RANKED MASSES IN TWO-PARAMETER FLEMING–VIOT DIFFUSIONS*

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Previous work constructed Fleming–Viot-type measure-valued diffusions (and diffusions on a space of interval partitions of the unit interval $[0,1]$) that are stationary with respect to the Poisson–Dirichlet random measures with parameters $\alpha \in (0,1)$ and $\theta > -\alpha$. In this paper, we complete the proof that these processes resolve a conjecture by Feng and Sun (2010) by showing that the processes of ranked atom sizes (or of ranked interval lengths) of these diffusions are members of a two-parameter family of diffusions introduced by Petrov (2009), extending a model by Ethier and Kurtz (1981) in the case $\alpha = 0$.

1. Introduction. In [13], Feng and Sun conjectured the existence of Fleming–Viot processes with neutral, parent-independent mutation, corresponding to Petrov's [29] two-parameter extension of Ethier and Kurtz's infinitely-many-neutral-alleles diffusion model [11]. Generally, Fleming–Viot processes with neutral, parent-independent mutation are measure-valued processes where the measure of a singleton set represents the proportion of a continuous population with a certain genetic type and, if the offspring of an individual in the population is born with a mutation, then its (random) genetic type is independent of the genetic type of its parent. There are several ways to make this precise, through martingale problems [12, 15], Dirichlet forms [26], or pathwise constructions using Poisson random measures [44]. The Fleming–Viot processes corresponding to the diffusions Petrov constructed must have two additional properties.

First, their stationary distribution must be a Poisson–Dirichlet random measure, $\texttt{PDRM}(\alpha,\theta)$, where $\overline{\Pi} \sim \texttt{PDRM}(\alpha,\theta)$ if $\overline{\Pi} \overset{d}{=} \sum_{i\geq 1} P_i \delta(U_i)$ with independent $(P_i,\, i \geq 1) \sim \texttt{PD}(\alpha,\theta)$ and $U_i \sim \texttt{Unif}[0,1]$, $i \geq 1$, and $\texttt{PD}(\alpha,\theta)$ is the well-known two-parameter Poisson–Dirichlet distribution [34]. This family of random measures is also known as the Dirichlet or Pitman–Yor process in Bayesian statistics [23, 47].

Second, the ranked sequence of their atom sizes must evolve like the diffusions Petrov constructed in [29]. Specifically, Petrov constructed diffusions on the Kingman simplex

$$\nabla_\infty := \left\{ \mathbf{x} = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0,\ \sum_{i\geq 1} x_i = 1 \right\} \tag{1.1}$$

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with the following pre-generator acting on the unital algebra $\mathcal{F}$ of symmetric functions generated by $q_m(\mathbf{x}) = \sum_{i\geq 1} x_i^{m+1}$, $m \geq 1$:

$$\mathcal{B} = \sum_{i\geq 1} x_i \frac{\partial^2}{\partial x_i^2} - \sum_{i,j\geq 1} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i\geq 1} (\theta x_i + \alpha) \frac{\partial}{\partial x_i}. \tag{1.2}$$

The one-parameter family of diffusions with $\alpha = 0$ is due to Ethier and Kurtz [11], while the extension to two parameters is due to Petrov [29]. Petrov [29] proved that for $\alpha \in [0, 1)$ and $\theta > -\alpha$ the operator $\mathcal{B}$ acting on $\mathcal{F}$ is closable and that its closure generates a Feller diffusion. We call these diffusions $\mathtt{EKP}$ diffusions and denote the laws of this two-parameter family by $\mathtt{EKP}(\alpha, \theta) = (\mathtt{EKP}_{\mathbf{x}}(\alpha, \theta), \mathbf{x} \in \nabla_\infty)$. The stationary distribution of $\mathtt{EKP}(\alpha, \theta)$ is $\mathtt{PD}(\alpha, \theta)$. The two-parameter $\mathtt{EKP}\,(\alpha, \theta)$ processes have been widely studied over the last decade using a variety of methods including Dirichlet forms, generators and discrete approximation, see [5, 10, 13, 14, 22, 37, 39, 40, 41]. The construction of Fleming–Viot processes corresponding to $\mathtt{EKP}(\alpha, \theta)$ diffusions was one of the longest-standing open problems in the field [13].

Recently, we and a collaborator constructed two-parameter measure-valued processes in [18] for $\theta \geq 0$ using methods that can be thought of as extending the pathwise construction in [44]. The methods from [18] were extended to $\theta > -\alpha$ in [43]. In [18, 43] these processes were called the Fleming–Viot processes associated to Petrov's diffusions, however, in [18, 43] we deferred verifying that these processes possess the second additional property mentioned above. This is a challenge because the pathwise constructions of [18, 43] do not immediately yield analytic information about these processes. In this paper, we initiate the study of the analytic properties of the processes constructed in [18, 43].

Our main result is that we identify the evolution of the sequence of ranked atom masses in the processes constructed in [18, 43] with parameters $0 < \alpha < 1$ and $\theta > -\alpha$ as the corresponding $\mathtt{EKP}$ diffusion. As noted in [18, 43], identifying the evolution of ranked atom masses of the process constructed there with the $\mathtt{EKP}(\alpha, \theta)$ diffusion completes the argument that those processes indeed are the conjectured associated Fleming–Viot processes.

This result opens the door to two-parameter extensions of the numerous applications of Fleming–Viot processes in the one-parameter $\alpha = 0$ case. Of these, we highlight two that we think would be particularly interesting to pursue. First, shortly after the $\mathtt{EKP}$ diffusions were introduced, a body of literature developed regarding the biological interpretation of the parameter α, see e.g. [5, 40]. In the one-parameter case, θ can be interpreted as the mutation rate of the population being modeled and, as discussed in [5], one of the main arguments for this interpretation comes from the role θ plays in defining the mutation operator in the generator of the corresponding Fleming–Viot process. Although our main results completes the identification of the two parameter Fleming–Viot processes, identifying the generator of these processes remains an open problem whose resolution would shed light on the biological interpretation of the parameter α. Second, there has been substantial recent interest in using dependent Poisson–Dirichlet random measures in nonparametric Bayesian inference [1, 4, 35, 46]. A novel approach in the one-parameter setting [1] uses Fleming–Viot processes to construct the dependent random measures because known formulas for the transition distributions of these processes makes predictive inference tractable. A natural question is to investigate if the two-parameter Fleming–Viot processes can be used to extend these methods to problems that require both parameters, such as those involving power-law decay [4].

Our arguments are related to those that appear in the study of polynomial processes [6, 9], which have recently drawn significant interest in mathematical finance for their balance of generality and computational tractability. Recall that a (classical) polynomial process is a Markov process on $\mathbb{R}^d$ whose semigroup preserves, for each m, the set of polynomials of degree at most m. Recently there have also been efforts to extend the study of polynomial processes to the infinite-dimensional setting [2, 7, 8], where the appropriate notion of "polynomial" depends on the context. Jacobi diffusions and Wright–Fisher diffusions are two classical examples of polynomial processes, and the key step in our argument is to identify statistics of the Fleming–Viot processes constructed in [18, 43] that evolve as Jacobi diffusions and Wright–Fisher diffusions. Using the action of the generators of these diffusions on a class of symmetric polynomials, we are able to compute the generator of the ranked sequence of atom masses in the Fleming–Viot processes. A consequence of our calculations is that these Fleming–Viot processes are $\mathcal{F}$-polynomial processes in the sense of [2]. We conjecture that they are polynomial processes in the sense of [7, 8], but have thus far been unable to compute the generator on the polynomials considered there.

Let us define the measure-valued processes of [18] that we called two-parameter Fleming–Viot processes $\mathtt{FV}(\alpha,\theta)$ in the parameter range $0<\alpha<1$ and $\theta\geq 0$. In this parameter range, the construction can be done in two steps. We first use explicit transition kernels identified in [18] to define purely-atomic-measure-valued self-similar superprocesses $\mathtt{SSSP}(\alpha,\theta)$ with a branching property. For the second step, we apply Shiga's [44] time-change/normalization, which we call de-Poissonization. Specifically, the *branching property* of $\mathtt{SSSP}(\alpha,\theta)$ means that each atom evolves independently (in size) and generates further atoms during its lifetime. De-Poissonization destroys the independence of the branching property. A similar approach applies in the full parameter range, but explicit transition kernels of $\mathtt{SSSP}(\alpha,\theta)$ are unknown in this case, making the construction and analysis of the $\mathtt{FV}(\alpha,\theta)$ processes more complicated. Nonetheless, our proofs for the $\theta\geq 0$ case apply with only minor changes. Thus, for the sake of simplicity, we carry out our construction and proofs first when $\theta\geq 0$ and then indicate what changes must be made in the general case.

To construct the transition kernel for an $\mathtt{SSSP}(\alpha,\theta)$ process when $0<\alpha<1$ and $\theta\geq 0$, there are three cases for the masses arising at a later time $s>0$ from a given atom in the initial state: (i) this atom survives to time s, as do infinitely many descendant atoms; (ii) the atom does not survive, but its descendants do; or (iii) neither the atom nor its descendants survive. The transitions of $\mathtt{SSSP}(\alpha,\theta)$ can thus be described via a probabilistic mixture of these three cases, independently for each atom.

Rather than separate out these cases entirely, we combine cases (i) and (ii), which yields a nicer formula for the law of the mass of either the surviving initial atom or one of its descendants. For this purpose, consider random variables $L_{b,r}^{(\alpha)}$ with Laplace transforms

$$\mathbf{E}\left[e^{-\lambda L_{b,r}^{(\alpha)}}\right]=\left(\frac{r+\lambda}{r}\right)^{\alpha}\frac{e^{br^2/(r+\lambda)}-1}{e^{br}-1},\quad \lambda\geq 0,\ b>0,\ r>0. \tag{1.3}$$

For an atom location $u\in[0,1]$, also consider a new location $U_0\sim\mathtt{Unif}[0,1]$ and mixing probabilities

$$p_{b,r}^{(\alpha)}(c)=\frac{I_{1+\alpha}(2r\sqrt{bc})}{I_{-1-\alpha}(2r\sqrt{bc})+\alpha(2r\sqrt{bc})^{-1-\alpha}/\Gamma(1-\alpha)}\quad\text{and}\quad 1-p_{b,r}^{(\alpha)}(c),$$

where I_v is the modified Bessel function of the first kind of index $v\in\mathbb{R}$. Independently of $L_{b,r}^{(\alpha)}$ and U_0, consider $\overline{\Pi}\sim\mathtt{PDRM}(\alpha,\alpha)$ and $G\sim\mathtt{Gamma}\,(\alpha,r)$ to define the random measure $\Pi:=G\overline{\Pi}$ of random mass G. Then

$$\begin{aligned}Q_{b,u,r}^{(\alpha)}:=e^{-br}\delta_0+\left(1-e^{-br}\right)\int_0^\infty\Big(&p_{b,r}^{(\alpha)}(c)\mathbf{P}\{c\delta(u)+\Pi\in\cdot\}\\&+\left(1-p_{b,r}^{(\alpha)}(c)\right)\mathbf{P}\{c\delta(U_0)+\Pi\in\cdot\}\Big)\mathbf{P}\left\{L_{b,r}^{(\alpha)}\in dc\right\}.\end{aligned}\tag{1.4}$$

is the distribution of a random measure that we will use to generate descendants at time $s=1/2r$ for an initial atom $b\delta(u)$. We remark that δ_0 refers to a Dirac mass at the zero measure, 0, in the space of measures on $[0,1]$, whereas $\delta(u)$ and $\delta(U_0)$ are Dirac masses in the interval $[0,1]$.

More precisely: $Q^{(\alpha)}_{b,u,r}$ yields no descendants with probability e^{-br} (case (iii)); or else the atoms of Π are descendants of the initial atom, and there is one additional atom of size $L^{(\alpha)}_{b,r}$. This special atom is located either at allelic type u (in case (i)) with conditional probability $p^{(\alpha)}_{b,r}(c)$ given $L^{(\alpha)}_{b,r}=c$, or at U_0 otherwise (in case (ii)). In this last case, the atom $c\delta(U_0)$ is an additional descendant of $b\delta(u)$.

DEFINITION 1.1 (Transition kernel $K^{\alpha,\theta}_s$). Let $\alpha\in(0,1)$ and $\theta\ge 0$. For a time $s>0$ and a finite measure $\mu=\sum_{i\ge1}b_i\delta(u_i)$ with distinct u_i, $i\ge 1$, we consider the random measure $G_0\overline{\Pi}_0+\sum_{i\ge1}\Pi_i$ for independent $G_0\sim \mathtt{Gamma}(\theta,1/2s)$, $\overline{\Pi}_0\sim\mathtt{PDRM}(\alpha,\theta)$ and $\Pi_i\sim Q^{(\alpha)}_{b_i,u_i,1/2s}$, $i\ge1$. We denote its distribution by $K^{\alpha,\theta}_s(\mu,\,\cdot\,)$.

