

RESEARCH ARTICLE

Higher order Lipschitz Sandwich theorems

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Technology Commission**Abstract**

We investigate the consequence of two $\text{Lip}(\gamma)$ functions, in the sense of Stein, being close throughout a subset of their domain. A particular consequence of our results is the following. Given $K_0 > \varepsilon > 0$ and $\gamma > \eta > 0$, there is a constant $\delta = \delta(\gamma, \eta, \varepsilon, K_0) > 0$ for which the following is true. Let $\Sigma \subset \mathbb{R}^d$ be closed and $f, h : \Sigma \rightarrow \mathbb{R}$ be $\text{Lip}(\gamma)$ functions whose $\text{Lip}(\gamma)$ norms are both bounded above by K_0 . Suppose $B \subset \Sigma$ is closed and that f and h coincide throughout B . Then, over the set of points in Σ whose distance to B is at most δ , we have that the $\text{Lip}(\eta)$ norm of the difference $f - h$ is bounded above by ε . More generally, we establish that this phenomenon remains valid in a less restrictive Banach space setting under the weaker hypothesis that the two $\text{Lip}(\gamma)$ functions f and h are only close in a pointwise sense throughout the closed subset B . We require only that the subset Σ be closed; in particular, the case that Σ is finite is covered by our results. The restriction that $\eta < \gamma$ is sharp in the sense that our result is false for $\eta := \gamma$.

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1 | INTRODUCTION

The notion of $\text{Lip}(\gamma)$ functions, for $\gamma > 0$, introduced in [28] provides an extension of γ -Hölder regularity that is both non-trivial and meaningful even when $\gamma > 1$. This notion of regularity is the appropriate one for the study of *rough paths* instigated by the first author in [19]; an introductory overview to this theory may be found in [20], for example. Moreover, $\text{Lip}(\gamma)$ regularity underpins the efforts made to extend the theory of rough paths to the setting of manifolds [3, 6]. Further the flow of $\text{Lip}(\gamma)$ vector fields is utilised to investigate the *accessibility problem* regarding the use of classical ordinary differential equations (ODEs) to obtain the terminal solution to a *rough differential equations* driven by *geometric rough paths* in [1, 2]. The notion of $\text{Lip}(\gamma)$ regularity is well defined for functions defined on arbitrary closed subsets including, in particular, finite subsets.

The origin of $\text{Lip}(\gamma)$ functions go back at least as far as the original extension work of Whitney in [29]. This work considered the following extension problem. Given an integer $m \in \mathbb{Z}_{\geq 1}$ and a closed subset $A \subset \mathbb{R}^d$, when can a real-valued function $f : A \rightarrow \mathbb{R}$ be extended to a $C^m(\mathbb{R}^d)$ function $F : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $F|_A \equiv f$. Whitney introduces a definition of f being m -times continuously differentiable on an arbitrary closed subset $A \subset \mathbb{R}^d$, which we denote by $f \in C^m(A)$, and subsequently establishes that this condition is sufficient to ensure that f admits an extension to an element $F \in C^m(\mathbb{R}^d)$ (see Section 3 and Theorem I in [29]). A variant of this extension result with quantified estimates may be found in [31].

Whitney's definition of $f \in C^m(A)$ involves assigning a family of 'derivatives' for f on A . Hence, applying Whitney's extension theorem requires one to first fix such an assignment of derivatives. Numerous works have subsequently considered the extension problem proposed by Whitney with the additional constraint of avoiding such an assignment, that is, using only the values of the function f throughout A . For the case that $d = 1$, Whitney himself provided an answer to this problem using the *divided differences* of f ; see [30].

The general case that $d \geq 1$ was fully resolved by Fefferman in [9, 10]. His resolution builds upon the reformulation of Whitney's result in [30] as a *finiteness principle* by Brudnyi and Shvartsman [4, 5]. The key point is that the finiteness principle no longer involves the divided differences of f . The subsequent finiteness principle established by Fefferman in [8] underlies his resolution to the Whitney extension problem (see also [11]). Subsequent algorithmic approaches to computing an extension have been considered by Fefferman et al. in [12–16].

Returning to Whitney's original extension theorem in [29], analogous results have been established in non-Euclidean settings where the domain of f is *not* a subset of \mathbb{R}^d for some $d \in \mathbb{Z}_{\geq 1}$. A C^1 version of Whitney's extension theorem was established for real valued mappings defined on subsets of the sub-Riemannian Heisenberg group in [17]. Mappings taking their values in the Heisenberg group have also been considered; a version of Whitney's extension theorem has been established for horizontal C^m curves in the Heisenberg group [24, 33]. Moreover, a finiteness principle for horizontal curves in the Heisenberg group is proven in [34]. Whitney-type extensions for horizontal C^1 curves in general Carnot groups and sub-Riemannian manifolds have been considered in [18, 27].

In this article, we focus on Stein's notion of $\text{Lip}(\gamma)$ functions in the Euclidean setting. Motivated by Whitney's original definition of $C^m(A)$ for a closed subset $A \subset \mathbb{R}^d$ in [29], Stein's definition of a $\text{Lip}(\gamma)$ is an extension of the classical notion of Lipschitz (or Hölder) continuity. Indeed, for $\gamma \in (0, 1]$, Stein's definition coincides with the classical notion of a function being bounded and γ -Hölder continuous. For $\gamma > 1$, Stein's definition provides a non-trivial extension of Hölder regularity to higher orders.

Stein's definition of a $\text{Lip}(\gamma)$ function is a refined weaker assignment of a family of 'derivatives' to a function f than the assignment proposed in Whitney's definition of $C^m(A)$. As in Whitney's original work [29], Stein's definition of $\text{Lip}(\gamma)$ requires, for each point $x \in A$, the prescription of a polynomial P_x based at x . These polynomials act as proposals for how the function should look at points distinct from x , and, similarly to Whitney's definition of $C^m(A)$, these polynomials are required to satisfy Taylor-like expansion properties. In particular, it is required that for every $x \in A$, we have that $P_x(x) = f(x)$, and that the remainder term $R(x, y) := f(y) - P_x(y)$ is bounded above by $C|y - x|^\gamma$ for some constant $C > 0$ and every pair $x, y \in A$. This remainder term bound is weaker than the remainder term bound imposed in Whitney's definition of a $C^m(A)$ function in [29].

To illustrate the sense in which $\text{Lip}(\gamma)$ is weaker than $C^m(A)$, let $\gamma \in \mathbb{Z}_{\geq 1}$ and $O \subset \mathbb{R}^d$ be a non-empty open subset. If $f : O \rightarrow \mathbb{R}$ is $C^\gamma(O)$ in the sense of Whitney [29], then f is $C^\gamma(O)$ in the classical sense. However, if $f : O \rightarrow \mathbb{R}$ is $\text{Lip}(\gamma)$ in the sense of Stein [28], then f is $C^{\gamma-1}(O)$ in the classical sense with its $(\gamma - 1)$ th derivative $D^{\gamma-1}f$ only guaranteed to be Lipschitz rather than C^1 .

Analogously to Whitney's extension theorem in [29], Stein proves an extension theorem establishing that, given any $\gamma > 0$, if $A \subset \mathbb{R}^d$ is closed and $f : A \rightarrow \mathbb{R}$ is $\text{Lip}(\gamma)$, then there exists a function $F : \mathbb{R}^d \rightarrow \mathbb{R}$ that is $\text{Lip}(\gamma)$ and satisfies that $F|_A \equiv f$; see Theorem 4 in Chapter VI of [28]. Recall that for $\gamma \in (0, 1]$ Stein's notion of a $\text{Lip}(\gamma)$ function coincides with the classical notion of a bounded γ -Hölder continuous function. Consequently, for $\gamma \in (0, 1]$, Stein's extension theorem recovers earlier extension results independently established by Whitney in [29] and McShane in [22]. Stein's extension theorem allows us to make the following conclusion regarding the original Whitney extension problem. Given an integer $k \in \mathbb{Z}_{\geq 0}$ and any $\varepsilon > 0$, the assumption that $f : A \rightarrow \mathbb{R}$ is $\text{Lip}(k + \varepsilon)$ ensures that f admits an extension to $F \in C^k(\mathbb{R}^d)$. Finally, Stein's extension theorem is essential in the first author's work on rough paths in [19].

Whilst our focus within this article is on Stein's notion of $\text{Lip}(\gamma)$ functions within a Euclidean framework, it is worth mentioning some recent works proposing generalisations of Stein's definition to non-Euclidean frameworks. A definition of $\text{Lip}(\gamma)$ functions for functions on *Carnot groups* that is consistent with Stein's original definition in [28] is proposed in [25]; further, the authors establish extension theorems analogous to Stein's extension theorem for this notion of $\text{Lip}(\gamma)$ functions on *Carnot groups* in [25]. More generally, building on the rough integration theory of cocyclic one-forms developed in [21] and [32] generalising the work of the first author in [19], a definition of $\text{Lip}(\gamma)$ functions on unparameterised paths is proposed in [23]. It is

established, in particular, in [23] that an extension property analogous to the Stein–Whitney extension property is valid for this notion of $\text{Lip}(\gamma)$ functions on unparameterised path space.

It is clear that, in general, there is no uniqueness associated to Stein’s extension theorem; for example, the mapping $[0, 1] \rightarrow \mathbb{R}$ given by $t \mapsto |t|$ is $\text{Lip}(1)$ (i.e. Lipschitz in the classical sense), and there are numerous distinct ways it can be extended to a bounded Lipschitz continuous mapping $\mathbb{R} \rightarrow \mathbb{R}$. Nevertheless, it seems intuitively clear that any two extensions must be, in some sense, *close* for points $x \in \mathbb{R}$ that are not, in some sense, too far away from the interval $[0, 1]$.

The aim of this article is to establish precise results realising this intuition for $\text{Lip}(\gamma)$ functions for any $\gamma > 0$. If $\gamma \in (0, 1]$, which we recall means that Stein’s notion of $\text{Lip}(\gamma)$ functions [28] coincides with the classical notion of bounded γ -Hölder continuous functions, then such results are well known. In particular, they arise as immediate consequences of the maximal and minimal extensions of Whitney [29] and McShane [22], respectively. The main novelty of this paper is establishing our main results in the case that $\gamma > 1$ (cf. the *Lipschitz Sandwich Theorem 3.1*, the *Single-Point Lipschitz Sandwich Theorem 3.8*, and the *pointwise Lipschitz Sandwich Theorem 3.11* in Section 3), for which we have been unable to locate formal statements of such properties within the existing literature. Moreover, our results are presented in a more general setting than restricting to working within \mathbb{R}^d for some $d \in \mathbb{Z}_{\geq 1}$. In particular, we consider the Banach space framework utilised in both [19] and [1], for example.

A particular consequence of our results is that $\text{Lip}(\gamma)$ functions can be cost-effectively approximated. Loosely, if $0 < \eta < \gamma$, then on compact sets the $\text{Lip}(\eta)$ -behaviour of a $\text{Lip}(\gamma)$ function is determined, up to an arbitrarily small error $\varepsilon > 0$, via knowledge of an upper bound for the $\text{Lip}(\gamma)$ norm of the function on the entire compact set, and the knowledge of the value of the $\text{Lip}(\gamma)$ function at a finite number of points. The number of points at which it is required to know the value of the $\text{Lip}(\gamma)$ function depends on the upper bound of its $\text{Lip}(\gamma)$ norm on the entire compact set, the regularity parameters η and γ , the desired error size $\varepsilon > 0$ and the compact subset (cf. Corollary 4.1 in Section 4).

The remainder of the paper is structured as follows. In Section 2, we provide Stein’s definition of a $\text{Lip}(\gamma)$ function within a Banach space framework (cf. Definition 2.2) and fix the notation and conventions that will be used throughout the article. In Section 3, we state our main results; the *Lipschitz Sandwich Theorem 3.1*, the *Single-Point Lipschitz Sandwich Theorem 3.8* and the *Pointwise Lipschitz Sandwich Theorem 3.11*. In Section 4, we illustrate how the *Lipschitz Sandwich Theorem 3.1* and the *Pointwise Lipschitz Sandwich Theorem 3.11* allow one to cost-effectively approximate $\text{Lip}(\gamma)$ functions defined on compact subsets (cf. Corollaries 4.1 and 4.5 and Remarks 4.3, 4.4, 4.7 and 4.8). In Section 5, we establish explicit pointwise remainder term estimates for a $\text{Lip}(\gamma)$ function. In Section 6, we record that, for $\gamma_1 > \gamma_2 > 0$, any $\text{Lip}(\gamma_1)$ function is also a $\text{Lip}(\gamma_2)$ function. Additionally, we establish an explicit constants $C \geq 1$ for which we have that $\|\cdot\|_{\text{Lip}(\gamma_2)} \leq C \|\cdot\|_{\text{Lip}(\gamma_1)}$ (cf. the *Lipschitz Nesting Lemma 6.1*). In Section 7, we record, given a $\text{Lip}(\gamma)$ function f defined on a subset Σ , a point $p \in \Sigma$, and $\eta \in (0, \gamma)$, quantified estimates for the $\text{Lip}(\eta)$ -norm of f over a neighbourhood of the point p in terms of the value of f at the point p (cf. Lemmas 7.2 and 7.3). Sections 8–10 contain the proofs of the main results stated in Section 3. In Section 8, we prove the *Pointwise Lipschitz Sandwich Theorem 3.11*. In Section 9, we prove the *Single-Point Lipschitz Sandwich Theorem 3.8*. Finally, in Section 10, we establish the full *Lipschitz Sandwich Theorem 3.1*.

Several of our intermediary lemmata record properties of $\text{Lip}(\gamma)$ functions that appear elsewhere in the literature. Nevertheless, there are two main benefits of including these results. Firstly, the variants we record are in specific forms that are particularly useful for our purposes. Secondly, their inclusion makes our article fully self-contained.

2 | MATHEMATICAL FRAMEWORK AND NOTATION

In this section, we introduce the mathematical framework considered throughout the remainder of the article, provide Stein’s definition of a $\text{Lip}(\gamma)$ function within this framework (cf. Definition 2.2), and fix the notation and conventions that will be used throughout the article. Throughout the remainder of this article, when referring to metric balls, we use the convention that those denoted by \mathbb{B} are taken to be open and those denoted by $\overline{\mathbb{B}}$ are taken to be closed.

Let V and W be real Banach spaces and assume that $\Sigma \subset V$ is a closed subset. The first goal of this section is to define the space $\text{Lip}(\gamma, \Sigma, W)$ of $\text{Lip}(\gamma)$ functions with domain Σ and target W . This will require a choice of norms for the tensor powers of V . We restrict to considering norms that are *admissible* in the sense originating in [7]. The precise definition is the following.

Definition 2.1 (Admissible norms on tensor powers). Let V be a Banach space. We say that its tensor powers are endowed with admissible norms if for each $n \in \mathbb{Z}_{\geq 2}$, we have equipped the tensor power $V^{\otimes n}$ of V with a norm $\|\cdot\|_{V^{\otimes n}}$ such that the following conditions hold.

- For each $n \in \mathbb{Z}_{\geq 1}$, the symmetric group S_n acts by isometries on $V^{\otimes n}$, that is, for any $\rho \in S_n$ and any $v \in V^{\otimes n}$, we have

$$\|\rho(v)\|_{V^{\otimes n}} = \|v\|_{V^{\otimes n}}. \tag{2.1}$$

The action of S_n on $V^{\otimes n}$ is given by permuting the order of the letters, that is, if $a_1 \otimes \dots \otimes a_n \in V^{\otimes n}$ and $\rho \in S_n$, then $\rho(a_1 \otimes \dots \otimes a_n) := a_{\rho(1)} \otimes \dots \otimes a_{\rho(n)}$, and the action is extended to the entirety of $V^{\otimes n}$ by linearity.

- For any $n, m \in \mathbb{Z}_{\geq 1}$ and any $v \in V^{\otimes n}$ and $w \in V^{\otimes m}$, we have

$$\|v \otimes w\|_{V^{\otimes(n+m)}} \leq \|v\|_{V^{\otimes n}} \|w\|_{V^{\otimes m}}. \tag{2.2}$$

- For any $n, m \in \mathbb{Z}_{\geq 1}$ and any $\phi \in (V^{\otimes n})^*$ and $\sigma \in (V^{\otimes m})^*$, we have

$$\|\phi \otimes \sigma\|_{(V^{\otimes(n+m)})^*} \leq \|\phi\|_{(V^{\otimes n})^*} \|\sigma\|_{(V^{\otimes m})^*}. \tag{2.3}$$

Here, given any $k \in \mathbb{Z}_{\geq 1}$, the norm $\|\cdot\|_{(V^{\otimes k})^*}$ denotes the dual-norm induced by $\|\cdot\|_{V^{\otimes k}}$.

It turns out (see [26]) that having *both* the inequalities (2.2) and (2.3) ensures that we, in fact, have equality in both estimates. Hence, if the tensor powers of V are equipped with admissible norms, we have equality in both (2.2) and (2.3).

