Mathematical Methods for Valuation and Risk Assessment of Investment Projects and Real Options

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Abstract

In this thesis, we study the problems of risk measurement, valuation and hedging of financial positions in incomplete markets when an insufficient number of assets are available for investment (real options). We work closely with three measures of risk: Worst-Case Scenario (WCS) (the supremum of expected values over a set of given probability measures), Value-at-Risk (VaR) and Average Value-at-Risk (AVaR), and analyse the problem of hedging derivative securities depending on a non-traded asset, defined in terms of the risk measures via their acceptance sets. The hedging problem associated to VaR is the problem of minimising the expected shortfall. For WCS, the hedging problem turns out to be a robust version of minimising the expected shortfall; and as AVaR can be seen as a particular case of WCS, its hedging problem is also related to the minimisation of expected shortfall.

Under some sufficient conditions, we solve explicitly the minimal expected shortfall problem in a discrete-time setting of two assets driven by correlated binomial models.

In the continuous-time case, we analyse the problem of measuring risk by WCS, VaR and AVaR on positions modelled as Markov diffusion processes and develop some results on transformations of Markov processes to apply to the risk measurement of derivative securities. In all cases, we characterise the risk of a position as the solution of a partial differential equation of second order with boundary conditions. In relation to the valuation and hedging of derivative securities, and in the search for explicit solutions, we analyse a variant of the robust version of the expected shortfall hedging problem. Instead of taking the loss function $l(x) = [x]^+$ we work with the strictly increasing, strictly convex function $L_\varepsilon(x) = \varepsilon \log\left(\frac{1+\exp\left\{-\frac{x}{\varepsilon}\right\}}{\exp\left\{-\frac{x}{\varepsilon}\right\}}\right)$. Clearly $\lim_{\varepsilon\to 0} L_\varepsilon(x) = l(x)$. The reformulation to the problem for $L_\varepsilon(x)$ also allow us to use directly the dual theory under robust preferences recently developed in [82]. Due to the fact that

the function $L_\varepsilon(x)$ is not separable in its variables, we are not able to solve explicitly, but instead, we use a power series approximation in the dual variables. It turns out that the approximated solution corresponds to the robust version of a utility maximisation problem with exponential preferences ($U(x) = -\frac{1}{\gamma}e^{-\gamma x}$) for a preferences parameter $\gamma = \frac{1}{\varepsilon}$. For the approximated problem, we analyse the cases with and without random endowment, and obtain an expression for the utility indifference bid price of a derivative security which depends only on the non-traded asset.

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Contents

Introduction

With the dissemination of quantitative methods in the financial sector and advent of complex derivative products, mathematical models have come to play an increasingly important role in financial decision making, especially in the context of pricing, hedging and risk management of derivative instruments.

In view of the recent treatment of the quantification of risk (initiated in [3] and further developed in [16], [29], and [35]; see also [32]) based on a set of desired axioms that every risk measure should satisfy, defining in such a way the class of coherent and convex risk measures, the "fair" pricing of derivative securities or risk-neutral valuation becomes a particular case of measuring risk under an arbitrage-free condition.

The other aspect inherent in measuring risk is the hedging of financial securities. Hedging and measuring risk are two faces of one procedure, as the same three elements to defining risk: a system of prices, a class of permitted actions and a criterion of acceptability are needed for both of them.

It is a well known fact that pricing and hedging of a given contingent claim has a unique solution in a complete market framework, but when some incompleteness is introduced the problem becomes more difficult and an extra criterion is needed in order to pick one price between all arbitrage-free prices.

One alternative method of valuation and hedging in incomplete markets is to use a "superhedging strategy" (see [20] and [52]). But from a practical point of view the cost of superhedging is often too high. Also perfect (super-) hedging takes away the opportunity of making a profit together with the risk of a loss. Suppose the investor is unwilling to put up the initial amount of capital required for a superhedge and is ready to accept some risk. Another set of criteria to pricing and hedging in incomplete markets is called utility maximisation, and it is perhaps, one of the most popular ones. Proposed by Hodges and Neuberger (1989), the price of the contingent claim is obtained as the smallest (resp. largest) amount leading the agent indifferent between selling (resp. buying) the claim and doing nothing. The price obtained is the indifference seller's (resp. buyer's) price. Typically the utility function is assumed to be a strictly increasing and strictly concave function on the real line, but ideas can be extended to the cases when the utility function is just increasing and concave and maybe state dependent. Examples of criteria like these are what are called: expected shortfall, and maximising the probability of a perfect hedge.

Although most of the criteria above for pricing and hedging of financial securities were initially formulated in a specific context, all of them can be reinterpreted as measuring risk (valuation) and finding the hedging strategy for a corresponding risk measure. This point of view is adopted in the present thesis.

Organisation of the thesis and contributions

In this thesis, we study the problems of risk measurement, valuation and hedging of financial positions in incomplete markets when insufficient number of assets are available for complete hedging. One application is to real options.

Chapter 1 contains some background material needed throughout the thesis. The first part discusses the axiomatic approach of risk measures introduced in [3] and further developed in [16], [29] and [30]. We then define the three main measures of risk analysed throughout the thesis, namely: Worst-Case-Scenario risk measure (WCS) (the supremum of the expected values over a set of given probability measures), Value-at-Risk (VaR), and Average value-at-Risk (AVaR). We conclude Chapter 1 with the connection between measuring risk and the associated hedging problems (defining acceptability of the positions via the risk measure).

Before describing the rest of the thesis, we briefly set the mathematical scene (the practical applications will be described later). Assume we work in a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{0\leq t\leq T}\,,\ \mathbb{P})$, and that an investor faces a random liability $H \geq 0$ at time T. If the market is complete and free of arbitrage opportunities, under mild conditions, any contingent claim H with fixed payoff at time T can be replicated or hedged by a trading strategy (v, π) consisting of an initial capital $v \geq 0$ and a dynamic portfolio process $\pi \in \mathcal{A}(v)$[1]. When the market is not complete, then the existence of a replicating process (v, π) cannot always be guaranteed, unless the investor is prepared to hold an initial capital equal to the super-replicating price

$$\sup_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}_{\mathbb{Q}} \left[H_T\right],$$

where $\mathcal{M}_e$ denotes the set of all equivalent martingale measures with respect to the probability $\mathbb{P}$. In this case, the risk involved in the payment H can be completely

[1] $\mathcal{A}(v)$ denotes the set of admissible portfolios. It will be defined in detail in the later chapters.

eliminated because a super-hedging strategy can be performed. On the other hand, when the investor is only willing to put up a smaller amount of the initial capital

$$\tilde{v} \in \left(0, \sup_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}_{\mathbb{Q}}\left[H_T\right]\right),$$

then a non-hedgeable risk will be involved and any hedging strategy (v, π) will be "partial" in the sense that its shortfall

$$(H_T - V_T)^+$$

may be non-zero with positive probability, where V_T denotes the value at time T of the wealth associated to the replicating strategy (v, π). This situation induces the so-called **shortfall risk minimisation** problem: For a fixed initial capital $\tilde{v} \in \left(0, \sup_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}_{\mathbb{Q}}\left[H_T\right]\right)$, find a trading strategy (v, π), with $v \leq \tilde{v}$ and $\pi \in \mathcal{A}(v)$ such that the expected shortfall

$$\mathbb{E}_{\mathbb{P}}\left[(H_T - V_T)^+\right]$$

is minimal under the physical probability measure $\mathbb{P}$.

This problem has been studied in the context of semimartingale processes and general Itô diffusions (see [9], [28] and [31, p. 341]) in the sense that the authors have shown existence and general characterisation of the solution (the trading strategy and the minimal expected shortfall). It turns out that the solution to the minimal expected shortfall problem can be divided into two parts: The first is the solution to a "static-hedging" problem, of minimising

$$\mathbb{E}_{\mathbb{P}}\left[(H_T - Y)^+\right]$$

among all $\mathcal{F}_T$-measurable random variables $Y \geq 0$ which satisfy the constraint

$$\sup_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}_{\mathbb{Q}}\left[Y\right] \leq \tilde{v}.$$

If Y^* denotes the solution in the first part, then the second part consists of fitting the terminal value V_T of an admissible strategy to the optimal solution Y^*. Although it has been shown that the solution exists and is characterised via the above two-step procedure, few explicit solutions or approximating algorithms have been studied in the literature. They will rely of course on the particular model assumed. In relation to the discrete-time settings, [23] has studied the problem with a single asset under binomial dynamics with model uncertainty leading to an incomplete-market situation; [79] provides an algorithm for the trinomial model of one asset, and [83] presents some general results for the multi-state case for a single risky asset. The multi-assets case in a complete financial market has been solved in [79].

As our interest is in real options situations (when the incompleteness of the market comes from insufficient number of assets available for investment), in Chapter 2 we analyse the problem of minimising the expected shortfall of a random liability faced at a fixed future time T in an incomplete market consisting of one riskless asset and two risky assets S and Y, but only one of them (S) is tradable in the market. We model S and Y in discrete-time as two correlated N-period binomial trees, and assume that the liability payoff at time T is a function only on the non traded asset of the form $H(Y_T)$. This setting can be seen as the simpler Markov-chain approximation to its continuous-time counterpart. Using dynamic programming techniques, we are able to find explicitly the minimal expected shortfall and the optimal strategy that solve the problem under a set of sufficient conditions. In the general case, we find upper and lower bounds for the minimal shortfall.

In the second part of the thesis, we focus on continuous-time models. We start Chapter 3 by computing the three measures of risk of interest (WCS, VaR and AVaR) for positions whose models are given by continuous Markov diffusion processes. The main idea to compute risk given by VaR or AVaR of a position X modelled as a diffusion process is to exploit the Markov property and characterise it as the solution to a second-order partial differential equation (PDE) with boundary conditions. In the case of the WCS risk measure, the approach is similar but less direct, as the WCS is defined as the supremum over a set of probability measures. In order to obtain WCS also as the solution to a boundary value PDE we need to state conditions on the set of measures, so the supremum in the definition of WCS is finite.

Motivated by practical application, firstly, we analyse in detail the case when under each measure on the definition of WCS the process X remains a Markov diffusion process. This involves the study of properties of what is called an exponential change of measure transformation of Markov processes and to adapt some results to our present situation. In most of the cases, the PDEs that characterise the risk measures do not have explicit solutions and series expansion or numerical methods need to be applied. In the few cases that do allow explicit solution, solving for the risk PDEs or solving for the transition probability density of the process X are equivalent. This is shown in the last part of Chapter 3.

When the restriction on the Markov property is lifted, we establish conditions so the computation of WCS can be formulated as a stochastic control problem and then as the solution to a nonlinear PDE of second order with boundary condition and a terminal condition. We end the chapter with several examples.

In Chapter 4 we study the problem of computing risk as WCS, VaR and AVaR for derivative securities that depend on an underlying asset given by a Markov diffusion processes as in the preceding chapter. This is, we assume that the derivative security is defined by a positive payoff function $H(S_T)$ on the final value of a security S_t. By the Markov property and our assumptions on the process S_t the random variable $H(S_T)$ may be written as a function $C \in C^{1,2}$ on the process S_t at the current time t. Defining a process given by

$$X_t = C(t, S_t),$$

one can apply Itô's lemma to obtain the dynamics of X. Then our problem reduces to the one studied in the previous chapter of computing the risk for the position X. When the function C is not injective, the dynamics of the process X_t may be degenerate. In order to analyse this situation in detail, we look at the process X_t from the point of view of a local transformation of the process S_t. In particular, we establish conditions and analyse when the transition probability density of a transformed process X can be expressed in terms of the transition probability density of our original process S. In other words, we find how to reduce the solution to the risk-PDEs for the position X to the solution of simpler PDEs corresponding to the solution to the risk-PDEs for the position S.

We apply our results on local transformation of Markov diffusion processes to the computation of risk for derivative securities and illustrate them with examples. We discuss briefly also the relation to this method with the approaches known as delta- and delta-gamma approximation for the computation of risk of derivatives.

Concerning the hedging problems corresponding to the WCS, VaR and AVaR, in Chapter 5 we analyse a variant of the robust version of the expected shortfall hedging problem:

For an initial capital $x \geq 0$, find a hedging strategy (x, π), $\pi \in A(x)$ with terminal value $X_T^{(x,\pi)}$ which optimises

$$\inf_{\pi \in A(x)} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\left(H_T - X_T^{(x,\pi)} \right)^+ \right], \tag{1}$$

in a continuous-time model consisting of two risky assets S_t and Y_t, $0 \leq t \leq T$ (given by geometric Brownian motion) and a risk-free bond B_t, $0 \leq t \leq T$. The asset S_t is assumed to be traded in the financial market but Y_t is not traded.

We consider a random payoff H_T to be function of the underlying process Y_t at time T, that is, $H_T = H(Y_T)$ and the set of measures (priors) $\mathcal{P}$ to be a subset of the equivalent probability measures $\mathcal{M}_e$.

The problem in (1) corresponds to the hedging problem for the $\mathrm{WCS}_{\mathcal{P}}$ risk measure discussed in Chapter 1 Section 1.5.1.1. For the particular choice of priors $\mathcal{P} = \{\mathbb{P}\}$ and[2] $\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}_a : \frac{d\mathbb{Q}}{d\mathbb{P}} \text{ is } \mathbb{P}\text{-a.s. bounded by } \frac{1}{\alpha}\}$ introduced in Proposition 30, we recover the solution to the hedging problems corresponding to VaR_α and AVaR_α, respectively.

In view of the fact that the theory for the primal-dual formulation to the robust versions of expected utility problems has only been recently developed in [82] and under the assumptions that the utility function is a strictly increasing and strictly concave function, we reformulate our original problem in (1) to fit into these assumption by considering an ε-approximation of the shortfall utility function $[x]^+$ for $0 \leq \varepsilon \leq 1$ by considering the following problem:

$$\inf_{\pi \in A(x)} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[U_\varepsilon \left(H(Y_T) - X_T^{(x,\pi)} \right) \right]. \tag{2}$$

with

$$U_\varepsilon(x) = \varepsilon \log \left(\frac{1 + \exp\left\{-\frac{x}{\varepsilon}\right\}}{\exp\left\{-\frac{x}{\varepsilon}\right\}} \right).$$

This is, for $\varepsilon \longrightarrow 0$ we would recover the original expected shortfall problem.

Due to the fact that the utility function $U_\varepsilon(x)$ is not separable in its variables, we are not able to solve explicitly in (2), but instead, we use a power series approximation in the dual variables. It turns out that the approximated solution to (2) is the solution to the corresponding robust version of an utility maximisation problem with exponential preferences ($U(x) = -\frac{1}{\gamma} e^{-\gamma x}$) for a preferences parameter $\gamma = \frac{1}{\varepsilon}$. Then the original expected shortfall problem when $\varepsilon \longrightarrow 0$ would correspond to $\gamma \to \infty$. For the approximated problem, we analyse the cases with and without random endowment, and obtain an expression for the utility indifference bid price corresponding to the liability $H_T = H(Y_T)$.

The study of whether the solution to the problem (value function and optimal strategies) in (2) converges to the optimal solution to the minimal expected shortfall problem when $\varepsilon \longrightarrow 0$ (resp. the convergence to the solution in the utility max. problem with exponential preferences when $\gamma \to \infty$) is left for future research among some other topics derived from this thesis, as described in the final Chapter 6.

[2] $\mathcal{M}_a$ denotes the set of all absolutely continuous probability measures to $\mathbb{P}$.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we present some background material needed throughout the thesis. In the first part, we discuss risk factors and exposures to uncertainty that are the core elements in defining risk measures. We then introduce a risk measure following the axiomatic approach developed by the seminal paper [3] and further developed in [16], [29] and [30]. The key aspect of this axiomatic approach is to define a risk measure from the point of view of a supervising agency as a **capital requirement**: *we are looking for the minimal amount of capital which, if added to the position and invested in a risk-free manner, makes the position acceptable*. In brief, a risk measure is a mapping from the a set of all possible positions to the real line that satisfy the properties of monotonicity and translation invariance. The interpretation of a risk measure as minimal capital required is related to the above properties of monotonicity and translation invariance. If furthermore, the risk measure satisfies a convexity property (respectively homogeneity) it is called a convex (resp. coherent) risk measure. It turns out (see [3] and [29]) that any convex measure of risk can be represented as the supremum over a set of all probability measures of a functional depending on the position and the probability measures. These results and general properties of convex and coherent risk measures are reviewed in the second part of this chapter.

Throughout the thesis we focus our attention on three risk measures, namely: Worst Conditional Scenario (WCS), Value-at-Risk (VaR) and Average Value-at-Risk (AVaR). We define and discuss some of their properties in the third part of the chapter.

By the interpretation of a risk measure as capital requirement, computing the risk of a given position only answers the question: What is the minimal amount of capital needed so that added to the position makes it acceptable? it says nothing about the way the capital needs to be invested. Thus, in the last part of this chapter, we study for our three risk measures the related hedging problem of finding an "optimal" trading strategy that renders the position riskless in terms of WCS, VaR, or AVaR, respectively.

1.2 Risk factors and exposure to uncertainty in risk assessment

Suppose a risk manager needs to carry a risk assessment program for a given portfolio of financial securities. Most of the time, even before the selection of an adequate risk measure, managers have to ask themselves three main questions:

1. What are the risk factors that affect the desired portfolio?
2. What is the right time horizon to measure risk?
3. How should the exposure to uncertainty of these risk factors be measured?

Most of the recent literature on risk measures starts by assuming that all of the above questions have been answered and that the answers are clear to managers. Standard assumptions are considering a fixed time horizon for risk measurements and that the exposure to uncertainty is given by a random variable on a given probability space.

The right answer to the three questions above may be crucial for risk managers when implementing any risk measurement program, and any of them may be a topic for research by itself. Before establishing the mathematical setting for the study of risk measures, we briefly set out some details about risk factors and exposure to uncertainty.[1]

Assume t is the current time for analysis and $T > t$ a fixed future end time such that if Y represents the value of our portfolio of securities, the interval $[t, T]$ belongs to the lifespan of Y. The difference $T - t$ will be called **risk horizon**, and correspondingly the interval $[t, T]$ will be referred as the **risk interval**.

A common assumption is that the portfolio Y is kept fixed until the end of the risk horizon.

Assume we work under a complete and filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\leq\tau\leq T}, \mathbb{P})$ where Ω is the non-empty set of all possible outcomes and take $\mathfrak{X}$ to be the space of

[1]For a more detailed discussion about risk factors and exposure to uncertainty see for example [22].

all real-valued functions on Ω. A function $a : [t,T] \times \Omega \to \mathbb{R}$ is called a **risk factor** over $[t,T]$ (e.g. interest rate, exchange rates, etc). Denote by $A_{t,T}$ the set of all risk factors over $[t,T]$. Assume we have a mapping X of the form

$$X^{t,T} : A_{t,T} \longrightarrow \mathfrak{X}.$$

This map assigns to each risky factor $a \in A_{t,T}$ a unique random variable $X_a^{t,T}(\omega) := X^{t,T}(a) \in \mathfrak{X}$, which we call the **exposure to uncertainty** over the horizon $[t,T]$ and due to the risk factor a. When no confusion arises, we will simply write $X^{t,T}$ omitting the dependence on the risk factor a.

Remark 1 *1. For each fixed $\tau \in [t,T]$, the mapping $X^{\tau,T}$ is determined at time τ based on the information $\mathcal{F}_\tau$, so that $X^{\tau,T}(a)$ is $\mathcal{F}_T$-measurable for each $a \in A_{t,T}$.*

2. $X_a^{\tau,T} := X^{\tau,T}(a)$ can be interpreted as the random loss over the time horizon $[t,T]$.

Example 2 (Exposure to uncertainty for a given portfolio) *Consider $T = t + \theta$ with $0 < \theta \in \mathbb{R}$ fixed. Assume that the constant risk-free rate for discounting cash-flows is r and that we are interested in measuring the risk of a given portfolio Y whose current value is Y_t. In this example, our risk factor is the portfolio Y itself, i.e., $a = Y$. We now show three examples of exposure to uncertainty maps.*

Future net worth *and its expected value are given by:*

$$X^{t,t+\theta}(Y) = Y_{t+\theta} - e^{r\theta}Y_t \text{ and } \mathbb{E}\left[X^{t,t+\theta}(Y)\right] = \mathbb{E}\left[Y_{t+\theta}\right] - e^{r\theta}Y_t.$$

Discounted net worth *and its expected value are given by*

$$X^{t,t+\theta}(Y) = e^{-r\theta}Y_{t+\theta} - Y_t \quad \text{and} \quad \mathbb{E}\left[X^{t,t+\theta}(Y)\right] = e^{-r\theta}\mathbb{E}\left[Y_{t+\theta}\right] - Y_t.$$

Profit and Loss *(P&L) and its expected value are given by*

$$X^{t,t+\theta}(Y) = Y_{t+\theta} - Y_t \quad \text{and} \quad \mathbb{E}\left[X^{t,t+\theta}(Y)\right] = \mathbb{E}\left[Y_{t+\theta}\right] - Y_t.$$

The three examples of exposure to uncertainty are naturally related.

Example 3 (Future net worth for a derivative security) *Assume we are interested in measuring the risk of a European derivative security with maturity time T on an underlying asset S, and that the current price at time t of the derivative is given by the function $u(t, S_t)$. In this case, our risk factor is the underlying asset S; this is $a = S$. Suppose we want to measure the risk of the future net worth of the derivative security value at time T. Then the exposure to uncertainty map and its expected value are given by*

$$X^{t,T}(S) = u(T, S_T) - e^{r(T-t)}u(t, S_t) \text{ and } \mathbb{E}\left[X^{t,T}(S)\right] = \mathbb{E}\left[u(T, S_T)\right] - e^{r(T-t)}u(t, S_t).$$

This is, the exposure to uncertainty measured as the future net worth is the uncertain future value of the derivative, less the risk-free time-T value of the cost now of buying the derivative.

Remark 4 *Note that if, in Example 2, we assume $r = 0$, then all three exposures to uncertainty coincide in value. The only difference is the time at which these variables are considered. Assuming $r = 0$ is equivalent to assuming that there exists a riskless asset and all prices are in discounted terms using the riskless asset as numeraire. Then without loss of generality, we can make the following assumptions.*

Assumption 5 *The risk-free rate r is zero.*

Assumption 6 *For the current time t, and the risk horizon $[t, t + \theta]$ for a given constant $\theta \geq 0$, we denote by X^θ to be the position for measuring risk relative to the discounted net worth of its current value, this is, X^θ is given by*

$$X^\theta := X_{t+\theta} - X_t. \tag{1.1}$$

Remark 7 *When a position X^θ is measured as the discounted net worth $X^\theta = X_{t+\theta} - X_t$, then scenarios for which the position $X^\theta > 0$ represents no risk at all, as this will mean $X_{t+\theta} - X_t > 0$ and no loss will be incurred. Then the only scenarios to care about are those for which $X^\theta < 0$. This is why sometimes measures of risk are defined in terms of $-X^\theta$ instead of X^θ.*

1.3 Static risk measures

In this section we will introduce the definition of a monetary risk measure in the spirit of the axiomatic approach initiated by [3], and followed by [29]. We will follow closely [31] in the exposition.

Monetary risk measures and their representation properties have been defined on a financial model consisting only of two dates (single-period static approach). The extension to the multi-period case and to the dynamical setting is very recent and still a subject of ongoing research. But many of the main ideas and properties in the single-period model hold in the multi-period and dynamical setting. In order to fix ideas, for the rest of this section, we assume we are in a model with only two dates: the current date t and a final date $T = t + \theta$, for $0 < \theta \in \mathbb{R}$.

The key aspect of this axiomatic approach is to define a risk measure from the point of view of a supervising agency, this is, as a **capital requirement**: *we are looking for the minimal amount of capital which, if added to the position and invested in a risk-free manner, makes the position acceptable.*

Similarly as in [3] and [29], we first define a monetary risk measure in its most generality, and then clarify the meaning of "acceptability" of a position by introducing what is called the acceptance set A_ρ of a risk measure ρ.

Note that by the above interpretation of the risk $\rho(X)$ as capital requirement; computing the risk $\rho(X)$ of a given position X only answers the question: What is the minimal amount needed so that added to the position makes it acceptable?, but says nothing about the way the capital $\rho(X)$ needs to be invested. Thus a related hedging problem to measuring risk is to find an "optimal" trading strategy[2] $(\rho(X), \hat{\pi})$, which makes the position X to be riskless in the sense of the acceptance set A_ρ (acceptable).

Let as before, $\mathfrak{X}$ be the space of all financial positions.

Definition 8 ***Monetary risk measure****. A monetary risk measure ρ is a mapping $\rho : \mathfrak{X} \to \mathbb{R}$ such that, for all $X, Y \in \mathfrak{X}$, we have:*

1. *Monotonicity: If $X \leqslant Y$, then $\rho(X) \geqslant \rho(Y)$.*

2. *Translation invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.*

The monotonicity property means that if the payoff profile is increased, then the risk (downside risk) is reduced. This is in accordance with the interpretation of a risk measure as capital requirement. On the other hand, the monotonicity property says that the lower the payoff, the more capital is needed in order for the position to become acceptable. The property of translation invariance tell us about the amount

[2]A trading strategy $(\rho(X), \hat{\pi})$ consists of a given initial amount of capital to invest $\rho(X)$, and $\hat{\pi}$ the monetary amount for investment over a set of given financial assets.

of money which should be added to the position in order to make it acceptable from the point of view of a supervisory agency. Thus, if the amount m is added to the position and invested in a risk-free manner, the capital requirement is reduced by the same amount. Many authors define a risk measure only as the corresponding map ρ without the monotonicity and translation invariance properties. But it turns out that most of the risk measures in practise, particularly the ones analysed in this thesis, satisfy these two properties, thus the equivalence in the definitions.

Remark 9 *For a position $X^\theta := X_{t+\theta} - X_t$ and a risk interval $[t, t+\theta]$, the second term on the right-hand side of X^θ is a deterministic quantity. Then the randomness of X^θ is only due to the term $X_{t+\theta}$, and by the translation invariance property of the risk measures, there is no difference in analysing X^θ or $X_{t+\theta}$. Therefore, without loss of generality, throughout this document we concentrate our analysis as if $X^\theta = X_{t+\theta}$, unless otherwise made explicit. Also, whenever there is no room for confusion, we will also omit the explicit dependence of X on θ, writing X when we mean X^θ.*

Remark 10 *The cash invariance property implies $\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0$, and $\rho(m) = \rho(0) - m$ for all $m \in \mathbb{R}$. This suggests assuming a normalisation whereby $\rho(0) = 0$.*

If a monetary risk measure has the additional property of being convex, then we have the following definition.

Definition 11 (Convex risk measure) *A monetary risk measure $\rho : \mathfrak{X} \to \mathbb{R}$ is called a **convex** risk measure if it satisfies the convexity property*

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \text{ for } 0 \leq \lambda \leq 1.$$

The convexity property is related to the notion of diversification in the sense that diversification in a portfolio should not increase the risk.

Definition 12 *A convex risk measure ρ is said to be **coherent** if it satisfies the following positive homogeneity property:*

$$\text{If } \lambda \geq 0, \text{ then } \rho(\lambda X) = \lambda\rho(X).$$

Remark 13 *A measure that satisfies positive homogeneity is always normalised so that $\rho(0) = 0$, and under this assumption, convexity is equivalent to the following subadditivity property:*

$$\rho(X + Y) < \rho(X) + \rho(Y).$$

Remark 14 *The homogeneity property implies that the risk grows in a linear way as the size of the position increases. This may not be the case for many risk measures.*

1.3.1 Acceptance sets and risk measures

We now introduce the notion of acceptability of a position given by a risk measure.

Definition 15 *Given a risk measure ρ, define the set A_ρ by*

$$\mathrm{A}_\rho := \{X \in \mathfrak{X} : \rho(X) \leq 0\}.$$

The set A_ρ will be called the acceptance set of ρ.

Note that all positions in A_ρ are acceptable in the sense that they do not require additional capital. Conversely, one can also induce a risk measure given an acceptance set $A \subset \mathfrak{X}$.

Definition 16 *For a position $X \in \mathfrak{X}$, and a given set $\mathrm{A} \subset \mathfrak{X}$ we define the related risk measure ρ_A as the minimal capital m for which $m + X$ becomes acceptable:*

$$\rho_\mathrm{A} := \inf\{m \in \mathbb{R} : m + X \in \mathrm{A}\}.$$

In order to understand the connection between acceptance sets, the risk measures associated to them and their hedging problems, consider the following examples.

Example 17 (Sec. 2.1 and 2.2 in [81]) *Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0,...,T}, \mathbb{P})$ and a market with one bond B and a positive risky asset S. We assume that the risk-free rate is zero, therefore $B_t = 1$ for $t = 0, ..., T$.*

Let π be a predictable process and π_t corresponds to the number of shares held of the asset during the trading period $(t-1) \longrightarrow (t)$. It is well known that if we impose the condition that the portfolio is a self-financing one (see Section 5.3), then we define completely the wealth process by the pair (v, π), with v the initial capital and $\pi \in \mathcal{A}(v)$, where $\mathcal{A}(v)$ is the set of all admissible strategies.[3] The associated value process for an initial investment v is given by

$$V_t = v + G_t(\pi) := v + \sum_{k=1}^{t} \pi_k \cdot (S_k - S_{k-1}). \tag{1.3}$$

[3] For the case of positions in L^∞, the set of admissible strategies $\mathcal{A}(v)$ is so that there is a constant $c = c(\pi)$ such that the related gain process satisfies

$$\sum_{k=1}^{t} \pi_k \cdot (S_k - S_{k-1}) \geq -c \;\; \mathbb{P} - \text{a.s.} \tag{1.2}$$

Assume we define a financial position $X \in L^\infty$ to be acceptable if satisfies $X \geq 0$ $\mathbb{P}$-a.s. (if the risky part of X can be hedge at no additional cost). This means, we can find a suitable hedging portfolio π such that

$$X + G_T(\pi) \geq 0 \;\; \mathbb{P} - a.s.$$

This acceptability condition defines the acceptance set

$$\mathcal{A}_0 := \{X \in L^\infty : \exists\, \pi \text{ with } X + G_T(\pi) \geq 0 \; \mathbb{P}\text{-}a.s.\},$$

and the corresponding risk measure ρ_0 defined as

$$\rho_0(X) := \rho_{\mathcal{A}_0}(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}_0\}.$$

Furthermore, if we assume that the market model is arbitrage-free, given the condition $inf\{m \in \mathbb{R} : m \in \mathcal{A}_0\} > -\infty$ (see [81, Theorem 2.1]), then ρ_0 can be represented in terms of the set $\mathcal{M}_e(\mathbb{P})$ of equivalent martingale measures for the price process S, this is,

$$\rho_0(X) = \sup_{\mathbb{Q} \in \mathcal{M}_e(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[-X].$$

Assume our investor is short in $H \geq 0$ at time T (she must deliver the amount H at time T). On one hand, if we define $p_{sup}(H)$ as

$$p_{sup}(H) := \rho_0(-H) = \sup_{\mathbb{Q} \in \mathcal{M}_e(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[H],$$

and provided the right-hand side is finite, then $p_{sup}(H)$ is equal to the cost of super-replicating H, i.e., there exists a trading strategy π such that

$$p_{sup}(H) + G_T(\pi) \geq H \;\; \mathbb{P} - a.s. \tag{1.4}$$

On the other hand, by (1.3) for a given initial capital v and a trading strategy $\pi \in \mathcal{A}(v)$

$$V_T^{(v,\pi)} = v + G_T(\pi). \tag{1.5}$$

Using ρ_0, the risk of the short position is $\rho_0(-H)$. And from the interpretation of a risk measure as the minimum amount of capital, the updated position $\rho_0(-H) - H$ belongs to $\mathcal{A}_0$, this means, there exists a hedging portfolio $\pi \in \mathcal{A}(\rho_0(-H))$ such that

$$\rho_0(-H) - H + G_T(\pi) = V_T^{(\rho_0(-H),\pi)} - H \geq 0 \;\; \mathbb{P} - a.s., \tag{1.6}$$

which is equivalent to the expression in (1.4).

Although by performing such a superhedging strategy, the investor eliminates completely the corresponding risk, the disadvantage is that the initial amount $p_{sup}(H)$ is most of the time too high from a practical point of view. There is a disadvantage even in the case where the claim is attainable, as the elimination of the risk goes together with the elimination of the possibility of making any profit.

Let us therefore suppose that the investor is unwilling to put up the capital $\rho_0(-H)$ and is ready to accept some risk. For a fixed $\tilde{v} \in (0, p_{sup}(H))$, this imply that for any $\pi \in \mathcal{A}(\tilde{v})$ there would exist some $\omega \in \Omega$ such that

$$V_T^{(\tilde{v},\pi)}(\omega) - H(\omega) \geq 0 \quad \textit{(superreplication/replication)} \tag{1.7}$$

and that some $\omega \in \Omega$ where

$$V_T^{(\tilde{v},\pi)}(\omega) - H(\omega) < 0 \quad \textit{(no-replication).} \tag{1.8}$$

Then any hedging strategy will be "partial" in the sense of replication/superreplication.

In order to make the most of the previous situation, we can formulate a sensible "partial" hedging problem by noting that it is desirable to find a hedging portfolio π which deals only with the problematic events -those in (1.8)- and such that $V_T^{(\tilde{v},\pi)}$ is as closest as possible to H. This is achieved by using the shortfall function

$$\left(H - V_T^{(\tilde{v},\pi)}\right)^+$$

as it assigns zero to the superreplication/replication events and a positive quantity to the no- replication events. And as the goal is to make this shortfall small, the general "partial" hedging problem to solve is:[4]

Find a hedging strategy $\pi \in \mathcal{A}(v)$ which attains the infimum in

$$\inf_{\pi \in \mathcal{A}(v)} \left(H - V_T^{(v,\pi)}\right)^+,$$

with $v \leq \tilde{v}$.

The above provided we give sufficient conditions so the random variable $H - V_T^{(v,\pi)} < \infty$; for example, guaranteeing that $H - V_T^{(v,\pi)} \in L^0$.

[4] The problem can also be generalised as in [81, Sec. 2.2] considering another suitable risk measure ρ. Find a hedging strategy $\pi \in \mathcal{A}(v)$ which attains the infimum in

$$\inf_{\pi \in \mathcal{A}(v)} \rho\left(-\left(H - V_T^{(v,\pi)}\right)^+\right),$$

with $v \leq \tilde{v}$.

Later in this chapter, we review briefly the hedging problems associated to the risk measures Worst Conditional Scenario (WCS), Value-at-Risk (VaR) and Average Value-at-Risk (AVaR), and in Chapter 5 we study in more detail the solution to the hedging problem associated to the WCS risk measure.

1.3.2 Robust representation of convex risk measures

We recall now some important characterisations of coherent and convex risk measures and their acceptance sets. For the case when $\mathfrak{X} := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, it has been proved in [81] that any coherent risk measure measure can be interpreted as a sort of worst-case scenario over a set of probability measures. This result is recalled in the following proposition. For similar results in spaces other than L^∞ or generalisations see for example [3],[16], [15], [34], [35] [29] and [32].

Denote by $\mathcal{M}_a := \mathcal{M}_a(\mathbb{P}) := \mathcal{M}_a(\Omega, \mathcal{F}, \mathbb{P})$ the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$; and by $\mathcal{M}_{a,f} := \mathcal{M}_{a,f}(\mathbb{P}) := \mathcal{M}_{a,f}(\Omega, \mathcal{F}, \mathbb{P})$ the set of all finitely additive set functions $\mathbb{Q} : \mathcal{F} \to [0, 1]$ which are normalised to $\mathbb{Q}[\Omega] = 1$ and absolutely continuous with respect to $\mathbb{P}$ in the sense that $\mathbb{Q}[A] = 0$ if $\mathbb{P}[A] = 0$.

Proposition 18 (Prop. 4.6 and 4.14 in [32] and Corollary 1.17 in [81]) *The following statements are equivalent.*

1. *A functional $\rho : \mathfrak{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is a coherent risk measure.*
2. *The acceptance set of ρ, A, is a cone.*
3. *ρ is a continuous from below: $X_n \nearrow X$ then $\rho(X_n) \searrow \rho(X)$.*
4. *There exists a subset $\mathcal{P} \subset \mathcal{M}_a(\mathbb{P})$ representing ρ such that the supremum is attained in*
$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[-X\right], \text{ for all } X \in L^\infty. \tag{1.9}$$

The next proposition shows the analogous representation for convex risk measures.

Proposition 19 (Prop. 4.6 and Thm. 4.15 in [32] and Thm 1.10 in [81]) *The following statements are equivalent:*

1. *A functional $\rho : \mathfrak{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is a convex risk measure.*
2. *The acceptance set of ρ, A is convex.*

3. ρ can be represented as

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{M}_{a,f}(\mathbb{P})} \left\{ \mathbb{E}_{\mathbb{Q}}\left[-X\right] - \alpha_{\min}(\mathbb{Q}) \right\}, \quad X \in L^{\infty}, \tag{1.10}$$

where the penalty function $\alpha_{\min}$ is given by

$$\alpha_{\min}(\mathbb{Q}) := \sup_{X \in \mathrm{A}_{\rho}} \mathbb{E}_{\mathbb{Q}}\left[-X\right] \text{ for } \mathbb{Q} \in \mathcal{M}_{a,f}(\mathbb{P}).$$

Moreover, $\alpha_{\min}$ is the minimal penalty function which represents ρ, i.e., any penalty function α for which (1.10) holds satisfies $\alpha(\mathbb{Q}) \geq \alpha_{\min}(\mathbb{Q})$ for all $\mathbb{Q} \in \mathcal{M}_{a,f}(\mathbb{P})$.

The difference between the representation of a coherent and a convex risk measure is that in the latter, the supremum is taken over a finer set of probability measures but the effect that each measure $\mathbb{Q}$ has on the risk measure ρ is captured via the penalty function α. For each measure $\mathbb{Q}$, the penalty function $\alpha(\mathbb{Q})$ can be interpreted as the worst value among all the acceptable positions computed under the measure $\mathbb{Q}$.

We omit the proofs of the previous propositions as it is out of the scope of this chapter, but we refer to [3],[16], [15], [34], [35] [29] and [32]. See also [31] for a general account on monetary risk measures, their robust representation and properties.

1.4 Dynamic risk measures

The definition of a monetary risk measure and the axiomatic approach in the previous section has been presented in a single-period model. A natural extension of this framework to the multi-period setting, or more generally to the continuous-time setting, is to replace the expectation operator by a conditional expectation operator. Thus, for any $t \leq \tau \leq T$, the dynamical version (in continuous-time) of a convex risk measure ρ_τ on the risk horizon $[t, T]$ will have the following representation

$$\rho_\tau(X) = ess.sup_{\mathbb{Q} \in \mathcal{M}_{a,f}(\mathbb{P})} \left\{ \mathbb{E}_{\mathbb{Q}}\left[-X|\mathcal{F}_\tau\right] - \alpha_{\min}(\mathbb{Q}) \right\}, \quad X \in L^{\infty},$$

where the penalty function $\alpha_{\min}$ is given by

$$\alpha_{\min}(\mathbb{Q}) := \sup_{X \in A_{\rho}(\tau)} \mathbb{E}_{\mathbb{Q}}\left[-X|\mathcal{F}_\tau\right] \text{ for } \mathbb{Q} \in \mathcal{M}_{a,f}(\mathbb{P}).$$

In a continuous-time setting, the dynamical version of a risk measure suggests the introduction of the following time-consistency property.

Definition 20 *A dynamic risk measure is said to be time-consistent on the risk horizon* $[t, T]$*, if for any* $t \leq T_1 \leq T$ *and any position* $X \in \mathfrak{X}$ *we have*

$$\rho_t(X) = \rho_t(-\rho_{T_1}(X)). \tag{1.11}$$

The time-consistency property in the multi-period setting can be analogously defined.

From the interpretation of a risk measure as a minimal capital requirement, the time-consistency property implies that if at time t a position X is accepted with respect to the risk measure ρ on the horizon $[t, T]$, then the position must also be accepted at any other intermediate time T_1, $t \leq T_1 \leq T$, but with the risk measured on the time horizon $[T_1, T]$. The minus sign in (1.11) is required because at time t we need to measure the risk of a short position of value $\rho_{T_1}(X)$. For more on risk measures and their properties see [5] or [32].

Remark 21 *Without the minus sign in front of* $\rho_{T_1}(X)$*, the property of time-consistency in (1.11) corresponds to the Bellman principle in dynamic programming.*

In the next section, we introduce the three dynamic risk measures which we are interested in, namely: **the Worst-Case-Scenario measure (WCS), Value-at-Risk (VaR), and Conditional Value-at-Risk (CVaR)**, their acceptance sets, some properties and the related hedging problems.

1.5 The risk measures: WCS, VaR and AVaR

1.5.1 Worst-Case-Scenario

Assume we have fixed a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\mathcal{M}_1 := \mathcal{M}_1(\mathbb{P}) := \mathcal{M}_1(\Omega, \mathcal{F}, \mathbb{P})$ the set of all probability measures on $(\Omega, \mathcal{F})$.

Definition 22 ***Worst-case scenario.*** *Let* $\mathcal{P}$ *be a subset of* $\mathcal{M}_1$*. The worst-case scenario risk measure over* $\mathcal{P}$ *for a position* X *is defined as:*

$$WCS_{\mathcal{P}}(X) = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[-X] \tag{1.12}$$

i.e., the supremum of expected losses over a set of probability measures.

Its acceptance is given by

$$A_{\mathrm{WCS}_{\mathcal{P}}} := \{X \in \mathfrak{X} : \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}[-X] \leq 0\}. \tag{1.13}$$

It is direct to see that it is a coherent risk measure.

One interpretation of $\mathrm{WCS}_{\mathcal{P}}(X)$ is to measure risk on stress-test scenarios, this is, imagine one needs to know the effect that a set of chosen scenarios (turmoil situations, new model estimations, etc.) has on the position X. This is done by computing the expected value on the worst possible situation among the chosen scenarios $\mathcal{P}$. Another interpretation of $\mathrm{WCS}_{\mathcal{P}}$ is that by assuming $\mathbb{P} \in \mathcal{P}$, then we can interpret $\mathrm{WCS}_{\mathcal{P}}$ as the risk measure that incorporates uncertainty in the model, this is, when there is no full knowledge of the probability structure of the model, but instead an approximation in terms of a set of probability measures (robust preferences). A particular case of the previous situation is assuming a model which may not be fully specified (e.g. a parameter may only be known to lie in a given range). Then in order to be on the safe side, one defines the expected values in terms of the worst possible case among the models in $\mathcal{P}$.

Note that the risk given by $\mathrm{WCS}_{\mathcal{P}}$ depends directly on the choice of the set $\mathcal{P} \subset \mathcal{M}_1$. In order to distinguish some important cases, define as before $\mathcal{M}_a := \mathcal{M}_a(\mathbb{P}) := \mathcal{M}_a(\Omega, \mathcal{F}, \mathbb{P})$ and analogously $\mathcal{M}_e := \mathcal{M}_e(\mathbb{P}) := \mathcal{M}_e(\Omega, \mathcal{F}, \mathbb{P})$ as the set of absolutely continuous and equivalent measures to the reference measure $\mathbb{P}$, respectively, this is,

$$\mathcal{M}_a := \left\{\mathbb{Q} \in \mathcal{M}_1 : \exists \text{ a Radon-Nikodym derivative } \frac{d\mathbb{Q}}{d\mathbb{P}}\right\},$$

and

$$\mathcal{M}_e := \left\{\mathbb{Q} \in \mathcal{M}_1 : \exists \text{ a Radon-Nikodym derivative } \frac{d\mathbb{Q}}{d\mathbb{P}} > 0 \ \ \mathbb{P}\text{-a.s.}\right\}.$$

Some special cases of interest are taking $\mathcal{P}$ equal to $\mathcal{M}_1, \mathcal{M}_a, \mathcal{M}_e$, and $\{\mathbb{Q}\}$ for a given $\mathbb{Q} \in \mathcal{M}_1$.

When $\mathcal{P} = \mathcal{M}_1$, the corresponding risk measure is called worst-case risk measure as shown in the next example.

Example 23 **($\mathcal{P} = \mathcal{M}_1$)** *Define the risk measure $\rho_{\max}$ called the worst-case risk measure by*

$$\rho_{\max}(X) := -\inf_{\omega\in\Omega} X(\omega) = \inf\{m \in \mathbb{R} : m + X \geq 0\}.$$

This measure is coherent and can be represented as

$$\rho_{\max}(X) = \sup_{\mathbb{Q}\in\mathcal{M}_1} \mathbb{E}_{\mathbb{Q}}[-X].$$

For a given position X, the worst-case risk measure, as its name suggests, give us an upper bound of the risk of the position measures by any other risk measure. It gives the largest value we can get.

We now give some remarks on the rest of the special cases.

Remark 24 *1. When $\mathcal{P} = \mathcal{M}_e$, the set $\mathcal{M}_e$ is convex but not compact; then if for the position X we have $\mathbb{E}_{\mathbb{Q}}[X] < \infty$ for each $\mathbb{Q} \in \mathcal{M}_e$, the measure where the supremum is attained will belong to $\mathcal{M}_a$.*

2. In the case where the set consists of only one measure, this is, $\mathcal{P} = \{\mathbb{Q}\}$ for $\mathbb{Q} \in \mathcal{M}_1$, then the problem reduces to find the expected value of the position $-X$ under the measure $\mathbb{Q}$. A particular situation is when taking $\mathcal{P} = \{\mathbb{P}\}$. In this case, the risk measure represents the expected value of the position $-X$ under the physical (real) probability measure.

In Chapter 3 we will be specially interested in computing $WCS_{\mathcal{P}}$ when $\mathcal{P} \subset \mathcal{M}_e$ as it has the interpretation of model risk.

1.5.1.1 The hedging problem

Consider a single-period financial market model on the time-horizon $[t, T]$, which consists of a risky asset S and a bond B. We assume the risk-free rate is zero, therefore $B_t = B_T = 1$. The current price of the asset S is denoted by S_t, and its price at time T is modelled as a nonnegative random variable S_T on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

A trading strategy is a predictable random vector (π^B, π), where π corresponds to the number of assets S held during the trading period $[t, T]$, and π^B is the number of assets invested in the bond B. For an initial capital $v \geq 0$ the value of the wealth v at time t defined by the trading strategy (π^B, π) is

$$v = \pi^B + \pi S_t.$$

As the quantities π^B and π are held constant during the time-period $[t, T]$, by time T, the value of the wealth has changed to

$$V_T = \pi^B + \pi S_T.$$

A portfolio is called *self-financing* if the only changes in the portfolio are due to changes in the asset values. In terms of the wealth values we have $V_T - v = \pi(S_T - S_t)$.

In order to define the gain process $G_T(\pi)$ as in Example 17, and to make explicitly the dependence of V_T on v and π, we write

$$V_T^{(v,\pi)} = v + \pi(S_T - S_t) =: v + G_T(\pi). \tag{1.14}$$

As we want a market model free of arbitrage opportunities (see [51, Ch. 5.8]) we assume the portfolio is such that

$$V_T^{(v,\pi)} \geq 0. \tag{1.15}$$

Denote by $A(v)$ the set of predictable random variables π that define a wealth as in (1.14) and satisfy (1.15) for an initial capital $v \geq 0$. Thus, any self-financing portfolio V can be fully described by a pair (v, π), $\pi \in A(v)$.

Assume the investor needs to pay the random amount $H_T \geq 0$ at time T. The risk, measured by $\mathrm{WCS}_{\mathcal{P}}$, of the short position in H_T is $\mathrm{WCS}_{\mathcal{P}}(-H_T)$. By the interpretation of risk as capital requirement, $\mathrm{WCS}_{\mathcal{P}}(-H_T)$ is the minimal capital so that the total position $\mathrm{WCS}_{\mathcal{P}}(-H_T) - H_T$ is acceptable, i.e.,

$$\mathrm{WCS}_{\mathcal{P}}(-H_T) - H_T \in \mathcal{A}_{\mathrm{WCS}_{\mathcal{P}}}.$$

We are particularly interested in linking hedging strategies with the measurement of risk, then by our assumption of an arbitrage-free model (i.e., $G_T(\pi) \geq 0$ $\mathbb{P}$-a.s. implies $G_T(\pi) = 0$ $\mathbb{P}$-a.s.), we want to find hedging portfolios $\pi \in \mathcal{A}(\mathrm{WCS}_{\mathcal{P}}(-H_T))$ such that satisfy

$$\mathrm{WCS}_{\mathcal{P}}(-H_T) - H_T + G_T(\pi) \in \mathcal{A}_{\mathrm{WCS}_{\mathcal{P}}}.$$

Using the equality in (1.14), the above expression can also be rewritten as

$$V_T^{(\mathrm{WCS}_{\mathcal{P}}(-H_T),\pi)} - H_T \in \mathcal{A}_{\mathrm{WCS}_{\mathcal{P}}},$$

or by the characterisation of the acceptance set for $\mathrm{WCS}_{\mathcal{P}}$ in (1.13), this is similar to find a hedging portfolios $\pi \in \mathcal{A}(\mathrm{WCS}_{\mathcal{P}}(-H_T))$ that satisfy

$$\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[-(V_T^{(\mathrm{WCS}_{\mathcal{P}}(-H_T),\pi)} - H_T) \right] = \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[H_T - V_T^{(\mathrm{WCS}_{\mathcal{P}}(-H_T),\pi)} \right] \leq 0$$

Assume for a moment that the investor is only willing to put up an initial capital $\tilde{v}$ less than $\mathrm{WCS}_{\mathcal{P}}(-H_T)$, then for any $\pi \in \mathcal{A}(\tilde{v})$ the position $V_T^{(\tilde{v},\pi)} - H_T \notin \mathcal{A}_{\mathrm{WCS}_{\mathcal{P}}}$, or equivalently

$$\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[H_T - V_T^{(\tilde{v},\pi)} \right] > 0.$$

In general, for any initial capital v and any hedging portfolio $\pi \in \mathcal{A}(\tilde{v})$, we may distinguish four cases regarding possible events, namely,

1. $H_T - V_T^{(\tilde{v},\pi)} > 0 \;\; \mathbb{P}-\text{a.s.}$

2. $\left.\begin{array}{l} H_T(\omega) - V_T^{(\tilde{v},\pi)}(\omega) > 0 \text{ for some } \omega \in \Omega, \\ H_T(\tilde{\omega}) - V_T^{(\tilde{v},\pi)}(\tilde{\omega}) \leq 0 \text{ for some } \tilde{\omega} \in \Omega, \end{array}\right\}$ but $\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[H_T - V_T^{(\tilde{v},\pi)}\right] > 0$

3. $\left.\begin{array}{l} H_T(\omega) - V_T^{(\tilde{v},\pi)}(\omega) > 0 \text{ for some } \omega \in \Omega, \\ H_T(\tilde{\omega}) - V_T^{(\tilde{v},\pi)}(\tilde{\omega}) \leq 0 \text{ for some } \tilde{\omega} \in \Omega, \end{array}\right\}$ but $\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[H_T - V_T^{(\tilde{v},\pi)}\right] \leq 0$

4. $H_T - V_T^{(\tilde{v},\pi)} \leq 0 \;\; \mathbb{P}-\text{a.s.}$

Case (1) and (2) are typical situations where acceptability w.r.t $\mathcal{A}_{\mathrm{WCS}_{\mathcal{P}}}$ does not hold, and the problematic events are precisely those where

$$H_T(\omega) - V_T^{(\tilde{v},\pi)}(\omega) > 0 \text{ for some } \omega \in \Omega.$$

Then, similarly as in the Example 17, we can formulate a partial hedging problem that deals primarily with these problematic events by introducing the shortfall function $\left(H_T - V_T^{(\tilde{v},\pi)}\right)^+$, and making its expected value, under robust preferences, as small as possible. The general partial hedging problem associated with the risk measure $\mathrm{WCS}_{\mathcal{P}}$ is:

> For an initial capital $v \geq 0$, find a hedging strategy (v, π), $\pi \in A(v)$ which attains the infimum in
>
> $$\inf_{\pi\in A(v)} \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[\left(H_T - V_T^{(v,\pi)}\right)^+\right].$$

Assume the supremum is attained by the measure $\mathbb{Q}^* \in \mathcal{P}$, the problem reduces to the hedging problem called *minimisation of expected shortfall* when the reference measure is $\mathbb{Q}^*$.

When $\mathcal{P} = \{\mathbb{P}\}$, the problem has been studied in [28] in a general semimartingale setting using the Neyman-Pearson lemma, in [94] in a general semimartingale setting as well but using duality methods, and in [9] in a model of general Itô diffusions. In the discrete-time setting, [23] has studied the problem in the binomial case under model uncertainty leading to an incomplete-market situation; [79] provides an algorithm for the trinomial model of one asset, and [83] presents some general results for the multi-state case for one asset.

An interesting related hedging problem is to find the strategy (v, π), $\pi \in A(v)$ that attains the infimum in

$$\sup_{\mathbb{Q}\in\mathcal{P}} \inf_{\pi\in A(v)} \mathbb{E}_{\mathbb{Q}}\left[\left(H_T - V_T^{(v,\pi)}\right)^+\right].$$

We note that the relation

$$\begin{aligned}\underline{V} \quad : &= \sup_{\mathbb{Q}\in\mathcal{P}} \inf_{\pi\in A(v)} \mathbb{E}_{\mathbb{Q}}\left[\left(H_T - V_T^{(v,\pi)}\right)^+\right] \\ &\leq \inf_{\pi\in A(v)} \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[\left(H_T - V_T^{(v,\pi)}\right)^+\right] =: \overline{V}\end{aligned}$$

always holds. The quantity $\underline{V}$ can be interpreted as the the risk measured as expected shortfall from the point of view of an agent who needs to take into account some chosen worst-case scenarios, and the quantity $\overline{V}$ is the risk measured as expected shortfall viewed from the perspective of a regulator who needs to assess the agent's efforts using "worst that can happen".

Existence of the optimal trading strategy for the case when

$$\mathcal{P} = \left\{\mathbb{Q} \in \mathcal{M}_e(\mathbb{P}) : \frac{d\mathbb{Q}}{d\mathbb{P}} \text{ is bounded}\right\}$$

has been studied in the complete market case in [10] and in [9] in incomplete markets. We will come back to this problem in Chapter 5.

1.5.2 Value-at-Risk

A common way to measure risk of a position X in the financial sector is by looking at a quantile of the distribution of X under the given probability $\mathbb{P}$. For a $\alpha \in (0, 1)$, the α-quantile of a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ is any real number q with

$$P[X < q] \leq \alpha \leq P[X \leq q].$$

We then can define the lower quantile function of X as

$$q_X^-(\alpha) = \sup\{m \in \mathbb{R} : \mathbb{P}[X < m] < \alpha\} = \inf\{m \in \mathbb{R} : \mathbb{P}[X \leq m] \geq \alpha\},$$

and the upper quantile function of X by

$$q_X^+(\alpha) = \inf\{m \in \mathbb{R} : \mathbb{P}[X \leq m] > \alpha\} = \sup\{m \in \mathbb{R} : \mathbb{P}[X < m] \leq \alpha\}.$$

The set of all α-quantiles of X is the interval $[q_X^-(\alpha),\ q_X^+(\alpha)]$.

Definition 25 ***VaR.*** *Given $\alpha \in (0,1)$, the value at risk at a level α of a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ is given by*

$$\mathrm{VaR}_\alpha(X) = \inf\{m \in \mathbb{R} : \mathbb{P}[X + m < 0] \le \alpha\} = -q_X^+(\alpha) = q_{-X}^-(1-\alpha).$$

VaR_α can be interpreted as the "smallest" value such that the probability of the absolute loss being at most this value is at least $1-\alpha$. Then 95% and 99% VaR corresponds to taking $\alpha = 0.05$ and $\alpha = 0.01$, respectively. Note that VaR is blind toward risks that create large losses with a very small probability (below the critical probability level α). For a good general account of VaR and its estimation methods with discrete data see for example [19], for some properties and pitfalls of VaR see [66], [74], [72], [93] [48], and [91].

In term of risk measures as capital requirement, VaR_α can be also interpreted as the minimal amount of capital that an investor needs to reserve in order to cover for potential losses with a confidence given by α. In order to see more clearly how VaR_α works, assume the position X has zero risk measured as VaR_α then we have $\mathbb{P}[X_T < X_t] \le \alpha$. It means that among the events of sure loss (those with $X_T - X_t < 0$), we only take as acceptable the events that have lower or equal probability than the chosen level α.

One can show that VaR_α satisfies the property of translation invariance, it is positive homogeneous, monotone decreasing but not a convex risk measure (for examples showing that VaR is not convex see [3], [15], [16] or [31]). The fact that VaR_α is not convex means that VaR_α penalise diversification instead of encouraging it in some models.

We have defined VaR_α only for positions, but as we will extensively be using the notation $X^\theta := X_{t+\theta} - X_t$ to represent a position for measuring risk on the interval $[t, t+\theta]$, and as the only random component on X^θ comes from $X_{t+\theta}$; it is useful to relate $\mathrm{VaR}_\alpha(X^\theta)$ with the value of the upper α-quantile of $X_{t+\theta}$ (i.e., $q^+_{X_{t+\theta}}(\alpha)$). See also Figure 1.1

Proposition 26 *Given $\alpha \in (0,1)$ fixed, the VaR_α of the position X^θ can be related with $q^+_{X_{t+\theta}}(\alpha)$ as follows:*

$$\mathrm{VaR}_\alpha(X^\theta) = X_t - q^+_{X_{t+\theta}}(\alpha).$$

Proof. It follows from the definition. □

The acceptance set for VaR_α is

$$\mathcal{A}_{\mathrm{VaR}_\alpha} := \left\{X \in L^0 : \mathrm{VaR}_\alpha(X) \le 0\right\} = \left\{X \in L^0 : q_X^+(\alpha) \ge 0\right\}. \tag{1.16}$$

The next characterisation of the acceptance set for VaR_α will be useful in the formulation to the hedging problem.

Proposition 27 *Given $\alpha \in (0,1)$, then*

$$\mathcal{A}_{\mathrm{VaR}_\alpha} = \left\{X \in L^0 : \mathbb{P}[X < 0] \leq \alpha\right\} \tag{1.17}$$

Furthermore, if the position is of the form $X^\theta := X_{t+\theta-X_t}$, then

$$\mathcal{A}_{\mathrm{VaR}_\alpha} = \left\{X \in L^0 : X_t \leq q^+_{X_{t+\theta}}(\alpha)\right\}.$$

Proof. Assume $\mathrm{VaR}_\alpha(X) \leq 0$. If $q^+_X(\alpha) \geq 0$ then obviously $\mathbb{P}[X < 0] \leq \alpha$. If $q^+_X(\alpha) < 0$, then it follows that $\mathrm{VaR}_\alpha(X) > 0$, which is a contradiction. Now assume $\mathbb{P}[X < 0] \leq \alpha$, then $q^+_X(\alpha) \geq 0$, which is equivalent to $\mathrm{VaR}_\alpha(X) \leq 0$. The second equality follows immediately from the definition of $\mathcal{A}_{\mathrm{VaR}_\alpha}$ and Proposition 26. $\square$

1.5.2.1 The hedging problem

In order to formulate the partial hedging problem related to the risk measure VaR_α, we consider the same assumptions and proceed similarly as in Section 1.5.1.1.

The risk of a future payment $H_T \geq 0$ at time T, measured by VaR_α, is $\mathrm{VaR}_\alpha(-H_T)$. We need to find hedging portfolios $\pi \in \mathcal{A}(\mathrm{VaR}_\alpha(-H_T))$ such that satisfy $\mathrm{VaR}_\alpha(-H_T) - H_T + G_T(\pi) = V_T^{(\mathrm{VaR}_\alpha(-H_T),\pi)} - H_T \in \mathcal{A}_{\mathrm{VaR}_\alpha}$. Or using the characterisation of $\mathcal{A}_{\mathrm{VaR}_\alpha}$ in (1.17), this is similar to finding hedging portfolios satisfying

$$\mathbb{P}[V_T^{(\mathrm{VaR}_\alpha(-H_T),\pi)} < H_T] \leq \alpha.$$

Again as in Section 1.5.1.1, for any initial capital v and any hedging portfolio $\pi \in \mathcal{A}(v)$, the problematic events are those which $H_T(\omega) - V_T^{(v,\pi)}(\omega) > 0$ for some $\omega \in \Omega$; and are captured by introducing the shortfall function $\left(H_T - V_T^{(v,\pi)}\right)^+$. Then the problem is to find hedging portfolios which minimise the probability that the shortfall is bigger than zero. This is, the general partial hedging problem associated with the risk measure VaR_α is:

For an initial capital $v \geq 0$, find a hedging strategy (v,π), $\pi \in \mathcal{A}(v)$ which attains the infimum in

$$\inf_{\pi \in \mathcal{A}(v)} \mathbb{P}\left[\left(H_T - V_T^{(v,\pi)}\right)^+ > 0\right].$$

Or equivalently that attains the supremum in

$$\sup_{\pi \in \mathcal{A}(v)} \mathbb{P}\left[\left(H_T - V_T^{(v,\pi)}\right)^+ = 0\right].$$

Note that we could also have formulated the following less restrictive partial hedging problem

For an initial capital $v \geq 0$, find a hedging strategy (v, π), $\pi \in \mathcal{A}(v)$ which attains the infimum in

$$\inf_{\pi \in \mathcal{A}(v)} \mathbb{P}\left[V_T^{(v,\pi)} < H_T\right].$$

Or equivalently that attains the supremum in

$$\sup_{\pi \in \mathcal{A}(v)} \mathbb{P}\left[V_T^{(v,\pi)} \geq H_T\right].$$

This hedging criteria are useful when the investor is interested in finding a hedging strategy that overcomes a future value liability but on the most possible scenarios. This fact is captured when maximising the probability that the final value of the wealth process is larger than the liability value.

The latter hedging problem is known in the literature as *maximising the probability of success*. It has been studied in [27] in a general semimartingale setting using the Neyman-Pearson lemma, in [85] in a model of general Itô diffusions, and in [44] in an incomplete market with two correlated assets given by geometric Brownian motions.

Another related hedging problem of interest is the so called *minimising the cost for a given probability of success:*

Find the minimal initial capital v such that

$$\mathbb{P}\left[V_T^{(v,\pi)} \geq H_T\right] \geq 1 - \alpha$$

holds.

1.5.3 Average Value-at-Risk

Given $\alpha \in (0, 1)$, one of the mayor drawbacks of VaR_α is that it does not put any attention to the losses that occur with probability smaller than the critical level α. A natural alternative to overcome this problem is to define a risk measure by taking the average of losses with probability levels less or equal to the critical level α. The resulting measure is sometimes called Expected Shortfall, Conditional Value-at-Risk, or Average Value-at-Risk. We adopt the latter name.

It has been shown, see for example [74], [72], [93], [89], and [2], that AVaR is a risk measure that possesses better qualities than VaR. It is defined as follows.

Definition 28 *The **Average Value-at-Risk** at a level* $\alpha \in (0, 1]$ *of a position* X *is given by*

$$\mathrm{AVaR}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_\gamma(X) d\gamma.$$

Similarly, for a r.v. $X_{t+\theta}$ *that comes from a position with representation* $X^\theta := X_{t+\theta} - X_t$*, we define the* **average upper** α**-quantile** $\overline{q^+_{X_{t+\theta}}(\alpha)}$ *by*

$$\overline{q^+_{X_{t+\theta}}(\alpha)} := \frac{1}{\alpha} \int_0^\alpha q^+_{X_{t+\theta}}(\gamma) d\gamma.$$

In terms of capital requirement, AVaR_α can be interpreted as the amount of capital that needs to be reserved in order to cover in average the potential losses that have a probability of occurrence of α or below.

The integral appearing in the definition of AVaR_α is very inconvenient for computation purposes, therefore we need to recall some other characterisations that are easier to handle.

Let $[x]^+$ represent the positive part of x, and $[x]^-$ its negative part.

Proposition 29 (Lemma 1.31 in [81]) ***Characterisation for*** AVaR_α. *Given* $\alpha \in (0, 1)$ *fixed, and* q *an* α*-quantile of* X*, we have the following characterisations for* AVaR_α*:*

$$\begin{aligned} \mathrm{AVaR}_\alpha(X) &= \frac{1}{\alpha}\mathbb{E}\left[(q - X)^+\right] - q \\ &= \frac{1}{\alpha}\mathbb{E}\left[(-\mathrm{VaR}_\alpha(X) - X)^+\right] + \mathrm{VaR}_\alpha(X). \end{aligned}$$

Furthermore, if the position is of the form $X^\theta := X_{t+\theta} - X_t$*, then*

$$\mathrm{AVaR}_\alpha(X) = X_t - \overline{q^+_{X_{t+\theta}}(\alpha)}.$$

Proof. Take $q = q^+_X(\alpha)$, we have

$$\begin{aligned} \frac{1}{\alpha}\mathbb{E}\left[(q - X)^+\right] - q &= \frac{1}{\alpha}\int_0^1 \left(q^+_X(\alpha) - q^+_X(t)\right)^+ \mathrm{dt} - q^+_X(\alpha) \\ &= \frac{1}{\alpha}\int_0^\alpha \max(-q^+_X(\alpha), -q^+_X(t))\mathrm{dt} + \frac{1}{\alpha}\int_\alpha^1 \max(q^+_X(\alpha) - q^+_X(t), 0)\mathrm{dt} \\ &= \frac{1}{\alpha}\int_0^\alpha - \min\left(q^+_X(\alpha), q^+_X(t)\right)\mathrm{dt} + \frac{1}{\alpha}\int_\alpha^1 \max(q^+_X(\alpha) - q^+_X(t), 0)\mathrm{dt} \\ &= \frac{1}{\alpha}\int_0^\alpha -q^+_X(t)\mathrm{dt} \\ &= \mathrm{AVaR}_\alpha(X). \end{aligned}$$

The rest of the equalities follow directly from the definition of $\overline{q^{+}_{X_{t+\theta}}(\alpha)}$, X^{θ} and the fact that $\frac{1}{\alpha}\mathbb{E}\left[\left(q^{+}_{X_{t+\theta}}(\alpha) - X_{t+\theta}\right)^{+}\right] - q^{+}_{X_{t+\theta}}(\alpha) = -\overline{q^{+}_{X_{t+\theta}}(\alpha)}$. $\square$

For more details on different characterisations for AVaR_{α} see [74], [93], [89], [2] and [29].

Note that the original definition of AVaR_{α} is to take an average of the Value-at-Risk of the position X, over all the critical levels $\lambda > 0$ up to α. The characterisation in the previous proposition exploits the fact that in AVaR_{α} the only scenarios that matter are those where X falls below $\mathrm{VaR}_{\alpha}(X)$ in average, but this is exactly the same as taking the expectation of the random variable $(X - \mathrm{VaR}_{\alpha}(X))^{-}$.

It turns out that AVaR_{α} is a coherent risk measure as shown in the next proposition (see [32, Theo. 4.47 and Rmk. 4.84] and [81, Theo. 1.32 and Rmk. 1.34]).

Proposition 30 *For $\alpha \in (0,1)$, AVaR_{α} is a coherent risk measure which is continuous from below. It has the representation*

$$\mathrm{AVaR}_{\alpha}(X) = \max_{\mathbb{Q}\in\mathcal{P}_{\alpha}} \mathbb{E}_{\mathbb{Q}}[-X], \quad X \in L^{1}, \tag{1.18}$$

where $\mathcal{P}_{\alpha}$ is the set of all probability measures $\mathbb{Q} \in \mathcal{M}_{a}$ whose density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is $\mathbb{P}$-a.s. bounded by $\frac{1}{\alpha}$. Furthermore, the maximum in (1.18) is attained by a measure $\mathbb{Q}_{\mathrm{AVaR}_{\alpha}} \in \mathcal{M}_{a}$, whose density is given by

$$\frac{d\mathbb{Q}_{\mathrm{AVaR}_{\alpha}}}{d\mathbb{P}} = \frac{1}{\alpha}\left(1_{\{X<q\}} + k1_{\{X=q\}}\right), \tag{1.19}$$

where q is a α-quantile of X, and where k is defined as

$$k := \begin{cases} 0 & \text{if} \quad \mathbb{P}[X=q] = 0 \\ \frac{\alpha - \mathbb{P}[X<q]}{\mathbb{P}[X=q]} & \text{otherwise.} \end{cases} \tag{1.20}$$

Corollary 31 (Cor. 4.49 in [32] and Cor. 1.35 in [81]) *For all $X \in L^{\infty}$,*

$$\begin{aligned} \mathrm{AVaR}_{\alpha}(X) &\geq \mathbb{E}[-X : -X \geq \mathrm{VaR}_{\alpha}(X)] \\ &\geq \sup\{\mathbb{E}[-X : A] : \mathbb{P}[A] > \alpha\} \\ &\geq \mathrm{VaR}_{\alpha}(X). \end{aligned}$$

The first two inequalities are identities if $\mathbb{P}\left[X \leq q^{+}_{X}(\alpha)\right] = \alpha$.

Remark 32 *The measure AVaR_{α} is just a particular case of the $WCS_{\mathcal{P}}(X)$ risk measure by taking $\mathcal{P} = \mathcal{P}_{\alpha}$.*

The acceptance set corresponding to AVaR_α is

$$\begin{aligned}
\mathcal{A}_{\mathrm{AVaR}_\alpha} &:= \left\{X \in L^1 : \mathrm{AVaR}_\alpha(X) \leq 0\right\} \\
&= \left\{X \in L^1 : \frac{1}{\alpha}\mathbb{E}\left[(-X - \mathrm{VaR}_\alpha(X))^+\right] + \mathrm{VaR}_\alpha(X) \leq 0\right\} \\
&= \left\{X \in L^1 : \max_{\mathbb{Q}\in\mathcal{P}_\alpha} \mathbb{E}_\mathbb{Q}[X] \geq 0\right\} \\
&= \left\{X \in L^1 : \mathbb{E}_{\mathbb{Q}_{\mathrm{AVaR}_\alpha}}[X] \geq 0\right\} \\
&= \left\{X \in L^1 : \mathbb{E}\left[\frac{X}{\alpha}\left(1_{\{X<q\}} + k1_{\{X=q\}}\right)\right] \geq 0\right\},
\end{aligned}$$

for q an α-quantile of X and k defined in (1.20).

