Brain Multiphysics 4 (2023) 100070

Contents lists available at ScienceDirect

Brain Multiphysics

journal homepage: www.sciencedirect.com/journal/brain-multiphysics

Research article

Generalised Kuramoto models with time-delayed phase-resetting for k-dimensional clocks

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ARTICLE INFO

Keywords:
Kuramoto models
Range-dependent networks
High dimensional clocks
Phase-resetting maps
The human cortex
Consciousness

ABSTRACT

We consider a class of Kuramoto models, with an array of N individual k-dimensional clocks ($k > 1$), coupled within a directed, range dependent, network. For each directed connection, a signal triggered at the sending clock incurs a (real valued) time delay before arriving at the receiving clock, where it induces an instantaneous phase reset affecting all k-phases. Instantaneous phase resetting maps (PRMs) for k-dimensional clocks have received little attention. The system may be treated as *open* and subject to periodic, or other types of, PRM forcing at any individual clock, as a result of external forcing stimuli. We show how the full system, with Nk phase variables, responds to such stimuli, as the impact spreads across the entire network. Beyond simulations, we employ methods to reverse engineer the dynamical behaviour of the whole: estimating the intrinsic dimensions of the responses to different experiments; and by analysing pairwise comparisons between those responses. This shows that the system's responses are governed by a hierarchy of internal dynamical *modes*, existing across both the Nk phases and over time.

We argue that this Kuramoto system is a model for the human cortex, where each k-dimensional clock models the dynamics of a single *neural column*, which contains 10,000 densely inter-connected neurons. The Kuramoto model exploits the natural *network of networks* architecture of the human cortex. An array of N =1M such columns/clocks is at the 10B neuron scale of the human cortex. However its simulation is far more accessible than very large scale (VLS) simulations of neuron-to-neuron systems on supercomputers. The latent modes may have important implications for cognition (information processing) and for consciousness (the existence of internal phenomenological experiences). We argue that the existence of the latter plays a key role in preconditioning the former, reducing the decision sets and the cognitive load, and thus enabling a fast-thinking evolutionary advantage.

This is the first time that systems of k-dimensional clocks ($k > 1$), coupled via time-lagged PRMs, within range dependent networks, have been deployed to demonstrate the basic internal phenomenological elements (of consciousness) and their potential role within immediate cognition.

1. Introduction

When one speaks of coupled clocks one evokes two important research threads that have been developed over the past fifty years or so. One is the Kuramoto model and its generalisations [1–4] and the other is instantaneous phase-resetting behaviour for oscillatory systems and coupled one-dimensional clocks [3,5]. Both of these concepts are foundational for this paper.

First, Kuramoto systems usually contain simple one-dimensional clock dynamics that are usually coupled in a continuum or else via a discrete grid (of *automata*): it is a model for the behaviour of a large set of coupled oscillators. That formulation was motivated by systems of chemical and biological oscillators, and has found applications in areas including neuroscience. However Kuramoto models have not previously considered coupling-up k-dimensional oscillators, for $k \gg 1$. Moreover, the phase-coupling is most usually between near neighbours on a grid (as automata, or in a continuum as a PDE), though occasionally via more complex networks [6], and, for applications in neuroscience, via a random range-dependent networks with successively rarer longer range directed connections between *regions of the cortex* (see [4] for example), just as we adopt here (see Section 3).

The Kuramoto model has previously been extended to include coupling time-delays, and consequently more exotic dynamical phenomena are observed [4]: an important conclusion.

Second, there is a long history of instantaneous phase-resetting maps (PRMs) being applied to one-dimensional clocks, and much of this

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https://doi.org/10.1016/j.brain.2023.100070
Received 18 January 2023; Received in revised form 24 April 2023; Accepted 24 April 2023

is summarised in [5] and by the references reviewed in [7]. As observed in [7], there has been almost no research on phase-resetting for k-dimensional clocks, and, as outlined there and in the Appendix, things may become very distinct from the one-dimensional case. Therefore our use of a directed network of (time-delayed) couplings between a set of k-dimensional clocks, providing instantaneous phase-resets for each other is far from an implementation of the existing body of work on phase-resetting for coupled one-dimensional oscillators.

For many decades Kuramoto models have been associated with oscillations within neural circuits in the brain (see [8] and the references there in). What is different here is that more recent work [9] has now shown that larger versions of such circuits, corresponding to the densely connected neural columns of the cortex (each with 10,000 or so neurons) are more complex: they should be represented by higher-dimensional clocks, winding maps on tori, $\mathbb{T}^k$, each with $k > 1$ phases, and not by simple oscillators ($k = 1$, which is quite common in Kuramoto models). Such clocks and their phase resetting (homotopic) classification are introduced in Section 2. This insight is important and nontrivial, when, in Section 3, we deploy range-dependent directed networks of k-dimensional clocks. It allows us to compare the behaviour of this class of Kuramoto systems, modelling the human cortex up to the 1M column-to-column level, with very large scale (VLS) simulations, at the $\sim$ 10B neuron-to-neuron level, that is only carried out on supercomputers. We discuss this application in Section 4.

In fact, generalisations of the Kuramoto model to couple dynamics on higher-dimensional spheres, $\mathbb{S}^k$, rather than on tori, $\mathbb{T}^k$, have previously been considered, see [10,11] and references therein. However for the application to neural columns considered here, the observed [9] free-running attractor is dynamically equivalent to a k-dimensional clock, on a torus, $\mathbb{T}^k$.

The most important point for whole cortex simulations, deploying a range dependent directed network of k-dimensional clocks, is that we may reverse engineer them. This aspiration concerns the explanation of the *inner life* of the whole system as it responds to a very wide variety of possible incoming stimuli. Just as in the VLS simulations [12] we show that the Kuramoto systems here exhibit internal dynamical *modes* of response, which compete and, once established, can precondition the system's response to immediately incoming stimuli. This in turn suggests an evolutionary fast-thinking advantage is conferred by such modes. The emergence of competing dynamical modes is also in line with modelling attempts to bridge cognition and the mind [13]. It also suggests that the internal experiences of the modes' internal temporal instantiation are viable candidates for phenomenological sensations, and thus offer a response to the *hard problem of consciousness*. This last matter is an essential goal for modellers, as even the possibility of a mathematical response is questioned by some philosophers [14,15].

In fact there is rather similar argument already set out in [16], without simulations such as those presented here, that emotional processing (including the role of phenomenological sensations) appears to be interlocked with cognition, and that this interaction is a functional consequence of cortex architecture that characterising the brain in terms of its (dynamic) network. Nevertheless we concede that there still remains an explanatory gap: we cannot be sure that the dynamical modes cause those sensations, yet they have not been observed before (since they exist over time and stretch over non-local regions of the cortex and hence they are not discussed within the philosophical literature.

This is the first time that such a systems of k-dimensional clocks (for $k > 1$), coupled via time-lagged PRMs, within range dependent networks, have been deployed to demonstrate basic internal phenomenological elements (of consciousness) and their potential role within immediate cognition.

Traditional networks of coupled (single) oscillators, on $\mathbb{T}^1 \equiv \mathbb{S}^1$, interact continuously, outputting their present phase in a way that influences the receiving oscillators' phases according to some coupling function. Thus instantaneous PRM-coupling (of k-dimensional oscillators) is rather distinct from that [5,7].

Within a neural column (with a k-dimensional clock attractor) there is no single neuron that is constantly firing and outputting such signals to an individual neuron within a neighbouring column. So signals sent to a neighbouring column are occasional and fast (like a PRM), relative to the periodicities of the k-clock, rather than providing a (near continuous) sequential read-out of the phases of the sending column: the k-dimensional toroidal attractor is an abstract function of the whole column.

Since wish to examine what goes on within a network of such columns, we will take a pragmatic approach and adopt the simplest possible type of PRMs within the column-to column couplings. From Section 2 it is clear that these "Type 0" PRMs might be generalised. Yet even for these PRMs we shall observe a rich internal structure of input–output behaviours for the whole systems. However we must bear in mind that such pragmatism may mislead: simple oscillators with instantaneous PRMs have been analysed in the context of neuroscience and the reset rules can have dynamical consequences, see for example [17].

