization term will occur automatically.

The above four methods have their own pros and cons and are all constrained in the sub-critical case. My research investigates paracontrolled distribution method and the energy solution method with different models.

### Parabolic Anderson Model and Rough Super Brownian Motion

In the context of branching processes, the limit of empirical measures of some scaled branching random walks or branching diffusion processes are called super processes. The description of the super processes always goes to the Laplace functional, which relates to the solution of PDEs in the sense

$$\mathbb{E}[e^{-\langle \mu_t, \varphi \rangle}] = e^{-\langle \mu_0, U_t(\varphi) \rangle}, \quad (1.0.3)$$

where  $U_t(\varphi)$  is a solution to some PDEs. By introducing white noise as the random environment for the branching processes, Perkowski and Rosati [78] generalized the

super Brownian Motion to the rough super Brownian Motion, where  $U_t(\varphi)$  in equation (1.0.3) is given by non-linear PAM.

The intuition of the relation of white noise environment and PAM comes from the paper [40], where they relate the lattice Parabolic Anderson Model to the mass of the branching random walk with a homogeneous branching random environment. Also there is a further result in [2] which shows the relation between higher order moments and the Anderson Hamiltonian  $\mathcal{H} := \Delta + \xi$ . The relation of the Laplace function and the moments then suggests some hidden relation between PAM and the measure itself.

In collaboration with Nicolas Perkowski, we investigate the compact support property of the rough super Brownian Motion and then I generalize the rough super Brownian Motion to some branching random walk whose off-spring distribution has infinite variance. The tool we used is the paracontrolled distributions.

### Fractional stochastic Landau-Lifshitz Navier-Stokes equations

The fluctuating hydrodynamics equation of Landau and Lifshitz [5, 63] is given by

$$\partial_t u = \nu \Delta u - \nabla p - \hat{\lambda} \operatorname{div}(u \otimes u) + \nabla \cdot \tau, \quad \nabla \cdot u = 0,$$

with noise term given by  $\nabla \cdot \tau$ , where  $\tau$  is a Gaussian field with covariance

$$\mathbb{E}[\tau_{ij}(t, x)\tau_{kl}(t', x')] = \frac{2\nu k_B T}{\rho} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}) \delta(x - x') \delta(t - t'),$$

where  $k_B$  is the Boltzmann constant,  $T$  is the temperature, and  $\rho$  is the density. By a direct calculation, it can be seen that above equation is equivalent to the equation in the distribution sense with  $\theta = 1$ ,

$$\partial_t u = -(-\Delta)^\theta u - \nabla p - \lambda \operatorname{div}(u \otimes u) + \sqrt{2}(-\Delta)^{\frac{\theta}{2}} \xi, \quad \nabla \cdot u = 0,$$

Above equation is not well-posed in the classical case and requires techniques developed in the study of singular SPDEs. In collaboration with Nicolas Perkowski, we use the energy method to investigate the martingale formulation of above equation and also the scaling limit at the critical exponent.

### Outline

The thesis is structured into four chapters following this one.

Chapter 2 derives Itô-type change of variable formulas for smooth functionals of irregular paths with non-zero  $p$ -th variation along a sequence of partitions, where  $p \geq 1$  is arbitrary (in particular non-integer). Using fractional derivative operators,

we extend results obtained by Cont and Perkowski [24] in the case  $p \in \mathbb{N}$  to the non-integer case. These results are then extended to functionals of paths with non-zero  $\phi$ -variation and multi-dimensional paths. An isometry property for the pathwise Föllmer integral in terms of  $\phi$ -variation is also proved. This chapter is based on the paper [23].

Chapter 3 proves the compact support property of the rough super Brownian motion constructed in [78] using some interior estimation methods from [73, 74].

Chapter 4 then generalize the rough super Brownian motion to some branching random walk whose off-spring distribution has infinite variance. The idea of proof comes from [27, 26] and is quite different from the techniques in [78] since the resulting super processes are not continuous and not in  $L^2$ , which prevents the use of most square-integrable martingale methods. We also show the compact support property of the generalized rough super Brownian motion with the method in Chapter 3 and the super exponentially persistence property.

Chapter 5 investigate fractional stochastic Navier-Stokes equations in  $d \geq 3$ , driven by the random force  $(-\Delta)^{\frac{\theta}{2}}\xi$ . We obtain the existence and uniqueness of martingale solutions on the torus  $\mathbb{T}^d$  for  $\theta > \frac{d}{2}$ . For  $\theta \leq 1$  the equation is supercritical and we regularize the problem by introducing a Galerkin approximation and we study the large scale behavior of the truncated model on  $\mathbb{R}^d$ . We show that the nonlinear term in the Galerkin approximation vanishes on large scales when  $\theta < 1$  and the model converges to the linearized equation. For  $\theta = 1$  the nonlinear term gives a nontrivial contribution to the large scale behaviour, and we conjecture that the large scale behavior is given by a linear model with strictly larger effective diffusivity compared to simply dropping the nonlinear term. The effective diffusivity is explicitly given in terms of the model parameters.

# Chapter 2

## Fractional Ito-Föllmer calculus

### 2.1 Introduction

Hans Föllmer derived [38] a pathwise formulation of the Itô formula and laid the grounds for the development of a pathwise approach to Itô calculus, which has been developed in different directions [3, 6, 16, 17, 19, 24, 20, 25, 68, 84].

Föllmer's original approach focuses on functions of paths with finite *quadratic* variation along a sequence of partitions. In a recent work Cont and Perkowski [24] extended the Föllmer-Itô formula [38] to function(al)s of paths with variation of order  $p \in 2\mathbb{N}$  along a sequence of partitions and obtained functional change of variable formulas, applicable to functionals of fractional Brownian motion and other fractional processes with arbitrarily low regularity (i.e. any Hurst exponent  $H > 0$ ). These results involve pathwise integrals defined as limits of compensated left Riemann sums, which are in turn related to rough integrals associated with a "reduced" rough path [24].

As the notion of  $p$ -th order variation may be defined for any  $p > 0$ , an interesting question is to investigate how the results in [24] extend to 'fractional' case  $p \notin \mathbb{N}$ . In particular one may ask whether the change of variable formula contains a fractional remainder term in this case and whether the definition of the compensated integral needs to be adjusted.

We investigate these questions using the tools of fractional calculus [83]. Given that fractional derivative operators are (non-local) integral operators, one challenge is to obtain non-anticipative, 'local' formulas which have similar properties to those obtained in the integer case [24]. We are able to do so using a 'local' notion of fractional derivative and exhibit conditions under which these change of variable formulas contain (or not) a 'fractional Itô remainder term'. In most cases there is no

remainder term; we also discuss some cases where a non-zero remainder term appears and give a representation for this term.

These results are first derived for smooth functions then extended to functionals, using the concept of vertical derivative [22]. We extend these results to the case of paths with finite  $\phi$ -variation [57] for a class of functions  $\phi$  and we obtain an isometry formula for the pathwise integral in terms of  $\phi$ -variation, extending the results of [3, 24] to the fractional case. Finally, we extend these results to the multi-dimensional case.

Our change of variable formulas are purely analytical and pathwise in nature, but applicable to functionals of fractional Brownian motions and other fractional processes with arbitrary Hurst exponent, leading in this case to non-anticipative 'Itô' formulas for functionals of such processes. However, as probabilistic assumptions play no role in the derivation of our results, we have limited the discussion of such examples.

Zähle [86] defined a pathwise integral using a Young-type condition based on fractional regularity of the integrand and the integrator. In the case of Hölder continuous functions this corresponds to Hölder exponents with sum strictly greater than 1 i.e. a Young-type condition. Our approach extends beyond the domain of validity of Young integration and we are able to treat borderline cases where the sum of Hölder exponents is one.

**Outline** Section 2.2 recalls some results on pathwise calculus for functions of irregular paths (Sec. 2.2.1) and fractional derivative operators and associated fractional Taylor expansions (Sec. 2.2.2). Section 2.3 contains our main results on change of variable formulas for function(al)s of paths with fractional regularity. We first give a change of variable formula without remainder term for time-independent functions (Theorem 2.3.3), followed by a discussion of an example where a remainder term may appear (Example 2.3.5). We then provide a formula for computing this fractional remainder term using an auxiliary space (Theorem 2.3.7). Section 2.3.2 extends these results to the case of path-dependent functionals using the Dupire derivative, to the case where the  $p$ -th variation is replaced by the more general concept of  $\phi$ -variation [57]. In Section 2.4 we derive a pathwise isometry formula extending a result of [3] to the case of  $\phi$ -variation. Finally, in Section 2.5 we discuss extensions to the multidimensional case. These extensions are not immediate, as the space  $V_p(\pi)$  is not a vector space.

## 2.2 Preliminaries

### 2.2.1 Pathwise calculus for paths with finite $p$ -th variation

We define, following [24, 38], the concept of  $p$ -th variation along a sequence of partitions  $\pi_n = \{t_0^n, \dots, t_{N(\pi_n)}^n\}$  with  $t_0^n = 0 < \dots < t_k^n < \dots < t_{N(\pi_n)}^n = T$ . Define the *oscillation* of  $S \in C([0, T], \mathbb{R})$  along  $\pi_n$  as

$$\text{osc}(S, \pi_n) := \max_{[t_j, t_{j+1}] \in \pi_n} \max_{r, s \in [t_j, t_{j+1}]} |S(s) - S(r)|.$$

We write  $[t_j, t_{j+1}] \in \pi_n$  to indicate that  $t_j$  and  $t_{j+1}$  are immediate successors in  $\pi_n$  (i.e.  $t_j < t_{j+1}$  and  $\pi_n \cap (t_j, t_{j+1}) = \emptyset$ ).

**Definition 2.2.1** ( $p$ -th variation along a sequence of partitions). *Let  $p > 0$ . A continuous path  $S \in C([0, T], \mathbb{R})$  is said to have a  $p$ -th variation along a sequence of partitions  $\pi = (\pi_n)_{n \geq 1}$  if  $\text{osc}(S, \pi_n) \rightarrow 0$  and the sequence of measures*

$$\mu^n := \sum_{[t_j, t_{j+1}] \in \pi_n} \delta(\cdot - t_j) |S(t_{j+1}) - S(t_j)|^p$$

*converges weakly to a measure  $\mu$  without atoms. In that case we write  $S \in V_p(\pi)$  and  $[S]^p(t) := \mu([0, t])$  for  $t \in [0, T]$ , and we call  $[S]^p$  the  $p$ -th variation of  $S$ .*

**Remark 2.2.2.** Functions in  $V_p(\pi)$  do not necessarily have finite  $p$ -variation in the usual sense. Recall that the  $p$ -variation of a function  $f \in C([0, T], \mathbb{R})$  is defined as [29]

$$\|f\|_{p\text{-var}} := \left( \sup_{\pi \in \Pi([0, T])} \sum_{[t_j, t_{j+1}] \in \pi} |f(t_{j+1}) - f(t_j)|^p \right)^{1/p},$$

where the supremum is taken over the set  $\Pi([0, T])$  of all partitions  $\pi$  of  $[0, T]$ . A typical example is the Brownian motion  $B$ , which has quadratic variation  $[B]^2(t) = t$  along any refining sequence of partitions almost surely while at the same time having infinite 2-variation almost surely [29, 85]:

$$\mathbb{P}(\|B\|_{2\text{-var}} = \infty) = 1.$$

If  $S \in V_p(\pi)$  and  $q > p$ , then  $S \in V_q(\pi)$  with  $[S]^q \equiv 0$ .

$S \in C([0, T], \mathbb{R})$  belongs to  $V_p(\pi)$  if and only if there exists a continuous function  $[S]^p$  such that

$$\forall t \in [0, T], \quad \sum_{\substack{[t_j, t_{j+1}] \in \pi_n: \\ t_j \leq t}} |S(t_{j+1}) - S(t_j)|^p \xrightarrow{n \rightarrow \infty} [S]^p(t). \quad (2.2.1)$$

If this property holds, then the convergence in (2.2.1) is uniform.

**Example 2.2.3.** *If  $B$  is a fractional Brownian motion with Hurst index  $H \in (0, 1)$  and  $\pi_n = \{kT/n : k \in \mathbb{N}_0\} \cap [0, T]$ , then  $B \in V_{1/H}(\pi)$  and  $[B]^{1/H}(t) = t\mathbb{E}[|B_1|^{1/H}]$ , see [79, 82].*

For  $p \in 2\mathbb{N}$ , the following change of variable formula for  $f \in C^p(\mathbb{R}, \mathbb{R})$  was shown in [24]:

$$\forall S \in V_p(\pi), \quad f(S(t)) - f(S(0)) = \int_0^t f'(S(s))dS(s) + \frac{1}{p!} \int_0^t f^{(p)}(S(s))d[S]^p(s),$$

where the integral

$$\int_0^t f'(S(s))dS(s) := \lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!} (S(t_{j+1} \wedge t) - S(t_j \wedge t))^k \quad (2.2.2)$$

is defined as a (pointwise) limit of compensated Riemann sums.

**Remark 2.2.4** (Relation with Young integration and rough integration). As p-variation can be infinite for  $S \in V_p(\pi)$ , the pathwise integral appearing in the formula cannot be defined as a Young integral. Compensated Riemann sums such as (2.2.2) also appear in the construction of ‘rough integrals’ [39, 46]. For  $X \in C^\alpha([0, T], \mathbb{R})$  with  $\alpha \in (0, 1)$ ,  $q = \lfloor \alpha^{-1} \rfloor$  one can define the pathwise integral (2.2.2) as a rough integral with respect to a ‘reduced’ rough path  $(\mathbb{X}_{s,t}^1, \mathbb{X}_{s,t}^2, \dots, \mathbb{X}_{s,t}^q)_{0 \leq s \leq t \leq T}$ , where  $\mathbb{X}_{s,t}^k := (X(t) - X(s))^k/k!$ .

## 2.2.2 Fractional derivatives and fractional Taylor expansions

Several different notions of fractional derivatives exist in the literature [65, 80, 83] and it is not clear which ones are the right tools for a given context. Our goal here is to shed some light on the advantages of different notions of fractional derivative. Much of this material may be found in the literature [83]. We have provided proofs for some useful properties whose proof we have not been able to find in the literature.

**Definition 2.2.5** (Riemann-Liouville fractional integral). *Let  $\alpha > 0$ . The left Riemann-Liouville fractional integral of order  $\alpha$  is defined by*

$$I_{a+}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

for real functions  $f$  for which the integral is well-defined for  $x > a \in \mathbb{R}$ . Similarly, the right Riemann-Liouville fractional integral is given by

$$I_{b-}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt$$

for  $x < b \in \mathbb{R}$ .

**Remark 2.2.6.** *The fractional integral is always well-defined for  $f \in L^1(a, b)$  with  $I_{a+}^\alpha f \in L^1(a, b)$ . This allows for blow up at the boundaries  $a, b$ .*

This may be used to define a (non-local) fractional derivative associated with some base point [65, 80]:

**Definition 2.2.7** (Riemann-Liouville fractional derivative). *Let  $a < b$ . Let  $f \in L^1([a, b])$  and  $n \leq \alpha < n + 1$  for some integer  $n \in \mathbb{N}$ . Then left Riemann-Liouville fractional derivative of order  $\alpha$  with base point  $a$  on  $[a, b]$  is defined by*

$$D_{a+}^\alpha f := \left( \frac{d}{dx} \right)^{n+1} I_{a+}^{n+1-\alpha} f$$

*if  $n$ -th order derivative of  $I_{a+}^{n+1-\alpha} f$  exists and is absolutely continuous on  $[a, b]$ . Similarly, the right Riemann-Liouville fractional derivative of order  $\alpha$  with base point  $b$  on  $[a, b]$  is given by*

$$D_{b-}^\alpha f := \left( -\frac{d}{dx} \right)^{n+1} I_{b-}^{n+1-\alpha} f.$$

**Remark 2.2.8.** *When  $\alpha$  is an integer, the fractional derivative coincides with the classical derivative, i.e.  $D_{a+}^n f = f^{(n)}$ .*

The Riemann-Liouville derivative has several shortcomings. One of them is that the fractional derivative of a constant is not zero. To overcome this, one can consider a modification of Riemann-Liouville fractional derivative which is called the *Caputo derivative*.

**Definition 2.2.9** (Caputo derivative). *Suppose  $f$  is a real function and  $n+1 \geq \alpha > n$ . We define the left and right Caputo fractional derivatives of order  $\alpha$  at  $x \in (a, b)$  by*

$$C_{a+}^\alpha f(x) = D_{a+}^\alpha \left[ f(t) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (t-a)^k \right] (x),$$

$$C_{b-}^\alpha f(x) = D_{b-}^\alpha \left[ f(t) - \sum_{k=0}^n \frac{(-1)^k f^{(k)}(b)}{k!} (b-t)^k \right] (x)$$

*if exists. Here  $f^{(k)}$  denotes the  $k$ -th derivative of  $f$ .*

We provide some basic properties of fractional derivatives in the appendix for the completeness. Associated with the Caputo derivative is a fractional Taylor expansion:

**Proposition 2.2.10.** *Let  $f \in C^n([a, b])$  and  $n + 1 \geq \alpha > n$ . If  $f$  admits a Caputo derivative of order  $\alpha$  on  $[a, b]$  and  $C_{a+}^\alpha f \in L^1([a, b])$ , then we have*

$$f(x) = \sum_{k=0}^n f^{(k)}(a) \frac{(x-a)^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_a^x C_{a+}^\alpha f(t) (x-t)^{\alpha-1} dt$$

and

$$f(x) = \sum_{k=0}^n f^{(k)}(b) \frac{(x-b)^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_x^b C_{b-}^\alpha f(t) (t-x)^{\alpha-1} dt.$$

As we were not able to find a proof of this expansion in the literature, we provide a detailed proof in the appendix.

### 2.2.3 Local fractional derivative

The above derivative operators are non-local operators. We now introduce the concept of *local* fractional derivative:

**Definition 2.2.11** (Local fractional derivative). *Suppose  $f$  is left fractional differentiable of order  $\alpha$  on  $[a, a + \delta]$  for some positive  $\delta$ , then the left local fractional derivative of order  $\alpha$  of function  $f$  at point  $a$  is given by*

$$f^{(\alpha+)}(a) = \lim_{y \rightarrow a, y \geq a} C_{a+}^\alpha(y),$$

when the limit exists. We can similarly define the right local fractional derivative of order  $\alpha$  of function  $f$  at a point  $a$  by

$$f^{(\alpha-)}(a) = \lim_{y \rightarrow a, y \leq a} C_{a-}^\alpha(y),$$

when the limit exists.

**Example 2.2.12.** *We give a simple example of Caputo fractional derivative and local derivative here. Consider  $f(x) = |x|^\alpha$  and  $0 < \beta \leq \alpha < 1$ . Then we have*

$$\begin{aligned} & C_{a+}^\beta f(x) \\ &= \frac{d}{dx} \left( \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{|t|^\alpha - |a|^\alpha}{(x-t)^\beta} dt \right) \\ &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \left( \int_a^x |t|^\alpha (x-t)^{-\beta} dt - |a|^\alpha \frac{(x-a)^{1-\beta}}{1-\beta} \right) \\ &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \left| \int_{\frac{a}{x}}^1 |t|^\alpha |1-t|^{-\beta} dt |x|^{\alpha-\beta+1} \right| - \frac{1}{\Gamma(1-\beta)} |a|^\alpha (x-a)^{-\beta} \\ &= \frac{1}{\Gamma(1-\beta)} \left( |a|^\alpha (x-a)^{-\beta} \frac{a-x}{x} + (\alpha - \beta + 1) |x|^{\alpha-\beta} \int_{\frac{a}{x}}^1 |t|^\alpha |1-t|^{-\beta} dt \right). \end{aligned}$$

So we can see directly that  $f^{(\beta+)}(a) = 0$  for any  $a \neq 0$  or  $\beta < \alpha$  but  $f^{(\alpha+)}(0) = \Gamma(\alpha+1)$ . It can be seen further that for all  $a \geq 0$ ,  $C_{a+}^{\alpha}f$  is continuous on  $[a, \infty]$  but for  $a < 0$ ,  $C_{a+}^{\alpha}f$  has singularity at point 0. In particular, for  $\beta = \alpha$ , we have

$$C_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(1-\beta)} \int_{\frac{a}{x}}^1 g(t) - g\left(\frac{a}{x}\right) dt \quad \text{with} \quad g(t) = \left| \frac{t}{1-t} \right|^{\alpha}.$$

**Remark 2.2.13.** Using integration by parts formula, we actually have

$$D_{a+}^{\alpha}f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^{k-\alpha-n}}{\Gamma(k+2-\alpha)} + D_{a+}^{(\alpha-n)}f^{(n)}(x).$$

Hence  $C_{a+}^{\alpha}f(x) = C_{a+}^{(\alpha-n)}f^{(n)}(x)$  so the existence of one side will imply the existence of the other side. Taking limits yields  $f^{(\alpha+)}(x) = (f^{(n)})^{((n-\alpha)+)}(x)$ . This is important in the proofs below.

**Corollary 2.2.14** (Fractional Taylor formula). *Let  $n+1 \geq \alpha > n$  and  $f \in C^n([a, b])$  admitting a left (resp. right) local fractional derivative of order  $\alpha$  at  $a$ . There for  $x \in [a, b]$*

$$f(x) = \sum_{k=0}^n f^{(k)}(a) \frac{(x-a)^k}{k!} + \frac{1}{\Gamma(\alpha+1)} f^{(\alpha+)}(a)(x-a)^{\alpha} + o(|x-a|^{\alpha}) \quad (2.2.3)$$

$$f(x) = \sum_{k=0}^n f^{(k)}(b) \frac{(x-b)^k}{k!} + \frac{1}{\Gamma(\alpha+1)} f^{(\alpha-)}(b)(b-x)^{\alpha} + o(|x-b|^{\alpha}). \quad (2.2.4)$$

The proof is given in Appendix A.2. A similar mean value theorem holds for the non-local fractional derivative. The following result, which we state for completeness, is a consequence of Proposition 2.2.10:

**Corollary 2.2.15.** *Let  $f \in C^n([a, b])$  and  $n+1 \geq \alpha > n$ . Suppose Caputo fractional derivative of order  $\alpha$  of  $f$  is continuous on  $[a, b]$ . Then for  $x \in [a, b]$ , there exists  $\xi \in [a, x]$  such that*

$$f(x) = \sum_{k=0}^n f^{(k)}(a) \frac{(x-a)^k}{k!} + \frac{1}{\Gamma(\alpha+1)} C_{a+}^{\alpha}f(\xi)(x-a)^{\alpha}.$$

A similar formula holds for the Caputo right derivative.

**Proposition 2.2.16.** *Let  $\alpha \notin \mathbb{N}$  and  $f \in C^{\alpha}([a, b])$ . If  $f^{(\alpha+)}$  exists everywhere on  $[a, b]$  then  $f^{(\alpha+)} = 0$  almost everywhere on  $[a, b]$ .*

A rather complex proof of this proposition is given in [15, Corollary 3]. We here give a simple proof of this property using only properties of monotone functions.

*Proof.* We first consider the case  $0 < \alpha < 1$ . By corollary 2.2.14, we actually have

$$f(y) = f(x) + g(x)(y - x)^\alpha + o(|y - x|^p) \quad \text{with} \quad g = \frac{1}{\Gamma(\alpha + 1)} f^{(\alpha+)}.$$

Hence for any sequence  $y_n \rightarrow x, y_n > x$ , we have

$$\lim_{n \rightarrow \infty} \frac{f(y_n) - f(x)}{(y_n - x)^\alpha} = g(x).$$

Now let  $E^+ = \{x \in \mathbb{R} : g(x) > 0\}$ . If  $\mathcal{L}(E^+) > 0$ , then there exists a compact set  $K \in E^+$  with  $\mathcal{L}(K) > 0$  where  $\mathcal{L}$  denotes the Lebesgue measure on the real line.

Consider the open set  $\cup_{x \in K}(x, x + \delta_x)$  in  $\mathbb{R}$ , where  $\delta_x$  is the largest number so that  $f(y) > f(x), \forall y \in (x, x + \delta_x)$ . We can then write  $\cup_{x \in K}(x, x + \delta_x) = \cup_{k=1}^\infty I_k$  for some open interval  $I_k$ . There exists  $I_r$  such that  $\mathcal{L}(K \cap \bar{I}_r) > 0$ . In fact, for each point in  $K$ , it is either in the set  $\cup_{k=1}^\infty I_k$  or it is a boundary point of some interval  $I_k$ . We may augment  $I_k$  to an interval  $\tilde{I}_k$  including zero or one of both of its boundary points so that  $\tilde{I}_k$  are disjoint and  $K \subset \cup_{k=1}^\infty \tilde{I}_k$ , and such that the right boundary point is not in  $\tilde{I}_k$  if it belongs to  $K$ . Then we find  $I_r$  such that  $\mathcal{L}(K \cap \tilde{I}_r) > 0$ . Hence  $\mathcal{L}(K \cap \bar{I}_r) > 0$ . We may also assume that there are no points in  $K$  isolated from the right, since the set of all such points is of measure zero.

For any  $\bar{x} < \bar{y} \in K \cap \bar{I}_r$ , there exists  $x_0 \in K \cap [\bar{x}, \bar{y}]$  such that

$$x_0 = \operatorname{argmax}_{x \in K \cap [\bar{x}, \bar{y}]} \{f(x)\}.$$

If  $x_0 \neq \bar{y}$ , then there exists  $y_0 \in (x_0, x_0 + \delta_{x_0}) \cap K \cap [\bar{x}, \bar{y}]$  since we assume there are no right-isolated points in  $K$ . Then we have  $f(y_0) > f(x_0)$ , which is a contradiction. Hence  $x_0 = \bar{y}$  and we have  $f(\bar{y}) > f(\bar{x}), \forall \bar{x} < \bar{y} \in K \cap \bar{I}_r$ .

Now define  $\bar{f} : \bar{I}_r \rightarrow \mathbb{R}$  such that  $\bar{f} = f$  on  $K \cap \bar{I}_r$  and  $\bar{f}$  is linear outside of  $K \cap \bar{I}_r$ . In fact, let  $\tilde{x} \in \bar{I}_r$ , if  $\tilde{x} \notin K$ , then  $\tilde{x} \in (z_0, z_1)$  with  $z_0, z_1 \in K$ . Thus  $\bar{f}(z_1) \geq \bar{f}(z_0)$ , and we can linearly interpolation to define  $\bar{f}(\tilde{x})$  as an increasing function. Hence this extension  $\bar{f}$  is differentiable almost everywhere on  $\bar{I}_r$ .

Now we go back to the function  $f$  on set  $K$ .  $\bar{f}$  is a.e differentiable on  $K$  and we can simply suppose here  $\bar{f}$  is differentiable and  $f = \bar{f}$  on  $K$  whose Lebesgue measure is positive. Now choose a point  $x_1 \in K$ , either  $x_1$  is right-isolated in  $K$  which is eliminated from  $K$  before or it is a right-accumulation point in  $K$ . For the latter case, we would have

$$g(x_1) = \lim_{n \rightarrow \infty} \frac{f(y_n) - f(x_1)}{(y_n - x_1)^p} = \lim_{n \rightarrow \infty} \frac{\bar{f}(y_n) - \bar{f}(x_1)}{(y_n - x_1)^p} = 0,$$

for any sequence  $\{y_n\}$  in  $K$  with  $y_n \rightarrow x_1$ . We then get  $g = 0$  a.e. on  $K$ , which is a contradiction. Hence  $\mathcal{L}(E^+) = 0$ . Similarly,  $\mathcal{L}(E^-) = 0$ . Hence  $f^{(p^+)} = 0$  a.e. on  $\mathbb{R}$ .

If  $m < \alpha < m + 1$  for some integer  $m > 0$ , since we have  $f^{(\alpha^+)} = (f^{(m)})^{(\alpha-m^+)}$ , we may conclude again that  $f^{(\alpha^+)}(x) = 0$  a.e. on  $\mathbb{R}$ .  $\square$

**Remark 2.2.17.** For the above proposition we actually only need a weaker condition on the function  $f$ , namely that

$$\lim_{y \rightarrow x, y > x} \frac{f^{(m)}(y) - f^{(m)}(x)}{|y - x|^\beta}$$

exists for  $x \in \mathbb{R}$  with  $m = \lfloor \alpha \rfloor$  and  $\alpha = m + \beta$ , which may be thought as classical definition of  $\alpha$ -th order derivative. We call

$$\lim_{x \rightarrow a, x > a} \frac{f^{(m)}(x) - f^{(m)}(a)}{|x - a|^\beta}$$

the 'classical' left fractional derivative of order  $\alpha$  if it exists. However, when this limit exists, it is not guaranteed that  $f$  admits a local fractional derivative.

We can actually state a stronger result:

**Proposition 2.2.18.** For  $f \in C^\alpha$ , the Hausdorff dimension of the set

$$E_f = \left\{ x \in \mathbb{R} : \liminf_{y \rightarrow x, y > x} \frac{|f^{(m)}(y) - f^{(m)}(x)|}{|y - x|^\beta} > 0 \right\}$$

is at most  $\beta = \alpha - \lfloor \alpha \rfloor$ .

*Proof.* WLOG, we suppose  $m = 0$  and  $0 < \alpha < 1$ , then  $\beta = \alpha$ . For each  $\epsilon, \delta$ , we define the set  $E_{\epsilon+}^\delta$  to be the subset of  $E$  such that for all  $x \in E_{\epsilon+}^\delta$ , we have

$$f(y) > f(x) + \epsilon(y - x)^\alpha, \forall x < y < x + \delta.$$

We only need to show the Hausdorff dimension of the set  $E_{\epsilon+}^\delta$  is at most  $\alpha$ . Furthermore, we can restrict the set  $E_{\epsilon+}^\delta$  on the interval  $[0, \delta]$ . Hence we will work on  $[0, \delta]$  with  $E_{\epsilon+}^\delta$ .

First of all, suppose the Hausdorff dimension of set  $E_{\epsilon+}^\delta$  is larger than  $\alpha$ . In this case, we consider partitions  $\{0, \frac{\delta}{k}, \dots, \frac{(k-1)\delta}{k}, \delta\}$  for  $k \in \mathbb{N}$ . In each interval  $[\frac{i\delta}{k}, \frac{(i+1)\delta}{k}]$ , we choose the leftmost and rightmost points of the set  $E_{\epsilon+}^\delta$  and denote this interval as  $I_k^i$  (if exists). Then  $\cup_{i=0}^{k-1} I_k^i$  covers the set  $E_{\epsilon+}^\delta$  and by the definition of Hausdorff measure, we know that  $\sum_{i=0}^{k-1} |I_k^i|^\alpha \rightarrow \infty$  as  $k \rightarrow \infty$ . Now let  $a_k^i$  be the left boundary point of  $I_k^i$  and  $b_k^i$  be the right boundary point of  $I_k^i$ . Now for any two points  $a < b \in \{a_k^i, b_k^i, i = 0 \dots, k-1\}$ , we have

$$f(b) \geq f(a) + \epsilon(b - a)^\alpha$$

since  $f$  is continuous and we can approximate  $a, b$  by points in  $E_{\epsilon_+}^\delta$ . Then we have

$$\begin{aligned} f(b_k^{k-1}) - f(a_k^0) &= \sum_{i=1}^{k-1} f(b_k^i) - f(a_k^i) + f(a_k^i) - f(b_k^{i-1}) + f(b_k^0) - f(a_k^0) \\ &\geq \epsilon \sum_{i=0}^{k-1} |I_k^i|^\alpha \rightarrow \infty, \end{aligned}$$

which gives a contradiction when  $k \rightarrow \infty$ .  $\square$

## 2.3 Fractional Itô Calculus

We now consider non-integer  $p > 2$  and derive Itô-type change of variable formulas for paths with non-zero  $p$ -th variation along a sequence of partitions  $\{\pi_n\}_{n \in \mathbb{N}}$ , first for functions (Sections 2.3.1) then for path-dependent functionals (Section 2.3.2). In each case the focus is the existence or not of a 'fractional' Itô remainder term: we will see that the existence of a non-zero Itô term depends on the fine structure of the function and its fractional derivative.

### 2.3.1 Change of variable formula

We first derive an Itô-type change of variable formula for functions of paths with  $p$ -th variation along a sequence of partitions  $\{\pi_n\}_{n \in \mathbb{N}}$ .

Let  $S \in V_p(\pi)$  be a path which admits  $p$ -th order variation along some sequence of partitions  $\{\pi_n\}_{n \in \mathbb{N}}$ . We make the following assumptions:

**Assumption 2.3.1.** *For any  $k \in \mathbb{R}$ , we have*

$$\int_0^T \mathbb{1}_{\{S(t)=k\}} d[S]_\pi^p = 0.$$

Let  $m = \lfloor p \rfloor$ .

**Assumption 2.3.2.**  *$f \in C^m(\mathbb{R})$  and admits a left local fractional derivative of order  $p$  everywhere.*

**Assumption 2.3.3.** *The complement of the set*

$$\begin{aligned} \Gamma_f &= \{x \in \mathbb{R} : \exists U \ni x \text{ open}, (a, b) \mapsto (C_{a^+}^p f)(b) \\ &\quad \text{is continuous on } \{(\bar{a}, \bar{b}) \in U \times U : \bar{a} \leq \bar{b}\}\} \end{aligned}$$

*is locally finite, i.e. for any compact set  $K \in \mathbb{R}$ , the set  $\Gamma_f^c \cap K$  has only finite number of points.*

We first give a simple lemma regarding the set  $\Gamma_f$ :

**Lemma 2.3.1.** *The left local fractional derivative of order  $p$  of  $f$  is equal to zero on  $\Gamma_f$ :  $\forall x \in \Gamma_f, f^{(p+)}(x) = 0$ .*

*Proof.* Let  $x \in \Gamma_f$ . There exists  $x \in U$  open such that  $(a, b) \mapsto C_{a+}^p f(b)$  is continuous on  $\{(a, b) \in U \times U : a \leq b\}$ . Hence  $f^{(p+)}$  is continuous on  $U$ . Since  $f^{(p+)}$  is zero a.e.,  $f^{(p+)}(x) = 0$ .  $\square$

Assumption 2.3.1 will be satisfied if the weighted occupation measure  $\gamma_S$  defined by

$$\gamma_S(A) = \int_0^T \mathbb{1}_{\{S(t) \in A\}} d[S]_{\pi}^p(t).$$

is atomless. Assumption 2.3.1 is satisfied in particular if  $\gamma_S$  has a Lebesgue density [41], which corresponds to a local time of order  $p$  [24]. However, as the following example shows, Assumption 2.3.1 may fail to be satisfied even if the path has a non-zero  $p$ -th order variation.

**Example 2.3.2** (A counterexample). *We now give an example of path failing to satisfy Assumption 2.3.1, in the spirit of [25, Example 3.6].<sup>1</sup>*

Let  $p > 2$  and define the intervals

$$I_1^1 = \left(\frac{1}{3}, \frac{2}{3}\right), \quad I_2^1 = \left(\frac{1}{9}, \frac{2}{9}\right), \quad I_2^2 = \left(\frac{7}{9}, \frac{8}{9}\right), \dots, I_j^i, j = 1, \dots, 2^{i-1}, \quad (2.3.1)$$

and let  $C = [0, 1] \setminus \cup_{i=1}^{\infty} \cup_{j=1}^{2^{i-1}} I_j^i$ , which is the Cantor ternary set [13]. Let  $c : [0, 1] \rightarrow \mathbb{R}_+$  be the associated Cantor function, which is defined by

$$c(x) = \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2^n}, & x = \sum_{n=1}^{\infty} \frac{2a_n}{3^n} \in C, \quad a_n \in \{0, 1\} \\ \sup_{y \leq x, y \in C} c(y), & x \in [0, 1] \setminus C. \end{cases}$$

We can see it is a non-decreasing function increasing only on the Cantor set  $C$ .

Consider the function

$$S(t) = |2^{\log_3(2 \cdot \min_{u \in C} |t-u|)}|^{\frac{1}{p}}.$$

We are going to construct a sequence of partitions such that the  $p$ -th variation of  $S$  along this sequence will be the Cantor function  $c$ . In this case, we will see

$$\int_0^1 \mathbb{1}_{\{S(t)=0\}} d[S]^p = \int_0^1 \mathbb{1}_{t \in C} dc(t) = c(1) - c(0) = 1,$$

<sup>1</sup>[25, Example 3.6] aims to construct a path which admits quadratic variation but does not possess local time along some sequence of partitions. There seems to be an issue with the construction in [25] but the underlying idea is still useful for our construction. In fact, the third equality of equation (3.12) in [25] should be  $\sum_{i=1}^n (\epsilon_n)^i = \frac{\epsilon_n - \epsilon_n^{n+1}}{1 - \epsilon_n} \rightarrow 0$  not 1.

which shows Assumption 2.3.1 is not satisfied.

We begin with the partition  $\{\pi_{j,n}^i\}$ , which denotes the  $n$ -th partition in the interval  $I_j^i$  and let  $\pi_n = \cup_{i=1}^n \cup_j \pi_{j,n}^i$ . Define  $t_{j,n}^{i,0} = \inf I_j^i$  and

$$t_{j,n}^{i,k+1} = \inf \left\{ t > t_{j,n}^{i,k} : S(t) \in \left( \frac{1}{k_n} \sup_{t \in I_j^i} S(t) \right) \mathbb{Z} \right\},$$

then  $t_{j,n}^{i,2k_n} = \sup I_j^i$ .  $k_n$  is an integer to be determined. We then do the calculation

$$\sum_{k=0}^{2k_n-1} |S(t_{j,n}^{i,k+1}) - S(t_{j,n}^{i,k})|^p = \sum_{k=0}^{2k_n-1} \left| \frac{1}{k_n} 2^{-\frac{i}{p}} \right|^p = 2^{1-i} k_n^{1-p}.$$

Then the sum over the  $n$ -th partition will be

$$\sum_{i=1}^n \sum_{j=1}^{2^{i-1}} 2^{1-i} k_n^{1-p} = n \cdot k_n^{1-p}.$$

Choosing  $k_n = \lfloor n^{\frac{1}{p-1}} \rfloor$  we obtain

$$n k_n^{1-p} \geq n \cdot (n^{\frac{1}{p-1}})^{1-p} = 1$$

and

$$n k_n^{1-p} \leq n \cdot (n^{\frac{1}{p-1}} - 1)^{1-p} = \left( 1 - n^{-\frac{1}{p-1}} \right)^{1-p} \xrightarrow{n \rightarrow \infty} 1.$$

Hence we see

$$[S]_{\pi}^p(1) = 1.$$

Furthermore, since

$$2^{1-i} k_n^{1-p} \xrightarrow{n \rightarrow \infty} 0,$$

we see that  $[S]_{\pi}^p$  will not change on the interval  $I_j^i$  and by symmetry, we finally can show that  $[S]^p = c$ , the Cantor function.

The following theorem extends the result of [24] to the case of functions with fractional regularity:

**Theorem 2.3.3.** *Let  $S \in V_p(\pi)$  satisfy Assumption 2.3.1. If  $f$  is a continuous function satisfying Assumptions 2.3.2 and 2.3.3 for  $m = \lfloor p \rfloor$ , then*

$$\forall t \in [0, T], \quad f(S(t)) = f(S(0)) + \int_0^t f'(S(u)) dS(u),$$

where the last term is a limit of compensated Riemann sums of order  $m = \lfloor p \rfloor$ :

$$\int_0^t f'(S(u)) dS(u) = \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n} \sum_{j=1}^m \frac{f^{(j)}(S(t \wedge t_i))}{j!} (S(t \wedge t_{i+1}) - S(t \wedge t_i))^j.$$

*Proof.* Using Taylor's formula with integral remainder of integer order  $m$ ,

$$\begin{aligned}
f(S(t)) - f(S(0)) &= \sum_{[t_i, t_{i+1}] \in \pi_n} f(S(t \wedge t_{i+1})) - f(S(t \wedge t_i)) \\
&= \sum_{[t_i, t_{i+1}] \in \pi_n} (f(S(t \wedge t_{i+1})) - f(S(t \wedge t_i)) - \dots \\
&\quad - \frac{f^{(m)}(S(t \wedge t_i))}{m!} (S(t \wedge t_{i+1}) - S(t \wedge t_i))^m) \\
&\quad + \sum_{[t_i, t_{i+1}] \in \pi_n} \sum_{j=1}^m \frac{f^{(j)}(S(t \wedge t_i))}{j!} (S(t \wedge t_{i+1}) - S(t \wedge t_i))^j \\
&= \sum_{[t_i, t_{i+1}] \in \pi_n} \frac{1}{\Gamma(m)} \int_{S(t \wedge t_i)}^{S(t \wedge t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t \wedge t_i))) (S(t \wedge t_{i+1}) - r)^{m-1} dr \\
&\quad + L^n,
\end{aligned}$$

where  $L^n = \sum_{[t_i, t_{i+1}] \in \pi_n} \sum_{j=1}^m \frac{f^{(j)}(S(t \wedge t_i))}{j!} (S(t \wedge t_{i+1}) - S(t \wedge t_i))^j$ . Our aim is to estimate the limit of the quantity

$$R^n := \sum_{[t_i, t_{i+1}] \in \pi_n} \frac{1}{\Gamma(m)} \int_{S(t \wedge t_i)}^{S(t \wedge t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t \wedge t_i))) (S(t \wedge t_{i+1}) - r)^{m-1} dr,$$

and show that this quantity converges to 0. In order to obtain an estimate, we divided the partitions into three parts:  $C_\epsilon^n = \{[t_i, t_{i+1}] \in \pi_n : |S(t \wedge t_i) - k| \leq \epsilon \text{ for some } k \in \Gamma_f^c\}$ ,  $C_{1,\epsilon}^n = \{[t_i, t_{i+1}] \in \pi_n \setminus C_\epsilon^n : S(t \wedge t_i) \leq S(t \wedge t_{i+1})\}$  and  $C_{2,\epsilon}^n = \{[t_i, t_{i+1}] \in \pi_n \setminus C_\epsilon^n : S(t \wedge t_{i+1}) < S(t \wedge t_i)\}$  for arbitrary  $\epsilon > 0$ .

1. On  $C_\epsilon^n$ : Denote

$$R_\epsilon^n := \sum_{[t_i, t_{i+1}] \in C_\epsilon^n} \frac{1}{\Gamma(m)} \int_{S(t \wedge t_i)}^{S(t \wedge t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t \wedge t_i))) (S(t \wedge t_{i+1}) - r)^{m-1} dr.$$

By Hölder continuity of  $f$  and continuity of  $S$ , there exists a constant  $M > 0$  such that

$$\begin{aligned}
|R_\epsilon^n| &\leq \sum_{[t_i, t_{i+1}] \in C_\epsilon^n} M |S(t \wedge t_{i+1}) - S(t \wedge t_i)|^p \\
&\leq \sum_{[t_i, t_{i+1}] \in \pi_n} M g_\epsilon(S(t \wedge t_i)) |S(t \wedge t_{i+1}) - S(t \wedge t_i)|^p.
\end{aligned}$$

For example  $M = \frac{\Gamma(p-m+1) \|f\|_{C^p}}{\Gamma(p+1)}$ . Here  $g_\epsilon$  is a continuous function taking value 1 on  $[k - \epsilon, k + \epsilon]$  for each  $k \in \Gamma_f^c$  and value 0 outside of  $\cup_{k \in \Gamma_f^c} [k - 2\epsilon, k + 2\epsilon]$  with

$\|g_\epsilon\|_\infty \leq 1$ . When  $\epsilon$  is small enough, we may assume  $[k - 2\epsilon, k + 2\epsilon]$  are disjoint. Then by definition of the  $p$ -th order variation, we see that

$$\limsup_{n \rightarrow \infty} |R_\epsilon^n| \leq \int_0^t M g_\epsilon(S(r)) d[S]^p(r).$$

Letting  $\epsilon \rightarrow 0$ , we obtain from assumption 2.3.1 that

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} |R_\epsilon^n| \leq \sum_{k \in \Gamma_f^c} \int_0^t M \mathbb{1}_{\{S(r)=k\}} d[S]^p(r) = 0.$$

2. Outside  $C_\epsilon^n$ : since  $S \in V^p(\pi)$ , there exists  $N_\epsilon \in \mathbb{N}$  such that for all  $n > N_\epsilon$ , we have for all  $[t_i, t_{i+1}] \in \pi_n$

$$|S(t_{i+1}) - S(t_i)| \leq \frac{\epsilon}{2}.$$

Thus for  $[t_i, t_{i+1}] \notin C_\epsilon^n$ , we have  $(S(t_{i+1}) - k)(S(t_i) - k) > 0$  for all  $k \in \Gamma_f^c$  and  $S(t_i), S(t_{i+1})$  is away from  $\Gamma_f^c$  with at least  $\frac{\epsilon}{2}$  distance, which means we have the following estimate

$$|C_{a_1^+}^p f(b_1) - C_{a_2^+}^p f(b_2)| \leq \omega_\epsilon(\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}) \quad (2.3.2)$$

for some modulus of continuity  $\omega_\epsilon$ ,  $a_1, a_2, b_1, b_2 \in [S(t_i), S(t_{i+1})]$  or  $[S(t_{i+1}), S(t_i)]$  and  $a_1 \leq b_1, a_2 \leq b_2$  by assumption 2.3.2 and 2.3.3.

3. We now consider the set  $C_{1,\epsilon}^n$ . Let  $\alpha = p - m$  and denote

$$R_{1,\epsilon}^n := \sum_{[t_i, t_{i+1}] \in C_{1,\epsilon}^n} \frac{1}{\Gamma(m)} \int_{S(t \wedge t_i)}^{S(t \wedge t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t \wedge t_i)))(S(t \wedge t_{i+1}) - r)^{m-1} dr.$$

We have by proposition 2.2.10,

$$\begin{aligned} & R_{1,\epsilon}^n \\ &= \sum_{[t_i, t_{i+1}] \in C_{1,\epsilon}^n} \frac{1}{\Gamma(m)} \int_{S(t \wedge t_i)}^{S(t \wedge t_{i+1})} (S(t \wedge t_{i+1}) - r)^{m-1} \frac{1}{\Gamma(\alpha)} \int_{S(t \wedge t_i)}^r \frac{C_{S(t \wedge t_i)+}^p f(s)}{(r-s)^{1-\alpha}} ds dr \\ &= \sum_{[t_i, t_{i+1}] \in C_{1,\epsilon}^n} \frac{1}{\Gamma(m)\Gamma(\alpha)} \int_{S(t \wedge t_i)}^{S(t \wedge t_{i+1})} \int_s^{S(t \wedge t_{i+1})} (S(t \wedge t_{i+1}) - r)^{m-1} \frac{C_{S(t \wedge t_i)+}^p f(s)}{(r-s)^{1-\alpha}} dr ds \\ &= \sum_{[t_i, t_{i+1}] \in C_{1,\epsilon}^n} \frac{1}{\Gamma(p)} \int_{S(t \wedge t_i)}^{S(t \wedge t_{i+1})} (S(t \wedge t_{i+1}) - s)^{p-1} C_{S(t \wedge t_i)+}^p f(s) ds. \end{aligned}$$

As  $S(t_i) \in \Gamma_f$  by Lemma 2.3.1 we have  $f^{(p+)}(S(t_i)) = C_{S(t_i)+}^p f(S(t_i)) = 0$ . Then by nequality (2.3.2) we have

$$\begin{aligned} & \left| \int_{S(t \wedge t_i)}^{S(t \wedge t_{i+1})} (S(t \wedge t_{i+1}) - s)^{p-1} C_{S(t \wedge t_i)+}^p f(s) ds \right| \\ & \leq \int_{S(t \wedge t_i)}^{S(t \wedge t_{i+1})} (S(t \wedge t_{i+1}) - s)^{p-1} \omega_\epsilon(s - S(t \wedge t_i)) ds \\ & \leq \frac{1}{p} \omega_\epsilon(S(t_{i+1}) - S(t_i)) |S(t_{i+1}) - S(t_i)|^p. \end{aligned}$$

Thus we obtain

$$|R_{1,\epsilon}^n| \leq \sum_{[t_i, t_{i+1}] \in \pi_n} \frac{\omega_\epsilon(S(t_{i+1}) - S(t_i))}{\Gamma(p+1)} |S(t_{i+1}) - S(t_i)|^p.$$

Since  $osc(S, \pi_n) \rightarrow 0$  as  $n \rightarrow \infty$ , we have  $\lim_{n \rightarrow \infty} |R_{1,\epsilon}^n| = 0$ .

4. On the set  $C_{2,\epsilon}^n$ . Denote

$$R_{2,\epsilon}^n = \sum_{[t_i, t_{i+1}] \in C_{2,\epsilon}^n} \frac{1}{\Gamma(m)} \int_{S(t \wedge t_i)}^{S(t \wedge t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t \wedge t_i))) (S(t \wedge t_{i+1}) - r)^{m-1} dr.$$

By proposition 2.2.10,  $R_{2,\epsilon}^n$  can be expressed as

$$\sum_{[t_i, t_{i+1}] \in C_{2,\epsilon}^n} \frac{1}{\Gamma(m)\Gamma(\alpha)} \int_{S(t \wedge t_{i+1})}^{S(t \wedge t_i)} \int_{S(t \wedge t_{i+1})}^s \frac{(S(t \wedge t_{i+1}) - r)^{m-1} \cdot C_{r+}^p f(s)}{(S(t \wedge t_i) - s)^{1-\alpha}} dr ds.$$

Again by the inequality (2.3.2), we will have

$$\begin{aligned} |C_{r+}^p f(s)| & \leq \omega_\epsilon(\sqrt{(r - S(t \wedge t_{i+1}))^2 + (s - S(t \wedge t_{i+1}))^2}) \\ & \leq \omega_\epsilon(\sqrt{2} |S(t \wedge t_{i+1}) - S(t \wedge t_i)|). \end{aligned}$$

Hence, we can obtain the estimate

$$\begin{aligned} & \left| \int_{S(t \wedge t_{i+1})}^{S(t \wedge t_i)} \int_{S(t \wedge t_{i+1})}^s (S(t \wedge t_{i+1}) - r)^{m-1} \frac{C_{r+}^p f(s)}{(S(t \wedge t_i) - s)^{1-\alpha}} dr ds \right| \\ & \leq \int_{S(t \wedge t_{i+1})}^{S(t \wedge t_i)} \int_{S(t \wedge t_{i+1})}^s (S(t \wedge t_{i+1}) - r)^{m-1} \frac{\omega_\epsilon(\sqrt{2} |S(t \wedge t_{i+1}) - S(t \wedge t_i)|)}{(S(t \wedge t_i) - s)^{1-\alpha}} dr ds \\ & \leq \frac{1}{m\alpha} \omega_\epsilon(\sqrt{2} |S(t \wedge t_{i+1}) - S(t \wedge t_i)|) |S(t_{i+1}) - S(t_i)|^p. \end{aligned}$$

Thus we obtain that  $\lim_{n \rightarrow \infty} |R_{2,\epsilon}^n| = 0$ .

Thus, we have  $R^n = R_\epsilon^n + R_{1,\epsilon}^n + R_{2,\epsilon}^n$  and

$$\lim_{n \rightarrow \infty} |R^n| \leq \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} |R_\epsilon^n| + |R_{1,\epsilon}^n| + |R_{2,\epsilon}^n| = 0.$$

Hence we see  $\lim_{n \rightarrow \infty} L^n$  exists and we denote it as  $\int_0^t (f(S(u))) dS(u)$ , which gives the result.  $\square$

**Remark 2.3.4.** We can not expect the Itô formula in the fractional case to have the same form as in the integer case, i.e. there might be no Itô term in the fractional case even if Assumption 2.3.1 does not hold. For Example 2.3.2, we can show that for  $2 < p < 3$  and  $f(x) = |x|^p$ , the Itô term still vanishes even though we have

$$\int_0^1 f^{(p+)}(S(t)) d[S]^p(t) = \frac{1}{\Gamma(p+1)} \int_0^1 \mathbb{1}_{\{S(t)=0\}} dc(t) = \frac{c(1) - c(0)}{\Gamma(p+1)} = \frac{1}{\Gamma(p+1)},$$

since  $\{S(t) = 0\} = C$  is the support of the function  $c$ . In this case, we will show that

$$0 = f(S(1)) - f(S(0)) = \int_0^1 f'(S(u)) dS(u).$$

In fact, we will need to calculate

$$\lim_{n \rightarrow \infty} \sum_{[t_i, t_{i+1}] \in \pi_n} f'(S(t_i))(S(t_{i+1}) - S(t_i)) + \frac{1}{2} f''(S(t_i))(S(t_{i+1}) - S(t_i))^2,$$

which is in fact the 'rough integral' of  $f'$  along the reduced order- $p$  Itô rough path associated with  $S$  [24]. Splitting the terms across each  $I_j^i$  we have

$$\begin{aligned} & \sum_{k=0}^{k_n-1} p(S(t_{j,n}^{i,k}))^{p-1} \frac{1}{k_n} 2^{-\frac{i}{p}} - \sum_{k=k_n}^{2k_n-1} p(S(t_{j,n}^{i,k}))^{p-1} \frac{1}{k_n} 2^{-\frac{i}{p}} \\ &= -p(S(t_{j,n}^{i,k_n}))^{p-1} \frac{1}{k_n} 2^{-\frac{i}{p}} = -p \cdot (2^{-\frac{i}{p}})^{p-1} \cdot \frac{2^{-\frac{i}{p}}}{k_n}. \end{aligned}$$

Thus

$$\sum_{[t_i, t_{i+1}] \in \pi_n} f'(S(t_i))(S(t_{i+1}) - S(t_i)) = \sum_{i=1}^n \sum_{j=1}^{2^{i-1}} -p \cdot \frac{2^{-i}}{k_n} = -\frac{np}{2k_n}.$$

Now we consider the term involving the second derivative. On  $I_j^i$ , we have

$$\begin{aligned} & \sum_{[t_{j,n}^{i,k}, t_{j,n}^{i,k+1}] \in \pi_{j,n}^i} \frac{1}{2} f''(S(t_{j,n}^{i,k})) (S(t_{j,n}^{i,k+1}) - S(t_{j,n}^{i,k}))^2 \\ &= \sum_{[t_{j,n}^{i,k}, t_{j,n}^{i,k+1}] \in \pi_{j,n}^i} \frac{p(p-1)}{2} (S(t_{j,n}^{i,k}))^{p-2} \frac{2^{-\frac{2i}{p}}}{k_n^2} \\ &= \frac{1}{2} \left( \sum_{k=0}^{k_n-1} p(p-1) k^{p-2} \frac{2^{-i}}{k_n^p} + \sum_{k=1}^{k_n} p(p-1) k^{p-2} \frac{2^{-i}}{k_n^p} \right). \end{aligned}$$

Summing the second order terms over  $\pi_n$  we obtain:

$$\frac{np(p-1)}{4} \frac{2 \sum_{k=1}^{k_n} k^{p-2} - k_n^{p-2}}{k_n^p}.$$

Adding together the first order term  $-\frac{np}{2k_n}$ , we need to calculate

$$\lim_{n \rightarrow \infty} \frac{np}{4} \left( (p-1) \frac{2 \sum_{k=1}^{k_n} k^{p-2} - k_n^{p-2}}{k_n^p} - \frac{2}{k_n} \right).$$

We thus observe that

$$\lim_{n \rightarrow \infty} \frac{np}{4} (p-1) \frac{1}{k_n^2} = \lim_{n \rightarrow \infty} \frac{np}{4} (p-1) \frac{1}{n^{\frac{2}{p-1}}}.$$

Since  $\frac{2}{p-1} > 1$  this implies

$$\lim_{n \rightarrow \infty} \frac{np}{4} (p-1) \frac{1}{k_n^2} = 0.$$

Hence it remains to consider

$$\lim_{n \rightarrow \infty} \frac{np}{2} \left( (p-1) \frac{\sum_{k=1}^{k_n} k^{p-2}}{k_n^p} - \frac{1}{k_n} \right).$$

Using the inequality

$$\frac{k_n^{p-1}}{p-1} = \int_0^{k_n} x^{p-2} dx \leq \sum_{k=1}^{k_n} k^{p-2} \leq \int_1^{k_n+1} x^{p-2} dx = \frac{(k_n+1)^{p-1} - 1}{p-1},$$

we obtain

$$\lim_{n \rightarrow \infty} \frac{np}{2} \left( (p-1) \frac{\sum_{k=1}^{k_n} k^{p-2}}{k_n^p} - \frac{1}{k_n} \right) \geq \lim_{n \rightarrow \infty} \frac{np}{2} \left( (p-1) \frac{k_n^{p-1}}{(p-1)k_n^p} - \frac{1}{k_n} \right) = 0,$$

and also

$$\lim_{n \rightarrow \infty} \frac{np}{2} \left( (p-1) \frac{\sum_{k=1}^{k_n} k^{p-2}}{k_n^p} - \frac{1}{k_n} \right) \leq \lim_{n \rightarrow \infty} \frac{np}{2} \left( \frac{(k_n+1)^{p-1} - 1 - k_n^{p-1}}{k_n^p} \right). \quad (2.3.3)$$

Notice that  $\lim_{n \rightarrow \infty} nk_n^{1-p} = 1$  and  $k_n \rightarrow \infty$ , we see the right hand side of the inequality (2.3.3) equals to

$$\lim_{k_n \rightarrow \infty} p \frac{k_n^{p-1}}{2} \left( \frac{(k_n+1)^{p-1} - 1 - k_n^{p-1}}{k_n^p} \right) = \frac{p}{2} \lim_{x \rightarrow \infty} \frac{(x+1)^{p-1} - 1 - x^{p-1}}{x}.$$

By l'Hopital's rule, above limit equals to

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{p(p-1)}{2} ((x+1)^{p-2} - x^{p-2}) \\ &= \frac{p(p-1)}{2} \lim_{x \rightarrow \infty} x^{p-2} \left( \left(1 + \frac{1}{x}\right)^{p-2} - 1 \right) = \frac{p(p-1)}{2} \lim_{x \rightarrow \infty} x^{p-2} \frac{p-2}{x} = 0 \end{aligned}$$

since  $p - 2 < 1$ . Thus in this case we see the remainder term is zero, which is different from the integer case. Hence we see that **the pathwise change of variable formula in Theorem 2.3.3 may hold even if Assumption 2.3.1 does not hold.**

We now give an example of path with the same  $p$ -th variation as above but leading to a non-zero remainder term in the change of variable formula.

**Example 2.3.5.** Define the intervals  $I_j^i$  as in (2.3.1). Then we define  $S|_{I_j^i}$  by induction on  $i$ . Let  $g(t) = 2\min\{t, 1 - t\}$  on  $[0, 1]$  and 0 otherwise. For  $a < b$ , we define  $g(t, [a, b]) = g\left(\frac{t-a}{b-a}\right)$ .

First for  $i = 1$ , we divide the interval  $I_j^i$  into  $r_i$  smaller intervals  $I_{j,k}^i, k = 1, \dots, r_i$  such that  $|I_{j,k}^i| = \frac{|I_j^i|}{r_i}$  and two of each are non-intersecting. On each interval  $I_{j,k}^i$ , we define  $S(t) = 2^{-i}g(t, I_{j,k}^i)$  for  $k = 1, \dots, r_i, j = 1, \dots, 2^{i-1}$ . In other words,  $S$  is defined as the limit

$$S(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} \sum_{k=1}^{r_i} 2^{-i} g(t, I_{j,k}^i).$$

Let  $\pi_n = (\tau_l^n)$  be the dyadic Lebesgue partition associated with  $S$ :

$$\tau_0^n = 0, \quad \tau_{l+1}^n = \inf\{t > \tau_l^n, |S(t) - S(\tau_l^n)| > 2^{-n}\}.$$

Hence, we have the calculation

$$\begin{aligned} [S]^p(1) &= \lim_{n \rightarrow \infty} \sum_{\pi_n} |S(\tau_{l+1}^n) - S(\tau_l^n)|^p = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^{2^{i-1}} \sum_{k=1}^{r_i} 2 \times 2^{-np} \times 2^{n-i} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n r_i 2^{-np+n}. \end{aligned}$$

By choosing

$$r_i = \lfloor 2^{(i-1)(p-1)}(2^{p-1} - 1) \rfloor,$$

so that

$$1 = \lim_{n \rightarrow \infty} \sum_{i=1}^n r_i 2^{-np+n},$$

together with similar to what was discussed in the above remark, we have  $[S]^p(t) = c(t)$ , the Cantor function. Let  $f(x) = |x|^p, 2 < p < \frac{5}{2}$  and  $T = 1$ , we are going to calculate the Itô remainder term for  $f(S(1)) - f(S(0))$ . We calculate the limit

$$\begin{aligned} &\lim_{n \rightarrow \infty} \sum_{[t_i, t_{i+1}] \in \pi_n} p(S(t_i))^{p-1} (S(t_{i+1}) - S(t_i)) + \frac{p(p-1)}{2} (S(t_i))^{p-2} (S(t_{i+1}) - S(t_i))^2 \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^{2^{i-1}} r_i \left( -p2^{-n}(2^{-i})^{p-1} + \frac{p(p-1)}{2} 2^{-2n} c_n^i \right), \end{aligned}$$

where

$$c_n^i = 2 \sum_{k=1}^{2^{n-i}-1} 2^{-n(p-2)} k^{p-2} + 2^{-n(p-2)} (2^{n-i})^{p-2}.$$

We see directly

$$c_n^i < 2^{1-n(p-2)} \int_0^{2^{n-i}} x^{p-2} dx + 2^{-n(p-2)} (2^{n-i})^{p-2}.$$

Thus,

$$\frac{p(p-1)}{2} 2^{-2n} c_n^i < p 2^{-n-ip+i} + \frac{p(p-1)}{2} 2^{-np+(n-i)(p-2)}.$$

This leads to

$$\begin{aligned} & \sum_{i=1}^n \sum_{j=1}^{2^{i-1}} r_i \left( -p 2^{-n} (2^{-i})^{p-1} + \frac{p(p-1)}{2} 2^{-2n} c_n^i \right) \\ & < \sum_{i=1}^{n-1} 2^{i-1} r_i \left( \frac{p(p-1)}{2} 2^{-2n-ip+2i} \right) + 2^{n-1} r_n \left( -p 2^{-np} + \frac{p(p-1)}{2} 2^{-np} \right) \\ & < \sum_{i=1}^{n-1} 2^{-p} (2^{p-1} - 1) \left( \frac{p(p-1)}{2} 2^{-2n+2i} \right) + 2^{-p} (2^{p-1} - 1) \left( -p + \frac{p(p-1)}{2} \right) \\ & = p 2^{-p} (2^{p-1} - 1) \left( \frac{p-1}{2} \frac{1 - (\frac{1}{4})^{n-1}}{3} + -1 + \frac{p-1}{2} \right). \end{aligned}$$

Let  $n \rightarrow \infty$ , we have

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^{2^{i-1}} r_i \left( -p 2^{-n} (2^{-i})^{p-1} + \frac{p(p-1)}{2} 2^{-2n} c_n^i \right) \leq p 2^{-p} (2^{p-1} - 1) \left( \frac{2(p-1)}{3} - 1 \right).$$

So if  $2 < p < \frac{5}{2}$ , then  $\frac{2(p-1)}{3} - 1 < 0$  and

$$\int_0^1 f'(S(t)) dS(t) < 0.$$

The equality  $f(S(1)) - f(S(0)) = 0$  shows there should be a non-zero remainder.

However, in this case, we still have

$$\int_0^1 \mathbb{1}_{\{S(t)=0\}} d[S]_{\pi}^p(t) = 1.$$

So Assumption 2.3.1 is not satisfied.

In fact, we can provide a formula for the Itô remainder term for this path and function  $f = |x|^p$ . Take  $T = 1$ ,  $m = \lfloor p \rfloor$  and  $\alpha = p - m$ . Let

$$G_f^p(a, b) = \frac{1}{(m-1)! |b-a|^p} \int_a^b (f^{(m)}(x) - f^{(m)}(a)) (b-x)^{m-1} dx$$

for  $a \neq b$  and take the limit value whenever it exists for  $a = b$ . For the function  $f(x) = |x|^p$ , we can see  $G_f^p$  is defined on  $\mathbb{R}^2/\{(0,0)\}$  and for  $k > 0$ ,

$$G_f^p(ka, kb) = G_f^p(a, b),$$

which suggests to consider  $G_f^p$  as a function on the unit circle  $S^1$ . We define the projection map  $P : \mathbb{R}^2/\{(0,0)\} \rightarrow S^1$  by  $p(x) = \frac{x}{\|x\|}$  and define a sequences of measures by

$$\tilde{\nu}_n := \frac{1}{N_n} \sum_{i=0}^{N_n-1} \delta_{P(S(t_i), S(t_{i+1}))},$$

where  $N_n$  is the number of intervals in the partition  $\pi_n$ . Furthermore, we define

$$\hat{G}_f^p(\theta) = G_f^p(\cos(\theta), \sin(\theta))$$

for  $\theta \in [0, 2\pi)$ . Since the distance between two successive points in partition  $\pi_n$  is  $2^{-n}$ , the remainder term  $\sum_{[t_i, t_{i+1}] \in \pi_n} G_f^p(S(t_{i+1}), S(t_i)) |S(t_{i+1}) - S(t_i)|^p$  for the partition  $\pi_n$  can be rewritten as

$$N_n \cdot 2^{-np} \int \hat{G}_f^p(x) \tilde{\nu}_n(dx)$$

and we have  $N_n = \sum_{i=1}^n \sum_{j=1}^{2^{i-1}} \sum_{k=1}^{r_i} 2^{n-i+1} = 2^n \sum_{i=1}^n r_i$ . Hence the remainder term is

$$2^{n-np} \sum_{i=1}^n r_i \int \hat{G}_f^p(x) \tilde{\nu}_n(dx).$$

We have  $\lim_{n \rightarrow \infty} 2^{n-np} \sum_{i=1}^n r_i = 1$ . It is easy to see that if we have the weak convergence of  $\tilde{\nu}_n$ , we can obtain the limit expression.

