

# Nonparametric Engel Curves and Revealed Preference

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## Abstract

This paper applies revealed preference theory to the nonparametric statistical analysis of consumer demand. Knowledge of expansion paths is shown to improve the power of nonparametric tests of revealed preference. They are used to derive tight bounds on indifference surfaces and welfare measures. Nonparametric Engel curves are used to estimate expansion paths and provide a stochastic structure within which to examine the consistency of household level data and revealed preference theory. An application is made to a long time series of repeated cross-sections from the Family Expenditure Survey for Britain. The consistency of this data with revealed preference theory is examined. Where rejections do occur, suitable adjustments to prices for quality or taste changes are explored. For periods of consistency with revealed preference, tight bounds are placed on true cost of living indices.

**Keywords:** Consumer demands, nonparametric regression, revealed preference.

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# 1. Introduction<sup>1</sup>

The attraction of revealed preference theory is that it allows an assessment of the empirical validity of the usual integrability conditions without the need to impose particular functional forms on preferences. Although developed by Afriat (1973) and Diewert (1973) to describe individual demands, following the seminal work of Samuelson (1938) and Houthakker (1950), it has usually been applied to aggregate data but this presents a number of problems<sup>2</sup>. First, on aggregate data, ‘outward’ movements of the budget line are often large enough, and relative price changes are typically small enough, that budget lines rarely cross (see Varian (1982), Bronars (1987) and Russell (1992)). This means that aggregate data may lack power to reject revealed preference (RP) conditions. Second, if we do reject RP conditions on aggregate data we have no way of assessing whether this is due to a failure at the micro level or to the inappropriate aggregation across households that do satisfy the integrability conditions but who have different non-homothetic preferences. By combining nonparametric statistical methods with a revealed preference analysis of micro data we can overcome the problems we have described.

We also have a number of other motivations for this study. First, parametric demand studies on micro data often reject Slutsky symmetry which is one of the implications of utility maximisation subject to a linear budget constraint. Amongst the many possible explanations for this rejection are that either we have the ‘wrong’ functional form or that there exists no well-behaved form of preferences

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<sup>2</sup>See Manser and McDonald (1988), and references therein.

which can rationalise the data. Nonparametric analysis allows us to check this. Second, it has proven difficult to test for (global) negative semi-definiteness of the Slutsky matrix in parametric demand models. Using nonparametric revealed preference analysis we can simultaneously test for both symmetry and negative semi-definiteness. Third, if the integrability conditions are not rejected, we often wish to go on and use demand estimates for policy analysis. Using parametric analysis there is always some uncertainty as to how much the welfare conclusions are driven by functional form. If we employ nonparametric techniques then we can obtain bounds on welfare effects and use these bounds to judge the importance of the choice of functional form on welfare conclusions. Fourth, the nonparametric analysis can aid in the development of new and parsimonious parametric demand systems. Finally, we can extend the nonparametric analysis to investigate revealed preference for conditional demands.

The layout of the paper is as follows. In Section 2 a method for choosing a sequence of total expenditures that maximise the power of the test of GARP with respect to a given preference ordering is developed. We then consider the use of conditional demands and separability in tests of the Generalised Axiom of Revealed Preference (GARP) when preferences for a particular good, or group of goods, may be changing over time. In this section we also develop a method of bounding true cost of living indices. Two algorithms are presented which give ‘tightest’ upper and lower bounds to a level set of utility passing through any point in commodity space chosen.

Section 3 presents a framework for implementing our procedures by using nonparametric Engel curves for each commodity. To do this we assume that households in the same time period and location face the same relative prices. Under this assumption, the nonparametric Engel curves correspond to expansion paths for each price regime. In estimation we address two key issues that arise when placing local average demands in a structural economic context. First, we consider the problem of pooling nonparametric Engel curves across households of

different demographic composition. We show that a partially linear model that allows for demographic variation (see, for example, Robinson (1988)) has the very unattractive property that it reduces to Piglog demands (budget shares are linear in log total outlay) under homogeneity and symmetry. We then show that the shape invariant model of Härdle and Marron (1990) provides a theory consistent generalisation to the partially linear semiparametric method of pooling nonparametric Engel curves across household of different composition. Second, we allow for the endogeneity of log total expenditure in the nonparametric budget share equations. We do this by completing the model with a reduced form specification for log total expenditure in terms of log disposable income. The residual from this reduced form regression is added to the nonparametric Engel curve regressions to control for the endogeneity of total expenditure. This augmented regression equation from this control function approach has a partially linear form and can be estimated using our semiparametric estimator. Finally in section 3 we discuss the issues surrounding unobservable preference heterogeneity. Even controlling for demographic composition taking two households that are similar in time, place and total expenditure we usually find that demand patterns are quite different. This makes the application of (RP) nonparametric techniques to micro data problematic. Even taking a small number of households in different price regimes usually leads to a rejection of the nonparametric conditions (see Koo (1963), Mossin (1972) and Mattei (1994), for example, and the recent paper by Sippel (1997) on the use of experimental data). We discuss the usefulness of working with local average demands in the presence of unobserved heterogeneity and derive a measure of the bias that results in measuring the welfare cost of finite price changes.

In Section 4 we discuss the data and present an empirical investigation of revealed preference using British Family Expenditure Survey data from 1974 to 1993. From this long time series of cross-sections we estimate the associated nonparametric Engel curves for 22 goods, adjusted for endogeneity and demographic

composition, to examine whether revealed preference theory can be rejected for particular sub-periods of the data. From the asymptotic distribution theory for nonparametric regression we are able to provide a statistical structure within which to examine the consistency of data with revealed preference theory without imposing a global parametric structure to preferences. The approach we adopt provides an alternative to the Afriat inefficiency measure explored in Famulari (1995) and Mattei (1994). We find that GARP is not rejected for long periods of our data for most income groups. Finally, we investigate how far failures of GARP can be attributed to changes in preferences for tobacco. We find that rejections of GARP for the median maximum power path can be attributed to a single comparison between 1985 and 1986. Over this period tobacco demand on this path fell by nearly 9%. A 10% price adjustment for tobacco in 1986 is period is shown to be sufficient for GARP to be satisfied over the whole 20 year period. This series is then used in Section 5 to compute bounds for the true cost of living. These are shown to provide large improvements on classical revealed preference bounds. Section 6 concludes with a summary of our results and a consideration of future directions.

## 2. Individual Data and Revealed Preference

### 2.1. Revealed Preference and Observed Demands

Suppose we wished to test experimentally whether a particular agent had ‘rational’ and stable preferences. In the context of demand, we could do this by facing them with a series of prices and total expenditures and testing whether their demand responses satisfy the Slutsky conditions. Specifically, if we have  $T$  time periods and given an  $n$ -vector of (positive) prices  $\mathbf{p}_t$  in each period  $t$  we could present the agent with a series of (positive) total expenditures  $x_t$ . A critical assumption we make is that the agent will always respond with a unique demand for each price vector and outlay:

**Assumption 1.** There exists a set of demand functions  $\mathbf{q}(\mathbf{p}, x) : \mathfrak{R}_{++}^{n+1} \rightarrow \mathfrak{R}_+^n$

which satisfy adding-up:  $\mathbf{p}'\mathbf{q}(\mathbf{p}, x) = x$  for all prices  $\mathbf{p}$  and total outlays  $x$ .

Denote the corresponding  $n$ -valued function of  $x$  for a given price vector  $\mathbf{p}_t$  as  $X(\mathbf{p}_t)$  which we will refer to an *expansion path* for the given prices. We shall refer to the  $n$ -vector of demands at expenditure level  $x$  under price  $n$ -vector  $\mathbf{p}_t$  as  $\mathbf{q}_t(x)$ . This reflects demands at expenditure level  $x$  on expansion path  $X(\mathbf{p}_t)$ . We shall also have need of the following assumption below:

**Assumption 2.** Weak normality: if  $x > x'$  then  $\mathbf{q}_t^j(x) \geq \mathbf{q}_t^j(x')$  for all  $j$  and all  $\mathbf{p}_t$ .

Adding up and weak normality imply that at least one of the inequalities in this assumption is strict (and that expansion paths are continuous).

For our hypothetical experiment we would observe the demands for the given prices and total outlays and test whether the resulting time series of  $n$ -vector demands  $\mathbf{q}_t$  satisfy revealed preference tests. To do this, construct a  $(T \times T)$  matrix  $M$  in which, for each pairwise comparison the  $(t, s)$  element defines an indicator variable:

$$m^{ts} = 1[\mathbf{p}'_t \mathbf{q}_t \geq \mathbf{p}'_t \mathbf{q}_s] \text{ for all } t, s = 1, \dots, T. \quad (2.1)$$

which is one when the revealed preference comparison in parentheses is satisfied (see Varian (1982)) and zero otherwise. We say that  $\mathbf{q}_t$  is *directly revealed weakly preferred* to  $\mathbf{q}_s$ ,  $(\mathbf{q}_t R^0 \mathbf{q}_s)$  if the period  $s$  vector of quantities is affordable at period  $t$  prices and total expenditure. If the inequality in (2.1) is strict then we say that  $\mathbf{q}_t(x_t)$  is *directly revealed strictly preferred* to  $\mathbf{q}_s$   $(\mathbf{q}_t P^0 \mathbf{q}_s)$  since the agent could have obtained the latter more cheaply (at the prices  $\mathbf{p}_t$ ) but chose not to.

Now consider a sub-sequence of periods  $\{s, t, u, \dots, v, w\}$  where the order matters. We say that the sub-sequence of total expenditures  $\{x_s, x_t, x_u, \dots, x_v, x_w\}$  is *preference ordered* if  $\{m^{st}, m^{tu}, \dots, m^{vw}\} = \{1, 1, \dots, 1\}$ . Thus a sub-sequence of total expenditures is preference ordered if the demand associated with any total outlay is revealed at least as good as the next one. Given this, we define an *indirect revealed preference relationship*:  $\mathbf{q}_s(x_s)$  is *indirectly revealed weakly preferred* to

$\mathbf{q}_w(x_s)$  if there is a preference ordered sub-sequence starting in  $s$  and ending in  $w$ ; we denote this by  $\mathbf{q}_s(x_s) R \mathbf{q}_w(x_w)$ . Given a matrix  $M$  of direct comparisons we can construct a matrix  $\widetilde{M}$  of indirect comparisons by taking the transitive closure of  $m$ ; Varian (1982) shows that this can be achieved inexpensively using Warshall's algorithm. Suppose now that we have a preference ordered sub-sequence  $\{x_s, x_t, x_u, \dots x_v, x_w\}$  and that we also have that  $\mathbf{q}_w(x_w)$  is directly revealed strictly preferred to  $\mathbf{q}_s(x_s)$  so that:

$$\mathbf{p}'_w \mathbf{q}_w(x_w) > \mathbf{p}'_w \mathbf{q}_s(x_s) \quad (2.2)$$

In this case we say that this sub-sequence fails GARP, the general axiom of revealed preference. We shall say that the set of total outlays  $\{x_1, x_2, \dots x_T\}$  fails GARP if some sub-sequence (for example,  $\{x_4, x_1, x_7\}$ ) fails GARP.

Below, we shall also make use of the Afriat numbers. In terms of the Afriat inequalities (Varian (1982, p. 949)),  $\mathbf{q}_s R \mathbf{q}_w$  implies that there exist numbers  $U_s, U_w, \lambda_w > 0$  such that

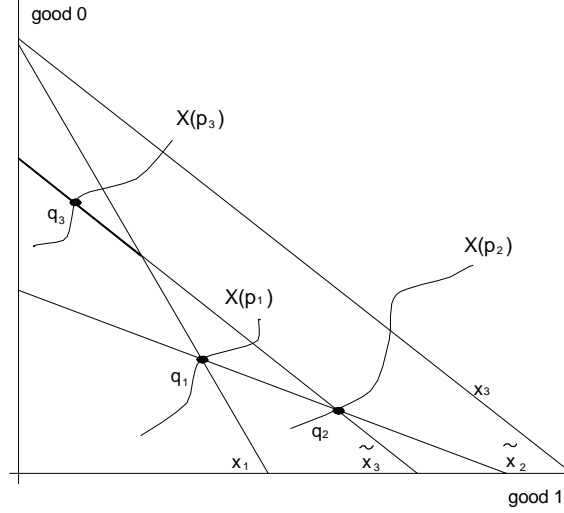
$$U_s \leq U_w + \lambda_w \mathbf{p}'_w (\mathbf{q}_s - \mathbf{q}_w). \quad (2.3)$$

If (2.2) holds then  $\mathbf{p}'_w (\mathbf{q}_s - \mathbf{q}_w) < 0$ , and since  $\lambda_w > 0$  it must be that  $U_s < U_w$  which is not consistent with  $\mathbf{q}_s R \mathbf{q}_w$  and consequently a failure of GARP.

## 2.2. Choosing a Path for Comparison Points

The choice of the sequence of total expenditures  $x_t$  used in the comparisons above requires some discussion. There is a well known problem with applying GARP tests to data to which Varian (1982) refers in his applied work. This problem arises since, particularly with annual data, income growth over time can swamp variations in relative prices (which are what we are interested in). This is because real income growth induces outward movements of the budget constraint and, combined with typically small period-to-period relative price movements, this means that budget lines may seldom cross. As a result, data often lacks

Figure 2.1: Testing GARP with Expansion Paths



power to reject GARP. Indeed, if we choose the  $x_t$ 's so that budget lines never cross then we can never violate the GARP conditions. Clearly then, with a given set of relative prices the power of a revealed preference test will depend critically on the choice of the outlay path  $(x_1, x_2, \dots, x_T)$ .

One possible solution is to choose a sequence of constant 'real' total expenditures. Thus given  $x_1$  and a set of price indices  $(P_1(\mathbf{p}_1), P_2(\mathbf{p}_2), \dots, P_T(\mathbf{p}_T))$  we could choose  $x_t = x_1 P_t / P_1$ . Although superficially attractive this begs the question of what price index to use. More importantly, even if the series of demands generated in this way did satisfy GARP, we cannot be sure that any other series of total expenditures starting from  $x_1$  would also satisfy GARP. Instead of this, we devise a simple algorithm for determining a sequence of  $x_t$  points through the data which maximises the chance of finding a rejection given a particular preference ordering of the data.

Suppose we have a sequence of demands and consider any preference ordered



sequence  $\{x_s, x_t, x_u, \dots, x_v, x_w\}$ <sup>3</sup>. The algorithm for choosing the most powerful path for this preference ordered sequence is a recursive scheme. Given total expenditure in the last period in the sequence,  $x_w$ , total outlay in the second to last period,  $v$ , is chosen so that the period  $w$  bundle is just affordable at the period  $v$  prices; denote this  $\tilde{x}_v = \mathbf{p}'_v \mathbf{q}_w(x_w)$ . Thus  $\mathbf{q}_v(\tilde{x}_v)$  is directly revealed weakly preferred to  $\mathbf{q}_w(x_w)$ . Then total outlay in the previous period is chosen so that  $\mathbf{q}_v(\tilde{x}_v)$  is just affordable and so on. Thus the *sequential maximum power* (SMP) path for the preference ordered sequence  $\{x_s, x_t, x_u, \dots, x_v, x_w\}$  is given by:

$$\{\tilde{x}_s, \tilde{x}_t, \tilde{x}_u, \dots, \tilde{x}_v, x_w\} = \{\mathbf{p}'_s \mathbf{q}_t(\tilde{x}_t), \mathbf{p}'_t \mathbf{q}_u(\tilde{x}_u), \dots, \mathbf{p}'_v \mathbf{q}_w(\tilde{x}_w), x_w\} \quad (2.4)$$

By construction, any SMP path is itself preference ordered. It is defined by the starting point  $\mathbf{q}_w(x_w)$  and the sequence  $\{s, t, u, \dots, v, w\}$  of relative prices.