We showed in [18, Theorem 1.2], that $(K^{\alpha,\theta}_s,\ s\ge0)$ is the transition semigroup of a path-continuous measure-valued Hunt process, which we refer to as $\mathtt{SSSP}_\mu(\alpha,\theta)$ when starting from any finite purely atomic measure $\mu_0=\mu$ on $[0,1]$. To obtain a probability-measure-valued process, we considered $(\mu_s,\ s\ge0)\sim\mathtt{SSSP}_\pi(\alpha,\theta)$ starting from any purely atomic probability measure $\mu_0=\pi$, its total mass process $\|\mu_s\|:=\mu_s([0,1])$ and the time-change

$$\rho(t)=\inf\left\{s\ge0\colon\ \int_0^s\frac{dv}{\|\mu_v\|}>t\right\},\quad t\ge0. \tag{1.5}$$

We called $\pi_t:=\|\mu_{\rho(t)}\|^{-1}\mu_{\rho(t)}$, $t\ge0$, a $\mathtt{FV}_\pi(\alpha,\theta)$ and showed in [18, Theorem 1.7] that it is a Hunt process in the space $\mathcal{M}^a_1$ of purely atomic probability measures on $[0,1]$ and, moreover, it has as its stationary distribution $\mathtt{PDRM}(\alpha,\theta)$.

For a probability measure $\pi=\sum_{i\ge1}p_i\delta(u_i)$ with $p_1\ge p_2\ge\cdots$, we denote by $\mathtt{RANKED}(\pi):=(p_i,\ i\ge1)\in\nabla_\infty$ its ranked sequence of atom sizes. The main result of this paper is the following connection to $\mathtt{EKP}$ diffusions.

The construction above is for $\theta\ge0$. We postpone the extension to $\theta\in(-\alpha,0)$ to Section 3.4, but state our main results here in full generality.

THEOREM 1.2. *Let $\alpha\in(0,1)$, $\theta>-\alpha$ and $\pi\in\mathcal{M}^a_1$. For a Fleming–Viot process $(\pi_t,\ t\ge0)\sim\mathtt{FV}_\pi(\alpha,\theta)$, the associated ranked process is an $\mathtt{EKP}$ diffusion with pre-generator (1.2), up to a linear time-change. More precisely, $\big(\mathtt{RANKED}(\pi_{t/2}),\ t\ge0\big)\sim\mathtt{EKP}_{\mathtt{RANKED}(\pi)}(\alpha,\theta)$.*

As noted in [18], this theorem is the last step in showing that $\mathtt{FV}(\alpha,\theta)$ may be viewed as a labeled variant of $\mathtt{EKP}\,(\alpha,\theta)$. Consequently, it completes the argument that the processes constructed in [18, 43] solve the open problem of Feng and Sun [13].

This theorem allows us to prove a number of properties of $\mathtt{EKP}(\alpha,\theta)$ processes based on our understanding of Fleming–Viot processes. For example, following [30], for $\alpha\in(0,1)$, the *α-diversity* is

$$\mathscr{D}_\alpha(\mathbf{x}):=\lim_{h\downarrow0}\Gamma(1-\alpha)h^\alpha\#\{i\ge1\colon\ x_i>h\}\qquad\text{for }\mathbf{x}\in\nabla_\infty, \tag{1.6}$$

if this limit exists. This may be understood as a continuum analogue to the number of blocks in a partition of n. A constant multiple of this is sometimes called the *local time* of $\mathbf{x}$ [34]. These quantities arise in a variety of contexts [31, 32]. Ruggiero et al. [41] have studied processes related to $\mathtt{EKP}$ diffusions for which α-diversity evolves as a diffusion. Then

$$\mathscr{D}_\alpha(\pi):=\lim_{h\downarrow0}\Gamma(1-\alpha)h^\alpha\#\{u\in[0,1]\colon\ \pi\{u\}>h\}=\mathscr{D}_\alpha(\mathtt{RANKED}(\pi)),$$

in the sense that either neither limit exists or they are equal. Since the path-continuity of the diversity process $t\mapsto\mathscr{D}_\alpha(\pi_t)$ was shown in [18, Theorem 1.12], our Theorem 1.2 here immediately implies the following.

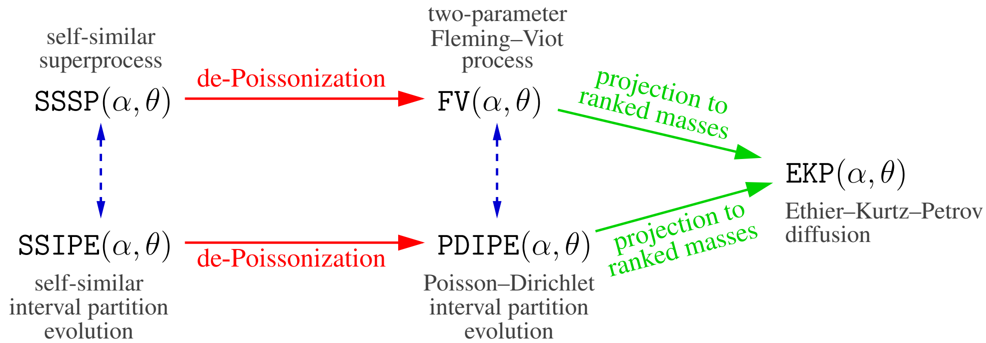

FIG 1.1. *Relationships between the main processes of this paper. Vertical arrows represent that these pairs of processes can be coupled in such a way that at every time, atoms of the superprocess are in bijection with blocks of equal mass in the interval partition evolution. This coupling is evident in the shared scaffolding-and-spindles constructions of both processes [18, Section 1.2] and [19, Section 2.3 and Proposition 3.4].* $\mathtt{FV}(\alpha,\theta)$*,* $\mathtt{PDIPE}(\alpha,\theta)$*, and* $\mathtt{EKP}(\alpha,\theta)$ *have respective stationary distributions* $\mathtt{PDRM}(\alpha,\theta)$*,* $\mathtt{PDIP}(\alpha,\theta)$*, and* $\mathtt{PD}(\alpha,\theta)$*.*

COROLLARY 1.3. *Let* $\mathbf{x}\in\nabla_\infty$*. Suppose the limit* $\mathscr{D}_\alpha(\mathbf{x})$ *in (1.6) exists. Then there exists an* EKP *diffusion* $(\mathbf{V}_t,\, t\geq 0)$ *starting from* $\mathbf{x}$ *such that* $t\mapsto\mathscr{D}_\alpha(\mathbf{V}_t)$ *is continuous.*

Since EKP diffusions are reversible [29, Main theorem (3), p. 280], the evolving ranked sequence of atom sizes in $\mathtt{FV}(\alpha,\theta)$ is reversible as well. We make the following conjecture.

CONJECTURE 1. $\mathtt{FV}(\alpha,\theta)$ *is reversible with respect to* $\mathtt{PDRM}(\alpha,\theta)$*.*

On the other hand, there is a loss of symmetry in the corresponding interval-partition-valued diffusions (except when $\theta=\alpha$), which we recall in the appendix. This means that reversibility fails for those diffusions.

We prove Theorem 1.2 in two steps. The first is to calculate relevant parts of the generator of $\mathtt{FV}(\alpha,\theta)$, which allows us to identify the semigroups of EKP (α,θ) and ranked $\mathtt{FV}(\alpha,\theta)$ on the Hilbert space $\mathbf{L}^2[\alpha,\theta]$ of functions on ∇_∞ that are square integrable with respect to the measure $\mathtt{PD}(\alpha,\theta)$. The second step involves interval partition evolutions, $\mathtt{PDIPE}(\alpha,\theta)$ [16, 17, 19, 42, 43], which can be coupled with $\mathtt{FV}(\alpha,\theta)$-processes to have the same ranked masses, as indicated in Figure 1.1. We use these couplings to establish sufficient regularity to identify the semigroups also as operators acting on the space of bounded continuous functions.

The structure of this paper is as follows. In Section 2 we collect some material about $\mathtt{SSSP}(\alpha,\theta)$ and $\mathtt{FV}(\alpha,\theta)$ for $\theta\geq 0$ from [18] and strengthen connections to Jacobi and Wright–Fisher diffusions that will facilitate the generator calculations of $\mathtt{FV}(\alpha,\theta)$. In Section 3, we carry out the first step in the proof of Theorem 1.2. In Section 4, we obtain a version of Theorem 1.2 for the interval partition evolutions of [19, 42] and use it to carry out the second step in the proof of Theorem 1.2.

2. Fleming–Viot, Jacobi and Wright–Fisher processes. This section recalls material on $\mathtt{BESQ}_b(2r)$, $\mathtt{JAC}_b(r,r')$ and $\mathtt{WF}_{\mathbf{b}}(\mathbf{r})$ processes and their connections due to Warren and Yor [48] and Pal [28]. In particular, we discuss the domains of their infinitesimal generators. Finally, we recall from [18] some more details about the construction and properties of $\mathtt{SSSP}(\alpha,\theta)$ and $\mathtt{FV}(\alpha,\theta)$ for $\theta\geq 0$, and we go beyond [18] by extracting from $\mathtt{FV}(\alpha,\theta)$ several "subprocesses" that are Jacobi diffusions or Wright–Fisher processes.

2.1. *Squared Bessel processes* $\mathtt{BESQ}_b(2r)$. Let $b \geq 0$, $r \in \mathbb{R}$, and consider a Brownian motion B. The squared Bessel process is the unique strong solution of

$$dZ_s = 2rds + 2\sqrt{|Z_s|}dB_s, \quad Z_0 = b,$$

see [21, 36]. For $r \geq 0$, we denote its distribution by $\mathtt{BESQ}_b(2r)$. These processes are $[0,\infty)$-valued and have 0 as an inaccessible boundary for $r \geq 1$, a reflecting boundary for $0 < r < 1$ and are absorbed at 0 for $r = 0$. For $r < 0$, the strong solution becomes negative after $S = \inf\{s \geq 0 \colon Z_s = 0\}$ and we denote by $\mathtt{BESQ}_b(2r)$ the distribution of the absorbed process $(Z_{s\wedge S},\, s \geq 0)$. The infinitesimal generator of $\mathtt{BESQ}(2r)$ is

$$2z\frac{d^2}{dz^2} + 2r\frac{d}{dz}$$

on a domain that includes all twice continuously differentiable functions $f \colon [0,\infty) \to \mathbb{R}$ with compact support in $(0,\infty)$.

2.2. *Jacobi diffusions* $\mathtt{JAC}_b(r,r')$. Let $r, r' \geq 0$, $b \in [0,1]$. Warren and Yor [48] take independent $Z \sim \mathtt{BESQ}_b(2r)$, $Z' \sim \mathtt{BESQ}_{1-b}(2r')$ and the time-change

$$\rho(t) = \inf\left\{s \geq 0 \colon \int_0^s \frac{dv}{Z_v + Z'_v} > t\right\}, \quad t \geq 0. \tag{2.1}$$

The time-changed proportion $X := \big((Z_{\rho(t)} + Z'_{\rho(t)})^{-1} Z_{\rho(t)},\, t \geq 0\big)$ is shown to be a $[0,1]$-valued Markov process, a Jacobi diffusion [25], which we denote by $\mathtt{JAC}_b(r,r')$. Jacobi diffusions satisfy the SDE

$$dX_t = 2\sqrt{X_t(1-X_t)}dB_t + 2\big(r - (r+r')X_t\big)dt, \quad X_0 = b,$$

and have infinitesimal generator

$$\mathcal{A}^{r,r'}_{\mathtt{JAC}} = 2x(1-x)\frac{d^2}{dx^2} + 2\big(r - (r+r')x\big)\frac{d}{dx}. \tag{2.2}$$

With some care at the boundaries of $[0,1]$, this all extends to general $r, r' \in \mathbb{R}$ up to the time S (or S' or $S \wedge S'$) if $r < 0$ (or $r' < 0$ or both), when our definition of $\mathtt{BESQ}$ leads, after time-change, to the absorption of $\mathtt{JAC}_b(r,r')$ in 0 (or in 1 or in either). We do, however, only absorb at 0 if $r \leq 0$ and at 1 if $r' \leq 0$, allowing reflection when $0 < r < 1$ or $0 < r' < 1$, respectively. The respective boundary is inaccessible for $r \geq 1$ or $r' \geq 1$.