Let us now compute the limit of the sequence  $\tilde{\nu}_n$ . Since  $S(t_i) = k2^{-n}$  for some positive number  $k$  and  $t_i \in \pi_n$ , the result  $\tan(P(S(t_i), S(t_{i+1})))$  will equal to either  $\frac{k+1}{k}$ , which means the limit  $\tilde{\nu}$  will be supported on these points. And

$$\tilde{\nu} \left( \arctan \left( \frac{k+1}{k} \right) \right) = \lim_{n \rightarrow \infty} \tilde{\nu}_n \left( \arctan \left( \frac{k+1}{k} \right) \right) = \frac{1}{2^{\lceil \log_2(k+1) \rceil p}} \frac{2^{p-1} - 1}{2^p - 1}.$$

By symmetry, we have

$$\tilde{\nu} \left( \arctan \left( \frac{k}{k+1} \right) \right) = \tilde{\nu} \left( \arctan \left( \frac{k+1}{k} \right) \right) = \frac{1}{2^{\lceil \log_2(k+1) \rceil p}} \frac{2^{p-1} - 1}{2^p - 1}.$$

Hence we have a change variable formula for the path  $S$  defined in Example 2.3.5 and  $f(x) = |x|^p$ :

$$f(S(1)) - f(S(0)) = \int_0^1 f'(S(u)) dS(u) + \int_{S^1} \hat{G}_f^p(x) \tilde{\nu}(dx). \quad (2.3.4)$$

The above calculations show that the remainder term may be non-standard when assumption 2.3.1 is not satisfied.

We can give in fact a general method for calculating the remainder term for paths satisfying certain assumptions.

For a function  $f$  and  $p = m + \alpha$ , where  $m = \lfloor p \rfloor$ , we consider the map  $G_f^p : \{(x, y) \in \mathbb{R}^2, x \neq y\} \mapsto \mathbb{R}$  defined by

$$G_f^p(a, b) = \frac{1}{(m-1)!|b-a|^p} \int_a^b (f^{(m)}(x) - f^{(m)}(a))(b-x)^{m-1} dx. \quad (2.3.5)$$

If  $f^{(m)}$  is continuous, it is easy to see  $G_f^p$  is a continuous function on  $\{(x, y) \in \mathbb{R}^2, x \neq y\}$ .

In order to compute the remainder term in the fractional Itô formula, we 'stratify' the increments across the partition by the values of  $G_f^p$ , to build an auxiliary quotient space  $X$  which is required to have certain properties:

**Assumption 2.3.4** (Auxiliary quotient space). *There exists a space  $X$  and a map*

$$P_f^p : \{(x, y) \in \mathbb{R}^2, x \neq y\} \rightarrow X$$

such that

- (i) For all  $x \in X$ , the map  $G_f^p$  defined by (2.3.5) is constant on  $(P_f^p)^{-1}(x)$ ;
- (ii) The sequence of measures on  $[0, T] \times X$  defined by

$$\nu_n(dt, dx) := \sum_{[t_i, t_{i+1}] \in \pi_n} |S(t_{i+1}) - S(t_i)|^p \delta_{t_i}(t) \delta_{P_f^p(S(t_i), S(t_{i+1}))}(x)$$

converges weakly to a measure  $\nu$  on  $[0, T] \times X$ .

**Remark 2.3.6.** *The measure  $\tilde{\nu}$  in Example 2.3.5 is actually the measure  $\nu([0, 1], dx)$  on  $X$  defined in Assumption 2.3.4. In fact, we have*

$$\tilde{\nu}(dx) = \frac{1}{[S]_n^p(1)} \nu_n([0, 1], dx).$$

where  $[S]_n^p(1) = \sum_{[t_i, t_{i+1}] \in \pi_n, t_i \leq 1} |S(t_{i+1}) - S(t_i)|^p$ . Letting  $n \rightarrow \infty$ , weak convergence implies

$$\tilde{\nu}(dx) = \frac{1}{[S]^p(1)} \nu([0, 1], dx) = \nu([0, 1], dx).$$

Hence Assumption 2.3.4 is a reasonable condition which covers Example 2.3.5.

Furthermore, as  $S \in V_p(\pi)$  is equivalent to the weak convergence of

$$\nu_n(dt, X) = \sum_{[t_i, t_{i+1}] \in \pi_n} |S(t_{i+1}) - S(t_i)|^p \delta_{t_i}(t),$$

Assumption 2.3.4 is in fact stronger than the existence of  $p$ -th variation along the sequence  $\{\pi_n\}$ .

One can always choose  $X = \mathbb{R}$  but this choice may not be the most tractable for the calculation of the remainder term.

**Theorem 2.3.7.** *Let  $S \in V_p(\pi)$  such that there exists  $(X, P_f^p)$  satisfying Assumption 2.3.4. Then exists a unique map  $\hat{G}_f^p : X \rightarrow \mathbb{R}$  such that  $\hat{G}_f^p \circ P_f^p = G_f^p$  and*

$$f(S(t)) - f(S(0)) = \int_0^t f'(S(u)) dS(u) + \int_0^t \int_X \hat{G}_f^p(x) \nu(ds, dx),$$

where  $\int_0^t f'(S(u)) dS(u)$  is defined as in theorem 2.3.3.

*Proof.* The existence of  $\hat{G}_f^p$  is given by universality of the quotient map construction. Using a Taylor expansion, we have actually

$$\begin{aligned} f(S(t)) - f(S(0)) &= \sum_{[t_i, t_{i+1}] \in \pi_n, t_i \leq t} \sum_{j=1}^m \frac{f^{(j)}(S(t \wedge t_i))}{j!} (S(t \wedge t_{i+1}) - S(t \wedge t_i))^j \\ &+ \sum_{[t_i, t_{i+1}] \in \pi_n, t_i \leq t} G_f^p(S(t \wedge t_i), S(t \wedge t_{i+1})) |S(t \wedge t_{i+1}) - S(t \wedge t_i)|^p \end{aligned}$$

By definition of  $\nu_n$ , we have

$$\begin{aligned} f(S(t)) - f(S(0)) &= \sum_{[t_i, t_{i+1}] \in \pi_n, t_i \leq t} \sum_{j=1}^m \frac{f^{(j)}(S(t \wedge t_i))}{j!} (S(t \wedge t_{i+1}) - S(t \wedge t_i))^j \\ &+ \int_0^t \int_X \hat{G}_f^p(x) \tilde{\nu}_n(ds dx). \end{aligned}$$

Using the weak convergence of  $\nu_n$  in the Assumption 2.3.4, yields the desired result.  $\square$

We can see from Theorem 2.3.7 that when the Itô remainder term is non-zero, the formula is not in the classical form. Hence we will need more information to describe the Itô remainder term in the fractional case.

Let us go back to zero remainder term case. Consider the set  $(E^p)'$  of all functions satisfying Assumptions 2.3.2 and 2.3.3. Then  $(E^p)'$  is a subset of  $C^p(\mathbb{R})$ . Let  $E^p$  be the closure of  $(E^p)'$  in  $C^p(\mathbb{R})$  equipped with the semi-norms

$$\sup_{x \in K} \{f^{(k)}(x) : k = 0, \dots, m\} + \sup_{x, y \in K} \left\{ \frac{|f^{(m)}(y) - f^{(m)}(x)|}{|y - x|^\alpha} \right\},$$

where  $p = m + \alpha$  with  $m < p < m + 1$ . Then we have the following theorem

**Theorem 2.3.8.** Let  $S \in V_p(\pi)$  satisfying Assumption 2.3.1 and  $f \in E^p$ . Then for any  $t \in [0, T]$

$$f(S(t)) = f(S(0)) + \int f'(S(u))dS(u),$$

where the integral is a limit of compensated Riemann sums of order  $m = \lfloor p \rfloor$ :

$$\int f'(S(u))dS(u) = \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n} \sum_{j=1}^m \frac{f^{(j)}(S(t \wedge t_i))}{j!} (S(t \wedge t_{j+1}) - S(t \wedge t_j))^j.$$

*Proof.* We prove the result for  $t = T$  for convenience. The case  $t \neq T$  can be obtained by changing all  $t_i$  to  $t \wedge t_i$ . Suppose  $f_k \in (E^p)'$  and  $f_k \rightarrow f$  in  $C^p$ . We have

$$f(S(T)) - f(S(0)) = L^n + \sum_{t_i \in \pi_n} \frac{1}{\Gamma(m)} \int_{S(t_i)}^{S(t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t_i)))(S(t_{i+1}) - r)^{m-1} dr.$$

Let  $K$  be a compact set containing the path of  $S([0, T])$ . Since  $f_k \rightarrow f$  in  $C^p$ , we have for every  $\epsilon > 0$ , there exists  $N > 0$  and for all  $k > N$ ,

$$\sup_{x, y \in K} \frac{|(f^{(m)} - f_k^{(m)})(y) - (f^{(m)} - f_k^{(m)})(x)|}{|y - x|^\alpha} \leq \epsilon.$$

Hence

$$\begin{aligned} & \left| \sum_{[t_i, t_{i+1}] \in \pi_n} \frac{1}{\Gamma(m)} \int_{S(t_i)}^{S(t_{i+1})} ((f^{(m)} - f_k^{(m)})(r) - (f^{(m)} - f_k^{(m)})(S(t_i)))(S(t_{i+1}) - r)^{m-1} dr \right| \\ & \leq \epsilon \sum_{[t_i, t_{i+1}] \in \pi_n} \frac{1}{\Gamma(m)} \int_{S(t_i)}^{S(t_{i+1})} |r - S(t_i)|^\alpha |S(t_{i+1}) - r|^{m-1} dr \\ & \leq \epsilon \sum_{[t_i, t_{i+1}] \in \pi_n} \frac{1}{m!} |S(t_{i+1}) - S(t_i)|^p. \end{aligned}$$

Hence we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \sum_{[t_i, t_{i+1}] \in \pi_n} \frac{1}{\Gamma(m)} \int_{S(t_i)}^{S(t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t_i)))(S(t_{i+1}) - r)^{m-1} dr \right| \\ & \leq \limsup_{n \rightarrow \infty} \left| \sum_{[t_i, t_{i+1}] \in \pi_n} \frac{1}{\Gamma(m)} \int_{S(t_i)}^{S(t_{i+1})} (f_k^{(m)}(r) - f_k^{(m)}(S(t_i)))(S(t_{i+1}) - r)^{m-1} dr \right| \\ & \quad + \frac{\epsilon}{m!} [S]^p(t) \end{aligned}$$

for all  $k > N$ . Since  $f_k \in E$ , we then have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \sum_{[t_i, t_{i+1}] \in \pi_n} \frac{1}{\Gamma(m)} \int_{S(t_i)}^{S(t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t_i)))(S(t_{i+1}) - r)^{m-1} dr \right| \\ & \leq \frac{\epsilon}{m!} [S]^p(t). \end{aligned}$$

Finally letting  $\epsilon$  tend to 0, we have

$$\lim_{n \rightarrow \infty} \left| \sum_{[t_i, t_{i+1}] \in \pi_n} \frac{1}{\Gamma(m)} \int_{S(t_i)}^{S(t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t_i)))(S(t_{i+1}) - r)^{m-1} dr \right| = 0.$$

Hence  $L^n$  converges when  $n$  tends to infinity and we have the desired result.  $\square$

**Remark 2.3.9.** *It can be seen in the proof of the theorem 2.3.3 that we only need*

$$\int_0^T \mathbb{1}_{\{S(t)=k\}} d[S]_{\pi}^p(t) = 0 \quad (2.3.6)$$

for  $k \in \Gamma_f^c$ . Denote  $P_f$  the set of paths satisfying (2.3.6) on  $\Gamma_f^c$ .

Given a continuous path  $S \in V_p(\pi)$ , by Fubini's theorem, we actually have

$$\int_{\mathbb{R}} \int_0^T \mathbb{1}_{\{S(t)=k\}} d[S]_{\pi}^p(t) dk = 0,$$

which means (2.3.6) is satisfied by almost all  $k \in \mathbb{R}$ . Hence we can consider the set  $(E_S^p)'$  of all functions  $f$  such that  $\Gamma_f$  contains all points that do not satisfy condition (2.3.6). We can then construct the closure  $E_S^p$  for a given path  $S$ .

**Example 2.3.10** (Examples of functions belonging to  $E^p$ ).

1. All functions  $f \in C^{m+1}(\mathbb{R})$ .
2. The function  $f(x) = |x - k|^p$  for some  $k \in \mathbb{R}$ . It can be seen in the example 2.2.12 that  $C_{a+}^p f(x)$  is continuous on  $\{(\bar{a}, \bar{x}) \in U \times U : \bar{a} \leq \bar{x}\}$  for any compact set  $U$  contained in  $(k, \infty)$  or  $(-\infty, k)$ , which means  $\Gamma_f = \mathbb{R} \setminus \{k\}$  and hence is a function in  $E_p' \subset E_p$ .
3. The linear combinations of  $|x - k|^p$ . For example  $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} |x - x_n|^p$  with  $\{x_n\}$  an ordered set of rationals in  $[0, 1]$ .

**Remark 2.3.11.** *Smooth functions belong to  $E^p$ . Denote by  $C^{p+}(\mathbb{R}) \subset E^p$  the completion of the smooth function under Hölder norm i.e. the set of functions  $f \in C^p$  such that on every compact set  $K$ :*

$$\lim_{\delta \rightarrow 0} \sup_{\substack{x, y \in K \\ |x-y| \leq \delta}} \left\{ \frac{|f^{(m)}(x) - f^{(m)}(y)|}{|x-y|^\alpha} \right\} = 0.$$

Then for any  $q \notin \mathbb{N} > p$ , we have  $E^q \subset C^q \subset C^{p+} \subset E_S$  for any path  $S \in V_p(\pi) \cap C([0, T], \mathbb{R})$ .

**Example 2.3.12** (Example for  $P_f$  with  $f(x) = |x|^p$ ). Let  $B^H$  be a fractional Brownian motion with Hurst index  $H$  on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let  $p = 1/H$ . Then

$$\mathbb{E} \left[ \int_0^T \mathbb{1}_{\{B^H(t)=0\}} dt \right] = 0 \quad \text{so} \quad \mathbb{P}(B^H \in P_f^p) = 1.$$

**Remark 2.3.13** (Time-dependent case). Along the lines of the proofs of Theorems 2.3.3 and 2.3.8, we can generalize these formulas to the time-dependent case with a modified definition of the set  $\Gamma_f$ . For  $f \in C^{1,p}([0, T] \times \mathbb{R})$ , define the open set

$$\Gamma_f = \left\{ (t, x) \in [0, T] \times \mathbb{R} : \exists \text{ open set } U \ni (t, x) \right. \\ \left. \text{such that } C_{a+}^p f(s, b) \text{ is continuous on } U \tilde{\times} U \right\}$$

where  $U \tilde{\times} U = \{(s, a, b) : (s, a), (s, b) \in U, a \leq b\}$ . Then under the assumption

$$\int_0^T \mathbb{1}_{\{(t, S(t)) \notin \Gamma_f\}} d[S]_{\pi}^p(t) = 0,$$

we have the change of variable formula

$$f(t, S(t)) - f(0, S(0)) = \int_0^t \partial_t f(u, S(u)) du + \int_0^t D_x f(u, S(u)) dS(u),$$

where  $m = \lfloor p \rfloor$  and

$$\int_0^t D_x f(u, S(u)) dS(u) = \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n} \sum_{j=1}^m \frac{f^{(j)}(t \wedge t_i, S(t \wedge t_i))}{j!} (S(t \wedge t_{i+1}) - S(t \wedge t_i))^j.$$

**Remark 2.3.14.** Our argument also applies to the integer case. For example, when  $p = 2$ , we have  $m = \lfloor p \rfloor = 2$ . For any function  $f \in C^2$ ,  $\Gamma_f = \mathbb{R}$ . Hence we have

$$\int_0^T \mathbb{1}_{\{S(t) \in \Gamma_f^c\}} d[S]_{\pi}^p = 0.$$

Thus we have the result

$$f(S(t)) - f(S(0)) = \int_0^t f'(S(u)) dS(u),$$

where

$$\int_0^t f'(S(u)) dS(u) := \lim_{n \rightarrow \infty} \sum_{[t_i, t_{i+1}] \in \pi_n} f'(S(t_i))(S(t_{i+1}) - S(t_i)) \\ + \frac{1}{2} f''(S(t_i))(S(t_{i+1}) - S(t_i))^2.$$

However, by changing the definition of  $\int_0^t f'(S(u))dS(u)$  to

$$\lim_{n \rightarrow \infty} \sum_{[t_i, t_{i+1}] \in \pi_n} f'(S(t_i))(S(t_{i+1}) - S(t_i)),$$

we recover the result in [38]. Similarly, for even integers  $p$ , a slight change of definition of  $\int_0^t f'(S(u))dS(u)$  recovers the result in [24]. Additionally, we can also explore the odd integer case using a similar argument. A similar change of definition of  $\int_0^t f'(S(u))dS(u)$  gives the remainder term

$$\lim_{n \rightarrow \infty} \sum_{[t_i, t_{i+1}] \in \pi_n} \frac{1}{p!} f^{(p)}(S(t_i))(S(t_{i+1}) - S(t_i))^p.$$

The condition  $S \in V_p(\pi)$  means that this expression converges absolutely. However, as indicated in the appendix in [24], the limit would be typically zero.

### 2.3.2 Extension to path-dependent functionals and $\phi$ -variation

The above results may be extended to path functionals using the Dupire derivative [30] and the associated non-anticipative functional calculus [21, 17, 19, 22]. We recall here some concepts from the non-anticipative functional calculus [17, 22, 19].

Denote by  $S_t(s) = S(s \wedge t)$  the path stopped at  $t$ . We consider the space  $D([0, T], \mathbb{R})$  of càdlàg paths from  $[0, T]$  to  $\mathbb{R}$ . Let

$$\Lambda_T = \{(t, \omega_t) : (t, \omega) \in [0, T] \times D([0, T], \mathbb{R})\}$$

be the space of stopped paths. This is a complete metric space equipped with

$$d_\infty((t, \omega), (t', \omega')) := \sup_{s \in [0, T]} |\omega(s \wedge t) - \omega'(s \wedge t')| + |t - t'|.$$

We will also need to stop paths 'right before' a given time, and set for  $t > 0$

$$\omega_{t-}(s) := \begin{cases} \omega(s), & s < t, \\ \lim_{r \uparrow t} \omega(r), & s \geq t, \end{cases}$$

while  $\omega_{0-} = \omega_0$ .

**Definition 2.3.15.** A non-anticipative functional is a map  $F : \Lambda_T \rightarrow \mathbb{R}$ . Let  $F$  be a non-anticipative functional.

- We write  $F \in \mathbb{C}_i^{0,0}$  if for all  $t \in [0, T]$  the map  $F(t, \cdot) : D([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  is continuous and if for all  $(t, \omega) \in \Lambda_T$  and all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $(t', \omega') \in \Lambda_T$  with  $t' < t$  and  $d_\infty((t, \omega), (t', \omega')) < \delta$ , we have  $|F(t, \omega) - F(t', \omega')| < \epsilon$ .

- We write  $F \in \mathbb{B}(\Lambda_T)$  if for every  $t_0 \in [0, T)$  and every  $K > 0$  there exists  $C_{K, t_0} > 0$  such that for all  $t \in [0, t_0]$  and all  $\omega \in D([0, T], \mathbb{R})$  with  $\sup_{s \in [0, t]} |\omega(s)| \leq K$ , we have  $|F(t, \omega)| \leq C_{K, t_0}$ .
- $F$  is horizontally differentiable at  $(t, \omega) \in \Lambda_T$  if its horizontal derivative

$$\mathcal{D}F(t, \omega) := \lim_{h \downarrow 0} \frac{F(t+h, \omega_t) - F(t, \omega_t)}{h}$$

exists. If it exists for all  $(t, \omega) \in \Lambda_T$ , then  $\mathcal{D}F$  is a non-anticipative functional.

- $F$  is vertically differentiable at  $(t, \omega) \in \Lambda_T$  if its vertical derivative

$$\nabla_\omega F(t, \omega) := \lim_{h \rightarrow 0} \frac{F(t, \omega_t + h\mathbb{1}_{[t, T]}) - F(t, \omega_t)}{h}$$

exists. If it exists for all  $(t, \omega) \in \Lambda_T$ , then  $\nabla_\omega F$  is a non-anticipative functional. In particular, we define recursively  $\nabla_\omega^{k+1} F := \nabla_\omega \nabla_\omega^k F$  whenever this is well defined.

- For  $p \in \mathbb{N}_0$ , we say that  $F \in \mathbb{C}_b^{1,p}(\Lambda_T)$  if  $F$  is horizontally differentiable and  $p$  times vertically differentiable in every  $(t, \omega) \in \Lambda_T$ , and if  $F, \mathcal{D}F, \nabla_\omega^k F \in \mathbb{C}_l^{0,0}(\Lambda_T) \cap \mathbb{B}(\Lambda_T)$  for  $k = 1, \dots, p$ .

There is no straightforward extension of fractional Caputo derivative for non-anticipative functionals. We choose an alternative approach to obtain an extension of our change of variable formulas to the path-dependent case.

**Definition 2.3.16** (Vertical Hölder continuity). *Let  $0 < \alpha < 1$ .*

- $F$  is vertically  $\alpha$ -Hölder continuous on  $E \subset \Lambda_T$  if

$$L := \lim_{\epsilon \rightarrow 0} \sup_{(t, \omega) \in E} \sup_{|h| \leq \epsilon} \frac{|F(t, \omega_t + h\mathbb{1}_{[t, T]}) - F(t, \omega_t)|}{|h|^\alpha} < \infty.$$

We call  $L$  the vertical Hölder coefficient of  $F$  over  $E$  and we denote  $F \in \mathbb{C}^{0,\alpha}(E)$ .

- Let  $E_n = \{(t, \omega) \in \Lambda : d((t, \omega), (0, 0)) < n\}$ .  $F$  is locally vertically  $\alpha$ -Hölder continuous on  $E \subset \Lambda_T$  if  $F$  is vertically  $\alpha$ -Hölder continuous on  $E_n$  for each  $n$ . We denote  $F \in \mathbb{C}_{loc}^{0,\alpha}(E)$ .
- For  $m < p < m + 1$ , we say  $F \in \mathbb{C}_{loc}^{0,p}$  if  $\nabla_\omega^m F \in \mathbb{C}_{loc}^{0,p-m}$ . For  $E \subset \Lambda_T$ , we say  $F \in \mathbb{C}^{0,p}(E)$  if  $\nabla_\omega^m F \in \mathbb{C}^{0,p-m}(E)$ .

We now give assumptions on the functional  $F \in \mathbb{C}_b^{1,m} \cap \mathbb{C}_{loc}^{0,p}$ , where  $m = \lfloor p \rfloor$ . Given a path  $S \in V_p(\pi)$ , we make the following assumptions:

**Assumption 2.3.5.** *There exists a sequence of continuous functions  $g_k \in \mathbb{C}(\Lambda_T, \mathbb{R})$  with support  $E_k := \text{supp}(g_k)$  such that*

1.  $E_{k+1} \subset E_k$ ,  $g_k|_{E_{k+1}} = 1$ ,  $g_k \leq 1$  and  $F \in \mathbb{C}_{loc}^{0,p}(\Lambda_T \setminus E_k)$  for each  $k$  with vertical Hölder coefficient  $L = 0$ .

2. There exists a bounded function  $g$  on  $\Lambda_T$  such that  $g_k \rightarrow g$  pointwise and

$$\int_0^T g((t, S_t)) d[S]_t^p = 0.$$

Define the piecewise-constant approximation  $S^n$  to  $S$  along the partition  $\pi_n$ :

$$S^n(t) := \sum_{[t_i, t_{i+1}] \in \pi_n} S(t_{j+1}) \mathbb{1}_{[t_j, t_{j+1})}(t) + S(T) \mathbb{1}_{\{T\}}(t).$$

Then  $\lim_{n \rightarrow \infty} \|S^n - S\|_\infty = 0$  whenever  $\text{osc}(S, \pi_n) \rightarrow 0$ .

**Theorem 2.3.17.** *Let  $p \notin \mathbb{N}$ ,  $m = \lfloor p \rfloor$  and  $S \in V_p(\pi)$ . If  $F \in \mathbb{C}_b^{1,m} \cap \mathbb{C}_{loc}^{0,p}$  satisfies Assumption 2.3.5 then the limit*

$$\int_0^T \nabla_\omega F(t, S_{t-}) d^\pi S(t) := \lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \sum_{k=1}^m \frac{1}{k!} \nabla_\omega^k F(t_j, S_{t_j-}^n) (S(t_{j+1} \wedge t) - S(t_j \wedge t))^k$$

exists and we have

$$F(t, S_t) = F(0, S_0) + \int_0^t \mathcal{D}F(s, S_s) ds + \int_0^t \nabla_\omega F(s, S_s) dS(s).$$

*Proof.* We write

$$\begin{aligned} F(t, S_t^n) - F(0, S_0^n) &= \sum_{[t_j, t_{j+1}] \in \pi_n} (F(t_{j+1} \wedge t, S_{t_{j+1} \wedge t-}^n) - F(t_j \wedge t, S_{t_j \wedge t-}^n)) \\ &\quad + F(t, S_t^n) - F(t, S_{t-}^n) \\ &= \sum_{[t_j, t_{j+1}] \in \pi_n} (F(t_{j+1} \wedge t, S_{t_{j+1} \wedge t-}^n) - F(t_j \wedge t, S_{t_j \wedge t-}^n)) + o(1). \end{aligned}$$

We only need to consider  $j$  with  $t_{j+1} \leq t$  since the remainder is  $o(1)$  as  $n \rightarrow \infty$  by left continuity of  $F$ . We split the difference into two parts:

$$F(t_{j+1}, S_{t_{j+1}-}^n) - F(t_j, S_{t_j-}^n) = (F(t_{j+1}, S_{t_{j+1}-}^n) - F(t_j, S_{t_j}^n)) + (F(t_j, S_{t_j}^n) - F(t_j, S_{t_j-}^n)).$$

Now since  $S_{t_j}^n = S_{t_{j+1}-}^n$  on  $[0, t_{j+1}]$ , we have

$$F(t_{j+1}, S_{t_{j+1}-}^n) - F(t_j, S_{t_j}^n) = \int_{t_j}^{t_{j+1}} \mathcal{D}F(u, S_{t_j}^n) du = \int_{t_j}^{t_{j+1}} \mathcal{D}F(u, S^n) du,$$

since  $\mathcal{D}F$  is non-anticipative functional. Hence we have

$$\lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} (F(t_{j+1}, S_{t_{j+1}-}^n) - F(t_j, S_{t_j}^n)) = \int_0^t \mathcal{D}F(u, S_u) du$$

by continuity. It remains to consider the term

$$F(t_j, S_{t_j}^n) - F(t_j, S_{t_j-}^n) = F(t_j, S_{t_j-}^{n, S_{t_j}, t_{j+1}}) - F(t_j, S_{t_j-}^n),$$

where  $S_{t_j, t_{j+1}} = S(t_{j+1}) - S(t_j)$  and  $S_{t_j-}^{n, x}(s) := S_{t_j-}^n(s) + \mathbb{1}_{[t_j, T]}(s)x$ . Now from Taylor's formula and definition of vertical derivative, we obtain

$$\begin{aligned} & F(t_j, S_{t_j-}^{n, S_{t_j}, t_{j+1}}) \\ &= F(t_j, S_{t_j-}^n) + \sum_{k=1}^m \frac{\nabla_{\omega}^k F(t_j, S_{t_j-}^n)}{k!} (S(t_{j+1}) - S(t_j))^k \\ &+ \frac{1}{(m-1)!} \int_0^{S_{t_j, t_{j+1}}} (\nabla_{\omega}^m F(t_j, S_{t_j-}^{n, u}) - \nabla_{\omega}^m F(t_j, S_{t_j-}^n)) (S_{t_j, t_{j+1}} - u)^{m-1} du. \end{aligned}$$

From the definition of  $\mathbb{C}^{0,p}$ , we see that there exists  $M > 0$  such that

$$|\nabla_{\omega}^m F(t_j, S_{t_j-}^{n, u}) - \nabla_{\omega}^m F(t_j, S_{t_j-}^n)| \leq M|u|^{p-m},$$

for each  $t_j$  when  $n$  is large enough. This shows that

$$\begin{aligned} & \left| \frac{1}{(m-1)!} \int_0^{S_{t_j, t_{j+1}}} (\nabla_{\omega}^m F(t_j, S_{t_j-}^{n, u}) - \nabla_{\omega}^m F(t_j, S_{t_j-}^n)) (S_{t_j, t_{j+1}} - u)^{m-1} du \right| \\ & \leq \tilde{M} |S(t_{j+1}) - S(t_j)|^p. \end{aligned}$$

for some constant  $\tilde{M}$ . Fix  $k \in \mathbb{N}$ , we now divide the partitions into two parts. The first part  $\pi_n^1$  contains  $[t_j, t_{j+1}]$  such that  $(t_j, S_{t_j-}^n) \in E_k$  and the second part  $\pi_n^2$  contains  $[t_j, t_{j+1}]$  such that  $(t_j, S_{t_j-}^n) \notin E_k$ . Then we have

$$\begin{aligned} & \sum_{[t_j, t_{j+1}] \in \pi_n} \left| \frac{1}{(m-1)!} \int_0^{S_{t_j, t_{j+1}}} (\nabla_{\omega}^m F(t_j, S_{t_j-}^{n, u}) - \nabla_{\omega}^m F(t_j, S_{t_j-}^n)) (S_{t_j, t_{j+1}} - u)^{m-1} du \right| \\ & \leq \sum_{[t_j, t_{j+1}] \in \pi_n^1} \tilde{M} |S(t_{j+1}) - S(t_j)|^p + \sum_{[t_j, t_{j+1}] \in \pi_n^2} \tilde{M} L(S_{t_j-}^n, \text{osc}(S, \pi_n)) |S(t_{j+1}) - S(t_j)|^p, \end{aligned}$$

where

$$K(\omega, \epsilon) := \sup_{|h| \leq \epsilon} \frac{|\nabla_{\omega}^m F(t, \omega_t + h \mathbb{1}_{[t, T]}) - \nabla_{\omega}^m F(t, \omega_t)|}{|h|^{p-m}}.$$

From Assumption 2.3.5(1) we have  $\lim_{\epsilon \rightarrow 0} \sup_{\omega \in E \setminus E_k} K(\omega, \epsilon) = 0$  for each  $k$  with  $E$  a bounded subset of  $\Lambda_T$  containing all stopped paths alluded to in this proof. Hence

$$\lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n^2} \tilde{M} K(S_{t_j^-}^n, \text{osc}(S, \pi_n)) |S(t_{j+1}) - S(t_j)|^p = 0.$$

As for the first part, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n^1} \tilde{M} |S(t_{j+1}) - S(t_j)|^p \\ & \leq \lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \tilde{M} g_k((t_j, S_{t_j^-}^n) |S(t_{j+1}) - S(t_j)|^p \\ & = \int_0^t \tilde{M} g_k((t, S_t)) d[S]_t, \end{aligned}$$

by assumption 2.3.5. Letting  $k$  tend to infinity yields the result.  $\square$

The notion of  $p$ -th order variation can be extended to the more general concept of  $\phi$ -variation [57] for more general functions  $\phi : \mathbb{R} \mapsto \mathbb{R}$ :

**Definition 2.3.18** ( $\phi$ -variation along a partition sequence). *Let  $\phi : \mathbb{R} \mapsto \mathbb{R}$  be an even function. A continuous path  $S \in \mathbb{C}([0, T], \mathbb{R})$  is said to have a  $\phi$ -variation along a sequence of partitions  $\pi = (\pi_n)_{n \geq 1}$  if  $\text{osc}(S, \pi_n) \rightarrow 0$  and the sequence of measures*

$$\mu^n := \sum_{t_i \in \pi_n} |\phi(S(t_{i+1}) - S(t_i))| \delta(\cdot - t_i)$$

*converges weakly to a measure  $\mu$  without atoms. In this case we write  $S \in V_\phi(\pi)$  and  $[S]^\phi(t) = \mu([0, t])$  for  $t \in [0, T]$ , and we call  $[S]^\phi$  the  $\phi$ -variation of  $S$ .*

**Remark 2.3.19.** *If we consider an expansion of  $f$  of the form*

$$f(y) = f(x) + \sum_{k=1}^m \frac{f^{(k)}(x)}{k!} (y-x)^k + g(x) \phi(y-x) + o(\phi(y-x)).$$

*then following the steps in the proof of Proposition 2.2.16, we can show that  $g = 0$  almost everywhere. Hence a regular remainder should not appear in this case.*

Under the following condition on the function  $f$  we can obtain an extension of Theorem 2.3.17 to the case of paths with finite  $\phi$ -variation.

**Assumption 2.3.6.** *There exists a sequence of open sets  $U_i$  such that  $\bar{U}_{i+1} \subset U_i$  for  $i = 1, 2, \dots$  and  $\lim_{\delta \rightarrow 0} \sup_{|x-y| \leq \delta, x, y \in K} \frac{f^{(m)}(y) - f^{(m)}(x)}{|\phi^{(m)}(y-x)|} = 0$  for all compact sets  $K \in \mathbb{R} \setminus U_i$  and, denoting  $C = \bigcap_i^\infty \bar{U}_i$ , we have*

$$\int_0^T \mathbb{1}_{\{S(t) \in C\}} d[S]_\pi^\phi(t) = 0.$$

We state the following result without proof, the proof being similar to that of Theorem 2.3.17:

**Theorem 2.3.20.** *Let  $m \in \mathbb{N}$  and  $\phi \in C^m(\mathbb{R}^+, \mathbb{R}^+)$  such that*

$$\lim_{x \rightarrow 0} \frac{\phi(x)}{x^m} = 0, \quad \lim_{x \rightarrow 0} \frac{\phi(x)}{x^{m+1}} = \infty.$$

*If  $f \in C^m(\mathbb{R})$  is such that Assumption 2.3.6 is satisfied, we have*

$$f(S(t)) - f(S(0)) = \int_0^t f'(S(u)) dS(u),$$

*where the integral is defined as a (pointwise) limit of compensated Riemann sums:*

$$\int_0^t f'(S(u)) dS(u) = \lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \sum_{k=1}^m \frac{f^{(k)}(S(t_j))}{k!} (S(t_{j+1} \wedge t) - S(t_j \wedge t))^k.$$

## 2.4 An isometry formula for the pathwise integral

Ananova and Cont [3] proved a pathwise analogue of the Itô isometry for the Föllmer integral with respect to paths with finite quadratic variation. This relation was extended to  $p \geq 1$  in [24]. In the same flavor we derive here an isometry relation for the pathwise integral in terms of the  $\phi$ -variation, where  $\phi : \mathbb{R} \mapsto \mathbb{R}$  is an even function satisfying the following assumptions:

**Assumption 2.4.1.** 1.  $\phi$  is strictly increasing, continuous and  $\phi(0) = 0$ .

2.  $\phi$  is convex on  $[0, \infty)$  and  $u \in \mathbb{R} \mapsto \log \phi(\exp(u))$  is convex.

3. For  $x > 0$ , the limit

$$\varphi(x) := \lim_{y \rightarrow 0} \frac{\phi(xy)}{\phi(y)}$$

exists and the convergence is uniform on bounded sets.

4.  $\infty > p(\phi) = \sup \left\{ p : \lim_{x \rightarrow 0} \frac{\phi(x)}{|x|^p} = 0 \right\} > 0$ .

We first introduce a generalized version of Minkowski's inequality here [75]:

**Lemma 2.4.1.** *If  $\phi$  satisfies first two conditions in Assumption 2.4.1, then for any positive sequences  $\{a_n\}$  and  $\{b_n\}$  we have*

$$\phi^{-1} \left( \sum \phi(a_n + b_n) \right) \leq \phi^{-1} \left( \sum \phi(a_n) \right) + \phi^{-1} \left( \sum \phi(b_n) \right).$$

**Theorem 2.4.2.** *Let  $\phi$  be an even function satisfying Assumption 2.4.1 and  $\pi_n$  a sequence of partitions of  $[0, T]$  with vanishing mesh. Let  $S \in V_\phi(\pi) \cap C^\alpha([0, T], \mathbb{R})$  for some  $\alpha > \frac{\sqrt{1+\frac{4}{p(\phi)}}-1}{2}$ . Let  $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$  be Lipschitz-continuous with respect to  $d_\infty$  with  $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T)$ . Then  $F(\cdot, S) \in V_\phi(\pi)$  and*

$$[F(\cdot, S)]_\pi^\phi(t) = \int_0^t \varphi(|\nabla_\omega F(s, S_s)|) d[S]_\pi^\phi(s).$$

*Proof.* We have the same

$$R_F(s, t) := F(t, S_t) - F(s, S_t) - \nabla_\omega F(s, S_s)(S(t) - S(s)), \quad |R_F(s, t)| \leq C|t - s|^{\alpha+\alpha^2}$$

Let  $\gamma_F(s, t) := \nabla_\omega F(s, S_s)(S(t) - S(s))$ , we have from lemma 2.4.1 that

$$\begin{aligned} & \phi^{-1} \left( \sum_{\substack{[t_j, t_{j+1}] \in \pi_n \\ t_{j+1} \leq t}} \phi(|F(t_{j+1}, S_{t_{j+1}}) - F(t_j, S_{t_j})|) \right) \\ & \leq \phi^{-1} \left( \sum_{\substack{[t_j, t_{j+1}] \in \pi_n \\ t_{j+1} \leq t}} \phi(|R_F(t_j, t_{j+1})|) \right) + \phi^{-1} \left( \sum_{\substack{[t_j, t_{j+1}] \in \pi_n \\ t_{j+1} \leq t}} \phi(|\gamma_F(t_j, t_{j+1})|) \right) \\ & \leq 2\phi^{-1} \left( \sum_{\substack{[t_j, t_{j+1}] \in \pi_n \\ t_{j+1} \leq t}} \phi(|R_F(t_j, t_{j+1})|) \right) \\ & \quad + \phi^{-1} \left( \sum_{\substack{[t_j, t_{j+1}] \in \pi_n \\ t_{j+1} \leq t}} \phi(|F(t_{j+1}, S_{t_{j+1}}) - F(t_j, S_{t_j})|) \right) \end{aligned} \tag{2.4.1}$$

and since  $R_F(s, t) \leq C|t - s|^{\alpha^2+\alpha}$ , we have

$$\phi^{-1} \left( \sum_{\substack{[t_j, t_{j+1}] \in \pi_n \\ t_{j+1} \leq t}} \phi(|R_F(t_j, t_{j+1})|) \right) \leq \phi^{-1} \left( \sum_{\substack{[t_j, t_{j+1}] \in \pi_n \\ t_{j+1} \leq t}} \phi(C|t_{j+1} - t_j|^{\alpha+\alpha^2}) \right).$$

Furthermore due to  $\alpha > \frac{\sqrt{1+\frac{4}{p(\phi)}}-1}{2}$ , we know  $\alpha + \alpha^2 > \frac{1}{p(\phi)}$ . Hence there exists  $\epsilon > 0$  such that

$$\phi(C|t_{j+1} - t_j|^{\alpha+\alpha^2}) \leq \phi(C|t_{j+1} - t_j|^{\frac{1}{p(\phi)-\epsilon}}).$$

Then by the definition of  $p(\phi)$ , we see that

$$\phi(C|t_{j+1} - t_j|^{\frac{1}{p(\phi)-\epsilon}}) \leq \omega(|t_{j+1} - t_j|) C^{p(\phi)-\epsilon/2} |t_{j+1} - t_j|^{\frac{p(\phi)-\epsilon/2}{p(\phi)-\epsilon}},$$

where  $\omega$  is continuous on  $\mathbb{R}^+$  and  $\omega(0) = 0$ . Combine all above, we get

$$\lim_{n \rightarrow \infty} \phi^{-1} \left( \sum_{\substack{[t_j, t_{j+1}] \in \pi_n \\ t_{j+1} \leq t}} \phi(|R_F(t_j, t_{j+1})|) \right) = 0$$

and we also have

$$\lim_{n \rightarrow \infty} \phi^{-1} \left( \sum_{\substack{[t_j, t_{j+1}] \in \pi_n \\ t_{j+1} \leq t}} \phi(|\gamma_F(t_j, t_{j+1})|) \right) = \phi^{-1} \left( \int_0^t \varphi(|\nabla_\omega F(s, S_s)|) d[S]^\phi(s) \right).$$

In fact, since  $\nabla_\omega F(t, S_t) \in \mathbb{B}(\Lambda_T)$ , there exist  $M > 0$  such that  $|\nabla_\omega F(u, S_u)| \leq M$ . For each  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\left| \frac{\phi(xy)}{\phi(y)} - \varphi(x) \right| \leq \epsilon.$$

for all  $|x| \leq M, |y| \leq \delta$ . Then we have for  $n$  large enough (i.e.  $\text{osc}(S, \pi_n) \leq \delta$ )

$$\begin{aligned} & \sum_{[t_j, t_{j+1}] \in \pi_n, t_{j+1} \leq t} \phi(|\nabla_\omega F(t_j, S_{t_j})(S(t_{j+1}) - S(t_j))|) \\ & \leq \sum_{[t_j, t_{j+1}] \in \pi_n, t_{j+1} \leq t} \varphi(|\nabla_\omega F(t_j, S_{t_j})|) \phi(|S(t_{j+1}) - S(t_j)|) + \epsilon \phi(|S(t_{j+1}) - S(t_j)|). \end{aligned}$$

Letting  $n$  tend to infinity and  $\epsilon$  tend to 0, we obtain

$$\lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n, t_{j+1} \leq t} \phi(|\nabla_\omega F(t_j, S_{t_j})(S(t_{j+1}) - S(t_j))|) = \int_0^t \varphi(|\nabla_\omega F(s, S_s)|) d[S]^\phi(s).$$

Then taking limit in (2.4.1), we obtain the result.  $\square$

In particular, if we set  $\phi(x) = |x|^p$  for  $p > 0$ , we have  $\varphi(x) = \lim_{y \rightarrow 0} \frac{|xy|^p}{|y|^p} = |x|^p$ . Hence, we have the corollary

**Corollary 2.4.3.** *Let  $p \in \mathbb{R}^+$ , and  $\alpha > ((1 + \frac{4}{p})^{\frac{1}{2}} - 1)/2$ ,  $\pi_n$  be a sequence of partitions with vanishing mesh, and  $S \in V_p(\pi) \cap C^\alpha([0, T], \mathbb{R})$ . If  $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$  such that  $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T)$  such that  $F$  is Lipschitz-continuous with respect to  $d_\infty$ , then  $F(\cdot, S) \in V_p(\pi)$  and*

$$[F(\cdot, S)]^p(t) = \int_0^t |\nabla_\omega F(s, S_s)|^p d[S]^p(s).$$

**Remark 2.4.4.** Notice that there always exists  $\alpha > \frac{\sqrt{1+\frac{4}{p(\phi)}-1}}{2}$  such that  $V_\phi \cap C^\alpha([0, T], \mathbb{R})$  is not empty.

**Example 2.4.5.** We can see  $\phi(x) = \frac{x}{\sqrt{-\log x}}$  satisfies Assumption (2.4.1). In fact, it is strictly increasing and continuous and  $\phi(0) = 0$ . Furthermore,

$$\phi'(x) = \frac{1}{\sqrt{-\log x}} + \frac{1}{2(\sqrt{-\log x})^3} > 0,$$

and

$$\phi''(x) = \frac{1}{2x(\sqrt{-\log x})^3} + \frac{3}{4x(\sqrt{-\log x})^5} > 0.$$

Hence  $\phi$  is convex. And we have

$$\log \phi(x) = \log x - \frac{\log(-\log x)}{2}.$$

It is easy to check that  $x \rightarrow x - \frac{\log(-x)}{2}$  is convex. And

$$\varphi(x) = \lim_{y \rightarrow 0} \frac{\phi(|x||y|)}{\phi(|y|)} = \lim_{y \rightarrow 0} |x| \sqrt{\frac{-\log y}{-\log x - \log y}} = |x|.$$

Hence we have

$$[F(\cdot, S)]^\phi(t) = \int_0^t |\nabla_\omega F(s, S_s)| d[S]_\pi^\phi(s).$$

## 2.5 Multi-dimensional extensions

The extension of the above results to the multidimensional case is not entirely straightforward, as the space  $V_p(\pi)$  is not a vector space [84].

In the case of integer  $p$ , some definitions may be extended to the vector case by considering symmetric tensor-valued measures as in [24] but this is not convenient for the fractional case. Our definition below is equivalent to the definition in [24] when  $p$  is an integer.

**Definition 2.5.1.** Let  $p \geq 1$  and  $S = (S^1, \dots, S^d) \in C([0, T], \mathbb{R}^d)$  be a continuous path. Let  $\{\pi_n\}$  be a sequence of partition of  $[0, T]$ . We say that  $S$  has a  $p$ -the order variation along  $\pi = \{\pi_n\}$  if  $\text{osc}(S, \pi_n) \rightarrow 0$  and for any linear combination  $S_a := \sum_{i=1}^d a_i S^i$ , we have  $S_a \in V_p(\pi)$ . Furthermore, we denote  $\mu_{S_a}$  to be the weak limit of measures

$$\mu_{S_a}^n := \sum_{[t_j, t_{j+1}] \in \pi_n} \delta(\cdot - t_j) |S_a(t_{j+1}) - S_a(t_j)|^p.$$

We denote  $S \in V_p(\pi)$  if  $S$  satisfies this property.

**Remark 2.5.2.** It can be easily seen [29] that multi-dimensional fractional Brownian motion satisfies this property along sequences of partitions with fine enough mesh.

**Theorem 2.5.3.** Let  $p = m + \alpha$  with  $m = \lfloor p \rfloor$  and  $S \in V_p(\pi)$ . Assume  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies

1.  $\nabla^m f(x) \in C_{loc}^\alpha(\text{Sym}_m(\mathbb{R}^d))$  and there exists a sequence of open sets  $U_k$  such that  $\bar{U}_{k+1} \subset U_k$  and

$$\lim_{y \rightarrow x} \frac{\|\nabla^m f(y) - \nabla^m f(x)\|}{\|y - x\|^\alpha} = 0,$$

locally uniformly on  $U_k^c$ .

2. Setting  $C = \bigcap_k U_k$ , for all  $a = (a_1, \dots, a_d) \in \mathbb{R}^d$  we have

$$\int_0^T \mathbb{1}_{\{S(t) \in C\}} d\mu_{S_a} = 0,$$

where  $\mu_{S_a}$  is defined as in Definition 2.5.1.

Then we have

$$f(S(t)) - f(S(0)) = \int_0^t \nabla f(S(u)) dS(u),$$

where

$$\int_0^t \nabla f(S(u)) dS(u) := \lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \sum_{k=1}^m \frac{1}{k!} \langle \nabla^k f(S(t_j)), (S(t_{j+1}) - S(t_j))^{\otimes k} \rangle.$$

Before we prove the theorem, we give a lemma here.

**Lemma 2.5.4.** Let  $\alpha_1, \dots, \alpha_d$  be positive numbers such that  $p = \sum_{i=1}^d \alpha_i > 1$ . Suppose  $S \in V_p(\pi)$  in the sense of definition 2.5.1. Then the limit of

$$\sum_{[t_j, t_{j+1}] \in \pi_n} \prod_{i=1}^d |S^i(t_{j+1}) - S^i(t_j)|^{\alpha_i}$$

is bounded by a sum of  $p$ -th variations of  $2^d$  different linear combinations of components of  $S$ .

*Proof.* For any positive number  $\alpha_1, \dots, \alpha_d$  such that  $p = \sum_{i=1}^d \alpha_i > 1$ , Young's inequality implies

$$\prod_{i=1}^d |S^i(t_{j+1}) - S^i(t_j)|^{\frac{\alpha_i}{p}} \leq \sum_{i=1}^d \frac{\alpha_i}{p} |S^i(t_{j+1}) - S^i(t_j)|,$$

which shows that

$$\prod_{i=1}^d |S^i(t_{j+1}) - S^i(t_j)|^{\alpha_i} \leq \sum_{\epsilon_i = \pm 1} \left| \sum_{i=1}^d \epsilon_i \frac{\alpha_i}{p} (S^i(t_{j+1}) - S^i(t_j)) \right|^p.$$

By taking sum over the partition  $\pi_n$  and taking the limit of  $n$  we then obtain desired result.  $\square$

*Proof of theorem 2.5.3.* The technique for the proof is the same as the previous theorems. We only consider the situation  $t = T$ , the case  $t < T$  is similar with an  $o(1)$  additional term. Using a Taylor expansion,

$$\begin{aligned} f(S(T)) - f(S(0)) &= \sum_{[t_j, t_{j+1}] \in \pi_n} \sum_{k=1}^m \frac{1}{k!} \langle \nabla^m f(S(t_j)), (S(t_{j+1}) - S(t_j))^{\otimes k} \rangle \\ &\quad + \frac{1}{(m-1)!} \int_0^1 \langle \nabla^m f(S(t_j) + \lambda(S(t_{j+1}) - S(t_j))) - \nabla^m f(S(t_j)), \\ &\quad (S(t_{j+1}) - S(t_j))^{\otimes m} \rangle (1-\lambda)^{m-1} d\lambda. \end{aligned}$$

We denote last integral by  $R(t_j, t_{j+1})$ . By Hölder continuity of  $\nabla^m f$ , there exists  $M > 0$  such that

$$\frac{\|\nabla^m f(S(t_j) + \lambda(S(t_{j+1}) - S(t_j))) - \nabla^m f(S(t_j))\|}{\lambda^\alpha \|S(t_{j+1}) - S(t_j)\|^\alpha} \leq M.$$

This leads to

$$\begin{aligned} &|R(t_j, t_{j+1})| \\ &\leq \frac{M}{(m-1)!} \int_0^1 \lambda^\alpha \|S(t_{j+1}) - S(t_j)\|^\alpha \|(S(t_{j+1}) - S(t_j))^{\otimes m}\| (1-\lambda)^{m-1} d\lambda. \end{aligned}$$

For  $k \in \mathbb{N}$ , we divide the partition  $\pi_n$  into two components  $\pi_n^1 := \{[t_j, t_{j+1}] \in \pi_n : S(t_j) \in U_k\}$  and  $\pi_n^2 := \pi_n \setminus \pi_n^1$ . On  $\pi_n^1$ , we have

$$\sum_{[t_j, t_{j+1}] \in \pi_n^1} |R(t_j, t_{j+1})| \leq \sum_{[t_j, t_{j+1}] \in \pi_n^1} \tilde{M} \|S(t_{j+1}) - S(t_j)\|^\alpha \|(S(t_{j+1}) - S(t_j))^{\otimes m}\|$$

with  $\tilde{M} = \frac{M}{(m-1)!} \int_0^1 \lambda^\alpha (1-\lambda)^{m-1} d\lambda$  and on  $\pi_n^2$ , for each  $\epsilon > 0$ , when  $n$  is large enough,

$$\sum_{[t_j, t_{j+1}] \in \pi_n^2} |R(t_j, t_{j+1})| \leq \sum_{[t_j, t_{j+1}] \in \pi_n^2} \epsilon \|S(t_{j+1}) - S(t_j)\|^\alpha \|(S(t_{j+1}) - S(t_j))^{\otimes m}\|.$$

In order to give our result, we need to give a bound for

$$\lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \|S(t_{j+1}) - S(t_j)\|^\alpha \|(S(t_{j+1}) - S(t_j))^{\otimes m}\|.$$

First we have

$$\|S(t_{j+1}) - S(t_j)\|^\alpha \leq C \sum_{i=1}^d |S^i(t_{j+1}) - S^i(t_j)|^\alpha$$

for some constant  $C$ . By Lemma 2.5.4, we know that each component of

$$\lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \|S(t_{j+1}) - S(t_j)\|^\alpha (S(t_{j+1}) - S(t_j))^{\otimes m}$$

can be bounded by the sum of  $2^d$  different  $p$ -th variations of linear combinations of path  $S$ . Hence

$$\lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \|S(t_{j+1}) - S(t_j)\|^\alpha \|(S(t_{j+1}) - S(t_j))^{\otimes m}\|$$

is bounded by some constant  $C(S, \pi)$  related to path  $S$ . This measure is dominated by a sum of measures  $\mu_r$  with enough terms  $r = 1, \dots, R$ . Then we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} |R(t_j, t_{j+1})| \\ & \leq \lim_{n \rightarrow \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \tilde{M}g_k(S(t_j)) \|S(t_{j+1}) - S(t_j)\|^\alpha \|(S(t_{j+1}) - S(t_j))^{\otimes m}\| + \epsilon C(S, \pi) \\ & \leq \sum_{r=1}^R \int_0^T g_k(S(t)) d\mu_r + \epsilon C(S, \pi). \end{aligned}$$

Here  $g_k \leq 1$  is a positive smooth function with support in  $U_{k-1}$  and  $g_k|_{U_k} = 1$ . Letting  $k \rightarrow \infty$  and  $\epsilon \rightarrow 0$ , from the assumption on  $f$  we obtain the result.  $\square$

**Remark 2.5.5.** *This result may be extended to time-dependent functions as discussed above.*

**Remark 2.5.6.** *One cannot expect to obtain a multi-variate Taylor-type expansion up to order  $p \notin \mathbb{N}$  in general. The simplest function  $f(x) = \|x\|^\alpha$  with  $0 < \alpha < 1$  does not admit a Taylor-type expansion up to order  $\alpha$  at 0 since the limit*

$$\lim_{x \rightarrow 0} \frac{\|x\|^\alpha}{|x_1|^\alpha + \dots + |x_d|^\alpha}$$

*does not exist.*

# Chapter 3

## Compact Support property of Rough Super Brownian Motion

### 3.1 Introduction

The aim of this chapter is to give an affirmative answer to the compact support property of rough super Brownian motion introduced in the paper [78] when studying a branching random walk in a random environment (BRWRE).

Very similar to the recent path-wise construction of solutions to singular SPDEs, such as the regularity structure introduced by Hairer [54] and paracontrolled distribution introduced by Gubinelli, Imkeller and Perkowski [49], the rough super Brownian motion can be viewed as a 'path-wise' version of the limit process of a BRWRE with potential  $V^n$  defined by

$$V^n(x) = \xi_e^n(x) = \xi^n(x) - c_n, \quad \{\xi^n(x)\}_{x \in \mathbb{Z}_n^2} \text{i.i.d} \sim n\Phi$$

for  $c_n \simeq \log(n)$ , a given random variable  $\Phi$  with mean 0 and variance 1 and  $\mathbb{Z}_n^2 = \frac{1}{n}\mathbb{Z}^2$ .

In the 'path-wise' sense, with a deterministic environment  $\xi$  which exhibits similar property with the samples of spatial white noise, the rough super Brownian motion  $\mu$  is characterized by the following log-Laplace equation

$$\mathbb{E}[e^{-\langle \mu(t), \varphi_0 \rangle}] = e^{-\langle \mu(0), U_t \varphi_0 \rangle} \quad (3.1.1)$$

for non-negative  $\varphi_0 \in C_c^\infty(\mathbb{R}^2)$  and  $U_t \varphi_0$  defined in Section 3.3. It is also mentioned in [78] that  $\mu$  is super-exponentially persistent, i.e.

$$\mathbb{P}[\lim_{t \rightarrow \infty} e^{-t\lambda} \langle \mu(t), \varphi \rangle = \infty] > 0$$

for all  $\lambda > 0$  and nonzero positive function  $\varphi \in C_c^\infty(\mathbb{R}^2)$ . This indicates that with positive probability, the mass of the rough super Brownian motion should grow

exponentially and eventually it will spread out to the whole space  $\mathbb{R}^2$ . This is partially due to the growth of spatial white noise  $\xi$  at infinity which accelerates the branching speed. Thus it is not clear if the rough super Brownian motion will stay in some compact set in finite time or not. This is the object that we consider in this chapter and it turns out that the rough super Brownian motion possesses the compact support property.

On the compact support property of the measure-valued process, there are several classical results by János Engländer and Ross G. Pinsky [32, 33]. They consider measure-valued processes satisfying the log-Laplace equation

$$\mathbb{E}[e^{-\langle \mu(t), f \rangle}] = e^{-\langle \mu(0), u_f \rangle},$$

where  $u_f$  is the minimal non-negative solution to the evolution equation

$$\begin{cases} u_t = Lu + \beta u - \alpha u^2 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u(x, 0) = f(x) & \text{in } \mathbb{R}^d, \\ u \geq 0 & \text{in } \mathbb{R}^d \times [0, \infty), \end{cases} \quad (3.1.2)$$

where  $L = \frac{1}{2} \sum_{i,j=1}^d a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$  and  $d$  is the dimension of the space.

Under certain condition ( $\beta$  is bounded from above with some smoothness property of  $\alpha, \beta$ ), they relate the compact support property of measure-valued process  $\mu$  to the uniqueness of the non-negative solution of equation (3.1.2). However, it is impossible with rough super Brownian motion since the coefficient  $\beta = \xi$ , which is only a distribution.

Our study then takes a detailed look at the proofs of the classical results. It is found that the difficulty lies in the understanding of limit behaviour of solutions to the equation of zero initial value

$$\partial_t \varphi = \mathcal{H} \varphi - \frac{\kappa}{2} \varphi^2 + \phi \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \quad \mathcal{H} := \Delta + \xi$$

when  $\phi$  stays 0 in some compact domain, tends to infinity around this compact domain and when this compact domain becomes larger and larger. To control these solutions, we utilize the method developed in [74, 73, 14], which obtains interior estimation with parabolic shrinkage of distance for  $\Phi_3^4$  equation. By adapting the method to our case, we are able to describe the limit solution in the mild solution formulation and give the affirmative answer to the compact support property of the rough super Brownian motion.

## 3.2 Notations

We set  $\mathcal{P}_n := [-n, n]^d$  throughout this chapter and we will use  $z = (t, x)$  for space-time points, while  $t$  represents time and  $x$  or  $y$  represent spatial points only.

For any region  $D \subset \mathbb{R}^d$ , and  $0 < T_1 < T_2$  we set  $D^{T_1, T_2} := [T_1, T_2] \times D$ . When  $T_1 = 0$ , we simply write  $D^T = D^{0, T}$ . For  $r > 0$ , we define  $D_r := \{x \in D : d(x, \partial D) \geq r\}$ , where  $d$  means spatial distance:

$$d(x, \bar{x}) = |x - \bar{x}| := \max_{i=1, \dots, d} \{|x_i - \bar{x}_i|\}. \quad (3.2.1)$$

We also write  $d'$  for the space-time distance with parabolic scaling,

$$d'((t, x), (\bar{t}, \bar{x})) := \max \left\{ \sqrt{|t - \bar{t}|}, |x - \bar{x}| \right\}. \quad (3.2.2)$$

Furthermore, for space-time points  $z \in \mathbb{R}_+ \times \mathbb{R}^d$  and  $R > 0$ , we define the ball  $B'(z, R) := \{\bar{z} \in \mathbb{R}_+ \times \mathbb{R}^d : d'(\bar{z}, z) \leq R, \bar{t} \leq t\}$ . We also define  $B(x, R) := \{\bar{x} \in \mathbb{R}^d : |x - \bar{x}| \leq R\}$  and  $B(x, R, T) = B^T(x, R) := [0, T] \times B(x, R)$ .

The Fourier transform of  $f \in L^1(\mathbb{R}^d)$  is defined as

$$\mathcal{F}f(k) = \int_{\mathbb{R}^d} e^{-2\pi i k \cdot x} f(x) dx,$$

and it is extended to (ultra-)distributions  $f$  by duality.

### 3.2.1 Notations on the regularity

The symbol  $\xi$  will always denote a “typical realization” (i.e. satisfying the regularity requirements mentioned below) of a spatial white noise or a mollified spatial white noise on  $\mathbb{R}^d$ . We also define  $\mathcal{J}\xi$  by the equation

$$-\Delta \mathcal{J}\xi = \chi(\mathcal{D})\xi \quad (3.2.3)$$

for a smooth function  $\chi$  which equals 1 outside of  $(-\frac{1}{4}, \frac{1}{4})^d$  and equals 0 on  $(-\frac{1}{8}, \frac{1}{8})^d$ . And we consider the two-variable distributions (functions in the case of a mollified white noise) formally defined as

$$(\xi X)(x, \bar{x}) := \xi(\bar{x})(X(\bar{x}) - X(x)), \quad (3.2.4)$$

where  $X(x) = x$ , and

$$((\mathcal{J}\xi)\xi)(x, \bar{x}) := (\mathcal{J}\xi(\bar{x}) - \mathcal{J}\xi(x))\xi(\bar{x}) - C, \quad (3.2.5)$$

where  $C$  is a renormalization constant. We will discuss below that for a.e. realization of the (mollified) space white noise in  $d = 2$  these distributions are well-defined.

To measure the regularity of distributions, we follow the approach of [14, 73, 74], but we are more restrictive in the choice of  $\Phi$  below because this allows us to compare the regularity defined here with classical notions of regularity, which is not so clear in [14, 73, 74]. We fix a non-negative (and non-vanishing) smooth and symmetric function  $\tilde{\Phi}$  on  $\mathbb{R}^d$  with support contained in  $B(0, \frac{1}{2})$ . Define

$$\Phi := (\tilde{\Phi} * \tilde{\Phi})^2.$$

Then  $\Phi$  is a non-negative symmetric smooth function with support contained in  $B(0, 1)$ , and  $\mathcal{F}\Phi$  is strictly positive on  $\mathbb{R}^d$ . In fact,  $\mathcal{F}\Phi = ((\mathcal{F}\tilde{\Phi})^2) * ((\mathcal{F}\tilde{\Phi})^2)$  and  $(\mathcal{F}\tilde{\Phi})^2$  is non-negative and real analytic due to the symmetry and compact support property of  $\tilde{\Phi}$ . Lojaciwicz's structure theorem [60] tells us that the set of zeros of a  $d$  dimensional real analytic function is of Hausdorff dimension  $d - 1$ , hence a null set. However, if  $\mathcal{F}\Phi = 0$  for some point  $x$ , it requires the zero set of  $\mathcal{F}\tilde{\Phi}$  to be of infinite Lebesgue measure. This is a contradiction, which means  $\mathcal{F}\Phi$  is strictly positive. Multiplying with a positive constant if necessary, we may assume that the integral of  $\Phi$  over  $\mathbb{R}^d$  equals 1.

Next, for  $\delta > 0$  we set  $\Phi^\delta(x) = \delta^{-d}\Phi(\frac{x}{\delta})$  and define

$$\Psi^{\delta,n} := \Phi^{\delta 2^{-1}} * \Phi^{\delta 2^{-2}} * \dots * \Phi^{\delta 2^{-n}} \quad \text{and} \quad \Psi^\delta = \lim_{n \rightarrow \infty} \Psi^{\delta,n}.$$

Then we have  $\Psi^\delta = \Phi^{\frac{\delta}{2}} * \Psi^{\frac{\delta}{2}}$  and so the Fourier transform of  $\Psi^\delta$  is still strictly positive. Indeed, if  $\mathcal{F}\Psi^\delta(z) = 0$ , then from  $\mathcal{F}\Psi^\delta = \mathcal{F}\Phi^{\frac{\delta}{2}}\mathcal{F}\Psi^{\frac{\delta}{2}}$  and  $\mathcal{F}\Phi^{\frac{\delta}{2}} > 0$  we deduce that also  $\mathcal{F}\Psi^{\delta/2}(z/2) = \mathcal{F}\Psi^{\delta/2}(z) = 0$ . Iterating this, we get  $\mathcal{F}\Psi^\delta(2^{-n}z) = 0$  for all  $n \in \mathbb{N}$  and therefore  $\mathcal{F}\Psi^\delta(0) = 0$  by continuity. But this is impossible because  $\mathcal{F}\Psi^\delta(0) = 1$  by construction.

With the function  $\Psi$ , we define local norms of distributions of negative regularity. We use  $(\cdot)_\delta$  to denote the convolution with  $\Psi^\delta$ , and  $(\cdot)_{\delta,n}$  for the convolution with  $\Psi^{\delta,n}$  for  $n \geq 1$ . For example,  $f_\delta = f * \Psi^\delta$ . On a set  $D \subset \mathbb{R}^d$  and for  $\alpha < 0$ , the local  $\alpha$ -Hölder seminorm of a distribution  $f$  is defined as

$$\|f\|_{\alpha,D} := \sup_{\delta \in (0,1]} \delta^{-\alpha} \|f_\delta\|_D, \quad (3.2.6)$$

where  $\|\cdot\|_D$  is the supremum norm over  $D$ . Note that  $\|f\|_{\alpha,D}$  depends on  $f$  on the set  $B(D, 1)$ , but this will not influence our result since we will only estimate functions that are defined on the whole space. In particular, for the space white noise  $\xi$ , its

multiplication with spatial variable  $\xi X$  and for the related distribution  $(\mathcal{J}\xi)\xi$  we have a.s.

$$\|\xi\|_{n,-1-\epsilon} := \|\xi\|_{\mathcal{P}_n,-1-\epsilon} = \sup_{x \in \mathcal{P}_n} \sup_{\delta \in (0,1]} \delta^{1+\epsilon} |\xi_\delta(x)| < \infty, \quad (3.2.7)$$

$$\|\xi X\|_{n,-\epsilon} := \sup_{x \in \mathcal{P}_n} \sup_{\delta \in (0,1]} \delta^\epsilon \left| \int (\xi X)(x, \bar{x}) \Psi^\delta(x - \bar{x}) d\bar{x} \right| < \infty, \quad (3.2.8)$$

$$\|(\mathcal{J}\xi)\xi\|_{n,-2\epsilon} := \sup_{x \in \mathcal{P}_n} \sup_{\delta \in (0,1]} \delta^{2\epsilon} \left| \int ((\mathcal{J}\xi)\xi)(x, \bar{x}) \Psi^\delta(x - \bar{x}) d\bar{x} \right| < \infty, \quad (3.2.9)$$

for all  $n \in \mathbb{N}$  and all  $\epsilon > 0$ .