Figure 2.1 illustrates a three period, two good example in which the order of the sub-sequence is  $\{\mathbf{q}_3 R^0 \mathbf{q}_2 R^0 \mathbf{q}_1\}$ . In figure 2.1 the shaded part of the period 3 budget line gives the demand points on  $\tilde{x}_3$  which result in a rejection of GARP. Intuitively, one can see that this path is going to maximise the probability of finding some rejection in the sense that ‘pushing out’ either of the period 2 or 3 budget lines will reduce the length of the rejection region. If demands are normal then this reduces the chance of observing a demand in that region. More formally, we have:

**Proposition 1.** *Suppose that the budget sequence  $\{x_s, x_t, x_u, \dots, x_v, x_w\}$  has a preference ordered sub-sequence that rejects GARP. If demands are weakly normal then the SMP path for that sub-sequence also rejects GARP.*

**Proof.** See Appendix A. ■

Thus, if we test for GARP along a given SMP path from a given starting point,  $\mathbf{q}_w(\hat{x}_w)$  say, and we do *not* reject, then we can be confident that we would

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<sup>3</sup>We are only interested in sequences over the whole data period in which there is some possibility of rejecting GARP. This implies that there must be at least one preference ordered sub-sequence.

not reject for any other path which starts from the same total expenditure and maintains the utility non-decreasing ordering implied by the SMP path.

Finally, note that to implement the SMP procedure we need expansion paths for each price regime. In the empirical work we first estimate these nonparametrically and then use the estimated expansion paths to generate SMP paths starting at different points in the total outlay distribution; full details are given below.

### **2.3. The Power of Parametric and Revealed Preference Tests of Integrability.**

The previous sub-section considered one aspect of power. When considering tests of integrability, whether parametric or nonparametric, other power considerations are also important. In considering the relative power of the two modes of analysis we must be careful to allow that there are some alternatives against which both modes of test will have low power. To illustrate with a well known example, suppose we draw a large independent sample each period from a large population of agents. If each agent in each period chooses demands on their budget surface by drawing from a uniform distribution on the budget surface then in general no individual path of demands will be integrable. However the (population and sample) mean data will appear to be generated by a Cobb-Douglas utility function with weights equal to the inverse of the number of commodities (see Becker (1962) and Grandmont (1992)). Parametric and revealed preference tests are unlikely to reject the integrability conditions for such data but it is not clear that we would wish to characterise them as the outcome of a ‘rational’ procedure. Equally there will be paths of relative prices which lead to low power tests of the integrability conditions under certain alternatives. The extreme case is if we have no variation in relative prices in which case, of course, we cannot estimate price effects for parametric models and we have only one expansion path for our GARP tests<sup>4</sup>.

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<sup>4</sup>Interestingly, it turns out that the power of tests of integrability are not necessarily increasing in relative price variability but since we take the relative price path as given and this problem applies to both modes of testing we do not pursue this issue here.

Thus many of the concerns with the power of tests of the integrability conditions are common to both parametric and revealed preference tests. There is also a concern, however, that revealed preference tests are inherently lacking in power (as compared with parametric tests) and will fail to reject ‘too often’. This view is based on the belief that if we allow ourselves almost unlimited latitude in capturing observed (mean) demands, except for satisfying the revealed preference conditions, then in some cases we will only be able to rationalise the data with ‘convoluted’ preferences which could only be approximated by a parametric utility function with very many parameters. In this view some flexibility is a good thing but we can also have too much of this good thing. This is, of course, the flip side of designing tests that do not impose any functional form. We emphasise again that one of our concerns regarding currently used parametric models is that they may be too inflexible and in particular they may unduly restrict differences in price effects between rich and poor. Nevertheless, the concern remains.

To investigate this issue we consider three alternative generating processes that produce non-integrable demands: a random procedure, an integrable path with measurement error and a path generated by a slow adjustment model. Note that all of the calculations below use the actual sequence of relative prices observed in our data which is the relevant set of relative prices. For the random alternative we suppose that the demand at any price/income configuration is a draw from a uniform distribution on the budget surface (just as we assumed for individual agents in the illustration above but without the averaging). The SMP procedure with a given sequence of prices is: choose  $x_1$  and draw the vector  $\mathbf{q}_1$  from a uniform distribution on the budget surface given by  $(\mathbf{p}_1, x_1)$ . Then set  $\hat{x}_2 = \mathbf{p}_2' \mathbf{q}_1$  and draw  $\hat{\mathbf{q}}_2$  from a uniform distribution on the new budget surface. Continue for all  $T$  periods. We can show analytically that if we have only two periods and two goods then GARP will only reject half of the time. This indicates low power. On the other hand, as the number of periods grows the probability of rejecting grows. The actual rejection probability depends on the number of periods and

the relative price variability. To illustrate this we take the actual sequence of relative prices we have in our data (for 22 goods over 20 years; details are given in the empirical section below) and generate demands according to this alternative. We found that in 10,000 random simulated SMP paths we reject GARP every time. This indicates that our procedure does have considerable power against this particular alternative. However, a sceptic might argue that any procedure that failed to reject the rationality of such an unstructured alternative would be very poor indeed. Thus we also consider two other alternatives which are 'close' to integrable.

Our second procedure is to take a set of demands that are integrable and to incorporate a multiplicative measurement error.<sup>5</sup> Specifically, in each period we draw a  $(22 \times 1)$  vector of budget shares from a joint distribution in which each budget share has a probability limit equal to the average budget shares over the whole sample; see Appendix B for details. This is equivalent to taking a Cobb-Douglas utility function over 22 goods with the average budget shares as utility weights and then multiplying by a measurement error with a unity probability limit. Clearly, if we set the variance of the budget share draws in this procedure to zero then we have a path of Cobb-Douglas demands which satisfy GARP. Conversely, if we allow for a great deal of measurement error then we shall almost certainly reject GARP. The critical issue, then is how much measurement error is it reasonable to allow for? We calibrate this to the variance of the budget shares in our data which gives an upper bound on measurement error. We then take different proportions of these variances and simulate 1000 times and record the proportion of rejections. We find that even very modest amounts of noise cause rejection. For example, if we allow for only 0.5% (respectively 1% and 2%) of the total variance to be due to noise and use these in our simulations then we reject 61%, (respectively, 87% and 97%) of the time. Thus the GARP/SMP procedure

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<sup>5</sup> An alternative interpretation is that for each price regime we generate a sample which is an independent draw from the same population with a given distribution of heterogeneity over the preference parameters.

has considerable power against this alternative.

The third demand generating process we consider is a ‘naive’ adjustment model. In this we assume that households adjust slowly to the optimum for the prices in that period. Specifically, if we take (integrable) demands  $\mathbf{q}(\mathbf{p}, x)$ , we set the period  $t$  demand  $\tilde{\mathbf{q}}(\mathbf{p}_t, x_t)$  to:

$$\tilde{\mathbf{q}}(\mathbf{p}_t, x_t) = \lambda \mathbf{q}(\mathbf{p}_t, x_t) + (1 - \lambda) \tilde{\mathbf{q}}(\mathbf{p}_{t-1}, x_{t-1}) \text{ for } t = 2, \dots, T$$

Thus the sequence of demands will be integrable if we set  $\lambda = 1$  but as adjustment becomes slower, the likelihood of rejecting GARP increases. Note that this system satisfies ‘long run’ integrability. For the demand functions we use a Quadratic Almost Ideal System (QUAIDS) (see Banks *et al* (1997)) with parameters estimated on our sample and the homogeneity and symmetry conditions imposed<sup>6</sup>. For the first period demands at a given outlay  $x_1$  we set  $\mathbf{q}_1 = \mathbf{q}(\mathbf{p}_1, x_1)$  and, as before, subsequent total expenditures are chosen using the SMP path for our price data. Doing this, we find that for the path starting at median first period total outlay, we reject GARP if and only if  $\lambda < 0.26$ . Such a low figure suggests that our testing procedure is unlikely to have good power against an alternative that satisfies the integrability conditions in the long run. Once again, we emphasise that the same may be true of alternative parametric procedures.

## 2.4. Preference Change, Conditional Demands, Separability and GARP

Often we suspect there is some good, or group of goods, which is considered to be rationed or subject to some unmeasured change in quality, preference or habit formation, and is also not separable from the group of goods under study. For example, demands for tobacco consumption are very likely to be subject to changes in preference (and/or quality) following government health announcements over the period of study. It is unlikely that the level and participation of tobacco

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<sup>6</sup>We do not impose the negativity conditions on our parameter estimates but we note that the ‘full adjustment’ paths generated by our simulations starting at median total expenditure do pass GARP.

consumption is therefore fully rationalisable by a set of stable preferences over this period. However, it is also likely that preferences over certain other goods of interest, such as beer, wine, spirits and entertainment are directly affected by tobacco consumption; that is, they do not form a subgroup which is separable from tobacco. In this case it is common in empirical demand analysis to work with conditional demands (see for example Browning and Meghir (1991)). That is, demands conditional on the level of tobacco consumption may be rationalised even though for the (unconditional) set of goods with tobacco included this would not be the case. Similarly, the (unconditional) set of goods excluding tobacco would also not be rationalised in the case where they were not separable from tobacco consumption. If there is an argument that preferences for tobacco may have changed over the period, then there is good reason to expect that a data set which includes tobacco will fail a test of GARP. If this is so, then it means that separability is formally, as well as intuitively, rejected and we cannot simply omit tobacco from the set of goods considered<sup>7</sup>.

Consider instead the case of  $n$  goods in which the ‘conditioning’ good  $q^1$  is subject to some ration or quality change and preferences over the remaining ‘goods of interest’  $q^2, \dots, q^n$  ( $= \mathbf{q}^{n-1}$ ) are thought to behave according to rational consumer theory. Note that if preferences are non-separable from the conditioning good. and we do not observe a price (quantity), we can always find a price (quantity) which will rationalise the set of prices and quantities for the goods of interest (see Varian (1986)). Note further that a ‘missing’ good makes it impossible to test GARP.

Define  $\mu_t p_t^1$  as an adjusted price which is decomposed into the observed market component  $p_t^1$  and an adjustment factor  $\mu_t$  (with  $\mu_1$  normalised to unity). The restrictions imposed by GARP in this case can be shown to imply a set of concavity conditions for the maximisation of some continuous, concave, monotonic and non-satiated utility function defined over  $\mathbf{q}$  conditional on  $q^1$ :

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<sup>7</sup>Varian (1983) provides a discussion of GARP tests of separability.

**Proposition 2.** *The data  $(q_1^1, \dots, q_T^1, \mathbf{q}_1^{n-1}, \dots, \mathbf{q}_T^{n-1}; p_1^1, \dots, p_T^1, \mathbf{p}_1^{n-1}, \dots, \mathbf{p}_T^{n-1})$  can be rationalised iff there exist numbers  $U_s, U_t, \lambda_s > 0$  and  $\mu_s$  such that*

$$U_t \square U_s + \lambda_s \mathbf{p}_s^{n-1'} (\mathbf{q}_t^{n-1} - \mathbf{q}_s^{n-1}) + \lambda_s \mu_s p_s^1 (q_t^1 - q_s^1) \quad (2.5)$$

**Proof.** See Appendix A. ■

Since the preference change model can be rewritten as a stable preference model with virtual prices the usual GARP restrictions apply with the actual price of the conditioning good replaced by the adjusted price. Thus allowing for a conditioning good is as though we can choose a price for this good that is different from the observed market price. If we can find  $\mu_s$ 's for each period that equal unity then we can rationalise the data on all  $n$  goods. But if GARP is rejected for the full set of goods, the addition of the extra free variables  $\mu_s$  may make it possible to rationalise the conditional demands for the goods of interest. Formally,  $\mu_s p_s^1$  is the virtual price for the conditioning good in period  $s$ . If agents like the conditioning good less over time then we would expect to find that  $\mu_t > \mu_s$  for  $t > s$ ; that is, it is as though the *virtual* price of the conditioning good is rising over time. Adding more conditioning goods further relaxes the restrictions GARP places on the observed data. Suppose that we have the following rejection of GARP:  $\mathbf{q}_t R \mathbf{q}_s$  and  $\mathbf{q}_s P^0 \mathbf{q}_t$ . The minimum adjustment to the price of the conditioning good in period  $s$  such that is  $\mathbf{q}_s$  not  $P^0 \mathbf{q}_t$  is given by setting

$$\mu_s p_s^1 = \frac{\mathbf{p}_s^{n-1'} (\mathbf{q}_t^{n-1} - \mathbf{q}_s^{n-1})}{q_s^1 - q_t^1} \quad (2.6)$$

If  $(q_s^1 - q_t^1) < (>) 0$  then this is a lower (upper) limit for the adjusted price (denoted by  $\underline{p}_s^1$  ( $\bar{p}_s^1$ )). If  $(q_s^1 - q_t^1)$  and  $\mathbf{p}_s^{n-1'} (\mathbf{q}_t^{n-1} - \mathbf{q}_s^{n-1})$  have different signs then the minimum necessary adjustment will result in a negative price. If this is an upper limit then no positive price can rationalise the data<sup>8</sup>.

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<sup>8</sup>Suppose that there are two rejections:  $\mathbf{q}_r P^0 \mathbf{q}_s$  and  $\mathbf{q}_r P^0 \mathbf{q}_t$ , while the SMP path induces

**Proposition 3.** *Suppose we observe  $\mathbf{q}_t R \mathbf{q}_s$  and  $\mathbf{q}_s P^0 \mathbf{q}_t$ . If  $(q_s^1 - q_t^1) > 0$  any  $p_s^1 > \bar{p}_s^1$  will violate GARP.*

**Proof.** See Appendix A. ■

Similarly if  $(q_s^1 - q_t^1) < 0$  any  $p_s^1 < \underline{p}_s^1$  will violate GARP (the proof is identical to that for Proposition 3.)

## 2.5. Computing Tight Bounds on Welfare Measures

Afriat (1977) showed how revealed preference restrictions can be used to provide information on the curvature of indifference surfaces in commodity space and used to set bounds on the welfare effects of a price change. This is further developed in Varian (1982) and Manser and McDonald (1988). However, these sorts of ideas only work well when budget surfaces cross and this, for the reasons discussed above, may be rare. In practice the bounds tend to be wide<sup>9</sup>. Knowledge of expansion paths can vastly improve these bounds. In this section we consider an indifference surface passing through some base bundle  $\mathbf{q}_0$ . We first characterise the upper and lower bounds on the indifference surface using the notion of the SMP path discussed earlier. We then present an algorithm which searches efficiently for these bounds.