There are, in some sense, two main examples of admissible tensor norms. The first is the *projective tensor norm*. This is defined, for $n \geq 2$, on $V^{\otimes n}$ by setting, for $v \in V^{\otimes n}$,

$$\|v\|_{\text{proj},n} := \inf \left\{ \sum_{i=1}^{\infty} \prod_{k=1}^n \|a_{i_k}\|_V : v = \sum_{i=1}^{\infty} a_{i_1} \otimes \dots \otimes a_{i_n} \text{ and } \sum_{i=1}^{\infty} \prod_{k=1}^n \|a_{i_k}\|_V < \infty \right\}. \tag{2.4}$$

The second is the *injective tensor norm*. This is defined, for $n \geq 2$, on $V^{\otimes n}$ by setting, for $v \in V^{\otimes n}$,

$$\|v\|_{\text{inj},n} := \sup \{ |\varphi_1 \otimes \dots \otimes \varphi_n(v)| : \varphi_1, \dots, \varphi_n \in V^* \text{ and } \|\varphi_1\|_{V^*} = \dots = \|\varphi_n\|_{V^*} = 1 \}. \tag{2.5}$$

The injective and projective tensor norms are the main two examples in the following sense. If we equip the tensor powers of V with *any* choice of admissible tensor norms in the sense of

Definition 2.1, then for every $n \in \mathbb{Z}_{\geq 2}$, if $\|\cdot\|_{V^{\otimes n}}$ denotes the admissible tensor norm chosen for $V^{\otimes n}$, we have that for every $v \in V^{\otimes n}$ that (cf. Proposition 2.1 in [26])

$$\|v\|_{\text{inj},n} \leq \|v\|_{V^{\otimes n}} \leq \|v\|_{\text{proj},n}. \tag{2.6}$$

In the case that the norm $\|\cdot\|_V$ is induced by an inner product $\langle \cdot, \cdot \rangle_V$, in the sense that $\|\cdot\|_V = \sqrt{\langle \cdot, \cdot \rangle_V}$, we can equip the tensor powers of V with admissible norms in the sense of Definition 2.1 by extending the inner product $\langle \cdot, \cdot \rangle_V$ to the tensor powers of V .

That is, suppose $n \in \mathbb{Z}_{\geq 2}$. Then, we define an inner product $\langle \cdot, \cdot \rangle_{V^{\otimes n}}$ on $V^{\otimes n}$ as follows. Firstly, if $u, w \in V^{\otimes n}$ are given by $u = u_1 \otimes \dots \otimes u_n$ and $w = w_1 \otimes \dots \otimes w_n$ for elements $u_1, \dots, u_n, w_1, \dots, w_n \in V$, define

$$\langle u, w \rangle_{V^{\otimes n}} := \prod_{s=1}^n \langle u_s, w_s \rangle_V. \tag{2.7}$$

Subsequently, extend $\langle \cdot, \cdot \rangle_{V^{\otimes n}}$ defined in (2.7) to the entirety of $V^{\otimes n} \times V^{\otimes n}$ by linearity so that the resulting function $\langle \cdot, \cdot \rangle_{V^{\otimes n}} : V^{\otimes n} \times V^{\otimes n} \rightarrow \mathbb{R}$ defines an inner product on $V^{\otimes n}$. For each $n \in \mathbb{Z}_{\geq 2}$, equip $V^{\otimes n}$ with the norm $\|\cdot\|_{V^{\otimes n}} := \sqrt{\langle \cdot, \cdot \rangle_{V^{\otimes n}}}$ induced by the inner product $\langle \cdot, \cdot \rangle_{V^{\otimes n}}$. Then the tensor powers of V are all equipped with admissible norms in the sense of Definition 2.1.

Returning to the general setting that V and W are merely assumed to be real Banach spaces, we now define a $\text{Lip}(\gamma, \Sigma, W)$ function.

Definition 2.2 ($\text{Lip}(\gamma, \Sigma, W)$ functions). Let V and W be Banach spaces, $\Sigma \subset V$ a closed subset and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Let $\gamma > 0$ with $k \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$. Suppose that $\psi^{(0)} : \Sigma \rightarrow W$ is a function, and that for each $l \in \{1, \dots, k\}$, we have a function $\psi^{(l)} : \Sigma \rightarrow \mathcal{L}(V^{\otimes l}; W)$ taking its values in the space of symmetric l -linear forms from V to W . Then, the collection $\psi = (\psi^{(0)}, \psi^{(1)}, \dots, \psi^{(k)})$ is an element of $\text{Lip}(\gamma, \Sigma, W)$ if there exists a constant $M \geq 0$ for which the following conditions hold:

- For each $l \in \{0, \dots, k\}$ and every $x \in \Sigma$, we have that

$$\left\| \psi^{(l)}(x) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq M. \tag{2.8}$$

- For each $j \in \{0, \dots, k\}$, define $R_j^\psi : \Sigma \times \Sigma \rightarrow \mathcal{L}(V^{\otimes j}; W)$ for $z, p \in \Sigma$ and $v \in V^{\otimes j}$ by

$$R_j^\psi(z, p)[v] := \psi^{(j)}(p)[v] - \sum_{s=0}^{k-j} \frac{1}{s!} \psi^{(j+s)}(z)[v \otimes (p - z)^{\otimes s}]. \tag{2.9}$$

Then, whenever $l \in \{0, \dots, k\}$ and $x, y \in \Sigma$, we have

$$\left\| R_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq M \|y - x\|_V^{\gamma-l}. \tag{2.10}$$

We sometimes say that $\psi \in \text{Lip}(\gamma, \Sigma, W)$ without explicitly mentioning the functions $\psi^{(0)}, \dots, \psi^{(k)}$. Furthermore, given $l \in \{0, \dots, k\}$, we introduce the notation that $\psi_{[l]} := (\psi^{(0)}, \dots, \psi^{(l)})$. The

$\text{Lip}(\gamma, \Sigma, W)$ norm of ψ , denoted by $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)}$, is the smallest $M \geq 0$ satisfying the requirements (2.8) and (2.10).

In Definition 2.2, we use that $V^{\otimes 0} := \mathbb{R}$ to observe that $\mathcal{L}(V^{\otimes 0}; W) = W$. Thus, we implicitly assume that $\mathcal{L}(V^{\otimes 0}; W) = W$ is taken to be equipped with the norm $\|\cdot\|_W$ on W ; that is, $\|\cdot\|_{\mathcal{L}(V^{\otimes 0}; W)} = \|\cdot\|_W$ in both (2.8) and (2.10). Thus, a consequence of Definition 2.2 is that $\psi^{(0)} \in C^0(\Sigma; W)$ with $\|\psi^{(0)}\|_{C^0(\Sigma; W)} \leq \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)}$. Here, given $f \in C^0(\Sigma; W)$, we take $\|f\|_{C^0(\Sigma; W)} := \sup\{\|f(x)\|_W : x \in \Sigma\}$.

Further, we implicitly assume in Definition 2.2 that, for each $j \in \{1, \dots, k\}$, norms have been chosen for the spaces $\mathcal{L}(V^{\otimes j}; W)$ of symmetric j -linear forms from V to W . Observe that $\mathcal{L}(V^{\otimes j}; W) \subset L(V^{\otimes j}; W)$ where $L(V^{\otimes j}; W)$ denotes the space of linear maps $V^{\otimes j}$ to W . There are, of course, numerous possible choices for such norms. Throughout this article, we will always assume the following choice. Given a norm $\|\cdot\|_{V^{\otimes j}}$ on $V^{\otimes j}$ and a norm $\|\cdot\|_W$ on W , we equip $L(V^{\otimes j}; W)$ with the corresponding operator norm. That is, for any $\mathbf{A} \in L(V^{\otimes j}; W)$, we have

$$\|\mathbf{A}\|_{L(V^{\otimes j}; W)} := \sup\{\|\mathbf{A}[v]\|_W : v \in V^{\otimes j} \text{ and } \|v\|_{V^{\otimes j}} = 1\}. \tag{2.11}$$

When $\mathbf{A} \in \mathcal{L}(V^{\otimes j}; W)$, we will denote the norm defined in (2.11) by $\|\mathbf{A}\|_{\mathcal{L}(V^{\otimes j}; W)}$. This choice of norm means that our definition of a $\text{Lip}(\gamma, \Sigma, W)$ function differs from the definition used in [1]. Indeed, we require the bounds in (2.8) and (2.10) to hold for the operator norms, whilst in [1] estimates of the analogous form are only required to be valid for *rank-one* elements in $V^{\otimes j}$; that is, for elements $v = v_1 \otimes \dots \otimes v_j$ where $v_1, \dots, v_j \in V$. Consequently, our definition of a $\text{Lip}(\gamma, \Sigma, W)$ function is more restrictive than the notion in [1].

A good way to understand a $\text{Lip}(\gamma, \Sigma, W)$ function is as a function that ‘locally looks like a polynomial function’. Given any point $x \in \Sigma$, consider the polynomial $\Psi_x : V \rightarrow W$ defined for $y \in V$ by

$$\Psi_x(y) := \sum_{s=0}^k \frac{1}{s!} \psi^{(s)}(x) [(y-x)^{\otimes s}]. \tag{2.12}$$

The polynomial Ψ_x defined in (2.12) gives a proposal, based at the point $x \in \Sigma$, for how the function ψ behaves away from x . The remainder term estimates in (2.10) of Definition 2.2 ensure that for every $y \in \Sigma$, we have that

$$\|\psi^{(0)}(y) - \Psi_x(y)\|_W \leq \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \|y-x\|_V^\gamma. \tag{2.13}$$

It follows from (2.13) that, for a given $\varepsilon > 0$, if we take $\delta := (\varepsilon/\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)})^{1/\gamma}$, then the polynomial Ψ_x is within ε of $\psi^{(0)}$, in the $\|\cdot\|_W$ norm sense, throughout the neighbourhood $\overline{\mathbb{B}}_V(x, \delta) \cap \Sigma$ of the point x .

The collection of functions $\psi^{(0)}, \dots, \psi^{(k)}$ are related to Ψ_x in the following sense. For each $l \in \{0, \dots, k\}$, the element $\psi^{(l)}(x) \in \mathcal{L}(V^{\otimes l}; W)$ is the l^{th} derivative of $\Psi_x(\cdot)$ at x . The proposal functions Ψ_x for points $x \in \Sigma$ enable one to view a $\text{Lip}(\gamma, \Sigma, W)$ function in a more traditional manner as the single function $\Sigma \times V \rightarrow W$ defined by the mapping $(x, y) \mapsto \Psi_x(y)$. The remainder term estimates in (2.10) of Definition 2.2 ensure that this mapping exhibits Hölder regularity in a classical sense.

To illustrate assume that $\gamma > 1$ so that $k \geq 1$, and suppose that we have basepoints $x, w \in \Sigma$ and $y, z \in V$ such that $\|x - w\|_V \leq 1$ and $\|y - z\|_V \leq 1$. Let $L_{z-x, y-x}, L_{z-w, z-x} \subset V$ denote the straight lines connecting $z - x$ to $y - x$ and $z - w$ to $z - x$, respectively. Define $r_1 := \max\{\|x - z\|_V, \|x - y\|_V\}$ and $r_2 := \max\{\|z - w\|_V, \|z - x\|_V\}$ so that, in particular, we have both the inclusions $L_{z-x, y-x} \subset \overline{\mathbb{B}}_V(0, r_1)$ and $L_{z-w, z-x} \subset \overline{\mathbb{B}}_V(0, r_2)$. Then, we have the Hölder-type estimate that

$$\|\Psi_x(y) - \Psi_w(z)\|_W \leq \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \left(e^{r_1} \|y - z\|_V + (e^{r_2} + e^{1+\|z-x\|_V}) \|x - w\|_V^{\gamma-k} \right). \tag{2.14}$$

To see this, we first observe that

$$\|\Psi_x(y) - \Psi_x(z)\|_W \stackrel{(2.12)}{\leq} \sum_{s=0}^k \frac{1}{s!} \left\| \psi^{(s)}(x) [(y-x)^{\otimes s} - (z-x)^{\otimes s}] \right\|_W \leq e^{r_1} \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \|y - z\|_V, \tag{2.15}$$

where the last inequality follows from applying the mean value theorem to each of the mappings $v \mapsto (v - x)^{\otimes s}$ for $s \in \{1, \dots, k\}$ and recalling that $L_{z-x, y-x} \subset \overline{\mathbb{B}}_V(0, r_1)$. Next observe, via (2.9) and (2.12), that

$$\begin{aligned} \Psi_x(z) - \Psi_w(z) &= \psi^{(0)}(x) - \psi^{(0)}(w) + \sum_{s=1}^k \frac{1}{s!} R_s^\psi(w, x) [(z-x)^{\otimes s}] + \\ &\sum_{s=1}^k \sum_{j=1}^{k-s} \frac{1}{s!j!} \psi^{(s+j)}(w) [(z-x)^{\otimes s} \otimes (x-w)^{\otimes j}] + \sum_{s=1}^k \frac{1}{s!} \psi^{(s)}(w) [(z-x)^{\otimes s} - (z-w)^{\otimes s}]. \end{aligned} \tag{2.16}$$

To estimate the first term on the RHS of (2.16), we compute

$$\begin{aligned} \|\psi^{(0)}(x) - \psi^{(0)}(w)\|_W &\stackrel{(2.9)}{\leq} \sum_{s=1}^k \frac{1}{s!} \left\| \psi^{(s)}(w) [(x-w)^{\otimes s}] \right\|_W + \|R_0^\psi(w, x)\|_W \\ &\stackrel{(2.8) \ \& \ (2.10)}{\leq} \left(\sum_{s=1}^k \frac{1}{s!} \|x - w\|_V^s + \|w - x\|_V^\gamma \right) \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq e \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \|w - x\|_V, \end{aligned} \tag{2.17}$$

where the last inequality uses that $\|w - x\|_V \leq 1$.

For the second term on the RHS of (2.16), we compute that

$$\begin{aligned} \sum_{s=1}^k \frac{1}{s!} \left\| R_s^\psi(w, x) [(z-x)^{\otimes s}] \right\|_W &\stackrel{(2.10)}{\leq} \sum_{s=1}^k \frac{1}{s!} \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \|w - x\|_V^{\gamma-s} \|z - x\|_V^s \\ &\leq (e^{\|z-x\|_V} - 1) \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \|w - x\|_V^{\gamma-k}, \end{aligned} \tag{2.18}$$

where the last inequality uses that $\|w - x\|_V \leq 1$.

For the third term on the RHS of (2.16), we compute that

$$\begin{aligned} & \sum_{s=1}^k \sum_{j=1}^{k-s} \frac{1}{s!j!} \left\| \psi^{(s+j)}(w) [(z-x)^{\otimes s} \otimes (x-w)^{\otimes j}] \right\|_W \\ & \stackrel{(2.8)}{\leq} \sum_{s=1}^k \sum_{j=1}^{k-s} \frac{1}{j!s!} \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \|z-x\|_V^s \|x-w\|_V^j \\ & \leq (e^{\|z-x\|_V} - 1)(e-1) \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \|x-w\|_V, \end{aligned} \tag{2.19}$$

where the last inequality uses that $\|w-x\|_V \leq 1$.

For the fourth term on the RHS of (2.16), we compute that

$$\sum_{s=1}^k \frac{1}{s!} \left\| \psi^{(s)}(w) [(z-x)^{\otimes s} - (z-w)^{\otimes s}] \right\|_W \leq e^{r_2} \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \|x-w\|_V, \tag{2.20}$$

where, for each $s \in \{1, \dots, k\}$, we have applied the mean value theorem to the mapping $v \mapsto (z-v)^{\otimes s}$ and recalled the inclusion $L_{z-w, z-x} \subset \overline{\mathbb{B}}_V(0, r_2)$.

The combination of (2.16)–(2.20) yields that

$$\|\Psi_x(z) - \Psi_w(z)\|_W \leq (e^{r_2} + e^{1+\|z-x\|_V}) \|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \|x-w\|_V^{\gamma-k}, \tag{2.21}$$

where we have used that $\|x-w\|_V \leq 1$ means $\|x-w\|_V \leq \|x-w\|_V^{\gamma-k}$ since $\gamma-k \in (0, 1]$. The combination of (2.15) and (2.21) then establishes the estimate claimed in (2.14).

Returning to considering the collection $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$, on the interior of Σ , the functions $\psi^{(1)}, \dots, \psi^{(k)}$ are determined by $\psi^{(0)}$. The remainder term estimates in (2.10) for $\psi^{(l)}$ for each $l \in \{0, \dots, k\}$ ensure, for each $j \in \{1, \dots, k\}$, that $\psi^{(j)}$ is the classical j^{th} Fréchet derivative of $\psi^{(0)}$ on the interior of Σ . Thus, on the interior of Σ , $\psi^{(0)}$ is k times continuously differentiable, and its k^{th} derivative is $(\gamma-k)$ -Hölder continuous.

3 | MAIN RESULTS

In this section, we state our main results and discuss some of their consequences. Suppose that V and W are real Banach spaces, that $\Sigma \subset V$ is a closed subset and that all the tensor powers of V are equipped with admissible norms (cf. Definition 2.1). Our starting point is to observe that Stein’s extension theorem (Theorem 4 in Chapter VI of [28]) remains valid for functions in $\text{Lip}(\gamma, \Sigma, W)$, provided the Banach space V is finite dimensional. Indeed, following the method proposed by Stein in [28], one uses the Whitney cube decomposition of $V \setminus \Sigma$ (originating in [29]) to define an appropriately weighted average of the collection $\{\Psi_x(\cdot) : x \in \Sigma\}$ to give an extension Ψ of ψ to the entirety of V . Provided that V is finite dimensional, this approach and the corresponding estimates carry across to our setting verbatim from Chapter VI in [28]. Only the given values of $\psi^{(0)}, \dots, \psi^{(k)}$ at points $x \in \Sigma$ are used to define this extension; consequently, there is no dependence on the dimension of the target space W .