We will also work with functions  $U(z, \bar{z})$  of two variables, and we will write  $U(t, x, \bar{x}) := U((t, x), (t, \bar{x}))$ . For  $\alpha \in (1, 2)$ , the  $\alpha$ -Hölder semi-norm for  $U(t, \cdot)$  on  $D$  for points with distance less than  $r$  is defined as

$$[U(t, \cdot)]_{\alpha, D, r} := \sup_{x \in D} \inf_{\nu \in \mathbb{R}^d} \sup_{\substack{\bar{x} \in D \setminus \{x\} \\ |\bar{x} - x| < r}} \frac{|U(t, x, \bar{x}) - \nu \cdot (\bar{x} - x)|}{|x - \bar{x}|^\alpha}. \quad (3.2.10)$$

Note that  $[U(t, \cdot)]_{\alpha, D, r} < \infty$  requires that  $U(t, x, x) = 0$  for all  $x \in D$ . And if  $U(t, \cdot)$  is smooth in both its variables, then at the point  $x$  the optimal  $\nu$  is the spatial gradient of  $U(t, x, \cdot)$  at the point  $x$ . We also define  $\bar{U}(z, \bar{z}) = U(z, \bar{z}) - \nu(x) \cdot (\bar{x} - x)$  when the spatial gradient of  $U(t, x, \cdot)$  exists at point  $x$  and is denoted by  $\nu(x)$ . As shorthand notation we define:

$$[U(t, \cdot)]_{\alpha, D} := [U(t, \cdot)]_{\alpha, D, \infty}.$$

In addition, for  $\alpha \in (0, 1)$ , we use  $\|\cdot\|_{\alpha, D}$  (resp.  $[\cdot]_{\alpha, D}$ ) to denote the  $\alpha$ -Hölder norm (resp. semi-norm) on  $D$ , and  $\|\cdot\|_{\alpha, D, r}$  (resp.  $[\cdot]_{\alpha, D, r}$ ) to denote the  $\alpha$ -Hölder norm on  $D$  within spatial distance  $r$ . For space-time functions  $f$  on  $[0, T] \times D$ , we define the semi-norm

$$[f]_{C_T(\alpha, D, r)} := \sup_{0 \leq t \leq T} [f(t, \cdot)]_{\alpha, D, r} = \sup_{0 \leq t \leq T} \sup_{\substack{x \neq \bar{x} \in D, \\ |x - \bar{x}| < r}} \frac{|f(t, x) - f(t, \bar{x})|}{|x - \bar{x}|^\alpha}.$$

To be consistent with the negative Hölder-norm defined above, we write

$$[\cdot]_{\alpha, D} := [\cdot]_{\alpha, D, 1}.$$

This also extends to norms and to the weighted (semi-)norms that we introduce below. On the whole space  $\mathbb{R}^d$ , we will simply write

$$\|\cdot\|_\alpha := \|\cdot\|_{\alpha, \mathbb{R}^d}.$$

Similarly, for functions of two variables we define

$$[U]_{C_T(\alpha, D, r)} := \sup_{0 \leq t \leq T} [U(t, \cdot)]_{\alpha, D, r}.$$

For simplicity, we will drop the  $\alpha$  when we are valuing the supremum spatial norm.

### 3.2.2 Notations on weights

We use the same weights as in [71].

**Definition 3.2.1.** *We write*

$$\omega^{pol}(x) := \log(1 + |x|), \quad \omega_{\sigma}^{exp}(x) := |x|^{\sigma},$$

where  $x \in \mathbb{R}^d, \sigma \in (0, 1)$ . For  $\omega \in \boldsymbol{\omega} := \{\omega^{pol}\} \cup \{\omega_{\sigma}^{exp} | \sigma \in (0, 1)\}$ , we denote by  $\boldsymbol{\rho}(\omega)$  the set of measurable, strictly positive  $\theta : \mathbb{R}^d \rightarrow (0, \infty)$  such that for some  $\lambda = \lambda(\theta) > 0$  we have

$$\theta(x) \lesssim \theta(y)e^{\lambda\omega(x-y)}, \quad x, y \in \mathbb{R}^d.$$

We also write  $\boldsymbol{\rho}(\boldsymbol{\omega}) := \bigcup_{\omega \in \boldsymbol{\omega}} \boldsymbol{\rho}(\omega)$ . The objects  $\theta \in \boldsymbol{\rho}(\boldsymbol{\omega})$  are called weights.

**Remark 3.2.2.** *We will mainly consider weights*

$$p(a)(x) := (1 + |x|)^a, \quad e(l)(x) := e^{|x|^l},$$

for non-negative  $a, l$ .

For any norm defined in the previous subsection, we may add a weight  $\theta \in \boldsymbol{\rho}(\boldsymbol{\omega})$  and consider a corresponding weighted norms. For example, we define

$$[f]_{C_T(\alpha, D, r, \theta)} := \sup_{t \leq T} \sup_{\substack{x, y \in D, \\ |x-y| \leq r}} \frac{|f(t, x) - f(t, y)|}{\theta(x)|x-y|^{\alpha}},$$

for  $\alpha \in (0, 1)$ , and for  $\alpha < 0$

$$\|f\|_{C_T(\alpha, D, \theta)} := \sup_{t \leq T} \sup_{0 \leq \delta \leq 1} \delta^{-\alpha} \|f(t, \cdot)_{\delta} \theta^{-1}\|_D.$$

This also applies to special norms that include noise  $\xi$ . For example, we have the definition

$$\|(\mathcal{J}\xi)\xi\|_{-2\epsilon, \mathbb{R}^d, \theta} := \sup_x \sup_{\delta \in (0, 1]} \delta^{2\epsilon} \left| \theta^{-1}(x) \int ((\mathcal{J}\xi)\xi)(x, \bar{x}) \Psi^{\delta}(x - \bar{x}) d\bar{x} \right|, \quad (3.2.11)$$

In addition, we will consider weighted Besov spaces. Let  $\rho_{-1}, \rho_0 \in C_c(\mathbb{R}^d)$  be two non-negative and radial functions such that the support of  $\rho_{-1}$  is contained in a ball  $B \subset \mathbb{R}^d$ , the support of  $\rho_0$  is contained in an annulus  $\{x \in \mathbb{R}^d : 0 < a \leq |x| \leq b\}$  and such that with

$$\rho_j = \rho_0(2^{-j}\cdot), \quad j \in \mathbb{N}_0,$$

the following conditions are satisfied:

1.  $\sum_{j=-1}^{\infty} \rho_j(x) = 1$  for all  $x \in \mathbb{R}^d$ ;
2.  $\text{supp}(\rho_i) \cap \text{supp}(\rho_j) = \emptyset$  whenever  $|i - j| > 1$ .

To deal with weights in  $\boldsymbol{\varrho}(\omega_\sigma^{\text{exp}})$  we need to consider ultra-distributions, and in particular we need to assume (which is possible) that the partition of unity is in  $\mathcal{S}_{\omega_\sigma^{\text{exp}}}$ , the space of smooth functions  $f$  such that  $f$ ,  $\mathcal{F}f$  and all their derivatives decrease faster than  $e^{-\lambda\omega_\sigma^{\text{exp}}}$  at infinity, for any  $\lambda > 0$ . The topological dual of  $\mathcal{S}_\omega$  is denoted by  $\mathcal{S}'_\omega$  and it is called the space of ultra-distributions. Here we do not need to know much about ultra-distributions and we refer to [71, Section 2.2] for a more detailed discussion.

Given  $\omega \in \boldsymbol{\omega}$  and a partition of unity with  $\rho_{-1}, \rho_0 \in \mathcal{S}_\omega$  we define for any ultra-distribution  $f \in \mathcal{S}'_\omega$  the Littlewood-Paley blocks of  $f$  as

$$\Delta_j f = \mathcal{F}^{-1}(\rho_j \mathcal{F}(f)), \quad j \geq -1.$$

**Definition 3.2.3.** For  $\alpha \in \mathbb{R}$ ,  $p, q \in [1, \infty]$  and  $\theta \in \boldsymbol{\varrho}(\omega)$ , we define the weighted Besov space  $B_{p,q}^\alpha(\mathbb{R}^d, \theta)$  by

$$B_{p,q}^\alpha(\mathbb{R}^d, \theta) := \{f \in \mathcal{S}'_\omega : \|f\|_{B_{p,q}^\alpha(\mathbb{R}^d, \theta)} := \|(2^{j\alpha} \|\theta^{-1} \Delta_j f\|_{L^p})\|_{\ell^q} < \infty\}.$$

Note that we consider  $\|\theta^{-1} \Delta_j f\|_{L^p}$ , while in [71] it would be  $\|\theta \Delta_j f\|_{L^p}$ . In the notation of [71] our weighted Besov space would be called  $B_{p,q}^\alpha(\mathbb{R}^d, \theta^{-1})$ .

Next, let us compare the different notions of regularity.

**Lemma 3.2.4.** Let  $\alpha \in (-\infty, 1) \setminus \{0\}$ ,  $\theta \in \boldsymbol{\varrho}(\omega)$ , and let  $f \in \mathcal{S}'_\omega$  be an ultra-distribution on  $\mathbb{R}^d$  such that  $\|f\|_{\alpha, \theta} < \infty$ . Then  $f \in B_{\infty, \infty}^\alpha(\mathbb{R}^d, \theta)$  and

$$\|f\|_{B_{\infty, \infty}^\alpha(\mathbb{R}^d, \theta)} \lesssim \|f\|_{\alpha, \theta}.$$

*Proof.* We divide the proof into two parts. When  $\alpha > 0$  and  $k > -1$ , we have

$$\Delta_k f(x) = (\mathcal{F}^{-1} \rho_k) * (f - f(x))(x),$$

since  $\int \mathcal{F}^{-1} \rho_k(x) dx = \rho_k(0) = 0$ . Using the definition of  $\|f\|_{\alpha, \theta}$ , we estimate

$$\begin{aligned} & |(\mathcal{F}^{-1} \rho_k) * (f - f(x))(x) \theta^{-1}(x)| \\ &= \left| \int (\mathcal{F}^{-1} \rho_k)(y) (f(x-y) - f(x)) \theta^{-1}(x) dy \right| \\ &\lesssim \left( \left| \int_{|y| \leq 1} |(\mathcal{F}^{-1} \rho_k)(y)| \cdot |y|^\alpha dy \right| + \left| \int_{|y| \geq 1} |(\mathcal{F}^{-1} \rho_k)(y)| (1 + e^{\lambda(\theta)\omega(y)}) dy \right| \right) \|f\|_{\alpha, \theta} \\ &\lesssim 2^{-k\alpha} \left( 1 + \left| \int (\mathcal{F}^{-1} \rho_0)(y) |y|^\alpha e^{\lambda\omega(y)} dy \right| \right) \|f\|_{\alpha, \theta} \lesssim 2^{-k\alpha} \|f\|_{\alpha, \theta}. \end{aligned}$$

The last inequality holds because  $\rho_0 \in \mathcal{S}_\omega$ . The estimate for  $k = -1$  is similar and therefore the result is true for  $\alpha > 0$ .

Next, we consider the case  $\alpha < 0$ . The  $k$ -th block of  $f$  is given by

$$\begin{aligned}\Delta_k f &= \mathcal{F}^{-1}(\rho_k \mathcal{F}f) \\ &= \mathcal{F}^{-1}\left(\frac{\rho_k}{\mathcal{F}\Psi^\delta} \mathcal{F}f \mathcal{F}\Psi^\delta\right) \\ &= \mathcal{F}^{-1}\left(\frac{\rho_k}{\mathcal{F}\Psi^\delta}\right) * f_\delta, \quad \forall 0 < \delta \leq 1,\end{aligned}$$

where we used that  $\mathcal{F}\Psi^\delta$  is strictly positive. With  $\delta = 2^{-k}$ , we have for  $k \geq 0$ :

$$\begin{aligned}\|\Delta_k f \cdot \theta^{-1}\|_{L^\infty} &\leq \left\| \mathcal{F}^{-1}\left(\frac{\rho_k}{\mathcal{F}\Psi^\delta}\right) e^{\lambda\omega} \right\|_{L^1} \|f_\delta \theta^{-1}\|_{L^\infty} \\ &\lesssim \left\| \mathcal{F}^{-1}\left(\frac{\rho_0}{\mathcal{F}\Psi^1}\right) e^{\lambda\omega} \right\|_{L^1} 2^{-k\alpha},\end{aligned}$$

where we used that  $e^{\lambda\omega(2^{-k}z)} \leq e^{\lambda\omega(z)}$ . Let us assume  $\left\| \mathcal{F}^{-1}\left(\frac{\rho_0}{\mathcal{F}\Psi^1}\right) e^{\lambda\omega} \right\|_{L^1} < \infty$  for now, which we will show at the end of this proof. Then

$$\sup_{k \geq 0} 2^{k\alpha} \|\Delta_k f \cdot \theta^{-1}\|_{L^\infty} \lesssim \left\| \mathcal{F}^{-1}\left(\frac{\rho_0}{\mathcal{F}\Psi^1}\right) e^{\lambda\omega} \right\|_{L^1} < \infty.$$

For  $k = -1$ , we choose  $\delta = 1$  and obtain

$$\|\Delta_{-1} f \cdot \theta^{-1}\|_{L^\infty} \lesssim \left\| \mathcal{F}^{-1}\left(\frac{\rho_{-1}}{\mathcal{F}\Psi^1}\right) e^{\lambda\omega} \right\|_{L^1} < \infty.$$

Thus,  $f \in B_{\infty, \infty}^\alpha(\mathbb{R}^d, \theta)$ .

Now, let us prove that  $\left\| \mathcal{F}^{-1}\left(\frac{\rho_0}{\mathcal{F}\Psi^1}\right) e^{\lambda\omega} \right\|_{L^1} < \infty$ . The case  $\omega(x) = \log(1 + |x|)$  follows directly from the fact that  $\rho_{-1}, \rho_0 \in C_c^\infty$  have compact support. For the case  $\omega(x) = |x|^\sigma$ , we only need to show that for any  $\hat{\lambda} > 0$

$$\mathcal{F}^{-1}\left(\frac{\rho_0}{\mathcal{F}\Psi^1}\right) \lesssim_{\hat{\lambda}} e^{-\hat{\lambda}|x|^\sigma},$$

and by Lemma 3.7 in [71] this follows if for all  $\delta > 0$ , there is  $C > 0$  such that for all  $l \geq 0$  and  $i = 1, 2$ , we have

$$\left\| D_i^l \left( \frac{\rho_0}{\mathcal{F}\Psi^1} \right) \right\|_{L^1} \lesssim_\delta \delta^l C^l (l!)^{\frac{1}{\sigma}}.$$

Now on the support of  $\rho_0$ , we have, by applying Leibniz's rule to  $1 = \mathcal{F}\Psi^1 \cdot \frac{1}{\mathcal{F}\Psi^1}$ ,

$$0 = \sum_{k=0}^l \binom{l}{k} D_i^k (\mathcal{F}\Psi^1) D_i^{l-k} \left( \frac{1}{\mathcal{F}\Psi^1} \right).$$

By induction (make use of  $\|D_i^k(\mathcal{F}\Psi^1)\|_{L^\infty} \leq C_0^k$  for some constant  $C_0$  since  $\Psi^1$  is compactly supported), we can show that on the support of  $\rho_0$ ,

$$\left\| D_i^k \left( \frac{1}{\mathcal{F}\Psi^1} \right) \right\|_{L^\infty} \lesssim_\delta \delta^k C_1^k (k!)^{\frac{1}{\sigma}},$$

for  $\delta < 1$ , and the constant in the inequality can be chosen proportional to  $e^{\frac{1}{\delta}}$ . Hence, again by applying Leibniz's rule to  $D_i^l \left( \frac{\rho_0}{\mathcal{F}\Psi^1} \right)$ , it remains to bound

$$\left\| \sum_{k=0}^l \binom{l}{k} (D_i^k \rho_0) \delta^{l-k} C_1^{l-k} ((l-k)!)^{\frac{1}{\sigma}} B_1 \right\|_{L^1}.$$

Now we use that  $\rho_0$  is compactly supported and that  $\rho_0 \in \mathcal{S}_\omega(\mathbb{R}^d)$ , which gives

$$\|D_i^k \rho_0\|_{L^1} \lesssim \|D_i^k \rho_0\|_{L^\infty} \lesssim \| |x|^k \hat{\rho} \|_{L^1} \lesssim \int |x|^k e^{-|x|^\sigma} dx \lesssim \left( \frac{k+d}{\sigma} \right)^{\lfloor \frac{k}{\sigma} \rfloor} \lesssim C_2^k (k!)^{\frac{1}{\sigma}}.$$

Thus, we obtain the result.  $\square$

**Remark 3.2.5.** *The reverse inequality  $\|f\|_{\alpha, \theta} \lesssim \|f\|_{B_{\infty, \infty}^{\alpha, \theta}(\mathbb{R}^d)}$  also holds. But we do not need this, so we do not include the proof of it.*

We will also measure the time regularity, so we consider also the norm

$$\|f\|_{\mathcal{L}_T^\alpha(\mathbb{R}^d, \theta)} = \|f\|_{C_T(\alpha, \mathbb{R}^d, \theta)} + \|f\|_{C_T^{\frac{\alpha}{2}}(\mathbb{R}^d, \theta)},$$

where

$$\|f\|_{C_T^{\frac{\alpha}{2}}(\mathbb{R}^d, \theta)} := \sup_{0 \leq s < t \leq T} \frac{\|f(t) - f(s)\|_{\mathbb{R}^d, \theta_t}}{|t - s|^{\frac{\alpha}{2}}}.$$

and  $\theta$  is a time-dependent weight. If we do not specify the time-dependence, then we choose  $\theta_t \equiv \theta$  and just write  $\mathcal{L}_T^\alpha(\mathbb{R}^d, \theta)$ .

### 3.3 Rough Super Brownian Motion

Here we recall and extend the definition of the rough super-Brownian motion from [78]. From now on, we will work in dimension 2 and in what follows the letter  $d$  will not represent the dimension. While our arguments are dimension independent they do need regularity requirements that the white noise only satisfies in  $d = 2$ . In higher dimensions we could consider a slightly mollified white noise and apply the same arguments, or we could treat a  $3d$  white noise with the same approach but at the price of more technicalities. In  $d \geq 4$  the parabolic Anderson model with white noise is scaling (super-)critical and there does not exist any solution theory for it,

and thus in particular there exists no rough super-Brownian motion with white noise environment in  $d \geq 4$ .

We fix a parameter  $\kappa > 0$ , which will describe the strength of branching in the rough super-Brownian motion. Let  $\varphi_0 \in C_c^\infty(\mathbb{R}^2)$ ,  $\varphi_0 \geq 0$  and  $\phi \in C_c(\mathbb{R}^2)$ ,  $\phi \geq 0$ . We use  $U_t^\phi \varphi_0$  to denote the solution to the equation

$$\begin{cases} \partial_t \varphi = \mathcal{H}\varphi - \frac{\kappa}{2}\varphi^2 + \phi, & \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ \varphi(0, \cdot) = \varphi_0, & \text{on } \mathbb{R}^2, \end{cases} \quad (3.3.1)$$

where  $\mathcal{H} = \Delta + \xi$  is the so-called Anderson Hamiltonian; see the next section for the solution theory, where it is also shown that  $U_t^\phi \varphi_0 \geq 0$  for all  $t \geq 0$ . Just to clarify the terminology, note that for  $\kappa = 0$  and  $\phi = 0$  this equation is the parabolic Anderson model. We will discuss the solution theory of (3.3.1) in the next section. In particular, we use  $U_t \varphi_0$  to denote the solution for  $\phi = 0$ , i.e.

$$\begin{cases} \partial_t \varphi = \mathcal{H}\varphi - \frac{\kappa}{2}\varphi^2, & \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ \varphi(0, \cdot) = \varphi_0, & \text{on } \mathbb{R}^2. \end{cases} \quad (3.3.2)$$

The definition of the rough super Brownian motion only requires making sense of equation (3.3.2), which can be solved as long as the resonant product  $\mathcal{J}\xi \odot \xi$  is provided. Here we write

$$f \otimes g = \sum_{i < j-1} \Delta_i f \Delta_j g, \quad f \odot g = \sum_{|i-j| \leq 1} \Delta_i \Delta_j g,$$

for the paraproduct and the resonant product; see [49] for the estimates on  $\otimes$  and  $\odot$  that we will need.

We let  $\epsilon > 0$  be small enough ( $\epsilon < 1/3$  suffices) and write the regularities of the (2d) white noise and related distributions as  $-1 - \epsilon$ ,  $-2\epsilon$ , etc. We make the following assumption on the noise  $\xi$ .

**Assumption 3.3.1.** *Let  $\xi_\alpha$  to be a sequence of smooth function, we assume there exist distributions  $\xi, \mathcal{J}\xi \odot \xi$  and constants  $C_\alpha$  such that, with some  $\epsilon' < \epsilon$ , and*

$$\lim_{\alpha \rightarrow 0} \|\xi_\alpha - \xi\|_{-1-\epsilon', \mathbb{R}^2, p(a)} = 0, \quad \lim_{\alpha \rightarrow 0} \|\mathcal{J}\xi_\alpha \odot \xi_\alpha - C_\alpha - \mathcal{J}\xi \odot \xi\|_{-2\epsilon', \mathbb{R}^2, p(a)} = 0,$$

for all  $a > 0$ , where  $-\Delta \mathcal{J}\xi_\alpha = \chi(\mathcal{D})\xi_\alpha$ .

**Lemma 3.3.1.** *Suppose  $\{\xi_\alpha\}_{\alpha \in (0,1)}$  is a sequence of smooth functions indexed by  $\alpha$  satisfying assumption 3.3.1, then we have that*

$$\|\xi_\alpha\|_{-1-\epsilon', \mathbb{R}^2, p(a)} < \infty, \quad \|(\mathcal{J}\xi_\alpha)\xi_\alpha\|_{-2\epsilon', \mathbb{R}^2, p(a)} < \infty$$

uniformly over  $\alpha$  for all  $a > 0$ . Conversely, it is also true. So these two conditions are equivalent.

*Proof.* Suppose the assumption 3.3.1 holds. It is obvious that  $\|\xi_\alpha\|_{-1-\epsilon', \mathbb{R}^2, p(a)} < \infty$  by the equivalence of two definitions of Besov norm. For the latter one, we have the decomposition

$$\begin{aligned} & (\mathcal{J}\xi_\alpha(\bar{x}) - \mathcal{J}\xi_\alpha(x))\xi_\alpha(\bar{x}) - C_\alpha \\ &= \mathcal{J}\xi_\alpha \otimes \xi_\alpha(\bar{x}) + \mathcal{J}\xi_\alpha \odot \xi_\alpha(\bar{x}) + \xi_\alpha \otimes \mathcal{J}\xi_\alpha(\bar{x}) - \mathcal{J}\xi_\alpha(x)\xi_\alpha(\bar{x}) - C_\alpha. \end{aligned}$$

From the theory of paracontrolled product [49], we know that  $\xi \otimes \mathcal{J}\xi$  has the regularity  $-2\epsilon'$ , together with the equivalence of two definitions of Hölder space, we know

$$\sup_{x \in \mathbb{R}^2} \sup_{\delta \in (0,1]} \left| \int (\mathcal{J}\xi_\alpha \odot \xi_\alpha(\bar{x}) - C_\alpha + \xi_\alpha \otimes \mathcal{J}\xi_\alpha(\bar{x})) \Psi^\delta(x - \bar{x}) d\bar{x} \right| \delta^{2\epsilon'} < \infty.$$

uniformly over  $\alpha \in (0, 1)$ . It remains to check

$$\sup_{x \in \mathbb{R}^2} \sup_{\delta \in (0,1]} \left| \int (\mathcal{J}\xi \otimes \xi(\bar{x}) - \mathcal{J}\xi_\alpha(x)\xi_\alpha(\bar{x})) \Psi^\delta(x - \bar{x}) d\bar{x} \right| \delta^{2\epsilon'} < \infty \quad (3.3.3)$$

We now consider a modelled distribution  $f^\pi(x) = \mathcal{J}\xi(x)\Xi$  with regularity structure of PAM such that  $\Pi_x \Xi = \xi$ , see [54] for more detail. Due to theorem 6.10 in [49], since the regularity of the modelled distribution is small than 0, the reconstruction operator can be taken by the bony product  $\otimes$ . Thus, the inequality (3.3.3) holds. The converse is basically the same, so we have the equivalence.  $\square$

This assumption is an alternative version of one of the assumption in the Assumption 2.3 (“Deterministic Environment”) in [78], which guarantees the well-posedness of the PAM and hence the existence of the rough super-Brownian motion. The assumption is satisfied for almost all sample paths of the space white noise on  $\mathbb{R}^2$ , see [78].

Let  $\mathcal{M}(\mathbb{R}^2)$  be the space of finite positive Borel measures on  $\mathbb{R}^2$ , equipped with the topology of weak convergence.

**Definition 3.3.2.** *Let  $\xi$  satisfies Assumption 3.3.1. Let  $\kappa > 0$  and let  $\mu$  be a process with values in the space  $C([0, \infty), \mathcal{M}(\mathbb{R}^2))$ , such that  $\mu(0)$  is compact supported. Write  $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, \infty]}$  for the completed and right-continuous filtration generated by  $\mu$ . We call  $\mu$  a rough super-Brownian motion (rSBM) with parameter  $\kappa$  if it satisfies one of the two following equivalent properties:*

1. *For any  $t \geq 0$  and  $\varphi_0 \in C_c^\infty(\mathbb{R}^2)$ ,  $\varphi_0 \geq 0$  and for  $U.\varphi_0$  the solution to equation (3.3.2) with initial condition  $\varphi_0$ , the process*

$$N_t^{\varphi_0}(s) = e^{-\langle \mu(s), U_{t-s}\varphi_0 \rangle}, \quad s \in [0, t]$$

*is a bounded continuous  $\mathcal{F}$  – martingale with respect to  $s$  on  $[0, t]$ .*

2. For any  $t \geq 0$  and  $\varphi_0 \in C_c^\infty(\mathbb{R}^2)$  and  $f \in C([0, t]; C^\zeta(\mathbb{R}^2, e(l)))$  for some  $\zeta > 0$  and  $l < -t$ , and for  $\varphi_t$  solving

$$\partial_s \varphi_t + \mathcal{H}\varphi_t = f, \quad s \in [0, t], \quad \varphi_t(t) = \varphi_0. \quad (3.3.4)$$

it holds that

$$s \rightarrow M_t^{\varphi_0, f}(s) := \langle \mu(s), \varphi_t(s) \rangle - \langle \mu(0), \varphi_t(0) \rangle - \int_0^s dr \langle \mu(r), f(r) \rangle,$$

defined for  $s \in [0, t]$ , is a continuous square-integrable  $\mathcal{F}$ -martingale with quadratic variation

$$\langle M_t^{\varphi_0, f} \rangle_s = \kappa \int_0^s dr \langle \mu(r), (\varphi_t)^2(r) \rangle.$$

In order to discuss the compact support property of the rough super-Brownian motion, we will need to generalize the first definition of the process.

**Lemma 3.3.3.** *The definition of the rough super-Brownian motion is equivalent to the following property: For any  $t \geq 0$  and  $\varphi_0 \in C_c^\infty(\mathbb{R}^2)$ ,  $\phi \in C_c^\infty(\mathbb{R}^2)$ ,  $\varphi_0, \phi \geq 0$  and for  $U_t^\phi \varphi_0$  the solution to equation (3.3.1) with initial condition  $\varphi_0$ , the process*

$$N_t^{\varphi_0, \phi}(s) = e^{-\langle \mu(s), U_{t-s}^\phi \varphi_0 \rangle - \int_0^s dr \langle \mu(r), \phi \rangle}, \quad s \in [0, t]$$

is a bounded continuous  $\mathcal{F}$ -martingale. In particular, the rough super-Brownian motion satisfies

$$\mathbb{E} \left[ e^{-\langle \mu(t), \varphi_0 \rangle - \int_0^t \langle \mu(r), \phi \rangle dr} \right] = e^{-\langle \mu(0), U_t^\phi \varphi_0 \rangle}. \quad (3.3.5)$$

*Proof.* Clearly the condition is sufficient, because it is stronger than our first characterization of the rough super-Brownian motion. To see that it is also necessary, consider  $\varphi_t(s) = U_{t-s}^\phi \varphi_0$  with time independent function  $\phi \in C_c^\infty(\mathbb{R}^d)$ . Then  $\varphi_t$  satisfies equation (3.3.4) with  $f = \frac{\kappa}{2}(\varphi_t)^2 - \phi$ . Hence, Itô's formula applied to  $F = e^{-x} \in C^2(\mathbb{R}^d)$  yields

$$\begin{aligned} N_t^{\varphi_0, \phi}(s) &= F \left( \langle \mu(s), \varphi_t(s) \rangle + \int_0^s dr \langle \mu(r), \phi \rangle \right) \\ &= N_t^{\varphi_0, \phi}(0) - \int_0^s N_t^{\varphi_0, \phi}(r) \langle \mu(r), \frac{\kappa}{2}(\varphi_t)^2(r) \rangle dr \\ &\quad + \frac{1}{2} \int_0^s N_t^{\varphi_0, \phi}(r) d\langle M_t^{\varphi_0, f} \rangle(r) - \int_0^s N_t^{\varphi_0, \phi}(r) dM_t^{\varphi_0, f}(r). \end{aligned}$$

Now since

$$\langle M_t^{\varphi_0, f} \rangle_s = \kappa \int_0^s dr \langle \mu(r), (\varphi_t)^2(r) \rangle,$$

the drift terms vanish and  $N^{\varphi_0, \phi}$  is a local martingale. As  $N^{\varphi_0, \phi}$  is bounded it is a true martingale.  $\square$

The compact support property of a measure-valued process is formulated in the following definition.

**Definition 3.3.4.** *Suppose  $\mu$  is a stochastic process with values in  $C(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^2))$ . We say that  $\mu$  possesses the compact support property if for all  $t \geq 0$*

$$\mathbb{P} \left[ \left( \bigcup_{0 \leq s \leq t} \text{supp } \mu(s) \right) \text{ is bounded} \right] = 1.$$

Now we are ready to state our main result.

**Theorem 3.3.5.** *The rough super-Brownian motion on  $\mathbb{R}^2$  possesses the compact support property.*

The proof is given in Section 3.5. Let us formulate several necessary steps now, following Engländer and Pinsky [32, 33]. Recall that  $\mathcal{P}_n = [-n, n]^2$  and consider regions  $\mathcal{P}_n^m := (-n - \frac{1}{m}, n + \frac{1}{m})^2$ . Let  $\phi_n^m \in C_c^\infty$  be such that

$$\phi_n^m(x) = \begin{cases} 0, & x \in \mathcal{P}_n \cup \mathcal{P}_{n+2}^c, \\ m, & x \in (\mathcal{P}_n^m)^c \cap \mathcal{P}_{n+1}, \\ \phi_n^m(x) \in [0, m], & \text{elsewhere.} \end{cases}$$

Then consider  $\varphi_n^m(t) := U_t^{\phi_n^m} 0$ , where 0 is the function with value 0 everywhere. Define  $A_t^n := \{\mu(s)((\mathcal{P}_n)^c) = 0, s \leq t\}$ . Since the rough super-Brownian motion has continuous trajectories in  $\mathcal{M}(\mathbb{R}^2)$ , it follows from (3.3.5) that

$$\begin{aligned} \mathbb{P}[A_t^n] &= \lim_{m \rightarrow \infty} \mathbb{E} \left[ \exp \left( - \int_0^t \langle \mu(s), \phi_n^m \rangle ds \right) \right] \\ &= \lim_{m \rightarrow \infty} \exp(-\langle \mu(0), \varphi_n^m(t) \rangle). \end{aligned}$$

Hence

$$\begin{aligned} &\mathbb{P} \left[ \left( \bigcup_{0 \leq s \leq t} \text{supp } \mu(s) \right) \text{ is bounded} \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{P}[A_t^n] = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \exp(-\langle \mu(0), \varphi_n^m(t) \rangle). \end{aligned}$$

Thus, our goal is to show that  $\lim_n \lim_m \varphi_n^m = 0$ . In [32] this is based on explicit supersolutions, which works because they assume  $\xi$  to be bounded. In our case we do not know any explicit supersolutions, and instead we will derive nonlinear interior estimates for (3.3.1) by adapting the methods of [73] for  $\Phi_3^4$  to our setting. We leave the discussion of the well-posedness of (3.3.1) to the section 3.5 and discuss the interior estimates first.

### 3.4 Interior estimates

Here we consider positive solutions to the equation

$$\begin{cases} \partial_t u = (\Delta + \xi - C)u - \frac{\kappa}{2}u^2, & \text{in } \mathbb{R}_+ \times \mathcal{P}_n, \\ u(0, \cdot) = 0, & \text{in } \{0\} \times \mathcal{P}_n, \end{cases} \quad (3.4.1)$$

where  $\xi$  is a typical realization of a mollified white noise on  $\mathbb{R}^2$ ,  $C$  is a renormalization constant, and we do not specify any boundary conditions on  $\partial\mathcal{P}_n$ . We will use the nonlinearity  $-\frac{\kappa}{2}u^2$  to derive uniform bounds for  $u$  on the interior of  $\mathcal{P}_n$ . Most ideas in this section come from the works of Chandra, Moinat and Weber, [73, 14, 74], with some changes to enable estimations near time 0. We define the two-variable functions

$$U(z, \bar{z}) := u(\bar{z}) - u(z) - u(z)(\mathcal{J}\xi(\bar{x}) - \mathcal{J}\xi(x)), \quad (3.4.2)$$

and

$$\bar{U}(z, \bar{z}) := U(z, \bar{z}) - \nu(z)(\bar{x} - x), \quad (3.4.3)$$

where  $\nu(z)$  is the spatial derivative of  $U(z, \cdot)$  at  $z$ . Note that we are working with regularized noise, so all the functions we encounter are smooth and the derivative  $\nu(z)$  exists. Of course, our goal is to derive estimates that are uniform in the mollification parameter.

The main estimate in this section is:

**Theorem 3.4.1.** *Let  $n \in \mathbb{N}$ ,  $0 < l < n$ . If  $u$  solves equation (3.4.1) in  $[0, T] \times \mathcal{P}_n$ , then we have:*

$$\|u\|_{C_T \mathcal{P}_{n-l}} \lesssim \max \left\{ \frac{1}{l^2}, \|\tau\|_{n, |\tau|}^{\frac{2}{n_\tau(1-\epsilon)}} : \tau \in \mathcal{T} \right\} \quad (3.4.4)$$

for  $\mathcal{T} = \{\xi, (\mathcal{J}\xi)\xi, \xi X\}$  and  $\mathcal{P}_{n-l} = [-n+l, n-l]^2$ . Here  $|\tau|$  is the regularity of  $\tau$  and  $n_\tau$  is the number of nodes in the tree. The implicit constant in “ $\lesssim$ ” only depends on  $\kappa, \Psi$  and  $\epsilon$ .

The proof is inspired by the paper [74]. We divide the proof into several lemmas. As a direct consequence, we get a bound for weighted norms of  $u$ .

**Corollary 3.4.2.** *Let  $r > 0$  and let  $\theta \in \mathfrak{g}(\omega)$  be a weight such that  $\theta$  is a radial function that is increasing in  $|x|$  and such that  $\theta(0) \geq 1$ . Let  $u$  solve the equation (3.4.1) in  $[0, T] \times \mathcal{P}_{r+1}$ . Then*

$$\|u\|_{C_T(\mathcal{P}_r, \theta^{\frac{2}{1-\epsilon}})} \lesssim 1 + \sum_{\tau=\xi, (\mathcal{J}\xi)\xi} \|\tau\|_{r+1, |\tau|, \theta}^{\frac{2}{n_\tau(1-\epsilon)}} \quad (3.4.5)$$

*Proof.* We take  $n = r + 1$  in the inequality (3.4.4), then we have

$$\|u\|_{C_T \mathcal{P}_r} \lesssim \max \left\{ 1, \|\tau\|_{r+1, |\tau|}^{\frac{2}{n_\tau(1-\epsilon)}} : \tau \in \mathcal{J} \right\}$$

Then we check the  $\theta^{\frac{2}{1-\epsilon}}$  weighted supremum of  $u$  between the radius  $r$  and  $r + 1$ , which should be controlled by

$$e^{\frac{2}{1-\epsilon}\lambda\omega(1)} \max \left\{ \theta(r+1)^{-1}, \theta(r+1)^{-\frac{2}{n_\tau(1-\epsilon)}} \|\tau\|_{r+1, |\tau|}^{\frac{2}{n_\tau(1-\epsilon)}} : \tau \in \mathcal{J} \right\}$$

Since  $\theta$  is increasing, we obtain the result.  $\square$

**Remark 3.4.3.** *In the proof of Theorem 3.4.1 we will not only derive bounds on norms of  $u$ , but also bounds on norms of related functions such as  $U, \nu$  (see for example (3.4.21), which is true without assumption (3.4.31), and choose carefully the quantity  $d_0$  to get an estimate on  $[U]_{C_T(2-2\epsilon, D_d, d)}$ ). Moreover, we can also control the regularity of  $u$  and we get for  $\tilde{\theta} = \theta^{\frac{4}{1-\epsilon}}$*

$$\|u\|_{C_T(1-\epsilon, \mathcal{P}_r, \tilde{\theta})} \lesssim 1 + \sum_{\tau=\xi, \mathcal{J}\xi\xi} \|\tau\|_{r+1, |\tau|, \theta}^{\frac{4}{n_\tau(1-\epsilon)}} \quad (3.4.6)$$

The same bound holds for  $\|\nu\|_{C_T(1-2\epsilon, \mathcal{P}_r, \tilde{\theta})}$ ,  $\|U\|_{C_T(\mathcal{P}_r, \tilde{\theta})}$  and  $[U]_{C_T(2-2\epsilon, \mathcal{P}_r, \tilde{\theta})}$ .

*Proof.* Let  $C = \max \left\{ \|\tau\|_{n, |\tau|}^{\frac{2}{n_\tau(1-\epsilon)}} : \tau = \xi X, \mathcal{J}\xi\xi, \xi \right\}$ , choose  $d_0 = c_0 C^{-\frac{1}{2}}$  and substitute it into the inequality (3.4.21) with  $D = \mathcal{P}_{n-\frac{1}{2}}$ , we obtain that

$$c_0^{2-2\epsilon} C^{-1+\epsilon} [U]_{C_T(2-2\epsilon, D_{d_0}, d_0)} \lesssim_{c_0} \|u\|_{C_T(D)},$$

for  $c_0 < c_1$ , where  $c_1$  is defined in lemma 3.4.12 and does not depend on  $n$ . To estimate the Hölder norm of  $u$ , we combine equations (3.4.23), (3.4.26) and equation (3.4.25) to get

$$[u]_{C_T(1-\epsilon, D_d, d)} \lesssim [U]_{C_T(2-2\epsilon, D_d, d)} d^{1-\epsilon} + \|u\|_{C_T D_d} [\mathcal{J}\xi]_{n, 1-\epsilon} + \|u\|_{C_T D_d} d^{-1+\epsilon}$$

for any  $d \leq d_0$ . Now substitute  $d_0 = c_0 C^{-\frac{1}{2}}$ , we have

$$[u]_{C_T(1-\epsilon, D_{d_0}, d_0)} \lesssim (c_0^{-1+\epsilon} C^{\frac{1-\epsilon}{2}} + C^{\frac{1-\epsilon}{2}}) \|u\|_{C_T(D)}.$$

Thus, we have actually

$$[u]_{C_T(1-\epsilon, D_{d_0})} \lesssim (c_0^{-1+\epsilon} + 1) C^{\frac{1-\epsilon}{2}} \|u\|_{C_T(D)} + d_0^{-1+\epsilon} \|u\|_{C_T(D)}$$

This then gives the result.  $\square$

The proof of Theorem 3.4.1 will be based on a series of lemmas whose proofs are in Appendix B.1. To start with, we give an interior supremum norm estimate for a simplified equation.

**Lemma 3.4.4.** *Let  $T > 0$  and let  $u \in C^\infty$  solve*

$$\begin{cases} (\partial_t - \Delta)u = -u^2 + g & \text{in } [0, T] \times \mathcal{P}_n, \\ u = 0 & \text{in } \{0\} \times \mathcal{P}_n, \\ u \geq 0, \end{cases} \quad (3.4.7)$$

where  $g$  is a smooth and bounded function. Then the following point-wise bound on  $u$  holds for all  $z = (t, x) \in [0, T] \times \mathcal{P}_n$ :

$$u(t, x) \leq 28 \cdot \max \left\{ \frac{1}{\min\{(n - x_i)^2, (n + x_i)^2, i = 1, 2\}}, \sqrt{\|g\|_{C_T \mathcal{P}_n}} \right\}. \quad (3.4.8)$$

The proof is on p. 142 in the appendix. We then have a direct consequence of this lemma for the equation (3.4.1).

**Lemma 3.4.5.** *Let  $u$  be solution of the equation (3.4.1) with  $\kappa = 2$ . Suppose  $R > R'$  and  $r + R < n$ , we have for any  $0 < \delta < 1$ ,*

$$\begin{aligned} \|u\|_{C_T \mathcal{P}_{n-r-R}} &\lesssim \max \left\{ \frac{1}{(R - R')^2}, \delta^{1-\epsilon} [u]_{C_T(1-\epsilon, \mathcal{P}_{n-r-R'+\delta}, \delta)}, \right. \\ &\quad \|u\|_{C_T(\mathcal{P}_{n-r-R'+\delta})}^{\frac{1}{2}} \delta^{\frac{1-\epsilon}{2}} [u]_{C_T(1-\epsilon, \mathcal{P}_{n-r-R'+\delta}, \delta)}^{\frac{1}{2}}, \\ &\quad \left. \|(u(\xi - C))_\delta\|_{C_T(\mathcal{P}_{n-r-R'})}^{\frac{1}{2}} \right\}. \end{aligned} \quad (3.4.9)$$

*Proof.* Let's first convolute the equation (3.4.1) with  $\Psi^\delta$  to obtain

$$(\partial_t - \Delta)u_\delta = (u\xi)_\delta - Cu_\delta - (u^2)_\delta. \quad (3.4.10)$$

Rewrite the equation (3.4.10) to

$$(\partial_t - \Delta)u_\delta = -u_\delta^2 + u_\delta^2 - (u^2)_\delta + (u\xi)_\delta - Cu_\delta.$$

Then the Lemma 3.4.4 shows

$$\|u_\delta\|_{C_T(\mathcal{P}_{n-r-R})} \lesssim \max \left\{ (R - R')^{-2}, \|u_\delta^2 - (u^2)_\delta\|_{C_T(\mathcal{P}_{n-r-R'})}^{\frac{1}{2}}, \|(u(\xi - C))_\delta\|_{C_T(\mathcal{P}_{n-r-R'})}^{\frac{1}{2}} \right\}.$$

Since for any region  $D \subset \mathbb{R}^2$ ,

$$\|u_\delta - u\|_{C_T(D_\delta)} \lesssim_\Psi \delta^{1-\epsilon} [u]_{C_T(1-\epsilon, D, \delta)},$$

and

$$\begin{aligned}
& (u_\delta^2 - (u^2)_\delta)(t, x) \\
&= \int \Psi^\delta(x-y)(u_\delta^2(t, x) - u^2(t, y))dy \\
&= 2 \int \Psi^\delta(x-y)(u_\delta(t, x) - u(t, y)) \int (\lambda u_\delta(t, x) + (1-\lambda)u(t, y))d\lambda dy \\
&\lesssim \|u(t, \cdot)\|_{B(x, \delta)} \int \Psi^\delta(x-y)(u_\delta(t, x) - u(t, x) + u(t, x) - u(t, y))dy \\
&\lesssim \|u(t, \cdot)\|_{B(x, \delta)} \int \Psi^\delta(x-y)(\delta^{1-\epsilon} + d(x, y)^{1-\epsilon})[u(t, \cdot)]_{1-\epsilon, B(x, \delta), \delta} dy \\
&\lesssim \|u\|_{C_T(B(x, \delta))} \delta^{1-\epsilon} [u]_{C_T(1-\epsilon, B(x, \delta), \delta)},
\end{aligned}$$

we then have the result.  $\square$

In order to estimate the norms, we need to adjust two lemmas from [74] to our setting. The first one is a variant of Hairer's reconstruction theorem [54]. Recall the notation  $B^T(x, R) = [0, T] \times B(x, R)$ , where  $B(x, R)$  is the closed ball with radius  $R$  and center  $x$ , and that  $\Psi^\delta$  is the function from Section 3.2.1 with which we measure regularity.

**Lemma 3.4.6** ([74], Theorem 2.8). *Let  $\gamma > 0$  and let  $A$  be a finite subset of  $(-\infty, \gamma]$ . Let  $T > 0, \delta \in [0, 1)$  and  $x \in \mathbb{R}^2$ . Let  $F : B^T(x, \delta)^2 \rightarrow \mathbb{R}$  be continuous and such that for all  $\beta \in A$  there exist constants  $C_\beta > 0$  and  $\gamma_\beta \geq \gamma$  such that for all  $\bar{\delta} \in (0, \delta)$ , for all  $x_1 \in B(x, \delta - \bar{\delta})$  and  $x_2 \in B(x_1, \bar{\delta})$*

$$\left| \int \Psi^{\bar{\delta}}(x_2 - y)(F(t, x_1, y) - F(t, x_2, y))dy \right| \leq \sum_{\beta \in A} C_\beta d(x_1, x_2)^{\gamma_\beta - \beta} \bar{\delta}^\beta, \quad (3.4.11)$$

uniformly in  $t \in [0, T]$ . Then  $f(t, x) := F(t, x, x)$  satisfies for all  $t \in [0, T]$

$$\left| \int \Psi^\delta(x-y)(F(t, x, y) - f(t, y))dy \right| \lesssim \sum_{\beta \in A} C_\beta \delta^{\gamma_\beta}, \quad (3.4.12)$$

where the implicit constant in " $\lesssim$ " only depends on  $A$  and  $\gamma$ .

To be precise, the formulation of this lemma is slightly different than in [74] because the time variable appears differently. But the proof from [74] works verbatim in our case, we just have to freeze  $t \in [0, T]$ . As a consequence, we get the following estimate,

where we recall that

$$[u(t, \cdot)]_{\alpha, D, r, \theta} = \sup_{\substack{x \neq \bar{x} \in D, \\ |x - \bar{x}| < r}} \frac{|u(t, x) - u(t, \bar{x})|}{\theta(x)|x - \bar{x}|^\alpha},$$

$$[U(t, \cdot)]_{\beta, D, r, \theta} := \sup_{x \in D} \inf_{\nu \in \mathbb{R}^d} \sup_{\substack{\bar{x} \in D \setminus \{x\} \\ |\bar{x} - x| < r}} \frac{|U(t, x, \bar{x}) - \nu \cdot (\bar{x} - x)|}{\theta(x)|x - \bar{x}|^\beta},$$

for  $\alpha \in (0, 1)$  and for  $\beta \in (1, 2)$ .

**Lemma 3.4.7.** *Let  $u$  be the solution to the equation (3.4.1). The following bounds holds for any weight  $\theta \in \mathbf{\rho}(\omega)$  and for  $z = (t, x)$  with  $x \in \mathcal{P}_n$ :*

$$\begin{aligned} & |((u - u(z))\xi)_\delta(z) - Cu_\delta(z)|\theta^{-2}(x) \\ & \lesssim e^{2\lambda(\theta)\omega(\delta)}\delta^{1-3\epsilon} \left( [u(t, \cdot)]_{1-\epsilon, B(x, \delta), \delta, \theta} \|\mathcal{J}\xi\xi\|_{n, -2\epsilon, \theta} \right. \\ & \quad + [U(t, \cdot)]_{2-2\epsilon, B(x, \delta), \delta, \theta} \|\xi\|_{n, -1-\epsilon, \theta} \\ & \quad \left. + [\nu(t, \cdot)]_{1-2\epsilon, B(x, \delta), \delta, \theta} \|\xi X\|_{n, -\epsilon, \theta} \right) \\ & + |u(z)|\theta^{-1}(x) \|\mathcal{J}\xi\xi\|_{n, -2\epsilon, \theta} \delta^{-2\epsilon} + |\nu(z)|\theta^{-1}(x) \|\xi X\|_{n, -\epsilon, \theta} \delta^{-\epsilon}, \end{aligned} \quad (3.4.13)$$

where, by a small abuse of notation, we write  $\omega(\delta) = \omega(x)$  for some (and thus all)  $x$  with  $|x| = \delta$ .

*Proof of Lemma 3.4.7.* Recall that  $U(z, \cdot) = u - u(z) - u(z)(\mathcal{J}\xi - \mathcal{J}\xi(x))$  and  $\bar{U}(z, \cdot) = U(z, \cdot) - \nu(z)(\cdot - x)$  and therefore

$$\begin{aligned} & ((u - u(z))\xi)_\delta(z) - Cu_\delta(z) \\ & = (\bar{U}(z, \cdot)\xi)_\delta(z) + C(u(z) - u_\delta(z)) + u(z)((\mathcal{J}\xi - \mathcal{J}\xi(z))\xi)_\delta - C \\ & \quad + \nu(z)((\cdot - x)\xi)_\delta. \end{aligned} \quad (3.4.14)$$

For  $z = (t, x)$  with  $x \in \mathcal{P}_n$  we bound the last two terms on the right hand-side by

$$\theta^{-2}(x)|u(z)|((\mathcal{J}\xi - \mathcal{J}\xi(z))\xi)_\delta - C(z) \leq \theta^{-1}(x)|u(z)|\|\mathcal{J}\xi\xi\|_{n, -2\epsilon, \theta} \delta^{-2\epsilon} \quad (3.4.15)$$

and

$$\theta^{-2}(x)|\nu(z)|((\cdot - x)\xi)_\delta(z) \leq \theta^{-1}(x)|\nu(z)|\|\xi X\|_{n, -\epsilon, \theta} \delta^{-\epsilon}. \quad (3.4.16)$$

To control the remaining terms in (3.4.14) let

$$F(t, x, \bar{x}) = \bar{U}(z, \bar{z})\xi(\bar{x}) + C(u(z) - u(\bar{z})),$$

where  $\bar{z} = (t, \bar{x})$ . Then

$$\begin{aligned} F(t, x_1, y) - F(t, x_2, y) & = (u(z_2) - u(z_1))((\mathcal{J}\xi(y) - \mathcal{J}\xi(x_2))\xi(y) - C) \\ & \quad + \bar{U}(t, x_1, x_2)\xi(y) + (\nu(z_2) - \nu(z_1))(y - x_2)\xi(y). \end{aligned}$$

Hence, for  $\bar{\delta} \in (0, \delta)$  and for  $x_1 \in B(x, \delta - \bar{\delta}), x_2 \in B(x_1, \bar{\delta})$ , we have

$$\begin{aligned} & \int \Psi^{\bar{\delta}}(x_2 - y)(F(t, x_1, y) - F(t, x_2, y))dy \\ & \leq \theta(x_2)^2 \left( [u(t, \cdot)]_{1-\epsilon, B(x, \delta), \delta, \theta} d(x_1, x_2)^{1-\epsilon} \|\mathcal{J}\xi\xi\|_{n, -2\epsilon, \theta} \bar{\delta}^{-2\epsilon} \right. \\ & \quad + [U(t, \cdot)]_{2-2\epsilon, B(x, \delta), \delta, \theta} d(x_1, x_2)^{2-2\epsilon} \|\xi\|_{n, -1-\epsilon, \theta} \bar{\delta}^{-1-\epsilon} \\ & \quad \left. + [\nu(t, \cdot)]_{1-2\epsilon, B(x, \delta), \delta, \theta} d(x_1, x_2)^{1-2\epsilon} \|\xi X\|_{n, -\epsilon, \theta} \bar{\delta}^{-\epsilon} \right) \end{aligned}$$

Thus, using that  $f(t, x) = F(t, x, x) = 0$ , we obtain from Lemma 3.4.6 the bound

$$\begin{aligned} & \theta^{-2}(x) |(\bar{U}(z, \cdot)\xi)_\delta(z) + C(u(z) - u_\delta(z))| \\ & = \theta^{-2}(x) \left| \int \Psi^\delta(x - y)F(t, x, y)dy \right| \\ & \leq e^{2\lambda(\theta)\omega(\delta)} \delta^{1-3\epsilon} \left( [u(t, \cdot)]_{1-\epsilon, B(x, \delta), \delta, \theta} \|\mathcal{J}\xi\xi\|_{n, -2\epsilon, \theta} \right. \\ & \quad + [U(t, \cdot)]_{2-2\epsilon, B(x, \delta), \delta, \theta} \|\xi\|_{n, -1-\epsilon, \theta} \\ & \quad \left. + [\nu(t, \cdot)]_{1-2\epsilon, B(x, \delta), \delta, \theta} \|\xi X\|_{n, -\epsilon, \theta} \right). \end{aligned}$$

Plugging this back into (3.4.14), together with (3.4.15) and (3.4.16), we obtain the claimed bound.  $\square$

As a direct consequence, we have

**Corollary 3.4.8.** *Let  $u$  be the solution to the equation (3.4.1) and  $\theta \in \boldsymbol{\rho}(\omega)$ , we have*

$$\begin{aligned} & \|(u\xi - Cu)_\delta\|_{C_T(D_\delta, \theta^2)} \\ & \lesssim_\theta \delta^{1-3\epsilon} ([u]_{C_T(1-\epsilon, D, \delta, \theta)} \|\mathcal{J}\xi\xi\|_{n, -2\epsilon, \theta} \\ & \quad + [U]_{C_T(2-2\epsilon, D, \delta, \theta)} \|\xi\|_{n, -1-\epsilon, \theta} + [\nu]_{C_T(1-2\epsilon, D, \delta, \theta)} \|\xi X\|_{n, -\epsilon, \theta}) \\ & \quad + \|u\|_{C_T(D_\delta, \theta)} (\|\mathcal{J}\xi\xi\|_{n, -2\epsilon, \theta} \delta^{-2\epsilon} + \|\xi\|_{n, -1-\epsilon, \theta} \delta^{-1-\epsilon}) \\ & \quad + \|\nu\|_{C_T(D_\delta, \theta)} \|\xi X\|_{n, -\epsilon, \theta} \delta^{-\epsilon}. \end{aligned} \tag{3.4.17}$$

*Proof.* We have

$$\begin{aligned} & \|(u\xi - Cu)_\delta\|_{C_T(D_\delta, \theta^2)} \\ & \lesssim \|((u - u(z))\xi)_\delta(z) - Cu_\delta(z)\|_{C_T(D_\delta, \theta^2)} + \|u\xi_\delta\|_{C_T(D_\delta, \theta^2)} \end{aligned}$$

then apply lemma 3.4.7, we obtain the result.  $\square$

We also need to adapt a Schauder estimate for functions of two variables from [74] (a variant of Hairer's Schauder estimates for modelled distributions [54]):

**Lemma 3.4.9.** *Let  $T > 0$ ,  $\kappa \in (1, 2)$  and let  $A \subset (-\infty, \kappa]$  be finite. Let  $U$  be a bounded function of two variables defined on a domain  $D^T \times D^T$  such that  $U(z, z) = 0$  for all  $z \in D^T$  and  $U(0, \cdot, \cdot) = 0$ . Let  $d_0 > 0$  and assume that for any  $0 < d \leq d_0$  and  $L \leq \frac{d}{4}$  there exists a constant  $M_{D_d, L}^{(1)}$  such that for all base points  $x \in D_d$  and length scales  $\delta \leq L$ , it holds uniformly in  $t$  that*

$$\delta^2 \|(\partial_t - \Delta)U_\delta(t, x, \cdot)\|_{B(x, L)} \leq M_{D_d, L}^{(1)} \sum_{\beta \in A} \delta^\beta L^{\kappa - \beta}. \quad (3.4.18)$$

*Assume furthermore, that for  $L_1, L_2 \leq \frac{d}{4}$  there exists a constant  $M_{D_d, L_1, L_2}^{(2)}$  such that for any  $x \in D_d$ , for any  $x_1 \in B(x, L_1)$  for any  $x_2 \in B(x_1, L_2)$  the following 'three-point continuity' holds:*

$$|U(t, x, x_2) - U(t, x, x_1) - U(t, x_1, x_2)| \leq M_{D_d, L_1, L_2}^{(2)} \sum_{\beta \in A} d(x_1, x)^\beta d(x_2, x_1)^{\kappa - \beta}. \quad (3.4.19)$$

*Additionally, define*

$$M^{(1)} := \sup_{d \leq d_0} d^\kappa M_{D_d, \frac{d}{4}}^{(1)}, \quad M^{(2)} := \sup_{d \leq d_0} d^\kappa M_{D_d, \frac{d}{4}, \frac{d}{4}}^{(2)}.$$

*Then*

$$\sup_{d \leq d_0} d^\kappa [U]_{C_T(\kappa, D_d, d)} \lesssim M^{(1)} + M^{(2)} + \sup_{d \leq d_0} \|U\|_{D_d^T, d}. \quad (3.4.20)$$

*Here, ' $\lesssim$ ' denotes a bound that holds up to a multiplicative constant that only depends on  $\kappa$  and  $A$  and  $T$ .*

The proof is on p. 145. As a direct consequence we get an estimate for  $\nu$ , with the same proof as in [74]:

**Corollary 3.4.10.** *Fix  $0 \leq d \leq d_0$  such that  $D_d \neq \emptyset$ . Assume that  $D_d$  satisfies an interior cone condition with parameter  $d \geq r_d > 0$  and  $\lambda \in (0, 1)$ , i.e. for all  $r \in [0, r_d]$ , for all  $x \in D_d$ , for any vector  $\nu \in \mathbb{R}^2$ , there exists  $y \in D_d$  such that  $d(x, y) = r$  and*

$$|\nu \cdot (y - x)| \geq \lambda |\nu| d(x, y).$$

*Then for the optimal function  $\nu$  in inequality (3.4.20), for all  $r \in [0, r_d]$ ,*

$$\lambda \|\nu(t, \cdot)\|_{D_d} \leq [U(t, \cdot)]_{\kappa, D_d, d} r^{\kappa - 1} + \|U(t, \cdot)\|_{D_d, r} r^{-1}$$

*If inequality (3.4.19) holds for all  $t \in [0, T]$ ,  $x, x_1, x_2 \in D_d$ , we have for  $r \leq r_d$ ,*

$$[\nu(t, \cdot)]_{\kappa - 1} \lesssim [U(t, \cdot)]_{\kappa, D_d, d} + M_{D_d, \frac{d}{4}, \frac{d}{4}}^{(2)} + r^{-\kappa} \|U(t, \cdot)\|_{D_d, r}$$

*Here ' $\lesssim$ ' denotes a bound that holds up to a multiplicative constant that only depends on  $\lambda, \kappa$  and  $A$ .*

Recall the definition (3.4.2) and the definition (3.4.3)

$$U(z, \bar{z}) := u(\bar{z}) - u(z) - u(z)(\mathcal{J}\xi(\bar{x}) - \mathcal{J}\xi(x)),$$

we apply lemma 3.4.9 and the corollary 3.4.10 to the function  $U$  and obtain the estimation

**Lemma 3.4.11.** *Let  $u$  be solution of the equation (3.4.1) with  $\kappa = 2$ . For any region  $D \subset \mathcal{P}_n$ , and  $d_0 > 0$  such that  $D$  contains a ball of radius  $d_0$ , we have*

$$\sup_{d \leq d_0} d^{2-2\epsilon} [U]_{C_T(2-2\epsilon, D_d, d)} \lesssim \sup_{d \leq d_0} \{A \|u\|_{C_T D} + B d^{2-2\epsilon} [U]_{C_T(2-2\epsilon, D_d, d)}\}, \quad (3.4.21)$$

where

$$\begin{aligned} A = & d^2 \|u\|_{C_T D} + d^2 J(\xi, \chi) + d^{1-\epsilon} (\|\xi X\|_{n, -\epsilon} + [\mathcal{J}\xi]_{n, 1-\epsilon} + \|\xi\|_{n, -1-\epsilon}) \\ & + d^{2-2\epsilon} ([\mathcal{J}\xi]_{n, 1-\epsilon} \|\xi\|_{n, -1-\epsilon} + [\mathcal{J}\xi]_{n, 1-\epsilon}^2 + \|\xi X\|_{n, -\epsilon} [\mathcal{J}\xi]_{n, 1-\epsilon} + \|\mathcal{J}\xi\xi\|_{n, -2\epsilon}) \\ & + d^{3-3\epsilon} ([\mathcal{J}\xi]_{n, 1-\epsilon} \|\mathcal{J}\xi\xi\|_{n, -2\epsilon} + [\mathcal{J}\xi]_{n, 1-\epsilon}^2 \|\xi X\|_{n, -\epsilon}) + 1, \end{aligned}$$

and

$$\begin{aligned} B = & d^{1-\epsilon} (\|\xi\|_{n, -1-\epsilon} + [\mathcal{J}\xi]_{n, 1-\epsilon} + \|\xi X\|_{n, -\epsilon}) \\ & + d^{2-2\epsilon} (\|\mathcal{J}\xi\xi\|_{n, -2\epsilon} + \|\xi X\|_{n, \epsilon} [\mathcal{J}\xi]_{n, 1-\epsilon}). \end{aligned}$$

*Proof.* For any  $L > 0, 0 < \delta < 1$ , let  $x \in D_{L+\delta}, \bar{x} \in B(x, L) \subset D_\delta$ . We have

$$\begin{aligned} & (\partial_t - \Delta)U(t, x, \cdot)_\delta(\bar{x}) \\ = & \int \Psi^\delta(\bar{x} - \tilde{x})(\partial_t - \Delta_{\tilde{x}})U(t, x, \tilde{x})d\tilde{x} \\ = & (u(\xi - C))_\delta(\bar{z}) - u(z)\xi_\delta(\bar{z}) - (u^2)_\delta(\bar{z}) + u(z)((1 - \chi(\mathcal{D}))\xi)_\delta(\bar{z}) \\ = & ((u - u(\bar{z}))\xi - Cu)_\delta(\bar{z}) - (u^2)_\delta(\bar{z}) + (u(\bar{z}) - u(z))\xi_\delta(\bar{z}) \\ & + u(z)((1 - \chi(\mathcal{D}))\xi)_\delta(\bar{z}). \end{aligned}$$

Since  $(1 - \chi)$  is compactly supported,  $(1 - \chi(\mathcal{D}))\xi$  is smooth and the supremum norm is bounded by a constant  $J(\xi, \chi)$ . For other terms, we have estimations

$$|(u^2)_\delta(\bar{z})| \leq \|u(t, \cdot)\|_{B(\bar{x}, \delta)}^2 \leq \|u\|_{C_T D}^2,$$

and

$$\begin{aligned} |(u(\bar{z}) - u(z))\xi_\delta(\bar{z})| & \leq d(x, \bar{x})^{1-\epsilon} [u(t, \cdot)]_{1-\epsilon, B(x, L), L} \|\xi\|_{n, -1-\epsilon} \delta^{-1-\epsilon} \\ & \leq L^{1-\epsilon} [u(t, \cdot)]_{1-\epsilon, B(x, L), L} \|\xi\|_{n, -1-\epsilon} \delta^{-1-\epsilon}. \end{aligned}$$

For  $x, L, \delta, d$ , s.t.  $x \in D_d, \delta \leq L \leq \frac{d}{4}, d \leq d_0$ , we have

$$\begin{aligned}
& \|(\partial_t - \Delta)U(t, x, \cdot)\|_{B(x, L)} \\
& \lesssim \|u\|_{C_T D}^2 + L^{1-\epsilon} [u]_{C_T(1-\epsilon, D_\delta, L)} \|\xi\|_{n, -1-\epsilon} \delta^{-1-\epsilon} \\
& \quad + \delta^{1-3\epsilon} ([u]_{C_T(1-\epsilon, B(x, L+\delta), \delta)} \|\mathcal{J}\xi\xi\|_{n, -2\epsilon} \\
& \quad + [U]_{C_T(2-2\epsilon, B(x, L+\delta), \delta)} \|\xi\|_{n, -1-\epsilon} + [\nu]_{C_T(1-2\epsilon, B(x, L+\delta), \delta)} \|\xi X\|_{n, -\epsilon}) \\
& \quad + \|u\|_{C_T B(x, L)} \|\mathcal{J}\xi\xi\|_{n, -2\epsilon} \delta^{-2\epsilon} + \|\nu\|_{C_T(B(x, L))} \|\xi X\|_{n, -\epsilon} \delta^{-\epsilon} + J(\xi, \chi) \|u\|_{C_T D}.
\end{aligned}$$

For  $L_1, L_2 \leq \frac{d}{4}, x \in D_d, x_1 \in B(x, L_1), x_2 \in B(x_1, L_2)$ , we have

$$\begin{aligned}
& |U(t, x, x_2) - U(t, x, x_1) - U(t, x_1, x_2)| \\
& = |(u(t, x_1) - u(t, x))(\mathcal{J}\xi(x_2) - \mathcal{J}\xi(x_1))| \\
& \leq [u(t, \cdot)]_{1-\epsilon, D_{\frac{d}{2}, \frac{d}{2}}} [\mathcal{J}\xi]_{n, 1-\epsilon} d(x_1, x)^{1-\epsilon} d(x_1, x_2)^{1-\epsilon}.
\end{aligned}$$

With  $\kappa = 2 - 2\epsilon$  in the Lemma 3.4.9, we obtain

$$\begin{aligned}
& \sup_{d \leq d_0} d^{2-2\epsilon} [U]_{\mathcal{M}_T(2-2\epsilon, D_d, d)} \\
& \lesssim \sup_{d \leq d_0} \{d^2 \|u\|_{C_T D}^2 + d^2 J(\xi, \chi) \|u\|_{C_T D} + [u]_{C_T(1-\epsilon, D_{\frac{d}{2}, \frac{d}{2}})} \|\xi\|_{n, -1-\epsilon} d^{2-2\epsilon} \\
& \quad + d^{3-3\epsilon} ([u]_{C_T(1-\epsilon, D_{\frac{d}{2}, \frac{d}{2}})} \|\mathcal{J}\xi\xi\|_{n, -2\epsilon} + [U]_{C_T(2-2\epsilon, D_{\frac{d}{2}, \frac{d}{2}})} \|\xi\|_{n, -1-\epsilon} \\
& \quad + [\nu]_{C_T(1-2\epsilon, D_{\frac{d}{2}, \frac{d}{2}})} \|\xi X\|_{n, -\epsilon}) + d^{2-\epsilon} \|\nu\|_{C_T(D_{\frac{d}{2}, \frac{d}{2}})} \|\xi X\|_{n, -\epsilon} \\
& \quad + d^{2-2\epsilon} \|u\|_{C_T(D_{\frac{d}{2}})} \|\mathcal{J}\xi\xi\|_{n, -2\epsilon}\} + \sup_{d \leq d_0} \|U\|_{C_T D_d, d}.
\end{aligned} \tag{3.4.22}$$

By corollary 3.4.10, we have inequalities for  $d \leq d_0$

$$\|\nu\|_{C_T D_d} \lesssim [U]_{C_T(2-2\epsilon, D_d, d)} d^{1-2\epsilon} + \|U\|_{C_T(D_d, d)} d^{-1}, \tag{3.4.23}$$

and

$$\begin{aligned}
[\nu]_{C_T(1-2\epsilon, D_d, d)} & \lesssim [U]_{C_T(2-2\epsilon, D_d, d)} + d^{-2+2\epsilon} \|U\|_{C_T(D_d, d)} \\
& \quad + [u]_{C_T(1-\epsilon, D_{\frac{d}{4}, \frac{d}{4}})} [\mathcal{J}\xi]_{n, 1-\epsilon}.
\end{aligned} \tag{3.4.24}$$

We have furthermore,

$$[u]_{C_T(1-\epsilon, D_d, d)} \lesssim [U]_{C_T(2-2\epsilon, D_d, d)} d^{1-\epsilon} + \|u\|_{C_T D_d} [\mathcal{J}\xi]_{n, 1-\epsilon} + \|\nu\|_{C_T D_d} d^\epsilon, \tag{3.4.25}$$

and

$$\|U\|_{C_T(D_d, d)} \lesssim 2\|u\|_{C_T D_d} + \|u\|_{C_T D_d} [\mathcal{J}\xi]_{n, 1-\epsilon} d^{1-\epsilon}. \tag{3.4.26}$$

Now substitute the above four inequalities (3.4.23 - 3.4.26) for equation (3.4.22), and we obtain the result.  $\square$

With the help of these lemmas, we are now ready to give the proof of Theorem 3.4.1. We start with a lemma

**Lemma 3.4.12.** *Suppose for some  $r > 0$  we have*

$$c_0 \|u\|_{C_T(\mathcal{P}_{n-r})} \geq \max \left\{ \|\tau\|_{n,|\tau|}^{\frac{2}{n\tau(1-\epsilon)}} : \tau = \xi, \xi X, \mathcal{J}\xi\xi \right\} \geq c_0$$

for some  $c_0 < c_1 < 1$ , where  $c_1$  is a constant depends on the equation itself (does not depend on  $n$ ). Then we have for any  $\|u\|_{C_T(\mathcal{P}_{n-r})}^{-\frac{1}{2}} < R < n - r$ ,

$$\max \left\{ \frac{C_0^2}{\left(R - \|u\|_{C_T(\mathcal{P}_{n-r})}^{-\frac{1}{2}}\right)^2}, \frac{1}{2} \|u\|_{C_T \mathcal{P}_{n-r}} \right\} \quad (3.4.27)$$

*Proof.* Let  $D = \mathcal{P}_{n-r}$ . Start with investigation of  $A, B$  in lemma 3.4.11, we first have the

$$A \lesssim 1 + J(\xi, \chi) + c_0^{\frac{1-\epsilon}{2}} + c_0^{1-\epsilon} + c_0^{\frac{3(1-\epsilon)}{2}} \lesssim 1 + J(\xi, \chi) + c_0^{\frac{1-\epsilon}{2}},$$

and

$$B \lesssim c_0^{\frac{1-\epsilon}{2}},$$

by choosing  $d_0 = \|u\|_{C_T D}^{-\frac{1}{2}} \leq 1$ . There then exists a constant  $c_1 < 1$  such that whenever  $c_0 \leq c_1$  we have the inequality

$$\sup_{d \leq d_0} d^{2-2\epsilon} [U]_{C_T(2-2\epsilon, D_d, d)} \lesssim (1 + J(\xi, \chi) + c_0^{\frac{1-\epsilon}{2}}) \|u\|_{C_T D} \quad (3.4.28)$$

where  $' \lesssim'$  does not depends on the value of  $c_0$  as long as  $c_0 \leq c_1$ . Then by inequalities (3.4.23 – 3.4.26), we have the estimation

$$\sup_{d \leq d_0} \{ \|U\|_{D_d, d}, d^{1-\epsilon} [u]_{C_T(1-\epsilon, D_d, d)}, d \|\nu\|_{C_T D_d}, d^{2-2\epsilon} [\nu]_{C_T(1-2\epsilon, D_d, d)} \} \lesssim \|u\|_{C_T D} \quad (3.4.29)$$

Combine lemma 3.4.5, lemma 3.4.7 with  $\theta = 1$  and equations (3.4.29), (3.4.28) with  $D_d = \mathcal{P}_{n-r-R'+\delta}$ ,  $R' = d_0$ ,  $\delta = \frac{d_0}{k}$ ,  $d = d_0 \frac{k-1}{k}$  for some  $k > 2$ , we obtain

$$\begin{aligned} \|u\|_{C_T \mathcal{P}_{n-r-R}} &\lesssim \max \left\{ \frac{1}{(R - R')^2}, \|u\|_{C_T \mathcal{P}_{n-r}} \left( (k-1)^{\epsilon-1} + (k-1)^{\frac{\epsilon-1}{2}} \right), \right. \\ &\quad \|u\|_{C_T \mathcal{P}_{n-r}} \cdot \left\{ c_0^{\frac{1-\epsilon}{2}} (k-1)^{\frac{\epsilon-1}{2}} k^\epsilon + c_0^{\frac{1-\epsilon}{4}} k^{\frac{1+\epsilon}{4}} (k-1)^{\epsilon-1} + c_0^{\frac{1-\epsilon}{2}} k^\epsilon \right. \\ &\quad \left. \left. + c_0^{\frac{1-\epsilon}{4}} k^{\frac{1+\epsilon}{2}} + c_0^{\frac{1-\epsilon}{4}} (k-1)^{-\frac{1}{2}} k^{\frac{1+\epsilon}{2}} \right\} \right\} \end{aligned} \quad (3.4.30)$$

Now first choose  $k$  large enough and then  $c_0$  small enough such that we finally have for a constant  $C_0 > 0$

$$\|u\|_{C_T \mathcal{P}_{n-r-R}} \leq \max \left\{ \frac{C_0^2}{(R - R')^2}, \frac{1}{2} \|u\|_{C_T \mathcal{P}_{n-r}} \right\}$$

Here  $c_0, k, C_0$  only depends on  $\epsilon, J(\xi, \chi), \Psi, T$  and the structure of PAM. In particular,  $C_0$  does not depend on  $n, r, R$ .  $\square$

Now we can finish the proof of the theorem 3.4.1.

