Polytopes, Scattering Amplitudes, and Cohomology



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Abstract

We introduce a new definition of abstract polytope in any dimension and answer the question of defining and calculating its area. The entire discussion is motivated by studying the expressions arising from super-BCFW recursion on tree-level, n-point NMHV amplitudes in the planar limit of $N = 4$ Super Yang-Mills (SYM) theory. It has recently been discovered that such expressions should be interpreted as volumes of complex projective polytopes in some sense, and here we give this interpretation explicit meaning. In doing so, we uncover a novel calculus for calculating the volumes of such polytopes. This calculus can be used on the polytopes corresponding to these BCFW representations of amplitudes to obtain new expressions that make manifest all of the known properties of the polytopes. Additionally, we show how this new calculus can be used to readily obtain the volumes of any lower-dimensional face of an arbitrary polytope.

We begin by reviewing the known results for n-point NMHV amplitudes in planar $N = 4$ SYM. In Chapter 2 we introduce our new definition of abstract (oriented) polytopes in terms of new objects called cyclic lists, and in Chapter 3 we define the volumes of these polytopes using cyclic lists in a natural way. Chapter 4 introduces a useful calculus for calculating not only the volumes of these polytopes, but also the volumes of any of their lower dimensional bounding polytopes. Finally, in Chapter 5, we briefly explore the twistor and cohomology theory that motivates the calculus and which may provide important insight for generalization of this whole discussion to more exotic spaces.

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Chapter 1

Introduction

In recent years there has been much progress in finding new, simple expressions for scattering amplitudes in various quantum field theories. Quite generally, this progress is both due to various discoveries of more natural variables to use when expressing amplitudes as well as more natural ways of expressing amplitudes as a perturbative expansion than that which Feynman promoted. In many cases, the newly found simplicity of the expressions comes with an increased simplicity in the interpretation of these expressions.

One important example of this process, and that which we will be solely focused on here, is the expression of NMHV tree amplitudes in planar $N = 4$ Super Yang-Mills (SYM). It has been found [1] that one BCFW representation of the 6-point NMHV tree amplitude M_6^{NMHV} is, up to an overall MHV factor, given by

$$M_6^{NMHV} = [12345] + [12356] + [13456] \tag{1.1}$$

where

$$[abcde] = \frac{\delta^4(\eta_a\langle bcde\rangle + ... + \eta_e\langle abcd\rangle)}{\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle\langle abcd\rangle} \tag{1.2}$$

and the remaining terms in the δ-function are obtained from cyclic permutation of a, b, c, d, e. The brackets $\langle abcd\rangle$ are defined as

$$\langle abcd\rangle \equiv \epsilon_{\alpha\beta\gamma\delta} Z_a^\alpha Z_b^\beta Z_c^\gamma Z_d^\delta \tag{1.3}$$

and the Z_a^α are the bosonic homogeneous coordinates of the supertwistor $\mathcal{Z}_a^{\mathcal{I}} = (Z_a^\alpha; \eta_a^\beta) \in \mathbb{CP}^{3|4}$ encoding the external data of the particles labelled by $a \in 1, 2, ..., 6$.

Expression (1.1) is the result of performing a super-BCFW shift on a particular pair of external legs. By performing such a shift on a different pair, one finds, for example, the result

$$M_6^{NMHV} = [23456] + [23461] + [24561]. \tag{1.4}$$

Since both (1.1) and (1.4) are found by doing super-BCFW recursion on the same amplitude, they must be equal. However seeing the equality of (1.1) and (1.4) independent of the knowledge of their "BCFW-equivalence" is by no means obvious. Namely, if handed the right hand sides of these two expressions, it would in general be difficult to see if they are equivalent. Moreover, as the number n of external points in the amplitude increases, expressions similar to (1.1) and (1.4) but with many more terms arise as the BCFW representation of M_n^{NMHV}, and seeing the equivalence between any two distinct BCFW representations becomes a formidable task. It thus became an important problem to understand more deeply and/or intuitively from where these seemingly non-obvious equivalences amongst different BCFW represenations were arising.

It turns out rather remarkably that the correct interpretation of expressions like (1.1) and (1.4) is to view them as "triangulations" of a certain space. Namely, as we will see later, the additivite properties of the objects $[abcde]$ allows us to view them as "volumes" of "4-simplices". With this view in mind, an expression like (1.1) is nothing but the "volume" of a "polytope" expressed as the sum of volumes of simplices[1]. Namely, we triangulate our polytope with three simplices and add up their volumes to get the volume of the full polytope. Then, the equivalence between (1.1) and (1.4) can be seen as nothing but choosing two different "triangulations" of the same polytope. It is then the polytope itself (or rather, its volume), that represents M_n^{NMHV} in a "BCFW-invariant" way, and any particular BCFW representation simply corresponds to a particular choice of triangulation.

As it stands, however, the interpretation of expressions like (1.1) as volumes of polytopes is loose at best. We tend to think of polytopes as living in some Euclidean space where the "inside" and "outside" of the polytope are well-defined—one is a compact space and the other is not. As we will see, the polytopes that we are interested in with regards to the above expressions for scattering amplitudes generally live in projective spaces, where there is no notion of "inside" or "outside". Additionally, the projective spaces we will be concerned with will be complex, thus changing the (real-)dimensionality of what we call edges, faces, and hyperfaces. Accordingly, we will only define our polytopes combinatorially, thus glossing over the topological details of our polytopes as these details are largely (but not entirely) unimportant in the formalism to be developed. Namely, our polytopes will more truly be high-dimensional contours with boundary of a particular homology class, as we will see in Chapter 5,

[1]Inverted commas are used to reflect the fact that words like "volume" and "polytope" are meant only as analogies. The rest of this work will be spent showing how to make these ideas more precise.

but we will first see that they can be given a very simple definition in terms of things which we call cyclic lists.

While some of these considerations will prove fruitful for our understanding of what is "really" going on and for motivating certain definitions, the topological nature of the polytopes we are concerned with for expressions like (1.1) is of little importance. In fact, since it is only the additive properties of the algebraic objects $[abcde]$ and their higher and lower dimensional analogues that motivated the original polyhedral interpretation in the first place, we seek to develop a more combinatorial and/or algebraic picture of these "polytopes" and their "volumes". In what follows we will be **defining** what we mean by a polytope and its volume in a novel way, using only vertices specified by the external data and new objects that we define and call "cyclic lists".

Before embarking, we need to establish the sufficiency of working with purely bosonic variables and thus making no further reference to supertwistors or anything of the like. Let us follow [1] and take N supertwistors $\{\mathcal{Z}_i^{\mathcal{I}} = (Z_i^\alpha; \eta_i^\beta) \in \mathbb{CP}^{3|4}\}_{1\leq i\leq N}$ as our external data, an auxiliary set of four Grassmann variables $\{\phi_\alpha\}$, and let us define the "extended twistors" $\{\hat{Z}_i^I = (Z_i^\alpha, \phi\cdot\eta_i) \in \mathbb{CP}^4\}$. We now notice that our extended twistors[2] have only bosonic homogenous coordinates. Moreover, by defining the reference extended twistor $\hat{Z}_0^I = (0,0,0,0,1)$, we can define the following object

$$\mathcal{V}[\hat{Z}_a,...,\hat{Z}_e] = \frac{1}{4!}\frac{\langle \hat{Z}_a\hat{Z}_b\hat{Z}_c\hat{Z}_d\hat{Z}_e\rangle^4}{\langle \hat{Z}_0\hat{Z}_a\hat{Z}_b\hat{Z}_c\hat{Z}_d\rangle\langle \hat{Z}_0\hat{Z}_b\hat{Z}_c\hat{Z}_d\hat{Z}_e\rangle...\langle \hat{Z}_0\hat{Z}_e\hat{Z}_a\hat{Z}_b\hat{Z}_c\rangle} \tag{1.5}$$

with the brackets $\langle\cdots\cdots\rangle$ being the natural contraction with the 5-index anti-symmetric tensor $\epsilon_{\alpha\beta\gamma\delta\lambda}$. One readily finds that

$$\mathcal{V}[\hat{Z}_a,...,\hat{Z}_e] = \frac{1}{4!}\frac{(\phi\cdot\eta_a\langle bcde\rangle + ... + \phi\cdot\eta_e\langle abcd\rangle)^4}{\langle abcd\rangle\langle bcde\rangle...\langle eabc\rangle}$$

so that we immediately have

$$[abcde] = \int d^4\phi\mathcal{V}[\hat{Z}_a,...,\hat{Z}_e]. \tag{1.6}$$

In what follows, it will be $\mathcal{V}[\hat{Z}_a,...,\hat{Z}_e]$ that we interpret as the volume of a 4-simplex in $\mathbb{CP}^4$. Accordingly, expression (1.5) shows us that by properly understanding what we mean by polytopes and their volumes in purely bosonic $\mathbb{CP}^4$ (and more generally in $\mathbb{CP}^n$), we will immediately be able to view expressions like (1.1) in the

[2]We only temporarily use the hats $\hat{}$ on our extended twistors. Later we will drop the hats and generalize the discussion from twistors to general elements of $\mathbb{CP}^n$.

same way after some fermionic integrations. This is because (1.6) makes it clear that the objects $[abcde]$ and $\mathcal{V}[...]$ have the same (linear) additive properties.

In this work we introduce a definition of polytopes in $\mathbb{CP}^n$ that is abstract, and fundamentally based on objects referred to as cyclic lists—we therefore refer to them, for lack of better terminology, as **abstract listed polytopes**. We define what we mean by the volume of such a polytope and we introduce a novel calculus for calculating these volumes. In doing so we provide a rigorous and self-contained account of what we really mean when we say that expressions like (1.1) and (1.4) are volumes of polytopes. In doing so, we will uncover a framework for calculations that is much richer than that needed for $n-$point NMHV amplitudes, which is combinatorial in nature, and which is motivated by some rather surprising cohomological calculations. Thus, such a large amount of focus on rigorously understanding such a small class of scattering amplitudes[3] is not for naught. In particular, the framework developed here not only produces new expressions for known results that make manifest more of the desired properties than any previous expressions, it is also the case that the mathematics developed here seems to be general enough to be applicable to other types of amplitudes in ways that are not yet known.

All of the work presented here is original by the author, except for those facts and expressions that are explicitly cited, as well as the content in sections 5.1 and 5.2. There is also a part of section 5.3 that is not due to the author, but are rather due to Dr. Andrew Hodges, and we alert the reader at the relevant times of the ideas and results that originally belong to him.

[3]Namely, NMHV tree amplitudes in planar $N = 4$ SYM.

Chapter 2

Abstract Listed Polytopes

2.1 Setup

In order to see precisely how we can interpret an expression like (1.5) (resp. its higher/lower dimensional analogues) as the volume of a 4-simplex (resp. the volume of higher/lower dimensional simplices), and to see precisely how sums of these objects correspond to volumes of more complicated polytopes, we first must make precise what we mean by a polytope in the first place. As mentioned above, the polytopes that we define will be abstract in nature, hence their rather clunky name "abstract listed polytopes".[1]

Our definition of polytopes will differ dramatically from the definitions of polytopes made in the standard literature, which often construct polytopes as intersections of half-planes in $\mathbb{R}^n$ or as convex hulls of a finite number of points [12]. The primary reason for not adopting these definitions is that the results we obtain hold for polytopes that need not be convex, nor even connected. Additionally, we want to view each "connected component"[2] of our polytopes as having orientations, where we allow for "going around" each connected component any finite number of times. The notions of disconnectedness and "many-fold orientations", which will become clearer below, are not contained in the current formalism of polytopes in a natural way, but are very natural in the formalism that we present.

Our formalism is more in line with Coxeter's (and others') definition of 3-polytopes as a set of connected 2-polytopes "glued together" in a particular way (and similarly defining d-polytopes as a collection of $(d-1)$-polytopes glued together) [13]. Again, however, our formalism necessarily differs from Coxeter's. First, as mentioned above,

[1] Suggestions are welcomed.

[2] Inverted commas are again used here since the notion of connectivity is not yet defined, but will become clearer later.

our formalism will allow us to "glue together" disconnected 2-polytopes to define arbitrarily complicated (and disconnected) 3-polytopes, if we so desire. Secondly, Coxeter and others sought to "exclude anomalies such as two pyramids with a common apex" [13] from their study of polytopes, whereas these are precisely the types of anomalies that our formalism will allow for. In fact, these "anomalies" will not only be allowed for, they will be treated on quite equal footing as those "less anomalous" polytopes, such as convex and/or regular polytopes. Therefore, in order to simultaneously and naturally allow for arbitrarily complicated (i.e., disconnected) polytopes of any dimension as well as the ability to "wrap around" each connected component of a polytope an arbitrary number of times, we introduce a notion of polytope that is very different from those in the literature.

We will define our polyhedra with respect to complex projective space $\mathbb{CP}^n$, though the entire discussion can be carried through directly by replacing $\mathbb{CP}^n$ with $\mathbb{RP}^n$ (or by viewing that latter as embedded in the former). Accordingly, we will review the basic facts about $\mathbb{CP}^n$ that we need, while simultaneously introducing the notation that we will use.

We consider two copies of $\mathbb{CP}^n$ and we view each as being dual to the other in the following sense. We do not label the $\mathbb{CP}^n$'s themselves, but rather let the upper/lower index structure of their respective elements determine if we are working in "the" $\mathbb{CP}^n$ or the dual. An element $Z^\alpha \in \mathbb{CP}^n$ (where α can be thought of as an abstract index as in [2]) is represented by $(n+1)$ homogeneous coordinates $Z^\alpha = (Z^0, ..., Z^n)$ (with $\alpha \in 0, ..., n$). An element in the dual $\mathbb{CP}^n$ is written $W_\alpha \in \mathbb{CP}^n$ and similarly has $(n+1)$ homogeneous coordinates with lower indices. We define $W \cdot Z \equiv W_\alpha Z^\alpha$ to be the usual linear pairing of W and Z's homogeneous coordinates. The expression $W \cdot Z = 0$ is projectively well-defined, and thus each $Z^\alpha \in \mathbb{CP}^n$ determines a linearly embedded $\mathbb{CP}^{n-1}$ in the dual space, by placing one linear, homogeneous constraint on the dual homogeneous coordinates:

$$H_Z \equiv \{W_\alpha \in \mathbb{CP}^n | W \cdot Z = 0\}, \tag{2.1}$$

where H_Z denotes the hyperplane in the dual $\mathbb{CP}^n$ defined by Z^α. Similarly, each element $W_\alpha \in \mathbb{CP}^n$ defines a linearly embedded $\mathbb{CP}^{n-1}$ in the dual space in the exact same way. Moreover, each linearly embedded $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ is obtained as the H_Z (or H_W) of some Z (or W). Thus, we can view either of the copies of $\mathbb{CP}^n$ as the space of linearly embedded $\mathbb{CP}^{n-1}$'s in the other. In general, we call elements of $\mathbb{CP}^n$ points, linearly embedded $\mathbb{CP}^1$'s lines, linearly embedded $\mathbb{CP}^2$'s planes, and linearly embedded $\mathbb{CP}^k$'s with $3 \leq k \leq n-1$ k-planes. We may at times refer to lines as

1-planes, etc., and we note that a 0-plane is just a point.

We now quote some useful facts about the incidence of points, lines, planes, and k-planes in $\mathbb{CP}^n$, making reference to [3].

Theorem 2.1. *i) A p-plane and a q-plane, in general position, meet in $\mathbb{CP}^n$ in a $(p+q-n)$-plane if $p+q \geq n$, and if $p+q < n$ then they do not meet (though under special circumstances—i.e. if the two planes are not in general position—they may still meet).*

ii) Given a k_1-plane, a k_2-plane, ..., and a k_r-plane (to make r planes of various dimensions) in general position, they will meet in a $(k_1+k_2+...+k_r-(r-1)n)$-plane if this sum is non-negative. If this sum is negative then, except for some particular special cases, these r planes will not have any point in common.

iii) Through any $k+1$ distinct points in $\mathbb{CP}^n$ there is a unique k-plane.

□

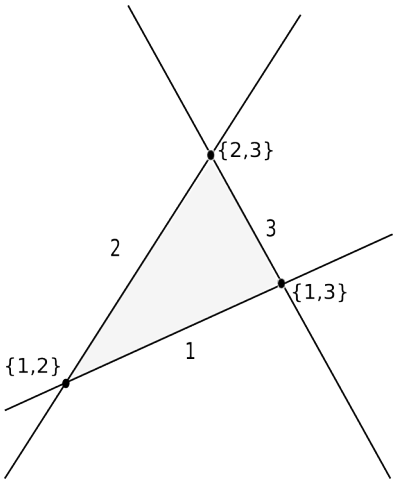
In this language (2.1) expresses the fact that each $Z^\alpha \in \mathbb{CP}^n$ determines a unique $(n-1)$-plane in the dual $\mathbb{CP}^n$. Therefore n distinct points $Z_1^\alpha, ..., Z_n^\alpha \in \mathbb{CP}^n$ determine a unique point in the dual $\mathbb{CP}^n$. This can be seen either by noticing that the collection of Z's determines n $(n-1)$-planes and using *ii)* in Theorem 2.1, or by using *iii)* of the same theorem to see that the collection of Z's determine a unique $(n-1)$-plane in $\mathbb{CP}^n$, each of which we know arises as the H_W for some unique W_α in the dual space. We are now ready to examine the simplest polytopes to visualize, which are 2-polytopes, or polygons.

2.2 2-Polytopes

Consider three distinct elements $Z_1^\alpha, Z_2^\alpha, Z_3^\alpha \in \mathbb{CP}^2$. These define three distinct 1-planes in the dual $\mathbb{CP}^2$, and any pair $\{i,j\}$, $i \neq j$, determines a unique point in the dual $\mathbb{CP}^2$ as the unique intersection point of H_{Z_i} and H_{Z_j}. We also note that we have $\{i,j\} = \{j,i\}$, as both intersections define the same point[3]. We represent this situation pictorially in Figure 2.1. In this figure we label the line defined by Z_i^α simply by i, and the unique intersection point of the lines defined by Z_i^α and Z_j^α simply by $\{i,j\}$.

[3]We will repeatedly use the analogous fact in higher dimensions without noting it explicitly.

Figure 2.1

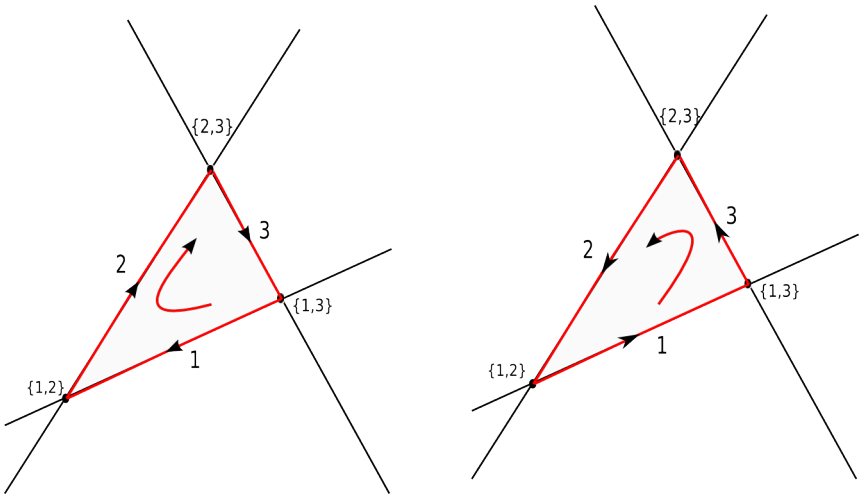


We note that this graphical representation of the situation has only combinatorial meaning, as opposed to topological meaning. Namely, the lines $1, 2, 3$ are not topologically (segments of) $\mathbb{R}$ as drawn, but are rather (portions of) linearly embedded $\mathbb{CP}^1$'s and therefore have the topology of S^2. It is only the intersection structure of the various lines that this figure correctly reflects, but fortunately this is the only information that we will need (as we will see later). Moreover, even if we restrict ourselves to working over $\mathbb{RP}^n$ where Figure 2.1 does reflect the correct local topology of the lines, there is still no notion of "inside" or "outside" any particular region, since $\mathbb{RP}^2$ is compact. Thus, even though Figure 2.1 invites us to view whatever object is defined by Z_1^α, Z_2^α, and Z_3^α as a "triangle", we must note the shortcomings of the drawing and be a bit more careful in defining what we mean. All is not lost, though, as the intersection structure (which is correctly reflected in the drawing) will allow us to define a meaningful notion of abstract 2-polytopes, which we now set out to do.

Throughout this discussion we keep an eye towards our future desire to be able to associate a "volume" to whatever we call a polytope, and thus we use the various desired properties of volumes to guide our definitions. In particular, the first property we want of our volumes will be a notion of orientation, so that each volume has two opposite orientations. This is easiest to visualize for 2-polytopes where the two different orientations correspond to "going around the polytope in opposite ways". Namely, however we define our polytopes, we want, in some sense, the two "triangles" in Figure 2.2 to be "opposite" each other.

This is enough to motivate our general definition. Namely, despite the topological difficulties of viewing these objects as triangles, there is one inherently combinatorial aspect of these objects that is reflected by our usual notion of polygon. That is,

Figure 2.2



we are still able to move from one vertex to the next along one of the bounding edges (whether the bounding edge is a line or a sphere makes little difference in this regard). Any connected component[4] of a polygon will then have a natural orientation defined by which way it is traversed. Thus, we look to **define** our 2-polytopes by this property—namely, the property of which (oriented) edges are connecting which vertices.

In short, we define our 2-polytopes via the "connecting the dot" information that specifies the combinatorics of edges connecting vertices. The "triangle" on the left of Figure 2.2 is specified by the following combinatorial data:

$$\{1,2\} \to \{2,3\} \to \{3,1\} \to \{1,2\} \tag{2.2}$$

where an expression like $\{i,j\} \to \{j,k\}$ means "start at vertex $\{i,j\}$ and go to vertex $\{j,k\}$ **along the line whose label is common to the vertices** (in this case, the line j)". What makes the set of instructions (2.2) into a 2-polytope is its closure: it ends where it begins, so that it "wraps all the way around". More abstractly, there is clearly some notion of "the boundary of the boundary" of the polytope vanishing. We can immediately generalize this idea into what we will call a 2-polytope.

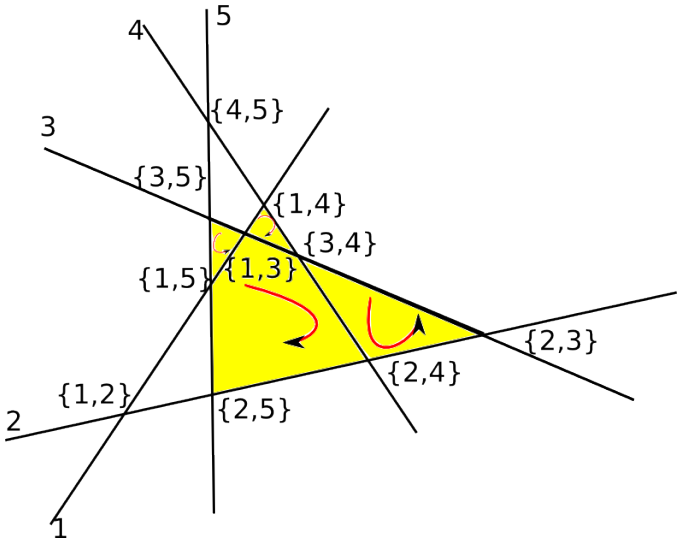
Let $\{Z_1^\alpha, ..., Z_N^\alpha\}$ be a set of N distinct elements in $\mathbb{CP}^2$, thus defining N distinct lines in the dual $\mathbb{CP}^2$. We refer to the set of Z's themselves, as well as the set of lines they define in the dual space, as well as the set of labels $\{1, ..., N\}$, interchangeably as a set of N boundaries. From here on we will usually abbreviate the above statement

[4]As we will see, our definition of a 2-polytope allows for pictures like Figure 2.1 which are disconnected, and which may have various connected components traversed more than once.

simply as "let $\{1, ..., N\}$ be a set of N boundaries". Any set of instructions $\{i_1, i_2\} \to \{i_2, i_3\} \to ... \to \{i_{n-1}, i_n\} \to \{i_n, i_1\} \to \{i_1, i_2\}$ then defines a 2-polytope. As an example, Figure 2.3 shows the orientation[5] of each closed region of a 2-polytope defined by 5 boundaries and the following instructions:

$$\begin{array}{ccccc} \{2,3\} & & \{1,4\} \to \{4,2\} & & \{1,3\} \to \{3,4\} \\ \downarrow & & \uparrow & \downarrow & \uparrow & \downarrow \\ \{3,5\} \to \{5,1\} & & \{2,5\} \to \{5,1\} & & \{4,2\} \to \{2,3\}. \end{array} \tag{2.3}$$

Figure 2.3



We now are ready to introduce the main object of interest: the cyclic list. In short, a cyclic list encodes the instructions like that in (2.2) and (2.3) in a much more succinct way. We have the following definition.

Definition 2.2. Let $S = \{1, ..., N\}$ be a set of N boundaries. A cyclic list l is a formal object of the form $l = (i_1, ..., i_n)$ where each $i_k \in S$, and it denotes the instructions

$$\{i_1, i_2\} \to \{i_2, i_3\} \to ... \to \{i_{n-1}, i_n\} \to \{i_n, i_1\} \to \{i_1, i_2\}, \tag{2.4}$$

where n is simply the length of the list.

Example 2.3. The cyclic list that defines the "triangle" on the left in Figure 2.2 is $l = (123)$ whereas the cyclic list that defines the "triangle" on the right in the same figure is $l = (132)$. The cyclic list that defines the "polygon" in Figure 2.3 is $l = (2351425134)$.

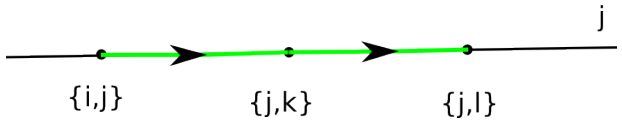
[5]If a region were to be traversed in some given direction more than once then we would endow that region with accordingly many arrows, though in Figure 2.3 there is no such region.

It is clear that many different cyclic lists can define the same instructions. In particular, any cyclic permutation of the elements in $l = (i_1, ..., i_n)$ will clearly give equivalent instructions, i.e., it will define the same polytope. However, since we have yet to define a polytope, we cannot say that two lists are equivalent if they define the same polytope. Instead, we need to define an equivalence relation amongst cyclic lists that reflects the notion of "equivalent polytopes" in a natural way. To do this, we need to establish some ideas and set up some notation that will both make the definition of an abstract listed 2-polytope clear as well as generalize to higher dimensions. In what follows we let $S = \{1, ..., N\}$ be a set of N boundaries.

Definition 2.4. An **oriented edge** is a formal object $[i(j)k]$ with $i, j, k \in S$, denoting the instructions $\{i, j\} \to \{j, k\}$. We place a formal additive structure on oriented edges as by defining $-[i(j)k] \equiv [k(j)i]$ and $[i(j)k] + [k(j)l] \equiv [i(j)l]$. An oriented edge of the form $[i(j)i]$ $(i \neq j)$ is then seen to be 0 in this additive structure, as desired. An oriented edge of the form $[i(j)j]$ $(i \neq j)$ denotes the formal instructions "$\{i, j\} \to$" while an oriented edge of the form $[j(j)i]$ $(i \neq j)$ denotes the formal instructions "$\to \{j, i\}$". Oriented edges of this form are called **oriented vertices** and we see from the additive structure that a sum of oriented vertices is an oriented edge: $[i(j)j] + [j(j)k] = [i(j)k]$, as desired.

We note that addition of oriented edges is only defined for edges that share their connecting edge (i.e., the j in $[i(j)k]$). This additive structure reflects the intuitive process of simply moving back and forth along the same line, and wanting to only keep track of the "net motion". This can be seen by expanding the formal objects $[i(j)k] + [k(j)l]$ as $\{i, j\} \to \{j, k\} + \{j, k\} \to \{j, l\}$, where all of the motion is along the line j. We clearly want this to reflect the net motion $\{i, j\} \to \{j, l\}$, which is precisely what the above additive structure formalises. This situation is depicted in Figure 2.4.

Figure 2.4

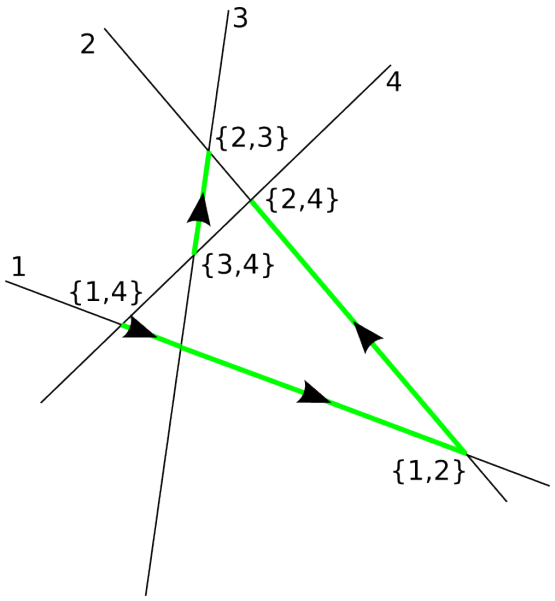


By allowing formal sums of oriented edges with different connecting edges, we can start to encapsulate what we mean by our abstract 2-polytopes.

Definition 2.5. An **edge set** E is any formal sum of oriented edges.

Note that not all edge sets give rise to what we would want to call a 2-polytope, as shown in Figure 2.5. It is clear that even though we have no notion of "inside" or "outside", we certainly do not want the edge set $E = [4(1)2] + [1(2)4] + [4(3)2]$ to be an example of what we call a polytope.

Figure 2.5



The defining characteristic of an edge set that gives rise to what we shall call a 2-polytope is the vanishing of the edge set's boundary. Namely, a shape is closed if and only if the boundary of its boundary vanishes. It is natural to want to view the edge set defining our 2-polytope as the boundary of our 2-polytope, and thus, for an edge set to define a closed polygon, the edge set's boundary must vanish. By defining the boundary $\partial[i(j)k]$ of the oriented edge $[i(j)k]$ to be $\{j,k\} - \{i,j\}$, by extending ∂ to formal sums in the obvious way, and by defining an additive structure on the vertices $\{i,j\}$ in the obvious way (namely, $\{i,j\} + (-\{i,j\}) = 0$), we can say that an edge set $E = \sum[i(j)k]$ defines a 2-polytope if and only of $\partial E = 0$. A cyclic list $l = (i_1, \cdots i_n)$ determines an edge set $E(l)$ in a canonical way, and the cyclicity of l ensures that $\partial E(l) = 0$.

Definition 2.6. Let $S = \{1, ..., N\}$ be a set of N boundaries and $l = (i_1, \cdots, i_n)$ be a cyclic list. The **edge set of** l, denoted by $E(l)$, is defined to be the formal sum

$$E(l) = \sum_{k=1}^{n} [i_{k-1}(i_k)i_{k+1}]$$

where $i_0 = i_n$ and $i_{n+1} = i_1$.

It is clear that $\partial E(l) = 0$ for any cyclic list l, and once we show that there is a natural way to add cyclic lists to form new cyclic lists in such a way as to reflect the additivity of 2-polytopes, we will see that the converse is true: any edge set E such that $\partial E = 0$ is of the form $E(l)$ for some cyclic list l. This result will motivate our definition of a 2-polytope purely in terms of its boundaries (which determine the vertices) and a cyclic list (which determines the "connecting the dot" data).