Let the  $\overline{\mathbf{Q}}$  denote the set of *all* SMP paths which *begin* at  $\mathbf{q}_0$  and which represent cycles of *all* of the periods (all the possible SMP paths defined over subperiods emanating from  $\mathbf{q}_0$  are automatically included in this set): all bundles in  $\overline{\mathbf{Q}}$  are revealed preferred to  $\mathbf{q}_0$ . Similarly let the  $\underline{\mathbf{Q}}$  denote the set of *all* SMP paths which end at  $\mathbf{q}_0$  and which represent cycles of *all* of the periods:  $\mathbf{q}_0$  is revealed preferred to all other bundles in  $\underline{\mathbf{Q}}$ . We now define the set  $\mathbf{Q}_P$  to be the

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that  $\mathbf{q}_t R^0 \mathbf{q}_s R^0 \mathbf{q}_r$ . If one rejection requires  $p_r^1 \geq \underline{p}_r^1$ , and the other requires  $p_r^1 \leq \bar{p}_r^1$  and if  $\bar{p}_r^1 > \underline{p}_r^1$  then no value for  $p_r^1$  in the interval will cause a violation of GARP. If  $\bar{p}_r^1 < \underline{p}_r^1$  then there exists no value for  $p_r^1$  which gives GARP. Under some circumstances a single adjustment designed to address one rejection may either introduce rejections elsewhere and/or correct others.

<sup>9</sup>Varian (1983) and Manser and McDonald (1988) tighten the bounds using a maintained assumption of homotheticity.



subset of  $\overline{\mathbf{Q}}$  consisting of bundles which are not directly revealed strictly preferred to any other bundle lying on the same expansion path; we also define the set to be the subset of  $\underline{\mathbf{Q}}$  consisting of bundles which are directly revealed strictly preferred to all other bundles lying on the same expansion path.

**Definition.** We define the lower bound on  $\overline{\mathbf{Q}}$  by  $\mathbf{Q}_P$  where

$$\mathbf{Q}_P = \{ \mathbf{q}_i \in \overline{\mathbf{Q}} : \mathbf{p}'_i \mathbf{q}_i \square \mathbf{p}'_i \tilde{\mathbf{q}}_i \ \forall \ \tilde{\mathbf{q}}_i \in \overline{\mathbf{Q}} \}$$

and we define the upper bound on  $\underline{\mathbf{Q}}$  by  $\mathbf{Q}_W$  where

$$\mathbf{Q}_W = \{ \mathbf{q}_i \in \underline{\mathbf{Q}} : \mathbf{p}'_i \mathbf{q}_i > \mathbf{p}'_i \tilde{\mathbf{q}}_i \ \forall \ \tilde{\mathbf{q}}_i \in \underline{\mathbf{Q}} \}.$$

The main property of  $\mathbf{Q}_P$  and  $\mathbf{Q}_W$  is set out in Propositions 4 below.

**Proposition 4.** (i) *Iff the data  $(\mathbf{p}, \mathbf{Q}_P)$  do not violate GARP and demands are weakly normal then  $\mathbf{Q}_P$  is the lowest upper bound on the indifference surface referenced at  $u(\mathbf{q}_0)$ .* (ii) *Iff the data  $(\mathbf{p}, \mathbf{Q}_W)$  do not violate GARP and demands are weakly normal then  $\mathbf{Q}_W$  is the highest lower bound on the indifference surface referenced at  $u(\mathbf{q}_0)$*

**Proof.** See Appendix A. ■

Figure 2.2 illustrates the sets  $\mathbf{Q}_P$  and  $\mathbf{Q}_W$  in a two-good, four-period example in which

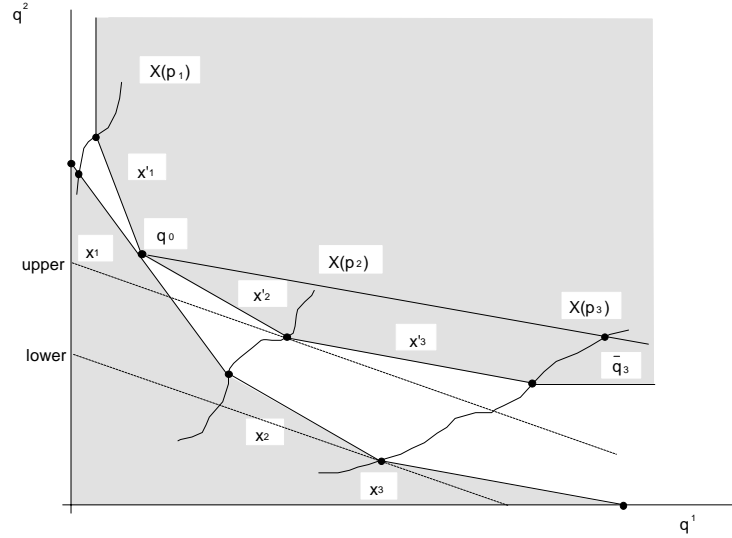
$$\mathbf{Q}_P = \{ \mathbf{q}_0, \mathbf{q}_1(x'_1), \mathbf{q}_2(x'_2), \mathbf{q}_3(x'_3) \}$$

and

$$\mathbf{Q}_W = \{ \mathbf{q}_0, \mathbf{q}_1(x_1), \mathbf{q}_2(x_2), \mathbf{q}_3(x_3), x_3/p_3^1, x_1/p_1^2 \}.$$

In principle we can proceed in three stages: Firstly calculate the sets  $\overline{\mathbf{Q}}$  and  $\underline{\mathbf{Q}}$ ; secondly extract the subsets  $\mathbf{Q}_P$  and  $\mathbf{Q}_W$ ; finally test  $(\mathbf{p}, \mathbf{Q}_P)$  and  $(\mathbf{p}, \mathbf{Q}_W)$  for violations of GARP. However, if there are  $T + 1$  periods including period 0 then  $\overline{\mathbf{Q}}$  will have  $(T!T) + 1$  elements :  $T$  on each of  $T!$  SMP paths plus  $\mathbf{q}_0$  itself. Instead, we present two algorithms designed to search efficiently for  $\mathbf{Q}_P$  and  $\mathbf{Q}_W$ . Algorithm A.1 searches for  $\mathbf{Q}_P$  and Algorithm A.2 searches for  $\mathbf{Q}_W$ . Both are presented in detail with illustration in Appendix C.

Figure 2.2: Improving Bounds with Expansion Paths



The algorithm which searches for  $\mathbf{Q}_P$  begins by computing the first points on all of the SMP paths emanating from  $\mathbf{q}_0$ . There are  $T$  of these, one on each expansion path. If any of these bundles are revealed strictly preferred to any other it is discarded as all subsequent points on all subsequent SMP paths which continue from this point must be indirectly revealed strictly preferred to  $\mathbf{q}_0$  and hence cannot be elements of  $\mathbf{Q}_P$ <sup>10</sup>. The subset which are all revealed preferred to  $\mathbf{q}_0$  (but not strictly so) are saved. In the next iteration all possible first steps on SMP paths continuing from each point in this set are calculated (there are  $T$  for each member of the set of point saved in the first iteration). Again, any which are revealed strictly preferred to  $\mathbf{q}_0$  (either directly or indirectly) are discarded from this set. At the end of each iteration demand bundles with the main properties of  $\mathbf{Q}_P$  are saved: all are on SMP path from  $\mathbf{q}_0$  and hence<sup>11</sup> revealed preferred to  $\mathbf{q}_0$  and none are revealed strictly preferred to  $\mathbf{q}_0$  (a property of  $\mathbf{Q}_P$  iff the data pass

<sup>10</sup>For a proof of this see the proof of Proposition 4(i) in Appendix A

<sup>11</sup>For a proof of this see also the proof of Proposition 4(i) in Appendix A.

GARP<sup>12</sup>). An illustration of the algorithms can be found in the Appendix. Finally, note the stopping condition for both algorithms is that the datasets  $(\mathbf{p}, \mathbf{Q}_P)$  and  $(\mathbf{p}, \mathbf{Q}_W)$  contain no revealed strict preferences. As a corollary of Proposition 4 we have the following.

**Corollary 4.1.** (i) *Iff the dataset  $(\mathbf{p}, \mathbf{Q}_P)$  violates GARP then Algorithm A.1 will fail to converge on  $\mathbf{Q}_P$ .* (ii) *Iff the dataset  $(\mathbf{p}, \mathbf{Q}_W)$  violates GARP then Algorithm A.2 will fail to converge on  $\mathbf{Q}_W$ .*

**Proof.** See Appendix A. ■

A consequence of these corollaries is that these bounds (by providing tightest bounds on the indifference curve) will also provide the tightest bounds on the welfare effects of a price change. For example, bounds can be placed on the true cost-of-living index based at  $u(\mathbf{q}_0)$  by finding the maximum expenditure level such that the price vector  $\mathbf{p}_z$  is tangent to the upper bound (i.e.  $\min \{\mathbf{p}'_z \mathbf{q}_t | \mathbf{q}_t \in \mathbf{Q}_P\}$ ), and the minimum expenditure level such that it is tangent to the lower bound (i.e.  $\min \{\mathbf{p}'_z \mathbf{q}_t | \mathbf{q}_t \in \mathbf{Q}_W\}$ ). Hence

$$\frac{\min \{\mathbf{p}'_z \mathbf{q}_t | \mathbf{q}_t \in \mathbf{Q}_W\}}{\mathbf{p}'_0 \mathbf{q}_0} \square \frac{c(\mathbf{p}_z, u(\mathbf{q}_0))}{c(\mathbf{p}_0, u(\mathbf{q}_0))} \square \frac{\min \{\mathbf{p}'_z \mathbf{q}_t | \mathbf{q}_t \in \mathbf{Q}_P\}}{\mathbf{p}'_0 \mathbf{q}_0}. \quad (2.7)$$

These bounds, labelled ‘upper’ and ‘lower’ are illustrated in figure ???. In section 5 we use these results together with nonparametric estimates of Engel curves to compute upper and lower bounds on the true cost of living index using British household survey data. These are then compared to more standard cost of living indices.

### 3. Nonparametric Engel Curves

#### 3.1. Kernel Estimation of the Budget Share System

To estimate the expansion paths for each price regime we employ nonparametric regression methods. The precise transformation of the expansion path

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<sup>12</sup>For a proof of this see also the proof of proposition 4(i) in Appendix A.

relationship used in estimation is chosen with respect to prior knowledge on the shape of Engel curves. The earlier empirical investigations by Working (1943) and Leser (1963) established the Piglog form, in which the budget share for each commodity is a linear function of log total budget, as a reasonable specification. Subsequent models of consumer behaviour have typically followed this specification for Engel curves, although many have pointed out the restrictive nature of this specification for many commonly used commodities (see Banks, Blundell and Lewbel (1997), for example).

Let  $\{(\ln x_i, w_{ij})\}_{i=1}^N$  represents a sequence of  $N$  household observations on the log of total expenditure  $\ln x_i$  and on the  $j$ th budget share  $w_{ij}$ , for each household  $i$  facing the same relative prices. For each commodity  $j$ , budget shares and total outlay are related by the stochastic Engel curve

$$w_{ij} = g_j(\ln x_i) + \varepsilon_{ij} \quad (3.1)$$

where we assume that, for each household  $i$ , the unobservable term  $\varepsilon_{ij}$  satisfies

$$E(\varepsilon_{ij} | \ln x) = 0 \text{ and } Var(\varepsilon_{ij} | \ln x) = \sigma_j^2(\ln x) \quad \forall \text{ goods } j = 1, \dots, J \quad (3.2)$$

so that the nonparametric regression of budget shares on log total expenditure estimates  $g_j(\ln x)$ .<sup>13</sup> If preferences are Piglog,  $g_j$  is linear in  $\ln x$  for all goods  $j = 1, \dots, J$ .

In our empirical application we use the following unrestricted Nadaraya-Watson kernel regression estimator for the  $j$ th budget share:

$$\hat{g}_j(\ln x) = \frac{\hat{r}_j^h(\ln x)}{\hat{f}^h(\ln x)} \equiv \hat{w}_j(\ln x) \quad (3.3)$$

in which

$$\hat{r}_j^h(\ln x) = \frac{1}{N} \sum_{l=1}^N K_h(\ln x - \ln x_l) w_{lj}, \quad (3.4)$$

---

<sup>13</sup>Below we discuss how we allow for the endogeneity of  $\ln x$  in the Engel curve regression equation.

and

$$\widehat{f}^h(\ln x) = \frac{1}{N} \sum_{l=1}^N K_h(\ln x - \ln x_l), \quad (3.5)$$

where  $h$  is the bandwidth and  $K_h(\cdot) = h^{-1}K(\cdot/h)$  for some symmetric kernel weight function  $K(\cdot)$  which integrates to one. We assume the bandwidth  $h$  satisfies  $h \rightarrow 0$  and  $Nh \rightarrow \infty$  as  $N \rightarrow \infty$ . Under standard conditions the estimator (3.3) is consistent and asymptotically normal, see Härdle (1990) and Härdle and Linton (1994). Provided the same bandwidth is used to estimate each  $g_j(\ln x)$ , adding-up across the share equations will be automatically satisfied for each  $\ln x$  and there is no efficiency gain from combining equations. This mirrors the standard invariance result for SURE results for systems with identical regressors (see Deaton (1983), for example).

### 3.2. Demographic Composition and Semiparametric Estimation

Household expenditures typically display a large variation with demographic composition. Let  $\mathbf{z}$  represent a vector of discrete household composition variables. A general specification might have the form

$$w_{ij} = G_j(\ln x_i, \mathbf{z}_i) + \varepsilon_{ij} \quad (3.6)$$

with

$$E(\varepsilon_{ij}|\mathbf{z}_i, \ln x_i) = 0 \text{ and } Var(\varepsilon_{ij}|\mathbf{z}_i, \ln x_i) = \sigma_j^2(\mathbf{z}_i, \ln x_i). \quad (3.7)$$

One approach to estimation would be to stratify by each distinct discrete outcome of  $\mathbf{z}_i$ . Inevitably, however, some pooling across household types will occur. One strategy for pooling is to assume additivity of  $G_j$  and work with the popular Robinson (1988) partially linear specification

$$w_{ij} = g_j(\ln x_i) + \mathbf{z}_i' \boldsymbol{\gamma}_j + \varepsilon_{ij} \quad (3.8)$$

in which  $\boldsymbol{\gamma}_j$  represents a finite parameter vector of household composition effects for commodity  $j$  and  $g_j(\ln x_i)$  is some unknown function as in (3.1).

Taking expectations of (3.8) conditional on  $\ln x_i$ , and eliminating  $g_j(\ln x_i)$  yields

$$w_{ij} - E(w_{ij} | \ln x_i) = (\mathbf{z}_i - E(\mathbf{z}_i | \ln x_i))' \boldsymbol{\gamma}_j + \varepsilon_{ij}. \quad (3.9)$$

A  $\sqrt{n}$  consistent and asymptotically normal estimator for  $\boldsymbol{\gamma}_j$  can be constructed from a weighted instrumental variable regression in which  $E(w_{ij} | \ln x_i)$  and  $E(\mathbf{z}_i | \ln x_i)$  are replaced by their nonparametric estimators, denoted  $\hat{w}_j^h(\ln x_i)$  and  $\hat{\mathbf{z}}^h(\ln x_i)$  respectively. For example, Robinson (1988), suggests regressing  $w_{ij} - \hat{w}_j^h(\ln x_i)$  on  $\mathbf{z}_i - \hat{\mathbf{z}}^h(\ln x_i)$  using  $I[\hat{f}^{h*}(\ln x_i) > b_N] \cdot \mathbf{z}_i$  as instruments, where  $\hat{f}^{h*}(\ln x_i)$  is defined in (3.5),  $I[\hat{f}^{h*}(\ln x_i) > b_N]$  is an indicator function that trims out observations for which  $\hat{f}^{h*}(\ln x_i) < b_N$ , for some sequence of trimming constants  $b_N$  which tend to zero at some rate. An alternative estimator, due to Powell (1987), is to use  $\hat{f}^{h*}(\ln x_i) \cdot \mathbf{z}_i$  as instruments. This effectively removes the random denominators from the kernel regression estimators  $\hat{w}_j^h(\ln x_i)$  and  $\hat{\mathbf{z}}^h(\ln x_i)$ . The income function in the additive specification (3.8) can then be estimated by

$$\hat{g}_j^h(\ln x_i) = \hat{w}_j^h(\ln x_i) - \hat{\mathbf{z}}^h(\ln x_i)' \hat{\boldsymbol{\gamma}}_j. \quad (3.10)$$

Since  $\hat{\boldsymbol{\gamma}}_j$  converges at  $\sqrt{n}$ , the asymptotic distribution results for  $\hat{g}_j^h(\ln x_i)$  remain unaffected by estimation of  $\boldsymbol{\gamma}_j$ , and follow from the distribution of  $\hat{w}_j^h(\ln x_i) - \hat{\mathbf{z}}^h(\ln x_i)' \boldsymbol{\gamma}_j$ .