LEMMA 2.1. *For all $r, r' \in \mathbb{R}$, the domain of $\mathcal{A}^{r,r'}_{\mathtt{JAC}}$ includes all twice continuously differentiable functions f on $[0,1]$ that further satisfy $f'(0) = 0$ if $r < 0$ and $f'(1) = 0$ if $r' < 0$.*

PROOF. Let $b \in (0,1)$. As for the squared Bessel SDE, the Jacobi SDE has a unique strong solution up to the absorption time S, which is infinite if $r > 0$ and $r' > 0$, exhibiting reflection at 0 and/or 1 if $0 < r < 1$ and/or $0 < r' < 1$. By the (local) Itô formula and a change of variables,

$$\mathbf{E}[f(X_s)] = f(b) + s\mathbf{E}\left[\int_0^{1\wedge S/s} \big(2(r-(r+r')X_{us})f'(X_{us}) + 2X_{us}(1-X_{us})f''(X_{us})\big)\, du\right]$$

for $X \sim \mathtt{JAC}_b(r,r')$. We conclude by path-continuity and dominated convergence that

$$s^{-1}\big(\mathbf{E}[f(X_s)] - f(b)\big) \to 2(r-(r+r')b)f'(b) + 2b(1-b)f''(b) =: g(b)$$

as $s \to 0+$. For $b = 0$, the same argument applies if $r > 0$. If $r \leq 0$ absorption yields a zero limit, which extends g continuously if and only if $f'(0) = 0$ or $r = 0$. The analogous argument at $b = 1$ requires $r' \geq 0$ or $f'(1) = 0$. □

2.3. *Wright–Fisher processes* $\mathtt{WF}_{\mathbf{b}}(\mathbf{r})$. Consider parameters $\ell\ge 2$, $\mathbf{r}=(r_1,\ldots,r_\ell)\in\mathbb{R}^\ell$ and initial state $\mathbf{b}=(b_1,\ldots,b_\ell)\in\Delta_\ell:=\left\{(x_1,\ldots,x_\ell)\in[0,1]^\ell\colon \sum_{i\in[\ell]}x_i=1\right\}$, where we wrote $[\ell]:=\{1,\ldots,\ell\}$. Set $r_+:=\sum_{i\in[\ell]}r_i$. Pal [27, 28] adapted the Warren–Yor construction of Jacobi diffusions to construct diffusions on the simplex Δ_ℓ. Specifically, consider independent $Z^{(i)}\sim\mathtt{BESQ}_{b_i}(2r_i)$, $i\in[\ell]$, and denote by

$$S_0=\inf\left\{s\ge 0\colon Z^{(i)}_s=0 \text{ for some } i\in[\ell] \text{ with } r_i<0\right\}$$

the first absorption time of a $\mathtt{BESQ}$ with negative parameter. On $[0,S_0)$, consider $Z^{(+)}:=\sum_{i\in\ell}Z^{(i)}$ and the time-change

$$\rho(t)=\inf\left\{s\ge 0\colon \int_0^s\frac{dv}{Z^{(+)}_v}>t\right\},\quad 0\le t<T:=\int_0^{S_0}\frac{dv}{Z^{(+)}_v}. \tag{2.3}$$

Then $\left(\big(Z^{(+)}_{\rho(t\wedge T)}\big)^{-1}Z^{(1)}_{\rho(t\wedge T)},\ldots,\big(Z^{(+)}_{\rho(t\wedge T)}\big)^{-1}Z^{(\ell)}_{\rho(t\wedge T)}\right)$, $t\ge 0$, the stopped and time-changed proportions of $(Z^{(1)},\ldots,Z^{(\ell)})$, form a Δ_ℓ-valued diffusion, whose distribution we denote by $\mathtt{WF}_{\mathbf{b}}(\mathbf{r})$. When $r_1,\ldots,r_\ell\ge 0$, this is (up to a linear time-change) the well-known Wright–Fisher diffusion, see e.g. [12]. In particular, $\mathbf{W}=\left(W^{(1)},\ldots,W^{(\ell)}\right)\sim\mathtt{WF}_{\mathbf{b}}(\mathbf{r})$ satisfies the SDEs

$$dW^{(i)}_t=2\left(1-W^{(i)}_t\right)\sqrt{W^{(i)}_t}dB^{(i)}_t-2W^{(i)}_t\sum_{j\in[\ell]\setminus\{i\}}\sqrt{W^{(j)}_t}dB^{(j)}_t+2\left(r_i-r_+W^{(i)}_t\right)dt,$$

with $W^{(i)}_0=b_i$, for all $i\in[\ell]$, where $(B^{(1)},\ldots,B^{(\ell)})$ is a vector of independent Brownian motions. Also, $\mathtt{WF}(\mathbf{r})$ has infinitesimal generator

$$\mathcal{A}^{\mathbf{r}}_{\mathtt{WF}}=2\sum_{i\in[\ell]}w_i\frac{\partial^2}{\partial w_i^2}-2\sum_{i,j\in[\ell]}w_iw_j\frac{\partial^2}{\partial w_i\partial w_j}-2\sum_{i\in[\ell]}(r_+w_i-r_i)\frac{\partial}{\partial w_i}. \tag{2.4}$$

The extension to negative parameters was observed by Pal [28]. The arguments are also valid for $\mathbf{r}$ with both negative and nonnegative entries.

LEMMA 2.2. *The domain of $\mathcal{A}^{\mathbf{r}}_{\mathtt{WF}}$ includes all functions $f\colon\Delta_\ell\to\mathbb{R}$ that possess an extension to $\mathbb{R}^\ell$ that is twice continuously differentiable and further satisfies*

$$\frac{\partial}{\partial w_i}f(\mathbf{w})=0\quad \textit{for all } \mathbf{w}\in\Delta_\ell \textit{ with } w_i=0, \textit{ if } r_i<0\quad \textit{for all } i\in[\ell].$$

The proof of Lemma 2.1 is easily adapted to this ℓ-dimensional setting.

2.4. *Properties of* $\mathtt{SSSP}(\alpha,\theta)$ *and* $\mathtt{FV}(\alpha,\theta)$ *for* $\theta\ge 0$. Throughout this subsection we assume that $\theta\ge 0$. The reader will have observed the parallels between the de-Poissonization time-change constructions of $\mathtt{FV}(\alpha,\theta)$ from $\mathtt{SSSP}(\alpha,\theta)$ and of $\mathtt{JAC}(r,r')$ and $\mathtt{WF}(\mathbf{r})$ from vectors of $\mathtt{BESQ}$ processes. Let us here recall from [18] some properties of $\mathtt{SSSP}(\alpha,\theta)$ and $\mathtt{FV}(\alpha,\theta)$ that shed more light on these parallels.

For $\mathtt{FV}(\alpha,\theta)$, the time-change $t\mapsto\rho(t)$ only depends on $\left(\|\mu_s\|,s\ge 0\right)$, the total mass process of the $\mathtt{SSSP}(\alpha,\theta)$. For $\mathtt{JAC}(r,r')$ and $\mathtt{WF}(\mathbf{r})$, the corresponding quantity is the sum of all independent $\mathtt{BESQ}$ processes, which is a $\mathtt{BESQ}(2r+2r')$ or $\mathtt{BESQ}(2r_+)$ by the well-known [45] additivity of $\mathtt{BESQ}$ when all parameters are nonnegative or natural extensions (subject to suitable stopping) when some parameters are negative, as noted previously by the present authors [16], see also [33].

PROPOSITION 2.3 (Theorem 1.5 of [18]). *For* $\big(\mu_s,\, s\geq 0\big)\sim \mathtt{SSSP}_\mu(\alpha,\theta)$*, we have* $\big(\|\mu_s\|,\, s\geq 0\big)\sim \mathtt{BESQ}(2\theta)$.

By definition, this total mass process is the sum of countably many atom sizes at all times, but the additivity of BESQ enters in a more subtle way. The transition semigroup $(K_s^{\alpha,\theta}, s\geq 0)$ of $\mathtt{SSSP}(\alpha,\theta)$ stated in Definition 1.1 leaves implicit the evolution of atoms and sheds little light on the creation of new atoms. In [18], we provide a Poissonian construction that explicitly specifies independent $\mathtt{BESQ}(-2\alpha)$ evolutions for each atom size and creates new atoms at times corresponding to pre-jump levels in a $\mathtt{Stable}(1+\alpha)$ Lévy process. We do not need the details of this construction in the present paper and refer the reader to [18], but the following consequence of the Poissonian construction is important for us.

PROPOSITION 2.4 (Corollary 5.11 of [18]). *Let* $\big(\overleftarrow{\mu}_s, s\geq 0\big)\sim \mathtt{SSSP}_0(\alpha,\alpha)$ *and, independently, consider* $Z\sim \mathtt{BESQ}_b(-2\alpha)$ *with absorption time* S*. Set* $\mu_s = Z_s\delta(u)+\overleftarrow{\mu}_s$, $0\leq s\leq S$*. Conditionally given* $\big(\mu_s,\, 0\leq s\leq S\big)$ *with* $\mu_S=\lambda$*, let* $\big(\mu_{S+v},\, v\geq 0\big)\sim \mathtt{SSSP}_\lambda(\alpha,0)$*. Then* $\big(\mu_s,\, s\geq 0\big)\sim \mathtt{SSSP}_{b\delta(u)}(\alpha,0)$.

Recall from the introduction that the transition kernels $K_s^{\alpha,\theta}$, $s\geq 0$, of $\mathtt{SSSP}(\alpha,\theta)$ stated in Definition 1.1 possess a branching property that suggests an ancestral relationship between any time-0 atom $b_i\delta(u_i)$ and the time-s atoms of Π_i, for each $i\geq 1$. We can interpret the remaining time-s atoms of $G_0\overline{\Pi}_0$ as *immigration*. The following result expresses this split at a fixed time, via the Markov property, in terms of independent superprocesses.

PROPOSITION 2.5 (Proposition 1.4 and Theorem 1.10 of [18]). *For any finite measure* $\mu=\sum_{i\geq 1} b_i\delta(u_i)$*, consider independent* $\mu^{(0)}\sim \mathtt{SSSP}_0(\alpha,\theta)$ *and* $\mu^{(i)}\sim \mathtt{SSSP}_{b_i\delta(u_i)}(\alpha,0)$*. Then* $(\mu_s,\, s\geq 0):=\sum_{i\geq 0}\mu^{(i)}\sim \mathtt{SSSP}_\mu(\alpha,\theta)$.

2.5. *Jacobi and Wright–Fisher processes associated with* $\mathtt{FV}(\alpha,\theta)$. The following result records the consequences for $\mathtt{FV}(\alpha,\theta)$ of the BESQ processes associated with $\mathtt{SSSP}(\alpha,\theta)$ by combining Propositions 2.3–2.5.

PROPOSITION 2.6. *In the setting of Proposition 2.5, with an initial probability measure* $\mu=\sum_{i\geq 1}p_i\delta(u_i)$*, denote by* $M_t:=\big(\|\mu_t\|,\, t\geq 0\big)$ *the total mass process and by* $(\rho(t),\, t\geq 0)$ *the time-change of* (1.5)*. Let* $k\geq 1$.

(i) *Then* $X^{(k)}:=\big((M_{\rho(t)})^{-1}\|\mu^{(k)}_{\rho(t)}\|, t\geq 0\big)\sim \mathtt{JAC}_{p_k}(0,\theta)$*,* $X^{(0)}:=\big((M_{\rho(t)})^{-1}\|\mu^{(0)}_{\rho(t)}\|, t\geq 0\big)\sim \mathtt{JAC}_0(\theta,0)$*, and*

$$\Big(X^{(1)},\ldots,X^{(k)},1-\sum\nolimits_{i\in[k]}X^{(i)}\Big)\sim \mathtt{WF}_{(p_1,\ldots,p_k,1-\sum_{i\in[k]}p_i)}(0,\ldots,0,\theta).$$

(ii) *We also have* $W^{(k)}:=\big((M_{\rho(t)})^{-1}\mu^{(k)}_{\rho(t)}\{u_i\}, t\geq 0\big)\sim \mathtt{JAC}_{p_k}(-\alpha,\theta+\alpha)$*, and for*

$$W^{(-k)}:=\big((M_{\rho(t\wedge T_k)})^{-1}\mu^{(k)}_{\rho(t\wedge T_k)}\big([0,1]\setminus\{u_k\}\big), t\geq 0\big),$$

where T_k *is the absorption time of* $W^{(k)}$*, we also have*

$$\big(W^{(k)},W^{(-k)},1-W^{(k)}-W^{(-k)}\big)\sim \mathtt{WF}_{(p_k,0,1-p_k)}(-\alpha,\alpha,\theta).$$

Furthermore, for $T=\min\{T_1,\ldots,T_k\}$*, we have*

$$\Big(\Big(W^{(1)}_{t\wedge T},\ldots,W^{(k)}_{t\wedge T},1-\sum\nolimits_{i\in[k]}W^{(i)}_{t\wedge T}\Big),\, t\geq 0\Big)\sim \mathtt{WF}_{\mathbf{b}}(\mathbf{r})$$

with $\mathbf{r}=(-\alpha,\ldots,-\alpha,\theta+k\alpha)$ *and* $\mathbf{b}=(p_1,\ldots,p_k,1-\sum_{i\in[k]}p_k)$.

PROOF. (i) Proposition 2.3 applied to each $\mu^{(i)}$, $i\geq 0$, yields independent $M^{(0)}:=\big(\|\mu_s^{(0)}\|,\, s\geq 0\big)\sim\mathtt{BESQ}_0(2\theta)$ and $M^{(k)}:=\big(\|\mu_s^{(k)}\|,\, s\geq 0\big)\sim\mathtt{BESQ}_{p_k}(0)$, $k\geq 1$. By the additivity of BESQ, we further note that $N^{(k)}:=\sum_{i\in\mathbb{N}_0\setminus\{k\}}M^{(i)}\sim\mathtt{BESQ}_{1-p_k}(2\theta)$, and that $M^{(k)}$ and $N^{(k)}$ are independent, for each $k\geq 1$. Similarly, $R^{(k)}:=\sum_{i\in\mathbb{N}_0\setminus[k]}M^{(i)}\sim\mathtt{BESQ}_{1-\sum_{i\in[k]}p_i}(2\theta)$ is independent of $\big(M^{(1)},\ldots,M^{(k)}\big)$.