We are now in the position to state the equivalence relation amongst cyclic lists more rigorously.

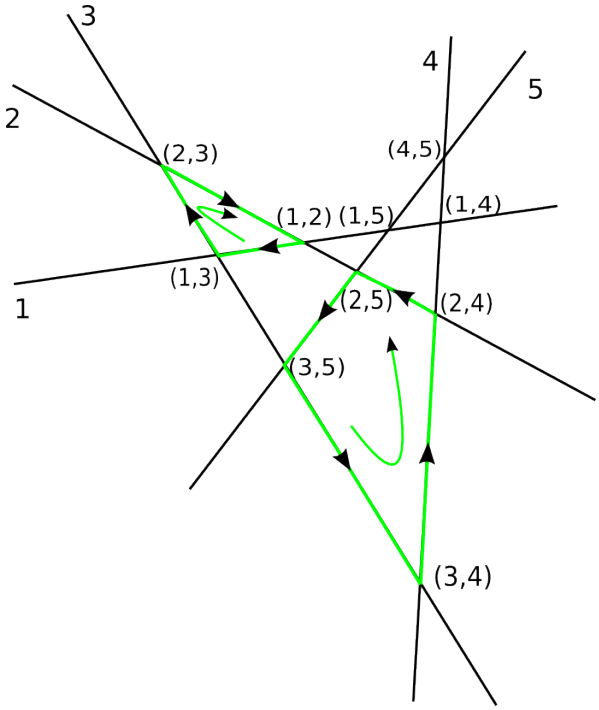
Definition 2.7. Let S and S' be two sets of boundaries, let l be a cyclic list taking values in S and let l' be a cyclic list taking values in S'. We say that $l \sim l'$ if $E(l) = E(l')$. $\sim$ is clearly an equivalence relation amongst lists.

To see that any edge set E such that $\partial E = 0$ can be written as $E(l)$ for some order cyclic list l, we first notice the following. Any edge set E such that $\partial E = 0$ will be able to be split up into a sum of a finite number of edge sets of the form $E(l)$. In other words, if $\partial E = 0$ for some edge set E, it is clear that there is a finite set of lists $\{l_i\}$ with $1 \leq i \leq M < \infty$ such that $E = \sum_{i=1}^{M} E(l_i)$. As an example, if we let E be the edge set of the polygon in Figure 2.6, then we clearly have $\partial E = 0$ and that $E = E(l_1) + E(l_2)$ where $l_1 = (1, 3, 2)$ and $l_2 = (3, 4, 2, 5)$. It is then easy to see that if we have an edge set E such that $\partial E = 0$, so that E eventually "closes up", there must be a finite set of lists with which we can decompose E.

Then all that remains to be shown is that for any two cyclic lists l_1 and l_2, there is a cyclic list l such that $E(l) = E(l_1) + E(l_2)$. Without loss of generality we can assume both l_1 and l_2 take values in the same set S of boundaries, by simply taking a union of their respective boundary sets and renumbering them accordingly.

Proposition 2.8. Let $S = \{1, ..., N\}$ be a set of N boundaries and let $l_1 = (i_1, \cdots, i_n)$ and $l_2 = (j_1, \cdots, j_m)$ be two cyclic lists with values in S. Then the cyclic list $l = (i_1, \cdots, i_n, i_1, j_1, \cdots, j_m, j_1)$ is such that $E(l) = E(l_1) + E(l_2)$.

Figure 2.6



Proof:

$$\begin{aligned}
E(l) =& [j_1(i_1)i_2] + \sum_{k=2}^{n-1}[i_{k-1}(i_k)i_{k+1}] + [i_{n-1}(i_n)i_1] + [i_n(i_1)j_1] \\
&+ [i_1(j_1)j_2] + \sum_{l=2}^{m-1}[j_{l-1}(j_l)j_{l+1}] + [j_{m-1}(j_m)j_1] + [j_m(j_1)i_1] \\
=& \sum_{k=2}^{n-1}[i_{k-1}(i_k)i_{k+1}] + [i_{n-1}(i_n)i_1] + [i_n(i_1)i_2] \\
&+ \sum_{l=2}^{m-1}[j_{l-1}(j_l)j_{l+1}] + [j_{m-1}(j_m)j_1] + [j_m(j_1)j_2] \\
=& E(l_1) + E(l_2),
\end{aligned}$$

where in the second equality we used the fact that $[i(j)k]+[k(j)l] = [i(j)l]$. Note that with the definitions in place for adding edges, the case where $i_1 = j_1$ is no different. □

Proposition 2.8 simply says that we can "bridge" two polygons defined by l_1 and l_2 by drawing first going around one list, walking along some connecting edge (the "bridge") to the second polygon, going around the second polygon, and walking in the opposite direction along the bridge to cancel it back out. One can readily check that

the list $l = (1, 3, 2, 1, 3, 4, 2, 5, 3)$, obtained by adding $l_1 = (1, 3, 2)$ and $l_2 = (3, 4, 2, 5)$ according to Proposition 2.8, corresponds to the polytope in Figure 2.6. In this case, the "bridge" that we introduce (and which is then cancelled out) is the edge [1(3)5], and had we used different representatives for l_1 or l_2, the "bridge" could just as well have been the edge [1(2)5].

Corollary 2.9. For any edge set E such that $\partial E = 0$, there exists a cyclic list l such that $E = E(l)$. $\square$

It is this corollary that makes our definition of a 2-polytope both reasonable, as well as completely general in the sense that we are not limiting ourselves to polygons that we would view as convex, or connected, or even "gone around" only once..

Definition 2.10. An **abstract listed 2-polytope** P is equivalent to the following data.

i) A set $S = \{1, ..., N\}$ of N boundaries in the above sense.

ii) A cyclic list $l = (i_1, ..., i_n)$.

We note that our ability to add lists together and deal with "disconnected" polygons using a single list—as our discussion of Figure 2.6 made clear—shows us that this definition of polygon is indeed extremely general.

Definition 2.11. We say that two abstract listed 2-polytopes P and P', defined respectively by the cyclic lists l and l' (possibly with two different sets of boundaries), are equivalent (or the same) if $l \sim l'$ as in Definition 2.7.

It should be noted that Definition 2.10 not only encodes the edge and vertex structure of our polytopes in a reasonable way, but also encodes the orientation (and if the polytope can be viewed as containing many "connected components", then the orientation of each component is encoded by the cyclic list as well). The real utility of this definition lies in its ability to generalize to higher dimensions, which we turn to in the next section, and in its ability to naturally encode what we will later define to be the volume of the 2-polytope P, which we turn to in the next chapter.

2.3 3-Polytopes

In the previous section we defined an abstract listed 2-polytope by a set of boundaries which, via the uniqueness of the point of intersection of any two such boundaries,

gives the set of possible vertices for our polytope. Thus, if there are N distinct boundaries, there are $\binom{N}{2}$ vertices "in play". There are infinitely many different 2-polytopes consisting of the same vertices, and we distinguish these by the equivalence class of the cyclic list which defines the "connecting the dot" instructions for how we connect the vertices.

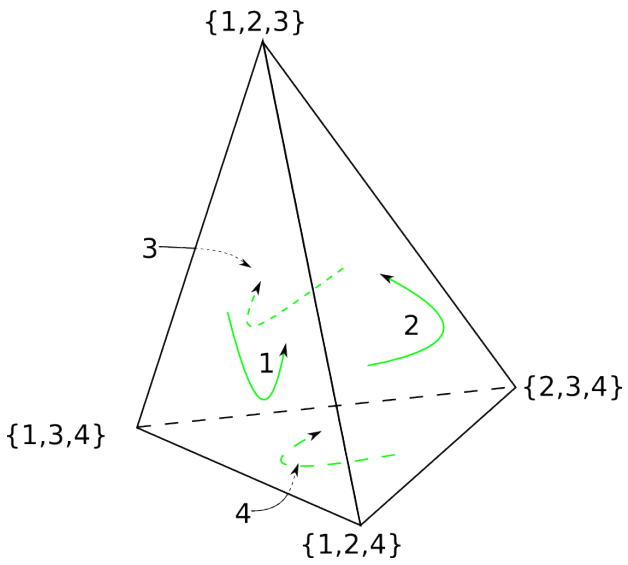
The definition of an abstract listed 2-polytope has a natural generalization to an n-polytope, but it is not an obvious one at this moment. In fact, it will be instructive to see both how to go from two to three dimensions, as well as from three to four, before defining a general abstract listed n-polytope. This is because the jump from two to three dimensions brings with it some very new concepts, as does that from three to four.

We aim to be able to define in a meaningful way a combinatorial 3-polytope purely in terms of its vertices and its "connecting the dot" information, as in the 2-dimensional case, because as we will see in the next chapter this description of a 2-polytope lends itself immediately to a simple algorithmic calculation of its volume. We can immediately see how to generalize the interaction between boundaries and vertices in 3-dimensions. We note that our boundaries now are linearly embedded $\mathbb{CP}^2$'s defined by N elements $Z_1^\alpha, ..., Z_N^\alpha$ in the dual $\mathbb{CP}^3$ in the usual way. Then, from Theorem 2.1, the intersection of any 3 distinct boundaries $i, j, k \in S = \{1, ..., N\}$ will uniquely define a point, which we call a vertex and denote by $\{i, j, k\}$.[6] The difficulty then lies in generalizing the "connecting the dots" information, since there is no canonical way in which a cyclic list could reflect what we want to view as a 3-polytope. In fact, as we will soon see, we can indeed define a meaningful notion of an abstract listed 3-polytope using cyclic lists, we just simply need to use more than one.

The key observation is that any notion of a 3-polytope can be described as a collection of 2-polytopes "glued together along their 1-boundaries" in such a way that the resulting object is viewed as a closed, three-dimensional volume. Thus, for any notion of a 3-polytope in $\mathbb{CP}^3$, it must be the case that there is a 2-polytope (possibly an empty one) sitting on each face, i.e., each 2-boundary. Thus, we expect that for each $i \in S$, there is a cyclic list $l_i = (j_{i1}, ..., j_{in_i})$, with each $j_{ik} \in S - \{i\}$, describing the 2-polytope on that plane. Here, n_i is just the length of the i^{th} list. Seeing as vertices are now labeled by three boundaries $\{i, j, k\}$, and seeing as each vertex in the

[6]We note that $\{i, j, k\}$ is independent of the order in which we write i, j, k, since the point of intersection of three planes is independent of the order in which we intersect them.

Figure 2.7



2-polytope lying on the i^{th} face will, by definition, be of the form $\{i, \cdot, \cdot\}$, the cyclic lists $\{l_i\}$ define the two-dimensional instructions

$$\{i, j_{i1}, j_{i2}\} \to \{i, j_{i2}, j_{i3}\} \to ... \to \{i, j_{in}, j_{i1}\} \to \{i, j_{i1}, j_{i2}\}, \tag{2.5}$$

where $\{i, j, k\} \to \{j, k, l\}$ means travel from the vertex $\{i, j, k\}$ to the vertex $\{j, k, l\}$ along the line defined by the intersection of two common labels (in this case, $j \cap k$).

In order to view the object defined by the N lists $\{l_i\}_{1 \leq i \leq N}$ as a closed 3-polytope, there must be some conditions on these lists that reflect the "gluing together" of these boundary 2-polytopes, so that the "boundary of the boundary" of our object vanishes as desired. For example, the object in Figure 2.7, which we clearly want to view as a tetrahedron, will be defined by the following cyclic lists (one for each of the $N = 4$ faces)

$$\begin{aligned} l_1 &= (2, 3, 4) \\ l_2 &= (1, 4, 3) \\ l_3 &= (1, 2, 4) \\ l_4 &= (1, 3, 2), \end{aligned} \tag{2.6}$$

where, for example, l_1 corresponds to the instructions

$$\{1, 2, 3\} \to \{1, 3, 4\} \to \{1, 4, 2\} \to \{1, 2, 3\}.$$

These lists are clearly not arbitrary, in the sense that certain lists "know about" certain other lists. Once we generalize the notion of oriented edges and vertices and

extend the additive structure of these objects to three dimensions, these conditions will become clear. In three dimensions, we have the following definition.

Definition 2.12. An **oriented edge** is a formal object $[i(jk)l]$ with $i, j, k, l \in S$, $j \neq k$, denoting the instructions $\{i, j, k\} \rightarrow \{j, k, l\}$ along the line defined by the intersection of the two boundaries common to both vertices (in this case, the line $j \cap k$). We place a formal additive structure on oriented edges by defining $-[i(jk)l] \equiv [l(jk)i]$ and $[i(jk)l]+[l(jk)m] \equiv [i(jk)m]$. An oriented edge of the form $[i(jk)i]$ (i, j, k pairwise distinct) is then seen to be 0 in this additive structure, as desired. An oriented edge of the form $[i(jk)j]$ (i, j, k pairwise distinct) denotes the formal instructions "$\{i, j, k\} \rightarrow$" while an oriented edge of the form $[j(j, k)i]$ (i, j, k pairwise distinct) denotes the formal instructions "$\rightarrow \{j, k, i\}$". Oriented edges of this form are called **oriented vertices** and we see from the additive structure that a sum of oriented vertices is an oriented edge: $[i(j, k)j] + [j(j, k)l] = [i(j, k)l]$, as desired. We note that since the intersection of two distinct boundaries j, k define a unique point regardless of the "order" in which we intersect them, we have $[i(jk)l] = [i(kj)l]$.

We note that for an oriented edge $[i(jk)l]$ we require $j \neq k$ because part of the (formal) instructions defined by this formal object is that we travel from $\{i, j, k\}$ to $\{j, k, l\}$ along the line defined by the intersection of two **distinct** boundaries j and k. A single boundary ($j = k$) does not define a line and so these instructions would be ill-defined in this case.

As before, an **edge set** is defined to be any formal sum of oriented edges. We now, however, have an edge set "derived from" each cyclic list.

Definition 2.13. Let $S = \{1, ..., N\}$ be a set of N boundaries and $\{l_i\}$ be a collection $\{l_i = (j_{i1}, ..., j_{in_i})\}$ of N cyclic lists. The **edge set derived from** l_i, $E_i \equiv E(l_i)$, is defined by the following formal sum:

$$E_i \equiv E(l_i) \equiv \sum_{k=1}^{n_i} [j_{i(k-1)}(ij_{ik})j_{i(k+1)}], \tag{2.7}$$

which is cyclic on k in that $j_{i0} = j_{in_i}$ and $j_{i(n_i+1)} = j_{i1}$.

Comparing this with Definition 2.6, we see that this is nothing but the set of oriented edges "lying on" the i^{th} face.

In the two-dimensional case the condition that we put on the edge sets to reflect the notion of "the boundary of the boundary vanishes" was that $\partial E = 0$. In the

three-dimensional case, however, the boundary of the boundary is the formal sum of all of the oriented edges (not their boundaries), so the necessary constraint will be on the edge sets themselves (as opposed to their boundaries). In other words, we want to constrain the edges sets $E(l_i)$ using

$$\sum_{i=1}^{N} E_i = 0. \tag{2.8}$$

One can readily check that the lists in (2.6) satisfy this condition.

There is a simpler constraint—or at least one that generalizes to all higher dimensions—which can be derived from (2.8), and which is stated in the following proposition. Before stating it, however, we need to introduce the last bit of notation that we will need.

Definition 2.14. The **edge set derived from l_i with respect to $s \in S$** $E_{i;s}$ is defined by the following formal sum

$$E_{i;s} = \sum_{k|j_{ik}=s} [j_{i(k-1)}(ij_{ik})j_{i(k+1)}] = \sum_{k|j_{ik}=s} [j_{i(k-1)}(is)j_{i(k+1)}]. \tag{2.9}$$

I.e., the sum is over only the j_{ik} such that $j_{ik} = s$.

The edge set $E_{i;k}$ is therefore nothing but the edges arising from l_i lying on the line $i \cap k$. We note that since i never makes an appearance in l_i, we have $E_{i;i} = 0$ identically. We can now state the result that motivates the definition of an abstract listed 3-polytope.

Proposition 2.15. Let $S = \{1, ..., N\}$ be a set of N boundaries and $\{l_i\}$ be a collection $\{l_i = (j_{i1}, ..., j_{in_i})\}$ of N cyclic lists. Then

$$\sum_{i=1}^{N} E_i = 0 \quad \Leftrightarrow \quad \forall i, k \in S, \qquad E_{i;k} = -E_{k;i} \tag{2.10}$$

Proof. Showing that the right hand side of (2.10) implies the left hand side is straightforward, so we must show the converse. Suppose $\sum_{i=1}^{N} E_i = 0$, and consider an edge $[j(is)k] \in E_i$ for some i. Since we can only add edges $[a(bc)d]$ and $[e(fg)h]$ when $(bc) = (fg)$, the edge that cancels $[j(is)k]$ in the sum must come either from $E_{i;s}$ or from $E_{s;i}$, so that $E_{i;s} + E_{s;i} = 0$. Symbolically we have

$$0 = \sum_{i=1}^{N} E_i = \sum_{i=1}^{N} \sum_{k \neq i} E_{i;k} = \frac{1}{2} \sum_{i=1}^{N} \sum_{i \neq k} (E_{i;k} + E_{k;i}) \tag{2.11}$$

and we see that each $E_{i;k} + E_{k;i}$ term must vanish individually since these provide the only edges along the line $i \cap k$. □

We now state the definition of an abstract listed 3-polytope.

Definition 2.16. An **abstract listed 3-polytope** P is equivalent to the following data.

i) A set $S = \{1, ..., N\}$ of N boundaries in the above sense.

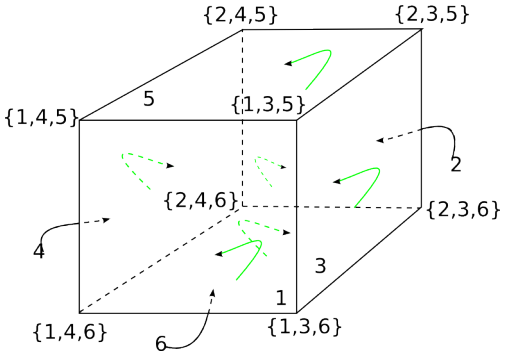
ii) A collection of N cyclic lists $\{l_i = (j_{i1}, ..., j_{in_i})\}$ such that for each $i, k \in S$, $E_{i;k} = -E_{k;i}$.

Definition 2.17. We say that two combinatorial 3-polytopes P and P', defined respectively by the boundary sets $S = \{1, ..., N\}$ and $S' = \{1, ..., N'\}$ and the lists $\{l_i\}_{1 \leq i \leq N}$ and $\{l'_i\}_{1 \leq i \leq N'}$, are equivalent (or the same) if $l_i \sim l'_i$ for all i. Namely, without loss of generality we can take $N' > N$, in which case we take l_i to be the empty list for any $i > N$, so that if $P \sim P'$ then it must be the case that l'_i is equivalent to the empty list for all $N' \geq i > N$.

We have seen how this definition applies to the "tetrahderon" of Figure 2.7, so let us look at two slightly more non-trivial examples that we will explore again in Chapter 4.

Example 2.18. We consider a "cube", depicted in Figure 2.8 and defined by six boundaries, so that $S = \{1, 2, 3, 4, 5, 6\}$. We read off from the picture that the follow-

Figure 2.8



ing six lists (one for each 2-face) are the lists that reflect the (oriented) combinatorial

properties of the cube:

$$\begin{aligned} l_1 &= (4,6,3,5) \\ l_2 &= (5,3,6,4) \\ l_3 &= (1,6,2,5) \\ l_4 &= (5,2,6,1) \\ l_5 &= (1,3,2,4) \\ l_6 &= (4,2,3,1). \end{aligned} \tag{2.12}$$

We note, though, that the figure should not be taken too literally (as a cube, that is), since it is only one possible way in which six planes in general position can intersect. The main point is that the lists in (2.12) do not care about the particular way in which these six boundaries intersect, and from these lists we can compute the corresponding volume regardless of the details of how the boundaries intersect (as we will see later).

For completeness, let us compute some edge sets derived from the lists in (2.12). By following through our definitions, we see that

$$E_1 = [4(16)3] + [6(13)5] + [3(15)4] + [5(14)6]. \tag{2.13}$$

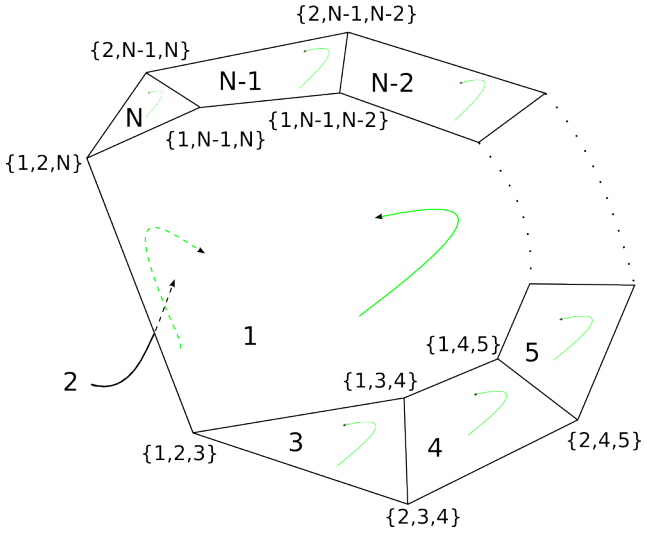
We then see that $E_{1;4} = [5(14)6]$, and we check that $E_{4;1} = [6(14)5] = -E_{1;4}$, as necessary.

Example 2.19. Consider the polytope shown in Figure 2.9, defined by $N \geq 5$ boundaries. Two boundaries 1 and 2 meet along the left-most line and are joined by a triangle at each end (defined respectively by the boundaries 3 and N) with $N-4$ quadrilaterals completing the prism. Again, this picture is only representative—from it we read off our cyclic lists, which then take on a life much more general than the shape in Figure 2.9. The N cyclic lists for this 3-polytope can be read off as:

$$\begin{aligned} l_1 &= (2,3,4,...,(N-1),N) \\ l_2 &= (N,(N-1),...,4,3,1) \\ l_3 &= (1,2,4) \\ l_s &= (1,s-1,2,s+1), \quad 4 \leq s \leq N-1 \\ l_N &= (1,N-1,2). \end{aligned} \tag{2.14}$$

In dimensions that are low enough for us to visualize, there is little need to explicitly check that $E_{i;k} = -E_{k;i}$ for all $i,k \in S$, as this constraint was motivated by the very pictures that we drew in the first place. Nonetheless, one can readily check that the lists in (2.14) do indeed satisfy this constraint.

Figure 2.9



We have thus successfully defined an abstract listed 3-polytope purely from its vertices and cyclic lists. As mentioned above, phrasing everything in terms of cyclic lists will be important for us when we introduce a new method for calculating what we define as the volume of such a polytope. As we will see, this method has several benefits not seen in other methods, as well as some surprising cohomological motivations that warrant us to believe that these methods can have even further generalizations. On this note, we move on to defining 4-polytopes, and by considering these we will see the pattern that allows us to define abstract listed n-polytopes for any $n \geq 2$.

2.4 4-Polytopes

In what follows we let $S = \{1, ..., N\}$ be a set of N boundaries, or elements $\{Z_i^\alpha\}$ in the dual $\mathbb{CP}^4$, each defining a 3-plane in the $\mathbb{CP}^4$ in which we view our polytope. In four dimensions a vertex is determined as the unique intersection of four distinct boundaries i, j, k, l and so the discussion of oriented edges carries directly over.

Definition 2.20. An **oriented edge** is a formal object $[i(jkl)m]$ with $i, j, k, l, m \in S$, j, k, l pairwise distinct, denoting the instructions $\{i, j, k, l\} \to \{j, k, l, m\}$ along the line defined by the intersection of the three boundaries common to both vertices (in this case, the line $j \cap k \cap l$). We place a formal additive structure on oriented edges by defining $-[i(jkl)m] \equiv [m(jkl)i]$ and $[i(jkl)m] + [m(jkl)n] \equiv [i(jkl)n]$. An oriented edge of the form $[i(jkl)i]$ (i, j, k, l pairwise distinct) is then seen to be 0 in this additive structure, as desired. An oriented edge of the form $[i(jkl)j]$ (i, j, k, l

pairwise distinct) denotes the formal instructions "$\{i,j,k,l\} \rightarrow$" while an oriented edge of the form $[j(j,k,l)i]$ (i,j,k,l pairwise distinct) denotes the formal instructions "$\rightarrow \{j,k,l,i\}$". Oriented edges of this form are called **oriented vertices** and we see from the additive structure that a sum of oriented vertices is an oriented edge: $[i(j,k,l)j]+[j(j,k,l)m] = [i(j,k,l)m]$, as desired. We note that since the intersection of three distinct boundaries j,k,l define a unique point regardless of the "order" in which we intersect them, we have $[i(jkl)m] = [i(\sigma(jkl))m]$ with σ being any permutation of 3 objects.

Definition 2.21. An **edge set** is any formal sum of oriented edges.

In order to define an abstract listed 4-polytope P in terms of cyclic lists, we observe that any meaningful definition of a 4-polytope will be one such that by restricting ourselves to the i^{th} boundary (with $i \in S$), we find ourselves with a 3-polytope P_i obtained as the restriction of P to its i^{th} face. Then, as before, by restricting P_i to its j^{th} boundary (with $j \neq i$), we should find ourselves with a 2-polytope P_{ij} defined by a cyclic list $l_{ij} = (k_{ij1},...,k_{ijn_{ij}})$ with each $k_{ijl} \in S - \{i,j\}$ and with n_{ij} being the length of the list l_{ij}. Thus, we suppose that a combinatorial 4-polytope is determined by N^2 cyclic lists $\{l_{ij} = (k_{ij1},...,k_{ijn_{ij}}\}$ with $i,j \in S = \{1,...,N\}$. We immediately observe that l_{ii} must be the empty list (since the intersection $i \cap i$ does not define a 2-face) so that we can consider $N(N-1)$ lists, however it will prove useful to leave the discussion more general in this regard.

Definition 2.22. Let $S = \{1,...,N\}$ be a set of N boundaries and $\{l_{ij}\}$ be a collection $\{l_{ij} = (k_{ij1},...,k_{ijn_{ij}})\}$ of N^2 cyclic lists. The **edge set derived from** l_{ij}, $E_{ij} \equiv E(l_{ij})$, is defined by the following formal sum:

$$E_{ij} \equiv E(l_{ij}) \equiv \sum_{l=1}^{n_{ij}} [k_{ij(l-1)}(ijk_{ijl})k_{ij(k+1)}], \tag{2.15}$$

where the sum on l is cyclic in the usual sense. The **edge set derived from** l_{ij} **with respect to** $s \in S$, $E_{ij;s}$, is defined by

$$E_{ij;s} \equiv \sum_{k_{ijl}|k_{ijl}=s}^{n_{ij}} [k_{ij(l-1)}(ijk_{ijl})k_{ij(k+1)}] = \sum_{k_{ijl}|k_{ijl}=s}^{n_{ij}} [k_{ij(l-1)}(ijs)k_{ij(k+1)}]. \tag{2.16}$$

We define an abstract listed 4-polytope P to have the properties that i) the restriction to any boundary i, by considering only the lists of the form l_{is} for some $s \in S$ produces a 3-polytope P_i, and that ii) the 3-polytopes P_i are "glued together" along

their two-dimensional faces in such a way to make the "boundary of the boundary" of our 4-polytope vanish. The first condition is easily satisfied once we recall our three-dimensional definition, for this condition states that for any $i \in S$, the lists $\{l_{ij}\}_{1\leq j\leq N}$ define a 3-polytope, which means that for each $j, k \in S$ we need

$$E_{ij;k} = -E_{ik;j}. \tag{2.17}$$

In order to satisfy the second condition we must think about our analogy to genuine 4-polytopes. By restricting P to the 3-polytopes P_i that bounds it on its i^{th} boundary, and then by restricting P_i to its j^{th} face, we would be "looking at" a 2-polytope P_{ij} where each "connected component" has a particular orientation (and possibly has been gone around more than once). However, had we restricted P first to its j^{th} boundary to get to the 3-polytope P_j, and then restricted this 3-polytope to its i^{th} face, we would be "looking at" a 2-polytope P_{ji}. In order for the boundary of the boundary of P to vanish, P_{ij} must be glued to P_{ji} in such a way that the sum of their oriented edges vanishes. In other words, it must be that P_{ij} is the "mirror image" of, or "reverse" of P_{ji} since in these two different orders of restriction, we would be "looking at" this 2-polytope from two different sides. This is reflected in the condition that for each $i, j \in S$,

$$E_{ij} = -E_{ji}, \tag{2.18}$$

since E_{ij} is the edge set that describes P_{ij}. Equations (2.17) and (2.18) can be nicely combined and motivate our definition of an abstract listed 4-polytope.

Definition 2.23. An **abstract listed 4-polytope** P is equivalent to the following data.

i) A set $S = \{1, ..., N\}$ of N boundaries in the above sense.

ii) A collection of N^2 cyclic lists $\{l_{ij} = (k_{ij1}, ..., k_{ijn_{ij}})\}$ such that for each $i, j, k \in S$, $E_{ij;k} = (-1)^{|\sigma|}E_{\sigma(ij;k)}$ where $\sigma \in S_3$ is any permutation of 3 objects.

Definition 2.24. We say that two abstract listed 4-polytopes P and P', defined respectively by the boundary sets $S = \{1, ..., N\}$ and $S' = \{1, ..., N'\}$ and the lists $\{l_{ij}\}_{1\leq i,j\leq N}$ and $\{l'_{ij}\}_{1\leq i,j\leq N'}$, are equivalent (or the same) if $l_{ij} \sim l'_{ij}$ for all i, j. Namely, without loss of generality we can take $N' > N$, in which case we take l_{ij} to be the empty list if either $i > N$ or $j > N$, so that if $P \sim P'$ then it must be the case that l'_{ij} is equivalent to the empty list for all $N' \geq i > N$ or $N' \geq j > N$.

We give an example of how to specify an abstract listed four-simplex in Appendix A.

2.5 $2 \leq d$-Polytopes

The above considerations have made the definition of an abstract listed d-polytope with $d \geq 2$, all but obvious. First, though, we need to set up the notation that generalizes all of the above considerations. Other than an explosion of indices, there is no new conceptual complexity.

Let d be the dimension of the polytope that we want to define and let $S = \{1, ..., N\}$ be a set of $N \geq d+1$ boundaries, i.e., a set $\{Z_1^\alpha, ..., Z_N^\alpha\} \subset \mathbb{CP}^d$. A vertex in d dimensions is specified uniquely by the intersection of d distinct boundaries and is denoted by $\{i_1, ..., i_d\}$ with $\{i_k \in S\}$ pairwise distinct. A line is specified uniquely by the intersection of $(d-1)$ boundaries, and so an oriented edge is denoted by $[j(i_1, ..., i_{d-1})k]$ with $i_1, ..., i_{(d-1)}, j, k \in S$ and with $\{i_1, ..., i_{(d-1)}\}$ pairwise distinct, and denotes the instruction $\{j, i_1, ..., i_{(d-1)}\} \to \{k, i_1, ..., i_{(d-1)}\}$ along the line $i_1 \cap ... \cap i_{(d-1)}$. We place the usual additive structure on formal sums of oriented edges and define oriented vertices in the usual way.