Moreover, the operator $A : \text{Lip}(\gamma, \Sigma, W) \rightarrow \text{Lip}(\gamma, V, W)$ defined by mapping $\phi \in \text{Lip}(\gamma, \Sigma, W)$ to its corresponding weighted average $\Phi \in \text{Lip}(\gamma, V, W)$ of the collection $\{\Phi_x(\cdot) : x \in \Sigma\}$ is a

bounded linear operator whose norm depends only on γ and the dimension of V . That is, there is a constant $C = C(\gamma, \dim(V)) \geq 1$ such that for any $\phi \in \text{Lip}(\gamma, \Sigma, W)$, we have that $A[\phi] \in \text{Lip}(\gamma, V, W)$ satisfies $\|A[\phi]\|_{\text{Lip}(\gamma, V, W)} \leq C\|\phi\|_{\text{Lip}(\gamma, \Sigma, W)}$. This is again a consequence of a verbatim repetition of the arguments of Stein in Chapter VI of [28].

Suppose that $B \subset \Sigma$ is a non-empty closed subset. A particular consequence of Stein’s extension theorem (Theorem 4 in Chapter VI of [28]) is that any element in $\text{Lip}(\gamma, B, W)$ can be extended to an element in $\text{Lip}(\gamma, \Sigma, W)$. Recall that it is unreasonable to expect uniqueness for such an extension. We are interested in understanding when extensions of an element in $\text{Lip}(\gamma, B, W)$ to $\text{Lip}(\gamma, \Sigma, W)$ are forced to remain, in some sense, close throughout Σ . We consider the following problem. Given elements $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$ and $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)})$ in $\text{Lip}(\gamma, \Sigma, W)$, when does knowing that ψ and φ are, in some sense, ‘close’ on B ensure that ψ and φ remain ‘close’, in some possibly different sense, throughout Σ .

The following *Lipschitz Sandwich Theorem* gives a condition for the subset B and precise meanings for the notions of closeness to be considered between ψ and φ on B and Σ , respectively, under which this problem has an affirmative answer.

Theorem 3.1 (Lipschitz Sandwich theorem). *Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Sigma \subset V$ is non-empty and closed. Let $\varepsilon > 0$, $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$, and $\gamma > \eta > 0$ with $k, q \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$ and $\eta \in (q, q + 1]$. Then, there exist constants $\delta_0 = \delta_0(\varepsilon, K_1 + K_2, \gamma, \eta) > 0$ and $\varepsilon_0 = \varepsilon_0(\varepsilon, K_1 + K_2, \gamma, \eta) > 0$ for which the following is true.*

Suppose $B \subset \Sigma$ is a closed subset that is a δ_0 -cover of Σ in the sense that

$$\Sigma \subset \bigcup_{x \in B} \overline{B}_V(x, \delta_0) = B_{\delta_0} := \{v \in V : \text{There exists } z \in B \text{ such that } \|v - z\|_V \leq \delta_0\}. \tag{3.1}$$

Suppose $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$, $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfy the $\text{Lip}(\gamma, \Sigma, W)$ norm estimates $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Further suppose that for every $l \in \{0, \dots, k\}$ and every $x \in B$, the difference $\psi^{(l)}(x) - \varphi^{(l)}(x) \in \mathcal{L}(V^{\otimes l}; W)$ satisfies the bound

$$\left\| \psi^{(l)}(x) - \varphi^{(l)}(x) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \varepsilon_0. \tag{3.2}$$

Then, we may conclude that

$$\left\| \psi_{[q]} - \varphi_{[q]} \right\|_{\text{Lip}(\eta, \Sigma, W)} \leq \varepsilon, \tag{3.3}$$

where $\psi_{[q]} := (\psi^{(0)}, \dots, \psi^{(q)})$ and $\varphi_{[q]} := (\varphi^{(0)}, \dots, \varphi^{(q)})$.

Remark 3.2. The condition $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$ is imposed to avoid having $K_1 = K_2 = 0$. Since $K_1 = K_2 = 0$ means that $\psi \equiv 0 \equiv \varphi$, we see that in this case the content of Theorem 3.1 is vacuous.

Remark 3.3. Assume the notation as in Theorem 3.1. For any $\delta > 0$, we have that Σ is a subset of its own δ -fattening $\Sigma_\delta := \{v \in V : \exists p \in \Sigma \text{ with } \|v - p\|_V \leq \delta\}$. Consequently, Theorem 3.1 is valid for the choice $B := \Sigma$. For this choice of B , Theorem 3.1 tells us that there exists a constant $\varepsilon_0 = \varepsilon_0(\varepsilon, K_1 + K_2, \gamma, \eta) > 0$ for which the following is true. If $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$,

$\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ with $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$ satisfy, for every $x \in \Sigma$ and every $l \in \{0, \dots, k\}$, that $\|\psi^{(l)}(x) - \varphi^{(l)}(x)\|_{\mathcal{L}(V^{\otimes l}, W)} \leq \varepsilon_0$, then we may, in fact, conclude that $\|\psi_{[q]} - \varphi_{[q]}\|_{\text{Lip}(\gamma, \Sigma, W)} \leq \varepsilon$ where $\psi_{[q]} := (\psi^{(0)}, \dots, \psi^{(q)})$ and $\varphi_{[q]} := (\varphi^{(0)}, \dots, \varphi^{(q)})$.

Remark 3.4. Using the same notation as in Theorem 3.1, by taking $\varphi \equiv 0$, we may conclude from Theorem 3.1 that if $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfies both that $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and, for every $l \in \{0, \dots, k\}$ and every $x \in B$, that $\|\psi^{(l)}(x)\|_{\mathcal{L}(V^{\otimes l}, W)} \leq \varepsilon_0$, then we have that $\|\psi_{[q]}\|_{\text{Lip}(\gamma, \Sigma, W)} \leq \varepsilon$.

Remark 3.5. Using the same notation as in Theorem 3.1, the estimates (3.2) throughout B are a weaker condition than $\|\psi - \varphi\|_{\text{Lip}(\gamma, B, W)} \leq \varepsilon_0$. The bound $\|\psi - \varphi\|_{\text{Lip}(\gamma, B, W)} \leq \varepsilon_0$ implies that the pointwise estimates in (3.2) are valid. But the converse is *not* true since the pointwise estimates in (3.2) alone are insufficient to establish the required estimates for the remainder terms associated to the difference $\psi - \varphi$ (cf. Definition 2.2).

Remark 3.6. The restriction that $\eta \in (0, \gamma)$ in Theorem 3.1 is necessary; the theorem is *false* for $\eta := \gamma$. As an example, fix $K_0, \varepsilon > 0$ with $\varepsilon < 2K_0$, let $\delta > 0$ and consider a fixed $N \in \mathbb{Z}_{\geq 1}$ for which $1/N < \delta$. Define $\Sigma := \{0, 1/N\} \subset \mathbb{R}$ and $B := \Sigma \setminus \{1/N\} = \{0\} \subset \mathbb{R}$. Then, we have that $\Sigma \subset [-\delta, \delta]$, and so, B is a δ -cover of Σ as required in (3.1). Define $\psi, \varphi : \Sigma \rightarrow \mathbb{R}$ by $\psi(0) := 0, \psi(1/N) := K_0/N$ and $\varphi(0) := 0, \varphi(1/N) := -K_0/N$. Then, $\psi, \varphi \in \text{Lip}(1, \Sigma, \mathbb{R})$ with $\|\psi\|_{\text{Lip}(1, \Sigma, \mathbb{R})} = \|\varphi\|_{\text{Lip}(1, \Sigma, \mathbb{R})} = K_0$ and $\psi - \varphi \equiv 0$ throughout B , establishing the validity of the bounds (3.2) for any $\varepsilon_0 \geq 0$. However, $|(\psi - \varphi)(1/N) - (\psi - \varphi)(0)| = 2K_0/N = 2K_0|1/N - 0|$, which means that $\|\psi - \varphi\|_{\text{Lip}(1, \Sigma, \mathbb{R})} = 2K_0 > \varepsilon$.

Remark 3.7. It may initially appear that the theorem should be valid for any fixed ε_0 with $\varepsilon_0 < \varepsilon$ by suitably restricting δ_0 , rather than having to allow ε_0 to depend on $\varepsilon, K_1 + K_2, \gamma$ and η . But this is *not* the case. If we only assume $\varepsilon_0 < \varepsilon$, then the estimates in (3.2) can even be insufficient to establish that $\|\psi - \varphi\|_{\text{Lip}(\eta, B, W)} \leq \varepsilon$. For example, let $\gamma := 1, \eta := 1/2$, and fix $0 < \varepsilon_0 < \varepsilon < 1 < K_0$ such that $2\varepsilon_0 K_0 > \varepsilon^2$. Define $x_0 := 2\varepsilon_0/K_0 > 0$ and consider $\Sigma = B := \{0, x_0\}$. Define $\psi, \varphi : \Sigma \rightarrow \mathbb{R}$ by $\psi(0) := -\varepsilon_0, \psi(x_0) := \varepsilon_0$ and $\varphi(0) := 0 = \varphi(x_0)$. Then, $\psi, \varphi \in \text{Lip}(1, \Sigma, \mathbb{R})$, with $\|\varphi\|_{\text{Lip}(1, \Sigma, \mathbb{R})} = 0$ and $\|\psi\|_{\text{Lip}(1, \Sigma, \mathbb{R})} = K_0$. Moreover, $|\psi - \varphi| = |\psi| \leq \varepsilon_0$ throughout $\Sigma = B$ so that the estimates (3.2) are valid. However, we may also compute that $|(\psi - \varphi)(x_0) - (\psi - \varphi)(0)| = 2\varepsilon_0 = 2\varepsilon_0 \sqrt{1/x_0} \sqrt{|x_0 - 0|} = \sqrt{2\varepsilon_0 K_0} \sqrt{|x_0 - 0|}$ so that $\|\psi - \varphi\|_{\text{Lip}(1/2, B, \mathbb{R})} = \sqrt{2\varepsilon_0 K_0} > \varepsilon$.

The issue described in Remark 3.7 is only present when the cardinality of the subset B is greater than 1, that is, when B contains at least two distinct points. When B consists of a single point, we can, in fact, allow for an arbitrary $\varepsilon_0 \in [0, \varepsilon)$ in Theorem 3.1 rather than having to allow ε_0 to depend on $\varepsilon, K_1 + K_2, \gamma$ and η . The precise statement is recorded in the following theorem.

Theorem 3.8 (Single-point Lipschitz Sandwich theorem). *Let V and W be Banach spaces and assume that the tensor powers of V are all equipped with admissible tensor norms (cf. Definition 2.1). Assume that $\Sigma \subset V$ is closed and non-empty. Let $\varepsilon > 0, (K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}, \gamma > \eta > 0$ with $k, q \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$ and $\eta \in (q, q + 1]$, and $0 \leq \varepsilon_0 < \min\{K_1 + K_2, \varepsilon\}$. Then, there exists a constant $\delta_0 = \delta_0(\varepsilon, \varepsilon_0, K_1 + K_2, \gamma, \eta) > 0$ for which the following is true.*

Suppose $p \in \Sigma$ and that $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$ and $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)})$ are elements in $\text{Lip}(\gamma, \Sigma, W)$ with $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Further suppose that for every $l \in \{0, \dots, k\}$, the

difference $\psi^{(l)}(p) - \varphi^{(l)}(p) \in \mathcal{L}(V^{\otimes l}; W)$ satisfies the bound

$$\left\| \psi^{(l)}(p) - \varphi^{(l)}(p) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \varepsilon_0. \tag{3.4}$$

Then, we may conclude that

$$\left\| \psi_{[q]} - \varphi_{[q]} \right\|_{\text{Lip}(\gamma, \bar{\mathbb{B}}_V(p, \delta_0) \cap \Sigma, W)} \leq \varepsilon, \tag{3.5}$$

where $\psi_{[q]} := (\psi^{(0)}, \dots, \psi^{(q)})$ and $\varphi_{[q]} := (\varphi^{(0)}, \dots, \varphi^{(q)})$.

Remark 3.9. The condition $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$ is imposed to avoid having $K_1 = K_2 = 0$. Since $K_1 = K_2 = 0$ means that $\psi \equiv 0 \equiv \varphi$, we see that in this case the content of Theorem 3.8 vacuous.

Remark 3.10. Using the same notation as in Theorem 3.8, by taking $\varphi \equiv 0$, we may conclude from Theorem 3.8 that if $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfies both that $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and, for every $l \in \{0, \dots, k\}$, that $\left\| \psi^{(l)}(p) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \varepsilon_0$, then we have that $\left\| \psi_{[q]} \right\|_{\text{Lip}(\gamma, \bar{\mathbb{B}}_V(p, \delta_0) \cap \Sigma, W)} \leq \varepsilon$.

Establishing Theorem 3.8 will form the first step in our proof of Theorem 3.1.

Returning our attention to Theorem 3.1, the Lipschitz estimates obtained in the conclusion (3.3) yield pointwise estimates for the difference $\psi^{(0)} - \varphi^{(0)} : \Sigma \rightarrow W$. In particular, we may conclude that $\left\| \psi^{(0)} - \varphi^{(0)} \right\|_{C^0(\Sigma; W)} \leq \varepsilon$. However, such pointwise estimates can be established directly without needing to appeal to Theorem 3.1. Moreover, this direct approach allows us to obtain estimates for the difference $\psi^{(l)} - \varphi^{(l)} : \Sigma \rightarrow \mathcal{L}(V^{\otimes l}; W)$ for every $l \in \{0, \dots, k\}$. An additional benefit is that we are able to provide a more explicit constant δ_0 for which we require the subset $B \subset \Sigma$ to be a δ_0 -cover of Σ . The precise result is recorded in the following theorem.

Theorem 3.11 (Pointwise Lipschitz Sandwich theorem). *Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Sigma \subset V$ is closed. Let $\varepsilon > 0$, $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$, $\gamma > 0$ with $k \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$, and $0 \leq \varepsilon_0 < \min \{K_1 + K_2, \varepsilon\}$. Then, given any $l \in \{0, \dots, k\}$, there exists a constant $\delta_0 = \delta_0(\varepsilon, \varepsilon_0, K_1 + K_2, \gamma, l) > 0$, defined by*

$$\delta_0 := \sup \left\{ \theta > 0 : (K_1 + K_2)\theta^{\gamma-l} + \varepsilon_0 e^\theta \leq \min \{K_1 + K_2, \varepsilon\} \right\} > 0, \tag{3.6}$$

for which the following is true.

Suppose $B \subset \Sigma$ is a δ_0 -cover of Σ in the sense that

$$\Sigma \subset \bigcup_{x \in B} \bar{\mathbb{B}}_V(x, \delta_0) = B_{\delta_0} := \{v \in V : \text{There exists } z \in B \text{ such that } \|v - z\|_V \leq \delta_0\}. \tag{3.7}$$

Suppose $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$, $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfy the $\text{Lip}(\gamma, \Sigma, W)$ norm estimates that $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Further suppose that for every $j \in \{0, \dots, k\}$

and every $x \in B$, the difference $\psi^{(j)}(x) - \varphi^{(j)}(x) \in \mathcal{L}(V^{\otimes j}; W)$ satisfies the bound

$$\left\| \psi^{(j)}(x) - \varphi^{(j)}(x) \right\|_{\mathcal{L}(V^{\otimes j}; W)} \leq \varepsilon_0. \tag{3.8}$$

Then, we may conclude that for every $s \in \{0, \dots, l\}$ and every $x \in \Sigma$ that

$$\left\| \psi^{(s)}(x) - \varphi^{(s)}(x) \right\|_{\mathcal{L}(V^{\otimes s}; W)} \leq \varepsilon. \tag{3.9}$$

Remark 3.12. The condition $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$ is imposed to avoid having $K_1 = K_2 = 0$. Since $K_1 = K_2 = 0$ means that $\psi \equiv 0 \equiv \varphi$, we see that in this case the content of Theorem 3.8 vacuous.

Remark 3.13. In contrasts to Theorem 3.1, we are able to deal with arbitrary $\varepsilon_0 < \varepsilon$ by suitably restricting δ_0 . The issue outlined in Remark 3.7 is no longer a problem in this setting since the same notion of closeness is used in both the assumption (3.8) and the conclusion (3.9).

Remark 3.14. Assume the notation as in Theorem 3.11. For any $\delta > 0$, we have that Σ is a subset of its own δ -fattening $\Sigma_\delta := \{v \in V : \exists p \in \Sigma \text{ with } \|v - p\|_V \leq \delta\}$. Consequently, Theorem 3.11 is valid for the choice $B := \Sigma$. For this choice of B , Theorem 3.11 recovers the following trivial statement. Let $\varepsilon_0 < \min\{K_1 + K_2, \varepsilon\}$, $l \in \{0, \dots, k\}$ and $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$, $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$. Suppose that $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Further suppose, for every point $x \in \Sigma$ and every integer $j \in \{0, \dots, k\}$, that $\left\| \psi^{(j)}(x) - \varphi^{(j)}(x) \right\|_{\mathcal{L}(V^{\otimes j}; W)} \leq \varepsilon_0$. Then, we have, for every point $x \in \Sigma$ and every $s \in \{0, \dots, l\}$, that $\left\| \psi^{(s)}(x) - \varphi^{(s)}(x) \right\|_{\mathcal{L}(V^{\otimes s}; W)} \leq \varepsilon$.