The time-change $(\rho(t),\, t\geq 0)$ of (1.5) is based on the total mass process $\big(\|\mu_s\|, s\geq 0\big)=: M$. But since $M=M^{(k)}+N^{(k)}=R^{(k)}+\sum_{i\in[k]}M^{(i)}$ for all $k\geq 1$, this is the same time-change as (2.1) to construct $X^{(k)}\sim\mathtt{JAC}_{p_k}(0,\theta)$ from $Z:=M^{(k)}$ and $Z':=N^{(k)}$. This time-change is also the same as (2.3) to construct $\mathtt{WF}_{\mathbf{b}}(\mathbf{r})$ with $\mathbf{b}=(p_1,\ldots,p_k,1-\sum_{i\in[k]}p_i)$ and $\mathbf{r}=(0,\ldots,0,\theta)$, $M^{(i)}$ as $Z^{(i)}$, $i\in[k]$, and $R^{(k)}$ as $Z^{(k+1)}$. In particular, the first k components of the $\mathtt{WF}_{\mathbf{b}}(\mathbf{r})$ are indeed $\big(X^{(1)},\ldots,X^{(k)}\big)$, and the last component is as required to add to 1.

(ii) We refine the setting of (i). If we furthermore construct each $\mu^{(i)}$, $i\geq 1$, as in Proposition 2.4, we instead obtain a countable family of independent $Z^{(i)}:=\big(\mu_s^{(i)}\{u_i\},\, s\geq 0\big)\sim\mathtt{BESQ}_{p_i}(-2\alpha)$, $i\geq 1$. Now applying Proposition 2.3 to $\mu^{(0)}$ and $\overleftarrow{\mu}^{(i)}$, we also have independent $Z^{(0)}:=\big(\|\mu_s^{(0)}\|,\, s\geq 0\big)\sim\mathtt{BESQ}_0(2\theta)$ and $Z^{(-i)}:=\big(\|\overleftarrow{\mu}_s^{(i)}\|,\, s\geq 0\big)\sim\mathtt{BESQ}_0(2\alpha)$, $i\geq 1$.

Recall notation $N^{(j)}$ from the proof of (i). Note that the independence of $\mu^{(j)}$ and $\sum_{i\in\mathbb{N}_0\setminus\{j\}}\mu^{(i)}$ entails that $Z^{(j)}\sim\mathtt{BESQ}_{p_j}(-2\alpha)$ is independent of $Z^{(-j)}\sim\mathtt{BESQ}_0(2\alpha)$ and $N^{(j)}\sim\mathtt{BESQ}_{1-p_j}(2\theta)$, and hence independent of their sum $L^{(j)}:=Z^{(-j)}+N^{(j)}\sim\mathtt{BESQ}_{1-p_j}(2(\theta+\alpha))$, as required to get $W^{(j)}:=\big((M_{\rho(t)})^{-1}Z^{(j)}_{\rho(t)}, t\geq 0\big)\sim\mathtt{JAC}_{p_j}(-\alpha,\theta+\alpha)$, and indeed as required to construct $(W^{(j)},W^{(-j)},1-W^{(j)}-W^{(-j)})\sim\mathtt{WF}_{(p_j,0,1-p_j)}(-\alpha,\alpha,\theta)$, where we recall that this process is stopped at the time that the left-most component hits 0.

Assembling several $\mathtt{JAC}_{p_i}(-\alpha,\theta+\alpha)$ to $\mathtt{WF}_{\mathbf{b}}(-\alpha,\ldots,-\alpha,\theta+k\alpha)$ can be done as in (i), with the caveat that having k negative parameters makes this construction (and the definition of WF) only valid/useful up to the random time

$$T=\inf\big\{t\geq 0\colon \exists_{i\in[k]}\,\pi_t\{u_i\}=0\big\}=\inf\big\{t\geq 0\colon \exists_{i\in[k]}\,W_t^{(i)}=0\big\}=\min\{T_1,\ldots,T_k\}.\qquad\square$$

COROLLARY 2.7. *Consider* $(\pi_t,t\geq 0)\sim\mathtt{FV}_\mu(\alpha,\theta)$ *starting from any probability measure* $\mu=\sum_{i\geq 1}p_i\delta(u_i)$, *then for any* $j\geq 1$, *we have* $(\pi_t\{u_j\},t\geq 0)\sim\mathtt{JAC}_{p_j}(-\alpha,\theta+\alpha)$. *Also,*

$$\big((\pi_{t\wedge T}\{u_1\},\ldots,\pi_{t\wedge T}\{u_k\},\pi_{t\wedge T}([0,1]\setminus\{u_1,\ldots,u_k\}),\, t\geq 0\big)\sim\mathtt{WF}_{\mathbf{b}}(\mathbf{r}),$$

for any $k\geq 1$, *where* $T=\inf\{t\geq 0\colon \exists_{i\in[k]}\,\pi_t\{u_i\}=0\}$, $\mathbf{b}=(p_1,\ldots,p_k,1-\sum_{i\in[k]}p_i)$ *and* $\mathbf{r}=(-\alpha,\ldots,-\alpha,\theta+k\alpha)$.

PROOF. In the setting of Proposition 2.6(ii), we have $(\pi_t,\, t\geq 0):=\big(\|\mu_{\rho(t)}\|^{-1}\mu_{\rho(t)}\big)\sim\mathtt{FV}_\mu(\alpha,\theta)$, and $W^{(j)}=\big(\pi_t\{u_j\},\, t\geq 0\big)$ a.s., so $(\pi_t\{u_j\},\, t\geq 0)\sim\mathtt{JAC}_{p_j}(-\alpha,\theta+\alpha)$. This also entails the $\mathtt{WF}_{\mathbf{b}}(\mathbf{r})$ claim. $\square$

3. Generators and semigroups on $\mathbf{L}^2[\boldsymbol{\alpha},\boldsymbol{\theta}]$. Recall that Theorem 1.2 claims that for $(\pi_t,t\geq 0)\sim\mathtt{FV}_\pi(\alpha,\theta)$, the projection $\big(\mathtt{RANKED}(\pi_t),t\geq 0\big)$ is $\mathtt{EKP}_{\mathtt{RANKED}(\pi)}(\alpha,\theta)$. The aim of this section is to identify the $\mathbf{L}^2[\alpha,\theta]$-semigroup of the projected process. We first treat the case when $\theta\geq 0$ in Sections 3.1-3.3. Specifically, for $\theta\geq 0$ we establish the Markov property in Section 3.1, compute the infinitesimal generator on Petrov's algebra $\mathcal{F}$ in Section 3.2, and in Section 3.3 we conclude that the $\mathbf{L}^2[\alpha,\theta]$-semigroups of the projected process and of $\mathtt{EKP}_{\mathtt{RANKED}(\pi)}(\alpha,\theta)$ coincide. We then sketch the modifications needed for the extension to $\theta>-\alpha$ in Section 3.4.

3.1. *Markov property of* $\big(\mathtt{RANKED}(\pi_t),\,t\geq 0\big)$ *for* $\theta\geq 0$. To study the projection of $\mathtt{FV}(\alpha,\theta)$ to ∇_∞, let us introduce notation $\mathcal{M}_1^a$ for the set of all purely atomic probability measures on the Borel sigma-algebra $\mathcal{B}([0,1])$ of the interval $[0,1]$. We consider two topologies on $\mathcal{M}_1^a$, the weak topology, which is separable, and the topology induced by the total variation distance

$$d_{\mathtt{TV}}(\pi,\pi')=\sup_{B\in\mathcal{B}([0,1])}\left|\pi(B)-\pi'(B)\right|,$$

which is not separable. We will denote by $\mathbb{P}_\pi^{\alpha,\theta}$ a probability measure under which $(\pi_t,t\geq 0)\sim\mathtt{FV}_\pi(\alpha,\theta)$, and by $\mathbb{E}_\pi^{\alpha,\theta}$ associated expectations.

Let us also clarify the topology on the Kingman simplex

$$\nabla_\infty:=\left\{\mathbf{x}=(x_1,x_2,\ldots):x_1\geq x_2\geq\cdots\geq 0,\ \sum_{i\geq 1}x_i=1\right\}. \tag{3.1}$$

This is a metric space under ℓ^∞. Its closure under ℓ^∞, denoted by $\overline{\nabla}_\infty$, is the set of non-increasing sequences in $[0,1]$ with sum at most 1. Petrov [29] established $\mathtt{EKP}(\alpha,\theta)$ as path-continuous Markov processes that can start anywhere in $\overline{\nabla}_\infty$. It has been shown in [10] that the processes, starting at $\mathbf{x}\in\nabla_\infty$, never leave ∇_∞. Therefore, it is already known that $\mathtt{EKP}(\alpha,\theta)$ diffusions can be considered as diffusions on ∇_∞. Since the $\mathtt{RANKED}$ map takes values in ∇_∞ only and is surjective onto ∇_∞, this also follows from our Theorem 1.2.

LEMMA 3.1. $\mathtt{RANKED}\colon\mathcal{M}_1^a\to\nabla_\infty$ *is Borel measurable with respect to the weak topology and is* $d_{\mathtt{TV}}$*-continuous.*

PROOF. It is well-known – see e.g. [24, Lemma 1.6] – that there are measurable enumeration maps that associate with $\pi\in\mathcal{M}_1^a$ the countable sequence of all location/size pairs of atoms, which can then be ranked measurably. Furthermore, $\mathtt{RANKED}$ is Lipschitz with respect to d_{TV} and ℓ^∞. □

PROPOSITION 3.2. *Let* $\alpha\in(0,1)$, $\theta\geq 0$, $\pi\in\mathcal{M}_1^a$ *and* $(\pi_t,t\geq 0)\sim\mathtt{FV}_\pi(\alpha,\theta)$. *Then* $\big(\mathtt{RANKED}(\pi_t),t\geq 0\big)$ *is a path-continuous* ∇_∞*-valued Markov process that is stationary with respect to the* $\mathtt{PD}(\alpha,\theta)$ *law.*

PROOF. We will show that for any two $\pi',\pi''\in\mathcal{M}_1^a$ with $\mathtt{RANKED}(\pi')=\mathtt{RANKED}(\pi'')$, we can couple $\mathtt{FV}_{\pi'}(\alpha,\theta)$ and $\mathtt{FV}_{\pi''}(\alpha,\theta)$ processes that have the same projection under $\mathtt{RANKED}$.

First consider $\mu'=b\delta(u')$ and $\mu''=b\delta(u'')$. By Proposition 2.4, we can construct $\mathtt{SSSP}_{b\delta(u)}(\alpha,0)$ from independent $\overleftarrow{\mu}\sim\mathtt{SSSP}_0(\alpha,\alpha)$ and $Z\sim\mathtt{BESQ}_b(-2\alpha)$, with absorption time $S=\inf\{s\geq 0\colon Z_s=0\}$, and from $(\widetilde{\mu}_v,\,v\geq 0)$, an $\mathtt{SSSP}(\alpha,0)$ starting from $\overleftarrow{\mu}_S$. None of this depends on u and we can set

$$\mu'_s:=\begin{cases}Z_s\delta(u')+\overleftarrow{\mu}_s, & 0\leq s<S,\\ \widetilde{\mu}_{s-S}, & s\geq S,\end{cases}\qquad \mu''_s:=\begin{cases}Z_s\delta(u'')+\overleftarrow{\mu}_s, & 0\leq s<S,\\ \widetilde{\mu}_{s-S}, & s\geq S,\end{cases}$$

to couple $(\mu'_s,\,s\geq 0)\sim\mathtt{SSSP}_{b\delta(u')}(\alpha,0)$ and $(\mu''_s,\,s\geq 0)\sim\mathtt{SSSP}_{b\delta(u'')}(\alpha,0)$.

In general, we can write $\pi'=\sum_{i\geq 1}b_i\delta(u'_i)$ and $\pi''=\sum_{i\geq 1}b_i\delta(u''_i)$ for the same sequence $(b_i,\,i\geq 1)$. We construct an $\mathtt{SSSP}_{\pi'}(\alpha,\theta)$ and an $\mathtt{SSSP}_{\pi''}(\alpha,\theta)$ as in Proposition 2.5, with each pair of $\mathtt{SSSP}_{b_i\delta(u'_i)}(\alpha,0)$ and $\mathtt{SSSP}_{b_i\delta(u''_i)}$, $i\geq 1$, coupled as above, and using the same $\mathtt{SSSP}_0(\alpha,\theta)$.

This coupling is such that all ranked masses at all times coincide, for the $\mathtt{SSSP}_{\pi'}(\alpha,\theta)$ and the $\mathtt{SSSP}_{\pi''}(\alpha,\theta)$. In particular, they have the same total mass processes and the same time-change (1.5), and therefore the associated $\mathtt{FV}_{\pi'}(\alpha,\theta)$ and $\mathtt{FV}_{\pi''}(\alpha,\theta)$ share the same ranked mass processes.