Our d-polytopes will then be specified by N^{d-2} cyclic lists

$$\{l_{i_1,...,i_{(d-2)}} = (j_{i_1,...,i_{(d-2)},1}, ..., j_{i_1,...,i_{(d-2)},n_{i_1,...,i_{(d-2)}}})\} \tag{2.19}$$

with $i_1, ..., i_{(d-2)} \in S$ and each $j_{i_1,...,i_{(d-2)},l} \in S - \{i_1, ..., i_{(d-2)}\}$. Here, $n_{i_1,...,i_{(d-2)}}$ is just the length of the list $l_{i_1,...,i_{(d-2)}}$. The edge set $E_{i_1,...,i_{(d-2)}}$ derived from the cyclic list $l_{i_1,...,i_{(d-2)}}$ is defined to be

$$E_{i_1,...,i_{(d-2)}} = \sum_{l=1}^{n_{i_1,...,i_{(d-2)}}} [j_{i_1,...,i_{(d-2)},(l-1)}(j_{i_1,...,i_{(d-2)},l}, i_1, ..., i_{(d-2)})j_{i_1,...,i_{(d-2)},(l+1)}], \tag{2.20}$$

where the sum is cyclic in the usual sense, and the edge set $E_{i_1,...,i_{(d-2)};s}$ derived from the cyclic list $l_{i_1,...,i_{(d-2)}}$ with respect to $s \in S$ is defined as

$$\begin{aligned} E_{i_1,...,i_{(d-2)};s} &= \sum_{l|j_{i_1,...,i_{(d-2)},l}=s}^{n_{i_1,...,i_{(d-2)}}} [j_{i_1,...,i_{(d-2)},(l-1)}(i_1, ..., i_{(d-2)}, j_{i_1,...,i_{(d-2)},l})j_{i_1,...,i_{(d-2)},(l+1)}] \\ &= \sum_{l|j_{i_1,...,i_{(d-2)},l}=s}^{n_{i_1,...,i_{(d-2)}}} [j_{i_1,...,i_{(d-2)},(l-1)}(i_1, ..., i_{(d-2)}, s)j_{i_1,...,i_{(d-2)},(l+1)}]. \end{aligned} \tag{2.21}$$

We can now easily generalize Definitions 2.10, 2.16, and 2.23 to a d-polytope for any $d \geq 2$.

Definition 2.25. An **abstract listed d-polytope** with $d \geq 2$ is equivalent to the following data:

i) A set $S = \{1, ..., N\}$ of N boundaries,

ii) A collection of N^{d-2} cyclic lists

$$\{l_{i_1,...,i_{(d-2)}} = (j_{i_1,...,i_{(d-2)},1}, ..., j_{i_1,...,i_{(d-2)},n_{i_1,...,i_{(d-2)}}})\}$$

as defined above, such that for any $i_1, ..., i_{d-1} \in S$,

$$E_{i_1,...,i_{(d-2)};i_{(d-1)}} = (-1)^{|\sigma|} E_{\sigma(i_1,...,i_{(d-2)};i_{(d-1)})}$$

where $\sigma \in S_{(d-1)}$ is any permutation of $(d-1)$ objects. This constraint is the natural generalization of the requirement that the "boundary of the boundary" vanish.

We note that this definition makes it clear that the restriction of a d-polytope to any number of boundaries (say, p distinct boundaries) gives a $(d-p)$-polytope. Namely, if we are given a d-polytope, then the $(d-1)$-polytope P_I obtained by restricting to the I^{th} face is indeed a polytope, since then we have

$$E_{I,...,i_{(d-3)};i_{(d-2)}} = (-1)^{|\sigma|} E_{I\sigma(i_1,...,i_{(d-3)};i_{(d-2)})}. \tag{2.22}$$

The analogous statement can be said after restricting to the boundaries $I_1, ..., I_p$ with $p \leq d-2$.

Now that we have a working definition of an abstract listed polytope in any dimension, we now turn our attention to defining and calculating what we want to interpret as their volumes. In this way we relate the work of this chapter to the initial motivations discussed in Chapter 1.

Chapter 3

Volumes of Abstract Listed Polytopes

3.1 A Fresh Look at Triangles

We begin our discussion of volumes (i.e., areas) of abstract listed polygons by considering a genuine triangle in $\mathbb{R}^2$. In Figure 3.1 is a triangle with vertices labelled by 1, 2, and 3. By placing the triangle in $\mathbb{R}^2$ we can assign these three vertices the coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3), respectively.

The "one-half base times height" formula for the area A of this triangle is then nicely expressed [4] as the determinant of the following 3×3 matrix:

$$A = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} . \tag{3.1}$$

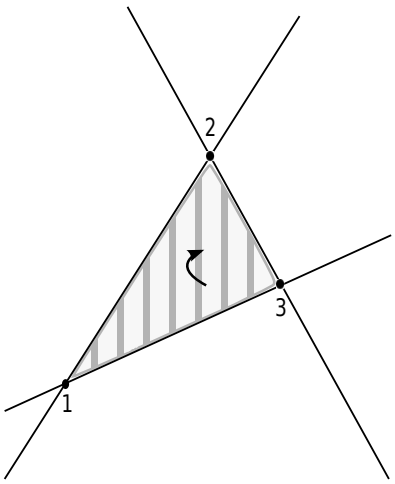
The above expression for the area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is the starting point for all that is to follow, as it makes manifest many aspects of the volume that we wish to preserve when defining the volume of an abstract listed polytope. Namely, we see that it encodes the two possible orientations of the volume via the minus sign that one picks up by swapping any two columns (corresponding to going around the triangle "the other way"). It also makes manifest the vanishing of the area when any two of the three vertices become linearly dependent, thus making the 3×3 matrix degenerate.

While (3.1) is a nice form for the area of a triangle, we can indeed do better[1]. Let us introduce the following 3-vectors:

$$W_{i\alpha} = \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} , \quad P_0^\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.2}$$

[1] At this point it is unclear why one would even desire to do better, but this will become more clear later.

Figure 3.1



with $\alpha = 1, 2, 3$ labeling the components of the vectors. Let us also denote by $W_i \cdot P_0$ the usual Euclidean dot product $W_{i\alpha} \cdot P_0^\alpha$ with summation on α. In our case, we rather trivially get $W_i \cdot P_0 = 1$ for each $i = 1, 2, 3$. Finally, let us denote[2] by $\langle 123 \rangle$ the contraction $\epsilon^{\alpha\beta\gamma} W_{1\alpha} W_{2\beta} W_{3\gamma}$, with $\epsilon^{\alpha\beta\gamma}$ the alternating symbol. We can now write

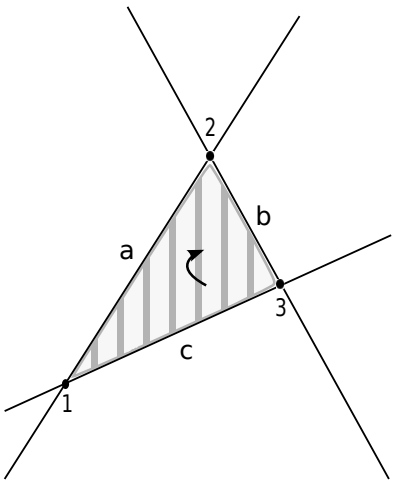
$$A = \frac{1}{2} \frac{\langle 123 \rangle}{(W_1 \cdot P_0)(W_2 \cdot P_0)(W_3 \cdot P_0)}. \tag{3.3}$$

Although (3.3) appears to be a highly redundant form for A, as each term in the denominator is identically 1, this expression is actually surprisingly useful. It clearly maintains the antisymmetry of orientation reversal as well as the vanishing of A when any two vertices become linearly dependent, and it also makes apparent an invariance under rescaling of the W's that was not apparent in (3.1).

Accordingly, we can interpret (3.3) as giving the area of a triangle in $\mathbb{RP}^2$ with vertices given by the homogeneous coordinates of the three W_i's, and with edges being the unique linear embeddings of $\mathbb{RP}^1$'s connecting the vertices. The standard formula (3.1) can then be recovered when the three vertices lie in the same canonical coordinate patch of $\mathbb{RP}^2$. In other words, the W_i's in (3.2) then lie in the portion of $\mathbb{RP}^2$ that can be covered by the homogeneous coordinates $(x, y, 1)$, which is linearly

[2]When there are more than 9 possible values for the subscript i of W we will write these brackets with commas separating the arguments so that, e.g. $\langle 1, 2, 14 \rangle$ can be distinguished from $\langle 12, 1, 4 \rangle$, but when there is no possibility of confusion we drop the commas.

Figure 3.2



isomorphic to $\mathbb{R}^2$. The area A is indeed not invariant under rescaling P_0, and it is the particular choice of P_0 in (3.2) that recovers the usual volume form. Accordingly, we should include a subscript of P_0 on the area A to remind us that the scaling of A is dependent on P_0, but since this overall scaling will not present any problems or confusions we choose to leave it implicit.

By allowing the W_i's and P_0 to have homogeneous coordinates in $\mathbb{CP}^2$ instead of $\mathbb{RP}^2$, all of the above expressions remain perfectly well-defined and we get an interpretation of the "complex area" of the triangle defined by the three W_i's. In what follows, we take all of our quantities to live in complex projective space, viewing real projective space as embedded within it (though we will have no reason to limit ourselves to real projective spaces again throughout).

It is important to note that P_0 lives in the $\mathbb{CP}^2$ that is dual to that in which the W_i's are defined. In fact, we can find an even more symmetric form for A by describing the W_i's by the dual information that uniquely defines them. As described in Chapter 2, each linearly embedded $\mathbb{CP}^1$ in $\mathbb{CP}^2$ is in one-to-one correspondence with the elements in a dual copy of $\mathbb{CP}^2$, via the relation

$$Z^\alpha \in \mathbb{CP}^2 \simeq \{W_\alpha \in \mathbb{CP}^2 | W \cdot Z = 0\}. \tag{3.4}$$

Now, if we label the lines (each of which denotes a linearly embedded $\mathbb{CP}^1$) connecting the vertices of the triangle in Figure 3.1 by a, b, and c, as in Figure 3.2, we

see that W_1 is the unique intersection point of the lines a and c, W_2 is the unique intersection point of the lines a and c, and W_3 is the unique intersection point of the lines b and c. If we denote the elements in the dual $\mathbb{CP}^2$ that define the lines a, b, and c respectively as Z_a^α, Z_b^α, and Z_c^α (with α labelling the homogeneous coordinates), then we can summarize the above as

$$W_1 \cdot Z_c = W_1 \cdot Z_a = 0 \tag{3.5}$$

$$W_2 \cdot Z_a = W_2 \cdot Z_b = 0 \tag{3.6}$$

$$W_3 \cdot Z_b = W_3 \cdot Z_c = 0. \tag{3.7}$$

These can be solved for the W_i's in terms of the Z's as

$$W_{1\alpha} = \epsilon_{\alpha\beta\gamma} Z_c^\beta Z_a^\gamma \tag{3.8}$$

$$W_{2\alpha} = \epsilon_{\alpha\beta\gamma} Z_a^\beta Z_b^\gamma \tag{3.9}$$

$$W_{3\alpha} = \epsilon_{\alpha\beta\gamma} Z_b^\beta Z_c^\gamma. \tag{3.10}$$

Plugging this into (3.3) and using the analogous notation $\langle abc\rangle \equiv \epsilon_{\alpha\beta\gamma} Z_a^\alpha Z_b^\beta Z_c^\gamma$ then leads to the following expression for A:

$$A = \frac{1}{2} \frac{\langle abc\rangle^2}{\langle abP_0\rangle\langle bcP_0\rangle\langle caP_0\rangle}. \tag{3.11}$$

As discussed in Chapter 2, the expression for A can now only loosely be interpreted as the area of a triangle, since there is very little about the region defined by the intersection of the boundaries a, b, and c in $\mathbb{CP}^2$ that resembles an area (in any topological sense). Therefore, we simply **define** (3.11) to be the area of the abstract listed 2-polytope defined by the corresponding cyclic list. Namely, by renaming our variables $1 \leftrightarrow a, 2 \leftrightarrow b, 3 \leftrightarrow c$, we see that the combinatorial polytope that we are describing here is that which is defined by the cyclic list $l = (1, 2, 3)$. We therefore make the following definition.

Definition 3.1. Let $\{Z_i^\alpha\}_{1\leq i\leq 3}$ be three distinct elements in $\mathbb{CP}^2$ defining three distinct boundaries $S = \{1, 2, 3\}$ in the dual $\mathbb{CP}^2$, and let $P_0^\alpha \notin \{Z_i^\alpha\}_{1\leq i\leq 3}$ be a fixed element in $\mathbb{CP}^2$ (the same copy of $\mathbb{CP}^2$ that contains the Z's). The **area** of the abstract listed 2-polytope defined by the cyclic list $l = (1, 2, 3)$ is defined to be

$$A \equiv A(l) \equiv A((1, 2, 3)) = \frac{1}{2} \frac{\langle 123\rangle^2}{\langle 12P_0\rangle\langle 23P_0\rangle\langle 31P_0\rangle} \equiv [123]. \tag{3.12}$$

We note that this definition satisfies the desired property that the area of the 2-polytope defined by the cyclic list $l = (1, 2, 3)$ is negative that of the 2-polytope defined by the cyclic list $l' = (1, 3, 2)$, as the latter involves going around the 2-polytope in the opposite direction. As we have seen, by choosing P_0 as in (3.2), by taking all of our coordinates to be real, and by assuming that the vertices defined by the intersections of pairs of lines lie in the same affine patch of $\mathbb{RP}^2$, we recover the **genuine** area of the triangle. Thus, Definition 3.1 is nothing but the generalization of genuine area to projective space.

As we will see, Definition 3.1 extends to arbitrary 2-polytopes in a natural way due to various non-trivial additive properties amongst the objects $[ijk]$ just defined. Seeing as the 2-polytope defined by the list $(1, 2, 3)$ can and should be viewed as a "triangle", we say that the expression $[ijk]$ (the area of the 2-polytope defined by the list (i, j, k)) is the area of the 2-simplex defined by the same boundaries. Our work in Chapter 2 has then given unambiguous meaning to this statement. After we extend Definition 3.1 to arbitrary 2-polytopes—i.e. arbitrary cyclic lists—we will work to extend it to arbitrary 3-polytopes, 4-polytopes, and d-polytopes in the subsequent sections.

3.2 Volumes of Abstract Listed 2-Polytopes

The volume of any general polygon can be expressed as a sum of volumes of oriented triangles, since any general polygon can be triangulated. There are, however, (infinitely) many different triangulations of the same polygon, and we will use one in particular to define what we mean by the area of a general abstract listed 2-polytope. The following additive property of the $[ijk]$ objects defined in (3.12) shows that "areas of triangles" add in the way that we would expect.

Claim 3.2. For any boundaries $1, 2, 3$, and 4, we have

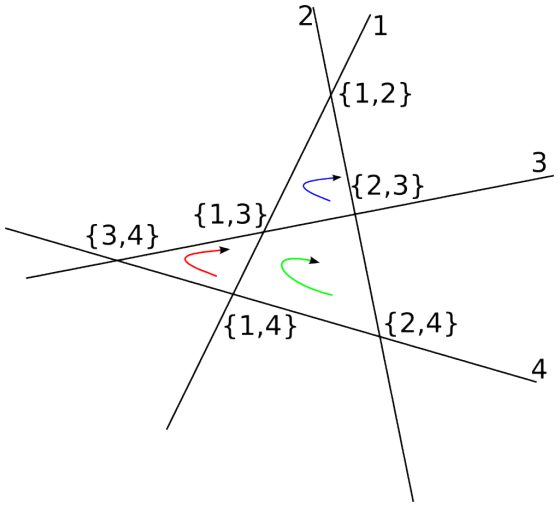
$$[123] = [124] + [234] + [314]. \tag{3.13}$$

Proof: We can prove this claim algebraically be tirelessly using the identity

$$\langle ABP\rangle\langle BCD\rangle - \langle BCP\rangle\langle ABD\rangle = \langle ABC\rangle\langle BPD\rangle \tag{3.14}$$

for any boundaries Z_A, Z_B, Z_C, Z_D, Z_P in the definition of $[123]$ in (3.12). This identity is a simple consequence of the fact that any totally anti-symmetric 4-index object vanishes in a 3-dimensional space (the space of homogeneous coordinates of the

Figure 3.3



boundaries). □

Instead of bothering ourselves with the algebraic proof, let us instead focus on the geometric interpretation of Claim 3.2. In Figure 3.3 we have drawn a general configuration of four boundaries.

It is reasonable to expect that the "area" of the region defined by the list $(1, 3, 2, 4)$ can be expressed as the difference between the area of the triangle defined by the list $(1, 2, 4)$ and that the triangle defined by the list $(1, 2, 3)$. Thus

$$A((1, 3, 2, 4)) = A((1, 2, 4)) - A((1, 2, 3)).$$

However, $A((1, 3, 2, 4))$ should also be expressed as the difference between $A((2, 4, 3))$ and $A((1, 4, 3))$, i.e.

$$A((1, 3, 2, 4)) = A((2, 4, 3)) - A((1, 4, 3)).$$

Thus

$$A((1, 2, 4)) - A((1, 2, 3)) = A((2, 4, 3) - A((1, 4, 3)) \tag{3.15}$$

$$\Rightarrow [124] - [123] = [243] - [143] \tag{3.16}$$

$$\Rightarrow [123] = [124] + [234] + [314], \tag{3.17}$$

(where we used the antisymmetry of $[ijk]$) as desired. We note that the expression $A((1, 3, 2, 4))$ is not yet defined, as we have yet to define the area of a polygon defined by a list with more than three entries. The preceding discussion is meant to be

of a motivational sort. Let us now find an expression for the area of an arbitrary combinatorial 2-polytope and thus make the expression $A((1,3,2,4))$ well-defined. First, we need the following definition of the sum of cyclic lists.

Definition 3.3. Let $l_1 = (i_1, ..., i_n)$ and $l_2 = (j_1, ..., j_m)$ be two cyclic lists. We define the sum $l_1 + l_2$ of these two lists as the list $l = l_1 + l_2 \equiv (i_1, ..., i_n, i_1, j_1, ..., j_m, j_i)$.

It is easy to see that the equivalence class of $l_1 + l_2$ depends only on the equivalence classes of l_1 and l_2. By Proposition 2.8 we then have that $E(l_1 + l_2) = E(l_1) + E(l_2)$, and thus the polygon defined by $l_1 + l_2$ is precisely the "superposition" of the polygons defined by l_1 and l_2. Accordingly, we want it to be the case that $A(l_1 + l_2) = A(l_1) + A(l_2)$, which again is an identity that depends only on the equivalence classes of the lists involved (clearly if $E(l) = E(l')$, then l and l' define the same polygon, so we want $A(l) = A(l')$). In fact, it is this constraint alone which will canonically define our area. Namely, by requiring the function $A(l)$ to behave as

$$A(l_1 + l_2) = A(l_1) + A(l_2), \tag{3.18}$$

we will be able to easily generalize (3.12) to any cyclic list. First, let us establish the following claim, which is a straightforward application of our definitions.

Claim 3.4. Let $l_1 = (i, k_1, ..., k_n, j)$ and $l_2 = (j, l_1, ...l_m, i)$. Then

$$l_1 + l_2 \sim (i, k_1, ..., k_n, j, l_1, ..., l_m).$$

□

We now have the machinery to prove the following.

Proposition 3.5. Let $l = (i_1, ..., i_n)$ be an cyclic list. Then

$$A(l) = \sum_{k=1}^{n} A((i_k, i_{k+1}, B)) = \sum_{k=1}^{n} [i_k i_{k+1} B] \tag{3.19}$$

for any reference boundary $B^\alpha \in \mathbb{CP}^2$, and is independent of the choice of B^α. The sum on k is cyclic in the sense that $i_{n+1} = i_1$.

Proof:

$$\begin{aligned}
\sum_{k=1}^{n} A((i_k, i_{k+1}, B)) &= A((i_1, i_2, B)) + A((i_2, i_3, B)) + \sum_{k=3}^{n} A((i_k, i_{k+1}, B)) \\
&= A((B, i_1, i_2)) + A((i_2, i_3, B)) + \sum_{k=3}^{n} A((i_k, i_{k+1}, B)) \\
&= A((B, i_1, i_2, i_3)) + \sum_{k=3}^{n} A((i_k, i_{k+1}, B)) \\
&= A((B, i_1, i_2, i_3)) + A((i_3, i_4, B)) + \sum_{k=4}^{n} A((i_k, i_{k+1}, B)) \\
&= A((B, i_1, i_2, i_3, i_4)) + \sum_{k=4}^{n} A((i_k, i_{k+1}, B)) \\
&\vdots \\
&= A((B, i_1, i_2, i_3, i_4, ..., i_n)) + A((i_n, i_1, B)) \\
&= A((B, i_1, i_2, i_3, i_4, ..., i_n, i_1)) \\
&= A((i_1, i_2, ..., i_n)) \\
&= A(l)
\end{aligned}$$

where in the third, fifth, and third-to-final equality we used Claim 3.4, and where we used the anti-symmetry of $A((i, j, k))$ in i, j, k. This proof also shows the manifest independence of the reference boundary B^α. $\square$

We will end up using Proposition 3.5 to motivate our definition of volumes of abstract listed polytopes in higher dimensions. Before doing so, however, we will take a look at a simple yet enlightening example, in which we will see the appearance of "spurious poles" [5] and describe some ways of dealing with them.

Example 3.6. Consider again the cyclic list $l = (1, 3, 2, 4))$. Proposition 3.5 says that the area of the polygon defined by this list is

$$A((1, 3, 2, 4)) = [13B] + [32B] + [24B] + [41B] \tag{3.20}$$

for some reference boundary B^α. However, since this result is independent of our choice of B^α, we can make the choice $B^\alpha = Z_2^\alpha$ to find (using the antisymmetry of $[\cdots]$)

$$A((1, 3, 2, 4)) = [124] - [123] \tag{3.21}$$

just as expected from the discussion of Figure 3.3. As discussed above, the list $(1, 3, 2, 4)$ can be drawn as in Figure 3.3 if we pick boundaries with the depicted intersection structure (otherwise we may not get a quadrilateral but rather two triangles connected at a vertex). Assuming we've chosen our boundaries as in Figure 3.3, we can also suppose we choose our boundaries 1 and 2 to be parallel in some affine patch. Working in this affine patch and choosing P_0^α as in (3.2), we see that the expressions [124] and [123] are both infinite. Of course, their **difference**, which is what we are interested in, is finite. The lack of finiteness of [124] and [123] individually arises simply because they express the volume of triangles which are infinitely large in this affine patch, since the vertex $\{1, 2\}$ defined by the intersection of the two parallel lines 1 and 2 is out at infinity. These infinities both come from the vanishing of $\langle 12P_0\rangle$ in the definition of [123] and [124], but since our polygon of interest does not involve the vertex $\{1, 2\}$ it must be the case that this term cancels in the sum $[124] - [123]$. Indeed, by making use of (3.14), we can show that it does, and express $[124] - [123]$ in terms of $\langle\cdot\rangle$ brackets that are manifestly finite. Namely, the terms that arise in the denominators in this sum are of the form $\langle ijP_0\rangle$ with $i, j \in S = \{1, ..., N\}$. Our cyclic list $l = (j_1, ..., j_n)$ gives us a list of vertices $\{j_k, j_{k+1}\}$ that define our polytope, and any term $\langle ijP_0\rangle$ such that $\{i, j\}$ is not in our list of vertices are called **spurious poles** and must vanish in the sum (3.19).

Alternatively, we note two other ways that we can get manifestly finite expression for this particular case (namely, the list $l = (1, 3, 2, 4)$ with 1 and 2 parallel in some affine patch). If it is the case that 3 and 4 are **not** parallel, then we can simply use the triangulation $[134] - [234]$ obtained by letting $B^\alpha = Z_4^\alpha$ in (3.20). If we are indeed dealing with a parallelogram in this affine patch (so that 1 and 2 are parallel, as are 3 and 4), then we can simply deform the vertex $\{2, 3\}$, say, carry out the calculation in the relevant triangulation, and then set the deformation to zero. All this said, the more general approach is to simply calculate the sum (3.19), possibly make a smart choice of B^α, get a sum of terms which may be individually infinite with denominators of the form $\langle ijP_0\rangle$ that vanish, use our cyclic list to determine which of these denominator factors **must** vanish, and repeatedly use (3.14) until this has been accomplished. These issues, however, have little relevance for us as we are primarily interested in the sum (3.19) purely as a projective expression.

3.3 Volumes of Abstract Listed d-Polytopes

We begin by introducing the generalization of the area of a 2-simplex. The form of the area of a 2-simplex used in (3.12) is easily generalized [4] to give the volume of an n-simplex bounded by $n+1$ boundaries to be

$$A_{n-simplex} = \frac{1}{n!}\frac{\langle 12...(n+1)\rangle^n}{\langle 12...nP_0\rangle\langle 23...n(n+1)P_0\rangle...\langle (n+1)1...(n-1)P_0\rangle} \equiv [12...(n+1)] \tag{3.22}$$

where $\langle 12...(n+1)\rangle$ is the $(n+1)$-dimensional epsilon contraction $\epsilon_{\alpha_1...\alpha_{n+1}}Z_1^{\alpha_1}...Z_{n+1}^{\alpha_{n+1}}$, and where $A_{n-simplex}$ depends on the choice of $P_0 \neq Z_1, ..., Z_{n+1}$. In particular, in three dimensions, we have

$$A_{3-simplex} = \frac{1}{6}\frac{\langle 1234\rangle^3}{\langle 123P_0\rangle\langle 234P_0\rangle\langle 341P_0\rangle\langle 412P_0\rangle} \equiv [1234] \tag{3.23}$$

for the volume of a 3-simplex bounded by the 2-faces $Z_1^\alpha, ..., Z_4^\alpha$.

Using (3.19) as motivation, we make the following definitions for the volume of an abstract listed 3-polytope, the volume of an abstract listed 4-polytope, and the volume of an abstract listed d-polytope.

Definition 3.7. Let $S = \{1, ..., N\}$ be a set of N boundaries and let $\{l_i = (j_{i1}, ..., j_{in_i})\}$ define an abstract listed 3-polytope. Let $B^\alpha \in \mathbb{CP}^3$ be a fixed reference boundary. Then the volume $A(\{l_i\})$ of this abstract listed 3-polytope is defined to be

$$A(\{l_a\}) \equiv \frac{1}{3}\sum_{i=1}^{N}\sum_{k=1}^{n_i}[ij_{ik}j_{i(k+1)}B] = \frac{2}{3!}\sum_{i=1}^{N}\sum_{k=1}^{n_i}[ij_{ik}j_{i(k+1)}B] \tag{3.24}$$

where the sum on k (but not on i) is cyclic in the usual sense.

This definition is clearly analogous to (3.19) and is also analogous to the "conical" triangulation of a 3-polytope, where we choose a reference vertex, triangulate each face, and construct the cone from this base-triangle to the reference vertex and sum over all simplices. The difference here is that our "reference vertex" is actually a reference **boundary** (B^α). This definition does indeed reflect what we mean by the genuine volume of a 3-polytope when we confine ourselves to a particular affine patch of $\mathbb{RP}^3$ and choose the scaling of P_0 correctly, and we will see this checked on various non-trivial examples in the next chapter after we introduce our calculus. The pre-factor $\frac{2}{3!}$ comes as a result of the fact that, due to the combinatorics of our cyclic lists, the triangulation (3.24) is summing over each simplex $\frac{3!}{2}$ times, one for each even permutation of the objects $\{i, j_{ik}, j_{i(k+1)}\}$. It is easy to see how this definition generalizes.

Definition 3.8. Let $S = \{1, ..., N\}$ be a set of N boundaries and let $\{l_{ij} = (k_{ij1}, ..., k_{ijn_{ij}})\}$ define an abstract listed 4-polytope. Let $B^\alpha \in \mathbb{CP}^4$ be a fixed reference boundary. Then the volume $A(\{l_{ij}\})$ of this abstract listed 4-polytope is defined to be

$$A(\{l_{ab}\}) \equiv \frac{2}{4!} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{n_{ij}} [ijk_{ijl}k_{ij(l+1)}B] \tag{3.25}$$

where the sum on l (but not on i or j) is cyclic in the usual sense, and where the objects $[abcde]$ are as defined in (3.22).

Although we will only look at explicit examples of abstract listed d-polytopes with $d \leq 4$, for completeness we write the definition of the volume of an abstract listed d-polytope for arbitrary d.

Definition 3.9. Let $S = \{1, ..., N\}$ be a set of N boundaries and let

$$\{l_{i_1,...,i_{(d-2)}} = (k_{i_1,...,i_{(d-2)},1}...k_{i_1,...,i_{(d-2)},n_{i_1,...,i_{(d-2)}}})\}$$

define an abstract listed d-polytope. Let $B^\alpha \in \mathbb{CP}^d$ be a fixed reference boundary. Then the volume $A(\{l_{i_1,...,i_{(d-2)}}\})$ of this abstract listed d-polytope is defined to be

$$A(\{l_{a_1,...,a_{(d-2)}}\}) \equiv \frac{2}{d!} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} ... \sum_{i_{(d-2)}=1}^{N} \sum_{l=1}^{n_{i_1,...,i_{(d-2)}}} [i_i...i_{(d-2)}k_{i_1,...,i_{(d-2)}l}k_{i_1,...,i_{(d-2)}(l+1)}B] \tag{3.26}$$

where the sum on l (but not on $i_1, ..., i_{(d-2)}$) is cyclic in the usual sense, and where the objects $[a_1a_2...a_{(d+1)}]$ are as defined in (3.22).

We note that though it is by no means clear, the expressions (3.24), (3.25), and (3.26) are all independent of the choice of B^α[3].

These definitions give the volumes of abstract listed d-polytopes a specific meaning, however as we saw in Chapter 1 we are much less interested in the expression of a given volume as a particular triangulation. Rather, we are more interested in seeing how and if different triangulations correspond to the same volume. The expressions (3.19), (3.24), (3.25), and (3.26) do not help us in this regard. Moreover, nature does not give us the cyclic lists in which we have framed our description of abstract listed polytopes. What nature does hand us (via super-BCFW recursion) is a **particular** triangulation of a polytope, one that usually in no way resembles those in the above definitions. It would therefore be advantageous to have a way to encode the information of the volume **intrinsically**, without reference to any particular triangulation,

[3]A proof of this fact is to come.

and in such a way that any particular triangulation can be derived. Additionally, it would be nice to be able to recover the information of the cyclic lists, which define a combinatorial/geometric object, from a particular triangulation, which is simply one of infinitely many ways to represent the volume of this object. At first sight this may seem like a tall order, but indeed there is a formalism in which all of these desires are satisfied, and in such a way that all computation is purely algebraic and reliant on only a few simple results. Developing this formalism is what the next chapter aims to do.

Chapter 4

The Calculus

4.1 Definitions and Identities

Many of the definitions that we make in deriving the following calculus have motivations that will not be apparent until the next chapter where we discuss the cohomology that motivates this whole story. Thus, some of these definitions will appear ad hoc, though the utility of these definitions will hopefully become clear as we go. We will not prove many of the identities that arise from our definitions, though we will discuss the combinatorial-geometric significance of many of them. That said, any identity that we state can be readily (and rigorously, though tediously) proved algebraically using identities like (3.14) and its higher dimensional analogues.