Remark 3.15. Using the same notation as in Theorem 3.11, by taking $\varphi \equiv 0$, we may conclude from Theorem 3.11 that if $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfies both that $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and, for every $j \in \{0, \dots, k\}$ and every $x \in B$, that $\left\| \psi^{(j)}(x) \right\|_{\mathcal{L}(V^{\otimes j}; W)} \leq \varepsilon_0$, then we have, for every $s \in \{0, \dots, l\}$ and every point $x \in \Sigma$, that $\left\| \psi^{(s)}(x) \right\|_{\mathcal{L}(V^{\otimes s}; W)} \leq \varepsilon$.

Remark 3.16. It follows from (3.6) that, for every $l \in \{0, \dots, k\}$, the constant $\delta_0 = \delta_0(\varepsilon, \varepsilon_0, K_1 + K_2, \gamma, l) > 0$ is bounded above by 1. Moreover, if the constants ε , ε_0 , $K_1 + K_2$, and γ remain fixed, we may conclude that the constant δ_0 specified in (3.6) is decreasing with respect to the argument $l \in \{0, \dots, k\}$ in the sense that the mapping $l \mapsto \delta_0(\varepsilon, \varepsilon_0, K_1 + K_2, \gamma, l)$ is a decreasing function on $\{0, \dots, k\}$.

4 | COST-EFFECTIVE APPROXIMATION APPLICATION

In this section, we use the results presented in Section 3 to establish that, when the closed subset $\Sigma \subset V$ is compact, an element $\psi \in \text{Lip}(\gamma, \Sigma, W)$ can be, in some to be detailed sense, well approximated using only its values at a finite number of points in Σ . We start with the following corollary of the Lipschitz Sandwich Theorem 3.1 establishing that, when $\Sigma \subset V$ is compact, ψ can be well approximated in the $\text{Lip}(\eta, \Sigma, W)$ -norm sense using only the values of ψ at a finite number of points in Σ . The precise result is the following corollary.

Corollary 4.1 (Consequence of the *Lipschitz Sandwich Theorem 3.1*). *Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Sigma \subset V$ is compact. Let $\varepsilon > 0$, $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$, and $\gamma > \eta > 0$ with $k, q \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$ and $\eta \in (q, q + 1]$. Let $\delta_0 = \delta_0(\varepsilon, K_1 + K_2, \gamma, \eta) > 0$ and $\varepsilon_0 = \varepsilon_0(\varepsilon, K_1 + K_2, \gamma, \eta) > 0$ denote the constants arising from Theorem 3.1, and let $N = N(\Sigma, \varepsilon, K_1 + K_2, \gamma, \eta) \in \mathbb{Z}_{\geq 0}$ denote the δ_0 -covering number of Σ . That is,*

$$N := \min \left\{ d \in \mathbb{Z} : \text{There exists } x_1, \dots, x_d \in \Sigma \text{ such that } \Sigma \subset \bigcup_{j=1}^d \overline{\mathbb{B}}_V(x_j, \delta_0) \right\}. \tag{4.1}$$

Then, there is a finite subset $\Sigma_N = \{z_1, \dots, z_N\} \subset \Sigma$ for which the following is true.

Suppose $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$, $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfy the $\text{Lip}(\gamma, \Sigma, W)$ norm estimates that $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Further suppose that for every $l \in \{0, \dots, k\}$ and every $j \in \{1, \dots, N\}$, the difference $\psi^{(l)}(z_j) - \varphi^{(l)}(z_j) \in \mathcal{L}(V^{\otimes l}; W)$ satisfies the bound

$$\left\| \psi^{(l)}(z_j) - \varphi^{(l)}(z_j) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \varepsilon_0. \tag{4.2}$$

Then, we may conclude that

$$\left\| \psi_{[q]} - \varphi_{[q]} \right\|_{\text{Lip}(\eta, \Sigma, W)} \leq \varepsilon, \tag{4.3}$$

where $\psi_{[q]} := (\psi^{(0)}, \dots, \psi^{(q)})$ and $\varphi_{[q]} := (\varphi^{(0)}, \dots, \varphi^{(q)})$.

Remark 4.2. In a similar spirit to Remark 3.4 and using the same notation as in Corollary 4.1, by taking $\varphi \equiv 0$, we conclude from Corollary 4.1 that if $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfies both $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and, for every $l \in \{0, \dots, k\}$ and every $x \in \Sigma_N$, that $\left\| \psi^{(l)}(x) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \varepsilon_0$, then we have that $\left\| \psi_{[q]} \right\|_{\text{Lip}(\eta, \Sigma, W)} \leq \varepsilon$.

Remark 4.3. When $N \in \mathbb{Z}_{\geq 1}$ defined in (4.1) is less than the cardinality of Σ , Corollary 4.1 guarantees that we are able to identify a strictly smaller collection of points at which the behaviour of a $\text{Lip}(\gamma, \Sigma, W)$ function determines the functions $\text{Lip}(\eta)$ -behaviour up to an arbitrarily small error over the entire set Σ . That is, using the notation of Corollary 4.1, if $F \in \text{Lip}(\gamma, \Sigma_N, W)$, then any two extensions ψ and φ of F to elements in $\text{Lip}(\gamma, \Sigma, W)$, with $\text{Lip}(\gamma, \Sigma, W)$ -norms bounded above by K_1 and K_2 respectively, can differ, in the $\text{Lip}(\eta)$ -sense, by at most ε throughout Σ .

A particular consequence of this is that a function in $\text{Lip}(\gamma, \Sigma, W)$ can be cost-effectively approximated. That is, let $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ with $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and suppose that we want to approximate ψ in a $\text{Lip}(\eta, \Sigma, W)$ -norm sense. Then, Corollary 4.1 guarantees us that any $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ with $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$ will satisfy that $\|\psi_{[q]} - \varphi_{[q]}\|_{\text{Lip}(\eta, \Sigma, W)} \leq \varepsilon$ provided we have, for every point $x \in \Sigma_N$ and every $l \in \{0, \dots, k\}$, that $\left\| \psi^{(l)}(x) - \varphi^{(l)}(x) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \varepsilon_0$. Thus, the task of approximating ψ throughout Σ in the $\text{Lip}(\eta, \Sigma, W)$ -norm sense can be reduced to needing only to approximate ψ in a pointwise sense at the finite number of points in the subset Σ_N .

Remark 4.4. We illustrate the content of Remark 4.3 via an explicit example. For this purpose, let $\gamma > 0$ with $k \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$. Fix a choice of $\eta \in (0, \gamma)$ and let $q \in \mathbb{Z}_{\geq 0}$ such that $\eta \in (q, q + 1]$. Consider fixed $K_0, \varepsilon > 0$ and $d \in \mathbb{Z}_{\geq 1}$. Take $V := \mathbb{R}^d$ equipped with its usual Euclidean norm $\|\cdot\|_2$, take $\Sigma := [0, 1]^d \subset \mathbb{R}^d$ to be the unit cube in \mathbb{R}^d and take $W := \mathbb{R}$. Observe that the norm $\|\cdot\|_2$ is induced by the usual Euclidean dot product $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ on \mathbb{R}^d . Equip the tensor powers of \mathbb{R}^d with admissible norms in the sense of Definition 2.1 by extending the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ to the tensor powers, and subsequently taking the norm induced by the resulting inner product on the tensor powers (cf. Section 2). Introduce the notation, for $x \in \mathbb{R}^d$ and $r > 0$, that $\mathbb{B}^d(x, r) := \{y \in \mathbb{R}^d : \|x - y\|_2 < r\}$.

Retrieve the constants $\delta_0 = \delta_0(\varepsilon, K_0, \gamma, \eta) > 0$ and $\varepsilon_0 = \varepsilon_0(\varepsilon, K_0, \gamma, \eta) > 0$ arising in Corollary 4.1 for these choices of ε, γ and η , and for both the constants K_1 and K_2 there are K_0 here. Let $N = N([0, 1]^d, \varepsilon, K_0, \gamma, \eta) \in \mathbb{Z}_{\geq 0}$ denote the δ_0 -covering number of $[0, 1]^d$. That is,

$$N := \min \left\{ m \in \mathbb{Z} : \text{There exists } x_1, \dots, x_m \in [0, 1]^d \text{ such that } [0, 1]^d \subset \bigcup_{j=1}^m \mathbb{B}^d(x_j, \delta_0) \right\}. \tag{4.4}$$

We first claim that N defined in (4.4) satisfies that

$$N \leq \frac{2^d}{\omega_d} \left(1 + \frac{1}{\delta_0} \right)^d, \tag{4.5}$$

where ω_d denotes the Euclidean volume of the unit ball $\mathbb{B}^d(0, 1) \subset \mathbb{R}^d$.

To see this, observe that the δ_0 -covering number of $[0, 1]^d$ is bounded from above by the δ_0 -packing number of $[0, 1]^d$ defined by

$$N_{\text{pack}}(\delta_0, [0, 1]^d, \mathbb{R}^d) := \max \{ m \in \mathbb{Z} : \text{There exists } x_1, \dots, x_m \in [0, 1]^d \text{ such that } \|x_i - x_j\|_2 > \delta_0 \text{ whenever } i \neq j \}. \tag{4.6}$$

Suppose $x_1, \dots, x_{N_{\text{pack}}(\delta_0, [0, 1]^d, \mathbb{R}^d)} \in [0, 1]^d$ satisfy the condition specified in (4.6), that is, that whenever $i, j \in \{1, \dots, N_{\text{pack}}(\delta_0, [0, 1]^d, \mathbb{R}^d)\}$ with $i \neq j$ we have $\|x_i - x_j\|_2 > \delta_0$. A consequence of this is that the collection of balls $\{\mathbb{B}^d(x_i, \delta_0/2) : i \in \{1, \dots, N_{\text{pack}}(\delta_0, [0, 1]^d, \mathbb{R}^d)\}\}$ are pairwise disjoint. Moreover, the disjoint union of this collection of balls is a subset of the cube $[-\delta_0/2, 1 + \delta_0/2]^d$. Hence, a volume comparison argument yields that $N_{\text{pack}}(\delta_0, [0, 1]^d, \mathbb{R}^d)$ defined in (4.6) satisfies that

$$N_{\text{pack}}(\delta_0, [0, 1]^d, \mathbb{R}^d) \leq \frac{2^d}{\omega_d} \left(1 + \frac{1}{\delta_0} \right)^d. \tag{4.7}$$

The estimate claimed in (4.5) is now a consequence of $N \leq N_{\text{pack}}(\delta_0, [0, 1]^d, \mathbb{R}^d)$ and (4.7).

Define $m \in \mathbb{Z}_{\geq 1}$ by

$$m := \min \left\{ n \in \mathbb{Z} : n \geq \frac{2^d}{\omega_d} \left(1 + \frac{1}{\delta_0} \right)^d \right\}. \tag{4.8}$$

Choose distinct points $x_1, \dots, x_m \in [0, 1]^d$. Then, via (4.5) and (4.8), we see that $[0, 1]^d \subset \cup_{j=1}^m \overline{\mathbb{B}}^d(x_j, \delta_0)$. Let $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, [0, 1]^d, \mathbb{R})$ with $\|\psi\|_{\text{Lip}(\gamma, [0, 1]^d, \mathbb{R})} \leq K_0$. Then, Corollary 4.1 tells us that if $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, [0, 1]^d, \mathbb{R})$ satisfies both that $\|\varphi\|_{\text{Lip}(\gamma, [0, 1]^d, \mathbb{R})} \leq K_0$ and, for every $j \in \{1, \dots, m\}$ and every $l \in \{0, \dots, k\}$, that $\|\psi^{(l)}(x_j) - \varphi^{(l)}(x_j)\|_{\mathcal{L}((\mathbb{R}^d)^{\otimes l}; \mathbb{R})} \leq \varepsilon_0$, then $\|\psi_{[q]} - \varphi_{[q]}\|_{\text{Lip}(\eta, [0, 1]^d, \mathbb{R})} \leq \varepsilon$.

Therefore, in order to approximate ψ up to an error of ε in the $\text{Lip}(\eta, [0, 1]^d, \mathbb{R})$ -norm sense, we need only find $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, [0, 1]^d, \mathbb{R})$ satisfying both that $\|\varphi\|_{\text{Lip}(\gamma, [0, 1]^d, \mathbb{R})} \leq K_0$ and, for every $j \in \{1, \dots, m\}$ and every $l \in \{0, \dots, k\}$, that

$$\left\| \psi^{(l)}(x_j) - \varphi^{(l)}(x_j) \right\|_{\mathcal{L}((\mathbb{R}^d)^{\otimes l}; \mathbb{R})} \leq \varepsilon_0. \tag{4.9}$$

That is, up to an error of magnitude $\varepsilon > 0$, the $\text{Lip}(\eta)$ -behaviour of ψ throughout the entire cube $[0, 1]^d$ is captured by the pointwise values of ψ at the finite number of points $x_1, \dots, x_m \in [0, 1]^d$, and we have the explicit upper bound resulting from (4.8) for the number of points m that are required.

We now provide a short proof of Corollary 4.1 using the *Lipschitz Sandwich Theorem 3.1*.

Proof of Corollary 4.1. Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Sigma \subset V$ is compact. Let $\varepsilon > 0$, $\gamma > \eta > 0$ with $k, q \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$ and $\eta \in (q, q + 1]$, and $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$. Retrieve the constants $\delta_0 = \delta_0(\varepsilon, K_1 + K_2, \gamma, \eta) > 0$ and $\varepsilon_0 = \varepsilon_0(\varepsilon, K_1 + K_2, \gamma, \eta) > 0$ arising from Theorem 3.1 for these choices of $\varepsilon, K_1, K_2, \gamma$ and η . Note that we are not actually applying Theorem 3.1, but simply retrieving constants in preparation for its future application. Define $N = N(\Sigma, \varepsilon, K_1 + K_2, \gamma, \eta) \in \mathbb{Z}_{\geq 0}$ to be the δ_0 -covering number for Σ . That is,

$$N := \min \left\{ a \in \mathbb{Z} : \text{There exists } x_1, \dots, x_a \in \Sigma \text{ such that } \Sigma \subset \bigcup_{j=1}^a \overline{\mathbb{B}}_V(x_j, \delta_0) \right\}. \tag{4.10}$$

The compactness of Σ ensures that N defined in (4.10) is finite. Let $z_1, \dots, z_N \in \Sigma$ be any collection of N points in Σ for which

$$\Sigma \subset \bigcup_{j=1}^N \overline{\mathbb{B}}_V(z_j, \delta_0). \tag{4.11}$$

Set $\Sigma_N := \{z_1, \dots, z_N\}$.

Let $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$, $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfy the $\text{Lip}(\gamma, \Sigma, W)$ norm estimates that $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Suppose that for every $l \in \{0, \dots, k\}$ and every $j \in \{1, \dots, N\}$, the difference $\psi^{(l)}(z_j) - \varphi^{(l)}(z_j) \in \mathcal{L}(V^{\otimes l}; W)$ satisfies the bound

$$\left\| \psi^{(l)}(z_j) - \varphi^{(l)}(z_j) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \varepsilon_0. \tag{4.12}$$

Then, (4.11) and (4.12) enable us to appeal to Theorem 3.1, with $B := \Sigma_N$, to conclude $\|\psi_{[q]} - \varphi_{[q]}\|_{\text{Lip}(\eta, \Sigma, W)} \leq \varepsilon$ where $\psi_{[q]} := (\psi^{(0)}, \dots, \psi^{(q)})$ and $\varphi_{[q]} := (\varphi^{(0)}, \dots, \varphi^{(q)})$. This is precisely the estimate claimed in (4.3). This completes the proof of Corollary 4.1. \square

If we weaken the sense in which we aim to approximate ψ to the pointwise notion considered in the *Pointwise Lipschitz Sandwich Theorem 3.11*, then we are able to establish the following consequence of the *Pointwise Lipschitz Sandwich Theorem 3.11* when the subset $\Sigma \subset V$ is compact.

Corollary 4.5 (Consequence of the *Pointwise Lipschitz Sandwich Theorem 3.11*). *Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Sigma \subset V$ is compact. Let $\varepsilon, \gamma > 0$, with $k \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$, $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$, and $0 \leq \varepsilon_0 < \min \{K_1 + K_2, \varepsilon\}$. Given $l \in \{0, \dots, k\}$, let $\delta_0 = \delta_0(\varepsilon, \varepsilon_0, K_1 + K_2, \gamma, l) > 0$ denote the constant arising from Theorem 3.11 (cf. (3.6)). Let $N = N(\Sigma, \varepsilon, \varepsilon_0, K_1 + K_2, \gamma, l) \in \mathbb{Z}_{\geq 0}$ denote the δ_0 -covering number of Σ . That is,*

$$N := \min \left\{ d \in \mathbb{Z} : \text{There exists } x_1, \dots, x_d \in \Sigma \text{ such that } \Sigma \subset \bigcup_{j=1}^d \overline{\mathbb{B}}_V(x_j, \delta_0) \right\}. \tag{4.13}$$

Then there is a finite subset $\Sigma_N = \{z_1, \dots, z_N\} \subset \Sigma$ for which the following is true.