Let $F\colon \nabla_\infty \to [0,\infty)$ be bounded measurable. For the coupled processes $(\pi'_t,\, t\geq 0)\sim \mathtt{FV}_{\pi'}(\alpha,\theta)$ and $(\pi''_t,\, t\geq 0)\sim \mathtt{FV}_{\pi''}(\alpha,\theta)$, we have $\mathbb{E}\big[F(\mathtt{RANKED}(\pi'_t))\big] = \mathbb{E}\big[F(\mathtt{RANKED}(\pi''_t))\big]$. In particular, $\mathbb{E}^{\alpha,\theta}_{\pi}\big[F(\mathtt{RANKED}(\pi_t))\big]$ is a function of $\mathtt{RANKED}(\pi)\in\nabla_\infty$. By Dynkin's criterion (e.g. [38, Lemma I.14.1]), mapping $\mathtt{FV}(\alpha,\theta)$ via $\mathtt{RANKED}$ yields a Markov process. By [18, Corollary 5.5], $\mathtt{SSSP}(\alpha,\theta)$ and hence $\mathtt{FV}(\alpha,\theta)$ are $d_{\mathtt{TV}}$-path-continuous. Since $\mathtt{RANKED}$ is $d_{\mathtt{TV}}$-continuous, by Lemma 3.1, mapping $\mathtt{FV}(\alpha,\theta)$ under $\mathtt{RANKED}$ yields a path-continuous process in ∇_∞. Mapping a stationary $\mathtt{FV}(\alpha,\theta)$, with $\mathtt{PDRM}(\alpha,\theta)$ stationary distribution clearly yields a process that has stationary distribution $\mathtt{PD}(\alpha,\theta)$. □

The same method allows us to prove that the projected process is a Hunt process, which will also follow from our identification with the $\mathtt{EKP}$ diffusion.

3.2. *The infinitesimal generator of* $(\mathtt{RANKED}(\pi_t),\, t\geq 0)$ *on the algebra* $\mathcal{F}$ *for* $\theta\geq 0$. Throughout this subsection we assume $\theta\geq 0$. We will frequently employ an (arbitrary) inclusion map $\iota\colon\nabla_\infty\to\mathcal{M}^a_1$:

$$\iota(\mathbf{x}) = \sum_{i\geq 1} x_i\delta(u_i), \qquad \text{where } u_i = 1/i,\ i\geq 1.$$

We will abuse notation and write $\mathtt{FV}_{\mathbf{x}}(\alpha,\theta) := \mathtt{FV}_{\iota(\mathbf{x})}(\alpha,\theta)$, $\mathbb{P}^{\alpha,\theta}_{\mathbf{x}} := \mathbb{P}^{\alpha,\theta}_{\iota(\mathbf{x})}$, and $\mathbb{E}^{\alpha,\theta}_{\mathbf{x}} := \mathbb{E}^{\alpha,\theta}_{\iota(\mathbf{x})}$. We will also follow the convention of including finite-dimensional unit simplices in ∇_∞ by appending zeros.

PROPOSITION 3.3. *For every $q\in\mathcal{F}$ we have*

$$\lim_{t\to 0+}\frac{\mathbb{E}^{\alpha,\theta}_{\mathbf{x}}\left[q\left(\mathtt{RANKED}(\pi_t)\right)\right]-q(\mathbf{x})}{t} = 2\mathcal{B}q(\mathbf{x}),\quad \textit{for every } \mathbf{x}\in\nabla_\infty, \tag{3.2}$$

where $\mathcal{B}$ is (the restriction to $\mathcal{F}$ of) the generator (1.2) of $\mathtt{EKP}\,(\alpha,\theta)$. The above convergence also holds in $\mathbf{L}^2$ with respect to the law of $\mathtt{PD}\,(\alpha,\theta)$.

Proposition 3.3 is proved in two steps: first we prove (3.2) when $q=q_m$ for some $m\geq 1$, and then for the general case. Recall that $q_m(\mathbf{x}) = \sum_{i\geq 1} x_i^{m+1}$. In general, $q_m(\mathtt{RANKED}(\pi_t))$ is a sum over many atoms, cf. Definition 1.1 for the transition kernel before the de-Poissonization time-change/normalization. We will work with lower and upper bounds on $\mathbb{E}^{\alpha,\theta}_{\mathbf{x}}\left[q\left(\mathtt{RANKED}(\pi_t)\right)\right]-q(\mathbf{x})$ that separate the main contributions and asymptotically negligible contributions.

To prepare this, we first establish three lemmas. In these lemmas we use the setting of Propositions 2.5 and 2.6, with $(\pi_t, t\geq 0)\sim\mathtt{FV}_{\mathbf{x}}(\alpha,\theta)$ constructed from independent $\mu^{(i)}$, $i\geq 0$, which are one $\mathtt{SSSP}(\alpha,0)$ starting from from each initial atom and one more $\mathtt{SSSP}_0(\alpha,\theta)$, "immigration". We denote by $M_t := \sum_{i\geq 0}\big\|\mu^{(i)}_t\big\|$ the total mass process so that

$$\pi_t = \sum_{i\geq 0}\pi^{(i)}_t,\quad \text{where } \pi^{(i)}_t := \frac{\mu^{(i)}_{\rho(t)}}{M_{\rho(t)}} \text{ and } \rho(t) = \inf\left\{s\geq 0\colon \int_0^s \frac{dv}{M_v} > t\right\}. \tag{3.3}$$

In particular, this gives access to two Jacobi processes for each $i\geq 1$:

$$W^{(i)}_t := \pi_t\{u_i\} = \big(M_{\rho(t)}\big)^{-1}\mu^{(i)}_{\rho(t)}\{u_i\} \leq \big(M_{\rho(t)}\big)^{-1}\big\|\mu^{(i)}_{\rho(t)}\big\| =: X^{(i)}_t.$$

LEMMA 3.4. *For each $i\geq 1$, we have, as $t\to 0+$,*

$$\frac{1}{t}\left(\mathbf{E}\left[\big(W^{(i)}_t\big)^{m+1}\right] - x_i^{m+1}\right) \to 2(m+1)(m-\alpha)x_i^m - 2(m+1)(m+\theta)x_i^{m+1}$$

$$\textit{and } \frac{1}{t}\left(\mathbf{E}\left[\big(X^{(i)}_t\big)^{m+1}\right] - x_i^{m+1}\right) \to 2(m+1)mx_i^m - 2(m+1)(m+\theta)x_i^{m+1}.$$

PROOF. By Proposition 2.6, $W^{(i)}\sim \mathtt{JAC}_{x_i}(-\alpha,\theta+\alpha)$ and $X^{(i)}\sim \mathtt{JAC}_{x_i}(0,\theta)$. By Lemma 2.1, we may apply the generator (2.2) to $f(x)=x^{m+1}$. ☐

Since $2\mathcal{B}q_m(\mathbf{x})=2(m+1)(m-\alpha)\sum_{i\geq 1}x_i^m-2(m+1)(m+\theta)\sum_{i\geq 1}x_i^{m+1}$, the first of these captures the main contributions. We also need some uniform bounds on these quantities.

LEMMA 3.5. *For each $i\geq 1$ and all $m\geq 1$, $\mathbf{x}\in\nabla_\infty$ and $t\geq 0$, we have*

$$-2(m+1)(m+\theta)x_i\leq \frac{1}{t}\left(\mathbf{E}\left[\left(W_t^{(i)}\right)^{m+1}\right]-x_i^{m+1}\right)$$
$$\leq \frac{1}{t}\left(\mathbf{E}\left[\left(X_t^{(i)}\right)^{m+1}\right]-x_i^{m+1}\right)\leq 2(m+1)mx_i.$$

PROOF. By Proposition 2.6, $X^{(i)}\sim \mathtt{JAC}_{x_i}(0,\theta)$. By Itô's formula, we have $\mathbf{E}\big[\big(X_v^{(i)}\big)^m\big]\leq \mathbf{E}\big[X_v^{(i)}\big]\leq x_i$, so applying Itô's formula again yields

$$\frac{1}{t}\left(\mathbf{E}\left[\left(X_t^{(i)}\right)^{m+1}\right]-x_i^{m+1}\right)\leq \frac{1}{t}\mathbf{E}\left[\int_0^t 2(m+1)m\big(X_v^{(i)}\big)^m dv\right]\leq 2(m+1)mx_i.$$

Similarly, we bound $t^{-1}\big(\mathbf{E}\big[\big(W_t^{(i)}\big)^{m+1}\big]-x_i^{m+1}\big)$ below by

$$-\frac{1}{t}\mathbf{E}\left[\int_0^t 2(m+1)(m+\theta)\big(W_v^{(i)}\big)^{m+1}dv\right]\geq -2(m+1)(m+\theta)x_i.$$ ☐

Finally, we control some asymptotically negligible contributions, using notation $X_t^{(0)}=\big(M_{\rho(t)}\big)^{-1}\|\mu^{(0)}_{\rho(t)}\|$ and $W_t^{(-i)}=X_t^{(i)}-W_t^{(i)}$ of Proposition 2.6.

LEMMA 3.6. *We have $t^{-1}\mathbf{E}\Big[\big(X_t^{(0)}\big)^{m+1}\Big]\to 0$ and $t^{-1}\mathbf{E}\Big[\big(W_t^{(-i)}\big)^{m+1}\Big]\to 0$ as $t\to 0+$, and $t^{-1}\mathbf{E}\Big[\big(W_t^{(-i)}\big)^{m+1}\Big]\leq 4(2+\theta)x_i$ for all $t>0$, for all $i\geq 1$.*

PROOF. By Proposition 2.6, $X^{(0)}\sim \mathtt{JAC}_0(\theta,0)$, so we can apply the generator to $f(x)=x^{m+1}$ and evaluate at $x=0$. Since $W^{(-i)}$ is not itself a $\mathtt{JAC}_0(\alpha,\theta-\alpha)$ as it has been stopped when $W^{(i)}$ vanishes, we consider $(W^{(i)},W^{(-i)},1-W^{(i)}-W^{(-i)})\sim \mathtt{WF}_{(x_i,0,1-x_i)}(-\alpha,\alpha,\theta)$. By Lemma 2.2, we can apply the generator (2.4) to $f(\mathbf{w})=w_2^{m+1}$ and evaluate at $w_2=0$.

Finally, $\big(W_t^{(-i)}\big)^{m+1}\leq \big(X_t^{(i)}-W_t^{(i)}\big)^2\leq \big(\big(X_t^{(i)}\big)^2-x_i^2\big)-\big(\big(W_t^{(i)}\big)^2-x_i^2\big)$, so the bound follows by taking $m=1$ in Lemma 3.5. ☐

PROOF OF THE $q=q_m$ CASE OF PROPOSITION 3.3. By only retaining atoms of π_t at the initial atom locations u_i of $\iota(\mathbf{x})$, we can bound the LHS of (3.2) below by

$$\sum_{i\geq 1}\frac{1}{t}\left(\mathbf{E}\left[\left(\pi_t\{u_i\}\right)^{m+1}\right]-x_i^{m+1}\right)=\sum_{i\geq 1}\frac{1}{t}\left(\mathbf{E}\left[\left(W_t^{(i)}\right)^{m+1}\right]-x_i^{m+1}\right). \tag{3.4}$$

By Lemmas 3.4–3.5 and dominated convergence we find the lower bound

$$\liminf_{t\to 0+}\frac{\mathbb{E}_{\mathbf{x}}^{\alpha,\theta}\left[q_m\left(\mathtt{RANKED}(\pi_t)\right)\right]-q_m(\mathbf{x})}{t}$$
$$\geq \sum_{i\geq 1}\left(2(m+1)(m-\alpha)x_i^m-2(m+1)(m+\theta)x_i^{m+1}\right)=2\mathcal{B}q_m(\mathbf{x}).$$

For the upper bound, split π_t as in (3.3) and bound above the sums of $(m+1)$st powers for each $\pi_t^{(i)}$, $i\geq 1$, by the $(m+1)$st power of the sums $W_t^{(-i)}=\pi_t^{(i)}\big([0,1]\setminus\{u_i\}\big)$ or $X^{(i)}=\big\|\pi_t^{(i)}\big\|$, so that for all $n\geq 0$

$$q_m(\mathtt{RANKED}(\pi_t))\leq\sum_{i\in[n]}\Big(\big(W_t^{(i)}\big)^{m+1}+\big(W_t^{(-i)}\big)^{m+1}\Big)+\sum_{i\in\mathbb{N}_0\setminus[n]}\big(X_t^{(i)}\big)^{m+1}. \tag{3.5}$$

By Lemmas 3.4–3.6, this yields the upper bounds

$$\limsup_{t\to 0+}\frac{\mathbb{E}_{\mathbf{x}}^{\alpha,\theta}\left[q_m\left(\mathtt{RANKED}(\pi_t)\right)\right]-q_m(\mathbf{x})}{t}$$
$$\leq\sum_{i\in[n]}\Big(2(m{+}1)(m{-}\alpha)x_i^m-2(m{+}1)(m{+}\theta)x_i^{m+1}\Big)+2(m{+}1)m\sum_{i\in\mathbb{N}\setminus[n]}x_i.$$

These upper bounds converge to $2\mathcal{B}q_m(\mathbf{x})$ as $n\to\infty$, so limsup and liminf coincide and we identify the limit. Note that $\mathbf{E}\big[\big(X_t^{(0)}\big)^{m+1}\big]$ does not depend on $\mathbf{x}\in\nabla_\infty$. Using Lemma 3.5 on (3.4) and (3.5) for $n{=}0$, we also see that

$$\sup_{\mathbf{x}\in\nabla_\infty}\left|\frac{\mathbb{E}_{\mathbf{x}}^{\alpha,\theta}\left[q_m\left(\mathtt{RANKED}(\pi_t)\right)\right]-q_m(\mathbf{x})}{t}\right|\leq\frac{1}{t}\mathbb{E}\big[\big(X_t^{(0)}\big)^{m+1}\big]+2(m+1)(m+\theta).$$

By dominated convergence, (3.2) holds in $\mathbf{L}^2$ with respect to $\mathtt{PD}(\alpha,\theta)$. □

We now generalize this argument to all $q{\in}\mathcal{F}$. Let $\mathbf{m}{=}(m_1,\ldots,m_k){\in}\mathbb{N}^k$, $k{\geq}1$. We will use notation

$$q_{\mathbf{m}}(\mathbf{x})=\prod_{j\in[k]}q_{m_j}(\mathbf{x})=\sum_{(i_1,\ldots,i_k)\in\mathbb{N}^k}\prod_{j\in[k]}x_{i_j}^{m_j}$$

and generalize Lemmas 3.4–3.5 to corresponding products.