We begin by recalling the result (3.13) relating the objects $[ijk]$ as defined in (3.12):

$$[124] + [234] + [314] = [123]. \tag{4.1}$$

There are two equivalent geometrical interpretations of this. The first is as described after the proof of Claim 3.2, by viewing this identity as a result of being able to triangulate the same quadrilateral two different ways. Equivalently, we can view this identity as being the result of the vanishing of a sum of a particular set of four cyclically related triangles. One can readily check that we get similar identities in n dimensions amongst the objects $[1, ..., n+1]$ as defined in (3.22). In particular, we get the following identities in three and four dimensions.

Claim 4.1. Let $Z_1^\alpha, Z_2^\alpha, Z_3^\alpha, Z_4^\alpha, Z_5^\alpha \in \mathbb{CP}^3$, then

$$[1235] - [2345] + [3415] - [4125] = [1234]. \tag{4.2}$$

□

Claim 4.2. Let $Z_1^\alpha, Z_2^\alpha, Z_3^\alpha, Z_4^\alpha, Z_5^\alpha, Z_6^\alpha \in \mathbb{CP}^4$, then

$$[12346] + [23456] + [34516] + [45126] + [51236] = [12345]. \tag{4.3}$$

□

The generalization of these identities to higher dimensions is clear, and since we will not be explicitly describing the calculus for dimensions higher than four, we leave the general n-dimensional identities out for brevity's sake. We note that the relative minus signs that appear in (4.2) are simply a reflection of the parity of cyclic permutations of n objects[1].

We now introduce the following notation for the area of a polygon defined by the list $l = (1, 4, 2, 3)$:

$$A((1, 4, 2, 3)) \equiv V[12][34] \equiv [123] - [124]. \tag{4.4}$$

Thus, in general, $V[ij][kl] = [ijk] - [ijl]$. The objects $V[\cdot\cdot][\cdot\cdot]$ are clearly antisymmetric under swapping any terms within a given $[\cdot\cdot]$, so that for example $V[12][34]$ is antisymmetric both under $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. Using (4.1) one can readily show that $V[\cdot\cdot][\cdot\cdot]$ is also antisymmetric under swapping the entire $[\cdot\cdot]$'s, so that $V[12][34] = -V[34][12]$. Therefore, despite the seemingly special treatment of the boundaries 1 and 2 in (4.4), we see that the pair $1, 2$ is actually on equal footing with the pair $3, 4$. Therefore, by working with the $V[\cdot\cdot][\cdot\cdot]$'s, the combinatorics of expressions like (4.4) become more apparent. The following fact is vital for performing calculations, and can be readily proved using the definition (4.4) and the identity (4.1).

Claim 4.3. Let $Z_1^\alpha, Z_2^\alpha, Z_3^\alpha, Z_4^\alpha, P^\alpha \in \mathbb{CP}^2$ be arbitrary. Then

$$V[12][3P] + V[12][P4] = V[12][34]. \tag{4.5}$$

□

This result reflects the fact that by slicing a quadrilateral with a line P^α, we can express the area of the original quadrilateral as the sum of the areas of the two new quadrilaterals that each have P^α as a face. We note that due to the total antisymmetry of the V's in both their individual brackets as well as in the swapping of entire brackets, one obtains an identity identical to (4.5) for each "slot" in $V[\cdot\cdot][\cdot\cdot]$. Manipulating expression at the level of the V's is then seen to be much simpler than doing so with the individual $[ijk]$'s. Let us now make the analogous definitions in three and four dimensions.

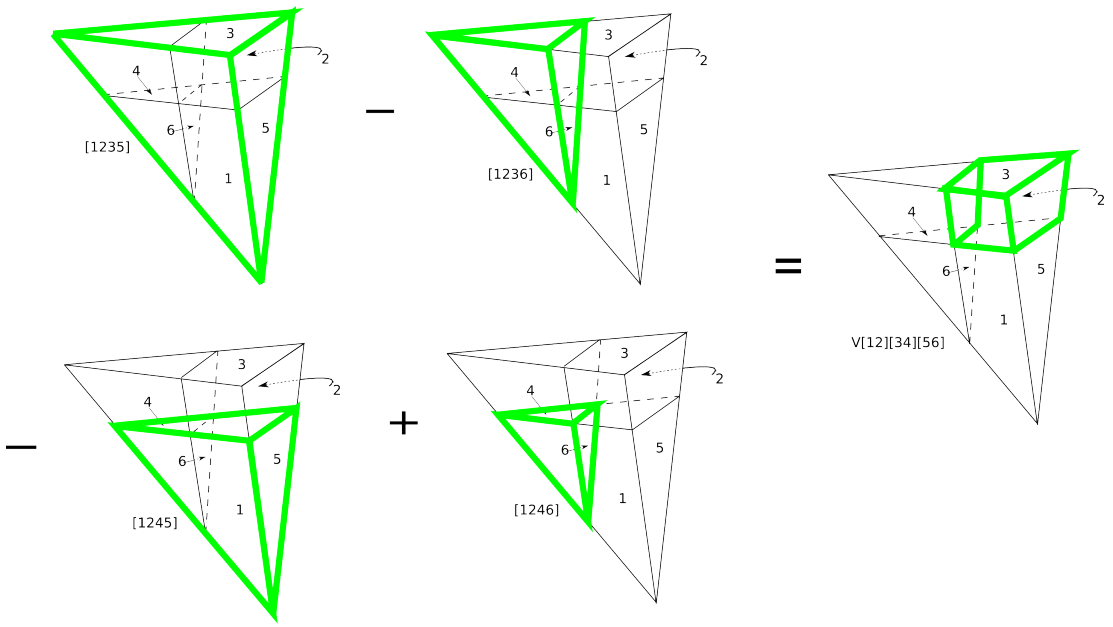
[1] I.e., a cyclic permutation of an odd number of objects is even, whereas that of an even number of objects is odd

Definition 4.4. Let $Z_1^\alpha, Z_2^\alpha, Z_3^\alpha, Z_4^\alpha, Z_5^\alpha, Z_6^\alpha \in \mathbb{CP}^3$, then

$$V[12][34][56] \equiv [1235] - [1236] - [1245] + [1246]. \tag{4.6}$$

The pattern here is to take the entries in the first $[\cdot\cdot]$ and put them in the first two entries of each $[\cdot\cdot\cdot]$, then taking one entry from each subsequent $[\cdot\cdot]$ and multiplying by -1 for each such entry that is on the right hand side of its respective $[\cdot\cdot]$. The motivation for this definition is the same as in two dimensions. Namely, $V[12][34] = [123] - [124]$ is viewed as the area of a "parallelogram" via the difference of two triangles (both of which not "contained in" the parallelogram), and the identity $V[12][34] = -V[34][12]$ comes from the fact that we can triangulate this parallelogram two different ways (see Figure 3.3 with the boundaries 3 and 4 switched). Similarly, (4.6) should be viewed as the volume of a "cube" bounded by "parallel" pairs of boundaries $\{1,2\}$, $\{3,4\}$, and $\{5,6\}$. The triangulation used in (4.6) is depicted in Figure 4.1.

Figure 4.1



We note that this is a triangulation of the cube using only four tetrahedra, which is impossible when we restrict our tetrahedra to lie **within** the cube. We also note that Figure 4.1 is a rather idealized picture, since the six faces can intersect in many different ways while remaining in general position. As we will soon see, (4.6) really defines the volume of the combinatorial 3-polytope defined by the lists in (2.12), which are well-defined regardless of how the six boundaries actually intersect. It is

also worth pointing out that if we choose a genuine cube in some affine patch of $\mathbb{R}^3$ for which opposite faces are genuinely parallel, then the line of intersection of the faces 1 and 2 depicted in Figure 4.1 is out at infinity. Thus, each term in (4.6) **individually** is infinite, even though the sum itself is finite (and gives the volume of the cube, under the correct choice of P_0 in (3.22)). Despite this presence of "spurious poles", these expressions are projectively well-defined.

The expression (4.6) is clearly antisymmetric under swapping any entries in a particular $[\cdot\cdot]$ bracket, and it is clearly antisymmetric under swapping the second and third brackets. I.e., $V[12][34][56] = -V[12][56][34]$. Though it is less clear, we do in fact have total antisymmetry in the $V[\cdot\cdot][\cdot\cdot][\cdot\cdot]$ objects, as can be readily proved using the identity (4.2). In particular, one can use (4.2) to prove the following claim, and thus prove the total antisymmetry in $V[\cdot\cdot][\cdot\cdot][\cdot\cdot]$ in both the entries within a given $[\cdot\cdot]$ as well as in the swapping of any $[\cdot\cdot] \leftrightarrow [\cdot\cdot]$.

Claim 4.5.

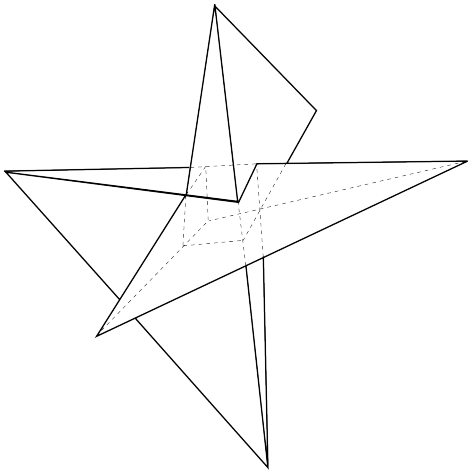
$$V[12][34][56] = -V[34][12][56]. \tag{4.7}$$

□

In two dimensions the total antisymmetry of $V[\cdot\cdot][\cdot\cdot]$ came from the existence of two particular triangulations of the quadrilateral. The analogous statement in three dimensions is that there are indeed three particular triangulations—analogues to that in Figure 4.1—that give rise to the volume of the same "cube". This situation is depicted in Figure 4.2 where we chose an intersection structure of the six faces that make the geometry most apparent.

We now give the three-dimensional analogue of (4.5).

Figure 4.2



Claim 4.6. Let $Z_1^\alpha, Z_2^\alpha, Z_3^\alpha, Z_4^\alpha, Z_5^\alpha, Z_6^\alpha, P^\alpha \in \mathbb{CP}^3$ be arbitrary, then

$$V[12][34][5P] + V[12][34][P6] = V[12][34][56]. \tag{4.8}$$

□

As before, the total antisymmetry of $V[\cdot\cdot][\cdot\cdot][\cdot\cdot]$ gives an identical statement for each "slot" of V.

We now use the same procedure to define the four-dimensional analogue of the $V[\cdot\cdot][\cdot\cdot]$ objects. The total antisymmetry of the object we define follows from

(4.3) in the analogous way as in two and three dimensions, and the results (4.5) and (4.8) carry over immediately.

Definition 4.7. Let $Z_1^\alpha, Z_2^\alpha, Z_3^\alpha, Z_4^\alpha, Z_5^\alpha, Z_6^\alpha, Z_7^\alpha, Z_8^\alpha \in \mathbb{CP}^4$, then

$$\begin{aligned} V[12][34][56][78] \equiv &[12357] - [12358] - [12367] + [12368] \\ &- [12457] + [12458] + [12467] - [12468]. \end{aligned} \tag{4.9}$$

Claim 4.8. $V[ij][kl][mn][pq]$ is antisymmetric under swapping any pairs within the same $[\cdot\cdot]$ as well as under swapping any pair of brackets $[\cdot\cdot] \leftrightarrow [\cdot\cdot]$.
□

Claim 4.9. Let $Z_1^\alpha, Z_2^\alpha, Z_3^\alpha, Z_4^\alpha, Z_5^\alpha, Z_6^\alpha, Z_7^\alpha, Z_8^\alpha, P^\alpha \in \mathbb{CP}^4$ be arbitrary, then

$$V[12][34][56][7P] + V[12][34][56][P8] = V[12][34][56][78]. \tag{4.10}$$

□

We introduce the brackets { } to represent one-half times the antisymmetrization on the indices that they contain in a given $V[ij]...[kl]$, and we leave out indices by surrounding them with vertical lines | |. Thus, for example,

$$V[\{ij][k\}P][QR] \equiv V[ij][kP][QR] + V[jk][iP][QR] + V[ki][jP][QR] \tag{4.11}$$

and

$$\begin{aligned} V[\{ij][k|P|][l\}Q] \equiv &\, V[\{ij][k\}P][lQ] - V[\{jk][l\}P][iQ] \\ &+ V[\{kl][i\}P][jQ] - V[\{li][j\}P][kQ]. \end{aligned} \tag{4.12}$$

We define this antisymmetrization operator with a factor of one-half because every such antisymmetrization that we will consider will involve antisymmetrization on two indices in the same $[\cdot\cdot]$, and due to the antisymmetry of the $[\cdot\cdot]$'s we would pick up an unwanted factor of two. This is why (4.11) has three terms instead of six, and why (4.12), when fully expanded, would have twelve terms instead of twenty-four.

We now state some identities amongst the various $V[\cdot\cdot]...[\cdot\cdot]$ objects, where for each identity the number of brackets $[\cdot\cdot]$ will determine the dimension of the space in which the corresponding boundaries are defined. All of the following results can be obtained readily from our definitions and the various identities already stated. Additionally, any identities stated in terms of particular boundaries $1, 2, 3.., P, Q, R, ... \in \mathbb{CP}^n$ are implied to hold for any such choice of boundaries. We outline the proof for the first two claims to show how such computations go, and we leave out the others.

Claim 4.10.

$$V[\{ij][k\}P][QR] = 2([ijkQ] - [ijkR]). \tag{4.13}$$

Proof:

$$\begin{aligned} V[\{ij][k\}P][QR] &= V[ij][kP][QR] + V[jk][iP][QR] + V[ki][jP][QR] \\ &= V[ij][kP][QR] + V[jk][iP][QR] - V[jP][ki][QR] \\ &= 2([ijkQ] - [ijkR]) \end{aligned}$$

where the last equality comes from simply expanding out the V's according to their definition. $\square$

In particular, we note that $V[\{ij][k\}P][QR]$ is independent of P^α.

Claim 4.11. For any boundaries i, j, k, l, P, Q, we have

$$V[\{ij][k|P|][l\}Q] = 6[ijkl], \tag{4.14}$$

and is thus independent of P^α, and Q^α.

Proof:

$$\begin{aligned} V[\{ij][k|P|][l\}Q] &= V[\{ij][k\}P][lQ] - V[\{jk][l\}P][iQ] + V[\{kl][i\}P][jQ] - V[\{li][j\}P][kQ] \\ &= 2([ijkl] - [ijkQ] - [jkli] + [jklQ] + [klij] - [kliQ] - [lijk] + [lijQ]) \\ &= 2(4[ijkl] - ([ijkQ] - [jklQ] + [kliQ] - [lijQ])) \\ &= 6[ijkl] \end{aligned}$$

where in the second equality we used Claim 4.10 and in the final equality we used (4.2). $\square$

The higher dimensional analogues of these identities are readily verified in a similar way.

Claim 4.12.

$$V[\{ij][k\}P][QR][ST] = -2([ijQRS]+[jkQRS]+[kiQRS]-[ijQRT]-[jkQRT]-[kiQRT]), \tag{4.15}$$

and is thus independent of P^α. $\square$

Claim 4.13.

$$V[\{ij][k|P|][l\}Q][RS] = 6([ijklR] - [ijklS]), \tag{4.16}$$

and is thus independent of P^α and Q^α. $\square$

Claim 4.14.

$$V[\{ij][k|P|][l|Q|][m\}R] = 4![ijklm]. \tag{4.17}$$

and is thus independent of $P^\alpha, Q^\alpha, R^\alpha$. $\square$

We now have the notation and the results needed to introduce the key objects of our calculus as well as prove the fundamental result involving these objects. If the previous definitions have not appeared sufficiently unmotivated, then the following certainly will. As mentioned before, however, there lies at the heart of this story a cohomological discussion that gives sufficient motivation for these definitions, and we will explore this part of the story in the next chapter. In the next section we will see how these objects relate to our calculus and look at some explicit examples in two, three, and four dimensions.

We begin by defining the central objects for our two-dimensional calculus.

Definition 4.15. Let $S = \{1, ..., N\} = \{Z_1^\alpha, ..., Z_N^\alpha\}$ be a set of N distinct boundaries in $\mathbb{CP}^2$, and let $Q^\alpha \in \mathbb{CP}^2$ be a fixed reference boundary. We denote by f_{ij} (with $i, j \in S$), the following sum

$$f_{ij} \equiv \sum_{k \neq i,j}^{N} V[ij][kQ]. \tag{4.18}$$

We note that Q^α being a **fixed** reference boundary means that it is the same for each f_{ij}.

This definition requires a few remarks. First, it is clear that each f_{ij} is antisymmetric in its indices. Second, we leave off any Q^α subscripts on the f's because when we express volumes of polygons in terms of the f_{ij}'s, we will prove that the result is independent of our choice of Q^α. The following result is what makes our calculus useful, as we will soon see.

Claim 4.16. Let S and Q^α be as in Definition 4.15. Then for any $i, j, k \in S$,

$$\rho_{[k} f_{ij]} \equiv f_{ij} + f_{jk} + f_{ki} = 2[ijk] \tag{4.19}$$

Proof: If any of the i, j, k are equal, then both sides of (4.19) are identically zero. Thus, without loss of generality we can take the i, j, k to be pairwise distinct. We

then have

$$\begin{aligned} f_{ij} + f_{jk} + f_{ki} &= \sum_{l\neq i,j}^{N} V[ij][lQ] + \sum_{l\neq j,k}^{N} V[jl][kQ] + \sum_{l\neq k,i}^{N} V[ki][lQ] \\ &= \sum_{l\neq i,j,k}^{N} (V[ij][lQ] + V[jk][lQ] + V[ki][lQ]) \\ &\quad + V[ij][kQ] + V[jk][iQ] - V[jQ][ki] \\ &= 2[ijk] \end{aligned}$$

where in the last equality we used the fact that $V[ij][lQ] + V[jk][lQ] = V[ik][lQ] = -V[ki][lQ]$ in the sum, and simply expanded out the three remainder terms according to the definition of the V's. $\square$

We note that (4.19) is very similar to the Cěch cocycle condition [6] for the first cohomology group: $f_{ij} + f_{jk} + f_{ki} = 0$. As we will see in the next chapter , this is no accident. It should also be noted that (4.19) is rather surprising. In particular, even though one can think of each f_{ij} as being defined by the vertex $\{i, j\}$, it is defined in a very "non-local" way. Namely, each f_{ij} involves a sum over all the boundaries "in play" **except for** the boundaries $\{i, j\}$, and so each individual f_{ij} knows about all of the boundaries defining a polygon. Yet, remarkably, the simple sum in (4.19) reduces to the very "local" object $[ijk]$, which is just the area of a triangle.

Being motivated by the Cěch cohomology condition mentioned above, we now define the higher dimensional analogues of f_{ij}.

Definition 4.17. Let $S = \{1, ..., N\} = \{Z_1^\alpha, ..., Z_N^\alpha\}$ be a set of N boundaries in $\mathbb{CP}^3$, and let $P^\alpha, Q^\alpha \in \mathbb{CP}^3$ be fixed reference boundaries (reference planes, in this dimension). Then for any $i, j, k \in S$,

$$f_{ijk} \equiv \sum_{l\neq i,j,k}^{N} V[\{ij][k\}P][lQ]. \tag{4.20}$$

We note that due to Claim 4.10 we could equally well write

$$f_{ijk} = \sum_{l\neq i,j,k}^{N} 2([ijkl] - [ijkQ]), \tag{4.21}$$

to bring out the manifest P^α independence of each f_{ijk}, however we choose to use (4.20) as it is more in line with the cohomological description of the $f_{...}$ objects. Moreover, even though each individual f_{ijk} is independent of P^α, they **individually**

dependent on our choice of Q^α just as each f_{ij} was dependent of our choice of $Q^\alpha \in \mathbb{CP}^2$. As we will see, though, the combinations of the f_{ijk} objects that we will be interested in **are** independent of our choice of Q^α.

The key identity for the f_{ijk} objects, and that which makes our three-dimensional calculus useful, is the natural analogue to (4.19).

Claim 4.18. Let S, P^α and Q^α be as in Definition 4.17. Then for any $i, j, k, l \in S$, we have

$$\rho_{[l} f_{ijk]} \equiv f_{ijk} - f_{jkl} + f_{kli} - f_{lij} = 6[ijkl]. \tag{4.22}$$

Proof:

$$\begin{aligned} f_{ijk} - f_{jkl} + f_{kli} - f_{lij} = &\sum_{m\neq i,j,k}^{N} V[\{ij][k\}P][mQ] - \sum_{m\neq j,k,l}^{N} V[\{jk][l\}P][mQ] \\ &+ \sum_{m\neq k,l,i}^{N} V[\{kl][i\}P][mQ] - \sum_{l\neq l,i,j}^{N} V[\{li][j\}P][mQ] \\ = &\sum_{m\neq i,j,k,l}^{N} \Big(V[\{jk][l\}P][mQ] - V[\{kl][i\}P][mQ] \\ &\qquad + V[\{li][j\}P][mQ] - V[\{ij][k\}P][mQ] \Big) \\ &+ V[\{ij][k|P][l\}Q] \\ \\ =&6[ijkl] \end{aligned}$$

where the sum in the second equality is straightforwardly checked to vanish using Claim 4.10. □

Once again, we see a particular (cohomology-inspired) highly "non-local" sum combining to form a completely "local" expression. The definition and subsequent results involving the four-dimensional objects f_{ijkl} is now straightforward.

Definition 4.19. Let $S = \{1, ..., N\} = \{Z_1^\alpha, ..., Z_N^\alpha\}$ be a set of N boundaries in $\mathbb{CP}^4$ and let $P_1^\alpha, P_2^\alpha, Q^\alpha \in \mathbb{CP}^4$ be fixed reference boundaries. Then for any $i, j, k, l \in S$ we define

$$f_{ijkl} \equiv \sum_{m\neq i,j,k,l}^{N} V[\{ij][k|P_1|][l\}P_2][mQ]. \tag{4.23}$$

Claim 4.13 shows that each individual f_{ijkl} is independent of P_1^α and P_2^α, and as we will soon see the combinations of the f_{ijkl} that we will be interested in are independent of our choice of Q^α. We now have the following key identity, on which the entire utility of our four-dimensional calculus relies.

Claim 4.20. Let $S, P_1^\alpha, P_2^\alpha$, and Q^α be as in Definition 4.19. Then for any $i, j, k, l, m \in S$, we have

$$\rho_{[m} f_{ijkl]} \equiv f_{ijkl} + f_{jklm} + f_{klmi} + f_{lmij} + f_{mijk} = 4![ijklm]. \tag{4.24}$$

□

The proof of this claim is algebraically identical to those in two and three dimensions, so we omit it, but we do point out that we again see the same interaction between "non-local" and "local" expressions.

Now that we have introduced the $f_{...}$ objects and established the key identities amongst them, we are ready to introduce the calculus and explore its utility. In the subsequent sections we will prove the general validity of the calculus in two, three, and four dimensions, as well as examine explicit examples and discuss the utility of this new way of calculating volumes of abstract listed polytopes.

4.2 The Two-Dimensional Calculus

We now turn our attention to using the f_{ij} objects defined in the previous section to calculate the areas of general abstract listed 2-polytopes. Before stating the main result, let us first show that the combinations of f_{ij} objects that we will be interested in are indeed independent of our choice of Q^α reference boundary used to define the individual f_{ij}'s. The following proposition does this.

Proposition 4.21. Let $S = \{1, ..., N\}$ be a set of N boundaries in $\mathbb{CP}^2$ in the usual sense, let $Q^\alpha \in \mathbb{CP}^2$ be a fixed reference boundary, and let $\{f_{ij}\}$ be defined with respect to S and Q^α as in Definition 4.15. Let $l = (i_1, ..., i_n)$ be a cyclic list taking values in S, i.e., each $i_k \in S$. Let $Q'^\alpha \in \mathbb{CP}^2$ be another choice of reference boundary and let f'_{ij} be defined as f_{ij} with Q' replacing Q. Let $Vol \equiv \sum_{k=1}^n f_{i_k i_{k+1}}$ and $Vol' \equiv \sum_{k=1}^n f'_{i_k i_{k+1}}$, where the sums on k are cyclic, i.e. $i_{n+1} = i_1$. Then $Vol = Vol'$. (The notation Vol is meant to be suggestive).

Proof:

$$\begin{aligned} Vol - Vol' &= \sum_{k=1}^{n} (f_{i_k i_{k+1}} - f'_{i_k i_{k+1}}) \\ &= \sum_{k=1}^{n} \sum_{l \neq i_k, i_{k+1}} (V[i_k i_{k+1}][lQ] - V[i_k i_{k+1}][lQ']) \\ &= \sum_{k=1}^{n} \sum_{l \neq i_k, i_{k+1}} V[i_k i_{k+1}][Q'Q] \\ &= (N-2) \sum_{k=1}^{n} V[i_k i_{k+1}][Q'Q] \\ &= 0, \end{aligned}$$

where the factor of $(N-2)$ comes from the fact that the dependence on l drops out and that for each k, $i_k = i_{k+1} \Rightarrow V[i_k i_{k+1}][Q'Q] = 0$. The third equality comes from Claim 4.3, and the final equality comes from the same claim and the fact that the sum is cyclic. $\square$

The preceding proposition shows that even though each individual f_{ij} depends on our choice of Q^α, the particular combination

$$Vol \equiv \sum_{k=1}^{n} f_{i_k i_{k+1}} \tag{4.25}$$

for some cyclic list $l = (i_1, ..., i_n)$ is independent of this choice. The next theorem is the statement of the two-dimensional calculus, and uses the particular combination of f_{ij}'s in (4.25).

Theorem 4.22. *Let $S = \{1, ..., N\}$ be a set of N boundaries, and let $l = (i_1, ..., i_n)$ be a cyclic list taking values in S. Let $A(l)$ be the area of the combinatorial 2-polytope defined by l, as in Proposition 3.5, with the reference boundary B^α. Let $S' = S \cup \{B^\alpha\}$ be the set of boundaries $\{1, ..., N, B\}$, and let $\{f_{ij}\}$ be defined with respect to S' as in Definition 4.15. Then $A(l) = \frac{1}{2} \sum_{k=1}^{n} f_{i_k i_{k+1}}$. In other words, the volume of the polygon defined by l is equal to $\frac{1}{2} Vol$ as defined above.*

Proof: Let $B^\alpha \neq Q^\alpha$ be some reference boundary. Then

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^{n} f_{i_k i_{k+1}} &= \frac{1}{2}\sum_{k=1}^{n}(f_{i_k i_{k+1}} + f_{i_{k+1}B} + f_{Bi_{k+1}}) \\ &= \frac{1}{2}\sum_{k=1}^{n}(f_{i_k i_{k+1}} + f_{i_{k+1}B} + f_{Bi_k}) \\ &= \frac{1}{2}\sum_{k=1}^{n} 2[i_k i_{k+1} B] \\ &= \sum_{k=1}^{n}[i_k i_{k+1} B] = A(l)\end{aligned}$$

where in the first equality we simply added $0 = f_{i_{k+1}B} + f_{Bi_{k+1}}$ to each term in the sum. In the second equality we took advantage of the fact that the sum on k is cyclic to relabel the index on the third term, and in the third equality we used Claim 4.16. □

We note that we had to introduce the set of boundaries $S' = \{1, ..., N, B\}$ simply to make the expression f_{iB} well-defined. In practice, however, we only need to use f_{ij} objects whose subscripts take values in $S = \{1, ..., N\}$. We can extend this set of boundaries however we wish since Claim 4.16, which is the main result for our calculus, is independent of the set of boundaries we choose. This can be seen explicitly in the second equality of the proof of this claim, where the sum vanishes regardless of what N is. Equivalently, we could also choose B^α to be within the set S itself, since Proposition 3.5 makes it clear that $A(l)$ does not depend on our choice of B^α. The following example will allow us to see this fact in practice, as well as give some geometric insight into what this calculus is all about.

Example 4.23. We consider yet again the triangle, defined by the list $l = (1, 2, 3)$, so that the set of boundaries is $S = \{1, 2, 3\}$. To define our f_{ij}'s, we must introduce a reference boundary Q^α. In Figure 4.3 we give a possible picture of these choices.

With S as above, i.e., not adding in another reference boundary B^α as in Theorem 4.22, we have

$$f_{12} = V[12][3Q] \tag{4.26}$$

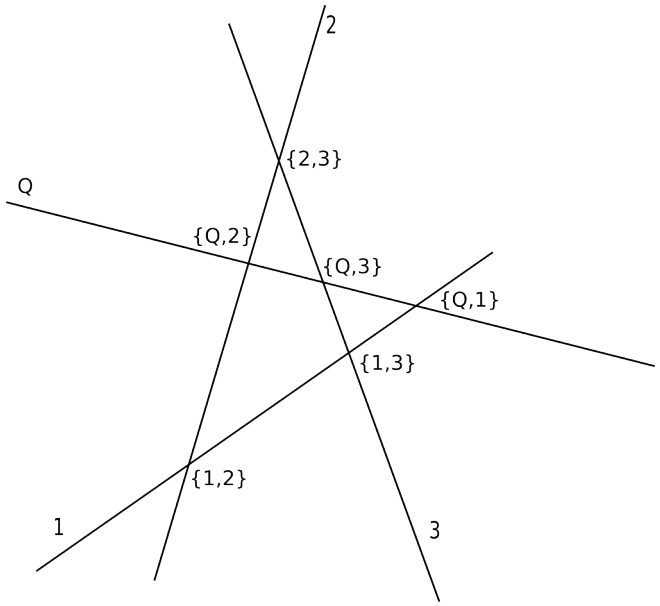
$$f_{23} = V[23][1Q] \tag{4.27}$$

$$f_{31} = V[31][2Q]. \tag{4.28}$$

Theorem 4.22 says that we should consider the sum

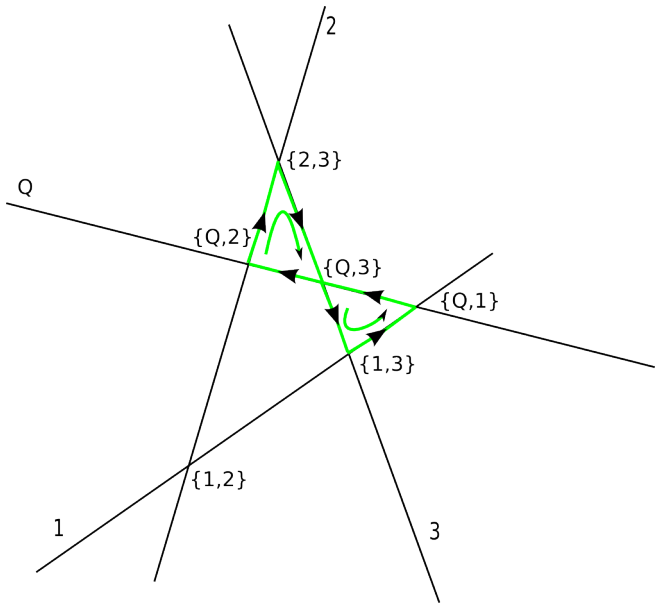
$$\frac{1}{2}(f_{12} + f_{23} + f_{31}) \tag{4.29}$$

Figure 4.3



to get the area of the polygon defined by the list $l = (1, 2, 3)$. Let us see what each of these terms corresponds to. By definition of the $V[\cdot\cdot][\cdot\cdot]$'s in (4.4), we have that f_{12} is the area of the polygon defined by the list $(1, Q, 2, 3)$. This polygon is depicted in Figure 4.4. We note that the actual volume corresponding to f_{12} is completely "non-

Figure 4.4



local", in the sense that the vertex $\{1, 2\}$ is not included in the resulting polygon. We now see that f_{23} is the area of the polygon defined by the list $(2, Q, 3, 1)$, which is

depicted in Figure 4.5. Finally, we see that f_{31} is the area of the polygon defined by the list $(3, Q, 1, 2)$, depicted in Figure 4.6.