2. Preliminaries

2.1. K-clocks with phase-resetting impulses

Consider a single k-dimensional clock which has a state vector of phase-variables, $\boldsymbol{\phi}(t)$, on the k–dimensional torus, $\mathbb{T}^k$, and is a function of time. Each component of $\phi = (\phi_1, \phi_2 ..., \phi_k)^T$ is identified $\mod 2\pi$.

We assume there is an unperturbed (free) autonomous winding dynamic over $\mathbb{T}^k$:

$$\dot{\phi} = \omega,$$

where $\omega > 0$ is a given vector of phase velocities. This dynamic holds so long as there are no external perturbations. Note, we expect that ω will vary (within a suitable distribution) from clock to clock.

If $\phi(t_i)$ is known, and the free dynamics remain unperturbed until $t = t_{i+1} > t_i$ we have

$$\phi(t) = \phi(t_i) + \omega(t - t_i), \qquad \mod 2\pi \quad t_i \le t < t_{i+1},$$

and the orbit just winds around the k–torus until t_{i+1}. Now suppose that at $t = t_{i+1}$ the state is subject to an instantaneous impulse, meaning a very fast (instantaneous) perturbation, in the form of a **Phase-Resetting Map (PRM)** which maps $\mathbb{T}^k$ into itself. The homotopy type of this map is the equivalent of the usual Type 0/Type 1 classification of PRMs for a simple clock [5], defined over $\mathbb{T}^1 \equiv \mathbb{S}$.

Specifically let us suppose that we have a PRM, which maps ϕ_{old} onto ϕ_{new}, in the form

$$\phi_{\text{new}} = A\phi_{\text{old}} + \mathbf{f}(\phi_{\text{old}}) \qquad \mod 2\pi. \tag{1}$$

Here A is a $k \times k$ binary matrix representing the homotopy type of the PRM (there are 2^{k^2} such types), and $\mathbf{f}$ is continuous, bounded, and 2π-periodic in all its arguments.

If $A_{ij} = 1$ then as $(\phi_{\text{old}})_j$, the jth component of ϕ_{old}, cycles once through $[0, 2\pi)$ then so does $(\phi_{\text{new}})_i$, the ith component of ϕ_{new}. Thus A holds the winding dependencies of the PRM. The nonlinear part, $\mathbf{f}$, is continuous and bounded and is thus deformable to any constant without changing the homotopy type of the PRM. The PRM is fully specified by the pair $(A, \mathbf{f})$.

Suppose that $\phi_0 = \phi(0)$ is given, and that the same PRM is applied at a sequence of times $\{t_1, t_2, t_3, \dots\}$. Then we have $\phi(t) = \phi(t_i) + \omega(t - t_i)$ $\mod 2\pi$, for $t_i \le t < t_{i+1}$; and $\phi(t_{i+1}) = A\phi(t_i + \omega(t_{i+1} - t_i)) + \mathbf{f}\phi(t_i + \omega(t_{i+1} - t_i))$ $\mod 2\pi$. So $\phi(t)$ is completely determined by the phase reset time sequence, the nature of the PRM, and the free winding dynamic.

Consider the example in Fig. 1, where

$$k = 2, \ \omega = (1, 1.41), \ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ \mathbf{f}(\phi) = 0.05 \begin{bmatrix} \sin[\phi_1]\sin[\phi_2] \\ \cos[\phi_1 + 1]\sin[\phi_2 + .2] \end{bmatrix}.$$

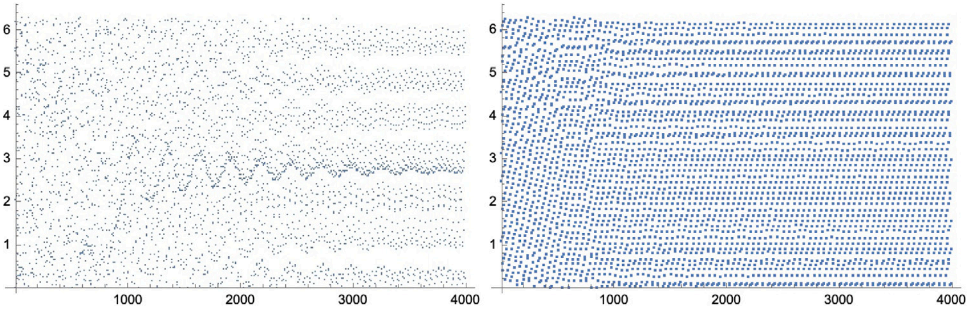

Fig. 1. Plots of ϕ_1 (left) and ϕ_2 (right), $\mod 2\pi$, versus time, immediately after each PRM is applied at integer values of time (the abscissa).

Setting $\phi(0) = (0, \pi)^T$, we applied the PRM periodically at each integer value of time.

In Fig. 1 the system becomes almost entrained by a very large multiple of the period of the forcing PRM, resulting in a quasi-periodic motions with much longer periods. In [7] there is a discussion about PRMs applied periodically on $\mathbb{T}^2$: for the above choices of A and ω [7] (Fig. 4) shows that one-to-one periodic entrainment of the system with the forcing PRM is not possible. The results in [7] sought to generalise the prevalent focus on periodic forcing for one-dimensional clocks (limit cycles), noting that there is no literature when $k > 2$ (see the references in both [5,7]). See the Appendix, where we discuss why periodic entertainment is not necessarily possible.

In Fig. 2 we show a second example with PRM given by

$$k = 3,\ \ A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ \ \mathbf{f}(\phi) = 0.1 \begin{bmatrix} \sin[\phi_1]\sin[\phi_2]\sin[\phi_3] \\ \cos[\phi_1 + 1.0]\sin[\phi_2 + .2] \\ \cos[\phi_1 + 0.5]\sin[\phi_3 + 2.0] \end{bmatrix}.$$

Setting $\phi(0) = (0, 0, 0)^T$, we again applied the PRM periodically at each integer value of time. The dynamics appear to be sensitive to the choice of ω see Fig. 2. Such sensitivity may be an essential feature of a full-scale systems in the next section.

In what follows, particularly in Sections 2.2 and 3, to keep things simple we will deploy PRMs where $A \equiv 0$ and $\mathbf{f}(\phi)$ is a given constant (so the post-reset phase is independent of the pre-reset phase). This "simple Type 0" PRM is the strongest possible type of PRM.

2.2. Coupling clocks via range dependent networks

Imagine the ensemble of k-dimensional clocks set out in an array, flat over the plane. Each clock may connect to, or be connected from, nearby clocks, according to a suitable directed range dependent network, meaning that near neighbours are more likely to be connected by a directed edge. We will arrange this as follows, using a two dimensional version of range dependent (a type of small world) networks [18].

Each clock is assumed to be $k > 1$ dimensional with k independent phase variables, as in Section 2.1. Whenever its k-phase satisfies a given edge-dependent output condition, a clock is able to generate a phase resetting signal (triggering a simply Type 0 PRM), to be sent out along a given directed edge to one of its immediate downstream neighbours. The signal incurs a positive, real, time-delay before it arrives at the receiving clock. Once it arrives, the PRM signal instantaneously resets the k-phase of the receiving clock to some post-reset state, which is again edge dependent. In this way, once started-up, the array of clocks chatter to each other, passing around successive phase resets. If there are N clocks arranged in a two dimensional grid, then the total system has Nk phases and could be conceived as a dynamical system defined on the state space $\mathbb{T}^{Nk}$.

In Fig. 3 we depict typical range dependent directed networks for the ensemble of phase-resetting k-dimensional clocks. These are drawn from a random directed graph as follows. For any pair of distinct clocks, at $v_{i,j}$ and $v_{i',j'}$, there is a directed edge present from $v_{i,j}$ to $v_{i',j'}$ with probability $\exp(-z^2/2)$, where the range is simply given by the range $z = \sqrt{(i - i')^2 + (j - j')^2}$. This may be easily generalised to deploy any probability map $h(i - i', j - j') \to [0, 1]$ which is decreasing in both arguments [18], so that longer range directed edge are successively more rare.

For each directed edge, e_{AB}, pointing from clock A to clock B say, we set up a number of system parameters.