Although the partially linear model (3.8) motivates the approach taken in this paper, consideration of the integrability conditions indicate that some modification is required. This is because the additive structure underlying (3.8) together with the Slutsky symmetry conditions

$$\frac{\partial w_j}{\partial \ln p_k} + w_k \frac{\partial w_j}{\partial \ln x} = \frac{\partial w_k}{\partial \ln p_j} + w_j \frac{\partial w_k}{\partial \ln x}, \quad (3.11)$$

requires that  $g(\cdot)$  be linear.

**Proposition 5.** *Suppose that budget shares have a form that is additive in functions of  $\ln x$  and demographics*

$$w_j(\ln \mathbf{p}, \ln x, \mathbf{z}) = m_j(\ln \mathbf{p}, \mathbf{z}) + g_j(\ln \mathbf{p}, \ln x) \quad (3.12)$$

If

(i) Slutsky symmetry (3.11) holds

(ii) the effects of demographics on budget shares are unrestricted in the sense that  $m_j$  can vary in any way with  $\mathbf{z}$

then  $g_j(\cdot)$  is linear in  $\ln x$ :

**Proof.** See Appendix A. ■

This proposition demonstrates that the additive form given in (3.12) will only be consistent with utility maximisation if we restrict the way in which demographics affect budget shares, or if preferences are Piglog. That is  $g_j(\ln x)$  is linear in  $\ln x$  for all  $j$ .

An alternative specification for pooling across demographic types, and one that we adopt, is the following generalisation of the partially linear model

$$w_{ij} = g_j(\ln x_i - \phi(\mathbf{z}'_i \boldsymbol{\theta})) + \mathbf{z}'_i \boldsymbol{\alpha}_j + \varepsilon_{ij} \quad (3.13)$$

in which  $\phi(\mathbf{z}'_i \boldsymbol{\theta})$  is some known function of a finite set of parameters  $\boldsymbol{\theta}$  and can be interpreted as the log of a general equivalence scale for household  $i$ <sup>14</sup>. Notice that  $\phi(\mathbf{z}'_i \boldsymbol{\theta})$  is common across share equations for all goods  $j$ . Consequently there are nonlinear cross-equation restrictions in this more general system. Interestingly, this extended partially linear model is precisely the shape invariance specification considered in the work on pooling nonparametric regression curves in Härdle and Marron (1990) and Pinske and Robinson (1995).

Normalise  $\phi(\mathbf{z}'_i \boldsymbol{\theta}) = 0$  and  $\mathbf{z}'_i \boldsymbol{\alpha}_j = 0$  for  $\mathbf{z}_i = \mathbf{z}^0$ , where  $\mathbf{z}^0$  is some base household composition (in the remainder of this subsection we suppress the bandwidth parameter and use superscripts represent the different demographic groups). For this base group 0 we write the share equation

$$w_{ij}^0 = g_j^0(\ln x_i) + \varepsilon_{ij}^0. \quad (3.14)$$

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<sup>14</sup>See Blundell, Duncan and Pendakur (1998) and Pendakur (1998).

Noting that  $\mathbf{z}$  is discrete, we define  $s = 1, \dots, S$  distinct composition groups over and above the base group, so that for each group the shape invariance restrictions become

$$g_j^s(\ln x_i) = g_j^0(\ln x_i - \phi(\mathbf{z}_i^{s'}\boldsymbol{\theta})) + \mathbf{z}_i^{s'}\boldsymbol{\alpha}_j \quad (3.15)$$

for  $s = 1, \dots, S$ .

By analogy with the partially linear specification, one possible method of estimation would be to replace each  $g_j^s(\ln x_i)$  by its unrestricted Nadaraya-Watson kernel regression estimator<sup>15</sup> and choose  $\boldsymbol{\alpha}_j$  and  $\boldsymbol{\theta}$  so as to minimise some weighted quadratic loss. This is the estimator developed by Härdle and Marron (1990). To eliminate the random denominators in this quadratic loss the restrictions, (3.15) could be weighted by the product of densities  $f^s f^0$  where  $f^s$  is evaluated at  $\ln x_i$  and  $f^0$  at  $(\ln x - \phi(\mathbf{z}^{s'}\boldsymbol{\theta}))$ . We define  $(\hat{\boldsymbol{\alpha}}_j, \hat{\boldsymbol{\theta}})$  as the value of  $(\boldsymbol{\alpha}_j, \boldsymbol{\theta})$  that minimises the integrated squared loss function

$$L(\boldsymbol{\theta}, \boldsymbol{\alpha}) = \sum_{s=1}^S \sum_{j=1}^{N-1} \int_{\underline{x}}^{\bar{x}} (\Lambda_{jk}(\ln x; \boldsymbol{\theta}, \boldsymbol{\alpha}_j))^2 \pi_k \varpi_j(\ln x) d \ln x \quad (3.16)$$

denoting  $\boldsymbol{\alpha}' = (\boldsymbol{\alpha}'_1, \dots, \boldsymbol{\alpha}'_{N-1})$  and where  $\underline{x}$  and  $\bar{x}$  are integration limits on the log of expenditure. The term in the squared loss is given by

$$\Lambda_{jk}(\ln x; \boldsymbol{\theta}, \boldsymbol{\alpha}_j) = r_j^s f^0 - f^{s'} r_j^0(\ln x - \phi(\mathbf{z}^{s'}\boldsymbol{\theta})) - \mathbf{z}^{s'}\boldsymbol{\alpha}_j f^s f^0(\ln x - \phi(\mathbf{z}^{s'}\boldsymbol{\theta}))$$

where  $\pi_s$  is a group specific weight ( $n^s/N$  in our specification) and  $\varpi_j(\ln x)$  is an equation-specific weighting function.<sup>16</sup> This is equivalent to using  $(f^s f^0(\ln x - \phi(\mathbf{z}^{s'}\boldsymbol{\theta})))^2$  as a weighting scheme for the Härdle and Marron (1990) estimator for random designs as suggested by Pinske and Robinson (1995).

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<sup>15</sup>We assume  $h \rightarrow 0$  and  $n_s h \rightarrow \infty$  as  $n_s \rightarrow \infty$  where  $n_s$  is the number of observations in each demographic group  $s$ .

<sup>16</sup>Note

$$\begin{aligned} g_j^1(\ln x) &= \mathbf{z}_i^{1'}\boldsymbol{\alpha}_j + g_j^0(\ln x - \phi(\mathbf{z}_i^{1'}\boldsymbol{\theta})) \iff \\ \hat{f}^0(\ln x - \phi(\mathbf{z}_i^{1'}\boldsymbol{\theta}))\hat{r}_j^1(\ln x) &= \hat{f}^1(\ln x)\hat{r}_j^0(\ln x - \phi(\mathbf{z}_i^{1'}\boldsymbol{\theta})) \\ &\quad + \hat{f}^1(\ln x)\hat{f}^0(\ln x - \phi(\mathbf{z}_i^{1'}\boldsymbol{\theta}))\mathbf{z}_i^{1'}\boldsymbol{\alpha}_j \end{aligned} \quad (3.17)$$

for all  $x$ .



Following Pinske and Robinson (1995) we have the following result:

**Proposition 6**

(i) Under assumptions A1 - A8 (see Appendix A)

$$N^{\frac{1}{2}}((\alpha', \theta') - (\alpha'_0, \theta'_0)) \rightarrow N(0, \Sigma)$$

for some finite matrix  $\Sigma$ .

(ii) Under assumptions A1-A10 hold (see Appendix A), for every  $j$

$$\hat{g}_j^0(\ln x - \phi(\mathbf{z}'\boldsymbol{\theta})) - g^0(\ln x - \phi(\mathbf{z}'\boldsymbol{\theta})) = o_p(N^{-\frac{2}{5}})$$

for all  $x \in \Theta$ .

**Proof.** See Appendix A. ■

>From Pinske and Robinson (1995, Theorem 3), it also follows that pooling Kernel estimates across demographic groups in this way leads to a smaller asymptotic mean squared error.

### 3.3. Endogeneity of Total Expenditure

To adjust for endogeneity we adapt the control function or augmented regression technique (see Holly and Sargan (1982), for example) to the semiparametric Engel curve framework. Suppose  $\ln x$  is endogenous in the sense that for each commodity  $j$

$$E(\varepsilon_{ij} | \ln x_i) \neq 0 \text{ or } E(w_{ij} | \ln x_i) \neq g_j(\ln x_i). \quad (3.18)$$

In this case the nonparametric estimator will not be consistent for the function of interest. To be precise, it will not provide the appropriate counterfactual: how expenditure share patterns change for some *ceterus paribus* change in total expenditure?

Suppose there exist instrumental variables  $\zeta_i$  such that

$$\ln x_i = \pi' \zeta_i + v_i \text{ with } E(v_i | \zeta_i) = 0. \quad (3.19)$$

In the application below we take the log of disposable income as the excluded instrumental variable for log total expenditure,  $\ln x$ . Further, we make the following

key assumption

$$E(w_j | \ln x_i, \zeta_i) = E(w_{ij} | \ln x_i, v_i) \quad (3.20)$$

$$= g_j(\ln x_i) + v_i \rho_j \quad \forall j. \quad (3.21)$$

This implies the augmented regression model

$$w_{ij} = g_j(\ln x_i) + v_i \rho_j + \varepsilon_{ij} \quad \forall j \quad (3.22)$$

with

$$E(\varepsilon_{ij} | \ln x_i) = 0 \quad \forall j. \quad (3.23)$$

By analogy with (3.9) eliminating  $g_j(\ln x_i)$  yields

$$w_{ij} - E(w_{ij} | \ln x_i) = (v_i - E(v_i | \ln x_i)) \rho_j + \varepsilon_{ij} \quad (3.24)$$

which suggests a weighted instrumental variable estimator for  $\rho_j$  by replacing the conditional means  $E(w_{ij} | \ln x_i)$  and  $E(v_i | \ln x_i)$  by their Nadaraya-Watson kernel regression estimators  $\hat{w}(\ln x_i)$  and  $\hat{v}(\ln x_i)$  respectively. Suitable instruments would be  $I[\hat{f}(\ln x_i) > b] \cdot v_i$ .

The resulting estimator of  $g(\ln x_i)$  is given by

$$\hat{g}(\ln x_i) = \hat{w}(\ln x_i) - \hat{v}(\ln x_i) \hat{\rho}_j. \quad (3.25)$$

Note that the unobservable error component  $v$  in (3.24) is unknown. In estimation  $v$  is replaced with the first stage reduced form residuals

$$\tilde{v}_i = \ln x_i - \zeta_i \hat{\pi} \quad (3.26)$$

where  $\hat{\pi}$  is the least squares estimator of  $\pi$ . Since  $\hat{\pi}$  and  $\hat{\rho}$  converge at  $\sqrt{n}$  the asymptotic distribution for  $\hat{g}(\ln x_i)$  follows the distribution of  $\hat{w}(\ln x_i) - \hat{v}(\ln x_i) \rho_j$ . Moreover, a test of the exogeneity null  $H_0 : \rho_j = 0$ , can be constructed from this least squares regression.<sup>17</sup>

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<sup>17</sup>Newey, Powell and Vella (1998) introduce a generalisation of this control function approach

### 3.4. Unobserved Heterogeneity

We turn now to the relationship between the (nonparametric) Engel curves above and the average demands for a set of heterogeneous agents. There are two alternative ways of interpreting the impact of heterogeneity on the average demands estimated from Kernel Engel curve regression. We could assume individual demands are rational and then ask for conditions on preferences and/or heterogeneity that imply rationality for average demands. This is the approach of McElroy (1987), Brown and Walker (1991) and Lewbel (1996). Alternatively, we could make no rationality assumptions on individual demands and simply ask what conditions enable average demands to satisfy rationality properties. This is the approach of Becker (1962), Grandmont (1992) and Hildenbrand (1994).

Suppose for each good  $j$  we write average budget shares as

$$E[w_j | \ln x, \mathbf{p}] = g_j(\ln x, \mathbf{p}) \quad (3.27)$$

then, if we let  $\varepsilon$  represent a vector of unobserved heterogeneity terms, a necessary condition for the average budget shares recovered by the nonparametric analysis discussed above to be equal to average budget shares is that

$$w_j = g_j(\ln x, \mathbf{p}) + \phi_j(\ln x, \mathbf{p})' \varepsilon \quad (3.28)$$

where  $E[\varepsilon | \ln \mathbf{x}, \mathbf{p}] = 0$ . Given this combination of functional form restrictions and distributional assumptions, our nonparametric analysis recovers  $g_t^j(x) = g_j(\ln x, \mathbf{p}^t)$ . Notice this allows for quite different tastes across agents. In particular, the first-order price and income responses for agents can vary in any way. Thus a good may be a luxury for one person and a necessity for another.

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for triangular simultaneous equation systems of the type considered here. They adopt a series estimator for the regression of  $w$  on  $\ln x$  and  $v$ . This generalises the form of (3.22) and allows an assessment of the additive structure. They also use a nonparametric regression for the reduced form in place of the linear model (3.19). In our application we extended the model along these lines by including second order terms in the residuals  $\hat{v}$  and then testing the partially linear specification (3.22) against this more general additive recursive alternative. The second order terms were not found to be significant.

This aggregation structure is very different to those used in Gorman (1954) and Muellbauer (1976). In particular, we are not aggregating across different total outlays. Additionally, we are not assuming that individual demands are integrable; that is, for given  $\epsilon$  we can have that the Slutsky conditions may fail for  $w_j(\ln x, \mathbf{p}, \epsilon)$ . In this respect, our structure is closer to that of Hildenbrand (1994) and Grandmont (1992). However, their analysis shows conditions for average demands to satisfy the Weak Axiom of Revealed Preference (WARP, see Varian (1982)) but GARP requires more. In particular, GARP implies the Slutsky symmetry conditions. In the heterogeneity structure given in (3.28) above we do not impose that individual demands satisfy the Slutsky conditions. If, however, we wish to impose integrability at the individual level then there are restrictions on the  $\phi_j(x, \mathbf{p})$  and the distribution of the heterogeneity terms (see McElroy (1987) and Brown and Walker (1989)). If all preference parameters are to be heterogeneous then preferences are essentially restricted to the class of Piglog demands (see Lewbel (1996), for example).