*Proof of Theorem 3.4.1.* Without loss of generality we take  $\kappa = 2$  in the equation (3.4.1).

Suppose for some  $D \subset \mathcal{P}_n$ , we have

$$c_0 \|u\|_{C_T(D)} \geq \max \left\{ \|\tau\|_{n, |\tau|}^{\frac{2}{n\tau(1-\epsilon)}} : \tau = \xi, \xi X, \mathcal{J}\xi\xi \right\} \quad (3.4.31)$$

for some  $c_0 < c_1$  as mentioned in lemma 3.4.12 and this lemma gives us

$$\|u\|_{C_T \mathcal{P}_{n-R}} \leq \max \left\{ \frac{C_0^2}{\left(R - \|u\|_{C_T(\mathcal{P}_n)}^{-\frac{1}{2}}\right)^2}, \frac{1}{2} \|u\|_{C_T \mathcal{P}_n} \right\}.$$

if  $D = \mathcal{P}_n$  satisfies (3.4.31). In particular, if  $C_0 > 1 + \frac{\sqrt{2}}{2}$  and  $R = \frac{2C_0}{\|u\|_{C_T \mathcal{P}_n}^{-\frac{1}{2}}}$ , then we have

$$\|u\|_{C_T \mathcal{P}_{n-R}} \leq \frac{1}{2} \|u\|_{C_T \mathcal{P}_n}$$

which is true for all  $n \in \mathbb{R}^+$  and  $D = \mathcal{P}_n$  satisfies (3.4.31). Now we begin with  $R_0 = 0$  and choose  $R_i$  recursively

$$R_{i+1} - R_i = 2C_0 \|u\|_{C_T \mathcal{P}_{n-R_i}}^{-\frac{1}{2}}$$

stop when  $R_i > n$  (set  $R_i = n$  in this case) or  $D = \mathcal{P}_n$  does not satisfy (3.4.31). Suppose we obtain the sequence  $0 = R_0 < R_1 < \dots < R_N$ . We have for all  $k \leq i \leq N - 1$ ,

$$\|u\|_{C_T \mathcal{P}_{n-R_i}} \leq \left(\frac{1}{2}\right)^{i-k} \|u\|_{C_T \mathcal{P}_{n-R_k}}$$

Thus for  $i \leq N - 1$ , we have

$$\begin{aligned} R_i &= \sum_{k=0}^{i-1} R_{i+1} - R_i = 2C_0 \sum_{k=0}^{i-1} \|u\|_{C_T \mathcal{P}_{n-R_k}}^{-\frac{1}{2}} \\ &\leq 2C_0 \sum_{k=0}^{i-1} \|u\|_{C_T \mathcal{P}_{n-R_i}}^{-\frac{1}{2}} 2^{-\frac{i-k}{2}} \\ &\lesssim 2C_0 \|u\|_{C_T \mathcal{P}_{n-R_i}}^{-\frac{1}{2}} \end{aligned}$$

Thus we know  $\|u\|_{C_T \mathcal{P}_{n-R_i}} \lesssim \frac{1}{R_i^2}$ . Now for  $R \in (R_i, R_{i+1})$ , we have

$$R \leq R_{i+1} = R_{i+1} - R_i + R_i \leq 2C_0 \|u\|_{C_T \mathcal{P}_{n-R_i}}^{-\frac{1}{2}} + R_i \lesssim \|u\|_{C_T \mathcal{P}_{n-R_i}}^{-\frac{1}{2}} \leq \|u\|_{C_T \mathcal{P}_{n-R}}^{-\frac{1}{2}}$$

This ends the proof.  $\square$

### 3.5 Proof of the main theorem

In this section, we will discuss the existence and uniqueness of the equation (3.3.1), complete the logic above, and close the whole proof of Theorem 3.3.5. To achieve both, we have to give a space-time regularity rather than the space regularity only, which can be achieved by following Schauder estimation.

**Lemma 3.5.1.** *Suppose  $u$  is a mild solution to the heat equation*

$$\begin{cases} (\partial_t - \Delta)u = f, & \text{in } [0, T] \times \mathbb{R}^2, \\ u(0, \cdot) = u_0, & \text{on } \{0\} \times \mathbb{R}^2. \end{cases} \quad (3.5.1)$$

Let  $\theta \in \boldsymbol{\rho}(\boldsymbol{\omega})$  be a weight such that  $\theta(x) \geq 1$ . Then we have estimation for  $\alpha \in (0, 2)$ :

$$\|u\|_{\mathcal{L}^\alpha(\mathbb{R}^2, \theta)} \lesssim \|f\|_{C_T(\alpha-2, \mathbb{R}^2, \theta)} + \|u_0\|_{\alpha, \mathbb{R}^2, \theta}. \quad (3.5.2)$$

*Proof.* See proposition 3.6 and lemma 3.10 in [71] for a proof in the discrete case, which leads to the continuous case directly via a limit. Also, see lemma 2.10 in [48] for a slightly different equation.  $\square$

Apart from the lemma (3.5.1), another difficulty we meet in solving the singular SPDE is the meaninglessness of the products of two distributions. For example, here the product of  $\xi$  and  $u$ , which are of regularities  $-1-$  and  $1-$ , is undefinable. We hereby need the concept of paracontrolled distribution to extract the singular part and solve the equation on the singular part.

**Definition 3.5.2.** *For a weight  $\theta \in \boldsymbol{\rho}(\boldsymbol{\omega})$ , we say  $u$  is paracontrolled if  $u \in C^{1-\epsilon}(\mathbb{R}^2, \theta)$  and*

$$u^\sharp = u - u \otimes \mathcal{J}\xi \in C^{1+2\epsilon}(\mathbb{R}^2, \theta).$$

For such  $u$  we set

$$\mathcal{H}u = \Delta u + \xi \otimes u + u \otimes \xi + u^\sharp \odot \xi + C_1(u, \mathcal{J}\xi, \xi) + u(\mathcal{J}\xi \odot \xi).$$

where  $C_1(u, \mathcal{J}\xi, \xi) = (u \otimes \mathcal{J}\xi) \odot \xi - u(\mathcal{J}\xi \odot \xi)$ . From [78] we know that  $\mathcal{H}u$  is a well-defined distribution in  $C^{-1-\epsilon}(\mathbb{R}^2, p(a)\theta)$  for all  $a > 0$ .

By analysing the equation of singular part  $u^\sharp$ , we could obtain the solution theory of any regularized version of PAM. Here we will take constants  $C_\alpha$  in the assumption 3.3.1 and consider the regularized equation

$$\begin{cases} \partial_t u^\alpha = \mathcal{H}^\alpha u^\alpha + f^\alpha & \text{in } \mathbb{R}_+ \times \mathbb{R}^2 \\ u^\alpha(0, \cdot) = u_0^\alpha & \text{on } \{0\} \times \mathbb{R}^2 \end{cases} \quad (3.5.3)$$

where  $\mathcal{H}^\alpha = \Delta + \xi_\alpha - C_\alpha$ . We also denote  $\mathcal{T}^\alpha$  to be the semi-group generated by the operator  $\mathcal{H}^\alpha$ , then solution  $u_\alpha$  to the above have a mild solution representation

$$u_\alpha = \int_0^t \mathcal{T}_{t-s}^\alpha f_s^\alpha ds + \mathcal{T}_t u_0^\alpha$$

.

We here give a simple version of the estimations of solutions of equation (3.5.3) and their limit. See [71] for more detail.

**Theorem 3.5.3.** *For  $\epsilon \in (0, \frac{1}{4})$ , taking the weight  $\theta$  to be  $e(l)$  for some  $l \in \mathbb{R}$ , if  $f^\alpha \in C_T(-1 + 2\epsilon, \mathbb{R}^2, e(l))$ ,  $u_0^\alpha \in C^{1+2\epsilon}(\mathbb{R}^2, e(l))$  and*

$$f^\alpha \rightarrow f \text{ in } C_T(-1 + 2\epsilon, \mathbb{R}^2, e(l)), \quad u_0^\alpha \rightarrow u_0 \text{ in } C^{1+2\epsilon}(\mathbb{R}^2, e(l + \cdot))$$

then we have

$$u^\alpha \rightarrow u = \int_0^t \mathcal{T}_{t-s} f_s ds + \mathcal{T}_t u_0 \text{ in } \mathcal{L}^{1-\epsilon}(\mathbb{R}^2, e(l + \cdot)),$$

where  $T$  is the limit operator of  $T^\alpha$  and

$$\|u\|_{\mathcal{L}^{1-\epsilon}(\mathbb{R}^2, e(l+\cdot))} \lesssim \|u_0\|_{1+2\epsilon, \mathbb{R}^2, e(l)} + \|f\|_{C_T(-1+2\epsilon, \mathbb{R}^2, e(l+\cdot))}.$$

The proof of existence and uniqueness of solutions to equation (3.3.1) uses similar arguments as Proposition 4.5 in [78].

**Theorem 3.5.4.** *Let  $T > 0$  and  $\epsilon \in (0, \frac{1}{4})$ , and let  $l_0 < -T$ . For a non-negative function  $\varphi_0 \in C^{1+2\epsilon}(\mathbb{R}^2, e(l_0))$  and a non-negative function  $\phi \in C^{-1+2\epsilon}(\mathbb{R}^2, e(l_0))$ , the solution of equation (3.3.1) exists and is unique in the space  $\mathcal{L}_T^{1-\epsilon}(\mathbb{R}^2, e(l))$  with  $l = l_0 + T$ . Moreover, the solution is non-negative.*

*Proof.* We define the map  $\mathcal{K}(\psi) = g$ , where  $g$  is the solution to

$$\begin{cases} \partial_t g = (\mathcal{H} - \frac{\kappa}{2}\psi)g + \phi & \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ g(0) = \varphi_0 & \text{in } \{0\} \times \mathbb{R}^2. \end{cases} \quad (3.5.4)$$

We have a similar estimate as in [78]:

$$\begin{aligned} & \|\mathcal{K}\psi\|_{\mathcal{L}_T^{1-\epsilon}(\mathbb{R}^2, e(l_0+\cdot))} \\ & \lesssim (\|\varphi_0\|_{1+2\epsilon, \mathbb{R}^2, e(l_0+\cdot)} + \|\phi\|_{C_T(-1+2\epsilon, \mathbb{R}^2, e(l_0+\cdot))}) e^{C\|\psi\|_{C_T(\mathbb{R}^2, e(l_0+\cdot))}}. \end{aligned} \quad (3.5.5)$$

Indeed, starting with  $L^\infty$  norm estimation, we have

$$\begin{aligned} \|g\|_{C_T(\mathbb{R}^2, e(l_0+\cdot))} & \lesssim \|\varphi_0\|_{1+2\epsilon, \mathbb{R}^2, e(l_0+\cdot)} + \|\phi\|_{C_T(-1+2\epsilon, \mathbb{R}^2, e(l_0+\cdot))} \\ & \quad + \frac{\kappa}{2} \sup_{t \in [0, T]} \left\{ \int_0^t \|\mathcal{T}_{t-s}\psi_s g_s\|_{\mathbb{R}^2, e(l_0+t)} ds \right\} \end{aligned}$$

When  $l_0 < -T$ , we are able to estimate

$$\|\mathcal{T}_{t-s}\psi_s g_s\|_{\mathbb{R}^2, e(l_0+t)} \lesssim \|\psi\|_{\mathbb{R}^2, e(l_0+\cdot)} \|g\|_{\mathbb{R}^2, e(l_0+\cdot)}.$$

Hence, the Grownall's inequality gives

$$\begin{aligned} & \|g\|_{C_T(\mathbb{R}^2, e(l_0+\cdot))} \\ & \lesssim (\|\varphi_0\|_{1+2\epsilon, \mathbb{R}^2, e(l_0+\cdot)} + \|\phi\|_{C_T(-1+2\epsilon, \mathbb{R}^2, e(l_0+\cdot))}) e^{C\|\psi\|_{C_T(\mathbb{R}^2, e(l_0+\cdot))}}. \end{aligned}$$

Use again the estimation of 3.5.3, we now obtain

$$\|g\|_{\mathcal{L}_T^{1-\epsilon}(\mathbb{R}^2, e(l_0+\cdot))} \lesssim \|\varphi_0\|_{1+2\epsilon, \mathbb{R}^2, e(l_0+\cdot)} + \|\phi - \frac{\kappa}{2}\psi g\|_{C_T(-1+2\epsilon, \mathbb{R}^2, e(l_0+\cdot))},$$

which gives equation (3.5.5). Furthermore, the maximum principle and the comparison principle[77, 10] show that  $T_t\varphi_0 + \int_0^t T_{t-s}\phi ds \geq \mathcal{K}(\psi)(t) \geq 0$ . Thus, we start with  $\varphi^0 = T_t\varphi_0 + \int_0^t T_{t-s}\phi ds$  and then let  $\varphi^m = \mathcal{K}\varphi^{m-1}$ . We have the bounds

$$\|\mathcal{K}\psi\|_{\mathcal{L}^{1-\epsilon}(\mathbb{R}^2, e(l_0+\cdot))} \lesssim C(\varphi_0, \phi) e^{C\|T_t\varphi_0 + \int_0^t T_{t-s}\phi ds\|_{C_T(\mathbb{R}^2, e(l_0+\cdot))}}.$$

To show the existence of the fixed points, we are going to use Schauder's fixed point theorem. To start with, we consider the convex set

$$\mathcal{E} := \{\mathcal{K}(\psi) : \psi \in \mathcal{L}_T^{1-\epsilon}\}.$$

Then  $\mathcal{K}(\mathcal{E}) \subset \mathcal{E}$  and  $\mathcal{K}(\mathcal{E})$  is a pre-compact set in  $\mathcal{L}_T^{1-\epsilon}$ . It remains shown  $\mathcal{E}$  and  $\mathcal{K}(\mathcal{E})$  are closed. Taking  $\psi_1, \psi_2$  and consider  $g_1 = \mathcal{K}(\psi_1), g_2 = \mathcal{K}(\psi_2)$ , we have the equation

$$\partial_t(g_1 - g_2) = \left(\mathcal{H} - \frac{\kappa}{2}\right)(g_1 - g_2) - \frac{\kappa}{2}(\psi_1 - \psi_2)g_2.$$

Then it is obvious that  $\mathcal{E}$  and  $\mathcal{K}(\mathcal{E})$  are closed. This then gives us the existence of the fixed point. Hence the existence of the solution to equation (3.3.1).

The uniqueness follows the same methods in the proof of Proposition 4.5 in [78].  $\square$

**Remark 3.5.5.** For equation (3.3.1), our existence proof only works for  $l_0 < -T$ . But for proving uniqueness we have to consider the difference of two solutions  $g = \varphi^1 - \varphi^2$ , which solves equation (3.5.4) with initial condition 0 and  $\phi = 0$ , and for  $\psi = \varphi^1 + \varphi^2$ . If  $\varphi^i \in \mathcal{L}_T^{1-\epsilon}(\mathbb{R}^2, p(a))$  for sufficiently small  $a > 0$ , then we can use the same arguments as in [56, 71] to show that  $g = 0$ : By giving up a bit of regularity or by introducing a blow up at time 0 we are able to kill the polynomial weight by the Schauder estimates for the heat equation. Thus we are able to prove the uniqueness to the equation (3.3.1) in  $\mathcal{L}_T^{1-\epsilon}(\mathbb{R}^2, p(a))$  if  $a > 0$  is sufficiently small.

Together with corollary 3.4.2 and above estimations, we can give a space-time weighted norm estimation of solution of equation (3.4.1). For each  $r > 2$ , we define the auxiliary function  $\eta_r$  by

$$\eta_r = \begin{cases} 1, & \text{in } \mathcal{P}_{r-2}, \\ 0, & \text{out side of } \mathcal{P}_{r-1}, \\ [0, 1], & \text{elsewhere.} \end{cases} \quad (3.5.6)$$

**Lemma 3.5.6.** *Let  $r > 2$  and a weight  $\theta \in \boldsymbol{\rho}(\boldsymbol{\omega})$  such that  $\theta(x) = \theta(|x|) \geq 1$  and increases with respect to radius. If  $u$  solves equation (3.4.1) in  $[0, T] \times \mathcal{P}_r$ , then we have for  $\tilde{\theta} = \theta^{1+\frac{4}{1-\epsilon}}$ :*

$$\|u\eta_r\|_{\mathcal{L}^{1-\epsilon}(\mathbb{R}^2, \tilde{\theta})} \lesssim 1 + \sum_{\tau=\xi, \mathcal{J}\xi\xi} [\tau]_{r, |\tau|, \theta}^{1+\frac{4}{n_\tau(1-\epsilon)}}. \quad (3.5.7)$$

*Proof.* The equation of  $u\eta_r$  is given by

$$(\partial_t - \Delta)(u\eta_r) = (\xi - C)u\eta_r - \frac{\kappa}{2}\eta_r u^2 - 2\nabla(u \cdot \nabla\eta_r) + u\Delta\eta_r.$$

By Lemma 3.4.7 and the step 2 in the proof of Theorem 3.4.1, we know the Holder norm  $\|u(\xi - C)\|_{C_T(-1-\epsilon, \mathcal{P}_{r-1}, \tilde{\theta})}$  is bounded by

$$1 + \sum_{\tau=\xi, \mathcal{J}\xi\xi} [\tau]_{r, |\tau|, \theta}^{1+\frac{4}{n_\tau(1-\epsilon)}}, \quad (3.5.8)$$

up to a constant. Since  $\eta_r$  is supported on  $\mathcal{P}_{r-1}$ , we know  $(\xi - C)u\eta_r \in C_T(-1-\epsilon, \mathbb{R}^2, \tilde{\theta})$  and has the same bound (3.5.8) up to a constant. It is directly from remark 3.4.3 that we have

$$\begin{aligned} & \|\eta_r u^2\|_{C_T(\mathbb{R}^2, \tilde{\theta})}, \|\nabla(u \cdot \nabla\eta_r)\|_{C_T(-\epsilon, \mathbb{R}^2, \tilde{\theta})}, \|u\Delta\eta_r\|_{C_T(1-\epsilon, \mathbb{R}^2, \tilde{\theta})} \\ & \lesssim 1 + \sum_{\tau=\xi, \mathcal{J}\xi\xi} [\tau]_{r, |\tau|, \theta}^{\frac{4}{n_\tau(1-\epsilon)}}. \end{aligned}$$

Thus, from Lemma 3.5.1, we obtain the desired result. □

By approximating the equation (3.3.1) by regularized equations, we are able to see the function  $u\eta_r$  in the above theorem has a mild solution representation.

**Lemma 3.5.7.** *Let  $l_0 < -T$  and  $\epsilon \in (0, \frac{1}{4})$ . Suppose  $u$  is a solution of equation (3.3.1) with non-negative function  $\varphi_0 \in C^{1+2\epsilon}(\mathbb{R}^2, e(l_0))$  and non-negative function*

$\phi \in C^{-1+2\epsilon}(\mathbb{R}^2, e(l_0))$ . Let  $\eta \in C_c^\infty(\mathbb{R}^2)$  be a compactly supported smooth function. Then we have

$$(u\eta)(t, x) = \int_0^t T_{t-s} \left( -\frac{\kappa}{2}\eta u^2 + \phi\eta - 2\nabla(u \cdot \nabla\eta) + u\Delta\eta \right)_s(x) ds + T_t(\varphi_0\eta)(x). \quad (3.5.9)$$

*Proof.* Since  $u$  is a solution of equation 3.3.1, it is a limit of  $u_\alpha$  solving equation

$$\begin{cases} \partial_t u^\alpha = \mathcal{H}^\alpha u^\alpha - \frac{\kappa}{2}(u^\alpha)^2 + \phi^\alpha, & \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ u^\alpha(0, \cdot) = \varphi_0^\alpha, & \text{on } \{0\} \times \mathbb{R}^2. \end{cases} \quad (3.5.10)$$

as  $\alpha \rightarrow 0$  where  $\varphi_0^\alpha$  converges to  $\varphi_0$  in  $C^{1+2\epsilon}(\mathbb{R}^2, e(l_0))$  and  $\phi^\alpha$  converges to  $\phi$  in  $C^{-1+2\epsilon}(\mathbb{R}^2, e(l_0))$ . Since the regularized equation can be solved in the classical sense, the function  $u^\alpha\eta$  actually solves the equation

$$\begin{cases} \partial_t(u^\alpha\eta) = \mathcal{H}^\alpha(u^\alpha\eta) - \frac{\kappa}{2}(u^\alpha)^2\eta + \phi\eta - 2\nabla(u^\alpha \cdot \nabla\eta) + u^\alpha\Delta\eta, & \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ u^\alpha(0, \cdot)\eta = \varphi_0^\alpha\eta, & \text{on } \{0\} \times \mathbb{R}^2. \end{cases} \quad (3.5.11)$$

Thus, we have

$$u_\alpha\eta = \int_0^t T_{t-s} f_s^\alpha ds + T_t(\varphi_0^\alpha\eta),$$

where  $f^\alpha = -\frac{\kappa}{2}(u^\alpha)^2\eta + \phi\eta - 2\nabla(u^\alpha \cdot \nabla\eta) + u^\alpha\Delta\eta$ . By checking the regularity of  $u^\alpha$ , we know  $f^\alpha$  is of regularity  $-1 + 2\epsilon$  with weight  $e(2(l_0 + \cdot))$  and  $\varphi_0^\alpha\eta$  is of regularity  $1 + 2\epsilon$  with weight  $e(l_0)$ . Thus we can let  $\alpha \rightarrow 0$  and obtain the result.  $\square$

Now let's give the proof of the main result, Theorem 3.3.5.

*Proof of Theorem 3.3.5.* As in Section 3.3, we consider solutions  $\varphi_n^m$  of equation (3.3.1) with  $\phi_n^m$  defined in Section 3.3. Lemma 3.5.7 tells us that  $\varphi_n^m\eta_n$  has the representation

$$\varphi_n^m\eta_n = \int_0^t T_{t-s} (-(\varphi_n^m)^2\eta_n - 2\nabla(\varphi_n^m \cdot \nabla\eta_n) + \varphi_n^m\Delta\eta_n)_s ds. \quad (3.5.12)$$

By the nature of space white noise, we actually have, by Lemma 3.5.6, that for any  $a > 0$ ,

$$\|\varphi_n^m\eta_n\|_{\mathcal{L}^{1-\epsilon}(\mathbb{R}^2, p(a))} \lesssim 1. \quad (3.5.13)$$

Now since  $\varphi_n^m\eta_n$  is bounded in  $\mathcal{L}^{1-\epsilon}(\mathbb{R}^2, p(a))$ , the compact embedding theorem shows that  $\varphi_n^m\eta_n$  converges to  $\varphi_n\eta_n$  in  $\mathcal{L}^{1-\epsilon}(\mathbb{R}^2, p(a))$  as  $m$  tends to infinity, passing to some sub-sequence. The convergence indicates the representation of  $\varphi_n\eta_n$ :

$$\varphi_n\eta_n = \int_0^t T_{t-s} (-(\varphi_n)^2\eta_n - 2\nabla(\varphi_n \cdot \nabla\eta_n) + \varphi_n\Delta\eta_n)_s ds. \quad (3.5.14)$$

The same argument applies to  $\varphi_n \eta_n$  shows that  $\varphi := \lim_{n \rightarrow \infty} \varphi_n \eta_n$  actually is a mild solution of the equation (3.3.2) with 0 initial condition. Hence  $\varphi$  is identically 0 and we have, for compact supported measure  $\mu(0)$ , that

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \exp(-\langle \mu(0), \varphi_n^m(t) \rangle) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \exp(-\langle \mu(0), \varphi_n^m \eta_n(t) \rangle) = 1.$$

Thus, the compact support property of rough super Brownian motion holds. □

# Chapter 4

## A Generalized Rough Super Brownian Motion

### 4.1 Introduction

The branching random walk in a random environment (BRWRM) is a random walk with stochastic branching mechanism and branching rate. This work explores the large-scale behaviour of some BRWRM whose branching mechanism and rate depend on the random environment at each spatial points.

The scaling limit of branching random walk has been discussed thoroughly in [26]. The aim of this paper is to generalize the results in [26] to BRWRM by considering rough environments.

Given a random potential

$$V(x) = \xi(x), \quad \{\xi(x)\}_{x \in \mathbb{Z}^2} \text{ i.i.d. } \sim \Phi,$$

with  $\mathbb{E}[\Phi] = 0, \mathbb{E}[\Phi^2] = 1$  on lattice  $\mathbb{Z}^2$ , it is shown in [78] that certain scaled branching random walks in this random environment with birth (one particle) rate  $\xi_+$  and die rate  $\xi_-$  will converge to a measure-valued Markov process whose Laplace transformation is associated to the stochastic PDE.

$$\partial_t \varphi = (\Delta + \xi) \varphi - \varphi^2, \quad \varphi(0) = \varphi_0, \quad (4.1.1)$$

where  $\xi$  is the white noise.

The off-spring distribution of The BRWRM considered in [78] has a finite variance and a general question is what happens if the variance is infinite. It is discussed in [26, 34] that the Laplace functionals of the limit super processes given by classical branching random walks should satisfy the PDE

$$\partial_t \varphi = L\varphi - b(x)\varphi - c(t, x)\varphi^2 + \int_0^\infty (1 - e^{-\theta\varphi(t, x)} - \theta\varphi(t, x))n(x, d\theta),$$

where  $A$  is a generator of a Feller semi-group,  $b$  is bounded and  $c > 0$  and  $n$  is some sort of transition kernel.

Due to the result in [78], it is then natural to think that the super-process should exist when  $b$  is only a distribution, for example, white noise  $\xi$ . We do not try to answer this general question in the paper since we are lacking knowledge of how to deal with the part with transition kernel  $n$ . However, we start with a non-trivial but simpler case when  $n(x, d\theta) \sim \theta^{\beta+2}$  for  $0 < \beta < 1$ . In other words, we try to deduce the existence and uniqueness of the super-process whose Laplace functional is given by the SPDE

$$\partial_t \varphi = (\Delta + \xi)\varphi - \varphi^{1+\beta}. \quad (4.1.2)$$

To this goal, we consider the BRWRM whose branching mechanism is given by

$$\frac{2\xi_{e,+}}{(1+\beta)|\xi_e|}(1-s)^{1+\beta} - \frac{\xi_e}{|\xi_e|} + \frac{2\xi_{e,+}}{|\xi_e|}s,$$

where  $\xi_e$  is the renormalized random potential defined below and then show the scaling limit of this branching random walk gives us exactly the desired super process.

It is worth mentioning that there is actually no general existence and uniqueness theory of equation (4.1.2) in the literature. The main difficulty of solving equation (4.1.2) is the Anderson Hamiltonian  $\mathcal{H} = \Delta + \xi$ . Due to the lack of regularity, it is not possible to define the product  $\xi\varphi$  directly in dimension 2 or 3. The study of equation (4.1.2) then requires recently built techniques such as regularity [54] or the para-controlled distributions [49]. In dimension 2, all of the solution theories require a renormalization, which means we need to remove a diverging constant in the Anderson Hamiltonian  $\mathcal{H}$  and consider actually the equation  $\partial_t \varphi = (\mathcal{H} - \infty)\varphi - \varphi^{1+\beta}$ . It is understood as the limit of renormalized and discretized equation with

$$\xi_e^n = \xi^n - c_n, c_n \simeq \log(n).$$

The second difficulty of solving equation (4.1.2) arises from the weight operation when solving the corresponding linear equation  $\partial_t \varphi = \mathcal{H}\varphi$  as in [71, 56]. The solution theory of PAM uses heavily the logarithm growth of  $\xi$  on the  $\mathbb{R}^2$ . When we view (4.1.2) as

$$\partial_t \varphi = (\mathcal{H} - \varphi^\beta)\varphi.$$

The growth of  $\xi - \varphi^\beta$  will be exponential and thus we can not apply directly the solution theory of PAM on  $\mathbb{R}^2$ . However, we will see in section 4.4 that the minus sign helps us build the existence and uniqueness results with positive initial condition.

In recent work by Perkowski and Rosati [78], they only obtain a partial existence and uniqueness result of equation (4.1.2) but this is enough for their application due to the  $L^2$  bounded property from the finite variance off-spring distribution. However, their methods rely strongly on the  $L^2$  martingale theory and can not be used directly for the infinite variance case. We, therefore, search for an alternative way, combining the idea of [78] and the classical approach via Laplace functional, to give the convergence in the infinite variance case. Finally, although we work in dimension 2, all results (with some change of parameters) are valid in dimension 1 with a much easier argument.

**Structure of the work.** We only consider a deterministic environment with assumption 4.3.1 since the construction of random environment from the deterministic environment is given in [78]. We introduce our model and the definition of  $\beta$ -rough super Brownian motion in section 4.3 and state our main results, i.e. the convergence results of the scaled branching random walk, a martingale problem formulation and the compact support property of the  $\beta$ -super Brownian motion.

We study a variant of PAM ( $\mathcal{H} \rightarrow \mathcal{H} - \phi$  for some  $\phi$ ) and the non-linear equation like 4.1.2 in section 4.4 and show the existence and uniqueness of solution in the space with weight  $e^{l|x|^\sigma}$  for any  $l \in \mathbb{R}, 0 \leq \sigma < 1$ . The result allows us to implement freely with the  $\varphi$  in the Laplace functional  $\mathbb{E}[e^{-\langle \mu_t, \varphi \rangle}]$ .

We then give the detailed proof of convergence of scaled branching random walk to the  $\beta$ -rough super Brownian motion by estimating the moments of  $\sup_{0 \leq s \leq t} \langle \mu_s, \varphi \rangle$ . This estimation requires controlling branching random walks from a coupling method and passing the estimations to a simpler branching random walk. Then we apply the Aldous criterion to obtain the tightness.

Section 4.6 is devoted to giving a martingale problem formulation of  $\beta$ -rough super Brownian motion. We finally then prove the compact support property and the super-exponentially persistence that is proved in [59, 78] for the finite variance case.

## 4.2 Notation

### 4.2.1 Notation on regularity

We define the lattice  $\mathbb{Z}_n^2 = \frac{1}{n}\mathbb{Z}^2$  and also denote  $\mathbb{Z}_\infty^2 := \mathbb{R}^2$  for convenience. Similarly, we define  $\mathbb{T}_n^2 := (\mathbb{R}/n\mathbb{Z})^2$  and  $\tilde{\mathbb{T}}_n^2 := n \left(-\frac{1}{2}, \frac{1}{2}\right]^2$ . We will consider weighted Besov space on  $\mathbb{Z}_n^2$  introduced in [71].

To motivate the definition, let's first introduce the definition of weight.

**Definition 4.2.1.** We denote by

$$\omega^{pol}(x) := \log(1 + |x|), \quad \omega_{\sigma}^{exp}(x) := |x|^{\sigma},$$

where  $x \in \mathbb{R}^d, \sigma \in (0, 1)$ . For  $\omega \in \boldsymbol{\omega} := \{\omega^{pol}\} \cup \{\omega_{\sigma}^{exp} | \sigma \in (0, 1)\}$ , we denote by  $\boldsymbol{\rho}(\omega)$  the set of measurable, strictly positive  $\rho : \mathbb{R}^d \rightarrow (0, \infty)$  such that

$$\rho(x) \lesssim \rho(y)e^{\lambda\omega(x-y)},$$

for some  $\lambda = \lambda(\rho) > 0$ . We also introduce the notation  $\boldsymbol{\rho}(\boldsymbol{\omega}) := \bigcup_{\omega \in \boldsymbol{\omega}} \boldsymbol{\rho}(\omega)$ . The objects  $\rho \in \boldsymbol{\rho}(\boldsymbol{\omega})$  will be called weights.

The weight we considered in this paper is of the following form:

$$p(a)(x) := (1 + |x|)^a, \quad e(l)(x) := e^{|x|^{\sigma}},$$

for non-negative  $a, l$  and  $0 < \sigma < 1$ . We drop the index  $\sigma$  since it does not influence the proof.

For function  $\varphi : \mathbb{Z}_n^2 \rightarrow \mathbb{R}$ , we define its Fourier transformation as function on torus  $\mathbb{T}_n^2$  by

$$\mathcal{F}_n \varphi(k) := \frac{1}{n^2} \sum_{x \in \mathbb{Z}_n^2} \varphi(x) e^{-2\pi i k \cdot x}, \quad k \in \mathbb{T}_n^2,$$

and the inverse Fourier transformation for functions  $\varphi : \mathbb{T}_n^2 \rightarrow \mathbb{R}$ ,

$$\mathcal{F}_n^{-1} \varphi(x) := \int_{\mathbb{T}_n^2} \varphi(k) e^{2\pi i k \cdot x} dk, \quad x \in \mathbb{Z}_n^2.$$

Let  $\omega \in \boldsymbol{\omega}$ , it is defined in [71] the space  $\mathcal{S}_{\omega}$  consisting of all functions such that all derivatives of itself and its Fourier transformation decay like  $e^{-\lambda\omega}$  for any  $\lambda > 0$ . The dual space of  $\mathcal{S}'_{\omega}$  is called ultra-distribution which will be the space the Besov spaces are built on.

Suppose  $\rho_{-1}, \rho_0 \in \mathcal{S}'_{\omega}(\mathbb{R}^2)$  are two non-negative and radial functions such that the support of  $\rho_{-1}$  is contained in a ball  $B \subset \mathbb{R}^2$ , the support of  $\rho_0$  is contained in an annulus  $\{x \in \mathbb{R}^2 : 0 < a \leq |x| \leq b\}$  and such that with

$$\rho_j = \rho_0(2^{-j}\cdot),$$

the following conditions are satisfied:

1.  $\sum_{i=-1}^{\infty} \rho_j(x) = 1, \forall x \in \mathbb{R}^2$
2.  $\text{supp}(\rho_i) \cap \text{supp}(\rho_j) = \emptyset$  if  $|i - j| > 1$

For each  $\omega \in \boldsymbol{\omega}$ , such a partition of unity always exists in  $\mathcal{S}'_\omega(\mathbb{R}^2)$ . We define furthermore an index  $j_n$  to be the minimum index such that the support of  $\phi_j$  intersects with  $\tilde{\mathbb{T}}_n^2$ . Update the partition of unity of  $\tilde{\mathbb{T}}_n^2$  by defining periodic functions

$$\rho_j^n = \rho_j, \quad j < j_n, \quad \rho_{j_n}^n = 1 - \sum_{j=-1}^{j_n-1} \rho_j.$$

For a ultra-distribution  $f \in \mathcal{S}'_\omega(\mathbb{Z}_n^2)$ , the Littlewood-Paley blocks of  $f$  as

$$\Delta_j^n f = \mathcal{F}^{-1}(\rho_j^n \mathcal{F}(f)).$$

for  $j = -1, 0, \dots, j_n$ .

For  $\alpha \in \mathbb{R}, p, q \in [1, \infty]$  and  $\rho \in \boldsymbol{\rho}(\boldsymbol{\omega})$ , we define the weighted Besov space  $B_{p,q}^\alpha(\mathbb{Z}_n^2, \rho)$  by

$$B_{p,q}^\alpha(\mathbb{Z}_n^2, \rho) := \{f \in \mathcal{S}'_\omega : \|f\|_{B_{p,q}^\alpha(\mathbb{Z}_n^2, \rho)} := \|(2^{j\alpha} \|\rho^{-1} \Delta_j^n f\|_{L^p})\|_{\ell^q} < \infty\}.$$

In particular, the discrete Hölder space is defined by  $C^\alpha(\mathbb{Z}_n^2, \rho) := B_{\infty,\infty}^\alpha(\mathbb{Z}_n^2, \rho)$  for  $\alpha$  not an integer. The extension operator  $\mathcal{E}^n$  defined in [71] will be helpful when we consider the convergence from the discrete equation to the continuum equation.

We also consider the space concerning the regularity with respect to time. Fix a time horizon  $T > 0$ , we define the space  $C_T C^\alpha(\mathbb{Z}_n^2, e(l))$  to be all continuous in time function  $f : [0, T] \rightarrow C^\alpha(\mathbb{Z}_n^2, e(l+T))$  such that the norm

$$\|f\|_{C_T C^\alpha(\mathbb{Z}_n^2, e(l))} := \sup_{0 \leq t \leq T} \|f(t)\|_{C^\alpha(\mathbb{Z}_n^2, e(l+t))},$$

is finite. For  $0 < \beta < 1$ , the space  $C_T^\beta C^\alpha(\mathbb{Z}_n^2, e(l))$  is defined by

$$\|f\|_{C_T^\beta C^\alpha(\mathbb{Z}_n^2, e(l))} := \|f\|_{C_T C^\alpha(\mathbb{Z}_n^2, e(l))} + \sup_{0 \leq s < t \leq T} \frac{\|f(t) - f(s)\|_{C^\alpha(\mathbb{Z}_n^2, e(l+t))}}{t - s}.$$

We also define the space  $\mathcal{M}^\gamma C^\alpha(\mathbb{Z}_n^2, e(l))$  to be all function  $f$  such that  $t^\gamma f(t) \in C_T C^\alpha(\mathbb{Z}_n^2, e(l))$ . And the space  $\mathcal{L}^{\gamma,\alpha}(\mathbb{Z}_n^2, e(l))$  consists of all functions  $f$  such that

$$f \in \mathcal{M}^\gamma C^\alpha(\mathbb{Z}_n^2, e(l)), \quad t^\gamma f \in C^{\frac{\alpha}{2}} L^\infty(\mathbb{R}^2, e(l)).$$

We simply write  $\mathcal{L}^\alpha$  when  $\gamma = 0$ .

## 4.2.2 Notation on operators

We give the definition of some discrete operators here. We define the discrete Laplace on lattice  $\mathbb{Z}_n^2$  by

$$\Delta^n f(x) := n^2 \sum_{y \sim x} (f(y) - f(x)).$$

We will use  $P_t^n$  to denote the discrete heat kernel, i.e. for any function  $\varphi$ , we denote  $P_t^n \varphi$  to be the mild solution of the equation

$$\begin{cases} \partial_t w_t^n = \Delta^n w_t^n, \\ w_0^n = \varphi. \end{cases} \quad (4.2.1)$$

We also use  $P_t$  for the continuum equation. Also, for a deterministic environment  $\xi^n$  (which will be introduced in assumption 4.3.1), we define the Anderson Hamiltonian  $\mathcal{H}^n = \Delta^n + \xi_e^n$ , where  $\xi_e^n = \xi^n - c_n$  with  $c_n$  a renormalization constant to be introduced in the definition of deterministic environment. We then use the operator  $T_t^n \varphi$  to indicate the solution to the equation

$$\begin{cases} \partial_t w_t^n = \mathcal{H}^n w_t^n, \\ w_0^n = \varphi. \end{cases} \quad (4.2.2)$$

if the solution exists and is unique. All above definitions move to the continuous case with symbols  $P_t, \Delta$  and  $T_t, \mathcal{H}$ . There is another definition of  $\mathcal{H}^n$  which links to the continuous equation by the para-product.

**Definition 4.2.2.** For any two distributions  $\varphi, \phi$  on  $\mathbb{Z}_n^2$ , the product can be decomposed as  $\varphi \cdot \phi = \varphi \otimes \phi + \varphi \odot \phi + \phi \ominus \varphi$  such that

$$\varphi \otimes \phi = \sum_{j=-1}^{j_n} \Delta_{<j-1}^n \varphi \Delta_j^n \phi, \quad \varphi \odot \phi = \sum_{\substack{|i-j| \leq 1 \\ -1 \leq i, j \leq j_n}} \Delta_i^n \varphi \Delta_j^n \phi,$$

where  $\Delta_{<i}^n = \sum_{j=-1}^i \Delta_j^n$ . When  $n = \infty$ , it is exactly the same definition on  $\mathbb{R}^2$ . This builds the relation between discrete and continuum cases.

The definition of para-product allows us to define the para-controlled distribution.

**Definition 4.2.3.** Suppose  $\kappa > 0$ ,  $\xi^n$  and  $\mathcal{J}\xi^n$  satisfy the assumption 4.3.1. We say  $\varphi^n$  is para-controlled if  $\varphi^n \in C^{1-\kappa}(\mathbb{Z}_n^2, e(l))$  for some  $l \in \mathbb{R}$  and

$$\varphi^{n,\#} := \varphi^n - \varphi^n \otimes \mathcal{J}\xi^n \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l)).$$

We define the para-controlled space  $\mathcal{D}(\mathbb{Z}_n^2, e(l))$  for all paracontrolled functions  $\varphi^n$  such that

$$\|\varphi^n\|_{\mathcal{D}(\mathbb{Z}_n^2, e(l))} = \|\varphi^n\|_{C^{1-\kappa}(\mathbb{Z}_n^2, e(l))} + \|\varphi^{n, \#}\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))} < \infty.$$

We also define the domain  $\mathcal{D}_{\mathcal{H}}$  of the operator  $\mathcal{H}$  on  $\mathbb{R}^2$ , which is given by

$$\mathcal{D}_{\mathcal{H}} := \left\{ \int_0^t T_s u ds : t > 0, u \in C^\zeta(\mathbb{R}^2, e(l)), l \in \mathbb{R}, \zeta > 1 \right\}.$$

The property of this set is discussed in [78].

### 4.2.3 Other notations

We denote  $\mathcal{M}_F(\mathbb{R}^2)$  as the space of finite measure on  $\mathbb{R}^2$  equipped with weak topology defined by family  $C_b(\mathbb{R}^2)$ . We also let  $\mathcal{M}_F^\pm(\mathbb{R}^2)$  be the signed finite measure on  $\mathbb{R}^2$ .

For any Polish space  $E$ , we consider the space of  $E$ -valued càdlàg path  $\mathbb{D}([0, \infty), E)$  endowed with Skorohod topology.

## 4.3 Model Formulation

From now on, fix a parameter  $1 < \beta < 2$ . To simplify this paper, we only consider the deterministic environment  $\xi^n$ , which is simply a function from  $\mathbb{Z}_n^2$  to  $\mathbb{R}$ . We refer readers to [78] for how to deal with the situation when the environment  $\xi^n$  is random. Let  $\chi$  be a smooth function taking value 1 outside of  $(-\frac{1}{4}, \frac{1}{4})^2$  and 0 in  $(-\frac{1}{8}, \frac{1}{8})^2$ .

**Assumption 4.3.1.** *For a deterministic environment  $\xi^n$ , define  $\mathcal{J}\xi^n$  to be the solution to the equation  $-\Delta^n \mathcal{J}\xi^n = \mathcal{F}_n^{-1}(\chi \mathcal{F}_n \xi^n)$ . Consider a regularity parameter  $\kappa' \in (0, \frac{1}{4})$ . We assume*

1. *There exists  $\xi \in \cap_{a>0} C^{-1-\kappa'}(\mathbb{R}^2, p(a))$  such that, for all  $a > 0$ ,*

$$\sup_n \|\xi^n\|_{C^{-1-\kappa'}(\mathbb{Z}_n^2, p(a))} < \infty, \quad \mathcal{E}^n \xi^n \rightarrow \xi \text{ in } C^{-1-\kappa'}(\mathbb{R}^2, p(a))$$

2. *The quantities  $\frac{|\xi^n|}{n}$  is uniformly bounded over  $n$ . Furthermore, there exists  $\nu \geq 0$  such that the following convergence holds*

$$\mathcal{E}^n \frac{\xi^n}{n} \rightarrow \nu, \quad \mathcal{E}^n \frac{|\xi^n|}{n} \rightarrow 2\nu$$

*in  $C^{-\delta}(\mathbb{R}^2, p(a))$  for any  $\delta > 0, a > 0$ .*

3. There exists  $\{c_n\} \subset \mathbb{R}$  with  $\frac{c_n}{n} \rightarrow 0$  and there exists  $\mathcal{J}\xi \in \cap_{a>0} C^{1-\kappa'}(\mathbb{R}^2, p(a))$  and  $\mathcal{J}\xi \diamond \xi \in \cap_{a>0} C^{-2\kappa'}(\mathbb{R}^2, p(a))$  such that for all  $a > 0$ ,

$$\sup_n \|\mathcal{J}\xi^n\|_{C^{1-\kappa'}(\mathbb{Z}_n^2, p(a))} + \sup_n \|(\mathcal{J}\xi^n \odot \xi^n) - c_n\|_{C^{-2\kappa'}(\mathbb{Z}_n^2, p(a))} < \infty$$

,  $\mathcal{E}^n \mathcal{J}\xi^n \rightarrow \mathcal{J}\xi$  in  $C^{1-\kappa'}(\mathbb{R}^2, p(a))$  and  $\mathcal{E}^n(\mathcal{J}\xi^n \odot \xi^n - c_n) \rightarrow \mathcal{J}\xi \diamond \xi$  in  $C^{-2\kappa'}(\mathbb{R}^2, p(a))$ .

We call a distribution  $\xi$  on  $\mathbb{R}^2$  a deterministic environment if there exists  $\xi^n$  satisfying the assumption 4.3.1 and  $\xi$  is the limit. Given a deterministic environment with assumption 4.3.1, we establish the particle systems approximating desired super-process.

For each  $n \in \mathbb{N}$ , we consider a branching random walk on the lattice  $\mathbb{Z}_n^2$ . In the language of branching random walk, the underlying motion is given by random walk  $X^n$  with generator  $\Delta^n$ . The branching rate is given by

$$a^n(x, dt) = |\xi_e^n(x)| dt, \quad \xi_e^n := \xi^n - c_n$$

for all  $x \in \mathbb{Z}_n^2$ . The branching mechanism is given by the generating function

$$g^n(x, s) = \frac{2\xi_{e,+}^n}{(1+\beta)|\xi_e^n|} (1-s)^{1+\beta} - \frac{\xi_e^n}{|\xi_e^n|} + \frac{2\xi_{e,+}^n}{|\xi_e^n|} s = \sum_{i=0}^{\infty} p_i \cdot q_i(x) s^i, \quad (4.3.1)$$

where

$$\sum_{i=0}^{\infty} p_i s^i = \frac{1}{1+\beta} (1-s)^{1+\beta} + s := g(s). \quad (4.3.2)$$

One can verify that we have

$$q_0(x) = \frac{(1-\beta)\xi_{e,+}^n + (1+\beta)\xi_{e,-}^n}{|\xi_e^n|} \in (1-\beta, 1+\beta), \quad q_i(x) = \frac{2\xi_{e,+}^n}{|\xi_e^n|} \leq 2, i \geq 2.$$

Since  $\frac{\partial}{\partial s} g^n(x, 0) = 0 = p_1$ , we can set  $q_1(x)$  to be any number. For example, set  $q_1(x) = 1$ .

A detailed description can be given by defining the infinitesimal generator of a Markov process on the configuration space  $E^n = (\mathbb{N})^{\mathbb{Z}_n^2}$  endowed with product topology. For any configuration  $\eta \in E^n$ , we define

$$\eta^{x \rightarrow y}(z) = \mathbb{1}_{\{\eta(x) > 0\}}(\eta(z) + \mathbb{1}_{z=y} - \mathbb{1}_{z=x}), \quad \eta^{x+k}(z) = 0 \vee (\eta(z) + (k-1)\mathbb{1}_{z=x}).$$

Furthermore, for any  $F \in C_b(E^n)$ , we define

$$\Delta_x^n F(\eta) = n^2 \left( \sum_{y \sim x} F(\eta^{x \rightarrow y}) - F(\eta) \right), \quad d_x^k F(\eta) = F(\eta^{x+k}) - F(\eta).$$

Since any configuration on  $E^n$  can be viewed as a sigma finite point measure on  $\mathbb{R}^2$  and we eventually prove the tightness in the measure space, we will freely change from  $E^n$  to the point measure on  $\mathbb{R}^2$  in the following content.

To construct the approximation particle systems, we need furthermore the Poisson random measure introduced in appendix C.1.

**Definition 4.3.1.** Fix a compact supported measure  $\mu_0^n$  on  $\mathbb{Z}_n^2$  and an average parameter  $\varrho \leq \beta$ . Let  $\epsilon = n^{-\frac{1}{\varrho}}$ . We define an  $E^n$ -valued stochastic process  $Z^n(t)$  started at a Poisson random measure on  $\mathbb{Z}_n^2$  with intensity  $\frac{\mu_0^n}{\epsilon}$  with infinitesimal generator

$$\mathcal{L}^n F(\eta) = \sum_{x \in \mathbb{Z}_n^2} \eta_x \left[ \Delta_x^n F(\eta) + |a^n(x)| \sum_{k=0}^{\infty} p_k q_k(x) d_x^k F(\eta) \right], \quad (4.3.3)$$

for any  $F \in \mathcal{D}(\mathcal{L}^n)$  which consists of all  $F \in C_b(E^n)$  such that the right-hand side of (4.3.3) is finite. Finally, set  $\mu^n(t) = \epsilon Z^n(t)$  for all  $t \in [0, \infty)$ .

Our main results concern the limit behaviour of the measures  $\mu^n$  when  $\varrho = \beta$ , which will be called  $\beta$ -rough super Brownian motion. Here is the definition of  $\beta$ -super Brownian motion. We use  ${}_{\varkappa}U_t(\varphi)$  to denote the unique solution to the equation

$$\begin{cases} \partial_t w_t = \mathcal{H}w_t - \varkappa w_t^{1+\beta}, \\ w_0 = \varphi. \end{cases} \quad (4.3.4)$$

In most situation, we will drop the prescript  $\varkappa$  and write  $U_t(\varphi)$  if there is no confusion.

**Definition 4.3.2.** Let  $\xi$  be a deterministic environment satisfying assumption 4.3.1. Let  $\varkappa > 0$  and  $\mu_0$  be a compact support measure. Let  $\mu$  be a process with values in the space  $\mathbb{D}([0, \infty), \mathcal{M}_F(\mathbb{R}^2))$ , such that  $\mu(0) = \mu_0$ . Write  $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, \infty)}$  for the completed and right-continuous filtration generated by  $\mu$ . We call  $\mu$  a  $\beta$ -rough super Brownian motion with parameter  $\varkappa$  if it satisfies one of the three properties below:

1. For any  $t \geq 0$  and non-negative function  $\varphi \in C_c^\infty(\mathbb{R}^2)$ . The process

$$N_t^\varphi(s) := e^{-\langle \mu_s, U_{t-s}(\varphi) \rangle},$$

is a bounded martingale.

2. For any  $t \geq 0$ ,  $\varphi_0 \in C_c^\infty(\mathbb{R}^2)$  and  $f \in C_t C^\zeta(\mathbb{R}^2, e(l))$  for any  $\zeta > 0, l < -t$ . Let  $\varphi_t(s)$  solves

$$\partial_s \varphi_t(s) + \mathcal{H}\varphi(s) = f(s), \quad s \in [0, t], \varphi_t(t) = \varphi_0,$$

then

$$\mathbb{M}_t^{\varphi_0, f}(s) := \langle \mu_s, \varphi_t(s) \rangle - \langle \mu_0, \varphi_0 \rangle - \int_0^s \langle \mu_r, f(r) \rangle dr,$$

defined for  $s \in [0, t]$ , is a  $L^{1+\theta}$  bounded purely discontinuous martingale for any  $0 < \theta < \beta$ . In addition, the random point measure associated with  $\mathbb{M}_t^{\varphi_0, f}$  has the compensator

$$\langle \mu_s, \varkappa |\varphi_t(s)|^{1+\beta} \rangle \frac{\beta(1+\beta)}{\Gamma(1-\beta)x^{\beta+2}} \mathbb{1}_{x\varphi_t > 0} ds dx$$

3. For all function  $\phi \in \mathcal{D}_{\mathcal{H}}$ ,

$$L_t^\phi := \langle \mu_t, \phi \rangle - \langle \mu_0, \phi \rangle - \int_0^t \langle \mu_s, \mathcal{H}\phi \rangle ds,$$

is a  $L^{1+\theta}$  bounded purely discontinuous martingale for any  $0 < \theta < \beta$ . In addition, the random point measure associated with  $L_t^\phi$  has the compensator

$$\langle \mu_t, \varkappa |\phi|^{1+\beta} \rangle \frac{\beta(1+\beta)}{\Gamma(1-\beta)x^{\beta+2}} \mathbb{1}_{x\phi > 0} dt dx.$$

**Remark 4.3.3.** It will be seen in the proof of 4.3.4 that the second definition actually holds for  $f$  with any weight  $e(l)$  if we goes from 2. to 1. and obtain the moment estimation of  $\langle \mu_t, e(l) \rangle$  from definition 1. via similar argument as discrete case. This moments estimation then ensure the limit procedure in the proof from 3. to 2. when the weight of  $f$  is arbitrary.

Our main results are the following:

**Theorem 4.3.4.** The three definition in 4.3.2 are equivalent.

The proof is given in section 4.6

**Theorem 4.3.5.** For any  $\varkappa > 0$ , the  $\beta$ -rough super Brownian motion exists and has unique law. Furthermore, it can be realized as a limit of some particles systems.

The proof is given in section 4.5. For the case  $\varkappa < \frac{2}{1+\beta}$ , it is the limit of  $\mu_n$ . And for more general case, we do the mixing method as in the paper [78].

As a direct consequence of the method in [59] and a coupling method, we can show the compact support property and the super-exponentially persistence of the  $\beta$ -super Brownian motion.

**Definition 4.3.6.** We say a stochastic process with value in  $\mathcal{M}_F(\mathbb{R}^2)$  is super-exponentially persistent if, for any non-zero positive function  $\varphi \in C_c^\infty(\mathbb{R}^2)$  and for all  $\lambda > 0$ , it holds that

$$\mathbb{P} \left[ \lim_{t \rightarrow \infty} e^{-t\lambda} \langle \mu(t), \varphi \rangle = \infty \right] > 0.$$

**Theorem 4.3.7.** The  $\beta$ -rough super Brownian motion is super-exponentially persistent.

**Definition 4.3.8.** We say a stochastic process with value in  $\mathcal{M}_F(\mathbb{R}^2)$  possesses the compact support property if

$$\mathbb{P} \left[ \bigcup_{0 \leq s \leq t} \text{supp}(\mu_s) \text{ is compact} \right] = 1,$$

for any  $t \geq 0$ .

**Theorem 4.3.9.** The  $\beta$ -rough super Brownian motion possesses the compact support property.

## 4.4 On variants of the Parabolic Anderson Model

In this section, we discuss the required knowledge of SPDEs to construct the  $\beta$ -rough super Brownian motion. To start with, let's give the solution theory of PAM on  $\mathbb{Z}_n^2$  and  $\mathbb{R}^2$ . Details can be found in [71]. Let  $\kappa \in (0, \frac{1}{4})$  be a regularity parameter and  $T > 0$  be the time horizon.

**Theorem 4.4.1.** Consider  $f^n \in C_T C^{-1+2\kappa}(\mathbb{Z}_n^2, e(l))$ ,  $\varphi^n \in C^{1-\kappa}(\mathbb{Z}_n^2, e(l))$  and  $\varphi^{n,\#} \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))$ . Then the equation

$$\begin{cases} \partial_t w_t^n = \mathcal{H}^n w_t^n + f^n, \\ w_0^n = \varphi^n, \end{cases} \quad (4.4.1)$$

admits a unique solution with the estimations

$$\|w^n\|_{\mathcal{L}_T^{1-\kappa}(\mathbb{Z}_n^2, e(l+T))} \lesssim \|f^n\|_{C_T C^{-1+2\kappa}(\mathbb{Z}_n^2, e(l))} + \|\varphi^n\|_{\mathcal{D}(\mathbb{Z}_n^2, e(l))}. \quad (4.4.2)$$

The constant in ' $\lesssim$ ' is uniform over  $n$ . Furthermore, if the functions  $\mathcal{E}^n \varphi^n \rightarrow \varphi$  and  $\mathcal{E}^n f^n \rightarrow f$ , then we have the convergence of solution  $w^n$  to the solution of continuum PAM with the same estimation as (4.4.2).

**Remark 4.4.2.** When the initial condition  $\varphi$  is not paracontrolled but smooth, we could operate as follows. Consider  $\tilde{w}_t = w_t - \varphi^n$ , then the function  $w_t$  satisfies

$$\begin{cases} \partial_t \tilde{w}_t = \mathcal{H}^n \tilde{w}_t + f^n + \mathcal{H}^n \varphi^n \\ \tilde{w}_0 = 0 \end{cases} \quad (4.4.3)$$

Now if  $\varphi^n \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))$  for some  $l \in \mathbb{R}$ , then we see that  $\mathcal{H}^n \varphi^n$  is well-defined and belongs to the space  $C^{-1+2\kappa}(\mathbb{Z}_n^2, e(l)p(a))$  for any  $a > 0$ . Then theorem 4.4.1 can be applied to equation (4.4.3) and obtain estimations:

$$\|w^n\|_{\mathcal{L}_T^{1-\kappa}(\mathbb{Z}_n^2, e(l+T))} \lesssim \|f^n\|_{C_T C^{-1+2\kappa}(\mathbb{Z}_n^2, e(l))} + \|\varphi^n\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))}. \quad (4.4.4)$$

#### 4.4.1 Discrete case

In this section, we fix a positive integer  $n$  and consider equation on the discrete space-time  $\mathbb{R}_+ \times \mathbb{Z}_n^2$ . We now turn to the consideration of a non-linear discrete PAM.

$$\begin{cases} \partial_t w_t^n = \mathcal{H}^n w_t^n - f(w_t^n, \xi_e^n) w_t^n, & \text{in } (0, \infty] \times \mathbb{Z}_n^2, \\ w_0^n = \varphi^n \geq 0, & \text{on } \mathbb{Z}_n^2, \\ w_t^n \geq 0, & \forall t \geq 0, \end{cases} \quad (4.4.5)$$

where  $f$  is a non-negative function. We give assumptions on  $f$  here:

**Assumption 4.4.1.** There exists some  $\alpha > 0$ ,

$$|f(x, y)| \lesssim 1 + |x|^\alpha,$$

and the three-variables function

$$\frac{f(x_1, y)x_1 - f(x_2, y)x_2}{x_1 - x_2},$$

is locally bounded on  $(\mathbb{Z}_n^2)^3$ .

For (4.4.5), we have the following theorem.