1.5.3.1 The hedging problem

Assume an investor needs to pay the random amount $H_T \geq 0$ at time T, and the same assumptions in the Section 1.5.1.1 hold. As in the case for WCS, we can deduce similarly that the associated hedging problem for AVaR_α is the hedging problem of *minimisation of expected shortfall under robust preferences* when the set of measures is $\mathcal{P}_\alpha$. This is, the related *partial hedging problem* can be formulated as follows:

For an initial capital $v \geq 0$, find a hedging strategy (v, π), $\pi \in A(v)$ which attains the infimum in

$$\inf_{\pi\in A(v)} \sup_{\mathbb{Q}\in\mathcal{P}_\alpha} \mathbb{E}_\mathbb{Q}\left[\left(H_T - V_T^{(v,\pi)}\right)^+\right].$$

1.6 Superreplication and partial hedging

In the previous section we have formulated the hedging problems associated with the three risk measures WCS, VaR and AVaR without assuming anything on the financial market (complete or incomplete market model, etc.). Assume we work on a time horizon $[t, T]$ and under a complete and filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\leq\tau\leq T}, \mathbb{P})$. If the market is complete and free of arbitrage opportunities, any contingent claim H_T with fixed payoff at time T can be replicated or hedged by a trading strategy (v, π) consisting of an initial capital $v \geq 0$ and a dynamical portfolio process $\pi \in \mathcal{A}(v)$. When the market is not complete, for example, when insufficient number of assets are available for investment, then the existence of a replicating process (v, π) cannot always be guaranteed, unless the investor is prepared to hold an initial capital equal to the super-replicating price

$$\sup_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}_{\mathbb{Q}}\left[H_T\right],$$

where $\mathcal{M}_e$ denotes the set of all equivalent martingale measures with respect to the probability $\mathbb{P}$.

In this case, the risk involved in the investment H_T can be completely eliminated because a super-hedging strategy can be performed. On the other hand, when the investor is only willing to put up a smaller amount of the initial capital

$$v \in \left(0, \sup_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}_{\mathbb{Q}}\left[H_T\right]\right),$$

then a non-hedgeable risk will be involved and any hedging strategy (v, π) will be "partial" in the sense that its shortfall

$$(H_T - V_T)^+$$

may be non zero with positive probability.

In the more general setting one can formulate the partial hedging problem not only by looking at the shortfall $S := (H_T - V_T)^+$, but any other similar criterion. Popular choices of criterion for hedging in incomplete markets are: maximise the expected utility U of the difference $-D := V_T - H_T$, minimise the risk of D or of the shortfall S by a convex risk functional, this is $l(D)$ or $l(S)$, respectively (for some examples of risk measures in terms of the expected shortfall see Appendix A.2).

In relation to the hedging problems associated with WCS, VaR and AVaR the chosen criterion is $S := (H_T - V_T)^+$, and in order to specify a pure partial hedging situation one need to incorporate the initial capital constraints.

The partial hedging problem for WCS need to be rewritten as follows:

For a fixed amount $\tilde{v}$,

$$\tilde{v} \in \left(0, \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[H_T\right]\right),$$

and an initial capital $\tilde{v} \geq v \geq 0$, find a hedging strategy (v, π), $\pi \in A(v)$ which attains the infimum in

$$\inf_{\pi\in A(v)} \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[\left(H_T - V_T^{(v,\pi)}\right)^+\right].$$

For the risk measure VaR, we have

For a fixed amount $\tilde{v}$,

$$\tilde{v} \in \left(0, \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[H_T\right]\right),$$

and an initial capital $\tilde{v} \geq v \geq 0$, find a hedging strategy (v, π), $\pi \in \mathcal{A}(v)$ which attains the infimum in

$$\inf_{\pi\in\mathcal{A}(v)} \mathbb{P}\left[\left(H_T - V_T^{(v,\pi)}\right)^+ > 0\right].$$

Or equivalently that attains the supremum in

$$\sup_{\pi\in\mathcal{A}(v)} \mathbb{P}\left[\left(H_T - V_T^{(v,\pi)}\right)^+ = 0\right].$$

Or alternatively,

For a fixed amount $\tilde{v}$,

$$\tilde{v} \in \left(0, \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[H_T\right]\right),$$

and an initial capital $\tilde{v} \geq v \geq 0$, find a hedging strategy (v, π), $\pi \in \mathcal{A}(v)$ which attains the infimum in

$$\inf_{\pi\in\mathcal{A}(v)} \mathbb{P}\left[V_T^{(v,\pi)} < H_T\right].$$

Or equivalently that attains the supremum in

$$\sup_{\pi\in\mathcal{A}(v)} \mathbb{P}\left[V_T^{(v,\pi)} \geq H_T\right].$$

The AVaR partial hedging problem becomes

For a fixed amount $\tilde{v}$,

$$\tilde{v} \in \left(0, \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[H_T\right]\right),$$

and an initial capital $\tilde{v} \geq v \geq 0$, find a hedging strategy (v, π), $\pi \in A(v)$ which attains the infimum in

$$\inf_{\pi\in A(v)} \sup_{\mathbb{Q}\in\mathcal{P}_\alpha} \mathbb{E}_{\mathbb{Q}}\left[\left(H_T - V_T^{(v,\pi)}\right)^+\right].$$

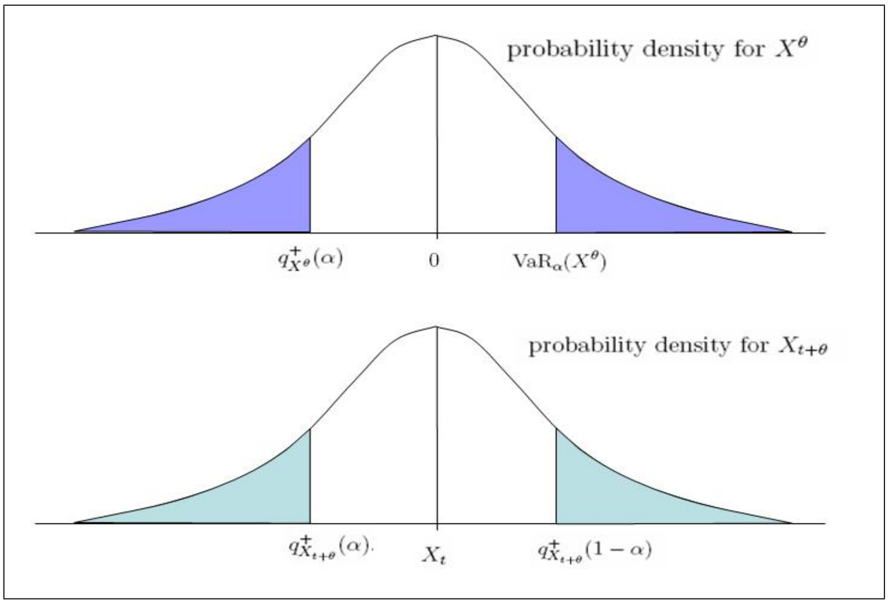


Figure 1.1: Example of continuous, symmetric around zero probability densities for the position X^θ and the r.v. $X_{t+\theta}$. In this case, the position X^θ is not acceptable with respect to $\mathcal{A}_{\mathrm{VaR}_\alpha}$.

Part I

Risk and Hedging in Discrete-time: A Two-factor Binomial Model

Chapter 2

Minimisation of Expected Shortfall

2.1 Introduction

Assume we work on the time horizon $[0, T]$ in a complete and filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{0\leq t\leq T}\,,\ \mathbb{P})$, and that an investor faces a random liability $H \geq 0$ at time T. In this chapter, we are interested in the following partial hedging problem: For a fixed initial capital $\tilde{v} \in \left(0, \sup_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}_{\mathbb{Q}}\left[H_T\right]\right)$, the problem is to find a trading strategy (v, π), with $v \leq \tilde{v}$ and $\pi \in \mathcal{A}(v)$ such that the expected shortfall

$$\mathbb{E}_{\mathbb{P}}\left[(H_T - V_T)^+\right] \tag{2.1}$$

is minimal under the physical probability measure $\mathbb{P}$.

This problem has been studied in the context of semimartingale processes and general Itô diffusions (see [9], [28] and [31, p. 341]) in the sense that the authors have shown existence and general characterisation of the solution (the trading strategy and the minimal expected shortfall). It turns out that the solution to the minimal expected shortfall problem can be divided into two parts: the solution to a "static-hedging" problem of minimising

$$\mathbb{E}_{\mathbb{P}}\left[(H_T - Y)^+\right]$$

among all $\mathcal{F}_T$-measurable random variables $Y \geq 0$ which satisfy the constraint

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[Y\right] \leq \tilde{v}.$$

If Y^* denotes the solution in the first part, then the second part consists in fitting the terminal value V_T of an admissible strategy to the optimal solution Y^* . This will be recalled in more detail in Section 2.2.

This two-steps solutions is intuitively clear as the optimisation criterion involves only values at time T, therefore, minimising only over $\mathcal{F}_T$-measurable random variables is equivalent, with the condition $\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[Y] \leq \tilde{v}$ needed to guarantee that we could match the value of the minimiser Y^* to a dynamical hedging strategy with initial value less than the constraint threshold $\tilde{v}$.

Although it has been shown that the solution exists and is characterised via the above two-step procedure, few explicit solutions or approximating algorithms have been studied in the literature. They will rely of course on the particular model assumed. In the discrete-time setting, [23] has studied the problem in the binomial case under model uncertainty leading to an incomplete-market situation; [79] provides an algorithm for the trinomial model of one asset, and [83] presents some general results for the multi-state case for one asset.

Because our interest is in real options (when there are non-traded assets in the market), in this chapter we analyse the problem in the basic setting of two correlated assets (one traded and one non-traded). Also, with the goal of understanding the nature of the solution and the optimal strategies, and in order to be able to compute explicitly the strategies and the expected shortfall, we will assume a discrete-time model of two correlated N-period binomial trees. We show how even in this the simplest discrete-time approximation of a continuous time-model it is difficult to find in general an explicit solution to the problem, as the key issue in the solution is that the value function preserves the same form at each time step.

2.2 The two-step procedure in the minimisation of expected shortfall

The core of the problem of minimising the expected shortfall in (2.1) is to find a *dynamic* self-financing trading strategy which solves a *static* optimisation problem (of a terminal value). This feature of *dynamically-hedging* a *static* position is also reflected in the shape of the solution to the problem in (2.1) (see [9], [28]). It suggests decomposing the problem into two parts:

1. The *static optimisation* problem: replace the terminal value of the (dynamic) trading strategy by an appropriate (static) random variable, and

2. The *dynamic-hedging* strategy: perform a dynamic replicating trading strategy on the modified claim.

In order to formulate the problem as a two-step procedure, we need some definitions beforehand.

Definition 33 (Def 1 in [83]) *Let*

$$\mathbf{V}_{\infty} := \{V : V \leq H_T \text{ and } \mathbb{E}_{\mathbb{Q}}[V] \leq v \text{ for all } \mathbb{Q} \in \mathcal{M}\}$$

denote the set of all modified contingent claims for which the price of their super-replicating strategy is less than or equal to the initial capital.

Definition 34 (Def 1 in [83]) *Let*

$$\mathbf{V}_b := \{V : H_T - b \leq V \leq H_T \text{ and } \mathbb{E}_{\mathbb{Q}}[V] \leq v \text{ for all } \mathbb{Q} \in \mathcal{M}\}$$

for all $b \in \mathbb{R}^+$, the set of all modified contingent claims for which b is an upper bound for the shortfall $(H_T - V^{v,\pi})^+$ when π is the super-hedging strategy of V.

The problem of minimising expected shortfall can be written as

Proposition 35 (Prop 2 in [83]) *Let $\hat{V} \in \mathbf{V}_b$ for $b \geq \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[H_T] - v$ denote a modified contingent claim which is optimal in the sense that*

$$\hat{V} = \arg\min_{V\in\mathbf{V}_b} \mathbb{E}[H_T - V_T].$$

Then the optimal expected shortfall strategy $\hat{\pi}$ of the problem in (2.1) is the super-replicating strategy for the claim $\hat{V}$:

$$\mathbb{E}\left[H_T - V_T^{v,\hat{\pi}}\right] = \min_{\mathcal{A}(v)} \mathbb{E}\left[(H_T - V^{v,\pi})^+\right] = \mathbb{E}\left[H_T - \hat{V}\right].$$

Proof. See [28] for the proof in continuous-time general semimartingale setting, [67] in a discrete-time setting, and [83] in the context of discrete time single asset. □

The above proposition justifies the following two-step procedure proposed in [83].

STEP 1 (static optimisation problem) Find an optimal modified contingent claim $\hat{V} \in \mathbf{V}_b$ with $\hat{V} = \arg\min_{V\in\mathbf{V}_b} \mathbb{E}[H_T - V]$.

STEP 2 (Representation problem) Determine the super-replicating strategy of $\hat{V}$.

Equivalently, STEP 1 above can be characterised as

Proposition 36 *The static optimisation problem in STEP 1 is equivalent to solving*

$$\max \mathbb{E}[V]$$

under the constraints

$$H_T - b \leq V \leq H_T \text{ and } \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[V] \leq v.$$

Remark 37 *The formulation of the problem of minimising expected shortfall in Proposition 36 highlights the strong relation with the problem of maximising the probability of a perfect hedge. This can be seen as follows. By the constraint $H_T - b \leq V \leq H_T$, the maximisation of $\mathbb{E}[V]$ will give an value V^* close to H_T in their expected values, but this is equivalent to maximising the probability that V is bigger than the payoff H_T, therefore establishing the relation between the two problems.*

In the rest of the section, we will analyse the problem in the basic setting of two correlated assets (one traded and one non-traded). Also, with the goal of understanding the nature of the solution and the optimal strategies, and in order to be able to compute explicitly the strategies and the expected shortfall, we will assume a discrete-time model of two correlated N-period binomial trees.

2.3 The two-factors N-period binomial model for the expected shortfall

Consider a N-period model on the horizon $[0, N]$ consisting of one riskless and two risky assets. Only one of the risky assets is considered to be traded in the market. Let us denote by S_0 and Y_0 the values of the traded (stock) and non traded asset at time zero, respectively. At the end of each period $[n, n+1]$, $n = 0, ..., N-1$, the traded asset can only take two values $S_{n+1} = S_n\xi_{n+1}$, where $\{\xi_{n+1}\}_{n=0,...,N-1}$ is a sequence of i.i.d. random variables taking values in the set $\{u, d\}$ with $0 < d < 1 < u$ for $n = 0, ..., N-1$. In a similar manner, the value of the non-traded asset satisfies $Y_{n+1} = Y_n\eta_{n+1}$, where as before, $\{\eta_{n+1}\}_{n=0,...,N-1}$ is a sequence of i.i.d. random variables with values in the set $\{h, l\}$, with $l < h$ for $n = 0, ..., N-1$.

We are interested in the two-dimensional stochastic process $(S_n, Y_n)_{0\leq n\leq N}$ defined on a probability space $(\Omega, (\mathcal{F}_n)_{0\leq n\leq N}, \mathbb{P})$, where the filtration $\mathcal{F}_n$ is generated by the random variables $S_{n+1}, Y_{n+1}, n = 0, ..., N-1$, or equivalently by the random variables

$\xi_{n+1}, \eta_{n+1}, n = 0, ..., N-1$. Also $\mathcal{F}_n$ is such that its marginal probabilities are constant (i.e., the probabilities do not depend on time), this is for $n = 0, ..., N-1$

$$(S_{n+1}, Y_{n+1}) = \begin{cases} (uS_n, hY_n) & \text{with probability} \quad p_1, \\ (uS_n, lY_n) & \text{with probability} \quad p_2, \\ (dS_n, hY_n) & \text{with probability} \quad p_3, \\ (dS_n, lY_n) & \text{with probability} \quad p_4. \end{cases}$$

We also assume that $p_1 + p_2 + p_3 + p_4 = 1$. Note that without loss of generality and simplicity we can consider our probability space $(\Omega, (\mathcal{F}_n)_{o \leq n \leq N}, \mathbb{P})$ to be the minimal one to support such conditions. Take $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}^N$. Denote by

$$\omega_n^{n_1,n_2,n_3} := (\omega_1)^{n_1} (\omega_2)^{n_2} (\omega_3)^{n_3} (\omega_4)^{N-n-(n_1+n_2+n_3)}, \ n = 0, ..., N-1, \tag{2.2}$$

for $0 \leq n_1, n_2, n_3 \leq N - n$ with $0 \leq n_1 + n_2 + n_3 \leq N - n$, representing a generic event in the space Ω, and each $\omega_i, i = 1, 2, 3, 4$ are the four possible states in each single-period marginal, i.e.,

$$\omega_1 := (u, h); \omega_2 := (u, l); \omega_3 := (d, h) \text{ and } \omega_4 := (d, l). \tag{2.3}$$

The exponents $n_i, i = 1, 2, 3, 4$ in (2.2) are the number of times the single-period events $\omega_i, i = 1, 2, 3, 4$ occurred from n to N.

We then take the σ-algebra to be $\mathcal{F} = 2^\Omega$ of all subsets of Ω, and the probability law $\mathbb{P}$ defined on each event $\omega_n^{n_1,n_2,n_3}$ as follows:

$$\mathbb{P}\left(\omega_n^{n_1,n_2,n_3}\right) = (p_1)^{n_1} (p_2)^{n_2} (p_3)^{n_3} (p_4)^{N-n-(n_1+n_2+n_3)}, \ n = 0, ..., N-1. \tag{2.4}$$

For simplicity, assume the interest rate is zero, and that the investor forms a portfolio consisting of ϑ_n units in the cash account and π_n units of the asset S at each time n, $n = 0, ..., N-1$, but is not allowed to invest in the correlated asset Y. Her self-financing wealth value evolves as

$$V_{n+1}^{v,\pi} = V_n + \pi_n(S_{n+1} - S_n), \ n = 0, ..., N-1. \tag{2.5}$$

Consider a contingent claim (liability) with maturity N and whose payoff $H_N \in \mathcal{F}_N$ is written as a function $H(Y_N)$ of the non-traded asset, therefore we are in a situation of an incomplete market.

2.3.1 Marginal martingale measures

We have assumed that the probability measure does not change during the different single-periods, i.e., the marginal probability measures are constant. For simplicity, we analyse some measures only in the single-period model, as we can extend the results on the measures to the N-period model by pasting together all the single-period marginals (see [17, Chapter 2]). In order to ease the notation and avoid confusion, assume our single-period model goes from 0 to T.

As the market is not complete, there are infinitely many martingale measures. Let $\mathbb{Q}$ be a generic marginal measure with $q_i = \mathbb{Q}\{\omega_i\} > 0$, for $i = 1, 2, 3, 4$ and $\omega_i,\ i = 1, 2, 3, 4$ the four possible events in (2.3).

The conditional probabilities for S and Y are:

$$\begin{array}{ll}
\mathbb{Q}\left[S_T = S_0u | Y_T = Y_0h\right] = \frac{q_1}{q_1+q_3}, & \mathbb{Q}\left[Y_T = Y_0h | S_T = S_0u\right] = \frac{q_1}{q_1+q_2}, \\
\mathbb{Q}\left[S_T = S_0d | Y_T = Y_0h\right] = \frac{q_3}{q_1+q_3}, & \mathbb{Q}\left[Y_T = Y_0l | S_T = S_0u\right] = \frac{q_2}{q_1+q_2}, \\
\mathbb{Q}\left[S_T = S_0u | Y_T = Y_0l\right] = \frac{q_2}{q_2+q_4}, & \mathbb{Q}\left[Y_T = Y_0h | S_T = S_0d\right] = \frac{q_3}{q_3+q_4}, \\
\mathbb{Q}\left[S_T = S_0d | Y_T = Y_0l\right] = \frac{q_4}{q_2+q_4}, & \mathbb{Q}\left[Y_T = Y_0l | S_T = S_0d\right] = \frac{q_4}{q_3+q_4}.
\end{array}$$

And for the measure $\mathbb{Q}$ to be a martingale measure it needs to satisfy

$$\begin{array}{l}
\mathbb{E}_{\mathbb{Q}}\left[S_T | Y_T = Y_0h\right] = \frac{q_1}{q_1+q_3} S_0u + \frac{q_3}{q_1+q_3} S_0d = S_0, \\
\mathbb{E}_{\mathbb{Q}}\left[S_T | Y_T = Y_0l\right] = \frac{q_4}{q_2+q_4} S_0u + \frac{q_2}{q_2+q_4} S_0d = S_0,
\end{array}$$

which together with the unity condition $q_1 + q_2 + q_3 + q_4 = 1$ give us the relation

$$q_1 + q_2 = \frac{1-d}{u-d} := q.$$

Then the generic martingale measure will be given by the two parameter vector[1]

$$(q - \alpha, \alpha, 1 - q - \beta, \beta) \text{ for } \alpha \in [0, q], \beta \in [0, 1 - q]. \tag{2.6}$$

Note also that $\alpha \notin \{0, q\}$ and $\beta \notin \{0, 1-q\}$ are necessary and sufficient conditions for the martingale measure to be equivalent to $\mathbb{P}$.

[1] This is equivalent to solve the following matrix-form 3×4 system of linear equations

$$\left(\begin{array}{cccc|c}
u-1 & 0 & d-1 & 0 & 0 \\
0 & u-1 & 0 & d-1 & 0 \\
1 & 1 & 1 & 1 & 1
\end{array}\right)$$

which for $q := \frac{1-d}{u-d}$ reduces to the 2×4 system

$$\left(\begin{array}{cccc|c}
1-q & 1-q & q & q & 0 \\
1 & 1 & 1 & 1 & 1
\end{array}\right).$$

This explains why a generic martingale measure in (2.6) is a two-parameter vector.

Remark 38 *The two free parameters in the expression for the generic martingale measure in (2.6) are associated with the fact that we have not imposed any restriction on the asset Y (it does not need be a martingale under $\mathbb{Q}$). We can interpret that one parameter is due to Y and the other determines the correlation between S and Y.*

2.3.2 Some important martingale measures

2.3.2.1 The impact measures

From the set of equivalent martingale measures defined in (2.6), the measures

$$\mathbb{Q}^q : \left(q^2, q(1-q), q(1-q), (1-q)^2\right) \tag{2.7}$$

and

$$\mathbb{Q}^{1-q} : \left(q(1-q), q^2, (1-q)^2, q(1-q)\right) \tag{2.8}$$

are the only measures with marginals (in the single-period model) that satisfy

$$\begin{array}{lclcl} \mathbb{Q}^q\left[Y_T = Y_0 h | S_T = S_0 u\right] & = & \mathbb{Q}^q\left[Y_T = Y_0 h | S_T = S_0 d\right] & = & q \\ \mathbb{Q}^{1-q}\left[Y_T = Y_0 l | S_T = S_0 u\right] & = & \mathbb{Q}^{1-q}\left[Y_T = Y_0 l | S_T = S_0 d\right] & = & q \\ \mathbb{Q}^q\left[Y_T = Y_0 l | S_T = S_0 u\right] & = & \mathbb{Q}^q\left[Y_T = Y_0 l | S_T = S_0 d\right] & = & 1-q \\ \mathbb{Q}^{1-q}\left[Y_T = Y_0 h | S_T = S_0 u\right] & = & \mathbb{Q}^{1-q}\left[Y_T = Y_0 h | S_T = S_0 d\right] & = & 1-q \end{array} \tag{2.9}$$

In the N-period model, the above conditions will be written as

$$\mathbb{Q}^q\left[Y_{n+1} | \mathcal{F}_n\right] = \mathbb{Q}^q\left[Y_{n+1} | \mathcal{F}_n^Y\right]$$

and

$$\mathbb{Q}^{1-q}\left[Y_{n+1} | \mathcal{F}_n\right] = \mathbb{Q}^{1-q}\left[Y_{n+1} | \mathcal{F}_n^Y\right],$$

where $\mathcal{F}_n^Y$ denotes the filtration generated only by the random variable Y_n and $\mathcal{F}_n$ the filtration generated by the pair (S_n, Y_n).

These conditions describe under $\mathbb{Q}^q$ and $\mathbb{Q}^{1-q}$ models in which the movements of the non-traded asset are not affected by the dynamics of the traded asset. Furthermore, the special interest in this measures, apart from the above condition (2.9) is that the conditional probabilities depend on the parameter q, which is related to our model under the physical probability $\mathbb{P}$ through the parameters u and d.

Note that if $0 < q < 1/2$ then $\mathbb{Q}^q$ assigns less probability to the events where Y goes up than the ones where Y goes down, conversely as $\mathbb{Q}^{1-q}$ does.

On the other hand, although on $\mathbb{Q}^q$ and $\mathbb{Q}^{1-q}$ we have imposed the condition of independence in Y to the information generated by S, the measures depend on the parameter q, which in itself is related to the model for S through the values for u

and d. Therefore we can interpret these measures as the measures that assess the "impact" of the parameters in the S-model (u and d) to the scenarios in Y (up h or down l). We also make the observation that the parameter q resembles the parameter that determines the risk-free probability measure in the binomial model where only one asset is considered.

In Section 2.3.5.1, we relate the measures $\mathbb{Q}^q$ and $\mathbb{Q}^{1-q}$ with the strategies in the minimal expected shortfall problem.

2.3.2.2 The upper and lower bound measures

It is a well known fact that the super-replication price of an European contingent claim on the non-traded asset with payoff $H(Y_T)$ is the infimum value of all initial capitals V_0^* for which there exists a self-financing strategy π as in (2.5) such that $\mathbb{P}\left[V_T^{v,\pi} \geq H(Y_T)\right] = 1$, and that its dual representation is given by

$$V_0^* = \sup_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}\left[H(Y_T)|\mathcal{F}_0\right],$$

where the supremum is taken with respect to all equivalent martingale measures. It is also the upper bound of the arbitrage-free prices interval.

For discrete time Markovian models and in a more general setting than the present, in [90] it has been shown that the super-replication price can also be computed as the solution of a stochastic control problem similar to the dual representation above, but the supremum taken over a large set of measures (not only over the equivalent ones). Applying these results to our present situation, when the number of assets is finite, the number of possible states (the number of values the random variables take) is finite as well, and the relation between the random variables and the assets is linear, the super-replication price can be characterised as follows.

Denote by $\mathbb{V}$ the set of of all extremal points of the set $\mathcal{M}$ of all martingale measures (consisting of all vertices of $\mathcal{M}$). By the finiteness assumption in the number of steps and the number of events, the set $\mathcal{M}$ is a polyhedron and $\mathbb{V}$ is a finite set (see [90]). Then, the super-replication price is given by

$$V_0^* = \sup_{\mathbb{Q}\in\mathbb{V}} \mathbb{E}_{\mathbb{Q}}\left[H(Y_T)|\mathcal{F}_0\right].$$

In a similar manner, it is possible to characterise the lower bound of the arbitrage-free prices.

Corresponding to our two-factor binomial model, we start by computing the extremal points of the set of all equivalent martingale measures. From the expression of the generic equivalent martingale measure in (2.6), we obtain four extremal points by taking $(0,0)$, $(0,1-q)$, $(q,0)$, and $(q,1-9)$ for pair of the form (α,β). They are

$$\begin{aligned} \mathbb{Q}^{ext1} &: (0,q,1-q,0), \\ \mathbb{Q}^{ext2} &: (q,0,0,1-q), \\ \mathbb{Q}^{-} := \mathbb{Q}^{ext3} &: (0,q,0,1-q), \\ \mathbb{Q}^{*} := \mathbb{Q}^{ext4} &: (q,0,1-q,0). \end{aligned} \tag{2.10}$$

The distinction of the last two extremal points $\mathbb{Q}^-$ and $\mathbb{Q}^*$ is because they are the lower and upper bound measures. This can be checked by computing their conditional probabilities and the expected values on a contingent claim $H(Y_T)$. This is, for $\mathbb{Q}^{ext1}$,

$$\mathbb{Q}^{ext1}\left[Y_T = Y_0 h \mid S_T = S_0 u\right] = \mathbb{Q}^{ext1}\left[Y_T = Y_0 l \mid S_T = S_0 d\right] = 0,$$
$$\mathbb{Q}^{ext1}\left[Y_T = Y_0 l \mid S_T = S_0 u\right] = \mathbb{Q}^{ext1}\left[Y_T = Y_0 h \mid S_T = S_0 d\right] = 1.$$

or

$$\begin{aligned} \mathbb{E}^{ext1}\left[H(Y_T)|S_T = S_0 u\right] &= H(Y_0 l) \\ \mathbb{E}^{ext1}\left[H(Y_T)|S_T = S_0 d\right] &= H(Y_0 h), \end{aligned}$$

This resembles an imperfect correlation situation. It assigns zero probability to the events (up, up) and $(down, down)$ and gives full probability to the events $(up, down)$ and $(down, up)$.

For $\mathbb{Q}^{ext2}$, we have

$$\mathbb{Q}^{ext2}\left[Y_T = Y_0 l \mid S_T = S_0 u\right] = \mathbb{Q}^{ext2}\left[Y_T = Y_0 h \mid S_T = S_0 d\right] = 0,$$
$$\mathbb{Q}^{ext2}\left[Y_T = Y_0 h \mid S_T = S_0 u\right] = \mathbb{Q}^{ext2}\left[Y_T = Y_0 l \mid S_T = S_0 d\right] = 1,$$

or

$$\begin{aligned} \mathbb{E}^{ext2}\left[H(Y_T)|S_T = S_0 u\right] &= H(Y_0 h) \\ \mathbb{E}^{ext2}\left[H(Y_T)|S_T = S_0 d\right] &= H(Y_0 l), \end{aligned}$$

This is a perfect correlation situation. It assigns zero probability to the events $(up, down)$ and $(down, up)$ and gives full probability to the events (up, up) and $(down, down)$.

For the measure $\mathbb{Q}^-$ we have

$$\mathbb{Q}^{-}\left[Y_T = Y_0 h \mid S_T = S_0 u\right] = \mathbb{Q}^{-}\left[Y_T = Y_0 h \mid S_T = S_0 d\right] = 0,$$
$$\mathbb{Q}^{-}\left[Y_T = Y_0 l \mid S_T = S_0 u\right] = \mathbb{Q}^{-}\left[Y_T = Y_0 l \mid S_T = S_0 d\right] = 1.$$

Note that the above relation is equal to

$$\mathbb{Q}^{-}\left[Y_T|\mathcal{F}_0\right] = \mathbb{Q}^{-}\left[Y_T|\mathcal{F}_0^Y\right], \tag{2.11}$$

which means that under this measure, the information on the asset S does not affect the dynamics of the non-traded asset Y, but the converse does not necessarily hold. In terms of the expected values, we have

$$\mathbb{E}^{-}\left[H(Y_T)|S_T = S_0 u\right] = \mathbb{E}^{-}\left[H(Y_T)|S_T = S_0 d\right] = H(Y_0 l),$$

which shows that for a contingent claim depending only on the non-traded asset Y its expected value is independent of the realisations of the process S. Similarly for $\mathbb{Q}^*$ we have

$$\begin{aligned}\mathbb{Q}^*\left[Y_T = Y_0 l \mid S_T = S_0 u\right] = \mathbb{Q}^*\left[Y_T = Y_0 l \mid S_T = S_0 d\right] = 0,\\ \mathbb{Q}^*\left[Y_T = Y_0 h \mid S_T = S_0 u\right] = \mathbb{Q}^*\left[Y_T = Y_0 h \mid S_T = S_0 d\right] = 1.\end{aligned}$$

This measure also satisfies the relation in (2.11) of independence of the information generated by S on the non-traded asset Y. Furthermore, the expected values are

$$\mathbb{E}^{*}\left[H(Y_T)|S_T = S_0 u\right] = \mathbb{E}^{*}\left[H(Y_T)|S_T = S_0 d\right] = H(Y_0 h).$$

The measures $\mathbb{Q}^-$ and $\mathbb{Q}^*$ resemble to the zero correlation situation. Under the assumption that the contingent claim payoff H is a convex increasing function, we identify $\mathbb{Q}^-$ as the lower bound measure and $\mathbb{Q}^*$ as the upper bound measure, or the other way around for a convex decreasing function H. Therefore we make the following assumption.

Assumption 39 *The payoff function $H(z)$ is a convex increasing function on z.*

Each extremal measure is absolutely continuous with respect to the marginal laws of the random variables $\omega_i, i = 1, 2, 3, 4$ under $\mathbb{P}$, but not equivalent. One by-product of this analysis is that the generic equivalent martingale measure described in (2.6) can be obtained by taking a strict convex combination of the extremal measures. We formulate this in the next lemma.

Lemma 40 *Any equivalent martingale measure $\mathbb{Q}$ in (2.6) can be represented as*

$$\mathbb{Q} = \theta_1 \mathbb{Q}^{ext1} + \theta_2 \mathbb{Q}^{ext2} + \theta_3 \mathbb{Q}^{-} + (1 - \theta_1 - \theta_2 - \theta_3)\mathbb{Q}^*,$$

for some $\theta_1, \theta_2, \theta_3 \in (0, 1)$.

Proof. Using the extremal measures, the convex representation is equivalent to

$$\mathbb{Q} : ((\theta_1 + \theta_3)q, (1 - \theta_1 - \theta_3)q, (\theta_1 + \theta_2)(1 - q), (1 - \theta_1 - \theta_2)(1 - q))$$

and by setting $\alpha = (1 - \theta_1 - \theta_3)q$ and $\beta = (1 - \theta_1 - \theta_2)(1 - q)$ we get

$$\mathbb{Q} : (q - \alpha, \alpha, 1 - q - \beta, \beta) \text{ for } \alpha \in (0, q), \beta \in (0, 1 - q),$$

which is the expression of the generic equivalent martingale measure we had before in (2.6). □

Remark 41 *If the constraints in the convex combination above are relaxed to take values in* $[0, 1]$*, we obtain the so called* ***linear pricing measures*** *introduced in [69].*

We recall that even though we have analysed the different measures in the single-period model, and by the assumption that the probabilities does not depend on time (the marginals remain constant at any time n, $n = 0, ..., N - 1$), all relations remain valid in the N-period model. It will be just a matter of pasting each single-period measure (each marginal measure) to obtain the corresponding N-period measure.

2.3.3 Minimising the expected shortfall

In order to analyse the problem of minimising the expected shortfall in the N-period model, define the following conditional expected values:

$$\begin{aligned} V_n^* &= \mathbb{E}^* \left[H(Y_N) | \mathcal{F}_n \right], \\ V_n^- &= \mathbb{E}^- \left[H(Y_N) | \mathcal{F}_n \right], \\ V_n^q &= \mathbb{E}^q \left[H(Y_N) | \mathcal{F}_n \right], \\ V_n^{1-q} &= \mathbb{E}^{1-q} \left[H(Y_N) | \mathcal{F}_n \right]. \end{aligned}$$

Making use of the Markov property of the process $(S_n, Y_n)_{0 \le n \le N}$, and specifically conditioning at time n for $S_n = S$ and $Y_n = Y$ we write,

$$V_n^*(Y) = \mathbb{E}^* \left[H(Y_N) | Y_n = Y \right] = H\left(Y \ h^{N-n} \right), \tag{2.12}$$

$$V_n^-(Y) = \mathbb{E}^- \left[H(Y_N) | Y_n = Y \right] = H\left(Y \ l^{N-n} \right),$$

$$V_n^q(Y) = \mathbb{E}^q \left[H(Y_N) | Y_n = Y \right] = \sum_{k=0}^{N-n} \binom{N-n}{k} q^k (1-q)^k H\left(Y \ h^k \ l^{N-n-k} \right),$$

$$V_n^{1-q}(Y) = \mathbb{E}^{1-q} \left[H(Y_N) | Y_n = Y \right] = \sum_{k=0}^{N-n} \binom{N-n}{k} (1-q)^k q^k H\left(Y \ h^k \ l^{N-n-k} \right).$$

We recall that the measures $\mathbb{Q}^*, \mathbb{Q}^-, \mathbb{Q}^q$ and $\mathbb{Q}^{1-q}$ used in the definitions above correspond to the upper and lower bound measures that define the arbitrage-free price interval and the impact measures. They have been defined in (2.10), (2.7) and (2.8), respectively.

The problem to solve is to find the minimal shortfall risk $J(0, S_0, Y_0, V_0)$, where $J(n, \cdot, \cdot, \cdot)$ is defined, for $n = 0, ..., N-1$ as

$$J(n, S_n, Y_n, V_n) = \inf_{\pi \in \mathcal{A}} \mathbb{E}\left[(H(Y_N) - V_N^{v,\pi})^+ | S_n, Y_n, V_n^{v,\pi} \right]. \tag{2.13}$$

This is, we need to find an admissible trading strategy π such that minimises the expected value of the shortfall between the claim payoff $H(Y_N)$ and the final wealth value $V_N^{v,\pi}$ under the measure $\mathbb{P}$ with the information up to time n.

2.3.4 Relation with the two-step procedure

In our discrete N-period setting, the two-step procedure presented in Section 2.2 simplifies enormously. By the discussion on the extreme measures in Section 2.3.2.2, the set of equivalent martingale measures $\mathcal{M}$ is a polyhedron and its extremal set V is finite. Then we have

Proposition 42 *The static optimisation problem in STEP 1 in Section 2.2 in the two-assets binomial N-period model is equivalent to solving*

$$\max \mathbb{E}[V_N]$$

under the constraints

$$H_N - (V_0^*(Y_0) - V_0) \leq V_N \leq H_N \text{ and } \max_{i=1,2,3,4} \mathbb{E}_{\mathbb{Q}^{exti}}[V_N] \leq V_0.$$

Proof. The proof follows from Proposition 36 and the characterisation of super-replicating prices in discrete-time models in [90]. □

Note that if the initial capital is greater than or equal to the super-replication price at time 0 ($V_0 = V_0^*(Y_0)$) then there is no problem to solve, as the minimal shortfall risk will be zero and the optimal strategy is just the super-replicating/replicating strategy for the contingent claim H. On the contrary, if $V_0 < V_0^*(Y_0)$, then the lower the initial capital V_0, the less likely it is we obtain a good hedge. We can also foresee that in the latter situation the problem becomes harder to solve, or, similarly, we will need more conditions to check in order to find the minimal shortfall and the optimal strategy.

Although Proposition 42 suggests the general algorithm for calculating the minimal expected shortfall, it involves solving at each time $n = 0, ..., N-1$ a discrete-time constrained stochastic control problem. Instead, we take a more direct approach by looking at the original formulation.

There are three main cases to analyse regarding the initial capital V_0. They are linked to the strategies, the impact and extreme measures as explained in the next section.

2.3.5 The single-period model

2.3.5.1 The strategies, the impact and the extreme measures

In this section, we relate the strategies (optimal strategies and candidate strategies) to the impact and extreme measures.

The key aspect in the solution to the problem in (2.13) of minimising the expected shortfall is to exploit the Markov property of the process $(S_n, Y_n)_{0\leq n\leq N}$ and make use of the Dynamic Programming Principle (see [6]) to solve via backward induction. At each step $N-1$, the goal is to find the minimum and the minimiser $\hat{\pi}$ in the equation

$$J(N-1, S_{N-1}, Y_{N-1}, V_{N-1}) = \inf_{\pi} \mathbb{E}\left[(H(Y_N) - V_N)^+ | S_{N-1}, Y_{N-1}, V_{N-1}\right].$$

Using the expression for the portfolio dynamics in (2.5) and assigning the corresponding probabilities to each of the four possible scenarios, the problem reduces to find at the step $N-1$ the value $\hat{\pi}$ that minimises the function f defined by

$$\begin{aligned} f(\pi) \;=\;& p_1\left[H(Y_{N-1}h) - V_{N-1} - \pi S_{N-1}(u-1)\right]^+ \\ &+p_2\left[H(Y_{N-1}l) - V_{N-1} - \pi S_{N-1}(u-1)\right]^+ \\ &+p_3\left[H(Y_{N-1}h) - V_{N-1} - \pi S_{N-1}(d-1)\right]^+ \\ &+p_4\left[H(Y_{N-1}l) - V_{N-1} - \pi S_{N-1}(d-1)\right]^+. \end{aligned} \tag{2.14}$$

As is customary in these situations, in order to search for the minimiser of f, we start by defining four candidates for optimal strategies by making each term of the above sum in (2.14) equal to zero. The four candidates to optimal strategies are:

$$\begin{aligned} &\pi^{uh} := \tfrac{H(Y_{N-1}h)-V_{N-1}}{S_{N-1}(u-1)}, \quad \pi^{ul} := \tfrac{H(Y_{N-1}l)-V_{N-1}}{S_{N-1}(u-1)}, \\ &\pi^{dh} := \tfrac{H(Y_{N-1}h)-V_{N-1}}{S_{N-1}(d-1)}, \quad \pi^{dh} := \tfrac{H(Y_{N-1}l)-V_{N-1}}{S_{N-1}(d-1)}. \end{aligned} \tag{2.15}$$

If by Assumption 39 the payoff function H is a convex increasing function on Y, and by the expressions of the candidate strategies in (2.15) we obtain the following relations:

$$\pi^{uh} > \pi^{ul}; \quad \pi^{dl} > \pi^{dh}; \quad \frac{q-1}{q}\pi^{uh} = \pi^{dh} \quad \text{and} \quad \frac{q-1}{q}\pi^{ul} = \pi^{dl}.$$

Furthermore, also by the convexity and increasing property of H we have for any π

$$\begin{aligned}
H(Y_{N-1}l) - V_{N-1} - \pi S_{N-1}(d-1) &< H(Y_{N-1}h) - V_{N-1} - \pi S_{N-1}(d-1), \\
H(Y_{N-1}l) - V_{N-1} - \pi S_{N-1}(u-1) &< H(Y_{N-1}h) - V_{N-1} - \pi S_{N-1}(u-1).
\end{aligned}$$

This helps to simplify the expressions for f in (2.14) in each of the four candidate strategies, yielding

$$\begin{aligned}
f(\pi^{uh}) &= \frac{p_3}{1-q}\left[V^*_{N-1}(Y_{N-1}) - V_{N-1}\right]^+ + \frac{p_4}{1-q}\left[V^q_{N-1}(Y_{N-1}) - V_{N-1}\right]^+, \quad (2.16) \\
f(\pi^{dh}) &= \frac{p_1}{q}\left[V^*_{N-1}(Y_{N-1}) - V_{N-1}\right]^+ + \frac{p_2}{q}\left[V^{1-q}_{N-1}(Y_{N-1}) - V_{N-1}\right]^+, \\
f(\pi^{dl}) &= \frac{p_1}{q}\left[V^q_{N-1}(Y_{N-1}) - V_{N-1}\right]^+ + \frac{p_2}{q}\left[V^-_{N-1}(Y_{N-1}) - V_{N-1}\right]^+ \\
&\quad + p_3\left[V^*_{N-1}(Y_{N-1}) - V^-_{N-1}(Y_{N-1})\right]^+, \\
f(\pi^{ul}) &= p_1\left[V^*_{N-1}(Y_{N-1}) - V^-_{N-1}(Y_{N-1})\right]^+ + \frac{p_3}{1-q}\left[V^{1-q}_{N-1}(Y_{N-1}) - V_{N-1}\right]^+ \\
&\quad + \frac{p_4}{1-q}\left[V^-_{N-1}(Y_{N-1}) - V_{N-1}\right]^+.
\end{aligned}$$

We are now in the position to relate the candidate strategies to the impact measures. Define $e1$ and $e2$ as the difference between the expected value of the claim H under the measure $\mathbb{Q}^q$ (resp. $\mathbb{Q}^{1-q}$) conditional to the information up to time $N-1$ and the wealth value at time $N-1$, this is,

$$\begin{aligned}
e1 &:= V^q_{N-1}(Y_{N-1}) - V_{N-1} \\
e2 &:= V^{1-q}_{N-1}(Y_{N-1}) - V_{N-1}.
\end{aligned}$$

Using the expressions of the candidate strategies in (2.15), the definition of the impact measures in (2.7) and (2.8), and after some algebra we have the following relations.

$$\begin{aligned} e1 &= qH(Y_{N-1}h) + (1-q)H(Y_{N-1}l) - V_{N-1} \\ &= \frac{(u-1)(1-d)}{u-d} S_{N-1} \left(\pi^{uh} - \pi^{dl}\right), \end{aligned} \tag{2.17}$$

$$\begin{aligned} e2 &= (1-q)H(Y_{N-1}h) + qH(Y_{N-1}l) - V_{N-1} \\ &= \frac{(u-1)(1-d)}{u-d} S_{N-1} \left(\pi^{ul} - \pi^{dh}\right). \end{aligned}$$

The quantity $e1$ involves only the parameters u, d, S_{N-1} and the difference between the candidate strategies $\pi^{uh} - \pi^{dl}$. As $0 < d < 1 < u$ and $S_{N-1} > 0$, then the sign in $e1$ depends on whether π^{uh} is bigger than π^{dl} or not. Similarly for $e2$, its sign depends on the values of $\pi^{ul}$and π^{dh}.

These relations between $e1$ and $e2$ highlights the importance of the parameter q in the decision of optimal strategy and optimal solution.

To see a more clear interpretation, define for a generic martingale measure $\mathbb{Q}$ as in (2.6) the functions:

$$u(\alpha) := \mathbb{E}_{\mathbb{Q}}\left[H(Y_T)|S_N = S_{N-1}u\right] \tag{2.18}$$
$$= \frac{q-\alpha}{q} H(Y_{N-1}h) + \frac{\alpha}{q} H(Y_{N-1}l), \tag{2.19}$$

$$w(\beta) := \mathbb{E}_{\mathbb{Q}}\left[H(Y_T)|S_N = S_{N-1}d\right] \tag{2.20}$$
$$= \frac{1-q-\beta}{1-q} H(Y_{N-1}h) + \frac{\beta}{1-q} H(Y_{N-1}l). \tag{2.21}$$

Both of this functions are decreasing in α and β, with

$$u(0) = H(Y_{N-1}h), \quad u(q) = H(Y_{N-1}l), \quad w(0) = H(Y_{N-1}h) \quad \text{and} \quad w(1-q) = H(Y_{N-1}l).$$

On the other hand, for $q \in [0, 1]$ the following inequalities always hold, (see Figure 2.1):

$$\begin{array}{lll} \text{if} \quad 0 \le q < \frac{3-\sqrt{5}}{2} & \text{then} & 1-q \ge (1-q)^2 > q \ge q(1-q) \ge q^2, \\ \text{if} \quad \frac{3-\sqrt{5}}{2} \le q < \frac{1}{2} & \text{then} & 1-q > (1-q)^2 \ge q > q(1-q) > q^2, \\ \text{if} \quad q = \frac{1}{2} & \text{then} & 1-q = q > (1-q)^2 = q(1-q) = q^2, \\ \text{if} \quad \frac{1}{2} < q \le \frac{-1+\sqrt{5}}{2} & \text{then} & q > 1-q \ge q^2 > q(1-q) > (1-q)^2, \\ \text{if} \quad \frac{-1+\sqrt{5}}{2} < q \le 1 & \text{then} & q \ge q^2 > 1-q \ge q(1-q) \ge (1-q)^2. \end{array}$$

Using this, together with the properties of the functions $u(\alpha)$ and $w(\beta)$, we obtain two important inequalities relating the impact and extreme measures at time $N-1$. They are illustrated in Figure 2.2.

For $0 < q < \frac{1}{2}$,

$$H(Y_{N-1}h) > (1-q)H(Y_{N-1}h)+qH(Y_{N-1}l) > qH(Y_{N-1}h)+(1-q)H(Y_{N-1}l) > H(Y_{N-1}l). \tag{2.22}$$

And for $\frac{1}{2} < q < 1$,

$$H(Y_{N-1}h) > qH(Y_{N-1}h) + (1-q)H(Y_{N-1}l) > (1-q)H(Y_{N-1}h) + qH(Y_{N-1}l) > H(Y_{N-1}l). \tag{2.23}$$

A consequence of the above inequalities, the expressions for candidate strategies in (2.15) and their relation in (2.17) we have the following ordering relation in the candidate strategies at time $N-1$,

$$\text{if} \quad e1 > 0 \quad \text{and} \quad e2 > 0 \quad \text{then} \quad \pi^{dh} < \pi^{ul} < \pi^{dl} < \pi^{uh} \text{ or } \pi^{dh} < \pi^{dl} < \pi^{ul} < \pi^{uh},$$

$$\text{if} \quad e1 > 0 \quad \text{and} \quad e2 < 0 \quad \text{then} \quad \pi^{ul} < \pi^{dh} < \pi^{dl} < \pi^{uh},$$

$$\text{if} \quad e1 < 0 \quad \text{and} \quad e2 > 0 \quad \text{then} \quad \pi^{dh} < \pi^{ul} < \pi^{uh} < \pi^{dl},$$

$$\text{if} \quad e1 < 0 \quad \text{and} \quad e2 < 0 \quad \text{then} \quad \pi^{ul} < \pi^{dh} < \pi^{uh} < \pi^{dl}.$$

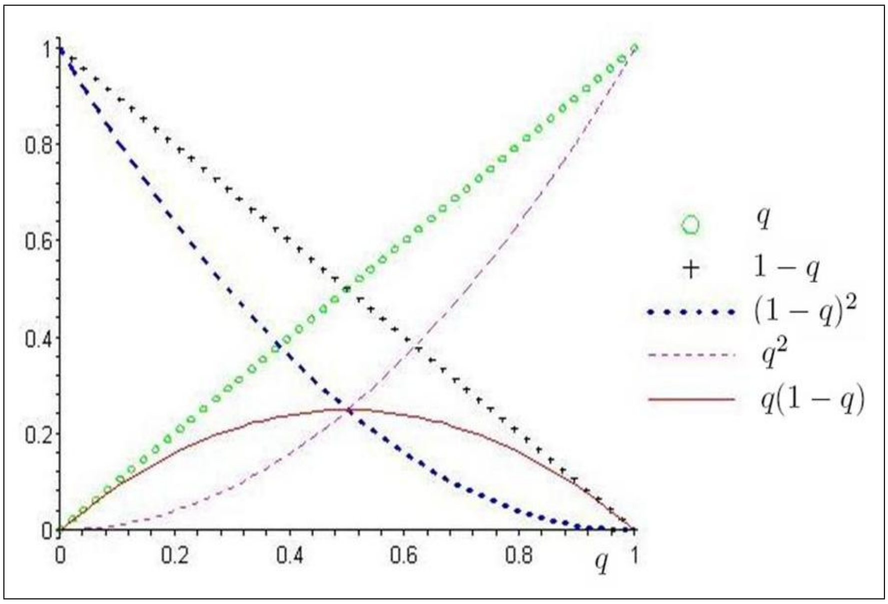
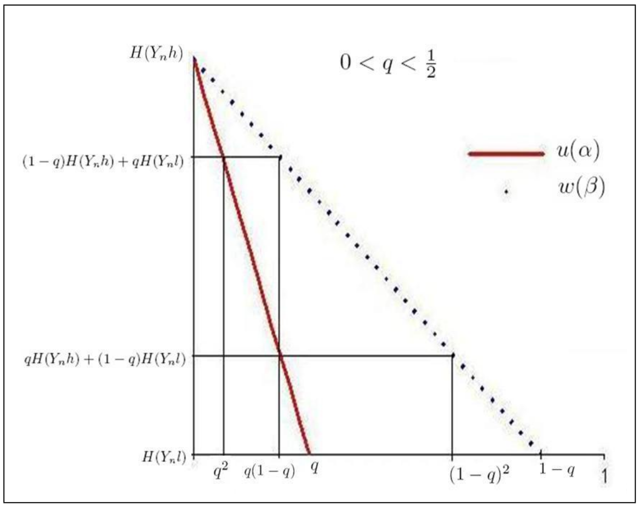
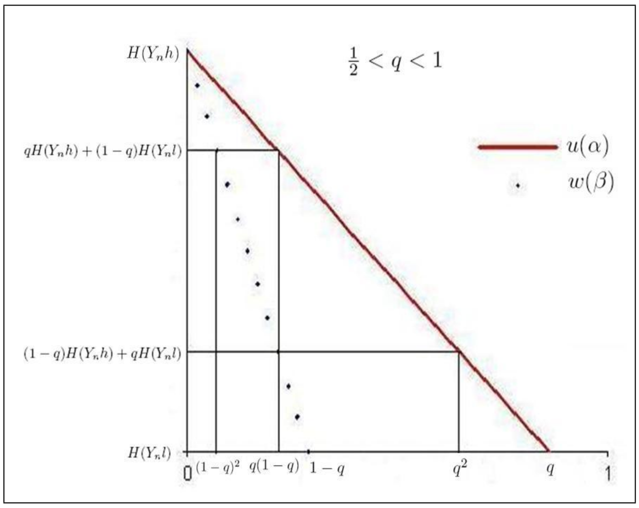


Figure 2.1: Comparison of the functions q, $1-q$, $(1-q)^2$, q^2 and $q(1-q)$ for $0 \le q \le 1$.



(a) The case $0 < q < \frac{1}{2}$.



(b) The case $\frac{1}{2} < q < 1$.

Figure 2.2: Comparison of values for the functions $u(\alpha)$ in (2.19) and $w(\beta)$ in (2.21) for $0 \leq q \leq 1$.

Our conclusion from this analysis is the following. The choice of the optimal strategy among the candidates is directly related to the relative position that the portfolio value V_{N-1} at time $N-1$ occupies in the inequalities (2.22) or (2.23). In order to clarify this, assume $0 < q < \frac{1}{2}$. The only three possibilities for the portfolio value V_{N-1} are:

$$\mathbb{E}^* \left[H(Y_T)|S_{N-1}\right] \;>\; V_{N-1} > \mathbb{E}^{1-q} \left[H(Y_T)|S_{N-1}\right], \tag{2.24}$$

$$\mathbb{E}^{1-q} \left[H(Y_T)|S_{N-1}\right] \;>\; V_{N-1} > \mathbb{E}^{q} \left[H(Y_T)|S_{N-1}\right], \tag{2.25}$$

$$\mathbb{E}^{q} \left[H(Y_T)|S_{N-1}\right] \;>\; V_{N-1} > \mathbb{E}^{-} \left[H(Y_T)|S_{N-1}\right]. \tag{2.26}$$

We will refer to these cases as the **Large**, **Medium** and **Low initial capital**. They will be analysed in the next sections, so as the structure of the corresponding optimal strategy $\hat{\pi}_{N-1}$ and the optimal solution.

From the similitude of the case when $\frac{1}{2} < q < 1$, without loss of generality through the rest of the section we can make the following assumption.

Assumption 43 *Assume* $0 < q < \frac{1}{2}$.

2.3.5.2 The three initial capital cases

In this section, we present the solution for the single-period model depending on the initial capital V_0 available.

The following lemma provides the solution in the **Large capital case**.

Lemma 44 (Large Capital Case) *In the single-period model* $[0, T]$*, assume we have constrained the initial capital by* $V_0^{1-q}(Y_0) < V_0 < V_0^*(Y_0)$ *then*

$$J^{(L)}(0, S_0, Y_0, V_0) = \left[\min\left(\frac{p_1}{q}, \frac{p_3}{1-q}\right)\right] \left[V_0^*(Y_0) - V_0\right]^+ \tag{2.27}$$

Moreover, the strategy corresponding to the risk in (2.27) is given by

$$\hat{\pi}_0^{(L)} = \begin{cases} \pi^{dh} := \frac{V_N^*(Y_0 h) - V_0}{S_0(d-1)} & \text{if } \frac{p_1}{q} \le \frac{p_3}{1-q} \\ \pi^{uh} := \frac{V_N^*(Y_0 h) - V_0}{S_0(u-1)} & \text{if } \frac{p_1}{q} \ge \frac{p_3}{1-q}. \end{cases} \tag{2.28}$$

which gives a final portfolio value $V_T^{\hat{\pi}^{(L)}}$ *such that*

$$[H(Y_N) - V_T^{\hat{\pi}^{(L)}}]^+ = \begin{cases} \left(\frac{p_1}{q_1^*}\right) \left[V_0^*(Y_0) - V_0\right]^+ \mathbf{1}_{\omega=\omega_1} & \text{if } \frac{p_1}{q_1^*} \le \frac{p_3}{q_3^*} \\ \left(\frac{p_3}{q_3^*}\right) \left[V_0^*(Y_0) - V_0\right]^+ \mathbf{1}_{\omega=\omega_3} & \text{if } \frac{p_1}{q_1^*} \ge \frac{p_3}{q_3^*}. \end{cases} \tag{2.29}$$

Proof. Recalling that

$$\begin{aligned} V_0^{1-q}(Y_0) &= (1-q)H(\ Y_0 h\ ) + qH(\ Y_0 l\ ) > qH(Y_0 h) + (1-q)H(Y_0 l) \\ V_0^*(Y_0) &= H(Y_0 h), \end{aligned}$$

then the constraint on V_0 implies $(1-q)H(Y_0h) + qH(Y_0l) - V_0 < 0 < H(Y_0h) - V_0$, and using the convexity property of the payoff function $H(y)$, the function f in (2.14) on the four candidate strategies. We have

$$f(\pi_0^{uh}) = \tfrac{p_3}{1-q}\left[H(Y_0h) - V_0\right]^+; \qquad f(\pi_0^{dh}) = \tfrac{p_1}{q}\left[H(Y_0h) - V_0\right]^+;$$

$$f(\pi_0^{dl}) = p_3\left[H(Y_0h) - H(Y_0l)\right]^+; \quad f(\pi_0^{ul}) = p_1\left[H(Y_0h) - H(Y_0l)\right]^+.$$

We obtain the optimal solution on a case by case basis.

- Case 1: $\frac{p_1}{q} < \frac{p_3}{1-q}$.
  1. The inequality $\frac{p_1}{q} < \frac{p_3}{1-q}$ rules out π^{dl} and π^{uh} as candidates for the optimal solution.
  2. The inequality $f(\pi^{dh}) - f(\pi^{ul}) = \frac{p_1}{q}\left[(1-q)H(Y_0h) + qH(Y_0l) - V_0\right] < 0$ rules out the strategy π^{ul}. Thus π^{dh} is the **optimal strategy**.
- Case 2: $\frac{p_1}{q} > \frac{p_3}{1-q}$
  1. The inequality $\frac{p_1}{q} > \frac{p_3}{1-q}$ dismiss the candidates π^{dl} and π^{dh} to be the optimal solution, and
  2. the inequality $f(\pi^{uh}) - f(\pi^{ul}) < \frac{p_1}{q}\left[(1-q)H(Y_0h) + qH(Y_0l) - V_0\right] < 0$ helps to rule out the strategy π^{ul}. Thus π^{uh} is the **optimal strategy**.

The relation in (2.29) for the final wealth $V_T^{\hat{\pi}^{(H)}}$ follows by the construction of the solution. ☐

The solution in the **Medium capital case** is presented in the next lemma.

Lemma 45 (Medium capital case) *In the single-period model* $[0, T]$*, assume we have constrained the initial capital by* $V_0^q(Y_0) < V_0 < V_0^{1-q}(Y_0)$ *then*

$$\begin{aligned} J^{(M)}(0, S_0, Y_0, V_0) \;=\; & \left[\min\left(\frac{p_1}{q}, \frac{p_3}{1-q}\right)\right]\left[V_0^*(Y_0) - V_0\right]^+ \\ & + \left[\min\left(\frac{p_2}{q}, \frac{p_3}{1-q}\right)\right]\left[V_0^{1-q}(Y_0) - V_0\right]^+ 1_{\left\{\frac{p_1}{q} < \frac{p_3}{1-q}\right\}} \\ & + \left\{p_1\left[V_0^*(Y_0) - V_0^-(Y_0)\right]^+ - \frac{p_1}{q}\left[V_0^*(Y_0) - V_0\right]^+\right\} 1_{\left\{\frac{p_1}{q} < \frac{p_3}{1-q} < \frac{p_1+p_2}{q}\right\}} \end{aligned} \tag{2.30}$$

Moreover, the strategy corresponding to the risk in (2.30) is given by

$$\hat{\pi}_0^{(M)} = \begin{cases} \pi^{uh} := \frac{V_N^*(Y_0h)-V_0}{S_0(u-1)} & \textit{if } \frac{p_1}{q} \geq \frac{p_3}{1-q} \\ \pi^{ul} := \frac{V_N^*(Y_0l)-V_0}{S_0(u-1)} & \textit{if } \frac{p_1}{q} < \frac{p_3}{1-q} < \frac{p_1+p_2}{q} \\ \pi^{dh} := \frac{V_N^*(Y_0h)-V_0}{S_0(d-1)} & \textit{if } \frac{p_1}{q} < \frac{p_1+p_2}{q} < \frac{p_3}{1-q}. \end{cases} \tag{2.31}$$

which gives a final portfolio value $V_T^{\hat{\pi}^{(M)}}$ such that[2]

$$[H(Y_T)-V_T^{\hat{\pi}^{(M)}}]^+ = \begin{cases} \left(\frac{1}{1-q}\right)[V_0^*(Y_0)-V_0]\,\mathbf{1}_{\omega=\omega_3} & w.p. & p_3 & \textit{if} & \frac{p_1}{q} \geq \frac{p_3}{1-q} \\ \left(\frac{1}{q}\right)\left[V_0^*(Y_0)-V_0^-(Y_0)\right]\mathbf{1}_{\omega=\omega_1} & w.p. & p_1 & \textit{if} & \frac{p_1}{q} < \frac{p_3}{1-q} < \frac{p_1+p_2}{q} \\ \left(\frac{1}{1-q}\right)\left[V_0^{1-q}(Y_0)-V_0\right]\mathbf{1}_{\omega=\omega_3} & w.p. & p_3 & \textit{if} & \frac{p_1}{q} < \frac{p_3}{1-q} < \frac{p_1+p_2}{q} \\ \left(\frac{1}{q}\right)[V_0^*(Y_0)-V_0]\,\mathbf{1}_{\omega=\omega_1} & w.p. & p_1 & \textit{if} & \frac{p_1}{q} < \frac{p_1+p_2}{q} < \frac{p_3}{1-q} \\ \left(\frac{1}{q}\right)\left[V_0^{1-q}(Y_0)-V_0\right]\mathbf{1}_{\omega=\omega_2} & w.p. & p_2 & \textit{if} & \frac{p_1}{q} < \frac{p_1+p_2}{q} < \frac{p_3}{1-q}. \end{cases} \tag{2.32}$$

Proof. The constraint on V_0 implies

$$qH(Y_0h)+(1-q)H(Y_0l)-V_0 < 0 < (1-q)H(Y_0h)+qH(Y_0l)-V_0,$$

and using the convexity property of the payoff function H, the function f in (2.14) on the four candidate strategies. We have

$$\begin{aligned} f(\pi^{dl}) &= p_3\,[H(Y_0h)-H(Y_0l)]^+ \\ f(\pi^{uh}) &= \frac{p_3}{1-q}\,[H(Y_0h)-V_0]^+ \\ f(\pi^{dh}) &= \frac{p_1}{q}\,[H(Y_0h)-V_0]^+ + \frac{p_2}{q}\,[(1-q)H(Y_0h)+qH(Y_0l)-V_0]^+ \\ f(\pi^{ul}) &= p_1\,[H(Y_0h)-H(Y_0l)]^+ + \frac{p_3}{1-q}\,[(1-q)H(Y_0h)+qH(Y_0l)-V_0]^+. \end{aligned}$$

In order to search for the optimal strategies, we compute the difference in values of the function f in (2.14) among the six possible combinations for the candidate strategies.

[2] w.p. means with probability.

This is,

$$f(\pi^{uh}) - f(\pi^{dh}) = [H(Y_0h) - V_0]\left\{\frac{p_3}{1-q} - \frac{p_1}{q}\right\} \tag{2.33}$$

$$+\frac{p_2}{q}[(1-q)H(Y_0h) + qH(Y_0l) - V_0] \tag{2.34}$$

$$f(\pi^{uh}) - f(\pi^{dl}) = \frac{p_3}{1-q}[qH(Y_0h) + (1-q)H(Y_0l) - V_0] \tag{2.35}$$

$$f(\pi^{uh}) - f(\pi^{ul}) = q[H(Y_0h) - H(Y_0l)]\left[\frac{p_3}{1-q} - \frac{p_1}{q}\right] \tag{2.36}$$

$$f(\pi^{dh}) - f(\pi^{dl}) = \frac{p_1}{q}[H(Y_0h) - V_0] \tag{2.37}$$

$$+\left[\frac{p_2}{q} - \frac{p_3}{1-q}\right][(1-q)H(Y_0h) + qH(Y_0l) - V_0] \tag{2.38}$$

$$f(\pi^{dh}) - f(\pi^{ul}) = [(1-q)H(Y_0h) + qH(Y_0l) - V_0]\left[\frac{p_1+p_2}{q} - \frac{p_3}{1-q}\right] \tag{2.39}$$

$$f(\pi^{dl}) - f(\pi^{ul}) = [H(Y_0l) - V_0]\left[p_1 - \frac{p_3}{1-q}\right] - p_1[H(Y_0h) - V_0]. \tag{2.40}$$

We use the above differences to obtain the optimal strategies. For $\frac{p_1}{q} > \frac{p_3}{1-q}$, we have,

- from (2.35) we get $f(\pi^{uh}) - f(\pi^{dl}) < 0$, so we rule out the strategy π^{dl};
- from (2.36) we get $f(\pi^{uh}) - f(\pi^{ul}) < 0$, and we rule out π^{ul} as the optimal strategy;
- from (2.39) we obtain $f(\pi^{dh}) - f(\pi^{ul}) > 0$. This rules out π^{dh} and then π^{uh} is the **optimal** strategy.

For the case when $\frac{p_1}{q} < \frac{p_3}{1-q}$, we obtain,

- from (2.35) we have $f(\pi^{uh}) - f(\pi^{dl}) < 0$, which rules out the strategy π^{dl},
- from (2.36) we get $f(\pi^{uh}) - f(\pi^{ul}) > 0$, so we rule out π^{uh} as optimal,
- from (2.39) the inequality $f(\pi^{dh}) - f(\pi^{ul}) < 0$ is valid only if $\frac{p_1+p_2}{q} < \frac{p_3}{1-q}$ is satisfied, then we can rule out the strategy π^{ul}. Thus the **optimal** strategy is π^{dh}, on the other hand,

- from (2.39), the inequality $f(\pi^{dh}) - f(\pi^{ul}) > 0$ holds if and only if the inequality $\frac{p_1+p_2}{q} > \frac{p_3}{1-q}$ is satisfied. This rules out the candidate π^{dh} leaving us with the **optimal strategy** π^{ul}.

The relation in (2.32) for the final wealth $V_T^{\hat{\pi}^{(M)}}$ follows by the construction of the solution. □

The **Low capital case** is presented in the following lemma.

Lemma 46 (low capital) *In the one-period model* $[0,T]$*, assume we have constrained the initial capital by* $V_0^-(Y_0) < V_0 < V_0^q(Y_0)$*. Then*

$$
\begin{aligned}
J^{(l)}(0,S_0,Y_0,V_0) &= \left[\min\left(\frac{p_1}{q},\frac{p_3}{1-q}\right)\right][V_0^*(Y_0)-V_0]^+ \\
&+ \left[\min\left(\frac{p_2}{q},\frac{p_3}{1-q}\right)\right]\left[V_0^{1-q}-V_0\right]^+ 1_{\left\{\frac{p_1}{q}<\frac{p_3}{1-q}\right\}} \\
&+ \left[\min\left(\frac{p_1}{q},\frac{p_4}{1-q}\right)\right][V_0^q-V_0]^+ 1_{\left\{\frac{p_1}{q}>\frac{p_3}{1-q}\right\}} \\
&+ \left[\min\left(\frac{p_1}{q},\frac{p_3}{1-q}\right)\right][V_0^*(Y_0)-V_0]^+ 1_{\left\{\frac{p_1}{q}<\frac{p_3}{1-q}<\frac{p_1+p_2}{q}\right\}\cup\left\{\frac{p_3}{1-q}<\frac{p_1}{q}<\frac{p_3+p_4}{1-q}\right\}} \\
&+ [\min(p_1,p_3)]\left[V_0^*(Y_0)-V_0^-(Y_0)\right]^+ 1_{\left\{\frac{p_1}{q}<\frac{p_3}{1-q}<\frac{p_1+p_2}{q}\right\}\cup\left\{\frac{p_3}{1-q}<\frac{p_1}{q}<\frac{p_3+p_4}{1-q}\right\}}.
\end{aligned}
\tag{2.41}
$$

Moreover, the strategy corresponding to the risk in (2.41) is given by

$$
\hat{\pi}_0^{(l)} = \begin{cases}
\pi^{uh} := \frac{H(Y_0h)-V_0}{S_0(u-1)} & \text{if } \frac{p_1}{q} \geq \frac{p_3+p_4}{1-q} \geq \frac{p_3}{1-q}, \\
\pi^{dl} := \frac{H(Y_0l)-V_0}{S_0(d-1)} & \text{if } \frac{p_3+p_4}{1-q} > \frac{p_1}{q} > \frac{p_3}{1-q}, \\
\pi^{ul} := \frac{H(Y_0l)-V_0}{S_0(u-1)} & \text{if } \frac{p_1}{q} < \frac{p_3}{1-q} < \frac{p_1+p_2}{q}, \\
\pi^{dh} := \frac{H(Y_0h)-V_0}{S_0(d-1)} & \text{if } \frac{p_1}{q} < \frac{p_1+p_2}{q} < \frac{p_3}{1-q}.
\end{cases}
\tag{2.42}
$$

This gives a final portfolio value $V_T^{\hat{\pi}^{(l)}}$ *such that*

$$[H(Y_T) - V_T^{\hat{\pi}^{(l)}}]^+ = \begin{cases} \left.\begin{array}{lll} \left(\frac{1}{1-q}\right)[V_0^*(Y_0) - V_0]\,\mathbf{1}_{\omega=\omega_3} & w.p. & p_3 \\ \left(\frac{1}{1-q}\right)[V_0^q(Y_0) - V_0]\,\mathbf{1}_{\omega=\omega_4} & w.p. & p_4 \end{array}\right\} & \textit{if } \frac{p_1}{q} \geq \frac{p_3+p_4}{1-q} \geq \frac{p_3}{1-q} \\ \left.\begin{array}{lll} \left(\frac{1}{q}\right)[V_0^q(Y_0) - V_0]\,\mathbf{1}_{\omega=\omega_1} & w.p. & p_1 \\ \left(\frac{1}{1-q}\right)\left[V_0^*(Y_0) - V_0^-(Y_0)\right]\mathbf{1}_{\omega=\omega_3} & w.p. & p_3 \end{array}\right\} & \textit{if } \frac{p_3+p_4}{1-q} > \frac{p_1}{q} > \frac{p_3}{1-q} \\ \left.\begin{array}{lll} \left(\frac{1}{q}\right)\left[V_0^*(Y_0) - V_0^-(Y_0)\right]\mathbf{1}_{\omega=\omega_1} & w.p. & p_1 \\ \left(\frac{1}{1-q}\right)\left[V_0^{1-q}(Y_0) - V_0\right]\mathbf{1}_{\omega=\omega_3} & w.p. & p_3 \end{array}\right\} & \textit{if } \frac{p_1}{q} < \frac{p_3}{1-q} < \frac{p_1+p_2}{q} \\ \left.\begin{array}{lll} \left(\frac{1}{q}\right)[V_0^*(Y_0) - V_0]\,\mathbf{1}_{\omega=\omega_1} & w.p. & p_1 \\ \left(\frac{1}{q}\right)\left[V_0^{1-q}(Y_0) - V_0\right]\mathbf{1}_{\omega=\omega_2} & w.p. & p_2 \end{array}\right\} & \textit{if } \frac{p_1}{q} < \frac{p_1+p_2}{q} < \frac{p_3}{1-q}. \end{cases} \tag{2.43}$$

Proof. By the assumption $V_0^-(Y_0) < V_0 < V_0^q(Y_0)$, the function f of the four candidate strategies is

$$f(\pi^{dh}) = \frac{p_1}{q}[H(Y_{N-1}h) - V_{N-1}]^+ + \frac{p_2}{q}[(1-q)H(Y_{N-1}h) + qH(Y_{N-1}l) - V_{N-1}]^+,$$

$$\begin{aligned} f(\pi^{ul}) = {} & \frac{p_1}{q}[H(Y_{N-1}h) - V_{N-1}]^+ + \frac{p_3}{1-q}[(1-q)H(Y_{N-1}h) + qH(Y_{N-1}l) - V_{N-1}]^+ \\ & + p_1[H(Y_{N-1}h) - H(Y_{N-1}l)]^+ - \frac{p_1}{q}[H(Y_{N-1}h) - V_{N-1}]^+, \end{aligned}$$

$$f(\pi^{uh}) = \frac{p_3}{1-q}[H(Y_{N-1}h) - V_{N-1}]^+ + \frac{p_4}{1-q}[qH(Y_{N-1}h) + (1-q)H(Y_{N-1}l) - V_{N-1}]^+,$$

$$\begin{aligned} f(\pi^{dl}) = {} & \frac{p_3}{1-q}[H(Y_{N-1}h) - V_{N-1}]^+ + \frac{p_1}{q}[qH(Y_{N-1}h) + (1-q)H(Y_{N-1}l) - V_{N-1}]^+ \\ & + p_3[H(Y_{N-1}h) - H(Y_{N-1}l)]^+ - \frac{p_3}{1-q}[H(Y_{N-1}h) - V_{N-1}]. \end{aligned}$$

In order to analyse the optimal strategies, we compute the difference in values of the function f in (2.14) among the six possible combinations for the candidate strategies. They are,

$$\begin{aligned} f(\pi^{uh}) - f(\pi^{dh}) = {} & [H(Y_{N-1}h) - V]\left\{\frac{p_3}{1-q} - \frac{p_1}{q} + \frac{q^2p_4 - (1-q)^2p_2}{q(1-q)}\right\} \\ & + [H(Y_{N-1}l) - V][p_4 - p_2], \end{aligned} \tag{2.44}$$

$$f(\pi^{uh}) - f(\pi^{dl}) = [qH(Y_{N-1}h) + (1-q)H(Y_{N-1}l) - V]\left[\frac{p_3+p_4}{1-q} - \frac{p_1}{q}\right], \tag{2.45}$$

$$\begin{aligned} f(\pi^{uh}) - f(\pi^{ul}) &= q\,[H(Y_{N-1}h) - H(Y_{N-1}l)]\left[\frac{p_3}{1-q} - \frac{p_1}{q}\right] \\ &\quad + \frac{p_4}{1-q}\,[qH(Y_{N-1}h) + (1-q)H(Y_{N-1}l) - V], \end{aligned} \tag{2.46}$$

$$\begin{aligned} f(\pi^{dh}) - f(\pi^{dl}) &= (1-q)\,[H(Y_{N-1}h) - H(Y_{N-1}l)]\left[\frac{p_1}{q} - \frac{p_3}{1-q}\right] \\ &\quad + \frac{p_2}{q}\,[(1-q)H(Y_{N-1}h) + qH(Y_{N-1}l) - V], \end{aligned} \tag{2.47}$$

$$f(\pi^{dh}) - f(\pi^{ul}) = [(1-q)H(Y_{N-1}h) + qH(Y_{N-1}l) - V]\left[\frac{p_1+p_2}{q} - \frac{p_3}{1-q}\right] \tag{2.48}$$

$$f(\pi^{dl}) - f(\pi^{ul}) = [H(Y_{N-1}l) - V]\left[\frac{p_1}{q} - \frac{p_3}{1-q}\right]. \tag{2.49}$$

We use the above differences to obtain the optimal strategies. For $\frac{p_1}{q} > \frac{p_3}{1-q}$, we get,

- from (2.47) we get $f(\pi^{dh}) - f(\pi^{dl}) > 0$, so we rule out the strategy π^{dh},
- from (2.49) we get $f(\pi^{dl}) - f(\pi^{ul}) < 0$, so we rule out π^{ul} as optimal strategy, and
- from (2.45) we obtain $f(\pi^{uh}) - f(\pi^{dl}) > 0$ only if $\frac{p_3+p_4}{1-q} > \frac{p_1}{q}$. This rules out π^{uh}, being the **optimal** strategy π^{dl}.
- from (2.45) $f(\pi^{uh}) - f(\pi^{dl}) < 0$ if and only if $\frac{p_3+p_4}{1-q} < \frac{p_1}{q}$, which means we can rule out the strategy π^{dl}. The **optimal** strategy is π^{uh}.