- A simple Type 0 PRM sending condition: this occurs when a specified trigger element of the k-phase vector, ϕ_A, for clock A say the j^*th element, for some $j^* \in \{1, \ldots, k\}$, is equal to a given phase, $\alpha_{AB} \in [0, 2\pi)$, called the trigger phase; so that at each time, $t = t^*$, that $(\boldsymbol{\phi}_A(t^*))_{j^*} = \alpha_{AB}$, a PRM signal is sent off to clock B. The particular trigger mechanism is uniquely defined for each directed edge.
- A time-delay: δ_{AB}, the time taken for the PRM signal to travel from clock A to clock B. Again, the delay is uniquely defined for each directed edge.
- A post-PRM k-phase: this is a given phase in $[0, 2\pi)^k$ to which the phase vector of clock B, that is $\boldsymbol{\phi}_B(t^* + \delta_{AB})$, will be instantly reset when the PRM signal arrives, at time $t = t^* + \delta_{AB}$. Again, the post PRM reset phase is uniquely defined for each directed edge.

In future work we will consider time-delays that, though random, are correlated with the range of the edge (or the direct the distance between the clocks), this being more physically realistic.

The sending out of a simple Type 0 PRM signal does not affect the dynamics of the sending clock (it is a consequence of it): it completely resets the state of the receiving clock on arrival. (Our use of simply Type 0 PRMs is pragmatic, and would be easily generalised). Thus, if a clock has exactly m incoming directed edges, from m distinct, and immediately upstream, clocks, then it may be subject to m distinct PRMs, one for each upstream clock, at various signal arrival times. Similarly, if a clock has m outgoing directed edges, to m distinct, immediately downstream, clocks, then it may send m distinct PRMs, one for each downstream clock, at various times, according to its own phase-state.

The motivation for this type of model is the *network-of-networks* architecture of the human cortex [19–21]. There each individual k-dimensional clock represents one of the inner, densely connected, networks of neurons, called *neural columns* with each containing 10,000 or so individual neurons. The neural columns are fairly uniform and are arranged over the two dimensional surface of the human cortex. We will discuss this possible application much more fully, in Section 4. To be clear, this is a generalisation of the *outer* network of neural columns, which is modelled by a range dependent network; while each of the neural columns is replaced by a winding maps on a k-dimensional tori. These are coupled within the directed outer network via PRMs. This is the direct result of the finding in [9], which considered a single column that was isolated from all other columns, and contained n excitable/refractory neurons all densely connected up by an irreducible directed network ("irreducible" meaning that it so cannot be decomposed). If started up with a single spike at any

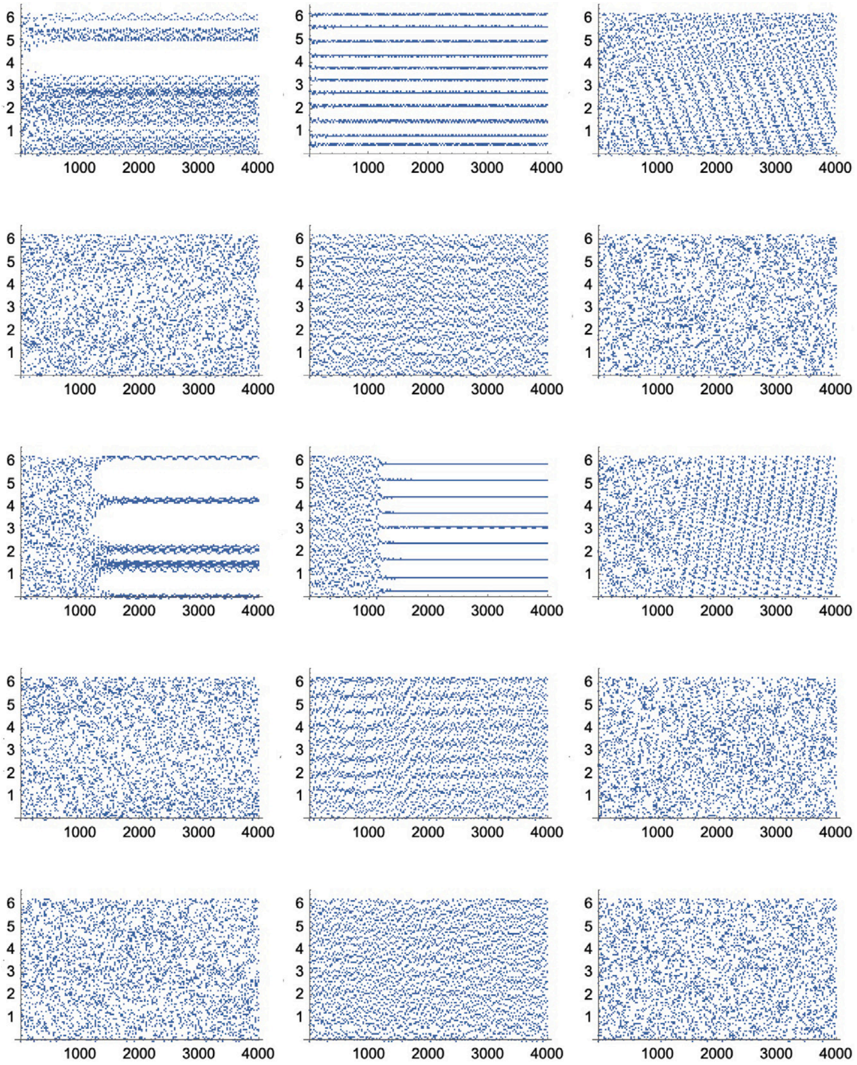

Fig. 2. Plots of ϕ_1 (left), ϕ_2 (centre), and ϕ_3 (right) $\mod 2\pi$ versus time, while ω varies by row: we have $\omega = (5.9 + 0.05\epsilon)(1., 0.578, 1.41)^T$ where $\epsilon = 0, 1, 2, 3, 4$ (top to bottom).

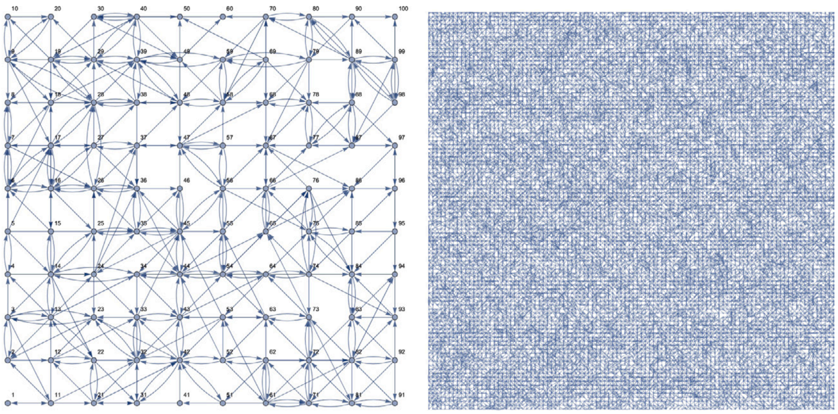

Fig. 3. Directed *range dependent* network models, with each vertex representing a k-dimensional clock, able to phase-reset its immediate outward adjacent neighbouring clocks. Left: $N = 100$ vertices arranged in a 10×10 grid. Right: Similar model with $N = 10{,}000$ vertices in a 100×100 grid.

neuron, the isolated column just chatters to itself in an a-periodic way forever. That repeated firing is reflective of existence the short and long directed cycles within the whole column's network (and the associated neuron-to-neuron time delays). Note that no single neuron is itself in any kind of oscillatory mode: the oscillations result from the directed cycles within the network; at least those that are viable with the time delays meaning that the total lap-time is long enough for each individual neuron to recover from its refractory mode after firing.