**Theorem 4.4.3.** Suppose  $f$  satisfies assumption 4.4.1. Let  $\varphi^n \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))$  for some  $l \in \mathbb{R}$ . Then the equation (4.4.5) admits a unique solution  $w_t^n$  in the space  $\cup_{\hat{l} \in \mathbb{R}, \sigma \in (0, 1)} \mathcal{L}^{1-\kappa}(\mathbb{Z}_n^2, e(\hat{l}))$  and we have the estimation

$$\|w_t^n\|_{\mathcal{L}^{1-\kappa}(\mathbb{Z}_n^2, e(l'(1+\alpha)t))} \lesssim \|\varphi^n\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))} (1 + \|\varphi^n\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))}^\alpha),$$

for any  $l' > (1 + \alpha)l$ .

To prove theorem 4.4.3, we first give a lemma on representing solutions of a larger class of equations by Dynkin's formula.

**Lemma 4.4.4.** *Suppose  $R : \mathbb{R}^3 \rightarrow \mathbb{R}$  is continuous and has at most polynomial growth. Let  $\phi \in C_T L^\infty(\mathbb{Z}_n^2, e(l))$  and  $v_t \in C_T L^\infty(\mathbb{Z}_n^2, e(l))$  be the solution to the equation*

$$\begin{cases} \partial_t v_t = (\Delta^n + \xi_e^n) v_t - R(v_t, \xi_e^n, \phi_t), & \text{in } (0, \infty] \times \mathbb{Z}_n^2, \\ v_0 = \varphi, & \text{on } \mathbb{Z}_n^2, \end{cases} \quad (4.4.6)$$

for some  $l \in \mathbb{R}$ . Then we have for  $x \in \mathbb{Z}_n^2$

$$v_t(x) = \mathbb{E} \left[ \varphi(X_t^n) + \int_0^t (v_{t-s} \cdot \xi_e^n)(X_s^n) - R(v_{t-s}, \xi_e^n, \phi_{t-s})(X_s^n) ds \mid X_0 = x \right]. \quad (4.4.7)$$

*Proof.* As said in [71], all the space  $C^\alpha(\mathbb{Z}_n^2, e(l))$  are equivalent when  $n$  and  $l$  are fixed. The assumption on  $\xi^n$  then tells us that the weighted supremum norm of  $\xi_e^n$  is bounded. Together with the assumption on  $R$ , we see that  $\partial_t v_t \in C^\alpha(\mathbb{Z}_n^2, e(\hat{l}))$  for some  $\hat{l}$  and any  $\alpha \in \mathbb{R}$ . In particular  $\nabla^n \partial_t v_t \in L^\infty(\mathbb{Z}_n^2, e(\hat{l}))$ .

For each  $N > 0$  and fixed  $t > 0$ , consider truncated function  $f^N(s, x) = v_{t-s}(x) \mathbb{1}_{|x| \leq N}$ . We also define  $f(s, x) = v_{t-s}(x)$ . Since  $X_s^n$  is a strong Markov process with generator

$$\Delta^n \varphi(x) = n^2 \sum_{y \sim x} (\varphi(y) - \varphi(x)),$$

from Dynkin's formula, we calculate

$$\begin{aligned} \mathbb{E} [f(t, X_t^n) - f(0, x)] &= \sum_i \lim_{N \rightarrow \infty} \mathbb{E} \left[ f^N(t_{i+1}, X_{t_{i+1}}^n) - f^N(t_i, X_{t_i}^n) \right] \\ &= \sum_i \lim_{N \rightarrow \infty} \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \Delta^n f^N(t_{i+1}, X_s^n) + \partial_t f^N(s, X_{t_i}^n) ds \right] \quad (4.4.8) \\ &= \sum_i \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \Delta^n f(t_{i+1}, X_s^n) + \partial_t f(s, X_{t_i}^n) ds \right]. \end{aligned}$$

The first and third equations hold because the bounds for  $v$ ,  $\partial_t v$  and  $\forall l_0 \in \mathbb{R}$

$$\mathbb{E} [e^{l_0 \sup_{0 \leq s \leq t} |X_s^n|}] < \infty.$$

The error terms can be estimated as

$$\begin{aligned} &\sum_i \mathbb{E} \left[ \left| \int_{t_i}^{t_{i+1}} \Delta^n (f(t_{i+1}, X_s^n) - f(s, X_s^n)) ds \right| \right] \\ &\lesssim \sum_i \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \int_s^{t_{i+1}} e^{l|X_s^n|^\sigma} dudv \right] \lesssim_n \sum_i (t_{i+1} - t_i)^2, \end{aligned}$$

and, let  $M_t^n := \sup_{0 \leq s \leq t} |X_s^n|$ ,

$$\begin{aligned}
& \sum_i \mathbb{E} \left[ \left| \int_{t_i}^{t_{i+1}} \partial_t f^N(s, X_s^n) - \partial_t f^N(s, X_{t_i}^n) ds \right| \right] \\
& \lesssim \sum_i \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} |X_{t_i}^n - X_s^n| e^{\hat{l} M_t^n |^\sigma} ds \right] \\
& \lesssim \sum_i \int_{t_i}^{t_{i+1}} \mathbb{E}[|X_{t_i}^n - X_s^n|^2] \mathbb{E}[e^{2\hat{l} M_t^n |^\sigma}] ds \lesssim \sum_i (t_{i+1} - t_i)^3.
\end{aligned} \tag{4.4.9}$$

Thus we obtain that

$$\mathbb{E} [f(t, X_t^n) - f(0, x)] = \mathbb{E} \left[ \int_0^t \Delta^n f(s, X_s^n) + \partial_t f(s, X_s^n) ds \right].$$

Insert the equation of  $v$ , we then obtain the result.  $\square$

Next, we need a lemma on a variant equation of PAM, namely

$$\begin{cases} \partial_t w_t^n = (\mathcal{H}^n - \phi_t^n) w_t^n, & \text{in } (0, \infty] \times \mathbb{Z}_n^2, \\ w_0^n = \varphi^n \geq 0, & \text{on } \mathbb{Z}_n^2, \\ w_t^n \geq 0, & \forall t \geq 0, \end{cases} \tag{4.4.10}$$

where  $\phi_t^n$  is a positive function.

**Lemma 4.4.5.** *Suppose non-negative functions  $\phi_t^n \in C_T L^\infty(\mathbb{Z}_n^2, e(l_0 + \cdot))$  and  $\varphi^n \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))$  for some  $l_0, l \in \mathbb{R}$ . Then the equation (4.4.10) admits a minimum non-negative solution given by mild formulation*

$$w_t^n = T_t^n(\varphi^n) + \int_0^t T_{t-s}^n(\phi_s^n w_s^n) ds,$$

with estimation

$$\|w_t^n\|_{\mathcal{L}^{1-\kappa}(\mathbb{Z}_n^2, e(l+l_0+2t))} \lesssim \|\varphi^n\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))} (1 + \|\phi_t^n\|_{L^\infty(\mathbb{Z}_n^2, e(l_0+t))}),$$

and

$$\|w_t^n\|_{L^\infty(\mathbb{Z}_n^2, e(l+t))} \lesssim \|\varphi^n\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))},$$

where ' $\lesssim$ ' is independent of  $n$ . Furthermore, the uniqueness of solution holds in the space  $\cup_{l \in \mathbb{R}} C_T L^\infty(\mathbb{Z}_n^2, e(\hat{l}))$ .

*Proof.* By similar argument in [40], we know the minimum solution of equation (4.4.10) is given by the following Feynman-Kac formula

$$w_t^n(x) = \mathbb{E} \left[ e^{\int_0^t (\xi_s^n - \phi_{t-s}^n)(X_s^n) ds} \varphi^n(X_t^n) | X_0^n = x \right], \tag{4.4.11}$$

if it is finite. By the positivity of  $\phi^n$  and  $\varphi^n$ , we see the integrand has the estimation

$$0 \leq e^{\int_0^t (\xi_e^n - \phi_{t-s}^n)(X_s^n) ds} \varphi^n(X_t^n) \leq e^{\int_0^t \xi_e^n(X_s^n) ds} \varphi^n(X_t^n).$$

Thus  $w_t^n$  is finite due to the solvability of discrete PAM. Furthermore, we then have the estimation

$$\|w_t^n\|_{L^\infty(\mathbb{Z}_n^2, e^{(l+t)})} \lesssim \|T_t^n \varphi^n\|_{L^\infty(\mathbb{Z}_n^2, e^{(l+t)})} \lesssim \|\varphi^n\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e^{(l)})}.$$

By the regularity estimation of PAM, we have

$$\begin{aligned} \|w^n\|_{\mathcal{L}^{1-\kappa}(\mathbb{Z}_n^2, e^{(l+l_0+2)})} &\lesssim \|\varphi^n\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e^{(l+l_0)})} + \|\phi^n w^n\|_{C_T L^\infty(\mathbb{Z}_n^2, e^{(l+l_0+2)})} \\ &\lesssim \|\varphi^n\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e^{(l)})} (1 + \|\phi_t^n\|_{L^\infty(\mathbb{Z}_n^2, e^{(l_0+t)})}). \end{aligned}$$

Now from the Feynman Kac formulation to mild formulation, the integration by parts tells us that

$$\begin{aligned} w_t^n(x) &= \mathbb{E}_x \left[ \left( 1 - \int_0^t d_s e^{\int_s^t (\xi_e^n - \phi_{t-r}^n)(X_r^n) dr} \right) \varphi^n(X_t^n) \right] \\ &= \mathbb{E}_x \left[ \varphi^n(X_t^n) + \int_0^t e^{\int_s^t (\xi_e^n - \phi_{t-u}^n)(X_u^n) du} (\xi_e^n - \phi_{t-s}^n)(X_s^n) ds \varphi^n(X_t^n) \right] \\ &= \mathbb{E}_x \left[ \varphi^n(X_t^n) + \int_0^t e^{\int_{t-s}^t (\xi_e^n - \phi_{t-u}^n)(X_u^n) du} \varphi^n(X_t^n) (\xi_e^n - \phi_s^n)(X_{t-s}^n) ds \right] \\ &= \mathbb{E}_x \left[ \varphi^n(X_t^n) + \int_0^t w_s^n(X_{t-s}^n) (\xi_e^n - \phi_s^n)(X_{t-s}^n) ds \right] \\ &= P_t^n(\varphi^n) + \int_0^t P_{t-s}^n(w_s^n(\xi_e^n - \phi_s^n)) ds, \end{aligned}$$

where we applied the Markov property of  $X^n$  at the fourth equality. Then basic argument goes to the formulation with operator  $T^n$ .

On the uniqueness, if the solution  $w^n \in C_T L^\infty(\mathbb{Z}_n^2, e^{\hat{l}})$  for some  $\hat{l} \in \mathbb{R}$ , we know by lemma 4.4.4 that the solution must be given by

$$w_t^n(x) = \mathbb{E} \left[ \varphi(X_t^n) + \int_0^t (w_{t-s}^n \cdot \xi_e^n)(X_s^n) - w_{t-s}^n(X_s^n) (\phi_{t-s}^n)(X_s^n) ds \mid X_0 = x \right],$$

which is exactly the mild formulation. Now we do the reverse procedure. The integration by parts formula gives us

$$\begin{aligned} &\int_0^t e^{\int_{t-s}^t (\xi_e^n - \phi_u^n)(X_u^n) du} (\xi_e^n - \phi_{t-s}^n)(X_s^n) \int_s^t w_{t-u}^n (\xi_e^n - \phi_{t-u}^n)(X_u^n) dud s \\ &= \int_0^t \int_s^t w_{t-u}^n (\xi_e^n - \phi_{t-u}^n)(X_u^n) dud \left( e^{\int_{t-s}^t (\xi_e^n - \phi_u^n)(X_u^n) du} \right) \\ &= \int_0^t e^{\int_{t-s}^t (\xi_e^n - \phi_u^n)(X_u^n) du} w_{t-s}^n (\xi_e^n - \phi_{t-s}^n)(X_s^n) ds \\ &\quad - \int_0^t w_{t-u}^n (\xi_e^n - \phi_{t-u}^n)(X_u^n) du. \end{aligned}$$

Thus we have by Markov property

$$\begin{aligned}
w_t^n(x) &= \mathbb{E}_x \left[ - \int_0^t e^{\int_{t-s}^t (\xi_e^n - \phi_u^n)(X_{t-u}^n) du} (\xi_e^n - \phi_{t-s}^n)(X_s^n) \{w_{t-s}^n(X_s^n) - \varphi^n(X_t^n)\} \right. \\
&\quad \left. - \int_s^t w_{t-u}^n(\xi_e^n - \phi_{t-u}^n)(X_u^n) du \right] ds + w_t^n(x) \\
&= \mathbb{E}_x \left[ w_t^n(x) - \int_0^t w_{t-s}^n(\xi_e^n - \phi_{t-s}^n)(X_s^n) ds \right. \\
&\quad \left. + \int_0^t \varphi^n(X_t^n) d \left( e^{\int_{t-s}^t (\xi_e^n - \phi_u^n)(X_{t-u}^n) du} \right) \right] \\
&= \mathbb{E}_x \left[ \varphi^n(X_t^n) e^{\int_0^t (\xi_e^n - \phi_{t-s}^n)(X_s^n) ds} \right].
\end{aligned}$$

Thus solution  $w_t^n$  is unique.  $\square$

**Remark 4.4.6.** Above lemma also holds when  $\varphi^n \in \mathcal{D}$  by theorem 4.4.1 and the estimation of  $w_t^n$  will change to

$$\|w_t^n\|_{\mathcal{L}^{1-\kappa}(\mathbb{Z}_n^2, e^{(l+l_0+2t)})} \lesssim \|\varphi^n\|_{\mathcal{D}(\mathbb{Z}_n^2, e^{(l)})} (1 + \|\phi_t^n\|_{L^\infty(\mathbb{Z}_n^2, e^{(l_0+t)})}).$$

Furthermore, by the equivalence between the mild representation and the Feynmann-Kac representation, we can see that the only solution to the equation (4.4.10) with initial 0 should be 0 even **without the positivity conditions (for  $\phi^n, w^n$ )**.

Now we are able to give a proof of theorem 4.4.3

*Proof of theorem 4.4.3.* We define the map  $\mathcal{K} : \phi^n \rightarrow \mathcal{K}(\phi^n)$  to be the minimum solution to the equation (4.4.10). Now consider a function  $w \in C_T(\mathbb{Z}_n^2, e^{((l+\cdot))})$ , we have

$$\|\mathcal{K}(f(w^n, \xi_e^n))_t\|_{L^\infty(\mathbb{Z}_n^2, e^{(l+t)})} \lesssim \|\varphi^n\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e^{(l)})},$$

This estimation shows that map  $\mathcal{K}(f(\cdot, \xi_e^n))$  maps the whole space  $C_T(\mathbb{Z}_n^2, e^{((l+\cdot))})$  to a bounded subspace of itself. We now equip the space with the weak\* topology, then the range of the map  $\mathcal{K}$  is contained in a compact subspace of the  $C_T(\mathbb{Z}_n^2, e^{((l+\cdot))})$ . From Schauder's fixed point theorem, we know there exists a fixed point for this map and the fixed point is exactly the mild solution to the equation (4.4.5). To show  $w_t^n$  is indeed a solution of equation (4.4.5). We could write  $w_t^n$  in the Feynman Kac formulation

$$w_t^n(x) = \mathbb{E}_x \left[ e^{\int_0^t (\xi_e^n - f(w_{t-s}^n, \xi_e^n))(X_s^n) ds} \varphi^n(X_t^n) \right],$$

then by the lemma 4.4.5, we know  $w_t^n$  is the minimum solution to the equation

$$\begin{cases} \partial_t u_t^n = (\mathcal{H}^n - f(w_t^n, \xi_e^n))u_t^n, \\ u_0^n = \varphi^n, \\ u_t^n \geq 0. \end{cases} \quad (4.4.12)$$

Thus is it a solution to the equation (4.4.5). Now we turn to the uniqueness property. Suppose we have two solutions to the equation (4.4.5), say  $\tilde{w}^n$  and  $w^n$ . Then the differences

$$\bar{w}^n := \tilde{w}^n - w^n,$$

satisfies the equation

$$\begin{cases} \partial_t \bar{w}_t^n = \mathcal{H}^n \bar{w}_t^n - \frac{f(\tilde{w}_t^n, \xi_e^n) w_t^n - f(w_t^n, \xi_e^n) w_t^n}{\bar{w}_t^n} \bar{w}_t^n, \\ \bar{w}_0^n = 0. \end{cases} \quad (4.4.13)$$

Above estimation shows that supremum norm of  $\tilde{w}^n$  is bounded, and combining with assumption 4.4.1, we know

$$\frac{f(\tilde{w}_t^n, \xi_e^n) w_t^n - f(w_t^n, \xi_e^n) w_t^n}{\bar{w}_t^n} \in C_T L^\infty(Z_n^2, e(\tilde{l})).$$

Then from remark 4.4.6, we know  $\bar{w}^n = 0$  and hence the uniqueness holds.  $\square$

## 4.4.2 Continuous case

For the continuous case, all the existence problems of mild solutions can be solved by sending  $n$  to infinity in the discrete since all the estimations in the discrete cases are uniform over  $n$ . We are then interested in the uniqueness of solutions.

As we can see in the discrete case, the keys to the uniqueness are the equivalence between Feynman-Kac formula and the mild formulation. It is not clear what  $e^{\int_0^t \xi(X_s) ds}$  means in Feynman-Kac formula for the continuous case. However, we can hide this quantity by considering the directed polymer measure directly.

The construction and convergence of polymer measure for spatial white noise by mollified or discrete white noise on  $\mathbb{T}^2$  are given in [7, 18]. Following the arguments in [18], the convergence of solutions to the corresponding Parabolic Anderson Models and the application of [71] on the whole  $\mathbb{R}^2$  will give the construction and convergence of polymer measure on  $\mathbb{R}^2$ .

Intuitively, we only need the Markov property of  $X^n$  when we prove the uniqueness of equation (4.4.10). And in [1], the authors show the polymer measure conditioned on the environment is actually a Markov process, which implies that the uniqueness in the continuous case should also hold. We start with the definition of polymer measure  $Q$  on the space  $C[0, T]$ .

**Definition 4.4.7.** *Let  $X_t$  be the canonical process of  $C[0, T]$ , define the finite dimensional distribution for  $0 = t_0 < t_1 < \dots < t_k \leq T$  and  $x_0 = x, x_i \in \mathbb{R}^2, i = 1, \dots, k$ :*

$$Q_x [X_{t_i} \in dx_i] = \frac{1}{\mathcal{Z}_T(x)} \prod_{i=0}^{k-1} \mathcal{Z}_{t_{i+1}-t_i}(x_i, x_{i+1}) \mathcal{Z}_{T-t_k}(x_k) dx_1 dx_2 \cdot dx_k,$$

where  $\mathcal{Z}_t(x, y)$  is the solution to the equation

$$\begin{cases} \partial_t \mathcal{Z}_t(x, y) = \mathcal{H}_x \mathcal{Z}_t(x, y), & \text{in } (0, \infty] \times \mathbb{R}^2, \\ \mathcal{Z}_0(x) = \delta_y(x), & \text{on } \mathbb{R}^2. \end{cases} \quad (4.4.14)$$

and  $\mathcal{Z}_t(x) := \int_{\mathbb{R}^2} \mathcal{Z}_t(x, y) dy$ . The measure  $Q_x$  generated by the above finite-dimensional distribution is called the directed polymer measure starting at point  $x$ .

**Remark 4.4.8.** The two-dimensional dirac function have the regularity of  $B_{p, \infty}^{-2(1-1/p)}$  and applying conclusion in [71] with  $p = 1$ , we see that equation (4.4.14) admits a unique solution in the paracontrolled space. The solution  $\mathcal{Z}_t(x, y)$  is actually a Green's function and we have the representation

$$T_t \varphi(x) = \int_{\mathbb{R}^2} \mathcal{Z}_t(x, y) \varphi(y) dy.$$

Furthermore, the consistency of finite-dimensional distribution is guaranteed by

$$\mathcal{Z}_{t+s}(x, y) = (T_{t+s} \delta_y)(x) = (T_t \mathcal{Z}_s(\cdot, y))(x) = \int_{\mathbb{R}^2} \mathcal{Z}_t(x, z) \mathcal{Z}_s(z, y) dz.$$

In addition, the markov property of polymer measure can be seen from its finite-dimensional distribution directly.

Apart from the Markov property, we need a result on the transition function of the polymer measure, which can actually be viewed as a version of definition of polymer measure.

**Lemma 4.4.9.** For  $0 < s < t$ , we have the following representation for  $\varphi \in C^{1+2\kappa}(\mathbb{R}^2, e(l))$ :

$$T_{t-s} \varphi(X_s) = \mathbb{E}_x^Q \left[ \mathcal{Z}_{T-s}(X_s) \frac{\varphi(X_t)}{\mathcal{Z}_{T-t}(X_t)} | X_s \right],$$

for  $X_s$  following the probability measure  $Q_x$ .

*Proof.* For any measurable function  $g$ , we calculate

$$\begin{aligned} & \mathbb{E}_x^Q \left[ \mathcal{Z}_{T-s}(X_s) \frac{\varphi(X_t)}{\mathcal{Z}_{T-t}(X_t)} g(X_s) \right] \\ &= \frac{1}{\mathcal{Z}_T(x)} \int_{\mathbb{R}^2} \mathcal{Z}_s(x, x_1) \mathcal{Z}_{t-s}(x_1, x_2) \mathcal{Z}_{T-t}(x_2) \mathcal{Z}_{T-s}(x_1) g(x_1) \frac{\varphi(x_2)}{\mathcal{Z}_{T-t}(x_2)} dx_1 dx_2 \\ &= \frac{1}{\mathcal{Z}_T(x)} \int_{\mathbb{R}^2} \mathcal{Z}_s(x, x_1) \mathcal{Z}_{t-s}(x_1, x_2) \mathcal{Z}_{T-s}(x_1) g(x_1) \varphi(x_2) dx_1 dx_2 \\ &= \frac{1}{\mathcal{Z}_T(x)} \int_{\mathbb{R}} T_{t-s} \varphi(x_1) g(x_1) \mathcal{Z}_s(x, x_1) \mathcal{Z}_{T-s}(x_1) dx_1 = \mathbb{E}_x^Q [T_{t-s} \varphi(X_s) g(X_s)]. \end{aligned}$$

Thus the result follows.  $\square$

Now consider equation

$$\begin{cases} \partial_t w_t = (\mathcal{H} - \phi_t)w_t, & \text{in } (0, \infty] \times \mathbb{R}^2, \\ w_0 = \varphi \geq 0, & \text{on } \mathbb{R}^2 \\ w_t \geq 0, & \forall t \geq 0. \end{cases} \quad (4.4.15)$$

**Lemma 4.4.10.** *Suppose  $\phi_t \in C_T L^\infty(\mathbb{R}^2, e(l_0 + \cdot))$  and  $\varphi \in C^{1+2\kappa}(\mathbb{R}^2, e(l))$  for some  $l_0, l \in \mathbb{R}$ . Then the equation (4.4.10) admits a solution given by mild formulation*

$$w_t = T_t(\varphi) + \int_0^t T_{t-s}(\phi_s w_s) ds,$$

with estimation

$$\|w_t\|_{\mathcal{L}^{1-\kappa}(\mathbb{R}^2, e(l_0+2t))} \lesssim \|\varphi\|_{C^{1+2\kappa}(\mathbb{R}^2, e(l))} (1 + \|\phi_t\|_{L^\infty(\mathbb{R}^2, e(l_0+t))}),$$

and

$$\|w_t\|_{L^\infty(\mathbb{R}^2, e(l+t))} \lesssim \|\varphi\|_{C^{1+2\kappa}(\mathbb{R}^2, e(l))}.$$

Same result for  $\varphi \in \mathcal{D}^{1-\epsilon}(\mathbb{R}^2, e(l))$ . Furthermore, the uniqueness of solution holds in the space  $\cup_{i \in \mathbb{R}, \sigma \in (0,1)} C_T C^0(\mathbb{R}^2, e(\hat{l}))$ .

*Proof.* The existence is a direct consequence of discrete case and taking the limit. On the uniqueness, the mild formulation is given by

$$\begin{aligned} w_t(x) &= T_t \varphi - \int_0^t T_{t-s}(\phi_s w_s) ds \\ &= \mathbb{E}_x^Q \left[ \mathcal{Z}_T(x) \frac{\varphi(X_t)}{\mathcal{Z}_{T-t}(X_t)} - \mathcal{Z}_T(x) \int_0^t \frac{\phi_{t-s} w_{t-s}(X_s)}{\mathcal{Z}_{T-s}(X_s)} ds \right]. \end{aligned}$$

Furthermore, let  $0 < s < t$ , we have

$$w_{t-s}(X_s) = \mathbb{E}_x^Q \left[ \mathcal{Z}_{T-s}(X_s) \frac{\varphi(X_t)}{\mathcal{Z}_{T-t}(X_t)} - \mathcal{Z}_{T-s}(X_s) \int_s^t \frac{\phi_{t-u} w_{t-u}(X_u)}{\mathcal{Z}_{T-u}(X_u)} du | X_s \right].$$

Thus, same calculation as in the lemma 4.4.5 gives

$$\begin{aligned} w_t(x) &= \mathbb{E}_x^Q \left[ w_t(x) + \mathcal{Z}_T(x) \int_0^t e^{\int_{t-s}^t -\phi_u(X_{t-u}) du} \frac{-\phi_{t-s}(X_s)}{\mathcal{Z}_{T-s}(X_s)} \{w_{t-s}(X_s) - \right. \\ &\quad \left. \mathcal{Z}_{T-s}(X_s) \frac{\varphi(X_t)}{\mathcal{Z}_{T-t}(X_t)} + \mathcal{Z}_{T-s}(X_s) \int_s^t \frac{\phi_{t-u}(X_u) w_{t-u}(X_u)}{\mathcal{Z}_{T-u}(X_u)} du \} ds \right] \\ &= \mathbb{E}_x^Q \left[ \mathcal{Z}_T(x) \frac{\varphi(X_t)}{\mathcal{Z}_{T-t}(X_t)} \left( 1 - \int_0^t e^{-\int_{t-s}^t \phi_u(X_{t-u}) dy} \phi_{t-s}(X_s) ds \right) \right] \\ &= \mathbb{E}_x^Q \left[ \mathcal{Z}_T(x) e^{-\int_0^t \phi_{t-u}(X_u) du} \frac{\varphi(X_t)}{\mathcal{Z}_{T-t}(X_t)} \right]. \end{aligned}$$

The continuity of  $w_t, X_t$  and the strict positivity of  $\mathcal{Z}$  ensure all the above calculations inside the expectation are valid. Thus we obtain the uniqueness of mild solution to the equation (4.4.15).  $\square$

It is then a direct consequence that the solution of equation

$$\begin{cases} \partial_t w_t = \mathcal{H}w_t - f(w_t)w_t, & \text{in } (0, \infty] \times \mathbb{R}^2, \\ w_0 = \varphi \geq 0, & \text{on } \mathbb{R}^2, \\ w_t \geq 0, & \forall t \geq 0, \end{cases} \quad (4.4.16)$$

is unique, where  $f$  is a non-negative function.

**Corollary 4.4.11.** *Suppose  $|f(x)| \leq 1 + |x|^\alpha$  for some  $\alpha > 0$ ,  $xf(x)$  is locally Lipschitz and  $\varphi \in C^{1+2\kappa}(\mathbb{R}^2, e(l))$ , then there exists a unique mild solution to the equation (4.4.16) and we have the estimation*

$$\|w_t\|_{\mathcal{L}^{1-\kappa}(\mathbb{R}^2, e(l'+(1+\alpha)t))} \lesssim \|\varphi\|_{C^{1+2\kappa}(\mathbb{R}^2, e(l))} (1 + \|\varphi\|_{C^{1+2\kappa}(\mathbb{R}^2, e(l))}^\alpha),$$

for any and  $l' > (1 + \alpha)l$ .

*Proof.* The existence comes from the approximation by discrete equation. Uniqueness comes from the similar argument as in the proof of theorem 4.4.3.  $\square$

**Remark 4.4.12.** *All the results apply to the inhomogeneous equations:*

$$\begin{cases} \partial_t w_t = (\mathcal{H} - \phi_t)w_t + g_t, & \text{in } (0, \infty] \times \mathbb{R}^2, \\ w_0 = \varphi \geq 0, & \text{on } \mathbb{R}^2, \\ w_t \geq 0 & \forall t \geq 0, \end{cases} \quad (4.4.17)$$

and

$$\begin{cases} \partial_t w_t = \mathcal{H}w_t - f(w_t)w_t + g_t, & \text{in } (0, \infty] \times \mathbb{R}^2, \\ w_0 = \varphi \geq 0, & \text{on } \mathbb{R}^2, \\ w_t \geq 0 & \forall t \geq 0, \end{cases} \quad (4.4.18)$$

with the additional condition  $g \in C_T L^\infty(\mathbb{R}^2, e(l))$  and non-negative. Since in these case, the mild solution of (4.4.17) can be expressed as

$$\begin{aligned} w_t(x) = \mathbb{E}_x^Q & \left[ \mathcal{Z}_T(x) e^{-\int_0^t \phi_{t-u}(X_u) du} \frac{\varphi(X_t)}{\mathcal{Z}_{T-t}(X_t)} \right. \\ & \left. + \mathcal{Z}_T(s) \int_0^t e^{-\int_0^s \phi_{t-u}(X_u) du} \frac{g_{t-s}(X_s)}{\mathcal{Z}_{T-s}(X_s)} ds \right]. \end{aligned} \quad (4.4.19)$$

Thus the comparison principle with respect to  $\phi$  still holds. The result is needed for the compact support property.

**Remark 4.4.13.** *All the above discussion is valid for the operator  $\mathcal{H}$  replaced by  $\Delta$  with much simpler argument and better regularity.*

## 4.5 Existence as limit of particle systems

### 4.5.1 For the case $\varkappa < \frac{2}{1+\beta}$

In this section, we will show that the family  $\{\mu^n\}$  defined in definition 4.3.1 is tight in  $\mathbb{D}([0, \infty), \mathcal{M}(\mathbb{R}^2))$  and its limit is the  $\beta$ -rough super Brownian motion with desired Laplace functional.

We will need auxiliary branching random walks given by the same branching rate  $a^n$  with branching mechanism  $g(s)$  defined in (4.3.2). We denote the empirical measures associated with auxiliary branching random walks as  $\{\tilde{\mu}^n\}$ .

Notice this branching mechanism  $g$  is independent with respect to the  $n$  and position  $x$  and is well studied in classical books[27, 26] when the branching rate is also independent of the position.

**Theorem 4.5.1.** *Suppose  $\mu_0$  is supported on a compact set. Following the definition 4.3.1, the limit  $\mu_t := \lim_{n \rightarrow \infty} \mu_t^n$  exists as stochastic processes in the space  $\mathbb{D}([0, \infty), \mathcal{M}(\mathbb{R}^2))$  and, when  $\varrho = \beta$ , for all non-negative function  $\varphi \in C^{1+2\kappa}(\mathbb{R}^2, e(l))$ ,  $l \in \mathbb{R}$ , the laplace transformation is given by*

$$\mathbb{E} [e^{-\langle \mu_t, \varphi \rangle}] = e^{-\langle \mu_0, U_t(\varphi) \rangle}, \quad (4.5.1)$$

where  $U_t(\varphi)$  is the unique non-negative solution to the equation (4.3.4) with  $\varkappa = \frac{2\nu}{1+\beta}$ . In particular, this shows  $\mu_t$  is markov and the first definition of definition 4.3.2 is satisfied. Thus the  $\beta$ -rough super Brownian motion exists and is unique in law. In addition, when  $\varrho < \beta$ , the limit  $\mu_t$  satisfies the equation PAM.

Let's first give a more detailed description of spatial dependent branching mechanism and branching rate here. Since all particles in the system move independently, it is sufficient to describe the motion of one particle. To describe the branching time of one particle, let's consider independent Poisson processes  $N_t^x$  with an arbitrary intensity  $a(x)$  attached at each site  $x$  of  $\mathbb{Z}_n^2$ . Let  $X_t^n$  be a simple random walk on  $\mathbb{Z}_n^2$ , then the branching time  $\tau$  can be defined by

$$\tau := \inf\{t \geq 0 : X_t^n \text{ triggers the clock of } N_t^{X_t^n} \text{ at time } t\}.$$

By the strong Markov property of Poisson process, we know that each time  $X^n$  changes site, say time  $t_0$ , there is a new Poisson process  $(N_t^x - N_{t_0}^x)_{t \geq t_0}$  which is independent of all past event determining the branching time. Now let's do a scaling: set  $\tilde{N}_t^x = N_{\frac{t}{a(x)}}^x$ , then  $\tilde{N}_t^x$  are Poisson processes with intensity 1. If we paste all the pieces of scaled

Poisson processes before  $X_t^n$  branches, we obtain a random variable following the law of exponential distribution with parameter 1. Hence, by the scaling time, we can give a new definition of branching time  $\tau$ :

$$\tau := \inf \left\{ t \geq 0 : \int_0^t a(X_s) ds \geq \tilde{\tau} \right\},$$

where  $\tilde{\tau}$  is of exponential distribution with parameter 1. This also leads to the law of branching time  $\tau$ , which is given by

$$\mathbb{P}[\tau \geq t] = e^{-\int_0^t a(X_s) ds}. \quad (4.5.2)$$

As the consequence of Poisson Cluster Random Measure [C.1.5](#), we are able to give the Laplace functional of  $\mu_t^n$  and  $\tilde{\mu}_t^n$ .

**Lemma 4.5.2.** *Let  $A = \mathcal{H}^n$  or  $\Delta^n$ ,  $B = \frac{2\xi_{e,+}^n + \epsilon^\beta}{1+\beta}$  or  $\frac{|\xi_e^n| \epsilon^\beta}{1+\beta}$  correspondingly. For each  $\varphi$ , there exists  $U_t^n(\varphi)$  or  $\tilde{U}_t^n(\varphi)$  being the unique solution to the equation*

$$\begin{cases} \partial_t v_t^n = Av_t - B(v_t^n)^{1+\beta}, & \text{in } (0, \infty] \times \mathbb{R}^2, \\ v_0^n = \frac{1 - e^{-\epsilon\varphi(x)}}{\epsilon}, & \text{on } \mathbb{R}^2. \end{cases} \quad (4.5.3)$$

such that the Laplace functional of  $\mu_t^n$  and  $\tilde{\mu}_t^n$  is given by

$$\mathbb{E}[e^{-\langle \mu_t^n, \varphi \rangle}] = e^{-\langle \mu_0^n, U_t^n(\varphi) \rangle}, \quad \mathbb{E}[e^{-\langle \tilde{\mu}_t^n, \varphi \rangle}] = e^{-\langle \mu_0^n, \tilde{U}_t^n(\varphi) \rangle}.$$

*Proof.* We only show the proof for  $\mu_t^n$ , the proof to  $\tilde{\mu}_t^n$  is similar.

Define

$$w_t^n(x) := \mathbb{E} \left[ e^{-\langle \mu_t^n, \varphi \rangle} | Z_0^n = \delta_x \right]. \quad (4.5.4)$$

Then lemma [C.3.2](#) and lemma [C.3.1](#) tell us that

$$w_t^n(x) := \mathbb{E} \left[ e^{-\epsilon\varphi(x)} + \int_0^t |\xi_e^n|(X_r^n) g^n(X_r^n, w_{t-r}^n(X_r)) dr - \int_0^t |\xi_e^n| w_{t-r}^n(X_r) dr \right]. \quad (4.5.5)$$

The Poisson cluster formula [C.1.5](#) then gives us that

$$\mathbb{E}[e^{-\langle \mu_t^n, \varphi \rangle}] = e^{-\langle \mu_0^n, \frac{1 - w_t^\epsilon}{\epsilon} \rangle}. \quad (4.5.6)$$

Now define

$$v_t^n(x) = \frac{1 - w_t^n(x)}{\epsilon}.$$

The calculation shows that

$$v_t^n(x) = \mathbb{E} \left[ \frac{1 - e^{-\epsilon\varphi(X_t^n)}}{\epsilon} + \int_0^t v_{t-r}^n \xi_e^n(Z_r) - \frac{1}{1+\beta} (v_{t-r}^n)^{1+\beta} (2\xi_{e,+}^n + \epsilon^\beta) dr \right]. \quad (4.5.7)$$

Thus,  $v_t^n$  is the unique solution to the equation [\(4.5.3\)](#) by lemma [4.4.5](#).  $\square$

Next goal is to prove the tightness of  $\mu^n$  in the space  $\mathbb{D}(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^2))$ . We need following lemmas.

**Lemma 4.5.3.** *For any  $0 < \beta < 1$ , there exists an integer  $N(\beta)$  such that there exists a coupling of process  $\mu_t^n$  and  $N(\beta)$  sum of independent  $\tilde{\mu}_t^{n,i}$  being copies of  $\tilde{\mu}_t^n$  so that we have*

$$\langle \mu_t^n, \varphi \rangle \leq \sum_{i=1}^{N(\beta)} \langle \tilde{\mu}_t^{n,i}, \varphi \rangle,$$

for any non-negative  $\varphi$ . Furthermore,  $\langle \tilde{\mu}_t^{n,i}, 1 \rangle$  are martingales.

*Proof.* Consider random variables  $Z_0^n(x)$  and  $Z_i^n$  whose generators are given by  $g^n(x, s)$  and  $g(s)$ . We show that there exists  $N(\beta)$  such that for any  $k \in \mathbb{N}$ , the inequality holds

$$\mathbb{P}[Z_0^n(x) \geq k] \leq \mathbb{P} \left[ \sum_{i=1}^{N(\beta)} Z_i^n \geq k \right]. \quad (4.5.8)$$

By the definition of generator  $g^n$ , we rewrite the generator into infinite series

$$g^n(x, s) = \sum_{k=0}^{\infty} p_k g_k(x) s^k,$$

and

$$g(s) = \sum_{k=0}^{\infty} p_k s^k.$$

We know furthermore that  $g_0 \geq \frac{1-\beta}{1+\beta}$  and  $g_k \leq 2$  for  $k \geq 1$ . Now consider  $Z_1^n + Z_2^n + Z_3^n$ , we have for all  $k \in \mathbb{N}$ ,

$$\mathbb{P}[Z_1^n + Z_2^n + Z_3^n \geq k] \geq 3\mathbb{P}[Z_1^n \geq k]\mathbb{P}[Z_1^n < k]^2.$$

Since  $\lim_{k \rightarrow \infty} \mathbb{P}[Z_1^n < k] = 1$ , we know there exists  $k_0$  such that

$$\mathbb{P}[Z_1^n + Z_2^n + Z_3^n \geq k_0] \geq 2\mathbb{P}[Z_1^n \geq k_0] \geq \mathbb{P}[Z_0^n(x) \geq k_0].$$

Now for the  $k < k_0$ , due to  $p_0 g_0(x) > 0$ , the central limit theorem tells us the existence of  $N(\beta)$  that (4.5.8) holds. Then the result follows since we can always produce more particles at each site using inequality (4.5.8).

The martingale property of  $\langle \tilde{\mu}_t^{n,i}, 1 \rangle$  follows from the fact that  $g'(1) = 1$ , which means that each time we branch, the expectation of number of off-spring is again 1. So the total number of the particle system should form a martingale.  $\square$

**Lemma 4.5.4.** *Let  $l$  be any real number. For  $\varphi \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))$ , the process*

$$M_t^{n,\varphi}(s) := \mu_s^n(T_{t-s}^n \varphi) - \epsilon Z_0^n(T_t^n \varphi), \quad \tilde{M}_t^{n,\varphi}(s) := \tilde{\mu}_s^n(P_{t-s}^n \varphi) - \epsilon Z_0^n(P_t^n \varphi), \quad (4.5.9)$$

*are martingales. Thus by the formula of Poisson random measure,*

$$\mathbb{E}[\langle \mu_t^n, \varphi \rangle] = \langle \mu_0^n, T_t^n \varphi \rangle, \quad \mathbb{E}[\langle \tilde{\mu}_t^n, \varphi \rangle] = \langle \mu_0^n, P_t^n \varphi \rangle.$$

*Proof.* We consider only  $\mu_t^n$  here, the argument for  $\tilde{\mu}_t^n$  is similar. Since we do not have the exponential estimation of the measure  $\mu_t^n$ , we need to consider first time dependent function  $\phi \in C_t^1 L^\infty(\mathbb{Z}_n^2, e(-t + \cdot))$ . Let function  $F_\phi^t : [0, t] \times (\mathbb{N}^{\mathbb{Z}_n^2}) \rightarrow \mathbb{R}$  be

$$F_\phi^t(s, \eta) = \sum_{x \in \mathbb{Z}_n^2} \phi_s(x) \eta_x.$$

Apply Dynkin's formula to  $F_\phi^t$  and do similar things as in the proof of lemma 4.4.4, we obtain that

$$M_t(s) := \mu_s^n(\phi_s) - \epsilon Z_0^n(\phi_0) - \int_0^s \partial_r F_\phi^t(r, \mu_r^n) + \mathcal{L}^n F_\phi^t(r, \mu_r^n) dr,$$

is a martingale. The calculation gives

$$\partial_r F_\phi^t(r, \mu_r^n) = \mu_r^n(\partial_r \phi_r), \quad \mathcal{L}^n F_\phi^t(r, \mu_r^n) = \mu_r^n(\mathcal{H}^n \phi_r).$$

Consider  $\phi_s = T_{t-s}^n \varphi$ , we have the equality

$$\partial_r \phi_r + \mathcal{H}^n \phi_r = 0.$$

When  $\varphi$  is compactly supported, the result follows since in this case, we have  $\phi \in C_t^1 L^\infty(\mathbb{Z}_n^2, e(-t + \cdot))$ . For more general  $\varphi \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))$ , we only need to consider when  $\varphi$  is non-negative since the general  $\varphi$  can be controlled by some non-negative function in the same space.

We can then approximate  $\varphi$  by an increasing sequence of compactly supported functions  $\varphi_m$  such that  $\varphi_m \rightarrow \varphi \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l+1))$ . This is visible since all regularities are the same for the discrete spaces and increasing weight a little bit can make  $\varphi$  to be 0 at the infinity under this weight. Then the solution theory of PAM or heat equation gives us that

$$T_t^n \varphi_m \rightarrow T_t^n \varphi \in C^{1-\kappa}(\mathbb{Z}_n^2, e(l+1)).$$

Then by monotone convergence theorem, we have

$$\mathbb{E}[\langle \mu_t^n, \varphi \rangle] = \lim_{m \rightarrow \infty} \mathbb{E}[\langle \mu_t^n, \varphi_m \rangle] = \lim_{m \rightarrow \infty} \langle \mu_0^n, T_t^n \varphi_m \rangle = \langle \mu_0^n, T_t^n \varphi \rangle < \infty,$$

since  $\mu_0^n$  is compact supported. Thus  $M_t$  is a martingale by the martingale convergence theorem and the result follows.  $\square$

Our next goal is to obtain the moment estimation of quantity  $\sup_{0 \leq s \leq t} \langle \mu_t^n, \varphi \rangle$  for some suitable  $\varphi$ . The direct operation on the  $\mu_t^n$  suffers some technique issues on applying  $\mathcal{H}^n$  to the functions in the domain of operator  $\mathcal{H}$ . It is non-trivial at all to see if the resulting functions  $\mathcal{H}^n \varphi^n$  with  $\varphi^n \rightarrow \varphi \in \mathcal{D}_{\mathcal{H}}$  are uniformly bounded over  $n$  at least in  $L^\infty$ . Thus, we turn to the estimation of sums of  $\tilde{\mu}_t^n$  which governs  $\mu_t^n$  and are much easier to manipulate. We will manipulate on  $\mu_t^n$  as much as possible and only consider  $\tilde{\mu}_t^n$  when there is some technical issue on  $\mu_t^n$  that we can not solve. We will follow the argument of lemma 5.5.4 in [27].

**Lemma 4.5.5.** *Suppose  $0 < \theta < \beta$  and  $\mu_0$  is compactly supported. For any function  $\varphi \in L^\infty(\mathbb{Z}_n^2, e(l))$  with some  $l \in \mathbb{R}$ , we have the moments estimation uniformly over  $n$*

$$\mathbb{E} [\langle \mu_t^n, \varphi \rangle^{1+\theta}] \lesssim_{\mu_0, l, \theta} \|\varphi\|_{L^\infty(\mathbb{Z}_n^2, e(l))}^{1+\theta}.$$

Furthermore, for any function  $\varphi \in L^\infty(\mathbb{Z}_n^2, e(l))$  such that  $\Delta^n \varphi \in L^\infty(\mathbb{Z}_n^2, e(l))$ , we have the uniform estimation of finite moments of supreme over  $n$ .

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} \langle \mu_s^n, \varphi \rangle^{1+\theta} \right] \lesssim_{\mu_0, l, \theta, \beta} \|\varphi\|_{L^\infty(\mathbb{Z}_n^2, e(l))} + \|\varphi\|_{L^\infty(\mathbb{Z}_n^2, e(l))}^2 + \|\Delta^n \varphi\|_{L^\infty(\mathbb{Z}_n^2, e(l))}^{1+\beta}. \quad (4.5.10)$$

*Proof.* **1.** Consider first a non-negative function  $\varphi \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))$  for some  $l \in \mathbb{R}$ , we have

$$\mathbb{E} [\langle \mu_s^n, \varphi \rangle^{1+\theta}] \lesssim 1 + (1 + \theta) \int_1^\infty r^{1+\theta} \int_0^{\frac{2}{r}} [\mathbb{E}[e^{-\langle \mu_s^n, u\varphi \rangle}] - 1 + \mathbb{E}[\langle \mu_s^n, u\varphi \rangle]] dudr, \quad (4.5.11)$$

by lemma C.4.1. We calculate

$$\begin{aligned} & \mathbb{E}[e^{-\langle \mu_s^n, u\varphi \rangle}] - 1 + \mathbb{E}[\langle \mu_s^n, u\varphi \rangle] \\ & \lesssim u^{1+\beta} \langle \mu_0^n, T_s^n(\varphi) \rangle^{1+\beta} + e^{-\langle \mu_0^n, U_s^n(u\varphi) \rangle} - e^{-\langle \mu_0^n, T_s^n(u\varphi) \rangle} \\ & \lesssim u^{1+\beta} \langle \mu_0^n, T_s^n(\varphi) \rangle^{1+\beta} + \langle \mu_0^n, |U_s^n(u\varphi) - T_s^n(u\varphi)| \rangle. \end{aligned}$$

From the representation

$$U_t^n(u\varphi) = T_t^n \left( \frac{1 - e^{-\epsilon u \varphi}}{\epsilon} \right) - \int_0^t T_{t-s}^n \left( \frac{2\xi_{e,+}^n + \epsilon^\beta}{1 + \beta} (U_s^n(u\varphi))^{1+\beta} \right) ds \leq u T_t^n \varphi,$$

we obtain

$$\begin{aligned} & |U_s^n(u\varphi) - T_s^n(u\varphi)| \\ & \lesssim \left| T_t^n \left( \frac{1 - e^{-\epsilon u \varphi} - \epsilon u \varphi}{\epsilon} \right) \right| + \left| \int_0^t T_{t-s}^n \left( \frac{2\xi_{e,+}^n + \epsilon^\beta}{1 + \beta} (U_s^n(u\varphi))^{1+\beta} \right) ds \right| \\ & \lesssim \epsilon^\beta u^{1+\beta} |T_t^n(\varphi^{1+\beta})| + u^{1+\beta} \left| \int_0^t T_{t-s}^n \left( \frac{2\xi_{e,+}^n + \epsilon^\beta}{1 + \beta} T_s^n(\varphi)^{1+\beta} \right) ds \right|. \end{aligned}$$

Let

$$I_t^n(\varphi) := \sup_{0 \leq s \leq t} \langle \mu_0^n, T_s^n(\varphi) \rangle^{1+\beta} + \langle \mu_0^n, \epsilon^\beta T_s^n(\varphi^{1+\beta}) \rangle \\ + \left\langle \mu_0^n, \left| \int_0^s T_{s-r}^n \left( \frac{2\xi_{e,+}^n \epsilon^\beta}{1+\beta} T_r^n(\varphi)^{1+\beta} \right) dr \right| \right\rangle.$$

Then we have the estimation

$$\mathbb{E}[\langle \mu_s^n, \varphi \rangle^{1+\theta}] \lesssim 1 + (1+\theta) \int_1^\infty r^{1+\theta} \int_0^{\frac{2}{r}} u^{1+\beta} I_s^n(\varphi) du dr \lesssim 1 + \frac{1+\theta}{\beta-\theta} I_s^n(\varphi).$$

The solution theory of PAM gives the bounds for any  $\varphi \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))$ ,

$$\|T_s^n(\varphi)\|_{C_T C^{1-\kappa}(\mathbb{Z}_n^2, e(l+\cdot))} \lesssim \|\varphi\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))},$$

and by lemma 3.10 in [71]

$$\left\| \int_0^s T_{s-r}^n \left( \frac{2\xi_{e,+}^n \epsilon^\beta}{1+\beta} T_r^n(\varphi)^{1+\beta} \right) dr \right\|_{C_T C^{1-\kappa}(\mathbb{Z}_n^2, e(\hat{l}+\cdot))} \\ \lesssim \left\| \frac{2\xi_{e,+}^n \epsilon^\beta}{1+\beta} \right\|_{C^{-\kappa}(\mathbb{Z}_n^2, p(b))} \|T_t^n(\varphi)^{1+\beta}\|_{C^{1-\kappa}(\mathbb{Z}_n^2, e(\hat{l}+\cdot))},$$

where we choose  $\hat{l} = ((1+\beta)l) \vee l$ . It then suggests the bounds

$$I_t^n(\varphi) \lesssim \langle \mu_0^n, e(\hat{l}+t) \|\varphi\| \rangle^{1+\beta} + \langle \mu_0^n, e(\hat{l}+t) \epsilon^\beta \|\varphi\|^{1+\beta} \rangle,$$

where  $\|\varphi\|$  indicates  $\|\varphi\|_{C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))}$ . Thus we have

$$\mathbb{E}[\langle \mu_t^n, \varphi \rangle^{1+\theta}] \lesssim 1 + (\langle \mu_0^n, e(\hat{l}+t) \rangle^{1+\beta} + \langle \mu_0^n, e(\hat{l}+t) \rangle) \|\varphi\|^{1+\beta},$$

for all non-negative  $\varphi \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))$  or  $\varphi \in \mathcal{D}^{1-\kappa}(\mathbb{Z}_n^2, e(l))$ . In particular, if we choose  $\varphi = e^{l\rho(x)}$ , where  $\rho(x)$  is a slight modification of  $|x|^\sigma$  so that  $\varphi \in C^{1+2\kappa}(\mathbb{Z}_n^2, e(l))$ , we know the quantity

$$\mathbb{E}[\langle \mu_t^n, e(l) \rangle^{1+\theta}]$$

is finite uniformly over  $n$  for all  $l \in \mathbb{R}$ . It is then easy to see that if  $\|\varphi\|_{L^\infty(\mathbb{Z}_n^2, e(l))}$  is finite, then we must have

$$\mathbb{E}[\langle \mu_t^n, \varphi \rangle^{1+\theta}] \leq 2 \|\varphi\|_{L^\infty(\mathbb{Z}_n^2, e(l))}^{1+\theta} \mathbb{E}[\langle \mu_t^n, e(l) \rangle^{1+\theta}] < \infty,$$

uniformly over  $n$ .

2. Now on the moment estimation of supreme, we have to use branching random walks  $\tilde{\mu}_t^{n,i}$ . Notice first that by the Laplace functional of  $\tilde{\mu}_t^n$ , we have the same results as the first step for  $\mu_t^n$ . Furthermore, by the proof of lemma 4.5.4, we know that

$$M_t := \langle \tilde{\mu}_t^n, \varphi \rangle - \langle \epsilon Z_0^n, \varphi \rangle - \int_0^t \langle \tilde{\mu}_s^n, \Delta^n \varphi \rangle ds$$

is a martingale if  $\Delta^n \varphi \in L^\infty(\mathbb{Z}_n^2, e(l))$  for some  $l \in \mathbb{R}$ . Thus by Doob's inequality, we can have the control

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} \langle \tilde{\mu}_s^n, \varphi \rangle^{1+\theta} \right] \lesssim \mathbb{E}[|M_t|^{1+\theta}] + \mathbb{E} \left[ \int_0^t |\langle \tilde{\mu}_s^n, \Delta^n \varphi \rangle|^{1+\theta} ds \right] + \langle \mu_0, \varphi \rangle + \langle \mu_0, \varphi \rangle^2.$$

Then by the coupling lemma 4.5.3, we know

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} \langle \mu_s^n, \varphi \rangle^{1+\theta} \right] \lesssim_{N(\beta)} \sum_{i=1}^{N(\beta)} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \langle \tilde{\mu}_s^{n,i}, \varphi \rangle^{1+\theta} \right],$$

which gives the desired result.  $\square$

Now we are able to show the convergence of the particle system to the super process.

*Proof of theorem 4.5.1.* By the definition of  $\mathbb{D}([0, \infty), \mathcal{M}(\mathbb{R}^2))$ , we only need to show the theorem for the path on  $[0, T]$  for any  $T > 0$ . Suppose  $\varphi \in C^2(\mathbb{R}^2, e(l))$  for some  $l \in \mathbb{R}$ . Consider  $Y_t^n = \langle \mu_t^n, \varphi \rangle$ . To prove the tightness of  $(\mu_t^n)_n$ , we apply Jakubowski's criterion [26]. By lemma 4.5.5, we know

$$\mathbb{E} \left[ \sup_{0 < t < T} \langle \mu_t^n, 1 + |\cdot|^2 \rangle \right]$$

is uniformly bounded over  $n$ . Furthermore, since for any  $K$ , the set  $\{\mu : \langle \mu, 1 + |\cdot|^2 \rangle < K\}$  is compact, we know compact containment property is satisfied by  $\mu_t^n$ .

Secondly, we apply Aldous's criterion [26]. We have to prove

$$Y_{\tau_n + \delta_n}^n - Y_{\tau_n}^n \rightarrow 0$$

in distribution, where  $\tau_n$  are stopping times and  $\delta_n$  decreasing to 0. The moment bound shows that  $\{Y_{\tau_n + \delta_n}^n, Y_{\tau_n}^n\}$  are tight. It then suffices to consider the difference of Laplace transform

$$\mathbb{E} \left[ e^{-s Y_{\tau_n + \delta_n}^n - t Y_{\tau_n}^n} \right] - \mathbb{E} \left[ e^{-(s+t) Y_{\tau_n}^n} \right].$$

Taking the conditional expectation with respect to  $\mu_{\tau_n}^n$ , we have to obtain the formula of  $\mathbb{E}[e^{-\langle \mu_{\tau_n + \delta_n}^n, \varphi \rangle} | \mu_{\tau_n}^n]$ . By lemma C.3.1 and lemma C.3.2, we know

$$\mathbb{E}[e^{-\langle \mu_{\tau_n + \delta_n}^n, \varphi \rangle} | Z_{\tau_n}^n = \delta_x] = 1 - \epsilon U_{\delta_n}^n(\varphi)(x).$$

We then obtain that

$$\mathbb{E}[e^{-\langle \mu_{\tau_n+\delta_n}^n, \varphi \rangle} | \mu_{\tau_n}^n] = \mathbb{E}[e^{-\langle Z_{\tau_n+\delta_n}^n, \epsilon \varphi \rangle} | Z_{\tau_n}^n] = e^{-\langle Z_{\tau_n}^n, -\log(1-\epsilon U_{\delta_n}^n(\varphi)) \rangle}.$$

Thus we have

$$\begin{aligned} & \mathbb{E} \left[ e^{-sY_{\tau_n+\delta_n}^n - tY_{\tau_n}^n} \right] - \mathbb{E} \left[ e^{-(s+t)Y_{\tau_n}^n} \right] \\ &= \mathbb{E} \left[ e^{-\langle \mu_{\tau_n}^n, -\frac{1}{\epsilon} \log(1-\epsilon U_{\delta_n}^n(s\varphi)) - \langle \mu_{\tau_n}^n, t\varphi \rangle \rangle} \right] - \mathbb{E} \left[ e^{-\langle \mu_{\tau_n}^n, (t+s)\varphi \rangle} \right] \\ &\lesssim \mathbb{E} \left[ \left\langle \mu_{\tau_n}^n, \left| -\frac{1}{\epsilon} \log(1-\epsilon U_{\delta_n}^n(s\varphi)) - s\varphi \right| \right\rangle \right] \\ &\lesssim \mathbb{E} \left[ \sup_{0 \leq s \leq t} \langle \mu_s^n, 1 \rangle \right] \left\| -\frac{1}{\epsilon} \log(1-\epsilon U_{\delta_n}^n(s\varphi)) - s\varphi \right\|_{L^\infty(\mathbb{Z}_n^2)} \\ &\lesssim \left\| -\frac{1}{\epsilon} \log(1-\epsilon U_{\delta_n}^n(s\varphi)) - U_{\delta_n}^n(s\varphi) \right\|_{L^\infty(\mathbb{Z}_n^2)} + \|U_{\delta_n}^n(s\varphi) - s\varphi\|_{L^\infty(\mathbb{Z}_n^2)}, \end{aligned}$$

which goes to zero as  $n \rightarrow \infty$  by estimation of SPDEs and lemma C.4.1. This tells us that we have the limit

$$(Y_{\tau_n+\delta_n}^n, Y_{\tau_n}^n) \rightarrow (Y_\infty, Y_\infty)$$

in distribution. Thus any linear combination of the above vectors also converges in distribution. In particular we have

$$Y_{\tau_n+\delta_n}^n - Y_{\tau_n}^n \rightarrow 0$$

in distribution. Thus we obtain the tightness of process  $Y_t^n$ . Since the family  $C_c^\infty(\mathbb{R}^2)$  separates points in  $\mathcal{M}_F(\mathbb{R}^2)$ , the Jakubowski's criterion gives the tightness of  $\mu^n$  in  $\mathbb{D}([0, \infty), \mathcal{M}_F(\mathbb{R}^2))$ . The discussion of SPDEs in section 4.4 and the tightness of  $Y_t$  tell us the limit process  $\mu_t$  has the Laplace functional as stated in the theorem. For the  $\rho < \beta$ , the limit  $\mu_t$  satisfies the Laplace functional

$$\mathbb{E}[e^{-\langle \mu_t, \varphi \rangle}] = e^{-\langle \mu_0, T_t(\varphi) \rangle} = e^{-\langle T_t \mu_0, \varphi \rangle},$$

which implies that  $\mu_t = T_t \mu_0$ . □

#### 4.5.2 For the general $\varkappa > \frac{2}{1+\beta}$

We do the mixing as in the paper [78].

**Definition 4.5.6.** Fix a compact supported measure  $\mu_0^n$  on  $\mathbb{Z}_n^2$ . Let  $\epsilon = n^{-\frac{1}{\beta}}$  and  $c > 0$ . We define a  $E^n$ -valued stochastic process  $Z^n(t)$  started at a Poisson random measure on  $\mathbb{Z}_n^2$  with intensity  $\frac{\mu_0^n}{\epsilon}$  with infinitesimal generator

$$\mathcal{L}^n F(\eta) = \sum_{x \in \mathbb{Z}_n^2} \eta_x \left[ \Delta_x^n F(\eta) + |\xi_\epsilon^n| \sum_{k=0}^{\infty} p_k q_k(x) d_x^k F(\eta) + c |\xi_\epsilon^n| \sum_{k=0}^{\infty} p_k d_x^k F(\eta) \right] \quad (4.5.12)$$

for any  $F \in \mathcal{D}(\mathcal{L}^n)$  which consists of all  $F \in C_b(E^n)$  such that the right-hand side of (4.3.3) is finite. Finally, set  $\mu^n(t) = \epsilon Z^n(t)$  for all  $t \in [0, \infty)$ .

It follows that the branching rate of the above branching random walk is

$$(1 + c)|\xi_e^n|$$

and the branching mechanism is given by

$$\frac{g^n + cg}{1 + c}$$

Then this is clear that we can still find some controlled branching mechanism and perform the exact same procedure as for the case  $\varkappa < \frac{2}{1+\beta}$ . The final parameter of the  $\beta$ -rough super Brownian motion given by this mixture will be

$$\varkappa = \frac{2\nu}{1 + \beta} + \frac{2\nu c}{1 + \beta}.$$

This finishes the proof of theorem 4.3.5.

## 4.6 Martingale problem

This section is devoted to proving the equivalence of three definitions of  $\beta$ -rough super Brownian motion. The  $\epsilon$  has no meaning from now on.

Let's start the first direction: from the Laplace functional to the martingale problem. We follow the method in book [26]. Let's begin with a lemma in [35] (corollary 3.3 in chapter 2)

**Lemma 4.6.1.** *Let  $X, Y$  be real-valued, adapted right continuous processes. Suppose that for each  $t$ ,  $\inf_{s \leq t} X_s > 0$ . Then*

$$X_t - \int_0^t Y_s ds$$

*is a local martingale if and only if*

$$X_t \exp\left(-\int_0^t \frac{Y_s}{X_s} ds\right)$$

*is a local martingale.*

**Lemma 4.6.2.** *Suppose  $\mu$  satisfies the first definition in definition 4.3.2, then for all non-negative  $\phi \in \mathcal{D}_{\mathcal{H}}$ , the process*

$$N_t(\phi) := e^{-\langle \mu_t, \phi \rangle + \int_0^t \langle \mu_s, \mathcal{H}\phi - \varkappa \phi^{1+\beta} \rangle ds},$$

*is a local  $\mathcal{F}$ -martingale.*

*Proof.* Taking  $B \in \mathcal{F}_s$ , we check

$$\mathbb{E} \left[ (e^{-\langle \mu_t, \phi \rangle} - e^{-\langle \mu_s, \phi \rangle}) \mathbb{1}_B \right] = \mathbb{E} \left[ - \int_s^t \langle \mu_r, \mathcal{H}\phi - \varkappa \phi^{1+\beta} \rangle e^{-\langle \mu_r, \phi \rangle} dr \mathbb{1}_B \right].$$

It is sufficient to prove that for  $t \geq s$ ,

$$\frac{d}{dt} \mathbb{E} \left[ e^{-\langle \mu_t, \phi \rangle} \mathbb{1}_B \right] = \mathbb{E} \left[ -\langle \mu_t, \mathcal{H}\phi - \varkappa \phi^{1+\beta} \rangle e^{-\langle \mu_t, \phi \rangle} \mathbb{1}_B \right].$$

By the definition, we have

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \left[ e^{-\langle \mu_t, \phi \rangle} \mathbb{1}_B \right] &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[ (e^{-\langle \mu_{t+\epsilon}, \phi \rangle} - e^{-\langle \mu_t, \phi \rangle}) \mathbb{1}_B \right] \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[ \mathbb{1}_B e^{-\langle \mu_t, \phi \rangle} \left( e^{-\langle \mu_t, \epsilon \frac{U_\epsilon(\phi) - \phi}{\epsilon} \rangle} - 1 \right) \right]. \end{aligned} \quad (4.6.1)$$

Now, in order to apply the dominant convergence theorem, we have to verify

$$\left\| \frac{U_\epsilon(\phi) - \phi}{\epsilon} \right\|_{L^\infty(\mathbb{R}^2, e(l))}$$

is uniformly bounded for some  $l \in \mathbb{R}$ . By definition of  $\mathcal{D}_{\mathcal{H}}$ , there exists  $\varphi \in C^{1+2\kappa}(\mathbb{R}^2, e(l_0))$  such that  $\phi = \int_0^t T_s \varphi ds$ . Then

$$U_t(\phi) = T_t(\phi) - \varkappa \int_0^t T_{t-s}(U_s(\phi))^{1+\beta} ds,$$

which implies that

$$\begin{aligned} \frac{1}{\epsilon} (U_\epsilon(\phi) - \phi) &= \frac{1}{\epsilon} \left( T_\epsilon(\phi) - \phi - \varkappa \int_0^\epsilon T_{\epsilon-s}(v_s(\phi))^{1+\beta} \right) \\ &= \frac{1}{\epsilon} \left( \int_t^{t+\epsilon} T_s \varphi ds - \int_0^\epsilon T_s \varphi ds - \varkappa \int_0^\epsilon T_{\epsilon-s}(U_s(\phi))^{1+\beta} ds \right). \end{aligned} \quad (4.6.2)$$

Thus we have

$$\frac{1}{\epsilon} \|U_\epsilon(\phi) - \phi\|_{L^\infty(\mathbb{R}^2, e(l))} \lesssim 1 + \|\varphi\|_{C^{1+2\kappa}(\mathbb{R}^2, e(l_0))}^{1+\beta},$$

for  $l > 1 + ((1 + \beta)l_0) \vee l_0 \vee (l_0 + t)$  uniformly over  $\epsilon$ . In addition, we have by the continuity of  $T_\cdot \varphi, U_\cdot \varphi$  that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (U_\epsilon(\phi) - \phi) = T_t \varphi - \varphi - \varkappa \phi^{1+\beta} = \mathcal{H}\phi - \varkappa \phi^{1+\beta}.$$

Thus dominant convergence theorem applies and we know

$$e^{-\langle \mu_t, \phi \rangle} + \int_0^t \langle \mu_r, \mathcal{H}\phi - \varkappa \phi^{1+\beta} \rangle e^{-\langle \mu_r, \phi \rangle} dr$$

is a local martingale. Lemma 4.6.1 then shows that  $N_t$  is a local martingale.  $\square$

We are now ready to prove the theorem 4.3.4

*Proof of theorem 4.3.4. 1.  $\Rightarrow$  3.* We first notice that since the Laplace functional uniquely determines the finite-dimensional law of the stochastic process, the  $1 + \theta$  moments estimation of  $\sup_{0 \leq s \leq t} \langle \mu_s, \phi \rangle$  for any  $\phi \in L^\infty(\mathbb{R}^2, e(l)), l \in \mathbb{R}$  and  $0 \leq \theta < \beta$  follows by the first definition. We therefore have the  $1 + \theta$  boundedness of  $L_t^\phi$  for any  $\phi \in \mathcal{D}_{\mathcal{H}}$ .