For the case when $\frac{p_1}{q} < \frac{p_3}{1-q}$ we have,

- From (2.49) we get$(\pi^{dl}) - f(\pi^{ul}) > 0$, which rules out π^{dl},
- from (2.45) then $f(\pi^{uh}) - f(\pi^{dl}) > 0$ and we rule out π^{uh},
- from (2.48) $f(\pi^{dh}) - f(\pi^{ul}) > 0$ if and only if $\frac{p_1+p_2}{q} > \frac{p_3}{1-q}$, this tells us to rule out π^{dh}, and the **optimal** strategy becomes π^{ul}.
- From (2.48) $f(\pi^{dh}) - f(\pi^{ul}) < 0$ is valid only if $\frac{p_1+p_2}{q} < \frac{p_3}{1-q}$, so we can rule out π^{ul} and the **optimal** strategy is π^{dh}.

The relation in (2.43) for the final wealth $V_T^{\hat{\pi}^{(M)}}$ follows by the construction of the solution. □

2.3.6 The large capital case for the N-period model

Intuitively, if the initial capital V_0 is just below the super-replication price, there will be just a few possible scenarios (events) that cannot be super-replicated. In analogy with the single-period model, in the next theorem we establish precisely what we mean by large initial capital, the minimal expected shortfall, and the optimal strategy. We discuss the details and interpretation of the solution after the Theorem 47 in some lemmas used in the proof.

Theorem 47 *Consider a European contingent claim on the non-traded asset Y, whose payoff at time N is defined by an increasing convex function H. Let $V_n^*(S_n), n = N-1, ..., 0$, be the super-replication price at time n defined in (2.12), corresponding to the super-replicating measure $\mathbb{Q}^*$ defined in (2.10) (we denote it by q_i^*, $i = 1..4$).*

If

$$V_0 \geq V_0^*(Y_0) - \begin{cases} (q_1^*)^N \{V_0^*(Y_0) - V_1^*(Y_0 l)\} & \text{if } \frac{p_1}{q_1^*} \leq \frac{p_3}{q_3^*} \\ q_1^* \, (q_3^*)^{N-1} \{V_0^*(Y_0) - V_1^*(Y_0 l)\} & \text{if } \frac{p_1}{q_1^*} \geq \frac{p_3}{q_3^*}. \end{cases} \tag{2.50}$$

then for $n = 0, ..., N-1$, the minimal shortfall risk in (2.13) becomes

$$J^{(Large)}(n, S_n, Y_n, V_n) = \left[\min\left(\frac{p_1}{q_1^*}, \frac{p_3}{q_3^*}\right)\right]^{N-n} [V_n^*(Y_n) - V_n]^+ \tag{2.51}$$

and the strategy corresponding to the risk in (2.51) is given by

$$\hat{\pi}_n^{(Large)} = \begin{cases} \pi_n^{dh} := \frac{V_{n+1}^*(Y_n h) - V_n}{S_n(d-1)} & \text{if } \frac{p_1}{q_1^*} \leq \frac{p_3}{q_3^*} \\ \pi_n^{uh} := \frac{V_{n+1}^*(Y_n h) - V_n}{S_n(u-1)} & \text{if } \frac{p_1}{q_1^*} \geq \frac{p_3}{q_3^*}, \end{cases} \tag{2.52}$$

which gives a final portfolio value $V_N^{\hat{\pi}^{(Large)}}$ such that

$$[H(Y_N) - V_N^{\hat{\pi}^{(Large)}}]^+ = \begin{cases} \left(\frac{p_1}{q_1^*}\right)^N [V_0^*(Y_0) - V_0]^+ \mathbf{1}_{\omega_n^{N-n,0,0}, \forall n=0,...,N-1} & \text{if } \frac{p_1}{q_1^*} \leq \frac{p_3}{q_3^*} \\ \left(\frac{p_3}{q_3^*}\right)^N [V_0^*(Y_0) - V_0]^+ \mathbf{1}_{\omega_n^{0,0,N-n}, \forall n=0,...,N-1} & \text{if } \frac{p_1}{q_1^*} \geq \frac{p_3}{q_3^*}. \end{cases} \tag{2.53}$$

To prove this theorem we need some results.

With the analysis of the optimal strategies in mind, the following discussion motivates the close relation among the optimal strategy and the super-replicating measure $\mathbb{Q}^*$.

2.3.6.1 Interpretation of the optimal expected shortfall

In the expression for the minimal expected shortfall in (2.27) in the single-period model, the only measure involved is $\mathbb{Q}^* : (q, 0, 1-q, 0)$, the super-replicating measure. Then we could equivalently write the minimum expected shortfall $J^{(L)}(0, S_0, Y_0, V_0)$ as

$$J^{(L)}(0, S_0, Y_0, V_0) = \left[\min\left(\frac{p_1}{q_1^*}, \frac{p_3}{q_3^*}\right)\right] \left[V_0^*(Y_0) - V_0\right]^+.$$

On the other hand, the Radon-Nikodym derivative of $\mathbb{P}$ with respect to $\mathbb{Q}^*$ on the four possible scenarios $\omega_i, i = 1, 2, 3, 4$ defined in (2.3) in the single-period model is

$$\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega_1) = \frac{p_1}{q_1^*}, \quad \frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega_3) = \frac{p_3}{q_3^*},$$

$$\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega_2) = \frac{p_2}{q_2^*}, \quad \frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega_4) = \frac{p_4}{q_4^*}.$$

This implies that the optimal decision is taken only in terms of the two scenarios $\omega_1 = (u, h)$ and $\omega_3 = (d, h)$, and we can rewrite the optimal value function in (2.27) as

$$J^{(L)}(0, S_0, Y_0, V_0) = \left[\min\left(\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega_1), \frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega_3)\right)\right] \left[V_0^*(Y_0) - V_0\right]^+.$$

The aim in the optimal strategy is to buy (or sell) $\hat{\pi}_0^{(L)}$ number of assets so that the expected shortfall is minimal. This can also be interpreted as follows. Buy (or sell) as many as possible Arrow-Debreu securities[3] on the **"most-favourable event"** $\tilde{\omega}$ to compensate for the outcomes on the other $H_T(\omega)$ Arrow-Debreu securities associated with all other events $\omega \in \Omega \backslash \{\tilde{\omega}\}$. On the other hand, as the initial capital is less than the super-replicating price, and by the expression for the final wealth in the solution to the single-period problem in (2.29), super-replication holds except in one event ω^*, the **"worst-case event"**.

When $\frac{p_1}{q_1^*} \leq \frac{p_3}{q_3^*}$ the **"most-favourable event"** is $\omega_3 = (d, h)$ and the **"worst-case event"** is $\omega_1 = (u, h)$. This explains the expression for the portfolio strategy $\hat{\pi}_0^{(L)}$ in (2.28). On the contrary, when $\frac{p_1}{q_1^*} \geq \frac{p_3}{q_3^*}$ the **"most-favourable event"** is $\omega_1 = (u, h)$ and the **"worst-case event"** is $\omega_3 = (d, h)$.

Then if $V_T^{\hat{\pi}^{(L)}}$ represents the terminal value of the optimal portfolio, the shortfall of the strategy will be $H_T(\omega^*) - V_T^{\hat{\pi}^{(L)}}(\omega^*)$, and the expected shortfall of the optimal strategy is

$$\left(H_T(\omega^*) - V_T^{\hat{\pi}^{(H)}}(\omega^*)\right) \mathbb{P}(\omega^*)$$

[3] A canonical Arrow Debreu security is a security that pays one unit of numeraire if a particular state of the world is reached and zero otherwise. As such, any derivatives contract whose terminal value is a function on an underlying whose value is uncertain at maturity date can be decomposed as linear combination of Arrow-Debreu securities.

if $V_T^{\hat{\pi}^{(L)}}(\omega^*) < H_T(\omega^*)$ and zero otherwise. In terms of the minimal short risk $J^{(L)}(0, S_0, Y_0, V_0)$ we have noticed that the Radon-Nikodym derivative $\frac{d\mathbb{P}}{d\mathbb{Q}^*}$ plays a crucial role. This quantity in fact is used as a selection procedure to pick the **"worst-case event"**. This is, the **"worst-case event"** ω^* is the one where the ratio between the expected payoff of an Arrow-Debreu security (w.r.t. $\mathbb{P}$) and its super-replication price (w.r.t. $\mathbb{Q}^*$) is minimal.

Assume for a moment that Theorem 47 for the N-period model holds. We can extrapolate ideas from the single-period model to note that for any measure $\mathbb{Q}$: (q_1, q_2, q_3, q_4), its Radon-Nikodym derivative $\frac{d\mathbb{P}}{d\mathbb{Q}}$ on a generic event $\omega_n^{n_1,n_2,n_3}$ defined in (2.2) is given by

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\left(\omega_n^{n_1,n_2,n_3}\right) = \left(\frac{p_1}{q_1}\right)^{n_1}\left(\frac{p_2}{q_2}\right)^{n_2}\left(\frac{p_3}{q_3}\right)^{n_3}\left(\frac{p_4}{q_4}\right)^{N-n-(n_1+n_2+n_3)}.$$

Then the two **"worst-case scenario"** paths in the N-period model at the current time n correspond to the paths of either always taking the one-period event $\omega_1 = (u, h)$ or $\omega_3 = (d, h)$ for the remaining time $N-n$. Using the notation in (2.2) for the generic event $\omega_n^{n_1,n_2,n_3}$, the two **"worst-case scenario"** paths correspond to

$$\omega_n^{N-n,0,0} \text{and } \omega_n^{0,0,N-n}.$$

Thus, for the events $\omega_n^{N-n,0,0}$ and $\omega_n^{0,0,N-n}$ and the measure $\mathbb{Q}^*$ we have

$$\frac{d\mathbb{P}}{d\mathbb{Q}^*}\left(\omega_n^{N-n,0,0}\right) = \left(\frac{p_1}{q_1^*}\right)^{N-n} \text{and } \frac{d\mathbb{P}}{d\mathbb{Q}^*}\left(\omega_n^{0,0,N-n}\right) = \left(\frac{p_3}{q_3^*}\right)^{N-n}.$$

And the optimal value function can be rewritten as

$$J^{(Large)}(n, S_n, Y_n, V_n) = \left[\min\left(\frac{d\mathbb{P}}{d\mathbb{Q}^*}\left(\omega_n^{N-n,0,0}\right), \frac{d\mathbb{P}}{d\mathbb{Q}^*}\left(\omega_n^{0,0,N-n}\right)\right)\right]\left[V_n^*(Y_n) - V_n\right]^+.$$

Remark 48 *It is important to highlight that the solution in this particular setting in incomplete markets has similar features to the solution in complete markets (see [9], [28]). The difference is that under a complete market, the reference measure is the risk-neutral measure and here the super-replicating measure. In reference to the dual formulation to the problem (see Appendix B), in this particular setting we have characterised the optimal pair* $(\tilde{\eta}, \tilde{\mathbb{Q}})$ *by* $\left(\min_{\omega\in\Omega}\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega)\right\}, \mathbb{Q}^*\right)$. *We will formulate more on this on the sequel.*

The previous discussion highlights the relation between the optimal strategy and the **"worst-case scenario"** path. We need to define precisely what we mean by **"worst-case scenarios"** and clarify the relation with the optimal strategy. This is done in the next section where we analyse the strategies.

2.3.6.2 Analysis of the strategies

Following the intuition given in Section 2.3.6.1, in this section we establish formally that the only strategies that cannot be super-replicated with an initial capital just below the super-replicating price in (2.50) are the paths corresponding to the events $\omega_n^{N-n,0,0}$ and $\omega_n^{0,0,N-n}$. In terms of the single-period model and when $\frac{p_1}{q_1^*} \leq \frac{p_3}{q_3^*}$, the **"worst-case scenario"** path (that cannot be super-replicated) consists of only movements of the type $\omega_1 = (u, h)$. For the contrary in the case $\frac{p_1}{q_1^*} \geq \frac{p_3}{q_3^*}$ the **"worst-case scenario"** path consists of only movements of the type $\omega_1 = (d, h)$.

Lemma 49 *Assume $\hat{\pi}_n$ is the optimal strategy according to (2.52); then*

$$[H(Y_N) - V_N^{\hat{\pi}}]^+ = \begin{cases} \left(\frac{1}{q}\right)^N [V_0^*(Y_0) - V_0]^+ \mathbf{1}_{\omega_n^{N-n,0,0}, \forall n=0,\ldots,N-1} & \text{if } \frac{p_1}{q} \leq \frac{p_3}{1-q} \\ \left(\frac{1}{1-q}\right)^N [V_0^*(Y_0) - V_0]^+ \mathbf{1}_{\omega_n^{0,0,N-n}, \forall n=0,\ldots,N-1} & \text{if } \frac{p_1}{q} \geq \frac{p_3}{1-q}. \end{cases}$$

Proof. Assume $\frac{p_1}{q} \leq \frac{p_3}{1-q}$. With $\hat{\pi}_n$ the optimal strategy we have

$$\begin{aligned} J(N-1, S_{N-1}, Y_{N-1}, V_{N-1}^{\hat{\pi}}) &= \mathbb{E}\left[J(N, S_N, Y_N, V_N^{\hat{\pi}}) | S_{N-1}, Y_{N-1}, V_{N-1}^{\hat{\pi}}\right] \\ &= \left(\frac{p_1}{q}\right)\left[V_N^*(S_{N-1}u, Y_{N-1}h) - V_{N-1}^{\hat{\pi}}\right]^+ \\ &= \left(\frac{p_1}{q}\right)\left[V_{N-1}^*(S_{N-1}, Y_{N-1}) - V_{N-1}^{\hat{\pi}}\right]^+ \mathbf{1}_{\omega_1=(u,h)}, \end{aligned}$$

or, using backward induction,

$$\mathbb{E}\left[(H(Y_N) - V_N^{v,\hat{\pi}})^+ | S_n, Y_n, V_n^{v,\hat{\pi}}\right] = \left(\frac{p_1}{q}\right)^{N-n} \left[V_n^*(Y_n) - V_n^{\hat{\pi}}\right]^+ \mathbf{1}_{\omega_n^{N-n,0,0}, \forall n=0,\ldots,N-1}$$

or, for $n = 0$,

$$\mathbb{E}\left[(H(Y_N) - V_N^{v,\hat{\pi}})^+\right] = \left(\frac{p_1}{q}\right)^N (V_0^*(Y_0) - V_0)^+ \mathbf{1}_{\omega_0^{N,0,0}}.$$

However, from the discussion in Section 2.3.6.1, the event $\{\ _{\omega=\omega_0^{N,0,0}}\ \}$ has under $\mathbb{P}$ a probability of p_1^N. From where can be concluded that

$$\mathbb{P}\left[H(Y_N) - V_N^{v,\hat{\pi}} = \left(\frac{1}{q}\right)^N (V_0^*(Y_0) - V_0)\right] = p_1^N,$$

or, equivalently $[H(Y_N) - V_N^{\hat{\pi}}]^+ = \left(\frac{1}{q}\right)^N [V_0^*(Y_0) - V_0]^+ \mathbf{1}_{\omega=\omega_0^{N,0,0}}$.

For the case $\frac{p_1}{q} \geq \frac{p_3}{1-q}$ we can conclude in a similar way that

$$\mathbb{P}\left[H(Y_N) - V_N^{v,\hat{\pi}} = \left(\frac{1}{1-q}\right)^N (V_0^*(Y_0) - V_0)\right] = p_3^N,$$

or equivalently $[H(Y_N) - V_N^{\hat{\pi}}]^+ = \left(\frac{1}{1-q}\right)^N [V_0^*(Y_0) - V_0]^+ \mathbf{1}_{\omega=\omega_0^{0,0,N}}$. $\square$

2.3.6.3 The initial capital condition

In this section, we formalise the relation between the condition (2.50) on the initial capital V_0 and a constraint on the optimal portfolio value $V_n^{v,\hat{\pi}}$.

Lemma 50 *If the inequality*

$$V_0 \geq V_0^*(Y_0) - \begin{cases} (q_1^*)^N \{V_1^*(Y_0h) - V_1^*(Y_0l)\} & \text{if } \frac{p_1}{q_1^*} \leq \frac{p_3}{q_3^*} \\ q_1^* \; (q_3^*)^{N-1} \{V_1^*(Y_0h) - V_1^*(Y_0l)\} & \text{if } \frac{p_1}{q_1^*} \geq \frac{p_3}{q_3^*} \end{cases} \tag{2.54}$$

is satisfied, then

$$(1-q)V_{n+1}^*(Y_nh) + qV_{n+1}^*(Y_nl) \leq V_n$$

holds for all $n = 0, .., N-1$.

Proof. The equality $(1-q)V_{n+1}^*(Y_nh) + qV_{n+1}^*(Y_nl) \leq V_n$ is equivalent to

$$r^n \left(V_n^*(Y_n) - V_n\right) \leq r^n q \left(V_n^*(Y_n) - V_{n+1}^*(Y_nl)\right) \tag{2.55}$$

for $r > 0$ and for all $n = 0, ..., N-1$. Assume $\frac{p_1}{q} \leq \frac{p_3}{1-q}$ and using some intermediate steps in the proof of Lemma 49 we have $q^n \left(V_n^*(Y_n) - V_n\right) = \{V_0^*(Y_0) - V_0\} \, \mathbf{1}_{\omega=\omega_0^{N,0,0}}$. Then for the relation (2.55) to be true we need the inequality $\{V_0^*(Y_0) - V_0\} \, \mathbf{1}_{\omega=\omega_0^{N,0,0}} \leq q^{n+1} \left(V_n^*(Y_n) - V_{n+1}^*(Y_nl)\right)$ for all $n = 0, ..., N-1$, or equivalently $\{V_0^*(Y_0) - V_0\} \leq q^{n+1} \left(V_n^*(Y_0 \; h^n) - V_{n+1}^*(Y_0 \; h^n \; l)\right)$ for all $n = 0, \ldots, N-1$. But this is equivalent to (2.50) when $\frac{p_1}{q} \leq \frac{p_3}{1-q}$. For the remaining case when $\frac{p_1}{q} \geq \frac{p_3}{1-q}$ we proceed similarly using the relation $(1-q)^n \left(V_n^*(Y_n) - V_n\right) = \{V_0^*(Y_0) - V_0\} \, \mathbf{1}_{\omega=\omega_0^{0,0,N}}$ to obtain the inequality $\{V_0^*(Y_0) - V_0\} \leq q(1-q)^n \left(V_n^*(Y_0 \; h^n) - V_{n+1}^*(Y_0 \; h^n \; l)\right)$ for all $n = 0, \ldots, N-1$, which give us the constraint in (2.50). ☐

In the interpretation of the optimal strategy as buying/selling Arrow-Debreu securities, the condition on the initial capital (2.54) guarantees that this capital is enough to buy/sell Arrow-Debreu securities on the most-favourable scenario events at every time step $n = 0, ..., N-1$.

When $\frac{p_1}{q_1^*} \leq \frac{p_3}{q_3^*}$ the initial capital condition (2.54) can also be written as

$$V_0 \geq (1-q^N)H(Y_0h^N) + q^N H(Y_0 \; l \; h^{N-1}) =: C^L, \tag{2.56}$$

and when $\frac{p_1}{q_1^*} \leq \frac{p_3}{q_3^*}$

$$V_0 \geq (1-q(1-q)^{N-1})H(Y_0h^N) + q(1-q)^{N-1}H(Y_0 \; l \; h^{N-1}) =: C^L. \tag{2.57}$$

This representation as a convex combination of the super-replicating price ($H(Y_0 h^N)$) and a slightly lower level ($H(Y_0\ l\ h^{N-1})$) gives us the notion of how narrow the condition on the initial capital is. Figure 2.3 shows the values for the condition C^L and the super-replication prices as function of the number of steps N for the numerical parameters in Example 53. Furthermore, when N grows large, the condition becomes tighter meaning that the optimal strategy given by Theorem 47 is valid only for initial capital values very close to the super-replication price. This is a very disadvantageous feature of the solution. It also represents problems if one would like to obtain conclusions in the continuous-time by passing to the limit. This issue needs further investigation and it is left to future research. Nevertheless, when the number of steps is not very large, it provides a manageable model and permits to get a lot of insight on the solution in this particular incomplete market situation. This is important as most of the incomplete market models are difficult to deal with.

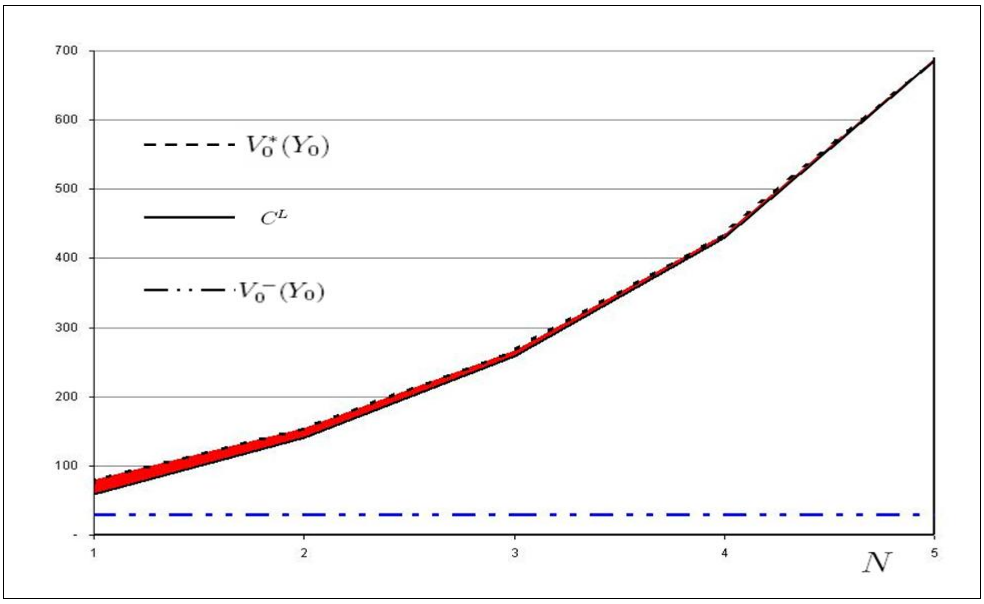


Figure 2.3: Change with respect to the number of steps N of the superreplication price V_0^*, the initial capital condition C^L in (2.56 and 2.57) and the lower bound of the arbitrage-free prices V_0^- for Example 53. For a small number of steps (5), the condition grows very fast to the superreplication price.

If the condition on the initial capital (2.54) is not satisfied, then always performing the same strategy as in Theorem 47 may not be optimal, as more than one scenario path could not be replicated (super-replicated). This will imply mixed-type strategies and will be more difficult to analyse. In any of these cases, we expect to have initial capital thresholds that characterise the optimal solutions.

If by analogy with the expression of C^L in (2.56) and (2.57) for the condition on the Large capital case, we denote by C^M the condition when two or less scenario paths can not be super-replicated, then $V_0^-(Y_0)$ will be the threshold that defines when three or less scenario paths are not super-replicated. The expected shortfall values as a function on the initial capital V_0 will be a piecewise linear function as in Figure 2.4. And the solution corresponding to the initial capital equal to C^L can be seen as a lower bound for the expected shortfall problem when $V_0 < C^L$. The lower the initial capital, the wider the lower bound will be.

Given these observations and by the form of the optimal solution in the single-period case, we conclude that if we denote by $ES_{V_0}(\pi^N)$ the expected shortfall at time 0 for the initial capital V_0 and strategy π performed at each of the N periods, the following bounds hold

$$\left(\frac{p_1}{q}\right)^N [V_0^*(Y_0) - V_0]^+ \leq ES_{V_0}(\text{optimal}) \leq \min \left\{ \begin{array}{c} ES_{V_0}\left(\left(\pi^{ul}\right)^N\right), \\ ES_{V_0}\left(\left(\pi^{dh}\right)^N\right) \end{array} \right\} \quad \text{if} \quad \frac{p_1}{q} \leq \frac{p_3}{1-q}$$

$$\left(\frac{p_3}{1-q}\right)^N [V_0^*(Y_0) - V_0]^+ \leq ES_{V_0}(\text{optimal}) \leq \min \left\{ \begin{array}{c} ES_{V_0}\left(\left(\pi^{uh}\right)^N\right), \\ ES_{V_0}\left(\left(\pi^{dl}\right)^N\right) \end{array} \right\} \quad \text{if} \quad \frac{p_1}{q} \geq \frac{p_3}{1-q}.$$

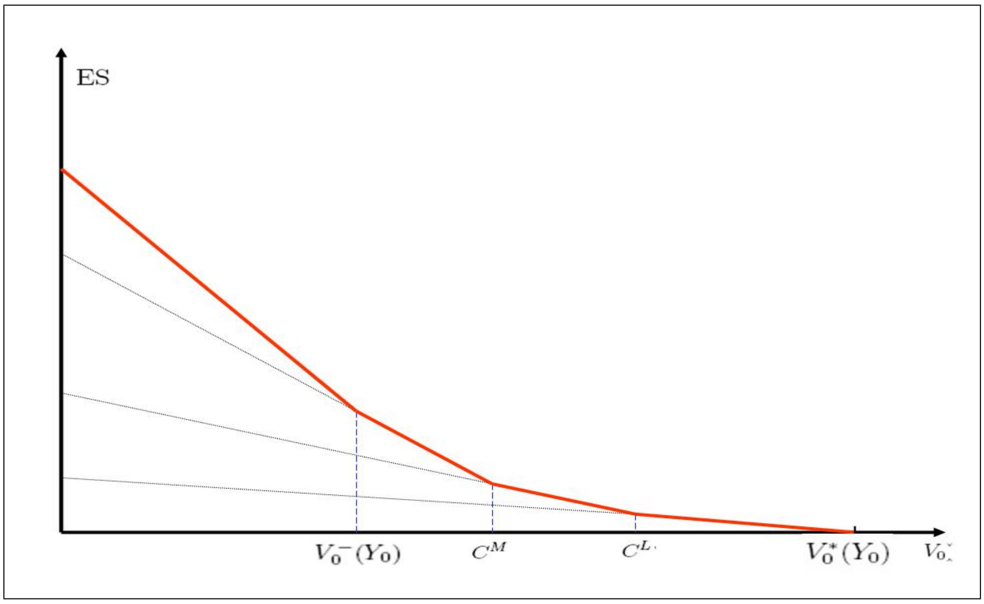


Figure 2.4: Piecewise linear shape of the minimal expected shortfall as a function of the initial capital V_0. The solution corresponding to the initial capital equal to C^L is a lower bound for the minimal expected shortfall when $V_0 < C^L$.

We can now turn to the proof of the main theorem. It is now direct that with Lemmas 44, 49 and 50 the Theorem 47 holds as shown below.

Proof of Theorem 47. For $n = N-1$, and by the Dynamic Programming Principle (see [6]),

$$J^{(L)}(N-1, S_{N-1}, Y_{N-1}, V_{N-1}) = \inf_{\pi} \mathbb{E}\left[(H(Y_N) - V_N)^+ | S_{N-1}, Y_{N-1}, V_{N-1}\right]. \quad (2.58)$$

The goal at the step $N-1$ is to find the minimum and the minimiser $\hat{\pi}$ in (2.58). Under the assumption $(1-q)V_N^*(Y_{N-1}h) + qV_N^*(Y_{N-1}l) \leq V_{N-1}$, we can use Lemma 44 and Section 2.3.6.1 to show that equations (2.51) and (2.52) hold for $n = N-1$. We now proceed by induction with respect to n. Assume $\frac{p_1}{q} \leq \frac{p_3}{1-q}$ and that Equation (2.51) is valid at $n+1$ this is $J(n+1, S_{n+1}, Y_{n+1}, V_{n+1}) = \left(\frac{p_1}{q_1^*}\right)^{N-n-1} \left[V_{n+1}^*(Y_{n+1}) - V_{n+1}\right]^+$ then

$$\begin{aligned}
&\mathbb{E}\left[J(n+1, S_{n+1}, Y_{n+1}, V_{n+1}) | S_n, Y_n, V_n\right] = \\
= &\left(\frac{p_1}{q_1^*}\right)^{N-n-1} \mathbb{E}\left[\left[V_{n+1}^*(Y_n\eta_{n+1}) - V_n - (\xi_{n+1} - 1)\left\{\frac{V_{n+1}^*(Y_nh) - V_n}{(d-1)}\right\}\right]^+\right] \\
= &\left(\frac{p_1}{q_1^*}\right)^{N-n-1} \left\{\frac{p_1}{q}\left[V_{n+1}^*(Y_n\ h) - V_n\right]^+ + \frac{p_2}{q}\left[(1-q)V_{n+1}^*(Y_nh) + qV_{n+1}^*(Y_nl) - V_n\right]^+\right\}
\end{aligned}$$

and assuming $(1-q)V_{n+2}^*(Y_{n+1}h) + qV_{n+2}^*(Y_{n+1}l) - V_{n+1} \leq 0$ it remains,

$$= \left(\frac{p_1}{q}\right)^{N-n} \left[V_{n+1}^*(Y_n\ h) - V_n\right]^+ = \left(\frac{p_1}{q}\right)^{N-n} \left[V_n^*(Y_n) - V_n\right]^+ = J(n, S_n, Y_n, V_n).$$

Finally, Equation (2.53) follows from Lemma 49. The validity of $(1-q)V_{n+1}^*(Y_nh) + qV_{n+1}^*(Y_nl) \leq V_n$ for all $n = 0, ..., N-1$ follows from Lemma 50.

The case $\frac{p_1}{q} \geq \frac{p_3}{1-q}$ can be proved similarly. □

2.3.6.4 The large capital case and the two-step procedure

We are now in a position to check whether the solution to the minimal expected shortfall for the large initial capital case in Theorem 47 is the same as the given by the two-step procedure in Proposition 42.

For STEP 1 in Proposition 42, we need to characterise the modified claim that solves the minimal expected shortfall problem. We use the intuition given in Section 2.3.6.1 summarised as follows:

1. The optimal portfolio minimises the expected shortfall.
2. The Radon-Nikodym derivative $\frac{d\mathbb{P}}{d\mathbb{Q}^*}$ acts as a selection procedure to the **"worst-case scenario"** path $\omega^{\circledcirc}$.

3. The optimal portfolio super-replicates the contingent claim except in the **"worst-case scenario"** path $\omega^{\circledcirc}$,

$$V_N^{V_0,\hat{\pi}}(\omega^{\circledcirc}) < H_N(\omega^{\circledcirc}) \text{ and } V_N^{V_0,\hat{\pi}}(\omega) \geq H_N(\omega) \text{ in } \omega \in \Omega \backslash \{\omega^{\circledcirc}\}.$$

4. The shortfall is non-zero at the **"worst-case scenario"** path $\omega^{\circledcirc}$ and zero otherwise.

These facts suggests a representation of the final value of the optimal portfolio as

$$\tilde{V}_N(\omega) := H_N(\omega)\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega)>\eta^*\right\}} + \gamma\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega)=\eta^*\right\}}$$

with $\eta^* := \min_{\omega\in\Omega}\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega)\right\}$, and γ such that matches the shortfall value. We can formulate the modified claim as a corollary to Theorem 47.

Corollary 51 *Define the modified claim* $\tilde{V}_N(\omega)$ *as*

$$\tilde{V}_N(\omega) := H_T(\omega)\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega)>\eta^*\right\}} + \gamma\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega)=\eta^*\right\}}$$

with

$$\gamma = \frac{V_0 - \mathbb{E}^*\left[H_N\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega)>\eta^*\right\}}\right]}{\mathbb{E}^*\left[\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega)=\eta^*\right\}}\right]}.$$

If $V_N^{V_0,\hat{\pi}}$ *represents the optimal portfolio strategy to the Large initial capital case in Theorem 47, then we have*

$$V_N^{V_0,\hat{\pi}} = \tilde{V}_N \quad \mathbb{P}\text{-}a.s.$$

Proof. Assume $\omega^{\circledcirc}$ represents the worst-case scenario path of Section 2.3.6.1. It is direct to see that $\{\omega = \omega^{\circledcirc}\} = \left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega) = \eta^*\right\}$, $\mathbb{P}$-a.s. Then the representation $\tilde{V}_N(\omega) := H_N(\omega)\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega)>\eta^*\right\}} + \gamma\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega)=\eta^*\right\}}$ is clear. It remains to show that γ selected in this way is the right choice. On the scenario path $\omega^{\circledcirc}$

$$H_T(\omega^{\circledcirc}) - \tilde{V}_T(\omega^{\circledcirc}) = H_T(\omega^{\circledcirc}) - \gamma = \begin{cases} \left(\frac{1}{q_1^*}\right)^N [V_0^*(Y_0) - V_0]\,\mathbf{1}_{\omega_0^{0,0,N}} & \text{if } \frac{p_1}{q_1^*} \leq \frac{p_3}{q_3^*} \\ \left(\frac{1}{q_3^*}\right)^N [V_0^*(Y_0) - V_0]\,\mathbf{1}_{\omega_0^{0,0,N}} & \text{if } \frac{p_1}{q_1^*} \geq \frac{p_3}{q_3^*} \end{cases}$$

or equivalently,

$$\begin{aligned}
\gamma &= H_T(\omega^{\circledcirc}) - \frac{1}{\mathbb{Q}^*\left[\omega^{\circledcirc}\right]}\left[V_0^*(Y_0) - V_0\right] \\
&= \frac{V_0 + \mathbb{Q}^*\left[\omega^{\circledcirc}\right] H_T(\omega^{\circledcirc}) - \mathbb{E}^*\left[H_N\right]}{\mathbb{Q}^*\left[\omega^{\circledcirc}\right]} \\
&= \frac{V_0 - (1 - \mathbb{Q}^*\left[\omega^{\circledcirc}\right])\mathbb{E}^*\left[H_N \mathbf{1}_{\omega \neq \omega^{\circledcirc}}\right]}{\mathbb{Q}^*\left[\omega^{\circledcirc}\right]} \\
&= \frac{V_0 - \mathbb{E}^*\left[H_N \mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega) > \eta^*\right\}}\right]}{\mathbb{E}^*\left[\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega) = \eta^*\right\}}\right]}.
\end{aligned}$$

□

In order to fully prove STEP 1 in Proposition 42 about the modified claim that solves the minimal expected shortfall problem we need to show that it indeed satisfies the conditions in Proposition 42. Furthermore, if STEP 2 also holds, then the solution to the N-period Large initial capital case in Theorem 47 is indeed equivalent to that obtained by the two-step algorithm (dual formulation to the minimal expected shortfall problem). These is guaranteed in the next proposition.

Proposition 52 *The solution to the N-period Large capital case in Theorem 47 is equivalent to the one given by the two-step procedure in Proposition 42. Furthermore, as the two-step procedure reflects the dual formulation to the problem (see Appendix B), in the present setting the dual pair is given by*

$$(\tilde{\eta}, \tilde{\mathbb{Q}}) := \left(\min_{\omega \in \Omega} \left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}(\omega)\right\}, \mathbb{Q}^*\right)$$

Proof. The proof follows immediately from Theorem 47 and the representation of the modified claim in Corollary 51. The inequalities $\tilde{V}_N \leq H_N$ and $\mathbb{E}^*\left[\tilde{V}_N\right] \leq V_0$ are a direct consequence of the representation of the modified claim. Then $\tilde{V}_N \in \mathbf{V}_\infty$. By Theorem 47, the fact that $\tilde{V}_N$ minimises the expected shortfall implies that $\tilde{V}_N = \arg\min_{V \in \mathbf{V}_b} \mathbb{E}\left[H_T - V_T\right]$. This proves that $\tilde{V}_N$ satisfies STEP1 in Proposition 42. For STEP 2, $\tilde{V}_N$ is a super-replicating strategy also by Theorem 47. Finally, the specification of the Dual pair is a consequence to the representation of the modified claim. □

2.3.6.5 Numerical example

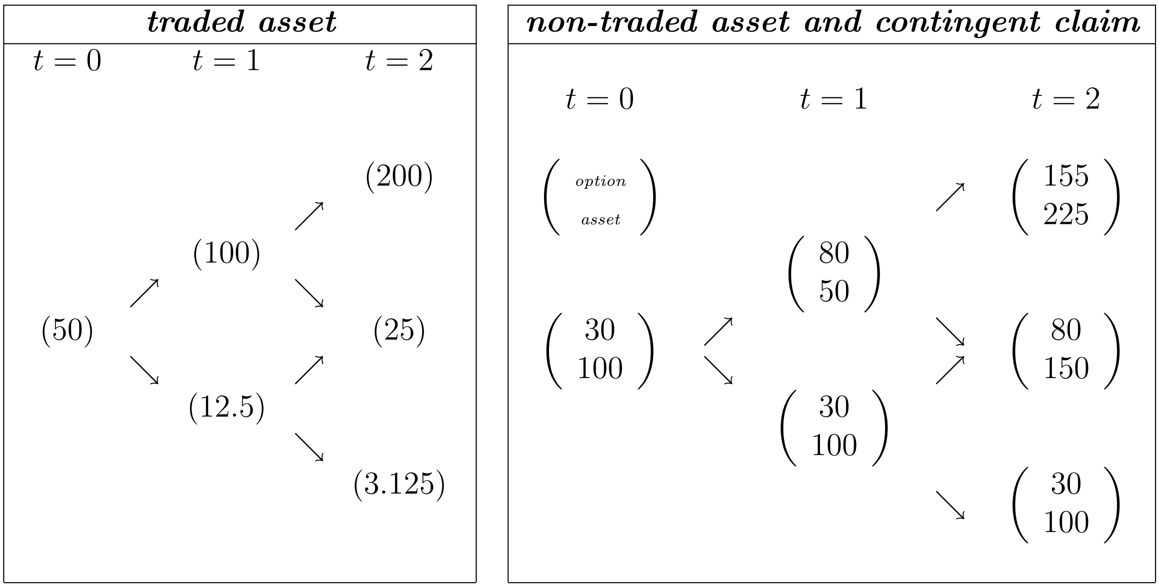
Example 53 *Consider the following example for $N = 2$, and*

$$\begin{array}{lllll}
S_0 = 50 & u = 2 & d = 0.25 & p_1 = 1/6 & p_2 = 5/16 \\
Y_0 = 100 & h = 1.5 & l = 1 & p_3 = 1/2 & p_4 = 1/48
\end{array}$$

With these parameters, we have

$$q \approx 0.429,$$
$$\frac{p_1}{q} \approx 0.389, \quad \frac{p_3}{1-q} \approx 0.875, \quad \frac{p_1+p_2}{q} \approx 1.118$$
$$\frac{p_2}{q} = 0.729, \quad \frac{p_4}{1-q} \approx 0.036, \quad \frac{p_3+p_4}{1-q} \approx 0.911.$$

We take the contingent claim to be a call option with strike $E = 70$*. Then the payoff is given by* $H(Y_T) = \max(Y_T - E, 0)$*. At time* 0*, the super-replicating price of the contingent claim* H *is* $V_0^*(Y_0) = 155$*, the lower bound price* $V_0^-(Y_0) = 30$ *and the initial capital threshold* $C^L = 141.2$*. The price trees are shown below.*



When $N = 1$ *(considering only the first step) Figure 2.5 shows the optimal expected shortfall and strategies as function of the initial capital* V_0*. One can notice only one change in strategy as expected by results in Lemma 44, 45 and 46.*

For $N = 2$*, we computed the expected shortfall values for different combinations of strategies* $(\pi_0, \pi_1) \in \Pi$ *with* $\Pi := \{(\pi_0, \pi_1) : \pi_0 \in \{\pi_0^{uh}, \pi_0^{ul}, \pi_0^{dh}, \pi_0^{dl}\}$ *and* $\pi_1 \in \{\pi_1^{uh}, \pi_1^{ul}, \pi_1^{dh}, \pi_1^{dl}\}\}$ *(*$\pi_i^{uh}, \pi_i^{ul}, \pi_i^{dh}, \pi_i^{dl}$ $i = 0, 1$ *defined in (51)), and for integer values for the initial capital from* 30 *to* 155*. The expected shortfall values as functions of the initial capital for the relevant strategies are shown in Figure 2.6. The solid line at the bottom of the figure shows that* $\left(\frac{p_1}{q}\right)^2 [V_0^*(Y_0) - V_0]$ *is indeed a lower bound for optimal expected shortfall. The triangle-style line corresponds to the strategy* (π_0^{dh}, π_1^{dh})*; the dotted line at the bottom of the Figure corresponds to the strategy* (π_0^{dh}, π_1^{ul})*; the dotted line at the top of the exhibit refers to the strategy* (π_0^{dl}, π_1^{uh}) *and the solid line to the strategy* (π_0^{uh}, π_1^{uh})*. For the parameters in this example, the minimal expected shortfall is not very far apart from the lower bound* $\left(\frac{p_1}{q}\right)^2 [V_0^*(Y_0) - V_0]$*.*

Figure 2.6 also shows the initial capital thresholds (Large, Medium and Low capital cases) that provoke a change in the expected shortfall value, and possible corresponding to a change in strategy. For the minimal shortfall strategy, this thresholds are $C^L = 141.2$, $C^M = 112$ *and* $C^{LOW} = 51$. *This can be seen in more detail in Figures 2.7 and 2.8, where we have computed the series*

$$1 - \frac{ES_{V_0 - 1}(\cdot)}{ES_{V_0}(\cdot)}$$

as functions of the initial capital V_0 *and for each strategy* $(\pi_0, \pi_1) \in \Pi$. *This series allow us to visualise more easily the levels that induce a change in expected shortfall. Figure 2.7 shows that for the strategy* (π_0^{dh}, π_1^{ul}) *the threshold levels are* 140 *and* 49*; for the strategy* (π_0^{dl}, π_1^{uh}) *the threshold levels are* 81 *and* 50*; and as illustrated in Figure 2.8 for the strategy* (π_0^{dh}, π_1^{dh}) *the threshold levels are* $140, 122$ *and* 112.

It turns out that the optimal strategies are: for $V_0 < 112$ *the one given by* (π_0^{dh}, π_1^{ul}) *and for* $V_0 \geq 112$ *the strategy* (π_0^{dh}, π_1^{dh}). *Note that this solution is in accordance to the solution in Theorem 47 for* $V_0 > 141.2 = C^L$. *This is illustrated in Figure 2.9 together as the solution to the maximal expected shortfall strategy for comparison purposes.*

2.3.7 The discrete-time as an approximation to a continuous time model

The two-asset model can be used to approximate a continuous time dynamics for the assets S_t and Y_t of the type

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dY_t &= \alpha Y_t dt + \beta Y_t \left(\rho dW_t + \sqrt{1-\rho^2} dZ_t \right), \end{aligned} \tag{2.59}$$

where $\mu, \sigma, \alpha, \delta, \beta$ and ρ are constant, (W, Z) are standard independent Brownian motions under $\mathbb{P}$. This is done by the right choice of parameters $u, d, h, l, p_1, p_2, p_3$ and p_4 so that match the distributional properties of the continuous time dynamics for the processes S_t and Y_t (see [57]). For a time interval $[0, T]$ divided into N subintervals with equal time steps $\Delta t = T/N$ we need to take the parameters such that

$$\begin{array}{ll} u = e^{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}} & h = e^{\left(\alpha - \frac{1}{2}\beta^2\right)\Delta t + \beta\sqrt{\Delta t}} \\ d = e^{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t - \sigma\sqrt{\Delta t}} & l = e^{\left(\alpha - \frac{1}{2}\beta^2\right)\Delta t - \beta\sqrt{\Delta t}}, \end{array} \tag{2.60}$$

and

$$\begin{aligned} p_1 &= p_4 = \frac{1+\rho}{4} \\ p_2 &= p_3 = \frac{1-\rho}{4}. \end{aligned} \tag{2.61}$$

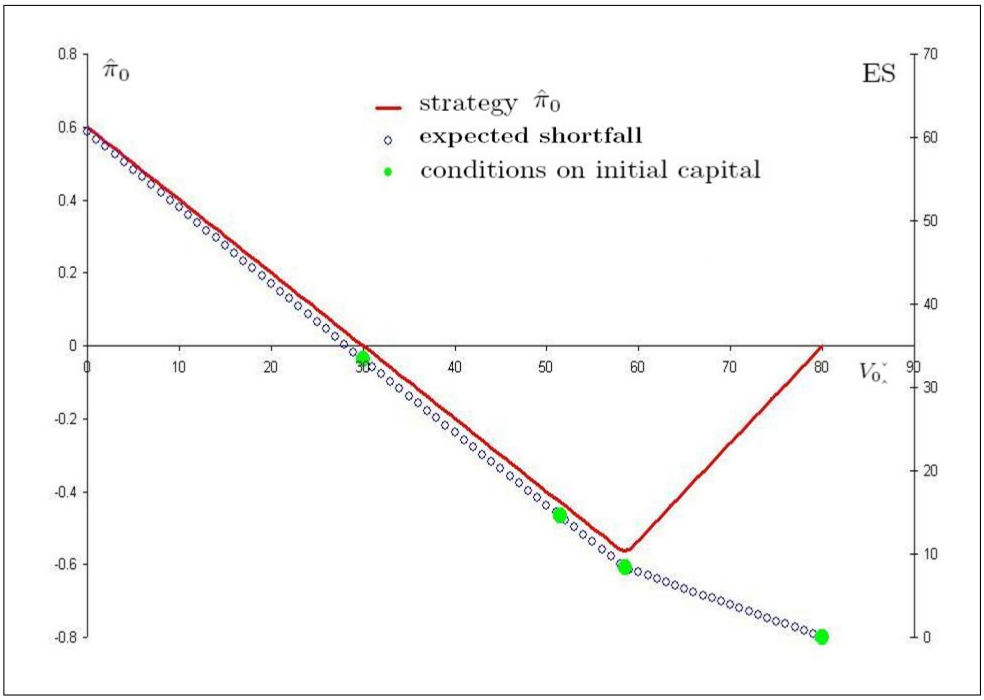


Figure 2.5: **The single-period model:** The right-hand side axis measures the optimal expected shortfall as a function of the initial capital V_0. The left-hand side axis corresponds to the values of the optimal strategies as a function of the initial capital V_0. The highlighted points on the expected shortfall function correspond to the initial capital thresholds (Large, Medium and Low capital cases).

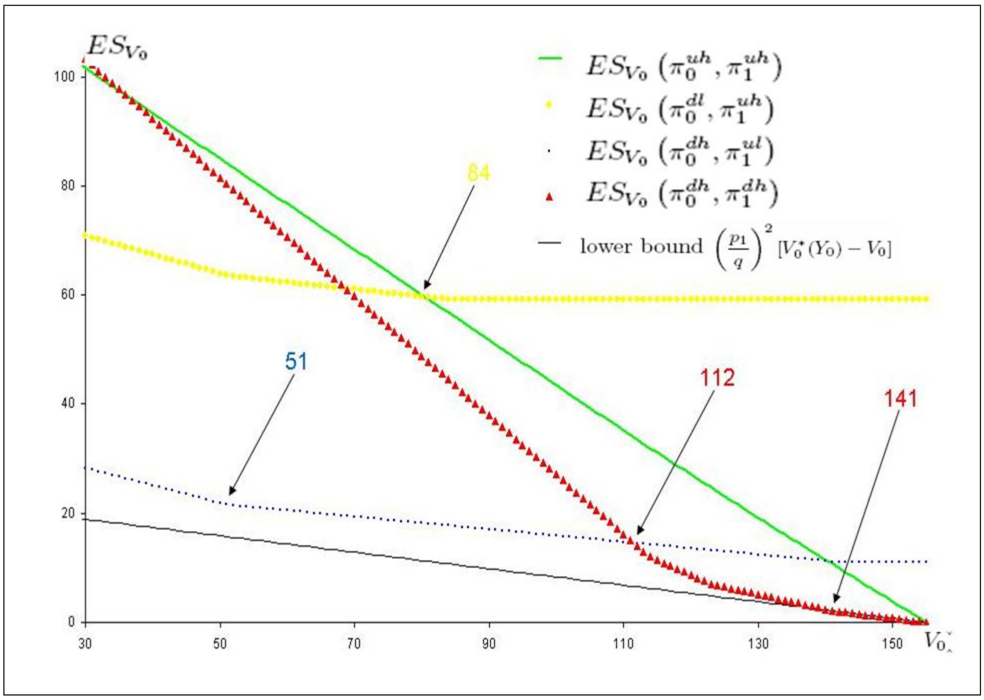


Figure 2.6: Comparison of expected shortfall values as function of the initial capital V_0 for several strategies. The solid line at the bottom of the figure shows that $\left(\frac{p_1}{q}\right)^2 [V_0^*(Y_0) - V_0]$ is indeed a lower bound for optimal expected shortfall. The triangle-style line corresponds to the strategy (π_0^{dh}, π_1^{dh}); the dotted line at the bottom of the figure corresponds to the strategy (π_0^{dh}, π_1^{ul}); the dotted line at the top of the exhibit refers to the strategy (π_0^{dl}, π_1^{uh}) and the solid line to the strategy (π_0^{uh}, π_1^{uh}). For the parameters in this example, the minimal expected shortfall is not very far apart from the lower bound $\left(\frac{p_1}{q}\right)^2 [V_0^*(Y_0) - V_0]$.

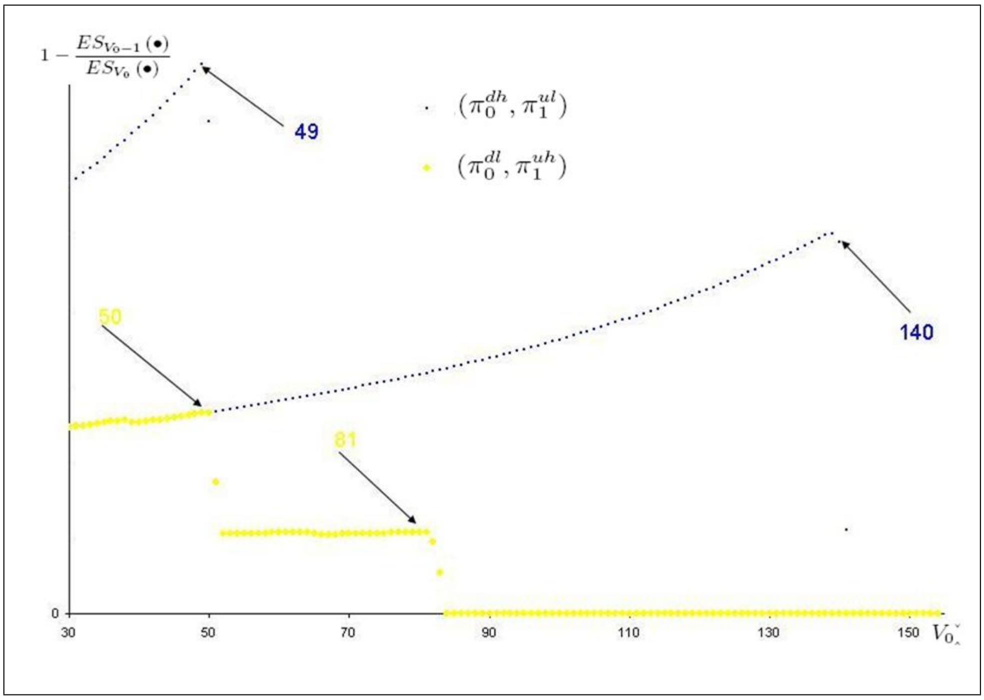


Figure 2.7: Threshold levels that induce changes in the expected shortfall values as function of the initial capital V_0. For the strategy (π_0^{dh}, π_1^{ul}) the threshold levels are 140 and 49; for the strategy (π_0^{dl}, π_1^{uh}) the threshold levels are 81 and 50.

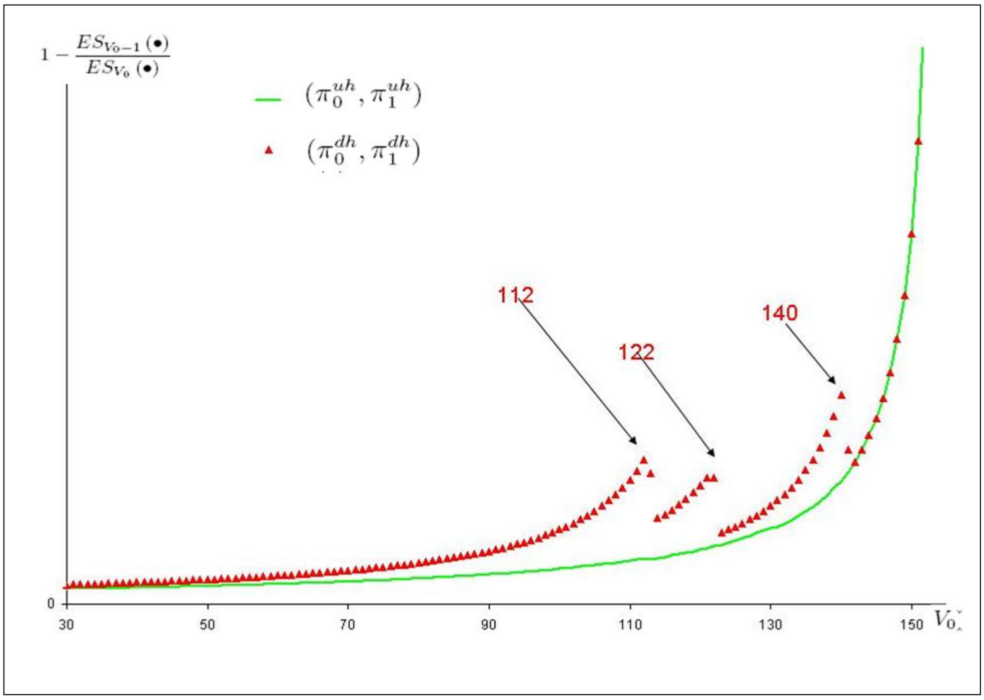


Figure 2.8: Threshold levels that induce changes in the expected shortfall values as function of the initial capital V_0. For the strategy (π_0^{dh}, π_1^{dh}) the threshold levels are $140, 122$ and 112.

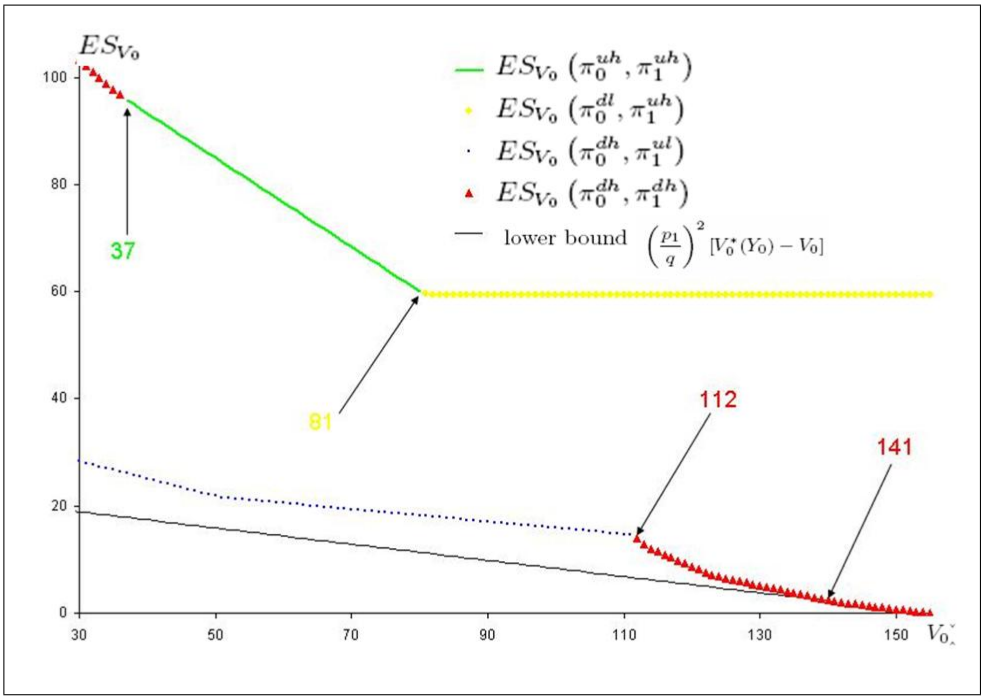


Figure 2.9: Comparison of minimal expected shortfall (mES) and Maximal expected shortfall (MES) strategies. The mES strategy is: (π_0^{dh}, π_1^{ul}) for $30 < V_0 < 112$ and (π_0^{dl}, π_1^{dl}) for $112 < V_0 \leq 141.2$; and the MES strategy corresponds to (π_0^{dh}, π_1^{dh}) for $30 < V_0 < 37$ and (π_0^{uh}, π_1^{uh}) for $37 < V_0 \leq 81$, and (π_0^{dl}, π_1^{uh}) for $81 < V_0 \leq 141.2$. The solid line at the bottom of the exhibit represents the lower bound for the expected shortfall given by $\left(\frac{p_1}{q}\right)^2 [V_0^*(Y_0) - V_0]$.

This choice of parameters give us a Markov chain approximation that converges in law to the continuous processes S_t and Y_t by matching the mean, variance and correlation.

2.3.7.1 Numerical example

In order to see how the above approximation to the continuous-time model works, consider the following numerical example.

Example 54 ***Continuous-time approximation and sensitivity analysis on the parameters.*** *Consider the the two-step approximation to the continuous-time dynamics in (2.59) for* $\mu = 0.1, \sigma = 0.4, \alpha = 0.1, \beta = 0.7,\ \rho = 0.9,\ S_0 = 50,$ *and* $Y_0 = 100$.

With these parameters, we have

$$\begin{array}{lll} q = 0.3772, & & \\ \frac{p_1}{q} = 1.259, & \frac{p_3}{1-q} = 0.040, & \frac{p_1+p_2}{q} = 1.325, \\ \frac{p_2}{q} = 0.066, & \frac{p_4}{1-q} = 0.0762, & \frac{p_3+p_4}{1-q} = 0.802. \end{array}$$

We take the contingent claim to be a call option with strike $E = 70$*. Then the payoff is given by* $H(Y_T) = \max(Y_T - E, 0)$*. At time* 0*, the super-replicating price of the contingent claim* H *is* $V_0^*(Y_0) = 233.4$*, the lower bound price* $V_0^-(Y_0) = 0$ *and the initial capital threshold* $C^L = 179.7$*. The price trees are shown below.*

traded asset		
$t=0$	$t=1$	$t=2$
		(115.81)
	(76.09) ↗	
(50)		(52.04)
	(34.19) ↘	
		(23.38)

non-traded asset and contingent claim		
$t=0$	$t=1$	$t=2$
$\begin{pmatrix} \text{option} \\ \text{asset} \end{pmatrix}$		$\begin{pmatrix} 233.43 \\ 303.43 \end{pmatrix}$
	$\begin{pmatrix} 104.19 \\ 174.19 \end{pmatrix}$	
$\begin{pmatrix} 30 \\ 100 \end{pmatrix}$		$\begin{pmatrix} 4.82 \\ 74.82 \end{pmatrix}$
	$\begin{pmatrix} 0 \\ 42.95 \end{pmatrix}$	
		$\begin{pmatrix} 0 \\ 18.45 \end{pmatrix}$

Figure 2.10 shows the expected shortfall values as functions of the initial capital V_0 *for the optimal strategy given by* $(\pi_0^{dl},\ \pi_1^{uh})$ *when* $V_0 < 139.4$ *and* $(\pi_0^{uh},\ \pi_1^{uh})$ *for*

$V_0 \geq 139.4$. *Note that the solution given by Theorem 47 is* (π_0^{uh}, π_1^{uh}) *for* $V_0 > C^L = 179.7$. *This shows that the condition* C^L *is not binding and could be improved..*

We now turn to analysing the sensitivity of the optimal solution to the volatilities and correlation coefficient. In Figure 2.11 and 2.12 we have plotted the minimal expected shortfall (mES) for values of the correlation coefficient ρ *from* -1 *to* 1 *by increments of size* 0.1. *We split the plot into two exhibits in order to see more clear that the mES increases as* ρ *increases from* -1 *to* -0.4 *and decreases for values from* -0.2 *to* 1. *This is due to the fact that the values of* ρ *affect the quantities* $\frac{p_1}{q}$ *and* $\frac{p_3}{1-q}$ *(see Figure 2.15(a)) and this values determine the optimal strategy and therefore, the mES.*

In relation to the sensitivity of the mES to the volatility to the traded asset S, *we computed the mES for values of* σ *from* 0.1 *to* 1. *The changes in* σ *affect directly the quantities* q *and the initial capital condition* C^L *(see Figure 2.15(b)). The Figure 2.13 shows that mES increases as the volatility* σ *increases. This is an expected behaviour, as the larger the volatility of the stock* S, *the larger the volatility on the hedging portfolio and more risk to cover.*

Figure 2.14 exhibits the values of mES for values of the non-traded asset volatility for β *from* 0.1 *to* 1. *The volatility* β *affects the values of the infimum and supremum of the arbitrage-free prices,* $V_0^*(Y_0)$ *and* $V_0^-(Y_0)$, *respectively, as well as the initial capital condition* C^L *(see Figure 2.15(c)). We have plotted the mES in pairs of values for* β *to distinguish that the volatility of* Y *has little effect on the mES.*

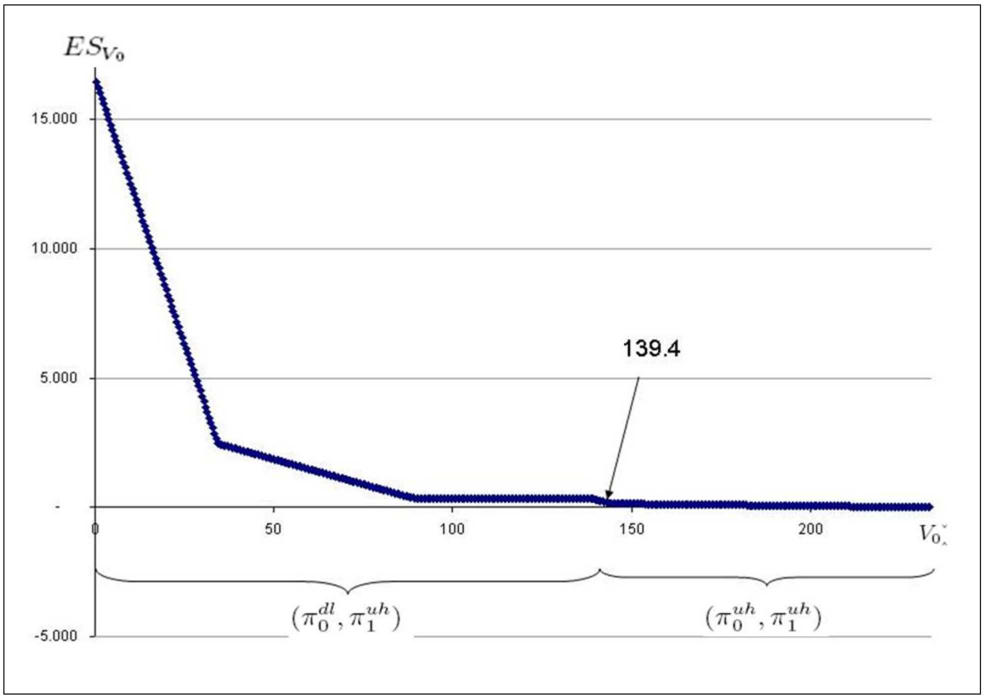


Figure 2.10: Minimal expected shortfall as function of the initial capital V_0. The corresponding optimal strategy is: (π_0^{dl}, π_1^{uh}) for $0 < V_0 < 139.4$ and (π_0^{uh}, π_1^{uh}) for $139.4 \leq V_0 < 233.4$.

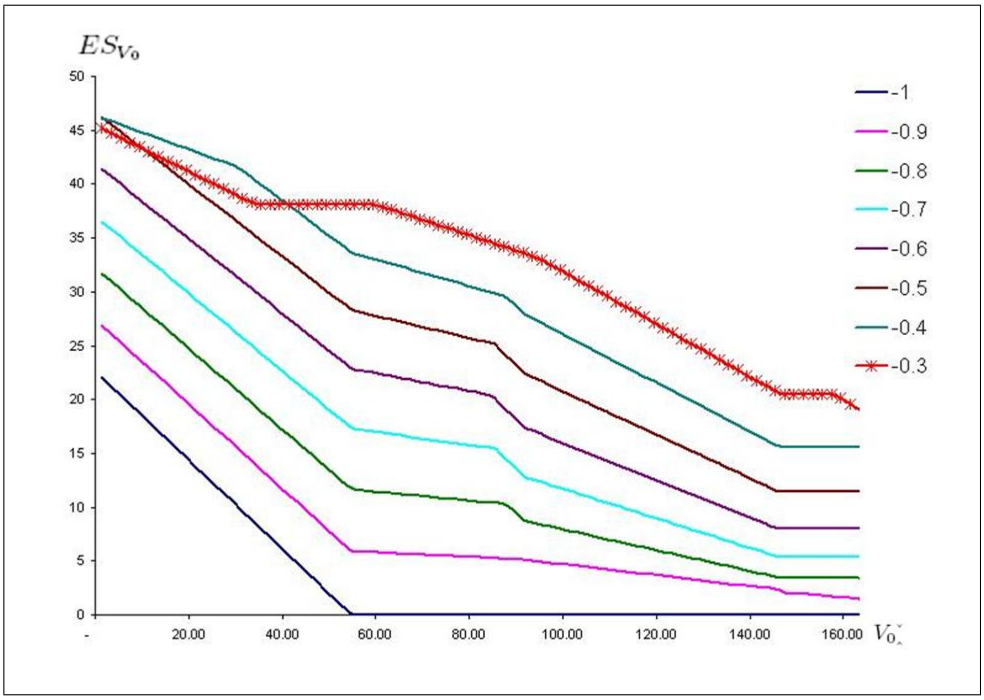


Figure 2.11: Comparison of minimal expected shortfall (mES) values as function of the initial capital V_0 by changing the correlation coefficient ρ from -1 to -0.4 by increments of size 0.1. The mES increases in the range.