LEMMA 3.7. *Let $k\geq 1$, $\mathbf{m}\in\mathbb{N}^k$ and consider distinct $i_1,\ldots,i_k\geq 1$. Then*

$$\frac{1}{t}\left(\mathbf{E}\Bigg[\prod_{j\in[k]}\big(W_t^{(i_j)}\big)^{m_j+1}\Bigg]-\prod_{j\in[k]}x_{i_j}^{m_j+1}\right)\to\mathcal{A}_k p_{\mathbf{m}}(x_{i_1},\ldots,x_{i_k}),$$

as $t\to 0+$, where $p_{\mathbf{m}}(w_1,\ldots,w_k)=\prod_{j\in[k]}w_j^{m_j+1}$ and

$$\mathcal{A}_k:=2\sum_{i\in[k]}w_i\frac{\partial^2}{\partial w_i^2}-2\sum_{i,j\in[k]}w_iw_j\frac{\partial^2}{\partial w_i\partial w_j}-2\sum_{i\in[k]}\big(\theta w_i{+}\alpha\big)\frac{\partial}{\partial w_i}. \tag{3.6}$$

PROOF. We use Proposition 2.6 and Lemma 2.2. Specifically, the quantity of interest is the generator of $\mathtt{WF}(\mathbf{r}(k))$ with $\mathbf{r}(k)=(-\alpha,\ldots,-\alpha,\theta+k\alpha)$ applied to the function $\overline{p}_{\mathbf{m}}(w_1,\ldots,w_k,w_{k+1})=p_{\mathbf{m}}(w_1,\ldots,w_k)$, and evaluated at $\big(x_{i_1},\ldots,x_{i_k},1-\sum_{j\in[k]}x_{i_j}\big)$. But $\mathcal{A}_{\mathtt{WF}}^{\mathbf{r}(k)}\overline{p}_{\mathbf{m}}(w_1,\ldots,w_k,w_{k+1})$ does not depend on w_{k+1} and, as a function of $(w_1,\ldots,w_k)$ coincides with $\mathcal{A}_k p_{\mathbf{m}}$. □

LEMMA 3.8. *Let $k\geq 1$ and $\mathbf{m}\in\mathbb{N}^k$. Then there is $c(\mathbf{m})>0$ such that*

$$-c(\mathbf{m})\prod_{j\in[k]}x_{i_j}\;\leq\;\frac{1}{t}\left(\mathbf{E}\Bigg[\prod_{j\in[k]}\big(W_t^{(i_j)}\big)^{m_j+1}\Bigg]-\prod_{j\in[k]}x_{i_j}^{m_j+1}\right)$$
$$\leq\;\frac{1}{t}\left(\mathbf{E}\Bigg[\prod_{j\in[k]}\big(X_t^{(i_j)}\big)^{m_j+1}\Bigg]-\prod_{j\in[k]}x_{i_j}^{m_j+1}\right)\;\leq\;c(\mathbf{m})\prod_{j\in[k]}x_{i_j},$$

for all $\mathbf{x}=(x_i,\,i\geq 1)\in\nabla_\infty$, all distinct $i_1,\ldots,i_k\geq 1$ and all $t>0$.

PROOF. Let $\overline{p}_{\mathbf{m}}(w_1,\ldots,w_k,w_{k+1}) := p_{\mathbf{m}}(w_1,\ldots,w_k) := \prod_{j\in[k]} w_j^{m_j+1}$ as in the proof of Lemma 3.7. By Proposition 2.6,

$$\left(X^{(1)},\ldots,X^{(k)},1-\sum_{i\in[k]}X^{(i)}\right)\sim \mathtt{WF}_{(x_{i_1},\ldots,x_{i_k},1-\sum_{i\in[k]}x_{i_j})}(0,\ldots,0,\theta).$$

To establish the last of the claimed inequalities, rewrite the expectation as in the proof of Lemma 3.5, here using the multi-dimensional Itô formula twice and dropping all negative terms to find an upper bound of the required form

$$\frac{1}{t}\mathbf{E}\left[\int_0^t 2\sum_{i\in[k]}(m_i+1)m_i p_{\mathbf{m}-e_i}\left(X_v^{(i_1)},\ldots,X_v^{(i_k)}\right)dv\right]\le 2\sum_{i\in[k]}(m_i+1)m_i\prod_{j\in[k]}x_{i_j},$$

where e_i denotes the ith unit vector in $\mathbb{R}^k$. For the lower bound, the same argument applies, based on $\mathtt{WF}(-\alpha,\ldots,-\alpha,\theta+k\alpha)$ instead of $\mathtt{WF}(0,\ldots,0,\theta)$, here dropping all positive terms to find a similar lower bound, which allows us to choose

$$c(\mathbf{m})=2\left(\sum_{i\in[k]}(m_i+1+\alpha)\right)\left(\sum_{j\in[k]}(m_j+1+\theta)\right).$$

□

PROOF OF PROPOSITION 3.3. By linearity, it suffices to consider functions $q(\mathbf{x}) = q_{\mathbf{m}}(\mathbf{x}) = \sum_{(i_1,\ldots,i_k)\in\mathbb{N}^k}\prod_{j\in[k]} x_{i_j}^{m_j}$. We use the lower bound

$$\mathbb{E}_{\mathbf{x}}^{\alpha,\theta}\left[q_{\mathbf{m}}\left(\mathtt{RANKED}(\pi_t)\right)\right]\ge\sum_{(i_1,\ldots,i_k)\in\mathbb{N}^k}\mathbf{E}\left[\prod_{j\in[k]}\left(W_t^{(i_j)}\right)^{m_j+1}\right]$$

and adapt the proof of the case $q=q_m$ for univariate m. Here, we split sums according to partitions $A=\{A_1,\ldots,A_r\}\in\mathcal{P}_{[k]}^{(r)}$ of $[k]$ with $r\in[k]$ parts

$$q_{\mathbf{m}}(\mathbf{x})=\sum_{(i_1,\ldots,i_k)\in\mathbb{N}^k}\prod_{j\in[k]}x_{i_j}^{m_j+1}=\sum_{r\in[k]}\sum_{A\in\mathcal{P}_{[k]}^{(r)}}\sum_{\substack{h_1,\ldots,h_r\\ \text{distinct}}}\prod_{\ell\in[r]}x_{h_\ell}^{m_\ell^A+1}, \tag{3.7}$$

where $m_\ell^A+1:=\sum_{j\in A_\ell}(m_j+1)$. Then Lemma 3.8 applies to yield for all $t>0$

$$\frac{1}{t}\left|\mathbf{E}\left[\prod_{\ell\in[r]}\left(W_t^{(h_\ell)}\right)^{m_\ell^A+1}\right]-\prod_{\ell\in[r]}x_{h_\ell}^{m_\ell^A+1}\right|\le c(\mathbf{m}^A)\prod_{\ell\in[r]}x_{h_\ell},$$

where $\mathbf{m}^A=(m_1^A,\ldots,m_r^A)$. The bounds are summable over distinct $h_1,\ldots,h_r$ so that we can apply dominated convergence and Lemma 3.7 to find

$$\begin{aligned}&\liminf_{t\to 0+}\frac{\mathbb{E}_{\mathbf{x}}^{\alpha,\theta}\left[q_{\mathbf{m}}\left(\mathtt{RANKED}(\pi_t)\right)\right]-q_{\mathbf{m}}(\mathbf{x})}{t}\\ &\qquad\ge\sum_{r\in[k]}\sum_{A\in\mathcal{P}_{[k]}^{(r)}}\sum_{\substack{h_1,\ldots,h_r\\ \text{distinct}}}\mathcal{A}_r p_{\mathbf{m}^A}\left(x_{h_1},\ldots,x_{h_r}\right)=2\mathcal{B}q_{\mathbf{m}}(\mathbf{x}).\end{aligned} \tag{3.8}$$

For the upper bounds, we use the same bounds as for (3.5), here making sure that every k-tuple of atoms of π_t is taken into account, to bound $\mathbb{E}_{\mathbf{x}}^{\alpha,\theta}\left[q_{\mathbf{m}}\left(\mathtt{RANKED}(\pi_t)\right)\right]$ above by

$$\begin{aligned}&\sum_{(i_1,\ldots,i_k)\in[n]^k}\mathbf{E}\left[\prod_{j\in[k]}\left(W_t^{(i_j)}\right)^{m_j+1}\right]+\sum_{(i_1,\ldots,i_k)\in\mathbb{N}_0^k\setminus[n]^k}\mathbf{E}\left[\prod_{j\in[k]}\left(X_t^{(i_j)}\right)^{m_j+1}\right]\\ &\quad+\sum_{r\in[k]}\sum_{(i_1,\ldots,i_k)\in\mathbb{N}_0^k:\,i_r\in[n]}\mathbf{E}\left[\left(W_t^{(-i_r)}\right)^{m_r+1}\prod_{j\in[k]\setminus\{r\}}\left(X_t^{(i_j)}\right)^{m_j+1}\right].\end{aligned} \tag{3.9}$$

We further bound the last term of (3.9) by $\sum_{r\in[k]}\sum_{i_r\in[n]}\mathbf{E}\big[\big(W_t^{(-i_r)}\big)^{m_r+1}\big]$. We split the middle term of (3.9) as in (3.7). Then Lemmas 3.6–3.8 this yield the upper bounds

$$\limsup_{t\to 0+}\frac{\mathbb{E}_{\mathbf{x}}^{\alpha,\theta}\left[q_{\mathbf{m}}\left(\mathtt{RANKED}(\pi_t)\right)\right]-q_{\mathbf{m}}(\mathbf{x})}{t}$$
$$\leq \sum_{r\in[k]}\sum_{A\in\mathcal{P}_{[k]}^{(r)}}\Bigg(\sum_{\substack{(h_1,\ldots,h_r)\in[n]^k\\ \text{distinct}}}\mathcal{A}_r p_{\mathbf{m}^A}(x_{h_1},\ldots,x_{h_r})+\sum_{\substack{(h_1,\ldots,h_r)\in\mathbb{N}^k\setminus[n]^k\\ \text{distinct}}}c(\mathbf{m}^A)\prod_{\ell\in[r]}x_{h_\ell}\Bigg).$$

These upper bounds converge to $2\mathcal{B}q_{\mathbf{m}}(\mathbf{x})$ as $n\to\infty$, and we conclude as in the proof of the case $q=q_m$ for univariate $m\in\mathbb{N}$. □

3.3. *Identifying the* $\mathbf{L}^2$*-semigroups of* $(\mathtt{RANKED}(\pi_t),t\geq 0)$ *and* $\mathtt{EKP}(\alpha,\theta)$ *when* $\theta\geq 0$. Let $\mathbf{L}^2[\alpha,\theta]$ refer to the Hilbert space of square integrable functions on ∇_∞ with respect to the measure $\mathtt{PD}(\alpha,\theta)$. Also, for this section, the corresponding norm will be denoted by $\|\cdot\|_{\alpha,\theta}$. Recall from the beginning of Section 3.2 the notation $\mathbb{P}_{\mathbf{x}}^{\alpha,\theta}$ for the distribution of a $\mathtt{FV}(\alpha,\theta)$-process starting from a measure $\iota(\mathbf{x})$ with atom sizes given by $\mathbf{x}$. Let $\mathrm{P}_{\mathbf{x}}$ denote the probability measure on $\mathcal{C}([0,\infty),\nabla_\infty)$ which is the distribution of $\big(\mathtt{RANKED}(\pi_t),\,t\geq 0\big)$ under $\mathbb{P}_{\mathbf{x}}^{\alpha,\theta}$. We will use the notation $\mathbf{V}=(\mathbf{V}_t,\,t\geq 0)$ for this canonical random process and notation $\mathrm{E}_{\mathbf{x}}$ for the expectation operator.