Let $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$, $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfy the $\text{Lip}(\gamma, \Sigma, W)$ norm estimates $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Suppose that for every $i \in \{0, \dots, k\}$ and every $j \in \{1, \dots, N\}$, the difference $\psi^{(i)}(z_j) - \varphi^{(i)}(z_j) \in \mathcal{L}(V^{\otimes i}; W)$ satisfies the bound

$$\left\| \psi^{(i)}(z_j) - \varphi^{(i)}(z_j) \right\|_{\mathcal{L}(V^{\otimes i}; W)} \leq \varepsilon_0. \tag{4.14}$$

Then, we may conclude that for every $s \in \{0, \dots, l\}$ and every $x \in \Sigma$ that

$$\left\| \psi^{(s)}(x) - \varphi^{(s)}(x) \right\|_{\mathcal{L}(V^{\otimes s}; W)} \leq \varepsilon. \tag{4.15}$$

Remark 4.6. In a similar spirit to Remark 3.15, and using the same notation as in Corollary 4.5, by taking $\varphi \equiv 0$, we may conclude from Corollary 4.5 that if $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfies both that $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and, for every $j \in \{0, \dots, k\}$ and every $x \in \Sigma_N$, that $\left\| \psi^{(j)}(x) \right\|_{\mathcal{L}(V^{\otimes j}; W)} \leq \varepsilon_0$, then we have, for every $s \in \{0, \dots, l\}$ and every point $x \in \Sigma$, that $\left\| \psi^{(s)}(x) \right\|_{\mathcal{L}(V^{\otimes s}; W)} \leq \varepsilon$.

Remark 4.7. When $N \in \mathbb{Z}_{\geq 1}$ defined in (4.13) is less than the cardinality of Σ , Corollary 4.5 guarantees that we are able to identify a strictly smaller collection of points Σ_N such that the behaviour of a $\text{Lip}(\gamma, \Sigma, W)$ function $F = (F^{(0)}, \dots, F^{(k)})$ on Σ_N determines the pointwise behaviour of $F_{[l]} = (F^{(0)}, \dots, F^{(l)})$ over the entire set Σ up to an arbitrarily small error. That is, using the notation of Corollary 4.5, if $F \in \text{Lip}(\gamma, \Sigma_N, W)$ and $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$ and $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)})$ are both extensions of F to $\text{Lip}(\gamma, \Sigma, W)$, with $\text{Lip}(\gamma, \Sigma, W)$ -norms bounded above by K_1 and K_2 , respectively, then for every $s \in \{0, \dots, l\}$, the functions $\psi^{(s)}$ and $\varphi^{(s)}$ may only differ, in the pointwise sense, by at most ε throughout Σ .

Similarly to Remark 4.3, a particular consequence is that a function in $\text{Lip}(\gamma, \Sigma, W)$ can be cost-effectively approximated in a pointwise sense. That is, let $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ with $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and suppose, for some $l \in \{0, \dots, k\}$, that we want to approximate the functions $\psi^{(0)}, \dots, \psi^{(l)}$ throughout Σ in a pointwise sense. Then, Corollary 4.5 guarantees that any $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ with $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$ will satisfy, for every $x \in \Sigma$ and

every $s \in \{0, \dots, l\}$, that $\left\| \psi^{(s)}(x) - \varphi^{(s)}(x) \right\|_{\mathcal{L}(V^{\otimes s}; W)} \leq \varepsilon$ provided we have, for every point $x \in \Sigma_N$ and every $j \in \{0, \dots, k\}$, that $\left\| \psi^{(j)}(x) - \varphi^{(j)}(x) \right\|_{\mathcal{L}(V^{\otimes j}; W)} \leq \varepsilon_0$. Thus, the task of approximating the functions $\psi^{(0)}, \dots, \psi^{(l)}$ throughout Σ in a pointwise sense can be reduced to needing only to approximate ψ in a pointwise sense at the finite number of points in the subset Σ_N .

Remark 4.8. We illustrate the content of Remark 4.7 via an explicit example. The explicit example is in the same setting considered in Remark 4.4. Let $d \in \mathbb{Z}_{\geq 1}$, take $V := \mathbb{R}^d$ equipped with its usual Euclidean norm $\| \cdot \|_2$, take $\Sigma := [0, 1]^d \subset \mathbb{R}^d$ to be the unit cube in \mathbb{R}^d and take $W := \mathbb{R}$. Observe that the norm $\| \cdot \|_2$ is induced by the usual Euclidean dot product $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ on \mathbb{R}^d . Equip the tensor powers of \mathbb{R}^d with admissible norms in the sense of Definition 2.1 by extending the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ to the tensor powers, and subsequently taking the norm induced by the resulting inner product on the tensor powers (cf. Section 2). As introduced in Remark 4.4, we use the notation, for $x \in \mathbb{R}^d$ and $r > 0$, that $\mathbb{B}^d(x, r) := \{y \in \mathbb{R}^d : \|x - y\|_2 < r\}$.

Let $\gamma > 0$ with $k \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$. Consider fixed $K_0, \varepsilon > 0$, $l \in \{0, \dots, k\}$ and $0 \leq \varepsilon_0 < \min\{2K_0, \varepsilon\}$. Retrieve the constant $\delta_0 = \delta_0(\varepsilon, \varepsilon_0, K_0, \gamma, l) > 0$ arising in Corollary 4.5 for these choices of ε , ε_0 , γ , and l , and for both the constants K_1 and K_2 there as K_0 here. Let $N = N([0, 1]^d, \varepsilon, \varepsilon_0, K_0, \gamma, l) \in \mathbb{Z}_{\geq 0}$ denote the δ_0 -covering number of $[0, 1]^d$. That is,

$$N := \min \left\{ m \in \mathbb{Z} : \text{There exists } x_1, \dots, x_m \in [0, 1]^d \text{ such that } [0, 1]^d \subset \bigcup_{j=1}^m \overline{\mathbb{B}^d}(x_j, \delta_0) \right\}. \tag{4.16}$$

Following the method used in Remark 4.4 to obtain (4.5) verbatim enables us to conclude that

$$N \leq \frac{2^d}{\omega_d} \left(1 + \frac{1}{\delta_0} \right)^d, \tag{4.17}$$

where ω_d denotes the Euclidean volume of the unit ball $\mathbb{B}^d(0, 1) \subset \mathbb{R}^d$. Define $m \in \mathbb{Z}_{\geq 1}$ by

$$m := \min \left\{ n \in \mathbb{Z} : n \geq \frac{2^d}{\omega_d} \left(1 + \frac{1}{\delta_0} \right)^d \right\}. \tag{4.18}$$

Choose distinct points $x_1, \dots, x_m \in [0, 1]^d$. Then, via (4.17) and (4.18), we see that $[0, 1]^d \subset \cup_{j=1}^m \overline{\mathbb{B}^d}(x_j, \delta_0)$. Let $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, [0, 1]^d, \mathbb{R})$ with $\|\psi\|_{\text{Lip}(\gamma, [0, 1]^d, \mathbb{R})} \leq K_0$. Then, Corollary 4.5 tells us that if $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, [0, 1]^d, \mathbb{R})$ satisfies both that $\|\varphi\|_{\text{Lip}(\gamma, [0, 1]^d, \mathbb{R})} \leq K_0$ and, for every $i \in \{1, \dots, m\}$ and every $j \in \{0, \dots, k\}$, that $\left\| \psi^{(j)}(x_i) - \varphi^{(j)}(x_i) \right\|_{\mathcal{L}((\mathbb{R}^d)^{\otimes j}; \mathbb{R})} \leq \varepsilon_0$, then, for every $x \in [0, 1]^d$ and every $s \in \{0, \dots, l\}$, we have $\left\| \psi^{(s)}(x) - \varphi^{(s)}(x) \right\|_{\mathcal{L}((\mathbb{R}^d)^{\otimes s}; \mathbb{R})} \leq \varepsilon$.

Therefore, in order to approximate $\psi^{(0)}, \dots, \psi^{(l)}$ up to an error of ε in a pointwise sense, we need only find $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, [0, 1]^d, \mathbb{R})$ satisfying both that $\|\varphi\|_{\text{Lip}(\gamma, [0, 1]^d, \mathbb{R})} \leq K_0$ and, for every $i \in \{1, \dots, m\}$ and every $j \in \{0, \dots, k\}$, that

$$\left\| \psi^{(j)}(x_i) - \varphi^{(j)}(x_i) \right\|_{\mathcal{L}((\mathbb{R}^d)^{\otimes j}; \mathbb{R})} \leq \varepsilon_0. \tag{4.19}$$

That is, up to an error of magnitude $\varepsilon > 0$, the pointwise behaviour of the functions $\psi^{(0)}, \dots, \psi^{(k)}$ throughout the entire cube $[0, 1]^d$ is captured by their pointwise values at the finite number of points $x_1, \dots, x_m \in [0, 1]^d$, and we have the explicit upper bound resulting from (4.18) for the number of points m that are required.

We end this section with a short proof of Corollary 4.5.

Proof of Corollary 4.5. Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Sigma \subset V$ is compact. Let $\varepsilon, \gamma > 0$, with $k \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1], (K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$, and $0 \leq \varepsilon_0 < \min\{K_1 + K_2, \varepsilon\}$. Given $l \in \{0, \dots, k\}$ let $\delta_0 = \delta_0(\varepsilon, \varepsilon_0, K_1 + K_2, \gamma, l) > 0$ denote the constant arising from Theorem 3.11 for these choices of $\varepsilon, \varepsilon_0, K_1, K_2, \gamma$ and l . Note that we are not actually applying Theorem 3.11, but simply retrieving a constant in preparation for its future application. Define $N = N(\Sigma, \varepsilon, \varepsilon_0, K_1 + K_2, \gamma, l) \in \mathbb{Z}_{\geq 0}$ to be the δ_0 -covering number of Σ . That is,

$$N := N_{\text{cov}}(\Sigma, V, \delta_0) = \min \left\{ a \in \mathbb{Z} : \text{There exists } x_1, \dots, x_a \in \Sigma \text{ such that } \Sigma \subset \bigcup_{j=1}^a \overline{\mathbb{B}}_V(x_j, \delta_0) \right\}. \tag{4.20}$$

The compactness of Σ ensures that the integer N defined in (4.20) is finite. Let $z_1, \dots, z_N \in \Sigma$ be any collection of N points in Σ for which

$$\Sigma \subset \bigcup_{j=1}^N \overline{\mathbb{B}}_V(z_j, \delta_0). \tag{4.21}$$

Set $\Sigma_N := \{z_1, \dots, z_N\}$.

Now suppose that both $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$, $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ have their norms bounded by K_1 and K_2 , respectively, that is, $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Further suppose that for every $i \in \{0, \dots, k\}$ and every $j \in \{1, \dots, N\}$, the difference $\psi^{(i)}(z_j) - \varphi^{(i)}(z_j) \in \mathcal{L}(V^{\otimes i}; W)$ satisfies the bound

$$\left\| \psi^{(i)}(z_j) - \varphi^{(i)}(z_j) \right\|_{\mathcal{L}(V^{\otimes i}; W)} \leq \varepsilon_0. \tag{4.22}$$

Together, (4.21) and (4.22) provide the hypotheses required to allow us to appeal to Theorem 3.11 with the subset B of that result as the subset Σ_N here. A consequence of doing so is that, for every $s \in \{0, \dots, l\}$ and $x \in \Sigma$, we have that $\left\| \psi^{(s)}(x) - \varphi^{(s)}(x) \right\|_{\mathcal{L}(V^{\otimes s}; W)} \leq \varepsilon$ as claimed in (4.15). This completes the proof of Corollary 4.5. \square

5 | REMAINDER TERM ESTIMATES

In this section, we establish the following remainder term estimates for a $\text{Lip}(\gamma)$ function which will be particularly useful in subsequent sections.

Lemma 5.1 (Remainder term estimates). *Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that*

$\Gamma \subset V$ is closed. Let $\rho > 0$ with $n \in \mathbb{Z}_{\geq 0}$ such that $\rho \in (n, n + 1]$, let $\theta \in (0, \rho)$ and suppose $\psi = (\psi^{(0)}, \dots, \psi^{(n)}) \in \text{Lip}(\rho, \Gamma, W)$. For $l \in \{0, \dots, n\}$, let $R_l^\psi : \Gamma \times \Gamma \rightarrow \mathcal{L}(V^{\otimes l}; W)$ denote the remainder term associated to $\psi^{(l)}$ (cf. (2.9) in Definition 2.2). If $\theta \in (n, \rho)$, then for every $l \in \{0, \dots, n\}$, we have that for every $x, y \in \Gamma$ with $x \neq y$ that

$$\frac{\|R_l^\psi(x, y)\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\theta - l}} \leq \min \left\{ \text{diam}(\Gamma)^{\rho - \theta}, G(\rho, \theta, l, \Gamma) \right\} \|\psi\|_{\text{Lip}(\rho, \Gamma, W)}, \tag{5.1}$$

where $G(\rho, \theta, l, \Gamma)$ is defined by

$$G(\rho, \theta, l, \Gamma) := \inf_{r \in (0, \text{diam}(\Gamma))} \left\{ \max \left\{ r^{\rho - \theta}, \frac{1}{r^{\theta - l}} \left(1 + \sum_{s=0}^{n-l} \frac{r^s}{s!} \right) \right\} \right\}. \tag{5.2}$$

If $\theta \in (0, n]$ (which is only possible if $n \geq 1$), then let $q \in \{0, \dots, n - 1\}$ be such that $\theta \in (q, q + 1]$. For each $l \in \{0, \dots, q\}$, let $\tilde{R}_l^\psi : \Gamma \times \Gamma \rightarrow \mathcal{L}(V^{\otimes l}; W)$ denote the alteration of the remainder term R_l^ψ defined for $x, y \in \Gamma$ and $v \in V^{\otimes l}$ by

$$\tilde{R}_l^\psi(x, y)[v] := R_l^\psi(x, y)[v] + \sum_{s=q-l+1}^{n-l} \frac{1}{s!} \psi^{l+s}(x)[v \otimes (y - x)^{\otimes s}]. \tag{5.3}$$

Then, for every $l \in \{0, \dots, q\}$ and every $x, y \in \Gamma$ with $x \neq y$, we have that

$$\frac{\|\tilde{R}_l^\psi(x, y)\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\theta - l}} \leq \min \left\{ \text{diam}(\Gamma)^{\rho - \theta} + \sum_{i=q+1}^n \frac{\text{diam}(\Gamma)^{i - \theta}}{(i - l)!}, H(\rho, \theta, l, \Gamma) \right\} \|\psi\|_{\text{Lip}(\rho, \Gamma, W)}, \tag{5.4}$$

where $H(\rho, \theta, l, \Gamma)$ is defined by

$$H(\rho, \theta, l, \Gamma) := \inf_{r \in (0, \text{diam}(\Gamma))} \left\{ \max \left\{ r^{\rho - \theta} + \sum_{i=q+1}^n \frac{r^{i - \theta}}{(i - l)!}, \frac{1}{r^{\theta - l}} \left(1 + \sum_{s=0}^{q-l} \frac{r^s}{s!} \right) \right\} \right\}. \tag{5.5}$$

Proof of Lemma 5.1. Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Gamma \subset V$ is closed and that $\rho > 0$ with $n \in \mathbb{Z}_{\geq 0}$ such that $\rho \in (n, n + 1]$. Suppose that, for $l \in \{0, \dots, n\}$, we have functions $\psi^{(l)} : \Gamma \rightarrow \mathcal{L}(V^{\otimes l}; W)$ such that $\psi = (\psi^{(0)}, \dots, \psi^{(n)})$ defines an element of $\text{Lip}(\rho, \Gamma, W)$. For each $l \in \{0, \dots, n\}$, define $R_l^\psi : \Gamma \times \Gamma \rightarrow \mathcal{L}(V^{\otimes l}; W)$ for $x, y \in \Gamma$ and $v \in V^{\otimes l}$ by

$$R_l^\psi(x, y)[v] := \psi^{(l)}(y)[v] - \sum_{s=0}^{n-l} \frac{1}{s!} \psi^{(l+s)}(x)[v \otimes (y - x)^{\otimes s}]. \tag{5.6}$$

We claim that the estimates (5.1) and (5.4) are immediate when $\|\psi\|_{\text{Lip}(\rho, \Gamma, W)} = 0$. To see this, note that if $\|\psi\|_{\text{Lip}(\rho, \Gamma, W)} = 0$, then for each $l \in \{0, \dots, n\}$ and any $x \in \Gamma$, we have that $\psi^{(l)}(x) \equiv 0$ in $\mathcal{L}(V^{\otimes l}; W)$. Consequently, for each $l \in \{0, \dots, n\}$ and any $x, y \in \Gamma$, we have via (5.6) that $R_l^\psi(x, y) \equiv 0$ in $\mathcal{L}(V^{\otimes l}; W)$. When $\theta \in (n, \rho)$, this tells us that the estimate (5.1) is true since both sides are zero.

When $\theta \in (0, n]$ (which is only possible if $n \geq 1$), then, if $q \in \{0, \dots, n - 1\}$ is such that $\theta \in (q, q + 1]$, for each $l \in \{0, \dots, q\}$ and any $x, y \in \Gamma$, we have via (5.3) that the alteration \tilde{R}_l^ψ of R_l^ψ satisfies that $\tilde{R}_l^\psi(x, y) \equiv 0$ in $\mathcal{L}(V^{\otimes l}; W)$. Hence, the estimate (5.4) is true since both sides are again zero.