Figure 4.5

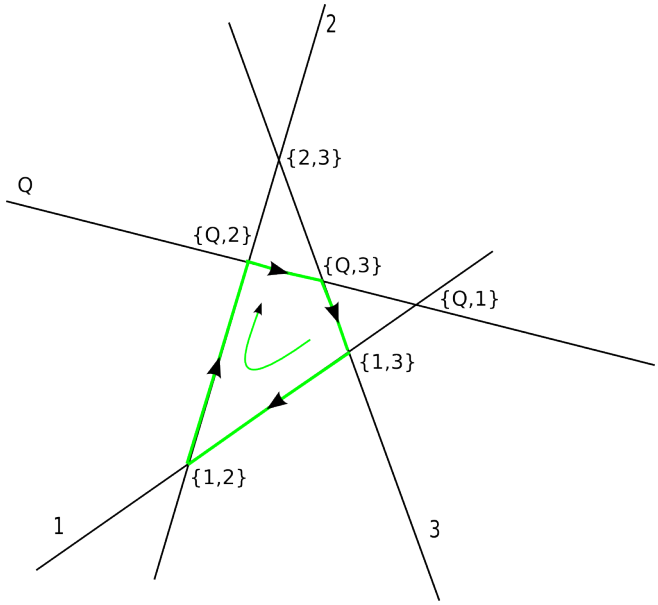
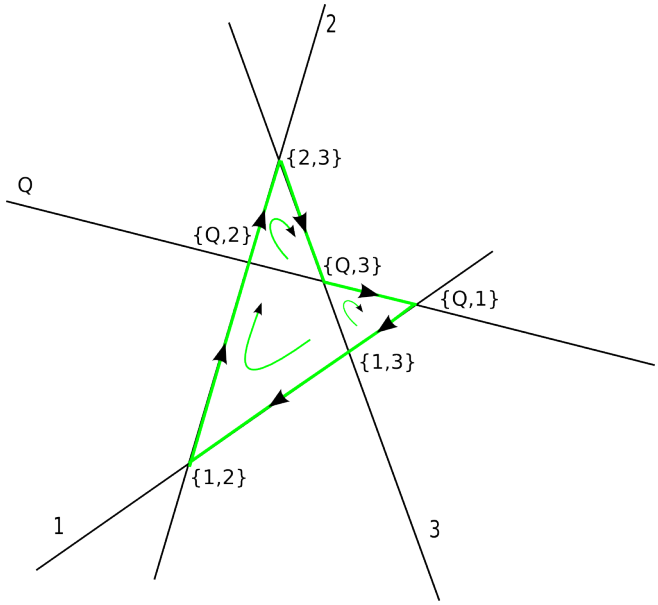


Figure 4.6



By "superposing" the three polygons depicted in Figures 4.4-6 and cancelling the areas that have opposite orientations, we see that we are left precisely with 2 times the area of the triangle defined by the list $(1, 2, 3)$, and the dependence on Q^α drops

out. Had we included some other reference boundary B^α in S to define $S' = S \cup \{B^\alpha\}$, then we would get new f_{ij}'s. Namely, we would have

$$f_{12} = V[12][3Q] + V[12][BQ] \tag{4.30}$$

$$f_{23} = V[23][1Q] + V[23][BQ] \tag{4.31}$$

$$f_{31} = V[31][2Q] + V[31][BQ], \tag{4.32}$$

and the corresponding pictures (the analogues of Figures 4.4-6) would be correspondingly more complex. However, we can see immediately that the sum

$$f_{12} + f_{23} + f_{31} \tag{4.33}$$

would be left unaffected, since

$$V[12][BQ] + V[23][BQ] + V[31][BQ] = 0 \tag{4.34}$$

using Claim 4.3. This is precisely the vanishing term in the second equality in the proof of Claim 4.16, and this is precisely why we can add (but not subtract) as many boundaries to S as we would like, or as is convenient, without having to worry about the sum of f_{ij}'s that we are interested in being affected. Thus, we add B^α to the set of boundaries S in Theorem 4.22 because considering the objects f_{iB} makes the proof simpler, but in practice we can deal only with S and define our f_{ij}'s with respect to it. We note that in this example the object f_{1B} (as well as f_{2B} and f_{3B}) can be defined using S', and we would have $f_{1B} = V[1B][2Q] + V[1B][3Q]$ (and similarly for f_{2B} and f_{3B}), but since B does not make an appearance in the list $l = (1, 2, 3)$ under consideration, we can disregard these objects.

The two-dimensional calculus for calculating the area of an abstract listed 2-polytope is therefore the following. Take the set of boundaries $S = \{1, ..., N\}$ and define the f_{ij} objects with respect to it. Then take the cyclic list $l = (i_1, ..., i_n)$ which defines our 2-polytope, and go down the list adding up $f_{i_k i_{k+1}}$ at each step. In practice, we never have to make reference to the actual structure of the f_{ij}'s, because we can just use (4.19) to simplify our resulting sum of f_{ij}'s into a sum of areas of triangles. By making different choices of simplification, we can recover any possible triangulation. Thus, our calculus does much more than simply recover the particular triangulation used in Proposition 3.5, and Theorem 4.22 simply says that **any** triangulation obtained from this sum of f_{ij}'s will correspond to the correct area. The following example will illustrate the two-dimensional calculus as well as show the additional benefit of how the f_{ij} calculus encodes all possible triangulations.

Example 4.24. Consider the cyclic list $l = (1, 4, 2, 3)$ again. Here, $N = 4$ and so we construct the $\binom{4}{2}$ objects f_{ij}. Theorem 4.22 tells us that

$$A(l) = \frac{1}{2}(f_{14} + f_{42} + f_{23} + f_{31}). \tag{4.35}$$

Using (4.19), which says that $f_{14} + f_{42} + f_{21} = 2[142]$, we can rewrite (4.35) as

$$A(l) = [142] + \frac{1}{2}(f_{23} + f_{31} - f_{21}). \tag{4.36}$$

Using the antisymmetry of the f_{ij}'s as well as the $[ijk]$'s, and using (4.19) again yields

$$A(l) = [123] - [124] = V[12][34], \tag{4.37}$$

which we know from the previous section to be correct. However, we also could have used (4.19) to say that $f_{31} + f_{14} + f_{43} = 2[314]$ and therefore simplified (4.35) as

$$A(l) = [314] + \frac{1}{2}(f_{42} + f_{23} - f_{43}) = [314] + [423]. \tag{4.38}$$

We therefore have $A(l) = [342] - [341] = -([341] - [342]) = -V[34][12] = V[12][34]$, as expected.

As it stands, we have gained nothing new in this example from the f_{ij} calculus. Suppose, however, that we were given the expression

$$(I) \equiv [145] + [425] + [236] + [316] + [156] + [265] \tag{4.39}$$

for some boundaries $\{1, ..., 6\}$ and we wanted to know if this expression corresponded to any other areas that we might know about. As we will see, $(I) = A((1, 4, 2, 3)) = A(l)$, but as things stood prior to knowing about the f_{ij} calculus, this would be an algebraic nightmare to uncover. Using (4.19) in reverse, we can expand out each $[ijk]$ in terms of f_{ij}'s, at which point cancellations become immediate.

$$\begin{aligned}(I) =& [145] + [425] + [236] + [316] + [156] + [265] \\ =& \frac{1}{2}\big(f_{14} + f_{45} + f_{51} + f_{42} + f_{25} + f_{54} + f_{23} + f_{36} + f_{62} \\ &+ f_{31} + f_{16} + f_{63} + f_{15} + f_{56} + f_{61} + f_{26} + f_{65} + f_{52}\big) \\ =& \frac{1}{2}(f_{14} + f_{42} + f_{23} + f_{31}) = A(l)\end{aligned} \tag{4.40}$$

where we have cancelled terms of the form $f_{ij} + f_{ji}$ due to the antisymmetry of each f_{ij}. Thus, (I) is seen to be both independent of the boundaries 5 and 6, as well as equal to $A((1, 4, 2, 3))$, both of which are highly non-obvious facts when presented

with (4.39). To go the other direction and recover (I) from (4.35) one simply adds zero in the form of

$$0 = f_{45} + f_{54} + f_{51} + f_{15} + f_{25} + f_{52} + f_{36} + f_{63} + f_{62} + f_{26} + f_{16} + f_{61} + f_{56} + f_{65}. \quad (4.41)$$

To do so more systematically, one simply notes that in order to get the expression [145] from (4.35), one must introduce the terms f_{45} and f_{51} (and then subsequently subtract them back off again). By dealing with the remaining terms we can "reconstruct" (I) term by term in a completely algorithmic fashion. Thus, any triangulation of $A(l)$ can be recovered from (4.35) by adding zero in various ways and simplifying the resulting sum using (4.19) in various ways, and the equality of any two triangulations can be checked in a similarly algorithmic way. This process is clearly not limited to the particular list $l = (1, 4, 2, 3)$, but rather carries directly over for any list $l = (i_1, ..., i_n)$.

We note that in the above example, we have implicitly extended our set of boundaries from $\{1, 2, 3, 4\}$ to $\{1, 2, 3, 4, 5, 6\}$, by introducing the "spurious boundaries" 5 and 6. Our discussion after the proof of Theorem 4.22 shows that this is indeed okay, since the only result that we used—Claim 4.16—is independent of which set of boundaries we choose.

Since we will end up looking at explicit examples in both three and four dimensions, we will leave the two-dimensional case with only one example and turn our attention now to the three-dimensional calculus.

4.3 The Three-Dimensional Calculus

As in the the two-dimensional case, we begin by showing that the combinations of the f_{ijk} objects that we will be interested in are independent of the reference boundary $Q^\alpha \in \mathbb{CP}^3$ used to define them as in Definition 4.17. The following proposition does this.

Proposition 4.25. Let $S = \{1, ..., N\}$ be a set of N boundaries in $\mathbb{CP}^3$ in the usual sense, let $P^\alpha, Q^\alpha \in \mathbb{CP}^3$ be fixed reference boundaries, and let $\{f_{ijk}\}$ be defined with respect to S, P^α, and Q^α as in Definition 4.17. Let $\{l_i = (j_{i1}, ..., j_{in_i})\}$ be a collection of cyclic lists with each $j_{ik} \in S$ that defines an abstract listed 3-polytope. Let $Q'^\alpha \in \mathbb{CP}^3$ be another choice of reference boundary and let f'_{ijk} be defined as f_{ijk} with Q' replacing Q. Let $Vol \equiv \sum_{i=1}^{N} \sum_{k=1}^{n_i} f_{ij_{ik}j_{i(k+1)}}$ and $Vol' \equiv \sum_{i=1}^{N} \sum_{k=1}^{n_i} f'_{ij_{ik}j_{i(k+1)}}$, where the sums on k (but not on i) are cyclic, i.e. $j_{i(n_i+1)} = j_{i1}$. Then $Vol = Vol'$.

Proof: As shown in Claim 4.10, each f_{ijk} is individually independent of P^α, so by proving this proposition we will have proved that the sum Vol is independent of both P^α and Q^α. We have

$$Vol - Vol' = \sum_{i=1}^{N}\sum_{k=1}^{n_i}(f_{ij_{ik}j_{i(k+1)}} - f'_{ij_{ik}j_{i(k+1)}}) \tag{4.42}$$

$$= \sum_{i=1}^{N}\sum_{k=1}^{n_i}\sum_{l\neq i,j_{ik},j_{i(k+1)}}^{N}(V[\{ij_{ik}][j_{i(k+1}\}P][lQ] - (Q \to Q')) \tag{4.43}$$

$$= \sum_{i=1}^{N}\sum_{k=1}^{n_i}\sum_{l\neq i,j_{ik},j_{i(k+1)}}^{N}(2[ij_{ik}j_{i(k+1)}l] - 2[ij_{ik}j_{i(k+1)}Q] - (Q \to Q')) \tag{4.44}$$

$$= \sum_{i=1}^{N}\sum_{k=1}^{n_i}\sum_{l\neq i,j_{ik},j_{i(k+1)}}^{N}2([ij_{ik}j_{i(k+1)}Q'] - [ij_{ik}j_{i(k+1)}Q]) \tag{4.45}$$

$$= 2(N-3)\sum_{i=1}^{N}\sum_{k=1}^{n_i}([ij_{ik}j_{i(k+1)}Q'] - [ij_{ik}j_{i(k+1)}Q]) \tag{4.46}$$

where the fourth equality holds because $[ij_{ik}j_{i(k+1)}l]$ is independent of Q and Q', and where the final equality holds because the l dependence in the sum drops out, and the sum is over $N-3$ terms unless $j_{ik} = j_{i(k+1)}$ in which case the summand vanishes identically. Since the sum on k is cyclic we can freely shift around the k indices, so we re-write the two terms in (4.46) with $k \to k-1$ and introduce a compensating factor of one-half to get

$$Vol - Vol' = (N-3)\sum_{i=1}^{N}\sum_{k=1}^{n_i}([ij_{ik}j_{i(k+1)}Q'] - [ij_{ik}j_{i(k+1)}Q] + [ij_{i(k-1)}j_{ik}Q'] - [ij_{i(k-1)}j_{i(k)}Q]) \tag{4.47}$$

$$= (N-3)\sum_{i=1}^{N}\sum_{k=1}^{n_i}V[ij_{ik}][j_{i(k+1)}j_{i(k-1)}][Q'Q]. \tag{4.48}$$

We now introduce another fixed reference boundary R, shift the k index of the first term as $k \to k-1$ and the k index of the second term as $k \to k+1$, and use Claim

4.6 to get rid of the i dependence in the sum:

$$Vol - Vol' = (N-3)\sum_{i=1}^{N}\sum_{k=1}^{n_i}(V[ij_{ik}][j_{i(k+1)}R][Q'Q] + V[ij_{ik}][Rj_{i(k-1)}][Q'Q]) \quad (4.49)$$

$$= (N-3)\sum_{i=1}^{N}\sum_{k=1}^{n_i}(V[ij_{i(k-1)}][j_{ik}R][Q'Q] + V[ij_{i(k+1)}][Rj_{ik}][Q'Q]) \quad (4.50)$$

$$= (N-3)\sum_{i=1}^{N}\sum_{k=1}^{n_i}V[j_{i(k+1)}j_{i(k-1)}][j_{ik}R][Q'Q]. \quad (4.51)$$

Now we must use the fact that we are dealing with a collection of lists that define an abstract listed 3-polytope, as opposed to an arbitrary collection of lists. Namely, we realize that in (4.51) we are really summing over all oriented edges $[j_{i(k-1)}(ij_{ik})j_{i(k+1)}]$. Additionally, the additive structure of terms of the form $V[j_{i(k+1)}j_{i(k-1)}][j_{ik}R][Q'Q]$ for a given value of $j_{ik} = s \in S$ is the same as the additive structure of the oriented edges in $E_{i;j_{ik}}$. Thus, if a sum of edges cancel within some $E_{i;j_{ik}}$, then their corresponding terms cancel in (4.51). We say that such a collection of edges cancel "within" the same $E_{i;j_{ik}}$. However, since we have a 3-polytope we know that for each i and j_{ik}, $E_{i;j_{ik}} = -E_{j_{ik};i}$. Thus, the edges that do not cancel within $E_{i;j_{ik}}$ are in one-to-one correspondence with those edges that do not cancel within $E_{j_{ik};i}$, and moreover this one-to-one correspondence switches the orientation of each edge (i.e., swaps starting and ending points). Since the terms that do cancel within any given $E_{i;j_{ik}}$ do not contribute to the second sum in (4.51), we see that we can rewrite (4.51) with $j_{ik} \to i$, $j_{i(k-1)} \to j_{i(k+1)}$, and $j_{i(k+1)} \to j_{i(k-1)}$. This is nothing but choosing a different (but equivalent) way to sum over the edges, so to speak. We thus have

$$Vol - Vol' = (N-3)\sum_{i=1}^{N}\sum_{k=1}^{n_i}V[j_{i(k-1)}j_{i(k+1)}][iR][Q'Q] \quad (4.52)$$

$$= 0 \quad (4.53)$$

where the final equality comes from the fact that each $l_i = (j_{i1}, ..., j_{in_i})$ is cyclic. Namely, if n_i is odd then the sum can be viewed as being over the cyclic list

$$(j_{i1}, j_{i3}, ..., j_{in_i}, j_{i2}, j_{i4}, ..., j_{i(n_i-1)}),$$

and if n_i is even then the sum can be viewed as being over the two cyclic lists $(j_{i1}, j_{i3}, ..., j_{i(n_i-1)}), (j_{i2}, j_{i4}, ..., j_{in_i})$. In both cases the sum on k identically vanishes.□

Now that we have shown that the particular combination of the f_{ijk} objects defined

by Vol is independent of our reference boundaries, we need to see how Vol relates to the volume of the abstract listed 3-polytope defined by the lists $\{l_i\}$. The following theorem does this, thus introducing the f_{ijk} calculus for combinatorial 3-polytopes.

Theorem 4.26. *Let $S = \{1, ..., N\}$ and $\{l_i = (j_{i1}, ..., j_{in_i})\}$ define an abstract listed 3-polytope as in Definition 2.16, and let $A(\{l_i\})$ be the volume of this abstract listed 3-polytope as defined in Definition 3.7, using some reference boundary $B^\alpha \in \mathbb{CP}^3$. Let $S' = S \cup \{B^\alpha\}$ and let $\{f_{ijk}\}$ be defined as in Definition 4.17 with respect to the set of boundaries S'. Then*

$$A(\{l_i\}) = \frac{1}{18} \sum_{i=1}^{N} \sum_{k=1}^{n_i} f_{ij_{ik}j_{i(k+1)}} \tag{4.54}$$

where the sum on k (but not on i) is cyclic in the usual sense.

Proof:

$$\frac{1}{18} \sum_{i=1}^{N} \sum_{k=1}^{n_i} f_{ij_{ik}j_{i(k+1)}} = \frac{1}{18} \sum_{i=1}^{N} \sum_{k=1}^{n_i} (f_{j_{i(k+1)}ij_{ik}} - f_{ij_{ik}B} - f_{iBj_{ik}}) \tag{4.55}$$

$$= \frac{1}{18} \sum_{i=1}^{N} \sum_{k=1}^{n_i} (6[j_{i(k+1)}ij_{ik}B] - f_{j_{ik}Bj_{i(k+1)}} + f_{Bj_{i(k+1)}i} - f_{iBj_{ik}}) \tag{4.56}$$

$$= \frac{1}{18} \sum_{i=1}^{N} \sum_{k=1}^{n_i} (6[j_{i(k+1)}ij_{ik}B] - f_{j_{ik}Bj_{i(k+1)}} + f_{Bj_{ik}i} - f_{iBj_{ik}}) \tag{4.57}$$

$$= \frac{1}{18} \sum_{i=1}^{N} \sum_{k=1}^{n_i} (6[j_{i(k+1)}ij_{ik}B] + f_{Bj_{ik}j_{i(k+1)}}) \tag{4.58}$$

$$= \frac{1}{3} \sum_{i=1}^{N} \sum_{k=1}^{n_i} [ij_{ik}j_{i(k+1)}B] + \frac{1}{18} \sum_{i=1}^{N} \sum_{k=1}^{n_i} f_{Bj_{ik}j_{i(k+1)}}. \tag{4.59}$$

Thus we only need to show that this second sum vanishes for any 3-polytope. We note that by simply changing the k index and introducing a factor of $\frac{1}{2}$ we can write

$$\sum_{i=1}^{N} \sum_{k=1}^{n_i} f_{Bj_{ik}j_{i(k+1)}} = \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{n_i} (f_{Bj_{i(k-1)}j_{ik}} + f_{Bj_{ik}j_{i(k+1)}}), \tag{4.60}$$

and just as in (4.51) we can view this sum as being a sum over the oriented edges defining the abstract listed 3-polytope (since the information $i, j_{i(k-1)}, j_{ik}, j_{i(k+1)}$ defines an edge), where the term in parentheses has the same additive structure as its

corresponding oriented edge. Similarly to (4.51), then, we can make the replacements $j_{ik} \to i$, $j_{i(k-1)} \to j_{i(k+1)}$, and $j_{i(k+1)} \to j_{i(k-1)}$. The sum can therefore be rearranged to give

$$\sum_{i=1}^{N}\sum_{k=1}^{n_i} f_{Bj_{ik}j_{i(k+1)}} = \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{n_i}(f_{Bj_{i(k-1)}i} + f_{Bij_{i(k+1)}}) \tag{4.61}$$

$$= 0 \tag{4.62}$$

after relabeling the k index again. $\square$

The same remarks can be made here about our lack of necessity to extend the boundary set S to S'. As we will see, using this f_{ijk} calculus does not require us to actually explore the "internal structure" of the f_{ijk}'s. Rather, we simply use Claim 4.18 to simplify our sums of f_{ijk}'s, and this result is independent of our set of boundaries (namely, in that claim, N can be arbitrarily large and the same result will hold). Thus, as in the two-dimensional case, we can freely extend our set of boundaries however we need. Let us now use this result to calculate the volumes of the abstract listed 3-polytopes defined in Examples 2.18 and 2.19.

Example 4.27. In Example 2.18 we introduced the abstract listed 3-polytope defined by the lists

$$\begin{aligned} l_1 &= (4, 6, 3, 5) \\ l_2 &= (5, 3, 6, 4) \\ l_3 &= (1, 6, 2, 5) \\ l_4 &= (5, 2, 6, 1) \\ l_5 &= (1, 3, 2, 4) \\ l_6 &= (4, 2, 3, 1). \end{aligned} \tag{4.63}$$

Theorem 4.26 then tells us that (noting that $n_i = 4$ for each i)

$$\begin{aligned}
A =& \frac{1}{18}\sum_{i=1}^{6}\sum_{k=1}^{4} f_{ij_{ik}j_{i(k+1)}} \\
=& \frac{1}{18}(f_{146} + f_{163} + f_{135} + f_{154} \\
&+ f_{253} + f_{236} + f_{264} + f_{245} \\
&+ f_{316} + f_{362} + f_{325} + f_{351} \\
&+ f_{452} + f_{426} + f_{461} + f_{415} \\
&+ f_{513} + f_{532} + f_{524} + f_{541} \\
&+ f_{642} + f_{623} + f_{631} + f_{614}) \\
=& \frac{1}{6}((f_{135} - f_{352}) - (f_{136} - f_{362}) - (f_{145} - f_{452}) + (f_{146} - f_{462})) \\
=& \frac{1}{6}((6[1352] - f_{521} + f_{213}) - (6[1362] - f_{621} + f_{213}) \\
&- (6[1452] - f_{521} + f_{214}) + (6[1462] - f_{621} + f_{214})) \\
=& [1235] - [1236] - [1245] + [1246] = V[12][34][56]
\end{aligned}$$

which we know from the discussion following Definition 4.4 to be the desired result.

Example 4.28. In Example 2.19 we introduced the abstract listed 3-polytope defined by the lists

$$\begin{aligned}
l_1 &= (2, 3, 4, ..., (N-1), N) \\
l_2 &= (N, (N-1), ..., 4, 3, 1) \\
l_3 &= (1, 2, 4) \\
l_s &= (1, s-1, 2, s+1), \quad 4 \leq s \leq N-1 \\
l_N &= (1, N-1, 2).
\end{aligned} \tag{4.64}$$

Theorem 4.26 then tells us that

$$\begin{aligned}
A =& \frac{1}{18}\sum_{i=1}^{N}\sum_{k=1}^{n_i} f_{ij_{ik}j_{i(k+1)}} \\
=& \frac{1}{18}\Big(\sum_{i=3}^{N-1}(f_{1i(i+1)} - f_{2i(i+1)}) + f_{1N2} + f_{123} + f_{21N} + f_{231} \\
&+ f_{312} + f_{324} + f_{341} + \sum_{s=4}^{N-1}(f_{s1(s-1)} + f_{s(s-1)2} + f_{s2(s+1)} + f_{s(s+1)1}) \\
&+ f_{N1(N-1)} + f_{N(N-1)2} + f_{N21}\Big) \\
=& \frac{1}{18}\Big(\sum_{i=3}^{N-1}(f_{1i(i+1)} - f_{2i(i+1)}) + 3f_{1N2} + 3f_{123} \\
&+ f_{324} + f_{341} + \sum_{s=3}^{N-2}(f_{(s+1)1s} + f_{(s+1)s2} + f_{(s+1)2(s+2)} + f_{(s+1)(s+2)1}) \\
&+ f_{N1(N-1)} + f_{N(N-1)2}\Big) \\
=& \frac{1}{18}\Big(\sum_{i=3}^{N-1}(f_{1i(i+1)} - f_{2i(i+1)}) + 3f_{1N2} + 3f_{123} \\
&+ \sum_{s=3}^{N-2}(2f_{(s+1)1s} + 2f_{(s+1)s2}) + 2f_{(N-1)2N} + 2f_{(N-1)N1}\Big) \\
=& \frac{1}{18}\Big(\sum_{i=3}^{N-1}(3f_{1i(i+1)} - 3f_{i(i+1)2}) + 3f_{1N2} + 3f_{123}\Big) \\
=& \frac{1}{18}\Big(\sum_{i=3}^{N-1}(18[1i(i+1)2] - 3f_{(i+1)21} + 3f_{21i}) + 3f_{1N2} + 3f_{123}\Big) \\
&= \sum_{i=3}^{N-1}[12i(i+1)]
\end{aligned}$$

where throughout we need to recall that the sums on i and s are **not** cyclic and so we pick up remainder terms when we shift indices. This result is in line with the known result from (27) of [3] with the appropriate relabelling of the boundaries.

The analogous statements can be said in the three-dimensional case, as in the two-dimensional case, about testing the equality of different triangulations and recovering any desired triangulation from the sum (4.54). Since we will take a close look at this process for a particular abstract listed 4-polytope in the next section, and since the discussion is precisely the same in four dimensions, we will omit it from the three-dimensional case and simply turn our attention to the four-dimensional calculus.

4.4 The Four-Dimensional Calculus

As before, we first introduce the specific combination of f_{ijkl} objects that we will be considering and show that this combination is independent of the Q^α reference boundary that is used to define the individual f_{ijkl}'s. Claim 4.13 shows that each individual f_{ijkl} is independent of the P_1^α and P_2^α that defines it, so the following proposition shows that the sums we will be interested in are completely independent of all reference boundaries.

Proposition 4.29. Let $S = \{1, ..., N\}$ be a set of N boundaries in $\mathbb{CP}^4$ in the usual sense, let $P_1^\alpha, P_2^\alpha, Q^\alpha \in \mathbb{CP}^4$ be fixed reference boundaries, and let $\{f_{ijkl}\}$ be defined with respect to $S, P_1^\alpha, P_2^\alpha$, and Q^α as in Definition 4.19. Let $\{l_{ij} = (k_{ij1}, ..., k_{ijn_{ij}})\}$ be a collection of cyclic lists with each $k_{ijl} \in S$ that defines an abstract listed 4-polytope. Let $Q'^\alpha \in \mathbb{CP}^4$ be another choice of reference boundary and let f'_{ijkl} be defined as f_{ijkl} with Q' replacing Q. Let $Vol \equiv \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^{n_{ij}} f_{ijk_{ijl}k_{ij(l+1)}}$ and $Vol' \equiv \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^{n_{ij}} f'_{ijk_{ijl}k_{ij(l+1)}}$, where the sums on l (but not on i or j) are cyclic, i.e. $k_{ij(n_i+1)} = k_{ij1}$. Then $Vol = Vol'$.

Proof:

$$
\begin{aligned}
Vol - Vol' &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}(f_{ijk_{ijl}k_{ij(l+1)}} - f'_{ijk_{ijl}k_{ij(l+1)}}) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}\sum_{m\neq i,j,k_{ijl},k_{ij(l+1)}}^{N}(V[\{ij][k)ijl|P_1|][k_{ij(l+1)}\}P_2][mQ] - (Q \to Q')) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}\sum_{m\neq i,j,k_{ijl},k_{ij(l+1)}}^{N}(6[ijk_{ijl}k_{ij(l+1)}m] - 6[ijk_{ij}k_{ij(l+1)}Q] - (Q \to Q')) \\
&= 6(N-4)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}([ijk_{ijl}k_{ij(l+1)}Q'] - [ijk_{ijl}k_{ij(l+1)}Q]) \\
&= 3(N-4)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}([ijk_{ijl}k_{ij(l+1)}Q'] - [ijk_{ijl}k_{ij(l+1)}Q] \\
&\qquad + [ijk_{ij(l-1)}k_{ijl}Q'] - [ijk_{ij(l-1)}k_{ijl}Q])
\end{aligned}
$$

Now for each of the four terms in this triple sum we introduce the reference boundary

S^α and employ the identity (4.3). We get some cancellations and find

$$Vol - Vol' = 3(N-4)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}\Big(V[Sk_{ijk}][ij][k_{ij(l+1)}k_{ij(l-1)}][R'R] - [k_{ij(l+1)}ijRS] + [k_{ij(l-1)}ijRS]\Big) \tag{4.65}$$

$$= 3(N-4)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}} V[k_{ij(l-1)}k_{ij(l+1)}][k_{ijk}S][ij][R'R] \tag{4.66}$$

since the last two terms in the first equality cancel in the cyclic sum. Now, for each i, (4.66) can be viewed as a sum over the edges restricted to the i^{th} face. Since we have assumed that we are dealing with an abstract listed 4-polytope (as opposed to an arbitrary collection of cyclic lists), we know that the edges restricted to any fixed i form an abstract listed 3-polytope. Thus, just as in the proof of Proposition 4.25, we can make the replacement $j \leftrightarrow k_{ijl}$, $k_{ij(l-1)} \leftrightarrow k_{ij(l+1)}$, which is nothing but choosing a different way to sum over all the edges of the abstract listed 3-polytope P_i (the restriction of the 4-polytope P to its i^{th} face). We thus have

$$Vol - Vol' = 3(N-4)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}} V[k_{ij(l+1)}k_{ij(l-1)}][jS][ik_{ijl}][R'R]. \tag{4.67}$$

We then introduce another reference boundary T^α, relabel the l indices, and use Claim 4.9 to make the i-dependence in (4.67) drop out.

$$Vol - Vol' = 3(N-4)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}(V[k_{ij(l+1)}T][jS][ik_{ijl}][R'R] + V[Tk_{ij(l-1)}][jS][ik_{ijl}][R'R]) \tag{4.68}$$

$$= 3(N-4)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}(V[k_{ijl}T][jS][ik_{ij(l-1)}][R'R] + V[Tk_{ijl}][jS][ik_{ij(l+1)}][R'R]) \tag{4.69}$$

$$= 3(N-4)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}} V[k_{ij(l+1)}k_{ij(l-1)}][k_{ijl}T][jS][R'R]. \tag{4.70}$$

Now since $E_{ij} = -E_{ji}$, we can make the replacements $j \to i$ and $k_{ij(l-1)} \leftrightarrow k_{ij(l+1)}$ in the sum, using the same reasoning as in the three-dimensional case about the one-to-one correspondence between edges that don't cancel "within" their respective edge sets, to get

$$Vol - Vol' = 3(N-4)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}} V[k_{ij(l-1)}k_{ij(l+1)}][k_{ijl}T][iS][R'R]. \tag{4.71}$$

Finally, we use again the fact that for each i, P_i is an abstract listed 3-polytope to make the same replacement $k_{ijl} \to j$, $k_{ij(l-1)} \leftrightarrow k_{ij(l+1)}$ to find

$$Vol - Vol' = 3(N-4)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}} V[k_{ij(l-1)}k_{ij(l+1)}][iS][jT][R'R] \tag{4.72}$$

$$= 0 \tag{4.73}$$

where the final equality comes from the cyclicity of each l_{ij}. Thus Vol is independent of our choice of Q^α. We note that this proof exactly mirrors the proof of the analogous statements in two and three dimensions, and that we explicitly used both the $(d-1)$-dimensional constraints that $E_{ij;k} = -E_{ik;j}$ for all i, j, k and the $(d-2)$-dimensional constraints that $E_{ij} = -E_{ji}$. $\square$

The following theorem introduces the four-dimensional f_{ijkl} calculus.