LEMMA 3.9. *Let* $(T_t,\,t\geq 0)$ *denote the transition semigroup of the process* $\mathbf{V}$. *Then, for every* $t>0$, T_t *is an operator on* $\mathbf{L}^2[\alpha,\theta]$ *and the semigroup is strongly continuous as a semigroup.*

PROOF. By definition, $T_tf(\mathbf{x})=\mathrm{E}_{\mathbf{x}}[f(\mathbf{V}_t)]=\int_{\nabla_\infty}f(\mathbf{v})p_t(\mathbf{x},d\mathbf{v})$, where $p_t(\mathbf{x},d\mathbf{v})$ is the transition kernel of $\mathbf{V}$. We first show that T_t is an operator on $\mathbf{L}^2[\alpha,\theta]$ in the sense that

(i) if f is square integrable with respect to $\mathtt{PD}(\alpha,\theta)$ then so is T_tf,
(ii) if $f=0$ $\mathtt{PD}(\alpha,\theta)$-a.e. then so is T_tf.

The second condition shows that the $\mathtt{PD}(\alpha,\theta)$-equivalence class of T_tf is determined by the $\mathtt{PD}(\alpha,\theta)$-equivalence class of f, so that we may consider $T_t\colon\mathbf{L}^2[\alpha,\theta]\to\mathbf{L}^2[\alpha,\theta]$. From Jensen's inequality we see that

$$\begin{aligned}\int_{\nabla_\infty}(T_tf(\mathbf{x}))^2\mathtt{PD}(\alpha,\theta)(d\mathbf{x})&\leq\int_{\nabla_\infty}T_tf^2(\mathbf{x})\mathtt{PD}(\alpha,\theta)(d\mathbf{x})\\&=\int_{\nabla_\infty}f^2(\mathbf{v})\mathtt{PD}(\alpha,\theta)(d\mathbf{v})\end{aligned}\tag{3.10}$$

since $\mathtt{PD}(\alpha,\theta)$ is the stationary distribution of $\mathbf{V}$. Both claims follow immediately.

It is easy to see that every element in the unital algebra $\mathcal{F}$ is in $\mathbf{L}^2[\alpha,\theta]$. As a corollary of the $\mathbf{L}^2$ part of Proposition 3.3, $\lim_{t\to 0+}\|T_tq-q\|_{\alpha,\theta}=0$ for any $q\in\mathcal{F}$. As noted in [29, Section 2.2], functions in $\mathcal{F}$ have continuous extensions to the ℓ^∞-closure $\overline{\nabla}_\infty$ of ∇_∞, and $\mathcal{F}$ is dense in the space of bounded continuous functions on $\overline{\nabla}_\infty$, and hence also in $\mathbf{L}^2[\alpha,\theta]$. Consider any $f\in\mathbf{L}^2[\alpha,\theta]$. Then, there exists a sequence $\{q_n\}\subseteq\mathcal{F}$ such that $\lim_{n\to\infty}q_n=f$ in $\mathbf{L}^2[\alpha,\theta]$. By the triangle inequality and Equation (3.10)

$$\begin{aligned}\|(T_t-I)f\|_{\alpha,\theta}&\leq\|(T_t-I)q_n\|_{\alpha,\theta}+\|(T_t-I)(q_n-f)\|_{\alpha,\theta}\\&\leq\|(T_t-I)q_n\|_{\alpha,\theta}+2\|q_n-f\|_{\alpha,\theta}.\end{aligned}$$

Consequently, we get $\lim_{t\to 0+}T_tf=f$ in $\mathbf{L}^2[\alpha,\theta]$. This proves strong continuity of the semigroup. □

Hence, by [12, Corollary 1.1.6], the $\mathbf{L}^2[\alpha,\theta]$ generator $\mathcal{A}$ of $(T_t,\ t\geq 0)$ is closed and has a dense domain in $\mathbf{L}^2[\alpha,\theta]$. Moreover, by [12, Proposition 1.2.1], for any $\lambda>0$, the resolvent $(\lambda-\mathcal{A})^{-1}$ exists as a bounded operator on $\mathbf{L}^2[\alpha,\theta]$ and is one-to-one and has dense range. Specifically, the elementary argument of [3, Step 2 in the proof of Proposition 1.4] gives the following result. See also [29, Proposition 4.3].

LEMMA 3.10. *For any $\lambda>0$, we have $(\lambda-\mathcal{A})\mathcal{F}=\mathcal{F}$.*

We are now able to complete the first step of the proof of Theorem 1.2.

STEP 1 OF THE PROOF OF THEOREM 1.2. Lemmas 3.9–3.10 together with [12, Proposition 1.3.1] imply that $\mathcal{F}$ is a core for $\mathcal{A}$. Letting $(\widetilde{T}_t,\ t\geq 0)$ be the $\mathbf{L}^2[\alpha,\theta]$-semigroup considered by Feng and Sun [13], this shows that $(T_t,\ t\geq 0)$ and $(\widetilde{T}_{2t},\ t\geq 0)$ have the same generator (given by the closure of $(\mathcal{A},\mathcal{F})$) and thus are equal as semigroups on $\mathbf{L}^2[\alpha,\theta]$. □

Petrov [29] constructs $\mathtt{EKP}(\alpha,\theta)$ as a Feller process on the closure

$$\overline{\nabla}_\infty := \left\{\mathbf{x}=(x_1,x_2,\ldots) : x_1\geq x_2\geq\cdots\geq 0,\ \sum_{i\geq 1} x_i\leq 1\right\}$$

of our state space ∇_∞. One might wonder if our method enables construction of the Feller process (rather than its $\mathbf{L}^2[\alpha,\theta]$-semigroup). This is not so clear in our measure-valued setting, where $\mathtt{FV}(\alpha,\theta)$ cannot be extended to a Feller process, cf. [18, below Proposition 3.6]. In an interval-partition-valued setting, we have extended corresponding processes $\mathtt{PDIPE}(\alpha,\theta)$, which we recall in Section 4. This yields a stronger regularity of semigroups that allows us to complete the proof of Theorem 1.2. While we could try to avoid the $\mathbf{L}^2$-theory in that setting, this does not appear to save any effort. We provide some further pointers on this in Section 4.

3.4. *Fleming–Viot processes with parameters $\alpha\in(0,1)$ and $\theta\in(-\alpha,0)$.* In this section we recall from [43, Section 5.1] the definition of $\mathtt{FV}(\alpha,\theta)$ when $\theta\in(-\alpha,0)$, and we prove Theorem 1.2 (Step 1) for these processes. The main idea is simple: each atom in $\mathtt{SSSP}(\alpha,\theta)$ still evolves independently as $\mathtt{BESQ}(-2\alpha)$. New atoms are still created both as descendants of existing atoms and as some further immigration. However, to achieve net "emigration" at rate $|\theta|$, one atom does not produce descendants, and this absence of descendants is partially compensated by an immigration rate of $\theta+\alpha>0$. Here is a more formal definition.

DEFINITION 3.11. Let $\alpha\in(0,1)$, $\theta\in(-\alpha,0)$, $\mu\in\mathcal{M}^a$, $H_0=0$ and $\mu_0=\mu$. Inductively given $(\mu_s, 0\leq s\leq H_n)$ for some $n\geq 0$, there are two cases. If $\mu_{H_n}=0$, let $H_{n+1}=H_n$. Otherwise, write $\mu_{H_n}=\sum_{i\geq 1} b_i^{(n)}\delta(u_i^{(n)})$ with $b_1^{(n)}\geq b_2^{(n)}\geq\cdots\geq 0$, consider independent

$$Z^{(n)}\sim\mathtt{BESQ}_{b_1^{(n)}}(-2\alpha)\quad\text{and}\quad \nu^{(n)}\sim\mathtt{SSSP}_{\nu_0^{(n)}}(\alpha,\theta+\alpha),\quad\text{where } \nu_0^{(n)}=\sum_{i\geq 2} b_i^{(n)}\delta(u_i^{(n)}),$$

and set $H_{n+1}=H_n+\inf\{s\geq 0\colon Z_s^{(n)}=0\}$ and $\mu_s=Z_{s-H_n}^{(n)}\delta(u_1^{(n)})+\nu_{s-H_n}^{(n)}$, $s\in(H_n,H_{n+1}]$.

Finally, let $H_\infty=\lim_{n\to\infty}H_n$. Given $(\mu_s,\ 0\leq s<H_\infty)$, let $\mu_s=0$ for all $s\geq H_\infty$. We refer to $(\mu_s,\ s\geq 0)$ as an $\mathtt{SSSP}_\mu(\alpha,\theta)$.

It was shown in [43, Theorems 5.3 and 5.5] that $\mathtt{SSSP}_\mu(\alpha,\theta)$ is (well-defined and) a path-continuous Hunt process, and that de-Poissonization as in and below (1.5) yields an $\mathcal{M}_1^a$-valued Hunt process extending $\mathtt{FV}(\alpha,\theta)$ to $\theta\in(-\alpha,0)$, with stationary distribution $\mathtt{PDRM}(\alpha,\theta)$. Let us revisit the main steps of our proof of Theorem 1.2 in the case $\theta\in(-\alpha,0)$.

- Proposition 2.3 holds by [43, Theorem 5.3]: $\mathtt{SSSP}(\alpha,\theta)$ has $\mathtt{BESQ}(2\theta)$ total mass process.
- In Proposition 2.5, replacing $\mu^{(1)}$ by $(Z_s^{(0)}\delta(u_1^{(0)}),\, s\geq 0)$ yields a process that is $\mathtt{SSSP}_\mu(\alpha,\theta)$ until $Z^{(0)}$ hits 0 and continues as $\mathtt{SSSP}(\alpha,\theta+\alpha)$. This is a consequence of Proposition 2.5, applied the $\mathtt{SSSP}(\alpha,\theta+\alpha)$ without $\mu^{(1)}$, and of Definition 3.11.
- Proposition 2.6 holds subject to some modifications. Specifically, due to the replacement of $\mu^{(1)}$, there is no $X^{(1)}$ here, and the Wright–Fisher part of (i) holds if $X^{(1)}$ is replaced by $W^{(1)}$, with first and last parameter changed to $-\alpha$ and $\theta+\alpha$, respectively. In (ii), there is no $W^{(-1)}$, but all claims not involving $W^{(-1)}$ continue to hold, as does Corollary 2.7.
- Proposition 3.2 continues to hold. For the proof, only the third paragraph needs revisiting: a $\mathtt{BESQ}_{b_1}(-2\alpha)$ can replace the coupled $\mathtt{SSSP}_{b_1\delta(u_1')}$ and $\mathtt{SSSP}_{b_1\delta(u_1'')}$. This establishes the required coupling up to time H_1. An induction extends it to time H_∞, which suffices.
- Proposition 3.3 continues to hold. The proof involves several lemmas. Of these, Lemma 3.4 continues to hold for each $i\geq 2$ and the first convergence also for $i=1$. For the remaining argument, set $\widetilde{W}^{(-1)}:=0$ and $\widetilde{W}^{(i)}:=W^{(i)}$ for all $i\geq 1$ and for all $i\leq -2$. Then $\widetilde{X}^{(i)}:=\widetilde{W}^{(i)}+\widetilde{W}^{(-i)}$, $i\geq 1$, is such that $\widetilde{X}^{(1)}=W^{(1)}$ and $\widetilde{X}^{(i)}=X^{(i)}$ for all $i\geq 2$, and all further statements remain valid with all $W^{(\cdot)}$ and $X^{(\cdot)}$ replaced by $\widetilde{W}^{(\cdot)}$ and $\widetilde{X}^{(\cdot)}$. The proofs are also still valid subject to slightly adjusting some $\mathtt{WF}$-parameters and domination bounds, also since θ is now negative. We leave the details to the reader.
- The remainder of the proof in Section 3.3 holds verbatim.