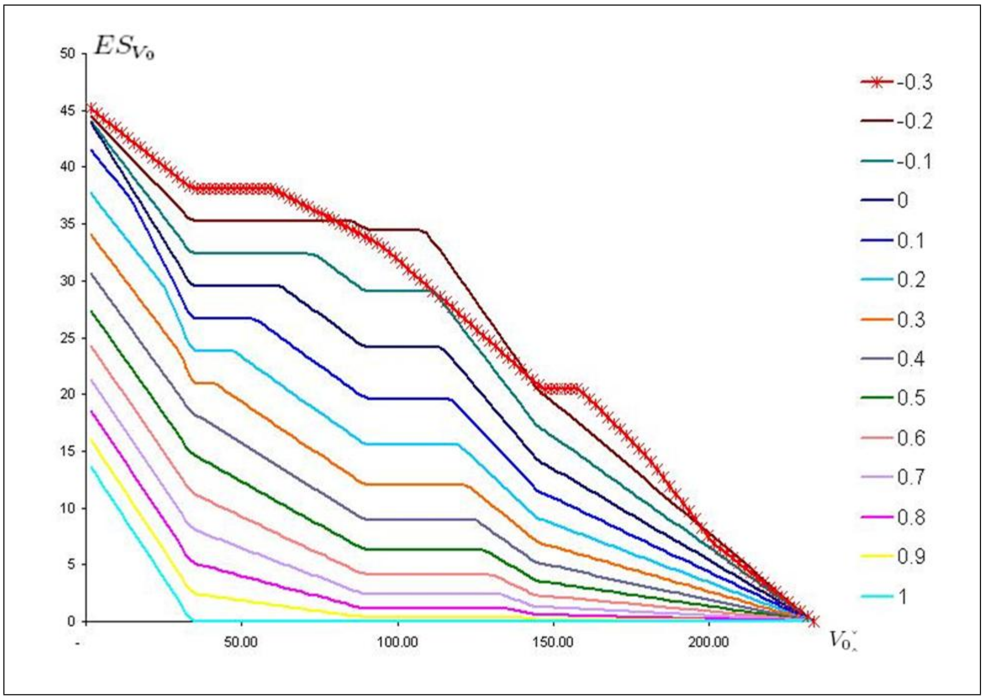


Figure 2.12: Comparison of minimal expected shortfall (mES) values as function of the initial capital V_0 by changing the correlation coefficient ρ from -0.3 to 1 by increments of size 0.1. The mES decreases in the range.

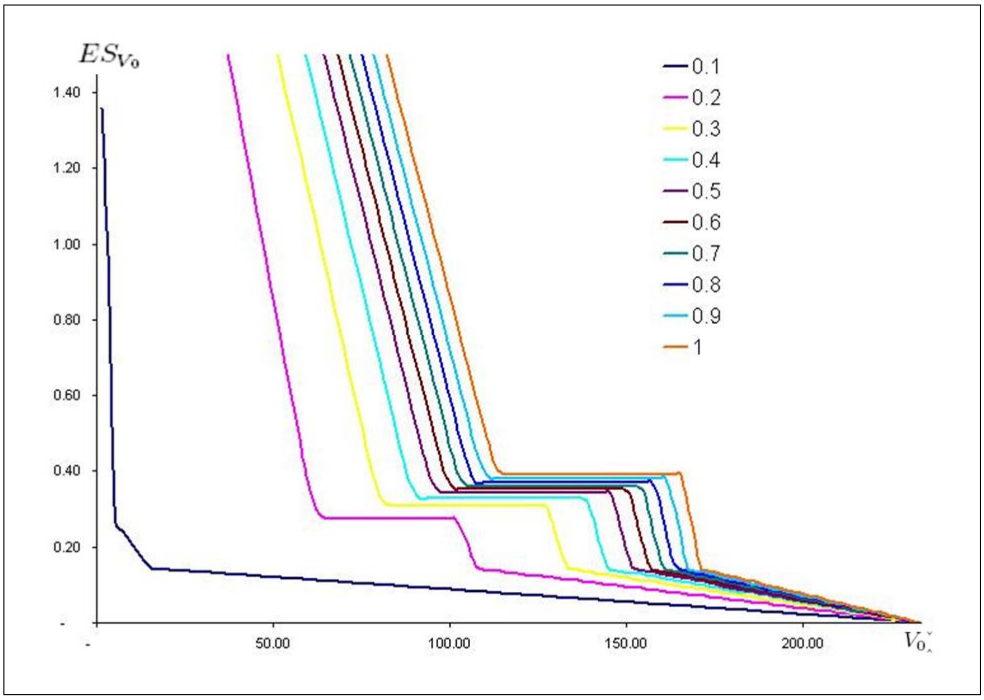
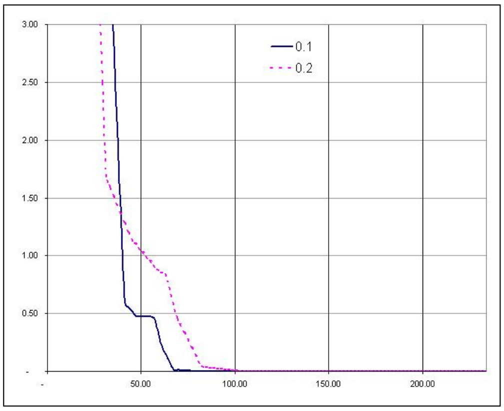
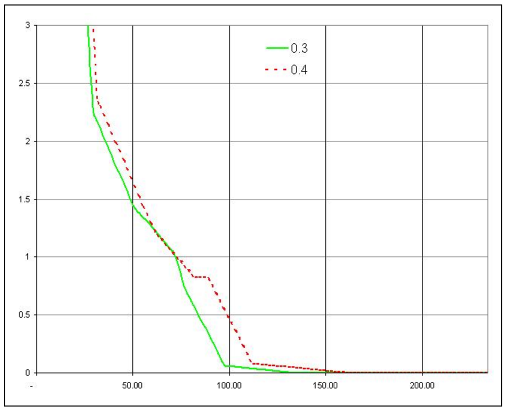


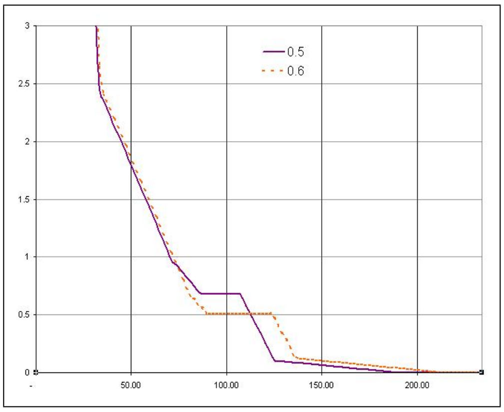
Figure 2.13: Comparison of minimal expected shortfall (mES) values as function of the initial capital V_0 by changing the volatility σ of the asset S from 0.1 to 1 by increments of size 0.1. The mES increases as volatility σ increases.



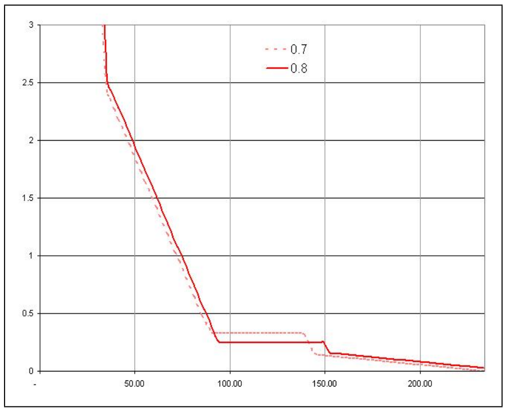
(a) $\beta = 0.1$ and $\beta = 0.2$.



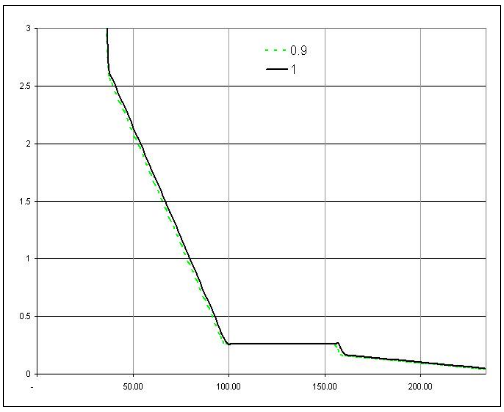
(b) $\beta = 0.3$ and $\beta = 0.4$.



(c) $\beta = 0.5$ and $\beta = 0.6$.

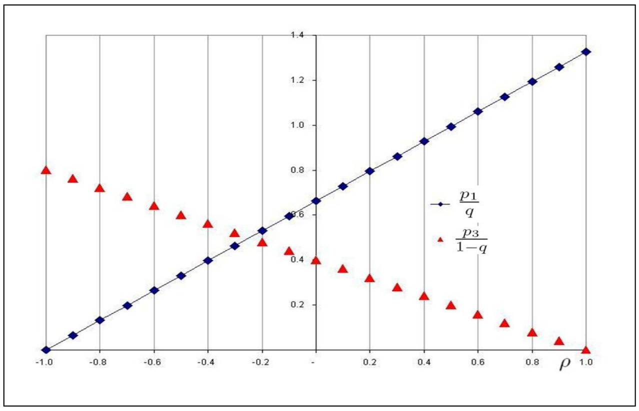


(d) $\beta = 0.7$ and $\beta = 0.8$.

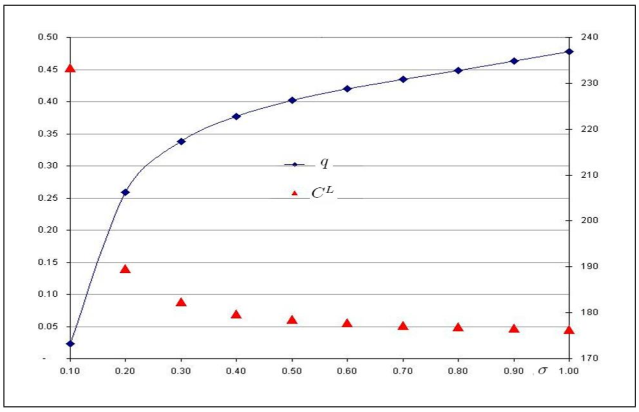


(e) $\beta = 0.9$ and $\beta = 1$.

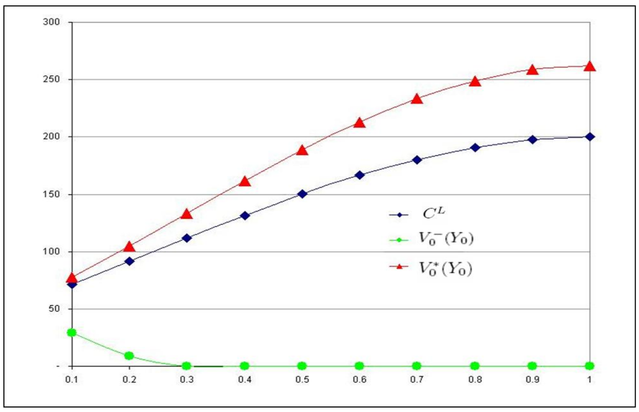
Figure 2.14: Comparison of minimal expected shortfall (mES) values as function of the initial capital V_0 by changing the volatility β of the non-traded asset Y from 0.1 to 1 by increments of size 0.1. The mES increases as volatility β increases but not significantly.



(a) Change in $\frac{p_1}{q}$ and $\frac{p_3}{1-q}$ by values of ρ.



(b) Change in q and C^L by values of σ.



(c) Change in inf and sup of prices and large capital initial condition C^L by values in β.

Figure 2.15: Behaviour of some of the strategy determining quantities by changes in the correlation coefficient ρ, and the volatilities σ and β of the assets S and Y, respectively.

Part II

Risk and Hedging in Continuous-time: Itô Diffusion Models

Chapter 3

WCS, VaR and AVaR

3.1 Introduction

Assume we work under a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the probability $\mathbb{P}$ is assumed to be the "real" or "physical" probability measure (data generating), which is objective and assumed unique. Let also assume our space supports a random variable (or process) X that represents a risky asset for which we need to compute risk. When X is model-dependent, this is, when the probability $\mathbb{P}$ on Ω is explicitly given, we can in many cases compute WCS, VaR, and AVaR. A special case of model-dependence is when X is given by a Markov diffusion process, and as they are the most common models used in finance, we devote this chapter to the study of the computation of our three measures of risk (WCS, VaR, and AVaR) for positions driven by one-dimensional continuous Markov diffusion processes.

Note that although we carry our analysis only for WCS, VaR, and AVaR, most of our procedures can also be applied to other similar measures of risk.

The main idea to compute risk given by VaR or AVaR of a position X modelled as a continuous diffusion process is to exploit its Markov property and characterise it as the solution to a second order partial differential equation (PDE) with boundary conditions. In the case of the WCS risk measure, the approach is similar but less direct as WCS is defined as the supremum over a set of probability measures. In order to obtain WCS also as the solution to a boundary value PDE we need to state conditions on the set of measures, so the supremum in the definition of WCS is finite.

Motivated by practical applications, firstly, we analyse in detail the case when under each measure on the definition of WCS the process X remains a Markov diffusion process. This involves the study of properties of what are called exponential change of measure transformations of Markov processes.

In most of the cases, the PDEs that characterise the risk measures do not have explicit solutions and series expansions or numerical methods need to be applied. In the few cases that do allow explicit solutions, solving for the risk PDEs and solving for the transition probability density of the process X are equivalent. This is shown in the last part of the chapter.

When the restriction on the Markov property is lifted, we establish conditions so the computation of WCS can be formulated as a stochastic control problem and then as the solution to a nonlinear PDE of second order with boundary conditions. We complement the chapter with several examples.

3.2 The model

For simplicity in the notation, we assume through this chapter that our time horizon is $[t_0, T]$, with $0 \leq t_0 \leq T$, unless otherwise stated.

Let the process $X_t \in \mathbb{R}$ be such that its dynamics are given by the following one-dimensional SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + a(X_t)\mathrm{dW}_t, \quad X_{t_0} = x_0 \qquad (X_t\text{-SDE})$$

with a, b given deterministic functions satisfying $b : \mathbb{R} \to \mathbb{R}$, $a : \mathbb{R} \to (0, \infty)$, and W_t a one-dimensional Brownian motion. We need the following assumption on the process X.

Assumption 55 *We assume that the functions a and b have sufficient regularity properties to ensure that the stochastic differential equation in (X_t-SDE) for the process X_t has a path-wise unique solution, non-explosive strong solution (cf [51, Section 5.3 p. 300] or [77, Chapter 5]), and $\{\mathcal{F}_t\}_{t_0 \leq t \leq T}$ is its natural filtration.*

Assumption 56 *We assume the process X_t has a transition probability density $p(x, t; \tau, y)$. This is the case for some conditions on the functions a, and b (see [51] p. 369, [92, p. 500] or [86, Theo 3.2.1, Cor. 3.22 and Theo 3.2.6 p. 71-77]). Furthermore, we also assume the conditions needed by the Feynman-Kac representation theorem hold (see [51, Theo 7.6 p. 366] or [92, Theo 3.33 p. 497]) so $p(x, t; \tau, y)$ allows a unique stochastic representations.*

The homogeneous infinitesimal generator associated with the SDE in (X_t-SDE) is given by

$$\mathcal{G}^{b,a}[f](x) := \frac{1}{2}a^2(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x}.$$

The main idea for computing $\mathrm{WCS}_{\mathcal{P}}$, VaR_α and AVaR_α for processes X_t as in (X_t-SDE) is to exploit their Markov property and formulate the risk measures as the solution of a PDE.

3.3 Worst Conditional Scenario risk measure (WCS)

By the definition of $\mathrm{WCS}_{\mathcal{P}}$ in Chapter 1, recalled below

$$\mathrm{WCS}_{\mathcal{P}}(X) = \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[-X\right]$$

we fix a priori a subset $\mathcal{P}$ (called set of priors) of the set of all probability measures on $(\Omega, \mathcal{F})$. It is necessary that each measure $\mathbb{Q} \in\mathcal{P}$ respects $\mathbb{P}$-null sets, for otherwise a stochastic integral defined with respect to $\mathbb{P}$ might make no sense under $\mathbb{Q}$. Therefore we make the following assumption.

Assumption 57 *The set of reference measures $\mathcal{P}$ used in $WCS_{\mathcal{P}}$ is a subset of $\mathcal{M}_a(\mathbb{P})$.*

Note that $\mathrm{WCS}_{\mathcal{P}}$ is "well defined" if the set $\mathcal{P}$ is such that $\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[-X\right] < \infty$. In particular, this is satisfied if $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ on $[t_0, T]$, $\mathcal{P}$ is convex, and the set $\mathcal{D}_{\mathcal{P}} := \left\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} : \mathbb{Q} \in\mathcal{P}\right\}$ is weakly compact in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. In other words, for a bounded position X, and $Z \in \mathcal{D}_{\mathcal{P}}$ we have

$$J_{-X}(Z) = \mathbb{E}\left[-XZ\right] = \mathbb{E}_{\mathbb{Q}}\left[-X\right] < \infty.$$

$J_{-X}(Z)$ is a linear functional and by the assumption that the set $\mathcal{D}_{\mathcal{P}}$ is weakly compact, the above relation also holds for the infimum and supremum of the elements in the set $\mathcal{D}_{\mathcal{P}}$. This is,

$$\sup_{Z\in\mathcal{D}_{\mathcal{P}}} J_{-X}(Z) = \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[-X\right] < \infty$$

and

$$\inf_{Z\in\mathcal{D}_{\mathcal{P}}} J_{-X}(Z) = \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[-X\right] < \infty.$$

On the other hand, if there exist sets $A \subset \Omega$ for which $\mathbb{Q}\left[A\right] = 0$ for all $\mathbb{Q} \in\mathcal{P}$ and $\mathbb{P}\left[A\right] \neq 0$ they may cause problems in the definition of WCS. Therefore, we assume, as follows:

Assumption 58 *We assume $\mathcal{P} \subset \mathcal{M}_a(\mathbb{P})$ is convex, the set $\mathcal{D}_{\mathcal{P}} := \left\{\frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in\mathcal{P}\right\}$ is weakly compact in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{P}\left[A\right] = 0$ if and only if $\mathbb{Q}\left[A\right] = 0$ for all $\mathbb{Q} \in\mathcal{P}$.*

In the case our risk horizon is $[t_0, t]$, so for computing $\mathrm{WCS}_{\mathcal{P}}$ on the position $X^{t-t_0} = X_t - X_{t_0}$, we need to know the dynamics of the process X under each of the elements in $\mathcal{P}$.

3.3.1 Change of measure

Change of measures are characterised by Girsanov theorem, which says (see [51, Sec. 3.5] or [77, Sec. 27]) that if $\mathbb{Q}$ is an absolutely continuous probability measure with respect to $\mathbb{P}$, then there exist an adapted processes $\{\varphi_t\}_{t_0 \le t \le T}$ with $\int_{t_0}^{T} \varphi_s^2 \mathrm{ds} < \infty$ (called Girsanov kernels) such that for

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = Z_t = Z(\varphi)_t := \exp\left[\int_{t_0}^{t} \varphi_u a(X_u)\mathrm{dW}_u - \frac{1}{2}\int_{t_0}^{t} (\varphi_u)^2 a(X_u)^2 \mathrm{du}\right]. \tag{3.1}$$

Under the change of measure $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = Z_t$ the process X_t under $\mathbb{Q}$ remains

$$\mathrm{dX}_t = \hat{\gamma}_t \mathrm{dt} + a(X_t)\mathrm{d}\tilde{\mathrm{W}}_t \tag{3.2}$$

with $X_{t_0} = x_0 \in \mathbb{R}$, and

$$\hat{\gamma}_t = b(X_t) + a^2(X_t)\varphi_t. \tag{3.3}$$

And $\tilde{W}$ is a Brownian motion under the measure $\mathbb{Q}$

Remark 59 *In the case Z_t is a uniformly integrable $\mathbb{P}$-martingale, the measure $\mathbb{Q}$ is not only absolutely continuous with respect to $\mathbb{P}$, but equivalent to $\mathbb{P}$. A sufficient condition for Z_t to be uniformly integrable is the Novikov criterion $\mathbb{E}\left[\exp\left(\frac{1}{2}\int \varphi_s^2 a^2(X_s)ds\right)\right] < \infty$. See [51] for conditions on φ_t for Z_t to be a uniformly integrable $\mathbb{P}$-martingale.*

Remark 60 *It is a well known fact that an absolutely continuous change of measure affects the process only in the drift term. From this observation, it is direct to see that the $WCS_{\mathcal{P}}$ risk measure can be interpreted as the risk due to uncertainty in the drift. This is a special case of* ***model risk****.*

One particular case of interest for computing WCS is when, under each measure in the set $\mathcal{P}$, the process X_t remains a Markov process. This includes the situation when the process X_t is assumed to have a fixed functional structure (e.g. geometric Brownian motion, Ornstein-Uhlenbeck processes, etc.) depending on some parameters. But the "true" parameters are unknown, and instead, one considers some estimation intervals on them. Computing WCS is then similar to considering the "worst-case scenario" case of the expected value under the estimation intervals.

We analyse first the particular case when each of the measures $\mathbb{Q} \in \mathcal{P}$ defines a Markov process, and then we explore some generalisations. If the set of measures $\mathcal{P}$ is so that the Markov property is preserved on the process X_t under each $\mathbb{Q} \in \mathcal{P}$, then we can formulate the problem of computing WCS as a solution of a PDE. This will be analysed in the next section.

3.3.2 Change of measure that preserves the Markov property

From the expression of the dynamics of X_t under the measure $\mathbb{Q} \in \mathcal{P}$ in (3.2) and (3.3) we notice that for the process X_t to be Markov for $\mathbb{Q} \in \mathcal{P}$, we need to impose conditions on the Girsanov kernel φ_t in (3.1). The Markov property will be preserved if φ_t only depends on the actual value of[1] X_t, this is, φ_t needs to be of the form $\varphi_t = \psi(t, X_t)$, for some $\mathcal{F}_t$-measurable function $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$.

In terms of the density process Z_t, any change of measure can also be expressed (cf [47, Prop 3.8 p.155] or [51, Lemma 5.3 p.193]) as

$$\mathbb{E}_{\mathbb{Q}}[X^{t-t_0}] = \frac{1}{Z(\varphi)_{t_0}} \mathbb{E}[X^{t-t_0} Z(\varphi)_t].$$

Thus the dynamics of X_t under $\mathbb{Q}$ can be rewritten as,

$$\mathrm{d}X_t = \gamma(t, X_t)\mathrm{d}t + a(X_t)\mathrm{d}\tilde{\mathrm{W}}_t, \tag{3.4}$$

for $\gamma(t, x) = b(x) + a^2(x)\psi(t, x)$.

Remark 61 *Note that defining the density process $Z_t = Z(\psi(\cdot, X.))_t$ in (3.1) via a time-dependent Girsanov kernel $\psi(t, X_t)$ makes the process time-dependent as the drift $\gamma(t, x) = b(x) + a^2(x)\psi(t, x)$ will depend on time.*

This type of measure change that preserves the Markov property is called an exponential change of measure for Markov processes (see [12], [65] and [8]).

3.3.2.1 Exponential change of measure for Markov processes

Before we motivate the exponential change of measure, we need to introduce some notation.

Define by $\mathcal{E}(y).$ the Doléans exponential for a martingale process $y.$ on $[0, \infty)$ by

$$\mathcal{E}(y)_t := \exp\left[y_t - \frac{1}{2}[y]_t\right].$$

One can also think of $\mathcal{E}(y).$ as the process $z_t = \mathcal{E}(y)_t$ such that defines the unique solution to the equation

$$\mathrm{d}z_t = z_t \mathrm{d}y_t, \quad z_0 = 1.$$

The equation above shows that $\mathcal{E}(y)_t$ represents a martingale that has exponential form (see [77, Ch. IV Sec. 19 p. 29] on Doléans exponential to define exponential martingales).

[1] In general, a Girsanov kernel φ_t may depend on the past, or the whole path of the process X.

From the previous section on changes of measure that preserve the Markov property, the Radon-Nikodym derivative remains

$$\begin{aligned}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} &= \exp\left[\int_0^t \psi(u,X_u)a(X_u)\mathrm{dW}_u - \frac{1}{2}\int_0^t (\psi(u,X_u))^2 a(X_u)^2\mathrm{du}\right] \\ &= \mathcal{E}\left(\int_0^t \psi(u,X_u)a(X_u)\mathrm{dW}_u\right) \\ &= Z_t \\ &= Z(\psi(\cdot,X.))_t \\ &= h(t,X_t)\end{aligned} \quad (3.5)$$

with $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ a $\mathcal{F}_t$-measurable function (the function h can be seen as the composition of the function Z by ψ).

The right-hand side on the first and second equalities in the set of equations in (3.5) are local-martingales by construction, which is a condition for a density process to define a proper change of measure. This implies that the function $h(t,x)$ cannot be just any function. We require conditions on h such that the process $h(t,X_t)$ is a $\mathbb{P}$-martingale.

In the rest of the section, we study conditions for the function h to define a Markov change of measure and use it to formulate the problem of computing $\mathrm{WCS}_{\mathcal{P}}$ of a position X^{t-t_0} with dynamics for X_t in (X_t-SDE) as the solution of a PDE.

For $f \in \mathcal{C}^{1,2}$ define the time-dependent infinitesimal generator $\mathcal{L}^{\gamma,a}$ associated with the dynamics of X_t in (3.4) by

$$\begin{aligned}\mathcal{L}^{\gamma,a}[f](t,x) &\triangleq \frac{\partial f}{\partial t} + \frac{1}{2}a^2(x)\frac{\partial^2 f}{\partial x^2} + \gamma(t,x)\frac{\partial f}{\partial x} \\ &= \frac{\partial f}{\partial t} + \mathcal{G}^{\gamma,a}[f](x).\end{aligned}$$

Remark 62 *We will indistinguishably denote the partial derivative with respect to the second variable of a function $f(t,x)$ by $\frac{\partial f}{\partial x}(t,x)$ or $f_x(t,x)$. Similarly for any other partial derivative. We will also omit their explicit dependency on the variables when no confusion arises.*

For $f \in \mathcal{C}^{1,2}$ define the process

$$\mathcal{J}(f)_t := f(t,X_t) - f(t_0,x_0) - \int_{t_0}^t \mathcal{L}^{b,a}[f](s,X_s)\mathrm{ds}. \quad (3.6)$$

It is a well-known fact that $\mathcal{J}(f)_t$ is a continuous local martingale for each f (see [12]). Similarly, we can also define the following processes:

$$\mathcal{K}(f)_t := \int_{t_0}^{t} f(s, X_s)\mathrm{d}X_s - \int_{t_0}^{t} f(s, X_s)b(X_s)\mathrm{d}s \tag{3.7}$$

$$\tag{3.8}$$

$$\mathcal{Z}(f)_t := \left\{ \begin{array}{l} \exp\{f(t, X_t) - f(t_0, x_0)\} \times \\ \exp\left\{-\int_{t_0}^{t} \mathcal{L}^{b,a}[f](s, X_s) + \frac{1}{2}a(X_s)^2(f_x(s, X_s))^2\mathrm{d}s\right\} \end{array} \right\} \tag{3.9}$$

and if $f(t, x) \neq 0$ for all t and x, define

$$\mathcal{ZH}(f)_t := \frac{f(t, X_t)}{f(t_0, x_0)} \exp\left\{-\int_{t_0}^{t} \frac{\mathcal{L}^{b,a}[f](s, X_s)}{f(s, X_s)}\mathrm{d}s\right\}. \tag{3.10}$$

They also are continuous $\mathbb{P}$-local martingales for each f (see [12]). These processes will be used to define exponential change of measure transformations for Markov processes, as we see next.

Remark 63 *The processes $\mathcal{J}(f)_t, \mathcal{K}(f)_t, \mathcal{Z}(f)_t$ and $\mathcal{ZH}(f)_t$ defined in (3.6), (3.7), (3.9) and (3.10), respectively, depend on a function $f \in \mathcal{C}^{1,2}$ and the process X_t, whose dynamics are given by (X_t-SDE). They are in fact functionals on the class $\mathcal{C}^{1,2}$. We highlight this fact as we will use extensively this notation in the sequel. We will also use the notation $\mathcal{J}(f(t,x))_t$ to emphasise the fact that the function f depends on t and x.*

The following lemma give us some conditions on the function f to define a change of measure which makes the dynamics of the process X under the new measure $\mathbb{Q}$ a homogeneous Markov process.

Lemma 64 *Let X_t be a Markov diffusion process on $(\Omega, \mathcal{F}, \mathbb{P})$ with dynamics in (X_t-SDE) and $f \in \mathcal{C}^{1,2}$; then*

$$\begin{aligned}
\mathcal{ZH}(\exp\{f\})_t &= \mathcal{Z}(f)_t \\
&= \mathcal{E}(\mathcal{K}(f)_t) \\
&= \mathcal{E}\left(\int_{t_0}^{t} f_x(t, X_s)a(X_s)dW_s\right) \\
&= \exp\left[\int_{t_0}^{t} f_x(t, X_s)dX_s - \int_{t_0}^{t} f_x(t, X_s)b(X_s)ds - \frac{1}{2}\int_{t_0}^{t} [f_x(t, X_s)]^2 a^2(X_s)ds\right].
\end{aligned}$$

Furthermore, if the function f is additively separable in the form

$$f(t, x) = u(t) + v(x),$$

then

$$\mathcal{Z}\left(f\right)_t = \alpha(X_t)\exp\left\{-\int_{t_0}^{t}\beta(X_s)ds\right\}, \tag{3.11}$$

with α and β computable functions depending only on v, a and b (see Proof for details). Thus defining the change of variable

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(f(t,x)\right)_t = \mathcal{Z}\left(v(x)\right)_t,$$

makes the process X_t under $\mathbb{Q}$ a homogeneous Markov process with differential form

$$dX_t = \left[b(X_t) + a^2(X_t)v_x(X_s)\right]dt + a(X_t)dW_t^{\mathbb{Q}}, \text{ with } X_{t_0} = x_0, \tag{3.12}$$

and $W^{\mathbb{Q}}$ a $\mathbb{Q}$-Brownian motion.

Proof. The first and second equalities follow from the expression of exponential change of measure for Markov processes, see [12], [65] and [8].

If $f(t,x) = u(t) + v(x)$, take $U(t) = \exp\{u(t)\}$ and $V(x) = \exp\{v(x)\}$, so $\exp\{f(t,x)\} = U(t)V(x)$. Easy computations give

$$\begin{aligned}
\mathcal{Z}\left(u(t)+v(x)\right)_t &= \mathcal{Z}\left(v(x)\right)_t \\
&= \mathcal{Z}\mathcal{H}\left(U(t)V(x)\right)_t \\
&= \mathcal{Z}\mathcal{H}\left(V(x)\right)_t \\
&= \exp\{v(X_t)-v(x_0)\}\exp\left\{-\int_{t_0}^{t}\left[\mathcal{G}_x^{b,a}[v](X_s)+\frac{1}{2}a^2(X_s)(v_x)^2(X_s)\right]\mathrm{ds}\right\} \\
&= \frac{V(X_t)}{V(x_0)}\exp\left\{-\int_{t_0}^{t}\frac{\mathcal{G}_x^{b,a}[V](X_s)}{V(X_s)}\mathrm{ds}\right\}.
\end{aligned} \tag{3.13}$$

which shows the particular time-dependence structure in (3.11). That X_t under the new measure $\mathbb{Q}$ is homogeneous and Markov also follows from the expression of the exponential change of measure. □

Remark 65 *Assume as before that $f(t,x) = u(t) + v(x)$ and $V(x) = \exp\{v(x)\}$; define*

$$d(x) = \frac{\mathcal{G}_x^{b,a}[V](x)}{V(x_0)},$$

so

$$\mathcal{Z}\mathcal{H}\left(V(x)\right)_t = \frac{V(X_t)}{V(x)}\exp\left\{-\int_{t_0}^{t}d(X_s)ds\right\}.$$

The case $d(x) = 0$ is called the homogeneous Doob h-transform for the process X_t. Furthermore, when $d(x) = 0$ we have that the function $V(x)$ satisfies

$$\frac{1}{2}a^2(x)V''(x) + b(x)V'(x) = 0,$$

and if the assumptions

$$\textit{for all } x \in \mathbb{R}, \exists\, \varepsilon > 0 \quad \textit{such that} \int_{x-\varepsilon}^{x+\varepsilon} \frac{|b(y)|dy}{a^2(y)} < \infty$$

of non-degeneracy and local integrability hold, then $V(x)$ has the form of the scale function of X_t (up to an affine transformation). See [51, Sec. 5.5] or [77, Sec. 28 and Sec. 46] for the definition and properties of the scale function for one-dimensional Markov processes.

The case $d(x) = \rho \in \mathbb{R}$ is called the inhomogeneous Doob h-transform and the density takes the form

$$\mathcal{ZH}\left(V(x)\right)_t = \frac{V(X_t)}{V(x_0)} \exp\left\{-\rho(t-t_0)\right\}.$$

From previous proposition, we see that one way to get a homogeneous Markov process under a change of measure $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(f\right)_t$ is to take functions f that depend only on the variable x. In order to explore more properties of X_t under an exponential change of measure, define for $f \in \mathcal{C}^2$

$$F(x) = (\mathcal{G}^{b,a}[f])(x) + \frac{1}{2}a(x)^2 \left(\frac{\partial f}{\partial x}\right)^2 (x) \tag{3.14}$$

and

$$\mathbf{Z}^f(t_0, x_0; t, x) = \exp\left\{f(x) - f(x_0)\right\} \mathbb{E}\left[\exp\left\{-\int_{t_0}^t F(X_t)\mathrm{ds}\right\} |X_t = x, X_{t_0} = x_0\right]. \tag{3.15}$$

The next proposition tell us the shape of the infinitesimal generator and the transition probability function for the transformed process X_t under the measure $\mathbb{Q}$ given by $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(f(t,x)\right)_t$.

Proposition 66 *Assume X_t is given as in (X_t-SDE) and let $\mathcal{Z}\left(f(t,x)\right)_t$ be as in (3.9) for some $f(t,x) = u(t) + v(x)$. Define the conditional probability $\mathbb{P}_{t_0,x_0}(dx) := \mathbb{P}\left[dx|X_{t_0} = x_0\right]$. Then for each x, $t > 0$ the probability measure $\mathbb{Q}^f_{t_0,x_0}$ given by the Radon-Nikodym derivative*

$$\frac{\mathbb{Q}^f_{t_0,x_0}(dx)}{\mathbb{P}_{t_0,x_0}(dx)} = \mathbf{Z}^f(t_0, x_0; t, x)$$

is absolutely continuous w.r.t. $\mathbb{P}_{t_0,x_0}$ for each x_0. If X^f_t represents the process under the measure $\mathbb{Q}^f$, then X^f_t on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}^f_{t_0,x_0})$ is a Markov process with generator given in terms of the infinitesimal generator of X_t under $\mathbb{P}$ by

$$\mathcal{G}^{(f)b,a}[\cdot](t,x) = \mathcal{G}^{b,a}[\cdot](t,x) + \frac{1}{2}a(x)^2 \frac{\partial f}{\partial x}\frac{\partial \cdot}{\partial x}$$

and transition probability density

$$q^f(t_0, x_0; t, x) = \mathbf{Z}^f(t_0, x_0; t, x)p(t_0, x_0; t, x),$$

where $p(t_0, x_0; t, x)$ *is the transition probability density of the original process* X_t *under* $\mathbb{P}$.

Proof. See [12] or [71, p. 350-352], and also in the context for general Markov processes see [65]. $\square$

3.3.3 Finding an exponential change of measure that gives a specific homogeneous drift

Assume X_t is given as in (X_t-SDE), recalled below

$$dX_t = b(X_t)\mathrm{dt} + a(X_t)\mathrm{dW}_t, \text{ with } X_{t_0} = x_0, \tag{X_t-SDE}$$

and we need to find the exponential change of measure

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(f(t,x)\right)_t,$$

such that it give us the dynamics

$$\mathrm{dX}_t = e(X_t)dt + a(X_t)\mathrm{dW}^{\mathbb{Q}}_t, \text{ with } X_{t_0} = x_0 > 0, \tag{3.16}$$

with $e : \mathbb{R} \to \mathbb{R}$, $e \in \mathcal{C}^1$ a deterministic and given function and $\mathrm{W}^{\mathbb{Q}}$ a $\mathbb{Q}$-Brownian motion.

For simplicity we write $h(t, X_t) = \mathcal{Z}\left(f(t,x)\right)_t$. A necessary condition for h to define a change of measure is to be a $\mathbb{P}$-local martingale, then it must satisfy

$$\frac{\partial h}{\partial t}(t,x) + \frac{1}{2}a^2(x)\frac{\partial^2 h}{\partial x^2}(t,x) + b(x)\frac{\partial h}{\partial x}(t,x) = 0,$$

but under the measure $\mathbb{Q}$ given by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(f(t,x)\right)_t = h(t, X_t)$, the process X_t has a drift of $\gamma(X_t)$, where

$$\gamma(x) = b(x) + a^2(x)\frac{1}{h(t,x)}\frac{\partial h}{\partial x}(t,x).$$

Therefore, in order to get the desired drift $e(x)$ we need the condition

$$b(x) + a^2(x)\frac{\partial h}{\partial x}\frac{1}{h}(t,x) = e(x).$$

Both conditions give us the equation

$$\frac{\partial h}{\partial t}(t,x) + \frac{1}{2}a^2(x)\frac{\partial^2 h}{\partial x^2}(t,x) + b(x)\frac{(e(x)-b(x))}{a^2(x)}h(t,x) = 0. \tag{3.17}$$

As the desired drift $e(x)$ does not depend on time, and by Lemma 64, we can assume without loss of generality that the function h is of separable in variables of the form[2]

$$h(t,x) = e^{\kappa t}V(x).$$

The equation in (3.17) remains

$$\frac{1}{2}a^2(x)V''(x) + \left\{\kappa + b(x)\frac{(e(x)-b(x))}{a^2(x)}\right\}V(x) = 0. \tag{3.18}$$

The equation will have two independent solutions V^1 and V^2 depending on the value κ. Write $V := V(x,\kappa) = c_1V^1(x,\kappa) + c_2V^1(x,\kappa)$ for $c_1, c_2 \in \mathbb{R}$. The problem reduces to find a parameter $\kappa \neq 0$ and two constants $c_1$and c_2 that satisfy the equality[3]

$$e(x) - b(x) - \frac{a^2(x)}{\kappa}\frac{V'(x,\kappa)}{V(x,\kappa)} = 0. \tag{3.19}$$

Writing κ^* as the solution to (3.19), the exponential change of measure that generates the dynamics in (3.16) is given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = e^{\kappa^* t}V(X_t,\kappa^*).$$

3.3.4 WCS under Markovian priors

Let $\mathcal{M}_a(\mathbb{P})$ be as before the set of all absolutely continuous probability measures w.r.t. $\mathbb{P}$ and denote by $\mathcal{M}_e(\mathbb{P})$ the set of equivalent probability measures to $\mathbb{P}$.

In order to highlight the dynamic nature of the risk measure $\mathrm{WCS}_{\mathcal{P}}$ we return to our original setting assuming that the current time is t and we want to measure risk on the horizon $[t, t+\theta]$ for a given $\theta > 0$ of a position $X = X^\theta = X_{t+\theta} - X_t$.

Definition 67 *Let $\mathcal{M}_{hM}(\mathbb{P}) \subset \mathcal{M}_a(\mathbb{P})$ be the set of all measures in $\mathcal{M}_a(\mathbb{P})$ that preserve the homogeneity and the Markov property for a given process X_t with dynamics in (X_t-SDE) under $\mathbb{P}$. This is*

$$\mathcal{M}_{hM}(\mathbb{P}) = \left\{\mathbb{Q} \in \mathcal{M}_a(\mathbb{P}) \,\middle|\, \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(f(t,x)\right)_t \text{ as in (3.9) for } f(t,x) = u(t) + v(x),\ f \in \mathcal{C}^{1,2}\right\}.$$

[2]This is equivalent to taking $f(t,x) = \kappa t + \log(V(x))$ for $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(f(t,x)\right)_t$ as in Lemma 64. This can also be seen as applying Laplace transformation in the variable t.

[3]This will be similar to fixing the two constants c_1 and c_2 (not both zero) and solving for κ (κ will represent an eigenvalue in the Laplace transformation interpretation). Sometimes the choice $c_1 = 1$ and $c_2 = 0$ is sufficient.

A subset $\mathcal{P} \subset \mathcal{M}_{hM}(\mathbb{P})$ satisfying the Assumption 58 will be called a ***set of homogeneous Markov absolutely continuous probability measures*** *to $\mathbb{P}$. Equivalently a set $\mathcal{P} \subset \mathcal{M}_{hM}^{b,a}(\mathbb{P}) \cap \mathcal{M}_e(\mathbb{P})$ satisfying the Assumption 58 will be called a* ***set of homogeneous Markov equivalent probability measures*** *to $\mathbb{P}$.*

Lemma 68 *$\mathcal{M}_{hM}(\mathbb{P})$ is not empty.*

Proof. Note that $\mathbb{P} \in \mathcal{M}_{hM}(\mathbb{P})$ because for $v(x) = 0$ $\frac{d\mathbb{P}}{d\mathbb{P}}|_{\mathcal{F}_t} = 1$. $\square$

Remark 69 *Let $\mathcal{P} \subset \mathcal{M}_{hM}(\mathbb{P})$ be a reference set satisfying Assumptions 58. Then we can define the set of drifts $\Upsilon_{\mathcal{P}}$ by*

$$\Upsilon_{\mathcal{P}} = \left\{ \gamma \,\middle|\, \gamma(x) := b(x) + a(x)v_x(x) \text{ where } \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(u(t) + v(x)\right)_t \text{ as in (3.9) for } \mathbb{Q} \in \mathcal{P} \right\}. \tag{3.20}$$

We have a one-to-one correspondence between measures $\mathbb{Q} \in \mathcal{P}$ and processes $\gamma \in \Upsilon_{\mathcal{P}}$. In order to make this dependence explicit we sometime write $\mathbb{Q}^{\gamma} \in \mathcal{P}$. Thus, the optimisation problem in $\mathrm{WCS}_{\mathcal{P}}$

$$WCS_{\mathcal{P}}(X) = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[-X\right],$$

becomes an equivalent problem of the form

$$WCS_{\Upsilon_{\mathcal{P}}}(X) = \sup_{\gamma \in \Upsilon_{\mathcal{P}}} \mathbb{E}_{\mathbb{Q}^{\gamma}}\left[-X\right].$$

The next proposition establishes some basic aspects of the computation of $\mathrm{WCS}_{\mathcal{P}}$.

Proposition 70 *Let $\mathcal{P} \subset \mathcal{M}_{hM}(\mathbb{P})$ be a given set satisfying Assumptions 58. Suppose X_t is a Markov diffusion process on $(\Omega, \mathcal{F}, \mathbb{P})$ as in (X_t-SDE). Then for a risk horizon $[t, t+\theta]$, $\theta \geq 0$, and the position $X = X_{t+\theta} - X_t$, there exist measures $\mathbb{Q}^+, \mathbb{Q}^- \in \mathcal{P}$ such that*

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[-X\right] = \mathbb{E}_{\mathbb{Q}^+}[-X] \text{ and } \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[-X\right] = \mathbb{E}_{\mathbb{Q}^-}[-X],$$

and therefore $\mathbb{P}$-a.s two uniquely defined functions $\gamma^+, \gamma^- \in \Upsilon_{\mathcal{P}}$ given by

$$\begin{aligned} \gamma^-(x) &= b(x) + a^2(x)v_x^-(x) \in \Upsilon_{\mathcal{P}} \\ \gamma^+(x) &= b(x) + a^2(x)v_x^+(x) \in \Upsilon_{\mathcal{P}} \end{aligned}$$

for

$$\begin{aligned} f^-(t,x) &= u^-(t) + v^-(x) \in \mathcal{C}^{1,2}, \\ f^+(t,x) &= u^+(t) + v^+(x) \in \mathcal{C}^{1,2}, \end{aligned}$$

related to $\mathbb{Q}^+, \mathbb{Q}^- \in \mathcal{P}$ by

$$\frac{d\mathbb{Q}^+}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(f^+\right)_t \text{ and } \frac{d\mathbb{Q}^-}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(f^-\right)_t.$$

Proof. The existence of $\mathbb{Q}^+, \mathbb{Q}^- \in \mathcal{P}$ follows from Assumptions 57 and 58. As $\mathcal{P} \subset \mathcal{M}_{hM}^{b,a}(\mathbb{P})$, for each measure, $\mathbb{Q}^i \in \mathcal{P}$, $i = \{+,-\}$, there exist functions $v^i(x)$ which define the density process for $\mathbb{Q}^i$; this is, $\frac{\mathrm{d}\mathbb{Q}^i}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(f^i(t,x)\right)_t$, $i = \{+,-\}$ for $f^-(t,x) = u^-(t) + v^-(x) \in \mathcal{C}^{1,2}$ and $f^+(t,x) = u^+(t) + v^+(x) \in \mathcal{C}^{1,2}$. To see that they define uniquely drift functions $\gamma^i(x)$, $i = \{+,-\}$, assume there are two homogeneous Markov absolutely continuous measures w.r.t $\mathbb{P}$, say, $\mathbb{Q}^1$ and $\mathbb{Q}^2 \in \mathcal{P}$ such that

$$\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[-X\right] = \mathbb{E}_{\mathbb{Q}^1}[-X] = \mathbb{E}_{\mathbb{Q}^2}[-X];$$

then there exist functions v^j, $j = 1,2$ coming from $f^j(t,x) = u^j(t) + v^j(x) \in \mathcal{C}^{1,2}$, $j = 1,2$ such that

$$\begin{aligned} \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} &= \mathcal{Z}\left(f^j(t,x)\right)_t \\ &= \exp\left[\int_t^{t+\theta} v_x^i(X_s)\mathrm{dX}_s - \int_t^{t+\theta} v_x^i(X_s)b(X_s)\mathrm{ds} - \frac{1}{2}\int_t^{t+\theta} \left[v_x^i(X_s)\right]^2 a^2(X_s)\mathrm{ds}\right], \end{aligned}$$

for $j = \{1,2\}$. Thus for $\tau = t+\theta$

$$\mathbb{E}\left[\left\{\frac{\mathcal{Z}\left(f^1(t,x)\right)_\tau}{\mathcal{Z}\left(f^1(t,x)\right)_t} - \frac{\mathcal{Z}\left(f^2(t,x)\right)_\tau}{\mathcal{Z}\left(f^2(t,x)\right)_t}\right\} X_\tau\right] = 0.$$

Substituting the expression for $\mathcal{Z}\left(f^j(t,x)\right)_t$, $j = \{1,2\}$ we get $\mathbb{P}$-a.s.

$$\int_t^\tau \left\{v_x^1(X_s) - v_x^2(X_s)\right\}\left[\mathrm{dX}_s - b(X_s)\mathrm{ds} - \frac{1}{2}a^2(X_s)\left\{v_x^1(X_s) + v_x^2(X_s)\right\}\mathrm{ds}\right] = 0$$

which implies $v_x^1(x) = v_x^2(x)$ $\mathbb{P}$-a.s., as the right-hand side term inside the integral is not equal to zero $\mathbb{P}$-a.s. And this guarantees the uniqueness in the expressions $\gamma^i(x)$, $i = \{+,-\}$. $\square$

Remark 71 *Proposition 70 guarantee the existence of measures $\mathbb{Q}^+, \mathbb{Q}^- \in \mathcal{P}$, but it does not specify how to find them. The method to find $\mathbb{Q}^+, \mathbb{Q}^-$ for a given set $\mathcal{P}$ will depend strongly on the specific dynamics of X_t and the particular structure of $\mathcal{P}$.*

Assume we can single out $\mathbb{Q}^+$ from the set $\mathcal{P}$. Then from the Markov structure of the exponential change of measure, we are able to formulate the value of the $\mathrm{WCS}_{\mathcal{P}}$ risk measure as the solution of a PDE as shown in the next proposition.

Proposition 72 ***WCS-PDE*** *Let X_t be a Markov diffusion process on $(\Omega, \mathcal{F}, \mathbb{P})$ as in (X_t-SDE). Let $\mathcal{P} \subset \mathcal{M}_{hM}(\mathbb{P})$ be a given set satisfying Assumptions 58. For a risk horizon $[t, t+\theta]$, $\theta \geq 0$, and the position $X = X_{t+\theta} - X_t$, $WCS_{\mathcal{P}}(X)$ is given by*

$$WCS_{\mathcal{P}}(X) = X_t - H(t, X_t) \tag{3.21}$$

where $H(t,x)$ is the solution of the following boundary value problem:

$$\begin{aligned} \mathcal{L}^{\gamma^+,a}[H](\tau, x) &= 0, \quad (\tau, x) \in [t, t+\theta) \times \mathbb{R}, \\ H(t+\theta, x) &= x. \end{aligned} \tag{3.22}$$

The operator $\mathcal{L}^{\gamma^+,a}$ given by

$$\mathcal{L}^{\gamma^+,a}[\cdot](t,x) = \frac{\partial \cdot}{\partial t} + \frac{1}{2}a^2(x)\frac{\partial^2 \cdot}{\partial x^2} + \gamma^+(x)\frac{\partial \cdot}{\partial x},$$

with

$$\gamma^+(x) = b(x) + a^2(x)v_x^+(x) \in \Upsilon_{\mathcal{P}}$$

and $u^+(t) + v^+(x) \in \mathcal{C}^{1,2}$ satisfying

$$\frac{d\mathbb{Q}^+}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(u^+(t) + v^+(x)\right)_t$$

for

$$\sup_{\gamma^+ \in \Upsilon_{\mathcal{P}}} \mathbb{E}_{\mathbb{P}^\gamma}\left[-X^\theta\right] = \mathbb{E}_{\mathbb{Q}^+}\left[-X^\theta\right].$$

Equivalently,

$$WCS_{\mathcal{P}}(X) = X_t - \hat{H}(t, X_t) \tag{3.23}$$

where $\hat{H}(t,x)$ is the solution of the following boundary value problem

$$\begin{aligned} \mathcal{L}^{b,a}[\hat{H}](\tau, x) &= c(x)\hat{H}(\tau, x), (\tau, x) \in [t, t+\theta) \times \mathbb{R}, \\ \hat{H}(t+\theta, x) &= x\exp\{v^+(x)\}. \end{aligned} \tag{3.24}$$

The operator $\mathcal{L}^{b,a}$ given by

$$\mathcal{L}^{b,a}[\cdot](t,x) = \frac{\partial \cdot}{\partial t} + \frac{1}{2}a^2(x)\frac{\partial^2 \cdot}{\partial x^2} + b(x)\frac{\partial \cdot}{\partial x},$$

and $c(x) = \frac{\mathcal{G}^{b,a}[\exp\{v^+(x)\}](x)}{\exp\{v^+(x)\}}$.

Proof. Proposition 70 guarantees the existence of $\gamma^+, \gamma^- \in \Upsilon_{\mathcal{P}}$. That WCS is given by the solution of the PDE follows from the Markov property of the process X_t on $(\Omega, \mathcal{F}, \mathbb{Q})$, $\mathbb{Q} \in \mathcal{P}$. For the second characterisation, we use the fact that for $H(t,x) \in \mathcal{C}^{1,2}$

$$\frac{\mathcal{L}^{b,a}[H\exp\{v\}](t,x)}{\exp\{v\}} = \mathcal{L}^{b+a^2 z_x,a}[H](t,x) + H(t,x)\frac{\mathcal{G}^{b,a}[\exp\{v(x)\}](x)}{\exp\{v(x)\}},$$

with $\mathcal{G}^{b,a}[\cdot](x) = \frac{1}{2}a^2(x)\frac{\partial^2 \cdot}{\partial x^2} + b(x)\frac{\partial \cdot}{\partial x}$. □

Proposition 72 assume a priori the knowledge of the measure $\mathbb{Q}^+ \in \mathcal{P}$ that attains the supremum in the expected values that define $\text{WCS}_{\mathcal{P}}$. If this is not the case, the alternative is to compute the expected values for the measures in $\mathcal{P}$ (by using the PDE methods above or any other method) and then look for the supremum. If the set of measures is finite (see Example 73 below), the problem reduces to find the measure that gives the maximum expected value.

Example 73 *Let X_t be a geometric Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ given by*

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

In terms of the dynamics in (X_t-SDE) the coefficients are $b(x) = \mu x$ and $a(x) = \sigma x$.

For the risk horizon $[t,T], t > 0$ consider the following set of drifts $\Upsilon_{\mathcal{P}} = \{\gamma_i\}$ $i = 1, .., 4$ with

$$\begin{aligned} \gamma_1(x) &= 0 \\ \gamma_2(x) &= \delta x \\ \gamma_3(x) &= (\eta - \xi \log(x))\, x, \;\; \xi > 0 \\ \gamma_4(x) &= \left(\frac{1}{t} + \lambda\right) x. \end{aligned}$$

We have chosen these four drifts because they allow explicit computation, but the choice could be interpreted as follows. Assume the right parametric form of the drift is unknown, and that the standard model to start with is a geometric Brownian motion process. Furthermore, there is evidence (perhaps by numerical samples, etc.) that the drift is nearly linear in the variable x, and that the time variable t plays a minor role. Instead of looking into a general and complicated functional form, the proposal is to compute risk with the four drifts proposed above. The first drift $\gamma_1(x)$ assume there is no drift, which is an extreme case given the evidence. The second drift $\gamma_2(x)$ does not assume a different structure as the base model, but allow to recalibrate the constant *term μ in the geometric Brownian motion. The third drift $\gamma_3(x)$ incorporate the possibility of a functional structure of order bigger than linear but less than quadratic, this is, of order $x\log(x)$, but still time-independent. The fourth drift $\gamma_4(x)$ includes a simple decreasing time-dependency with the term $\frac{1}{t}$, in accordance with the evidence of a very small effect by the time.*

We take as $\mathcal{P}$ the corresponding set of measures that generate the dynamics for the process X_t with drifts $\gamma_i, i = 1, 2, 3, 4$. Without further knowledge of the parameters in the drifts (δ, η, ξ, and λ), it is not clear a priori which of the four measures gives us the maximum expected value. This is, we need to compute the WCS risk measure for holding an asset driven by a geometric Brownian motion with a set of priors consisting of the measures given by the four drifts above. Thus, we compute the expected values given by each of the measures using the PDE formulation in (3.22) for the corresponding drifts $\gamma_i, i = 1, 2, 3, 4$. Note that we do not really need to compute the density processes that define the measures. But they could be calculated using the procedure in Section 3.3.3 for the homogeneous drifts.

For the drifts γ_1 and γ_2 (see Example 81) their expected values are

$$\begin{aligned} H^1(t, X_t) &= \mathbb{E}_{\mathbb{Q}^1}[X_T] = X_t \\ H^2(t, X_t) &= \mathbb{E}_{\mathbb{Q}^2}[X_T] = X_t e^{\delta(T-t)}. \end{aligned}$$

And for γ_3, the expected value is given by

$$H^3(t, X_t) = \mathbb{E}_{\mathbb{Q}^3}[X_T] = X_t^{\exp\{-\xi(T-t)\}} \exp\left\{ \frac{\eta}{\xi}\left(1 - e^{-\xi(T-t)}\right) - \frac{1}{4}\frac{\sigma^2}{\xi}\left(1 - e^{-2\xi(T-t)}\right) \right\}.$$

The process with drift γ_3 can also be seen as a transformation of an Ornstein-Uhlenbeck process Y_t with drift $\eta - 1/2\sigma^2 - \xi x$ and diffusion coefficient σ by a function $u(y) = e^y$ (see [8]).

The method in Section 3.3.3 cannot be applied directly for the drift γ_4 as it is time-dependent. But the PDE in (3.22) has an explicit solution. Its expected value is given by

$$H^4(t, X_t) = \mathbb{E}_{\mathbb{Q}^4}[X_T] = \frac{T}{t} X_t e^{\lambda(T-t)}.$$

This process can be seen as the transformation of a geometric Brownian motion with drift λx and diffusion coefficient σx by the function $u(t, x) = tx$. Thus its transition probability density can be computed in terms of the transition probability density of the underlying geometric Brownian motion (see [8] Example 81).

Table 3.1 shows the conditions on the parameters δ, η, ξ , and λ for the measures $\mathbb{Q}^i$, $i = 1,2,3,4$, to attain the maximum in the definition of $WCS_{\mathcal{P}}(X)$, the corresponding value for $WCS_{\mathcal{P}}(X)$ and the conditions to decide whether the position is acceptable or not[4].

$\mathbb{Q}^i$	$WCS_{\mathcal{P}}(X\|\mathcal{F}_t)$	*CONDITIONS FOR* $\mathbb{Q}^i$ *TO ATTAIN THE MAX*	*POSITION ACCEPTABLE FOR*
$\mathbb{Q}^1$	0	$\delta \geq 0,\ \lambda \geq \Xi$ *and* $X_t \leq 1$	*always*
$\mathbb{Q}^2$	$X_t\left(1 - e^{\delta(T-t)}\right)$	$\delta \leq 0,\ \lambda \geq \delta + \Xi$ *and* $X_t \leq \exp\left\{\Lambda - \frac{\delta(T-t)}{\Psi}\right\}$	$\delta \geq 0$
$\mathbb{Q}^3$	$X_t\left(1 - X_t^{-\Psi}\right)e^{\Lambda\Psi}$	$X_t \geq \exp\left\{\left[\Lambda - \min(\delta, \lambda - \Xi)\frac{(T-t)}{\Psi}\right]^+\right\}$	$X_t \leq 1$
$\mathbb{Q}^4$	$X_t\left(1 - \frac{T}{t}e^{\lambda(T-t)}\right)$	$\lambda \leq [\delta]^- + \Xi$ *and* $X_t \leq \exp\left\{\Lambda + (\Xi - \lambda)\frac{(T-t)}{\Psi}\right\}$	$\lambda \geq \Xi$

Table 3.1: $WCS_{\mathcal{P}}(X)$ for Example 73.

Here

$$\Psi = 1 - e^{-\xi(T-t)}, \quad \Lambda = \frac{\eta - \sigma^2/4\Psi}{\xi} \quad \textit{and} \quad \Xi = -\frac{1}{T-t}\log\left(\frac{T}{t}\right).$$

3.3.5 WCS as a stochastic control problem

In this section, we consider sets $\mathcal{P}$ with measures in $\mathcal{M}_a(\mathbb{P})$ (absolutely continuous w.r.t $\mathbb{P}$) in a wider class than the $\mathcal{M}_{hM}(\mathbb{P})$ (set of homogeneous Markov absolutely continuous probability measures to $\mathbb{P}$), in the sense that we will not restrict them to generate Markov processes. Instead, we make the following assumption.

Recall that a measure $\mathbb{Q} \in \mathcal{M}_a(\mathbb{P})$ is of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = Z(\varphi)_t := \exp\left[\int_{t_0}^t \varphi_u dW_u - \frac{1}{2}\int_{t_0}^t (\varphi_u)^2 du\right], \tag{3.25}$$

for $(\varphi_t)_{t_0 \leq t \leq T}$ an adapted process with $\int_{t_0}^T \varphi_s^2 ds < \infty$. We take sets $\mathcal{P}$ such that the

[4]For a given risk measure ρ, a position X is acceptable if $\rho(X) \leq 0$ and it represents a risky position if $\rho(X) > 0$.

set[5]

$$\Psi_{\mathcal{P}}(\mathbb{P}) := \left\{ \varphi_t \in \mathcal{PM} : \int_{t_0}^{T} \varphi_s^2 \mathrm{ds} < \infty \text{ and } \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = Z(\varphi)_t \text{ as in (3.25) for } \mathbb{Q} \in \mathcal{P} \right\}$$

is compact. This is, we make the following assumption:

Assumption 74 *The set of priors $\mathcal{P}$ comes from a compact set of progressively measurable and square integrable Girsanov kernels.*

This assumptions is necessary for the solution of $\mathrm{WCS}_{\mathcal{P}}(X)$ using stochastic control theory (see [26]).

Assume the current time is t and we are interested in computing $\mathrm{WCS}_{\mathcal{P}}$ on a position X for the risk horizon $[t, t+\theta]$, $\theta \geq 0$, where the process $(X_t)_{0 \leq t \leq T}$ has dynamics as in (X_t-SDE) under the measure $\mathbb{P}$. Under Assumption 74, we can think and work with the process X_t from the point of view of a controlled one-dimensional SDE. In this case, the control variable φ_t only affects the drift term. Our controlled SDE is given by

$$\mathrm{dX}_t^{\varphi} = \left\{ b(X_t^{\varphi}) + a^2(X_t^{\varphi})\varphi_t \right\} \mathrm{d}t + a(X_t^{\varphi})\mathrm{dW}_t.$$

And the problem of computing $\mathrm{WCS}_{\mathcal{P}}(X)$ is equivalent to

$$\begin{aligned} \mathrm{WCS}_{\mathcal{P}}(X) &= \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[-X \right] \\ &= \sup_{\varphi \in \Psi_{\mathcal{P}}(\mathbb{P})} \mathbb{E} \left[-X^{\varphi} \right]. \end{aligned}$$

Then using our assumptions on the drift function b, the volatility function a and the fact that $\Psi_{\mathcal{P}}(\mathbb{P})$ is bounded, we apply usual arguments in stochastic control theory (see [26, Chap. 6 p.155], [63, Chap. XI p.227] or [25, Chap. VI p.151]) to reduce $WCS_{\mathcal{P}}(X)$ to the solution of a nonlinear PDE of second order. This is described in the following proposition.

Proposition 75 *Let X_t be a Markov diffusion process on $(\Omega, \mathcal{F}, \mathbb{P})$ as in (X_t-SDE). Let $\mathcal{P}$ satisfy the Assumption (74) and let $\Psi_{\mathcal{P}}(\mathbb{P})$ be the corresponding compact Girsanov kernel set. For a risk horizon $[t, t+\theta]$, $\theta \geq 0$, and the position $X = X_{t+\theta} - X_t$, $WCS_{\mathcal{P}}(X)$ is given by*

$$WCS_{\mathcal{P}}(X) = X_t - \check{H}(t, X_t)$$

[5] $\mathcal{PM}$ means progressively measurable process.

where $\check{H}(t,x)$ is the solution of the following boundary value problem:

$$\begin{aligned} \frac{\partial \check{H}}{\partial t} + \frac{1}{2}a^2(x)\frac{\partial^2 \check{H}}{\partial x^2} + b(x)\frac{\partial \check{H}}{\partial x} + \sup_{\varphi \in \Psi_{\mathcal{P}}(\mathbb{P})} \left\{ a^2(x)\varphi \frac{\partial \check{H}}{\partial x} \right\} &= 0, \ (\tau, x) \in [t, t+\theta) \times \mathbb{R}, \\ \check{H}(t+\theta, x) &= x. \end{aligned}$$

As the set $\Psi_{\mathcal{P}}(\mathbb{P})$ is compact, the optimum φ^* is attained. Then by a measurable selection argument (see Appendix C) one can find a (measurable) function $\bar{\varphi}(t,x)$ such that $\varphi_t^* = \bar{\varphi}(t,x)$ for a.a. x.

The fact that the optimal control can be chosen as a measurable function only on the current value of the process X_t means that the optimal control is Markov. In this general setting, unfortunately the only thing we can say about the function $\bar{\varphi}$ is that it is measurable, this is, it may not be even be differentiable as required in the previous section.

3.4 Value-at-Risk (VaR)

In this section, we focus on the computation of VaR for processes given in (X_t-SDE).

Note that the definition of VaR in Definition 25 can be equivalently rewritten as

$$\mathrm{VaR}_\alpha(X) = \inf\{m \in \mathbb{R} : \mathbb{E}[1_{-X \le m}] \ge 1 - \alpha\}.$$

This representation of VaR admits the following interpretation (see [48]):

"VaR is similar to finding the lowest strike price $-m$ of a digital call option on the position X under the probability $\mathbb{P}$, such that its price is larger or equal than $1-\alpha$."

Similarly to the WCS risk measure, the value of VaR can be characterised using a PDE as shown in the next proposition.

Proposition 76 ***VaR-PDE.*** *Let X_t be a Markov diffusion process on $(\Omega, \mathcal{F}, \mathbb{P})$ as in (X_t-SDE). For the risk horizon $[t, t+\theta]$, $\theta > 0$, and the position $X = X_{t+\theta} - X_t$, $VaR_\alpha(X)$ is given by*

$$\begin{aligned} VaR_\alpha(X) &= X_t - q^+_{X_{t+\theta}}(\alpha) \\ &= X_t - \sup\{q \in \mathbb{R} : V^q(t, X_t) \le \alpha\}. \end{aligned} \tag{3.26}$$

where $V^q(t,x)$ is the solution of the following boundary value problem:

$$\begin{aligned}\mathcal{L}^{b,a}[V^q](\tau,x) &= 0, (\tau,x) \in [t, t+\theta) \times \mathbb{R}, \\ V^q(t+\theta, x) &= 1_{\{x<q\}}\end{aligned} \tag{3.27}$$

for a fixed real number q and the operator $\mathcal{L}^{b,a}$ given by

$$\mathcal{L}^{b,a}[\cdot](t,x) = \frac{\partial \cdot}{\partial t} + b(x)\frac{\partial \cdot}{\partial x} + \frac{1}{2}a(x)\frac{\partial^2 \cdot}{\partial x^2}.$$

Proof. By the Markov property of the process X_t, $q^+_{X_{t+\theta}}(\alpha) = \sup\{q \in \mathbb{R} : F(t, X_t; t+\theta, q) \leq \alpha\}$ with $F(t_0, x_0; t, x)$ the distribution function for the process X_t. Under our assumptions, $F \in \mathcal{C}^{1,2}$ in the back variables (t_0, x_0), so we can apply Ito's formula and characterise VaR as the solution of a PDE. □

Remark 77 *Let $\bar{V}(t,x,q) := V^q(t,x)$ with V^q be the solution of (3.27), the function $q \longmapsto \bar{V}(t,x,q)$ satisfy the properties of a distribution function, i.e.*

1. *The function $\bar{V}(t,x,q)$ is monotonic increasing and right continuous in q,*

2. $\lim_{q\to\infty} \bar{V}(t,x,q) = 1$ *and* $\lim_{q\to-\infty} \bar{V}(t,x,q) = 0$.

3.5 Average Value-at-Risk (AVaR)

From the equivalence expressions of AVaR_α in Proposition 29 and using the definition of VaR in Definition 25, we have

$$\text{AVaR}_\alpha(X) = X_t - \overline{q^+_{X_{t+\theta}}(\alpha)} \tag{3.28}$$

$$= \frac{1}{\alpha}\mathbb{E}_{\mathbb{P}}\left[\left(q^+_{X_{t+\theta}}(\alpha) - X_{t+\theta}\right)^+\right] + X_t - q^+_{X_{t+\theta}}(\alpha) \tag{3.29}$$

Then, AVaR can be interpreted as (see [48]):

"The price of a (standard European) put option with time to maturity θ with a strike equal to the upper α-quantile of the position $X_{t+\theta}$ divided by the price of the corresponding digital put option plus the actual value of the asset X_t minus the value of the upper α-quantile of the position $X_{t+\theta}$."

This interpretation is useful for calculating AVaR when VaR is already known, otherwise, one can compute it via a minimisation problem and recover VaR simultaneously. We refer to [93] for more details on this optimisation approach in a static setting.

In a similar manner as WCS and VaR; AVaR can be computed as the solution to a PDE as shown in the next proposition.

Proposition 78 . ***AVaR PDE*** *Let X_t be a Markov diffusion process on $(\Omega, \mathcal{F}, \mathbb{P})$ as in (X_t-SDE). For the risk horizon $[t, t+\theta]$, $\theta > 0$, and the position $X = X_{t+\theta} - X_t$, $AVaR_\alpha(X)$ is given by*

$$AVaR_\alpha(X) = \frac{1}{\alpha}\tilde{U}(t, X_t, q^+_{X_{t+\theta}}(\alpha)) + X_t - q^+_{X_{t+\theta}}(\alpha). \tag{3.30}$$

where $\tilde{U}(t, x, q) := U^q(t, x)$ and $U^q(t, x)$ is the solution of the following boundary value problem:

$$\begin{aligned} \mathcal{L}^{b,a}[U^q](\tau, x) &= 0, (\tau, x) \in (t, t+\theta) \times \mathbb{R}, \\ U^q(t+\theta, x) &= [q - x]^+, \end{aligned} \tag{3.31}$$

for a fixed real number q and the operator $\mathcal{L}^{b,a}$ given by

$$\mathcal{L}^{b,a}[\cdot](t, x) = \frac{\partial \cdot}{\partial t} + b(t, x)\frac{\partial \cdot}{\partial x} + \frac{1}{2}a(t, x)\frac{\partial^2 \cdot}{\partial x^2}.$$

Proof. That AVaR is given by the solution of the PDE follows from the Markov property of the process X_t. □

In this section, we have characterised the computation for WCS, VaR and CVaR via the solution of PDEs with final conditions. In most of the cases, we do not expect the PDEs to have explicit solutions, but suitable numerical methods or series expansion solutions can be applied to solve them. Whether the PDEs have explicit solutions or not will depend strongly on the coefficients in the process X_t. If this is the case, it means that one can compute explicitly the transition probability density for the process X_t, as we will explore in the next section.

3.6 Computation of risk measures when the transition density is known

In this section, we show how to compute WCS, VaR and AVaR when the transition probability density $p(t, x; \tau, y)$ for the process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ is known. At the end of the section we present some examples.

3.6.1 WCS

We recall from Section 3.3.4 that for $\mathcal{P} \subset \mathcal{M}^{b,a}_{hM}(\mathbb{P})$ satisfying the Assumption 58, we have a one-to-one correspondence between measures $\mathbb{Q}^\gamma \in \mathcal{P}$ and processes $\gamma \in \Upsilon_{\mathcal{P}}$. Furthermore, we also know from Proposition 70 that for each position X with dynamics in (X_t-SDE) there exists a measure $\mathbb{Q}^+ \in \mathcal{P}$ and therefore a drift $\gamma^+(x) = b(x) + a^2(x)v_x^+(x) \in \Upsilon_{\mathcal{P}}$.

If we assume that the transition probability density $p(t, x; \tau, y)$ for the process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ is known, and also the transition probability densities $p^\gamma(t, x; \tau, y)$ for X_t under each of the measures $\mathbb{Q}^\gamma \in \mathcal{P}$, then WCS is related to p^γ as follows.

From the definition of WCS and the equivalence relation in Remark 69, we have

$$\begin{aligned}
\mathrm{WCS}_{\mathcal{P}}(X) &= \sup_{\gamma \in \Upsilon_{\mathcal{P}}} \mathbb{E}_{\mathbb{Q}^\gamma}[-X] \\
&= X_t + \sup_{\mathbb{P}^\gamma \in \mathcal{P}} \int_{\mathbb{R}} -p^\gamma(t, X_t; t+\theta, y)\; y\; dy \\
&= X_t - \int_{\mathbb{R}} p^{\gamma^+}(t, X_t; t+\theta, y)\; y\; dy,
\end{aligned}$$

with $p^\gamma(t, x; \tau, y)$ the transition probability density for the process X_t under the measure $\mathbb{P}^\gamma \in \mathcal{P}$.