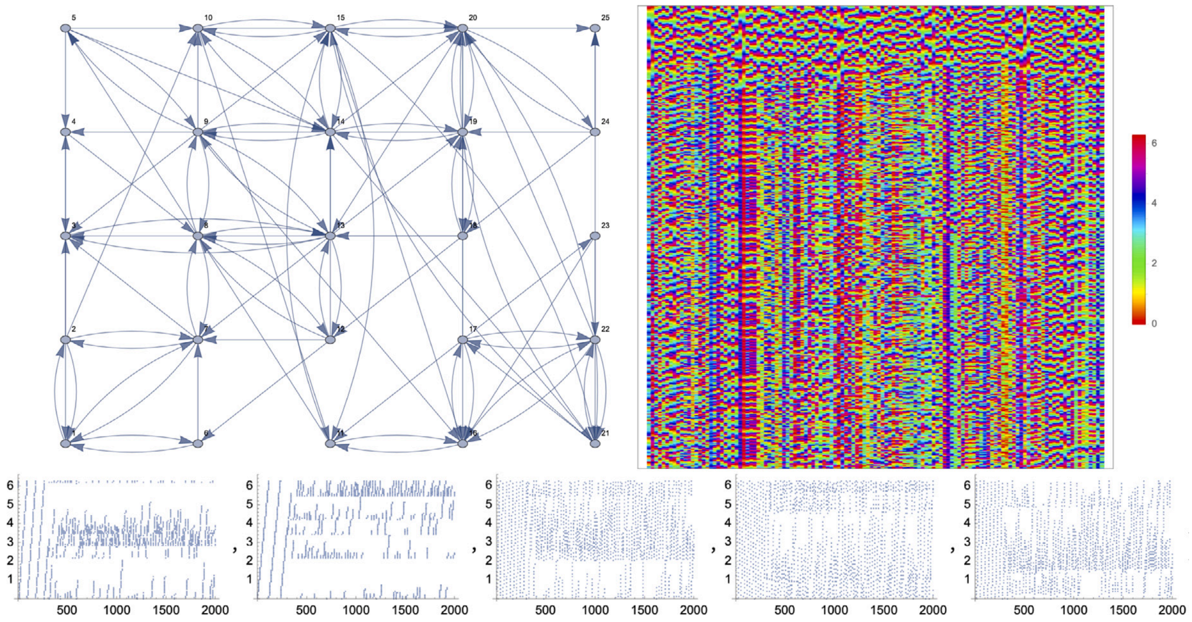

Fig. 4. Upper left: the directed range dependent network for $N = 25$ clocks. Upper right: the $k = 5$ phases for all $N = 25$ clocks (across 125 columns) at successive uniform time steps, $t = 0, 0.005, 0.010, \ldots, 10$ (in 2001 successive rows). Below: the $k = 5$ phases for clock 1, versus time step, which is reset in four different ways, by PRMs arriving from clocks 2,3,.6 and 7.

If isolated those individual cycles would just produce a corresponding cyclic periodic firing pattern. However, the separate cycles do interfere with (influence) each other (within the network). When the resulting dynamics for such free running systems was analysed [9] it was shown that the dimensionality of the free running dynamic grew like the log of the number, n, of neurons in the column. (Note that as n increased, they kept the degree distributions fixed). Thus [9] showed that the free running attractor for a single neural column is indeed a winding map on k-dimensional torus.

3. Kuramoto systems: A network of coupled *k*-clocks

We consider generalised Kuramoto systems composed of N mutually phase resetting k-clocks, coupled via a directed range dependent network, with transmission time delays.

3.1. *Small scale systems*

We begin with an example containing a 5×5 array of 25 k-clocks, where the dimension of the clocks is given by $k = 5$. The choice of $k > 1$ is to be motivated within particular applications (see Section 4, where this is motivated by the results of experiments [9]). The clocks' directed network is shown in Fig. 4 (upper left).

For each clock we generated an initial value for its phase at $t = 0$, independently and uniformly distributed on $[0, 2\pi)^k$; and a phase velocity vector ω independently and uniformly distributed in $[10, 50)^k$.

For each directed edge, e_{AB} we generated: (i) a random trigger element, $j^* \in \{1, \ldots, k\}$, and a trigger phase, α_{AB}, for clock A independently from a uniform distribution over $[0, 2\pi)$, so that a PRM is sent out whenever $(\boldsymbol{\phi}_A)_{j^*} = \alpha$; (ii) a time-delay, δ_{AB}, independently from a uniform distribution over [1,2]; and (iii) a post-reset phase for clock B independently from a uniform distribution over $[0, 2\pi)^k$.

The result of an initial *free running* experiment is shown in Fig. 4 (upper right). There is no external forcing applied. The clocks all free run until they are reset by one another: we show the output phases at 2001 successive uniform time steps between $t = 0.0$ and $t = 10.0$ (in the successive rows). Each clock has $k = 5$ phases, shown as 5 successive columns. There is a *burn-in* period, up to $t \sim 2$, followed by a dynamic, non periodic equilibrium.

Next we consider 25 separate forcing experiments.

In each case we apply a special PRM at single, chosen, clock, at $t = 1, 2, \ldots, 10$, resetting its phase vector to $(0, 0, 0, 0, 0)^T$. This is so that we can see the effect on all of the clocks as the forcing at a specific clock spreads through the directed network. In each case we calculate the $k = 5$ dimensional phase for all $N = 25$ clocks, and we take the difference between that output and the free running behaviour depicted in Fig. 4 (top right). As a result we see the forcing effects propagate: see Fig. 5.

Even though the forcing is fairly minimal (a PRM applied to a given clock, at 10 times) we can measure its effect by estimating the intrinsic dimension of the resulting orbit embedded in the state space, $\mathbb{T}^{125}$. We used the two-nearest-neighbour method introduced in [22]. The results are shown in Fig. 6. The dimensional estimates for all of the orbits lies between 16 and 18.5, compared to the state space dimension of $Nk = 125$. Clearly the behaviour of the whole is very far away from any simple nonlinear dynamic winding over $\mathbb{T}^{125}$: the repeated PRMs force the orbit to repeat certain segments for each clock. In general, this forcing reduces the dimension of the orbit, compared to the free running case. This is to be expected as any extra PRMs anywhere simply return the orbit to its reset phase, and thus it repeats. Hence the density of the whole, on each k-torus, will be lessened. Even when clock 25 is forced, which has no outgoing influence on any of the other clocks, there is a small effect (due to the resetting of clock 25 itself, whilst the others all carry on as free). The forcing of clock 10 has the largest effect in reducing the intrinsic dimension: a fact that might be deduced by eye from the 10th experiment's phase differences, depicted in Fig. 5. Hence the dimensional estimation from [22] corresponds well with our intuition about the wider impacts of the various distinct external forcing of the whole system. The out-degrees for each clock within the network are not correlated with the dimensional estimates, implying that the latter are likely to depend upon the edge-specific time-delays and PRM details.