The function  $g_j(\ln x, \mathbf{p})$  gives mean responses to changes in prices conditional on a given level of total expenditure. Thus we can use this function for positive analysis, for example to recover the revenue implications from a change in taxes. Additionally, the utility function that is associated with an integrable set of demands  $g_j(\ln x, \mathbf{p})$  is a prime candidate for use in equilibrium models that assume a representative agent. In our analysis below we apply the GARP tests to the mean function  $g_j(\ln x, \mathbf{p})$ . The reason that we are interested in testing for GARP using these mean responses is that without such a rationality condition holding, it is difficult to see how we would ever conduct coherent welfare analysis of price changes. The heterogeneity conditions for using the mean function for welfare analysis are, however, stronger than the conditions given in (3.28) which suffice for positive analysis.

To understand the biases that derive from using  $g_j(\ln x, \mathbf{p})$  to conduct welfare

analysis consider the second order approximation of the log cost function<sup>18</sup> for a non-marginal price change  $\Delta \ln p_j$ .

$$\frac{\Delta \ln c}{\Delta \ln p_j} = w_j + \frac{1}{2} \left( \frac{\partial w_j}{\partial \ln p_j} + \frac{\partial w_j}{\partial \ln x} w_j \right) \Delta \ln p_j. \quad (3.29)$$

Using (3.28) this becomes

$$\frac{\Delta \ln c}{\Delta \ln p_j} = w_j + \frac{1}{2} \left[ \left( \frac{\partial g_j}{\partial \ln p_j} + \frac{\partial g_j}{\partial \ln x} g_j \right) + \left( \frac{\partial \phi'_j}{\partial \ln p_j} + \frac{\partial \phi'_j}{\partial \ln x} \right) \varepsilon(g_j + \phi'_j \varepsilon) \right] \Delta \ln p_j. \quad (3.30)$$

Therefore the mean welfare measure has the form

$$E \left[ \frac{\Delta \ln c}{\Delta \ln p_j} | x, \mathbf{p} \right] = w_j + \frac{1}{2} \left( \frac{\partial g_j}{\partial \ln p_j} + \frac{\partial g_j}{\partial \ln x} g_j \right) \Delta \ln p_j + \frac{1}{2} \frac{\partial \phi'_j}{\partial \ln x} \Omega_\varepsilon \phi_j \Delta \ln p_j. \quad (3.31)$$

where  $E\{\varepsilon \varepsilon' | x, \mathbf{p}\} = \Omega_\varepsilon$ .

The first two terms on the right hand side of this expression can be computed using the mean function  $g_j(\cdot)$  so that our mean function gives an exact first order welfare effect. It also gives second order effects if the final bias term is zero. This will be the case if, for example, the heterogeneity term  $\phi(\ln x, \mathbf{p})$  is independent of total expenditure so that all households have the same marginal income effects. Note, however, that this condition is sufficient and not necessary; weaker assumptions suffice to make the bias term zero or small.

To illustrate, suppose that each household's preferences are Piglog. This covers the class of Almost Ideal (see Deaton and Muellbauer (1980)) and Translog (see Jorgenson, Lau and Stoker (1982) demand systems. The budget share for good  $j$  can be expressed as

$$w_j = \alpha_j + \Gamma_j(\mathbf{p}) + \beta_j (\ln x - \boldsymbol{\alpha}' \ln \mathbf{p} - \Gamma(\mathbf{p})' \ln \mathbf{p}) \text{ for } j = 1, \dots, n \quad (3.32)$$

where  $\alpha_j$  and  $\beta_j$  are preference parameters  $\Gamma(\mathbf{p})$  is a nonstochastic matrix of functions of prices (of which  $\Gamma_j(\mathbf{p})$  is the  $j$ 'th row) and  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)'$ . Allowing

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<sup>18</sup>See Banks, Blundell and Lewbel (1996), for example.

$\alpha_j$  and  $\beta_j$  to have additive random components  $v_j$  and  $\eta_j$  respectively results in a share model where the residual term is given by

$$u_j = v_j - \beta_j \mathbf{v}' \ln \mathbf{p} + \eta_j (\ln x - (\boldsymbol{\alpha} + \mathbf{v})' \ln \mathbf{p} - \boldsymbol{\Gamma}(\mathbf{p})' \ln \mathbf{p}) \quad (3.33)$$

If we assume  $E(\mathbf{v} | \ln x, \ln \mathbf{p}) = \mathbf{0}$ ,  $E(\boldsymbol{\eta} | \ln x, \ln \mathbf{p}) = \mathbf{0}$  and  $E(\mathbf{v}\boldsymbol{\eta}' | \ln x, \ln \mathbf{p}) = \mathbf{0}$  we have the heterogeneity structure given in (3.28). However, notice also that if the heterogeneity is restricted to the  $\mathbf{v}$  terms then there are no  $\ln x$  terms in the heterogeneity expression (3.33) and the bias term disappears.

An alternative structure is suggested in Heckman (1974) and Brown and Matzkin (1995). In both of these papers the heterogeneity is introduced so that it enters in to the first order conditions in a convenient way. Specifically, these authors allow for multiplicative and additive heterogeneity terms, respectively, on the marginal utility of each good. For example, Brown and Matzkin have:

$$v(\mathbf{q}, \boldsymbol{\varepsilon}) = \Psi(\mathbf{q}) + \mathbf{q}' \boldsymbol{\varepsilon}$$

where  $\Psi(\cdot)$  is the common utility function. This heterogeneity scheme makes the marginal utilities heterogeneous:

$$\lambda p_j = v_j(\mathbf{q}, \boldsymbol{\varepsilon}) = \Psi_j(\mathbf{q}) + \varepsilon_j$$

If heterogeneity of this form is introduced in to the simple AI model above it will be seen that the resulting error term does *not* fall in the structure given in (3.28). Thus we could not estimate consistently the average of budget shares using nonparametric techniques.<sup>19</sup> Brown and Matzkin suggest an alternative strategy.

In general the error term in (3.28) will represent measurement and optimisation error as well as preference heterogeneity so it would seem natural to work with local average demands. Averaging locally to each  $x$  eliminates unobserved heterogeneity, measurement error and (zero mean) optimisation errors in demands but preserves any nonlinearities in the Engel curve relationship for each price regime.

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<sup>19</sup>Heckman (1974) actually adds a heterogeneity term to the marginal rate of substitution but this can be modelled as given in the text.

## 4. An Empirical Investigation on Repeated Cross-Sections

### 4.1. Data

The data are drawn from the repeated cross-sections of household-level data in the British Family Expenditure Survey (1974 to 1993). The FES is a random sample of around 7,000 households per year from which a sub-sample of all the two-adult households with a car was drawn<sup>20</sup>. The first and last percentiles of the within-year total expenditure distribution in this sub-sample was then trimmed out. This leaves 75,753 households (between 3,386 and 4,086 in each year). Expenditures on non-durable goods by these households were aggregated into 22 commodity groups and chained Laspeyres price indices for these groups were calculated from the sub-indices of the UK Retail Price Index giving 20 annual price points for each group of goods.

The commodity groups are non-durable expenditures grouped into: beer, wine, spirits, tobacco, meat, dairy, vegetables, bread, other foods, food consumed outside the home, electricity, gas, adult clothing, children's clothing and footwear, household services, personal goods and services, leisure goods, entertainment, leisure services, fares, motoring and petrol<sup>21</sup>. Descriptive statistics for total nominal expenditure are given in Table D.1 in the Data Appendix.

### 4.2. Estimated Engel Curves and Normal Goods

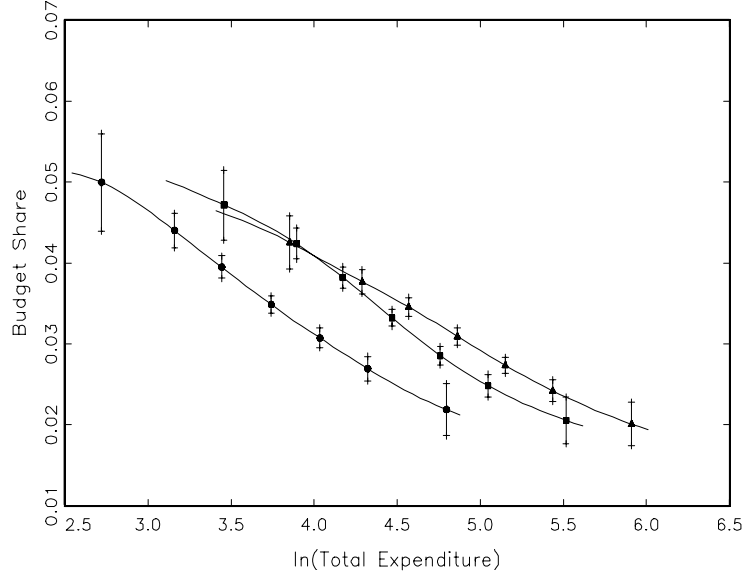
The nonparametric regression results are based on a Gaussian kernel estimation under the shape invariance restrictions (3.15). Bandwidths are chosen by crossvalidation [*cf.* Härdle (1990)]. The three figures below show the estimated Working-Leser Engel curves (budget share against log total nominal expenditure) for 3 of our 22 commodities, for 3 of our 20 periods (1975 (circles), 1980 (squares), 1985 (triangles)). These represent a typical necessity (bread), a luxury

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<sup>20</sup>This was in order to allow us to include motoring and particularly petrol as commodity groups.

<sup>21</sup>More precise descriptions of components of the commodity groups are available from the authors.

Figure 4.1: The Engel curve for Bread



(entertainment) and beer which displays a roughly quadratic logarithmic Engel curve behaviour. On each Engel curve we plot the points on the chronological SMP paths which correspond to the 1st, 10th, 25th, 50th, 75th, 90th and 99th percentile points in the base year (1974). Pointwise 95% confidence bands at these points are also drawn. Note that, as we would expect, the precision is much lower at the tails of the outlay distribution. The left to right drift of the Engel curves apparent in these figure illustrates the growth in nominal expenditure which took place between these periods.<sup>22</sup>

### 4.3. Testing GARP

At each stage in the empirical analysis of the GARP conditions we will be comparing weighted sums of kernel regressions. The pairwise comparison

$$\mathbf{p}'_t \mathbf{q}_t > \mathbf{p}'_t \mathbf{q}_s$$

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<sup>22</sup>A full set of non-parametric regression results are available from the authors on request. These results confirm the normal goods assumption used in the discussion above.



Figure 4.2: The Engel curve for Entertainment

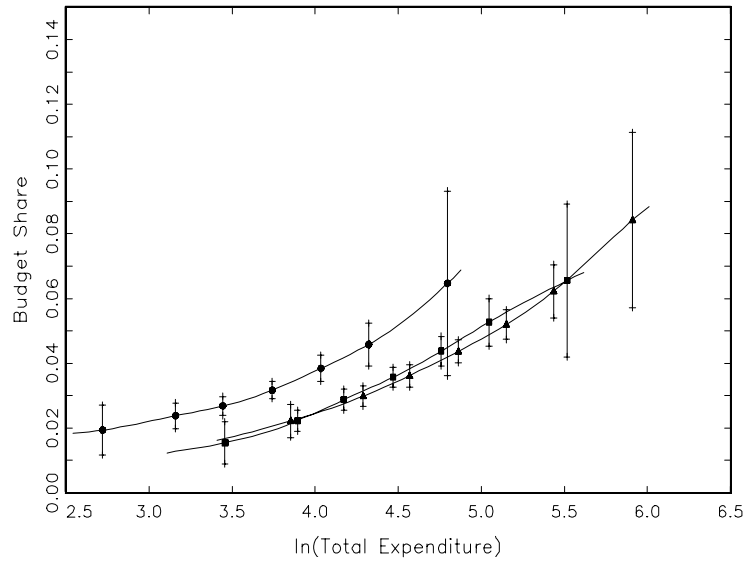
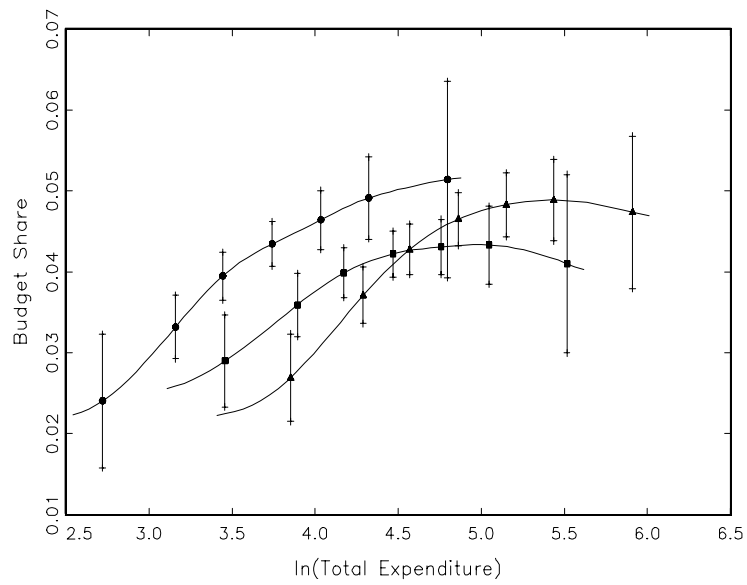


Figure 4.3: The Engel curve for Beer



can be written

$$x_t > \sum_{j=1}^n \frac{p_t^j}{p_s^j} g_s^j(x_s) x_s \text{ for } s \neq t. \quad (4.1)$$

where  $g_s^j(x_s)$  is the estimated budget share in equation (3.1). Noting that adding-up implies

$$\sum_{j=1}^n g_t^j(x_t) \equiv 1 \text{ for all } t$$

condition (4.1) conveniently reduces to the comparison

$$\delta_{ts} > \sum_{j=1}^{n-1} \gamma_{ts}^j g_s^j(x_s), \quad (4.2)$$

where  $\gamma_{ts}^j = \left( \frac{p_t^j}{p_s^j} - \frac{p_t^n}{p_s^n} \right)$  and  $\delta_{ts} = \left( \frac{x_t}{x_s} - \frac{p_t^n}{p_s^n} \right)$  are known constant weights in each price regime.

To test GARP we will need to evaluate the inequality (4.2) at particular points on an SMP path. Since the nonparametric Engel curve has a pointwise asymptotic standard error we can evaluate the distribution of each  $g_t^j(x)$  at any point  $x$ .<sup>23</sup> To evaluate (4.2) we need to find the distribution of the weighted sum of correlated kernel regression estimates  $\sum_{j=1}^{n-1} \gamma_{ts}^j g_s^j(x)$ . However, since on any SMP path in any period the  $g^j(x)$  kernel estimates for each good  $j$  are to be evaluated

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<sup>23</sup>Briefly, for bandwidth choice  $h$  and sample size  $N$  the variance can be well approximated at point  $x$  for large samples by

$$\text{var}(g^j(x)) \simeq \frac{\sigma_j^2(x) c_K}{N h f_h(x)}$$

where  $c_K$  is a known constant and  $f_h(x)$  is an (estimate) of the density of  $x$

$$\sigma_j^2(x) = N^{-1} \sum_{i=1}^N \omega_h^i(x) (w_{ij} - g_j(x))^2$$

with weights from the kernel function

$$\omega_h^i(x) = K_h(x - x_i^j) / f_h(x)$$

see Härdle (1990).

using the same kernel smoother and the same bandwidth, the expression for the asymptotic variance of the weighted sum simplifies. In particular, the constants associated with the kernel function and the density  $f_h(x)$  itself will be common to all variance and covariance terms. Pointwise standard errors and confidence bands for expression (4.2) are therefore tractable and are used extensively in the empirical application below.