From Lemma 64, the density process for $\mathbb{Q}^\gamma \in \mathcal{P}$ can be represented by

$$\frac{\mathrm{d}\mathbb{Q}^\gamma}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_\tau} = \mathcal{Z}\left(v^\gamma(y)\right)_\tau = \mathcal{Z}\mathcal{H}\left(V^\gamma(y)\right)_\tau = \frac{V^\gamma(X_\tau)}{V^\gamma(X_t)} \exp\left\{-\int_t^\tau \frac{\mathcal{G}_x^{b,a}[V^\gamma](X_s)}{V^\gamma(X_s)} \mathrm{ds}\right\},$$

for $V^\gamma(y) = \exp\{v^\gamma(y)\}$, so using the definition of $\mathbf{Z}^\bullet(t_0, x_0; t, x)$ in (3.15) evaluated at the function v from (t, x) to (τ, y) we have

$$\mathbf{Z}^{v^\gamma(y)}(t, x; \tau, y) = \frac{V^\gamma(y)}{V^\gamma(x)} \mathbb{E}_{\mathbb{P}}\left[\exp\left\{-\int_t^\tau \frac{\mathcal{G}_x^{b,a}[V^\gamma](X_s)}{V^\gamma(X_s)} \mathrm{ds}\right\} | X_\tau = y, X_t = x\right].$$

Applying Proposition 66, we can compute explicitly the transition probability density for X_t under $\mathbb{Q}^\gamma \in \mathcal{P}$ as

$$p^\gamma(t, x; \tau, y) = \mathbf{Z}^{v^\gamma(y)}(t, x; \tau, y)\; p(t, x; \tau, y). \tag{3.32}$$

The previous discussion suggests two alternatives to tackle the problem of computing WCS when the transition probability density $p(t, x; \tau, y)$ for the process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ is known. Each case will depend on the special situation on the dynamics of X_t and the particular set $\mathcal{P} \subset \mathcal{M}_{hM}^{b,a}(\mathbb{P})$.

Case 1: *From the dynamics of X_t* or directly from the set $\mathcal{P}$ we can deduce which is the measure $\mathbb{Q}^+ \in \mathcal{P}$ that maximises the expected value. Then WCS will be the expected value of the position $-X$ under the measure $\mathbb{Q}^+$. We have seen in Proposition 72 one alternative to compute the expected value via a PDE. We give here another one using the transition probability density $p(t, x; \tau, y)$ as follows:

1. Find the function $f^+(t,x) = u^+(t) + v^+(x)$ that defines the density

$$\frac{\mathrm{d}\mathbb{Q}^+}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{Z}\left(f^+(t,x)\right)_t.$$

2. For $V^+(y) = \exp\{v^+(y)\}$, compute the expected value

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{-\int_t^\tau \frac{\mathcal{G}_x^{b,a}[V^+](X_s)}{V^+(X_s)}\mathrm{ds}\right\}|X_\tau = y, X_t = x\right].$$

3. Using (3.32), the transition probability density $p^+(t,x;\tau,y)$ for the process X_t under $\mathbb{Q}^+$ is

$$p^+(t,x;\tau,y) = \mathbf{Z}^{v^+(y)}(t,x;\tau,y)\ p(t,x;\tau,y).$$

4. And the WCS will be given by

$$\mathrm{WCS}_{\mathcal{P}}(X) = X_t - \int_{\mathbb{R}} p^{\gamma^+}(t,X_t;t+\theta,y)y\mathrm{dy}.$$

Case 2: *We do not know the measure $\mathbb{Q}^+ \in \mathcal{P}$, but we can compute*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{-\int_t^\tau d(X_s)\mathrm{ds}\right\}|X_\tau = y, X_t = x\right]$$

for

$$d(x) = \frac{\mathcal{G}_x^{b,a}[V^\gamma](x)}{V^\gamma(x)}$$

and V^γ, $\gamma \in \Upsilon_{\mathcal{P}}$ are the functions that define the density processes

$$\frac{\mathrm{d}\mathbb{Q}^\gamma}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{LZ}\left(V^\gamma(x)\right)_t$$

for each $\mathbb{Q}^\gamma \in \mathcal{P}$. In this case, the procedure to compute WCS using $p(t,x;\tau,y)$ is as follows:

1. For each $\mathbb{Q}^\gamma \in \mathcal{P}$, find V^γ, $\gamma \in \Upsilon_{\mathcal{P}}$ such that

$$\frac{\mathrm{d}\mathbb{Q}^\gamma}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{LZ}\left(V^\gamma(x)\right)_t.$$

2. Compute

$$ED^\gamma(t,x;\tau,y) := \mathbb{E}_{\mathbb{P}}\left[\exp\left\{-\int_t^\tau d^\gamma(X_s)\mathrm{ds}\right\}|X_\tau = y, X_t = x\right]$$

with $d^\gamma(x) = \frac{\mathcal{G}_x^{b,a}[V^\gamma](x)}{V^\gamma(x)}$.

3. The transition probability density $p^\gamma(t,x;\tau,y)$ for the process X_t under $\mathbb{Q}^\gamma \in \mathcal{P}$ will be given by

$$p^\gamma(t,x;\tau,y) = \frac{V^\gamma(y)}{V^\gamma(x)} ED^\gamma(t,x;\tau,y)\ p(t,x;\tau,y).$$

4. Compute

$$EV^\gamma(t,x;\theta) := \int_{\mathbb{R}} -p^\gamma(t,x;t+\theta,y)\ y\ dy$$

5. The WCS will be given by

$$\mathrm{WCS}_{\mathcal{P}}(X) = X_t + \sup_{\gamma \in \Upsilon_{\mathcal{P}}} EV^\gamma(t,X_t;\theta).$$

3.6.2 VaR

In order to see how VaR can be expressed in terms of the transition probability density $p(t,x;\tau,y)$ for the process X_t on $(\Omega,\mathcal{F},\mathbb{P})$, define

$$\hat{V}(t,x,I) := \int_{\mathbb{R}} p(t,x;t+\theta,y) 1_{\{I\}} \mathrm{dy} = \int_I p(t,x;t+\theta,y) \mathrm{dy} \text{ for } I \subset \mathscr{B}(\mathbb{R}),$$

where $\mathscr{B}(\mathbb{R})$ is the Borel sigma-algebra on $\mathbb{R}$. We can think of I as an interval on $\mathbb{R}$. The function $\hat{V}(t,x,I)$ represents the conditional probability that $X_{t+\theta}$ belongs to a given interval I for the information up to the time t. Similarly, we could have defined it as

$$\hat{V}(t,x,I) = \mathbb{P}(X_{t+\theta} \in I | X_t = x).$$

For $m \in \mathbb{R}$ take $V(t,x,m) := \hat{V}(t,x,I)$ with $I = \{z : z < m\}$; then using the definition of VaR_α in Definition 25 we have

$$\begin{aligned}
\mathrm{VaR}_\alpha(X) &= X_t - q^+_{X_{t+\theta}}(\alpha) \\
&= X_t - \sup\{m \in \mathbb{R} : \mathbb{E}[1_{X_{t+\theta}<m}] \le \alpha\} \\
&= X_t - \sup\{m \in \mathbb{R} : \hat{V}(t,X_t,I) \le \alpha \text{ with } I = \{z : z < m\}\} \\
&= X_t - \sup\{m \in \mathbb{R} : V(t,X_t,m) \le \alpha\} \\
&= X_t - \sup\left\{m \in \mathbb{R} : \int_{\{y<m\}} p(t,X_t;t+\theta,y))\mathrm{dy} \le \alpha\right\}.
\end{aligned}$$

Remark 79 *For a general random variable X, if its distribution function $f(x)$ has no atoms and is continuous, the supremum on*

$$\sup\left\{m \in \mathbb{R} : \int_{\{y<m\}} f(y)\ dy \le \alpha\right\}$$

is attained at the value m^* such that

$$\int_{\{y<m^*\}} f(y)\ dy = \alpha.$$

For an account of VaR and AVAR for general distributions see [72].

In our present setting, $p(t,x;t+\theta,y)$ is a continuous function and atom-less, therefore we can express VaR for a position X as

$$VaR_\alpha(X) = X_t - q^+_{X_{t+\theta}}(\alpha) = X_t - m^*$$

with m^* obtained from the relation

$$\int_{\{y<m^*\}} p(t,x;t+\theta,y))\,dy = \alpha.$$

3.6.3 AVaR

We now describe how to express AVaR in terms of the transition probability density $p(t,x;\tau,y)$ for the process X_t on $(\Omega,\mathcal{F},\mathbb{P})$.

Define

$$U(t,x,q) \triangleq \int_{\mathbb{R}} p(t,x;t+\theta,y)\ [q-y]^+\mathrm{dy} = \int_{\{y\leqslant q\}} p(t,x;t+\theta,y)(q-y)\mathrm{dy} \text{ for } q\in\mathbb{R},$$

or equivalently

$$U(t,x,q) = \mathbb{E}_{\mathbb{P}}\left[[q-X_{t+\theta}]^+|X_t=x\right].$$

From the expression of AVaR_α in (3.29) we have

$$\begin{aligned}
\mathrm{AVaR}_\alpha(X) &= \frac{1}{\alpha}\mathbb{E}_{\mathbb{P}}\left[\left(q^+_{X_{t+\theta}}(\alpha) - X_{t+\theta}\right)^+\right] + \mathrm{VaR}_\alpha(X),\\
&= \frac{1}{\alpha}U(t,x,q^+_{X_{t+\theta}}(\alpha)) + \mathrm{VaR}_\alpha(X),\\
&= \frac{1}{\alpha}\int_{\left\{y\leqslant q^+_{X_{t+\theta}}(\alpha)\right\}} p(t,X_t;t+\theta,y)\left(q^+_{X_{t+\theta}}(\alpha)-y\right)\mathrm{dy} + \mathrm{VaR}_\alpha(X),\\
&= \mathrm{VaR}_\alpha(X_t) - \frac{1}{\alpha}\int_{\{y\leqslant 0\}} p\left(t,X_t;t+\theta,y+q^+_{X_{t+\theta}}(\alpha)\right)y\mathrm{dy}.
\end{aligned}$$

The computation of WCS, VaR and CVaR when there is available an explicit expression for the transition probabilities $p(t,x;\tau,y)$ of the process X_t under the appropriate measures is summarised in the following proposition.

Proposition 80 *For a given set* $\mathcal{P} \subset \mathcal{M}^{b,a}_{hM}(\mathbb{P})$ *satisfying Assumption 58, let* $p(t,x;\tau,y)$ *(resp.* p^γ*) be the transition probability density for a process* X_t *as in (*X_t*-SDE) under* $\mathbb{P}$ *(resp.* $\mathbb{P}^\gamma \in \mathcal{P}$*). Then WCS, VaR and CVaR for the position* $X = X_{t+\theta} - X_t$ *can be computed as follows:*

$$\begin{aligned} WCS_{\mathcal{P}}(X) &= X_t - \int_{-\infty}^{\infty} p^{\gamma^+}(t, X_t; t+\theta, y) y \, dy, \\ VaR_\alpha(X) &= X_t - \sup\left\{ m \in \mathbb{R} : \int_{-\infty}^{m} p(t, X_t; t+\theta, y) \, dy \leq \alpha \right\}, \\ CVaR_\alpha(X) &= VaR_\alpha(X) - \frac{1}{\alpha} \int_{-\infty}^{0} p\left(t, X_t; t+\theta, y + q^+_{X_{t+\theta}}(\alpha)\right) y \, dy. \end{aligned}$$

Proof. Follows from previous discussion. □

By the fact that $p(t,x;\tau,y)$ satisfies the Kolmogorov backward equation, it can be checked directly that the expressions given above are the same as the ones obtained using the PDE approach in sections 3.3, 3.4 and 3.5 given by the equations in (3.22), (3.26) and (3.30), respectively. Note that in the expression for WCS, we need to know the transition density function $p^+(t,x;\tau,y)$ for the probability measures $\mathbb{Q}^+ \in \mathcal{P}$ where the supremum in WCS is attained.

3.6.4 Example of risk measures for a single process

Example 81 (Geometric Brownian motion - Black-Scholes model.) *Geometric Brownian motion is one of the models most used in finance, and probably one of the few processes that permits explicit computations. Fix* $\theta > 0$*, and consider* t *to be the current time and* $[t, t+\theta]$ *the risk horizon. Assume we need to compute the risk of a position* $X^\theta = X_{t+\theta} - X_t$ *measured as WCS, VaR and CVaR, for* X_t *given by*

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

In the notation in (X_t*-SDE) we have* $b(x) = \mu x$ *and* $a(x) = \sigma x$*. The family of functions*

$$v^\delta(x) = \left(\frac{\delta - \mu}{\sigma^2}\right) \log(x), \;\; \delta \in \mathbb{R}$$

generates the family of drifts indexed by $\delta \in \mathbb{R}$ *given by* $\gamma(x) = \delta x$ *that preserve the Markov property and the structure of the model of having a linear drift. In this case the corresponding density processes are well defined on* $(0, \infty)$ *(are uniformly* *integrable $\mathbb{P}$-martingales) for each $\delta \in \mathbb{R}$ with zero a natural barrier, and given by*

$$\begin{aligned}\left.\frac{d\mathbb{Q}^{\delta}}{d\mathbb{P}}\right|_{\mathcal{F}_{t+\theta}} &= \mathcal{Z}\left(v^{\delta}(x)\right)_{t+\theta} \quad (3.33)\\ &= \frac{\exp\{v^{\delta}(X_{t+\theta})\}}{\exp\{v^{\delta}(X_t)\}}\exp\left\{-\int_t^{t+\theta}\frac{\mathcal{G}_x^{b,a}[\exp\{v^{\delta}\}](X_s)}{\exp\{v^{\delta}(X_s)\}}ds\right\}\\ &= \exp\left[\int_t^{t+\theta}\left(\frac{\delta-\mu}{\sigma^2}\right)\frac{dX_s}{X_s}-\int_t^{t+\theta}\left(\frac{\delta-\mu}{\sigma^2}\right)\mu ds-\frac{1}{2}\int_t^{t+\theta}\frac{(\delta-\mu)^2}{\sigma^2}ds\right]\\ &= \left(\frac{X_{t+\theta}}{X_t}\right)^{\frac{\delta-\mu}{\sigma^2}}\exp\left\{\frac{1}{2}\left(\frac{\delta-\mu}{\sigma^2}\right)\left(\sigma^2-\delta-\mu\right)\theta\right\}.\end{aligned}$$

Define

$$\mathcal{M}_{GBM}(\mathbb{P}) := \left\{\mathbb{Q}^{\delta}\in\mathcal{M}_{hm}^{\mu x,\sigma x}(\mathbb{P})\,\left|\,\frac{d\mathbb{Q}^{\delta}}{d\mathbb{P}}\right|_{\mathcal{F}_t} \text{ is given by (3.33) for } \delta\in\mathbb{R}\backslash\{-\infty,\infty\}\right\}.$$

Any $\mathbb{Q}^{\delta}\in\mathcal{M}_{GBM}(\mathbb{P})$ is equivalent to $\mathbb{P}$. Thus Assumption 57 is satisfied. On the other hand, in order to guarantee the Assumption 58 of weak compactness in the set of densities it is enough to take bounded intervals for the parameter δ.

Assume the set of measures for computing WCS are

$$\mathcal{P} := \left\{\mathbb{Q}^{\delta}\in\mathcal{M}_{GBM}(\mathbb{P})\,\middle|\,\delta\in[\delta^-,\delta^+] \text{ with } \delta^-\leqslant\mu,r\leqslant\delta^+\right\}. \quad (3.34)$$

The transition probabilities are

$$p(t,x;\tau,y)=\frac{1}{\sigma y\sqrt{2\pi(\tau-t)}}\exp\left\{-\frac{1}{2(\tau-t)\sigma^2}\left[\log\frac{y}{x}-(\mu-\sigma^2/2)(\tau-t)\right]^2\right\}$$

under $\mathbb{P}$ and

$$p^{\delta}(t,x;\tau,y)=\frac{1}{\sigma y\sqrt{2\pi(\tau-t)}}\exp\left\{-\frac{1}{2(\tau-t)\sigma^2}\left[\log\frac{y}{x}-(\delta-\sigma^2/2)(\tau-t)\right]^2\right\}$$

under $\mathbb{P}^{\delta}$, $\delta\in[\delta^-,\delta^+]$.

From the dynamics of X_t and the set of measures $\mathcal{P}$ it is clear that the supremum *in the expression of WCS is attained at the measure* $\mathbb{Q}^{\delta^+}$. *Then*

$$\begin{aligned}
WCS_{\mathcal{P}}(X) &= X_t\left(1-e^{\delta^+\theta}\right)\\
VaR_\alpha(X) &= X_t\left(1-e^{(\mu-\sigma^2/2)\theta+\Phi^{-1}(\alpha)\sigma\sqrt{\theta}}\right)\\
AVaR_\alpha(X) &= VaR_\alpha(X)+\frac{1}{\alpha}\boldsymbol{put}_{\mathbb{P}}(X_t,q^+_{X_{t+\theta}}(\alpha),\theta)\\
&= VaR_\alpha(X)+\frac{1}{\alpha}q^+_{X_{t+\theta}}(\alpha)e^{-\mu\theta}\Phi\left(-d^\mu_{2,q^+_{X_{t+\theta}}(\alpha)}\right)\\
&\quad -\frac{X_t}{\alpha}\Phi\left(-d^\mu_{1,q^+_{X_{t+\theta}}(\alpha)}\right)\\
&= X_t+X_te^{(\mu-\sigma^2/2)\theta+\Phi^{-1}(\alpha)\sigma\sqrt{\theta}}\left(e^{-\mu\theta}-1\right)-\frac{X_t}{\alpha}\Phi\left(\Phi^{-1}(\alpha)-\sigma\sqrt{\theta}\right).
\end{aligned}$$

where $\boldsymbol{put}_{\mathbb{P}}(S,K,\tau)$ *refers to the function that evaluates a Black-Scholes-type European put option with time to maturity* τ*, current price* S *and strike* K *but taking as pricing measure* $\mathbb{P}$*. The function* Φ *is the standard cumulative normal distribution and*

$$\begin{aligned}
d^\mu_{1,q} &= \frac{\log(x/q)+(\mu+\sigma^2/2)\theta}{\sigma\sqrt{\theta}}\\
d^\mu_{2,q} &= \frac{\log(x/q)+(\mu-\sigma^2/2)\theta}{\sigma\sqrt{\theta}}.
\end{aligned}$$

From the axiomatic of the risk measures, a position is acceptable (riskless) if its risk is negative or zero. Thus positive risk values mean potential losses and negative or zero risk values mean the position is acceptable. The critical levels of the parameters δ^+ *and* α *that define acceptability of the position (risk equal to zero) are*

	acceptable for	*risky position for*
$WCS_{\mathcal{P}}(X)$	$\delta^+\geq 0$	$\delta^+<0$
$VaR_\alpha(X)$	$\alpha\geq\Phi\left(\frac{-(\mu-\sigma^2/2)\sqrt{\theta}}{\sigma}\right)$	$\alpha<\Phi\left(\frac{-(\mu-\sigma^2/2)\sqrt{\theta}}{\sigma}\right)$
$AVaR_\alpha(X)$	$\alpha\geq\alpha_{cr}$	$\alpha<\alpha_{cr}$

where α_{cr} *is the solution to the equation*

$$1+e^{(\mu-\sigma^2/2)\theta+\Phi^{-1}(\alpha)\sigma\sqrt{\theta}}\left(e^{-\mu\theta}-1\right)-\frac{1}{\alpha}\Phi\left(\Phi^{-1}(\alpha)-\sigma\sqrt{\theta}\right)=0.$$

Consider the numerical values $\mu=0.1$*,* $\sigma=0.7,\theta=1,\delta^-=-0.2,\delta^+=0.7$ *and* $\alpha=0.05$*. Figure 3.1 shows the probability densities for the random variable* $X_{t+\theta}$

given the actual value $X_t = 1$ *corresponding to* $\delta \in \{\mu, \delta^+, \delta^-\}, \delta^- < \mu < \delta^+$. *The vertical line on the left-hand side corresponds to* $\overline{q^+_{X_{t+\theta}}(\alpha)} \approx 0.05714141622$. *The vertical line in the middle corresponds to* $q^+_{X_{t+\theta}}(\alpha) \approx 0.2735153487$ *and the line to the most right position corresponds to* $\mathbb{E}_{\mathbb{Q}^{\delta^+}}[X_{t+\theta}] \approx 2.013752707$. *For the same values of the parameters, Figure 3.2 exhibits the loss distributions for the position* X *and the respective risk values:* $WCS_{\mathcal{P}}(X) \approx -1.013752707, VaR_\alpha(X) \approx 0.7264846513$, *and* $AVaR_\alpha(X) \approx 0.7836260675$. *In order to see how the risk change for different values of* δ^+ *and* α, *Figure 3.3 plots the risk given by* $VaR_\alpha(X)$ *and* $AVaR_\alpha(X)$ *for* $\alpha \in [0, 1]$ *and* $WCS_{\mathcal{P}}(X)$ *corresponding to* $\delta^+ \in [-0.5, 1]$. *Because this model only allows positive values for* X_t, *the loss of the position is bounded by one, which is the current value* X_t. *The critical values related to each risk measure that make the position acceptable are:* $\delta^+_{cr} = 0$ *for* $WCS_{\mathcal{P}}(X)$, $\alpha_{cr} \approx 0.582051$ *for* $VaR_\alpha(X)$ *and* $\alpha_{cr} \approx 0.8993166$ *for* $AVaR_\alpha(X)$.

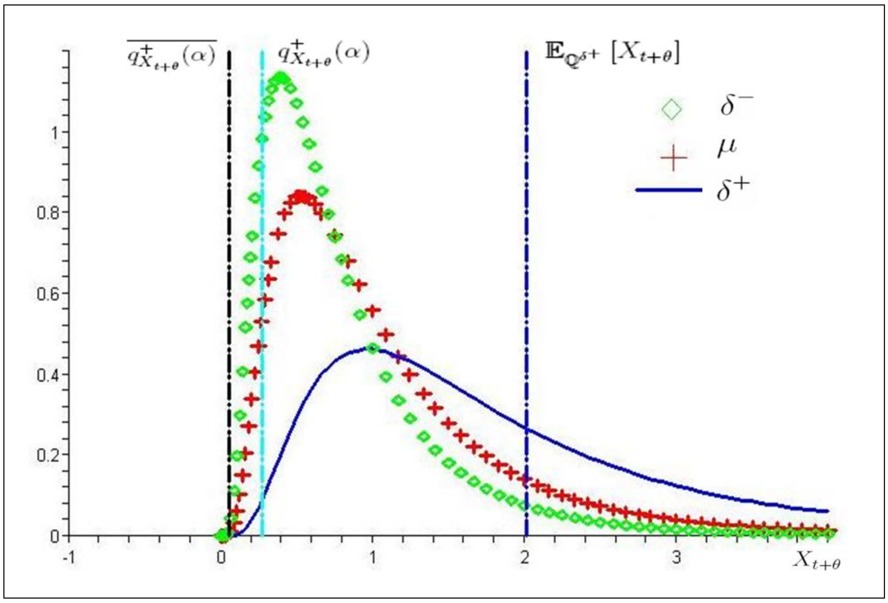


Figure 3.1: Example 81. Probability densities for $X_{t+\theta}$ given $X_t = 1$ for the parameters $\delta^- \leq \mu \leq \delta^+$, and some statistics.

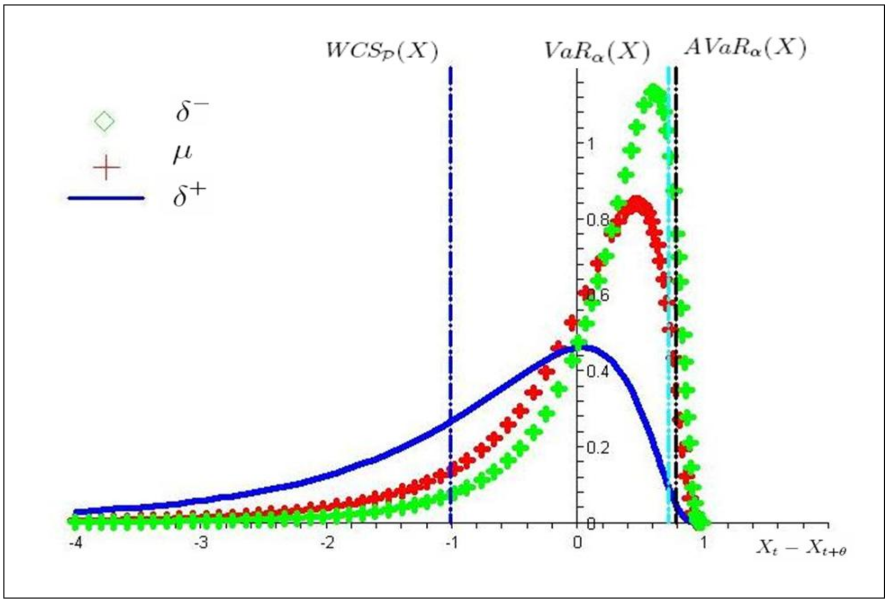


Figure 3.2: Example 81. Loss distribution for the position X and the risk measure values (WCS, VaR and AVaR).

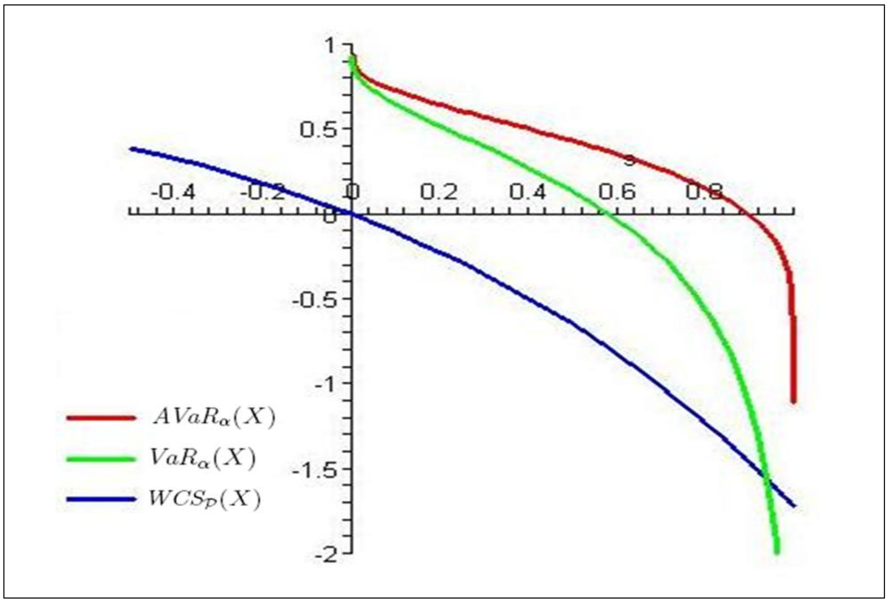


Figure 3.3: Example 81. Change in risk measure values (WCS, VaR and AVaR) by taking the parameters $\alpha \in [0, 1]$ and $\delta^+ \in [-0.5, 1]$.

Chapter 4

Risk for Derivatives

4.1 Introduction

In this chapter, we study the problem of computing risk as WCS, VaR and AVaR for derivative securities that depend on an underlying asset given by a Markov diffusion processes as in the preceding chapter. This is, we assume that the derivative security is defined by a positive function $C(t, s)$ on the security S_t. Defining a process given by

$$X_t = C(t, S_t),$$

one can apply Itô's lemma to obtain the dynamics of X. Then our problem reduces to the one studied in the previous chapter of computing the risk for the position X. When the function C is not injective, the dynamics of the process X_t may degenerate. In order to analyse this situation in detail, we look at the process X_t from the point of view of a local transformation of the Markov diffusion process S_t. In particular, we establish conditions and analyse when the transition probability density of a transformed process X_t can be expressed in terms of the transition probability density of our original process S_t. In other words, we find how to reduce the solution to the risk-PDEs for the position X to the solution of a simpler PDEs problem corresponding to the solution to the risk-PDEs for the position S.

We apply our results on local transformation of Markov diffusion processes to the computation of risk for derivative securities and complement with examples. We discuss briefly also the relation to this method with the approaches known as delta- and delta-gamma approximation for the computation of risk of derivatives and the case of American type-derivative securities.

4.2 Risk measures for European derivatives

Suppose the position X on $(\Omega, \mathcal{F}, \mathbb{P})$ of which we want to measure the risk is the discounted net worth of a European-type contingent claim (derivative security) with expiration time $\tilde{T}$ and payoff function $H(s) \geq 0$ on an underlying process S_t satisfying the assumptions of Chapter 3 and given by

$$\mathrm{dS}_t = \beta(S_t)\mathrm{dt} + \alpha(S_t)\mathrm{dW}_t, \text{ with } S_t = s. \tag{S-SDE}$$

Assume the derivative security may not be liquidated during the risk horizon $[t, t+\theta]$, for $0 < t \leq t+\theta < \tilde{T}$. By the Markov property and our assumptions on the process S_t the random variable $X_{t+\theta}$ may be written as a function $C \in C^{1,2}$ of the process S_t at time $t+\theta$, this is, $X_{t+\theta} = C(t+\theta, S_{t+\theta})$. Then, we can use Itô's lemma to obtain the dynamics of the process X_t and use the method in Chapter 3 to obtain the risk of the position via the solution of the corresponding PDE. For the particular relation between the process X_t and S_t we expect the corresponding risk PDEs to be related.

In this section, we focus on exploring the solution to the risk PDE for the derivative security X_t in terms of the solution to the risk PDE for the underlying process S_t. In order to set ideas in general rather than for a particular risk measure, we work from the point of view of the transition probability densities (t.p.d.); that is, we explore conditions for obtaining the t.p.d. of the derivative security X_t in terms of the t.p.d. of the underlying process S_t. Once we have obtained the t.p.d. of X_t we can apply the method in Section 3.6 for the computation of the risk of the position X measured as $\mathrm{WCS}_{\mathcal{P}}(X)$, $\mathrm{VaR}_\alpha(X)$ and $\mathrm{CVaR}_\alpha(X)$.

4.2.1 The derivative-dynamics

In general, for a function $C \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ define the process $X_t = C(S_t, t)$. By Itô's lemma the dynamics of the process X_t (the C-dynamics) under the measure $\mathbb{P}$ are given by

$$\mathrm{dX}_t = b(t, S_t)\mathrm{dt} + a(t, S_t)\mathrm{dW}_t, \text{ with } X_t = x = C(0, S_t), \tag{4.1}$$

for

$$b(t,s) := \frac{\partial C}{\partial t}(t,s) + \beta(t,s)\frac{\partial C}{\partial s}(t,s) + \frac{1}{2}\alpha^2(t,s)\frac{\partial^2 C}{\partial s^2}(t,s),$$

and

$$a(t,s) := \alpha(t,s)\frac{\partial C}{\partial s}(t,s).$$

In particular, if C represents the current value of a European derivative security with payoff function $H(s)$ and risk-free rate r (we assume momentarily the risk-free rate to be nonzero for completeness), then

$$C(t,S_t) = \mathbb{E}_{\mathbb{Q}}[e^{-r(\theta)}H(S_{t+\theta})],$$

where $\mathbb{Q}$ is the risk-neutral probability measure. We can use the Black-Scholes equation for the derivative security to further simplify the expression for $b(t,s)$ as

$$\begin{aligned} b(t,s) &= \frac{\partial C}{\partial t}(t,s) + \beta(t,s)\frac{\partial C}{\partial s}(t,s) + \frac{1}{2}\alpha^2(t,s)\frac{\partial^2 C}{\partial s^2}(t,s) \\ &= rC(t,s) + (\beta(t,s) - rs)\frac{\partial C}{\partial s}(t,s). \end{aligned}$$

In particular, assuming $r = 0$ we have

$$b(t,s) = \beta(t,s)\frac{\partial C}{\partial s}(t,s).$$

Remark 82 *If there exist points (t,x) in the domain of the function C at which $\frac{\partial C}{\partial x}(t,s) = 0$, then they have to be analysed carefully as they may induce degeneracy in the process $X_t = C(t,S_t)$. On the other hand, when the process X_t is regarded as a local change of variable transformation of the process S_t, then the analysis of the degeneracy points is similar to analysing what happen to a transformed process when the transformation is not injective. For that purpose, we borrow some results on the transformation of continuous Markov diffusions developed in [8]. In order to make this thesis self-contained, we recall all the results and proofs needed in the next section and we adapt them to the present context.*

4.2.2 Homogeneous local transformations

For completeness and in order to introduce ideas, before we analyse the case of $C(t,S_t)$ as a local transformation of the S_t process, we study what happen with local transformations that do not depend on time (homogeneous). For that purpose, we recall some key results that guarantee the Markov property for the transformed processes.

Lemma 83 ***Images of a Markov process.*** *[from [71, Ex.1.17 p.87]] Let X be a Markov process with transition function (P_t) and ϕ a Borel function from $(E,\mathcal{E})$ into a space $(\bar{E},\overline{\mathcal{E}})$ such that $\phi(A) \in \overline{\mathcal{E}}$ for every $A \in \mathcal{E}$. If moreover, for every t and every $\bar{A} \in \overline{\mathcal{E}}$*

$$P_{t,x}(\phi^{-1}(\bar{A})) = P_{t,\hat{x}}(\phi^{-1}(\bar{A})) \quad \text{whenever} \quad \phi(x) = \phi(\hat{x}),$$

then the process $\bar{Y}_t = \phi(X_t)$ under $P_{t,x}$, $x \in E$, is a Markov process with state space $(\bar{E},\overline{\mathcal{E}})$.

In order for a transformed Markov process to be again Markov, it is necessary that the conditional probability on the points where the transformation is not injective is well defined. This is, if two different points have the same image, then the conditional probability starting at any of those points of the inverse image of any event in the space must be equal. As we are interested in compute the t.p.d. of a transformed process in terms of the t.p.d. of the original process, the next corollary is an adapted version of the lemma above (see also [33, Theorem 271J, p. 343]).

Corollary 84 *Let S_t be a Markov diffusion process as in (S-SDE) with conditional probability $P_{t,s}(\cdot) := \mathbb{P}(\cdot|S_t = s)$ and denote by $p^S(t, s; t+\theta, z)$ its transition probability density. For a differentiable function $C : \mathbb{R} \to \mathbb{R}$ the process $X_t = C(S_t)$ is also a Markov process if either of the following conditions hold:*

1. *C is a diffeomorphism on $\mathbb{R}$. In this case, the transition probability density of the transformed process X_t under $\mathbb{P}$ is given by*

$$p^X(t, x; t+\theta, y) = \left| \left(\frac{dC}{ds}(\chi(y)) \right)^{-1} \right| p^X(t, \chi(x); t+\theta, \chi(y))$$

with $\chi(x)$ the corresponding inverse function of $C(s)$ such that $C(\chi(x)) = x$.

2. *For $D \subseteq dom[C]$ (the domain of the function C) with $|D^c| = 0$ (or equivalently $\mathcal{L}(s \in D) = 1$, and $\mathcal{L}$ the Lebesgue measure), the derivative of C does not vanish for each $s \in D$ and there exist a disjoint sequence of Borel sets $(D_k)_{k\in\mathbb{N}}$ with union D, such that $C \restriction D_k$ (C restricted to the set D_k) is injective for every $k \in \mathbb{N}$. And furthermore, for every t and $F \subset \mathbb{R}$ we have that*

$$P_{t,s}(C^{-1}(F)) = P_{t,\hat{s}}(C^{-1}(F)) \quad \text{whenever} \quad C(s) = C(\hat{s}). \tag{4.2}$$

In this case, the transition probability density of the transformed process X_t under $\mathbb{P}$ is given by

$$p^X(t, x; t+\theta, y) = \begin{cases} \sum_{k=0}^{\infty} \left| \left(\frac{dC}{ds}(\chi(y)) \right)^{-1} \right| p^S(t, \chi(x); t+\theta, \chi(y)) & \text{for} \quad y \in C(D_k \cap dom[p^S] \\ 0 & \text{for} \quad y \in \mathbb{R} \backslash C(D_k) \end{cases}$$

with $\chi(x)$ the corresponding inverse function of $C(s)$ such that $C(\chi(x)) = x$.

Proof. Note that Part 1 is a special case of Part 2. To prove Part 2 observe that we may allow the derivative of C to vanish on the sets $D_k \backslash \mathring{D}_k$ (boundary of D_k) for $k \in \mathbb{N}$. In such case, take $D' = \{s \in D | \frac{\partial C}{\partial s}(s) = 0\}$ and $E = D \backslash D'$. Clearly D' is a null set, therefore we still have $\mathcal{L}(s \in E) = 1$, $E = \bigcup_{k=0}^{\infty} D_k$ and $C \upharpoonright D_k$ is injective for each $k \in \mathbb{N}$.

If C is differentiable relative to its domain at every $s \in E$, then C is continuous, Borel measurable, so $C(S_t)$ is a well defined random variable.

Fix $k \in \mathbb{N}$ and $F \subseteq \mathbb{R}$, define the functions

$$g_k(y) = \begin{cases} \sum_{k=0}^{\infty} \left| \left(\frac{dC}{ds}(\chi(y))\right)^{-1} \right| p^S(t, \chi(x); t+\theta, \chi(y)) & \text{for} \quad y \in C(D_k \cap dom[p^S]) \\ 0 & \text{for} \quad y \in \mathbb{R} \backslash C(D_k) \end{cases}$$

then $p^S(t, s; t+\theta, z) = |\frac{dC}{ds}(z)| g_k(C(z))$ for every $z \in D_k \bigcap dom[p^S]$, and we have

$$\begin{aligned} \int_F g_k d\mathbb{P} &= \int_{C(D_k)} g_k \mathbb{I}_F d\mathbb{P} = \int_{D_k} \left| \frac{dC}{ds}(s) \right| g_k(C(s)) \mathbb{I}_F \mathbb{P}(ds) \\ &= \int_{D_k \cap C^{-1}(F)} f d\mathbb{P} = \mathbb{P}(s \in D_k \cap C^{-1}(F)). \end{aligned}$$

Now summing over k and by the fact that each g_k is non-negative for each k, $g = \sum_{k=0}^{\infty} g_k$ is finite almost everywhere on F, and

$$\int_F g d\mathbb{P} = \sum_{k=0}^{\infty} \int_F g_k d\mathbb{P} = \sum_{k=0}^{\infty} \mathbb{P}(s \in D_k \cap C^{-1}(F)) = \mathbb{P}(s \in C^{-1}(F)).$$

Now, applying the same reasoning but with conditional probabilities to the event $S_t = s$, and by the condition

$$P_{t,s}(C^{-1}(F)) = P_{t,\hat{s}}(C^{-1}(F)) \quad \text{whenever} \quad C(s) = C(\hat{s})$$

we have that

$$\mathbb{P}_{t,s}(s \in C^{-1}(F)) = \mathbb{P}_{t,s}(C(s) \in F).$$

As F was arbitrary, then g is the density function for $C(s)$ as needed. $\square$

The condition (4.2) in the Corollary 84 above may be a very restrictive one. To give an idea of this restrictiveness, consider the following proposition saying that the only valid homogeneous local trasnformations of Brownian motion (that satisfy the condition in (4.2)) are functions that are either invertible in all the real line, or are functions that are symmetric with respect to a vertical line crossing at the points where the the function is not injective (e.g. $\sin(x), \cos(x)$, etc.).

Proposition 85 *Let $S_t, 0 \leq t < \infty$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ starting at $s > 0$ for $t \in \mathbb{R}$. Then C is a change of variable transformations $(X_t = C(S_t))$ satisfying conditions in Corollary 84 if and only if*

1. *C is invertible in all its domain, or*

2. *C is a symmetric function.*

Before we give the proof, we need a somehow trivial but useful result.

Lemma 86 *For $a_1, b_1, a_2, b_2 \in \mathbb{R}$, the equation*

$$\Phi[b_1 + h] - \Phi[a_1 + h] + \Phi[b_2 + h] - \Phi[a_2 + h] = \Phi[b_1] - \Phi[a_1] + \Phi[b_2] - \Phi[a_2] \quad (4.3)$$

has nontrivial solution $h \in \mathbb{R}$ only in any of the following cases:

1. *if $a = a_1 = a_2$ and $b = b_1 = b_2$, then $h = -(a + b)$,*

2. *if $a_1 + b_1 = a_2 + b_2$ then $h = -(a_1 + b_1)$,*

3. *if $a_1 + b_2 = a_2 + b_1$ then $h = -(a_1 + b_2)$.*

And Φ is the standard cumulative normal distribution.

Remark 87 *That $h = -(a + b)$ satisfies the relation in (4.3) is very intuitive, it is perhaps surprising that it is the unique solution, therefore we decided to include a proof.*

Proof of Lemma 86. For the Part 1, we need to prove that $\hat{h} = -(a + b)$ is the only solution to $\Phi[b + \hat{h}] - \Phi[a + \hat{h}] = \Phi[-a] - \Phi[-b]$ for $a, b \in \mathbb{R}$. Define $f(h) = \Phi[b + h] - \Phi[a + h] = \int_{a+h}^{b+h} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\}dy$. It is direct to see that $f(h) > 0$, continuous for all $h \in \mathbb{R}$ and $\lim_{h\to-\infty} f(h) = 0 = \lim_{h\to\infty} f(h)$ with derivative $f'(h) = \frac{1}{\sqrt{2\pi}} \left[\exp\{-\frac{(b+h)^2}{2}\} - \exp\{-\frac{(a+h)^2}{2}\}\right]$. And the only point satisfying $f'(h) = 0$ is $h^* = -\frac{(b+a)}{2}$ with $f''(h^*) < 0$, then h^* is a maximum point for $f(h)$. All this together leads to the conclusion that f is increasing in $(-\infty, h^*)$ and decreasing in (h^*, ∞). Therefore for any $\alpha \in \mathbb{R}$, $\alpha \neq f(h^*)$, there exist real numbers $h^1 \neq h^2$ such that $\alpha = f(h^1) = f(h^2)$. In particular, if $h^* \neq 0$, there exist a unique real number $\hat{h}$ with the property $f(0) = f(\hat{h})$.

Now, in order to compute $\hat{h}$ note that $f'(h^* - \Delta) = -f'(h^* + \Delta)$ for any $\Delta > 0$, so the function f is symmetric. And for $\Delta = h^*$ we have $f'(0) = -f'(2h^*)$, so our natural candidate is $\hat{h} = 2h^* = -(a + b)$, which is confirmed to be the right choice by the fact that $f"(0) = f"(\hat{h})$ and $f(0) = f(\hat{h})$.

Parts 2 and 3 result from applying Part 1 to all different combinations of terms in (4.3). □

Proof of Proposition 85. The first characterisation is obvious from Part 1 of Corollary 84. For the second one, assume without loss of generality that C has a unique minimum point at s^* and that is decreasing in $(-\infty, s^*)$ and increasing in (s^*, ∞). Take $F = \{s \in \mathbb{R} : \alpha \leqslant s \leqslant \beta \text{ for } \alpha, \beta \in \mathbb{R}\}$, so $C^{-1}(F) = \{s \in \mathbb{R} : s'_\beta \leqslant s \leqslant s'_\alpha \text{ and } s_\alpha \leqslant s \leqslant s_\beta\}$. Then condition (4.2) is equivalent to check whether $P1$ is equal to $P2$ for:

$$\begin{aligned} P1 &= \mathbb{P}\left[s'_\beta \leqslant S_t \leqslant s'_\alpha | S_{t_0} = s_0\right] + \mathbb{P}\left[s_\alpha \leqslant S_t \leqslant s_\beta | S_{t_0} = s_0\right] \\ P2 &= \mathbb{P}\left[s'_\beta \leqslant S_t \leqslant s'_\alpha | S_{t_0} = s'_0\right] + \mathbb{P}\left[s_\alpha \leqslant S_t \leqslant s_\beta | S_{t_0} = s'_0\right] \end{aligned}$$

with s_0 and s'_0 such that $C(s_0) = C(s'_0)$. But by properties of Brownian motion we have

$$\begin{aligned} P1 &= \mathbb{P}\left[\frac{s'_\beta - s_0}{\sqrt{t-t_0}} \leqslant \xi \leqslant \frac{s'_\alpha - s_0}{\sqrt{t-t_0}}\right] + \mathbb{P}\left[\frac{s_\alpha - s_0}{\sqrt{t-t_0}} \leqslant \xi \leqslant \frac{s_\beta - s_0}{\sqrt{t-t_0}}\right] \\ &= \Phi\left[\frac{s'_\alpha - s_0}{\sqrt{t-t_0}}\right] - \Phi\left[\frac{s'_\beta - s_0}{\sqrt{t-t_0}}\right] + \Phi\left[\frac{s_\beta - s_0}{\sqrt{t-t_0}}\right] - \Phi\left[\frac{s_\alpha - s_0}{\sqrt{t-t_0}}\right] \\ P2 &= \mathbb{P}\left[\frac{s'_\beta - s_0}{\sqrt{t-t_0}} + h \leqslant \xi \leqslant \frac{s'_\alpha - s_0}{\sqrt{t-t_0}} + h\right] + \mathbb{P}\left[\frac{s_\alpha - s_0}{\sqrt{t-t_0}} + h \leqslant \xi \leqslant \frac{s_\beta - s_0}{\sqrt{t-t_0}} + h\right] \\ &= \Phi\left[\frac{s'_\alpha - s_0}{\sqrt{t-t_0}} + h\right] - \Phi\left[\frac{s'_\beta - s_0}{\sqrt{t-t_0}} + h\right] + \Phi\left[\frac{s_\beta - s_0}{\sqrt{t-t_0}} + h\right] - \Phi\left[\frac{s_\alpha - s_0}{\sqrt{t-t_0}} + h\right] \end{aligned}$$

with $\xi = \frac{S_t - s_0}{\sqrt{t-t_0}} \sim \mathcal{N}(0,1)$ ($\mathcal{N}(0,1)$ is the standard normal distribution and Φ the standard cumulative normal distribution) and $h = \frac{s_0 - s'_0}{\sqrt{t-t_0}}$. Then by Lemma 86 we have two possibilities in comparing $P1$ and $P2$

1. $$h = -\frac{s'_\alpha + s'_\beta}{\sqrt{t-t_0}} + \frac{2s_0}{\sqrt{t-t_0}} = -\frac{s_\beta + s_\alpha}{\sqrt{t-t_0}} + \frac{2s_0}{\sqrt{t-t_0}}$$
   and from here, the condition turns into $s'_0 = (s'_\alpha + s'_\beta) - s_0 = (s_\beta + s_\alpha) - s_0$. But this can not be true because, taking $s_0 = \frac{(s_\beta + s_\alpha)}{2}$ we would have $s'_0 = s_0$ meaning that $\frac{(s_\beta + s_\alpha)}{2} < s^*$ is a minimum point, contradicting the initial assumption that s^* is the unique minimum.

2. $$h = -\frac{s'_\alpha + s_\alpha}{\sqrt{t-t_0}} + \frac{2s_0}{\sqrt{t-t_0}} = -\frac{s_\beta - s'_\beta}{\sqrt{t-t_0}} + \frac{2s_0}{\sqrt{t-t_0}}$$
   and the condition turns into $s'_0 = (s'_\alpha + s_\alpha) - s_0 = (s_\beta + s'_\beta) - s_0$. Considering the last two equalities we have $s'_\alpha - s'_\beta = s_\beta - s_\alpha$ meaning that both intervals have to be of the same length. Now for $s_0 = \frac{s'_\alpha + s_\alpha}{2} + \Delta$ we need $s'_0 = \frac{s'_\alpha + s_\alpha}{2} - \Delta$ and this indicates that the function has to be symmetric around $\frac{s'_\alpha + s_\alpha}{2}$, this is, $s^* = \frac{s'_\alpha + s_\alpha}{2} = \frac{s_\beta + s'_\beta}{2}$.

□

4.2.3 The process $C(t, S_t)$ as a local transformation of the S_t process

In this section, we study the dynamics of the process X_t as a local change of variable transformation of the process S_t, defined as follows.

Definition 88 *Let S_t be a Markov diffusion process as in S-SDE. A local change of variable transformation is a map $C : [t, t+\theta] \times \mathbb{R} \longrightarrow \mathbb{R}$.*

In the case the transformation $C(t, s)$ is injective on s for all $t \in [t, t+\theta]$, the t.p.d. of the process X_t can be easily recovered from the t.p.d. of the process S_t as shown in the next proposition.

4.2.3.1 Injective local transformations

Proposition 89 *Let S_t be a Markov diffusion process as in (S-SDE) with conditional probability $P_{t,s}(\cdot) := \mathbb{P}(\cdot|S_t = s)$ and denote by $p^S(t, s; t+\theta, z)$ its transition probability density with $t < t+\theta$. For an injective differentiable function $C : [t, t+\theta] \times \mathbb{R} \to \mathbb{R}$ the process $X_t = C(t, S_t)$ is also a Markov process with transition probability density under $\mathbb{P}$ given by*

$$p^X(t, x; t+\theta, y) = \left| \left(\frac{\partial C}{\partial s}(t+\theta, \chi(t+\theta, y)) \right)^{-1} \right| p^S(t, \chi(t, x); t+\theta, \chi(t+\theta, y))$$

with $\chi(t, x)$ the corresponding inverse function of $C(t, s)$ such that $C(t, \chi(t, x)) = x$.

Proof. The proof is very similar to the first part in the homogeneous case. □

A direct application to the above proposition concerns the computation of the upper α-quantile of derivative securities whose values are strictly increasing (resp. strictly decreasing) functions on the current level of the underlying process S_t. Then using these results, one can easily compute VaR for positions in derivative securities with the help of the results in Section 3.4.

Corollary 90 *Let C be a local change of variable transformation with $C(t, \cdot)$ continuous and strictly increasing function. Then*

$$q^+_{C(t+\theta, S_{t+\theta})}(\alpha) = C(t, q^+_{S_{t+\theta}}(\alpha)).$$

If $C(t, \cdot)$ is a continuous strictly decreasing function, then

$$q^+_{C(t+\theta, S_{t+\theta})}(\alpha) = C(t, q^+_{S_{t+\theta}}(1-\alpha)).$$

Proof. This is a direct consequence of using Proposition 89 to find the t.p.d. for the transformed processes X_t and the application of the method in Section 3.6 for computing $q^+_{\bullet}(1-\alpha)$. $\square$

Remark 91 *Most of the common derivative securities and some simple exotic derivatives satisfy the assumptions in the above proposition, for example: plain vanilla calls and puts, common barrier options, etc.*

In order to fix ideas, consider the following example.

Example 92 (VaR$_\alpha$ of a down-and-out call) *Assume S_t is given by a geometric Brownian motion*

$$dS_t = rS_t dt + \sigma S_t dW_t$$

absorbed at a level $s \geq B > 0$. The t.p.d.of the process S_t for $t < \tau$ is given by (see Example 5 in [8])

$$p(t,s;\tau,z) = p^{GBM(r,\sigma)}(t,s;\tau,z) - \left(\frac{s}{B}\right)^{1-2r/\sigma^2} p^{GBM(r,\sigma)}\left(\frac{B^2}{s},t;z,\tau\right).$$

Assume the current time is t and we want to measure the VaR of a position on a down-and-out European call option with strike K, maturity $\tilde{T}$ and barrier $B < K$ for the risk horizon $[t,T]$, $T < \tilde{T}$. The price of the down-and-out call option $c^{d/o}(t,S_t)$ is an increasing function on S_t and given by

$$c^{d/o}(t,S_t) = c(t,S_t,K,\tilde{T}) - \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} c\left(t,\frac{B^2}{S_t},K,\tilde{T}\right),$$

with $c(t,S_t,K,\tilde{T})$ denoting the price of a European call option with strike K, and maturity $\tilde{T}$. In order to apply Corollary 90, we need first to compute $q^+_{S_T}(1-\alpha)$. Define for $m \in [B,\infty)$

$$y_m = -\frac{\log\left(\frac{S_t}{m}\right) + \left(r - 1/2\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

and for $y \in [y_B,\infty)$

$$\Sigma(y) := \Phi(y) + \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \Phi\left(-y - \frac{2\log\left(\frac{S_t}{B}\right)}{\sigma\sqrt{T-t}}\right), \tag{4.4}$$

with Φ the standard cumulative Normal distribution.

The function $\Sigma(y)$ is strictly increasing on $[y_B, \infty)$ with $\alpha_{\min} := \Sigma(y_B)$ and $\Sigma(\infty) = 1$. Thus using the t.p.d. for the process S_t we conclude that

$$q^+_{S_T}(\alpha) = \begin{cases} B & \text{if} \quad \alpha \leq \alpha_{\min} \\ m^*(\alpha) & \quad \alpha > \alpha_{\min} \end{cases}$$

with $m^*(\alpha)$ given by

$$m^*(\alpha) = \exp\left\{\Sigma^{-1}(\alpha)\sigma\sqrt{T-t} + \log(S_t) + \left(r - 1/2\sigma^2\right)(T-t)\right\}. \tag{4.5}$$

Note that $\Sigma^{-1}(\alpha)$ has not explicit expression but it can easily be computed numerically. And the quantity $\alpha_{\min}$ has the interpretation of the probability of hitting the barrier B.

For simplicity in the computations, consider $r = 0$. The case $r \neq 0$ can be treated similarly with minor changes.

Then using Proposition 26 and Corollary 90 for computing VaR_α on a long position in the derivative security (a down-and-out call) we have

$$\begin{aligned} VaR_\alpha(c^{d/o}(T, S_T) - c^{d/o}(t, S_t)) &= c^{d/o}(t, S_t) - q^+_{C(T,S_T)}(\alpha) \\ &= c^{d/o}(t, S_t) - c^{d/o}(t, q^+_{S_T}(\alpha)) \\ &= \begin{cases} c^{d/o}(t, S_t) & \text{if} \quad \alpha \leq \alpha_{\min} \\ c^{d/o}(t, S_t) - c^{d/o}(t, m^*(\alpha)) & \quad \alpha > \alpha_{\min}. \end{cases} \end{aligned}$$

Then the position will be acceptable when $\alpha \leq \alpha_{\min}$ only if $S_t = B$. And if $\alpha > \alpha_{\min}$ we have that the position in the derivative is acceptable if

$$c^{d/o}(t, S_t) \leq c^{d/o}(t, m^*(\alpha)),$$

or equivalently,

$$S_t \leq m^*(\alpha).$$

In order to see acceptability in terms of $\alpha \in (0,1)$, define $\alpha_{S_t} := \Sigma(S_t)$, using (4.4) or (4.5) one can easily see show that

$$\alpha_{S_t} = \Sigma\left(\frac{1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right).$$

Then the position on the derivative will be acceptable if $\alpha \geq \alpha_{S_t}$.

The VaR_α of a short position in the derivative is

$$\begin{aligned} VaR_\alpha(-c^{d/o}(T,S_T)+c^{d/o}(t,S_t)) &= -c^{d/o}(t,S_t)-q^+_{-C(T,S_T)}(\alpha) \\ &= -c^{d/o}(t,S_t)-c^{d/o}(t,q^+_{S_T}(1-\alpha)) \\ &= \begin{cases} -c^{d/o}(t,S_t) & \text{if } \; 1-\alpha \leq \alpha_{\min} \\ -c^{d/o}(t,S_t)-c^{d/o}(t,m^*(1-\alpha)) & 1-\alpha > \alpha_{\min}. \end{cases} \end{aligned}$$

In this case, the short position in the down-and-out call will always be acceptable.

4.2.3.2 Piecewise injective transformations

In this section, we consider the situation where there may exist points (t,s) on the domain of the local transformation C at which $\frac{\partial C}{\partial s}(t,s)=0$. This is, the transformation is not injective on the whole domain of C.

As one of our motivating applications is to be able to find explicit expressions for the t.p.d. when $C(t,s)$ represents the price of a European derivative security on a risky asset given by continuous Markov diffusion process, we expect C to be a nice behaving function. The next proposition shows that if the set of critical points given by $\frac{\partial C}{\partial s}(t,s)=0$ is a curve (or several disjoint curves), then we are able to solve as in the previous section.

Proposition 93 *Consider $C \in C^{1,2}([0,T]\times\mathbb{R})$ and for $t\in[t,t+\theta]$,*

1. *[CONTINUITY CONDITION I] if there is $s^*\in\mathbb{R}$ such that*

$$\frac{\partial C}{\partial s}(t,s^*)=0 \qquad \text{and} \qquad \frac{\partial^2 C}{\partial s^2}(t,s^*)\neq 0,$$

   then there exists an interval $[t,t+\Delta t]$, with $0<\Delta t\in\mathbb{R}_+$ and a unique function $s_m:[t,t+\Delta t]\to\mathbb{R}$ such that $s_m(t)=s^*$ and $\frac{\partial C}{\partial s}(\tau,s_m(\tau))=0$ for all $\tau\in[t,t+\Delta t)$. Furthermore,

2. *[CONTINUITY CONDITION II] if there exists a process S_t on $(\Omega,\mathcal{F},\mathbb{P})$ satisfying the Assumptions 55 and 56 given by the sde in (S-SDE), and functions Ψ,Φ, and k satisfying the conditions in the Feynman-Kac theorem for the stochastic representation*

$$C(t,s)=\mathbb{E}_{\mathbb{Q}}\left[\Psi(S_T)\Lambda(t)+\int_t^{t+\theta}\Phi(v,S_v)\Lambda(t)dv|S_t=s\right], \tag{4.6}$$

   with $\Lambda(t)=\exp\left\{-\int_t^{t+\theta}k(\theta,S_\theta)d\theta\right\}$ and $\mathbb{Q}$ a measure equivalent to $\mathbb{P}$. Then $s_m(\tau)$ and its derivative $\frac{ds_m}{dt}(\tau)$ are continuous for all $t\leqslant\tau\leqslant t+\theta$.

Proof. The first part of the result is an adapted version of the implicit function theorem to $\frac{\partial C}{\partial s}(t,s)$. See for example [78] or [1, p.101-115]. The second part follows from the fact that the infinitesimal generator associated to the SDE in expression (S-SDE) is an uniformly elliptic operator defined on the whole interval $[t, t+\theta]$, and the points $(\tau, s_m(\tau))$, $t \leqslant \tau < t+\Delta t$ define a set of optimal points (either maxima when $\frac{\partial^2 C}{\partial s^2}(\tau, s_m(\tau)) < 0$ or minima when $\frac{\partial^2 C}{\partial s^2}(\tau, s_m(\tau)) > 0$), then by the Strong Maximum Principle for elliptic operators this frontier needs to be continuous and smooth for all $t \leqslant \tau \leqslant t+\theta$. □

Remark 94 *We can summarize previous Proposition 93 as: Let* Υ *defined below*

$$\Upsilon = \begin{cases} t+\Delta t & \textit{if condition 1 in Preposition 93 is satisfied} \\ t+\theta & \textit{if condition 1 and 2 in Preposition 93 are satisfied.} \end{cases}$$

Then $s_m(\tau)$ *and its derivative* $\frac{ds_m}{dt}(\tau)$ *are continuous for all* $t \leqslant \tau \leqslant \Upsilon$.

The previous result says that if at time t, the function C has local maxima or minima, and C is sufficiently well behaved, then the region $\mathbb{R} \times [t, \Upsilon)$ is divided by a continuous smooth curve $s_m(\tau), \tau \in [t, \Upsilon)$ as shown in Figure 4.1.

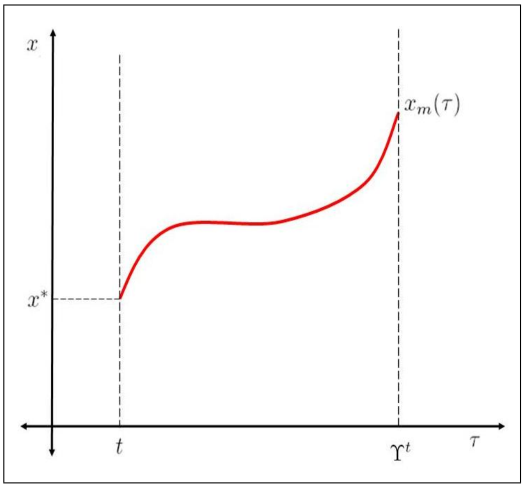


Figure 4.1: Division of the region $[t, t+\theta) \times \mathbb{R}$ by a continuous smooth curve $s_m(\tau)$, $\tau \in [t, t+\theta)$.

The aim is then to solve for the transition density on each sides of the curve $s_m(\tau)$ so that both solutions match appropriately on $s_m(\tau)$ and also satisfy the usual boundary conditions.

For simplicity in the exposition, we formulate our theorem for the case of a function $C(t,s)$ with only one minimum point $s^*(t)$ for each $t \in [t, \Upsilon)$ (see Fig 4.2(a)), as generalisations may be treated similarly (e.g. several local minima (resp. maxima) or when minima (resp. maxima) appear in time). See Example 99: Inhomogeneous transformations of BM.

The extension of Corollary 84 to the inhomogeneous case is considered in the next proposition.

For simplicity in the notation, in the following proposition consider the risk horizon to be $[t_0, t]$, corresponding to the pairs (t_0, x_0) and (t, x).

Proposition 95 *Let S_t be a Markov process on $(\Omega, \mathcal{F}, \mathbb{P})$ given by*

$$dS_t = b(t, S_t)\,dt + a(t, S_t)\,dW_t \qquad \textit{with} \qquad S_{t_0} = s_0$$

satisfying the Assumptions 55 and 56, and a function $C \in \mathcal{C}^{1,2}([t_0, t] \times \mathbb{R})$ satisfying the conditions in Proposition 93, with a unique minimal point $s^*(t)$ for each $t \in [t_0, \Upsilon)$. Write $s^* := s^*(t_0)$. Denote by $p^X(t_0, x_0; t, x)$ the transition probability density for the process $X_t = C(t, S_t)$ under $\mathbb{P}$ and define the function $\phi^{t,x}$ for $x_0 = C(t_0, s_0)$ as

$$\phi^{t,x}(t_0, x_0) = p^X(t_0, x_0; t, x).$$

For fixed $t < \Upsilon$ and x, $\phi^{t,x}$ satisfies the final condition backward equation

$$\mathcal{L}^{\beta,\alpha}\left[\phi^{t,x}\right](\tau, x_0) \;=\; 0, \qquad (\tau, x_0) \in (t_0, t) \times \mathbb{R}, \tag{4.7}$$

$$\lim_{\tau \nearrow t} \phi^{t,x}(\tau, x_0) \;=\; \delta(y_0 - y), \qquad y \in \mathbb{R}. \tag{4.8}$$

with

$$\beta(t, s) = \mathcal{L}^{b,a}[C](t, s) \text{ and } \alpha(t, s) = a(t, s)\frac{\partial C}{\partial s}(t, s).$$

Then solving for $\phi^{t,x}$ in the above system is equivalent to solve for $\Psi^{t,s}$ in

$$\begin{aligned}
\mathcal{L}^{b,a}\Psi^{t,s}(\tau, s_0) &= 0, \qquad (\tau, s_0) \in [t_0, t] \times \mathbb{R} \\
\lim_{\tau \nearrow t} \Psi^{t,s}(\tau, s_0) &= \delta(s_0 - s) + \delta(s_0 - \hat{s}), \\
\Psi(\tau, s_m(\tau)) &= \delta(s_m(\tau) - s) \text{ for all } \tau \in [t_0, t],
\end{aligned}$$

with $\hat{s} \leqslant s_m(t) \leqslant s$ such that $C(t, \hat{s}) = x = C(t, s)$, and $\delta(z - x)$ is the delta function of z centred at x. Furthermore, if for arbitrary $I \subset \mathbb{R}$ the following condition holds

$$\int_{C^{-1}(t,I)} \Psi^{t,s}(t_0, s_0)ds = \int_{C^{-1}(t,I)} \Psi^{t,s}(t_0, \hat{s}_0)dx \qquad \textit{whenever} \qquad C(t, s_0) = C(t, \hat{s}_0), \tag{4.9}$$

then $p^S(t_0, s_0; t, s) = \Psi^{t,s}(t_0, s_0)$ *is the transition probability density for the process* S_t *under* $\mathbb{P}$ *and* p^X *and* p^S *are related as follows*

$$p^X(t_0, x_0; t, x) = \begin{pmatrix} \left|\left(\frac{\partial C}{\partial s}(t, \chi^l(t,x))\right)^{-1}\right| p^S(t_0, \chi(t_0, x_0); t, \chi^l(t,x)) \\ + \\ \left|\left(\frac{\partial C}{\partial x}(t, \chi^r(t,x))\right)^{-1}\right| p^S(t_0, \chi(t_0, y_0); t, \chi^r(t,x)) \end{pmatrix} \text{ for } x \geq C(t, s_m(t))$$

with

$$\chi(t_0, x_0) = \begin{cases} \chi^l(t_0, x_0) & \text{if } s_0 \leq s_m(t_0) \\ \chi^r(t_0, x_0) & \text{if } s_0 > s_m(t_0) \end{cases}$$

for $\chi^l(t,x) \leq s_m(t) \leq \chi^r(t,x)$ *representing the inverse functions of* $C(t,s)$*, i.e.,* $C(t, \chi^l(t,x)) = x = C(t, \chi^r(t,x))$*, and the function* $s_m(\tau)$ *is the time parametrisation of a curve where* $\frac{\partial C}{\partial s}(\tau, s_m(\tau)) = 0$ *for* $\tau \in [t_0, t]$ *and* $s_m(t_0) = s^*$*. For details see the proof below.*

The proof is similar in spirit at the one for the homogeneous case, plus considering the conditions in Proposition 93 for guaranteeing the differentiability and continuity of the degeneracy points. We prove this time the result from a PDE point of view.

Proof. As the proof is long, we divide it into four parts:

We will omit the supraindices in Ψ and ϕ, replace (t_0, s_0) by (τ, ς) and (t_0, x_0) by (τ, ξ) for simplicity in the notation.

1. **Continuation of the solving region:** As the assumptions in Proposition 93 are satisfied, there exist a differentiable and continuous function $s_m : \mathbb{R}^+ \to \mathbb{R}^+$ such that $s_m(\tau) = s^*$ and $\frac{\partial C}{\partial s}(\tau, s_m(\tau)) = 0$ for all $\tau < \Upsilon$.