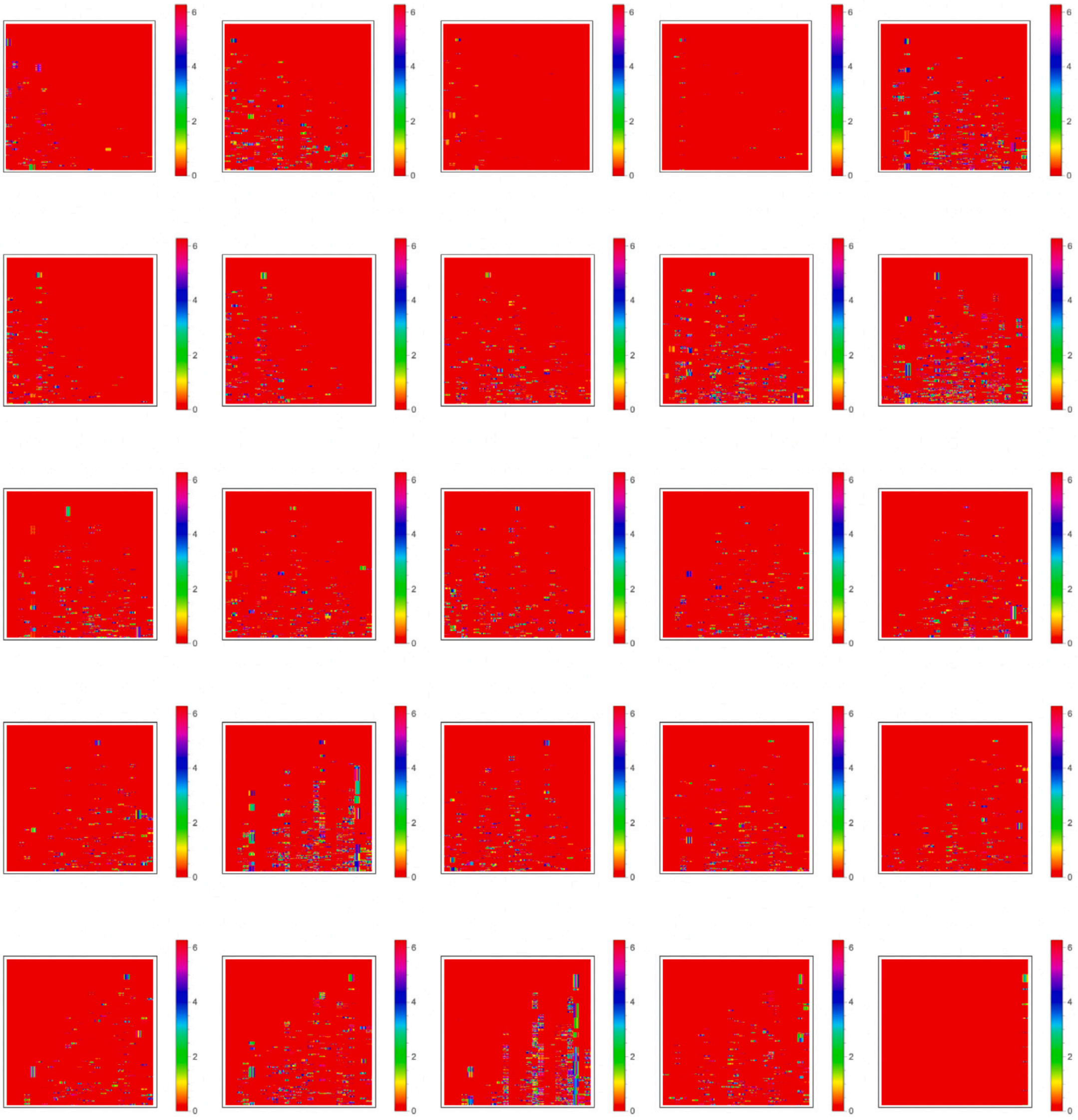

Fig. 5. 25 forcing experiments, with phases contrasted with the free running (unforced case). In each experiment we force Jth clock ($J = 1, 2, \ldots, 25$) to trigger an outgoing PRM at $t = 1, 2, \ldots, 10$. We plot all $Nk = 125$ phase differences between the forced output and the free running output (shown in Fig. 4, top right). As before the rows represent successive time steps, for $t = 0.0$, 0.005, 0.010, …,10.; and the columns represent the 5-phases for all 25 clocks.

3.2. Large scale numerical simulations

In this section we present a much larger scale numerical simulation of a system, where again all of the PRMs here are Simple Type 0.

We consider $N = 10{,}000$ k-clocks, with $k = 5$, arranged in the 100×100 range dependent graph shown in Fig. 3 (right). There are 51,785 directed edges. As in the previous section, the initial phases are all set randomly in $[0, 2\pi)^k$. The values for ω for each clock is chosen independently and randomly in $[1, 5]^k$. For each edge we generated PRM trigger condition for the sending clock, a time delay, and a post PRM reset phase vector for the receiving clock, just as we did for the small scale example in Section 3.1.

In Fig. 7 we plot the all of k-clocks' phases at 2000 successive time steps, each of length 0.00125, for $t \in [0, 2.5]$. The number of phase resetting events per unit time increases linearly with N (and thus the number of directed edges). We have $Nk = 50{,}000$ phase dimensions in total, and computations such as this are close to the limit of what is achievable on a standard laptop (Apple M1 chip and 8 GB memory), encoded in Mathematica™. Our aim, in the applications to simulation of the human cortex that we have in mind (see Section 4), is to analyse 1M such k-clocks and carry out a very large number of forcing experiments, similar to those in Section 3. We are already only a factor of 100 away. This example illustrates the feasibility of whole cortex simulation by addressing the meso-level of an array of neural columns, represented as a generalised Kuramoto model.

4. An application to dynamical cognition and consciousness within the human cortex

How might we model, and then reverse engineer, an information processing system that contains all of the features and dynamical behaviour of the human cortex?

In particular, can we do so in such a way that we can capture any hidden latent features of the dynamical behaviour that might give rise to conscious phenomenological experiences, and ascertain their role within cognition, and thus their evolutionary advantage?

The aspiration to identify dynamical properties supporting internal elements, such as subjective qualia, feelings, and phenomenological sensations [12,19], is a response to the *hard problem of consciousness* [14,15]. As *physicalists*, we argue that the latter are elements of the cognitive system: yet they are not passive consequences. Instead they play an active role in preconditioning immediate cognition, and their

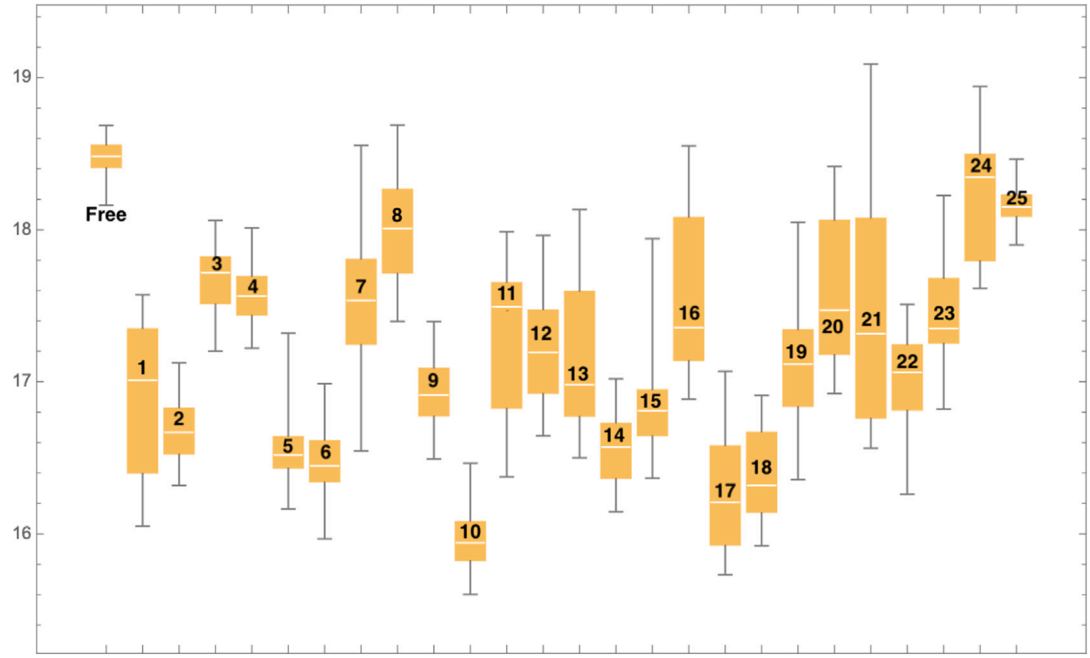

Fig. 6. 26 estimates of the intrinsic dimension of the orbits within $\mathbb{T}^{125}$: we show the estimates and uncertainties for the unforced, "Free" case, left, and then successively for the 25 forcing experiments depicted in Fig. 5. The general effect of forcing is to reduce the dimension of the orbits.