Theorem 4.30. *Let $S = \{1, ..., N\}$ and $\{l_{ij} = (j_{ij1}, ..., k_{ijn_{ij}})\}$ define an abstract listed 4-polytope as in Definition 2.23, and let $A(\{l_{ij}\})$ be the volume of this abstract listed 4-polytope as defined in Definition 3.8, using some reference boundary $B^\alpha \in \mathbb{CP}^4$. Let $S' = S \cup \{B^\alpha\}$ and let $\{f_{ijkl}\}$ be defined as in Definition 4.19 with respect to the set of boundaries S'. Then*

$$A(\{l_{ij}\}) = \frac{2}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}} f_{ijk_{ijl}k_{ij(l+1)}} \tag{4.74}$$

where the sum on l (but not on i or j) is cyclic in the usual sense.

Proof:

$$\frac{2}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}} f_{ijk_{ijl}k_{ij(l+1)}} = \frac{2}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}(f_{ijk_{ijl}k_{ij(l+1)}} + f_{jk_{ijl}k_{ij(l+1)}B} - f_{jk_{ijl}k_{ij(l+1)}B}) \tag{4.75}$$

$$= \frac{2}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}(4![ijk_{ijl}k_{ij(l+1)}B] - f_{k_{ijl}k_{ij(l+1)}Bi} - f_{k_{ij(l+1)}Bij} - f_{Bijk_{ijl}} - f_{jk_{ijl}k_{ij(l+1)B}}) \tag{4.76}$$

$$= \frac{2}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}(4![ijk_{ijl}k_{ij(l+1)}B] - f_{k_{ijl}k_{ij(l+1)}Bi} - f_{jk_{ijl}k_{ij(l+1)B}})$$

where the last equality comes from the fact that the third and fourth terms in (4.76) cancel in the cyclic sum. Thus we have

$$\frac{2}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}} f_{ijk_{ijl}k_{ij(l+1)}} = A(\{l_{ij}\}) + \frac{2}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}(f_{ik_{ijl}k_{ij(l+1)}B} - f_{jk_{ijl}k_{ij(l+1)}B}) \tag{4.77}$$

$$= A(\{l_{ij}\}) + \frac{1}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}((f_{ik_{ijl}k_{ij(l+1)}B} + f_{ik_{ij(l-1)}k_{ijl}B}) \tag{4.78}$$

$$- (f_{jk_{ijl}k_{ij(l+1)B}} + f_{jk_{ij(l-1)}k_{ijlB}})).$$

We now focus on

$$\frac{1}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}((f_{ik_{ijl}k_{ij(l+1)}B} + f_{ik_{ij(l-1)}k_{ijl}B})$$

and make the previously discussed substitution $k_{ijl} \to j$, $k_{ij(l-1)} \leftrightarrow k_{ij(l+1)}$, due to the additive structure of the summand, to find

$$\frac{1}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}((f_{ik_{ijl}k_{ij(l+1)}B} + f_{ik_{ij(l-1)}k_{ijl}B}) = \frac{1}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}((f_{ijk_{ij(l-1)}B} - f_{ijk_{ij(l+1)}B}) \tag{4.79}$$

$$= 0$$

in the cyclic sum. Now we focus on

$$\frac{1}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}(f_{jk_{ijl}k_{ij(l+1)B}} + f_{jk_{ij(l-1)}k_{ijlB}})$$

and make the substitution (for the same reasons) $j \to i$, $k_{ij(l-1)} \leftrightarrow k_{ij(l+1)}$ to get

$$\frac{1}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}(f_{jk_{ijl}k_{ij(l+1)B}} + f_{jk_{ij(l-1)}k_{ijlB}}) = \frac{1}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}}(f_{ik_{ijl}k_{ij(l-1)B}} + f_{ik_{ij(l+1)}k_{ijlB}}) \tag{4.80}$$

$$= 0$$

by (4.79).

Thus from (4.78), we have

$$\frac{2}{(4!)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{n_{ij}} f_{ijk_{ijl}k_{ij(l+1)}} = A(\{l_{ij}\}). \tag{4.81}$$

□

We have therefore proved abstractly that our $f_{...}$ calculus extends to four dimensions. However, since an abstract listed 4-polytope defined by N boundaries requires specifying N^2 cyclic lists, we omit an explicit example here and leave the example of calculating the volume of a four-simplex to Appendix B. Instead, we will focus on a different case of an abstract listed 4-polytope. In particular, we can now return to the expressions that motivated this whole discussion in the first place: the NMHV tree amplitude in the planar limit of $N = 4$ Super Yang-Mills theory. The following subsection generalizes the discussion of Chapter 1 and introduces how our new $f_{...}$ objects can be used to find nice expressions for the volume of this polytope.

4.4.1 The Cyclic Polytope

In chapter 1 we gave two different expressions for the $n = 6$ point NMHV amplitude—expressions (1.1) and (1.4). It is known [5] that the integrand M_n^{NMHV} of the n-point NMHV scattering amplitude is[2]

$$M_n^{NMHV} = \frac{1}{2}\sum_{i,j=1}^{n}[*i(i+1)j(j+1)] \tag{4.82}$$

where $*$ denotes some reference boundary and where the sum on both i and j is cyclic, so that $n + 1 = 1$. Expressions (1.1) and (1.4) can be readily recovered by letting $n = 6$ and $* = 1$ and $* = 2$, respectively. We call M_n^{NMHV} the cyclic polytope due to its cyclic dependence on $i, i + 1, j, j + 1$. It can be shown that M_n^{NMHV} is independent of our choice of $*$, however the expression (4.82) does not make this manifest. Equivalently, we could choose $*$ to be, say, 1, thus getting a manifestly $*$-independent expression, but then we lose the manifestly cyclic structure which gives M_n^{NMHV} its name. So as things stand, we have to make a compromise between manifest $*$-independence and manifest cyclicity. Using our calculus, however, there is

[2]Here, we are dealing with the purely bosonic brackets [...], relying on (1.6) to take us back to the expressions involving fermionic variables.

no need for a compromise, due to the following. Let us write $0 = *$. Then

$$M_n^{NMHV} = \frac{1}{2}\sum_{i,j=1}^{n}[0i(i+1)j(j+1)] \tag{4.83}$$

$$= \frac{1}{2\cdot 4!}\sum_{i,j=1}^{n}(f_{0i(i+1)j} + f_{i(i+1)j(j+1)} + f_{(i+1)j(j+1)0} + f_{j(j+1)0i} + f_{(j+1)0i(i+1)}) \tag{4.84}$$

$$= \frac{1}{2\cdot 4!}\sum_{i,j=1}^{n} f_{i(i+1)j(j+1)} \tag{4.85}$$

since any term with a 0 subscript cancels after relabelling of the i's and j's, which we are permitted to do due to the cyclicity of the sum. Thus the dependence on the reference boundary 0 manifestly drops out, while the fact that the cyclic polytope is defined by vertices of the form $\{i, i+1, j, j+1\}$ remains manifest. This last point can be seen by noticing that the $f_{...}$'s that appear in the expression of a volume (when fully simplified) have as subscripts the labels of genuine (i.e., not spurious) vertices. We note that the reference boundary 0 is very different in nature than the reference boundary Q^α used to define the $f_{...}$'s. The latter is "hidden" within each $f_{...}$, and we showed that any volume is independent of its choice. The boundary 0, however, is originally introduced to define the cyclic polytope itself. Thus, the independence of 0 that M_n^{NMHV} enjoys is a non-trivial fact made manifest with the new calculus.

As we have seen, the $f_{...}$ calculus in any dimension encodes the information about any triangulation of the polytope via the different ways of using Claim 4.20 (and its lower dimensional analogues). Let us look at this explicitly for the cyclic polytope with $n = 6$. Let us simplify (4.85) with $n = 6$. We have

$$M_6^{NMHV} = \frac{1}{2\cdot 4!}\sum_{i,j=1}^{6} f_{i(i+1)j(j+1)} \tag{4.86}$$

$$= \frac{1}{4!}(f_{1234} + f_{1245} + f_{1256} + f_{2345} + f_{2356} \tag{4.87}$$

$$+ f_{2361} + f_{3456} + f_{3461} + f_{4561}) \tag{4.88}$$

$$= \frac{1}{4!}((f_{1234} + f_{2345}) + (f_{2356} + f_{5612} + f_{6123}) \tag{4.89}$$

$$+ (f_{3456} + f_{4561} + f_{6134}) + f_{1245}) \tag{4.90}$$

$$= \frac{1}{4!}((4![12345] - f_{3451} - f_{4512} - f_{5123}) + (4![23561] - f_{3561} - f_{1235}) \tag{4.91}$$

$$+ (4![34561] - f_{5613} - f_{1345}) + f_{1245}) \tag{4.92}$$

$$= [12345] + [12356] + [13456], \tag{4.93}$$

which agrees with the choice $0 = 1$ leading to (1.1). However, in (4.89) we could have chosen to lump the $f_{...}$'s together differently to obtain

$$\begin{aligned} M_6^{NMHV} =& \frac{1}{4!}((f_{2136} + f_{1364}) + (f_{2154} + f_{1546} + f_{6215}) && (4.94)\\ &+ (f_{2345} + f_{3456} + f_{5623}) + f_{2143}) && (4.95)\\ =& \frac{1}{4!}(4![21364] - f_{3642} - f_{6421} - f_{4213}) + (4![21546] - f_{5462} - f_{4621}) && (4.96)\\ &+ (4![23456] - f_{4562} - f_{6234}) + f_{2143}) && (4.97)\\ =& [23456] + [23461] + [24561], && (4.98) \end{aligned}$$

and this corresponds to making the choice $0 = 2$, leading to (1.4). It is readily checked that any choice of $0 = i$ for $1 \leq i \leq 6$ can be recovered from (4.85) in a similar way. Thus one way to think of (4.85) is as encoding all possible choices of 0 in a cyclic manner. This is still an incomplete picture, however, since **any** triangulation of M_n^{NMHV}—and not just those obtained from different choices of 0—can be recovered from (4.85) by simply introducing fictitious boundaries analogously to what we did in (4.41).

In short, our calculus makes the $*$-independence of M_n^{NMHV} manifest while maintaining its manifest cyclic structure, thus allowing us to have our cake and eat it too, so to speak.

We have now been able to answer the questions that started us down this path in Chapter 1. We have successfully defined the spaces with which we are working as well as their volumes, and we have found a very useful way to calculate these volumes and relate the various representations of these volumes. However, there is one important piece of the story that has yet to be discussed, and that is the mathematical motivation for making the various definitions that we have made. This will be the subject of the next chapter, but before getting there we must introduce one very important aspect of the calculus that we have introduced. Namely, with things phrased as they are, we can readily obtain the volumes not only of the abstract listed polytope that we are given, but also the volumes of the polytopes that lie on their lower-dimensional faces. The following section discusses this process and will lead us naturally into the discussion of the following chapter.

4.5 Going to the Boundary

As we saw in Chapter 2, an abstract listed d-polytope P is defined in such a way that by restricting to, say, the I^{th} face, we are guaranteed to obtain a $(d-1)$-polytope

denoted by P_I. More generally, by restricting the polytope P first to the I_1^{th} face, then to the I_2^{th} face, and so on, and finally to the I_p^{th} face, we obtain a $(d-p)$-polytope denoted by $P_{I_1,...,I_p}$. If $\{l_{i_1,...,i_{(d-2)}}\}$ are the lists that define the d-polytope P, then the lists $\{l_{I_1,...,I_p,i_{(p+1)},...,i_{(d-2)}}\}$ are the lists that define the $(d-p)$-polytope $P_{I_1,...,I_p}$. It is therefore a natural question to ask whether or not we can obtain the volumes of these lower dimensional boundary polytopes using our calculus, and indeed we can answer this question positively.

Consider a 3-polytope with boundaries $S = \{1, ..., N\}$ and cyclic lists $\{l_i\}$. We saw above that the volume A of a 3-simplex with boundaries i, j, k, l is given by

$$A = \frac{1}{6} \frac{\langle ijkl \rangle^3}{\langle ijkP_0 \rangle \langle jklP_0 \rangle \langle kliP_0 \rangle \langle lijP_0 \rangle} \tag{4.99}$$

where A is homogeneously well-defined in i, j, k, l but does depend on the scaling of the "boundary at infinity" P_0^α. Accordingly, if we want to ask about the area of one of the 3-polytope's 2-faces, namely that 2-face that resides on the restriction of the 3-polytope to the boundary $I \in S$, then we expect the area of this 2-polytope to depend on the scaling of I. This is because the scaling of I determines the scaling of the affine patch $\mathbb{R}^2$ of the $\mathbb{RP}^2$ corresponding to Z_I^α in the same way that the scaling of P_0 determined the scaling of the affine patch $\mathbb{R}^3$ of $\mathbb{RP}^3$ (or with $\mathbb{R} \leftrightarrow \mathbb{C}$). We note, however, that we cannot write the area of a triangle (2-simplex) bounded by i, j, k restricted to the I^{th} face as

$$A_I = \frac{1}{2} \frac{\langle ijk \rangle^2}{\langle ijI \rangle \langle jkI \rangle \langle kiI \rangle}$$

for two reasons. First, this area no longer depends on the scaling of P_0^α, which we know must still be the case. Second and more seriously we see that the contraction of indices is ill-defined, since each boundary $i \in S$ corresponds to an object Z_i^α with four homogeneous coordinates. We therefore readily find the correct expression for the 2-simplex bounded by $i, j, k \in S$ (all not equal to I) restricted to the I^{th} face to be

$$A_I \equiv \frac{1}{2} \frac{\langle ijkI \rangle^2}{\langle ijIP \rangle \langle jkIP \rangle \langle kIP \rangle} \equiv [ijk]_I. \tag{4.100}$$

We note that A_I scales with degree -3 in P_0 and with degree -1 in I and so is dependent on both boundaries' scaling, and involves the correct index contractions, as desired. Thus, when the boundaries i, j, k all intersect in the same affine patch and when we choose the scaling of I and P_0 properly, we recover the usual Euclidean area of the triangle.

One can then show that all of the identities amongst the $[ijk]$ derived in Chapter

3 apply just as well to $[ijk]_I$ for any $I \in S$. Thus, the entire two-dimensional calculus can be carried directly over to the I^{th} boundary of our 3-polytope by defining the objects

$$f_{ij}^I \equiv \sum_{k \neq i,j,I}^{N} V_I[ij][kQ] \tag{4.101}$$

where $V_I[ij][kl] \equiv [ijk]_I - [ijl]_I$ in the obvious way. The key identity

$$f_{ij}^I + f_{jk}^I + f_{ki}^I = 2[ijk]_I \tag{4.102}$$

carries right over as well.

Thus, to get the volume of the polygon restricted to the I^{th} face of our polytope, we simply use the list $l_I = (j_{Ii}, ..., j_{In_I})$ given to us from the information of the 3-polytope. Explicitly, the area $A_I(l_I)$ of this polygon is

$$A_I(l_I) = \frac{1}{2} \sum_{k=1}^{n_I} f^I_{j_{Ik} j_{I(k+1)}} \tag{4.103}$$

with the sum on k being cyclic.

Example 4.31. Consider the polygon sitting on the 1 boundary in Example 2.18. The cyclic list of interest is $l_1 = (4635)$ and so we find the area $A_1(l_1)$ to be

$$\begin{aligned} A_1(l_1) &= \frac{1}{2} \sum_{k=1}^{4} f^1_{j_{1k} j_{1(k+1)}} && (4.104)\\ &= \frac{1}{2}(f^1_{46} + f^1_{63} + f^1_{35} + f^1_{54}) && (4.105)\\ &= \frac{1}{2}(2[463]_1 - f^1_{34} + 2[354]_1 - f^1_{43}) && (4.106)\\ &= [346]_1 - [346]_1 = V_1[34][65]. && (4.107) \end{aligned}$$

which we know from Chapter 3 to be the correct result.

Going to the boundary of a 4-polytope is exactly the same in our case as doing so for a 3-polytope, only now we have both 3-polytopes and 2-polytopes as boundaries. In what follows, we let $S = \{1, ..., N\}$ be a set of N boundaries, $\{l_{ij} = (k_{iji}, ..., k_{ijn_{ij}})\}$ be a set of N^2 cyclic lists defining a 4-polytope P, P_i be the 3-polytope that lies on the i^{th} boundary of P, and P_{ij} be the 2-polytope obtained by restricting first to P_i and then to P_i's j^{th} boundary. We note from our definition of abstract listed 4-polytopes that the volume of P_{ij} should be the negative of the volume of P_{ji}.

The volume A_I of a 3-simplex restricted to the I^{th} face of P bounded by boundaries $i, j, k, l \in S - \{I\}$ is given by

$$A_I \equiv \frac{1}{6} \frac{\langle ijklI\rangle^3}{\langle ijkIP\rangle\langle jklIP\rangle\langle kliIP\rangle\langle lijIP\rangle} \equiv [ijkl]_I \tag{4.108}$$

and the volume A_{IJ} of a 2-simplex restricted to the J^{th} face of P_I, and bounded by $i, j, k \in S - \{I, J\}$ is

$$A_{IJ} \equiv \frac{1}{2} \frac{\langle ijkJI\rangle^2}{\langle ijJIP\rangle\langle jkJIP\rangle\langle kiJIP\rangle} \equiv [ijk]_{IJ}. \tag{4.109}$$

We note that $A_{IJ} = -A_{JI}$, A_I depends on the scaling of I and P, and that A_{IJ} depends on the scaling of I, J, and P, all as desired. As before, it can be shown that $[ijkl]_I$ and $[ijk]_{IJ}$ both satisfy all the same identities as $[ijkl]$ and $[ijk]$, and thus the usual three- and two-dimensional calculi can be defined. Namely, we can define the objects f^I_{ijk} and f^{IJ}_{ij} for any boundaries $i, j, k \in S - \{I\}$ (and $i, j \in S - \{I, J\}$) analogously to (4.101) and carry through the calculus of the relevant dimension as usual. Thus, if we want the volume A_I of the 3-polytope P_I, we simply calculate

$$A_I = \frac{1}{18} \sum_{i=1}^{N} \sum_{l=1}^{n_{Ii}} f^I_{ik_{Iik}k_{Ii(l+1)}}. \tag{4.110}$$

Namely, we use the 3-dimensional calculus using the $\{f^I_{ijk}\}$ and the lists of the form l_{Ii}. Similarly, for the volume A_{IJ} of the 2-polytope P_{IJ}, we use the list l_{IJ} and the objects $\{f^{IJ}_{ij}\}$ to calculate

$$A_{IJ} = \frac{1}{2} \sum_{l=1}^{n_{IJ}} f^{IJ}_{k_{IJl}k_{IJ(l+1)}}. \tag{4.111}$$

4.5.1 Lists From Triangulations

One possible concern one might have for using the $f_{...}$ calculus that we have been describing is that it relies very heavily on knowing the cyclic lists defining our polytope. In particular, our discussion in Chapter 1 shows that BCFW recursion does not give us cyclic lists, but rather a particular triangulation of some polytope. Additionally, our previous discussion of calculating the volumes of polytopes that sit on the boundary of larger polytopes relies even more heavily on knowing the cyclic lists. As we have seen, it is very easy to calculate the volume of a bounding polytope when we know the cyclic lists of the larger polytope, but in general it is a very difficult problem to calculate the volume of a bounding polytope when we are just given a

particular triangulation.

For example, consider the triangulation of M_6^{NMHV} given in (1.1):

$$M_6^{NMHV} = [12345] + [12356] + [13456]. \tag{4.112}$$

By looking at (4.112) it is not at all obvious what the 3-polytope P_2 sitting on the 2^{nd} boundary of this 4-polytope even looks like, let alone what its volume is. Thus, it appears our calculus is pretty useless here, since BCFW recursion gives us expressions of the form (4.112), and not a tabulation of what the underlying cyclic lists are. Remarkably, however, for **any** sum of volumes of simplices $\sum[\cdot\ldots\cdot]$, one can recover the cyclic lists that correspond to the underlying polytope. Moreover, one can do so completely algorithmically, relying not at all on geometric or other insight. Let us see how this works for a simple example in three dimensions, since the same procedure will work in all dimensions and for arbitrarily complex sums of volumes of simplices.

Example 4.32. Consider once again the 3-polytope depicted by Figure 2.8 and defined by the lists (2.12). Let us denote the volume of this abstract listed 3-polytope by A and let us suppose for the moment that we do not actually know what the cyclic lists are. Rather, let us suppose we only know about one particular triangulation of this polytope's volume. Namely, let us suppose we know about the expression at the end of Example 4.27:

$$A = [1235] - [1236] - [1245] + [1246]. \tag{4.113}$$

We can use the key identity (4.22) to write this as

$$\begin{aligned} A =& \frac{1}{6}[f_{123} - f_{235} + f_{351} - f_{512} - f_{123} + f_{236} - f_{361} + f_{612} \\ & - f_{124} + f_{245} - f_{451} + f_{512} + f_{124} - f_{246} + f_{461} - f_{612}]. \end{aligned} \tag{4.114}$$

We can make a few immediate cancellations[3] and write

$$A = \frac{1}{6}[-f_{235} + f_{351} + f_{236} - f_{361} + f_{245} - f_{451} - f_{246} + f_{461}]. \tag{4.115}$$

Theorem 4.26 guarantees that A must be expressible as a sum of the form of (4.54), which is completely determined by certain cyclic lists, and vice versa. Thus, any terms in (4.115) with a 1 as a subscript must be coming from l_1, and the same goes

[3]We later comment on why even this step is not strictly necessary, but here we carry out these cancellations just for calculational simplicity

for all other subscripts. Let us therefore reconstruct l_1. We note first though that we will be constructing a particular **representative** of the equivalence class of l_1, and had we used (4.114) to reconstruct l_1 we would get a different representative belonging to the same equivalence class.

We collect the terms in (4.115) that contain a 1 as a subscript and we use the antisymmetry of the $f_{...}$'s to move the 1 to the left-most subscript, keeping track of the relative minus signs. The relevant terms are $\{f_{135}, -f_{136}, -f_{145}, f_{146}\}$. Using the remaining antisymmetry of the $f_{...}$ we rearrange the subscripts until all coefficients are positive, to get $\{f_{135}, f_{163}, f_{154}, f_{146}\}$. We then see that we can (as we always will be able to) form a cyclic list out of the last two subscripts of these objects, and one such list is $l_1 = (3546)$. Namely, it is clear that these four terms come from this cyclic list in the sum (4.54). We note that each f_{ijk} will appear in the sum (4.54) once for each list l_i, l_j, and l_k. This is where the normalizing factor comes from in (4.54). It is clear that this l_1 is in the same equivalence class as the l_1 in (2.12), as they are related by a cyclic permutation. The five other lists l_i with $2 \leq i \leq 6$ arise in the same way.

There is one subtlety in this process, and that is at the stage that we created l_1 from the terms $\{f_{135}, f_{163}, f_{154}, f_{146}\}$. The algorithm at this point was to start with f_{135} and place $3, 5$ in our list, so that our list reads $l_1 = (3, 5...)$. We then move on to an $f_{...}$ of the form f_{15k} and place k in our list so that $l_1 = (3, 5, k, ...)$. This process continues until we have "hit" all the $f_{...}$'s of the form $f_{1...}$. The subtlety is that there may not be a unique choice of "next" f_{15k}. This subtlety is easily seen not to be so subtle once we realize that it does not matter which f_{15k} we add to the list next. The only other subtlety then arises if our list "closes" before we have hit all of the $f_{1..}$'s. In this event, we simply start at any other $f_{1..}$ that we have not covered yet and start the process over, only considering the remaining $f_{1..}$'s. We then get our list l_1 as a sum of cyclic lists, one for each time the list had closed before we were done. Then $l_1 = l_{1a_1} + ... + l_{1a_n}$, if we covered the $f_{1..}$'s using n lists. However, thanks to Proposition 2.8, there is a canonical way to add these n lists to recover a single list l_1 in the correct equivalence class. We therefore see that there is indeed no subtlety or ambiguity in this process, and it is always possible to recover our (equivalence classes of) lists from any particular triangulation. One readily checks that this process produces equivalent lists for any choice of triangulation equal to that in (4.113).

4.5.2 Going to the Boundary of the Cyclic Polytope

Let us apply the discussion of the previous subsection to the polytope corresponding to M_6^{NMHV}, and then calculate the volume of the 3-polytope sitting on the 2^{nd} face. We start with (4.112) and then apply the key identity "in reverse" to get back to (4.87), which we reproduce here.

$$\begin{aligned} M_6^{NMHV} =& \frac{1}{4!}(f_{1234} + f_{1245} + f_{1256} + f_{2345} + f_{2356} \\ &+ f_{2361} + f_{3456} + f_{3461} + f_{4561}). \end{aligned} \tag{4.116}$$

We need to reproduce the five lists l_{2i} with $1 \leq i \leq 6,\ i \neq 2$. We do l_{21} completely here and then simply quote the results for the other four lists. We collect all the $f_{...}$ terms of the form $f_{21..}$ to get $\{f_{2143}, f_{2154}, f_{2165}, f_{2136}\}$ and see that $l_{21} = (4365)$. Altogether we find

$$\begin{aligned} l_{21} &= (4,3,6,5) \\ l_{23} &= (1,4,5,6) \\ l_{24} &= (3,1,5) \\ l_{25} &= (4,1,6,3) \\ l_{26} &= (5,1,3). \end{aligned} \tag{4.117}$$

Now, using the three-dimensional objects f^2_{ijk} corresponding to volumes of 3-polytopes restricted to the face defined by Z_2^α, our three-dimensional calculus tells us that the volume $A_2(\{l_{2i}\})$ of the 3-polytope P_2 is

$$A_2(\{l_{2i}\}) = \frac{1}{18}\sum_{i=1}^{6}\sum_{l=1}^{n_{2i}} f^2_{il(l+1)} \tag{4.118}$$

$$\begin{aligned} =& \frac{1}{18}\big(f^2_{143} + f^2_{136} + f^2_{165} + f^2_{154} \\ &+ f^2_{314} + f^2_{345} + f^2_{356} + f^2_{361} \\ &+ f^2_{431} + f^2_{415} + f^2_{453} \\ &+ f^2_{541} + f^2_{516} + f^2_{563} + f^2_{534} \\ &+ f^2_{651} + f^2_{613} + f^2_{635}\big) \\ =& \frac{1}{6}(f^2_{143} + f^2_{136} + f^2_{165} + f^2_{154} + f^2_{345} + f^2_{356}) \\ =& [1365]_2 - [1345]_2 \end{aligned} \tag{4.119}$$

after using (4.22). We note that this calculation is in line with the results of [5].

Let us now briefly summarize all the utility that this $f_{\dots}$ calculus provides. Most trivially, given the cylic lists defining a d-polytope, we can algorithmically obtain the volume of that polytope as well as the volume of any polytope lying on any of its $(d-p)$ dimensional faces. Additionally, we can algorithmically check the equality (or lack thereof) of any two sums of volumes of simplices. Perhaps most importantly, we can **start** with a particular triangulation and recover **all** of the information pertaining to the underlying polytope. Thus, given one triangulation of a polytope, we can algorithmically obtain the volumes of any of the polytopes lying on any of its lower dimensional faces.

Now that we have seen the various uses of the $f_{\dots}$ calculus and of phrasing things in terms of abstract listed polytopes defined by cyclic lists, we turn our attention in the next chapter to exploring the initial cohomological motivations for this whole story, as these may be the keys to various forms of generalizations of this calculus.

Chapter 5

Cohomological Origins of the Calculus

5.1 Elements of Twistor Theory

The cohomological origins of the calculus that we have spent the first four chapters introducing lie in twistor theory, and we will here assume an understanding of many of the main results in twistor theory and only give reference to those upon which we most heavily rely. This section will serve only to setup our notation and state the most relevant results of twistor theory.

In general, we use the conventions of [7]. Accordingly, we denote projective twistor space by $\mathbb{PT} \simeq \mathbb{CP}^3$ and dual projective twistor space by $\mathbb{PT}^* \simeq \mathbb{CP}^3$. We denote by $Z^\alpha = (Z^0, Z^1, Z^2, Z^3) \in \mathbb{CP}^3$ the homogenous coordinates of a projective twistor, and by $W_\alpha = (W_0, W_1, W_2, W_3) \in \mathbb{CP}^3$ the homogenous coordinates of a dual projective twistor. We use $W \cdot Z = W_\alpha Z^\alpha$ to represent the canonical inner product between twistors and dual twistors, and we interchangeably use "projective twistors" and "twistors" to mean the same thing, as we will have no use for non-projective twistors here. Given four twistors $Z_1^\alpha, Z_2^\alpha, Z_3^\alpha, Z_4^\alpha$, we use $\langle Z_1 Z_2 Z_3 Z_4 \rangle$ or simply $\langle 1234 \rangle$ to denote $\epsilon_{\alpha\beta\gamma\delta} Z_1^\alpha Z_2^\beta Z_3^\gamma Z_4^\delta$, with similar expressions for dual twistors. We also define the (normalized) projective volume form on twistor space as

$$DZ = \frac{1}{6} \epsilon_{\alpha\beta\gamma\delta} Z^\alpha \wedge dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta. \tag{5.1}$$

We now introduce some new notation for describing complex projective lines in (dual) twistor space. Suppose $Z_1^\alpha, Z_2^\alpha \in \mathbb{PT}$ are two distinct twistors. We denote the unique complex projective line joining these two points in $\mathbb{PT}$ by $L[Z_1, Z_2]$, so that $L[Z_1, Z_2] = \{Z^\alpha \in \mathbb{PT} | Z^\alpha = \sigma Z_1^\alpha + \tau Z_2^\alpha, (\sigma, \tau) \in \mathbb{C}^2 - (0,0)\} \subset \mathbb{PT}$. These two twistors also define a line in dual twistor space, via the intersection of the two $\mathbb{CP}^2$'s

defined by $\{W_\alpha \in \mathbb{PT}^* | W \cdot Z_i = 0\}$. We denote the linearly embedded $\mathbb{CP}^1 \subset \mathbb{PT}^*$ defined in this way by Z_1^α, Z_2^α, as $L^*[Z_1, Z_2] \subset \mathbb{PT}^*$. The same works if we replace the two twistors with two dual twistors $W_{1\alpha}, W_{2,\alpha} \in \mathbb{PT}^*$, so that we have $L[W_1, W_2] \subset \mathbb{PT}^*$ and $L^*[W_1, W_2] \subset \mathbb{PT}$ in the obvious way.