2. **Equation reduction via change of variable** Define

$$\Gamma = \{(\tau, s_m(\tau)) : \tau < \Upsilon\}$$

On Γ, note that

$$\beta(\tau, \varsigma) = \frac{\partial C}{\partial \tau} + \frac{1}{2} a(\tau, \varsigma)^2 \frac{\partial^2 C}{\partial \varsigma^2} \text{ and } \alpha(\tau, \varsigma) = 0.$$

Take $\Psi(\tau, \varsigma) = \phi(\tau, \xi) = \phi(\tau, C(\tau, \varsigma))$ then we have

$$\frac{\partial \Psi}{\partial \tau} = \frac{\partial \phi}{\partial \tau} + \frac{\partial C}{\partial \tau}\frac{\partial \phi}{\partial \varsigma}, \qquad \frac{\partial \Psi}{\partial \varsigma} = \frac{\partial C}{\partial \varsigma}\frac{\partial \phi}{\partial \xi} = 0$$

and

$$\frac{\partial^2 \Psi}{\partial \varsigma^2} = \left(\frac{\partial C}{\partial \varsigma}\right)^2 \frac{\partial \phi}{\partial \xi^2} + \frac{\partial^2 C}{\partial \varsigma^2}\frac{\partial \phi}{\partial \xi} = \frac{\partial^2 C}{\partial \varsigma^2}\frac{\partial \phi}{\partial \xi}$$

and as $\frac{\partial C}{\partial s}(\tau, s_m(\tau)) = 0$, equation (4.7) reduces to

$$\frac{\partial \Psi}{\partial t} + \frac{1}{2}a(\tau,\varsigma)^2\frac{\partial^2 \Psi}{\partial s^2} + b(\tau,\varsigma)\frac{\partial \Psi}{\partial s} = 0.$$

Using the fact that $s_m(\tau)$ for $\tau \in [t_0, t]$ is a minimal point for C, as illustrated in Figure 4.2(a), the expression in (4.8) becomes

$$\lim_{\tau \nearrow t} \Psi(\tau,\varsigma) = \delta(\xi - x) = \delta(C(\tau,\varsigma) - C(t,s)) = \delta(\varsigma - s) + \delta(\varsigma - \hat{s})$$

for $\hat{s} \leqslant s_m(t) \leqslant s$ such that $C(t,\hat{s}) = x = C(t,s)$. But if $s = s_m(t)$ then $s = s_m(t) = \hat{s}$. Thus

$$\lim_{\tau \nearrow t} \Psi(\tau,\varsigma) = \delta(s_m(t) - s).$$

In particular, if condition 2 in Proposition 93 is satisfied, then

$$\beta(\tau,\varsigma) = 0 = \alpha(\tau,\varsigma)$$

(as $\frac{\partial u}{\partial t} + \frac{1}{2}a(\tau,\varsigma)^2\frac{\partial^2 C}{\partial s^2} + b(\tau,\varsigma)\frac{\partial C}{\partial s} = -a(\tau,\varsigma)^2\frac{h_x}{h}\frac{\partial C}{\partial s}$ with $h(t,S_t)$ a $\mathbb{P}$-martingale), then the Equation (4.7) remains

$$\frac{\partial \Psi}{\partial t}(\tau, s_m(\tau)) = 0 \quad \text{for all} \quad \tau \in [t_0, t),\, t < T \quad \text{with} \quad \Psi(t) = \delta\left(x_m(t) - s\right),$$

which implies that

$$\Psi(\tau) = \delta\left(s_m(\tau) - s\right) \ \text{for all } \tau \in [t_0, t], \text{ for } t < T.$$

In order to solve the PDE in (4.7) not including Γ, define

$$D := \mathbb{R}_+ \times \mathbb{R} \backslash \Gamma$$

and take

$$C^r(\tau,\varsigma) = \begin{cases} C(\tau,\varsigma) & \text{if} \quad s > s_m(t) \\ 0 & \text{else} \end{cases} \quad \text{and} \quad C^l(\tau,\varsigma) = \begin{cases} C(\tau,\varsigma) & \text{if} \quad s < s_m(t) \\ 0 & \text{else} \end{cases}.$$

Then in D we have $C(\tau,\varsigma) = C^l(\tau,\varsigma) + C^l(\tau,\varsigma)$ with C^l and C^r invertible functions, therefore the following change of variable

$$\Psi(\tau,\varsigma) = \phi(\tau,\xi) = \phi(t, C(\tau,\varsigma))$$

is well defined, and using the properties of change of variable transformation, the first part of the equation in (4.7) is transformed into

$$\mathcal{L}^{b,a}\Psi(\tau,\varsigma) = \mathcal{L}^{\mathcal{L}^{b,a}[C],a\frac{\partial C}{\partial x}}[\phi](\tau,\xi) = 0, \quad (\tau,\varsigma) \in \times[t_0,t) \times \mathbb{R},$$

and the final condition remains

$$\lim_{\tau \nearrow t} \Psi(\tau, \varsigma) = \delta(C(t, \varsigma) - x), x \in \mathbb{R},$$

which for $\hat{s} \leqslant s_m(t) \leqslant s$ such that $C(t, \hat{s}) = x = C(t, s)$ the final condition above can be rewritten as

$$\lim_{\tau \nearrow t} \Psi(\tau, \varsigma) = \delta(C(\tau, \varsigma) - x) = \delta(\varsigma - s) + \delta(\varsigma - \hat{s}).$$

3. **Solution to the related problems** Summarising the steps above, solving equation (4.7) is equivalently to solve the following problem

$$\begin{aligned} \mathcal{L}^{b,a}\Psi(\tau, \varsigma) &= 0, \quad (\tau, \varsigma) \in \times[t_0, t) \times \mathbb{R} \\ \lim_{\tau \nearrow t} \Psi(\tau, \varsigma) &= \delta(\varsigma - s) + \delta(\varsigma - \hat{s}), \\ \Psi(\tau, s_m(\tau)) &= \delta(s_m(\tau) - s) \quad \text{for all} \quad \tau \in [t_0, t]. \end{aligned} \tag{4.10}$$

But the above system in (4.10) can be split into two problems as:

$$\Psi(\tau, \varsigma) = \Psi^l(\tau, \varsigma) + \Psi^r(\tau, \varsigma)$$

with $\Psi^l(\tau, \varsigma)$ and $\Psi^r(\tau, \varsigma)$ solutions of

$$\begin{aligned} \mathcal{L}^{b,a}_{\varsigma}\Psi^l(\tau, \varsigma) &= 0, \quad \tau \in [t_0, t) \quad \text{for} \quad \varsigma \leqslant s_m(\tau) \\ \lim_{\tau \nearrow t} \Psi^l(\tau, \varsigma) &= \delta(\varsigma - \hat{s}) \\ \Psi^l(\tau, s_m(\tau)) &= \delta(s_m(\tau) - \hat{s}) \quad \text{for all} \quad \tau \in [t_0, t]. \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^{b,a}\Psi^r(\tau, \varsigma) &= 0, \quad \tau \in \times[t_0, t) \quad \text{for} \quad \varsigma > s_m(\tau) \\ \lim_{\tau \nearrow t} \Psi^r(\tau, \varsigma) &= \delta(\varsigma - s) \\ \Psi^r(s_m(\tau), \tau) &= \delta(s_m(\tau) - s) \quad \text{for all} \quad \tau \in [t_0, t]. \end{aligned}$$

and using the stochastic representation formula we have that

$$\begin{aligned} \Psi^l(\tau, \varsigma) &= p^S(\tau, \varsigma; t, \hat{s}) \quad \text{with} \quad \varsigma \leqslant s_m(\tau) \quad \text{for all} \quad \tau \in [t_0, t] \\ \Psi^r(\tau, \varsigma) &= p^S(\tau, \varsigma; t, x) \quad \text{with} \quad \varsigma > s_m(\tau) \quad \text{for all} \quad \tau \in [t_0, t]. \end{aligned}$$

Therefore

$$\Psi^r(\tau, \varsigma) = p^S(\tau, \varsigma; t, \hat{s})\mathbb{I}_{\{\varsigma \leq s_m(t)\}} + p^S(\tau, \varsigma; t, x)\mathbb{I}_{\{s_m(t) < \varsigma\}}$$

4. **Back transformation of variables**. Now, transforming back into (t, x) variables, note that

$$\Psi(\tau, \varsigma) = p^X(\tau, C(\tau, \varsigma); t, x) = p^S(\tau, \varsigma; t, \hat{s})\mathbb{I}_{\{\varsigma \le s_m(t)\}} + p^S(\tau, \varsigma; t, s)\mathbb{I}_{\{s_m(t) < \varsigma\}},$$

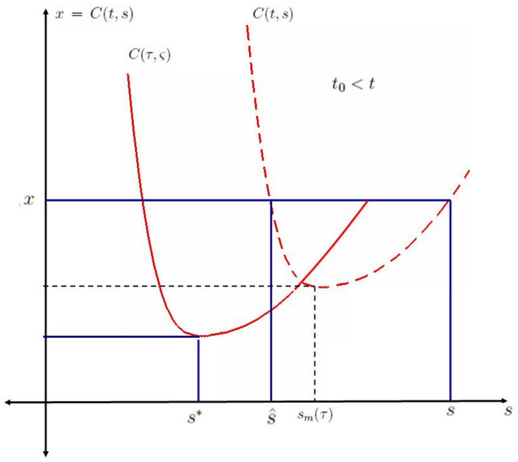
with $s = \chi^r(t, x)$, $\hat{s} = \chi^l(t, x)$ and for

$$\varsigma = \chi(\tau, \xi) = \begin{cases} \chi^l(\tau, \varsigma) & \text{if } \varsigma \le s_m(\tau) \\ \chi^r(\tau, \varsigma) & \text{if } \varsigma > s_m(\tau) \end{cases} \tag{4.11}$$

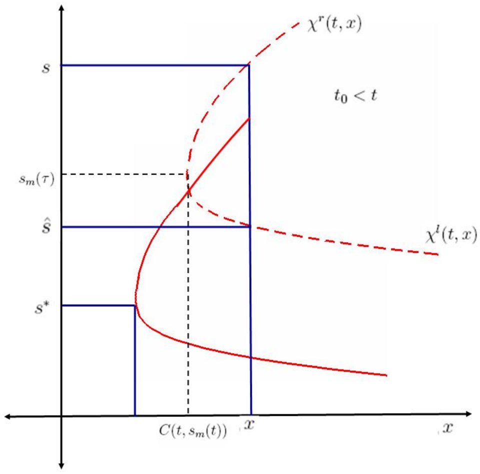
with $\chi^l(t, x) \le s_m(t) \le \chi^r(t, x)$ representing the inverse functions C^r and C^l defined above in (4.11), i.e., $C^r(t, \chi^r(t, x)) = x = C^l(t, \chi^l(t, x))$, as illustrated in Fig. 4.2(b). Using similar arguments as in Proposition 84 we get the desired expression

$$p^X(t_0, x_0; t, x) = \begin{pmatrix} \left| \left(\frac{\partial C}{\partial s}(t, \chi^l(t, x))\right)^{-1} \right| p^S(t_0, \chi(t_0, x_0); t, \chi^l(t, x)) \\ + \\ \left| \left(\frac{\partial C}{\partial s}(t, \chi^r(t, x))\right)^{-1} \right| p^S(t_0, \chi(t_0, x_0); t, \chi^r(t, x)) \end{pmatrix} \text{ for } x \ge C(t, s_m(t)).$$

□



(a) Change of variable $C(t,s)$ with only one local minimum $s^*(t)$ for $t \leq \Upsilon^t$.



(b) Inverse function for $C(t,s)$.

Figure 4.2:

Note that Proposition 93 and Proposition 95 have been formulated in general terms and not particularly for the case of transformations of continuous Markov processes that come from the valuation of derivative securities. But as our main interest is the computation of risk measures of derivative securities, we adapt previous results and make the following assumption.

Assumption 96 *The local transformation C is given by*

$$C(t, S_t) = \mathbb{E}_{\mathbb{Q}}[e^{-r(\tilde{T}-t)} H(S_{\tilde{T}})],$$

for a continuous positive function $H(s)$ and $\mathbb{Q}$ an equivalent probability measure to $\mathbb{P}$ (a risk-neutral probability measure).

This assumption, together with the initial assumptions on the process S_t, guarantees that $C \in C^{1,2}([t, t+\theta] \times \mathbb{R})$ and satisfies a uniformly elliptic PDE. Therefore the assumptions on Proposition 93 are satisfied and Proposition 95 tuns to be very useful when we need to compute risk measures of derivative securities that have minimum/maximum points. Some examples are combinations of plain vanilla options such as straddles, strangles, etc. Before we present the example of a straddle under the Black and Scholes model, we need the results in Proposition 97 that characterises all transformations of Brownian motion, but adapted to the inhomogeneous case. For simplicity, we rewrite the results below.

Proposition 97 *Let $(S_t, 0 \le t < \infty)$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ starting at $s_{t_0} > 0$ for $t_0 \in \mathbb{R}$. Then C is a change of variable transformation $(X_t = C(t, S_t))$ satisfying the condition (4.9) in Proposition 95 if and only if*

1. *C is invertible in all its domain, or*

2. *$C(\cdot, s)$ is injective and $C(t, \cdot)$ is symmetric.*

4.2.4 Example

Example 98 (VaR of a straddle) *Assume S_t is given by a geometric Brownian motion*

$$dS_t = \mu S_t dt + \sigma S_t dW_t \text{ with } S_t = s \tag{4.12}$$

and the derivative security is a straddle

$$C(t, s) = Call(t, s; K, \tilde{T}) + Put(t, s; K, \tilde{T})$$

with strike K and time to maturity $\tilde{T} - t$. Its payoff is given by

$$H(s) = \begin{cases} s - K & \text{if } s \ge K \\ K - s & \text{if } s < K. \end{cases}$$

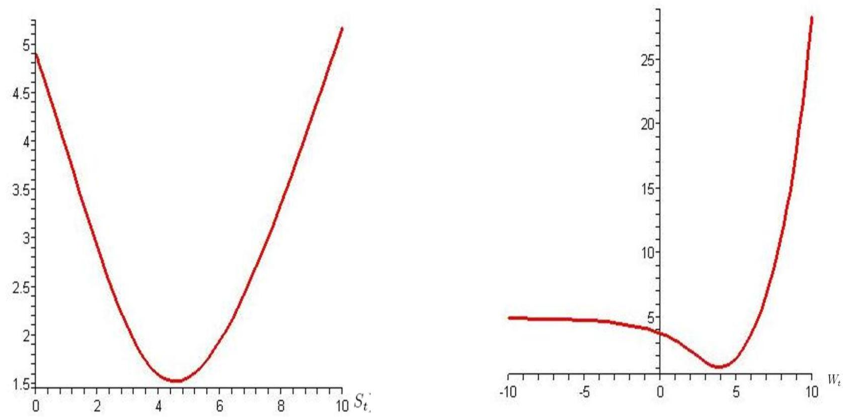
Figure 4.3(a) shows the value of a straddle for the numerical parameters $K = 6$, $\mu = 0.2$, $\sigma = 0.4$, $\tilde{T} - t = 1$.

Consider t to be the current time for analysis and $[t, T], T < \tilde{T}$ the risk horizon. Our goal is to compute VaR_α of a long position in the straddle.

Unfortunately, the assumptions in Proposition 95 are not satisfied as the condition in (4.9) does not hold. This can be seen by writing $C(t, S_t)$ in terms of the Brownian motion W_t defining the SDE in (4.12). Write

$$g(t, W_t) = C(t, S_t) = C\left(t, S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}\right).$$

By Proposition 97, if the function $g(\cdot, w)$ is injective and $g(t, \cdot)$ is symmetric, then the relation in (4.9) would hold. It is easy to see that $g(t, \cdot)$ is not a symmetric function. Figure 4.3(b) shows a plot of the function $g(t, w)$ against w for the numerical values $S_t = 1,\ K = 6,\ \mu = 0.2,\ \sigma = 0.4,\ \tilde{T} - t = 1,\ t = 0.5.$



(a) The value of a straddle $C(t, s)$ as a function of s.

(b) The function $g(t, W_t)$ against W_t.

Figure 4.3: Example: 98. VaR of a straddle.

Even though we cannot obtain an explicit expression for the t.p.d. of the straddle as a process using Proposition 95, we can apply the same method used in the Proposition 95 for solving the t.p.d. PDE to solve the VaR-PDE in (3.27) for the straddle. Thus the VaR_α of a long position in the derivative will be given by

$$\begin{aligned} VaR_\alpha\left(C(T, S_T) - C(t, S_t)\right) &= C(t, S_t) - q^+_{C(T,S_T)}(\alpha) \\ &= C(t, S_t) - \sup\left\{q \in \mathbb{R} : V^q(t, S_t) \le \alpha\right\} \end{aligned}$$

where $V^q(t, s)$ solves the following PDE

$$\begin{aligned} \frac{\partial V^q}{\partial t}(t, s) + \mu s \frac{\partial V^q}{\partial s}(t, s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V^q}{\partial s^2}(t, s) &= 0, (t, s) \in [t, t + \theta) \times \mathbb{R}, \quad (4.13) \\ V^q(T, s) &= 1_{\{C(T,s) < q\}}. \end{aligned}$$

The function $C(t,s)$ satisfies the elliptic PDE (Black-Scholes equation with $r=0$)

$$\begin{aligned} \frac{\partial C}{\partial t}(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C}{\partial s^2}(t,s) &= 0, (\tau,s) \in [t,\tilde{T}) \times \mathbb{R}, \\ C(\tilde{T},s) &= H(s), \end{aligned}$$

and $\frac{\partial C}{\partial s}(\tau,K) = 0$ *with* $\frac{\partial^2 C}{\partial s^2}(\tau,K) \neq 0$ *for all* $t \leqslant \tau \leqslant T$. *The function* $s_m : [t,T] \to \mathbb{R}$ *defined as*

$$s_m(t) = K \text{ with } \frac{\partial C}{\partial s}(\tau, s_m(\tau)) = 0 \quad \text{for all} \quad \tau \in [t,T),$$

is continuous with continuous derivative $\frac{ds_m}{dt}(\tau)$. *As in Proposition 95, we write the function* C *as the sum of two injective functions* $C(t,s) = C^r(t,s) + C^l(t,s)$ *with* $C^r(t,s) = Call(t,s;K,\tilde{T})$ *and* $C^l(t,s) = Put(t,s;K,\tilde{T})$.

Take

$$y_m(t) = \log\left(\frac{s_m(t)}{K}\right), \; y_q^l = \log\left(\frac{(C^l)^{-1}(q)}{K}\right), y_q^r = \log\left(\frac{(C^r)^{-1}(q)}{K}\right),$$

and by the usual dimensionalisation and change of variable to reduce the PDE in (4.13) to the heat equation, take

$$s = Ke^y, t = T - 2\tau/\sigma^2 \text{ and } u^q(\tau,y) = \exp\{\eta y + \beta\tau\} V^q\left(T - \frac{2\tau}{\sigma^2}, Ke^y\right).$$

Taking

$$\eta = -\frac{k-1}{2}, \; \beta = -\left(\frac{k-1}{2}\right)^2 \text{ and } k = \frac{2\mu}{\sigma^2},$$

the PDE becomes

$$u_\tau^q = u_{yy}^q \quad \text{for} \quad -\infty < y < \infty \tag{4.14}$$

with the initial condition

$$u^q(0,y) := u_0(y) = 1_{\{C(T,s)<q\}} = \begin{cases} e^{-\eta y} & \text{for } \; y_q^l < y < y_q^r \\ 0 & \text{else} \end{cases}, q > C(t,K)$$

and the moving boundary condition

$$u^q(\tau,y) = 0 \quad \text{for} \quad q = C\left(T - \frac{2\tau}{\sigma^2}, K\right) \text{ and for all} \quad \tau \in [0, T-t].$$

Due to the condition above we need to use the method of images to solve the equation in (4.14). Solving and substituting back we have

$$\begin{aligned} s_q^l &= Ke^{y_q^l} = (C^l)^{-1}(q), \\ s_q^r &= Ke^{y_q^r} = (C^r)^{-1}(q), \end{aligned}$$

and

$$V^q(t,s) = \Phi(d^{\mu}_{2,s^r_q}) - \Phi(d^{\mu}_{2,s^l_q}),$$

with Φ *the standard cumulative normal distribution and*

$$d^{\mu}_{2,E} = \frac{\log(S_t/E) + (\mu - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

One can easily check that choosing q *such that* $q = q_\alpha := q^+_{C(T,S_T)}(\alpha)$ *we have*

$$C^l\left(t, S^l_{q_\alpha}\right) = q_\alpha = C^r\left(t, S^l_{q_\alpha}\right),$$

and

$$s^r_q = q^+_{S_T}(\delta+\alpha) \text{ and } s^l = q^+_{S_T}(\alpha)$$

for $0 \le \delta + \alpha \le 1$ *with*

$$q^+_{S_T}(\alpha) = s\exp\{(\mu - \sigma^2/2)(T-t) + \Phi^{-1}(\delta)\sigma\sqrt{T-t}\},$$

then we have $V^q(t,s) = \alpha$. *Thus*

$$q^+_{C(T,S_T)}(\alpha) = C(t, q^+_{S_T}(\delta)),$$

for δ *satisfying the equality*

$$C(t, q^+_{S_T}(\alpha)) = C(t, q^+_{S_T}(\delta+\alpha)). \tag{4.15}$$

Therefore the VaR_α *of a long position in the derivative is*

$$\begin{aligned} VaR_\alpha(C(T,S_T) - C(t,S_t)) &= C(t,S_t) - C(t, q^+_{S_T}(\delta)) \\ &= C(t,S_t) - C(t, S_t\exp\{(\mu - \sigma^2/2)(T-t) + \Phi^{-1}(\delta)\sigma\sqrt{T-t}\}) \end{aligned}$$

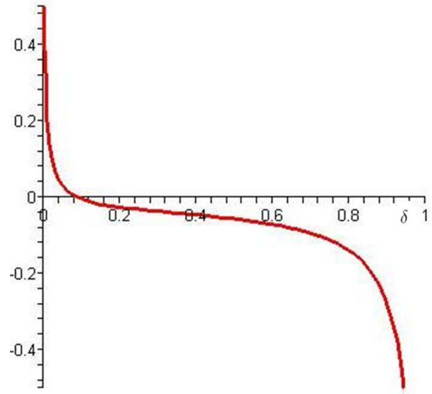
with δ *satisfying (4.15).*

This is, we need to find δ *and* α *such that* $DIF(\delta,\alpha) = 0$ *for*

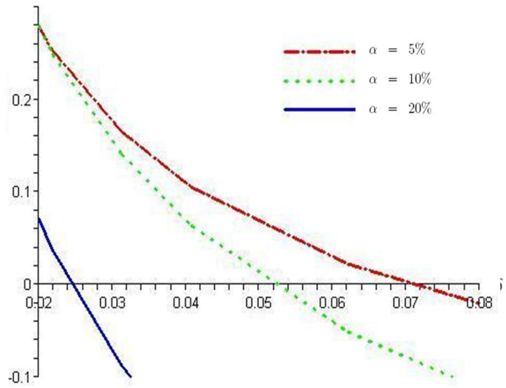
$$DIF(\delta,\alpha) := C(t, q^+_{S_T}(\delta), K, \tilde{T}) - C(t, q^+_{S_T}(\delta+\alpha), K, \tilde{T}).$$

By the shape of the function $C(t,S)$, *it is direct to see that there exist values of* $0 \le \delta \le 1$ *and* $0 \le \alpha \le 1$ *such that the function* $DIF(\delta,\alpha)$ *takes the value zero. We illustrate this fact with some numerical values for* $S_t = 1,\ K = 6,\ \mu = 0.2,\ \sigma = 0.4,\ \tilde{T} - t = 1$, *Figure 4.4(a) exhibits the value of the function* $DIF(\delta,\alpha)$ *for* $\alpha = 0.01$, *and Figure 4.4(b) plots the values of* $DIF(\delta,\alpha)$ *for* $\alpha = 0.05,\ 0.1$ *and* 0.2.

For the value $\alpha = 0.01$, *the corresponding values in* $\delta \approx 0.0882971$, *which gives the values* $q^+_{C(T,S_T)}(\alpha) = C(t, q^+_{S_T}(\delta)) \approx 1.708133948$, *and* $VaR_\alpha(C(T,S_T) - C(t,S_t)) \approx 13.291866787$



(a) The function $DIF(\delta, \alpha)$ for $\alpha = 0.01$.



(b) The function $DIF(\delta, \alpha)$ for $\alpha = 5\%,\ 10\%, and\ 20\%$.

Figure 4.4: Example: 98. VaR of a straddle.

In order to illustrate more details of the applicability of Proposition 95, consider the following examples.

Example 99 ***Inhomogeneous transformation of BM.*** *For $0 < t_0 < T$, let $b \in \mathbb{R}$ and*

$$dX_t = bdt + dB_t \quad and \quad X_{t_0} = x_0$$

be the base process with $B_t, 0 \le t \le T$ a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the inhomogeneous transformation

$$u(t,x) = \frac{1}{6}\left(x^2 + \sqrt{3}t\right)^2 - t\left(\left[1 + \frac{\sqrt{3}}{3}\right]x^2 + \frac{1}{2}C_1\right) - x\left(C_3 - \frac{1}{2}C_1 x\right) + C_2, \text{ with } C_1, C_2, C_3 \in \mathbb{R}.$$

For $h(t,x) = \exp\left\{-bx + \frac{1}{2}b^2 t\right\}$ one can easily check the following properties on u

1. $u(t,x) \in \mathcal{C}^{1,2}\left([0,T] \times \mathbb{R}\right)$,
2. $\frac{\partial u}{\partial x}(t,x) = -\frac{\partial u}{\partial x}(t,-x)$, $\frac{\partial^2 u}{\partial x^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,-x)$ *and if* $C_3 = 0$ $u(t,x) = u(t,-x)$, $\frac{\partial u}{\partial t}(t,x) = \frac{\partial u}{\partial t}(t,-x)$; *so the function u is symmetric respect to $x = 0$ for $C_3 = 0$, see Figure 4.5,*
3. *$h(t, X_t)$ is a $\mathbb{P}$-martingale, so the density process defined by $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = h(t, X_t)$ is well defined,*

4. *the process X under $\mathbb{Q}$ is a standard Brownian motion, furthermore, $u(t,x_t)$ is a $\mathbb{Q}$-martingale,*

5. *when $C_3=0$ the function $x_m:\mathbb{R}\to\mathbb{R}$*

$$x_m(t)=\begin{cases} x^-(t)=-\frac{1}{2}\sqrt{12t-6C_1} \\ x^0(t)=0 \\ x^+(t)=\frac{1}{2}\sqrt{12t-6C_1}\end{cases},$$

is such that $\frac{\partial u}{\partial x}(t,x_m(t))=0$ for $t\in\left[\frac{1}{2}C_1,T\right]$ and given by (see Figure 4.6(a) and 4.6(b)). Note that $x_m\left(\frac{1}{2}C_1\right)=0$, and $x^+(t)$ is an increasing function (resp. $x^-(t)$ decreasing).

Assumption 100 *We assume $C_3=0$.*

We can redefine u as $u(t,x)=u^1(t,x)+u^2(t,x)+u^3(t,x)+u^4(t,x)$ for

$$\begin{array}{rl} u^1(t,x)= & u(t,x)I_{\{-\infty<x\le x^-(t)\}} \\ u^2(t,x)= & u(t,x)I_{\{x^-(t)<x\le 0\}} \\ u^3(t,x)= & u(t,x)I_{\{0<x\le x^+(t)\}} \\ u^4(t,x)= & u(t,x)I_{\{x^+(t)<x<\infty\}}\end{array},$$

so the corresponding inverse functions are given by (see Figure 4.7)

$$\begin{array}{ll} \chi^1(t,y)=\Pi\left[f,g\right](t,y) & \text{for } u(t,x^-(t))\le y<\infty \\ \chi^2(t,y)=\Pi\left[f,-g\right](t,y) & \text{for } u(t,x^-(t))\le y\le u(t,x^0(t)) \\ \chi^3(t,y)=-\Pi\left[f,-g\right](t,y) & \text{for } u(t,x^+(t))\le y\le u(t,x^0(t)) \\ \chi^4(t,y)=-\Pi\left[f,g\right](t,y) & \text{for } u(t,x^+(t))\le y<\infty \end{array}$$

where

$$\begin{aligned} u(t,x^-(t)) &= u(t,x^+(t))=-t^2+C_1t-C_2-\frac{3}{8}C_1^2, \\ u(t,x^0(t)) &= \frac{1}{2}t^2-\frac{1}{2}C_1t-C_2, \end{aligned}$$

and

$$\begin{aligned} f(t) &= 12t-6C_1, \\ g(t,y) &= \sqrt{9C_1^2-24C_1t+24t^2+24C_2+24y} \\ \Pi\left[f,g\right](t,y) &= \frac{1}{2}\sqrt{f(t)+2g(t,y)}. \end{aligned}$$

Define

$$\chi(t_0,y_0)=\begin{cases} \chi^1(t_0,y_0) & \text{if } \infty<x_0\le x^-(t_0) \\ \chi^2(t_0,y_0) & \text{if } x^-(t_0)<x_0\le x^0(t_0) \\ \chi^3(t_0,y_0) & \text{if } x^0(t_0)<x_0\le x^+(t_0) \\ \chi^4(t_0,y_0) & \text{if } x^+(t_0)<x_0<\infty \end{cases}$$

and note that

$$\begin{aligned}
\left|\left(\frac{\partial u}{\partial x}(t,\chi^1(t,y))\right)\right|^{-1} &= \frac{6}{|2\Pi\,[f,g]\,(t,y)g(t,y)|} \\
\left|\left(\frac{\partial u}{\partial x}(t,\chi^2(t,y))\right)\right|^{-1} &= \frac{6}{|-2\Pi\,[f,-g]\,(t,y)g(t,y)|} \\
\left|\left(\frac{\partial u}{\partial x}(t,\chi^3(t,y))\right)\right|^{-1} &= \frac{6}{|2\Pi\,[f,-g]\,(t,y)g(t,y)|} \\
\left|\left(\frac{\partial u}{\partial x}(t,\chi^4(t,y))\right)\right|^{-1} &= \frac{6}{|-2\Pi\,[f,g]\,(t,y)g(t,y)|}.
\end{aligned}$$

All assumptions in Proposition 95 are satisfied, then we can compute the transition probability density function for $Y_t = u(t, X_t)$ as

$$p^Y(t_0,y_0;t,y) = \begin{cases} \sum\limits_{\{i=1,i=4\}} \frac{p^{BMD(0,1)}(\chi(t_0,t_0,y_0);t,\chi^i(t,y))\mathbb{I}_{\{y\in I_i\}}}{\left|\left(\frac{\partial u}{\partial x}(t,\chi^i(t,y))\right)\right|} & \text{for } 0\le t\le \frac{1}{2}C_1 \text{ and } C_1>0 \\ \sum_{i=1}^4 \frac{p^{BMD(0,1)}(t_0,\chi(t_0,y_0);t,\chi^i(t,y))\mathbb{I}_{\{y\in I_i\}}}{\left|\left(\frac{\partial u}{\partial x}(t,\chi^i(t,y))\right)\right|} & \text{else} \end{cases}$$

or the rather complicated expression

$$= \frac{6}{\sqrt{2\pi(t-t_0)}} \begin{cases} \frac{Z(f,-g)\mathbb{I}_{\{y\in I_1\}}}{\exp\{W(f,-g)\}}\left[\exp\{-V(f,-g)\}+\exp\{V(f,-g)\}\right] & \text{for } 0\le t\le \frac{1}{2}C_1 \text{ and } C_1>0 \\ \left[\begin{array}{c} \frac{Z(f,g)\mathbb{I}_{\{y\in I_1\}}}{\exp\{W(f,g)\}}\left[\exp\{-V(f,g)\}+\exp\{V(f,g)\}\right] \\ + \\ \frac{Z(f,-g)\mathbb{I}_{\{y\in I_2\}}}{\exp\{W(f,-g)\}}\left[\exp\{-V(f,-g)\}+\exp\{V(f,-g)\}\right] \end{array}\right] & \text{else} \end{cases}$$

for

$$\begin{aligned}
I_1 &= I_4 = [u(t,x^-(t)),\infty), \\
I_2 &= I_3 = [u(t,x^-(t)),u(t,x^0(t))],
\end{aligned}$$

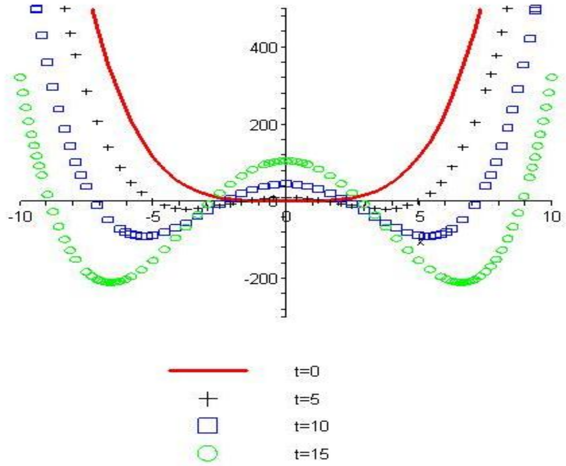
and

$$\begin{aligned}
W(f,g)(x_0,t_0;y,t) &= \frac{\left[\Pi\,[f,g]\,(t,y)\right]^2+\chi(t_0,y_0)^2}{2(t-t_0)}, \\
V(f,g)(x_0,t_0;y,t) &= \frac{2\Pi\,[f,g]\,(t,y)\chi(t_0,y_0)}{2(t-t_0)}, \\
Z(f,g)(x_0,t_0;y,t) &= \frac{1}{|2\Pi\,[f,g]\,(t,y)g(t,y)|}.
\end{aligned}$$

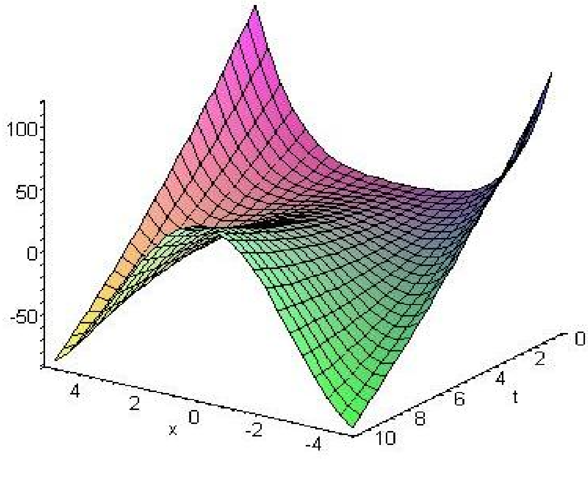
To see that this is indeed a transition probability function, one can easily check that

$$\int_{\mathbb{R}} \left| \left(\frac{\partial u}{\partial x}(t, \chi^i(t,y)) \right) \right|^{-1} p^{BMD(0,1)}(t_0, \chi(t_0, y_0); t, \chi^i(t,y)) \mathbb{I}_{\{y \in I_i\}} dy = \begin{cases} -\Xi + \frac{1}{2} & i = 1 \\ \Xi & i = 2 \\ \Xi & i = 3 \\ -\Xi + \frac{1}{2} & i = 4 \end{cases}$$

and Ξ *a real number, so* $\int_{\mathbb{R}} p(t_0, y_0; t, y) dy = 1$ *(in the particular case* $0 \leq t \leq \frac{1}{2}C_1, C_1 > 0$ *the terms for* $i = 2, 3$ *vanish and* $\Xi = 0$ *).*

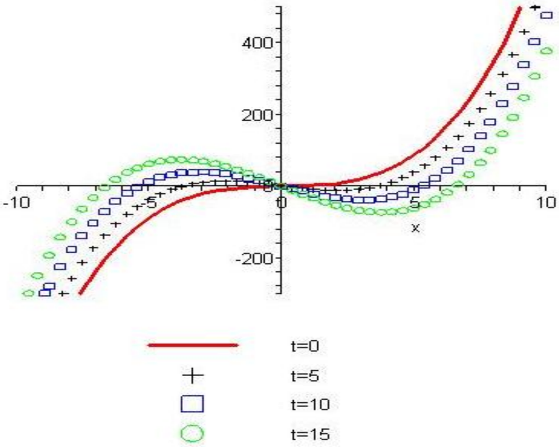


(a) The function $u(t,x)$ plotted against x for different values of t.

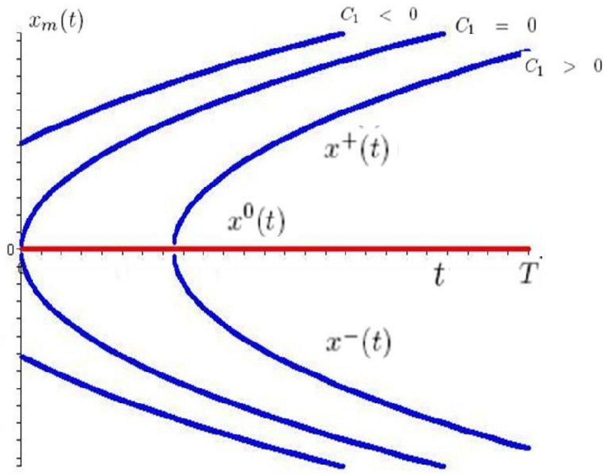


(b) Plot of $u(t,x)$ in both variables.

Figure 4.5: Functions $u(t,x) = \frac{1}{6}\left(x^2+\sqrt{3}t\right)^2 - t\left(\left[1+\frac{\sqrt{3}}{3}\right]x^2+\frac{1}{2}C_1\right) - x\left(C_3-\frac{1}{2}C_1x\right)+C_2$ for $C_1=1$, $C_2=0$, and $C_3=0$.



(a) Function $\frac{\partial u}{\partial x}(t,x) = \frac{2}{3}x^3 + C_1 x - 2xt$ for $C_1 = 1$, $C_2 = 0$, and $C_3 = 0$.



(b) Function $x_m(t)$.

Figure 4.6: Example **Inhomogeneous transformation of BM**.

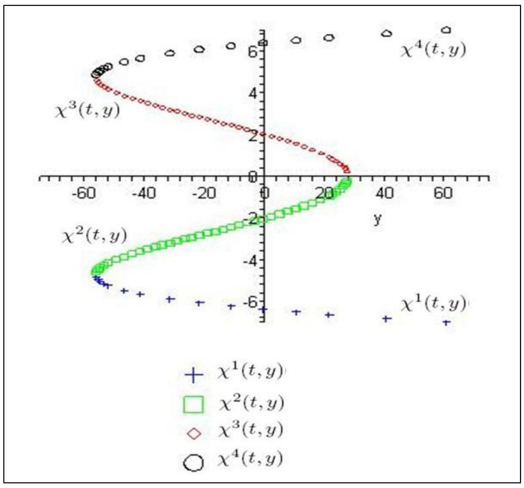


Figure 4.7: Inverses of the function $u(t, x)$.

Example 101 ***Solution of SDEs and their moments.*** *The method presented here can also be used to find the moments of solutions of SDEs. We recall an example appearing in [53].*

Let $(B_t, 0 \leq t \leq T)$ a standard Brownian motion and take the following SDE

$$X_t = x + \int_0^t \sqrt{1 + X_s^2} dB_s + \frac{1}{2} \int_0^t X_s ds. \tag{4.16}$$

As the coefficients in Equation (4.16) are Lipschitz (i.e., $|\sqrt{1+y^2}| - |\sqrt{1+z^2}| + \frac{1}{2}|y - z| \leqslant \frac{3}{2}|y - z|$), so the equation has a unique strong solution for every $x \in \mathbb{R}$. We check that this equation is obtained via a change of variables, this is, if there exist $\varphi \in \mathcal{C}^2$ such that $X_t = \varphi(B_t)$, then by Itô's lemma

$$d(\varphi(B_t)) = \varphi'(B_t) dB_t + \frac{1}{2} \varphi''(B_t) dt$$

and collecting terms we have

$$\varphi'(B_t) = \sqrt{1 + \varphi^2(B_t)} \quad \textit{and} \quad \varphi''(B_t) = \varphi(B_t)$$

and as for each $\omega \in \Omega$, $B_t(\omega) \in (-\infty, \infty)$, we can replace in the previous equation by $y \in \mathbb{R}$ and solve the corresponding deterministic equation, whose solution is

$$\varphi(y) = \sinh(y + c).$$

Now substituting back into our original SDE and using the initial value we have

$$X_t = \sinh(B_t + arcsinh(x)).$$

In order to find the n-th moment for X_t, and as the function $\sinh(y)$ *is a diffeomorphism of* $\mathbb{R}$*, as shown in Figure 4.8, we can use Part 1 of Corollary 84 (homogeneous diffeomorphism) to find its transition probability density and then its n-th moment. Thus,*

$$p^X(t_0, x_0; t, x) = \frac{1}{\sqrt{2\pi(x^2+1)(t-t_0)}} \exp\left\{\frac{-\left[arcsinh(x) - arcsinh(x_0)\right]^2}{2(t-t_0)}\right\}$$

and

$$\begin{aligned}
\mathbb{E}[X_t^n | X_{t_0} = x_0] &= \int_{-\infty}^{\infty} x^n p^X(t_0, x_0; t, x) dx \\
&= \int_{-\infty}^{\infty} \frac{x^n}{\sqrt{2\pi(x^2+1)(t-t_0)}} \exp\left\{\frac{-\left[arcsinh(x) - arcsinh(x_0)\right]^2}{2(t-t_0)}\right\} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2^n\sqrt{2\pi(t-t_0)}} \left(\exp\{2U\} - 1\right)^n \exp\left\{-\frac{U^2}{2(t-t_0)} - nU\right\} dU
\end{aligned}$$

for $U = \log\left(\frac{y+\sqrt{y^2+1}}{y_0+\sqrt{y_0^2+1}}\right)$*, expanding the power, calculating the integrals and collecting terms the expression remains*

$$\begin{aligned}
\mathbb{E}[X_t^n | X_{t_0} = x_0] &= \lim_{y\to\infty} A(n,t,t_0) \sum_{i=0}^{n} (-1)^i \binom{n}{i} \exp\{2i(i-n)(t-t_0)\} C(n,t,t_0,i,y,y_0) \\
&= \frac{1}{2^n} \exp\left\{\frac{n^2(t-t_0)}{2}\right\} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \exp\{2i(i-n)(t-t_0)\} \qquad (4.17)
\end{aligned}$$

with

$$\begin{aligned}
A(n,t,t_0) &= \frac{1}{2^n} \exp\left\{\frac{n^2(t-t_0)}{2}\right\} \\
B(n,t,t_0,i,y,y_0) &= \frac{\log\left(\frac{y+\sqrt{y^2+1}}{y_0+\sqrt{y_0^2+1}}\right) - (n-2i)(t-t_0)}{\sqrt{t-t_0}} \\
C(n,t,t_0,i,y,y_0) &= \Phi\left[B(n,t,t_0,i,y,y_0)\right] - \Phi\left[B(n,t,t_0,i,-y,y_0)\right]
\end{aligned}$$

as $\lim_{y\to\infty} C(n,t,t_0,i,y,y_0) = 1$ *and* Φ *the standard cumulative normal distribution function.*

Denote by

$$\Theta_i^n \triangleq (-1)^i \binom{n}{i} \exp\{2i(i-n)(t-t_0)\} \frac{1}{2^n} \exp\left\{\frac{n^2(t-t_0)}{2}\right\},$$

we can further simplify Equation (4.17) using the following remarks:

- ***For*** n ***an odd number*** $(n = 2k+1)$

  The sum in Equation (4.17) has an even number of terms $(n+1 = 2k+2)$

  $\Theta^{2k+1}_{2(k-j)+1} < 0$ *for* $j = 0 \dots k$

  $\Theta^{2k+1}_{2(k-j)} > 0$ *for* $j = 0 \dots k$

  $\Theta^{2k+1}_{2k+1-j} = -\Theta^{2k+1}_{j}$ *for* $j = 0 \dots k$

  by the above properties we find that

$$\sum_{j=0}^{2k+1} \Theta^{2k+1}_{2k+1-j} = \sum_{j=0}^{k} \Theta^{2k+1}_{2k+1-2j} + \sum_{j=0}^{k} \Theta^{2k+1}_{2k-2j} = -\sum_{j=0}^{k} \Theta^{2k+1}_{2j} + \sum_{j=0}^{k} \Theta^{2k+1}_{2j} = 0.$$

- ***For*** n ***an odd number*** $(n = 2k)$

  The sum in Equation (4.17) has an odd number of terms $(n+1 = 2k+1)$

  $\Theta^{2k}_{2(k-j)-1} < 0$ *for* $j = 0 \dots k$

  $\Theta^{2k}_{2(k-j)} > 0$ *for* $j = 0 \dots k$

  $\Theta^{2k}_{2k-j} = \Theta^{2k}_{j}$ *for* $j = 0 \dots k$

  the following relation holds:

$$\Theta^{2k}_{j} + \Theta^{2k}_{j+1} = \Upsilon(k,j) \quad \textit{for } j = 0 \dots k$$

$$\textit{for } \Upsilon(k,j) \triangleq \Theta^{2k}_{j} \left\{ 1 - \tfrac{(n-j)}{(j+1)} \exp\left[2(2j-n+1)(t-t_0)\right] \right\}.$$

Then, the expression in Equation (4.17) simplifies to

$$\mathbb{E}[X^n_t | X_{t_0} = x_0] = \begin{cases} 1 & \textit{for} \quad n = 0 \\ 0 & \textit{for} \quad n = 2k+1 \\ 2\sum_{i=0}^{m-1} \Upsilon(k,i) + \Theta^{2k}_{k} & \textit{for} \quad n = 2k \quad \textit{and} \quad k = 2m \\ 2\sum_{i=0}^{m-1} \Upsilon(k,i) + \Upsilon(k,k-1) + \Theta^{2k}_{k+1} & \textit{for} \quad n = 2k \quad \textit{and} \quad k = 2m+1. \end{cases}$$

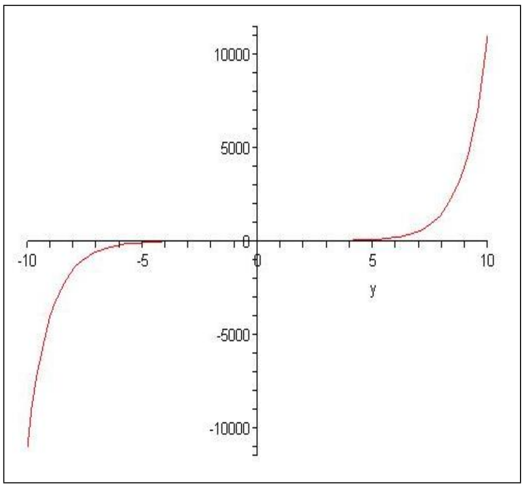


Figure 4.8: Example **Solutions of SDEs and their moments**: Function $\sinh(x)$.

4.3 Relation to the delta and delta-gamma approaches for risk of derivatives

One popular approach for computing the risk of positions on a derivative securities $C(t, S_t)$ is to use a first- or second-order approximation of the derivative function C around the current value S_t and for a small perturbation ΔS. This is equivalent to assuming that the risk horizon $T - t$ is small. In this section, we review these two approaches and make some comments with the relation to the method proposed in this thesis.

4.3.1 The delta-approach

For a small change in the underlying price ΔS, the first-order approximation to the price function $C(t, s)$ of the derivative security is taken as

$$C(t, s + \Delta s) \simeq C(t, s) + \frac{\partial C}{\partial s}(t, s)\Delta s + \varepsilon(1),$$

where $\varepsilon(1)$ is the first-order approximation error.[1]

Thus, for a risk horizon $T - t$ small enough, we can take $\Delta S = S_T - S_t$ and consider the firs-order approximation (delta-approach) to the position on the derivative security as

$$C(T, S_T) - C(t, S_t) \simeq \frac{\partial C}{\partial s}(t, S_t)(S_T - S_t) + \varepsilon(1).$$

[1]Note that the right way to take a Taylor's series approximation of the function C is by considering $C(t + \Delta t, S + \Delta S)$. Here, the time shift Δt is missing.

If in addition, we can assume that the factor $\frac{\partial C}{\partial s}(t, S_t)$ stays constant on $[t, T]$, then the risk of the position on the derivative security can be approximate as the risk of a linear function on the position on the underlying asset S_t, this is

$$\rho(C(T, S_T) - C(t, S_t)) \simeq \frac{\partial C}{\partial s}(t, S_t)\rho\left(S_T - S_t\right).$$

This is of course an easier problem to solve than the original one. As an example, consider as the risk measure ρ the VaR, and as the derivative security a European call option $C(t, S_t)$. Then

$$\mathrm{VaR}_\alpha(C(T, S_T) - C(t, S_t)) \simeq \frac{\partial C}{\partial s}(t, S_t)\mathrm{VaR}_\alpha\left(S_T - S_t\right).$$

This is, the delta-approximation to the VaR of a position on a call option is equal to the VaR of the underlying position $S_T - S_t$ times the delta of the call option.

4.3.2 The delta-gamma approach

When the first-order approximation of a derivative is not sufficient accurate, a second-order approximation may help.

For a small change in the underlying price ΔS, the second-order approximation to the price function $C(t, s)$ of the derivative security is taken as

$$C(t, s + \Delta s) \simeq C(t, s) + \frac{\partial C}{\partial s}(t, s)\Delta s + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}(t, s)\left(\Delta s\right)^2 + \varepsilon(2),$$

where $\varepsilon(2)$ is the second-order approximation error and assumed to be smaller, for sufficient small s, than the first-order error $\varepsilon(1)$.

Thus, for a risk horizon $T - t$ small enough, we can take $\Delta S = S_T - S_t$ and consider the second-order approximation (gamma-approach) to the position on the derivative security as

$$C(T, S_T) - C(t, S_t) \simeq \frac{\partial C}{\partial s}(t, s)(S_T - S_t) + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}(t, s)\left(S_T - S_t\right)^2.$$

In this case, we need to assume that the terms $\frac{\partial C}{\partial s}(t, S_t)$ and $\frac{\partial^2 C}{\partial s^2}(t, S_t)$ stay constant on $[t, T]$. Thus the risk of the position on the derivative security C is approximated as the risk of a quadratic function on the position on the underlying asset S_t, this is

$$\rho(C(T, S_T) - C(t, S_t)) \simeq \rho\left(\frac{\partial C}{\partial s}(t, s)(S_T - S_t) + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}(t, s)(S_T - S_t)^2\right).$$

One of the main drawbacks of the delta- and delta-gamma-approaches is the assumption that the movement in the underlying asset price is small (or equivalently small risk horizon $T - t$), and therefore the approximation may not be very accurate. Specially in the situation where the price function of the derivative security is far from linear or quadratic. The method presented in this chapter does not require to make the assumption of a small risk horizon $T - t$. But instead, it requires to solve a final value PDE, which, apart from few cases on the dynamics of the underlying process S_t and option prices functions $C(t, S_t)$, the PDE will not have explicit solution. In such a case, series expansion solutions or accurate numerical methods may be applied. Another alternative is to use Monte Carlo simulation, but this usually requires large computational capacity.

4.4 Risk measures for American derivatives

In the situation when the derivative security for which one needs to measure the risk is of American type, we can also formulate the the value of the risk measure (WCS, VaR and AVaR) as the solution of a PDE, but in this case, the system to solve will be a free-boundary problem with a final condition.

Chapter 5

Hedging and Derivative Pricing in the Robust ε-expected Shortfall Problem

5.1 Introduction

In this chapter, we analyse a variant of the robust version of the expected shortfall hedging problem:

For an initial capital $x \geq 0$, find a hedging strategy (x, π), $\pi \in A(x)$ with terminal value $X_T^{(x,\pi)}$ such that

$$\inf_{\pi \in A(x)} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\left(H_T - X_T^{(x,\pi)} \right)^+ \right], \tag{5.1}$$

in a continuous-time model consisting of two risky assets S_t and Y_t, $0 \leq t \leq T$ (given by Itô diffusions) and a risk-free bond B_t, $0 \leq t \leq T$. The asset S_t is assumed to be traded in the financial market but not Y_t.

We consider a random payoff H_T to be a function of the underlying process Y_t at time T, this is $H_T = H(Y_T)$ and the set of measures (priors) $\mathcal{P}$ to be a subset all equivalent probability measures $\mathcal{M}_e$.

The problem in (5.1) corresponds to the hedging problem for the WCS$_{\mathcal{P}}$ risk measure discussed in Chapter 1, Section 1.5.1.1. For the particular choice of priors $\mathcal{P} = \{\mathbb{P}\}$ and $\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}_a : \frac{d\mathbb{Q}}{d\mathbb{P}} \text{ is } \mathbb{P} \leq \frac{1}{\alpha}\}$ defined in Proposition 30, we recover the solution to the hedging problems corresponding to VaR$_\alpha$ and AVaR$_\alpha$, respectively.

In view of the fact that the theory for the primal-dual formulation to the robust versions of expected utility problems has only been recently developed in [82] and under the assumptions that the utility function is a strictly increasing and strictly concave function, we reformulate our original problem in (5.1) to fit with these assumptions by considering an ε-approximation of the shortfall utility function $(x)^+$ for $0 \leq \varepsilon \leq 1$ by considering the following problem:

$$\inf_{\pi \in A(x)} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[U_\varepsilon \left(H(Y_T) - X_T^{(x,\pi)} \right) \right]. \tag{5.2}$$

with

$$U_\varepsilon(x) = \varepsilon \log \left(\frac{1 + \exp\left\{-\frac{x}{\varepsilon}\right\}}{\exp\left\{-\frac{x}{\varepsilon}\right\}} \right).$$

This is, when $\varepsilon \longrightarrow 0$ we recover the original expected shortfall problem.

Due to the fact that the utility function $U_\varepsilon(x)$ is not separable in its variables, we are not able to solve explicitly in (5.2), but instead, we use a power series approximation in the dual variables. It turns out that the approximate solution to (5.2) is the solution to the corresponding robust version of a utility maximisation problem with exponential preferences ($U(x) = -\frac{1}{\gamma}e^{-\gamma x}$) for a preferences parameter $\gamma = \frac{1}{\varepsilon}$. Then the original expected shortfall problem recovered when $\varepsilon \longrightarrow 0$ will correspond to $\gamma \to \infty$. For the approximate problem, we analyse the cases with and without random endowment, and obtain an expression for the utility indifference bid price corresponding to the liability $H_T = H(Y_T)$.

5.2 The financial model

We consider an investment model of a single agent who manages her portfolio by investing in a bond and a risky asset S_t which is tradable in the market and we also consider a risky non-tradable asset Y_t.

The bond price B_t is given by

$$\mathrm{d}B_t = rB_t\mathrm{dt}, \quad B_0 = B \tag{5.3}$$

where $r \geq 0$ is the interest rate. The tradable risky asset is modelled as a diffusion process S_t solving

$$\mathrm{d}S_t = \mu S_t \mathrm{dt} + \sigma S_t \mathrm{dW}_t^s \tag{5.4}$$

with $S_0 = s_0 > 0$. The non-tradable risky process, which can be conceived as an "stochastic factor" is assumed to satisfy

$$\mathrm{d}Y_t = bY_t\mathrm{dt} + aY_t\mathrm{dW}_t^y \tag{5.5}$$

for $Y_0 = y$ and μ, b, σ and a constants. The processes W_t^s and W_t^y are Brownian motions correlated with correlation coefficient $\rho_{sy} \in [-1, 1]$.

The assumption of geometric Brownian motions for the dynamics for S_t and Y_t is basically to be able to have some explicit solutions, but this assumption can be relaxed.

It is convenient to express W^y as a linear combination of two independent Brownian motions W and W^s. Thus if

$$W_t^y = \rho_{sy} W_t^s + \bar{\rho_{sy}} W_t$$

for $\bar{\rho_{sy}} = \sqrt{1 - \rho_{sy}^2}$ the dynamics of Y remains

$$\mathrm{d}Y_t = bY_t \mathrm{dt} + \rho_{sy} a Y_t \mathrm{dW}_t^s + \bar{\rho_{sy}} a Y_t \mathrm{dW}_t$$

Remark 102 *Note that by the specific shape of the coefficients in the process S_t, the process Y_t does not depend on S_t. This assumption simplifies the computations.*

When $|\rho_{sy}| < 1$ we are in an incomplete market situation as the agent cannot trade in Y. If $\rho_{sy} = 1$ then we are in the complete market case and the coefficients in the SDEs for S_t and Y_t must be related as follows:[1]

$$b = r + \frac{a}{\sigma} (\mu - r) .$$

5.3 The wealth process

The investor starts at time t, with an initial capital x and re-balances her portfolio holdings by dynamically choosing at any time $s \in [t, T]$ and $0 \le t \le T$, the amounts (money) Π_s^0 and Π_s to be invested, respectively, in the bond and the risky asset S. Her total wealth process X_t satisfies the budget constraint

$$X_s = \Pi_s^0 + \Pi_s$$

and using the dynamics in (5.3) and (5.4) of the bond B and the risky asset S_t, the current wealth X_s satisfies the following controlled diffusion equation:

$$\mathrm{d}X_u = rX_u \mathrm{d}u + (\mu - r)\, \pi_u X_u \mathrm{d}u + \sigma \pi_u X_u \mathrm{d}W_u^s. \tag{5.6}$$

with initial value $X_t = x \ge 0,\ 0 \le t \le T$ (see [51, Ch. 5.8] for more detail on this). The quantity $\pi_t = \Pi_t / X_t$ represents the proportion of wealth invested in the risky asset.

In order to avoid arbitrage opportunities, the wealth process must also satisfy the usual state constraint,

$$X_u \ge 0 \text{ a.s. } t \le u \le T. \tag{5.7}$$

[1] The Sharpe ratios of the discounted price processes need to be equal to avoid arbitrage opportunities.

Assumption 103 *We assume the investor does not have the opportunity to consume part of her wealth nor to introduce any exogenous funds during the trading interval* $[t, T]$.

Apart from condition (5.7), a process π_s is considered to be admissible if it is $\mathcal{F}_s$-progressively measurable and satisfies the integrability condition $\mathbb{E}\left[\int_t^T \pi_s^2 ds\right] < \infty$ a.s. The set of admissible controls (or policies) given the initial capital x will be denoted by $\mathcal{A}(x)$.

5.4 Equivalent measures

Let us denote by $\mathcal{M}_e$ the set of measures equivalent[2] to $\mathbb{P}$, and by $\mathcal{PM}_b$ the set of progressively measurable process φ_t such that $\int_0^T \varphi_t^2 dt < \infty$ $\mathbb{P}$-a.s.

If $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ then there exists a vector process $\bar{\varphi}_t = (\varphi_{1t}, \varphi_{2t})$ whose components $\varphi_{1t}, \varphi_{2t} \in \mathcal{PM}_b$, $0 \le t \le T$ (called Girsanov kernels) are such that

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = D_t^{\bar{\varphi}} := \exp\left(\int_0^t \varphi_{1\tau}\mathrm{dW}_\tau^s - \frac{1}{2}\int_0^t \varphi_{1\tau}^2 d\tau + \int_0^t \varphi_{2\tau}\mathrm{dW}_\tau - \frac{1}{2}\int_0^t \varphi_{2\tau}^2 \mathrm{d}\tau\right). \tag{5.8}$$

Under the measure $\mathbb{Q}$ the processes $\tilde{W}_t^s$ and $\tilde{W}_t$ defined by

$$\begin{aligned} \tilde{W}_t^s &= W_t^s - \int_0^t \varphi_{1\tau}\mathrm{d}\tau \\ \tilde{W}_t &= W_t - \int_0^t \varphi_{2\tau}\mathrm{d}\tau \end{aligned}$$

are independent $\mathbb{Q}$- Brownian motions.

5.4.1 Local martingale measures

As S_t is the only traded asset, a measure $\mathbb{Q}$ given in (5.8) will be a risk-neutral measure if the discounted process $e^{-rt}S_t$ is a $\mathbb{Q}$-local martingale, but this is true if and only if

$$\varphi_{1t} = -\varrho \text{ with } \varrho := \frac{\mu - r}{\sigma}.$$

The quantity ϱ is called the "market price of risk".

In this case, the set $\mathcal{M}_e$ is in one-to-one correspondence with the set of integrands φ_{2t} in (5.8).

[2]Whenever a measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ will be denoted by $\mathbb{Q} \sim \mathbb{P}$.

Then under a risk-neutral measure $\mathbb{Q}$ the dynamics of our processes S_t and Y_t satisfy

$$\begin{aligned} \mathrm{d}S_t &= rS_t\mathrm{d}t + \sigma S_t\mathrm{d}\tilde{\mathrm{W}}_t^s \\ \mathrm{d}Y_t &= Y_t\left[b - \rho_{sy}a\varrho + \bar{\rho_{sy}}a\varphi_{2t}\right]\mathrm{d}t + aY_t\mathrm{d}\tilde{\mathrm{W}}_t^y \end{aligned}$$

where $\tilde{W}_t^y = \rho_{sy}\tilde{W}_t^s + \bar{\rho_{sy}}\tilde{W}_t$ is a Brownian motion under $\mathbb{Q}$. From the expression of the dynamics of Y_t under $\mathbb{Q}$, we see that Y_t can have arbitrary drift.

5.4.2 The minimal martingale measure

Denote by $\mathbb{Q}^0$ the risk-neutral measure corresponding to the special case in (5.8) when $\varphi_{2t} = 0$. Its Radon-Nikodym derivative remains

$$\frac{\mathrm{d}\mathbb{Q}^0}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \exp\left(-\int_0^t \varrho \mathrm{dW}_\tau^s - \frac{1}{2}\int_0^t \varrho^2 \mathrm{d}\tau\right). \tag{5.9}$$

The measure $\mathbb{Q}^0$ is called the minimal martingale measure, and it is the measure, that apart from making the discounted price process for the traded asset S a local-martingale, leaves unaffected the Brownian motion W.

5.5 The set of priors $\mathcal{P}$

Note that for a given hedging strategy (x, π), $\pi \in A(x)$, $x \geq 0$ the inner part of the robust hedging problem in (5.1),

$$\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[\left(H_T - X_T^{(x,\pi)}\right)^+\right],$$

is nothing else than computing WCS on a position $\left(H_T - X_T^{(x,\pi)}\right)^+$. Thus, similarly as in Section 3.3.5, this problem can be seen as a stochastic control problem, but here on two controls κ_{1t} and κ_{2t}, the Girsanov kernels (or equivalently the control variable is a two-dimensional vector κ_t). Based on similar arguments as in Section 3.3.5, in order to be able to formulate the problem as a solution of a PDE, we need to restrict the controls to a compact set. Therefore our class of priors will be given by[3]

$$\mathcal{P} := \left\{ \mathbb{Q} \sim \mathbb{P} \;\middle|\; \begin{array}{c} \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = D_t^\kappa \text{ as in (5.8)},\ \kappa = (\kappa_1, \kappa_2) \in \mathcal{K}, \\ \mathcal{K} \subset \mathbb{R}^2 \text{ fixed compact convex set} \\ \text{and } \kappa_1, \kappa_2 \in \mathcal{PM}_b \end{array} \right\}. \tag{5.10}$$

For simplicity in the notation, we make the following assumption.

[3] $\mathcal{PM}_b$ is the set of progressively measurable process φ_t such that $\int_0^T \varphi_t^2 dt < \infty$ $\mathbb{P}$-a.s.

Assumption 104 *Assume $r = 0$, or equivalently, that the dynamics of S_t, Y_t and X_t are in discounted terms.*

5.6 The robust ε-expected shortfall hedging problem

In this section, we return to our problem in (5.1), the robust version of the expected shortfall hedging problem, and give the basis for the reformulation as a usual utility maximisation problem with robust preferences.

Assume the investor needs to pay the random amount $H(Y_T)$ at time T for H a continuous positive function. For a hedging strategy (x, π), $\pi \in A(x)$ that define a final value wealth X_T at time T as in (5.6) and a set of prior models as in (5.10), the robust version of the expected shortfall hedging problem is

$$\inf_{\pi \in A(x)} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\left(H(Y_T) - X_T^{(x,\pi)} \right)^+ \right].$$

One first step to the reformulation of the problem is to highlight the dependence on the state of the utility function. This is, for each $H(Y_T)$ define the state-dependent function $U^{H,\omega}(x) = (H(Y_T)(\omega) - x)^+$, and write the problem (5.1) as[4]

$$\inf_{\pi \in A(x)} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[U^H \left(X_T^{(x,\pi)} \right) \right].$$

When $\mathcal{P} = \{\mathbb{P}\}$ (no robust preferences), convex duality methods for solving expected utility of final wealth problems have been widely used in the last decade (see [52, Ch. 3 and 5]). It involves to solve the so-called primal and dual problems (see Appendix B). In view that the theory for the primal-dual formulation to the robust versions of expected utility problems has only been recently developed in [82] and under the assumptions that the utility function is a *strictly increasing* and *strictly concave* function, we consider a closely related problem to (5.1) in order to fit into these assumption by considering an ε-approximation of the utility function $U^{H,\omega}(x)$ for $0 \leq \varepsilon \leq 1$ by taking

$$U_\varepsilon^{H,\omega}(x) = \varepsilon \log \left(\frac{1 + \exp\left\{ -\frac{(H(Y_T)(\omega) - x)}{\varepsilon} \right\}}{\exp\left\{ -\frac{(H(Y_T)(\omega) - x)}{\varepsilon} \right\}} \right).$$

In this way, the function $U_\varepsilon^{H,\omega}(x)$ is strictly increasing and strictly convex in x.

[4] The function $U^{H,\omega}(x)$ is increasing and convex in x but not strictly.

The reformulation of our problem for $U_\varepsilon^{H,\omega}(x)$ into the usual maximisation of expected utility of terminal wealth is analysed in the next section. Once reformulated the problem, our aim is to use stochastic control techniques to solve the dual and primal problems.

5.6.1 Reformulation of the problem

Define the function (see Figure 5.1(a))

$$U_\varepsilon(x) = \varepsilon \log \left(\frac{1 + \exp\left\{-\frac{x}{\varepsilon}\right\}}{\exp\left\{-\frac{x}{\varepsilon}\right\}} \right).$$

The robust ε-expected shortfall hedging problem is

$$\inf_{\pi \in A(x)} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[U_\varepsilon \left(H(Y_T) - X_T^{(x,\pi)} \right) \right]. \tag{5.11}$$

Define

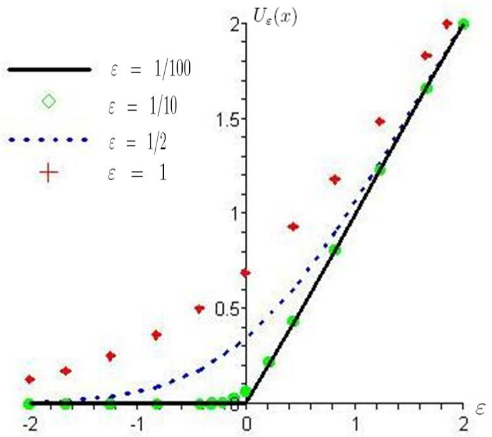
$$\tilde{U}_\varepsilon^H(x) = U_\varepsilon(H(Y_T)) - U_\varepsilon(H(Y_T) - x). \tag{5.12}$$

The function $\tilde{U}_\varepsilon^H(x)$ is strictly increasing and strictly concave in x (see Figure 5.1(b)). The hedging problem in (5.11) is equivalent to the following problem:

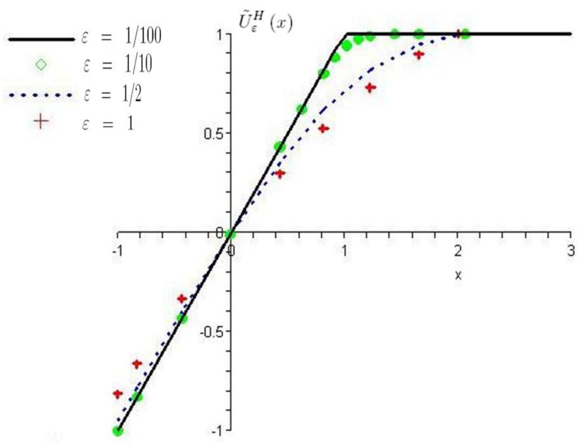
$$\sup_{\pi \in A(x)} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\tilde{U}_\varepsilon^H \left(X_T^{(x,\pi)} \right) \right] \tag{5.13}$$

$$= \sup_{\pi \in A(x)} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[U_\varepsilon(H(Y_T)) - U_\varepsilon \left(H(Y_T) - X_T^{(x,\pi)} \right) \right]. \tag{5.14}$$

The problem in (5.14) is the standard form of the problem treated in [82] (see Appendix B).



(a) The function $U_\varepsilon(x)$ for $\varepsilon = 1, 1/2, 1/10$ and $1/100$.



(b) The function $\tilde{U}_\varepsilon^H\,(x)$ for $\varepsilon = 1, 1/2, 1/10$ and $1/100$ and $h = 1$.

Figure 5.1:

5.6.2 Utility indifference pricing for the robust ε-ES hedging problem

The utility indifference buy (or bid) price p^b is the price at which the investor is indifferent (in the sense that her expected utility under optimal trading is unchanged) between paying nothing and not having to pay the claim $H(Y_T)$ at time T and paying p^b today in order to cover for the payment $H(Y_T)$ at time T. Assume the investor has an initial wealth x'. Define

$$R_H(x') = \sup_{\pi \in A(x')} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\tilde{U}_\varepsilon^H \left(X_T^{(x',\pi)} \right) \right]$$

where the supremum is taken over all wealths X_T which can be generated from the initial fortune x'. The utility indifference buy price p^b is the solution to

$$R_H(x' - p^b) = R_0(x').$$

Then, by solving the following two problems: 1) **Maximising utility with no random endowment**

$$\sup_{\pi \in A(x)} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\tilde{U}_\varepsilon^0 \left(X_T^{(x,\pi)} \right) \right] \tag{P1}$$

and 2) **Maximising utility with a random endowment**

$$\sup_{\pi \in A(x)} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\tilde{U}_\varepsilon^H \left(X_T^{(x,\pi)} \right) \right] \tag{P2}$$

for arbitrary initial capital x, we recover as a by-product the indifference bid price p^b.

5.6.3 Maximising utility with no random endowment

Assume the current time is zero. When there is no random endowment (no claim to be paid at time T), by (5.12) the term $\tilde{U}_\varepsilon^0 \left(X_T^{(x,\pi)} \right)$ in (P1) reduces to

$$U_\varepsilon(0) - U_\varepsilon \left(-X_T^{(x,\pi)} \right).$$

Define $\hat{U}_\varepsilon(x)$ by

$$\hat{U}_\varepsilon(x) := -U_\varepsilon(-x) = -\varepsilon \log \left(\frac{1 + \exp\left\{ \frac{x}{\varepsilon} \right\}}{\exp\left\{ \frac{x}{\varepsilon} \right\}} \right). \tag{5.15}$$

The function $\hat{U}_\varepsilon(x)$ is strictly increasing and strictly concave in x. The problem P1 is equivalent to

$$u(x) = \sup_{\pi \in A(x)} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\hat{U}_\varepsilon \left(X_T^{(x,\pi)} \right) \right], \tag{5.16}$$

plus a constant term of $\varepsilon \log 2$.

The main idea to solve (5.16) is to look at it from the perspective of a usual stochastic control problem with controls in a compact set. This can be accomplished by showing that all assumptions in [82] (see Appendix B) for the equality to the primal and dual problem are satisfied. The reformulation to the problem in (5.16) as a usual stochastic control problem results from working with the dual problem rather than with the primal problem itself. It turns out that the control variable is given by a vector formed of a triplet of Girsanov kernels that define equivalent probability measures to $\mathbb{P}$ on a compact set. This technique has already been used to solve utility maximisation problems in [13] for an exponential utility function with $\mathcal{P}=\{\mathbb{P}\}$ and recently in the robust case in [43] for a power utility function and in [42] for a logarithmic utility function with a penalty term.

In relation with utility maximisation problems including no traded assets but when no robust preferences are considered [39], [60], and [58] have treated the problem using exponential preferences. Also in incomplete markets, but in a stochastic volatility framework, [49] works with the HJB equation and the dual formulation to solve a problem of minimising expected shortfall.

We summarise the key results of the solution to the robust problem (5.16) in the following theorem.

Theorem 105 *Assume the current time is t, $0 \leq t \leq T$. The value function $u(t,x)$ of the robust utility maximisation problem (5.16) can be approximated by*

$$\begin{aligned} u^{approx}(t,x) &= -\varepsilon \exp\left\{-\frac{x}{\varepsilon}\right\} \exp\left\{-\frac{1}{2}\varrho^2(T-t)\right\} \times \\ & \exp\left\{-\varrho\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau}\, d\tau\right] - \frac{1}{2}\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau}^2\, d\tau\right]\right\}, \end{aligned}$$

with an error of

$$u(t,x) - u^{approx}(t,x) \leq \frac{1}{2}\left(1-\log(2)\right)\varepsilon \approx 0.1534264097\varepsilon.$$

Here $\hat{\kappa} = (\hat{\kappa}_1, \hat{\kappa}_2) \in \mathcal{K}$ is a pair of Girsanov kernels that solve

$$\min_{(\kappa_1,\kappa_2)\in\mathcal{K}} \mathbb{E}^{\kappa_2}\left[\int_t^T \left(\varrho + \kappa_{1\tau}\right)^2 d\tau\right].$$

The operator $\mathbb{E}^{\hat{\kappa}_2}[\cdot]$ represents the expected value under the measure $\mathbb{Q}^{\hat{\kappa}_2}$ given by

$$\frac{d\mathbb{Q}^{\hat{\kappa}_2}}{d\mathbb{P}}|_{\mathcal{F}_T} = Z_T^{\hat{\kappa}_2} := \mathcal{E}\left(-\int_0 \varrho\, dW_\tau^s + \int_0 \hat{\kappa}_{2\tau}\, dW_\tau\right)_T.$$

The corresponding approximate optimal strategy $\hat{\pi}^{approx}$ for the robust problem is

$$\hat{\Pi}_t^{approx} = \hat{\pi}_t^{approx} X_t^{x,\hat{\pi}} = \frac{\varepsilon}{\sigma}\left(\varrho + \hat{\kappa}_{1t}\right).$$

Define the measure $\hat{\mathbb{Q}} \in \mathcal{P}$ *via*

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}|_{\mathcal{F}_T} = D_T^{\hat{\kappa}} := \exp\left(\int_0^T \hat{\kappa}_{1\tau} dW_\tau^s - \frac{1}{2}\int_0^t \hat{\kappa}_{1\tau}^2 d\tau + \int_0^t \hat{\kappa}_{2\tau} dW_\tau - \frac{1}{2}\int_0^t \hat{\kappa}_{2\tau}^2 d\tau\right).$$

Then the pair $\left(\hat{\pi}_t^{approx}, \hat{\mathbb{Q}}\right)$ *is a saddle point for the problem*

$$u^{approx}(t,x) = \sup_{\pi \in \mathrm{A}(x)} \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[\mathcal{U}_\varepsilon\left(X_T^{(x,\pi)}\right)\right],$$

with $\mathcal{U}_\varepsilon(x) = -\varepsilon \exp\left\{-\frac{x}{\varepsilon}\right\}$.

The proof of the above theorem will be divided into several parts and developed through this section.

Remark 106 *The approximated value function* $u^{approx}(t,x)$ *solves the robust utility maximisation problem for exponential utility function. When* $\mathcal{P} = \{\mathbb{P}\}$ *(no robust preferences), we have* $(\hat{\kappa}_{1t},\ \hat{\kappa}_{2t}) = (0,0)$ *and we recover the usual solution (see [13], [60] and [39]).*

One of the key elements to the solution to the robust problem (5.16) is the use of the dual-primal relations and results developed in [82, Theorem 2.2] for robust utility maximisation problems (see Appendix B).

Lemma 107 *For the utility functions* $\hat{U}_\varepsilon(x) = -\varepsilon \log\left(\frac{1+\exp\left\{\frac{x}{\varepsilon}\right\}}{\exp\left\{\frac{x}{\varepsilon}\right\}}\right)$ *and* $\mathcal{U}_\varepsilon(x) = -\varepsilon\exp\left\{-\frac{x}{\varepsilon}\right\}$*, and the priors set* $\mathcal{P}$ *defined in (5.10), the assumptions in [82, Theorem 2.2] are satisfied.*

Proof. See Section B.4 in Appendix B on page 189. □

For simplicity in the notation, assume for this discussion that the current time is $t = 0$ and omit the time dependence in the primal value function u, i.e., we write only $u(x)$.