By repeating the argument in [26], we can obtain the martingale problem of  $\mu$  from definition 1. Since the argument is quite long, we give only a sketch here and refer interested readers to chapter 6.1 in [26]. First we choose a non-negative function  $\phi \in \mathcal{D}_{\mathcal{H}}$ . Let

$$N_t(\phi) := e^{-\langle \mu_t, \phi \rangle + \int_0^t \langle \mu_s, \mathcal{H}\phi - \varkappa\phi^{1+\beta} \rangle ds},$$

and

$$Y_t(\phi) = e^{-\langle \mu_t, \phi \rangle}, \quad H_t(\phi) = e^{-\int_0^t \langle \mu_s, \mathcal{H}\phi - \varkappa\phi^{1+\beta} \rangle ds},$$

then  $Y_t(\phi) = H_t(\phi)N_t(\phi)$  is a strict positive semi-martingale. The Itô's formula gives the representation

$$\begin{aligned} dY_t(\phi) &= H_t(\phi)dN_t(\phi) + N_{t-}(\phi)dH_t(\phi) \\ &= H_t(\phi)dN_t(\phi) - Y_{t-}(\phi)\langle \mu_t, \mathcal{H}\phi - \varkappa\phi^{1+\beta} \rangle dt. \end{aligned} \quad (4.6.3)$$

Again by Itô's formula, we know the process  $\langle \mu_t, \phi \rangle = -\log Y_t(\phi)$  is again a semi-martingale. Now let  $\mathcal{N}(ds, d\nu)$  be adapted random point measure on  $\mathbb{R}_+ \times \mathcal{M}_F(\mathbb{R}^2)$  given by  $\sum \delta_{(s, \mu_s - \mu_{s-})}$ ,  $\widehat{\mathcal{N}}$  denotes its compensator and  $\widetilde{\mathcal{N}}$  the corresponding martingale measure. See appendix C.1 for more detail on random point measure.

The unique canonical decomposition of semi-martingale (lemma C.2.1) gives that

$$\langle \mu_t, \phi \rangle = \langle \mu_0, \phi \rangle + B'_t(\phi) + M_t^C(\phi) + \widetilde{\mathcal{N}}_t(\phi) + \mathcal{N}_t(\phi),$$

where  $B'_t(\phi)$  is predictable, continuous and locally of bounded variation,  $M_t^C(\phi)$  is a continuous local martingale with predictable increasing process  $C_t(\phi)$  and

$$\begin{aligned} \widetilde{\mathcal{N}}_t(\phi) &:= \int_0^t \int_{\mathcal{M}_F^\pm(\mathbb{R}^2)} \mathbb{1}_{|\langle \nu, \phi \rangle| \leq \|\phi\|} \langle \nu, \phi \rangle \widetilde{\mathcal{N}}(ds, d\nu), \\ \mathcal{N}_t(\phi) &:= \int_0^t \int_{\mathcal{M}_F^\pm(\mathbb{R}^2)} \mathbb{1}_{|\langle \nu, \phi \rangle| > \|\phi\|} \langle \nu, \phi \rangle \mathcal{N}(ds, d\nu). \end{aligned}$$

The uniqueness of decomposition gives the relation

$$B'_t(\theta\phi) = \theta B'_t(\phi), \quad M_t^C(\theta\phi) = \theta M_t^C(\phi), \quad C_t(\theta\phi) = \theta^2 C_t(\phi),$$

for any  $\theta > 0$ . Applying Itô's lemma C.2.2 to  $Y_t$  and obtain the second representation

$$\begin{aligned} dY_t(\phi) = & Y_{t-}(\phi) \left( -dB_t(\phi) + \frac{1}{2}dC_t(\phi) + \int_{\mathcal{M}_{\mathbb{F}}^{\pm}(\mathbb{R}^2)} (e^{-\langle \nu, \phi \rangle} - 1 + \langle \nu, \phi \rangle) \widehat{\mathcal{N}}(dt, d\nu) \right) \\ & + d(\text{local martingale}), \end{aligned} \quad (4.6.4)$$

where

$$B_t(\phi) = B'_t(\phi) + \int_{\mathcal{M}_{\mathbb{F}}^{\pm}(\mathbb{R}^2)} \langle \nu, \phi \rangle \mathbb{1}_{|\langle \nu, \phi \rangle| > \|\phi\|} \widehat{\mathcal{N}}(dt, d\nu).$$

Now since  $Y_t$  is a special semi-martingale, the unique decomposition lemma C.2.3 can be used to identify the predictable locally bounded variation part of two representations (4.6.3) and (4.6.4), we have

$$\langle \mu_t, \mathcal{H}\phi - \varkappa\phi^{1+\beta} \rangle dt = dB_t(\phi) - \frac{1}{2}dC_t(\phi) - \int_{\mathcal{M}_{\mathbb{F}}^{\pm}(\mathbb{R}^2)} (e^{-\langle \nu, \phi \rangle} - 1 + \langle \nu, \phi \rangle) \widehat{\mathcal{N}}(dt, d\nu).$$

Replacing  $\phi$  with  $\theta\phi$  and comparing coefficient before different power of  $\theta$ , we obtain  $\int_0^t \langle \mu_s, \mathcal{H}\phi \rangle ds = B_t(\phi)$ ,  $C_t(\phi) = 0$  and

$$\begin{aligned} & \int_{\mathcal{M}_{\mathbb{F}}^{\pm}(\mathbb{R}^2)} (e^{-\langle \nu, \phi \rangle} - 1 + \langle \nu, \phi \rangle) \widehat{\mathcal{N}}(dt, d\nu) \\ = & \langle \mu_t, \varkappa\phi^{1+\beta} \rangle dt = \int_0^{\infty} \frac{\varkappa\beta(1+\beta)}{\Gamma(1-\beta)u^{\beta+2}} \langle \mu_t, e^{-u\phi} - 1 + u\phi \rangle du, \end{aligned}$$

where  $\Gamma(1-\beta)$  is the Gamma function. This shows that

$$\widehat{\mathcal{N}}(dt, d\nu) = \frac{\varkappa\beta(1+\beta)}{\Gamma(1-\beta)u^{\beta+2}} dt du \mu_t(dx) \delta_{u\delta_x}(d\nu).$$

Now for the general  $\phi \in \mathcal{D}_{\mathcal{H}}$ , we can decompose  $\phi = \phi_+ - \phi_-$  and approximate  $\phi_+$ ,  $\phi_-$  by non-negative functions in  $\mathcal{D}_{\mathcal{H}}$  and see that  $\langle \mu_t, \phi \rangle$  is again a semi-martingale. Then by Itô's formula, we have

$$\begin{aligned} \mathbb{M}_t^f := & f(\langle \mu_t, \phi \rangle) - f(\langle \mu_0, \phi \rangle) - \int_0^t f'(\langle \mu_s, \phi \rangle) \langle \mu_s, \mathcal{H}\phi \rangle ds - \frac{\varkappa\beta(1+\beta)}{\Gamma(1-\beta)} \int_0^t \\ & \left\langle \mu_t, \int_0^{\infty} \frac{1}{u^{\beta+2}} (f(\langle \mu_t, \phi \rangle + u\phi(\cdot)) - f(\langle \mu_t, \phi \rangle) - f'(\langle \mu_t, \phi \rangle)u\phi(\cdot)) du \right\rangle dt \end{aligned} \quad (4.6.5)$$

is a purely discontinuous local martingale for all  $\phi \in \mathcal{D}_{\mathcal{H}}$ . Now by comparing to the formula given by the compensator of  $\langle \mu_t, \phi \rangle$  on  $\mathbb{R}$ , we obtain the result.

**3.  $\Rightarrow$  2.** The proof is very similar to the proof in [78]. However, we have to be careful with non-square-integrable martingale. To start with, fix an  $l < 0$ , from lemma C.4.2, we know there exists a non-negative function  $\tilde{\phi} \in \mathcal{D}_{\mathcal{H}}$  such that  $\frac{\tilde{\phi}}{e^{(l)}}$  is strict

larger than a small positive number. We choose a stopping time  $T_n$  according to this  $\langle \mu_s, \tilde{\phi} \rangle$  describing the  $n$ -th jump times with jump bigger than 1. Then the formula

$$\mathbb{E} \left[ \sum_{0 \leq r \leq T_n} 1_{\langle \mu_r - \mu_{r-}, \tilde{\phi} \rangle > 1} \right] = \mathbb{E} \left[ \int_0^{T_n} \int_0^\infty \mathbb{1}_{u > 1} \langle \mu_r, \varkappa \phi^{1+\beta} \rangle \frac{\beta(1+\beta)}{\Gamma(1-\beta)u^{\beta+2}} du dr \right]$$

gives that

$$\mathbb{E} \left[ \int_0^{T_n} \langle \mu_r, \varkappa \tilde{\phi}^{1+\beta} \rangle dr \right] < \infty,$$

which implies that  $\mathbb{E} \left[ \int_0^{T_n} \langle \mu_r, e(l) \rangle \right] < \infty$  by the assumption of  $\tilde{\phi}$ . Then by the moments bounds of  $L_t^{\tilde{\phi}}$  and doob's maximal inequality, we know that we actually have

$$\mathbb{E} \left[ \sup_{0 \leq r \leq T_n} \langle \mu_r, e(l) \rangle \right] < \infty. \quad (4.6.6)$$

We will assume we have  $\mathbb{E} \left[ \sup_{0 \leq r \leq s} \langle \mu_r, e(l) \rangle \right] < \infty$  and derive the local martingale property of 2. first.

Now consider a piecewise function  $f : [0, t] \rightarrow \mathcal{D}_{\mathcal{H}}$  with weight  $e(l - t + \cdot)$  and  $\varphi_0 \in \mathcal{D}_{\mathcal{H}}$  with weight  $e(l - t)$ . Let  $\varphi_t$  be the solution to the equation

$$\partial_s \varphi_t(s) + \mathcal{H} \varphi_t(s) = f(s), \quad s \in [0, t], \quad \varphi_t(t) = \varphi_0,$$

which is given by  $\varphi_t(s) = T_{t-s} \varphi_0 + \int_s^t T_{r-s} f(r) dr$ . By assumption we have  $\varphi_t(s) \in \mathcal{D}_{\mathcal{H}}$  for all  $s \leq t$ . For two time step  $t_1 < t_2$  and a partition  $\pi$  of  $[t_1, t_2]$ , we have

$$\begin{aligned} & \mathbb{E} [\langle \mu_{t_2}, \varphi_t(t_2) - \langle \mu_{t_1}, \varphi_t(t_1) \rangle | \mathcal{F}_{t_1} ] \\ &= \mathbb{E} \left[ \int_{t_1}^{t_2} \langle \mu_{s'}, f(s) - \mathcal{H} \varphi_t(s) \rangle + \langle \mu_s, \mathcal{H} \varphi_t(s'') \rangle ds | \mathcal{F}_{t_1} \right], \end{aligned} \quad (4.6.7)$$

where  $s'$  is the smallest partition point in  $\pi$  bigger than  $s$  and where  $s''$  is the largest partition point in  $\pi$  smaller than  $s$ . The right continuous property of  $\mu_t$  in  $\mathcal{M}_F(\mathbb{R}^2)$  shows that  $\lim_{s' \rightarrow s, s' > s} \langle \mu_{s'}, (f(s) - \mathcal{H} \varphi_t(s)) \rangle = \langle \mu_s, (f(s) - \mathcal{H} \varphi_t(s)) \rangle$ , together with (4.6.6), we can send the mesh of partition of  $[t_1, t_2]$  go to zero to get the limit of the first term of (4.6.7)

$$\mathbb{E} \left[ \int_{t_1}^{t_2} \langle \mu_s, f(s) - \mathcal{H} \varphi_t(s) \rangle ds | \mathcal{F}_{t_1} \right].$$

It remains to check

$$\lim_{s'' \rightarrow s, s'' < s} \mathbb{E} [\langle \mu_s, \mathcal{H} \varphi_t(s'') - \mathcal{H} \varphi_t(s) \rangle | \mathcal{F}_{t_1} ] \rightarrow 0.$$

By the definition of  $\varphi_t(s)$ , we have

$$\begin{aligned} & |\mathcal{H}\varphi_t(s'') - \mathcal{H}\varphi_t(s)| \\ &= \left| T_{t-s''}\mathcal{H}\varphi_0 - T_{t-s}\mathcal{H}\varphi_0 + \int_s^{t-s''} T_{r-s''}f(r) - T_{r-s}f(r)dr + \int_{s''}^s T_{r-s''}f(r)dr \right| \\ &\lesssim (s - s'')^{\frac{1-\kappa}{2}} e(l). \end{aligned}$$

This gives the result and shows that  $\mathbb{M}_t^{\varphi_0, f}$  is a local martingale. And the density of  $\mathcal{D}_{\mathcal{H}}$  gives the result. We will see it is actually a martingale when we show the direction 2.  $\rightarrow$  1. and 1. implies the moments estimation of  $\langle \mu_t, \phi \rangle$ .

From the compensator of  $L_t^\phi$ , we can obtain the compensator of the random point measure  $\mathcal{N}$  defined in the first step. Now since the purely discontinuous martingale is determined by its jump part and the jump of  $\mathbb{M}_t^{\varphi_0, f}$  is given by  $\sum_{0 \leq s \leq t} \langle \mu_s - \mu_{s-}, \varphi_t(s) \rangle$ . Then from the compensator of  $\mathcal{N}$ , we know the compensator of  $\mathbb{M}_t^{\varphi_0, f}$  should be given by

$$\langle \mu_s, \varkappa \varphi_t(s)^{1+\beta} \rangle \frac{\beta(1+\beta)}{\Gamma(1-\beta)x^{\beta+2}} \mathbb{1}_{x>0} ds dx$$

**2.  $\Rightarrow$  1.** We apply Itô's formula to the semi-martingale  $\langle \mu_s, \varphi_t(s) \rangle$  and obtain

$$\begin{aligned} & f(\langle \mu_s, \varphi_t(s) \rangle) \\ &= f(\langle \mu_0, \varphi_t(0) \rangle) + \int_0^s f'(\langle \mu_r, \varphi_t(r) \rangle) \langle \mu_r, f(r) \rangle dr + \int_0^s \int_0^\infty \langle \mu_r, \varkappa \varphi_t(s)^{1+\beta} \rangle \\ &\quad \frac{\beta(1+\beta)}{\Gamma(1-\beta)u^{\beta+2}} (f(\langle \mu_{r-}, \varphi_t(r) \rangle + u) - f(\langle \mu_{r-}, \varphi_t(r) \rangle) - f'(\langle \mu_{r-}, \varphi_t(r) \rangle)u) du dr \\ &\quad + \text{local martingale.} \end{aligned}$$

Choose  $f(x) = e^{-x}$  and  $f(r) = -\varkappa(U_{t-s}(\varphi_0))^{1+\beta}$ , then  $\varphi_t(s) = U_{t-s}\varphi_0$  and

$$e^{-\langle \mu_s, U_{t-s}\varphi_0 \rangle} - e^{-\langle \mu_0, U_t\varphi_0 \rangle}$$

is a bounded local martingale. Thus it is a martingale, which gives the result.  $\square$

## 4.7 Properties of rough Super Brownian Motion

In this section, we discuss the compact support property and the super exponentially persistence of the  $\beta$ -super Brownian motion. The compact support property for the rough super Brownian motion with  $\beta = 1$  has been discussed in 3 and here we will give a sketch of how the method in 3 can be applied for the  $0 < \beta < 1$ .

Let  $U_t^\phi \varphi$  be the solution to the equation

$$\begin{cases} \partial_t w_t = \mathcal{H}w_t - w_t^{1+\beta} + \phi, & \text{in } (0, \infty] \times \mathbb{R}^2, \\ w_0 = \varphi, & \text{on } \mathbb{R}^2. \end{cases} \quad (4.7.1)$$

Then we have

**Lemma 4.7.1.** For any function  $\varphi \in C^{1+2\kappa}(\mathbb{R}^2, e(l))$  and  $\phi \in C^{-1+2\kappa}(\mathbb{R}^2, e(l))$  for some  $l \in \mathbb{R}$ , the process

$$N_t(s) = e^{-\langle \mu_s, U_{t-s}^\phi \varphi \rangle - \int_0^s \langle \mu_r, \phi \rangle dr}$$

is a  $\mathcal{F}$ -martingale. In particular,

$$\mathbb{E} \left[ e^{-\int_0^t \langle \mu_r, \phi \rangle dr} \right] = e^{-\langle \mu_0, U_t^\phi 0 \rangle}.$$

*Proof.* Let  $f(x) = e^{-x}$  and  $\varphi_t(s) = U_{t-s}^\phi \varphi$  in the Itô's formula of  $\langle \mu_s, \varphi_t(s) \rangle$  and then apply lemma 4.6.1, we can get the result.  $\square$

As discussed in 3, let  $\mathcal{P}_n = [-n, n]^2$  and  $\mathcal{P}_n^m := \left(-n - \frac{1}{m}, n + \frac{1}{m}\right)^2$ . Define  $\phi_n^m \in C_c^\infty$  such that

$$\phi_n^m \begin{cases} = 0 & \text{on } \mathcal{P}_n, \\ = m & \text{on } (\mathcal{P}_n^m)^c, \\ \in [0, m] & \text{elsewhere.} \end{cases}$$

Since the limit

$$\lim_{m \rightarrow \infty} \mathbb{E} \left[ e^{-\int_0^t \langle \mu_r, \phi_n^m \rangle dr} \right]$$

describes the probability that  $\{\mu_s\}_{0 \leq s \leq t}$  stays in the box  $\mathcal{P}_n$  and the limit over  $n$  describes the probability the compact support property holds. The compact support property holds if we show

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} (U_t^{\phi_n^m} 0) \eta_n = 0,$$

uniformly over the support of  $\mu_0$  for some cut-off function  $\eta_n = 1$  on  $\mathcal{P}_{n-1}$  and 0 on  $\mathcal{P}_n^c$ . We here give a uniform interior estimation lemma on  $U_t^{\phi_n^m} 0$  without proof. The similar result is proved in 3. Also in [73, 74] for similar results of  $\Phi_3^4$  model.

**Lemma 4.7.2.** Let  $n, m \in \mathbb{N}$  and  $a > 0$ , we have

$$\|(U_t^{\phi_n^m} 0) \eta_m\|_{\mathcal{L}_T^{1-\kappa}(\mathbb{R}^2, p(a))} \lesssim 1, \quad (4.7.2)$$

where  $' \lesssim'$  depends on time horizon  $T$ ,  $\|\xi\|_{C^{-1-\kappa}(\mathbb{R}^2, p(b))}$  and  $\|\mathcal{J}\xi \diamond \xi\|_{C^{-2\kappa}(\mathbb{R}^2, p(b))}$  for  $b = \frac{4a}{\beta}$ . Here the weight for the norm does not increase along time but remain  $p(a)$ .

*Proof of theorem 4.3.9.* As proved in 3, the bounds on  $\|(U_t^{\phi_n^m} 0) \eta_m\|_{\mathcal{L}_T^{1-\kappa}(\mathbb{R}^2, p(a))}$  tells us that, up to some subsequence, we have

$$U_t := \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (U_t^{\phi_n^m} 0) \eta_m$$

is the solution to the equation

$$\partial_t U_t = \mathcal{H}U_t - U_t^{1+\beta}, \quad U_0 = 0.$$

Thus the uniqueness theorem tells us that  $U_t = 0$ . And we get the result.  $\square$

**Remark 4.7.3.** *By using the coupling method, one may reduce the proof of compact support property into a much easier problem, i.e. the problem for the branching random walk with classical branching mechanism but stochastic branching rate.*

We now turn to the super-exponentially persistence of the generalized rough super Brownian motion. The super-exponentially persistence has been discussed in [78] using the approximation of super process in box of  $\mathbb{R}^2$ . We here will use the result in [78], together with the coupling method, to show the theorem 4.3.7.

*Proof of theorem 4.3.7.* We need to consider mixed super processes for the proof. Let  $\bar{\mu}^n$  be the mixed particle systems with branching rate  $(1+c)|\xi_e^n|$  and branching mechanism

$$\frac{h^n + ch}{1+c}, \quad h^n(x, s) = \frac{\xi_{e,+}^n}{|\xi_e^n|} s^2 - \frac{\xi_{e,-}^n}{|\xi_e^n|}, \quad h(s) = \frac{s^2 + 1}{2}.$$

For  $0 < \beta < 1$ , we define a sequence of independent probability measures  $\{\mu^{n,i}\}_{i \leq N(\beta)}$  corresponding to particle systems with branching rate  $(1+c)|\xi_e^n|$  and branching mechanism

$$\frac{g^n + cg}{1+c}.$$

For any  $\varphi \in C_c^\infty(\mathbb{R}^2)$ , we now need to compare the two quantities  $\langle \bar{\mu}^n, \varphi \rangle$  and  $\sum_{i=1}^{\tilde{N}(\beta)} \langle \mu^{n,i}, \varphi \rangle$  with some number  $\tilde{N}(\beta)$ .

Let  $Z_0$  be the random variable with generating function  $\frac{h^n+ch}{1+c}$  and  $Z_i$  be the random variable with generating function  $\frac{g^n+cg}{1+c}$  for  $i = 1, \dots$ , we need to find a number  $\tilde{N}(\beta)$  such that

$$\mathbb{P}[Z_0 \geq 2] \leq \mathbb{P} \left[ \sum_i^{N(\beta)} Z_i \geq 2 \right],$$

which is equivalent to showing that

$$\mathbb{P}[Z_0 = 0] \geq \mathbb{P} \left[ \sum_i^{N(\beta)} Z_i = 0 \right] = \mathbb{P}[Z_1 = 0]^{N(\beta)},$$

Since  $c > 0$ , we know  $\mathbb{P}[Z_0 = 0] > 0$  and  $\mathbb{P}[Z_1 = 0] < 1$ , the existence of  $N(\beta)$  is guaranteed. Thus, there exists  $N(\beta) > 0$  such that

$$\langle \bar{\mu}^n, \varphi \rangle < \sum_{i=1}^{N(\beta)} \langle \mu^{n,i}, \varphi \rangle.$$

Now consider event  $A_0 = \{\lim_{t \rightarrow \infty} e^{-t\lambda} \langle \bar{\mu}^n, \varphi \rangle\}$ ,  $A_i = \{\lim_{t \rightarrow \infty} e^{-t\lambda} \langle \bar{\mu}^n, \varphi \rangle\}$ , then

$$A_0 \subset \bigcup_{i=1}^{N(\beta)} A_i.$$

We also have  $\mathbb{P}[A_i] = \mathbb{P}[A_1]$  for any  $i = 1, \dots, N(\beta)$ . By the result in [78], we know

$$\mathbb{P}[A_0] > 0,$$

Hence we must have  $\mathbb{P}[A_1] > 0$ , which gives the result. □

# Chapter 5

## Fractional stochastic Landau-Lifshitz Navier-Stokes equations in dimension $d \geq 3$ : Existence and (non-)triviality

### 5.1 Introduction

The Navier-Stokes equation is a fundamental partial differential equation in fluid dynamics:

$$\partial_t u = \nu \Delta u - \nabla p - \operatorname{div}(u \otimes u) + f, \quad \nabla \cdot u = 0,$$

where  $\nu$  is the kinematic viscosity,  $f$  is an external force and  $p$  is the pressure. When the force  $f$  is random, one speaks of the stochastic Navier-Stokes equation [36, 37, 87] when the noise is not rougher than space-time white noise. Even rougher noise, say the derivative of space-time white noise, is considered in the recent works [53, 12] for the vorticity formulation of the two-dimensional fractional stochastic Navier-Stokes equation.

Here we consider a fractional stochastic Navier-Stokes equation in dimension  $d \geq 3$ , which is given by

$$\partial_t u = -(-\Delta)^\theta u - \nabla p - \lambda \operatorname{div}(u \otimes u) + \sqrt{2}(-\Delta)^{\frac{\theta}{2}} \xi, \quad \nabla \cdot u = 0, \quad (5.1.1)$$

where  $\theta > 0$  is a positive number and  $\xi$  is a white noise with covariance

$$\mathbb{E}[\xi_i(t, x) \xi_j(s, y)] = \delta_{i,j} \delta(t - s) \delta(x - y), \quad (t, x), (s, y) \in \mathbb{R}_+ \times \mathbb{T}^d.$$

For  $\theta = 1$ , we show that this equation is equivalent to the fluctuating hydrodynamics equation of Landau and Lifshitz [5, 63],

$$\partial_t u = \nu \Delta u - \nabla p - \hat{\lambda} \operatorname{div}(u \otimes u) + \nabla \cdot \tau, \quad \nabla \cdot u = 0, \quad (5.1.2)$$

with noise term given by  $\nabla \cdot \tau$ , where  $\tau$  is a Gaussian field with covariance

$$\mathbb{E}[\tau_{ij}(t, x)\tau_{kl}(t', x')] = \frac{2\nu k_B T}{\rho}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl})\delta(x - x')\delta(t - t'), \quad (5.1.3)$$

where  $k_B$  is the Boltzmann constant,  $T$  is the temperature, and  $\rho$  is the density.

Equations (5.1.1) and (5.1.2) are scaling supercritical in  $d \geq 3$ , and therefore they are outside of the scope of regularity structures [55] or paracontrolled distributions [49], and making rigorous sense of (5.1.2) is a long-standing problem, with [42] making recent progress on physical interpretations of truncated versions of the equation.

We do not attempt to make sense of (5.1.2) directly, and we expect that there is no meaningful solution theory for the non-truncated equation. Instead, we consider two simplified problems. First, we study a hyper-viscous version of the equation, in which the diffusion is replaced by  $-(-\Delta)^\theta$  with  $\theta > \frac{d}{2}$  and simultaneously the noise is replaced by  $\sqrt{2(-\Delta)^\theta}\xi$  to guarantee a fluctuation-dissipation relation that preserves the “energy measure”. Our first result concerns the generator of equation (5.1.1) and the corresponding martingale problem.

**Theorem 5.1.1.** *Let  $\theta > \frac{d}{2}$  and  $M > 0$  and consider an initial distribution  $\nu$  with  $\frac{d\nu}{d\mu^M} \in L^2(\mu)$ , where  $\mu^M$  is the divergence-free and mean-free space white noise on  $\mathbb{T}_M^d = (\mathbb{R}^d/(M\mathbb{Z}))^d$ . There exists a unique-in-law energy solution to (5.1.1) on  $\mathbb{R}_+ \times \mathbb{T}_M^d$  with initial distribution  $\nu$ .*

Energy solutions are defined in Section 5.4 below, where we also give the proof of Theorem 5.1.1.

For our second result we focus on the case  $\theta \leq 1$ , so that in particular  $\theta < \frac{d}{2}$  and we cannot apply the previous result. Therefore, we consider a truncated equation, with the mollifier  $\rho^1$  given by  $\mathcal{F}\rho^1 = \mathbb{1}_{B(0,1)}$  for simplicity

$$\partial_t u = -(-\Delta)^\theta u - \lambda \rho^1 * \mathbf{\Pi} \operatorname{div}((\rho^1 * u) \otimes (\rho^1 * u)) + \sqrt{2}(-\Delta)^{\frac{\theta}{2}} \mathbf{\Pi} \xi, \quad (5.1.4)$$

now on  $\mathbb{R}_+ \times \mathbb{R}^d$ ; here  $\mathbf{\Pi}$  is the Leray projection that will be introduced below. We drop the condition  $\nabla u = 0$  here but should remember that it holds throughout the paper. This is essentially a Galerkin type projection of the equation for  $u$ , except that we did not mollify the noise. As will become clear from the proof of Theorem 5.3.4 below, we could also mollify  $\xi$  with  $\rho^1$  without affecting the results, but for simplicity we do not do this. Our second main result concerns the large-scale behavior of  $u$ : With the scaling

$$u^N(t, x) = N^{\frac{d}{2}} u(N^{2\theta} t, Nx)$$

we obtain (with a new space-time white noise that we still denote by  $\xi$  for simplicity)

$$\partial_t u^N = -(-\Delta)^\theta u^N - \lambda N^{2\theta - \frac{d+2}{2}} \rho^N * \mathbf{\Pi} \operatorname{div}((\rho^N * u^N) \otimes (\rho^N * u^N)) + \sqrt{2}(-\Delta)^{\frac{\theta}{2}} \mathbf{\Pi} \xi, \quad (5.1.5)$$

where  $\rho^N(x) = N^d \rho^1(Nx)$ .

**Theorem 5.1.2.** *Let  $\theta \leq 1$  and  $d \geq 3$  and let  $u$  be a stationary solution to (5.1.4). Then  $u^N$  has weakly convergent subsequences in  $C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$ . For  $\theta < 1$ , the limit  $u^\infty$  is unique and it solves*

$$\partial_t u^\infty = -(-\Delta)^\theta u^\infty + \sqrt{2}(-\Delta)^{\frac{\theta}{2}} \mathbf{\Pi} \xi,$$

*i.e. the nonlinearity simply drops out. For  $\theta = 1$ , no limit solves this equation, i.e. the nonlinearity gives rise to a nontrivial contribution in the limit.*

This result is shown in Section 5.6, where we conjecture that for  $\theta = 1$  the limit solves a linear equation, with explicitly given effective diffusivity  $> 1$ . For the mollified Landau-Lifshitz equation (5.1.2) we would obtain the following linear equation for the limit

$$\partial_t u = G(\hat{\lambda}) \nu \Delta u + \sqrt{G(\hat{\lambda})} \sqrt{\frac{2\nu k_B T}{\rho}} \mathbf{\Pi} (-\Delta)^{\frac{1}{2}} \xi,$$

where

$$G(\hat{\lambda}) = \sqrt{1 + \frac{\hat{\lambda}^2 k_B T \omega_d}{4\nu^2 \rho \pi^2 (d-2)}},$$

with an explicit constant  $\omega_d$  that depends on the specific mollifier, see Section 5.6 for details.

Let us comment briefly on our methods. For Theorem 5.1.1 we use the energy solution approach introduced and developed in the works [43, 50, 52, 51]. This approach is generalized in [44] to provide a general framework to make sense of the infinitesimal generator of stochastic Burgers and other types of singular stochastic partial differential equations. The crucial feature of the equation which makes this approach works is that it contains a good symmetric part and the singularity is contained in the anti-symmetric part, and this allows us to apply the results of [44]. In fractional problems like the ones we study here it is expected that there is an “energy critical exponent”  $\theta$  above which the energy solution approach works, but below which the theory does not give any information. This parameter does not have to be equal to the scaling critical parameter, and in fact here it is  $\theta = \frac{d}{2}$ , while the scaling critical parameter is  $\theta = \frac{d+2}{4}$ . For  $d \geq 3$ , the energy critical exponent is thus bigger than the

scaling critical exponent, and it is not well understood what happens in between (or in fact for  $\theta \in (1, \frac{d}{2}]$ ).

For Theorem 5.1.2 we follow the approach of [8, 12]. In particular, the approach and results are almost the same as the ones in [12] for the vorticity formulation of the 2d stochastic Navier-Stokes equation, and we employ their method for the stochastic Navier-Stokes equation in all dimension  $d \geq 3$ .

While the mathematical methods for both our main results existed previously, we believe that our main contribution is the observation that the stochastic Navier-Stokes equation has an invariant energy measure in all dimensions, and that this allows to study Landau-Lifshitz type dynamics with energy solution methods.

## 5.2 Notation and preliminaries

For  $M > 0$  we will use  $\mathbb{M}_M$  to denote the torus  $\mathbb{T}_M^d = (\mathbb{R}/(M\mathbb{Z}))^d$  of length  $M$ . When  $M = \infty$ , we interpret  $\mathbb{M}_\infty$  as the space  $\mathbb{R}^d$ . When  $M$  is clear from context, we will drop the index  $M$  and simply write  $\mathbb{M}$ . We write  $\mathcal{S}(\mathbb{M}_M)$  for  $C^\infty(\mathbb{M}_M)$  if  $M < \infty$ , and for the Schwartz functions on  $\mathbb{R}^d$  if  $M = \infty$ , and  $\mathcal{S}'(\mathbb{M}_M)$  are the (Schwartz) distributions on  $\mathbb{M}_M$ . We will write  $\mathcal{S}(\mathbb{M}, \mathbb{R}^d)$  and  $\mathcal{S}'(\mathbb{M}, \mathbb{R}^d)$  to denote vector-valued test functions and distributions; if it is clear from context we may also omit  $\mathbb{R}^d$  and simply write  $\mathcal{S}(\mathbb{M})$  or  $\mathcal{S}'(\mathbb{M})$  even in the vector-valued case.

### 5.2.1 Fourier Analysis

For  $M > 0$  and  $\varphi \in \mathcal{S}(\mathbb{M}_M)$  we define the Fourier transform at  $k \in \mathbb{Z}_M^d = (\frac{1}{M}\mathbb{Z})^d$  (with  $\mathbb{Z}_\infty^d = \mathbb{R}^d$ ) with the normalization

$$\mathcal{F}(\varphi)(k) = \hat{\varphi}(k) := \int_{\mathbb{M}} \varphi(x) e_{-k}(x) dx,$$

where  $e_k(x) = e^{2\pi i k x}$ . The inverse Fourier transformation is

$$\varphi(x) = \frac{1}{M^d} \sum_{k \in \mathbb{Z}_M^d} \hat{\varphi}(k) e_k(x),$$

where the sum becomes an integral for  $M = \infty$ , and Parseval's identity is

$$\langle \varphi, \psi \rangle = \frac{1}{M^d} \sum_{k \in \mathbb{Z}_M^d} \hat{\varphi}(k) \hat{\psi}(-k).$$

The convolution of two functions  $\varphi, \psi$  on the space  $\mathbb{M}_M$  is defined by

$$\varphi *_M \psi(x) := \int_{\mathbb{M}_M} \varphi(y) \psi(x - y) dy,$$

and the convolution of two functions on  $\mathbb{Z}_M^d$  is defined by

$$\hat{\varphi} *_M \hat{\psi}(k) := \frac{1}{M^d} \sum_{l \in \mathbb{Z}_M^d} \hat{\varphi}(l) \hat{\psi}(k-l),$$

so that

$$\widehat{\varphi *_M \psi}(k) = \hat{\varphi}(k) \hat{\psi}(k), \quad \widehat{\varphi \psi}(k) = \hat{\varphi} *_M \hat{\psi}(k).$$

Derivatives and fractional Laplacian can be expressed as Fourier multipliers:

$$\widehat{\partial_i \varphi}(k) = 2\pi i k_i \hat{\varphi}(k), \quad \widehat{(-\Delta)^\theta \varphi}(k) = (2\pi |k|)^{2\theta} \hat{\varphi}(k),$$

and the Leray projection  $\mathbf{\Pi}$  maps vectors  $\varphi \in \mathcal{S}(\mathbb{M}, \mathbb{R}^d)$  to divergence free vectors,

$$\widehat{\mathbf{\Pi}(\varphi)}_i(k) = \sum_j \widehat{\mathbf{\Pi}}_{ij}(k) \hat{\varphi}_j(k),$$

where

$$\widehat{\mathbf{\Pi}}_{ij} = \delta_{ij} - \frac{k_i k_j}{|k|^2}, \quad 1 \leq i, j \leq n, \quad (5.2.1)$$

with the convention  $\widehat{\mathbf{\Pi}}_{ij}(0) = \delta_{ij}$ .

## 5.2.2 Gaussian Analysis

We consider a mean-free and divergence-free white noise: For  $M < \infty$ ,  $\mu^M$  is an  $\mathcal{S}'(\mathbb{M}, \mathbb{R}^d)$ -valued random variable such that  $\mu^M(f)$  is centered Gaussian for all  $f \in \mathcal{S}(\mathbb{M}, \mathbb{R}^d)$ , with covariance

$$\mathbb{E}[\mu^M(f_1) \mu^M(f_2)] = \left\langle \mathbf{\Pi} \left( f_1 - \int f_1 dx \right), \mathbf{\Pi} \left( f_2 - \int f_2 dx \right) \right\rangle_{L^2(\mathbb{M}, \mathbb{R}^d)},$$

for all  $f_1, f_2 \in \mathcal{S}(\mathbb{M}, \mathbb{R}^d)$ , where  $\int_{\mathbb{M}_M} \dots dx = \frac{1}{M^d} \int_{\mathbb{M}_M} \dots dx$ . It can be seen that the measure induced on  $\mathcal{S}'$ , which we again noted as  $\mu^M$ , is concentrated on the divergence-free and mean-free fields. We can also consider the Hilbert space

$$\mathbb{H}_M := \left\{ \varphi \in L^2(\{1, \dots, d\} \times \mathbb{M}_M) : \operatorname{div}(\varphi) = 0, \int_{\mathbb{M}} \varphi dx = 0 \right\},$$

equipped with the norm  $\|h\|_{\mathbb{H}_M}^2 = \sum_{i=1}^d \int_{\mathbb{M}_M} |h(x)|^2 dx$ , so that  $\mu^M$  is a white noise on  $\mathbb{H}_M$  with

$$\mathbb{E}[\mu^M(h_1) \mu^M(h_2)] = \langle h_1, h_2 \rangle_{\mathbb{H}_M},$$

for  $h_1, h_2 \in \mathbb{H}_M$ . For  $M = \infty$  we also write  $\mu := \mu^\infty$  for the divergence-free white noise on  $\mathbb{R}^d$ , i.e. the covariance is

$$\mathbb{E}[\mu^M(f_1) \mu^M(f_2)] = \langle \mathbf{\Pi} f_1, \mathbf{\Pi} f_2 \rangle_{L^2(\mathbb{M}, \mathbb{R}^d)}.$$

We write  $L^2(\mathcal{S}'(\mathbb{M}), \mu^M)$  (in short  $L^2(\mu^M)$ ) for the space of all square-integrable random variables with respect to measure  $\mu^M$ . The  $n$ -th Wiener Chaos is defined via

$$\mathcal{H}_{M,n} := \overline{\text{span}\{H_n(\mu^M(h)) : h \in \mathbb{H}_M, \|h\| = 1\}},$$

where  $H_n$  are the Hermite polynomials such that  $H_0 = 1$ ,  $H'_n = nH_{n-1}$  and  $\mathbb{E}[H_n(X)] = 0$ ,  $n \geq 1$ , for a standard normal random variable  $X$ . When no confusion arises, we drop the index  $M$  for simplicity.

For all non-negative integers  $n$ , let  $\Gamma L_n^2$  be the symmetric subspace of  $\mathbb{H}^{\otimes n}$ . The Fock space  $\Gamma L^2 = \bigoplus_{n=0}^{\infty} \Gamma L_n^2$  consists of symmetric functions with finite norm defined by

$$\|\varphi\|_{\Gamma L^2}^2 := \sum_{n=0}^{\infty} n! \|\varphi_n\|_{\mathbb{H}^{\otimes n}}^2,$$

for  $\varphi = (\varphi_n)_{n \geq 0}$  with  $\|h^{\otimes n}\|_{\mathbb{H}^{\otimes n}} := \|h\|_{\mathbb{H}}^n$ . The space  $\Gamma L^2$  can be described as

$$\Gamma L^2 := \{(\varphi_n)_{n \geq 0} : \varphi_n \in \mathbb{H}^{\otimes n}, \varphi_n \text{ symmetric and } \|\varphi\|_{\Gamma L^2} < \infty\}.$$

For convenience, we also may say that a non-symmetric function is in the Fock space, provided its symmetrization is in the Fock space. For any  $f \in L^2(\mu)$ , we have the chaos decomposition

$$f = \sum_{n=0}^{\infty} I_n(\varphi_n), \quad \varphi = (\varphi_n)_n \in \Gamma L^2,$$

where  $I_n$  is the continuous linear map from the symmetric subspace of  $\mathbb{H}^{\otimes n}$  to  $L^2(\mu)$  with  $I_n(h^{\otimes n}) := H_n(\mu(h))$  for  $h \in \mathbb{H}$  and  $\|h\|_{\mathbb{H}} = 1$ . Furthermore, the identity

$$\|f\|_{L^2(\mu)} = \|\varphi\|_{\Gamma L^2}$$

holds.

For future usage, we can also extend the definition to the symmetric subspace of  $\bigoplus_{n=0}^{\infty} (L^2(\mathbb{M}, \mathbb{R}^d))^{\otimes n}$ . For function  $\varphi_1, \dots, \varphi_n \in L^2(\mathbb{M}, \mathbb{R}^d)$ , we can define a generalized Leray Projection on the tensor products by

$$\mathbf{\Pi}(\varphi_1 \otimes \dots \otimes \varphi_n) := \mathbf{\Pi}\varphi_1 \otimes \dots \otimes \mathbf{\Pi}\varphi_n,$$

which is a linear operator and can be extended to the space  $(L^2(\mathbb{M}, \mathbb{R}^d))^{\otimes n}$ . Now, for any function  $\varphi_n \in (L^2(\mathbb{M}, \mathbb{R}^d))^{\otimes n}$ , we define

$$I_n(\varphi_n) := I_n(\mathbf{\Pi}\varphi_n).$$

By the nature of the projection, For any  $\varphi = (\varphi_n)_{n=0}^\infty$  and  $\varphi_n \in \text{Sym}((L^2(\mathbb{M}, \mathbb{R}^d))^{\otimes n})$ , we define

$$\|\varphi\|^2 := \|\mathbf{\Pi}\varphi\|_{\Gamma L^2}^2 = \sum_{n=0}^{\infty} n! \|\mathbf{\Pi}\varphi_n\|_{(L^2(\mathbb{M}, \mathbb{R}^d))^{\otimes n}}^2 \leq \sum_{n=0}^{\infty} n! \|\varphi_n\|_{(L^2(\mathbb{M}, \mathbb{R}^d))^{\otimes n}}^2.$$

Thus, the upper bounds can be calculated as without existence of the Leray projection.

Another tool we need is the identity

$$I_p(\varphi_p)I_q(\varphi_q) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(\widetilde{\varphi_p \otimes_r \varphi_q}), \quad (5.2.2)$$

where  $\widetilde{\cdot}$  is the symmetrization operator and

$$\begin{aligned} & \varphi_p \otimes_r \varphi_q((i_1, x_1), \dots, (i_{p+q-2r}, x_{p+q-2r})) \\ &= \langle \varphi_p((i_1, x_1), \dots, (i_{p-r}, x_{p-r}), \cdot), \varphi_q((i_{p-r+1}, x_{p-r+1}), \dots, (i_{p+q-2r}, x_{p+q-2r}), \cdot) \rangle_{\mathbb{H}^{\otimes r}}. \end{aligned}$$

We will also need ‘‘Sobolev-like’’ spaces  $\mathcal{H}_\beta^\alpha$  defined by

$$\mathcal{H}_\beta^\alpha := (1 + \mathcal{N})^{-\beta} (1 - \mathcal{L}_\theta)^{-\alpha} \Gamma L^2,$$

where the number operator  $\mathcal{N}$  maps  $\varphi = (\varphi_n)_n$  to  $\mathcal{N}\varphi = (n\varphi_n)_n$ , and where  $\mathcal{L}_\theta$  is defined in equation (5.3.7) below. For  $\theta = 1$  and  $\varphi \in \Gamma L_1^2$ , this is exactly the inhomogeneous  $H^\alpha$  Sobolev norm.

### 5.3 Infinitesimal generator of the truncated equation

Our goal is to solve the following stochastic Navier-Stokes Equation on the space  $\mathbb{M}$ :

$$\begin{cases} \partial_t u = -(-\Delta)^\theta u - \lambda \mathbf{\Pi}(\text{div}(u \otimes u)) + \sqrt{2}(-\Delta)^{\frac{\theta}{2}} \mathbf{\Pi}\xi, & \text{on } \mathbb{R}^+ \times \mathbb{M}, \\ \text{div}(u) = 0, & \text{on } \mathbb{R}^+ \times \mathbb{M}. \end{cases} \quad (5.3.1)$$

Here  $u : \mathbb{M} \rightarrow \mathbb{R}^d$  is a divergence-free vector function,  $\xi = (\xi_1, \dots, \xi_d)$  are independent space-time white noises on  $\{1, \dots, d\} \times \mathbb{M}$ , and we recall that  $\mathbf{\Pi}$  is the Leray projector.

We start by considering the following approximation: Let  $\chi$  be a smooth, even function on the whole space  $\mathbb{R}^d$  with support on the ball  $B(0, 1)$  and consider the kernel  $\rho^{MN}$  with Fourier transform  $\hat{\rho}^{MN}(k) = \chi(k)$  for  $k \in \mathbb{Z}_{MN}^d$  (where  $MN$  is the product  $M \cdot N$ ). Let  $u^{MN}$  be the divergence-free solution to the equation

$$\partial_t u^{MN} = -A^\theta u^{MN} - \lambda \rho^{MN} *_{MN} \mathbf{\Pi}(\text{div}((\rho^{MN} *_{MN} u) \otimes (\rho^{MN} *_{MN} u))) + \sqrt{2} A^{\frac{\theta}{2}} \mathbf{\Pi}\xi, \quad (5.3.2)$$

on  $\mathbb{R}^+ \times \mathbb{M}_{MN}$ , for divergence-free initial condition  $u_0^M$ . With the same scaling as in (5.1.5) we see that  $u^{N,M} := N^{\frac{d}{2}} u^{MN}(N^{2\theta}t, Nx)$  solves

$$\partial_t u^{N,M} = -A^\theta u^{N,M} - \lambda_{N,\theta} B^{N,M}(u^{N,M}) + \sqrt{2} A^{\frac{\theta}{2}} \mathbf{\Pi} \xi, \quad (5.3.3)$$

on  $\mathbb{R}^+ \times \mathbb{M}_M$  with divergence-free initial condition  $u^{N,M}(0) = u_0^{N,M}$ , where  $A = -\Delta$ ,  $\lambda_{N,\theta} = \lambda N^{2\theta - \frac{d+2}{2}}$  and

$$B^{N,M}(u) = \rho^{N,M} *_M \mathbf{\Pi}(\operatorname{div}((\rho^{N,M} *_M u) \otimes (\rho^{N,M} *_M u))), \quad (5.3.4)$$

where  $\widehat{\rho^{N,M}}(k) = \chi(\frac{k}{N})$  on  $\mathbb{Z}_M^d$ . Next, let us define solutions to equation (5.3.3).

**Definition 5.3.1.** *A divergence-free stochastic process  $u^{N,M} \in C(\mathbb{R}, \mathcal{S}'(\mathbb{M}, \mathbb{R}^d))$  is called a strong solution to (5.3.3) with  $\lambda_{N,\theta}$  replaced by  $\hat{\lambda} > 0$ , if for all test function  $\varphi \in \mathcal{S}(\mathbb{M}_M, \mathbb{R}^d)$ , we have*

$$u_t^{N,M}(\varphi) = u_0^{N,M}(\varphi) - \int_0^t u_s^{N,M}(A^\theta \varphi) ds - \hat{\lambda} \int_0^t B^{N,M}(u_s^{N,M})(\varphi) ds + M_t^\varphi, \quad (5.3.5)$$

where  $M_t^\varphi = \sqrt{2} \xi(\mathbb{1}_{[0,t]} \otimes A^{\frac{\theta}{2}} \mathbf{\Pi} \varphi)$  is a continuous martingale with quadratic variation

$$[M^\varphi]_t = 2t \|A^{\frac{\theta}{2}} \mathbf{\Pi} \varphi\|_{\mathbb{H}_M}^2.$$

**Remark 5.3.2.** *We may consider Schwartz functions as test functions. However, since the solution  $u^{N,M}$  is divergence-free, we actually have*

$$u_t^{N,M}(\varphi) = (\mathbf{\Pi} u_t^{N,M})(\varphi) = u_t^{N,M}(\mathbf{\Pi} \varphi).$$

*Thus we can restrict to divergence-free test functions. Furthermore, when  $M < \infty$ , the 0-th Fourier coefficient does not change in time. Thus, for convenience, we will also restrict the test function to have average 0 when  $M$  is finite. We will write  $\mathcal{S}_f(\mathbb{M})$  and  $\mathcal{S}'_f(\mathbb{M})$  to indicate that we choose the divergence-free functions.*

To start with, let us give an orthonormal basis of  $\mathbb{H}_M$ .

**Proposition 5.3.3.** *Let  $d$  be the dimension and let  $M < \infty$ . Then for each Fourier mode  $k \in \mathbb{Z}_M^d \setminus \{0\}$ , there are exactly  $d-1$  vectors  $(\tilde{e}_k^i)_{1 \leq i \leq d-1}$  such that  $\{\tilde{e}_k^i\}_{k \neq 0, i=0, \dots, d-1}$  forms an orthogonal basis of  $\mathbb{H}_M$ .*

*Proof.* We start with the  $L^2$  Fourier basis:

$$e_k^i(z) := \{(e_k(z) \delta_{ij})_{j=1, \dots, d}\}_{0 \neq k \in \mathbb{Z}_M^d, i=1, \dots, d},$$

with Fourier transformation

$$\delta_k^i(\bar{k}) := \{(\delta_k(\bar{k})\delta_{ij})_{j=1,\dots,d}\}_{0 \neq k \in \mathbb{Z}_M^d, i=1,\dots,d}.$$

The Fourier transform of the Leray projection of  $e_k^i$  is given by

$$\left(\widehat{\Pi}\delta_k^i\right)(\bar{k}) := \left(\widehat{\Pi}_{ij}\delta_k^i\right)_{j=1,\dots,d}(\bar{k}) = \left(\delta_{ij} - \frac{\bar{k}_i\bar{k}_j}{|k|^2}\right)_{j=1,\dots,d} \delta_k(\bar{k})$$

This means that the Leray projections for different  $k \in \mathbb{Z}_M^d$  are linearly independent. We now go to check the case when  $k$  is the same but  $i$  is different. We only need to compute the rank of the matrix  $\widehat{\Pi}(k)$  for any fixed non-zero  $k$ , where

$$\widehat{\Pi}(k) = \begin{pmatrix} |k|^2 - k_1^2 & -k_1k_2 & \dots & -k_1k_d \\ -k_1k_2 & |k|^2 - k_2^2 & \dots & -k_2k_d \\ \vdots & \vdots & \ddots & \vdots \\ -k_dk_1 & -k_dk_2 & \dots & |k|^2 - k_d^2 \end{pmatrix} = |k|^2 I_d - (k_i k_j)_{i,j}.$$

To show the rank is  $d - 1$ , we only need to show that  $(k_i k_j)_{i,j}$  has a one-dimensional eigenspace with respect to the eigenvalue  $|k|^2$ . One eigenvector for  $|k|^2$  is  $k$ . If  $(x_j)$  is another eigenvector for  $|k|^2$ , then we can take the inner product with  $x$  in the equality  $k_i \sum_j k_j x_j = |k|^2 x_i$ ,  $i = 1, \dots, d$ , and we obtain

$$|k|^2 |x|^2 = \left(\sum_j x_j k_j\right)^2.$$

By the Cauchy-Schwarz inequality, we know that  $x$  is a multiple of  $k$  and therefore the dimension of the eigenspace is exactly one. Thus we proved the proposition.  $\square$

Now we can prove the existence and uniqueness of strong solutions to the truncated equation:

**Theorem 5.3.4.** *For  $M < \infty$ , equation (5.3.3) admits a unique strong solution  $u^{N,M} \in C(\mathbb{R}_+, \mathbb{B}_{2,2}^{-\frac{d}{2}-})$  for any divergence-free initial condition  $u_0 \in \mathbb{B}_{2,2}^{-\frac{d}{2}-}$ , where*

$$\mathbb{B}_{2,2}^{-\frac{d}{2}-} = \bigcap_{\alpha < -\frac{d}{2}} \mathbb{B}_{2,2}^\alpha.$$

Furthermore, the solution is a strong Markov process and  $\mu^M$  is an invariant measure.

*Proof.* In this proof we drop the superscript  $\cdot^M$ . We can follow Section 4 of [50] or Proposition 2.1 of [12]. As indicated by Proposition 5.3.3, instead of considering Fourier series, we take the orthogonal basis of  $\mathbb{H}$  as test functions, i.e.  $\tilde{e}_k^i$  in Proposition

**5.3.3.** We can then decompose  $u^N = v^N + Z^N$  where  $v^N$  and  $Z^N$  are independent with disjoint spectral support. Indeed, for the decomposition we take  $V^N := \{0 \neq k \in \mathbb{Z}_M^d : \frac{k}{N} \in \text{supp}(\chi)\}$  and we let  $v^N$  be the orthogonal projection of  $u^N$  to the space generated by  $\{\tilde{e}_k^i\}_{k \in V^N, i=0, \dots, d-1}$ .

Furthermore,  $Z^N$  is the Leray projection of the infinite-dimensional Ornstein-Uhlenbeck process on the Fourier modes  $U^N := \{k \in \mathbb{Z}_M^d : \mathcal{F}(\rho^{N,M})(k) = 0\}$ , which is a strong Markov process in the space  $C(\mathbb{R}_+, \mathbb{B}_{2,2}^{-\frac{d}{2}-})$  with invariant measure given by the projection of  $\mu^M$  onto the Fourier modes in  $U^N$ .

On the other hand,  $v^N$  is spectrally supported on  $V_k^N$  and can therefore be identified with a finite-dimensional SDE with smooth coefficients. Thus, we only have to deal with the non-linear part  $B^N$  to rule out explosions. It suffices to show weak coercivity (see [67, Chapter 3]), which follows from

$$\langle u, B^N(u) \rangle_{L^2(\mathbb{M})} = 0. \quad (5.3.6)$$

This follows from the divergence-free property of  $u$  since

$$\begin{aligned} \langle u, B^N(u) \rangle &= \langle \rho^N * u, \text{div}((\rho^N * u) \otimes (\rho^N * u)) \rangle \\ &= \sum_{i,j} \langle \rho^N * u_i, \partial_j(\rho^N * u_i \cdot \rho^N * u_j) \rangle \\ &= - \sum_{i,j} \langle (\partial_j \rho^N * u_i) \cdot \rho^N * u_j, \rho^N * u_i \rangle \\ &= - \sum_{i,j} \langle \partial_j(\rho^N * u_i \cdot \rho^N * u_j), \rho^N * u_i \rangle = 0. \end{aligned}$$

This proves the existence and pathwise uniqueness of strong solutions in  $C(\mathbb{R}_+, \mathbb{B}_{2,2}^{-\frac{d}{2}-})$ .

To prove invariance of  $\mu^M$ , we can either directly use the divergence-free property of  $B^N$  when expressed in Fourier modes, or proceed as in [50, 12] to calculate the generator of equation (5.3.3) and use the general result in [31]. We postpone the argument until we introduced the generator of the equation (5.3.3).  $\square$

To proceed, we introduce the generator of equation (5.3.3) acting on cylinder functions. A cylinder function is a function with representation  $F(u) = f(u(\varphi_1), \dots, u(\varphi_n))$  for some  $n \in \mathbb{N}$ ,  $\varphi_1, \dots, \varphi_n \in \mathcal{S}_f$  and smooth  $f$  with at most polynomial growth at infinity. By Itô's formula, the generator  $\mathcal{L}^N$  of equation (5.3.3) is given by  $\mathcal{L}^N = \mathcal{L}_\theta + \mathcal{G}^N$ .

$$\mathcal{L}_\theta F(u) = - \sum_i \partial_i f \cdot u(A^\theta \varphi_i) + \sum_{i,j} \partial_{ij}^2 f \langle A^{\frac{\theta}{2}} \varphi_i, A^{\frac{\theta}{2}} \varphi_j \rangle, \quad (5.3.7)$$

and

$$\tilde{\mathcal{G}}^N F(u) = - \sum_i \partial_i f \langle B^N(u), \varphi_i \rangle, \quad \mathcal{G}^N = \lambda_{N,\theta} \tilde{\mathcal{G}}^N. \quad (5.3.8)$$

Using the approach of [52], we calculate the action of the generator on the space  $L^2(\mu^M)$ .

**Theorem 5.3.5.** *On the space  $L^2(\mu^M)$ , the operator  $\mathcal{L}_\theta$  is symmetric and the operator  $\tilde{\mathcal{G}}^N$  is anti-symmetric. Furthermore, we have the decomposition  $\tilde{\mathcal{G}}^N = \tilde{\mathcal{G}}_+^N + \tilde{\mathcal{G}}_-^N$  such that  $\tilde{\mathcal{G}}_+^N : \Gamma L_n^2 \rightarrow \Gamma L_{n+1}^2$  and  $\tilde{\mathcal{G}}_-^N : \Gamma L_n^2 \rightarrow \Gamma L_{n-1}^2$ .*

For sufficiently regular  $\varphi = (\varphi_n)_{n \in \mathbb{N}} \in \Gamma L^2$ , the operators have the following expressions in Fourier modes for  $n \geq 0$ :

$$\mathcal{F}(\mathcal{L}_\theta \varphi)_n((l_i, k_i)_{1:n}) = -(2\pi)^{2\theta} \sum_{i=1}^n |k_i|^{2\theta} \hat{\varphi}_n((l_i, k_i)_{1:n}), \quad (5.3.9)$$

and  $\tilde{\mathcal{G}}_+^N(\Gamma L_0^2) = 0$ ,  $\tilde{\mathcal{G}}_-^N(\Gamma L_0^2) = 0$ , while for  $n \geq 1$

$$\begin{aligned} \mathcal{F}(\tilde{\mathcal{G}}_+^{N,M} \varphi)_n(l_{1:n}, k_{1:n}) &= 2\pi i(n-1) \mathcal{R}_{k_{n-1}, k_n}^N(k_n + k_{n-1})_{l_n} \\ &\quad \cdot \hat{\varphi}_{n-1}((l_{n-1}, k_n + k_{n-1}), \dots), \end{aligned} \quad (5.3.10)$$

and

$$\begin{aligned} \mathcal{F}(\tilde{\mathcal{G}}_-^{N,M} \varphi)_n(l_{1:n}, k_{1:n}) &= \frac{2\pi i n(n+1)}{M^d} \sum_i \sum_{p+q=k_n} \mathcal{R}_{p,q}^N \left[ p_{l_n} \hat{\varphi}_{n+1}((i, p), (i, q), \dots) \right. \\ &\quad \left. - k_{n,i} \hat{\varphi}_{n+1}((l_n, p), (i, q), \dots) \right], \end{aligned} \quad (5.3.11)$$

where  $\mathcal{R}_{p,q}^N = \hat{\rho}^N(p) \hat{\rho}^N(q) \hat{\rho}^N(p+q)$ .

*Proof.* We divide the proof into three steps. The first two steps are used for the calculation of  $\mathcal{L}_\theta$ ,  $\tilde{\mathcal{G}}_+^N$  and  $\tilde{\mathcal{G}}_-^N$ . The final step is to prove the anti-symmetry of  $\tilde{\mathcal{G}}^N$ . Due to linearity, we only need to consider  $F(u) = H_n(u(h)) = I_n(h^{\otimes n})$  with  $h \in \mathcal{S}_f(\mathbb{M})$  such that  $\|h\|_{\mathbb{H}_M} = 1$ .

**Step 1.** We have for  $\mathcal{L}_\theta$ :

$$\begin{aligned} \mathcal{L}_\theta H_n(\mu^M(h)) &= -n H_{n-1}(\mu^M(h)) \mu^M(A^\theta h) + n(n-1) H_{n-2}(\mu^M(h)) \|A^{\frac{\theta}{2}} h\|_{\mathbb{H}}^2 \\ &= -n I_n(h^{\otimes(n-1)} \otimes A^\theta h), \end{aligned} \quad (5.3.12)$$

and thus for any  $\varphi \in \Gamma L^2$

$$(\mathcal{L}_\theta \varphi)_n = - \sum_{i=1}^n A_{x_i}^\theta \varphi_n,$$

which gives the desired result in Fourier modes.

**Step 2.** Defining  $\rho_{i,y}^N(x)$  as a vector with value  $\rho^N(x-y)$  at the  $i$ -th position and 0 at other positions, we have

$$\begin{aligned}
\tilde{\mathcal{G}}^{N,M} F &= -nH_{n-1}(u(h))\langle h, \rho^N * (\operatorname{div}(\rho^N * \mu^M) \otimes (\rho^N * \mu^M)) \rangle \\
&= n \sum_{i,j} \int \partial_j h_i(x) \rho^N(x-y) I_1(\rho_{i,y}^N) I_1(\rho_{j,y}^N) I_{n-1}(h^{\otimes(n-1)}) dx dy \\
&= n \sum_{i,j} \int \partial_j h_i(x) \rho^N(x-y) I_2(\rho_{i,y}^N \otimes \rho_{j,y}^N) I_{n-1}(h^{\otimes(n-1)}) dx dy \\
&= \tilde{\mathcal{G}}_+^{N,M} + \tilde{\mathcal{G}}_-^{N,M}.
\end{aligned} \tag{5.3.13}$$

The first equality holds because  $h$  is divergence-free and the Leray projection is a Fourier multiplier, hence it commutes with the convolution. The third equality is because  $I_1(\rho_{i,y}^N) I_1(\rho_{j,y}^N) = I_2(\rho_{i,y}^N \otimes \rho_{j,y}^N) + \langle \rho_{i,y}^N, \rho_{j,y}^N \rangle$  and

$$\sum_{i,j} \int \partial_j h_i(x) \rho^N(x-y) \langle \rho_{i,y}^N, \rho_{j,y}^N \rangle dx dy = \sum_i \int \partial_i h_i(x) \rho^N(x-y) \|\rho_y^N\|_{L^2}^2 dx dy = 0.$$

Using the contraction rule for Wiener-Itô integrals, see Proposition 1.1.3 of [76],  $\tilde{\mathcal{G}}_+^{N,M}$  and  $\tilde{\mathcal{G}}_-^{N,M}$  are given by (we drop the operator  $I$  here)

$$\begin{aligned}
\tilde{\mathcal{G}}_+^{N,M} F(l_{1:n+1}, x_{1:n+1}) &= \frac{n}{2} \int (\partial_{l_{n+1}} h_{l_n}(x) + \partial_{l_n} h_{l_{n+1}}(x)) h^{\otimes(n-1)}(l_{1:n-1}, x_{1:n-1}) \\
&\quad \cdot \rho_y^N(x) \rho_{x_n}^N(y) \rho_{x_{n+1}}^N(y) dx dy,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{G}}_-^{N,M} F(l_{1:n-1}, x_{1:n-1}) &= n(n-1) \int \sum_i (\partial_i h_{l_{n-1}}(x) + \partial_{l_{n-1}} h_i(x)) h^{\otimes(n-2)}(l_{1:n-2}, x_{1:n-2}) \\
&\quad \cdot h_i(z) \rho_y^N(x) \rho_z^N(y) \rho_{x_{n-1}}^N(y) dx dy dz \\
&= n(n-1) \int \sum_i (\partial_{l_{n-1}} h_i(x) \rho_{x_{n-1}}^N(y) - h_{l_{n-1}}(x) \partial_i \rho_{x_{n-1}}^N(y)) \\
&\quad \cdot h^{\otimes(n-2)}(l_{1:n-2}, x_{1:n-2}) h_i(z) \rho_y^N(x) \rho_z^N(y) dx dy dz.
\end{aligned}$$

The second equality holds because  $h$  is divergence-free and it is useful for the estimations below. There is also a vanishing term:

$$\begin{aligned}
\tilde{\mathcal{G}}_{-3}^{N,M} F(l_{1:n-3}, x_{1:n-3}) &= \sum_{i,j} \int \partial_j h_i(x) \rho_y^N(x) h^{\otimes(n-3)}(l_{1:n-3}, x_{1:n-3}) \langle h_i, \rho_y^N \rangle \langle h_j, \rho_y^N \rangle dx dy \\
&= - \sum_{i,j} \int h_i(x) \rho_y^N(x) h^{\otimes(n-3)}(l_{1:n-3}, x_{1:n-3}) \\
&\quad \cdot \partial_{y_j} (\langle h_i, \rho_y^N \rangle \langle h_j, \rho_y^N \rangle) dx dy \\
&= -\tilde{\mathcal{G}}_{-3}^{N,M} F(l_{1:n-3}, x_{1:n-3}),
\end{aligned}$$

using that  $h$  is divergence-free in the last step.

A direct calculation shows that  $\tilde{\mathcal{G}}_-^{N,M} = 0$  for  $n = 0, 1$ , and  $\tilde{\mathcal{G}}_+^{N,M} = 0$  for  $n = 0$ . Hence, we have for all  $\varphi \in \Gamma L^2$

$$\begin{aligned} \tilde{\mathcal{G}}_+^{N,M} \varphi_n &= \frac{n}{2} \int [\partial_{l_{n+1}} \varphi_n((l_n, x), \dots) + \partial_{l_n} \varphi_n((l_{n+1}, x), \dots)] \\ &\quad \cdot \rho_y^N(x) \rho_{x_n}^N(y) \rho_{x_{n+1}}^N(y) dx dy, \end{aligned} \quad (5.3.14)$$

and

$$\begin{aligned} \tilde{\mathcal{G}}_-^{N,M} \varphi_n &= n(n-1) \int \sum_i [\partial_{x_{l_{n-1}}} \varphi_n((i, x)(i, z), \dots) \rho_{x_{n-1}}^N(y) \\ &\quad - \varphi_n((l_{n-1}, x), (i, z), \dots) \partial_i \rho_{x_{n-1}}^N(y)] \rho_y^N(x) \rho_z^N(y) dx dy dz. \end{aligned} \quad (5.3.15)$$

This gives the desired result on the Fourier modes of  $\tilde{\mathcal{G}}^{N,M}$ .