For completeness, we define the operation of complex conjugation on twistors, which lets us then cover twistor space with three disjoint regions. We define the complex conjugate $\bar{Z}_\alpha$ of the twistor Z^α as

$$\bar{Z}_\alpha = (\overline{Z^2}, \overline{Z^3}, \overline{Z^0}, \overline{Z^1}), \tag{5.2}$$

and we define

$$\begin{aligned} \mathbb{PT}^+ &\equiv \{Z^\alpha \in \mathbb{PT} | \bar{Z} \cdot Z > 0\} \\ \mathbb{PN} &\equiv \{Z^\alpha \in \mathbb{PT} | \bar{Z} \cdot Z = 0\} \\ \mathbb{PT}^- &\equiv \{Z^\alpha \in \mathbb{PT} | \bar{Z} \cdot Z < 0\}. \end{aligned} \tag{5.3}$$

In the following, we rely heavily on the following result, known as the Penrose transform.

$$\begin{aligned} \check{H}^1(\hat{U}; \mathcal{O}(-n-2)) \simeq \{&totally\ symmetric\ \phi_{A'B'\ldots D'}\ of\ helicity\ \frac{n}{2} \\ &such\ that\ \nabla^{AA'}\phi_{A'B'\ldots D'} = 0 holomorphic\ on\ U \subset \mathbb{CM}\} \end{aligned} \tag{5.4}$$

$$\begin{aligned} \check{H}^1(\hat{U}; \mathcal{O}(n-2)) \simeq \{&totally\ symmetric\ \phi_{AB\ldots D}\ of\ helicity\ \frac{-n}{2} \\ &such\ that\ \nabla^{A'A}\phi_{AB\ldots D} = 0\ holomorphic\ on\ U \subset \mathbb{CM}\}. \end{aligned} \tag{5.5}$$

where $\mathbb{CM}$ denotes complexified compactified (conformally) flat Minkowski space, $\check{H}^1(\hat{U}; \mathcal{O}(m))$ denotes the first Cěch cohomology group, relative to $\hat{U} \subseteq \mathbb{PT}$, of the sheaf $\mathcal{O}(m)$ (the sheaf of holomorphic sections of H^m, the m-fold tensor product of the hyperplane section bundle), where $\hat{U}$ is the subset of $\mathbb{PT}$ derived from $U \subseteq \mathbb{CM}$ in the way described in [8].

For our purposes, (5.4) and (5.5) mean the following. If one is given a totally symmetric holomorphic spinor field $\phi_{A'B'\ldots D'}$ of helicity $\frac{n}{2}$ (i.e., with n spinor indices) satisfying the equation in (5.4), then there is an element $[f] \in \check{H}^1(\hat{U}; \mathcal{O}(-n-2))$, which is represented by a function $f(Z^\alpha)$ holomorphic in the twistor variable Z^α only on particular subsets of $\hat{U}$ and homogenous of degree $(-n-2)$ such that

$$\phi^{\alpha\ldots\beta}(X^{\alpha\beta}) = \frac{1}{2\pi i} \oint X^{\alpha\rho} W_\rho \ldots X^{\beta\sigma} W_\sigma f(X^{\xi\eta} W_\eta) X^{\mu\nu} W_\mu dW_\nu, \tag{5.6}$$

where $\phi^{\alpha\ldots\beta}$ is the manifestly conformally covariant representation of $\phi_{A'\ldots D'}$ and $X^{\alpha\beta}$ is the bitwistor defining the spacetime point at which we are evaluating the field. This is all described in more detail in [3]. Similarly, after defining the following notation:

$$f_{\alpha\ldots\beta}(Z^\mu) \equiv \frac{\partial^{|n|} f}{\partial Z^\alpha \ldots \partial Z^\beta}, \tag{5.7}$$

we have the statement that if one is given a holomorphic spinor field $\phi_{AB\ldots D}$ of helicity $\frac{-n}{2}$ satisfying the equation in (5.5), then there is an element $[f] \in \check{H}^1(\hat{U}; \mathcal{O}(n-2))$, which is represented by a function $f(Z^\alpha)$ holomorphic in the twistor variable Z^α only on particular subsets of $\hat{U}$ and homogenous of degree $(n-2)$ such that

$$\phi_{\alpha\ldots\beta}(X^{\alpha\beta}) = \frac{1}{2\pi i} \oint f_{\alpha\ldots\beta}(X^{\xi\eta} W_\eta) X^{\mu\nu} W_\mu dW_\nu. \tag{5.8}$$

In what follows we deal solely with spinor fields with unprimed spinor indices, i.e. those arising as in (5.8). It turns out that any such spinor field $\phi_{\alpha\ldots\beta}$, due to the total symmetry in its indices, can be faithfully represented by a single component as follows

$$\phi(U, V) = (-1)^n \frac{n!}{(2\pi i)^3} \oint \frac{f(Z)\ DZ}{(U \cdot Z)(V \cdot Z)^{n+1}}, \tag{5.9}$$

where $X^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta} U_\gamma V_\delta$, and where the remaining n components can be recovered from this scalar function by repeated application of the operator

$$V \cdot \partial_U \equiv V_\alpha \cdot \frac{\partial}{\partial U_\alpha}. \tag{5.10}$$

In particular, we will be interested in helicity -1 fields which, by the above discussion, can all be represented as

$$\phi(U, V) = \frac{2}{(2\pi i)^3} \oint \frac{f(Z)\ DZ}{(U \cdot Z)(V \cdot Z)^3} \tag{5.11}$$

for some $f(Z^\alpha)$ homogenous of degree zero and holomorphic on a region of $\mathbb{PT}$ determined by the region of holomorphicity of ϕ and the underlying cover used to define f as an element of $\check{H}^1(\hat{U}; \mathcal{O}(0))$. We note that given some ϕ, finding the corresponding $f(Z^\alpha)$ is not always straightforward. This is known as the problem of the "inverse Penrose transform".

Now that we have our notation established, we turn to a particularly useful example of this inverse problem. Indeed, in the next section, we will see that finding the corresponding f for a given ϕ is precisely what motivates the discussion of the previous chapters.

5.2 Boundary Defined Fields

We begin by considering the following boundary-defined integral[1], without worrying about its relevance to space-time states or physics in general:

$$(I) = \frac{1}{2\pi i} \int_{\substack{A\cdot Z=0 \\ B\cdot Z=0 \\ C\cdot Z=0}} \oint \frac{DZ}{(U\cdot Z)(V\cdot Z)^3}, \tag{5.12}$$

where A_α, B_α, and C_α are fixed dual twistors. The region of integration is understood to be as follows. We first go around the simple pole in $U \cdot Z$ to project the integrand onto the $\mathbb{CP}^2$ defined by $U \cdot Z = 0$. This $\mathbb{CP}^2$ will intersect each of the projective planes defined by $A \cdot Z = 0$, $B \cdot Z = 0$, and $C \cdot Z = 0$ in a $\mathbb{CP}^1$, denoted respectively by $L^*[A, U], L^*[B, U]$, and $L^*[C, U]$. Recalling our discussion from Chapter 2 each of these $\mathbb{CP}^1$'s will intersect in a point in the plane $U \cdot Z = 0$, and we denote these points by η_{AB}, η_{BC}, and η_{AC}. Therefore, when restricted to the plane $U \cdot Z = 0$, the dual twistors A_α, B_α, and C_α define a complex "triangle". We note that the two-real-dimensional contour (i.e., the remaining two-dimensions after going around the $U \cdot Z = 0$ pole) with boundary is specified only up to deformation so long as the deformation does not cross any singularities, and as we see from the integrand there are no singularities to worry about[2]. We also note that we must pick an orientation to this contour, and this whole scenario is depicted in Figure 5.1.

To begin to make a connection with our earlier work, it is clear that expressions of the form (5.12) have the same additive structure as the triangles that we discussed in Chapter 3, since the boundaries of the integral simply add linearly. Using the methods of [5], the integral in (5.12) can be readily carried out to give

$$(I) = \frac{1}{2} \frac{\langle ABCU\rangle^2}{\langle ABUV\rangle\langle BCUV\rangle\langle CAUV\rangle}. \tag{5.13}$$

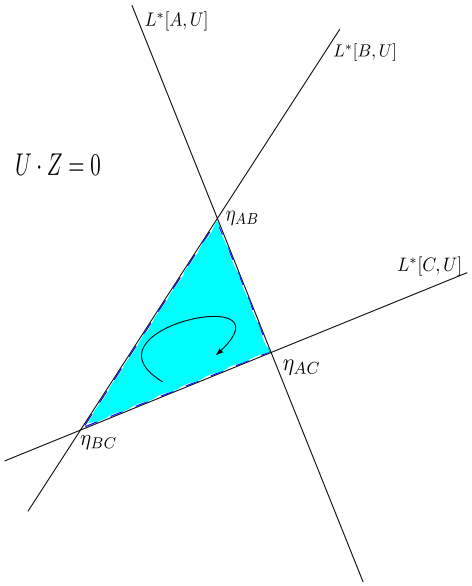
From our discussion in Chapters 3 and 4, we see that (I) is nothing but the area of a triangle with boundaries A_α, B_α, and C_α restricted to the plane $U \cdot Z = 0$, where we have chosen $P_0 = V$.

However, if we assume that $L[A, B], L[B, C], L[A, C] \subset \mathbb{PT}^{*+}$, then it is clear that

[1] Consideration of such boundary-defined integrals and the later identification of these with positive frequency helicity -1 spacetime fields is due to Dr. Andrew Hodges.

[2] This indeed gives us yet another—albeit less practical—way to think about the polytopes discussed in Chapter 2. Namely, we can think of them as equivalence classes of contours with boundary, all belonging to the same homology class.

Figure 5.1



(I) is holomorphic for all U, V such that $L[U,V] \subset \mathbb{PT}^{*-}$. Moreover, by differentiating (5.13) directly, one readily finds that (I) is the first component of a spinor field satisfying the conformally covariant equations for free fields, as discussed more thoroughly in [3] and [7]. Thus, with these assumptions on A_α, B_α, and C_α, we see that (I) can be also be interpreted as the $\phi(U,V)$ of a positive frequency helicity -1 field. This is a highly non-trivial fact, since it implies that there must be an element $[f] \in \check{H}^1(\mathbb{PT}^+; \mathcal{O}(0))$ such that

$$(I) = \phi(U,V) = \frac{2}{(2\pi i)^3} \oint \frac{f(Z)\ DZ}{(U \cdot Z)(V \cdot Z)^3}. \tag{5.14}$$

Strictly speaking, by writing (5.14) we are implying that $[f]$ can be represented by a single homogeneity zero function. This occurs if we can represent $[f]$ with a Leray cover $\mathcal{U}$ consisting of two open sets U_1 and U_2, with f being holomorphic on their intersection. The details of this discussion can be found in [8]. There is no reason to assume, however, that the element of $\check{H}^1(\mathbb{PT}^+; \mathcal{O}(0))$ that we will need will be of this form, so more generally we should allow for the possibility that

$$(I) = \phi(U,V) = \frac{2}{(2\pi i)^3} \sum_{i<j} \oint_{\gamma_{ij}} \frac{f_{ij}(Z)\ DZ}{(U \cdot Z)(V \cdot Z)^3}, \tag{5.15}$$

which arises in the case that $[f] \in \check{H}^1(\mathbb{PT}^+; \mathcal{O}(0))$ is defined with respect to a Leray cover involving many open sets $\{U_i\}$, with each f_{ij} being holomorphic on the inter-

sections $U_i \cap U_j$. Each contour γ_{ij} in (5.15) must then lie in the intersection $U_i \cap U_j$ as well. Fields ϕ described by elements $[f]$ represented in this way—namely with respect to many different non-trivial intersections—are said to be described by "branched contours", as described by Penrose in the very early days of twistor theory [9]. It is quite surprising that the correct $[f]$ that we need to satisfy (5.15) is an explicit example of such a branched contour integral, as there are very few (if any) explicit examples of branched contours in the twistor literature to date. We also note that satisfying (5.15) does not appear to be an easy task, nor one whose solution will be particularly enlightening. Namely, we see that (5.15) relates a contour with boundary on one side to a sum of entirely closed contours on the other. The fact that the solution is both simple and relies on the elegant machinery of branched contours suggests its utility.

We therefore now turn our attention to finding these f_{ij}'s and seeing how (5.15) can be satisfied. In doing so, we will see our two-dimensional calculus (as applied to polygons restricted to the face $U \cdot Z = 0$) fall right out. We emphasize, though, that at this point there is no reason to see any similarity between the f_{ij}'s discussed thus far in this chapter—which are homogeneity zero functions holomorphic on two-fold intersections of some Leray cover $\mathcal{U}$—and the f_{ij}'s used in our two-dimensional calculus, which were large sums of volumes of triangles.

5.3 Triangle Cohomology

We begin by considering the twistor function[3]

$$f_{12}(Z^\alpha) = \log\left(\frac{A \cdot Z}{B \cdot Z}\right) \log\left(\frac{C \cdot Z}{D \cdot Z}\right) \tag{5.16}$$

and the open subsets of $\mathbb{PT}$

$$\begin{aligned} U_1 &= \{Z^\alpha |\ [(1-u)A + uB] \cdot Z \neq 0, \quad u \in [0,1]\} \\ U_2 &= \{Z^\alpha |\ [(1-u)C + uD] \cdot Z \neq 0, \quad u \in [0,1]\}, \end{aligned} \tag{5.17}$$

where $A_\alpha, B_\alpha, C_\alpha$, and D_α are fixed dual twistors. f_{12} clearly is homogeneous in Z^α, and if we take the principle branches of the logarithms in (5.16), then f_{12} is holomorphic on $U_1 \cap U_2$, but not on U_1 and not on U_2. Thus, if $\mathcal{U} \equiv \{U_1, U_2\}$ is a cover of $\mathbb{PT}^+$, then f_{12} is a non-trivial representative of a cohomology class in

[3]Consideration of such a twistor function and its related contour integral, as well as the useful result in (5.22) below, is also due to Dr. Andrew Hodges.

$\check{H}^1(\mathbb{PT}^+, \mathcal{O})$. We note that the constraint that needs to be imposed on A, B, C, and D in order for $\mathcal{U}$ to indeed be a cover of $\mathbb{PT}^+$ is that

$$L^*[(1-u)A + uB, (1-v)C + vD] \subset \mathbb{PT}^- \text{ for all } (u,v) \in [0,1] \times [0,1]. \tag{5.18}$$

This will not be important for us, though, and we note it only to show that such choices are possible and thus that the following discussion is worthwhile.

Before calculating the space-time field generated from f_{12}, we first give a brief argument for why we do not need to make reference to the cover when writing $\check{H}^1(\mathbb{PT}^+, \mathcal{O})$—i.e., for why $\mathcal{U}$ is Leray. We begin by noting that $\mathcal{V} = \{V_1, V_2\}$ with $V_1 = \{Z^\alpha | A_1 \cdot Z \neq 0\}$ and $V_2 = \{Z^\alpha | A_2 \cdot Z \neq 0\}$, with $A_{1\alpha} \neq A_{2\alpha} \in \mathbb{PT}^*$, is a cover of $\mathbb{PT} - L^*[A_1, A_2]$ and is Leray [6]. We then notice that each element in $\mathcal{U}$ is homotopic to, and even deformation retractable to, an open set of the same form of each open set in $\mathcal{V}$. Since $\mathcal{V}$ is Leray, the cohomology of its constituent sets and their intersections is trivial, and since cohomology is in general preserved under deformation retracts, we can say that the cohomology of the constituent elements of $\mathcal{U}$ and their intersections is trivial, thus making $\mathcal{U}$ Leray. We will use a similar argument again when our covers become more convoluted.

To find the spacetime field generated by this f_{12} we need to consider (5.14) with $f(Z^\alpha)$ as in (5.16), i.e. we need to consider

$$\phi(U,V) = \frac{2}{(2\pi i)^3} \oint_\gamma \log\Big(\frac{A \cdot Z}{B \cdot Z}\Big) \log\Big(\frac{C \cdot Z}{D \cdot Z}\Big) \frac{DZ}{(U \cdot Z)(V \cdot Z)^3}. \tag{5.19}$$

The contour γ in (5.19) is meant to go around the $U \cdot Z = 0$ pole, then the $V \cdot Z = 0$ pole (picking up the residue of the term of the form $\log(1-z)\log(1-z)/z^3$) and thus restricting us to the line $L^*[U, V]$, and finally separating the branch cuts of the logarithms on each fixed $L^*[U, V]$. It is readily checked, after choosing a particular orientation for the contour γ, that we get

$$\phi(U,V) = \frac{\langle ABCU\rangle^2}{\langle ABUV\rangle\langle BCUV\rangle\langle CAUV\rangle} - \frac{\langle ABDU\rangle^2}{\langle ABUV\rangle\langle BDUV\rangle\langle DAUV\rangle}. \tag{5.20}$$

There are two things to note about (5.20). The first is that it is dangerously close to the expression that we want for $\phi(U, V)$, namely (5.13), in that we only need a way of getting rid of the second term. Second, and more importantly, we see that this expression is proportional to the volume of the "quadrilateral" defined by $A_\alpha, B_\alpha, C_\alpha$, and D_α restricted to the plane $U \cdot Z = 0$. Namely, we see that

$$\phi(U,V) = 2V[AB][CD]_U, \tag{5.21}$$

in the notation of section 4.5, where we have let $P_0 = V$. We note that in (5.21) we have placed dual twistor variables in the slots of $V[\cdot\cdot][\cdot\cdot]$, and this is simply the result of an overall swap in this chapter of dual and non-dual variables in relation to our discussion in Chapters 2-4.

One way of recovering (5.13) from (5.20) is to simply let $D_\alpha = \lambda A_\alpha + \mu B_\alpha$ with $(\lambda, \mu) \in \mathbb{C}^2 - (0,0)$ chosen so that D_α does not lie on the branch cut of the logarithm involving $A \cdot Z$ and $B \cdot Z$. We need to ensure this so that we can indeed choose a contour γ that separates the branch cuts from the two different logarithms. Therefore, the spacetime field derived from the twistor function

$$f_{12}(Z) = \log\Big(\frac{A \cdot Z}{B \cdot Z}\Big) \log\Big(\frac{C \cdot Z}{(\lambda A + \mu B) \cdot Z}\Big) \tag{5.22}$$

with the correct choice of λ, μ will reproduce (5.13).

This is not an ideal representation of the underlying $[f] \in \check{H}^1(\mathbb{PT}^+, \mathcal{O})$, though, for two reasons. First, (5.22) in no way reflects the (anti-)symmetry of the resulting spacetime field (or volume) in permuting A_α, B_α, and C_α. Secondly, and more importantly, the representative of $[f]$ given in (5.22) is not at all suited to reflect the additivity that we know such states should exhibit. Fortunately, our realization in (5.21) can solve both of these issues at once.

Let us begin by relabeling our dual twistors as $A_\alpha \to A_{1\alpha}$, $B_\alpha \to A_{2\alpha}$, $C_\alpha \to A_{3\alpha}$, and $D_\alpha \to Q_\alpha$. We then have that

$$f_{12}(Z^\alpha) = \log\Big(\frac{A_1 \cdot Z}{A_2 \cdot Z}\Big) \log\Big(\frac{A_3 \cdot Z}{Q \cdot Z}\Big). \tag{5.23}$$

Let us abbreviate the integral in (5.19) and simply write

$$\phi_{12}(U,V) \equiv \frac{2}{(2\pi i)^3} \oint_{\gamma_{12}} \log\Big(\frac{A_1 \cdot Z}{A_2 \cdot Z}\Big) \log\Big(\frac{A_3 \cdot Z}{Q \cdot Z}\Big) \frac{DZ}{(U \cdot Z)(V \cdot Z)^3} \equiv \oint_{\gamma_{12}} f_{12}, \tag{5.24}$$

where we have introduce the subscripts $\{12\}$ on both the field as well as the contour because we will be considering expressions with different subscripts shortly. It will always be implied that the contour γ_{ij} is chosen so as to give (5.20), as opposed to its negative. The result (5.21) can then be written

$$\oint_{\gamma_{12}} f_{12} = 2V[12][3Q]. \tag{5.25}$$

Let us now define the twistor functions $f_{23}(Z^\alpha)$ and $f_{31}(Z^\alpha)$ as follows:

$$f_{23}(Z^\alpha) = \log\Big(\frac{A_2 \cdot Z}{A_3 \cdot Z}\Big) \log\Big(\frac{A_1 \cdot Z}{Q \cdot Z}\Big) \tag{5.26}$$

and

$$f_{31}(Z^\alpha) = \log\Big(\frac{A_3\cdot Z}{A_1\cdot Z}\Big)\log\Big(\frac{A_2\cdot Z}{Q\cdot Z}\Big), \tag{5.27}$$

with $\phi_{23}(U,V)$ and $\phi_{31}(U,V)$ defined analogously to $\phi_{12}(U,V)$. Combining the above results then gives

$$\begin{aligned}\phi_{12}(U,V)+\phi_{23}(U,V)+\phi_{31}(U,V) &= 2(V[12][3Q]+V[23][1Q]+V[31][2Q]) \\ &= 4[123],\end{aligned} \tag{5.28}$$

where it is understood that $[123]=[123]_U$ with $P_0=V$. We see that the dependence on Q_α (formerly known as D_α) manifestly drops out, as desired, and we are left with something proportional to the expression in (5.13).

These results can be summarized in the following claim.

Claim 5.1. Let $A_\alpha, B_\alpha, C_\alpha$, and P_α be dual twistors such that $L[A,B], L[B,C], L[A,C] \subset \mathbb{PT}^{*+}$. The twistor functions

$$f_{12}(Z^\alpha) = \frac{1}{4}\log\Big(\frac{A\cdot Z}{B\cdot Z}\Big)\log\Big(\frac{C\cdot Z}{P\cdot Z}\Big) \tag{5.29}$$

$$f_{23}(Z^\alpha) = \frac{1}{4}\log\Big(\frac{B\cdot Z}{C\cdot Z}\Big)\log\Big(\frac{A\cdot Z}{P\cdot Z}\Big)$$

$$f_{31}(Z^\alpha) = \frac{1}{4}\log\Big(\frac{C\cdot Z}{A\cdot Z}\Big)\log\Big(\frac{B\cdot Z}{P\cdot Z}\Big)$$

represent an element $[f]\in\check{H}^1(\mathbb{PT}^+,\mathcal{O})$ whose Penrose transform

$$\phi(U,V) = \frac{2}{(2\pi i)^3}\sum_{i<j}\oint_{\gamma_{ij}}\frac{f_{ij}(Z)\;DZ}{(U\cdot Z)(V\cdot Z)^3} \tag{5.30}$$

with each γ_{ij} chosen as described above, gives the field

$$\phi(U,V) = \frac{1}{2}\frac{\langle ABCU\rangle^2}{\langle ABUV\rangle\langle BCUV\rangle\langle CAUV\rangle} \equiv [ABC]_U. \tag{5.31}$$

We note that it is implied that $f_{21}\equiv -f_{12}$, $f_{32}\equiv -f_{23}$, and $f_{13}\equiv -f_{31}$.

Proof: To prove this claim we need to show that everything in it is well-defined, for if everything is, then we know from our above discussion that the final result (5.31) holds. Thus, we first need to prove that these twistor functions do indeed represent a genuine $[f]$, and finally that our contours γ_{ij} actually do exist, i.e., that they can

separate the branch cuts in each f_{ij}. Before embarking on an only pseudo-rigorous discussion of why this should be the case, we first note the following more obvious fact. On the three-fold intersection $U_1 \cap U_2 \cap U_3$ (where each f_{ij} is holomorphic on $U_i \cap U_j$), we have

$$f_{12} + f_{23} + f_{31} = 0, \tag{5.32}$$

as required by the Cěch cocycle conditions [6]. Therefore, showing that these $\{f_{ij}\}$ define a good $[f]$ amounts to finding a good (i.e., Leray) cover $\{U_1, U_2, U_3\}$ such that f_{ij} is holomorphic on $U_i \cap U_j$, yet not holomorphic either on U_i or U_j (so as to get a non-trivial representative). It turns out that in finding such a cover, we need to primarily concern ourselves with ensuring that our cover will allow for the contours γ_{ij} to exist. Namely, we also need to make sure that we can separate the branch cuts of each f_{ij} with a contour γ_{ij} that lies solely in $U_i \cap U_j$. We will therefore **construct** a cover so as to guarantee the existence of these contours, and then later give an argument for why this cover is Leray. The following arguments will not be entirely rigorous, but hopefully they will convey the notion that it is possible to make them such.

In what follows, we will mostly concern ourselves with what goes on in any given $L^*[U, V]$. The reason for this is three-fold. First, we note that the integral in the Penrose transform (5.30) goes around the $U \cdot Z = 0$ and $V \cdot Z = 0$ poles, thus restricting the final contour to the complex projective line $L^*[U, V]$. Second, and accordingly, if we can't find the right cover of a given $L^*[U, V]$ to make the above arguments rigorous while ensuring that the necessary contours exist, then we certainly have no hope of doing this for every $L^*[U, V]$. Third, if we can find the right setup for a given $L^*[U, V]$, then doing so in general will be possible assuming our constraints on A_α, B_α, C_α, and P_α hold—namely, that these intersect in the "other half" of twistor space as U and V do.

It is perhaps most enlightening to see what the right answer **is not**, for this will lead us to what the right answer is. We know we want a cover involving only three sets, and we know that there are six branch cuts with which we must concern ourselves: the $A - B$, $A - C$, $B - C$, $A - P$, $B - P$, and $C - P$ cuts. We also know that f_{ij} needs to be holomorphic on $U_i \cap U_j$ but not holomorphic on either U_i or U_j. Thus, defining U_1 to be the complement of the $A - B$ and $C - P$ cut, for example, will not work because then f_{12} would be holomorphic on U_1. We therefore need to "distribute" the cuts involved in any given f_{ij} across U_i and U_j. This is progress, but one naive guess will also show us that we still need to do better.

It will help to assign colors to each U_i, then draw the cuts on a given $L^*[U, V]$

with this color, and define U_i to be the complement of that cut. Then the rules for constructing a cover that allows for the right contours to exist can be simply stated as follows. First, no point on any $L^*[U, V]$ can be colored by all three colors—this ensures that we indeed cover the space, as each set in the cover is defined as the **complement** of a specific color. Second, we need to be able to surround each cut with a contour that only crosses lines that are of a colour **not** corresponding to the given f_{ij}, so that if f_{ij} corresponds to red (for U_i) and blue (for U_j), then γ_{ij} must be able to separate the two branch cuts in f_{ij} without crossing any red or blue lines. This corresponds to the fact that each contour must lie entirely in $U_i \cap U_j$. For concreteness, let us associate U_1 with red, U_2 with blue, and U_3 with green. Then suppose we made the following definitions:

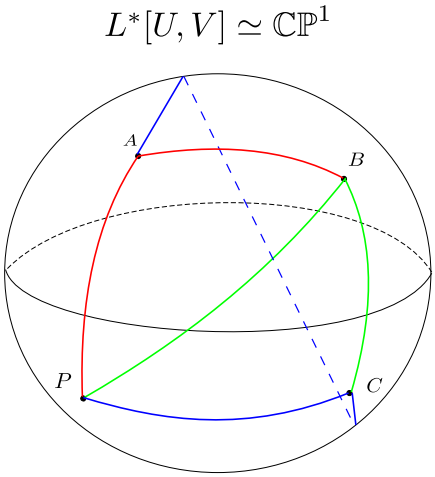
$$\begin{aligned} U_1 &= \{\text{complement of } A - B, \text{ and } A - P \text{ cuts}\} \\ U_2 &= \{\text{complement of } A - C, \text{ and } C - P \text{ cuts}\} \\ U_3 &= \{\text{complement of } B - C, \text{ and } B - P \text{ cuts}\}. \end{aligned} \tag{5.33}$$

We then have the picture given in Figure 5.2 on a fixed $L^*[U, V]$, and it is easily seen that each f_{ij} is holomorphic where it needs to be, and not holomorphic where it needs to be. In the figure, we label the points at which the planes $A \cdot Z = 0$, $B \cdot Z = 0$, $C \cdot Z = 0$, and $P \cdot Z = 0$ intersect $L^*[U, V]$ simply by A, B, C, and P, respectively.

We note that we define these U_i in terms of arbitrary branch cuts instead of the more explicit realization analogous to that given in (5.17) because we want to leave the option open for not taking the principle branch of the logarithms involved. Namely, we want to be able to ensure that our branch cuts do not intersect in too nasty of ways, and therefore we need to leave some flexibility in the definition of the logarithms. We note that this is fine since nothing in our discussion so far has relied on the fact that we are using the principle branch of the logarithm, but rather simply that we pick up certain residues by going around whatever branch cut there is.

There are two problems with (5.33), which are both made apparent by Figure 5.2. First, (5.33) is indeed not a cover, since the point P in Figure 5.2 is covered by all three colors, which corresponds to the fact that the plane $P \cdot Z = 0$ is not covered by (5.33). This can be easily remedied, however, by simply extending, say, the blue line through P and along the red line, so that P is only covered by blue and green and we do not lose any of the (lack of) holomorphicity that we need in the twistor functions. The second problem, which is that our contours do not exist with this cover, is more

Figure 5.2

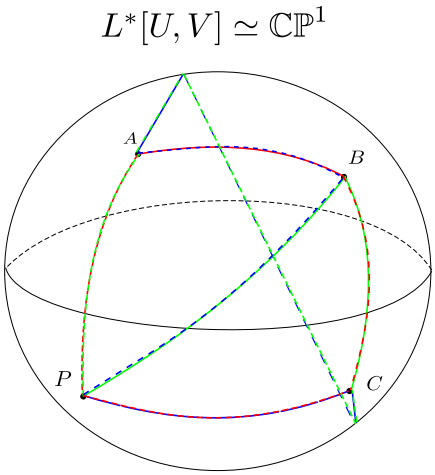


severe. For example, we need to be able to separate the $A-B$ cut from the $C-P$ cut with a contour that only crosses green (i.e., not blue or red) lines, and this is clearly impossible.

This problem can be fixed in many different ways. We will do so using a method that may at first seem overly complicated, but will prove useful for generalizing to arbitrary boundary-defined states in the next section. Again, our color-coding scheme will continue to prove useful, now more than ever. However, we must adopt a slightly different scheme. We still associate a color to each U_i, but now for each f_{ij} we draw overlapping dotted lines for the relevant cuts and using the relevant colors. In other words, we don't worry about distributing the cuts between different sets for now, but rather just draw all cuts with the colors that apply, and use dotted lines so that we can lay one color over the other. For now we don't write out an explicit cover, but rather use the new version of Figure 5.2, which is given in figure 5.3 (where we've used the same color scheme as above, namely 1 is red, 2 is blue, and 3 is green).