If $\|\psi\|_{\text{Lip}(\rho, \Gamma, W)} \neq 0$, then by replacing ψ by $\psi/\|\psi\|_{\text{Lip}(\rho, \Gamma, W)}$, it suffices to prove the estimates (5.1) and (5.4) under the additional assumption that $\|\psi\|_{\text{Lip}(\rho, \Gamma, W)} = 1$. As a consequence, whenever $l \in \{0, \dots, n\}$ and $x, y \in \Gamma$, we have the bounds (cf. (2.8) and cf. (2.10))

$$(I) \quad \left\| \psi^{(l)}(x) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq 1 \quad \text{and} \quad (II) \quad \left\| R_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \|y - x\|_V^{\rho-l}. \tag{5.7}$$

Firstly suppose $\theta \in (n, \rho)$ and let $l \in \{0, \dots, n\}$. For any $x, y \in \Gamma$, we have that

$$\left\| R_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \stackrel{(II) \text{ of } (5.7)}{\leq} \|y - x\|_V^{\rho-l} = \|y - x\|_V^{\rho-\theta} \|y - x\|_V^{\theta-l}. \tag{5.8}$$

A first consequence of (5.8) is

$$\left\| R_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \text{diam}(\Gamma)^{\rho-\theta} \|y - x\|_V^{\theta-l}. \tag{5.9}$$

A second consequence of (5.8) is that, for any fixed $r \in (0, \text{diam}(\Gamma))$, if $\|y - x\|_V \leq r$, then

$$\left\| R_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq r^{\rho-\theta} \|y - x\|_V^{\theta-l}. \tag{5.10}$$

If $\|y - x\|_V > r$, then we use (5.6) and that the tensor powers of V are equipped with admissible norms (cf. Definition 2.1) to compute for any $v \in V^{\otimes l}$ that

$$\begin{aligned} \left\| R_l^\psi(x, y)[v] \right\|_W &\stackrel{(5.6)}{\leq} \left\| \psi^{(l)}(y)[v] \right\|_W + \sum_{j=0}^{n-l} \frac{1}{j!} \left\| \psi^{(l+j)}(x)[v \otimes (y - x)^{\otimes j}] \right\|_W \\ &\stackrel{(I) \text{ of } (5.7)}{\leq} \|v\|_{V^{\otimes l}} + \sum_{j=0}^{n-l} \frac{1}{j!} \|y - x\|_V^j \|v\|_{V^{\otimes l}} \\ &\leq r^{-(\theta-l)} \left(1 + \sum_{j=0}^{n-l} \frac{r^j}{j!} \right) \|y - x\|_V^{\theta-l} \|v\|_{V^{\otimes l}}. \end{aligned}$$

In the last line, we have used that, for any $j \in \{0, \dots, n - l\}$, that $\|y - x\|_V^{j-(\theta-l)} < r^{j-(\theta-l)}$. This is itself a consequence of the facts that for any $j \in \{0, \dots, n - l\}$ that $j - (\theta - l) \leq n - \theta < 0$, and that $r < \|y - x\|_V$. By taking the supremum over $v \in V^{\otimes l}$ with unit $V^{\otimes l}$ norm, we may conclude that when $\|y - x\|_V > r$, we have

$$\left\| R_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq r^{-(\theta-l)} \left(1 + \sum_{s=0}^{n-l} \frac{r^s}{s!} \right) \|y - x\|_V^{\theta-l}. \tag{5.11}$$

By combining (5.10) and (5.11), we deduce that for every $x, y \in \Gamma$, we have

$$\left\| R_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \max \left\{ r^{\rho-\theta}, \frac{1}{r^{\theta-l}} \left(1 + \sum_{s=0}^{n-l} \frac{r^s}{s!} \right) \right\} \|y - x\|_V^{\theta-l}. \tag{5.12}$$

Recall that the choice of $r \in (0, \text{diam}(\Gamma))$ was arbitrary. Consequently, we may take the infimum over the choice of $r \in (0, \text{diam}(\Gamma))$ in (5.12) to obtain that whenever $x \neq y$, we have

$$\frac{\left\| R_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\theta-l}} \leq \inf_{r \in (0, \text{diam}(\Gamma))} \left\{ \max \left\{ r^{\rho-\theta}, \frac{1}{r^{\theta-l}} \left(1 + \sum_{s=0}^{n-l} \frac{r^s}{s!} \right) \right\} \right\}. \tag{5.13}$$

If we define

$$G(\rho, \theta, l, \Gamma) := \inf_{r \in (0, \text{diam}(\Gamma))} \left\{ \max \left\{ r^{\rho-\theta}, \frac{1}{r^{\theta-l}} \left(1 + \sum_{s=0}^{n-l} \frac{r^s}{s!} \right) \right\} \right\}, \tag{5.14}$$

then (5.9) and (5.13) yield that

$$\frac{\left\| R_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\theta-l}} \leq \min \left\{ \text{diam}(\Gamma)^{\rho-\theta}, G(\rho, \theta, l, \Gamma) \right\}. \tag{5.15}$$

The arbitrariness of $l \in \{0, \dots, n\}$ and the points $x, y \in \Gamma$ with $x \neq y$ mean that (5.15) establishes the estimates claimed in (5.1) for the case that $\|\psi\|_{\text{Lip}(\rho, \Gamma, W)} = 1$.

Now assume that $0 < \theta \leq n < \rho \leq n + 1$ which requires $n \geq 1$. Let $q \in \{0, \dots, n - 1\}$ be such that $\theta \in (q, q + 1]$. For each $l \in \{0, \dots, q\}$, let $\tilde{R}_l^\psi : \Gamma \times \Gamma \rightarrow \mathcal{L}(V^{\otimes l}; W)$ denote the alteration of the remainder term R_l^ψ defined for $x, y \in \Gamma$ and $v \in V^{\otimes l}$ by

$$\tilde{R}_l^\psi(x, y)[v] := R_l^\psi(x, y)[v] + \sum_{s=q-l+1}^{n-l} \frac{1}{s!} \psi^{(l+s)}(x)[v \otimes (y - x)^{\otimes s}]. \tag{5.16}$$

Let $l \in \{0, \dots, q\}$, $x, y \in \Gamma$ and $v \in V^{\otimes l}$. Recalling that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1), we may compute that

$$\begin{aligned} \left\| \tilde{R}_l^\psi(x, y)[v] \right\|_W &\stackrel{(5.16)}{\leq} \left\| R_l^\psi(x, y)[v] \right\|_W + \sum_{s=q-l+1}^{n-l} \frac{1}{s!} \left\| \psi^{(l+s)}(x)[v \otimes (y - x)^{\otimes s}] \right\|_W \\ &\stackrel{(5.7)}{\leq} \left(\|y - x\|_V^{\rho-l} + \sum_{s=q-l+1}^{n-l} \frac{1}{s!} \|y - x\|_V^s \right) \|v\|_{V^{\otimes l}} \\ &= \left(\|y - x\|_V^{\rho-\theta} + \sum_{i=q+1}^n \frac{1}{(i-l)!} \|y - x\|_V^{i-\theta} \right) \|y - x\|_V^{\theta-l} \|v\|_{V^{\otimes l}}. \end{aligned}$$

By taking the supremum over $v \in V^{\otimes l}$ with unit $V^{\otimes l}$ norm, we may conclude that

$$\left\| \tilde{R}_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \left(\|y - x\|_V^{\rho-\theta} + \sum_{i=q+1}^n \frac{\|y - x\|_V^{i-\theta}}{(i-l)!} \right) \|y - x\|_V^{\theta-l}. \tag{5.17}$$

A first consequence of (5.17) is that

$$\left\| \tilde{R}_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \left(\text{diam}(\Gamma)^{\rho-\theta} + \sum_{i=q+1}^n \frac{\text{diam}(\Gamma)^{i-\theta}}{(i-l)!} \right) \|y - x\|_V^{\theta-l}. \tag{5.18}$$

Now consider a fixed choice of constant $r \in (0, \text{diam}(\Gamma))$. If $\|y - x\|_V \leq r$, then a consequence of (5.17) is that

$$\left\| \tilde{R}_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \left(r^{\rho-\theta} + \sum_{i=q+1}^n \frac{r^{i-\theta}}{(i-l)!} \right) \|y - x\|_V^{\theta-l}. \tag{5.19}$$

If $\|y - x\|_V > r$, then we may first observe via (5.6) and (5.16) that for any $v \in V^{\otimes l}$, we have

$$\tilde{R}_l^\psi(x, y)[v] = \psi^{(l)}(y)[v] - \sum_{s=0}^{q-l} \frac{1}{s!} \psi^{(l+s)}(x)[v \otimes (y - x)^{\otimes s}]. \tag{5.20}$$

We may use (5.20) and that the tensor powers of V are equipped with admissible norms (cf. Definition 2.1) to compute for any $v \in V^{\otimes l}$ that

$$\begin{aligned} \left\| \tilde{R}_l^\psi(x, y)[v] \right\|_W &\stackrel{(5.20)}{\leq} \left\| \psi^{(l)}(y)[v] \right\|_W + \sum_{j=0}^{q-l} \frac{1}{j!} \left\| \psi^{(l+j)}(x)[v \otimes (y - x)^{\otimes j}] \right\|_W \\ &\stackrel{(5.7)}{\leq} \|v\|_{V^{\otimes l}} + \sum_{j=0}^{q-l} \frac{1}{j!} \|y - x\|_V^j \|v\|_{V^{\otimes l}} \leq r^{-(\theta-l)} \left(1 + \sum_{j=0}^{q-l} \frac{r^j}{j!} \right) \|y - x\|_V^{\theta-l} \|v\|_{V^{\otimes l}}. \end{aligned}$$

For the last inequality, we have used that, for any $j \in \{0, \dots, q-l\}$, that $\|y - x\|_V^{j-(\theta-l)} < r^{j-(\theta-l)}$. This is itself a consequence of the facts that for any $j \in \{0, \dots, q-l\}$ that $j - (\theta - l) \leq q - \theta < 0$, and that $r < \|y - x\|_V$. By taking the supremum over $v \in V^{\otimes l}$ with unit $V^{\otimes l}$ norm, we may conclude that when $\|y - x\|_V > r$, we have

$$\left\| \tilde{R}_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq r^{-(\theta-l)} \left(1 + \sum_{s=0}^{q-l} \frac{r^s}{s!} \right) \|y - x\|_V^{\theta-l}. \tag{5.21}$$

Together (5.19) and (5.21) give that for every $x, y \in \Gamma$ with $x \neq y$, we have

$$\frac{\left\| \tilde{R}_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\theta-l}} \leq \max \left\{ r^{\rho-\theta} + \sum_{i=q+1}^n \frac{r^{i-\theta}}{(i-l)!}, \frac{1}{r^{\theta-l}} \left(1 + \sum_{s=0}^{q-l} \frac{r^s}{s!} \right) \right\}. \tag{5.22}$$

Recall that the choice of $r \in (0, \text{diam}(\Gamma))$ was arbitrary. Consequently, we may take the infimum over the choice of $r \in (0, \text{diam}(\Gamma))$ in (5.22) to obtain that whenever $x \neq y$, we have

$$\frac{\left\| \tilde{R}_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\theta - l}} \leq H(\rho, \theta, l, \Gamma) \tag{5.23}$$

for $H(\rho, \theta, l, \Gamma)$ defined by

$$H(\rho, \theta, l, \Gamma) := \inf_{r \in (0, \text{diam}(\Gamma))} \left\{ \max \left\{ r^{\rho - \theta} + \sum_{i=q+1}^n \frac{r^{i - \theta}}{(i - l)!}, \frac{1}{r^{\theta - l}} \left(1 + \sum_{s=0}^{q-l} \frac{r^s}{s!} \right) \right\} \right\}. \tag{5.24}$$

Together (5.18) and (5.23) yield

$$\frac{\left\| \tilde{R}_l^\psi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\theta - l}} \leq \min \left\{ \text{diam}(\Gamma)^{\rho - \theta} + \sum_{i=q+1}^n \frac{\text{diam}(\Gamma)^{i - \theta}}{(i - l)!}, H(\rho, \theta, l, \Gamma) \right\}. \tag{5.25}$$

The arbitrariness of $l \in \{0, \dots, q\}$ and the points $x, y \in \Gamma$ with $x \neq y$ mean that (5.25) establishes the estimates claimed in (5.4) for the case that $\|\psi\|_{\text{Lip}(\rho, \Gamma, W)} = 1$. This completes the proof of Lemma 5.1. \square

6 | NESTED EMBEDDING PROPERTY

In this section, we establish that Lipschitz spaces are nested in the following sense. Let V and W be Banach spaces and assume that the tensor powers of V are equipped with admissible tensor norms (cf. Definition 2.1). Let $\rho \geq \theta > 0$ and $\Gamma \subset V$ be a closed subset. Then $\text{Lip}(\rho, \Gamma, W) \subset \text{Lip}(\theta, \Gamma, W)$. This nesting property is established by Stein in his original work [28], whilst Theorem 1.18 in [1] provides a formulation in our particular framework. To elaborate, if we let $\psi \in \text{Lip}(\rho, \Gamma, W)$, $q \in \mathbb{Z}_{\geq 0}$ such that $\theta \in (q, q + 1]$, and $\psi_{[q]} = (\psi^{(0)}, \dots, \psi^{(q)})$, then $\psi_{[q]} \in \text{Lip}(\theta, \Gamma, W)$. But it is *not* necessarily true that $\|\psi_{[q]}\|_{\text{Lip}(\theta, \Gamma, W)} \leq \|\psi\|_{\text{Lip}(\rho, \Gamma, W)}$.

For example, consider $\Gamma := [-1, 1] \subset \mathbb{R}$ and define functions $\psi^{(0)} : \Gamma \rightarrow \mathbb{R}$ and $\psi^{(1)} : \Gamma \rightarrow \mathcal{L}(\mathbb{R}; \mathbb{R})$ by $\psi^{(0)}(x) := x^2$ and $\psi^{(1)}(x)[v] := 2xv$, respectively. Let $\psi = (\psi^{(0)}, \psi^{(1)})$. Then, the associated remainder terms are $R_0^\psi(x, y) := \psi^{(0)}(y) - \psi^{(0)}(x) - \psi^{(1)}(x)[y - x] = (y - x)^2$ and $R_1^\psi(x, y)[v] := \psi^{(1)}(y)[v] - \psi^{(1)}(x)[v] = 2(y - x)v$. It follows that $\psi \in \text{Lip}(2, \Gamma, \mathbb{R})$ with $\|\psi\|_{\text{Lip}(2, \Gamma, \mathbb{R})} = 2$. However, $R_1^\psi(-1, 1)[v] = 4v = 2\sqrt{2}|1 - (-1)|^{\frac{1}{2}}v$, and so, $\|\psi\|_{\text{Lip}(3/2, \Gamma, \mathbb{R})} = 2\sqrt{2} = \sqrt{2}\|\psi\|_{\text{Lip}(2, \Gamma, \mathbb{R})}$.

In the following *Lipschitz Nesting Lemma 6.1*, we provide an explicit constant $C \geq 1$ for which the estimate $\|\cdot\|_{\text{Lip}(\theta, \Gamma, W)} \leq C\|\cdot\|_{\text{Lip}(\rho, \Gamma, W)}$ holds. The constant C is more finely attuned to the geometry of the domain Γ than the corresponding constant in Theorem 1.18 in [1].

Lemma 6.1 (Lipschitz nesting). *Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Gamma \subset V$ is closed. Let $\rho > 0$ with $n \in \mathbb{Z}_{\geq 0}$ such that $\rho \in (n, n + 1]$, and $\theta \in (0, \rho)$ with $q \in \{0, \dots, n\}$ such that $\theta \in (q, q + 1]$. Suppose that $\psi = (\psi^{(0)}, \dots, \psi^{(n)}) \in \text{Lip}(\rho, \Gamma, W)$. Then, $\psi_{[q]} = (\psi^{(0)}, \dots, \psi^{(q)}) \in$*

$\text{Lip}(\theta, \Gamma, W)$. Further, if $\theta \in (n, \rho)$, then we have the estimate that

$$\|\psi_{[q]}\|_{\text{Lip}(\theta, \Gamma, W)} \leq \max \{1, \min \{1 + e, \text{diam}(\Gamma)^{\rho-}\}\} \|\psi\|_{\text{Lip}(\rho, \Gamma, W)}. \tag{6.1}$$

If $\theta \in (0, n]$, then we have the estimate that

$$\|\psi_{[q]}\|_{\text{Lip}(\theta, \Gamma, W)} \leq \min \{C_1, C_2\} \|\psi\|_{\text{Lip}(\rho, \Gamma, W)} \tag{6.2}$$

where $C_1, C_2 > 0$ are constants, depending only on $\text{diam}(\Gamma)$, ρ and θ , defined by

$$C_1 := \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(\Gamma)^{\rho-} + \sum_{j=q+1}^n \frac{\text{diam}(\Gamma)^{j-}}{(j-q)!} \right\} \right\} \tag{6.3}$$

and

$$C_2 = \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(\Gamma)^{q+1-} \right\} \right\} (1 + \min \{e, \text{diam}(\Gamma)^{\rho-}\}) (1 + \min \{e, \text{diam}(\Gamma)\})^{n-(q+1)}. \tag{6.4}$$

Finally, as a consequence of (6.1) and (6.2), for any $\theta \in (0, \rho)$, we have the estimate

$$\|\psi_{[q]}\|_{\text{Lip}(\theta, \Gamma, W)} \leq (1 + e) \|\psi\|_{\text{Lip}(\rho, \Gamma, W)}. \tag{6.5}$$

Remark 6.2. The integers $n, q \in \mathbb{Z}_{\geq 0}$ are determined by ρ and θ , respectively. Consequently, any apparent dependence on n and q in (6.3) and (6.4) is really dependence on ρ and θ , respectively.