To implement this procedure we need to choose a set of SMP paths along which to evaluate GARP. To do this we select the starting points for each path to be at the 1st percentile, 1st decile, 1st quartile, median, 3rd quartile, 9th decile and 99th percentile points in the  $x$  distribution for 1974, the first year in our data set. The comparison points for the following years are chosen along the SMP path following (2.4) in section 2.2.<sup>24</sup> By Proposition 1 we know that if this path passes GARP then no path which preserves the same preference ordering will violate GARP. The annual median and mean (non-SMP) paths are also computed for comparison.

Following the procedure described in section 2.1, a  $(T \times T)$  an indicator matrix  $\widetilde{M}$  for each path. For comparisons on each SMP path this is particularly straightforward as every element in the lower triangle of this matrix must be one since either  $\mathbf{q}_t R \mathbf{q}_{t-i}$  or  $\mathbf{q}_t P \mathbf{q}_{t-i}$ . Rejections are then revealed by the corresponding direct and transitive comparisons. That is, if  $\mathbf{q}_t R \mathbf{q}_{t-i}$  (or  $\mathbf{q}_t P \mathbf{q}_{t-i}$ ) in the lower triangle then  $\mathbf{q}_{t-i} P^0 \mathbf{q}_t$  in the upper triangle indicates a rejection of GARP. Table 4.1 shows the number and pattern of rejections for the system of 22 goods.

Each column in Table 4.1 provides a count of the total number of rejections according to inequality (4.2) for size  $\alpha$ . We have not attempted to compute the size

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<sup>24</sup>The derivation of the  $\tilde{x}$  comparison points depend on estimated kernel regressions. The pointwise standard errors used in the computing the number of rejections below are adjusted to allow for the estimation error in constructing these comparison points.

Table 4.1: Number of rejections of GARP, by size of test.

Starting point for each comparison path	$\alpha$				
	1.0	0.30	0.20	0.10	0.05
<i>SMP paths:</i>					
1st percentile	1	0	0	0	0
1st decile	1	1	0	0	0
1st quartile	1	0	0	0	0
Median	1	1	1	1	0
3rd quartile	2	2	0	0	0
9th decile	11	6	3	1	0
99th percentile	28	21	1	0	0
<i>Median path</i>	0	0	0	0	0
<i>Mean path</i>	0	0	0	0	0

of the implicit joint test. The column headed  $\alpha = 1$  counts the number of rejects using inequality (4.2) without adjustment. That is using the raw data from the nonparametric regression without adjustment for estimation error in the  $g_s^j(x_s)$ . In the remaining columns each inequality is adjusted by a one sided interval of size  $1 - \alpha$  based on the pointwise asymptotic distribution of  $\sum_{j=1}^{n-1} \gamma_{ts}^j g_s^j(x_s)$ . GARP is rejected for a large number of points in the upper tail of the outlay distribution but these rejections are not very ‘significant’ statistically. Very little adjustment is needed to dramatically reduce the number of rejections.

It is also interesting to observe that there are no rejections, even in the raw data, for the median or mean (non-SMP) paths. This is consistent with the observation which arises in tests of GARP on aggregate data that if the budget constraint is allowed to shift much either way between comparison points, as it does for median or mean total expenditure, then there is little chance of being able to find demands that cannot be rationalised.

#### 4.4. Continuous Sub-Periods Which Satisfy GARP

In section 2 it was shown that the bounds algorithm will only converge if there are well-behaved indifference curves. Given the rejection of GARP in the raw data, the algorithms will not be able to find coherent indifference curves using data for the entire period. However, we can obtain convergence for non-rejecting sub-periods and the results of this are presented in Table 4.2.

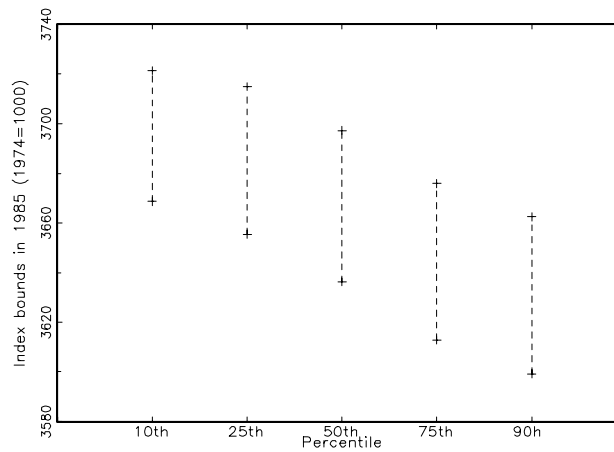
Table 4.2: Continuous periods of convergence.

	Periods																			
	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93
1st																				
10th																				
25th																				
50th																				
75th																				
90th																				
99th																				

The table shows the largest continuous sub-period in which the algorithms are able to bound the indifference curve. For example, for the starting point at median total outlay in 1974 we are able to bound a curve using the expansion paths and price data for 1974 to 1985 inclusive, and using any of the periods within the interval as the base. If we add 1986 to the set of admissible periods the algorithm fails to converge (we already know from the SMP path that 1985 and 1986 are not rationalisable). We then start again using the 1986 point on the median SMP path as our starting point. In all, for the median we find the entire period separates down into two sub-periods within which we are able to bound an indifference curve. Similarly the 1st and 9th decile paths break into two and four sub-periods respectively, while the 99th percentile breaks down into five.

In Figure 4.4 these results are used to bound the welfare effects of price changes over the period 1974 to 1985. This figure shows the GARP bounds for the decile, interquartile and median points of the 1974 total outlay distribution. It is interesting to note that the bounds for 10th and 90th percentile points do not overlap and indicate greater rise on the cost of living for poorer, compared to richer, households over this period. Before considering the relationship of these bounds to alternative cost of living indices, the next section asks whether there is some price adjustment that can be made so that the complete series from 1974 to 1993 can be used.

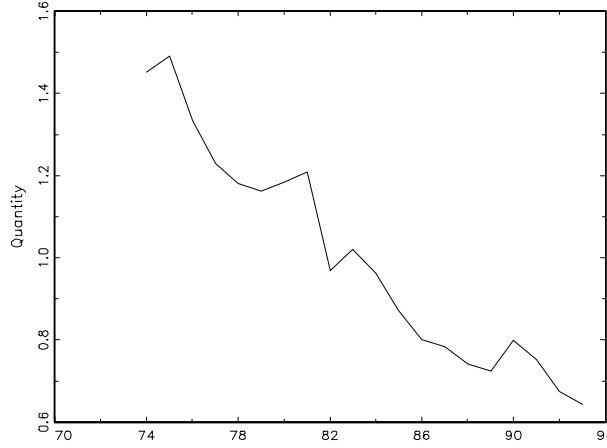
Figure 4.4: GARP Cost of living index bounds 1985 by percentile point.



#### 4.5. Allowing for Changes in Preferences

The GARP test itself gives us no clue as to the good or goods which are causing the rejection. We choose tobacco as our conditioning good and argue that we have reasonable prior belief that preferences for tobacco may have changed over the period with the arrival of new information on the health effects of smoking. We apply the conditioning procedure to the Median SMP path to the path of

Figure 4.5: The median SMP quantity path for tobacco.



demands illustrated below. This allows us to account for changes preferences by calculating an adjusted price for tobacco which is consistent with GARP and should, for example, allow the run of non-rejecting periods to be extended passed the first rejection on the median SMP path.

In this case the rejection is being caused by  $\mathbf{p}'_{85}\mathbf{q}_{85} > \mathbf{p}'_{85}\mathbf{q}_{86}$  while the chronological SMP path requires  $\mathbf{p}'_{86}\mathbf{q}_{86} = \mathbf{p}'_{86}\mathbf{q}_{85}$ . Figure 4.5 shows the quantity of tobacco demanded on the median SMP expenditure path. Between 1985 and 1986 quantity demand on this path fell by 8.9%. However, a 10.1% adjustment of the tobacco price in 1986 is sufficient to leave GARP satisfied over the complete 1974 - 1993 period under consideration.

Table 4.3 below report the results for the re-run of the GARP test with the actual price series for tobacco replaced by the virtual price in 1985. Of course this adjustment changes the relative prices not just between the years in question (85/86), but between 1985 and every other year. Further, given that the confidence intervals for the pairwise tests are based on price-weighted sums of kernel regressions, changes in the prices can affect the standard error of the price/demand bundles and changes the results of the statistical tests of GARP. The first column

in Table 4.3 reveals that this change has also corrected rejections in the raw data on the 1st and 3rd quartile SMP paths. This is because these rejections also referred to the 85/86 comparisons and in each case the minimum virtual price adjustment to the tobacco price was less than that required by the median path. The adjustment for the 1st quartile SMP path was such that, in order to rationalise the 7.6% demand fall, a minimum price increase of 9.9% was required. The adjustment for the 3rd quartile SMP path was such that, in order to rationalise the 11.5% demand fall, a minimum price increase of 12% was required.

Table 4.3: Number of rejections of GARP including virtual price of tobacco (1985).

Starting points for comparison paths	$\alpha$				
	1.00	0.30	0.20	0.10	0.05
<i>SMP paths:</i>					
1st percentile	1	0	0	0	0
1st decile	1	1	0	0	0
1st quartile	0	0	0	0	0
Median	0	0	0	0	0
3rd quartile	1	1	0	0	0
9th decile	10	6	3	1	0
99th percentile	28	15	1	0	0
<i>Median</i>	0	0	0	0	0
<i>Mean</i>	0	0	0	0	0

#### 4.6. Bounds on the True Cost-of-living Index

In this section we compare some of the more popular indices with the GARP bounds developed in this paper. Using the virtual price of tobacco for the median SMP path we derive GARP-based bounds on the cost-of-living index over the period 1974 to 1993 using welfare at 1974 median total expenditure as our reference welfare level. The results are reported in Table 4.4 which summarises some popular parametric indices, some of which represent first-order, and some of which



represent second-order approximations to true indices based on any arbitrary cost-function. These indices can also be thought of as corresponding exactly to true indices under various assumptions regarding the precise form of preferences<sup>25</sup>. Table 4.4 also reports various nonparametric bounds which have been suggested in the literature and the GARP-based bounds (in the final column) derived using the algorithms described above.

The nonparametric results reported represent the bounds on the true 1974 median welfare-based cost of living index which can be derived without *any* assumption on functional forms. The bounds provided by Lerner (1935-36) are simply that the true index (being a weighted average of price changes) must lie somewhere between the maximum and the minimum ratio of the price changes of all goods: i.e .

$$\min_i \left\{ \frac{p_t^i}{p_{74}^i} : i = 1, \dots, n \right\} \square \frac{c(u_{74}, p_t)}{c(u_{74}, p_{74})} \square \max_i \left\{ \frac{p_t^i}{p_{74}^i} : i = 1, \dots, n \right\}.$$

Pollak (1971) improves this by linking Lerner's result with the original Konüs (1924) result that the Laspeyres index approximates the true base-referenced cost of living index from above, i.e.

$$\min_i \left\{ \frac{p_t^i}{p_{74}^i} : i = 1, \dots, n \right\} \square \frac{c(u_{74}, \mathbf{p}_t)}{c(u_{74}, \mathbf{p}_{74})} \square \frac{\mathbf{p}'_t \mathbf{q}_{74}}{\mathbf{p}'_{74} \mathbf{q}_{74}}.$$

The bounds from classical revealed preference (GARP) restrictions of the type used by Varian (1982) (i.e. those which utilise the restrictions implied by prices and demands at observed annual mean expenditures to bound a level set of utility in commodity space) are also reported and the bounds derived from the procedure developed in this paper above are reported in bold type.

We find, confirming the results in Varian (1982) and Manser and McDonald (1988) that classical non-parametric/revealed preference bounds based on the

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<sup>25</sup>The Paasche and Laspeyres, for example, are exact for Leontief preferences, the Törnqvist is exact for translog.

Table 4.4: Popular price indices, nonparametric and GARP bounds, 1974 to 1993; virtual price of tobacco (1985).

Year	Price Indices			Nonparametric/RP bounds			
	P	L	T	GARP	Classical RP	Pollak	Lerner
74	1000	1000	1000	1000	1000	1000	1000
75	1215	1232	1223	[1215,1228]	[1206,1232]	[1025,1232]	[1025,1721]
76	1516	1530	1528	[1515,1530]	[1431,1530]	[1182,1530]	[1182,1985]
77	1762	1787	1783	[1763,1781]	[1700,1787]	[1239,1787]	[1239,2590]
78	1931	1957	1960	[1937,1957]	[1894,1957]	[1385,1957]	[1385,2513]
79	2086	2119	2121	[2094,2119]	[2058,2119]	[1461,2119]	[1461,2636]
80	2463	2514	2514	[2479,2509]	[2442,2514]	[1734,2514]	[1734,3142]
81	2780	2841	2841	[2802,2838]	[2687,2841]	[1770,2841]	[1770,4077]
82	3093	3189	3178	[3124,3172]	[2983,3189]	[1821,3189]	[1821,4287]
83	3260	3381	3371	[3316,3369]	[3197,3381]	[1828,3381]	[1828,4924]
84	3408	3558	3534	[3474,3530]	[3329,3558]	[1790,3558]	[1790,4921]
85	3541	3709	3685	[3622,3682]	[3228,3709]	[1836,3709]	[1836,5022]
86	3700	3911	3875	[3809,3873]	[3308,3911]	[1900,3911]	[1900,5463]
87	3825	4035	3990	[3919,3988]	[3300,4035]	[1920,4035]	[1920,6049]
88	3922	4163	4112	[4037,4109]	[3370,4163]	[1923,4163]	[1923,6143]
89	3413	4379	4321	[4241,4318]	[3356,4379]	[1996,4379]	[1996,6397]
90	4406	4669	4607	[4521,4603]	[3403,4669]	[2079,4669]	[2079,6637]
91	4723	5044	4966	[4870,4961]	[3911,5044]	[2109,5044]	[2109,7507]
92	4996	5437	5322	[5212,5316]	[3888,5437]	[2091,5437]	[2091,8353]
93	5177	5650	5498	[5379,5491]	[3841,5650]	[2066,5650]	[2066,9098]

Notes: P = Paasche. L = Laspeyres, T = Chained Törnqvist

average demand data gives little additional information on the curvature of the indifference curve through commodity space and hence the bounds on the true index are wide. However, by the use of expansion paths we can dramatically improve these bounds. The GARP bounds represent approximately  $\pm 2\%$  of the level of the centre of the bounds by 1993. In contrast, the Paasche and Laspeyres indices under- and over-state the true index by 5% and 4% of the level by 1993.

Perhaps the most compelling results in support of the approach we have developed here is presented in Figure 4.6. This figure illustrates the classical RP bounds of the type calculated by Varian (1982) and the GARP bounds. The

Figure 4.6: Classical RP bounds and GARP bounds, 1974 to 1993.