In this color scheme each cut is covered by two different colors, corresponding to the colors assigned to the subscripts i and j (corresponding to the open sets $\{U_i\}$) of the twistor function that the cut belongs to. The rule for drawing the contours is simply that each contour cannot cross a line that contains **both** the colors corresponding to its f_{ij}. Thus, for example, γ_{12} cannot cross any line that is

Figure 5.3

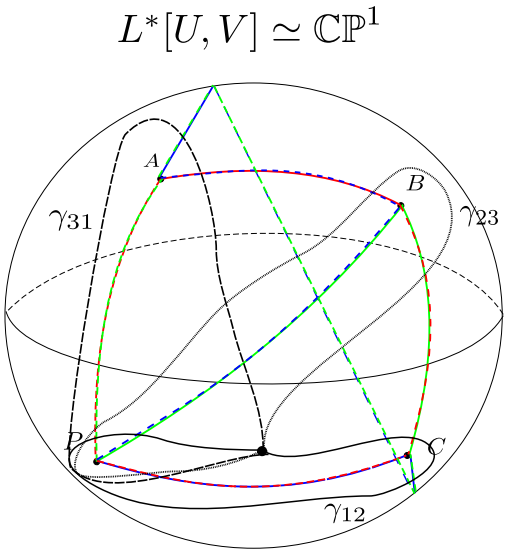


both red and blue. An example of a possible branched contour satisfying this rule while separating the cuts properly is given in Figure 5.4. We note that we can place the base point anywhere (in the white regions), since the white region corresponds to the complement of all cuts (which is by definition the intersection of all three $\{U_1, U_2, U_3\}$), and we have simply made an arbitrary choice in Figure 5.4. In this figure, γ_{12} is given by a solid black line, γ_{23} by a very finely dotted black line, and γ_{31} by a thickly dotted black line.

We note that although the contours are drawn going around the $C - P$, $B - P$, and $A - P$ cuts, on $\mathbb{CP}^1$ this is equivalent to going around the other set of cuts with the opposite orientation. The important quality of each contour is that they separate the sphere into two connected components, each containing precisely one of the two relevant cuts. The fact that we have used dotted colored lines now comes in handy when we try to finish our construction and actually define our covers. This involves a simple algorithm.

We first notice that there are clearly only finitely many intersections of contours with colored dotted lines, and finitely many points at which colored dotted lines intersect each other. Therefore there will only be finitely many steps to the algorithm. We simply go to each point at which a contour intersects a colored dotted line and choose the color that is allowed for that contour in a sufficiently small neighborhood

Figure 5.4



around the intersection point. For example, in Figure 5.4, γ_{12} intersects the $B-C$ cut near the point C. This contour is only allowed to cross green lines, since it must lie in $U_1 \cap U_2$ (which we are defining as we go, and which are red and blue respectively). Then, in a neighborhood of the cut around this intersection point, we get rid of the red. However, this neighborhood can't be too large, because γ_{23} also intersects this cut and must only cross red lines. So, in a neighborhood of the cut around this intersection point, there should be no green. What happens in between doesn't matter, so long as the whole cut is some combination of red and green. We continue this process for each contour-cut intersection, and our color coding scheme makes it clear that it is always possible to do this. We note that this does not affect the required domains of holomorphicity for the f_{ij}, since each cut will be covered by colors in such a way so as to make the relevant f_{ij} not holomorphic on U_i or U_j alone, but holomorphic on $U_i \cap U_j$.

We then do a similar procedure at all points at which cuts intersect each other, namely at the A, B, C, and P points in the figure. We do this by first noting that each such point (assuming the cuts do not intersect elsewhere) has exactly three overlapping dotted lines impinging on it, and all three colors make an appearance. We simply

choose any two of these colors, and let them determine the third. For example, we have lines impinging on A with colors RG, RB, and BG. If we choose red from the RG line and blue from the RB line, then we must choose blue from the BG line. Similarly, had we chosen green from the RG line and blue from the RB line, then we'd have to choose green from the BG line. This ensures that our construction indeed gives us a cover—i.e., that it does not leave out any points. We note that the discreteness in the choices being made and the continuity of the cuts, the contours, and the freedom to move the contours, ensures that this is always possible. In fact, this procedure will generalize immediately to cases where there is an arbitrary **finite** number of cuts and contours, for the same reasons (for a larger number of sets in the cover, we simply need to introduce more colors). Finally, we note that difficulties arise if cuts intersect other cuts at points which are not "vertices" in the sense of being at the beginning or end of a cut. However, it is reasonable to believe (though not rigorously proven) that our freedom in choosing different cuts for different logarithms should make this problem either obsolete, or at least reconcilable with this construction.

We must now remark on whether or not a cover constructed in this way is Leray, and in doing so we will simply reuse the arguments given between (5.18) and (5.19) with one small generalisation. From that argument we see that a cover of $\mathbb{PT}^+$ of the form $\mathcal{V} = \{V_1 = \{Z^\alpha | A_1 \cdot Z \neq 0\}, V_2 = \{Z^\alpha | A_2 \cdot Z \neq 0\}\}$ where $L^*[A_1, A_2] \subset \mathbb{PT}^-$ is Leray. This cover is obtained simply by deleting planes in $\mathbb{PT}^+$ that intersect only in $\mathbb{PT}^-$. However, we note that a cover with an arbitrary finite number of open sets, each defined by the deletion of a plane, such that all deleted planes intersect in $\mathbb{PT}^-$, is also Leray. This follows from how we add cohomology elements defined from different covers to obtain new cohomology elements, as discussed in [9], and viewing such a cover as that obtained from adding elements whose respective covers are of the form $\mathcal{V}$ above. We then run the argument through in exactly the same way, noting that the cover that we have defined abstractly in this "proof" with our color-coding scheme is such that each open set involved is deformation retractable to an open set with some finite number of planes removed. Therefore, such a cover will also be Leray. □

We note again that the above argument is in no way rigorous, but rather is meant to give the impression that such ideas can be made rigorous. Of course, it is the space-time fields themselves which we are after, and the details of their cohomology only serve to give them precise cohomological meaning. This is why we will not see the cohomolgocial details through to their bitter end here. The only constraints on A_α, B_α, and C_α of any physical relevance are that $L[A, B]$, $L[B, C]$, and $L[A, C]$

must be contained in $\mathbb{PT}^{*+}$ to ensure that the space-time field they define in (5.31) is holomorphic on $\mathbb{CM}^+$. These constraints—those that actually make a difference on space-time—are the ones that will concern us the most, and we will use the more general continuity arguments used here to establish existence results only when it is relevant not to the space-time field but only to its cohomological description. It is hoped that the above arguments show that such precision is indeed possible. Similar arguments will be used in the next section, with slight generalisations. However, much of those arguments will simply be referred back to here.

As a final remark, if we are only interested in this part of the story as giving a **motivation** for the calculus developed in Chapters 2-4, then clearly the details of these cohomological arguments are even more irrelevant. There is, however, an interesting set of results that come from dealing with these "triangular" states, such as the expressions one gets by taking inner products of various such geometric states. To develop this theory further we need some amount of cohomological rigour, but since we are not interested in this part of the story in this work, we will leave these details out.

We now turn our attention, just as in Chapters 3 and 4, to describing spacetime states defined by arbitrary polygons as opposed to simple triangles, and we will see the motivation for the rest of the two-dimensional calculus fall out of such considerations.

5.4 General Boundary-Defined States

In the previous section we found the $[f] \in \check{H}^1(\mathbb{PT}^+, \mathcal{O})$ whose Penrose transform gives the state

$$\phi(U, V) = [ABC]_U, \tag{5.34}$$

with $P_0 = V$ as described above. The natural next step, then, is to try to find the $[f] \in \check{H}^1(\mathbb{PT}^+, \mathcal{O})$ whose Penrose transform gives the state

$$\phi(U, V) = \sum_{k=1}^{n} [i_k i_{(k+1)} K]_U \tag{5.35}$$

for some cyclic list $l = (i_1, ..., i_n)$ and some set of boundaries $\{A_{1\alpha}, ..., A_{N\alpha}\}$. In other words, now that we have found the $[f] \in \check{H}^1(\mathbb{PT}^+, \mathcal{O})$ corresponding to a triangle, we want to find the $[f] \in \check{H}^1(\mathbb{PT}^+, \mathcal{O})$ corresponding to an arbitrary polygon. We note that in (5.35) we have used the reference boundary K_α—instead of the B_α that we used in Chapter 3—to define the area of the polygon defined by l simply because we used B in (5.34). We emphasize that a priori there is no reason to believe that

there must be a collection of functions $\{f_{ij}\}$ such that the sum of contour integrals in (5.15) gives the expression in (5.35), nor that such a collection of functions would be interesting. However, it is by realizing that if we put certain constraints on the boundares $\{A_{1\alpha}, ..., A_{N\alpha}\}$, then we can view (5.35) as a positive frequency helicity -1 spacetime field. This observation combined with knowledge of the Penrose transform then tells us not only that such a collection of functions must exist, but that they should also satisfy the Cěch constraint (5.32). Once we find these functions, then, we can forget about spacetime fields entirely, release the constraints on the boundaries, and study these functions for their own sake. This is how we move from the work in this chapter to that in the previous chapters. In particular, this is why we were motivated in Chapter 4 to search for objects f_{ij} that satisfied a certain cohomological identity. As we saw, it was this cohomological identity that made the whole calculus work—it allowed us to expand a particular triangulation in terms of f_{ij}'s and recover the underlying cyclic list(s), see which vertices and boundaries were "spurious", obtain new triangulations from old ones, and test the equivalence of different triangulations.

In what follows, we fix some set of distinct boundaries $S = \{A_{1\alpha}, ..., A_{N\alpha}\} \equiv \{1, ..., N\}$ and a cyclic list $l = (i_1, ..., i_n)$ where each $i_k \in S$. We also assume that the proper constraints have been placed on these boundaries so that (5.35) is indeed a positive frequency field. The details of these constraints will not concern us, however. Knowing the results in Chapter 4 as well as the previous section, we can immediately write down the twistor functions that will work for us. For each $i, j \in S$, we define

$$f_{ij}(Z^\alpha) = \frac{1}{4} \sum_{l=1, l\neq i,j}^{N} \log(\frac{A_i \cdot Z}{A_j \cdot Z}) \log(\frac{A_l \cdot Z}{P \cdot Z}) \tag{5.36}$$

for some fixed reference twistor P_α. We note that each f_{ij} is antisymmetric in its subscripts. The following claim shows that the cocycle condition for any $i, j, k \in S$ is automatically satisfied.

Claim 5.2. Let $\{f_{ij}\}$ be defined as in (5.36) with respect to the set of boundaries S. Then for any $i, j, k \in S$, we have

$$f_{ij} + f_{jk} + f_{ki} = 0. \tag{5.37}$$

Proof: We first assume that i, j, k are pairwise distinct, otherwise the result is

trivially satisfied. We then have

$$f_{ij}+f_{jk}+f_{ki}=\sum_{l=1,l\neq i,j}^{N}\log(\frac{A_l\cdot Z}{P\cdot Z})\log(\frac{A_i\cdot Z}{A_j\cdot Z})+\sum_{l=1,l\neq j,k}^{N}\log(\frac{A_l\cdot Z}{P\cdot Z})\log(\frac{A_j\cdot Z}{A_k\cdot Z}) \tag{5.38}$$
$$+\sum_{l=1,l\neq k,i}^{N}\log(\frac{A_l\cdot Z}{P\cdot Z})\log(\frac{A_k\cdot Z}{A_i\cdot Z})$$

$$=\Big\{\sum_{l=1,l\neq i,j,k}^{N}\log(\frac{A_l\cdot Z}{P\cdot Z})\Big[\log(\frac{A_i\cdot Z}{A_j\cdot Z})+\log(\frac{A_j\cdot Z}{A_k\cdot Z})+\log(\frac{A_k\cdot Z}{A_i\cdot Z})\Big]\Big\}$$

$$+\log(\frac{A_k\cdot Z}{P\cdot Z})\log(\frac{A_i\cdot Z}{A_j\cdot Z})+\log(\frac{A_i\cdot Z}{P\cdot Z})\log(\frac{A_j\cdot Z}{A_k\cdot Z})+\log(\frac{A_j\cdot Z}{P\cdot Z})\log(\frac{A_k\cdot Z}{A_i\cdot Z})$$
$$=0.$$

We note the similarity of this proof with that of Claim 4.16. We now introduce the following shorthand:

$$\frac{2}{(2\pi i)^3}\oint_{\gamma_{ij}}\sum_{l\neq i,j}^{N}\frac{1}{4}\log\Big(\frac{A_i\cdot Z}{A_j\cdot Z}\Big)\log\Big(\frac{A_l\cdot Z}{P\cdot Z}\Big)\frac{DZ}{(U\cdot Z)(V\cdot Z)^3}\equiv\oint_{\gamma_{ij}}f_{ij} \tag{5.39}$$

where the contour γ_{ij} is understood to separate the A_i-A_j branch cut from all of the other cuts in the sum, and to do so with the same orientation as the contour used to obtain (5.20). We will comment on the ability to always find such a contour shortly, and for now we simply assume that we can. From the result (5.25) and assuming we can always find such contours, we immediately have

$$\oint_{\gamma_{ij}}f_{ij}=\frac{1}{2}\sum_{l\neq i,j}V[ij][lP]. \tag{5.40}$$

From our results in Chapter 4, then, we immediately see that, for the $\phi(U,V)$ defined in (5.35), we have

$$\phi(U,V)=\sum_{k=1}^{n}\oint_{\gamma_{i_ki_{(k+1)}}}f_{i_ki_{(k+1)}}, \tag{5.41}$$

where the factor of two that we usually have to explicitly divide by has been absorbed into the f_{ij}'s, as seen in (5.40). We note that it is now trivial to express $\phi(U,V)$ by a sum of contour integrals as prescribed by (5.15). We simply need to count the appearances of each $(i_ki_{(k+1)})$ in the cyclic list l and treat this as the winding number of

a new contour, γ'_{ij}. This small detail is only relevant for expressing $\phi(U,V)$ explicitly as a Penrose transform. The key realization then, and that which motivates the entire calculus described in Chapters 2-4, is that for the $\{f_{ij}\}$ defined in (5.36), and with the notation of (5.39), we have that for all i,j,k,

$$f_{ij} + f_{jk} + f_{ki} = 0 \tag{5.42}$$

while

$$\oint_{\gamma_{ij}} f_{ij} + \oint_{\gamma_{jk}} f_{jk} + \oint_{\gamma_{ki}} f_{ki} = [ijk]_U. \tag{5.43}$$

To summarize, we have seen how the f_{ij} objects defined in Chapter 4 can be expressed as contour integrals of particular $\check{H}^1(\mathbb{PT}^+,\mathcal{O})$ representatives. There is reason to believe that the same can be done for the higher dimensional objects $\{f_{ijk}\}$ and $\{f_{ijkl}\}$ (using higher cohomology groups), though there are still some subtleties to be worked out in this regard. In this chapter we have seen how these f_{ij} objects can be used to describe "polygonal" spacetime fields of helicity -1, but as we have seen a great deal of the utility of this discussion relies on its ability to motivate a calculus for defining and calculating volumes of polytopes that can be defined completely independently of these considerations, and which is entirely self-contained. That said, the considerations of this chapter may still have an important role to play in generalizing these ideas in various ways, but more on this will be said in the next chapter.

Before turning our attention to future work that needs to be done, let us close this chapter by giving some arguments for why the functions defined in (5.36) genuinely represent an element in $\check{H}^1(\mathbb{PT}^+,\mathcal{O})$, and why the contours needed in (5.40) should always exist. Similarly to the argument that ended the last section, these two issues (finding the right cover for these f_{ij} as well as guaranteeing that these contours will exist) are closely related. The argument here will also be similar to that which ended the previous section in that it will both draw upon that argument immensely, as well as be equivalently lacking in rigour. In what follows, we restrict our attention to a particular $L^*[U,V]$ as before, and we use the same color coding idea that we used at the end of the last section.

We know that we need N open sets $\{U_i\}$, with N being the number of boundaries, since in (5.36) we have implicitly defined $\binom{N}{2}$ twistor functions. We also need each f_{ij} to be holomorphic on $U_i \cap U_j$ while not being holomorphic on either open set individually. We make the key (and currently unverified) assumption that each $A_i - A_j$ cut can be defined such that it does not intersect any $A_k - P$ cut when $k \neq i$ and

$k \neq j$. This assumption is made to make it possible for the necessary contours to always exist. We then assign to each open set a distinct color, so that there are N colors in total (compared with the three colors of the triangle state). For each U_i, we fill in all cuts found in each f_{ij} as j runs from 1 to N with a dotted line of the corresponding color. Then, for any given f_{ij}, the only cuts that will include both colors corresponding to i and j will be the $A_i - A_j$ cut and the $A_k - P$ cuts with $k \neq i$ and $k \neq j$ (which we note forms a connected set, since these cuts all intersect at P). Thus, if we allow γ_{ij} to only cross cuts that don't involve both the i and j colours, a contour will always exist separating the former cut from the latter cuts. Once all of the contours are drawn, there will only be a finite number of points (on a given $L^*[U, V]$) at which a contour intersects a cut, and we simply choose to fill in the cuts with their corresponding colors in such a way that a) no point is covered by all N colors and b) for each i, j the cuts $A_i - A_j$ and $A_k - P,\ k \neq i, j$ are covered by the two colors i and j, but not covered by any color single-handedly. As before, with our assumption of the lack of intersection between $A_k - P$ cuts and $A_i - A_j$ cuts, this will always be possible. Such a cover is immediately seen to be such that each open set U_i is deformation retractable to an open set with finitely many planes cut out, all intersecting in a region of twistor space that we are not interested in covering, and is therefore Leray.

Finally, we note that such detailed cohomological considerations are not important for our work in the previous chapters, as all of those definitions were made in a purely self-contained way, independent of any cohomological considerations.

Chapter 6

Future Directions

6.1 The Grassmannian

In this work we have introduced a new definition of abstract polyhedra by use of new objects called cyclic lists, and we have introduced an algorithmic way of calculating what we define to be the volumes of such polyhedra immediately from their defining cyclic lists. We have seen how to reconstruct these cyclic lists from any given triangulation of a polytope, and therefore how to algorithmically perform tasks like checking if two triangulations are equivalent, spotting spurious vertices and boundaries, and calculating the volumes of any of the lower dimensional polytopes sitting on a face of the larger polytope. These considerations are useful for a particular class of scattering amplitudes—namely, the tree-level n-point $NMHV$ amplitudes in the planar limit of $N = 4$ Super Yang-Mills.

It has recently been found that many other amplitudes in the planar limit of $N = 4$ SYM can be interpreted, albeit vaguely, as "some volumes" of "some region" of "some space". In this work we have made these statements rigorous for the case when "some space" is $\mathbb{CP}^n$ and "some region" is an abstract listed polytope. In doing so, we have found a calculus which is purely combinatorial in nature, and for which the sometimes difficult problems of spotting spurious vertices and/or calculating volumes of boundaries of these spaces becomes trivial. For more complicated amplitudes (i.e., loop order and/or higher $NMHV$ levels), the "some space" that one must consider is the positive Grassmannian [10], and sometimes even more exotic spaces [11]. Seeing as we can view $\mathbb{CP}^n$ as $G(1, n)$ (over the field of complex numbers), it is therefore natural to wonder how the methods and definitions developed here can be extended to these more interesting spaces.

We have reason to believe that an equally simple and largely combinatorial framework can be developed for the more general $G(k, n)$ (with positivity being considered

at a later stage). The main reason for believing this to be the case is that the polytopes that one considers for n-point $NMHV$ amplitudes are only a very particular class of the kinds of polytopes that the general framework introduced here can in theory deal with. Therefore, since the known polytopes M_n^{NMHV} sit as a tiny subset of the possible polytopes that this framework can handle, and since dealing with this polytope is most natural in this framework, there is hope that other physically relevant polytopes can be most naturally dealt with in the same (or a similar) framework. There is no doubt, however, that even making the step from $G(1,n)$ to $G(2,n)$ is not immediately obvious, and will likely require a good deal of further insight. Therefore, just as we have seen that genuine real (topological) polytopes in $\mathbb{R}^n$ are but a small subset of combinatorial polytopes (namely, those defined on some particular affine patch of a particular "real slice" of $\mathbb{CP}^n$), we hope to find how combinatorial polytopes in $\mathbb{CP}^n$ are but small subsets of some larger, hopefully largely combinatorial, space (or object).

6.2 Dolbeault Representations

One possible insight into the problem mentioned in the previous section may lie in the cohomological description of the $f_{...}$'s described in Chapter 5. Seeing as many of the algebraic results that motivated the initial calculus—namely, the fact that a point in one copy of $\mathbb{CP}^n$ defines an $(n-1)$-dimensional hyperplane in the other copy, and the particular intersection structure of such hyperplanes—do not in general carry over to $G(k,n)$ for $k>1$, one must look for possible generalizations elsewhere. The obvious possibility is that such a generalization will come from the cohomology of $G(k,n)$, since the calculus has a straightforward interpretation in $G(1,n)$ as contour integrals of representatives of certain cohomology classes (the independence of the contour integral under certain deformations of the cohomology representative is perfectly in line with the dependence of the corresponding $f_{...}$ on only the cohomology **class**, instead of the particular representative).

One possible step that could provide significant insight into this generalization would be to find the correct (i.e., simplest) Dolbeault represenatives of the $\check{H}^1$'s that we dealt with in Chapter 5. In particular, Dolbeault's theorem states that [8]

$$\check{H}^q(M,\Omega^p(E)) \simeq H^{p,q}(M,E) \quad p,q=0,1,2,..., \tag{6.1}$$

where the left hand side is the q^{th} Cěch cohomology group of the sheaf of holomorphic p-forms on the vector bundle E over the complex manifold M, and the right hand side

is the Dolbeault cohomology group of (p, q)-forms on E. We therefore know that there exist $(0, 1)$-forms on $\mathbb{CP}^3$ that give us, via the contour integrals described in Chapter 5, the two-dimensional calculus. Such $(0, 1)$-forms would be useful to find to see which of the two cohomology theories—Cěch or Dolbeault—is most natural. If it turns out that the latter is indeed more natural to work with than the Cěch representatives that we have used here, then this will not only make the cohomological description of this calculus more clear, but also provide more insight into possible generalizations to $G(k, n)$.

6.3 Higher Dimensional Cohomology

As discussed briefly at the end of Chapter 5, describing the $\check{H}^n$'s (with $n > 1$) that correspond to the higher dimensional $f_{...}$ objects (e.g. with three or four subscripts) is a messy issue. The biggest problem is dealing with our reference boundaries, as in higher dimensions it seems the only way to make the cohomology well-defined is to introduce an explosion of such boundaries[1]. As we know, the calculus itself is completely self-contained and independent of (though motivated by) such cohomological considerations, however if we want the cohomology to motivate possible generalizations, then these issues must be dealt with. Since these reference boundaries play no actual role in the final objects that we deal with in the calculus, and since the main difficulty in dealing with them lies in defining the right covers and finding the right contours, and since these issues are purely Cěch-related issues, it is hoped that a solution to the problem discussed in the previous section will also shed light on a possible solution to finding these $\check{H}^n$'s in higher dimensions.

6.4 Combinatorial/Abstract Polytopes

One other possibly fruitful future line of inquiry will be to translate the definitions and result established for the abstract listed polytopes defined here with known results from the fields of combinatorial and/or abstract polyhedra. The perspective taken here is significantly different than that taken in these two well-developed fields, as we focus solely on the cyclic lists that define a polytope as opposed to the polytope's face lattice. That said, the two pictures must be combinatorially equivalent, just phrased in a different language. The results we are interested in and obtained

[1]Even more than the reference boundaries being introduced to define the $f_{...}$ objects at the level of the calculus.

here are most naturally phrased in the language of cyclic lists, but this does not mean that new things can't be learned from uncovering some dictionary between the language introduced here and that which is already in use. There are, however, genuine differences between the languages. For example, we view a "disconnected" polytope as a single polytope as opposed to two (or several) connected polytopes, as is the case in the more conventional definitions of both combinatorial and abstract polytopes. Additionally, knowledge of a polytope's cyclic list(s) gives knowledge of the polytope's face lattice, but not vice versa, since defining a polytope in terms of cyclic lists also provides information about the orientation of each "connected component" of the polytope. Seeing how these languages interact with each other may be fruitful for all the languages involved.

Appendices

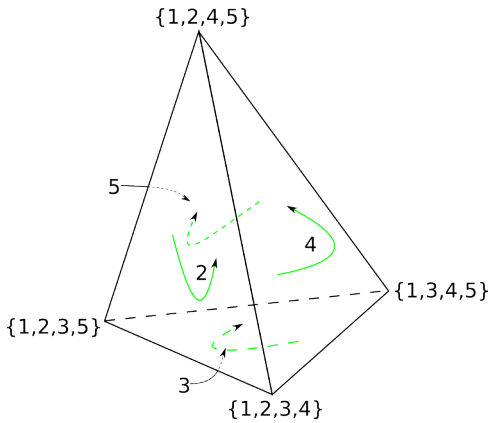
Appendix A

Four-dimensional Abstract Listed Simplex

In this section we will explicitly find the $20 = 5 \times 4$ non-trivial (i.e., non-empty) cyclic lists $\{l_{ij}\}$ with $i, j \in S = \{1, 2, 3, 4, 5\}$ needed to specify an abstract listed 4-simplex bounded by five faces.

We give the simplex an orientation by choosing an orientation of one of its bounding 3-simplices and letting that determine the orientation for the rest of the simplex. Namely, in Figure A.1 we show the 3-simplex sitting on the boundary 1, and we give it a particular (arbitrary) orientation.

Figure A.1



We note that the 3-simplex sitting on the boundary 1 is bounded by the faces 2,3,4, and 5, and the label 1 makes an appearance in each vertex because we should view Figure A.1 as lying entirely within the $\mathbb{CP}^3$ defined by Z_1^α. By recalling that a cyclic list for a combinatorial 4-polytope is labelled by two boundaries i, j, so that $l_{ij} = (...k, l, m...)$ gives the oriented edge $[k(ijl)m]$ which in turn gives the instructions

$\{k, i, j, l\} \to \{i, j, l, m\}$, we can then read off from this picture the following cyclic lists.

$$l_{12} = (3, 4, 5) \tag{A.1}$$

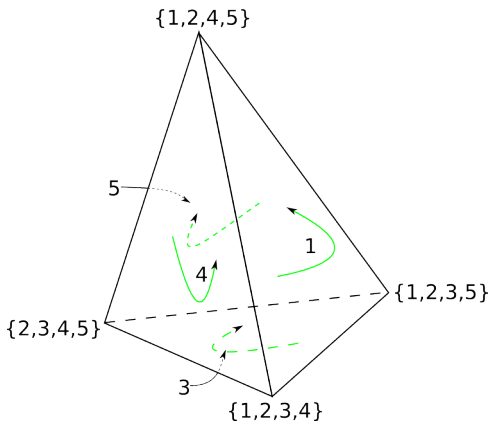
$$l_{13} = (4, 2, 5) \tag{A.2}$$

$$l_{14} = (2, 3, 5) \tag{A.3}$$

$$l_{15} = (4, 3, 2). \tag{A.4}$$

We note that by choosing, say, l_{12} to have the orientation it has, the other three lists are then completely determined. In fact, all of the remaining lists are completely determined. To see this, consider the simplex lying on the boundary 2. It will then be bounded by the four faces 1, 3, 4, and 5, and thus the 2-face defined by the intersection of boundaries 1 and 2 will be common to the 3-polytope sitting on boundary 1 and the 3-polytope sitting on boundary 2. Accordingly, we need these two 3-polytopes to be glued along this 2-face in a consistent way, namely such that one of these two-faces has the opposite orientation of the other. We can therefore immediately read off that $l_{21} = (3, 5, 4)$, which we note is an odd permutation of the list l_{12}. We can indeed read off the rest of the lists by requiring $E_{ij;k} = (-1)^{|\sigma|} E_{\sigma(ij;k)}$ for all $i, j, k \in \{1, 2, 3, 4, 5\}$ in a minimal way, however for the sake of pedagogy, we present in Figure A.2 the picture of the 3-simplex sitting on the boundary 2. In Figure A.2 we note that every

Figure A.2



vertex has a 2 included in it, as it should, and that the orientation of the 2-face sitting on boundary 1 is completely determined by its requirement for being opposite that of the 2-face sitting on boundary 2 in Figure A.1. We then read off from this figure

the following cyclic lists.

$$l_{21} = (3, 5, 4) \tag{A.5}$$
$$l_{23} = (4, 5, 1) \tag{A.6}$$
$$l_{24} = (3, 1, 5) \tag{A.7}$$
$$l_{25} = (1, 3, 4). \tag{A.8}$$

By repeating the process of ensuring that all 2-faces are properly glued, one readily obtains the remaining cyclic lists:

$$l_{31} = (4, 5, 2) \quad l_{32} = (5, 4, 1) \quad l_{34} = (1, 2, 5) \quad l_{35} = (1, 4, 2) \tag{A.9}$$
$$l_{41} = (2, 5, 3) \quad l_{42} = (3, 5, 1) \quad l_{43} = (1, 5, 2) \quad l_{45} = (2, 3, 1) \tag{A.10}$$
$$l_{51} = (4, 2, 3) \quad l_{52} = (1, 4, 3) \quad l_{53} = (1, 2, 4) \quad l_{54} = (2, 1, 3). \tag{A.11}$$

Thus, the 20 lists in (A.1-11) give one representation of the 20 equivalence classes of lists defining the 4-simplex (where the 5 lists of the form l_{ii}, $i \in \{1, 2, 3, 4, 5\}$ are all taken to be (equivalent to) the empty list). One can see that as the dimension of the polytope goes up, the number of lists needed to specify the polytope increases greatly (exponentially). However, there are only a few lists that are independent of the other lists. In the special case of a simplex (in any dimension), one single list determines the rest of them via the defining requirement for an abstract listed 4-polytope. These considerations motivate various definitions of connectivity and convexity for abstract listed polytopes, but these ideas are not yet fully developed and are outside the scope of this work at present.

Appendix B

Volume of the Four-dimensional Abstract Listed Simplex

We now show how to apply the four-dimensional calculus to the simple case of the four-dimensional simplex described in Appendix A. This will be more for practice with the four-dimensional calculus as well as a check on its validity, since the volume itself is immediately read off as [12345], by definition. Regardless, let us denote this volume by $A(\{l_{ij}\})$, and use the lists in (A.1-11). Theorem 4.30 tells us that

$$A(\{l_{ij}\}) = \frac{2}{(4!)^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{n_{ij}} f_{ijk_{ijl}k_{ij(l+1)}} \tag{B.1}$$

which, in our case, becomes

$$\begin{aligned} A(\{l_{ij}\}) =& \frac{2}{(4!)^2}\Big(f_{1234} + f_{1245} + f_{1253} + f_{1342} + f_{1325} + f_{1354} \\ &+ f_{1423} + f_{1435} + f_{1452} + f_{1543} + f_{1532} + f_{1524} \\ &+ f_{2135} + f_{2154} + f_{2143} + f_{2345} + f_{2351} + f_{2314} \\ &+ f_{2431} + f_{2415} + f_{2453} + f_{2513} + f_{2534} + f_{2541} \\ &+ f_{3145} + f_{3152} + f_{3124} + f_{3254} + f_{3241} + f_{3215} \\ &+ f_{3412} + f_{3425} + f_{3451} + f_{3514} + f_{3542} + f_{3521} \\ &+ f_{4125} + f_{4153} + f_{4132} + f_{4235} + f_{4251} + f_{4213} \\ &+ f_{4315} + f_{4352} + f_{4321} + f_{4523} + f_{4531} + f_{4512} \\ &+ f_{5142} + f_{5123} + f_{5134} + f_{5214} + f_{5243} + f_{5231} \\ &+ f_{5312} + f_{5324} + f_{5341} + f_{5421} + f_{5413} + f_{5432}\Big) \\ =& \frac{1}{4!}\big(f_{1234} + f_{2345} + f_{3451} + f_{4512} + f_{5123}\big) \\ =& [12345]. \end{aligned} \tag{B.2}$$

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