Remark 6.3. We can have equality in (6.1). To see this, let $\Gamma := [-1, 1] \subset \mathbb{R}$ and define $\psi^{(0)} : \Gamma \rightarrow \mathbb{R}$ by $\psi^{(0)}(x) := x^2$, $\psi^{(1)} : \Gamma \rightarrow \mathcal{L}(\mathbb{R}; \mathbb{R})$ by $\psi^{(1)}(x)[v] := 2xv$, $R_0(x, y) := \psi^{(0)}(y) - \psi^{(0)}(x) - \psi^{(1)}[y - x] = (y - x)^2$ and $R_1(x, y)[v] := \psi^{(1)}(y)[v] - \psi^{(1)}(x)[v] = 2(y - x)v$. Then, $\psi = (\psi^{(0)}, \psi^{(1)}) \in \text{Lip}(2, \Gamma, \mathbb{R})$ with $\|\psi\|_{\text{Lip}(2, \Gamma, \mathbb{R})} = 2$. However, $R_1(-1, 1)[v] = 4v = 2\sqrt{2}|1 - (-1)|^{\frac{1}{2}}v$ and so $\|\psi\|_{\text{Lip}(3/2, \Gamma, \mathbb{R})} = 2\sqrt{2} = \sqrt{2}\|\psi\|_{\text{Lip}(2, \Gamma, \mathbb{R})}$. Here, $\text{diam}(\Gamma) = 2$, $\rho = 2$ and $\theta = 3/2$. Thus, we observe that $1 < \text{diam}(\Gamma)^{2-} = \sqrt{2} < 1 + e$, which establishes equality in (6.1).

Remark 6.4. We can have equality in (6.2). As an example, let $\Gamma := \{0, 1\} \subset \mathbb{R}$ and define $\psi^{(0)} : \Gamma \rightarrow \mathbb{R}$ by $\psi^{(0)}(0) := -A$ and $\psi^{(0)}(1) := A$ for some $A > 0$, and define $\psi^{(1)} : \Gamma \rightarrow \mathcal{L}(\mathbb{R}; \mathbb{R})$ by $\psi^{(1)}(x)[v] := Av$ for every $x \in \Gamma$. Then, given $x, y \in \Gamma$

$$\psi^{(0)}(y) - \psi^{(0)}(x) - \psi^{(1)}[y - x] = \begin{cases} A & \text{if } x = 0, y = 1 \\ -A & \text{if } x = 1, y = 0 \\ 0 & \text{if } x = y. \end{cases} \tag{6.6}$$

It follows from (6.6) that $\psi = (\psi^{(0)}, \psi^{(1)}) \in \text{Lip}(2, \Gamma, \mathbb{R})$ with $\|\psi\|_{\text{Lip}(2, \Gamma, \mathbb{R})} = A$. Moreover, we also have that $\psi_{[0]} = \psi^{(0)} \in \text{Lip}(1, \Gamma, \mathbb{R})$ with $\|\psi_{[0]}\|_{\text{Lip}(1, \Gamma, \mathbb{R})} = 2A = 2\|\psi\|_{\text{Lip}(2, \Gamma, \mathbb{R})}$. Here, $\text{diam}(\Gamma) = 1$, $n = 1$, $\rho = 2$, $\theta = 1$ and $q = 0$. Consequently, both C_1 defined in (6.3) and C_2 defined in (6.4) are equal to 2. Hence, $\min \{C_1, C_2\} = 2$, and so, we have equality in (6.2).

Proof of Lemma 6.1. Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Gamma \subset V$ is closed. Let $\rho > 0$ with $n \in \mathbb{Z}_{\geq 0}$ such that $\rho \in (n, n + 1]$, and let $\theta \in (0, \rho)$ with $q \in \{0, \dots, n\}$ such that $\theta \in (q, q + 1]$. To deal with the case that $\theta \in (n, \rho)$, we first establish the following claim.

Claim 6.5. Suppose V and W are Banach spaces, and that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $D \subset V$ is closed. Let $\lambda > 0$ with $m \in \mathbb{Z}_{\geq 0}$ such that $\lambda \in (m, m + 1]$, and $\sigma \in (m, \lambda)$. If $\phi = (\phi^{(0)}, \dots, \phi^{(m)}) \in \text{Lip}(\lambda, D, W)$, then $\phi \in \text{Lip}(\sigma, D, W)$, and we have the estimate that

$$\|\phi\|_{\text{Lip}(\sigma, D, W)} \leq \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(D)^{\lambda - \sigma} \right\} \right\} \|\phi\|_{\text{Lip}(\lambda, D, W)}. \tag{6.7}$$

Proof of Claim 6.5. For each $l \in \{0, \dots, m\}$, define $R_l^\phi : D \times D \rightarrow \mathcal{L}(V^{\otimes l}; W)$ for $x, y \in D$ and $v \in V^{\otimes l}$ by

$$R_l^\phi(x, y)[v] := \phi^{(l)}(y)[v] - \sum_{s=0}^{n-l} \frac{1}{s!} \phi^{l+s}(x) \left[v \otimes (y - x)^{\otimes s} \right]. \tag{6.8}$$

Since the estimate (6.7) is trivial when $\|\phi\|_{\text{Lip}(\lambda, D, W)} = 0$, we need only establish the validity of (6.7) when $\|\phi\|_{\text{Lip}(\lambda, D, W)} \neq 0$. But in this case, by replacing ϕ by $\phi / \|\phi\|_{\text{Lip}(\lambda, D, W)}$, it suffices to prove (6.7) under the additional assumption that $\|\phi\|_{\text{Lip}(\lambda, D, W)} = 1$.

A consequence of $\phi \in \text{Lip}(\lambda, D, W)$ with $\|\phi\|_{\text{Lip}(\lambda, D, W)} = 1$ is that, whenever $l \in \{0, \dots, m\}$ and $x, y \in D$, we have the bounds (cf. (2.8) and (2.10))

$$\text{(I)} \quad \left\| \phi^{(l)}(x) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq 1 \quad \text{and} \quad \text{(II)} \quad \left\| R_l^\phi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \|y - x\|_V^{\lambda - l}. \tag{6.9}$$

Given any $l \in \{0, \dots, m\}$ and any point $x \in D$, we can conclude from (6.8) that $R_l^\psi(x, x) \equiv 0$ in $\mathcal{L}(V^{\otimes l}; W)$. Hence, controlling the $\mathcal{L}(V^{\otimes l}; W)$ norm of the remainder term R_l^ψ is trivial on the diagonal of $D \times D$.

Given any $l \in \{0, \dots, m\}$, we now estimate the $\mathcal{L}(V^{\otimes l}; W)$ norm of R_l^ψ off the diagonal of $D \times D$. For any points $x, y \in D$ with $x \neq y$, we apply Lemma 5.1, recalling that $\|\phi\|_{\text{Lip}(\lambda, D, W)} = 1$, to conclude that (cf. (5.1))

$$\frac{\left\| R_l^\phi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\sigma - l}} \leq \min \left\{ \text{diam}(D)^{\lambda - \sigma}, G(\lambda, \sigma, l, D) \right\}, \tag{6.10}$$

where $G(\lambda, \sigma, l, D)$ is defined by

$$G(\lambda, \sigma, l, D) := \inf_{r \in (0, \text{diam}(D))} \left\{ \max \left\{ r^{\lambda - \sigma}, \frac{1}{r^{\sigma - l}} \left(1 + \sum_{s=0}^{m-l} \frac{r^s}{s!} \right) \right\} \right\}. \tag{6.11}$$

We now prove that

$$\min \left\{ \text{diam}(D)^{\lambda - \sigma}, G(\lambda, \sigma, l, D) \right\} \leq 1 + e. \tag{6.12}$$

If $\text{diam}(D) \leq 1$, then (6.12) is obtained by observing that

$$\min \left\{ \text{diam}(D)^{\lambda^-}, G(\lambda, \sigma, l, D) \right\} \leq \text{diam}(D)^{\lambda^-} \leq 1 < 1 + e.$$

If $\text{diam}(D) > 1$, then (6.12) is obtained by observing that

$$\min \left\{ \text{diam}(D)^{\lambda^-}, G(\lambda, \sigma, l, D) \right\} \leq G(\lambda, \sigma, l, D) \leq \max \left\{ 1, 1 + \sum_{s=0}^{m-l} \frac{1}{s!} \right\} \leq (1 + e).$$

Hence, (6.12) is proven. Together (6.10) and (6.12) yield that

$$\frac{\|R_l^\phi(x, y)\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\sigma-l}} \leq 1 + e. \tag{6.13}$$

Thus, we may combine (6.10) and (6.13) to conclude that

$$\frac{\|R_l^\phi(x, y)\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\sigma-l}} \leq \min \left\{ \text{diam}(D)^{\lambda^-}, 1 + e \right\}. \tag{6.14}$$

Both the choice of $l \in \{0, \dots, m\}$ and the choice of points $x, y \in D$ with $x \neq y$ were arbitrary. Hence, we may conclude that the estimate (6.14) is valid for every $l \in \{0, \dots, m\}$ and all points $x, y \in D$ with $x \neq y$. The pointwise bounds for the functions $\phi^{(0)}, \dots, \phi^{(m)}$ given in (I) of (6.9) and the remainder term bounds (6.14) establish that

$$\|\phi\|_{\text{Lip}(\sigma, D, W)} \leq \max \left\{ 1, \min \left\{ \text{diam}(D)^{\lambda^-}, 1 + e \right\} \right\}. \tag{6.15}$$

The estimate (6.15) is precisely the estimate claimed in (6.7) for the case that $\|\phi\|_{\text{Lip}(\lambda, D, W)} = 1$. This completes the proof of Claim 6.5. \square

The estimate claimed in the case that $\theta \in (n, \rho)$ is an immediate consequence of Claim 6.5. Indeed, assuming that $\theta \in (n, \rho)$, we appeal to Claim 6.5 with $D := \Gamma, m := n, \lambda := \rho$ and $\sigma := \theta$ to conclude from (6.7) that

$$\|\psi\|_{\text{Lip}(\theta, \Gamma, W)} \leq \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(\Gamma)^{\rho^-} \right\} \right\} \|\psi\|_{\text{Lip}(\rho, \Gamma, W)}, \tag{6.16}$$

which is precisely the estimate claimed in (6.1).

It remains only to establish the estimate claimed in (6.2) for the case that $0 < \theta \leq n < \rho \leq n + 1$. Observe that this requires $n \geq 1$ and $q \in \{0, \dots, n - 1\}$. We begin by establishing the following claim.

Claim 6.6. Suppose that V and W are Banach spaces, and that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $D \subset V$ is closed. Let $\lambda > 1$ with $m \in \mathbb{Z}_{\geq 1}$ such that $\lambda \in (m, m + 1]$, and $\sigma \in (0, m]$ with $p \in \{0, \dots, m - 1\}$ such that $\sigma \in (p, p + 1]$. If $\phi = (\phi^{(0)}, \dots, \phi^{(m)}) \in \text{Lip}(\lambda, D, W)$ then $\phi_{[p]} = (\phi^{(0)}, \dots, \phi^{(p)}) \in \text{Lip}(\sigma, D, W)$, and we have the

estimate that

$$\|\phi_{[p]}\|_{\text{Lip}(\sigma, D, W)} \leq \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(D)^{\lambda^-} + \sum_{j=p+1}^m \frac{\text{diam}(D)^{j^-}}{(j-p)!} \right\} \right\} \|\phi\|_{\text{Lip}(\lambda, D, W)}. \tag{6.17}$$

Proof of Claim 6.6. For each $l \in \{0, \dots, m\}$, define $R_l^\phi : D \times D \rightarrow \mathcal{L}(V^{\otimes l}; W)$ for $x, y \in D$ and $v \in V^{\otimes l}$ by

$$R_l^\phi(x, y)[v] := \phi^{(l)}(y)[v] - \sum_{j=0}^{m-l} \frac{1}{j!} \phi^{(j+l)}(x) \left[v \otimes (y-x)^{\otimes j} \right]. \tag{6.18}$$

Since the estimate (6.17) is trivial when $\|\phi\|_{\text{Lip}(\lambda, D, W)} = 0$, we need only establish the validity of (6.17) when $\|\phi\|_{\text{Lip}(\lambda, D, W)} \neq 0$. But in this case, by replacing ϕ by $\phi/\|\phi\|_{\text{Lip}(\lambda, D, W)}$, it suffices to prove (6.17) under the additional assumption that $\|\phi\|_{\text{Lip}(\lambda, D, W)} = 1$.

A consequence of $\phi \in \text{Lip}(\lambda, D, W)$ with $\|\phi\|_{\text{Lip}(\lambda, D, W)} = 1$ is that, whenever $l \in \{0, \dots, m\}$ and $x, y \in D$, we have the bounds (cf. (2.8) and (2.10))

$$\text{(I)} \quad \|\phi^{(l)}(x)\|_{\mathcal{L}(V^{\otimes l}; W)} \leq 1 \quad \text{and} \quad \text{(II)} \quad \|R_l^\phi(x, y)\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \|y-x\|_V^{\lambda-l}. \tag{6.19}$$

Our goal is to show that $\phi_{[p]} = (\phi^{(0)}, \dots, \phi^{(p)})$ is in $\text{Lip}(\sigma, D, W)$. For this purpose, given $s \in \{0, \dots, p\}$, let $\tilde{R}_s^\phi : D \times D \rightarrow \mathcal{L}(V^{\otimes s}; W)$ be defined for all $x, y \in D$ and $v \in V^{\otimes s}$ by

$$\tilde{R}_s^\phi(x, y)[v] := \phi^{(s)}(y)[v] - \sum_{j=0}^{p-s} \frac{1}{j!} \phi^{(s+j)}(x) \left[v \otimes (y-x)^{\otimes j} \right]. \tag{6.20}$$

Together, (6.18) and (6.20) yield that

$$\tilde{R}_s^\phi(x, y)[v] = R_s^\phi(x, y)[v] + \sum_{j=p+1-s}^{m-s} \frac{1}{(k-j)!} \phi^{(s+j)}(x) \left[v \otimes (y-x)^{\otimes j} \right]. \tag{6.21}$$

Given any $l \in \{0, \dots, p\}$ and any point $x \in D$, we conclude from (6.21) that $\tilde{R}_l^\psi(x, x) \equiv 0$ in $\mathcal{L}(V^{\otimes l}; W)$. Hence, controlling the $\mathcal{L}(V^{\otimes l}; W)$ norm of \tilde{R}_l^ψ is trivial on the diagonal of $D \times D$.

Given any $l \in \{0, \dots, p\}$, we now estimate the $\mathcal{L}(V^{\otimes l}; W)$ norm of \tilde{R}_l^ψ off of the diagonal of $D \times D$. The alteration in (6.21) is exactly the same as the alteration defined in (5.3) of Lemma 5.1. Consequently, recalling that $\|\phi\|_{\text{Lip}(\lambda, D, W)} = 1$, we may apply that result (Lemma 5.1) to deduce that for every $x, y \in D$ with $x \neq y$, we have that (cf. (5.4))

$$\frac{\|\tilde{R}_l^\phi(x, y)\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y-x\|_V^{\sigma-l}} \leq \min \left\{ \text{diam}(D)^{\lambda^-} + \sum_{i=p+1}^m \frac{\text{diam}(D)^{i^-}}{(i-l)!}, H(\lambda, \sigma, l, D) \right\}, \tag{6.22}$$

where $H(\lambda, \sigma, l, D)$ is defined by (cf. (5.5))

$$H(\lambda, \sigma, l, D) := \inf_{r \in (0, \text{diam}(D))} \left\{ \max \left\{ r^{\lambda-\sigma} + \sum_{i=p+1}^m \frac{r^{i-\sigma}}{(i-l)!}, \frac{1}{r^{\sigma-l}} \left(1 + \sum_{s=0}^{p-l} \frac{r^s}{s!} \right) \right\} \right\}. \tag{6.23}$$

We now prove that

$$H := \min \left\{ \text{diam}(D)^{\lambda-} + \sum_{i=p+1}^m \frac{\text{diam}(D)^{i-}}{(i-l)!}, H(\lambda, \sigma, l, D) \right\} \leq 1 + e. \tag{6.24}$$

If $\text{diam}(D) \leq 1$, then we obtain (6.24) by observing, for every $i \in \{p+1, \dots, m\}$, that $(i-l)! \geq (i-p)!$, and hence,

$$H \leq \text{diam}(D)^{\lambda-} + \sum_{i=p+1}^m \frac{\text{diam}(D)^{i-}}{(i-l)!} \leq 1 + \sum_{i=p+1}^m \frac{1}{(i-p)!} < 1 + e.$$

If $\text{diam}(D) > 1$, then we obtain (6.24) by observing that

$$\begin{aligned} H \leq H(\lambda, \sigma, l, D) &\leq \max \left\{ 1 + \sum_{i=p+1}^m \frac{1}{(i-l)!}, 1 + \sum_{s=0}^{p-l} \frac{1}{s!} \right\} \\ &\leq \max \left\{ 1 + \sum_{i=p+1}^m \frac{1}{(i-p)!}, 1 + \sum_{s=0}^{p-l} \frac{1}{s!} \right\} < 1 + e. \end{aligned}$$

Hence, (6.24) is proven. Together (6.22) and (6.24) establish that

$$\frac{\left\| \bar{R}_l^\phi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\sigma-l}} \leq 1 + e. \tag{6.25}$$

Thus, we may combine (6.22), (6.25) and the observation that for every $i \in \{p+1, \dots, m\}$, we have $(i-l)! \geq (i-p)!$ to conclude that

$$\frac{\left\| \bar{R}_l^\phi(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)}}{\|y - x\|_V^{\sigma-l}} \leq \min \left\{ \text{diam}(D)^{\lambda-} + \sum_{i=p+1}^m \frac{\text{diam}(D)^{i-}}{(i-p)!}, 1 + e \right\}. \tag{6.26}$$