For any $\zeta \in \mathcal{PM}$ (progressively measurable process), define

$$Z_t^\zeta := \mathcal{E}\left(\int_0 -\varrho \mathrm{dW}_\tau^s + \int_0 \zeta_\tau \mathrm{dW}_\tau\right)_t. \tag{5.17}$$

Given $\mathbb{E}[Z_T] = 1$, each Z_t^ζ would correspond to a density process that defines an equivalent probability measure to $\mathbb{P}$.

By Lemma 107 and therefore using results in [82], the dual value function of the robust utility maximisation problem is given by (see Appendix B on page 189)

$$v(\lambda) := \inf_{\kappa\in\mathcal{K}} \inf_{\zeta\in\mathcal{PM}} \mathbb{E}\left[D_T^\kappa \hat{V}_\varepsilon\left(\lambda\frac{Z_T^\zeta}{D_T^\kappa}\right)\right], \ \lambda > 0 \tag{5.18}$$

where the the vector process κ and the set $\mathcal{K}$ are defined in (5.10), the parameter λ plays the role of a Lagrange multiplier in the optimisation problem and the function $\hat{V}(y)$ is the Legendre-Fenchel transform of $\hat{U}_\varepsilon(x)$ and given by[5]

$$\begin{aligned}\hat{V}_\varepsilon(y) &= \sup_{x\geq 0}\left(\hat{U}_\varepsilon(x) - yx\right)\\ &= \begin{cases}\infty & \text{if} \quad y \geq 1\\ -\varepsilon\log\left(\frac{1}{1-y}\right) - \varepsilon y\log\left(\frac{1-y}{y}\right) & \text{if} \quad 0 < y < 1.\end{cases}\end{aligned}$$

The primal and dual value functions are related as follows:

$$u(x) = \min_{\lambda>0}\left(v(\lambda) + \lambda x\right).$$

Assume for each $x > 0$ the solution to the above minimisation problem is well defined (it has a solution $0 < \lambda^{\min}(x) < \infty$, with $\left|v(\lambda^{\min}(x))\right| < \infty$), then the value function of the robust primal problem can be computed as

$$u(x) = v(\lambda^{\min}(x)) + \lambda^{\min}(x)x,$$

with $\lambda^{\min}(x)$ satisfying the relation

$$\frac{dv}{d\lambda}(\lambda^{\min}(x)) = -x.$$

As the expression

$$\hat{V}_\varepsilon\left(\lambda\frac{Z_T^\zeta}{D_T^\kappa}\right) = \begin{cases}\infty & \text{if} \quad \lambda\frac{Z_T^\zeta}{D_T^\kappa} \geq 1\\ -\varepsilon\log\left(\frac{1}{1-\lambda\frac{Z_T^\zeta}{D_T^\kappa}}\right) - \varepsilon\lambda\frac{Z_T^\zeta}{D_T^\kappa}\log\left(\frac{1-\lambda\frac{Z_T^\zeta}{D_T^\kappa}}{\lambda\frac{Z_T^\zeta}{D_T^\kappa}}\right) & \text{if} \quad 0 < \lambda\frac{Z_T^\zeta}{D_T^\kappa} < 1,\end{cases}$$

in (5.18) is not separable of variables in λ, in the next section we find a series approximation in λ to $\lambda^{\min}(x)$ and set conditions for the finiteness of the approximated primal value function.

[5]The maximum as attained at

$$x^* = \begin{cases}0 & \text{if} \quad z \geq 1\\ \varepsilon\log\left(\frac{1-z}{z}\right) & \text{if} \quad 0 < z < 1.\end{cases}$$

5.6.3.1 Optimal Lagrange multiplier

Define the function

$$\phi(\lambda,\xi) := \hat{V}_\varepsilon\left(\lambda \exp(\xi)\right),\ \xi \in \mathbb{R}$$

then we have

$$v(\lambda) = \inf_{\kappa\in\mathcal{K}} \inf_{\zeta\in\mathcal{PM}} \mathbb{E}\left[D_T^\kappa \phi\left(\lambda, \log\left(\frac{Z_T^\zeta}{D_T^\kappa}\right)\right)\right].$$

By Corollary 136 in the Appendix B, on one hand, the function $v(\lambda)$ is continuously differentiable on $(0,\infty)$ and strictly convex. On the other hand, the optimisation and expectation operator on the right-hand side does not depend on λ, the dependence is only on the function $\phi(\lambda,\xi)$. Thus, we can differentiate as

$$\frac{\mathrm{d}v}{\mathrm{d}\lambda}(\lambda) = \inf_{\kappa\in\mathcal{K}} \inf_{\zeta\in\mathcal{PM}} \mathbb{E}\left[D_T^\kappa \frac{\partial\phi}{\partial\lambda}\left(\lambda, \log\left(\frac{Z_T^\zeta}{D_T^\kappa}\right)\right)\right].$$

It is enough to analyse the behaviour of the function $\phi(\lambda,\cdot)$ in order to conclude for the function $v(\lambda)$.

By the definition of $\hat{V}_\varepsilon(z)$ we have for $\xi \in \mathbb{R}$ (see Figure 5.2 and Figure 5.3)

$$\phi(\lambda,\xi) = \begin{cases} \infty & \text{if} \quad \lambda\exp(\xi) \geq 1 \\ -\varepsilon \log\left(\frac{1}{1-\lambda\exp(\xi)}\right) - \varepsilon\lambda\exp(\xi)\log\left(\frac{1-\lambda\exp(\xi)}{\lambda\exp(\xi)}\right) & \text{if} \quad 0 < \lambda\exp(\xi) < 1, \end{cases}$$

and

$$\frac{\partial\phi}{\partial\lambda}(\lambda,\xi) = \begin{cases} \infty & \text{if} \quad \lambda\exp(\xi) \geq 1 \\ -\varepsilon\exp(\xi)\log\left(\frac{1-\lambda\exp(\xi)}{\lambda\exp(\xi)}\right) & \text{if} \quad 0 < \lambda\exp(\xi) < 1. \end{cases} \tag{5.19}$$

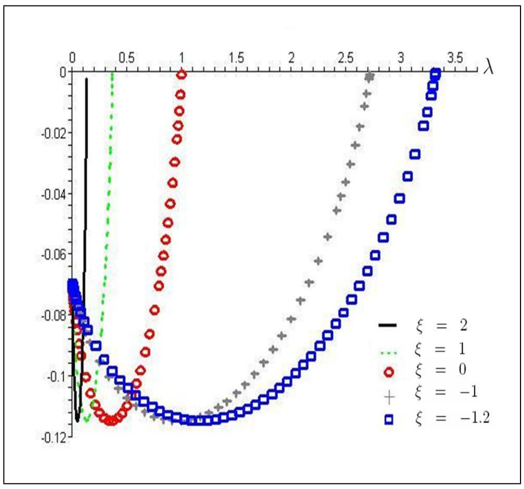
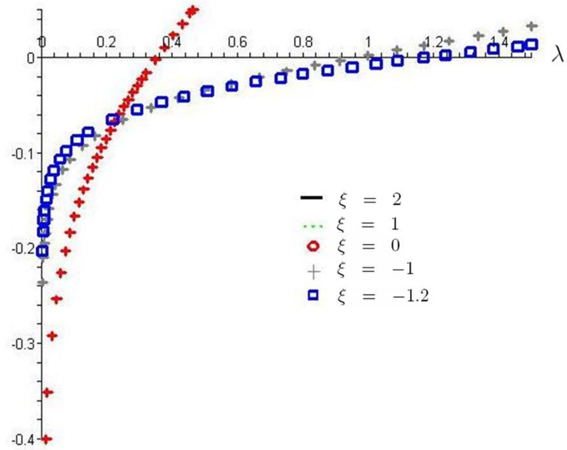
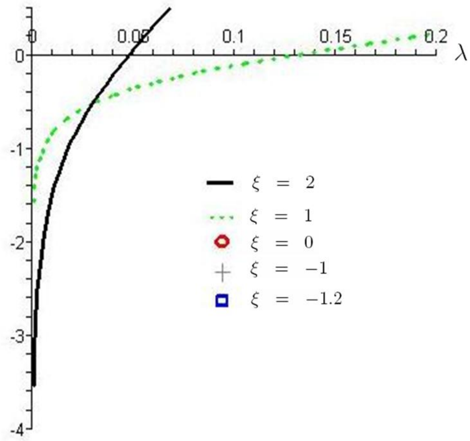


Figure 5.2: The function $\phi(\lambda,\xi)$ for $\xi = -1.2, -1, 0, 1$ and 2 and $\varepsilon = 1/10$.



(a)



(b)

Figure 5.3: The function $\frac{d\phi}{d\lambda}(\lambda, \xi)$ for $\xi = -1.2, -1, 0, 1$ and 2 and $\varepsilon = 1/10$.

On $0 < \lambda \exp(\xi) < 1$, the map $\frac{\partial\phi}{\partial\lambda}(\cdot, \xi)$ is strictly increasing and has an inflexion point at

$$\lambda^*(\xi) := \frac{1}{2}\exp(-\xi) < \exp(-\xi). \tag{5.20}$$

One can easily check that $\lambda^*(\xi)$ is a minimum point for the map $\phi(\cdot, \xi)$.[6] Thus $\frac{d\phi}{d\lambda}(\lambda, \xi) \in (-\infty, 0]$ for λ on the domain $(0, \lambda^*(\xi)]$. Furthermore, the threshold $\lambda^*(\xi)$

[6] The function $\lambda^*(\xi)$ is a minimum point for $\phi(\cdot, \xi)$ as $\frac{d\phi}{d\lambda}(\lambda, \xi)|_{\lambda=\lambda^*(\xi)} = 0$ and $\frac{d^2\phi}{d\lambda^2}(\lambda, \xi)|_{\lambda=\lambda^*(\xi)} = 4\varepsilon\exp\{2\xi\} > 0$.

is decreasing in $\xi \in \mathbb{R}$.

Now, if we define

$$\Psi_{\kappa,\zeta}(\lambda) = \mathbb{E}\left[D_T^\kappa \frac{\partial \phi}{\partial \lambda}\left(\lambda, \log\left(\frac{Z_T^\zeta}{D_T^\kappa}\right)\right)\right],$$

and substituting from (5.19) we obtain

$$\Psi_{\kappa,\zeta}(\lambda) = \mathbb{E}\left[-\varepsilon Z_T^\zeta \left\{\log\left(\frac{1-\lambda\left(\frac{Z_T^\zeta}{D_T^\kappa}\right)}{\lambda\left(\frac{Z_T^\zeta}{D_T^\kappa}\right)}\right)\right\}\right]. \tag{5.21}$$

The properties of $\frac{\partial \phi}{\partial \lambda}(\cdot, \xi)$ are inherited to the map $\Psi_{\kappa,\zeta}(\cdot)$, this is, $\Psi_{\kappa,\zeta}(\cdot)$ is strictly increasing and has an inflexion point at

$$\lambda^*_{\kappa,\zeta} := \mathbb{E}\left[\lambda^*\left(\log\left(\frac{Z_T^\zeta}{D_T^\kappa}\right)\right)\right] = \frac{1}{2}\mathbb{E}\left[\frac{D_T^\kappa}{Z_T^\zeta}\right] < \mathbb{E}\left[\frac{D_T^\kappa}{Z_T^\zeta}\right].$$

Thus on $(0, \lambda^*_{\kappa,\zeta}]$, the function $\Psi_{\kappa,\zeta}(\lambda) \in (-\infty, 0]$ is strictly convex and strictly increasing. Moreover, $\Psi_{\kappa,\zeta}(\lambda)$ is related to the dual value function of the robust problem as

$$\frac{\mathrm{d}v}{\mathrm{d}\lambda}(\lambda) = \inf_{\kappa \in \mathcal{K}} \inf_{\zeta \in \mathcal{PM}} \Psi_{\kappa,\zeta}(\lambda).$$

Finally, by the properties of $\Psi_{\kappa,\zeta}(\lambda)$ and for each $x \geq 0$ we can conclude that the equation

$$\inf_{\kappa \in \mathcal{K}} \inf_{\zeta \in \mathcal{PM}} \Psi_{\kappa,\zeta}(\lambda) = -x,$$

has a unique finite solution $\lambda^{\min}(x)$ with $0 < \lambda^{\min}(x) < \lambda^*_{\kappa,\zeta}$.

Let us now analyse how to compute $\lambda^{\min}(x)$. One one hand, note that if $\int_0^T \zeta_t dt < \infty$ then Z_T^ζ and $\log\left(\frac{D_T^\kappa}{Z_T^\zeta}\right)$ will be bounded, which would imply that $\mathbb{E}\left[Z_T^\zeta\right] = 1$, $\mathbb{E}\left[\frac{D_T^\kappa}{Z_T^\zeta}\right] = 1$ and $\mathbb{E}\left[\log\left(\frac{D_T^\kappa}{Z_T^\zeta}\right)\right] < \infty$. On the other hand, by the shape of $\Psi_{\kappa,\zeta}$ in (5.21) we cannot find directly an explicit expression for $\lambda^{\min}(x)$. Thus if $\int_0^T \zeta_t dt < \infty$, then

$$0 < \lambda^{\min}(x) < \lambda^*_{\kappa,\zeta} < \mathbb{E}\left[\frac{D_T^\kappa}{Z_T^\zeta}\right] = 1,$$

and as the function inside the expectation operator in (5.21) is of logarithmic type, then we could use a series expansion approximation on λ on the region $(0, \infty)$. This is, using

$$\log\left(\frac{1-\lambda z}{\lambda z}\right) \approx -\log(z) - \log(\lambda) + O(\lambda),$$

we have

$$\begin{aligned}\Psi_{\kappa,\zeta}(\lambda) &\approx \mathbb{E}\left[-\varepsilon Z_T^{\zeta}\left\{-\log\left(\frac{Z_T^{\zeta}}{D_T^{\kappa}}\right)-\log\left(\lambda\right)\right\}\right]\\ &= \varepsilon\mathbb{E}\left[Z_T^{\zeta}\log\left(\frac{Z_T^{\zeta}}{D_T^{\kappa}}\right)\right]+\varepsilon\left\{\log(\lambda)\right\}\mathbb{E}\left[Z_T^{\zeta}\right]\\ &= \varepsilon\mathbb{E}^{\zeta}\left[\log\left(\frac{Z_T^{\zeta}}{D_T^{\kappa}}\right)\right]+\varepsilon\left\{\log(\lambda)\right\}\mathbb{E}\left[Z_T^{\zeta}\right].\end{aligned}$$

In such a case, the function $\lambda^{\min}(x)$ can be approximated by

$$\lambda^{\min}(x)\approx\lambda^{\min approx}(x):=\exp\left\{-\frac{x}{\varepsilon}\right\}\exp\left\{-\inf_{\kappa\in\mathcal{K}}\inf_{\zeta\in\mathcal{PM}}\Lambda_{\kappa,\zeta}\right\}, \tag{5.22}$$

with

$$\Lambda_{\kappa,\zeta}:=\mathbb{E}^{\zeta}\left[\log\left(\frac{Z_T^{\zeta}}{D_T^{\kappa}}\right)\right]. \tag{5.23}$$

The following lemma supports the assumption that it is enough to take processes ζ_t with $\int_0^T\zeta_t^2dt$ bounded.

Define

$$\mathcal{PM}_b:=\left\{\zeta\in\mathcal{PM}:\int_0^T\zeta_t^2dt \text{ is } \mathbb{P}\text{-a.s. bounded}\right\}.$$

Lemma 108 *For fixed $\kappa\in\mathcal{K}$ we have*

$$\inf_{\zeta\in\mathcal{PM}}\Lambda_{\kappa,\zeta}=\inf_{\zeta\in\mathcal{PM}_b}\Lambda_{\kappa,\zeta}.$$

Proof. Firstly, note that for $\zeta\in\mathcal{PM}$, and under the measure $\mathbb{Q}^{\zeta}$ given by $\frac{d\mathbb{Q}^{\zeta}}{d\mathbb{P}}|_{\mathcal{F}_t}=Z_t^{\zeta}$, and Z_t^{ζ} defined in (5.17) and using D_t^{κ} given as in (5.8) the random variable $\frac{Z_T^{\zeta}}{D_T^{\kappa}}$ has the form

$$\frac{Z_T^{\zeta}}{D_T^{\kappa}}=\exp\left\{\begin{array}{c}-\int_0^T\left(\varrho+\kappa_{1\tau}\right)d\breve{W}_{\tau}^{s}+\frac{1}{2}\int_0^T\left(\varrho+\kappa_{1\tau}\right)^2d\tau\\ +\\ \int_0^T\left(\zeta_{\tau}-\kappa_{2\tau}\right)d\breve{W}_{\tau}+\frac{1}{2}\int_0^T\left(\zeta_{\tau}-\kappa_{2\tau}\right)^2d\tau\end{array}\right\},$$

with $\breve{W}_t^s$ and $\breve{W}_t$ two $\mathbb{Q}^{\zeta}$-Brownian motions given by

$$\begin{aligned}\breve{W}_t^s &= W_t^s+\int_0^t\varrho d\tau\\ \breve{W}_t &= W_t-\int_0^t\zeta_{\tau}d\tau.\end{aligned}$$

Thus,

$$\begin{aligned}\Lambda_{\kappa,\zeta} \quad : &= \mathbb{E}^{\zeta}\left[\log\left(\frac{Z_T^{\zeta}}{D_T^{\kappa}}\right)\right] \\ &= \mathbb{E}^{\zeta}\left[\frac{1}{2}\int_0^T (\varrho+\kappa_{1\tau})^2\,\mathrm{d}\tau + \frac{1}{2}\int_0^T (\zeta_\tau-\kappa_{2\tau})^2\,\mathrm{d}\tau\right].\end{aligned}$$

By the expression of $\Lambda_{\kappa,\zeta}$ we have for any $\kappa_2 \in \mathcal{PM}_b$ and any $\zeta \in \mathcal{PM}$ either

$$0 < \Lambda_{\kappa,\kappa_2} = \Lambda_{\kappa,\kappa_2}\cdot 1_{\{\kappa_2\in\mathcal{PM}_b\}} \leq \Lambda_{\kappa,\zeta} \leq \Lambda_{\kappa,0},$$

or

$$0 < \Lambda_{\kappa,\kappa_2}\Lambda_{\kappa,\kappa_2}\cdot 1_{\{\kappa_2\in\mathcal{PM}_b\}} \leq \Lambda_{\kappa,0} \leq \Lambda_{\kappa,\zeta}.$$

Then

$$0 \leq \inf_{\kappa_2\in\mathcal{PM}_b}\Lambda_{\kappa,\kappa_2} \leq \inf_{\zeta\in\mathcal{PM}}\Lambda_{\kappa,\zeta}.$$

On the other hand, as $\mathcal{PM}_b \subseteq \mathcal{PM}$ we have

$$\inf_{\zeta\in\mathcal{PM}}\Lambda_{\kappa,\zeta} \leq \inf_{\kappa_2\in\mathcal{PM}_b}\Lambda_{\kappa,\kappa_2} \leq \Lambda_{\kappa,0}.$$

Putting this both conditions together we obtain

$$0 \leq \inf_{\zeta\in\mathcal{PM}}\Lambda_{\kappa,\zeta} = \inf_{\kappa_2\in\mathcal{PM}_b}\Lambda_{\kappa,\kappa_2} \leq \Lambda_{\kappa,0}.$$

□

5.6.3.2 An approximation to the dual and primal value functions

In this section, we compute approximations to the primal value function $u(x)$ and the dual value function $v(\lambda)$ using a similar series expansion as in previous section.

Define

$$\phi^{approx}(\lambda,\xi) := -\varepsilon\lambda\exp\{\xi\}\left(1-\xi-\log(\lambda)\right), \tag{5.24}$$

then the series expansion for $\phi(\lambda,\xi)$ is given by

$$\phi(\lambda,\xi) \approx \phi^{approx}(\lambda,\xi) + O(\lambda^2).$$

The approximation error between the function $\phi(\lambda,\xi)$ and the series expansion approximation $\phi^{approx}(\lambda,\xi)$ only depends on ε, as it is stated in the following lemma. This property is important as one may use ε as a parameter to control the error and get the desired accuracy.

Lemma 109 *For any* $\xi \in \mathbb{R}$*, any* $0 < \lambda < \lambda^*(\xi)$ *and* $\varepsilon \geq 0$*, the maps* $\phi(\cdot, \xi)$ *and* $\phi^{approx}(\cdot, \xi)$ *are strictly decreasing and strictly convex. Furthermore,*

$$0 \leq \phi(\lambda, \xi) - \phi^{approx}(\lambda, \xi) \leq \phi(\lambda^*(\xi), \xi) - \phi^{approx}(\lambda^*(\xi), \xi) = \frac{1}{2}(1 - \log(2))\,\varepsilon \approx 0.1534264097\varepsilon.$$

Proof. The fact that the function $\phi(\cdot, \xi)$ is strictly decreasing and strictly convex follows from the properties of the function $\hat{V}_\varepsilon(z)$. That $\phi^{approx}(\cdot, \xi)$ satisfies similar properties can be directly checked by its definition.

In order to prove the inequalities, note that $\lim_{\lambda \downarrow 0^+} \phi(\lambda, \xi) = \lim_{\lambda \downarrow 0^+} \phi^{approx}(\lambda, \xi) = -\varepsilon \log(2)$. The function $\phi(\cdot, \xi)$ attains its minimum at $\lambda^*(\xi) = \frac{1}{2}\exp(-\xi)$ and the function $\phi^{approx}(\cdot, \xi)$ at $\lambda^{approx}(\xi) = e\exp(-\xi)$. But $\lambda^*(\xi) < \lambda^{approx}(\xi)$ for any $\xi \in \mathbb{R}$. On the other hand, for any $\xi \in \mathbb{R}$, any $0 < \lambda < \lambda^*(\xi)$ and $\varepsilon \geq 0$ we have

$$\frac{\partial \phi^{approx}}{\partial \lambda}(\lambda, \xi) \leq \frac{\partial \phi}{\partial \lambda}(\lambda, \xi),$$

which proves the left-hand side of the inequality. This also suggests that the maximum gap between the functions is reached at $\lambda^*(\xi)$. Direct substitution of $\lambda^*(\xi)$ in $\phi(\lambda, \xi)$ and $\phi^{approx}(\lambda, \xi)$ show that the difference $\phi(\lambda, \xi) - \phi^{approx}(\lambda, \xi)$ does not depend on ξ nor on λ for $0 < \lambda < \lambda^*(\xi)$. This concludes the proof. $\square$

Recall that the dual value function $v(\lambda)$ to the robust problem is given by

$$v(\lambda) = \inf_{\kappa \in \mathcal{K}} \inf_{\zeta \in \mathcal{PM}_b} \mathbb{E}\left[D_T^\kappa \phi\left(\lambda, \log\left(\frac{Z_T^\zeta}{D_T^\kappa}\right)\right)\right].$$

We use the series expansion of $\phi(\lambda, \xi)$ to approximate the term inside the expectation operator in $v(\lambda)$ as follows.

$$\begin{aligned}
\mathbb{E}\left[D_T^\kappa \phi\left(\lambda, \log\left(\frac{Z_T^\zeta}{D_T^\kappa}\right)\right)\right] &\approx \mathbb{E}\left[D_T^\kappa \phi^{approx}\left(\lambda, \log\left(\frac{Z_T^\zeta}{D_T^\kappa}\right)\right)\right] \\
&= \mathbb{E}\left[-\varepsilon\lambda Z_T^\zeta\left\{\left(1 - \log\left(\frac{Z_T^\zeta}{D_T^\kappa}\right) - \log(\lambda)\right)\right\}\right] \\
&= -\varepsilon\lambda(1 - \log(\lambda)) + \varepsilon\lambda\Lambda_{\kappa,\zeta},
\end{aligned}$$

with $\Lambda_{\kappa,\zeta}$ as defined in (5.23). Then the approximation to the dual value function $v(\lambda)$ is

$$v(\lambda) \approx -\varepsilon\lambda(1 - \log(\lambda)) + \varepsilon\lambda\left\{\inf_{\kappa \in \mathcal{K}} \inf_{\zeta \in \mathcal{PM}_b} \Lambda_{\kappa,\zeta}\right\} =: v^{approx}(\lambda). \tag{5.25}$$

We can now use the above approximation evaluated at $\lambda^{\min approx}(x)$ to obtain an expression for the approximation to the primal value function $u(x)$ to the robust problem. This is,

$$u^{approx}(x) = -\varepsilon \exp\left\{-\frac{x}{\varepsilon}\right\} \exp\left\{-\inf_{\kappa\in\mathcal{K}} \inf_{\zeta\in\mathcal{PM}_b} \Lambda_{\kappa,\zeta}\right\}. \tag{5.26}$$

5.6.3.3 The solution to the dual problem

In this section, we obtain the solution to

$$\inf_{\kappa\in\mathcal{K}} \inf_{\zeta\in\mathcal{PM}_b} \Lambda_{\kappa,\zeta}. \tag{5.27}$$

In order to capture the dynamical behaviour of the problem, assume now the starting time is $t \in [0, T]$.

Using the definition of $\Lambda_{\kappa,\zeta}$ in (5.23), take

$$\begin{aligned} J_t^{\kappa,\zeta} \quad : &= \mathbb{E}^{\zeta}\left[\log\left(\frac{Z_T^{\zeta}}{D_T^{\kappa}}\right)\right] \\ &= \mathbb{E}^{\zeta}\left[\frac{1}{2}\int_t^T (\varrho + \kappa_{1\tau})^2 \,\mathrm{d}\tau + \frac{1}{2}\int_t^T (\zeta_\tau - \kappa_{2\tau})^2 \,\mathrm{d}\tau\right]. \end{aligned}$$

The problem is to find processes $(\kappa, \zeta) \in \mathcal{K} \times \mathcal{PM}_b$ that solves

$$\inf_{\kappa\in\mathcal{K}} \inf_{\zeta\in\mathcal{PM}_b} J_t^{\kappa,\zeta}. \tag{5.28}$$

Assume $(\hat{\kappa}, \hat{\zeta})$ is optimal in (5.28), then we recover the solution to (5.27) as $\Lambda_{\hat{\kappa},\hat{\zeta}} = J_0^{\hat{\kappa},\hat{\zeta}}$. And the approximated value function $u^{approx}(x)$ to the primal problem in (5.16) for utility maximisation with zero random endowment is given by

$$u^{approx}(x) = -\varepsilon \exp\left\{-\frac{x}{\varepsilon}\right\} \exp\left\{-J_0^{\hat{\kappa},\hat{\zeta}}\right\}.$$

Note that the dual problem in (5.28) does not depend on any of the dynamics S_t, Y_t nor X_t, and the solution can be easily characterised as in the following proposition.

Proposition 110 *There exists $(\hat{\kappa}, \hat{\zeta}) \in \mathcal{K} \times \mathcal{PM}_b$, which attains the minimum for the dual problem (5.28). Furthermore, such a pair $(\hat{\kappa}, \hat{\zeta})$ is characterised by the fact that $\hat{\kappa}_{1t}$ and $\hat{\kappa}_{2t}$ solves*

$$\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T (\varrho + \hat{\kappa}_{1\tau})^2 d\tau\right] = \inf_{(\kappa_1,\kappa_2)\in\mathcal{K}} \mathbb{E}^{\kappa_2}\left[\int_t^T (\varrho + \kappa_{1\tau})^2 d\tau\right], \tag{5.29}$$

and $\hat{\zeta}_t = \hat{\kappa}_{2t}$*. Thus the value function to the dual problem in (5.28) is given by*

$$\begin{aligned} J_t^{\hat{\kappa},\hat{\zeta}} &= \frac{1}{2}\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T (\varrho + \hat{\kappa}_{1\tau})^2 \, d\tau\right] \\ &= \frac{1}{2}\varrho^2(T-t) + \varrho\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau}\, d\tau\right] + \frac{1}{2}\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau}^2\, d\tau\right]. \end{aligned}$$

Proof. By the compactness of the set $\mathcal{K}$ and as $\zeta \in \mathcal{PM}_b$, the existence of $(\hat{\kappa}, \hat{\zeta}) \in \mathcal{K} \times \mathcal{PM}_b$ is guaranteed. And by Lemma 108 (see proof) we know that $\Lambda_{\kappa,\zeta} \geq \Lambda_{\kappa,\kappa_2}$ for all $(\kappa_1, \kappa_2) \in \mathcal{K}$ and any $\zeta \in \mathcal{PM}_b$. Then the original problem in (5.28) is reduced to

$$\inf_{(\kappa_1,\kappa_2)\in\mathcal{K}} \mathbb{E}^{\kappa_2}\left[\int_t^T (\varrho + \kappa_{1\tau})^2 d\tau\right].$$

This proves the characterisation to the optimal solution. $\square$

Remark 111 *Define the measures* $\hat{\mathbb{Q}}$ *by*

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}|_{\mathcal{F}_T} = D_T^{\hat{\kappa}}.$$

In the case the set $\mathcal{K}$ *is a rectangle with deterministic edges of the form* $[k_1^-, k_1^+] \times [k_2^-, k_2^+]$, $k_1^-, k_1^+, k_2^-, k_2^+ \in \mathbb{R}$*, the optimal Girsanov kernels are*

$$\hat{\kappa}_1 = \begin{cases} \kappa_1^+ & \text{if } \kappa_1^+ < -\varrho \\ -\varrho & \text{if } \kappa_1^- \leq -\varrho \leq \kappa_1^+ \\ \kappa_1^- & \text{if } \kappa_1^- > -\varrho \end{cases}$$

and $\hat{\kappa}_2$ *is any* $\hat{\kappa}_2 \in \mathcal{PM}_b$ *with* $(\hat{\kappa}_1, \hat{\kappa}_2) \in \mathcal{K}$ *and* $\hat{\zeta}_1 = \hat{\kappa}_2$*. The dual value function in (5.28) is given by*

$$J_t^{\hat{\kappa},\hat{\zeta}} = \frac{1}{2}(\varrho + \hat{\kappa}_1)^2 (T-t).$$

Remark 112 *When* $\mathcal{P} = \{\mathbb{P}\}$ *(no robust preferences), we have* $(\hat{\kappa}_{1t}, \hat{\kappa}_{2t}) = (0,0)$ *or equivalently* $\frac{d\mathbb{Q}^{\hat{\kappa}}}{d\mathbb{P}} = 1$ $\mathbb{P}$*-a.s. then for* $\frac{d\mathbb{Q}^\zeta}{d\mathbb{P}} = Z_T^\zeta$, $\mathbb{Q}^\zeta \in \mathcal{M}_e$ *defined as in (5.17) we obtain*

$$\begin{aligned} \mathbf{\Lambda}_{0,\zeta} &= \mathbb{E}^\zeta\left[\log\left(\frac{Z_T^\zeta}{D_T^0}\right)\right] \\ &= \mathbb{E}\left[\frac{d\mathbb{Q}^\zeta}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}^\zeta}{d\mathbb{P}}\right)\right] = H(\mathbb{Q}^\zeta|\mathbb{P}). \end{aligned}$$

where

$$H(\mathbb{Q}|\mathbb{P}) := \begin{cases} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], & \text{if} \quad \mathbb{Q} \ll \mathbb{P} \text{ on } \mathcal{F}_T \\ +\infty & \text{otherwise} \end{cases}.$$

The problem

$$\inf_{\zeta \in \mathcal{PM}_b} \Lambda_{0,\zeta} \tag{5.30}$$

reduces to finding the minimal entropy martingale measure $\mathbb{Q}^E$ *defined by*

$$\mathbb{Q}^E := \arg \min_{\mathbb{Q} \in \mathcal{M}_e} H(\mathbb{Q}|\mathbb{P}).$$

But in our present setting with constant coefficients geometric Brownian motion processes for S_t *and* Y_t*, the minimal entropy martingale measure* $\mathbb{Q}^E$ *and the minimal martingale measure* $\mathbb{Q}^0$ *in (5.9) are the same, i.e.,* $\mathbb{Q}^E = \mathbb{Q}^0$ *(see [36]), then the optimal* ζ *in (5.30) is given by* $\hat{\zeta} = -\varrho$*.*

5.6.3.4 Approximation to the optimal strategy

By [82, Theorem 2.6], the process $M_t := Z_t^{\hat{\zeta}} X_t^{x,\hat{\pi}}$ is a $\mathbb{P}$-martingale. Hence as $\hat{\zeta} = \hat{\kappa}_2$ $\mathbb{P}$-a.s and using the dynamics of $X_t^{x,\hat{\pi}}$ in (5.6) and $Z_t^{\hat{\zeta}}$ in (5.17), we get

$$\begin{aligned} dM_t &= Z_t^{\hat{\zeta}} dX_t^{x,\hat{\pi}} + X_t^{x,\hat{\pi}} dZ_t^{\hat{\zeta}} + d < X_t^{x,\hat{\pi}}, Z^{\hat{\zeta}} >_t \\ &= Z_t^{\hat{\zeta}} X_t^{x,\hat{\pi}} \left(\mu\hat{\pi}_t dt + \sigma\hat{\pi}_t dW_t^s\right) + X_t^{x,\hat{\pi}} Z_t^{\hat{\zeta}} \left(-\varrho dW_t^s + \hat{\zeta}_\tau dW_t\right) - Z_t^{\hat{\zeta}} X_t^{x,\hat{\pi}} \varrho\sigma\hat{\pi}_t dt \\ &= M_t \left\{(\sigma\hat{\pi}_t - \varrho)\, dW_t^s + \hat{\kappa}_{2\tau} dW_t\right\}. \end{aligned} \tag{5.31}$$

On the other hand, for the optimal control processes $(\hat{\kappa}, \hat{\zeta}) \in \mathcal{K} \times \mathcal{PM}_b$ by [82, Theorem 2.6] there exists an optimal strategy $\hat{\pi} \in \mathcal{A}(x)$, whose terminal wealth is given by

$$X_T^{x,\hat{\pi}} = I\left(\frac{\lambda^{\min} Z_T^{\hat{\zeta}}}{D_T^{\hat{\kappa}}}\right),$$

where

$$\begin{aligned} I(y) &= -\hat{V}_\varepsilon'(y) \\ &= \varepsilon\left\{\log\left(\frac{1-y}{y}\right)\right\}. \end{aligned}$$

Using again a series expansion in λ for

$$I(\lambda z) = \varepsilon \log\left(\frac{1-\lambda z}{\lambda z}\right) \approx -\varepsilon \log(z) - \varepsilon \log(\lambda) =: I^{approx}(\lambda z),$$

and evaluating at $\lambda^{\min approx}(x) = \exp\{-\frac{x}{\varepsilon}\} \exp\left\{-\Lambda_{\hat{\kappa},\hat{\zeta}}\right\}$, we have

$$\begin{aligned} I\left(\frac{\lambda^{\min} Z_T^{\hat{\zeta}}}{D_T^{\hat{\kappa}}}\right) &\approx I^{approx}\left(\frac{\lambda^{\min approx} Z_T^{\hat{\zeta}}}{D_T^{\hat{\kappa}}}\right) \\ &= -\varepsilon\left\{\log\left(\frac{Z_T^{\hat{\zeta}}}{D_T^{\hat{\kappa}}}\right) + \log(\lambda^{\min approx})\right\} \\ &= -\varepsilon \log\left(\frac{Z_T^{\hat{\zeta}}}{D_T^{\hat{\kappa}}}\right) + x + \varepsilon\Lambda_{\hat{\kappa},\hat{\zeta}} \\ &= x - \varepsilon\left\{\log\left(\frac{Z_T^{\hat{\zeta}}}{D_T^{\hat{\kappa}}}\right) - \Lambda_{\hat{\kappa},\hat{\zeta}}\right\}. \end{aligned}$$

Recalling that $\frac{Z_T^{\hat{\zeta}}}{D_T^{\hat{\kappa}}}$ under $\mathbb{P}$ is given by

$$\frac{Z_T^{\hat{\zeta}}}{D_T^{\hat{\kappa}}}|\mathcal{F}_T = \exp\left\{-\int_0^T (\varrho + \hat{\kappa}_{1\tau})\,\mathrm{dW}_\tau^s - \frac{1}{2}\int_0^T (\varrho^2 - \hat{\kappa}_{1\tau}^2)\,\mathrm{d}\tau\right\},$$

and $\Lambda_{\hat{\kappa},\hat{\zeta}} = \frac{1}{2}\mathbb{E}^{\hat{\kappa}_2}\left[\int_0^T (\varrho + \hat{\kappa}_{1\tau})^2\, d\tau\right]$ we obtain

$$X_T^{x,\hat{\pi}} = x - \varepsilon \left\{ \begin{array}{c} -\int_0^T (\varrho + \hat{\kappa}_{1\tau})\,\mathrm{dW}_\tau^s - \frac{1}{2}\int_0^T (\varrho^2 - \hat{\kappa}_{1\tau}^2)\,\mathrm{d}\tau \\ -\frac{1}{2}\mathbb{E}^{\hat{\kappa}_2}\left[\int_0^T (\varrho + \hat{\kappa}_{1\tau})^2\,\mathrm{d}\tau\right] \end{array} \right\}. \tag{5.32}$$

Using again the fact that $M_t := Z_t^{\hat{\zeta}} X_t^{x,\hat{\pi}}$ is a $\mathbb{P}$-martingale (i.e., $M_t = \mathbb{E}\left[M_T|\mathcal{F}_t\right] = \mathbb{E}\left[Z_T^{\hat{\zeta}} X_T^{x,\hat{\pi}}|\mathcal{F}_t\right]$), but this time computing with the expression in (5.32) we obtain

$$M_t = Z_t^{\hat{\zeta}} x - \varepsilon Z_t^{\hat{\zeta}}\left\{-\int_t^T (\varrho + \hat{\kappa}_{1\tau})\,\mathrm{dW}_\tau^s - \frac{1}{2}\int_t^T (\varrho^2 - \hat{\kappa}_{1\tau}^2)\,\mathrm{d}\tau - \frac{1}{2}\mathbb{E}^{\hat{\kappa}_2}\left[\int_0^T (\varrho + \hat{\kappa}_{1\tau})^2\,\mathrm{d}\tau\right]\right\}$$

and after omitting the finite variation terms, we have

$$\mathrm{d}M_t = \varepsilon Z_t^{\hat{\zeta}} X_t^{x,\hat{\pi}} \frac{(\varrho + \hat{\kappa}_{1\tau})}{X_t^{x,\hat{\pi}}}\mathrm{dW}_\tau^s + Z_t^{\hat{\zeta}} X_t^{x,\hat{\pi}}\left\{-\varrho \mathrm{dW}_\tau^s + \hat{\kappa}_{2\tau}\mathrm{dW}_\tau\right\},$$

or

$$\mathrm{d}M_t = M_t\left(\varepsilon\frac{(\varrho + \hat{\kappa}_{1t})}{X_t^{x,\hat{\pi}}} - \varrho\right)\mathrm{dW}_t^s + M_t\hat{\kappa}_{2t}\mathrm{dW}_t.$$

Comparing here and in (5.31) all terms involving dW_t^s yields

$$\hat{\Pi}_t = \hat{\pi}_t X_t^{x,\hat{\pi}} = \frac{\varepsilon}{\sigma}(\varrho + \hat{\kappa}_{1t}).$$

5.6.4 Maximising utility with random endowment

When there is a random endowment to be paid at time T given by a claim $H(Y_T)$ on the nontraded asset Y_t the robust utility maximisation problem in (P2) is

$$u^H(x) = \sup_{\pi \in A(x)} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\tilde{U}_\varepsilon^H \left(X_T^{(x,\pi)} \right) \right]. \tag{5.33}$$

Recalling that

$$\tilde{U}_\varepsilon^H(x) = U_\varepsilon(H(Y_T)) - U_\varepsilon(H(Y_T) - x)$$

and

$$\hat{U}_\varepsilon(x) := -U_\varepsilon(-x) = -\varepsilon \log \left(\frac{1 + \exp\left\{\frac{x}{\varepsilon}\right\}}{\exp\left\{\frac{x}{\varepsilon}\right\}} \right),$$

then adapting Lemma 143 in Appendix B to our present situation, the problem P2 is an upper bound for the problems $(P2a)$ and $(P2b)$ defined below. This is,

$$u^H(x) \geq \underbrace{-\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\hat{U}_\varepsilon(-H(Y_T)) \right]}_{(P2a)} + \underbrace{\sup_{\pi \in A(x)} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\hat{U}_\varepsilon \left(X_T^{(x,\pi)} - H(Y_T) \right) \right]}_{(P2b)}. \tag{5.34}$$

By Lemma 144, the equality in (5.34) will be satisfied provided each of the problems $(P2a)$ and $(P2b)$ attains the infimum at the same measure $\mathbb{Q}^* \in \mathcal{P}$. Therefore we make the following assumption.

Assumption 113 *The set $\mathcal{P}$ and the payoff function $H(y)$ are such that the optimal measure $\mathbb{Q}^* \in \mathcal{P}$ in both problems $(P2a)$ and $(P2b)$ is the same, this is, there exist $\mathbb{Q}^* \in \mathcal{P}$ such that*

$$\mathbb{E}_{\mathbb{Q}^*} \left[\hat{U}_\varepsilon(-H(Y_T)) \right] = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\hat{U}_\varepsilon(-H(Y_T)) \right]$$

and

$$\sup_{\pi \in \mathrm{A}(x)} \mathbb{E}_{\mathbb{Q}^*} \left[\hat{U}_\varepsilon \left(X_T^{(x,\pi)} - H(Y_T) \right) \right] = \sup_{\pi \in \mathrm{A}(x)} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\hat{U}_\varepsilon \left(X_T^{(x,\pi)} - H(Y_T) \right) \right]$$

This assumption is useful as each of the two problems $(P2a)$ and $(P2b)$ above are simpler to solve than the original problem in (5.33). We proceed by solving them separately. But by the shape of the utility function $\hat{U}_\varepsilon(x)$, an explicit solution is not easy, instead, we use a series approximation to find upper bounds for the solution of $(P2a)$ and $(P2b)$.

Remark 114 *Assumption 113 is satisfied at least when no robust preferences are present, or when the optimal pair of Girsanov kernels is* $(0,0)$*. Further investigation on the conditions on* $\mathcal{K}$ *and* $H(y)$ *that satisfy Assumption 113 is left for further research.*

We summarise the key results in the solution to the robust problem (5.33) in the following theorem.

Theorem 115 *Assume the current time is* t*,* $0 \leq t \leq T$*,* $Y_t = y$ *and* $X_t = x$*, and that Assumptions 113 and 116 hold. The value function* $u^H(t,x)$ *of the robust utility maximisation problem (5.33) can be approximated by*

$$\begin{aligned} u^{H\ approx}(t,x) &= -u^a(t,y) - \varepsilon \exp\left\{-\frac{x}{\varepsilon}\right\} \exp\left\{-\frac{1}{2}\varrho^2(T-t) + \frac{1}{\varepsilon}h(t,y)\right\} \\ &\quad \times \exp\left\{-\varrho\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau}\,d\tau\right] - \frac{1}{2}\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau}^2\,d\tau\right]\right\}, \end{aligned}$$

with an error of

$$u^H(t,x) - u^{H\ approx}(t,x) \leq \frac{1}{2}\left(1 - \log(2)\right)\varepsilon \approx 0.1534264097\varepsilon.$$

And $\hat{\kappa} = (\hat{\kappa}_1, \hat{\kappa}_2) \in \mathcal{K}$ *is a pair of Girsanov kernels that solve*

$$\min_{(\kappa_1,\kappa_2)\in\mathcal{K}} \mathbb{E}^{\kappa_2}\left[\frac{1}{2}\int_t^T (\varrho + \kappa_{1\tau})^2\,d\tau - \frac{1}{\varepsilon}H(Y_T)\right].$$

The operator $\mathbb{E}^{\hat{\kappa}_2}[\cdot]$ *represents the expected value under the measure*

$$\frac{d\mathbb{Q}^{\hat{\kappa}_2}}{d\mathbb{P}}|_{\mathcal{F}_T} = Z_T^{\hat{\kappa}_2} := \mathcal{E}\left(-\int_0 \varrho\,dW_\tau^s + \int_0 \hat{\kappa}_{2\tau}\,dW_\tau\right)_T.$$

The function $u^a(t,y)$ *is the solution to the following PDE*

$$u_t^a + \frac{1}{2}a^2y^2u_{yy}^a + byu_y^a + \sup_{(\kappa_1,\kappa_2)\in\mathcal{K}}\left\{\left(\rho_{sy}\kappa_1 + \bar{\rho_{sy}}\kappa_2\right)ayu_y^a\right\} = 0,$$

with terminal condition

$$u^a(T,y) = \hat{U}_\varepsilon\left(-H(y)\right) = -\varepsilon\log\left(\frac{1+\exp\left\{\frac{-H(y)}{\varepsilon}\right\}}{\exp\left\{\frac{-H(y)}{\varepsilon}\right\}}\right).$$

And the function $h(t,y)$ *solves*

$$h_t + \frac{1}{2}a^2y^2h_{yy} + \left(b - \rho_{sy}a\varrho\right)yh_y + \inf_{(\kappa_1,\kappa_2)\in\mathcal{K}}\left\{\bar{\rho_{sy}}a\kappa_2yh_y\right\} = 0,$$

with terminal condition

$$h(T, y) = H(Y_T).$$

The corresponding approximate optimal strategy $\hat{\pi}_t^{approx}$ *for the robust problem is*

$$\hat{\Pi}_t^{approx} = \hat{\pi}_t^{approx} X_t^{x,\hat{\pi}} = \frac{\varepsilon}{\sigma}(\varrho + \hat{\kappa}_{1t}) + \rho_{sy}\frac{a}{\sigma}Y_t f_y(t, Y_t),$$

with $f(t, y)$ *given by*

$$f(t,y) = \exp\left\{-\frac{1}{2}\frac{b^2}{a^2}(T-t)\right\}\mathbb{E}\left[\exp\left\{\frac{b}{a}\sqrt{T-t}N\right\}H\left(y\exp\{a\sqrt{T-t}N\}\right)\right].$$

Define the measure $\hat{\mathbb{Q}} \in \mathcal{P}$ *via*

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}|_{\mathcal{F}_T} = D_T^{\hat{\kappa}} := \exp\left(\int_0^T \hat{\kappa}_{1\tau} dW_\tau^s - \frac{1}{2}\int_0^t \hat{\kappa}_{1\tau}^2 d\tau + \int_0^t \hat{\kappa}_{2\tau} dW_\tau - \frac{1}{2}\int_0^t \hat{\kappa}_{2\tau}^2 d\tau\right).$$

Then the pair $\left(\hat{\pi}_t^{approx}, \hat{\mathbb{Q}}\right)$ *is a saddle point for the problem*

$$u^{approx}(t,x) = \sup_{\pi\in\mathrm{A}(x)} \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[\mathcal{U}_\varepsilon\left(X_T^{(x,\pi)} - H(Y_T)\right)\right],$$

with $\mathcal{U}_\varepsilon(x) = -\varepsilon\exp\left\{-\frac{x}{\varepsilon}\right\}$.

5.6.4.1 The problem $(P2b)$

We analyse first the solution to the problem $(P2b)$ above. This is the random endowment counterpart of the problem solved in Section 5.6.3. We use the same technique of solving first the dual problem and express the primal value function in terms of the dual value function.

By Lemma 107 and therefore using results in [82], the dual value function of the robust utility maximisation with random endowment problem is given by (see Appendix B)

$$v^b(\lambda) := \inf_{\kappa\in\mathcal{K}} \inf_{\zeta\in\mathcal{PM}}\left\{\mathbb{E}\left[D_T^\kappa \hat{V}_\varepsilon\left(\lambda\frac{Z_T^\zeta}{D_T^\kappa}\right)\right] - \lambda\mathbb{E}\left[Z_T^\zeta H(Y_T)\right]\right\}, \ \lambda > 0 \tag{5.35}$$

where the vector process κ and the set $\mathcal{K}$ are defined in (5.10), the parameter λ plays the role of a Lagrange multiplier in the optimisation problem and the function $\hat{V}(y)$ is the Legendre-Fenchel transform of $\hat{U}_\varepsilon(x)$ and given by

$$\begin{aligned}\hat{V}_\varepsilon(y) &= \sup_{x\geq 0}\left(\hat{U}_\varepsilon(x) - yx\right)\\ &= \begin{cases}\infty & \text{if } \ y \geq 1\\ -\varepsilon\log\left(\frac{1}{1-y}\right) - \varepsilon y\log\left(\frac{1-y}{y}\right) & \text{if } \ 0 < y < 1.\end{cases}\end{aligned}$$

The primal and dual value functions are related as follows:

$$u^b(x) = \min_{\lambda>0} \left(v^b(\lambda) + \lambda x \right).$$

We use the same idea as in Section 5.6.3 of approximating the dual value function $v^b(\lambda)$ by a series expansion in λ. This is, we use the approximation of $\hat{V}_\varepsilon\left(\lambda \frac{Z_T^\zeta}{D_T^\kappa}\right)$ in (5.24) recalled below

$$\hat{V}_\varepsilon\left(\lambda \frac{Z_T^\zeta}{D_T^\kappa}\right) \approx -\varepsilon\lambda \frac{Z_T^\zeta}{D_T^\kappa}\left(1 - \log\left(\frac{Z_T^\zeta}{D_T^\kappa}\right) - \log(\lambda)\right).$$

Then the dual value function $v^b(\lambda)$ can be approximated by

$$v^b(\lambda) \approx -\varepsilon\lambda\left(1 - \log(\lambda)\right) + \lambda \inf_{\kappa\in\mathcal{K}} \inf_{\zeta\in\mathcal{PM}} \left\{\varepsilon\Lambda_{\kappa,\zeta} - \Delta_\zeta\right\} =: v^{b\ approx}(\lambda),$$

with

$$\Delta_\zeta := \mathbb{E}\left[Z_T^\zeta H(Y_T)\right]$$

and $\Lambda_{\kappa,\zeta}$ defined in (5.23) and given by

$$\Lambda_{\kappa,\zeta} := \mathbb{E}^\zeta\left[\log\left(\frac{Z_T^\zeta}{D_T^\kappa}\right)\right].$$

The approximation $v^{b\ approx}(\lambda)$ is valid provided $v^{b\ approx}(\lambda) < \infty$, but similar as in Section 5.6.3, a sufficient condition for $v^{b\ approx}(\lambda) < \infty$, is to have $\zeta \in \mathcal{PM}_b$ (ζ is progressively measurable with $\int_0^T \zeta_t^2 dt < \infty$) because

$$\inf_{\kappa\in\mathcal{K}} \inf_{\zeta\in\mathcal{PM}} \left\{\varepsilon\Lambda_{\kappa,\zeta} - \Delta_\zeta\right\} \geq \varepsilon \inf_{\kappa\in\mathcal{K}} \inf_{\zeta\in\mathcal{PM}} \Lambda_{\kappa,\zeta} + \inf_{\zeta\in\mathcal{PM}} \left\{-\Delta_\zeta\right\}.$$

For the term with $\Lambda_{\kappa,\zeta}$, Lemma 108 assures the choice of $\zeta \in \mathcal{PM}_b$ and in order to extend it to the term Δ_ζ, me make the following assumption.

Assumption 116 *The payoff $H(y)$ is bounded below.*

Hence we have

$$v^{b\ approx}(\lambda) = -\varepsilon\lambda\left(1 - \log(\lambda)\right) + \lambda \inf_{\kappa\in\mathcal{K}} \inf_{\zeta\in\mathcal{PM}_b} \left\{\varepsilon\Lambda_{\kappa,\zeta} - \Delta_\zeta\right\}. \qquad (5.36)$$

Assumption 116 is related to the fact that the approximated primal function $u^b(x)$ is the solution to a robust version of an utility maximisation problem with exponential utility preferences (see Theorem 105). In such a case, Assumption 116 is necessary for the finiteness of the primal value function when considering a random endowment (see [13] or [39]). As pointed out in [13], Assumption 116 implies the existence of constants y_0, c_1, c_2 such that $H(y) = c_1 + c_2 y$ for $y \geq y_0$. Then H is bounded below if $c_2 = 0$; otherwise $c_2 > 0$, i.e., H is either constant, or has a constant positive slope for large y. This assumption includes put options and some spread options but rules out short calls. Note that this restriction on the payoffs H is needed only in the approximated problem but not in the original problem (5.33). This is the trade-off of using the series approximation. We gain explicitness in the solutions but we lose generality in the type of claims.

Optimal Lagrange multiplier and approximation to the primal value function Given the approximation to the dual value function in (5.36), the approximated optimal Lagrange multiplier is given by

$$\begin{aligned}\lambda^{b\ approx}(x) &= \arg\min_{\lambda>0}\left(v^{b\ approx}(\lambda)+\lambda x\right)\\ &= \exp\left\{-\frac{x}{\varepsilon}\right\}\exp\left\{-\inf_{\kappa\in\mathcal{K}}\inf_{\zeta\in\mathcal{PM}_b}\left\{\Lambda_{\kappa,\zeta}-\frac{1}{\varepsilon}\Delta_\zeta\right\}\right\}.\end{aligned}$$

And then the approximated primal value function $u^{b\ approx}(x)$ is

$$\begin{aligned}u^{b\ approx}(x) &= v^{b\ approx}\left(\lambda^{b\ approx}(x)\right)+\lambda^{b\ approx}(x)x\\ &= -\varepsilon\exp\left\{-\frac{x}{\varepsilon}\right\}\exp\left\{-\inf_{\kappa\in\mathcal{K}}\inf_{\zeta\in\mathcal{PM}_b}\left\{\Lambda_{\kappa,\zeta}-\frac{1}{\varepsilon}\Delta_\zeta\right\}\right\}.\end{aligned}$$

The solution to the dual problem In this section, we obtain the solution to

$$\inf_{\kappa\in\mathcal{K}}\inf_{\zeta\in\mathcal{PM}_b}\left\{\Lambda_{\kappa,\zeta}-\frac{1}{\varepsilon}\Delta_\zeta\right\}. \tag{5.37}$$

In order to capture the dynamical behaviour of the problem, assume the starting time is $t \in [0, T]$.

As in Section 5.6.3, take

$$\begin{aligned}J_t^{\kappa,\zeta} &:= \mathbb{E}^\zeta\left[\log\left(\frac{Z_T^\zeta}{D_T^\kappa}\right)\right]\\ &= \mathbb{E}^\zeta\left[\frac{1}{2}\int_t^T(\varrho+\kappa_{1\tau})^2\,d\tau+\frac{1}{2}\int_t^T(\zeta_\tau-\kappa_{2\tau})^2\,d\tau\right],\end{aligned}$$

and

$$L_t^{\kappa,\zeta} := \mathbb{E}\left[Z_T^\zeta H(Y_T)\right].$$

The dynamic problem is to find processes $(\kappa, \zeta) \in \mathcal{K} \times \mathcal{PM}_b$ that solves

$$\begin{aligned} & \inf_{\kappa \in \mathcal{K}} \inf_{\zeta \in \mathcal{PM}_b} \left\{ J_t^{\kappa,\zeta} - \frac{1}{\varepsilon} L_t^{\kappa,\zeta} \right\} \\ = & \inf_{\kappa \in \mathcal{K}} \inf_{\zeta \in \mathcal{PM}_b} \left\{ \mathbb{E}^{\zeta} \left[\frac{1}{2} \int_t^T (\varrho + \kappa_{1\tau})^2 \, d\tau + \frac{1}{2} \int_t^T (\zeta_\tau - \kappa_{2\tau})^2 \, d\tau - \frac{1}{\varepsilon} H(Y_T) \right] \right\}. \end{aligned} \tag{5.38}$$

If $\hat{\zeta}$, $\hat{\kappa_1}$ and $\hat{\kappa_2}$ denote the optimisers in (5.38), note that $\hat{\zeta}$ and $\hat{\kappa_1}$ will be characterised similarly as in the no random endowment case in Proposition 110. This can be seen as $\hat{\zeta} \in \mathcal{PM}_b$ seeks to minimise $\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T (\zeta_\tau - \hat{\kappa_{2\tau}})^2 d\tau\right]$ independently of the other terms that do not involve ζ. Similarly for $\hat{\kappa_1}$, it needs to minimise $\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T (\varrho + \kappa_{1\tau})^2 d\tau\right]$, independently of the rest of the term. For $\hat{\kappa_2}$, the optimal choice in (5.38) may change with respect to the optimal solution in Proposition 110, and it will depend on the function $H(Y_T)$, as we state in the next proposition.

Proposition 117 *There exists $(\hat{\kappa}, \hat{\zeta}) \in \mathcal{K} \times \mathcal{PM}_b$, which attains the minimum for the dual problem (5.38). Furthermore, such a pair $(\hat{\kappa}, \hat{\zeta})$ is characterised by the fact that $\hat{\kappa}_{1t}$ and $\hat{\kappa}_{2t}$ solves*

$$\mathbb{E}^{\hat{\kappa}_2}\left[\frac{1}{2}\int_t^T (\varrho + \hat{\kappa}_{1\tau})^2 d\tau - \frac{1}{\varepsilon} H(Y_T)\right] = \inf_{(\kappa_1,\kappa_2)\in\mathcal{K}} \mathbb{E}^{\kappa_2}\left[\frac{1}{2}\int_t^T (\varrho + \kappa_{1\tau})^2 d\tau - \frac{1}{\varepsilon} H(Y_T)\right], \tag{5.39}$$

and $\hat{\zeta}_t = \hat{\kappa}_{2t}$. Thus the value function to the dual problem in (5.28) is given by

$$\begin{aligned} J_t^{\hat{\kappa},\hat{\zeta}} &= \mathbb{E}^{\hat{\kappa}_2}\left[\frac{1}{2}\int_t^T (\varrho + \hat{\kappa}_{1\tau})^2 \, d\tau - \frac{1}{\varepsilon} H(Y_T)\right] \\ &= \frac{1}{2}\varrho^2 (T-t) + \varrho \mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau} d\tau\right] + \frac{1}{2}\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau}^2 d\tau\right] - \frac{1}{\varepsilon}\mathbb{E}^{\hat{\kappa}_2}\left[H(Y_T)\right]. \end{aligned}$$

Proof. The proof is very similar to the no random endowment case in Proposition 117. □

In order to ease the notation, note that by a measurable selection argument (see Appendix C), one can always choose a measurable functions κ_2^* of t and y for which $\hat{\kappa}_{2t} := \kappa_2^*(t, Y_t)$, then by the Markov property we have for $(\kappa_1, \kappa_2) \in \mathcal{K}$

$$h(t, Y_t) = \mathbb{E}^{\hat{\kappa}_2}\left[H(Y_T)\right].$$

Then $h(t, Y_t)$ is the solution to the PDE

$$h_t + \frac{1}{2} a^2 y^2 h_{yy} + \left(b - \rho_{sy} a \varrho + \bar{\rho_{sy}} a \kappa_2^*(t, y)\right) y h_y = 0 \tag{5.40}$$

with terminal condition

$$h(T, y) = H(Y_T).$$

Remark 118 *The shape of the function $\kappa_2^*(t,y)$ will depend strongly on the set $\mathcal{K}$, in the special case $\kappa_2^*(t,y) = k_2$ for $k_2 \in \mathbb{R}$ constant (e.g. when $\mathcal{K}$ is a rectangle as in Remark 111), the equation (5.40) can be reduced to the Heat equation.*

Approximation to the optimal strategy We proceed similarly as in the no random endowment case. By [82, Theorem 2.6], the process $M_t := Z_t^{\hat{\zeta}} \left\{ X_t^{x,\hat{\pi}} - f(t,Y_t) \right\}$ is a $\mathbb{P}$-martingale, where we have defined $f(t,y) = \mathbb{E}\left[H(Y_T)\right]$. Hence as $\hat{\zeta} = \hat{\kappa}_2$ $\mathbb{P}$-a.s we get

$$\begin{aligned} \mathrm{d}M_t &= \left\{ \begin{array}{c} Z_t^{\hat{\zeta}} \mathrm{d}X_t^{x,\hat{\pi}} + X_t^{x,\hat{\pi}} \mathrm{d}Z_t^{\hat{\zeta}} + \mathrm{d} < X_t^{x,\hat{\pi}}, Z^{\hat{\zeta}} >_t \\ -Z_t^{\hat{\zeta}} \mathrm{d}\left(f(t,Y_t)\right) - f(t,Y_t)\mathrm{d}Z_t^{\hat{\zeta}} + \mathrm{d} < Z^{\hat{\zeta}}, f(t,Y_t) >_t \end{array} \right\} \qquad (5.41) \\ &= X_t^{x,\hat{\pi}} \frac{M_t}{X_t^{x,\hat{\pi}} - f(t,Y_t)} \left(\sigma \hat{\pi}_t \mathrm{dW}_t^s\right) + M_t \left(-\varrho \mathrm{dW}_t^s + \hat{\zeta}_\tau \mathrm{dW}_t \right) \\ &\quad - \frac{M_t}{X_t^{x,\hat{\pi}} - f(t,Y_t)} a Y_t f_y(t,Y_t) \left\{ \rho_{sy} \mathrm{dW}_t^s + \bar{\rho_{sy}} \mathrm{dW}_t \right\} \\ &= M_t \left\{ \frac{\sigma \hat{\pi}_t X_t^{x,\hat{\pi}}}{X_t^{x,\hat{\pi}} - f(t,Y_t)} - \varrho - \frac{\rho_{sy} a Y_t f_y(t,Y_t)}{X_t^{x,\hat{\pi}} - f(t,Y_t)} \right\} \mathrm{dW}_t^s + M_t \left\{ \hat{\kappa}_{2t} - \frac{\bar{\rho_{sy}} a Y_t f_y(t,Y_t)}{X_t^{x,\hat{\pi}} - f(t,Y_t)} \right\} \mathrm{dW}_t. \end{aligned}$$

However, for the optimal control processes $(\hat{\kappa}, \hat{\zeta}) \in \mathcal{K} \times \mathcal{PM}_b$ by [82, Theorem 2.6] there exists an optimal strategy $\hat{\pi} \in \mathcal{A}(x)$, whose terminal wealth is given by

$$X_T^{x,\hat{\pi}} - H(Y_T) = I\left(\frac{\lambda^{\min} Z_T^{\hat{\zeta}}}{D_T^{\hat{\kappa}}} \right),$$

where

$$\begin{aligned} I(y) &= -\hat{V}_\varepsilon'(y) \\ &= \varepsilon \left\{ \log\left(\frac{1-y}{y} \right) \right\}. \end{aligned}$$

Using again a series expansion in λ for

$$I(\lambda z) = \varepsilon \log\left(\frac{1-\lambda z}{\lambda z} \right) \approx -\varepsilon \log(z) - \varepsilon \log(\lambda) =: I^{approx}(\lambda z),$$

and evaluating at $\lambda^{b\ approx}(x) = \exp\{-\frac{x}{\varepsilon}\}\exp\left\{-\Lambda_{\kappa,\zeta} + \frac{1}{\varepsilon}\Delta_\zeta\right\}$, we have

$$\begin{aligned}
I\left(\frac{\lambda^{\min} Z_T^{\hat\zeta}}{D_T^{\hat\kappa}}\right) &\approx I^{approx}\left(\frac{\lambda^{\min approx} Z_T^{\hat\zeta}}{D_T^{\hat\kappa}}\right) \\
&= -\varepsilon\left\{\log\left(\frac{Z_T^{\hat\zeta}}{D_T^{\hat\kappa}}\right) + \log(\lambda^{\min approx})\right\} \\
&= -\varepsilon\log\left(\frac{Z_T^{\hat\zeta}}{D_T^{\hat\kappa}}\right) + x + \varepsilon\Lambda_{\hat\kappa,\hat\zeta} - \Delta_\zeta \\
&= x - \Delta_\zeta - \varepsilon\left\{\log\left(\frac{Z_T^{\hat\zeta}}{D_T^{\hat\kappa}}\right) - \Lambda_{\hat\kappa,\hat\zeta}\right\}.
\end{aligned}$$

Recalling that $\frac{Z_T^{\hat\zeta}}{D_T^{\hat\kappa}}$ under $\mathbb{P}$ is given by

$$\frac{Z_T^{\hat\zeta}}{D_T^{\hat\kappa}}|\mathcal{F}_T = \exp\left\{-\int_0^T(\varrho+\hat\kappa_{1\tau})\,\mathrm{d}W_\tau^s - \frac{1}{2}\int_0^T\left(\varrho^2+\hat\kappa_{1\tau}^2\right)\mathrm{d}\tau\right\},$$

that $\Lambda_{\hat\kappa,\hat\zeta} = \frac{1}{2}\mathbb{E}^{\hat\kappa_2}\left[\int_0^T(\varrho+\hat\kappa_{1\tau})^2\,\mathrm{d}\tau\right]$ and $\Delta_\zeta = h(0,y)$ we obtain

$$X_T^{x,\hat\pi} - H(Y_T) = x - h(0,y) - \varepsilon\left\{\begin{array}{c} -\int_0^T(\varrho+\hat\kappa_{1\tau})\,\mathrm{d}W_\tau^s - \frac{1}{2}\int_0^T\left(\varrho^2+\hat\kappa_{1\tau}^2\right)\mathrm{d}\tau \\ -\frac{1}{2}\mathbb{E}^{\hat\kappa_2}\left[\int_0^T(\varrho+\hat\kappa_{1\tau})^2\,\mathrm{d}\tau\right]\end{array}\right\}. \quad (5.42)$$

Using again the fact that $M_t := Z_t^{\hat\zeta}\left\{X_t^{x,\hat\pi} - f(t,Y_t)\right\}$ is a $\mathbb{P}$-martingale $\left(\text{i.e., } M_t = \mathbb{E}\left[M_T\right] = \mathbb{E}\left[Z_T^{\hat\zeta}\left\{X_T^{x,\hat\pi} - H(Y_T)\right\}\right]\right)$, but this time computing with the expression in (5.42) we obtain

$$\begin{aligned}
M_t &= Z_t^{\hat\zeta}(x - h(0,y)) \\
&\quad -\varepsilon Z_t^{\hat\zeta}\left\{-\int_t^T(\varrho+\hat\kappa_{1\tau})\,dW_\tau^s - \frac{1}{2}\int_t^T\left(\varrho^2+\hat\kappa_{1\tau}^2\right)d\tau - \frac{1}{2}\mathbb{E}^{\hat\kappa_2}\left[\int_0^T(\varrho+\hat\kappa_{1\tau})^2\,d\tau\right]\right\},
\end{aligned}$$

which yields, after omitting the finite variation terms,

$$\mathrm{d}M_t = \varepsilon\frac{M_t}{X_t^{x,\hat\pi} - f(t,Y_t)}(\varrho+\hat\kappa_{1t})\,\mathrm{d}W_t^s + M_t\left\{-\varrho\mathrm{d}W_\tau^s + \hat\kappa_{2\tau}\mathrm{d}W_\tau\right\},$$

or

$$\mathrm{d}M_t = M_t\left(\varepsilon\frac{(\varrho+\hat\kappa_{1t})}{X_t^{x,\hat\pi} - f(t,Y_t)} - \varrho\right)\mathrm{d}W_t^s + M_t\hat\kappa_{2t}\mathrm{d}W_t.$$

Comparing here and in (5.41) all terms involving dW_t^s yields

$$\hat\Pi_t = \hat\pi_t X_t^{x,\hat\pi} = \varepsilon\left(\frac{\mu}{\sigma^2} + \frac{\hat\kappa_{1t}}{\sigma}\right) + \rho_{sy}\frac{a}{\sigma}Y_t f_y(t,Y_t).$$

Moreover, $f(t,y)$ can be computed as follows. First note that $f(t,Y_t)$ is a $\mathbb{P}$-local martingale, then the function $f(t,y)$ satisfies the PDE

$$f_t + \frac{1}{2}a^2y^2f_{yy} + byf_y = 0$$

with terminal condition

$$f(T,y) = H(Y_T).$$

Taking the usual change of variable to reduce the PDE to the heat equation we have

$$f(t,y) = \exp\left\{-\frac{1}{2}\frac{b^2}{a^2}(T-t)\right\}\mathbb{E}\left[\exp\left\{\frac{b}{a}\sqrt{T-t}N\right\}H\left(y\exp\left\{a\sqrt{T-t}N\right\}\right)\right],$$

where N is a standard normal random variable.

5.6.4.2 The problem $(P2a)$

In this section, we solve the problem $(P2a)$ defined in (5.34) and given by

$$\sup_{\mathbb{Q}\in\mathcal{P}}\mathbb{E}_{\mathbb{Q}}\left[\hat{U}_\varepsilon\left(-H(Y_T)\right)\right].$$

Assume the current time is t, $0\le t\le T$. By the definition of the priors set $\mathcal{P}$ in (5.10) and the Markov property of the process Y_t, the above problem is equivalent to

$$u^a(t,y) = \sup_{(\kappa_1,\kappa_2)\in\mathcal{K}}\mathbb{E}_{\mathbb{Q}^\kappa}\left[\hat{U}_\varepsilon\left(-H(Y_T)\right)|Y_t=y\right],$$

where the measures $\mathbb{Q}^\kappa$, $\kappa=(\kappa_1,\kappa_2)\in\mathcal{K}$ are defined by

$$\frac{\mathrm{d}\mathbb{Q}^\kappa}{\mathrm{d}\mathbb{P}} = D_T^\kappa := \exp\left(\int_0^T\kappa_{1\tau}\mathrm{dW}_\tau^s - \frac{1}{2}\int_0^T\kappa_{1\tau}^2\mathrm{d}\tau + \int_0^T\kappa_{2\tau}\mathrm{dW}_\tau - \frac{1}{2}\int_0^T\kappa_{2\tau}^2\mathrm{d}\tau\right),$$

The dynamics of Y_t under a measure $\mathbb{Q}^\kappa$ is given by

$$\mathrm{d}Y_t = Y_t\left\{b+\rho_{sy}a\kappa_{1t}+\bar{\rho_{sy}}a\kappa_{2t}\right\}\mathrm{d}t + \rho_{sy}aY_t\mathrm{d}\breve{\mathrm{W}}_t^s + \bar{\rho_{sy}}aY_t\mathrm{d}\breve{\mathrm{W}}_t,$$

where $\breve{W}^s$ and $\breve{W}$ are $\mathbb{Q}^\kappa$-Brownian motions. By the fact that the set $\mathcal{K}$ is compact, standard control theory arguments (see [26]) suggests that the function $u^a(t,y)$ is formally a solution to the Hamilton-Jacobi-Bellman (HJB) equation

$$u_t^a + \frac{1}{2}a^2y^2u_{yy}^a + byu_y^a + ayu_y^a\left\{\sup_{(\kappa_1,\kappa_2)\in\mathcal{K}}\left(\rho_{sy}\kappa_1+\bar{\rho_{sy}}\kappa_2\right)\right\} = 0 \tag{5.43}$$

with terminal condition

$$u^a(T,y) = \hat{U}_\varepsilon\left(-H(y)\right) = -\varepsilon\log\left(\frac{1+\exp\left\{\frac{-H(y)}{\varepsilon}\right\}}{\exp\left\{\frac{-H(y)}{\varepsilon}\right\}}\right).$$

(we have used the following notation $u_t^a = \frac{\partial u^a}{\partial t}$, $u_y^a = \frac{\partial u^a}{\partial y}$ and $u_{yy}^a = \frac{\partial^2 u^a}{\partial y^2}$).