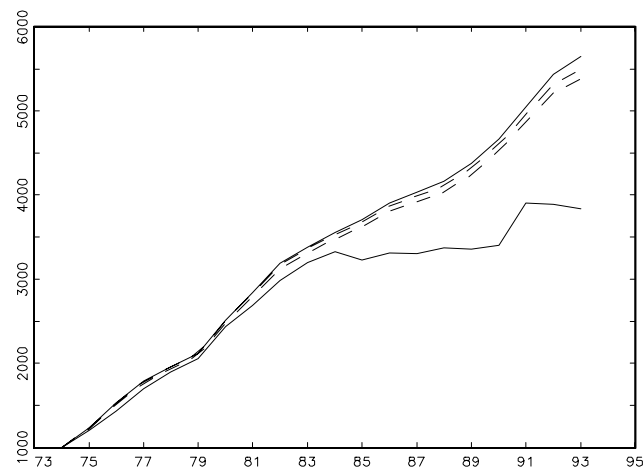
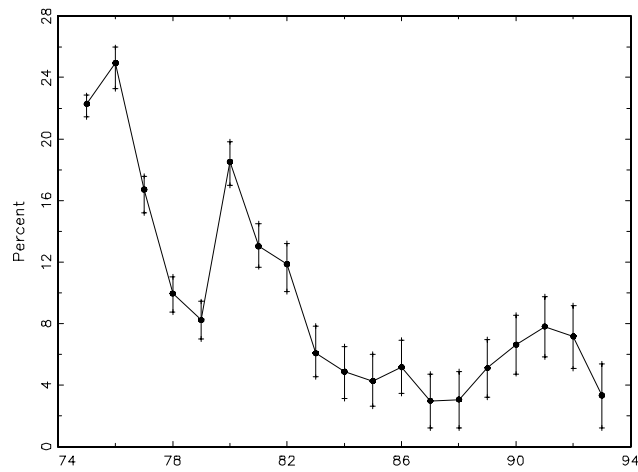


Figure 4.7: Annual inflation rates, GARP bounds and chained Törnqvist, 1975 to 1993.



improvement is dramatic, the lower classical RP bound capture only 70% of the increase in the true cost of living over the period. Without the use of expansion paths to tighten the bounds, the upper and lower bounds often reflect the extremes of perfect substitution and Leontief preferences.

Although it is not base period utility referenced<sup>26</sup>, in practice, the chained Törnqvist index performs well in approximating the true index for this sample. Figure 4.7 illustrates the GARP-derived bounds on the annual inflation rate for the period 1975 to 1993. The GARP bounds are shown by the vertical lines ending in a cross, the chained Törnqvist is shown by the dots connected by the solid line. The chained Törnqvist is everywhere within the GARP bounds on the true index.<sup>27</sup>

## 5. Summary and Conclusions

In this paper we have applied nonparametric demand theory to the nonparametric statistical analysis of consumer demand. We exploit the idea that price taking individuals in the same market face the same relative prices, in order to smooth across the demands of individuals for each common price regime. We show how this provides a conventional stochastic structure within which to examine the consistency of individual data and revealed preference theory. We present a method of maximising the power of these tests of revealed preference. We have discussed the way in which we might allow for taste changes for a good from which the other goods are not separable. We also present algorithms which allow us to place bounds on level sets of utility in commodity space. This allows the calculations of improved bounds on true cost-of-living indices. The heterogeneity conditions for carrying out both positive and welfare analysis were also discussed.

Using a long time series of repeated cross-sections from the 1974-1993 British Family Expenditure Surveys we were able to examine whether revealed prefer-

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<sup>26</sup>The Törnqvist index which links period  $t$  with period  $s$ , for example, is referenced at the utility level  $(u_s u_t)^{1/2}$ .

<sup>27</sup>Information on other price index formulae are available from the authors.

ence theory is rejected. We show that GARP is not rejected for long periods, particularly when we allow for sampling/stochastic variation. Allowing for taste changes in tobacco in one year is shown to reduce the number of the rejections further. We also show that we can derive bounds on cost of living indices from our analysis which appear to be much tighter than those based on the revealed preference restrictions implied by demands at, say, annual mean total expenditure. We also note that, (despite the fact that neither of these indices are themselves base period referenced) the Törnqvist and the chained Divisia indices perform well as empirical approximations to the true base-period referenced index.

## Appendices

### A. Proofs of Lemmas and Propositions

#### Proof of Lemma 1

By adding-up we have  $\hat{\mathbf{p}}'\mathbf{q}(\mathbf{p}^*, x) \rightarrow 0$  as  $x \rightarrow 0$  so that we can always find an  $x^l$  such that  $\hat{\mathbf{p}}'\mathbf{q}(\mathbf{p}^*, x^l) < \hat{x}$ . Similarly by adding-up we can always find an  $x^u$  such that  $\hat{\mathbf{p}}'\mathbf{q}(\mathbf{p}^*, x^u) > \hat{x}$ . Now define  $f(\lambda) = \hat{\mathbf{p}}'\mathbf{q}(\mathbf{p}^*, \lambda x^u + (1 - \lambda)x^l)$ . This function is continuous (from continuity of the expansion paths) and strictly increasing (from weak normality) with  $f(0) < \hat{x}$  and  $f(1) > \hat{x}$  so that there is a unique  $\lambda^*$  with  $f(\lambda^*) = \hat{x}$ . Now set  $x^* = \lambda^* x^u + (1 - \lambda^*)x^l$  which gives the desired result. ■

#### Proof of Proposition 1

Without loss of generality we take the length of the GARP rejecting preference ordered sub-sequence to be  $\{\hat{x}_s, \hat{x}_t, \hat{x}_u\}$ . We have:

- (1)  $\hat{x}_s = \mathbf{p}'_s \mathbf{q}_s(\hat{x}_s) \geq \mathbf{p}'_s \mathbf{q}_t(\hat{x}_t)$  and
- (2)  $\hat{x}_t = \mathbf{p}'_t \mathbf{q}_t(\hat{x}_t) \geq \mathbf{p}'_t \mathbf{q}_u(\hat{x}_u)$  and
- (3)  $\hat{x}_u = \mathbf{p}'_u \mathbf{q}_u(\hat{x}_u) > \mathbf{p}'_u \mathbf{q}_s(\hat{x}_s)$ .

We consider the SMP path for this preference ordered sub-sequence and show that it too rejects GARP. The SMP path  $(\tilde{x}_s, \tilde{x}_t, \tilde{x}_u)$  has:

- (4)  $\tilde{x}_t = \mathbf{p}'_t \mathbf{q}_t(\tilde{x}_t) = \mathbf{p}'_t \mathbf{q}_u(\hat{x}_u)$  and
- (5)  $\tilde{x}_s = \mathbf{p}'_s \mathbf{q}_s(\tilde{x}_s) = \mathbf{p}'_s \mathbf{q}_t(\tilde{x}_t)$ .

By construction this is a preference ordered sub-sequence  $(\mathbf{q}_t(\tilde{x}_t)R^0\mathbf{q}_u(\hat{x}_u)$  and  $\mathbf{q}_s(\tilde{x}_s)R^0\mathbf{q}_t(\tilde{x}_t))$  so that this sub-sequence rejects GARP if  $\mathbf{q}_u(\hat{x}_u)P^0\mathbf{q}_s(\tilde{x}_s)$ ; that is, if:

- (6)  $\mathbf{p}'_u \mathbf{q}_u(\hat{x}_u) > \mathbf{p}'_u \mathbf{q}_s(\tilde{x}_s)$ .

Conditions (2) and (4)  $\implies \mathbf{p}'_t \mathbf{q}_t(\hat{x}_t) \geq \mathbf{p}'_t \mathbf{q}_t(\tilde{x}_t) \implies \hat{x}_t \geq \tilde{x}_t$ .

This and conditions (1) and (5)  $\implies \mathbf{p}'_s \mathbf{q}_s(\hat{x}_s) \geq \mathbf{p}'_s \mathbf{q}_t(\hat{x}_t) \geq \mathbf{p}'_s \mathbf{q}_t(\tilde{x}_t) = \mathbf{p}'_s \mathbf{q}_s(\tilde{x}_s) \implies \hat{x}_s \geq \tilde{x}_s$ .

Finally, condition (3) and normality  $\implies \mathbf{p}'_u \mathbf{q}_u(\hat{x}_u) > \mathbf{p}'_u \mathbf{q}_s(\hat{x}_s) \geq \mathbf{p}'_u \mathbf{q}_s(\tilde{x}_s)$  which is condition (6); hence GARP is rejected for this sub-sequence. ■

#### Proof of Proposition 2

The proof is identical to that of Theorem 7 in Varian (1982). ■

#### Proof of Proposition 3

- (1) Denote  $\bar{\mathbf{p}}_s = (\bar{p}_s^1, \mathbf{p}_s^{n-1})$ .
- (2) By equation (2.5)  $\bar{p}_s^1$  is set such that  $\bar{\mathbf{p}}'_s \mathbf{q}_s = \bar{\mathbf{p}}'_s \mathbf{q}_t$ .
- (3) Suppose  $\tilde{p}_s^1 > \bar{p}_s^1$  and let  $\tilde{\mathbf{p}}_s = (\tilde{p}_s^1, \mathbf{p}_s^{n-1})$  so that  $\tilde{\mathbf{p}}_s > \bar{\mathbf{p}}_s$ .
- (4) Given  $(q_s^1 - q_t^1) > 0$  from (2) and (3)  $\tilde{\mathbf{p}}'_s \mathbf{q}_s > \bar{\mathbf{p}}'_s \mathbf{q}_s = \bar{\mathbf{p}}'_s \mathbf{q}_t \implies \tilde{\mathbf{p}}'_s \mathbf{q}_s > \tilde{\mathbf{p}}'_s \mathbf{q}_t$  i.e.  $\mathbf{q}_s P^0 \mathbf{q}_t$ , but since  $\mathbf{q}_t R \mathbf{q}_s$  this is a violation of GARP. ■

#### Proof of Proposition 4

We begin with part (i). This proof proceeds in stages. Firstly we need to show that

i)  $\mathbf{q}_i R \mathbf{q}_0$  for all  $\mathbf{q}_i \in \mathbf{Q}_P$ .

Suppose  $\exists$  some  $\mathbf{q}_j$  not  $R \mathbf{q}_0$ . Since  $\mathbf{q}_j \in \overline{\mathbf{Q}}$  and was on an SMP path emanating from  $\mathbf{q}_0$  this means that there must have been some  $\mathbf{q}_k P \mathbf{q}_0$  on that SMP path such that  $\mathbf{q}_j R \mathbf{q}_k P \mathbf{q}_0$  and that this  $\mathbf{q}_k$  has been removed because  $\exists$  some other bundle on  $\mathbf{q}^k(x)$ ,  $\tilde{\mathbf{q}}_k$  say such that  $\mathbf{q}_k P^0 \tilde{\mathbf{q}}_k$ . If demands are weakly normal then this means that there must also  $\exists$  some other bundle on  $\mathbf{q}^j(x)$ ,  $\tilde{\mathbf{q}}_j$  say, such that  $\mathbf{q}_j P^0 \tilde{\mathbf{q}}_j R^0 \tilde{\mathbf{q}}_k$ . Given the definition of  $\mathbf{Q}_P$  there cannot exist any such  $\mathbf{q}_j$  not  $R \mathbf{q}_0$ .

We now need to show that

ii) *Iff  $\exists$  some  $\mathbf{q}_j P^0 \mathbf{q}_i$ ,  $\mathbf{q}_i, \mathbf{q}_j \in \mathbf{Q}_P$  then the data  $(\mathbf{p}, \mathbf{Q}_P)$  violate GARP.*

Suppose  $\exists$  some  $\mathbf{q}_j \in \mathbf{Q}_P$  such that  $\mathbf{q}_j P^0 \mathbf{q}_i$ , where  $\mathbf{q}_i \in \mathbf{Q}_P$ . Note that  $\mathbf{q}_i R \mathbf{q}_0$  and  $\mathbf{q}_j R \mathbf{q}_0$ . If demands are weakly normal then  $\exists$  some other bundle on  $\mathbf{q}^j(x)$ ,  $\underline{\mathbf{q}}_j$  say, such that  $\underline{\mathbf{q}}_j R^0 \mathbf{q}_i R \mathbf{q}_0$  and hence for which  $\mathbf{q}_j P^0 \underline{\mathbf{q}}_j$ . However such a point would be on an SMP emanating from  $\mathbf{q}_0$  (hence  $\mathbf{q}_j \in \mathbf{Q}$ ) and since  $\mathbf{p}'_j \mathbf{q}_j > \mathbf{p}'_j \underline{\mathbf{q}}_j$  then  $\underline{\mathbf{q}}_j \in \mathbf{Q}_P$ . The fact that  $\underline{\mathbf{q}}_j \notin \mathbf{Q}_P$  means that  $\underline{\mathbf{q}}_j$  cannot be on an SMP path. This can only be the case iff  $\mathbf{q}_i$  and  $\mathbf{q}_j$  are on the same SMP path and iff  $\mathbf{q}_i R \mathbf{q}_j$ . This is because by the definition of an SMP path, each expansion path is only used once. The fact that  $\mathbf{q}_j P^0 \mathbf{q}_i$  and  $\mathbf{q}_i R \mathbf{q}_j$  gives the violation of GARP.

Finally we show that

iii) *Iff the data  $(\mathbf{p}, \mathbf{Q}_P)$  do not violate GARP then  $\mathbf{Q}_P$  is the lowest upper bound on the indifference surface referenced at  $u(\mathbf{q}_0)$ .*

Suppose  $\exists \tilde{\mathbf{q}}_t$  such that  $\tilde{\mathbf{q}}_t R \mathbf{q}_0$  and that  $\tilde{\mathbf{q}}_t$  is outside the bound of the  $\mathbf{Q}_P$  set. By normality  $\mathbf{q}_t P^0 \tilde{\mathbf{q}}_t$  where  $\mathbf{q}_t \in \overline{\mathbf{Q}}_P$  i.e.  $\mathbf{q}_t R \mathbf{q}_0 \Rightarrow \exists \mathbf{q}_i$  and  $\mathbf{q}_j$  such that  $\tilde{\mathbf{q}}_t R^0 \mathbf{q}_i R \mathbf{q}_0$  and  $\mathbf{q}_t R^0 \mathbf{q}_j R \mathbf{q}_0$ . But since by the definition of  $\mathbf{Q}_P$  we have  $\mathbf{q}_t \in \{\mathbf{q}_i \in \overline{\mathbf{Q}}: \mathbf{p}'_i \mathbf{q}_i \square \mathbf{p}'_i \tilde{\mathbf{q}}_i \forall \tilde{\mathbf{q}}_i \in \overline{\mathbf{Q}}\}$  there cannot exist any such  $\tilde{\mathbf{q}}_t R \mathbf{q}_0$  outside the set  $\mathbf{Q}_P$  unless (by Proposition 2.i.) the data violate GARP.

The proof of part (ii) is analogous.

■

### Proof of Corollary 4.1

Consider Algorithm A.1 after  $n$  iterations, denote the sets  $F, W$  and  $E$  at this point by  $F_n, W_n$  and  $E_n$ . Suppose further that  $F_n \equiv \mathbf{Q}_P$ . Suppose that the data in the set  $\mathbf{Q}_P$  violate GARP: hence  $\exists \mathbf{q}_i, \mathbf{q}_{jj} \in \mathbf{Q}_P$  such that  $\mathbf{q}_i P^0 \mathbf{q}_j$  and  $\mathbf{q}_j R \mathbf{q}_i$ . By Step (3) of the algorithm  $\mathbf{q}_j \in E_n$  where  $E_n \subset F_n$ . Hence  $E_n \neq \emptyset$  and the stopping condition is not met.

■

### Proof of Proposition 5

Applying (3.11) and (3.12) we have<sup>28</sup>

$$g^j(\ln \mathbf{p}, \ln x) = \tilde{g}^j(\ln \mathbf{p}) + \hat{g}^j(\ln \mathbf{p}) \ln x.$$

<sup>28</sup>In what follows let  $\mathbf{h}^j_z$  denote the vector of partial derivatives of  $\mathbf{h}^j$  with respect to the vector of demographics  $\mathbf{z}$ .