4. Continuity in the initial condition via interval partitions. In this section, we extend Theorem 1.2 to the setting of interval partition evolutions of [17, 19] and complete the proof of Theorem 1.2. The role of the $\mathtt{PDRM}(\alpha,\theta)$ stationary distributions is played by a natural two-parameter family of regenerative partitions of the unit interval $[0,1]$ that we call the Poisson–Dirichlet interval partitions, $\mathtt{PDIP}\,(\alpha,\theta)$; see Pitman and Winkel [32] for more details when $\theta\geq 0$. For $\theta\in(-\alpha,0)$, we refer to [43] for a three-parameter family $\mathtt{PDIP}^{(\alpha)}(\theta_1,\theta_2)$ with $\theta_1,\theta_2\geq 0$ and $\theta:=\theta_1+\theta_2-\alpha\geq -\alpha$. For example, in order to visualize $\mathtt{PDIP}\left(\frac{1}{2},\frac{1}{2}\right)$ consider a Brownian bridge during time $[0,1]$ and consider the intervals formed by the complement of the zero-set. This is distributed according to $\mathtt{PDIP}\left(\frac{1}{2},\frac{1}{2}\right)$; see [20, Example 3]. A similar construction for Brownian motion during time $[0,1]$ gives us $\mathtt{PDIP}\left(\frac{1}{2},0\right)$; see [20, Example 4]. The sequence of decreasing block masses is Poisson–Dirichlet distributed. Replacing Brownian motion by recurrent Bessel (or $\mathtt{BESQ}$) processes and their bridges similarly yields $\mathtt{PDIP}\,(\alpha,0)$ and $\mathtt{PDIP}\,(\alpha,\alpha)$ for all $\alpha\in(0,1)$.

An *interval partition* is a countable set $\beta=\{J_i, i\in I\}$ of disjoint open subintervals J_i of some $[0,M]$ such that the *complement* $C(\beta):=[0,M]\setminus\bigcup_{i\in I}J_i$ is Lebesgue-null. We write $\|\beta\|$ to denote the *total mass* M. We use notation $\mathcal{I}_H$ for the set of interval partitions and equip it with the metric $d_H(\beta_1,\beta_2)$ that applies the Hausdorff metric to $C(\beta_1),C(\beta_2)\subset[0,\infty)$.

We write $\beta_1\star\beta_2$ for the *concatenation* of $\beta_1,\beta_2\in\mathcal{I}_H$ that consists of all intervals of $(c,d)\in\beta_1$, and shifted versions $(\|\beta_1\|+c,\|\beta_1\|+d)$ of all intervals $(c,d)\in\beta_2$, with similar notation $\bigstar_{a\in A}\beta_a$ for the concatenation of a countable family of $\beta_a\in\mathcal{I}_H$ indexed by a totally ordered set $(A,\prec)$, with $\sum_{a\in A}\|\beta_a\|<\infty$. We write $g\beta:=\{(gc,gd)\colon (c,d)\in\beta\}$ for the interval partition that has all lengths scaled by $g>0$. The empty interval partition is denoted by $\emptyset$. In analogy to (1.4), we define here on $\mathcal{I}_H$ distributions

$$\widetilde{Q}_{b,r}^{(\alpha)}:=e^{-br}\delta_\emptyset+(1-e^{-br})\mathbf{P}\Big\{\big\{\big(0,L_{b,r}^{(\alpha)}\big)\big\}\star G\overline{\beta}\in\cdot\Big\},$$

where $G\sim\mathtt{Gamma}(\alpha,r)$, $\overline{\beta}\sim\mathtt{PDIP}(\alpha,\alpha)$ and $L_{b,r}^{(\alpha)}$ as in (1.3) are independent. In this framework, we can give the following analog of Definition 1.1.

DEFINITION 4.1 (Transition kernel $\widetilde{K}_s^{\alpha,\theta}$). Let $\alpha\in(0,1)$, $\theta\geq 0$. For any interval partition $\beta\in\mathcal{I}_H$ and any time $s>0$, we consider the interval partition $G_0\overline{\beta}_0\star\bigstar_{J\in\beta}\Pi_J$ for independent $G_0 \sim \texttt{Gamma}(\theta, 1/2s)$, $\overline{\beta}_0 \sim \texttt{PDIP}(\alpha,\theta)$ and $\Pi_J \sim \widetilde{Q}^{(\alpha)}_{\texttt{Leb}(J),1/2s}$, $J \in \beta$. We denote its distribution by $\widetilde{K}^{\alpha,\theta}_s(\beta, \cdot)$.

Compared with (1.4) and Definition 1.1, atom sizes such as $L^{(\alpha)}_{b,r}$ are now interval lengths (we refer to both as *masses*), and rather than atom locations in $[0,1]$ that were partly preserved ("survival") partly sampled from $\texttt{Unif}[0,1]$ in (1.4), we now record under $\widetilde{Q}^{(\alpha)}_{b,r}$ a left-to-right total order of intervals that places all "descendants" to the right of a left-most interval $\big(0, L^{(\alpha)}_{b,r}\big)$, and this order is further preserved under $\widetilde{K}^{\alpha,\theta}_s(\beta, \cdot)$ in that descendants of different ancestors inherit the order of their ancestors.

The family $\big(\widetilde{K}^{\alpha,\theta}_s, s\geq 0\big)$ is the transition semigroup of an $\mathcal{I}_H$-valued diffusion $(\beta_s, s\geq 0)$ that we call $\texttt{SSIPE}(\alpha,\theta)$. This was further extended in [42, Definition 1.3] to a three-parameter family $\texttt{SSIPE}^{(\alpha)}(\theta_1,\theta_2)$, $\theta_1,\theta_2\geq 0$, so that $\theta:=\theta_1+\theta_2-\alpha\geq-\alpha$. The time-change

$$\rho(t) = \inf\left\{ s \geq 0\colon \int_0^s \frac{dv}{\|\beta_v\|} > t\right\}, \tag{4.1}$$

and normalisation to unit mass yield $\gamma_t := \big\|\beta_{\rho(t)}\big\|^{-1}\beta_{\rho(t)}$, $t\geq 0$. If $\beta_0=\gamma\in\mathcal{I}_H$ has total mass $\|\gamma\|=1$, we write $(\gamma_t, t\geq 0)\sim\texttt{PDIPE}_\gamma(\alpha,\theta)$, respectively $\texttt{PDIPE}^{(\alpha)}_\gamma(\theta_1,\theta_2)$. We showed in [19, Theorems 1.3 and 1.6], [42, Theorem 1.4] and [43, Theorem 1.4], that all of these evolutions are interval partition diffusions, and that $\texttt{PDIPE}(\alpha,\theta)$ and $\texttt{PDIPE}^{(\alpha)}(\theta_1,\theta_2)$ have $\texttt{PDIP}(\alpha,\theta)$, respectively $\texttt{PDIP}^{(\alpha)}(\theta_1,\theta_2)$, as their stationary distribution.

THEOREM 4.2. *Let* $\alpha \in (0,1)$ *and* $\theta \geq 0$. *For* $(\gamma_t, t \geq 0) \sim \texttt{PDIPE}_\gamma(\alpha,\theta)$ *we have* $\big(\texttt{RANKED}(\gamma_{t/2}), t \geq 0\big) \sim \texttt{EKP}_{\texttt{RANKED}(\gamma)}(\alpha,\theta)$. *Similarly, for* $\theta_1,\theta_2 \geq 0$, $\theta := \theta_1+\theta_2-\alpha > -\alpha$ *and* $(\gamma_t, t\geq 0) \sim \texttt{PDIPE}^{(\alpha)}_\gamma(\theta_1,\theta_2)$, *we have* $\big(\texttt{RANKED}(\gamma_{t/2}), t \geq 0\big) \sim \texttt{EKP}_{\texttt{RANKED}(\gamma)}(\alpha,\theta)$.

PROOF OF THEOREM 4.2 AND STEP 2 OF THE PROOF OF THEOREM 1.2. Let $\gamma \in \mathcal{I}_H$ with $\|\gamma\| = 1$ and $\pi \in \mathcal{M}^a_1$ such that $\texttt{RANKED}(\pi) = \texttt{RANKED}(\gamma)$. Given that both $\texttt{PDRM}(\alpha,\theta)$ and $\texttt{PDIP}(\alpha,\theta)$ have $\texttt{PD}(\alpha,\theta)$ ranked masses, the semigroups $K^{\alpha,\theta}_s(\pi,\cdot)$ and $\widetilde{K}^{\alpha,\theta}_s(\gamma,\cdot)$, the processes $\texttt{SSSP}_\pi(\alpha,\theta)$ and $\texttt{SSIPE}_\gamma(\alpha,\theta)$, and the processes $\texttt{FV}_\pi(\alpha,\theta)$ and $\texttt{PDIPE}_\gamma(\alpha,\theta)$ can be perfectly coupled so that the latter two have identical ranked mass processes. We remark for readers who have seen scaffolding and spindles, that this is a consequence of the clade constructions in [18, Section 1.2] and [19, Section 2.3 and Proposition 3.4] of both processes, when $\theta \geq 0$. These references also discuss the connection between the measure-valued and interval-partition-valued settings. For $\theta_1,\theta_2 \geq 0$, Definition 4.1 of [43] of $\texttt{SSIP}^{(\alpha)}(\theta_1,\theta_2)$ similarly compares with Definition 3.11 of an $\texttt{SSSP}(\alpha,\theta)$ with associated parameter $\theta = \theta_1+\theta_2-\alpha$ to similarly couple these processes. Hence, the claims in Theorem 1.2 are equivalent to the claims in Theorem 4.2.

Let $\theta \geq 0$. We showed in [19, Theorem 1.8] that $\texttt{PDIPE}(\alpha,\theta)$ has a Hausdorff-continuous extension to a state space of generalised interval partitions of $[0,1]$ in which the requirement $\texttt{Leb}([0,1] \setminus \bigcup_{i\in I} J_i) = 0$ is dropped. We refer to $[0,1] \setminus \bigcup_{i\in I} J_i$ as "dust" and show that in the generalised setting, initial dust is not negligible, but starting $\texttt{PDIPE}(\alpha,\theta)$ from a state where dust has positive Lebesgue measure, the evolution immediately enters $\mathcal{I}_H$ and never leaves. Indeed, we showed in [19, Corollary 4.15] that mapping the generalised $\texttt{PDIPE}(\alpha,\theta)$ under $\texttt{RANKED}$ yields a $\overline{\nabla}_\infty$-valued Feller process. In particular, the projected ∇_∞-valued $\texttt{PDIPE}(\alpha,\theta)$ itself, $\mathbf{V}_t = \texttt{RANKED}(\gamma_t)$, $t \geq 0$, is continuous in the initial state for initial states in ∇_∞. For $\texttt{PDIPE}^{(\alpha)}(\theta_1,\theta_2)$, the corresponding continuity in the initial state was obtained in [43, Theorem 1.4].

Now recall that Step 1 of the proof of Theorem 1.2 yields the identification of $\mathbf{L}^2[\alpha,\theta]$-semigroups, $(T_t, t \geq 0)$ and $(\widetilde{T}_{2t}, t \geq 0)$. Let $(\widetilde{\mathbf{V}}_t, t \geq 0)$ be the diffusion associated with

$(\widetilde{T}_t,\ t\geq 0)$ constructed in [29] (which is the Feller version of the diffusion constructed in [13]). Let $\widetilde{\mathrm{P}}_{\mathbf{x}}$ denote the law of $\widetilde{\mathbf{V}}$, when starting from $\mathbf{x}$. Then we find that for every $f\in\mathbf{L}^2\left[\alpha,\theta\right]$ we have

$$\mathrm{E}_{\mathbf{x}}\left[f(\mathbf{V}_t)\right]=T_tf(\mathbf{x})=\widetilde{T}_{2t}f(\mathbf{x})=\widetilde{\mathrm{E}}_{\mathbf{x}}[f(\widetilde{\mathbf{V}}_{2t})]\qquad\text{for }\mathtt{PD}\left(\alpha,\theta\right)\text{-a.e. }\mathbf{x}\in\nabla_\infty.$$

Now consider $f\colon\nabla_\infty\to[0,\infty)$ bounded and continuous. Then $\mathbf{x}\mapsto\mathrm{E}_{\mathbf{x}}[f(\mathbf{V}_t)]$ is continuous, and $\mathbf{x}\mapsto\widetilde{\mathrm{E}}_{\mathbf{x}}[f(\widetilde{\mathbf{V}}_{2t})]$ is continuous by [29, Proposition 4.3]. As any set of full $\mathtt{PD}\left(\alpha,\theta\right)$-measure is dense in ∇_∞, we get $\mathrm{E}_{\mathbf{x}}[f(\mathbf{V}_t)]=\widetilde{\mathrm{E}}_{\mathbf{x}}[f(\widetilde{\mathbf{V}}_{2t})]$ for every bounded, continuous f and every $\mathbf{x}\in\nabla_\infty$. Together with path-continuity and the Markov property, this identifies the laws of the processes $(\mathbf{V}_t,\ t\geq 0)$ and $(\widetilde{\mathbf{V}}_{2t},\ t\geq 0)$. □

Since we showed in [19, Corollary 4.15] that for $\theta\geq 0$, mapping the generalised $\mathtt{PDIPE}(\alpha,\theta)$ under $\mathtt{RANKED}$ yields a $\overline{\nabla}_\infty$-valued Feller process, Theorem 4.2 also identifies this Feller process with Petrov's Feller version of $\mathtt{EKP}(\alpha,\theta)$. Returning to the question of whether this allows one to avoid the $\mathbf{L}^2$-theory used in Section 3.3, the answer is yes, but at a cost, as this would require further estimates of the type established in Lemmas 3.6 and 3.8 to handle the (mostly negligible) contribution of dust to the pre-generator on $\mathcal{F}$. We omit the details.

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