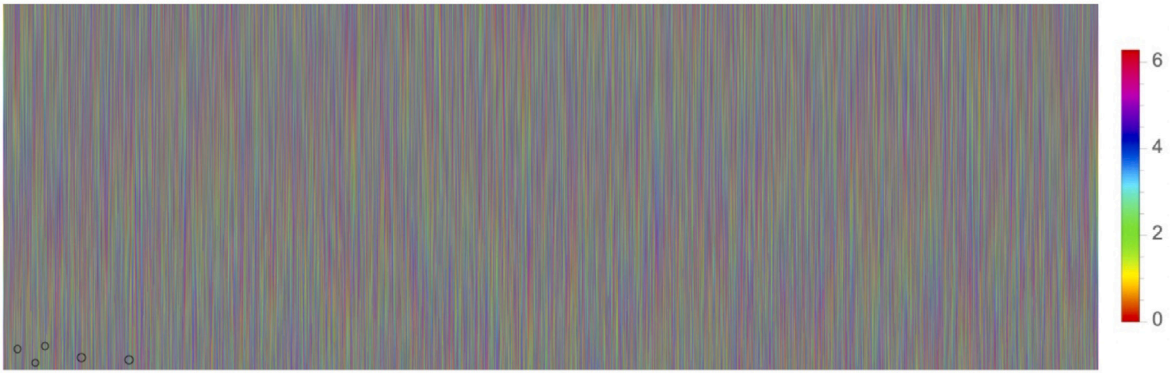

Fig. 7. 10,000 5-dimensional clocks, each with five phases (shown as columns), and 2,000 successive time steps (shown as rows descending in time order): $t = 0.0$, 0.00125, …, 2.5. Some later phase reset discontinuities are visible as discontinuities within the columns (examples are circled). This figure depicts 100M phases and is close to the limit for a MacBook.

evolutionary development that has enabled both efficient and effective information processing.

The human cortex itself contains around 10B neurons. They are arranged into approximately 1M neural columns, with each neural columns containing 10,000 rather densely connected neurons. The columns are arranged in a two-dimensional grid over the surface of the cortex. If the human cortex was ironed out flat it would be like a carpet with the columns forming the carpet pile. Some neurons within near-neighbouring modules are connected (all connections are directional). All neuron-to-neuron transmission of firing spikes (signals) incurs a variable (real valued) time delay.

In summary, the cortex has a *network-of-networks* architecture, with the inner networks representing the densely connected neurons within neural columns, and the range dependent outer network provided by some inter column neuron-to-neuron connections between neurons from near-neighbouring columns. There should be about 1M columns, each containing 10,000 neurons or so.

Previous work in [12] has considered very large scale (VLS) neuron-to-neuron simulations. This required the use of a bespoke multi-core computational facility (SpiNNaker [23]) in order to deal with the very large network of neurons reflecting, while the multi-column architecture.

The VLS neuron-to-neuron simulations in [9] of individual neural columns revealed that each actually behaves like a k-dimensional clock, just as described in Section 2.1; and that k is observed to proportional to the log of the number of neurons within a single column [9]. Typically, they show $k \sim O(10)$ for 10,000 or so neurons in a column. Note that k does not simply count the number of cycles within a directed network, since not all of those are viable (given the refractory nature of neuron spiking), and some cycles would cancel out one another. It might be better to think that k counts longish, yet mutually independent, cycles. In analysing simulations the number of degrees of freedom, k, exhibited in an neuron-to-neuron simulation of the individual neural column, may be directly estimated by state-space embedding techniques applied to neuronal spike trains (see [9] and the references therein).

Since a human cortex has only limited total volume and total energy, the corollary of this result is that neural columns should be uniform in size: any ancestors with a wide range of column sizes would have been performing sub-optimally and have been at an evolutionary disadvantage.

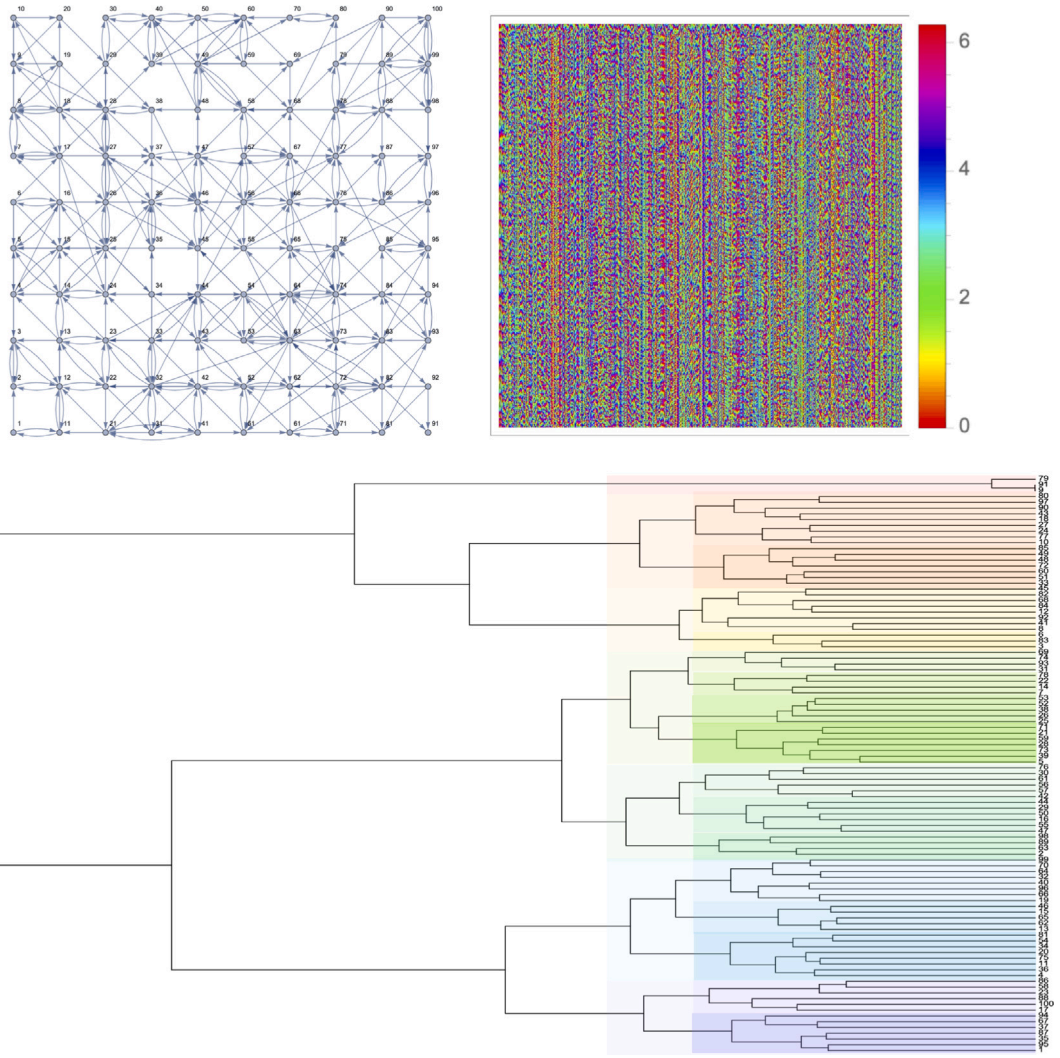

Fig. 8. Above left: The network of $N = 100$ clocks, each with $k = 5$. Above right: Unforced free running output for $Nk = 500$ phases (as columns) over 2000 successive time steps (as rows downwards), for $t = 0.0, 0.05, \ldots, 100.0$. Below: Ward clustering for the 100 forcing experiments. Labels indicate which k-clock is forced in each experiment (see text for details of the distance function). Clustered experiments possess similar internal responses over 5000 successive time steps (with possible time offsets), called "modes".

At a much larger scale, the VLS cortex simulations of many connected neural columns have also revealed the inner dynamics of the systems in response to various external patterns of stimulation [12], just as set out in Section 2.2. The system responds consistently in each case by exhibiting one of a number of distinct internal dynamical *modes*. The modes are internal patterns of firing behaviour and exist in time and across part of the cortex. They are arranged hierarchically and once in process will bias the immediately following information processing. So *love is blind* (at least for a while). Such internal sensations (modes) pre-condition our immediate cognition: that is likely to be their evolutionary advantage.

Though not discussed nor even anticipated by philosophers of mind, these dynamical, hierarchical, modes of behaviour are very strong candidates for subjective internal phenomenological sensations. They are inaccessible, unless one could reverse engineer a human brain. In order to discover such modes one must stimulate the whole system in many, many different ways and contrast the resultant patterns, modulo suitable time shifts [12].