By a measurable selection argument (see Appendix C) we can always choose as the optimal controls $\hat{\kappa}_{1t}$ and $\hat{\kappa}_{2t}$ Markov controls of the form

$$\hat{\kappa}_{1t} := \kappa_1^*(t, Y_t), \text{ and } \hat{\kappa}_{2t} := \kappa_2^*(t, Y_t).$$

Thus, after the transformation $x = \log(y)$, $\tau = T - t$ and taking $w^a(\tau, x) := u^a(T - \tau, e^x)$ and $\bar{\kappa}_i^*(\tau, x) := \kappa_i^*(t, e^x)$, $i = 1, 2$ the equation in (5.43) remains

$$\frac{1}{2}a^2 w_{xx}^a + \left[b + \rho_{sy} a \bar{\kappa}_1^*(\tau, x) + \bar{\rho_{sy}} a \bar{\kappa}_2^*(\tau, x)\right] w_x^a = w_\tau^a \tag{5.44}$$

with initial condition

$$w^a(0, x) = -\varepsilon \log\left(\frac{1 + \exp\left\{\frac{-H(\exp\{x\})}{\varepsilon}\right\}}{\exp\left\{\frac{-H(\exp\{x\})}{\varepsilon}\right\}}\right).$$

The solution to the equation (5.44) will depend strongly on the set $\mathcal{K}$, which will determine the shape of the functions $\kappa_1^*(t, y)$ and $\kappa_2^*(t, y)$. In the special case $\kappa_1^*(t, y) = k_1$ and $\kappa_2^*(t, y) = k_2$, for $k_1, k_2 \in \mathbb{R}$ constants (e.g. when $\mathcal{K}$ is a rectangle as in Remark 111), the equation (5.44) can be reduced by taking a transformation of the form

$$w^a(\tau, x) = \exp\{\alpha\tau + \beta x\} v^a(\tau, x).$$

Choosing

$$\alpha = \frac{1}{2}\beta \quad \text{and} \quad \beta = -\frac{b}{a^2} - \frac{\rho_{sy} k_1 + \bar{\rho_{sy}} k_2}{a} + \frac{1}{2},$$

we find that $v^a(\tau, x)$ solves

$$\frac{1}{2}a^2 v_{xx}^a = v_\tau^a$$

with initial condition

$$v^a(0, x) = -\varepsilon \exp\{-\beta x\} \log\left(\frac{1 + \exp\left\{\frac{-H(\exp\{x\})}{\varepsilon}\right\}}{\exp\left\{\frac{-H(\exp\{x\})}{\varepsilon}\right\}}\right).$$

This is the Heat equation, with solution

$$v^a(\tau, x) = \int_{-\infty}^{\infty} \frac{1}{a\sqrt{2\pi\tau}} \exp\left\{-\frac{z^2}{2a^2\tau}\right\} v^a(0, x + z) dz$$

so, transforming back into our original function $u^a(t,y)$ we have

$$\begin{aligned} u^a(t,y) &= \exp\left\{\frac{1}{2}\beta(T-t)+\beta\log(y)\right\} \\ &\quad \times \int_{-\infty}^{\infty} \frac{1}{a\sqrt{2\pi(T-t)}} \exp\left\{-\frac{z^2}{2a^2(T-t)}\right\} v^a(0,\log(y)+z)dz \\ &= y^\beta \exp\left\{\frac{1}{2}\beta(T-t)\right\} \times \mathbb{E}\left[v^a(0,\log(y)+a\sqrt{T-t}N)\right] \\ &= y^\beta \exp\left\{\frac{1}{2}\beta(T-t)\right\} \\ &\quad \times \mathbb{E}\left[\left[y\exp\left\{a\sqrt{T-t}N\right\}\right]^{-\beta} \hat{U}_\varepsilon\left(-H\left(y\exp\left\{a\sqrt{T-t}N\right\}\right)\right)\right], \end{aligned} \tag{5.45}$$

where N is a standard normal random variable. Substituting $v^a(0,x)$ we obtain

$$u^a(t,y) = \begin{cases} -\varepsilon\exp\left\{-\frac{1}{2}\left(\frac{b}{a^2}+\frac{\rho_{sy}k_1+\bar{\rho_{sy}}k_2}{a}-\frac{1}{2}\right)(T-t)\right\}\times \\ \mathbb{E}\left[\exp\left\{\left(\frac{b}{a}+\rho_{sy}k_1+\bar{\rho_{sy}}k_2-\frac{1}{2}a\right)\sqrt{T-t}N\right\}\log\left(\exp\left\{\frac{H(y\exp\{a\sqrt{T-t}N\})}{\varepsilon}\right\}+1\right)\right]. \end{cases}$$

Remark 119 *The term* $\mathbb{E}\left[\left[\bar{Y}_T\right]^{-\beta}\hat{U}_\varepsilon\left(-H\left(\bar{Y}_T\right)\right)\right]$ *from the last line in (5.45), recalled below:*

$$u^a(t,y) = y^\beta\exp\left\{\frac{1}{2}\beta(T-t)\right\}\times\mathbb{E}\left[\left[\bar{Y}_T\right]^{-\beta}\hat{U}_\varepsilon\left(-H\left(\bar{Y}_T\right)\right)\right],$$

with

$$\bar{Y}_T = y\exp\left\{a\sqrt{T-t}N\right\},$$

can be interpreted as the price of a claim on Y_T *with payoff*

$$F(y) = y^{-\beta}\hat{U}_\varepsilon\left(-H(y)\right)$$

but under the martingale measure that makes the process Y_t *driftless. This is, define the measure* $\mathbb{Q}^a$ *by*

$$\frac{d\mathbb{Q}^a}{d\mathbb{P}}|_{\mathcal{F}_T} = \exp\left(\int_0^T -\varrho dW_\tau^s - \frac{1}{2}\int_0^T \varrho^2 d\tau + \int_0^T\left(\frac{\rho_{sy}a\varrho-b}{\bar{\rho_{sy}}a}\right)dW_\tau - \frac{1}{2}\int_0^T\left(\frac{\rho_{sy}a\varrho-b}{\bar{\rho_{sy}}a}\right)^2 d\tau\right).$$

Under the measure $\mathbb{Q}^a$ the processes $\check{W}_t^s$ and $\check{W}_t$ defined by

$$\begin{aligned}\check{W}_t^s &= W_t^s - \int_0^t \varphi_{1\tau} d\tau \\ \check{W}_t &= W_t - \int_0^t \varphi_{2\tau} d\tau\end{aligned}$$

are $\mathbb{Q}^a$-independent Brownian motions. Under this measure $\mathbb{Q}^a$ the process Y_t has the dynamics

$$dY_t = aY_t \left(\rho_{sy} d\check{W}_t^s + \bar{\rho_{sy}} d\check{W}_t\right) \text{ with } Y_0 = y.$$

Then

$$\mathbb{E}\left[\left[\bar{Y}_T\right]^{-\beta} \hat{U}_\varepsilon \left(-H\left(\bar{Y}_T\right)\right)\right] = \mathbb{E}^a\left[F(Y_T)\right].$$

5.7 Utility indifference pricing for the approximated problems

In this section, we come back to the utility indifference pricing for the approximated problem under robust preferences. We recall from Section 5.6.2 that the utility indifference bid price p^b is the solution to

$$R_H(t, x' - p^b) = R_0(t, x'),$$

for

$$R_H(t, x') = \sup_{\pi \in A(x')} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[\tilde{U}_\varepsilon^H\left(X_T^{(x',\pi)}\right)\right],$$

where the supremum is taken over all wealths X_T which can be generated from initial fortune x'.

In Section 5.6.3 we have computed the approximated primal value function to the no random endowment problem and it is given by

$$\begin{aligned}u^{approx}(t,x) &= -\varepsilon \exp\left\{-\frac{x}{\varepsilon}\right\} \exp\left\{-\frac{1}{2}\varrho^2 (T-t)\right\} \times \\ &\quad \exp\left\{-\varrho \mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau} d\tau\right] - \frac{1}{2}\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau}^2 d\tau\right]\right\}.\end{aligned}$$

And adding the constant term that we were missing (see Section 5.6.3), we have

$$R_0(t,x) \approx \varepsilon \log(2) + u^{approx}(t,x).$$

On the other hand, the approximated primal value function to the random endowment problem was computed in Section 5.6.4 and given by

$$\begin{aligned} u^{H\ approx}(t,x) &= -u^a(t,y) - \varepsilon \exp\left\{-\frac{x}{\varepsilon}\right\} \exp\left\{-\frac{1}{2}\varrho^2(T-t) + \frac{1}{\varepsilon}h(t,y)\right\} \\ &\quad \times \exp\left\{-\varrho \mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau} d\tau\right] - \frac{1}{2}\mathbb{E}^{\hat{\kappa}_2}\left[\int_t^T \hat{\kappa}_{1\tau}^2 d\tau\right]\right\} \\ &= -u^a(t,y) + u^{approx}(t,x) \exp\left\{\frac{1}{\varepsilon}h(t,y)\right\}. \end{aligned}$$

Thus,

$$R_H(t,x) \approx u^{H\ approx}(t,x).$$

Note that $u^{approx}(t, x + x') = u^{approx}(t,x) \exp\left\{-\frac{x'}{\varepsilon}\right\}$. Then after some algebra manipulation, we obtain that the utility indifference bid price p^b for the approximated robust problem is given by

$$p^b = -h(t,y) + \varepsilon \log\left\{\frac{(\varepsilon \log(2) + u^a(t,y))}{u^{approx}(t,x)} + 1\right\}.$$

5.8 Conclusions

In this chapter, we have indirectly studied the problem of pricing and hedging of derivative securities on an incomplete financial market when insufficient number of assets are available for investment, and using as a criterion for selection the minimisation of the expected shortfall under robust preferences (the minimum expected shortfall over a set of probability measures called priors). We assume that the set of priors is given by a subset of all equivalent probability measures whose Girsanov kernels are progressively measurable processes that lie in a compact convex set.

In order to use recent results on utility maximisation problems under robust preferences developed in [82], we deal with an approximation to the original problem (the robust ε-expected shortfall hedging problem) by taking as minimisation criterion the function

$$U_\varepsilon(x) = \varepsilon \log\left(\frac{1 + \exp\left\{-\frac{x}{\varepsilon}\right\}}{\exp\left\{-\frac{x}{\varepsilon}\right\}}\right).$$

This function is strictly increasing and strictly convex. Translating the problem into an usual utility maximisation problem, we characterise the optimal hedging strategy, the value function and the indifference bid price by tackling mainly its associated dual problem. It turned out that the utility maximisation problem of the ε-approximation is similar to the problem of utility maximisation problem with exponential preferences but under robust preferences, extending previous results on the later problem.

Chapter 6

Future Research

In this chapter, we discuss briefly some lines of future research that arise naturally from the topics developed through the chapters in this thesis.

In regard to Chapters 3 and 4, a natural extension for future research is the analysis and computation with other measures of risk of interest, such that one-sided moments (see [24]), or some convex measures of risk. Another line of future research is to explore accurate numerical methods, Markov chain approximation or spectral approximation methods to the solution of the risk-PDEs.

On the other hand, under the perspective of model risk, it will also be interesting to look at measures of risk that capture "risk" in the volatility of the position, and/or combinations of drift and volatility risk.

With relation to Chapter 2 on the discrete-time approximation to the problem of expected shortfall, it will be of interest to investigate under which conditions we obtain convergence to the continuous-time case. A further direction of research is to combine ideas from Chapter 5 of looking at the problem of utility maximisation with exponential preferences as an approximation to the problem of minimising the expected shortfall. Musiela and Zariphopoulou in [61] have analysed the former problem in a similar set-up of Chapter 2 (discrete-time two-factor model). It is appealing to relate their results on the optimal value function and strategies wit the one obtained in Chapter 2 and also try to compare with the relations obtained in the continuous case in Chapter 5. This relations may give hints on the conditions to pass to the limit to the continuous case.

Another interesting case-study in the discrete-time model is to explore its robust version corresponding to WCS.

Concerning the continuous-time model in Chapter 5, it is worth while to examine the issue of the convergence of value functions, optimal strategies and indifference bid prices on the utility maximisation problem under exponential preferences ($U(x) =$

$-\varepsilon e^{-\frac{1}{\varepsilon}x}$) when the risk aversion parameter $\frac{1}{\varepsilon}$ increases to infinity ($\varepsilon \to 0$), as it is related to the fact that in the limit, we expect this problem to converge to the solution of the expected shortfall problem. Some recent research in this direction that motivates this line are: [87], [50] and [7].

Throughout the thesis, we have assumed a fixed and deterministic risk horizon. This transcribes into the valuation and hedging problem of only European-type derivative securities. It will be, of course, of interest to incorporate into the study American-type securities.

Appendix A

Some Important Examples of Risk Measures

Shortfall risk measures is a class of monetary measures of risk that contain many of the most popular measures of risk (VaR, AVaR and WCS when the set of measures $\mathcal{P}$ is a singleton). In this appendix, we present some general properties and some examples.

A.1 Shortfall risk measures

Assume the set of positions $\mathfrak{X}$ consists only of bounded measurable random variables (i..e., $\mathfrak{X} := L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$. Let $l : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function such that $\mathbb{E}\left[l\left(X\right)\right] < \infty$ for $X \in \mathfrak{X}$. We will call the function l the *loss function* associated to the shortfall risk.

For a given loss function l and a point x_0 in the interior of the range of l, consider the acceptance set

$$A_l(x_0) := \left\{X \in \mathfrak{X} : \mathbb{E}\left[l\left(X^{-}\right)\right] \leq x_0\right\}. \tag{A.1}$$

It defines a measure of risk ρ_{l,x_0} with the following characteristics.

Proposition 120 *The measure of risk ρ_{l,x_0} can be represented as*

$$\rho_{l,x_0}(X) = \max_{\mathbb{Q}\in\mathcal{M}_1} \left\{\mathbb{E}_{\mathbb{Q}}\left[-X\right] - \alpha_{\min}(\mathbb{Q})\right\},$$

where the minimal penalty function $\alpha_{\min}$ is given by

$$\alpha_{\min}(\mathbb{Q}) = \inf_{\lambda>0} \frac{1}{\lambda}\left(x_0 + \mathbb{E}_{\mathbb{Q}}\left[l^*\left(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right) \quad \textit{for } \mathbb{Q} \in\mathcal{M}_1(\mathbb{P}).$$

The function $l^*(y)$ is the Fenchel-Legendre transform of the convex function l defined by

$$l^*(y) = \sup_{x \in \mathbb{R}} (yx - l(x)) \,. \tag{A.2}$$

Furthermore, $l^*(y)$ is a convex function and takes finite values.

From the representation in the above proposition it is clear that ρ_{l,x_0} is a convex measure of risk.

Note that as the acceptance set $A_l(x_0)$ is defined only in terms of $X^- = \min(X, 0)$, without loss of generality we can assume

Assumption 121 *The loss function l is such that $l(x) = 0$ for $X \leq 0$.*

By the definition of a measure of risk as capital requirement: *"the risk of a position is the minimal amount of capital which, if added to the position and invested in a risk-free manner, makes the position acceptable"* we can associate a hedging problem to each shortfall risk measures as follows.

A.1.1 The hedging problem

Assume and investor needs to pay the random amount $H_T \geq 0$ at time T. We need to construct a riskless portfolio such that makes the position $-H_T$ acceptable. Assume such portfolio is (v, π), $\pi \in A(v)$. The portfolio is riskless using ρ_{l,x_0} if

$$\mathbb{E}\left[l\left(v - V_T^{(v,\pi)} \right) \right] \leq x_0,$$

and makes the position $-H_T$ acceptable if

$$\mathbb{E}\left[l\left(H_T - v \right) \right] \leq x_0.$$

The two previous conditions together imply that the shortfall of the total position $X = -H_T + V_T^{(v,\pi)}$ is acceptable, this is,

$$\mathbb{E}\left[l(H_T - V_T^{(v,\pi)}) \right] \leq x_0.$$

On the other hand, by the interpretation of a measure of risk as capital requirement, $\rho_{l,x_0}(-H_T)$ is the minimal amount such that makes the position $-H_T$ acceptable. Then there exists a portfolio $(\rho_{l,x_0}(-H_T), \hat{\pi})$ such that satisfies the above condition in the equality, i.e.,

$$\mathbb{E}\left[l(H_T - V_T^{(\rho_{l,x_0}(-H_T),\hat{\pi})}) \right] = x_0.$$

But this implies $\rho_{l,x_0}(-H_T) \le v$ and

$$\mathbb{E}\left[l\left(H_T - V_T^{(v,\pi)}\right)\right] \le \mathbb{E}\left[l\left(H_T - V_T^{(\rho_{l,x_0}(-H_T),\hat{\pi})}\right)\right] = x_0.$$

From previous expression, we see that the *related hedging problem* can be formulated as follows:

For a given initial capital x_0 find a hedging strategy (v, π), $\pi \in A(v)$ such that minimises the expected shortfall risk for the $\pi \in A(v)$ such that attains the infimum in loss function l under the cost constraint $v \le x_0$. Or, find $\pi \in A(v)$ which attains the infimum in

$$\inf_{\pi \in A(v)} \mathbb{E}\left[l\left(\left(H_T - V_T^{(v,\pi)}\right)^+\right)\right].$$

Note that if $x_0 \ge \sup_{\mathbb{Q}\in\mathcal{M}_1} \mathbb{E}_{\mathbb{Q}}[H_T]$ (the initial capital is larger than the super-replication price for H) then the problem becomes trivial as the shortfall becomes zero and the optimal strategy would be the super-replication strategy. Thus, in order for the problem to make sense, we need the stronger cost constraint

$$\sup_{\mathbb{Q}\in\mathcal{M}_1} \mathbb{E}_{\mathbb{Q}}[H_T] \le x_0.$$

A.1.2 Robust representation of shortfall risk measures

Assume that instead of defining the acceptance set in (A.1) only with respect to the physical probability $\mathbb{P}$, we would like to be more conservative and incorporate a given family $\mathcal{P} \subset \mathcal{M}_1$of measures. We define the acceptance set $A_l^{\mathcal{P}}(x_0)$ as

$$A_l^{\mathcal{P}}(x_0) := \left\{X \in \mathfrak{X} : \mathbb{E}_{\mathbb{Q}}\left[l\left(X^-\right)\right] \le x_0 \text{ for all } \mathbb{Q} \in \mathcal{P}\right\}.$$

The corresponding risk measure $\rho_{l,x_0}^{\mathcal{P}}$ will be given by the following proposition.

Proposition 122 (Cor. 4.110 in [31]) . *The corresponding measure of risk $\rho_{l,x_0}^{\mathcal{P}}$ associated to the acceptance set $\mathrm{A}_l^{\mathcal{P}}(x_0)$ is a convex measure of risk and can be represented in terms of the penalty function*

$$\alpha(\mathbb{Q}) = \inf_{\lambda>0} \frac{1}{\lambda}\left(x_0 + \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[l^*\left(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right) \text{ for } \mathbb{Q} \in \mathcal{M}_1(\mathbb{P}),$$

where l^* is the Fenchel-Legendre transform of l defined in (A.2). Thus

$$\begin{aligned}\rho_{l,x_0}^{\mathcal{P}}(X) &= \max_{\mathbb{Q}\in\mathcal{M}_1} \left\{\mathbb{E}_{\mathbb{Q}}[-X] - \alpha(\mathbb{Q})\right\} \\ &= \max_{\mathbb{Q}\in\mathcal{M}_1} \left\{\mathbb{E}_{\mathbb{Q}}[-X] - \inf_{\lambda>0}\frac{1}{\lambda}\left(x_0 + \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[l^*\left(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right)\right\}.\end{aligned}$$

Example 123 (Lower partial moments) *Consider the special case of shortfall risk measures defined in Section A.1 for the loss function*

$$l(x) = \begin{cases} \frac{1}{p}x^p & \text{if} \quad x \geq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{A.3}$$

for some $p > 1$ and point x_0 in the interior of the range of l. The Fenchel-Legendre transform for l in (A.3) is given by

$$l^*(x) = \begin{cases} \frac{1}{q}x^q & \text{if} \quad x \geq 0 \\ +\infty & \text{otherwise,} \end{cases}$$

with $q = \frac{p}{p-1}$.

For this case, the measure of risk ρ_{p,x_0} is given by

$$\begin{aligned} \rho_{p,x_0}(X) &= \max_{\mathbb{Q}\in\mathcal{M}_1} \left\{\mathbb{E}_{\mathbb{Q}}\left[-X\right] - \alpha^p_{\min}(\mathbb{Q})\right\} \\ &= \max_{\mathbb{Q}\in\mathcal{M}_1} \left\{\mathbb{E}_{\mathbb{Q}}\left[-X\right] - \inf_{\lambda>0}\frac{1}{\lambda}\left(x_0 + \mathbb{E}_{\mathbb{Q}}\left[l^*\left(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right)\right\}. \end{aligned} \tag{A.4}$$

Lemma 124 (Example 4.109 in [31].) *For $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q(\Omega, \mathcal{F}, \mathbb{P})$ the infimum in (A.4) is attained for*

$$\lambda_{\mathbb{Q}} = \left(\frac{px_0}{\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^q\right]}\right)^{1/q}.$$

Furthermore, as $\mathfrak{X} := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, it is enough to use $\mathcal{M}_a$ instead of $\mathcal{M}_1$ in (A.4) (see [31, Theorem 4.31]). Hence, the risk measure ρ_{p,x_0} for a position X is given by

$$\rho_{p,x_0}(X) = \max_{\mathbb{Q}\in\mathcal{M}_a} \left\{\mathbb{E}_{\mathbb{Q}}\left[-X\right] - (px_0)^{1/p}\,\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^q\right]^{1/q}\right\}.$$

When limit $p \downarrow 1$, it corresponds to the case $l(x) = x^+$ and the measure of risk reduces to[1]

$$\rho_{1,x_0}(X) = \max_{\mathbb{Q}\in\mathcal{M}_a} \left\{\mathbb{E}_{\mathbb{Q}}\left[-X\right] - x_0\left\|\frac{d\mathbb{Q}}{d\mathbb{P}}\right\|_\infty\right\}.$$

The acceptance set is given by

$$\mathrm{A}_p(x_0) := \left\{X \in \mathfrak{X} : \mathbb{E}\left[\frac{1}{p}\left(X^-\right)^p\right] \leq x_0\right\}. \tag{A.5}$$

[1] The norm $\|\cdot\|_\infty$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as

$$\|z\|_\infty = \inf\{c \geq 0 : \mathbb{P}[\ |\ z\ |\ > c] = 0\}.$$

And the hedging problem for a liability $H_T \geq 0$ at time T becomes: For a given initial capital x_0 find a hedging strategy (v, π), $\pi \in \mathrm{A}(v)$ such that minimises the expected shortfall risk for the loss function l under the cost constraint $v \leq x_0$. Or, find $\pi \in \mathrm{A}(v)$ such that attains the infimum in

$$\inf_{\pi \in \mathrm{A}(v)} \mathbb{E}\left[\frac{1}{p}\left(\left(H_T - V_T^{(v,\pi)}\right)^+\right)^p\right],$$

under the cost constraint

$$\sup_{\mathbb{Q} \in \mathcal{M}_1} \mathbb{E}_{\mathbb{Q}}[H_T] \leq x_0.$$

One popular approach for the valuation and hedging of contingent claims when markets are incomplete is the utility maximisation/indifference pricing approach. In order to show its relation with shortfall risk measures and their associated hedging problems, we present the example of the entropic measure of risk which corresponds to the exponential utility function.

Example 125 (Entropic risk measure and exponential utility maximisation) *Consider similar assumptions as in Example 2 but with an acceptance set given by*

$$\mathrm{A}_{\exp}(x_0) := \left\{X \in \mathfrak{X} : \text{such that } \mathbb{E}\left[e^{-\beta X}\right] \leq x_0\right\},$$

*with x_0 an interior point in the range of the function$e^{-\beta X}$. It defines the **entropic measure of risk** as*

$$\rho_{\exp}^{x_0}(X) := \inf\{m \in \mathbb{R} : \mathbb{E}\left[e^{-\beta(m+X)}\right] \leq x_0\} = \frac{1}{\beta}\left\{\log \mathbb{E}\left[e^{-\beta X}\right] - \log x_0\right\}.$$

This is a convex measure of risk with representation

$$\rho_{\exp}^{x_0}(X) = \sup_{\mathbb{Q} \in \mathcal{M}_1} \left\{\mathbb{E}_{\mathbb{Q}}[-X] - \alpha_{\min}(\mathbb{Q})\right\}, \tag{A.6}$$

where $\mathcal{M}_1$ denotes the set of all probability measures,

$$\alpha_{\min}(\mathbb{Q}) = \frac{1}{\beta}\left\{H(\mathbb{Q} \mid \mathbb{P}) - \log x_0\right\} = \frac{1}{\beta}\left\{\mathbb{E}_{\mathbb{Q}}\left[\log \frac{d\mathbb{Q}}{d\mathbb{P}}\right] - \log x_0\right\}$$

and $H(\mathbb{Q} \mid \mathbb{P})$ is the entropy between the measures $\mathbb{Q}$ and $\mathbb{P}$. The upper bound in (A.6) is attained by a measure with density

$$\frac{e^{-\beta X}}{\mathbb{E}\left[e^{-\beta X}\right]}.$$

Furthermore, one can show (see [31]) that the penalty function $\alpha_{\min}(\mathbb{Q})$ can be written as

$$\alpha_{\min}(\mathbb{Q}) = \sup_{X\in\mathfrak{X}}\left\{\mathbb{E}_{\mathbb{Q}}\left[-X\right] - \frac{1}{\beta}\log\mathbb{E}\left[e^{-\beta X}\right] - \frac{\log x_0}{\beta}\right\}$$

and that the dual identity

$$\log\mathbb{E}\left[e^{\beta X}\right] = \sup_{\mathbb{Q}\in\mathcal{M}_1}\left\{\mathbb{E}_{\mathbb{Q}}\left[X\right] - \frac{1}{\beta}H(\mathbb{Q}\mid\mathbb{P})\right\} \tag{A.7}$$

holds.

Assume we are interested in measuring the risk of a short position of a claim with payoff $H_T \geq 0$ at time T. Its risk is $\rho^l_{x_0}(-H_T)$, and by the way the acceptance set $\mathrm{A}^l(x_0)$ was defined, the quantity $\rho^l_{x_0}(-H_T)$ is the smallest amount m such that there exists an admissible strategy (m,π), $\pi\in\mathrm{A}(v)$ whose final value $V_T^{(m,\pi)}$ satisfies

$$\mathbb{E}\left[e^{-\beta\left(V_T^{(m,\pi)}-H_T\right)}\right] \leq x_0.$$

Using the identity in (A.7) for the final position $V_T^{(m,\pi)} - H_T$ we get the related hedging problem: Find a strategy (m,π), $\pi\in\mathrm{A}(v)$ such that

$$\inf_{V_T}\log\mathbb{E}\left[e^{\beta\left(V_T^{(m,\pi)}-H_T\right)}\right] = \sup_{\mathbb{Q}\in\mathcal{M}_1}\left\{\mathbb{E}_{\mathbb{Q}}\left[\left(V_T^{(m,\pi)} - H_T\right)\right] - H(\mathbb{Q}\mid\mathbb{P})\right\}. \tag{A.8}$$

The supremum and infimum above are achieved by $\mathbb{Q}^*$ and π^* related by

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = ce^{-\beta\left(V_T^{(\rho^l_{x_0}(-H_T),\pi^*)}-H_T\right)}.$$

From the formulation to the hedging problem in (A.8), we can see the similitude with the problem of utility maximisation and indifference pricing using the utility function $U(x) = -e^{-\beta x}$. See [40] for a general view on utility maximisation and indifference pricing.

Remark 126 *The measure of risk $WCS_{\mathcal{P}}$ defined in (1.12) can be seen as a particular case of the robust formulation of shortfall risk measures in the following way. For $\mathcal{P} = \{\mathbb{P}\}$, take $x_0 = 0$ and the loss function $l(x) = x$ for $x \geq 0$, then*

$$l^*(z) = \begin{cases} 0 & \text{if} \quad z = 1, \\ +\infty & \text{otherwise.} \end{cases}$$

One can show that the penalty function in this case is such that $\alpha_{\min}(\mathbb{Q}) = \infty$ if $\mathbb{Q}\neq\mathbb{P}$, thus the risk measure remains $\rho(X) = \mathbb{E}\left[-X\right]$. If now we incorporate the robust version for a set of probabilities $\mathcal{P}\subset\mathcal{M}_1$, the corresponding risk measure becomes

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{P}}\mathbb{E}\left[-X\right].$$

Which is exactly the expression we have for $WCS_{\mathcal{P}}$.

A.2 State-dependent utility functions derived from shortfall risk measures

The problem of minimising the expected shortfall risk with loss function l can be reformulated as a problem of maximising expected utility in the following way.

Introduce the state-dependent utility function

$$U_l(z,\omega) = l(H(\omega)) - l\left((H(\omega)-z)^+\right).$$

As the function $l(x)$ is increasing convex, the term $-l\left((H(\omega)-z)^+\right)$ becomes an increasing concave function on the variable z, exactly as it is needed in the usual formulation of a utility maximisation problem. The term $l(H(\omega))$ is needed to shift the values to their original level.

Then the primal problem becomes

$$\sup_{\pi\in\mathcal{A}(x)} \mathbb{E}\left[U_l\left(X_T^{(x,\pi)}(\omega),\omega\right)\right]$$

under the constraint

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[X_T^{(x,\pi)}\right] \le \tilde{x}.$$

A.2.1 Examples of loss functions, their corresponding utility functions and Fenchel-Legendre transforms

1. Probability of over perform: Here $U(z) = 1_{z\le 0}$, its Fenchel-Legendre transform is

$$V(\xi,\omega) = \xi H(\omega)\mathbb{1}_{\{\xi H(\omega)\ge 1\}} + \mathbb{1}_{\{0\le \xi H(\omega)<1\}}$$

2. Expected shortfall: Here $l(x) = x$ and the utility function is

$$U_1(z,\omega) = (H(\omega)) - \left((H(\omega)-z)^+\right) = H(\omega)\wedge z$$

and its Fenchel-Legendre transform is

$$V(\xi,\omega) = (1-\xi)^+ H(\omega).$$

Appendix B

Duality Theory for Optimal Investment Problems on Semimartingales

In this Appendix, we present a brief introduction to the duality theory for optimal investment problems. For the first part on the usual utility maximisation problem we follow closely [55], and [4]; and for the robust utility maximisation case we refer to [70] and [82].

B.1 The model

Assume the market model consists of $d+1$ assets, one bond and d stocks, and that they are already in discounted terms. Denote by $(S_i)_{1\leq i\leq d}$ the price process of the d stocks, that are assumed to be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{0\leq t\leq T}, \mathbb{P})$. The time horizon is T and finite. To simplify notation we assume that $\mathcal{F} = \mathcal{F}_T$.

A (self-financing) portfolio Π is defined as a pair (x, H) where the constant x is the initial value of the portfolio, and $(H_i)_{1\leq i\leq d}$ is a predictable S-integrable process, where H_t^i specifies how many units of asset i are held in the portfolio at time t. The value process $(X_t)_{0\leq t\leq T}$ of such a portfolio is given by

$$X_t = X_0 + \int_0^t H_u dS_u \quad 0 \leq t \leq T. \tag{B.1}$$

Let us denote by $\mathcal{X}(x)$ the family of wealth processes with nonnegative capital at any instant, that is,

$$\mathcal{X}(x) = \{X \geq 0 : X \text{ is defined by (B.1) with } X_0 = x\}$$

We make the following assumption on the family of equivalent local martingale measures $\mathcal{M}$,

$$\mathcal{M} \neq \emptyset. \tag{B.2}$$

This condition is intimately related to the absence of arbitrage opportunities on the security market. See [17] for precise statements and references.

We also consider that an investor models her preferences via a utility function $U : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ for wealth at maturity time T. The assumptions on the utility function are as follows.

Assumption 127 (Usual Regularity Conditions) *A utility function $U : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ satisfies the usual regularity conditions if it is increasing on $\mathbb{R}$, continuous on $\{U > -\infty\}$, differentiable and strictly concave on the interior of $\{U > -\infty\}$, and satisfies*

$$U'(\infty) = \lim_{x\to\infty} U'(x) = 0.$$

Denoting by dom(U) the interior of $\{U > -\infty\}$, we assume that we have one of the two following cases.

Case 128 (negative wealth not allowed) *In this case dom$(U) = (0, \infty)$ and assume that U satisfies the conditions*

$$U'(0) = \lim_{x\searrow 0} U'(x) = \infty.$$

Case 129 (negative wealth allowed) *In this case dom$(U) = \mathbb{R}$, we assume*

$$U'(-\infty) = \lim_{x\searrow -\infty} U'(x) = \infty.$$

B.2 The single prior case

B.2.1 The primal problem

For a given initial capital $x > 0$, the investor's objective is to to maximise the expected value of terminal utility. The value function of this problem is denoted by

$$u(x) = \sup_{X\in\mathcal{X}(x)} \mathbb{E}\left[U(X_T)\right]. \tag{B.3}$$

Intuitively speaking, the value function u plays the role of the utility function of the investor at time 0, if she subsequently invests in an optimal way. To exclude trivial cases in (B.3) we shall assume that the value function u is not degenerate:

Assumption 130

$$u(x) < \sup_{\xi} U(\xi), \ \textit{for some } x \in \textit{dom}(U).$$

B.2.2 The dual problem

A well-known tool to the study of optimisation problem is the use of duality relationships. In our setting, the dual relation is among the space of convex functions and semimartingales.

The Fenchel-Legendre transform (or conjugate function) of the utility function U is defined as

$$V(y) = \sup_{dom(U)} [U(x) - xy], \quad y > 0,$$

where $dom(U)$ denotes the domain of definition of the utility function U.

It is well known (see, e.g. [73]) that if U satisfies the hypotheses stated above then V is a continuously differentiable, decreasing, strictly convex function satisfying $V'(0) = -\infty$, $V'(\infty) = 0, V(0) = U(\infty)$ and $V(\infty) = U(0)$, and also the following relation holds true

$$U(x) = \inf_{y>0} [V(x) + xy], \quad x \in dom(U).$$

The derivative of U and V are related as

$$U'(x) = y \quad \Longleftrightarrow \quad x = -V'(y).$$

Define the set $\mathcal{Y}$ of nonnegative semimartingales, which represents the dual set of X in the following sense

$$\mathcal{Y} = \{Y \geq 0 : Y_0 = 1 \text{ and } XY \text{ is a supermartingale for all } X \in \mathcal{X}\}.$$

The set $\mathcal{Y}$ contains the density process for all $\mathbb{Q}\in\mathcal{M}$. For $y > 0$, define

$$\mathcal{Y}(y) = y\mathcal{Y} = \{y\mathcal{Y} : Y \in \mathcal{Y}\},$$

and consider the following optimisation problem called the Dual problem

$$v(y) = \sup_{Y\in\mathcal{Y}(y)} \mathbb{E}[V(Y_T)]. \tag{B.4}$$

B.2.3 Relation between the primal and dual problem

In this section, we present some results based on the first case in Assumption 127. For the general case when negative wealth is allowed we refer to [54, Theorem 2.2], and [64] when random endowment is permitted.

The primal and dual value functions $u(x)$ and $v(y)$, respectively, are related as conjugates. This is established in [54, Theorem 2.1] and recalled below.

Theorem 131 *Suppose that (B.2), Assumption 127 first case and*

$$u(x) < \infty \text{ for some } x > 0$$

hold. Then,

1. $u(x) < \infty$ *for all* $x > 0$*, and there exists* $y_0 \geq 0$ *such that* $v(y)$ *is finitely valued for* $y > y_0$*. The value functions* u *and* v *are conjugate*

$$\begin{aligned} v(y) &= \sup_{dom(u)} [u(x) - xy], \; y > 0 \\ u(x) &= \inf_{y>0} [v(y) + xy], \; x \in dom(u). \end{aligned}$$

The function u *is continuously differentiable on* $(0, \infty)$ *and the function* v *is strictly convex on* $\{v < \infty\}$*. The functions* u *and* v *satisfy*

$$u'(0) = \lim_{x \to 0} u'(x) = \infty \text{ and } v'(\infty) = \lim_{y \to \infty} v'(y) = 0.$$

The optimal solution $\hat{Y}(y) \in \mathcal{Y}(y)$ *to (B.4) exists and is unique provided that* $v(y) < \infty$*.*

Definition 132 *A utility function* U *satisfying Assumption 127 is said to have "reasonable asymptotic elasticity" if*

$$AE_{+\infty}(U) = \lim\sup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1$$

and, in case 2 of Assumption 127, we also have

$$AE_{-\infty}(U) = \lim\sup_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1.$$

Focused on the first case in Assumption 127, in [54, Theorem 2.2], it was shown that the assumption on "reasonable asymptotic elasticity" is sufficient for the existence of an optimal solution $X \in \mathcal{X}(x)$ to the primal problem (B.3) and the function u is increasing, strictly concave, continuously differentiable and such that $u'(0) = \infty$, and $u'(\infty) = 0$. Furthermore, the value function to the dual problem has the representation

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{M}_1} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \tag{B.5}$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}$ denotes the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}_T$.

Furthermore, in [55, Theorem 2] the necessary condition for the existence solution to the primal problem was also analysed and recalled below

Theorem 133 *Suppose that (B.2), Assumption 127 first case and*

$$v(y) < \infty \text{ for all } y > 0$$

hold. Then in addition to the assertions of Theorem 131, we have the following

1. *The value functions u and $-v$ are continuously differentiable, increasing and strictly concave on $(0, \infty)$ and satisfy*

$$\begin{aligned} u'(\infty) &= \lim_{x \to \infty} u'(x) = 0 \\ -v'(0) &= \lim_{y \to 0} -v'(y) = \infty. \end{aligned}$$

2. *The optimal solution $\hat{X}(x) \in \mathcal{X}(x)$ to (B.3) exists, for any $x > 0$, and is unique. In addition, if $y = u'(x)$ then*

$$U'(\hat{X}_T(x)) = \hat{Y}_T(y),$$

where $\hat{Y}(y) \in \mathcal{Y}(x)$ is the optimal solution to (B.4). Moreover, the process $\hat{X}(x)\hat{Y}(y)$ is a martingale.

3. *The dual value function v satisfies (B.5).*

B.3 The robust utility maximisation case

With the same model setup as in section B.1, in the robust version of the utility maximisation problem we need to formulate conditions on the subset $\mathcal{P}$ of all probability measures from which the robust utility functional will be defined.

Assumption 134 *The set $\mathcal{P}$ satisfies*

a) $\mathcal{P}$ is convex,

b) $\mathbb{P}[A] = 0$ for some $A \in \Omega$ if and only if $\mathbb{Q}[A] = 0$ for all $\mathbb{Q} \in \mathcal{P}$,

c) the set $\left\{\frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{P}\right\}$ is weakly compact in $L^1(\mathbb{P})$.

This set of assumptions are equivalent to the ones in [82] as we discuss next. In [82] the assumptions needed are the following set.

It is necessary that each measure $\mathbb{Q} \in \mathcal{P}$ respects $\mathbb{P}$-null sets, for otherwise a stochastic integral defined with respect to $\mathbb{P}$ might make no sense under $\mathbb{Q}$, this means we need,

1. $\mathbb{Q} << \mathbb{P}$ for all $\mathbb{Q} \in \mathcal{P}$ (this means if for $A \in \Omega$, $\mathbb{P}[A] = 0$ then $\mathbb{Q}[A] = 0$ and $\mathbb{Q} = f\mathbb{P}$ for some f).

   Without loss of generality we can assume,

2. $\mathcal{P}$ is convex.

   For practical purposes, one may need that the measure attaining the optimum in the robust utility case also belongs to the set $\mathcal{P}$. In this way, one could treat the problem as an usual utility maximisation problem on the least favourable measure. This is achieved if

3. $\mathcal{P}$ is closed in some reasonable topology such as total variation (the set $\left\{\frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{P}\right\}$ is closed in $L^0(\mathbb{P})$).

   If furthermore, we need to guarantee the existence of the least favourable measure, we need to assume

4. $\mathcal{P}$ is relatively compact in a reasonable topology.

   Finally, an assumption on the "sensitivity" in the set $\mathcal{P}$ is that

5. $\mathbb{Q}[A] = 0$ for all $\mathbb{Q} \in \mathcal{P}$ implies $\mathbb{P}[A] = 0$.

The equivalence in the set of assumptions becomes clear as, number 1 and number 5 imply part a) in Assumption 134; part b) in Assumption 134 is slightly weaker than $\mathbb{Q} \sim \mathbb{P}$ for $\mathbb{Q} \in \mathcal{P}$. On the other hand, assuming c) and b) in Assumption 134 implies number 1 and $\mathcal{M}_e \neq \emptyset$. And the parts a), b) and c) in Assumption 134 hold if and only if the set $\left\{\frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{P}\right\}$ is weakly compact in $L^1(\mathbb{P})$.

B.3.1 The multiple priors primal problem

By the assumptions in the preceding section for each $\mathbb{Q} \in \mathcal{P}$ one can associate a random variable $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_T}$ at the final time T, and thus by identification a density process $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$,$0 \leq t \leq T$ that defines the change of measure.

Denote by $D^i_T := \frac{d\mathbb{Q}^i}{d\mathbb{P}}|_{\mathcal{F}_T}$ (hence $(D^i_t)_{0\leq t\leq T}$) the density process for $\mathbb{Q}^i \in \mathcal{P}$ and by $Z^j_T := \frac{d\mathbb{Q}^j}{d\mathbb{P}}|_{\mathcal{F}_T}$ (hence $\left(Z^j_t\right)_{0\leq t\leq T}$) the density processes corresponding to $\mathbb{Q}^j \in \mathcal{M}_e$.

Take

$$Z^j_i(T) := \frac{Z^j_T}{D^i_T}.$$

The process $\left(Z^j_i(t)\right)_{0\leq t\leq T}$ is a $\mathbb{Q}^j$-local martingale.

Let $\mathbb{Q}^i \in \mathcal{P}$ be fix momentarily. The usual primal value function for the utility maximisation problem under the measure $\mathbb{Q}^i$ is given by

$$u_{\mathbb{Q}^i}(x) = \sup_{X\in\mathcal{X}(x)} \mathbb{E}_{\mathbb{Q}^i}\left[U(X_T)\right].$$

Now, as we want to incorporate the set of priors $\mathcal{P}$, instead of defining the primal problem under a specific $\mathbb{Q}^i \in \mathcal{P}$ we need to have it defined on the least favourable probability measure in $\mathcal{P}$, this is,

$$u(x) = \sup_{X\in\mathcal{X}(x)} \inf_{\mathbb{Q}^i\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}^i}\left[U(X_T)\right]. \tag{B.6}$$

B.3.2 The multiple priors dual problem

For the dual problem, consider as before the densities D_T^i, Z_T^j and $Z_i^j(T)$. The dual value function for the utility maximisation problem under a fixed measure $\mathbb{Q}^i \in \mathcal{P}$ is

$$\begin{aligned} v_{\mathbb{Q}^i}(\lambda) &= \inf_{\mathbb{Q}^j\in\mathcal{M}_e} \mathbb{E}\left[D_T^i V\left(\lambda \frac{Z_T^j}{D_T^i}\right)\right] \\ &= \inf_{\mathbb{Q}^j\in\mathcal{M}_e} \mathbb{E}\left[D_T^i V\left(\lambda Z_i^j(T)\right)\right]. \end{aligned}$$

Thus, the dual value function of the robust utility maximisation problem is given as

$$v(\lambda) := \inf_{\mathbb{Q}^i\in\mathcal{P}} v_{\mathbb{Q}^i}(\lambda) = \inf_{\mathbb{Q}^i\in\mathcal{P}} \inf_{\mathbb{Q}^j\in\mathcal{M}_e} \mathbb{E}\left[D_T^i V\left(\lambda Z_i^j(T)\right)\right],$$

where the function $V(y)$ is the Fenchel-Legendre transform of U.

B.3.3 Relation between the primal and dual problem

In previous section we recall that necessary and sufficient conditions for the existence of optimal strategies is the finiteness of the dual value function $v_{\mathbb{Q}^i}(\lambda)$ for $\mathbb{Q}^i \in \mathcal{M}_e$. As shown in [82], this condition translates in the robust setting as

$$v_{\mathbb{Q}^i}(\lambda) < \infty \text{ for all } y > 0 \text{ and each } \mathbb{Q}^i \in \mathcal{M}_e. \tag{B.7}$$

But this condition holds as soon as $v_{\mathbb{Q}^i}(\lambda)$ is finite for all $\mathbb{Q}^i \in \mathcal{M}_e$ and the asymptotic elasticity of the utility function $AE(U)$ is strictly less than one. While it is sufficient to assume (B.7) when all measures in $\mathcal{P}$ are equivalent to $\mathbb{P}$, we need to assume $AE(U) < 1$ to get some regularity results in the general case.

The main results in the robust case are exposed in the next theorem from [82, Theorem 2.6].

Theorem 135 *In addition to Assumption 134 assume (B.7). Then both value functions u and v take only finite values and satisfy*

$$u'(\infty-) = 0 \text{ and } v'(0+) = -\infty.$$

For any $x > 0$ there exists an optimal strategy $\hat{X} \in \mathcal{X}(x)$ and a measure $\hat{\mathbb{Q}} \in \mathcal{P}$ such that

$$u(x) = \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}\left[U(\hat{X}_T)\right] = \mathbb{E}_{\hat{\mathbb{Q}}}\left[U(\hat{X}_T)\right] = u_{\hat{\mathbb{Q}}}(x).$$

In particular, the supremum and infimum in the primal value function in (B.6) are attained. There also exist some $\hat{y}$ in the superdifferental of $u(x)$ and some $Y \in \mathcal{Y}_{\mathbb{P}}(\hat{y})$ such that,

$$v(\hat{y}) = \mathbb{E}\left[\hat{Z} V\left(\frac{Y_T}{\hat{Z}}\right)\right], \text{ and } \hat{X}_T = I\left(\frac{Y_T}{\hat{Z}}\right) \ \hat{\mathbb{Q}}\text{-a.s.},$$

where $\hat{Z} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}$ and $I = -V'$. Furthermore, $\hat{X}Y$ is a martingale under $\mathbb{P}$, and the dual value function satisfies

$$v(y) = \inf_{\mathbb{Q}^* \in \mathcal{M}_e} \inf_{\mathbb{Q} \in \mathcal{M}_e} \mathbb{E}\left[V\left(y \frac{d\mathbb{Q}^*}{d\mathbb{Q}}\right)\right].$$

If in addition $AE(U) < 1$ holds, then u is strictly concave and v is continuously differentiable. Moreover, $\hat{X}_T Y_T$ is supported by $\{\hat{Z} > 0\}$, i.e.,

$$\left\{\hat{X}_T Y_T > 0\right\} = \{\hat{Z} > 0\} \ \mathbb{P}\text{-a.s.}$$

And if all measures in $\mathcal{P}$ are equivalent to $\mathbb{P}$ we have from [82, Corollary 2.7].

Corollary 136 *In addition to the assumptions in previous theorem, let us assume (B.7) and that all measures in $\mathcal{P}$ are equivalent to $\mathbb{P}$. Then both value functions u and v take only finite values, u is strictly concave, and v is continuously differentiable. Then for each $y > 0$ such that $v(y) > \infty$ there exist $\hat{\mathbb{Q}} \in \mathcal{P}$ and $\hat{Y} \in \mathcal{Y}_{\hat{\mathbb{Q}}}(y)$ such that*

$$v(y) = \mathbb{E}_{\hat{\mathbb{Q}}}\left[V\left(\hat{Y}_T\right)\right].$$

Moreover, $\hat{Y}$ is unique: any other optimal pair $(\mathbb{Q}', Y') \in \{(\mathbb{Q}, Y) : \mathbb{Q} \in \mathcal{P}, Y \in \mathcal{Y}_{\mathbb{Q}}(y)\}$ satisfies $Y' = \hat{Y}$ $\mathbb{P}$-a.s.

On the other hand, for any $x > 0$, the optimal solution $\hat{X} \in \mathcal{X}(x)$ is unique and it is given by

$$\hat{X}_T = I(\hat{Y}_T),$$

where I is the inverse function of U' and $\hat{Y}$ is as above for $\hat{y}$ as in previous theorem.

B.4 Proof of Lemma 107

In this section we check that all conditions in [82, Theorem 2.2] are satisfied.

Proof. For the conditions on the utility function we have that the functions $\mathcal{U}_\varepsilon(x) : \mathbb{R} \to \mathbb{R}$, and $\hat{U}_\varepsilon(x) : \mathbb{R} \to \mathbb{R}$ are strictly increasing, strictly concave and continuously differentiable. In relation with the no random endowment case, our assumption in (5.7) about $X_t \geq 0$ makes sure the functions take only nonnegative values. For the random endowment case, the condition in (5.7) and Assumption 116 on the boundedness of the payoff function $H(y)$ guarantees that $X_T + H(Y_T) > 0$ (see [39]), then the functions also only take values in $(0, \infty)$.

The Inada conditions are:

$$\lim_{x \to -\infty^+} \mathcal{U}_\varepsilon\prime(x) = +\infty, \quad \lim_{x \to +\infty^-} \mathcal{U}_\varepsilon\prime(x) = 0 \quad \text{and} \quad \mathcal{U}_\varepsilon\prime(0) = 1.$$

and

$$\lim_{x \to -\infty^+} \hat{U}_\varepsilon\prime(x) = 1, \quad \lim_{x \to +\infty^-} \mathcal{U}_\varepsilon\prime(x) = 0 \quad \text{and} \quad \mathcal{U}_\varepsilon\prime(0) = 1/2.$$

For the function $\hat{U}_\varepsilon(x)$, the Inada condition at $-\infty$ (or zero) are not fully satisfied, but it is not needed as explained in [75, p. 13 and Sec. 6], arguing that we can always approximate a given utility function uniformly to within any given $\varepsilon > 0$ by one satisfying the Inada condition at $-\infty$ (or zero). With respect to the asymptotic elasticity defined as

$$\begin{aligned} AE_{-\infty}(U) &= \liminf_{x \to -\infty} \frac{xU\prime(x)}{U(x)} \\ AE_{+\infty}(U) &= \limsup_{x \to +\infty} \frac{xU\prime(x)}{U(x)}, \end{aligned}$$

we have

$$AE_{-\infty}(\mathcal{U}_\varepsilon) = +\infty > 1, \quad AE_{+\infty}(\mathcal{U}_\varepsilon) = -\infty < 1,$$

and

$$AE_{-\infty}(\hat{U}_\varepsilon) = 1, AE_{+\infty}(\hat{U}_\varepsilon) = -\infty < 1, \text{ but } \lim_{x \to 0^+} \frac{x\hat{U}_\varepsilon\prime(x)}{\hat{U}_\varepsilon(x)} < 1.$$

For the set of assumptions regarding the set $\mathcal{P}$, we have that $\mathbb{P}[A] = 0$ if and only if $\mathbb{Q}[A] = 0$ for all $\mathbb{Q} \in \mathcal{P}$, as $\mathcal{P}$ is a subset of $\mathcal{M}_e$. That the set $\left\{\frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{P}\right\}$ is convex and closed in $L^0(\mathbb{P})$ follows from [43, Lemma 3.1]. In [43] $\mathcal{P}$ is a subset of $\mathcal{M}_a$, which is more general than in our setting. $\square$

Appendix C

Selecting a Measurable Function

Let

$$z = (z_1, ..., z_p) \in \mathbb{R}^p \text{ and } u = (u_1, ..., u_m) \in \mathbb{R}^m,$$

for p and m two positive integers. Denote by (z, u) the elements in $\mathbb{R}^{p+m}$. For a set $D \subset \mathbb{R}^{p+m}$, let

$$D^z = \{u : (z, u) \in D\}, \quad \Delta = \{z : D^z \text{ is not empty}\}.$$

We call D to be σ-compact if $D = D_1 \cup D_2 \cup ...$ where $D_1, D_2, ...$ are compact sets (every open and closed set is σ-compact). By "almost all z" we mean except for a set of p-dimensional Lebesgue measure 0. A vector valued function $u(z) = (u_1(z), ..., u_m(z))$ is measurable if and only if each component u_i is measurable. By changing a Lebesgue measurable function u on a set of p-dimensional measure 0, we can arrange that u is Borel measurable.

Lemma 137 ([25, Lemma B]) *If D is σ-compact, then there exists a measurable function $u = u(z)$ with $(z, u(z)) \in D$ for almost all $z \in \Delta$.*

Proof. See [25, Lemma B]. □

Appendix D

Some Results Related with sup and inf of Sets and Functions

Definition 138 *Let A be a nonempty set of the affinely extended real numbers $\overline{\mathbb{R}} := \mathbb{R} \cup \pm\infty$, the supremum of A, denoted by $\sup A$ is defined as*

$$\sup A := min \left\{ r \in \overline{\mathbb{R}} : \forall a \in A, a \leq r \right\}.$$

Proposition 139 *Let $A \subseteq \overline{\mathbb{R}}$ nonempty. Then $\sup A$ satisfies $\forall a \in A$, $a \leq \sup A$ and for any arbitrary $\epsilon > 0$ there exists $\hat{a} \in A$ such that $\hat{a} > \sup A - \epsilon$.*

Proof. Suppose there does not exist $\hat{a} \in A$ for which $\hat{a} > \sup A - \epsilon$, but this implies that $\sup A - \epsilon$ is an upper bound for the set A and clearly $\sup A - \epsilon < \sup A$, which contradicts the fact that $\sup A$ is the least upper bound for the set A. Conversely, assume $M \in \overline{\mathbb{R}}$ is such that $M > \sup A$ and $\forall a \in A$, $a \leq M$ and $a \leq \sup A$. Take $\epsilon < \sup A - M$, then there exists an $\hat{a} \in A$ for which $\hat{a} > \sup A - \epsilon > M$, but this inequality contradicts the assumption that M is an upper bound for the set A. □

Lemma 140 *Let A and B two subsets of $\overline{\mathbb{R}}$, then*

$$\sup A + B = \sup A + \sup B$$

Proof. For any $a \in A$ and $b \in B$, we have $a \leq \sup A$ and $b \leq \sup B$, thus $a + b \leq \sup A + \sup B$. This means that $\sup A + \sup B$ is an upper bound for the set $A + B$. Now, take $\epsilon > 0$ and by By Proposition 139 there exist $a \in A$ and $b \in B$ such that $\sup A - \epsilon/2 < a$ and $\sup A - \epsilon/2 < a$, which together give us $\sup A + \sup B - \epsilon < a + b$. Using again Proposition 139 on the set $A + B$, we conclude that $\sup A + \sup B = \sup A + B$. □

Lemma 141 *For two nonempty sets $A \subseteq B \subseteq \overline{\mathbb{R}}$, we have*

$$\sup A \leq \sup B$$

Proof. Take any $a \in A$, and as a also belongs to B then $a \leq \sup B$, which implies that $\sup B$ is an upper bound of the set A, but from the Definition 138, $\sup A \leq \sup B$. $\square$

Lemma 142 *For two nonempty sets $A \subseteq B \subseteq \overline{\mathbb{R}}$, suppose there exists $a \in A$ such that $a = \sup B$, then*

$$\sup A = \sup B.$$

Proof. As $a \in A$ we have $a \leq \sup A$, thus $\sup B \leq \sup A$, and by Lemma 141 we get the equality. $\square$

Lemma 143 *Let f and g two real continuous functions with domain D, then*

$$\sup_{x \in D} \{f(x) + g(x)\} \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x).$$

Proof. Define

$$\begin{aligned} A &:= \{f(x) : x \in D\} \\ B &:= \{g(x) : x \in D\} \\ C &:= \{f(x) + g(x) : x \in D\}. \end{aligned}$$

clearly $C \subseteq A+B$, applying Lemma 141 and Lemma 140 we have $\sup C \leq \sup A+B = \sup A + \sup B$. $\square$

Lemma 144 *Let f and g two continuous real functions with compact domain D. If $\hat{x} \in D$ is such that*

$$f(\hat{x}) = \sup_{x \in D} f(x)$$
$$g(\hat{x}) = \sup_{x \in D} g(x).$$

Then

$$\sup_{x \in D} \{f(x) + g(x)\} = \sup_{x \in D} f(x) + \sup_{x \in D} g(x)$$

Proof. It follow immediately by using Lemma 141, Lemma 140 and Lemma 142. $\square$

References

[1] Abraham, R., Marsden, J. E., and Ratiu, T. (1988). *Manifolds, tensor analysis, and applications*, volume 75 of *Applied Mathematical Sciences*. Springer-Verlag, New York.

[2] Acerbi, C. and Tasche, D. (2002). On the coherence of expected shortfall. *arXiv:cond-mat/0104295 v5*.

[3] Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. *Math. Finance*, 9(3):203–228.

[4] Back, K., Bielecki, T. R., Hipp, C., Peng, S., and Schachermayer, W. (2004). *Stochastic methods in finance*, volume 1856 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

[5] Barrieu, P. and El Karoui, N. (2005). *Indifference Pricing*, chapter Pricing, hedging and optimally designing derivatives via minimization of risk measures. Princeton University Press.

[6] Bertsekas, D. P. and Shreve, S. E. (1978). *Stochastic optimal control*, volume 139 of *Mathematics in Science and Engineering*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York.

[7] Cheridito, P. and Summer, C. (2006). Utility maximization under increasing risk aversion in one-period models. *Finance Stoch.*, 10(1):147–158.

[8] Cisneros-Molina, M. (2005). On explicit computation of transition probability densities for some transformed markov diffusion processes. Working paper.

[9] Cvitanić, J. (2000). Minimizing expected loss of hedging in incomplete and constrained markets. *SIAM J. Control Optim.*, 38(4):1050–1066 (electronic).

[10] Cvitanić, J. and Karatzas, I. (1999). On dynamic measures of risk. *Finance Stoch.*, 3(4):451–482.

[11] Cvitanić, J., Schachermayer, W., and Wang, H. (2001). Utility maximization in incomplete markets with random endowment. *Finance Stoch.*, 5(2):259–272.

[12] Davis, M. (1998). A note on the forward measure. *Finance and Stochastics*, 2:19 – 28.

[13] Davis, M. (2000a). Optimal hedging with basis risk.

[14] Davis, M. (2000b). Option valuation and hedging with basis risk. In *System theory: modeling, analysis and control (Cambridge, MA, 1999)*, volume 518 of *Kluwer Internat. Ser. Engrg. Comput. Sci.*, pages 245–254. Kluwer Acad. Publ., Boston, MA.

[15] Delbaen, F. (2000). *Coherent risk measures*. Cattedra Galileiana. [Galileo Chair]. Scuola Normale Superiore, Classe di Scienze, Pisa.

[16] Delbaen, F. (2002). Coherent risk measures on general probability spaces. In *Advances in finance and stochastics*, pages 1–37. Springer, Berlin.

[17] Delbaen, F. and Schachermayer, W. (2005). *The Mathematics of Arbitrage*. Springer-Verlag.

[18] Detlefsen, K. and Scandolo, G. (2005). Conditional and dynamic convex risk measures. *Finance Stoch.*, 9(4):539–561.

[19] Duffie, D. and Pan, J. (1997). An overview of value at risk. *The Journal of Derivatives*, 1:7–49.

[20] El Karoui, N. and Quenez, M.-C. (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.*, 33(1):29–66.

[21] Elliot, R. J. and Kopp, P. E. (1999). *Mathematics of financial markets*. Springer.

[22] Elliott, R., Siu, T., and Yang, H. (2003). On a generalized form of risk measure. *University of Hong Kong Preprint Series*, (364 (7/03)).

[23] Favero, G. (2001). Shortfall risk minimization under model uncertainty in the binomial case: adaptive and robust approaches. *Math. Methods Oper. Res.*, 53(3):493–503.

[24] Fischer, T. (2003). Risk capital allocation by coherent risk measures based on one-sided moments. *Insurance Math. Econom.*, 32(1):135–146.

[25] Fleming, W. H. and Rishel, R. W. (1975). *Deterministic and stochastic optimal control*. Springer-Verlag, Berlin.

[26] Fleming, W. H. and Soner, H. M. (2006). *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability*. Springer, New York.

[27] Föllmer, H. and Leukert, P. (1999). Quantile hedging. *Finance Stoch.*, 3(3):251–273.

[28] Föllmer, H. and Leukert, P. (2000). Efficient hedging: cost versus shortfall risk. *Finance Stoch.*, 4(2):117–146.

[29] Föllmer, H. and Schied, A. (2002a). Convex measures of risk and trading constraints. *Finance Stoch.*, 6(4):429–447.

[30] Föllmer, H. and Schied, A. (2002b). Robust preferences and convex measures of risk. In *Advances in finance and stochastics*, pages 39–56. Springer, Berlin.

[31] Föllmer, H. and Schied, A. (2002c). *Stochastic finance*, volume 27 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin.

[32] Föllmer, H. and Schied, A. (2004). *Stochastic finance*, volume 27 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., 2nd. Edition, Berlin.

[33] Fremlin, D. H. (2002). *Measure Theory*, volume 3. Torres Fremlin publisher.

[34] Frittelli, M. and Gianin, R. (2002). Putting order in risk measures. *Journal of Banking and Finance*, 26:1473–1486.

[35] Frittelli, M. and Gianin, R. (2004). Dynamic convex risk measures. In Szegö, G., editor, *Risk Measures for the 21st Century*, pages 227–248. John Wiley and Sons.

[36] Goll, T. and L., R. (2001). Minimax and minimal distance martingele measures and their relationship to portfolio optimization. *Finance and Stochastics*, 5:557–581.

[37] Gundel, A. (2005). Robust utility maximization for complete and incomplete market models. *Finance Stoch.*, 9(2):151–176.

[38] Heath, D. and Schweizer, M. (2000). Martingales versus PDEs in finance: an equivalence result with examples. *J. Appl. Probab.*, 37(4):947–957.

[39] Henderson, V. (2002). Valuation of claims on nontraded assets using utility maximization. *Math. Finance*, 12(4):351–373.

[40] Henderson, V. and Hobson, D. (2004). Utility indifference pricing - an overview. In *Indifference pricing*. Princeton University Press.

[41] Henderson, V. and Hobson, D. G. (2002). Real options with constant relative risk aversion. *J. Econom. Dynam. Control*, 27(2):329–355.

[42] Hernández-Hernández, D. and Schied, A. (2006a). A control approach to robust utility maximization with logarithmic and time-consistent penalties.

[43] Hernández-Hernández, D. and Schied, A. (2006b). Robust utility maximization in a stochastic factor model. *Statistics and Decisions. Special Volume on Risk Measures*, 24.

[44] Hobson, D. and Jeremy, P. (2005). Maximizing the probability of a perfect hedge using an imperfectly correlated instrument. *International Journal of Theoretical and Applied Finance.*

[45] Hobson, D. G. (2005). Bounds for the utility-indifference prices of non-traded assets in incomplete markets. *Decis. Econ. Finance*, 28(1):33–52.

[46] Hubalek, F. and Sachermeyer, W. (1999). The limitations of no-arbitrage arguments for real options. *Adaptative Information Systems and Modelling in Economics and Management Science. Working paper series*, (58).

[47] Jacod, J. and Shiryaev, A. N. (2003). *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin.

[48] Jaschke, S. (2002). Quantile var is the wrong measure to quantify market risk for regulatory purposes. Research report, Weierstrass Institute.

[49] Jonsson, M. and Sircar, K. R. (2002). Partial hedging in a stochastic volatility environment. *Math. Finance*, 12(4):375–409.

[50] Jouini, E. and Napp, C. (2004). Convergence of utility functions and convergence of optimal strategies. *Finance Stoch.*, 8(1):133–144.

[51] Karatzas, I. and Shreve, S. E. (1988). *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

[52] Karatzas, I. and Shreve, S. E. (1998). *Methods of mathematical finance*, volume 39 of *Applications of mathematics: Stochastic modelling and applied probability*. Springer-Verlag.

[53] Kloeden, P. E. and Platen, E. (1992). *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin.

[54] Kramkov, D. and Schachermayer, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9(3):904–950.

[55] Kramkov, D. and Schachermayer, W. (2003). Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Ann. Appl. Probab.*, 13(4):1504–1516.

[56] Krätschmer, V. (2005). Robust representation of convex risk measures by probability measures. *Finance Stoch.*, 9(4):597–608.

[57] Lapeyre, B., Agnès, S., and Talay, D. (2006). *Simulation of Financial Models: Mathematical Foundations and Applications.*

[58] Monoyios, M. (2004). Performance of utility-based strategies for hedging basis risk. *Quant. Finance*, 4(3):245–255.

[59] Musiela, M. and Rutkowski, M. (1997). *Martingale Methods in Financial Modeling*. Springer-Verlag.

[60] Musiela, M. and Zariphopoulou, T. (2004a). An example of indifference prices under exponential preferences. *Finance Stoch.*, 8(2):229–239.

[61] Musiela, M. and Zariphopoulou, T. (2004b). A valuation algorithm for indifference prices in incomplete markets. *Finance Stoch.*, 8(3):399–414.

[62] Øksendal, B. (1991). Stochastic control theory—a brief summary. In *Stochastic models and option values (Loen, 1989)*, volume 200 of *Contrib. Econom. Anal.*, pages 19–29. North-Holland, Amsterdam.

[63] Øksendal, B. (1995). *Stochastic Differential Equations. An Introduction with Applications*. Springer Verlag.

[64] Owen, M. P. (2002). Utility based optimal hedging in incomplete markets. *Ann. Appl. Probab.*, 12(2):691–709.

[65] Palmowski, Z. and Rolski, T. (2002). A technique for exponential change of measure for Markov processes. *Bernoulli*, 8(6):767–785.

[66] Pflug, G. C. (2000). Some remarks on the value-at-risk and the conditional value-at-risk. In *Probabilistic constrained optimization*, volume 49 of *Nonconvex Optim. Appl.*, pages 272–281. Kluwer Acad. Publ., Dordrecht.

[67] Pham, H. (2000). Dynamic L^p-hedging in discrete time under cone constraints. *SIAM J. Control Optim.*, 38(3):665–682 (electronic).

[68] Pham, H. (2005). On some recent aspects of stochastic control and their applications. *Probability Surveys*, 2:506–549.

[69] Pliska, S. R. (1997). *Introduction to mathematical finance: discrete time models.* Blackwell, Oxford.

[70] Quenez, M.-C. (2004). Optimal portfolio in a multiple-priors model. In *Seminar on Stochastic Analysis, Random Fields and Applications IV*, volume 58 of *Progr. Probab.*, pages 291–321. Birkhäuser, Basel.

[71] Revuz, D. and Yor, M. (2005). *Continuous Martingales and Brownian Motion.* Grundlehren der Mathematischen Wissenchaften. Springer-Verlag, 3rd. edition.

[72] Rockafellar, R. and Uryaser, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking and Finance*, 26(7):1443–1471.

[73] Rockafellar, R. T. (1970). *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J.

[74] Rockafellar, R. T. and Uryaser, S. (2000). Optimization of conditional value-at-risk. *Journal of Risk*, 2:21–41.

[75] Rogers, L. C. G. (2001). Duality in constrained optimal investment and consumption problems: a synthesis.

[76] Rogers, L. C. G. and Williams, D. (2000a). *Diffusions, Markov processes, and martingales. Vol. 1.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2nd. edition.

[77] Rogers, L. C. G. and Williams, D. (2000b). *Diffusions, Markov processes, and martingales. Vol. 2.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2nd. edition.

[78] Rudin, W. (1953). *Principles of mathematical analysis.* McGraw-Hill Book Company, Inc., New York-Toronto-London.

[79] Scagnellato, C. and Vargiolu, T. (2002). Explicit solutions for shortfall risk minimization in multinomial models. *Decis. Econ. Finance*, 25(2):145–155.

[80] Schied, A. (2005). Optimal investments for robust utility functionals in complete market models. *Math. Oper. Res.*, 30(3):750–764.

[81] Schied, A. (2006). Risk measures and robust optimization problems. *Stoch. Models*, 22(4):753–831.

[82] Schied, A. and Wu, C.-T. (2005). Duality theory for optimal investments under model uncertainty. *Statistics and Decision*, 23(3):199–217.

[83] Schulmerich, M. and Trautmann, S. (2003). Local expected shortfall-hedging in discrete time. *European Finance Review*, 7:75–102.

[84] Siu, T. K. and Yang, H. (2001). A p.d.e. approach for measuring risk of derivatives. *Appl. Math. Finance*, 7(3):211–228.

[85] Spivak, G. and Cvitanić, J. (1999). Maximizing the probability of a perfect hedge. *Ann. Appl. Probab.*, 9(4):1303–1328.

[86] Stroock, D. W. and Varadhan, S. R. S. (1979). *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin.

[87] Summer, C. (2002). *Utility maximization and Increasing Risk Aversion.* PhD thesis, Vienna University of Technology.

[88] Tak Kuen Siu, H. T. and Yang, H. (2003). On p.d.e. approach for measuring tail risk of portfolios containing derivatives.

[89] Tasche, D. (2002). Expected shortfall and beyond. In *Statistical data analysis based on the L_1-norm and related methods (Neuchâtel, 2002)*, Stat. Ind. Technol., pages 109–123. Birkhäuser, Basel.

[90] Tessitore, G. and Zabczyk, J. (2002). Pricing options for Markovian models. In *Stochastic processes and related topics (Siegmundsburg, 2000)*, volume 12 of *Stochastics Monogr.*, pages 249–268. Taylor & Francis, London.

[91] Testuri, C. E. and Uryasev, S. (2004). On relation between expected regret and conditional value-at-risk. In *Handbook of computational and numerical methods in finance*, pages 361–372. Birkhäuser Boston, Boston, MA.

[92] Tudor, C. (1994). *Procesos estocásticos*, volume 2 of *Aportaciones Matemáticas: Textos [Mathematical Contributions: Texts]*. Sociedad Matemática Mexicana, México.

[93] Uryasev, S. and Rockafellar, R. (2001). Conditional value-at-risk: optimization approach. In *Stochastic optimization: algorithms and applications (Gainesville, FL, 2000)*, volume 54 of *Appl. Optim.*, pages 411–435. Kluwer Acad. Publ., Dordrecht.

[94] Xu, M. (2004). *Minimizing Shortfall Risk Using Duality Approach - An Application to Partial Hedging in Incomplete Markets*. PhD thesis, Carnegie Mellon University.

[95] Yang, H. and Siu, T. K. (2001). Coherent risk measures for derivatives under Black-Scholes economy. *Int. J. Theor. Appl. Finance*, 4(5):819–835.

[96] Yong, J. and Zhou, X. Y. (1999). *Stochastic controls*, volume 43 of *Applications of Mathematics (New York)*. Springer-Verlag, New York. Hamiltonian systems and HJB equations.