Taking derivatives of both sides with respect to  $\ln x$  then with respect to  $\mathbf{z}$  gives

$$\mathbf{h}_z^k g_{xx}^j = \mathbf{h}_z^j g_{xx}^k.$$

Invoking condition (ii) in the statement of the proposition we can set  $\mathbf{h}_z^j$  equal to zero and  $\mathbf{h}_z^k$  non-zero, which implies that  $g_{xx}^j = 0$  so that  $h^j(\ln \mathbf{p}, \ln x)$  is linear in  $\ln x$ .  
■

## Proof of Proposition 6

*Assumptions:*

A1:  $\varepsilon_{ji}^s$  are assumed mutually independent and have finite second moments

A2:  $E(\varepsilon_{ji}^s | \ln x) = 0$

A3:  $\ln x_i$  is independently distributed with density  $\hat{f}^s(\cdot)$  that is 2 times boundedly differentiable.

A4:  $\hat{f}^s(\cdot)(\hat{r}^s(\cdot))^2$  are 2 times boundedly differentiable functions.

A5:  $(\boldsymbol{\alpha}', \boldsymbol{\theta}')$  is in a bounded and open set.

A6: The twice boundedly differentiable weight function  $\varpi$ , is nonnegative and positive only on the interior of a compact interval  $\Theta$ . For all points  $x \in \Theta$  we have that  $f(x) > 0$  and that for all  $(\boldsymbol{\alpha}', \boldsymbol{\theta}'), x \in \Phi \times \Theta$  that  $f^1(\ln x) > 0$ .

A7: No parameter vector  $(\boldsymbol{\alpha}', \boldsymbol{\theta}') \neq (\boldsymbol{\alpha}'_0, \boldsymbol{\theta}'_0)$  exists such that for some  $j$ ,  $g_j^s(\ln x) = \mathbf{z}^{s'} \boldsymbol{\alpha}_j + g_j^0(\ln x - \phi(\mathbf{z}^{s'} \boldsymbol{\theta}))$  almost all  $x \in \Theta$ .

A8: The same kernel is used for all  $k = 1, \dots, r$  groups.

A9: A common bandwidth  $h$  is chosen such that  $nh^5 \rightarrow \infty$ ,  $nh^6 \rightarrow 0$ , as  $n \rightarrow \infty$ .

With assumptions A1 - A8 in place, proposition 6(i) follows directly from Lemmas 1-6 and Theorem 1 in Pinske and Robinson (1995). In particular, if we define  $A(\boldsymbol{\alpha}'_0, \boldsymbol{\theta}'_0)$  to be the nonsingular limit of the expectation of the Hessian of (3.16) and  $B(\boldsymbol{\alpha}'_0, \boldsymbol{\theta}'_0)$  be the corresponding outer product matrix, then

$$N^{\frac{1}{2}}((\boldsymbol{\alpha}', \boldsymbol{\theta}') - (\boldsymbol{\alpha}'_0, \boldsymbol{\theta}'_0)) \rightarrow N(0, A^{-1}(\boldsymbol{\alpha}'_0, \boldsymbol{\theta}'_0) B(\boldsymbol{\alpha}'_0, \boldsymbol{\theta}'_0) A^{-1}(\boldsymbol{\alpha}'_0, \boldsymbol{\theta}'_0)).$$

Furthermore, assuming

A10:  $Nh^3 \rightarrow \infty$  and  $h \rightarrow 0$ , as  $N \rightarrow \infty$ ,

then from Pinske and Robinson (1995), Theorem 2, we have Proposition 6(ii):

for every  $j$

$$\hat{g}_j^0(\ln x - \phi(z' \boldsymbol{\theta})) - g^0(\ln x - \phi(z' \boldsymbol{\theta})) = o_p(N^{-\frac{2}{5}}) \text{ for all } x \in \Theta.$$

■

## B. Welfare Bound Algorithms

### B.1. The Algorithms

**Algorithm A.1** Input: a set of  $T$  expansion paths  $\mathbf{q}^t(x)$  and price vectors  $\mathbf{p}_t$  for  $t = 1, 2, \dots, T$  and a base bundle  $\mathbf{q}_0$ . Output: the set  $\mathbf{Q}_P$  of boundary points of which  $\mathbf{q}_0$  is a member and which has  $T + 1$  elements where  $\mathbf{p}'_i \mathbf{q}_i \square \mathbf{p}'_i \mathbf{q}_j \forall \mathbf{q}_i, \mathbf{q}_j \in \mathbf{Q}_P$  and  $\mathbf{q}_i R \mathbf{q}_0$  for all  $\mathbf{q}_i \in \mathbf{Q}_P$ .

1) Set  $W = \{\mathbf{q}_0\}$ ,  $E = \emptyset$ .



- 2) For each  $t \neq 0$  we define  $\mathbf{q}_t = \mathbf{q}_t(\hat{x}_t)$  where  $\hat{x}_t = \arg \min\{x \mid \mathbf{p}'_t \mathbf{q}_t = \mathbf{p}'_t \mathbf{q}_w\}$  for all  $\mathbf{q}_w \in W$ . Call this set  $F$ .
- 3) Set  $E = \{\mathbf{q}_i \in F : \mathbf{p}'_i \mathbf{q}_i > \mathbf{p}'_i \mathbf{q}_j \text{ for } \mathbf{q}_j \in F\}$ .
- 4) Set  $F = F/E$ .
- 5) Set  $G = \{\mathbf{q}_i \in F : \mathbf{q}_i \text{ not } R \mathbf{q}_0\}$ .
- 6) Set  $W = F/G$ .
- 7) If  $E = \emptyset$ , set  $\mathbf{Q}_L = W$  and goto (8). Else if  $\mathbf{q}_0 \in E$  set  $W = \{\mathbf{q}_0\}$  and goto (2). Otherwise go to (2).
- 8) Stop.

**Algorithm A.2** Input: a set of  $T$  expansion paths  $\mathbf{q}_t(x)$  and price vectors  $\mathbf{p}_t$  for  $t = 0, 1, 2, \dots, T$  and a base bundle  $\mathbf{q}_0$ . Output: the set  $\mathbf{Q}_W$  of boundary points of which  $\mathbf{q}_0$  is a member and which has  $T + 1$  elements where  $\mathbf{p}'_i \mathbf{q}_i \square \mathbf{p}'_i \mathbf{q}_j \forall \mathbf{q}_i, \mathbf{q}_j \in \mathbf{Q}_W$  and  $\mathbf{q}_0 R \mathbf{q}_i$  for all  $\mathbf{q}_i \in \mathbf{Q}_W$ .

- 1) Set  $W = \{\mathbf{q}_0\}$ ,  $E = \emptyset$ .
- 2) For each  $t \neq 0$  we define  $\mathbf{q}_t = \mathbf{q}_t(\hat{x}_t)$  where  $\hat{x}_t = \arg \max\{x \mid \mathbf{p}'_w \mathbf{q}_w = \mathbf{p}'_w \mathbf{q}_t\}$  for all  $\mathbf{q}_w \in W$ . Call this set  $F$ .
- 3) Set  $E = \{\mathbf{q}_i \in F : \mathbf{p}'_j \mathbf{q}_j > \mathbf{p}'_j \mathbf{q}_i \text{ for } \mathbf{q}_j \in F\}$ .
- 4) Set  $F = F/E$ .
- 5) Set  $G = \{\mathbf{q}_i \in F : \mathbf{q}_0 \text{ not } R \mathbf{q}_i\}$ .
- 6) Set  $W = F/G$ .
- 7) If  $E = \emptyset$ , goto (8). Else if  $\mathbf{q}_0 \in E$  set  $W = \{\mathbf{q}_0\}$  and goto (2). Otherwise go to (2).
- 8) Set  $\mathbf{Q}_W = W \cup \max_j \{\mathbf{p}'_t \mathbf{q}_t / p_t^j : \forall \mathbf{q}_t \in W\}$ .
- 7) Stop.

## B.2. Illustration of Algorithm A.1.

Figure ?? illustrates the algorithm<sup>29</sup>. Begin with  $W_0 = \{\mathbf{q}_0\}$ ,  $F_0 = \emptyset$  and  $E_0 = \emptyset$ . In the first iteration budgets are set such that  $F_1 = \{\mathbf{q}_0, \mathbf{q}_1(x'_1), \mathbf{q}_2(x'_2), \bar{\mathbf{q}}_3\}$ . The point  $\bar{\mathbf{q}}_3 \in E_1$  since  $\mathbf{p}'_3 \bar{\mathbf{q}}_3 = \mathbf{p}'_3 \mathbf{q}_0$  but  $\mathbf{p}'_3 \bar{\mathbf{q}}_3 > \mathbf{p}'_3 \mathbf{q}_1$  and  $\mathbf{p}'_3 \bar{\mathbf{q}}_3 > \mathbf{p}'_3 \mathbf{q}_2$  imply  $\bar{\mathbf{q}}_3 P \mathbf{q}_0$  so  $\bar{\mathbf{q}}_3$  must be above the indifference curve and so can be improved. We then set  $W_1 = \{\mathbf{q}_0, \mathbf{q}_1(x'_1), \mathbf{q}_2(x'_2)\}$ . This completes the first iteration and since  $E_1 \neq \emptyset$  continue to the next iteration.

In the next iteration points are computed that are revealed preferred to each of the (now three) members of  $W_1$ . There are nine of these (one on each of the expansion paths for each of the members of  $W_1$ ). We select the bundle with minimum total expenditure on each path in step two. Now  $F_2 = \{\mathbf{q}_0, \mathbf{q}_1(x'_1), \mathbf{q}_2(x'_2), \mathbf{q}_3(x'_3)\}$  and  $\mathbf{q}_3(x'_3) R^0 \mathbf{q}_2(x'_2)$  and  $\mathbf{q}_2(x'_2) R^0 \mathbf{q}_0$  give  $\mathbf{q}_3(x'_3) R \mathbf{q}_0$ . The algorithm ends with the upper bound illustrated as the next iteration will find no improvements as  $E_2 = \emptyset$  and  $W_3 \equiv W_2$ . The budget lines using each price vector at the final total expenditure levels are denoted  $\{x'_1, x'_2, x'_3\}$  and  $\mathbf{Q}_P = \{\mathbf{q}_0, \mathbf{q}_1(x'_1), \mathbf{q}_2(x'_2), \mathbf{q}_3(x'_3)\}$ . Algorithm A.2 proceeds in a similar way with the

<sup>29</sup>We subscript the sets defined at each stage by the current iteration of the algorithms.

additional step which identifies the final two points on the  $q^1 = 0$  and  $q^0 = 0$  axes:  $\mathbf{Q}_W = \{\mathbf{q}_0, \mathbf{q}_1(x_1), \mathbf{q}_2(x_2), \mathbf{q}_3(x_3), x_3/p_3^1, x_1/p_1^2\}$ . The dashed lines marked ‘upper’ and ‘lower’ shows the bounds on  $c(\mathbf{p}_z, u(\mathbf{q}_0))$  given by  $\min\{\mathbf{p}'_z \mathbf{q}_t | \mathbf{q}_t \in \mathbf{Q}_W\}$  and  $\min\{\mathbf{p}'_z \mathbf{q}_t | \mathbf{q}_t \in \mathbf{Q}_P\}$  for some new set of relative prices  $\mathbf{p}_z$ .

### C. Simulating measurement error.

Given a period  $t$  total outlay  $x_t$  we set expenditure on good  $i$  in that period equal to  $\gamma_{it}x_t$  for  $i = 1, 2, \dots, n$ . We choose the  $\gamma_{it}$  weights in the following way. First, we draw  $\tilde{\gamma}_{it}$  from a Beta distribution with parameters  $(a_i, b_i)$  (where the distribution parameters are kept constant over time). To do this, we first have to calibrate the two parameters for each good. To fix one parameter we set  $E(\tilde{\gamma}_{it}) = \omega_i$  where the latter is a given budget share (see below). Given the usual expression for the mean of Beta distributed variable this implies that we must set:

$$b_i = \frac{(1 - \omega_i)a_i}{\omega_i} \quad (\text{C.1})$$

for each good  $i$ . It only remains to calibrate the  $a_i$  parameters. The variance of a Beta distributed random variable is given by:

$$\sigma_i^2 = \frac{a_i b_i}{(1 + a_i + b_i)(a_i + b_i)^2} = \frac{(1 - \omega_i)\omega_i}{a_i + \omega_i} \quad (\text{C.2})$$

so that if we take a value for the variance the associated value of  $a_i$  is given by:

$$a_i = \left( \frac{(1 - \omega_i)\omega_i}{\sigma_i^2} - 1 \right) \omega_i \quad (\text{C.3})$$

Thus we first choose  $(\omega_i, \sigma_i^2)$  for each good and then calculate  $(a_i, b_i)$  for  $i = 1, 2, \dots, n$ . Given these parameters we can simulate a set of budget weights for each period  $t$ ,  $(\tilde{\gamma}_{1t}, \tilde{\gamma}_{2t}, \dots, \tilde{\gamma}_{nt})$ . Since these will not normally sum to unity we set:

$$\gamma_{it} = \frac{\tilde{\gamma}_{it}}{\sum_{j=1}^n \tilde{\gamma}_{jt}} \quad (\text{C.4})$$

Although the marginals of the joint distribution of the  $\gamma_{it}$ ’s are now no longer a Beta distribution and the weights do not have the desirable property that  $E(\gamma_{it}) = \omega_i$  we do have  $\text{plim}(\gamma_{it}) = \omega_i$  and  $0 < \gamma_{it} < 1$ , which suffices for our purposes. It only remains to choose the mean and variances discussed above. We use data from one representative year (1974) and set the budget shares  $\omega_i$ ’s equal to the mean budget shares. For the variance, we first take the variance of each budget share, denoted  $\bar{\sigma}_i^2$  for good  $i$ . We take this to be an upper bound for noise in the measurement and then choose an attenuation factor  $\rho$  to give the calibrating variance  $\sigma_i^2 = \rho \bar{\sigma}_i^2$  for each good. Thus an attenuation factor of unity gives the maximum noise and an attenuation factor of zero gives no noise.

## D. Data

Table D.1: Total nominal expenditure: Annual descriptive statistics.

<b>Year</b>	<b>No. of Obs</b>	<b>Mean</b>	<b>Std Dev.</b>	<b>10%</b>	<b>50%</b>	<b>90%</b>
1974	3386	39.11	17.95	20.41	35.19	62.93
1975	3696	47.17	21.17	24.83	42.36	75.92
1976	3553	52.79	24.20	27.75	47.23	84.15
1977	3683	60.94	27.71	31.87	54.83	98.65
1978	3583	67.84	31.33	35.34	60.78	108.76
1979	3476	79.18	37.04	40.36	71.42	127.72
1980	3717	92.84	43.07	47.67	82.77	152.70
1981	4072	102.63	47.94	52.78	91.29	169.21
1982	3974	108.89	50.10	56.83	98.15	175.15
1983	3749	117.11	54.40	60.33	105.69	190.41
1984	3755	124.71	59.71	62.81	110.22	206.58
1985	3775	132.56	64.68	64.94	117.65	219.00
1986	3826	143.35	71.64	69.35	126.01	240.79
1987	3962	150.49	74.20	72.42	134.40	249.69
1988	4003	163.01	83.09	75.71	145.68	274.40
1989	4086	173.93	86.57	83.38	155.14	292.80
1990	3772	191.01	95.95	91.15	169.15	320.19
1991	3886	199.59	99.41	96.19	177.71	332.81
1992	3999	205.58	97.29	101.02	185.86	339.20
1993	3800	219.84	111.99	105.47	192.97	363.91

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