These investigations are a direct response to the *hard problem of consciousness* [14]. This relies on the uncovering of the dynamical modes (which have not featured in research prior to [12]). These modes would furnish a clear evolutionary (fast-thinking) advantage by acting as constraints/pre-conditioners upon the immediate cognitive processing (decision making, for example). This hypothesis, if true, would explain "why" the human brain has such internal sensations: suggesting that consciousness and cognitive processing are entwined, being coupled both ways. It always provides a dynamical systems basis for "fast-thinking". Counter arguments might suggest that the internal modes are not related to internal sensations, or else that their existence is irrelevant to cognition.

However, here we are also showing that by setting out an intermediate-scale model one does not need to focus on full neuron-to-neuron VLS simulations, such as in [12]. As we shall see, the existence of the dynamical (latent) modes remains unaffected by the Kuramoto approximation.

As a demonstration of principle, we may repeat the modal analysis of [12] for the internal behavioural responses for the class of Kuratmoto system introduced above.

Consider 100 separate forcing experiments each carried out on the same 10 by 10 array of $N = 100$ k-clocks, similar to the model in Section 3.1, with $k = 5$: see the architecture and the free running dynamics in Fig. 8. In each experiment we force exactly one of k-clocks, labelled 1 through 100, with the same PRM applied at integer units of time.

Following the method deployed in [12], we may compare the time-dependent outputs of those clock-forcing experiments pairwise. Each output contains $Nk = 500$ phases at 2000 time steps, each of duration 0.05 units of time (so over 100 units of time in total). In each case we choose a *window* of 500 successive time steps (a quarter of the whole – 25 units of time) from near to the end of the sequence, and allow the window to slide back and forth with an offset by up to by 40 time steps (2 units of time). Then we compare the windows for two experiments and choose the window offsets that minimise the sum of the absolute values of the phase differences (allowing for the 2π periodicity) for all phases, for all 500 time steps within the windows. The offsetting allows for some possible shifts in absolute time between the pair of windows, since it is the patterns across the array of clocks, and over relative time, that we wish to compare; that is, the internal behavioural responses to the corresponding forcing experiments. This procedure results in a 100×100 pairwise distance (or dis-similarity) matrix for the set of 100 experiments. Any two experiments with a small distance will exhibit similar (possibly time shifted) internal responses to the corresponding pair of forcing experiments. Following [12] we deploy Ward clustering to produce the dendrogram shown in Fig. 8.

Of course, a common problem with any hierarchical clustering is that when you seek it you will usually find it. So Fig. 8 is a *demonstration of the possibility* of defining “modes” as mutually exclusive clusters of internal (latent) response patterns, with similar internal behaviour. It is not a proof of their existence.

However, in [12] the authors calculated pairwise comparisons for $32^2 = 1024$ distinct experiments on a large neuron-to-neuron system (analogous to the smaller number of experiments in Fig. 8). They showed that the Ward clustering statistics achieved with the original distance matrix was distinct from those for matrices generated via a full or the fractional resampling (from the observed distribution of all pairwise distances). That type of demonstration removes some of the concerns about the relevance of the hierarchical clusters uncovered.

We suggest that this hierarchical modal structure as a candidate for latent sensations. Ideally one should avoid choosing a specific level as all may be valid; and the idea that modes are hierarchical is aligned with the idea that sensations (emotions down to qualia) are indeed hierarchical [24].

As pointed out previously, full VLS whole cortex simulations at the neuron-to-neuron level are prohibitively expensive, since one requires access to a multi-core super computer (such as SpiNNaker [23]), and, even then, those computations are very slow. On the other hand the simulations shown in Section 3.2, deploying a Kuramoto system are viable on a laptop and seem to have captured something of the inner life of the columns and therefore of the whole cortex. Since each clock/column models 10,000 neurons, the 10,000 clock system in Section 3.2 is dynamically equivalent to a 100M neuron cortex.

We suggest that one might dispense with the VLS simulation approach at the neuron-to-neuron level, to be carried out on supercomputers. Instead we wish to focus in future on a generalised Kuramoto model, with an array of k-dimensional clocks set in a range dependent directed network, with triggered, time-delayed, impulses causing instantaneous phase-resettings between adjacent neighbours.

5. Whatever next?

In this paper we have introduced a particular class of generalised Kuramoto system: one with a range dependent directed network of k-dimensional clocks, each triggering and sending time-lagged strong PRMs to reset their immediate neighbours. The resulting system just chatters amongst itself when left free-running. Typically we want to have $k \sim 5$ or 10 (since, as we have noted, k grows with the log of the number of neurons in a neural column, as exhibited by the experiments set out in [9]); with each clock having k phases that simply wind forwards at individual rates while left un-reset.

We have illustrated the behaviour such systems with a variety of scales with N clocks (N=25, 100, 10,000) arranged within a two dimensional array.

We consider how such a system responds to external forcing stimuli. We have carried out experiments where we force an individual clock (within the array) periodically. Then the corresponding internal response, across the full array ensemble of clocks, may be compared from experiment to experiment. While allowing for some time off-setting from experiment to experiment, we show that the internal responses are dominated by a set of internal dynamical modes (patterns of behaviour across the array and over time).

The application that we have in mind is to the human cortex (“what lies within?” [12]), where previous work [9] has shown that neural columns, simulating up to 10,000 coupled neurons, behave as a k-dimensional generalised clocks. Since the cortex has an array of 1M such columns, we have shown that scale-up is achievable even on a modest laptop.

These experiments and findings here mirror those of full VLS neuron-to-neuron simulations carried out previously on super computers [12], yet at a fraction of the computational cost. The internal modes are a property of the architecture and the dynamics, and they offer a response to the *hard problem of consciousness* [14,15] (why is there something internal, some set of phenomenological sensations, that it is *like* to be a human brain experiencing the world?). In turn, such dynamical modes may confer an evolutionary, fast-thinking, advantage to the cortex, by preconditioning the immediate system response to incoming stimuli. These conscious phenomena may play an active role within cognition, as opposed to being a passive consequence of the cognitive challenge [25]. These are the basic building blocks of the subjective, internal, human conscious experience: the foothills of something much larger and more sophisticated. In future work we will develop these ideas much further in the application to the human cortex. The system introduced here make this whole topic far more accessible than whole brain neuron-to neuron simulation (on super computers).

In very recent work the important (non philosophical) implications of this approach and subsequent leanings for both next-generation AI and novel neuromorphic computer chips are set out as one of a number of open challenges [26] for mathematics.

From a mathematical point of view, phase resetting for k-dimensional clocks, via instantaneous PRMs, represents a relatively un-addressed topic, generalising the research over decades for such phase resetting of simple clocks. The appendix, here in this paper, illustrates some of the issues to be addressed, even for weakly nonlinear PRMs.

Within future simulations we would like to relax the assumption that the PRMs are all Simple Type 0 PRMs, and allow for much a less entrainable directed resetting. In practice this would be straightforward, but the theory for the coupling is more opaque.

In response to [26] we will also develop a framework for both the range dependent architecture and parameters of the PRMs to evolve slowly over time, subject to various *experiences*, in order that the whole system might tune itself to display some simple input and output behaviours simultaneously (using multiple input and output sites).

CRediT authorship contribution statement

Martin Brennan: Simulations, Devised the experimental paradigm, Devised the next step challenges, Involved in the writing up. **Peter Grindrod CBE:** Devised the theory for the paper and the framework for the model, based on previous research, Devised the next step challenges, Involved in the writing up.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors acknowledge the interest and advice given by Clive E. Bowman, Desmond J. Higham and many colleagues in the Association for Mathematical Consciousness Science (AMCS) and the Oxford Mathematics of Consciousness and Applications Network (OMCAN).

Funding

PG was funded by EPSRC, United Kingdom grant EP/R018472/1, *Application Driven Topological Data Analysis*. For the purpose of Open Access, the authors have applied a CC BY public copyright licence to any Author Accepted Manuscript (AAM) version arising from this paper.

Declaration of competing interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Appendix. Entrainment with periodic phase-resetting

Consider the module system dynamics and phase-resetting introduced in Section 2.2.