**Step 3.** Finally, we prove the anti-symmetry of  $\tilde{\mathcal{G}}^{N,M}$ : We have for  $h, g \in \mathcal{S}_f$

$$\begin{aligned} &\frac{1}{(n+1)!} \langle \tilde{\mathcal{G}}_+^{N,M}(h^{\otimes n}), g^{\otimes(n+1)} \rangle \\ &= n \langle g, h \rangle^{n-1} \int \sum_{l_n: n+1} g_{l_n}(z_n) g_{l_{n+1}}(z_{n+1}) \rho_y^N(x) \rho_{z_n}^N(y) \rho_{z_{n+1}}^N(y) \\ &\quad \cdot \partial_{l_{n+1}} h_{l_n}(x) dx dy dz_{n:n+1} \\ &= -n \langle g, h \rangle^{n-1} \int \sum_{l_n: n+1} g_{l_n}(z_n) g_{l_{n+1}}(z_{n+1}) \rho_y^N(x) \partial_{y_{l_{n+1}}} (\rho_{z_n}^N(y) \rho_{z_{n+1}}^N(y)) \\ &\quad \cdot h_{l_n}(x) dx dy dz_{n:n+1} \\ &= -n \langle g, h \rangle^{n-1} \int \sum_{l_n, i} \partial_i g_{l_n}(x) g_i(z) \rho_y^N(x) \rho_{z_n}^N(y) \rho_z^N(y) h_{l_n}(x) dx dy dz dz_n + 0, \end{aligned} \quad (5.3.16)$$

where we applied integration by parts in the second and third equalities and the second term in the third equality vanishes because  $g$  is divergence-free. For the remaining part in the last line, we write  $i = l_{n+1}, z = z_{n+1}$  and exchange  $z_n$  and  $x$ . For the operator  $\tilde{\mathcal{G}}_-^{N,M}$ , we have

$$\begin{aligned} \frac{1}{(n+1)!} \langle \tilde{\mathcal{G}}_-^{N,M}(g^{\otimes(n+1)}), h^{\otimes n} \rangle &= n \langle g, h \rangle^{n-1} \int \sum_{l_n, i} h_{l_n}(z_n) g_i(z) \rho_y^N(x) \rho_z^N(y) \rho_{z_n}^N(y) \\ &\quad \cdot (\partial_{l_n} g_i + \partial_i g_{l_n})(x) dx dy dz dz_n \\ &= 0 + \left( -\frac{1}{(n+1)!} \langle \tilde{\mathcal{G}}_+^{N,M}(h^{\otimes n}), g^{\otimes(n+1)} \rangle \right). \end{aligned} \quad (5.3.17)$$

The first term after the second equality vanishes because

$$\begin{aligned} & \int \sum_{l_n, i} h_{l_n}(z_n) g_i(z) \rho_y^N(x) \rho_z^N(y) \rho_{z_n}^N(y) \partial_{l_n} g_i(x) dx dy dz dz_n \\ &= \int \sum_{l_n, i} (h_{l_n} * \rho^N)(y) \partial_{l_n} (g_i * \rho^N)^2(y) dy dz_n = 0, \end{aligned}$$

because  $h$  is divergence-free. Thus the operator  $\mathcal{G}^{N, M}$  is anti-symmetric.  $\square$

Now we can complete the proof of the Theorem 5.3.4, namely show that  $\mu^M$  is an invariant measure for  $u^{N, M}$ .

*Proof of the invariance of  $\mu^M$  in Theorem 5.3.4.* Let  $\Pi_K$  be the projection on the space generated by  $\{\tilde{e}_k^i\}_{0 \neq k \in \mathbb{Z}_M^d: |k| \leq K}$ . Then  $\Pi_K$  commutes with  $\mathcal{L}^N$  if  $K$  is sufficiently large, and the invariance of  $\mu \circ \Pi_K^{-1}$  for  $\Pi_K u^N$  follows from Echeverria's criterion [31] because

$$\mathbb{E}_{\mu \circ \Pi_K^{-1}}[\mathcal{L}^N \Pi_K \varphi] = \mathbb{E}_\mu[\mathcal{L}^N \Pi_K \varphi] = 0,$$

for any cylinder function  $F$ , which follows from Theorem 5.3.5.  $\square$

Next, we derive estimates for  $\tilde{\mathcal{G}}^N$ , along the lines of [52, 53]. We consider a function  $\omega : \mathbb{N} \rightarrow (0, \infty)$ , which we interpret as a weight.

**Theorem 5.3.6.** *The operators  $\tilde{\mathcal{G}}_+^N$  and  $\tilde{\mathcal{G}}_-^N$  satisfy for  $\zeta \in [0, 1]$  and  $\lambda_\theta^\zeta = \frac{d+2}{4\theta} + \zeta \frac{1 \vee \theta - 1}{2\theta}$*

$$\|\omega(\mathcal{N})(\lambda - \mathcal{L}_\theta)^{\beta - \lambda_\theta^\zeta} \tilde{\mathcal{G}}_-^N \varphi\| \lesssim \|\mathcal{N}^{1 + \frac{1-\zeta}{2}(1 - \frac{1}{\theta \vee 1})} \omega(\mathcal{N} - 1)(\lambda - \mathcal{L}_\theta)^\beta \varphi\|, \quad (5.3.18)$$

if  $\beta > \frac{d}{4\theta}$ , and

$$\|\omega(\mathcal{N})(\lambda - \mathcal{L}_\theta)^{\beta - \lambda_\theta^\zeta} \tilde{\mathcal{G}}_+^N \varphi\| \lesssim \|\mathcal{N}^{1 + \frac{1-\zeta}{2}(1 - \frac{1}{\theta \vee 1})} \omega(\mathcal{N} + 1)(\lambda - \mathcal{L}_\theta)^\beta \varphi\|, \quad (5.3.19)$$

if  $\beta < \lambda_\theta^\zeta - \frac{d}{4\theta}$ ; the implicit constants in (5.3.18) and (5.3.19) depend on  $M$  if  $\theta > 1$ , but they are independent of  $M$  for  $\theta \leq 1$ . Next, we also have for  $0 \leq \beta < \frac{d}{2\theta}$ ,

$$\|(\lambda - \mathcal{L}_\theta)^{-\frac{\beta}{2}} \mathcal{G}_+^N \varphi\|^2 \lesssim \lambda_{N, \theta}^2 N^{d-2\beta\theta} \|\mathcal{N}^{\frac{3}{2} - \frac{1}{2(\theta \vee 1)}} (-\mathcal{L}_\theta)^{\frac{1}{2\theta}} \varphi\|^2, \quad (5.3.20)$$

and for  $\mathcal{G}_-^N$ , we have the estimate

$$\|(\lambda - \mathcal{L}_\theta)^{-\frac{1}{2\theta}} \mathcal{G}_-^N \varphi\|^2 \lesssim \|\mathcal{N}(-\mathcal{L}_\theta)^{1 - \frac{1}{2\theta}} \varphi\|^2. \quad (5.3.21)$$

Finally, we have a non-uniform estimate of the operator  $\tilde{\mathcal{G}}_\pm^N$ .

$$\|\tilde{\mathcal{G}}^N \varphi\| \lesssim_N \|\mathcal{N}^{\frac{3}{2}} \varphi\|. \quad (5.3.22)$$

*Proof.* For  $\tilde{\mathcal{G}}_-^N$ , we have, by the symmetry of  $\hat{\varphi}$ ,

$$\begin{aligned}
& \|\omega(\mathcal{N})(\lambda - \mathcal{L}_\theta)^{\beta - \lambda_\theta^\zeta} \tilde{\mathcal{G}}_-^N \varphi\|^2 \\
& \lesssim \sum_{n \geq 0} n! \omega(n)^2 \sum_{l_{1:n}, k_{1:n}} (1 + (2\pi)^{2\theta} (|k_1|^{2\theta} + \dots + |k_n|^{2\theta}))^{2\beta - 2\lambda_\theta^\zeta} \frac{1}{M^{d(n+2)}} \\
& \quad \cdot n^2 (n+1)^2 \left| \sum_i \sum_{p+q=k_n} p l_n \hat{\varphi}_{n+1}((i, p), (i, q), \dots) - k_{n,i} \hat{\varphi}_{n+1}((l_n, p), (i, q), \dots) \right|^2 \\
& \lesssim \sum_{n \geq 0} \frac{n! n^2 (n+1)^2 \omega(n)^2}{M^{d(n+2)}} \sum_{l_{1:n}, k_{1:n}, i} (1 + |k_1|^{2\theta} + \dots + |k_n|^{2\theta})^{2\beta - 2\lambda_\theta^\zeta} \\
& \quad \cdot |k_n|^2 \left( \left| \sum_{p+q=k_n} \hat{\varphi}_{n+1}((l_n, p), (i, q), \dots) \right|^2 + \left| \sum_{p+q=k_n} \hat{\varphi}_{n+1}((i, p), (i, q), \dots) \right|^2 \right).
\end{aligned}$$

By multiplying the RHS by the dimension  $d$ , we notice that

$$\sum_{l_{1:n}, i} \left| \sum_{p+q=k_n} \hat{\varphi}_{n+1}((i, p), (i, q), \dots) \right|^2 \lesssim d \cdot \sum_{l_{1:n}, i} \left| \sum_{p+q=k_n} \hat{\varphi}_{n+1}((l_n, p), (i, q), \dots) \right|^2.$$

We have for any  $\alpha > \frac{d}{2\theta}$

$$\frac{1}{M^d} \sum_{p+q=k_n} (C + |p|^{2\theta} + |q|^{2\theta})^{-\alpha} \approx \int (C + |k_n|^{2\theta} + |p|^{2\theta})^{-\alpha} dp \lesssim (C + |k_n|^{2\theta})^{\frac{d}{2\theta} - \alpha}, \quad (5.3.23)$$

and therefore

$$\begin{aligned}
& \frac{1}{M^d} \left| \sum_{p+q=k_n} \hat{\varphi}_{n+1}((l_n, p), (i, q), \dots) \right|^2 \leq (\lambda + |k_1|^{2\theta} + \dots + |k_n|^{2\theta})^{\frac{d}{2\theta} - \alpha} \\
& \quad \cdot \sum_{p+q=k_n} (\lambda + |k_1|^{2\theta} + \dots + |k_{n-1}|^{2\theta} + |p|^{2\theta} + |q|^{2\theta})^\alpha |\hat{\varphi}_{n+1}|^2.
\end{aligned}$$

We then obtain, choosing  $\alpha = 2\beta > \frac{d}{2\theta}$ ,

$$\begin{aligned}
& \|\omega(\mathcal{N})(\lambda - \mathcal{L}_\theta)^{\beta - \lambda_\theta^\zeta} \tilde{\mathcal{G}}_-^N \varphi\|^2 \\
& \lesssim \sum_{n \geq 0} \frac{n! n^2 (n+1)^2 \omega(n)^2}{M^{d(n+1)}} \sum_{l_{1:n+1}, k_{1:n}} (\lambda + |k_1|^{2\theta} + \dots + |k_n|^{2\theta})^{-2\lambda_\theta^\zeta + \frac{d}{2\theta}} |k_n|^2 \\
& \quad \cdot \sum_{p+q=k_n} (\lambda + |k_1|^{2\theta} + \dots + |k_{n-1}|^{2\theta} + |p|^{2\theta} + |q|^{2\theta})^{2\beta} |\hat{\varphi}_{n+1}((l_n, p), (l_{n+1}, q), \dots)|^2 \\
& \lesssim \sum_{n \geq 0} \frac{1}{M^{dn}} n! n^{3-\zeta - \frac{1-\zeta}{\theta\sqrt{1}}} \omega(n-1)^2 \sum_{(l,k)_{1:n}} (\lambda + |k_1|^{2\theta} + \dots + |k_n|^{2\theta})^{2\beta} |\hat{\varphi}_n|^2 \\
& = \|\mathcal{N}^{1 + \frac{1-\zeta}{2}(1 - \frac{1}{\theta\sqrt{1}})} \omega(\mathcal{N}-1) (\lambda - \mathcal{L}_\theta)^\beta \varphi\|^2.
\end{aligned}$$

In the second inequality, we use symmetry of  $\hat{\varphi}_n$  and  $\sum_{i=1}^n |k_i|^2 \leq (\sum_{i=1}^n |k_i|^{2\theta})^{\frac{1}{\theta}}$  for  $\theta \leq 1$  while for  $\theta > 1$ , we use  $|k_n| \geq \frac{1}{M}$  and therefore  $|k_n|^2 \leq M^{2\theta-2}$  to obtain  $\sum_{i=1}^n |k_i|^2 \lesssim_M (\sum_{i=1}^n |k_i|^{2\theta})^\zeta (\sum_{i=1}^n |k_i|^{2\theta})^{\frac{1-\zeta}{\theta}} n^{(1-\zeta)(1-\frac{1}{\theta})}$ .

The estimate (5.3.19) for  $\tilde{\mathcal{G}}_+^N$  now follows by a duality argument from (5.3.18) because  $\langle \tilde{\mathcal{G}}_+^N \varphi, \psi \rangle = -\langle \varphi, \tilde{\mathcal{G}}_-^N \psi \rangle$  for all cylinder functions  $\varphi, \psi$ .

For  $\mathcal{G}_+^N$  (i.e. including  $\lambda_{N,\theta}$ ) and  $\beta < \frac{d}{2\theta}$  we have

$$\begin{aligned} \|(\lambda - \mathcal{L}_\theta)^{-\frac{\beta}{2}} \mathcal{G}_+^{N,M} \varphi\|^2 &\lesssim \sum_{n \geq 0} \sum_{(l,k)_{1:n}} |\mathcal{R}_{k_n, k_{n-1}}^N|^2 \frac{n! \lambda_{N,\theta}^2}{M^{nd}} (\lambda + |k_1|^{2\theta} + \dots + |k_n|^{2\theta})^{-\beta} \\ &\quad \cdot (n-1)^2 |k_n + k_{n-1}|^2 |\hat{\varphi}_{n-1}((l_{n-1}, k_n + k_{n-1}), \dots)|^2 \\ &\lesssim \lambda_{N,\theta}^2 N^{d-2\beta\theta} \|\mathcal{N}^{\frac{3}{2} - \frac{1}{2(\theta \vee 1)}} (-\mathcal{L}_\theta)^{\frac{1}{2\theta}} \varphi\|^2, \end{aligned}$$

where the estimation in the second inequality follows from

$$\frac{1}{M^d} \sum_{p+q=k} |\mathcal{R}_{p,q}^N|^2 (C + |p|^{2\theta} + |q|^{2\theta})^{-\beta} \lesssim \int_{|p| \leq N} (C + |p|^{2\theta})^{-\beta} dp \lesssim N^{d-2\beta\theta}. \quad (5.3.24)$$

Finally, for  $\mathcal{G}_-^N$ , we have

$$\begin{aligned} &\|(\lambda - \mathcal{L}_\theta)^{-\frac{1}{2\theta}} \mathcal{G}_-^{N,M} \varphi\|^2 \\ &\lesssim \sum_{n \geq 0} \frac{n! n^2 (n+1)^2}{M^{(n+2)d}} \sum_{(l,k)_{1:n}} \frac{\lambda_{N,\theta}^2 |\hat{\rho}^N(k_n)|^2}{(\lambda + |k_1|^{2\theta} + \dots + |k_n|^{2\theta})^{\frac{1}{\theta}}} \\ &\quad \cdot \left| \sum_i \sum_{\substack{p+q=k_n \\ |p|, |q| \leq N}} p l_n \hat{\varphi}_{n+1}((i, p), (i, q), \dots) - k_{n,i} \hat{\varphi}_{n+1}((i_{n-1}, p), (i, q), \dots) \right|^2 \\ &\lesssim \sum_{n \geq 0} \frac{n! n^2 (n+1)^2}{M^{(n+2)d}} \sum_{(l,k)_{1:n,i}} \frac{\lambda_{N,\theta}^2 |\hat{\rho}^N(k_n)|^2 |k_n|^2 \sum_{\substack{p+q=k_n \\ |p|, |q| \leq N}} |\hat{\varphi}_{n+1}(p, q)|^2}{(\lambda + |k_1|^{2\theta} + \dots + |k_n|^{2\theta})^{\frac{1}{\theta}}} \\ &\lesssim \sum_{n \geq 0} \frac{(n+1)! n^3}{M^{(n+1)d}} \sum_{(l,k)_{1:n}} \lambda_{N,\theta}^2 |\hat{\rho}^N(k_n)|^2 \frac{1}{M^d} \sum_{\substack{p+q=k_n \\ |p|, |q| \leq N}} |\hat{\varphi}_{n+1}(p, q, \dots)|^2. \end{aligned} \quad (5.3.25)$$

We note furthermore that

$$\begin{aligned} &\frac{1}{M^d} \sum_{\substack{p+q=k_n \\ |p|, |q| \leq N}} |\hat{\varphi}_{n+1}(p, q, \dots)|^2 \\ &\lesssim \frac{1}{M^d} \sum_{\substack{p+q=k_n \\ |p|, |q| \leq N}} (|p|^{2\theta} + |q|^{2\theta})^{\frac{1}{\theta}-2} \sum_{p+q=k_n} (|p|^{2\theta} + |q|^{2\theta})^{2-\frac{1}{\theta}} |\hat{\varphi}_{n+1}(p, q, \dots)|^2 \\ &\lesssim N^{d+2-4\theta} \sum_{p+q=k_n} (|p|^{2\theta} + |q|^{2\theta})^{2-\frac{1}{\theta}} |\hat{\varphi}_{n+1}(p, q, \dots)|^2, \end{aligned}$$

which gives the result since since  $\lambda_{N,\theta}^2 = N^{4\theta-d-2}$ . For the non-uniform estimation, we calculate as follows

$$\begin{aligned}
& \|\tilde{\mathcal{G}}_+^N \varphi\|^2 \\
& \lesssim \sum_{n \geq 0} \sum_{(l,k)_{1:n}} |\mathcal{R}_{k_n, k_{n-1}}^N|^2 \frac{n!(n-1)^2}{M^{nd}} |k_n + k_{n-1}|^2 |\hat{\varphi}_{n-1}((l_{n-1}, k_n + k_{n-1}), \dots)|^2 \\
& \lesssim \sum_{n \geq 0} \sum_{(l,k)_{1:n-1}} \frac{n!(n-1)^2}{M^{(n-1)d}} N^{d+2} |\hat{\varphi}_{n-1}((l_{n-1}, k_{n-1}), \dots)|^2 \\
& \lesssim_N \|\mathcal{N}^{\frac{3}{2}} \varphi\|^2.
\end{aligned} \tag{5.3.26}$$

And for the operator  $\tilde{\mathcal{G}}_-$ , we have

$$\begin{aligned}
& \|\tilde{\mathcal{G}}_-^N \varphi\|^2 \\
& \lesssim \sum_{n \geq 0} \frac{n!n^2(n+1)^2}{M^{(n+2)d}} \sum_{(l,k)_{1:n}} |\hat{\rho}^N(k_n)|^2 \\
& \quad \cdot \left| \sum_i \sum_{\substack{p+q=k_n \\ |p|, |q| \leq N}} p_{l_n} \hat{\varphi}_{n+1}((i, p), (i, q), \dots) - k_{n,i} \hat{\varphi}_{n+1}((i_{n-1}, p), (i, q), \dots) \right|^2 \\
& \lesssim_N \sum_{n \geq 0} \frac{n!n^2(n+1)^2}{M^{(n+1)d}} \sum_{(l,k)_{1:n+1}} |\hat{\varphi}_{n+1}|^2 \lesssim \|\mathcal{N}^{\frac{3}{2}} \varphi\|^2.
\end{aligned} \tag{5.3.27}$$

□

**Remark 5.3.7.** *The dependence of the implicit constant in (5.3.18) and (5.3.19) for  $\theta > 1$  should also appear in equation (11) of [53]. Replacing the estimate  $\sum_{i=1}^n |k_i|^2 \lesssim_M \sum_{i=1}^n |k_i|^{2\theta}$  by Hölder's inequality, we can get a uniform-in- $M$  estimate with  $\mathcal{N}^{\frac{3}{2}-\frac{1}{2\theta}}$  on the right hand side, which therefore extends to  $M = \infty$ . But to prove well-posedness we need  $\mathcal{N}^\gamma$  with some  $\gamma \leq 1$ , which requires  $\theta \leq 1$ . Therefore, the proof of well-posedness of energy solutions to the 2d hyperviscous Navier-Stokes equation in the plane in [53] seems incorrect. The more precise estimations in [11, Lemma 2.4] would give a bound with  $\mathcal{N}^{\frac{3}{2}-\frac{1}{\theta}}$ , thus  $\theta \leq 2$  would work. For  $\theta > 2$  we are currently lacking an argument to prove well-posedness in  $\mathbb{R}^d$ .*

## 5.4 Existence and uniqueness of energy solutions for $\theta > \frac{d}{2}$

To introduce energy solutions, we first introduce the crucial energy estimate.

**Definition 5.4.1.** We say that a process  $(u_t)_{t \geq 0}$  with trajectories in  $C(\mathbb{R}_+, \mathcal{S}'_f(\mathbb{M}))$  satisfies an energy estimate if for all  $p \geq 1$  there exists  $c_{2p} > 0$  (with  $c_2 = 1$ ) such that for all cylinder functions  $F$

$$\mathbf{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t F(s, u_s) ds \right|^p \right] \lesssim T^{\frac{p}{2}} \sup_{0 \leq s \leq T} \|c_{2p}^N (1 - \mathcal{L}_\theta)^{-\frac{1}{2}} F(s)\|^p. \quad (5.4.1)$$

**Lemma 5.4.2.** Let  $\theta > 0$  and let  $\nu \ll \mu$  be a probability measure with  $\eta = \frac{d\nu}{d\mu} \in L^2(\mu^M)$ . Let  $u^{N,M}$  be the solution to the equation (5.3.3) with initial condition  $\nu$ , then  $u^{N,M}$  satisfies an energy inequality with implicit constant that does not depend on  $N$ .

*Proof.* This follows by the Itô trick using the same arguments as in Section 4.1 of [52].  $\square$

**Remark 5.4.3.** The energy estimate holds also for the case  $M = \infty$  if we have the existence of cylinder function martingale solutions of the stochastic Navier-Stokes equation on the whole space.

As a simple consequence of the energy estimate we can make sense of additive functionals with distributional coefficients:

**Lemma 5.4.4.** For any horizon  $T$ , assume that a process  $(u_t)_{t \geq 0}$  satisfies (5.4.1) and let  $I_t(F) = \int_0^t F(u_s) ds$  for all cylinder functions  $F$  and  $t \in [0, T]$ . Then the map  $I$  can be uniquely extended as a bounded linear functional from  $\mathcal{H}_0^{-\frac{1}{2}}$  to  $L^1(\Omega, \mathcal{F}, \mathbb{P}, C[0, T])$ .

*Proof.* We simply apply (5.4.1) with  $p = 1$  and use that cylinder functions are dense in  $\mathcal{H}_0^{-\frac{1}{2}}$ .  $\square$

If  $\theta > \frac{d}{2}$  and  $\zeta = 1$ , then  $\lambda_\theta^\zeta = \frac{d}{4\theta} + \frac{1}{2} < 1$  in equations (5.3.18) and (5.3.19). Taking  $\beta \in (\frac{d}{4\theta}, \frac{1}{2})$  and  $\omega(n) = \frac{1}{1+n}$  and using that  $\mathcal{H}_\gamma^\alpha \subset \mathcal{H}_{\gamma'}^{\alpha'}$  for  $\alpha \geq \alpha'$  and  $\gamma \geq \gamma'$ , we obtain

$$\tilde{\mathcal{G}}_-^N : \mathcal{H}_1^{\frac{1}{2}} \rightarrow \mathcal{H}_0^{-\frac{1}{2}}, \quad \tilde{\mathcal{G}}_+^N : \mathcal{H}_1^{\frac{1}{2}} \rightarrow \mathcal{H}_0^{-\frac{1}{2}}. \quad (5.4.2)$$

with bounds on the operator norms that are uniform in  $N$ . With these two bounds, we obtain the existence of the integral

$$\int_0^t \tilde{\mathcal{L}} F(u_s) ds := \lim_{N \rightarrow \infty} \int_0^t (\mathcal{L}_\theta + \tilde{\mathcal{G}}^N) F(u_s) ds,$$

for all  $(u_s)$  satisfying the energy estimate (5.4.1) and for all cylinder functions  $F$ .

We now introduce two different martingale problems for our equation.

**Definition 5.4.5.** A process  $(u_t)_{t \geq 0}$  with trajectories in  $C(\mathbb{R}_+, \mathcal{S}'_f(\mathbb{M}))$  is an energy solution for  $\tilde{\mathcal{L}}$  with initial distribution  $\nu$  if  $u_0 \sim \nu$  and if the following conditions are satisfied:

- $(u_t)_{t \geq 0}$  is incompressible, i.e. for all  $T > 0$  there exists  $C(T) > 0$  with

$$\sup_{0 \leq t \leq T} \mathbb{E}|F(u_t)| \leq C(T)\|F\|,$$

for all cylinder functions  $F$ ;

- $u$  satisfies an energy estimate;
- for any cylinder function  $F$ , the process

$$M_t^F := F(u_t) - F(u_0) - \int_0^t \tilde{\mathcal{L}}F(u_s) ds, \quad t \geq 0,$$

is a continuous martingale with quadratic variation

$$\langle M_t^F \rangle = \int_0^t \mathcal{E}(F)(u_s) ds, \quad \text{with } \mathcal{E}(F) := 2 \int_{\mathbb{M}} |A_x^{\frac{\theta}{2}} D_x F|^2 dx,$$

where  $D_x$  is the Malliavin derivative.

For cylinder functions  $F$ , we saw that  $\tilde{\mathcal{L}}F \in \mathcal{H}_0^{-1/2}$  is only a distribution. But we will discuss below that there is a domain  $\mathcal{D}(\tilde{\mathcal{L}})$  such that  $\tilde{\mathcal{L}}\varphi \in \mathcal{H}_0^0$  for all  $\varphi \in \mathcal{D}(\tilde{\mathcal{L}})$ .

**Definition 5.4.6.** A process  $(u_t)_{t \geq 0}$  with trajectories in  $C(\mathbb{R}_+, \mathcal{S}'_f(\mathbb{M}))$  solves the martingale problem for  $\tilde{\mathcal{L}}$  with initial distribution  $\nu$  if  $u_0 \sim \nu$ ,  $\text{law}(u_t) \ll \mu^M$  for all  $t \geq 0$  and if for all  $\varphi \in \mathcal{D}(\tilde{\mathcal{L}})$  and  $t \geq 0$  we have  $\int_0^t |\tilde{\mathcal{L}}\varphi(u_s)| ds < \infty$  almost surely and the process

$$\varphi(u_t) - \varphi(u_0) - \int_0^t \tilde{\mathcal{L}}\varphi(u_s) ds, \quad t \geq 0,$$

is a martingale in the filtration generated by  $(u_t)_{t \geq 0}$ .

To construct the domain  $\mathcal{D}(\tilde{\mathcal{L}})$ , we follow the approach developed in Lukas Gräfner's Ph.D. thesis, which is presented in the lecture notes [44]. We need to show that  $\mathcal{G}$  is a bounded operator from  $\mathcal{H}_1^{1/2}$  to  $\mathcal{H}_0^{-1/2}$  (which follows from (5.4.2)) and that the commutator  $[\mathcal{N}, \mathcal{G}]$  is a bounded operator from  $\mathcal{H}_0^{1/2}$  to  $\mathcal{H}_{-1}^{-1/2}$ . The commutator estimate follows in a similar way as (5.4.2), using  $\omega(x) = 1$  this time as well as the fact that  $\tilde{\mathcal{G}}_+$  and  $\tilde{\mathcal{G}}_-$  only increase/decrease the chaos by one. Thus, we can apply Theorem 4.11 and Theorem 4.15 in [44] to obtain uniqueness for the two martingale problems and the existence of the domain  $(\mathcal{D}(\tilde{\mathcal{L}}), \tilde{\mathcal{L}})$  on which  $\tilde{\mathcal{L}}$  generates a contraction semigroup. Now we are ready to finish the proof of Theorem 5.1.1.

*Proof of Theorem 5.1.1.* From Theorem 4.15 in [44], it remains to show the existence of energy solutions to (5.3.3). The arguments are similar to Theorem 4.6 in [52] or Theorem 1 in [53]. Thus, we give only a sketch of the proof here.

1. We start with the solution  $u^N$  of the martingale problem of  $\mathcal{L}^N$ , with the same initial distribution as for  $u$ . Given a cylinder function  $F$  we consider the martingale

$$M_t^F := F(u_t^N) - F(u_0^N) - \int_0^t (\mathcal{L}_\theta + \tilde{\mathcal{G}}^N)F(u_s^N)ds, \quad t \geq 0,$$

with quadratic variation  $\int_0^t \mathcal{E}(F)(u_s^N)$ . We furthermore obtain the equality

$$\|\omega(\mathcal{N})(\mathcal{E}(F))^{\frac{1}{2}}\|^2 = 2\|\omega(\mathcal{N} - 1)(-\mathcal{L}_\theta)^{\frac{1}{2}}F\|^2.$$

2. From the decomposition

$$F(u_t^N) - F(u_s^N) = \int_s^t (\mathcal{L}_\theta + \tilde{\mathcal{G}}^N)(u_r^N)dr + M_t^F - M_s^F,$$

we can obtain the estimation

$$\mathbb{E}[|F(u_t^N) - F(u_s^N)|^p] \leq (t - s)^{\frac{p}{2}} \|c_{4p}^N (1 - \mathcal{L}_\theta)^{-\frac{1}{2}} F\|^p.$$

Thus, we can apply the Kolomogorov and Mitoma's criterion [72] to get the tightness of the sequence  $(u^N)_N$  in  $C(\mathbb{R}_+, \mathcal{S}'_f)$ .

3. The incompressibility and energy inequality for  $u^N$  follow directly from the Cauchy-Schwarz inequality and Lemma 5.4.2, respectively. The implicit constants are independent of  $N$ , and therefore any weak limit satisfies the same bounds.
4. Using the uniform estimates in (5.4.2), we obtain that the  $L^1$  distance between  $\int_0^t (\mathcal{L}_\theta + \tilde{\mathcal{G}}^N)F(u_s^N)ds$  and  $\int_0^t \tilde{\mathcal{L}}F(u_s^N)ds$  is  $o(1)$  with respect to  $N$ . Thus, and limit point of  $(u^N)_N$  as  $N \rightarrow \infty$  satisfies also the third point in the definition of an energy solution.

□

## 5.5 Solution on the whole space for $\theta \leq 1$ and $N < \infty$

We consider here the truncated equation on the whole space  $\mathbb{R}^d$ , i.e. when  $M = \infty$ , for  $\theta \leq 1$ . To begin with, we derive the limit of the stationary distributions  $\mu^M$ .

**Proposition 5.5.1.** *The periodic extension  $\tilde{\mu}^M$  of the white noise  $\mu^M$  to the whole space converges weakly to  $\mu$  as  $M \rightarrow \infty$  as Gaussian random variables on  $\mathcal{S}'_f(\mathbb{R}^d)$ .*

*Proof.* It is sufficient to prove that for each  $f \in \mathcal{S}_f(\mathbb{R}^d)$ , the random variable  $\tilde{\mu}^M(f)$  converges weakly to  $\mu(f)$ . Since both are centered Gaussian, it suffices to consider the variance:

$$\mathbb{E}[\tilde{\mu}^M(f)^2] = \int_{[-\frac{M}{2}, \frac{M}{2}]^d} |f_M|^2 dx - \frac{1}{M^d} \left( \int_{\mathbb{R}^d} f dx \right)^2,$$

where  $f_M(x) = \sum_{k \in \mathbb{Z}^d} f(x + kM)$ . As  $M \rightarrow \infty$ , the second term vanishes and the first term converges to  $\|f\|_{\mathbb{R}^d}^2$  by the dominated convergence theorem. Thus we get the result.  $\square$

To show the existence of weak solution to (5.3.3) when  $M = \infty$ , we need some estimations.

**Lemma 5.5.2.** *Let  $\theta \leq 1$  and consider a weak solution  $u^{N,M}$  to the equation (5.3.3). The following bounds hold uniformly in  $M > 0$ : For all  $\varphi \in \mathcal{S}_f(\mathbb{M})$*

$$\mathbf{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t u^{N,M}(A^\theta \varphi) ds \right|^p \right] \lesssim T^{\frac{p}{2}} \|\varphi\|_{H^\theta}^p, \quad (5.5.1)$$

and

$$\mathbf{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t B^N(u_s^{N,M})(\varphi) ds \right|^p \right] \lesssim N^{\frac{d-2\theta}{2} \cdot p} T^{\frac{p}{2}} \|\varphi\|_{H^1}^p. \quad (5.5.2)$$

*Proof.* From Lemma 5.4.2, we have

$$\mathbf{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t u^{N,M}(A^\theta \varphi) ds \right|^p \right] \lesssim T^{\frac{p}{2}} \|c_{2p}^N (\lambda - \mathcal{L}_\theta)^{-\frac{1}{2}} (A^\theta \varphi)\|^p \lesssim T^{\frac{p}{2}} \|\varphi\|_{H^\theta}^p.$$

Notice that for the functional  $B^N$ , we have

$$B^N(u^{N,M})(\varphi) = \tilde{\mathcal{G}}_+^N \varphi. \quad (5.5.3)$$

Then, we obtain from Theorem 5.3.6

$$\mathbf{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t B^N(u_s^{N,M})(\varphi) ds \right|^p \right] \lesssim T^{\frac{p}{2}} \|c_{2p}^N (\lambda - \mathcal{L}_\theta)^{-\frac{1}{2}} (\tilde{\mathcal{G}}_+^N \varphi)\|^p \lesssim N^{\frac{d-2\theta}{2} \cdot p} T^{\frac{p}{2}} \|\varphi\|_{H^1}^p.$$

$\square$

We are now able to show the existence of solutions.

**Definition 5.5.3.** Let  $\nu \ll \mu$ . We call a process  $u^N$  with trajectories in  $C(\mathbb{R}_+, \mathcal{S}'_f(\mathbb{R}^d))$  a weak solution of equation (5.3.3) with initial condition  $\nu$  if for all  $\varphi \in \mathcal{S}_f(\mathbb{R}^d)$

$$u_t^N(\varphi) = \nu(\varphi) - \int_0^t u_s^N(A^\theta \varphi) ds + \lambda_{N,\theta} \int_0^t B^N(u_s^N)(\varphi) ds + M_t^\varphi,$$

where  $M^\varphi$  is a continuous martingale with quadratic covariation

$$\mathbb{E}[M_t^\varphi M_s^\phi] = 2(t \wedge s) \langle A^\theta \Pi \varphi, A^\theta \Pi \phi \rangle_{\mathbb{H}},$$

for all  $\phi, \varphi \in \mathcal{S}_f(\mathbb{R}^d)$ . A weak solution is called an energy solution if it is also incompressible and it satisfies an energy estimate, as in Definition 5.4.5 but with respect to  $\mu$  instead of  $\mu^M$ .

**Theorem 5.5.4.** Let  $\theta \in (0, 1]$  and let  $\eta \in \Gamma L^2$  with  $\eta \geq 0$  and  $\int \eta d\mu = 1$ . There exists a unique in law energy solution to the equation (5.3.3) for  $M = \infty$  with initial condition  $d\nu = \eta d\mu$ .

*Proof.* The existence proof is basically the same as the proof given in [12] for the vorticity formulation of the two-dimensional case and very similar to the proof of Theorem 5.1.1. From the uniform-in- $M$  estimates in Lemma 5.5.2 and the Burkholder-Davis-Gundy inequality for the martingale, we deduce the tightness of  $u^{N,M}$  by Kolomorgov's criterion. Because  $u^{N,M}$  are energy solutions, we obtain the energy solution property of  $u^N$  directly by taking the limit.

The operator  $\mathcal{L}^N = \mathcal{L} + \mathcal{G}^N$  on the whole space has the same expression in Fourier modes as on the torus (see Theorem 5.3.5), if we now consider frequencies in  $\mathbb{R}^d$ . Because  $\theta \leq 1$ , the operator  $\mathcal{G}^N$  satisfies the same bounds as in Theorem 5.3.6. Uniqueness in law then follows again by Theorem 4.15 in [44].  $\square$

## 5.6 Large-scale behavior for $\theta \leq 1$

### 5.6.1 Limiting behaviour

Now we consider  $M = 1$  or  $M = \infty$ , representing the torus or the whole space,  $\theta \leq 1$ , and  $N \rightarrow \infty$ . For  $\theta < 1$ , the limit is trivial:

**Theorem 5.6.1.** For  $\theta < 1$ , let  $u^N$  be an energy solution to equation (5.3.3) with initial distribution  $\nu \ll \mu$  and  $\frac{d\nu}{d\mu} \in L^2(\mu)$ . As  $N \rightarrow \infty$ ,  $u^N$  converges to the unique-in-law weak solution of

$$\partial_t u = -(-\Delta)^\theta u + (-\Delta)^{\frac{\theta}{2}} \Pi \xi, \quad u_0 \sim \nu. \quad (5.6.1)$$

*Proof.* For any test function  $\varphi \in \mathcal{S}_f(\mathbb{R}^d)$ , the non-linear term is bounded by

$$\mathbf{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \lambda_{N,\theta} B^N(u_s^N)(\varphi) ds \right|^p \right] \lesssim N^{p(\theta-1)} T^{\frac{p}{2}} \|\varphi\|_{H^1}^p.$$

Since  $\theta < 1$ , this term will vanish when taking the limit  $N \rightarrow \infty$ . For the linear term, notice that we have tightness of  $u^N$  by similar arguments as above. Since the linear part of the equation does not depend on  $N$ , any limit solves (5.6.1). Uniqueness in law follows by similar arguments as for energy solutions but with  $\tilde{\mathcal{G}} = 0$ , or by testing against the heat kernel generated by  $-(-\Delta)^\theta$ .  $\square$

We now turn to the case  $\theta = 1$ . In this case, we have the following upper bound for the non-linear part:

$$\mathbf{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \lambda_{N,1} B^N(u_s^N)(\varphi) ds \right|^p \right] \lesssim T^{\frac{p}{2}} \|\varphi\|_{H^1}^p,$$

from where we can prove tightness as before. Following [8], the aim is then to derive a lower bound for the nonlinear part to show that its limit is non-zero. We first give a lemma.

**Lemma 5.6.2.** *Suppose  $a, b$  are vectors in a Hilbert space  $E$ , then we have*

$$2\|a\|^2\|b\|^2 \leq \|a \otimes b + b \otimes a\|^2 \leq 4\|a\|^2\|b\|^2.$$

*Proof.* The second inequality is a direct consequence of the triangle inequality. For the first inequality, we define  $e_A = \frac{a}{\|a\|}$  and let  $P_{e_A}$  be the orthogonal projection on the span of  $e_A$ . Then we have

$$e_A \otimes b + b \otimes e_A = e_A \otimes (b - P_{e_A}(b)) + (b - P_{e_A}(b)) \otimes e_A + 2\|P_{e_A}(b)\|(e_A \otimes e_A).$$

Now let's view  $e_A, \frac{b - P_{e_A}(b)}{\|b - P_{e_A}(b)\|}$  to be part of the basis of the Hilbert space  $E$ . Then we know that

$$e_A \otimes e_A, \left( \frac{b - P_{e_A}(b)}{\|b - P_{e_A}(b)\|} \right) \otimes e_A, e_A \otimes \left( \frac{b - P_{e_A}(b)}{\|b - P_{e_A}(b)\|} \right)$$

are orthogonal vectors of tensor space  $E^{\otimes 2}$ . Thus,

$$\|a \otimes b + b \otimes a\|^2 = \|a\|^2(2\|b - P_{e_A}(b)\|^2 + 4\|P_{e_A}(b)\|^2) \geq 2\|a\|^2\|b\|^2.$$

$\square$

**Proposition 5.6.3.** *Let  $\theta = 1$ ,  $\varphi \in \mathcal{S}_f$ , and let  $u^N$  be the stationary solution to the equation (5.3.3), there exists a constant  $C > 0$  such that*

$$\frac{C^{-1}}{\lambda^2} \|\varphi\|_{H^1}^2 \leq \int_0^\infty e^{-\lambda t} \mathbf{E} \left[ \left| \int_0^t \lambda_{N,1} B^N(u_s^N)(\varphi) ds \right|^2 \right] dt \lesssim \frac{C}{\lambda^2} \|\varphi\|_{H^1}^2.$$

*Proof.* We already showed the upper bound. For the lower bound, we apply Lemma 5.1 in [8] to rewrite

$$A = \int_0^\infty e^{-\lambda t} \mathbf{E} \left[ \left| \int_0^t \lambda_{N,1} B^N(u_s^N)(\varphi) ds \right|^2 \right] dt = \frac{2}{\lambda^2} \mathbb{E}[\lambda_{N,1} B^N(\varphi)(\lambda - \mathcal{L}_1^N)^{-1} \lambda_{N,1} B^N(\varphi)],$$

where  $\lambda_{N,1} B^N(\varphi) = \mathcal{G}_+^N \varphi$  lives in the second Wiener chaos. Lemma 5.2 in the same paper gives a variational representation of the right hand side:

$$A = \frac{2}{\lambda^2} \sup_G \{ 2\langle \mathcal{G}_+^N \varphi, G \rangle - \langle (\lambda - \mathcal{L}_1)G, G \rangle - \langle \mathcal{G}^N G, (\lambda - \mathcal{L}_1)^{-1} \mathcal{G}^N G \rangle \}.$$

For the third term, we divide  $\mathcal{G}^N = \mathcal{G}_-^N + \mathcal{G}_+^N$  and use the fact  $\mathcal{G}^N$  is anti-symmetric to obtain

$$\frac{2}{\lambda^2} \sup_G \left\{ 2\langle \mathcal{G}_+^N \varphi, G \rangle - \|(\lambda - \mathcal{L}_1)^{\frac{1}{2}} G\|^2 - \|(\lambda - \mathcal{L}_1)^{-\frac{1}{2}} \mathcal{G}_+^N G\|^2 - \|(\lambda - \mathcal{L}_1)^{-\frac{1}{2}} \mathcal{G}_-^N G\|^2 \right\}.$$

From now on we restrict to  $G$  in the second chaos. Then we have by Theorem 5.3.6

$$\|(\lambda - \mathcal{L}_1)^{-\frac{1}{2}} \mathcal{G}_+^N G\|^2 + \|(\lambda - \mathcal{L}_1)^{-\frac{1}{2}} \mathcal{G}_-^N G\|^2 \leq C \|(\lambda - \mathcal{L}_1)^{\frac{1}{2}} G\|^2,$$

and thus

$$A \geq \frac{2}{\lambda^2} \sup_G \left\{ 2\langle \mathcal{G}_+^N \varphi, G \rangle - (1 + C) \|(\lambda - \mathcal{L}_1)^{\frac{1}{2}} G\|^2 \right\}.$$

For

$$G = \delta(\lambda - \mathcal{L}_1)^{-1} \mathcal{G}_+^N \varphi,$$

for  $\delta > 0$ , we thus obtain the lower bound

$$\int_0^\infty e^{-\lambda t} \mathbf{E} \left[ \left| \int_0^t B^N(u_s^N)(\varphi) ds \right|^2 \right] dt \geq \frac{2}{\lambda^2} (2\delta - \delta^2(1 + C)) \|(\lambda - \mathcal{L}_1)^{-\frac{1}{2}} \mathcal{G}_+^N \varphi\|^2, \quad (5.6.2)$$

and for sufficiently small  $\delta > 0$  the prefactor on the right hand side becomes positive. It remains to give a lower bound for  $\|(\lambda - \mathcal{L}_1)^{-\frac{1}{2}} \mathcal{G}_+^N \varphi\|^2$ . Define

$$U(k, \hat{\varphi}(k)) := k \otimes \hat{\varphi}(k) + \hat{\varphi}(k) \otimes k,$$

and its component by

$$U_{i,j}(k, \hat{\varphi}(k)) := k_i \hat{\varphi}(j, k) + \hat{\varphi}(i, k) k_j.$$

Let  $\hat{\Pi}$  be the Leray projection in the Fourier mode. We have (note that for  $M = \infty$  we have to replace all sums by integrals)

$$\begin{aligned} \|(\lambda - \mathcal{L}_1)^{-\frac{1}{2}} \mathcal{G}_+^N \varphi\|^2 &\gtrsim \frac{\lambda_{N,1}^2}{M^{2d}} \sum_{l_1, l_2, k_1, k_2} (\lambda + |k_1|^2 + |k_2|^2)^{-1} |\mathcal{R}_{k_1, k_2}^N|^2 \\ &\quad \cdot \left| \sum_{i,j} \hat{\Pi}_{l_1, i}(k_1) \hat{\Pi}_{l_2, j}(k_2) U_{i,j}(k_1 + k_2, \hat{\varphi}(k_1 + k_2)) \right|^2 \end{aligned}$$

For each  $k_1 + k_2 \neq 0$  and  $\hat{\varphi}(k_1 + k_2) \neq 0$ , we know  $\|U(k_1 + k_2, \hat{\varphi}(k_1 + k_2))\| \neq 0$  and define

$$\begin{aligned} \eta(k_1, k_2, \hat{\varphi}(k_1 + k_2))^2 &= \frac{\sum_{l_1, l_2} |\sum_{i, j} \hat{\Pi}_{l_1, i}(k_1) \hat{\Pi}_{l_2, j}(k_2) U_{i, j}(k_1 + k_2, \hat{\varphi}(k_1 + k_2))|^2}{\sum_{l_1, l_2} |U(l_1, l_2, k_1 + k_2)|^2} \\ &= \frac{\|(\hat{\Pi}(k_1)) \otimes (\hat{\Pi}(k_2))(U(k_1 + k_2, \hat{\varphi}(k_1 + k_2)))\|^2}{\|U(k_1 + k_2, \hat{\varphi}(k_1 + k_2))\|^2}. \end{aligned}$$

The lera projection can ensure that  $1 \geq \eta(k_1, k_2, \hat{\varphi}(k_1 + k_2)) \geq 0$  in this case. Now since

$$\hat{\Pi}(k) = \hat{\Pi}(\tilde{\lambda}k),$$

for any non-zero  $\tilde{\lambda} \in \mathbb{R}$  and non zero vector  $k \in \mathbb{R}^d$ , we know that

$$\eta(k_1, k_2, \hat{\varphi}(k_1 + k_2))^2 = \eta(\tilde{\lambda}k_1, \tilde{\lambda}k_2, \hat{\varphi}(k_1 + k_2))^2, \quad (5.6.3)$$

for any non-zero  $\tilde{\lambda} \in \mathbb{R}$ .

For all other  $k_1, k_2$ , we define  $\eta(k_1, k_2, \hat{\varphi}(k_1 + k_2)) = 1$ . Then equation (5.6.3) holds for all  $k_1, k_2$ . We then obtain

$$\begin{aligned} \|(\lambda - \mathcal{L}_1)^{-\frac{1}{2}} \mathcal{G}_+^N \varphi\|^2 &\gtrsim \frac{\lambda_{N,1}^2}{M^{2d}} \sum_{k_1, k_2} (\lambda + |k_1|^2 + |k_2|^2)^{-1} |\mathcal{R}_{k_1, k_2}^N|^2 \\ &\quad \cdot \sum_{l_1, l_2} |U_{i, j}(k, \hat{\varphi}(k))|^2 \eta(k_1, k_2, \hat{\varphi}(k_1 + k_2))^2. \\ &\geq \frac{\lambda_{N,1}^2}{M^{2d}} \sum_{k_1, k_2} \frac{|\mathcal{R}_{k_1, k_2}^N|^2 \eta(k_1, k_2, \hat{\varphi}(k_1 + k_2))^2 |k_1 + k_2|^2 |\hat{\varphi}(k_1 + k_2)|^2}{\lambda + |k_1|^2 + |k_2|^2}. \end{aligned}$$

For each  $N, k$ , define  $k_N = \frac{k}{N}$  and

$$\vartheta^N(k, \hat{\varphi}(k)) = \sum_{l+m=k} |\mathcal{R}_{l, m}^N| \frac{N^{2-d} \eta(l, m, \hat{\varphi}(k))^2}{\lambda + |l|^2 + |m|^2}.$$

Then the lower bound can be expressed as

$$\int_{\mathbb{Z}_M^d} \vartheta^N(k, \hat{\varphi}(k)) |k|^2 |\hat{\varphi}(k)|^2 dk.$$

We define additionally

$$\Theta_r^N(k_N, \hat{\varphi}(k)) = \int \frac{\mathbb{1}_{B(0, r) \cap B(k_N, r)} \eta(Nx, k - Nx, \hat{\varphi}(k))^2}{\frac{\lambda}{N^2} + |x|^2 + |k_N - x|^2} dx.$$

Let us check the limit, by applying Lemma 5.6.2 and equation (5.6.3):

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \eta(Nx, k - Nx, \hat{\varphi}(k))^2 \\
&= \lim_{N \rightarrow \infty} \eta(x, k_N - x, \hat{\varphi}(k))^2 \\
&= \lim_{N \rightarrow \infty} \frac{\|\hat{\mathbf{\Pi}}(x) \otimes \hat{\mathbf{\Pi}}(k_N - x)(k \otimes \hat{\varphi}(k) + \hat{\varphi}(k) \otimes k)\|^2}{\|k \otimes \hat{\varphi}(k) + \hat{\varphi}(k) \otimes k\|^2} \\
&= \frac{\|\hat{\mathbf{\Pi}}(x) \otimes \hat{\mathbf{\Pi}}(-x)(k \otimes \hat{\varphi}(k) + \hat{\varphi}(k) \otimes k)\|^2}{\|k \otimes \hat{\varphi}(k) + \hat{\varphi}(k) \otimes k\|^2} \\
&\geq \frac{\|\hat{\mathbf{\Pi}}(x)(k)\|^2 \|\hat{\mathbf{\Pi}}(x)(\hat{\varphi}(k))\|^2}{2\|k\|^2 \|\hat{\varphi}(k)\|^2} \\
&= \frac{\sin(\theta_1(x))^2 \sin(\theta_2(x))^2}{2},
\end{aligned}$$

where  $\theta_1(x), \theta_2(x)$  are angles between  $x$  and  $k, \hat{\varphi}(k)$  respectively. It is easy to check that for large  $N$ , we have

$$\Theta_1^N(k_N, \hat{\varphi}(k)) \leq \vartheta^N(k, \hat{\varphi}(k)) \leq \Theta_{1+\frac{1}{N}}^N(k_N, \hat{\varphi}(k)),$$

which shows that

$$\begin{aligned}
\lim_{N \rightarrow \infty} \vartheta^N(k, \hat{\varphi}(k)) &\geq \int_{B(0,1)} \frac{\left(1 - \frac{\langle k, x \rangle^2}{\|k\|^2 \|x\|^2}\right) \left(1 - \frac{\langle \hat{\varphi}(k), x \rangle^2}{\|\hat{\varphi}(k)\|^2 \|x\|^2}\right)}{4|x|^2} dx \\
&= \int_{B(0,1)} \frac{\left(1 - \frac{x_1^2}{\|x\|^2}\right) \left(1 - \frac{x_2^2}{\|x\|^2}\right)}{4|x|^2} dx.
\end{aligned}$$

Then the dominated convergence theorem gives the desired lower bound.  $\square$

This proves the non-triviality of the non-linear term for the case  $\theta = 1$ .

**Theorem 5.6.4.** *Let  $\theta = 1$  and let  $u^N$  be the stationary solution to the equation (5.3.3), then  $u^N$  converges weakly to  $u$  satisfying*

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u(\Delta\varphi) + \mathcal{B}(\varphi)_t + M_t^\varphi, \quad (5.6.4)$$

where  $\mathcal{B}(\varphi)_t$  is such that

$$\frac{1}{\lambda^2} \|\varphi\|_{\mathcal{H}^1}^2 \lesssim \int_0^\infty e^{-\lambda t} \mathbf{E}[|\mathcal{B}_t(\varphi)|^2] dt \lesssim \frac{1}{\lambda^2} \|\varphi\|_{\mathcal{H}^1}^2.$$

Although so far we restricted our attention to dimension  $d \geq 3$ , the approach also works in dimension 2 and it yields the same conclusion as in [12], where the vorticity formulation of the same equation is studied, and triviality is shown for  $\theta < 1$ .

Next, we discuss possible approaches to describe the limit  $\mathcal{B}(\varphi)$  in the case  $\theta = 1$ . Using the same method in [9], we are able to determine the weak coupling limit for the two-dimensional stochastic Navier-Stokes equation.

**Theorem 5.6.5.** *Let  $d = 2$ ,  $\theta = 1$  and the support of the Fourier transformation of mollifier  $\rho^1$  is the indicator function of the unit ball. Let  $u^N$  be the stationary solution to the equation*

$$\partial_t u^N = \Delta u^N - \frac{\hat{\lambda}}{\log N} B^N(u^N) + \sqrt{2}(-\Delta)^{\frac{1}{2}} \mathbf{\Pi} \xi, \quad (5.6.5)$$

on the torus  $\mathbb{T}^2$ , then  $u^N$  converges weakly to the stationary solution  $u$  of

$$\partial_t u = \nu_{\text{eff}} \Delta u + \sqrt{2\nu_{\text{eff}}}(-\Delta)^{\frac{1}{2}} \mathbf{\Pi} \xi,$$

where  $\nu_{\text{eff}} = \sqrt{1 + \frac{\hat{\lambda}^2}{2\pi}}$ .

This can be deduced directly from the method introduced in the paper [9]. We only need to insert our formula for the operators  $\mathcal{G}_{\pm}^N$  into the replacement lemma therein to obtain the effective coefficient.

To generalize this result to higher dimensions, the method in [9] can no longer be applied due to the lack of control in the both off-diagonal part and diagonal part used for the proof of replacement lemma. We expect that the methods of [11] can be applied to prove the following result:

**Conjecture 5.6.6.** *Let  $d \geq 3$ ,  $\theta = 1$ ,  $\omega_d = \frac{d\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}$  and assume that the support of the Fourier transformation of the mollifier  $\rho^1$  is the indicator function of the unit ball. Let  $u^N$  be the stationary solution to the equation (5.3.3) on  $\mathbb{T}^d$  or  $\mathbb{R}^d$ , then it converges weakly to the stationary solution  $u$  of*

$$\partial_t u = \nu_{\text{eff}} \Delta u + \sqrt{2\nu_{\text{eff}}}(-\Delta)^{\frac{1}{2}} \mathbf{\Pi} \xi,$$

where  $\nu_{\text{eff}} = \sqrt{1 + \frac{\hat{\lambda}^2 \omega_d}{4\pi^2(d-2)}}$ .

## 5.6.2 Implication for fluctuating hydrodynamics

We revisit the fluctuating hydrodynamics equations introduced by Landau and Lifshitz in [63], which are given by

$$\partial_t u = \nu \Delta u - \nabla p - \hat{\lambda} \operatorname{div}(u \otimes u) + \nabla \cdot \tau, \quad \nabla \cdot u = 0,$$

where the centered Gaussian noise  $\tau$  has covariance

$$\mathbb{E}[\tau_{ij}(t, x) \tau_{kl}(t', x')] = \frac{2\nu k_B T}{\rho} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl}) \delta(x - x') \delta(t - t'). \quad (5.6.6)$$

Let us compute the covariance of  $(\nabla \cdot \tau)_i$  for  $i = 1, \dots, d$ . We have

$$\begin{aligned} & \mathbb{E} \left[ \sum_{j,k} \langle \partial_j \tau_{i_1,j}, \varphi_1 \rangle \langle \partial_k \tau_{i_2,k}, \varphi_2 \rangle \right] = \mathbb{E} \left[ \sum_{j,k} \langle \tau_{i_1,j}, \partial_j \varphi_1 \rangle \langle \tau_{i_2,k}, \partial_k \varphi_2 \rangle \right] \\ &= \int \sum_{j,k} \frac{2\nu k_B T}{\rho} \left( \delta_{i_1 i_2} \delta_{jk} + \delta_{i_1 k} \delta_{j i_2} - \frac{2}{3} \delta_{i_1 j} \delta_{i_2 k} \right) \partial_j \varphi_1(t, x) \partial_k \varphi_2(t, x) dt dx. \end{aligned}$$

So if  $i_1 \neq i_2$ , we have

$$\begin{aligned} & \mathbb{E} \left[ \sum_{j,k} \langle \partial_j \tau_{i_1,j}, \varphi_1 \rangle \langle \partial_k \tau_{i_2,k}, \varphi_2 \rangle \right] \\ &= \int \frac{2\nu k_B T}{\rho} \left( \partial_{i_2} \varphi_1 \partial_{i_1} \varphi_2 - \frac{2}{3} \partial_{i_1} \varphi_1 \partial_{i_2} \varphi_2 \right) dt dx \\ &= \int \frac{2\nu k_B T}{3\rho} \partial_{i_1} \varphi_1 \partial_{i_2} \varphi_2 dt dx \\ &= \frac{2\nu k_B T}{3\rho} \mathbb{E}[\langle \partial_{i_1} \tilde{\xi}, \varphi_1 \rangle \langle \partial_{i_2} \tilde{\xi}, \varphi_2 \rangle], \end{aligned}$$

where  $\tilde{\xi}$  is a real-valued white noise on  $\mathbb{R}_+ \times \mathbb{R}^d$ . When  $i_1 = i_2$ , we have

$$\begin{aligned} & \mathbb{E} \left[ \sum_{j,k} \langle \partial_j \tau_{i_1,j}, \varphi_1 \rangle \langle \partial_k \tau_{i_1,k}, \varphi_2 \rangle \right] \\ &= \int \frac{2\nu k_B T}{\rho} \left( \sum_j \partial_j \varphi_1 \partial_j \varphi_2 + \frac{1}{3} \partial_{i_1} \varphi_1 \partial_{i_1} \varphi_2 \right) dt dx \\ &= \frac{2\nu k_B T}{\rho} \mathbb{E}[\langle (-\Delta)^{\frac{1}{2}} \xi_{i_1}, \varphi_1 \rangle \langle (-\Delta)^{\frac{1}{2}} \xi_{i_1}, \varphi_2 \rangle + \frac{1}{3} \langle \partial_{i_1} \tilde{\xi}, \varphi_1 \rangle \langle \partial_{i_1} \tilde{\xi}, \varphi_2 \rangle], \end{aligned}$$

where  $\xi$  is a  $d$ -dimensional white noise on  $\mathbb{R}_+ \times \mathbb{R}^d$  that is independent of  $\tilde{\xi}$ . Thus, we conclude that

$$\nabla \cdot \tau \stackrel{d}{=} \sqrt{\frac{2\nu k_B T}{\rho}} (-\Delta)^{\frac{1}{2}} \xi + \sqrt{\frac{2\nu k_B T}{3\rho}} \nabla \cdot \tilde{\xi}. \quad (5.6.7)$$

Applying the Leray projection, we find that

$$\mathbf{\Pi} \nabla \cdot \tau \stackrel{d}{=} \sqrt{\frac{2\nu k_B T}{\rho}} (-\Delta)^{\frac{1}{2}} \mathbf{\Pi} \xi.$$

Thus, the solution to the equation of fluctuating hydrodynamics should have the same distribution as the solution to the equation

$$\partial_t u = \nu \Delta u - \nabla p - \hat{\lambda} \operatorname{div}(u \otimes u) + \sqrt{\frac{2\nu k_B T}{\rho}} (-\Delta)^{\frac{1}{2}} \xi, \quad \nabla \cdot u = 0,$$

but of course we have to truncate both equations to make sense of them. Before that, we apply a scaling transformation to normalize the coefficients: Let  $\tilde{u}(t, x) = Au(rt, x)$  with

$$A = \sqrt{\frac{\rho}{k_B T}}, \quad r = \frac{1}{\nu}.$$

The equation for  $\tilde{u}$  becomes

$$\partial_t \tilde{u} = \Delta \tilde{u} - \nabla \tilde{p} - \hat{\lambda} \sqrt{\frac{k_B T}{\rho \nu^2}} \operatorname{div}(\tilde{u} \otimes \tilde{u}) + \sqrt{2}(-\Delta)^{\frac{1}{2}} \xi, \quad \nabla \cdot \tilde{u} = 0,$$

and we consider the same truncation as before with  $\rho^1$  described in the Conjecture 5.6.6 (note that we did not rescale space yet and therefore the truncation appears in the same form in the equation for  $u$ ):

$$\partial_t \tilde{u} = \Delta \tilde{u} - \nabla \tilde{p} - \hat{\lambda} \sqrt{\frac{k_B T}{\rho \nu^2}} \rho^1 * \operatorname{div}(\rho^1 * \tilde{u} \otimes \rho^1 * \tilde{u}) + \sqrt{2}(-\Delta)^{\frac{1}{2}} \xi, \quad \nabla \cdot \tilde{u} = 0.$$

Thus, Conjecture 5.6.6 says that by considering the scaling

$$\hat{u}^N(t, x) = N^{\frac{d}{2}} \tilde{u}(N^2 t, Nx),$$

the limit  $u$  of  $\frac{1}{A} \hat{u}^N(\frac{t}{r}, x)$  will satisfy the equation

$$\begin{aligned} \partial_t u &= \frac{G(\hat{\lambda})}{r} \Delta u + \frac{\sqrt{2G(\hat{\lambda})}}{Ar^{\frac{1}{2}}} \mathbf{\Pi}(-\Delta)^{\frac{1}{2}} \xi \\ &= G(\hat{\lambda}) \nu \Delta u + \sqrt{G(\hat{\lambda})} \sqrt{\frac{2\nu k_B T}{\rho}} \mathbf{\Pi}(-\Delta)^{\frac{1}{2}} \xi, \end{aligned}$$

where

$$G(\hat{\lambda}) = \sqrt{1 + \frac{\hat{\lambda}^2 k_B T \omega_d}{4\nu^2 \rho \pi^2 (d-2)}}.$$

# Appendix A

## Appendix of Chapter 2

### A.1 Properties of Fractional Derivatives

We recall here some basic properties of fractional derivatives used in our proofs. Proofs may be found in [83].

**Proposition A.1.1.** *Suppose  $f$  is a real function,  $\alpha, \beta$  are two real numbers. We will also use the convention that  $I_{a+}^\alpha = D_{a+}^{-\alpha}$ . They are well-defined since there is always one non-negative superscript. Then we have*

1. *Fractional integration operators  $\{I_{a+}^\alpha, \alpha \geq 0\}$  ( $\{I_{b-}^\alpha, \alpha \geq 0\}$ ) form a semigroup in  $L^p(a, b)$  for every  $p \geq 1$ . It is continuous in uniform topology for  $\alpha > 0$  and strongly continuous for  $\alpha \geq 0$ .*

2. *In each of the following cases:*

- $\beta \geq 0, \alpha + \beta \geq 0, f \in L^1([a, b])$ .
- $\beta \leq 0, \alpha \geq 0, f \in I_{a+}^{-\beta}(L^1)(I_{b-}^{-\beta}(L^1))$ .
- $\alpha \leq 0, \alpha + \beta \leq 0, f \in I_{a+}^{-\alpha-\beta}(L_1)(I_{b-}^{-\alpha-\beta}(L_1))$ .

*we have*

$$I_{a+}^\alpha \circ I_{a+}^\beta f = I_{a+}^{\alpha+\beta} f, \quad I_{b-}^\alpha \circ I_{b-}^\beta f = I_{b-}^{\alpha+\beta} f.$$

3. *If  $n \leq \alpha < n + 1$  and  $f \in L^1(a, y)$  and  $D_{a+}^\alpha f$  exists on  $(a, y)$  for some  $y > x \in [a, b]$ , then we have*

$$I_{a+}^\alpha D_{a+}^\alpha f(x) = f(x) - \sum_{j=1}^{n+1} \frac{D_{a+}^{\alpha-j} f(a)}{\Gamma(\alpha - j - 1)} (x - a)^{\alpha-j}.$$

Similarly,

$$I_{b^-}^\alpha D_{b^-}^\alpha f(x) = f(x) - \sum_{j=1}^{n+1} \frac{(-1)^j D_{b^-}^{\alpha-j} f(b)}{\Gamma(\alpha - j - 1)} (b - x)^{\alpha-j}.$$

Here  $D_{a^+}^\kappa f(a) = \lim_{y \rightarrow a, y > a} D_{a^+}^\kappa f(y)$  for any  $\kappa \in \mathbb{R}$  and similar for the right Riemann-Liouville fractional derivative. Blow up at  $(x - a)^{\alpha-n-1}$  is allowed since it is integrable near  $a$  and  $I_{a^+}^\alpha D_{a^+}^\alpha f$  lives in the space  $L^1([a, y])$ .