Following [7], suppose that the instantaneous PRM $(A, \mathbf{f})$ in (1) is applied to the k-clock periodically in time, with period p. Then the phase, ϕ_i, immediately prior to the ith successive resetting satisfies

$$\phi_{i+1} = p\omega + A\phi_i + \mathbf{f}(\phi_i) \quad \mod (2\pi), \tag{2}$$

where ω is the vector of phase velocities, just as before.

Let us modify (2) to include a small positive parameter ϵ and write

$$\mathbf{f}(\phi_i) = \bar{\mathbf{f}} + \epsilon \mathbf{g}(\phi_i),$$

so that we may continuously deform $\mathbf{f}$ down to a single point value $\bar{\mathbf{f}} \in \mathbb{T}^k$.

If the system responses becomes entrained with the periodic forcing then we have $\phi_i = \phi^*$ for all i, which satisfies

$$(I - A)\phi^* = p\omega + \bar{\mathbf{f}} - 2\pi\mathbf{k} + \epsilon\mathbf{g}(\phi^*) \tag{3}$$

where $\mathbf{k} \in \mathbb{Z}^m$ is some vectors of integers.

We immediately have a number of distinct cases to consider in (3).

If A is the zero matrix then we may solve (3) asymptotically for small ϵ and obtain a solution for all forcing periods, p:

$$\phi^* = p\omega + \bar{\mathbf{f}} + \epsilon\mathbf{g}_m(p\omega + \bar{\mathbf{f}}) + O(\epsilon^2) \qquad \mod (2\pi).$$

Suppose that A is zero except for a unitary value at a single element on the diagonal, say at the rth element. Let $\mathbf{e}_r$ denote the rth unit vector. Then $(I - A)$ is singular with null-space spanned by $\mathbf{e}_r$ and, in the limit $\epsilon \to 0$, we must have p taking only specific "resonant" values

$$p = \mathbf{e}_r.(2\pi\mathbf{k} - \bar{\mathbf{f}})/\mathbf{e}_r.\omega$$

which is equivalent to

$$(\omega)_r p + (\bar{\mathbf{f}})_r \equiv 0 \quad \mod (2\pi).$$

In this case there is an Arnold tongue of entrained solution in the (p, ϵ) upper half-plane that reaches down to meet the p-axis at each resonant value.

Now suppose that A is zero except for having two unitary values on the diagonal, at say the rth element and the r'th element. Then $(I - A)$ is singular with both $\mathbf{e}_r$ and $\mathbf{e}_{r'}$ spanning its null-space. Unless the frequencies $(\omega)_r$ and $(\omega)_{r'}$ are rationally related we cannot satisfy both solvability conditions, in the limit $\epsilon \to 0$.

In this case [7] there are Arnold tongues of entrained solution in the (p, ϵ) upper half-plane that reach downwards but can never quite meet the p-axis, where $\epsilon = 0$ (with some stronger variation being required to entrain the behaviour to the periodic forcing).

There has been much research regarding phase-resetting behaviour for one-dimensional clocks ($k = 1$), and their applications, over the past forty years [5]. Yet there is a real need to understand more about such coupling for much higher values of k, following-on the from the observations and analysis given in [7].

References

[1] Y. Kuramoto, in: H. Araki (Ed.), International Symposium on Mathematical Problems in Theoretical Physics, in: Lecture Notes in Physics, vol. 39, Springer-Verlag, New York, 1975.

[2] Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence, Springer-Verlag, New York, NY, 1984.

[3] S.J. Strogatz, From kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators, Physica D 143 (1–4) (2000) 1–20.

[4] M.K. Stephen Yeung, S.H. Strogatz, Time delay in the Kuramoto model of coupled oscillators, Phys. Rev. Lett. 82 (648) (1999) Published 18 January 1999.

[5] L. Glass, M.C. Mackey, From Clocks To Chaos: The Rhythms of Life, Princeton, 1988.

[6] F.A. Rodrigues, T.K.D.M. Peron, P. Ji, J. Kurths, The kuramoto model in complex networks, Phys. Rep. (ISSN: 0370-1573) 610 (2016) 1–98, http://dx.doi.org/10.1016/j.physrep.2015.10.008.

[7] P. Grindrod, E.L. Patel, Phase locking to the n-torus, IMA J. Appl. Math. 81 (1) (2016) 152–164, http://dx.doi.org/10.1093/imamat/hxv031.

[8] M. Breakspear, S. Heitmann, A. Daffertshofer, Generative models of cortical oscillations: Neurobiological implications of the kuramoto model, Front. Hum. Neurosci. 4 (2010) https://www.frontiersin.org/articles/10.3389/fnhum.2010.00190.

[9] P. Grindrod, T.E. Lee, On strongly connected networks with excitable-refractory dynamics and delayed coupling, Roy. Soc. Open Sci. 4 (4) (2017) 160912, http://dx.doi.org/10.1098/rsos.160912.

[10] L. Mi, J. Gonçalves, J. Markdahl, Asymptotically stable polarization of multi-agent gradient flows over manifolds, 2023, https://arxiv.org/abs/2301.04877.

[11] M.A. Lohe, Non-Abelian Kuramoto models and synchronization, J. Phys. A 42 (39) (2009).

[12] P. Grindrod, C. Lester, Cortex-like complex systems: what occurs within? Front. Appl. Math. Stat. 7 (2021) 51, https://www.frontiersin.org/article/10.3389/fams.2021.627236.

[13] M.I. Rabinovich, A.N. Simmons, P. Varona, Dynamical bridge between brain and mind, Trends in Cognitive Sciences 19 (8) (2015) 453–461, http://dx.doi.org/10.1016/j.tics.2015.06.005.

[14] D.J. Chalmers, Facing up to the problem of consciousness, J. Conscious. Stud. 2 (1995) 200–219.

[15] D.J. Chalmers, The Conscious Mind: In Search of a Fundamental Theory, Oxford University Press, New York, 1996.

[16] L. Pessoa, Understanding emotion with brain networks, Curr. Opin. Behav. Sci. 19 (2018) 19–25.

[17] C. Kirst, T. Geisel, M. Timme, Sequential desynchronization in networks of spiking neurons with partial reset, Phys. Rev. Lett. 102 (6) (2009) 1–4.

[18] P. Grindrod, Range-dependent random graphs and their application to modeling large small-world proteome datasets, Phys. Rev. E 66 (6) (2002) 066702, American Physical Society, https://link.aps.org/doi/10.1103/PhysRevE.66.066702.

[19] P. Grindrod, On human consciousness: A mathematical perspective, Netw. Neurosci. 2 (1) (2018) 23–40.

[20] D. Meunier, R. Lambiotte, E.T. Bullmore, Modular and hierarchically modular organization of brain networks, Front Neurosci 4 (200) (2010) Available at: https://www.frontiersin.org/article/10.3389/fnins.2010.00200.

[21] R.F. Betzel, J.D. Medaglia, L. Papadopoulos, G.L. Baum, R. Gur, R. Gur, D. Roalf, T.D. Satterthwaite, D.S. Bassett, The modular organization of human anatomical brain networks: Accounting for the cost of wiring, Netw. Neurosci. 1 (1) (2017) 42–68, http://dx.doi.org/10.1162/netn_a_00002.

[22] E. Facco, M. d'Errico, A. Rodriguez, A. Laio, Estimating the intrinsic dimension of datasets by a minimal neighborhood information, Sci. Rep. 7 (2017) 12140, http://dx.doi.org/10.1038/s41598-017-11873-y.

[23] S.B. Furber, F. Galluppi, S. Temple, L.A. Plana, The SpiNNaker project, Proc. IEEE 102 (5) (2014) 652–665.

[24] R.D. Lane, in: A. Kaszniak (Ed.), Hierarchical Organisation of Emotional Experience and Its Natural Sub-Straights, Emotions, Qualia, and Consciousness, in: Series on Biophysics and Biocybernetics, vol. 10, World Scientific, Singapore, 2001.

[25] D.C. Dennett, Consciousness Explained. (P. Weiner, Illustrator), Little, Brown and Co, 1991.

[26] P. Grindrod, J. Kleiner, K. Kremnitzer, The Mathematics of Consciousness: Some Immediate Mathematical Challenges, Research Gate, 2023, Preprint.