Next, we enumerate below some basic properties of the Caputo derivative:

**Proposition A.1.2.** *Let  $f$  be a real function and  $\alpha, \beta > 0$ .*

1. *If  $\alpha = n \in \mathbb{N}$  and  $f$  is  $n$  times differentiable, then  $C_{a^+}^\alpha f = f^{(n)}$  and  $C_{b^-}^\alpha f = (-1)^n f^{(n)}$*
2. *Let  $0 < \alpha \leq \beta$  and  $f \in L^1[a, b]$ , then we have  $C_{a^+}^\alpha I_{a^+}^\beta f = I_{a^+}^{\beta-\alpha} f$  and  $C_{b^-}^\alpha I_{b^-}^\beta f = I_{b^-}^{\beta-\alpha} f$*
3.  *$C_{a^+}^\alpha C_{a^+}^n = C_{a^+}^{\alpha+n}$  and  $C_{b^-}^\alpha C_{b^-}^n = C_{b^-}^{\alpha+n}$  for all  $n \in \mathbb{N}$ .*

We provide a proof of the proposition 2.2.10 here. Let us start with a simple but useful lemma, which corresponds to Theorem 3.3 in [58] for the case  $\rho = 1$ :

**Lemma A.1.3.** *Suppose  $f \in C^n([a, b])$ , then for any  $0 < \alpha < n$  and  $\alpha \notin \mathbb{N}$ , we have actually  $C_{a^+}^\alpha f(a) = C_{b^-}^\alpha f(b) = 0$ .*

*Proof of Proposition 2.2.10.* Define

$$g(x) = f(x) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k.$$

From proposition A.1.1, we have

$$I_{a^+}^\alpha D_{a^+}^\alpha g(x) = g(x) - \sum_{j=1}^{n+1} \frac{D_{a^+}^{\alpha-j} g(a)}{\Gamma(\alpha - j + 1)} (x - a)^{\alpha-j}.$$

Now we have by linearity of Riemann-Liouville fractional derivative operator and definition of Caputo fractional derivative that

$$D_{a^+}^{\alpha-j} g(a) = C_{a^+}^{\alpha-j} f(a) - \sum_{k=n-j+1}^n \frac{f^{(k)}(a)}{k!} D_{a^+}^{\alpha-j} ((x - a)^k)(a).$$

Furthermore, for  $k \geq n - j + 1 > \alpha - j$ ,  $D_{a^+}^{\alpha-j}((x-a)^k)(a) = C_{a^+}^{\alpha-j}((x-a)^k)(a)$  by definition. Hence we have

$$D_{a^+}^{\alpha-j}g(a) = C_{a^+}^{\alpha-j}f(a) - \sum_{k=n-j+1}^n \frac{f^{(k)}(a)}{k!} C_{a^+}^{\alpha-j}((x-a)^k)(a).$$

Since  $f \in C^n$  and  $(\cdot - a)^k \in C^\infty$ , we see that actually  $D_{a^+}^{\alpha-j}g(a) = 0$  for every  $j = 1, \dots, n$  by lemma A.1.3. For the case  $j = n + 1$  and  $\alpha < n + 1$ , we actually have that

$$\begin{aligned} |D_{a^+}^{\alpha-n-1}g(a)| &= \lim_{y \rightarrow a, y > a} |I_{a^+}^{n+1-\alpha}g(y)| \\ &= \lim_{y \rightarrow a, y > a} \frac{1}{\Gamma(n+1-\alpha)} \left| \int_a^y \frac{g(t)}{(y-t)^{\alpha-n}} dt \right| \\ &\leq \lim_{y \rightarrow a, y > a} \frac{1}{\Gamma(n+1-\alpha)} \left| \int_a^y \frac{\max_{t \in [a, y]} \{|g(t)|\}}{(y-t)^{\alpha-n}} dt \right| \\ &= \lim_{y \rightarrow a, y > a} \frac{1}{\Gamma(n+1-\alpha)} \max_{t \in [a, y]} \{|g(t)|\} (y-a)^{n+1-\alpha} = 0. \end{aligned}$$

When  $j = n + 1 = \alpha$ , then  $D_{a^+}^{\alpha-n-1}g(a) = g(a) = 0$ . So actually we have

$$g(x) = I_{a^+}^\alpha D_{a^+}^\alpha g(x) = I_{a^+}^\alpha C_{a^+}^\alpha f(x),$$

hence the result. The derivation of the other formula is similar.  $\square$

## A.2 Proof of Corollary 2.2.14

*Proof.* We only prove for the left Caputo derivative case. WLOG, we assume  $C_{a^+}^\alpha f$  exists on the interval  $[a, x]$  since  $o(|x-a|^\alpha)$  only concerns with the  $x$  close to  $a$ .

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_a^x \frac{C_{a^+}^\alpha f(t)}{(x-t)^{1-\alpha}} dt &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f^{(\alpha+)}(a)}{(x-t)^{1-\alpha}} dt + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{C_{a^+}^\alpha f(t) - f^{(\alpha+)}(a)}{(x-t)^{1-\alpha}} dt \\ &= \frac{f^{(\alpha+)}(a)}{\Gamma(\alpha+1)} (x-a)^\alpha + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{C_{a^+}^\alpha f(t) - f^{(\alpha+)}(a)}{(x-t)^{1-\alpha}} dt. \end{aligned}$$

Furthermore,

$$\begin{aligned} \left| \frac{1}{\Gamma(\alpha)} \int_a^x \frac{C_{a^+}^\alpha f(t) - f^{(\alpha+)}(a)}{(x-t)^{1-\alpha}} dt \right| &\leq \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\max_{t \in [a, x]} \{|C_{a^+}^\alpha f(t) - f^{(\alpha+)}(a)|\}}{(x-t)^{1-\alpha}} dt \\ &= \frac{\max_{t \in [a, x]} \{|C_{a^+}^\alpha f(t) - f^{(\alpha+)}(a)|\}}{\Gamma(\alpha+1)} (x-a)^\alpha. \end{aligned}$$

Hence

$$\lim_{x \rightarrow a, x \geq a} \frac{\left| \frac{1}{\Gamma(\alpha)} \int_a^x \frac{C_{a^+}^\alpha f(t) - f^{(\alpha+)}(a)}{(x-t)^{1-\alpha}} dt \right|}{(x-a)^\alpha} \leq \lim_{x \rightarrow a, x \geq a} \frac{\max_{t \in [a, x]} \{|C_{a^+}^\alpha f(t) - f^{(\alpha+)}(a)|\}}{\Gamma(\alpha+1)} = 0.$$

$\square$

# Appendix B

## Appendix of Chapter 3

### B.1 Proofs of technical lemmas on interior estimates

Here we give proofs of some technical lemmas. First recall the lemma 3.4.4.

**Lemma.** *Let  $T > 0$  and let  $u \in C^\infty$  solve*

$$\begin{cases} (\partial_t - \Delta)u = -u^2 + g & \text{in } [0, T] \times \mathcal{P}_n, \\ u = 0 & \text{in } \{0\} \times \mathcal{P}_n, \\ u \geq 0, \end{cases} \quad (\text{B.1.1})$$

where  $g$  is a smooth and bounded function. Then the following point-wise bound on  $u$  holds for all  $z = (t, x) \in [0, T] \times \mathcal{P}_n$ :

$$u(t, x) \leq 28 \cdot \max \left\{ \frac{1}{\min\{(n - x_i)^2, (n + x_i)^2, i = 1, 2\}}, \sqrt{\|g\|_{C_T \mathcal{P}_n}} \right\}. \quad (\text{B.1.2})$$

*Proof.* We consider a  $C^2$  function  $\eta$  on  $\mathcal{P}_n$  such that  $\eta = 0$  on  $\partial\mathcal{P}_n$  and  $\eta$  is strictly positive on the interior  $(-n, n)^2$ . Then  $u\eta$  is 0 on the parabolic boundary of  $\mathbb{R}_+ \times \mathcal{P}_n$  and non-negative in  $\mathbb{R}_+ \times \mathcal{P}_n$ .

Let  $\mathcal{P}_n^t = [0, t] \times \mathcal{P}_n$  and let  $z_t$  be a maximum point of  $u\eta$  in the region  $\mathcal{P}_n^t$ . If  $z_t \in \partial_p \mathcal{P}_n^t$ , then  $u\eta = 0$  on  $\Omega_t$  and  $u = 0$  in  $\mathbb{R}_+ \times (-n, n)^2$ . Here  $\partial_p$  means parabolic boundary. The results holds in this case. Now suppose  $z_t \notin \partial_p \mathcal{P}_n^t$  and  $(u\eta)(z_t) > 0$ . Then we have  $\nabla(u\eta)(z_t) = 0$ ,  $\Delta(u\eta)(z_t) \leq 0$  and  $\partial_t(u\eta)(z_t) \geq 0$ . Hence, we obtain

$$\nabla u(z_t) = -\frac{u\nabla\eta}{\eta}(z_t).$$

Moreover, we have at  $z_t$ :

$$\begin{aligned} 0 &\leq (\partial_t - \Delta)(u\eta)(z_t) \\ &= (\eta(\partial_t - \Delta)u + u(\partial_t - \Delta)\eta - 2\nabla u \cdot \nabla\eta)(z_t) \\ &= \left( \eta(-u^2 + g) + u \left( -\Delta\eta + 2\frac{|\nabla\eta|^2}{\eta} \right) \right) (z_t), \end{aligned}$$

which yields

$$u(z_t) \leq \frac{g}{u}(z_t) - \frac{\Delta\eta}{\eta}(z_t) + 2\frac{|\nabla\eta|^2}{\eta^2}(z_t).$$

Denoting  $\tilde{\eta} := \frac{1}{\eta}$ , we obtain the inequality

$$u(z_t) \leq \frac{g}{u}(z_t) + \frac{\Delta\tilde{\eta}}{\tilde{\eta}}(z_t).$$

Consider now  $\tilde{\eta} = \sum_{i=1}^2 \left( \frac{1}{(n-x_i)^2} + \frac{1}{(n+x_i)^2} \right) + \sqrt{\|g\|_{[0,T] \times \mathcal{P}_n}}$ , for which  $\eta = \frac{1}{\tilde{\eta}}$  satisfies the assumptions made above. Moreover, we have  $\Delta\tilde{\eta} \leq 6\tilde{\eta}^2$  and  $g \leq \tilde{\eta}^2$ , and thus

$$\frac{u}{\tilde{\eta}}(z_t) \leq \frac{\tilde{\eta}^2}{u\tilde{\eta}}(z_t) + 6 = \frac{\tilde{\eta}}{u}(z_t) + 6.$$

The inequality  $x \leq \frac{1}{x} + 6$  for  $x \geq 0$  can only be satisfied if  $x \leq 7$ . Otherwise we would get  $\frac{1}{x} < 7$  and then  $x < \frac{1}{7} + 6 < 7$ , a contradiction. Therefore, using that  $z_t$  is a maximum point of  $\frac{u}{\tilde{\eta}}$  on  $\mathcal{P}_n^t$ , we get

$$u \leq 7 \cdot \tilde{\eta} \leq 28 \cdot \max \left\{ \frac{1}{\min\{(n-x_i)^2, (n+x_i)^2, i=1,2\}}, \sqrt{\|g\|_{C_T \mathcal{P}_n}} \right\},$$

as claimed.  $\square$

In order to give a proof of Lemma 3.4.9, we will need an interior gradient estimate for the heat equation.

**Lemma B.1.1.** *Suppose  $u$  satisfies the equation*

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}_+ \times B(x, L) \\ u = 0 & \text{on } \{0\} \times B(x, L) \end{cases} \quad (\text{B.1.3})$$

for some  $x \in \mathbb{R}^d$ . Let  $R < \frac{L}{2}$  and  $T > 0$ . Then we have the interior gradient estimate for  $T > 0$  for any  $f \in C(B(x, L))$ ,

$$\|\nabla u\|_{C_T B(x, R)} \lesssim \frac{1}{L} \|u - f\|_{C_T B(x, L)}$$

uniformly in  $T$  and  $f$ .

We suspect that this is well-known, but were unable to find a reference.

*Proof.* We first take  $f = 0$  and argue later how to treat general  $f$ . By scaling, it suffices to consider the case  $L = 1$  and  $R = \frac{1}{2}$ . Classical inner regularity theory for the heat equation gives (for  $T \geq \frac{1}{4}$ )

$$\sup_{t \in [\frac{1}{4}, T]} \|\nabla u(t, \cdot)\|_{B(x, \frac{1}{2})} \lesssim \|u\|_{C_T B(x, 1)},$$

see for example Theorem 8.4.4 in [61] or Theorem IV.4.8 in [64]. These estimates are independent of the initial condition, and we leverage the initial condition  $u(0) = 0$  to extend the estimate to  $t \in [0, \frac{1}{4}]$ .

In our argument we will consider two different spatial scales, and for that purpose it is convenient to reintroduce  $R$  and  $L$  in the notation. Let  $\eta \in C_c^\infty(\mathbb{R}^d)$  be such that  $\eta = 1$  on  $B(x, R)$ ,  $\eta = 0$  outside of  $B(x, \frac{R+L}{2})$  and  $\|\nabla\eta\|_\infty \leq \frac{4}{L-R}$ ,  $\|\Delta\eta\|_\infty \leq \frac{8}{(L-R)^2}$ . Let  $w = u\eta$ , so that

$$\begin{cases} (\partial_t - \Delta)w = -2\nabla(u\nabla\eta) + u\Delta\eta, \\ w(0, \cdot) = 0, \end{cases} \quad (\text{B.1.4})$$

on  $\mathbb{R}_+ \times \mathbb{R}^d$ , and therefore

$$w(t) = \int_0^t K_{t-s} * (-2\nabla(u(s)\nabla\eta) + u(s)\Delta\eta) ds,$$

where  $K$  is the heat kernel. Consider the difference operator  $D_h f(x) = f(x+h) - f(x)$ , and note that by interpolation for  $\alpha = \frac{3}{4}$  (the following argument works for any  $\alpha \in (\frac{1}{2}, 1)$ ):

$$\|D_h \nabla K_{t-s}\|_{L^1} \lesssim (h \|\nabla^2 K_{t-s}\|_{L^1})^\alpha (\|\nabla K_{t-s}\|_{L^1})^{1-\alpha} \lesssim h^\alpha (t-s)^{-\alpha} (t-s)^{-\frac{1-\alpha}{2}},$$

which yields

$$\begin{aligned} \left| \int_0^t (D_h K_{t-s}) * (-2\nabla(u(s)\nabla\eta)) ds \right| &= \left| 2 \int_0^t (D_h \nabla K_{t-s}) * (u\nabla\eta) ds \right| \\ &\lesssim \frac{|h|^\alpha t^{\frac{1-\alpha}{2}}}{L-R} \|u\|_{C_t B(x,L)}. \end{aligned}$$

Similarly, we have

$$\left| \int_0^t (D_h K_{t-s}) * (u(s)\Delta\eta) \right| \lesssim \frac{|h|^\alpha t^{1-\frac{\alpha}{2}}}{(L-R)^2} \|u\|_{C_t B(x,L)},$$

and therefore

$$\|D_h w(t)\|_\infty \lesssim |h|^\alpha t^{\frac{1-\alpha}{2}} \|u\|_{C_t B(x,L)} \left( \frac{1}{L-R} + \frac{t^{\frac{1}{2}}}{(L-R)^2} \right).$$

Recalling that  $w = u\eta$  and that  $\eta|_{B(x,R)} \equiv 1$ , this means

$$[u(t, \cdot)]_{\alpha, B(x,R)} \lesssim t^{\frac{1-\alpha}{2}} \left( \frac{1}{L-R} + \frac{t^{\frac{1}{2}}}{(L-R)^2} \right) \|u\|_{C_t B(x,L)}.$$

Since we had to take  $\alpha < 1$  to keep the singularity  $(t-s)^{-\frac{1+\alpha}{2}}$  integrable, this is not yet sufficient. Therefore, we iterate the argument. Let now  $R_1 \in (R, L)$  and let

$|h| \in (0, L - R_1)$ . Then  $\frac{D_h u}{|h|^\alpha}$  also solves the heat equation (B.1.3) on  $B(x, R_1)$ , so by the previous argument

$$\left[ \frac{D_h u}{|h|^\alpha}(t, \cdot) \right]_{\alpha, B(x, R)} \lesssim t^{\frac{1-\alpha}{2}} \left( \frac{1}{R_1 - R} + \frac{t^{\frac{1}{2}}}{(R_1 - R)^2} \right) \left\| \frac{D_h u}{|h|^\alpha} \right\|_{C_t B(x, R_1)}.$$

Now we take  $L = 1$ ,  $R_1 = \frac{3}{4}$  and  $R = \frac{1}{2}$ , and combine the above inequalities to obtain for any  $|h| < L - R_1 = \frac{1}{4}$ ,

$$\left[ \frac{D_h u}{|h|^\alpha}(t, \cdot) \right]_{\alpha, B(x, \frac{1}{2})} \lesssim t^{1-\alpha} (1 + \sqrt{t})^2 \|u\|_{C_t B(x, 1)}.$$

By Lemma 5.6 in [81], we have then (since  $\alpha = \frac{3}{4} > \frac{1}{2}$ )

$$\|\nabla u(t, \cdot)\|_{B(x, \frac{1}{2})} \lesssim t^{1-\alpha} (1 + t) \|u\|_{C_t B(x, 1)}$$

Let  $T > 1$  as an upper time horizon. Now when  $t \leq \frac{1}{4}$ , we have

$$\|\nabla u(t, \cdot)\|_{B(x, \frac{1}{2})} \lesssim \|u\|_{C_T B(x, 1)}, \quad (\text{B.1.5})$$

which is the desired result for  $f = 0$  (recall that we only have to consider  $R = \frac{1}{2}$ ,  $L = 1$ , and  $t \leq \frac{1}{4}$ ).

For general continuous function  $f$ , we simply use that  $u(0) = 0$  and that  $f$  does not depend on time to estimate

$$\begin{aligned} \|u\|_{C_T B(x, L)} &\leq \|u - f\|_{C_T B(x, L)} + \|f\|_{B(x, L)} \\ &= \|u - f\|_{C_T B(x, L)} + \|u(0, \cdot) - f\|_{B(x, L)} \\ &\leq 2\|u - f\|_{C_T B(x, L)}. \end{aligned}$$

□

Now we can give the proof of Lemma 3.4.9.

*Proof of Lemma 3.4.9.* The proof is very similar to that of Lemma 2.11 in the paper [74]. The only change is that we replace the parabolically shrunk region by the spatially shrunk cylinder. We include the proof for completeness.

*Step 1.* We claim that for all base points  $x$  and scales  $\delta, R$  and  $L$  with  $R \leq \frac{L}{2}$  and such that  $B(x, L) \subset D$ , it holds that for  $0 \leq t < T = kR^2, k \geq 1$

$$\begin{aligned} \inf_l \|U_\delta(t, x, \cdot) - l\|_{C_T B(x, R)} &\lesssim \frac{kR^2}{L^2} \inf_l \|U_\delta(t, x, \cdot) - l\|_{C_T B(x, L)} \\ &\quad + kL^2 M_{\{x\}, L}^{(1)} \sum_{\beta \in A} \delta^{\beta-2} L^{\kappa-\beta}, \end{aligned} \quad (\text{B.1.6})$$

where the infimum runs over all spatial affine functions  $l(y) = C \cdot (y - x) + c$ . To prove this, we define a decomposition  $U_\delta = u_> + u_<$  where  $u_>$  is the solution to

$$(\partial_t - \Delta)u_>(t, y) = \mathbb{1}_{B^T(x, L)}(\partial_t - \Delta_y)U_\delta(t, x, y),$$

with Dirichlet boundary conditions. By standard estimates for the heat equation [74] and assumptions in this lemma,

$$\begin{aligned} \|u_>\|_{C_TB(x, L)} &\lesssim L^2 \|(\partial_t - \Delta_y)U_\delta(t, x, y)\|_{y \in B(x, L)} \\ &\leq L^2 M_{\{x\}, L}^{(1)} \sum_{\beta \in A} \delta^{\beta-2} L^{\kappa-\beta}. \end{aligned}$$

Now  $(\partial_t - \Delta)u_< = 0$  on  $B^T(x, L)$  and  $u_<(0, \cdot) = 0$  on  $B(x, L)$ . By Lemma B.1.1, we know directly for  $\partial \in \{\partial_t, \partial_i, \partial_j\}$  a differential operator of order 1 in time and 2 in space,

$$\|\partial u_<\|_{C_TB(x, R)} \leq L^{-2} \|u_< - l_>\|_{C_TB(x, R)},$$

for any affine function  $l_>$  with  $R \leq \frac{L}{2}$ . Let  $l_<(y) = u_<(T, x) + \nabla u_<(T, x)(y - x)$ , the Taylor's formula show that

$$\|u_< - l_<\|_{C_TB(x, R)} \leq kR^2 \|Du_<\|_{C_TB(x, R)} \lesssim \frac{kR^2}{L^2} \|u_< - l_>\|_{C_TB(x, L)}.$$

Thus,

$$\begin{aligned} &\|U_\delta(t, x, \cdot) - l_<\|_{C_TB(x, R)} \\ &\leq \|u_< - l_<\|_{C_TB(x, R)} + \|u_>\|_{C_TB(x, R)} \\ &\lesssim \frac{kR^2}{L^2} \|U_\delta(t, x, \cdot) - l_>\|_{C_TB(x, L)} + k \|u_>\|_{C_TB(x, L)} \\ &\leq \frac{kR^2}{L^2} \|U_\delta(t, x, \cdot) - l_>\|_{C_TB(x, L)} + kL^2 M_{\{x\}, L}^{(1)} \sum_{\beta \in A} \delta^{\beta-2} L^{\kappa-\beta}, \end{aligned}$$

which implies equation (B.1.6).

*Step 2.* We claim that for all base point  $x$  and scales  $\delta$  and  $L$ , it holds that

$$\|U_\delta(t, x, \cdot) - U(t, x, \cdot)\|_{C_TB(x, R)} \leq M_{\{x\}, R, \delta}^{(2)} \sum_{\beta \in A} R^\beta \delta^{\kappa-\beta} + \delta^\kappa [U]_{C_T(\kappa, B(x, R), \delta)}. \quad (\text{B.1.7})$$

Indeed, by symmetry of  $\Psi$ ,

$$\begin{aligned}
& |U_\delta(t, x, y) - U(t, x, y)| \\
&= \left| \int \Psi^\delta(y - y_1)(U(t, x, y_1) - U(t, x, y)) dy_1 \right| \\
&= \inf_{\nu(t, y)} \left| \int \Psi^\delta(y - y_1)(U(t, x, y_1) - U(t, x, y) - U(t, y, y_1)) \right. \\
&\quad \left. + \int \Psi^\delta(y - y_1)(U(t, y, y_1) - \nu(t, y)(y_1 - y)) dy_1 \right| \\
&\leq M_{\{x\}, R, \delta}^{(2)} \sum_{\beta \in A} d(x, y)^\beta \int \Psi^\delta(y - y_1) d(y, y_1)^{\kappa - \beta} dy_1 \\
&\quad + \left( \sup_{y \in B(x, R)} \inf_{\nu(t, y)} \sup_{y_1 \in B(y, \delta)} d(y, y_1)^{-\kappa} |U(t, y, y_1) - \nu(t, y)(y_1 - y)| \right) \\
&\quad \times \int \Psi^\delta(y - y_1) d(y, y_1)^\kappa dy_1.
\end{aligned}$$

*Step 3.* We prove for small enough  $\epsilon(T)$ , we have

$$\begin{aligned}
& \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U(t, x, \cdot) - l\|_{C_TB(x, R)} \\
&\lesssim_{T, \epsilon} \sum_{\beta \in A} \left( M_{\{x\}, \frac{d}{2}}^{(1)} \epsilon^{-4+2\beta-\kappa} + M_{\{x\}, \frac{d}{2}, \frac{\epsilon^2 d}{2}}^{(2)} \epsilon^{\kappa-\beta} (1 + \epsilon^{\kappa-\beta}) \right) \\
&\quad + \epsilon^{2-2\kappa} \frac{d^{-\kappa}}{2^{-\kappa}} \|U(t, x, \cdot)\|_{C_TB(x, \frac{d}{2}(1+\epsilon^2))} + (\epsilon^\kappa + \epsilon^{2+\kappa}) [U]_{C_T(\kappa, B(x, \frac{\epsilon d}{2}), \frac{\epsilon^2 d}{2})}.
\end{aligned} \tag{B.1.8}$$

Multiplying equation (B.1.6) by  $R^{-\kappa}$  and fixing the length ratios  $R = \epsilon L = \epsilon^{-1} \delta$  for some  $\epsilon \leq \frac{1}{2}$  to be fixed below, we get for any point  $x \in D_d$  and length  $L \leq \frac{d}{2}$ ,

$$\begin{aligned}
& R^{-\kappa} \inf_l \|U_\delta(t, x, \cdot) - l\|_{C_TB(x, R)} \\
&\lesssim k \epsilon^{2-\kappa} L^{-\kappa} \inf_l \|U_\delta(t, x, \cdot) - l\|_{C_TB(x, L)} + k M_{D_d, L}^{(1)} \sum_{\beta \in A} \epsilon^{-4+2\beta-\kappa}.
\end{aligned}$$

Taking the supremum over  $L \leq \frac{d}{2}$  while keeping the ratios  $R = \epsilon L = \epsilon^{-1} \delta$  fixed, we get

$$\begin{aligned}
& \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U_\delta(t, x, \cdot) - l\|_{C_TB(x, R)} \\
&\lesssim k \epsilon^{2-\kappa} \sup_{L \leq \frac{d}{2}} L^{-\kappa} \inf_l \|U_\delta(t, x, \cdot) - l\|_{C_TB(x, L)} + k \sup_{L \leq \frac{d}{2}} M_{D_d, L}^{(1)} \sum_{\beta \in A} \epsilon^{-4+2\beta-\kappa} \\
&\leq k \epsilon^{2-\kappa} \sup_{L \leq \frac{\epsilon d}{2}} L^{-\kappa} \inf_l \|U_\delta(t, x, \cdot) - l\|_{C_TB(x, L)} + k \sup_{L \leq \frac{d}{2}} M_{D_d, L}^{(1)} \sum_{\beta \in A} \epsilon^{-4+2\beta-\kappa} \\
&\quad + k \epsilon^{2-\kappa} \sup_{\frac{\epsilon d}{2} < L \leq \frac{d}{2}} L^{-\kappa} \inf_l \|U_\delta(t, x, \cdot) - l\|_{C_TB(x, L)}.
\end{aligned}$$

The last term is bounded by

$$k\epsilon^{2-\kappa} \left(\frac{\epsilon d}{2}\right)^{-\kappa} \|U_\delta(t, x, \cdot)\|_{C_TB(x, \frac{d}{2})} \leq k\epsilon^{2-2\kappa} \frac{2^\kappa}{d^\kappa} |U(t, x, \cdot)|_{C_TB(x, \frac{d}{2}(1+\epsilon^2))}.$$

Hence we have

$$\begin{aligned} & \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U_\delta(t, x, \cdot) - l\|_{C_TB(x, R)} \\ & \lesssim k\epsilon^{2-\kappa} \sup_{L \leq \frac{\epsilon d}{2}} L^{-\kappa} \inf_l \|U_\delta(t, x, \cdot) - l\|_{C_TB(x, L)} + kM_{D_d, \frac{d}{2}}^{(1)} \sum_{\beta \in A} \epsilon^{-4+2\beta-\kappa} \\ & \quad + k\epsilon^{2-2\kappa} \frac{2^\kappa}{d^\kappa} \|U(t, x, \cdot)\|_{C_TB(x, \frac{d}{2}(1+\epsilon^2))}. \end{aligned}$$

Applying equation (B.1.7), we obtain

$$\begin{aligned} & \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U(t, x, \cdot) - l\|_{C_TB(x, R)} \\ & \lesssim \sum_{\beta \in A} \left( kM_{D_d, \frac{d}{2}} \epsilon^{4-2\beta-\kappa} + M_{\{x\}, \frac{\epsilon d}{2}, \frac{\epsilon^2 d}{2}} \epsilon^{\kappa-\beta} \right) + \epsilon^\kappa [U]_{C_T(\kappa, B(x, \frac{\epsilon d}{2}), \frac{\epsilon^2 d}{2})} \\ & \quad + k\epsilon^{2-2\kappa} \frac{2^\kappa}{d^\kappa} \|U(t, x, \cdot)\|_{C_TB(x, \frac{d}{2}(1+\epsilon^2))} \\ & \quad + k\epsilon^{2-\kappa} \sup_{L \leq \frac{\epsilon d}{2}} L^{-\kappa} \inf_l \|U_\delta(t, x, \cdot) - l\|_{C_TB(x, L)} \\ & \lesssim_T \sum_{\beta \in A} \left( M_{\{x\}, \frac{d}{2}}^{(1)} \epsilon^{-4+2\beta-\kappa} + M_{\{x\}, \frac{d}{2}, \frac{\epsilon^2 d}{2}}^{(2)} \epsilon^{\kappa-\beta} (1 + \epsilon^{\kappa-\beta}) \right) \\ & \quad + \epsilon^{2-2\kappa} \frac{d^{-\kappa}}{2^{-\kappa}} \|U(t, x, \cdot)\|_{C_TB(x, \frac{d}{2}(1+\epsilon^2))} + (\epsilon^\kappa + \epsilon^{2+\kappa}) [U]_{C_T(\kappa, B(x, \frac{\epsilon d}{2}), \frac{\epsilon^2 d}{2})} \\ & \quad + \epsilon^{2-\kappa} \sup_{L \leq \frac{\epsilon d}{2}} L^{-\kappa} \inf_l \|U(t, x, \cdot) - l\|_{C_TB(x, L)}. \end{aligned}$$

The last term of the right-hand side can be absorbed in the left-hand side when  $\epsilon$  is small enough. Thus we obtain the inequality (B.1.8).

*Step 4.* We prove that

$$\begin{aligned} & \sup_{d \leq d_0} d^\kappa [U]_{C_T(\kappa, D_d, d)} \\ & \lesssim \sup_{d \leq d_0} \sum_{\beta \in A} \left( M_{D_d, \frac{d}{2}}^{(1)} \epsilon^{-4+2\beta-\kappa} + M_{D_d, \frac{d}{2}, \frac{\epsilon^2 d}{2}} \epsilon^{\kappa-\beta} \right) + \epsilon^{2-2\kappa} \sup_{d \leq d_0} \|U\|_{C_T(D_d, d)}. \end{aligned} \tag{B.1.9}$$

we first argue that we can change the order of the supremum and the infimum in  $\sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U(t, x, \cdot) - l\|_{C_TB(x, R)}$ . To begin with, since  $U(t, x, x) = 0$ , we have

$$\begin{aligned} \|U(t, x, \cdot) - (l - l(x))\|_{C_TB(x, R)} & \leq \|U(t, x, \cdot) - l\|_{C_TB(x, R)} + |l(x)| \\ & \leq 2\|U(t, x, \cdot) - l\|_{C_TB(x, R)}. \end{aligned}$$

Hence

$$\sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_{l(x)=0} \|U(t, x, \cdot) - l\|_{C_T B(x, R)} \lesssim \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U(t, x, \cdot) - l\|_{C_T B(x, R)}.$$

Furthermore, for  $\frac{\epsilon d}{2} > R > 0$ , let  $l_R = C_R(y - x)$  such that

$$\|U(t, x, \cdot) - l_R\|_{C_T B(x, R)} \leq 2 \inf_l \|U(x, \cdot) - l\|_{C_T B(x, R)}.$$

Then we have

$$|C_R - C_{\frac{R}{2}}| R^{-(\kappa-1)} \lesssim \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U(t, x, \cdot) - l\|_{C_T B(x, R)}.$$

This shows that there exists a limit  $C_0 := \lim_{R \rightarrow 0} C_R$  and we have the bound

$$|C_R - C_0| R^{-(\kappa-1)} \lesssim_{\kappa} \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U(t, x, \cdot) - l\|_{C_T B(x, R)}.$$

Now consider  $\bar{l} = C_0(y - x)$ , we have

$$R^{-\kappa} \|U(t, x, \cdot) - \bar{l}\|_{C_T B(x, R)} \lesssim_{\kappa} \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U(t, x, \cdot) - l\|_{C_T B(x, R)}.$$

Thus for any  $0 \leq t \leq T$ ,

$$\begin{aligned} & \inf_{\nu(t, x)} \sup_{x \neq y \in B(x, \frac{\epsilon d}{2})} d(x, y)^{-\kappa} |U(t, x, y) - \nu(t, x)(y - x)| \\ & \leq \inf_{l(x)=0} \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \|U(t, x, \cdot) - l\|_{C_T B(x, R)} \\ & \lesssim \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U(t, x, \cdot) - l\|_{C_T B(x, R)}. \end{aligned}$$

Therefore, using the inequality (B.1.8), if we take the supremum over  $t \in [0, T]$  and then take supremum over  $x \in D_d$ , multiply it by  $d^\kappa$  and take the supremum over  $d$ , we will have

$$\begin{aligned} & \sup_{d \leq d_0} d^\kappa \sup_{x \in D_d} \sup_{0 \leq t \leq T} \inf_{\nu(t, x)} \sup_{x \neq y \in B(x, \frac{\epsilon d}{2})} d(x, y)^{-\kappa} |U(t, x, y) - \nu(t, x)(y - x)| \\ & \lesssim \sup_{d \leq d_0} d^\kappa \sum_{\beta \in A} \left( M_{D_d, \frac{d}{2}}^{(1)} \epsilon^{-4+2\beta-\kappa} + M_{D_d, \frac{d}{2}, \frac{\epsilon^2 d}{2}}^{(2)} \epsilon^{\kappa-\beta} \right) \\ & \quad + \epsilon^{2-2\kappa} \sup_{d \leq d_0} \|U\|_{C_T(D_d, d)} + \epsilon^\kappa \sup_{d \leq d_0} d^\kappa \sup_{x \in D_d} \sup_{y \in B(x, \frac{\epsilon d}{2})} \sup_{0 \leq t \leq T} \\ & \quad \left[ \inf_{\nu(t, y)} \sup_{y \neq y_1 \in B(y, \frac{\epsilon^2 d}{2})} d(y, y_1)^{-\kappa} |U(y, y_1) - \nu(t, y)(y_1 - y)| \right]. \end{aligned}$$

The last term can be absorbed into the left-hand side for  $\epsilon$  small enough since for  $y \in B(x, \frac{\epsilon d}{2})$ , we have  $y \in D_{(1-\frac{\epsilon}{2})d}$  with  $(1 - \frac{\epsilon}{2}) \leq d_0$ . And  $\frac{\epsilon^2 d}{2} \leq \frac{\epsilon}{2}(1 - \frac{\epsilon}{2})d$  for small enough  $\epsilon$ . Hence we have

$$\begin{aligned} \sup_{d \leq d_0} d^\kappa [U]_{C_T(\kappa, D_d, \frac{\epsilon d}{2})} &\lesssim_{T, \kappa} \sup_{d \leq d_0} \sum_{\beta \in A} \left( M_{D_d, \frac{d}{2}}^{(1)} \epsilon^{-4+2\beta-\kappa} + M_{D_d, \frac{d}{2}, \frac{\epsilon^2 d}{2}} \epsilon^{\kappa-\beta} \right) \\ &\quad + \epsilon^{2-2\kappa} \sup_{d \leq d_0} \|U\|_{C_T(D_d, d)}. \end{aligned}$$

Then it is directly to extend  $[U]_{C_T(\kappa, D_d, \frac{\epsilon d}{2})}$  to  $[U]_{C_T(\kappa, D_d)}$  by considering  $\leq \frac{\epsilon d}{2}$  and  $> \frac{\epsilon d}{2}$  parts.  $\square$

# Appendix C

## Appendix of Chapter 4

### C.1 Integer-valued random measures

We review some basics on point processes, integer-valued random measures and purely discontinuous martingales. We refer reader to [66] for more detail.

Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  be a probability space and  $(E, \mathcal{E})$  be a Polish space equipped with the Borel  $\sigma$ -algebra. Denote  $\mathbb{R}_+ = [0, \infty)$ . We give the definition of random measure on the space  $\mathbb{R}_+ \times E$  here

**Definition C.1.1.** *A family  $\mu = \{\mu(\omega, dt, dx), \omega \in \Omega\}$  of  $\sigma$  finite non-negative measure  $\mu(\omega, \cdot)$  on  $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E})$ , such that for each  $A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}$  the variable  $\mu(\cdot, A)$  is  $\mathcal{F}$  measurable and*

$$\mu(\omega, \{0\} \times E) = 0, \quad \forall \omega \in \Omega,$$

*is called a random measure on  $\mathbb{R}_+ \times E$ . We call  $\mu$  predictable if for any predictable random variable  $X(\omega, s, x)$ , the process*

$$\int_0^t \int_E X(\omega, s, x) \mu(\omega, ds, dx)$$

*is predictable.*

We then give the definition of the compensator of a random measure  $\mu$ .

**Definition C.1.2.** *A predictable random measure  $\nu$  is called the compensator of the random measure  $\mu_X$  if for any non-negative predictable random process  $X(\omega, s, x)$ , we have*

$$\mathbb{E} \left[ \int_0^t \int_E X(s, x) \mu(dsdx) \right] = \mathbb{E} \left[ \int_0^t \int_E X(s, x) \nu(dsdx) \right]$$

**Example C.1.3.** We will consider the random measure  $\mu_X$  associated to jumps of some process  $X_t$  with value in  $E$ . The random point measure is defined via

$$\mu_X = \sum_{0 \leq s \leq t} \delta_{s, X_s - X_{s-}}$$

Its compensator will play an important role in the Itô's formula for a purely discontinuous martingale.

Here is another example of random measure on the space  $E$  without time.

**Example C.1.4.** A Poisson random measure  $\Lambda$  on  $E$  with intensity  $\mu$  is an integer-valued random measure with Laplace functional given by

$$\mathbb{E}_\Lambda[e^{-\langle \cdot, \varphi \rangle}] = e^{-\int_E (1 - e^{-\varphi(x)}) \mu(dx)}$$

for any non-negative measurable function  $\varphi : E \rightarrow \mathbb{R}$ .

The Poisson random measure is the most well-known example of integer-valued random measure. We only need its Laplace characterization here which can also determine the Poisson random measure uniquely.

We also need a lemma to the Poisson cluster random measure.

**Lemma C.1.5.** Suppose we have a family of independent random measures  $(\mathbb{P}_x)_{x \in E}$  on the space  $\tilde{E}$  and a Poisson random measure  $\Lambda$  on  $E$  with intensity  $\mu$ . Then the Laplace functional of the random measure  $\mathbb{P}_\Lambda := \int_E \mathbb{P}_x \Lambda(dx)$  is given by

$$\mathbb{E}_\Lambda[e^{-\langle \cdot, \varphi \rangle}] = e^{-\int_E (1 - \mathbb{E}_x[e^{-\langle \cdot, \varphi \rangle}]) \mu(dx)}$$

for any non-negative measurable function  $\varphi : \tilde{E} \rightarrow \mathbb{R}$ .

*Proof.* For a fixed realization of Poisson random measure  $\Lambda$ , the conditional Laplace functional of  $\mathbb{P}_\Lambda$  is given by

$$\prod_{x \in \Lambda} \mathbb{E}_x [e^{-\langle \cdot, \varphi \rangle}] = e^{\int_E \log \mathbb{E}_x [e^{-\langle \cdot, \varphi \rangle}] \Lambda(dx)} = e^{-\langle \Lambda, -\log \mathbb{E}_x [e^{-\langle \cdot, \varphi \rangle}] \rangle}$$

Now take the expectation with respect to Poisson random measure  $\Lambda$  and use the Laplace functional of Poisson random measure, we obtain the result.  $\square$

## C.2 Canonical decomposition of a semi-martingale and Itô's formula

In this section, we say a stochastic process  $A$  is of locally bounded variation if almost surely, the total variation of  $A$ , denoted by  $\text{Var}(A)$ , is finite on every finite interval. We also say  $A$  is locally integrable if there exists a localization  $T_n$  such that  $\mathbb{E}[\text{Var}(A)_{T_n}] < \infty$  for all  $n \in \mathbb{N}$ .

It is well-known that a càdlàg stochastic process  $X_t$  on  $\mathbb{R}$  is semi-martingale if it admits the decomposition

$$X_t = X_0 + M_t + A_t$$

where  $M_t$  is a local martingale and  $A_t$  is a locally bounded variation process. We call a semi-martingale special if there exists stopping times  $\{T_n\}_n$  such that  $T_n \rightarrow \infty$  and  $E[A_{T_n}] < \infty$ . The canonical decomposition (Chapter 4, section 1 in [66]) of  $X_t$  is given by

**Theorem C.2.1.** *Suppose  $X_t$  is a semi-martingale and  $a > 0$ . Then  $X_t - \int_0^t \int_{|x|>a} x d\mu$  admits a unique decomposition*

$$X_t - \int_0^t \int_{|x|>a} x d\mu = X_0 + A_t^a + X_t^c + \int_0^t \int_{|x|\leq a} x d(\mu - \nu).$$

where  $X_t^c$  is the continuous martingale part of  $X_t$ ,  $A_t^a$  is a continuous predictable bounded variation process,  $\mu$  is the jump measure of  $X_t$  given by

$$\mu((0, t], C) = \sum_{0 \leq s \leq t} \mathbb{1}_{\{X_s - X_{s-} \in C\}}, \quad C \subset \mathbb{R}$$

and  $\nu$  is the compensator of  $\mu$ . In the other words, when we subtracts the jumps that bigger than some fixed  $a$ , the remaining part is uniquely determined by the above decomposition.

Now consider a much more general case when the jump measure  $\mu$  is defined on some Polish space  $E$  and  $\nu$  is again its compensator. Consider two predictable functions  $g_1, g_2$  on  $\mathbb{R}_+ \times E$  such that  $g_1 \cdot g_2 = 0$ ,

$$\int_0^t \int_E |g_2| d\mu < \infty \text{ a.s.}, \quad \mathbb{E} \left[ \int_0^t \int_E |g_1|^2 d\nu \right] < \infty.$$

We give an Itô's formula (section 6, chapter 3 in [66]) for the semi-martingale in the form

$$X_t = X_0 + A_t + X_t^c + \int_0^t \int_E g_1 d(\mu - \nu) + \int_0^t \int_E g_2 d\mu \quad (\text{C.2.1})$$

where  $A_t$  is a continuous locally bounded variation process and  $X_t^c$  is a continuous local martingale.

**Lemma C.2.2.** *Let  $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ , then for semi-martingale  $X_t$  with decomposition (C.2.1), we have the Itô's formula*

$$\begin{aligned}
& f(t, X_t) - f(0, X_0) \\
&= \int_0^t \partial_t f(s, X_s) ds + \int_0^t \partial_x f(s, X_s) dA_s + \frac{1}{2} \int_0^t \partial_{xx}^2 f(s, X_s) d\langle X^c \rangle_s \\
&+ \int_{(0,t]} \int_E (f(s, X_{s-} + g_2(s, x)) - f(s, X_{s-})) \mu(ds, dx) \\
&+ \int_{(0,t]} \int_E (f(s, X_{s-} + g_1(s, x)) - f(s, X_{s-}) - \partial_x f(s, X_{s-}) g_1(s, x)) \nu(ds, dx)
\end{aligned} \tag{C.2.2}$$

when the last term is of locally integrable.

We also need a unique decomposition of special semi-martingale.

**Lemma C.2.3.** *Let  $X_t$  be a special semi-martingale, then  $X_t$  admits a unique decomposition*

$$X_t = X_0 + A_t + M_t$$

where  $M_t$  is a local martingale and  $A_t$  is predictable and of locally bounded variation.

### C.3 Lemmas for Branching random walk

We here give some useful lemma on branching random walk.

**Lemma C.3.1.** *For any branching random walk  $Z_t$  on  $\mathbb{Z}_n^2$  with branching mechanism  $g(x, s) = \sum_{k=0}^{\infty} p_k(x) s^k$  and branching rate  $a(x, ds)$ , if*

$$\sum_{k=0}^{\infty} k p_k(x) < K$$

uniformly over  $t, x$  with some finite constant  $K$ , then the branching random walk exists (no explosion, especially  $\mathbb{E}[\langle Z_t, 1 \rangle]$  is finite) and we have the expression for the Laplace functional of  $Z_t$  for any test function  $\varphi$  such that

$$\begin{aligned}
w_t(x) &:= \mathbb{E} [e^{-\langle Z_t, \varphi \rangle} | Z_0 = \delta_x] \\
&= \mathbb{E} \left[ e^{-\varphi(X_t) - a(0,t)} + \int_0^t e^{-a(0,r)} g(r, X_r, w_{t-r}(X_r)) a(X_r, dr) | X_0 = x \right]
\end{aligned} \tag{C.3.1}$$

where  $X_t$  is the simple random walk on  $\mathbb{Z}_n^2$  and

$$a(s, t) = \int_s^t a(X_r, dr)$$

*Proof.* The non-explosion follows by the corollary 1, page 111 in [4]. Let  $\tau$  to be the first branching time of  $Z$  in the time interval  $[s, t]$  and  $M(x)$  are independent random variables with the generator function  $g(x, s)$ . We define filtrations  $\mathcal{F}_t = \sigma(Z_s, s \leq t)$  and  $\mathcal{G}_t = \sigma(X_s, s \leq t)$ . Then the random time  $\tau$  is both stopping time of filtrations  $\mathcal{F}_t$  and  $\mathcal{G}_t$ . Then we have

$$\begin{aligned} w_t(x) &= \mathbb{E}_x [e^{-\langle Z_t, \varphi \rangle} \mathbb{1}_{\tau \leq t} + e^{-\langle Z_t, \varphi \rangle} \mathbb{1}_{\tau > t}] \\ &= \mathbb{E}_x [\mathbb{E}[w_{t-\tau}(X_\tau)^{M(X_\tau)} | \mathcal{F}_\tau] \mathbb{1}_{\tau \leq t} + e^{-\langle X_t, \varphi \rangle} \mathbb{1}_{\tau > t}] \\ &= \mathbb{E}_x [e^{-\langle X_t, \varphi \rangle - a(s, t)} + g(\tau, X_\tau, w_{t-\tau}(X_\tau)) \mathbb{1}_{\tau \leq t}] \end{aligned} \quad (\text{C.3.2})$$

by (4.5.2) and the definition of moment generating function. Then by the law of  $\tau$ , we obtain the desired result.  $\square$

Here is a general lemma concerning the Feynman-Kac formula and the mild solution to some equation.

**Lemma C.3.2.** *For any measurable function  $g : [0, \infty] \times \mathbb{Z}_n^2 \times [0, 1] \rightarrow [0, 1]$ , following the notation in lemma C.3.1, the equation (C.3.1) is equivalent to*

$$w_t(x) = \mathbb{E} \left[ e^{-\varphi(X_t)} + \int_0^t g(r, X_r, w_{t-r}(X_r)) a(X_r, dr) - \int_0^t w_{t-r}(X_r) a(X_r, dr) \right] \quad (\text{C.3.3})$$

*Proof.* See lemma 4.3.4 in [26], a simple version is given in the proof of lemma 4.4.5 when we try to show the equivalence of Feynman-Kac formulation and the mild formulation.  $\square$

## C.4 Auxiliary lemmas

Here is an auxiliary lemma for our calculation of moment estimation.

**Lemma C.4.1.** *We have following inequalities:*

1.  $1 - x + \frac{x^2}{2} - e^{-x} \geq 0, x \geq 0$
2.  $\frac{x}{4} \leq 1 - x + \frac{x^2}{2} - e^{-x}$  for  $x \geq 2$ .
3.  $0 \leq e^{-x} - 1 + x \leq 2x^{1+\beta}$  for  $x \geq 0, 0 < \beta \leq 1$ .
4.  $0 \leq -\frac{1}{\epsilon} \log(1 - \epsilon x) - x \leq 2\epsilon x^2$  for  $\epsilon > 0, x \geq 0, \epsilon x \leq \frac{1}{2}$ .
5. For non-negative random variable  $X$ , we have  $\mathbb{E}[X^{1+\theta}] \leq (1 + \theta)\delta^{1+\theta} + (1 + \theta) \int_\delta^\infty r^\theta \mathbb{P}[X \geq r] dr$ , where  $\delta > 0, \theta \geq -1$ .

6.  $\mathbb{P}[X \geq r] \leq 2r \int_0^{\frac{2}{r}} \mathbb{E}[e^{-uX} - 1 + uX] du$ , if  $X \geq 0, \mathbb{E}[X] < \infty$ .

*Proof.* The first three inequalities are obvious and we omit the proofs. For the fourth inequality, we consider the expansion of  $\log(1 - x)$  when  $0 \leq x < 1$ , we have

$$\begin{aligned} -\frac{1}{\epsilon} \log(1 - \epsilon x) - x &= \sum_{i=2}^{\infty} \frac{\epsilon^{i-1} x^i}{i} = \epsilon x^2 \left( \frac{1}{2} + \sum_{i=1}^{\infty} \frac{\epsilon^i x^i}{i+2} \right) \\ &\leq \epsilon x^2 \left( \frac{1}{2} - \log(1 - \epsilon x) \right) \leq 2\epsilon x^2 \end{aligned}$$

The fifth inequality is from the basic probability theory.

$$\begin{aligned} \mathbb{E}[X^{1+\theta}] &= \int_0^{\infty} \int_0^x (1 + \theta)t^{\theta} dt \mathbb{P}[dx] \\ &= (1 + \theta) \int_0^{\infty} \int_t^{\infty} t^{\theta} \mathbb{P}[dx] dt \\ &\leq (1 + \theta) \left( \delta^{1+\theta} + \int_{\delta}^{\infty} t^{\theta} \mathbb{P}[X \geq t] dt \right) \end{aligned}$$

The final inequality is given by

$$\begin{aligned} r \int_0^{\frac{2}{r}} \mathbb{E}[e^{-uX} - 1 + uX] du &= r \int_0^{\frac{2}{r}} \int_0^{\infty} (e^{-ux} - 1 + ux) \mathbb{P}[dx] du \\ &= \int_0^{\infty} \frac{r}{x} \left( 1 - e^{-\frac{2x}{r}} - \frac{2x}{r} + \frac{1}{2} \frac{(2x)^2}{r^2} \right) \mathbb{P}[dx] \\ &\geq \int_r^{\infty} \frac{x}{2r} \mathbb{P}[dx] \geq \frac{1}{2} \mathbb{P}[X \geq r] \end{aligned}$$

We use the second inequality for the third inequality. □

We also need to find a special function  $\varphi_0 \in \mathcal{D}(\mathcal{H})$  such that we have some lower bounds.

**Lemma C.4.2.** *There exists positive function  $\varphi_0 \in \mathcal{D}_{\mathcal{H}}$  such that*

$$\varphi_0(x) \geq C e^{-l|x|^{\sigma}}$$

for some constant  $C$  and any  $l > 0$ .

*Proof.* Consider  $u_0 = 1$  and  $A_t u_0$  for some horizon  $t$ . Then we have the representation

$$T_s^n u_0 = P_s^n u_0 + \int_0^s P_{s-r}^n ((T_r^n u_0) \xi^n) ds$$

where  $P_t^n$  is the discrete heat semi-group. Since  $u_0 = 1$ , we know  $P_s^n u_0 = 1$  for all time  $s$ . Let

$$w_s^n = \int_0^s P_{s-r}^n ((T_r^n u_0) \xi^n) ds$$

we see by the solution theory of PAM that  $w_s^n \in C^{\frac{1-\epsilon}{2}} L^\infty(\mathbb{Z}_n^d, e(t))$  uniformly over  $n$ . Thus there exists  $C > 0$  such that

$$|w_s^n(x)| \leq C s^{\frac{1-\epsilon}{2}} e^{t|x|^\sigma}$$

for each  $x$ , set  $s(x) = \frac{e^{-\frac{2t}{1-\epsilon}|x|^\sigma}}{(2C)^{\frac{2}{1-\epsilon}}}$ , then for any  $0 \leq r \leq s(x)$ , we have

$$|w_r^n(x)| \leq \frac{1}{2}$$

Choosing  $C$  large enough such that  $s(x) \leq t$  for all  $x \in \mathbb{Z}_n^2$ . Then

$$A_t u_0(x) = \int_0^t T_s u_0(x) ds \geq \frac{1}{2} \int_0^{s(x)} dr = \frac{s(x)}{2}$$

This gives the result. □

# Bibliography

- [1] Tom Alberts, Konstantin Khanin, and Jeremy Quastel. The continuum directed random polymer. *Journal of Statistical Physics*, 154(1):305–326, 2014.
- [2] Sergio Albeverio, Leonid Bogachev, Stanislav Molchanov, and Elena Yarovaya. Annealed moment lyapunov exponents for a branching random walk in a homogeneous random branching environment. *Markov Processes and Related Fields*, 6:473–516, 2000.
- [3] Anna Ananova and Rama Cont. Pathwise integration with respect to paths of finite quadratic variation. *Journal de Mathématiques Pures et Appliquées*, 107:737–757, 2017.
- [4] Krishna B Athreya and Peter E Ney. *Branching processes*. Springer, 1972.
- [5] Dmytro Bandak, Nigel Goldenfeld, Alexei A Mailybaev, and Gregory Eyink. Dissipation-range fluid turbulence and thermal noise. *Physical Review E*, 105(6):065113, 2022.
- [6] Jean Bertoin. Temps locaux et intégration stochastique pour les processus de Dirichlet. In *Séminaire de Probabilités, XXI*, volume 1247 of *Lecture Notes in Math.*, pages 191–205. Springer, Berlin, 1987.
- [7] Giuseppe Cannizzaro and Khalil Chouk. Multidimensional sdes with singular drift and universal construction of the polymer measure with white noise potential. *The Annals of Probability*, 46(3):1710–1763, 2018.
- [8] Giuseppe Cannizzaro, Dirk Erhard, and Philipp Schönbauer. 2d anisotropic kpz at stationarity: Scaling, tightness and nontriviality. *The Annals of Probability*, 49(1):122–156, 2021.
- [9] Giuseppe Cannizzaro, Dirk Erhard, and Fabio Toninelli. Weak coupling limit of the anisotropic kpz equation. *arXiv preprint arXiv:2108.09046*, 2021.

- [10] Giuseppe Cannizzaro, Peter K Friz, and Paul Gassiat. Malliavin calculus for regularity structures: the case of gpm. *Journal of Functional Analysis*, 272(1):363–419, 2017.
- [11] Giuseppe Cannizzaro, Massimiliano Gubinelli, and Fabio Toninelli. Gaussian fluctuations for the stochastic burgers equation in dimension  $d \geq 2$ , 2023.
- [12] Giuseppe Cannizzaro and Jacek Kiedrowski. Stationary stochastic navier-stokes on the plane at and above criticality. *Stoch PDE: Anal Comp*, 12:247–280, 2024.
- [13] Georg Cantor. De la puissance des ensembles parfaits de points: Extrait d’une lettre adressée à l’éditeur. *Acta Mathematica*, 4:381–392, 1884.
- [14] Ajay Chandra, Augustin Moinat, and Hendrik Weber. A priori bounds for the  $\phi^4$  equation in the full sub-critical regime. *Archive for Rational Mechanics and Analysis*, 247(3):48, 2023.
- [15] Yan Chen, Ying Yan, and Kewei Zhang. On the local fractional derivative. *Journal of Mathematical Analysis and Applications*, 362(1):17–33, 2010.
- [16] Henry Chiu and Rama Cont. On pathwise quadratic variation for càdlàg functions. *Electron. Commun. Probab.*, 23:Paper No. 85, 12, 2018.
- [17] Henry Chiu and Rama Cont. Causal functional calculus. *Transactions of the London Mathematical Society*, 9:237–269, 2022.
- [18] Khalil Chouk, Jan Gairing, and Nicolas Perkowski. An invariance principle for the two-dimensional parabolic anderson model with small potential. *Stochastics and Partial Differential Equations: Analysis and Computations*, 5(4):520–558, 2017.
- [19] Rama Cont. Functional Ito Calculus and functional Kolmogorov equations. In *Stochastic Integration by Parts and Functional Ito Calculus (Lecture Notes of the Barcelona Summer School in Stochastic Analysis, July 2012)*, Advanced Courses in Mathematics, pages 115–208. Birkhauser Basel, 2016.
- [20] Rama Cont and Purba Das. Quadratic variation and quadratic roughness. *Bernoulli*, 29(1):496 – 522, 2023.
- [21] Rama Cont and David Fournié. A functional extension of the ito formula. *Comptes Rendus Mathématique*, 348(1-2):57–61, 2009.

- [22] Rama Cont and David-Antoine Fournié. Change of variable formulas for non-anticipative functionals on path space. *J. Funct. Anal.*, 259(4):1043–1072, 2010.
- [23] Rama Cont and Ruhong Jin. Fractional Ito calculus. *Transactions of the American Mathematical Society B*, 11:727–761, 2024.
- [24] Rama Cont and Nicolas Perkowski. Pathwise integration and change of variable formulas for continuous paths with arbitrary regularity. *Trans. Amer. Math. Soc. Ser. B*, 6:161–186, 2019.
- [25] Mark Davis, Jan Obloj, and Pietro Siorpaes. Pathwise stochastic calculus with local times. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(1):1–21, 02 2018.
- [26] Donald Dawson. Measure-valued markov processes. In *École d’été de probabilités de Saint-Flour XXI-1991*, pages 1–260. Springer, 1993.
- [27] Donald A Dawson. Infinitely divisible random measures and superprocesses. In *Stochastic analysis and related topics*, pages 1–129. Springer, 1992.
- [28] Paweł Duch. Flow equation approach to singular stochastic pdes. *arXiv preprint arXiv:2109.11380*, 2021.
- [29] R. M. Dudley and R. Norvaiša. *Concrete functional calculus*. Springer Monographs in Mathematics. Springer, New York, 2011.
- [30] Bruno Dupire. Functional Itô calculus. *Quantitative Finance*, 19(5):721–729, 2019.
- [31] Pedro Echeverría. A criterion for invariant measures of markov processes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 61(1):1–16, 1982.
- [32] János Engländer and Ross G. Pinsky. On the construction and support properties of measure-valued diffusions on  $D \subset \mathbb{R}^d$  with spatially dependent branching. *Annals of probability*, pages 684–730, 1999.
- [33] János Engländer and Ross G. Pinsky. The compact support property for measure-valued processes. In *Annales de l’IHP Probabilités et statistiques*, volume 42, pages 535–552, 2006.
- [34] Alison Etheridge. *An introduction to superprocesses*. American Mathematical Soc., 2000.

- [35] Stewart N Ethier and Thomas G Kurtz. *Markov processes: characterization and convergence*. John Wiley & Sons, 2009.
- [36] Franco Flandoli and Dariusz Gatarek. Martingale and stationary solutions for stochastic navier-stokes equations. *Probability Theory and Related Fields*, 102(3):367–391, 1995.
- [37] Franco Flandoli and Marco Romito. Markov selections for the 3d stochastic navier–stokes equations. *Probability Theory and Related Fields*, 140(3):407–458, 2008.
- [38] Hans Föllmer. Calcul d’Itô sans probabilités. In *Séminaire de Probabilités, XV (Univ. Strasbourg, 1979/1980)*, volume 850 of *Lecture Notes in Math.*, pages 143–150. Springer, Berlin, 1981.
- [39] Peter K. Friz and Martin Hairer. *A course on rough paths*. Universitext. Springer, 2014.
- [40] Jürgen Gärtner and Stanislav A Molchanov. Parabolic problems for the anderson model. *Communications in mathematical physics*, 132(3):613–655, 1990.
- [41] Donald Geman and Joseph Horowitz. Occupation densities. *Ann. Probab.*, 8(1):1–67, 02 1980.
- [42] Benjamin Gess, Daniel Heydecker, and Zhengyan Wu. Landau-Lifshitz-Navier-Stokes equations: Large deviations and relationship to the energy equality. *arXiv preprint arXiv:2311.02223*, 2023.
- [43] Patrícia Gonçalves and Milton Jara. Nonlinear fluctuations of weakly asymmetric interacting particle systems. *Arch. Ration. Mech. Anal.*, 212(2):597–644, 2014.
- [44] Lukas Gräfner and Nicolas Perkowski. Energy solutions and generators of singular SPDEs. *MPI MiS Leipzig Spring School 2023*, 2023.
- [45] Lukas Johannes Gräfner. Abstract fock space methods for weak well-posedness of critical and subcritical singular spdes. *Ongoing work*, 2023+.
- [46] Massimiliano Gubinelli. Controlling rough paths. *J. Funct. Anal.*, 216(1):86–140, 2004.
- [47] Massimiliano Gubinelli. Ramification of rough paths. *Journal of Differential Equations*, 248(4):693–721, 2010.

- [48] Massimiliano Gubinelli and Martina Hofmanová. Global solutions to elliptic and parabolic  $\phi^4$  models in Euclidean space. *Communications in Mathematical Physics*, 368(3):1201–1266, 2019.
- [49] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. In *Forum of Mathematics, Pi*, volume 3. Cambridge University Press, 2015.
- [50] Massimiliano Gubinelli and Milton Jara. Regularization by noise and stochastic burgers equations. *Stochastic Partial Differential Equations: Analysis and Computations*, 1(2):325–350, 2013.
- [51] Massimiliano Gubinelli and Nicolas Perkowski. Energy solutions of kpz are unique. *Journal of the American Mathematical Society*, 31(2):427–471, 2018.
- [52] Massimiliano Gubinelli and Nicolas Perkowski. The infinitesimal generator of the stochastic burgers equation. *Probability Theory and Related Fields*, 178(3):1067–1124, 2020.
- [53] Massimiliano Gubinelli and Mattia Turra. Hyperviscous stochastic navier–stokes equations with white noise invariant measure. *Stochastics and Dynamics*, 20(06):2040005, 2020.
- [54] Martin Hairer. A theory of regularity structures. *Inventiones mathematicae*, 198(2):269–504, 2014.
- [55] Martin Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.
- [56] Martin Hairer and Cyril Labbé. A simple construction of the continuum parabolic Anderson model on  $\mathbb{R}^2$ . *Electronic Communications in Probability*, 20:1–11, 2015.
- [57] Xiyue Han, Alexander Schied, and Zhenyuan Zhang. A probabilistic approach to the  $\phi$ -variation of classical fractal functions with critical roughness. *Statistics & Probability Letters*, 168:108920, 2021.
- [58] Fahd Jarad, Thabet Abdeljawad, and Dumitru Baleanu. On the generalized fractional derivatives and their Caputo modification. *J. Nonlinear Sci. Appl*, 10(5):2607–2619, 2017.

- [59] Ruhong Jin and Nicolas Perkowski. The compact support property of rough super brownian motion on  $\mathbb{R}^2$ , 2023.
- [60] Steven G Krantz and Harold R Parks. *A primer of real analytic functions*. Springer Science & Business Media, 2002.
- [61] Nikolai Vladimirovich Krylov. *Lectures on elliptic and parabolic equations in Holder spaces*. American Mathematical Soc., 1996.
- [62] Antti Kupiainen. Renormalization group and stochastic pdes. In *Annales Henri Poincaré*, volume 17, pages 497–535. Springer, 2016.
- [63] Lev Davidovich Landau and Evgenii Mikhailovich Lifshitz. *Fluid Mechanics: Landau and Lifshitz: Course of Theoretical Physics, Volume 6*, volume 6. Elsevier, 2013.
- [64] Gary M Lieberman. *Second order parabolic differential equations*. World Scientific, 1996.
- [65] Joseph Liouville. Sur le calcul des différentielles á indices quelconques. *Journal de l'Ecole Polytechnique*, 13:71–162, 1832.
- [66] Robert Liptser and Albert Nikolaevich Shiryaev. *Theory of martingales*, volume 49. Springer Science & Business Media, 2012.
- [67] Wei Liu and Michael Röckner. *Stochastic partial differential equations: an introduction*. Springer, 2015.
- [68] Rafał M. Łochowski, Jan Oblój, David J. Prömel, and Pietro Siorpaes. Local times and Tanaka-Meyer formulae for càdlàg paths. *Electron. J. Probab.*, 26:Paper No. 77, 29, 2021.
- [69] Terry Lyons, Zhongmin Qian, et al. *System control and rough paths*. Oxford University Press, 2002.
- [70] Terry J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoam.*, 14(2):215–310, 1998.
- [71] Jörg Martin and Nicolas Perkowski. Paracontrolled distributions on Bravais lattices and weak universality of the 2d parabolic Anderson model. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 55, pages 2058–2110. Institut Henri Poincaré, 2019.

- [72] Itaru Mitoma. Tightness of probabilities on  $c([0, 1]; y')$  and  $d([0, 1]; y')$ . *The Annals of Probability*, pages 989–999, 1983.
- [73] Augustin Moinat and Hendrik Weber. Local bounds for stochastic reaction diffusion equations. *Electronic Journal of Probability*, 25:1–26, 2020.
- [74] Augustin Moinat and Hendrik Weber. Space-time localisation for the dynamic  $\phi_3^4$  model. *Communications on Pure and Applied Mathematics*, 73(12):2519–2555, 2020.
- [75] H.P. Mulholland. On generalizations of Minkowski’s inequality in the form of a triangle inequality. *Proceedings of the London mathematical society*, 2(1):294–307, 1949.
- [76] David Nualart. Stochastic calculus with respect to fractional brownian motion. *Annales de la Faculté des sciences de Toulouse : Mathématiques*, 15(1):63–78, 2006.
- [77] Nicolas Perkowski. Spdes, classical and new. *Dimension*, 2:105, 2020.
- [78] Nicolas Perkowski and Tommaso Rosati. A rough super-Brownian motion. *The Annals of Probability*, 49(2):908–943, 2021.
- [79] Maurizio Pratelli. A remark on the  $1/H$ -variation of the fractional Brownian motion. In *Séminaire de Probabilités XLIII*, volume 2006 of *Lecture Notes in Math.*, pages 215–219. Springer, Berlin, 2011.
- [80] Bernhard Riemann. Versuch einer allgemeinen auffassung der integration und differentiation. In R. Dedekind, editor, *Gesammelte mathematische Werke*, pages 353–366. 1876.
- [81] Luis A Roberts, Luis A Caffarelli, and Xavier Cabré. *Fully nonlinear elliptic equations*, volume 43. American Mathematical Soc., 1995.
- [82] L Chris G Rogers. Arbitrage with fractional Brownian motion. *Mathematical Finance*, 7(1):95–105, 1997.
- [83] Stefan G Samko, A.A. Kilbas, and O.I. Marichev. Fractional integrals and derivatives, theory and applications. *Minsk; Nauka I Tekhnika*, 1987.
- [84] Alexander Schied. On a class of generalized Takagi functions with linear pathwise quadratic variation. *J. Math. Anal. Appl.*, 433(2):974–990, 2016.

- [85] S. J. Taylor. Exact asymptotic estimates of Brownian path variation. *Duke Math. J.*, 39(2):219–241, 06 1972.
- [86] Martine Zähle. Integration with respect to fractal functions and stochastic calculus. i. *Probability theory and related fields*, 111:333–374, 1998.
- [87] Rongchan Zhu and Xiangchan Zhu. Three-dimensional navier–stokes equations driven by space–time white noise. *Journal of Differential Equations*, 259(9):4443–4508, 2015.