The arbitrariness of $l \in \{0, \dots, p\}$ and the points $x, y \in D$ with $x \neq y$ ensure that the estimate (6.26) is valid for every $l \in \{0, \dots, p\}$ and every $x, y \in D$ with $x \neq y$. Together, the definitions (6.20), the bounds in (I) of (6.19) and the estimates (6.26) allow us to conclude that $\phi_{[p]} = (\phi^{(0)}, \dots, \phi^{(p)}) \in \text{Lip}(\sigma, D, W)$, and that

$$\left\| \phi_{[p]} \right\|_{\text{Lip}(\sigma, D, W)} \leq \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(D)^{\lambda-} + \sum_{j=p+1}^m \frac{\text{diam}(D)^{j-}}{(j-p)!} \right\} \right\}. \tag{6.27}$$

The estimate (6.27) is precisely the estimate claimed in (6.17) for the case that $\|\phi\|_{\text{Lip}(\lambda, D, W)} = 1$. This completes the proof of Claim 6.6. \square

Returning to the proof of Lemma 6.1 itself, suppose $\theta \in (0, n]$. A direct application of Claim 6.6 with $D := \Gamma, m := n, \lambda := \rho$ and $\sigma := \theta$ means that (6.17) yields that

$$\|\psi_{[q]}\|_{\text{Lip}(\theta, \Gamma, W)} \leq \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(\Gamma)^{\rho^-} + \sum_{j=q+1}^n \frac{\text{diam}(\Gamma)^{j^-}}{(j-q)!} \right\} \right\} \|\psi\|_{\text{Lip}(\rho, \Gamma, W)}, \tag{6.28}$$

where $q \leq n - 1$ since $\theta \in (0, n]$. By examining the definition of C_1 in (6.3), we see that (6.28) is the first part of the estimate claimed in (6.2). To derive the remaining estimate, we note that $\theta \in (q, q + 1]$. By appealing to Claim 6.6, with $D := \Gamma, m := n, \lambda := \rho$ and $\sigma := n$, we deduce via (6.17) that

$$\|\psi_{[n-1]}\|_{\text{Lip}(n, \Gamma, W)} \leq \min \{ 1 + e, 1 + \text{diam}(\Gamma)^{\rho^-} \} \|\psi\|_{\text{Lip}(\rho, \Gamma, W)}. \tag{6.29}$$

If we now appeal to Claim 6.6 for $D := \Gamma, m := n - 1, \lambda := n$ and $\sigma := n - 1$, then (6.17) and (6.29) give

$$\begin{aligned} \|\psi_{[n-2]}\|_{\text{Lip}(n-1, \Gamma, W)} &\stackrel{(6.17)}{\leq} \max \{ 1, \min \{ 1 + e, 1 + \text{diam}(\Gamma) \} \} \|\psi_{[n-1]}\|_{\text{Lip}(n, \Gamma, W)} \\ &\stackrel{(6.29)}{\leq} \min \{ 1 + e, 1 + \text{diam}(\Gamma)^{\rho^-} \} \min \{ 1 + e, 1 + \text{diam}(\Gamma) \} \|\psi\|_{\text{Lip}(\rho, \Gamma, W)} \\ &= (1 + \min \{ e, \text{diam}(\Gamma)^{\rho^-} \}) (1 + \min \{ e, \text{diam}(\Gamma) \}) \|\psi\|_{\text{Lip}(\rho, \Gamma, W)}. \end{aligned}$$

We can now appeal to Claim 6.6 for $D := \Gamma, m := n - 2, \lambda := n - 1$ and $\sigma := n - 2$. Proceeding inductively as $r = 0, 1, \dots, n - 1$ increases, we establish via applying Claim 6.6 for $D := \Gamma, m := n - r, \lambda := n - (r - 1)$, and $\sigma := n - r$ that for every $r \in \{0, 1, \dots, n - 1\}$, we have that

$$\|\psi_{[n-r-1]}\|_{\text{Lip}(n-r, \Gamma, W)} \leq (1 + \min \{ e, \text{diam}(\Gamma)^{\rho^-} \}) (1 + \min \{ e, \text{diam}(\Gamma) \})^r \|\psi\|_{\text{Lip}(\rho, \Gamma, W)}. \tag{6.30}$$

Taking $r := n - (q + 1)$ in (6.30) yields that

$$\|\psi_{[q]}\|_{\text{Lip}(q+1, \Gamma, W)} \leq (1 + \min \{ e, \text{diam}(\Gamma)^{\rho^-} \}) (1 + \min \{ e, \text{diam}(\Gamma) \})^{n-(q+1)} \|\psi\|_{\text{Lip}(\rho, \Gamma, W)}. \tag{6.31}$$

As $\theta \in (q, q + 1]$, we can appeal to Claim 6.5 with $D := \Gamma, m := q, \lambda := q + 1$ and $\sigma := \theta$ to deduce via (6.7) that

$$\|\psi_{[q]}\|_{\text{Lip}(\theta, \Gamma, W)} \leq \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(\Gamma)^{q+1^-} \right\} \right\} \|\psi_{[q]}\|_{\text{Lip}(q+1, \Gamma, W)}. \tag{6.32}$$

Together, (6.31) and (6.32) yield that

$$\|\psi_{[q]}\|_{\text{Lip}(\theta, \Gamma, W)} \leq C_2 \|\psi\|_{\text{Lip}(\rho, \Gamma, W)} \tag{6.33}$$

for $C_2 > 0$ defined by

$$C_2 := \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(\Gamma)^{q+1-} \right\} \right\} (1 + \min \{e, \text{diam}(\Gamma)^{\rho-}\}) \times (1 + \min \{e, \text{diam}(\Gamma)\})^{n-(q+1)}$$

as claimed in (6.4). The estimates (6.28) and (6.33) combine to give

$$\|\psi_{[q]}\|_{\text{Lip}(\theta, \Gamma, W)} \leq \min \{C_1, C_2\} \|\psi\|_{\text{Lip}(\rho, \Gamma, W)},$$

where

$$C_1 = \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(\Gamma)^{\rho-} + \sum_{j=q+1}^n \frac{\text{diam}(\Gamma)^{j-}}{(j-q)!} \right\} \right\}$$

and

$$C_2 := \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(\Gamma)^{q+1-} \right\} \right\} (1 + \min \{e, \text{diam}(\Gamma)^{\rho-}\}) \times (1 + \min \{e, \text{diam}(\Gamma)\})^{n-(q+1)}$$

as claimed in (6.2).

Finally, since $C_1 \leq 1 + e$, (6.1) and (6.2) combine to yield that, for any $\theta \in (0, \rho)$, we have the estimate $\|\psi_{[q]}\|_{\text{Lip}(\theta, \Gamma, W)} \leq (1 + e)\|\psi\|_{\text{Lip}(\rho, \Gamma, W)}$ as claimed in (6.5). This completes the proof of Lemma 6.1. \square

7 | LOCAL LIPSCHITZ BOUNDS

In this section, we establish some local estimates arising as consequences from knowing the Lipschitz norm of a function is small when the domain is taken to be a single point in a similar spirit to Lemma 1.13 in [1]. The constants appearing in our estimates are more convenient for our purposes.

We first record the following result relating the pointwise properties of a Lipschitz function at one point to its pointwise values at another. The precise result is the following.

Lemma 7.1 (Pointwise estimates). Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume $\Gamma \subset V$ is closed with $p \in \Gamma$. Let $A, \rho > 0, r_0 \geq 0$, and $n \in \mathbb{Z}_{\geq 0}$ such that $\rho \in (n, n + 1]$. Let $F = (F^{(0)}, \dots, F^{(n)}) \in \text{Lip}(\rho, \Gamma, W)$ with $\|F\|_{\text{Lip}(\rho, \Gamma, W)} \leq A$. For every $j \in \{0, \dots, n\}$, let $R_j^F : \Gamma \times \Gamma \rightarrow \mathcal{L}(V^{\otimes j}; W)$ denote the remainder term associated to $F^{(j)}$, defined for $x, y \in \Gamma$ and $v \in V^{\otimes j}$ by

$$R_j^F(x, y)[v] := F^{(j)}(y)[v] - \sum_{s=0}^{n-j} \frac{1}{s!} F^{(j+s)}(x)[v \otimes (y-x)^{\otimes s}]. \tag{7.1}$$

Then, for every $l \in \{0, \dots, n\}$, any $x, y \in \Gamma$ and any $\theta \in (n, \rho)$, we have that

$$\left\| R_l^F(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq A(\text{dist}(x, p) + \text{dist}(y, p))^{\rho-\theta} \|y - x\|_V^{\theta-l}. \quad (7.2)$$

Further, suppose $q \in \{0, \dots, n\}$ and that for every $s \in \{0, \dots, q\}$, we have $\|F^{(s)}(p)\|_{\mathcal{L}(V^{\otimes s}; W)} \leq r_0$. Then, for any $l \in \{0, \dots, q\}$ and any $x \in \Gamma$, we have that

$$\left\| F^{(l)}(x) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \min \left\{ A, A[\text{dist}(x, p)^{\rho-l} + S_{l,q}(x, p)] + r_0 \sum_{j=0}^{q-l} \frac{1}{j!} \text{dist}(x, p)^j \right\}, \quad (7.3)$$

where

$$S_{l,q}(x, p) := \begin{cases} \sum_{j=q+1-l}^{n-l} \frac{1}{j!} \text{dist}(x, p)^j & \text{if } q < n \\ 0 & \text{if } q = n. \end{cases} \quad (7.4)$$

Proof of Lemma 7.1. Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Gamma \subset V$ is closed and that $\rho > 0$ with $n \in \mathbb{Z}_{\geq 0}$ such that $\rho \in (n, n+1]$. Suppose that, for $l \in \{0, \dots, n\}$, we have functions $F^{(l)} : \Gamma \rightarrow \mathcal{L}(V^{\otimes l}; W)$ such that $F = (F^{(0)}, \dots, F^{(n)}) \in \text{Lip}(\rho, \Gamma, W)$ with $\|F\|_{\text{Lip}(\rho, \Gamma, W)} \leq A$. For each $j \in \{0, \dots, n\}$, let $R_j^F : \Gamma \times \Gamma \rightarrow \mathcal{L}(V^{\otimes j}; W)$ denote the remainder term associated to $F^{(j)}$, defined for $x, y \in \Gamma$ and $v \in V^{\otimes j}$ by

$$R_j^F(x, y)[v] := F^{(j)}(y)[v] - \sum_{s=0}^{n-j} \frac{1}{s!} F^{(j+s)}(x)[v \otimes (y-x)^{\otimes s}]. \quad (7.5)$$

As a consequence of $F \in \text{Lip}(\rho, \Gamma, W)$ with $\|F\|_{\text{Lip}(\rho, \Gamma, W)} \leq A$, whenever $l \in \{0, \dots, n\}$ and $x, y \in \Gamma$, we have the bounds (cf. (2.8) and (2.10))

$$\text{(I)} \quad \left\| F^{(l)}(x) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq A \quad \text{and} \quad \text{(II)} \quad \left\| R_l^F(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq A \|y - x\|_V^{\rho-l}. \quad (7.6)$$

If $l \in \{0, \dots, n\}$, $x, y \in \Gamma$, and $\theta \in (n, \rho)$, we use (II) of (7.6) to compute that

$$\begin{aligned} \left\| R_l^F(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} &\stackrel{\text{(II) of (7.6)}}{\leq} A \|y - x\|_V^{\rho-l} = A \|y - x\|_V^{\rho-\theta} \|y - x\|_V^{\theta-l} \\ &\leq A (\|x - p\|_V + \|y - p\|_V)^{\rho-\theta} \|y - x\|_V^{\theta-l} \\ &= A (\text{dist}(x, p) + \text{dist}(y, p))^{\rho-\theta} \|y - x\|_V^{\theta-l} \end{aligned}$$

as claimed in (7.2).

Now suppose that $q \in \{0, \dots, n\}$ and that for every $s \in \{0, \dots, q\}$, we have $\|F^{(s)}(p)\|_{\mathcal{L}(V^{\otimes s}; W)} \leq r_0$. Given $l \in \{0, \dots, q\}$, $x \in \Gamma$ and $v \in V^{\otimes l}$, recalling that the tensor powers of V are equipped with admissible norms (cf. Definition 2.1), we may use (7.5) and (II) of (7.6) to obtain that

$$\left\| F^{(l)}(x)[v] \right\|_W \leq \sum_{s=0}^{n-l} \frac{1}{s!} \left\| F^{(s)}(p) \right\|_{\mathcal{L}(V^{\otimes s}; W)} \|x - p\|_V^s \|v\|_V^{\otimes l} + A \|x - p\|_V^{\rho-l} \|v\|_V^{\otimes l}. \quad (7.7)$$

If $q = n$, then (7.7) tells us that

$$\|F^{(l)}(x)[v]\|_W \leq \left(r_0 \sum_{s=0}^{q-l} \frac{1}{s!} \|x - p\|_V^s + A \|x - p\|_V^{\rho-l} \right) \|v\|_{V^{\otimes l}}. \tag{7.8}$$

Whilst if $q < n$, we deduce from (7.7) that

$$\|F^{(l)}(x)[v]\|_W \leq \left(r_0 \sum_{s=0}^{q-l} \frac{1}{s!} \|x - p\|_V^s + A \sum_{s=q-l+1}^{n-l} \frac{1}{s!} \|x - p\|_V^s + A \|x - p\|_V^{\rho-l} \right) \|v\|_{V^{\otimes l}}. \tag{7.9}$$

If we let $S_{l,q}(x, p)$ be defined as in (7.4), then (7.8) and (7.9) combine to yield that

$$\|F^{(l)}(x)[v]\|_W \leq \left(r_0 \sum_{s=0}^{q-l} \frac{1}{s!} \text{dist}(x, p)^s + A [\text{dist}(x, p)^\rho + S_{l,q}(x, p)] \right) \|v\|_{V^{\otimes l}}. \tag{7.10}$$

Taking the supremum over $v \in V^{\otimes l}$ with unit $V^{\otimes l}$ norm in (7.10) yields the second estimate claimed in (7.3). The first estimate claimed in (7.3) follows from (I) in (7.6). This completes the proof of Lemma 7.1. \square

Our aim for the remainder of this section is to strengthen the pointwise estimates obtained in Lemma 7.1 to full Lipschitz norm bounds on a local neighbourhood of the point p . The first local Lipschitz norm bounds we can establish in a neighbourhood of a given point are recorded in the following result.

Lemma 7.2 (Local Lipschitz bounds I). Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Gamma \subset V$ is non-empty and closed, and that $z \in \Gamma$. Let $A, \rho > 0$ with $n \in \mathbb{Z}_{\geq 0}$ such that $\rho \in (n, n + 1]$, $r_0 \in [0, A]$, and $\theta \in (n, \rho)$. Suppose that $F = (F^{(0)}, \dots, F^{(n)}) \in \text{Lip}(\rho, \Gamma, W)$ satisfies that $\|F\|_{\text{Lip}(\rho, \Gamma, W)} \leq A$, and that for every $j \in \{0, \dots, n\}$, we have the bound $\|F^{(j)}(z)\|_{\mathcal{L}(V^{\otimes j}; W)} \leq r_0$. Then, for any $\delta \in [0, 1]$, we have that

$$\|F\|_{\text{Lip}(\theta, \Omega, W)} \leq \max \left\{ (2\delta)^{\rho-\theta} A, \min \{A, A\delta^{\rho-n} + r_0 e^\delta\} \right\}, \tag{7.11}$$

where $\Omega := \Gamma \cap \overline{\mathbb{B}}_V(z, \delta)$.

Proof of Lemma 7.2. Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Gamma \subset V$ is non-empty and closed, and that $z \in \Gamma$. Let $A, \rho > 0$ with $n \in \mathbb{Z}_{\geq 0}$ such that $\rho \in (n, n + 1]$, $r_0 \in [0, A]$, and $\theta \in (n, \rho)$. Suppose that $F = (F^{(0)}, \dots, F^{(n)}) \in \text{Lip}(\rho, \Gamma, W)$ satisfies that $\|F\|_{\text{Lip}(\rho, \Gamma, W)} \leq A$, and that for every $j \in \{0, \dots, n\}$, we have the bound $\|F^{(j)}(z)\|_{\mathcal{L}(V^{\otimes j}; W)} \leq r_0$. For each $l \in \{0, \dots, n\}$, let $R_l^F : \Gamma \times \Gamma \rightarrow \mathcal{L}(V^{\otimes l}; W)$ be defined for $x, y \in \Gamma$ and $v \in V^{\otimes l}$ by

$$R_l^F(x, y)[v] := F^{(l)}(y)[v] - \sum_{j=0}^{n-l} \frac{1}{j!} F^{(j+l)}(x)[v \otimes (y - x)^{\otimes j}]. \tag{7.12}$$

An application of Lemma 7.1, with A, r_0, ρ, n and θ here playing the same roles there, yields that for each $l \in \{0, \dots, n\}$ and any $x \in \Sigma$, we have (cf. (7.3) for $q = n$)

$$\left\| F^{(l)}(x) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \min \left\{ A, A \operatorname{dist}(x, p)^{\rho-l} + r_0 \sum_{s=0}^{n-l} \frac{1}{s!} \operatorname{dist}(x, p)^s \right\} \tag{7.13}$$

and (cf. (7.2))

$$\left\| R_l^F(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq A(\operatorname{dist}(x, p) + \operatorname{dist}(y, p))^{\rho-\theta} \|y - x\|_V^{\theta-l}. \tag{7.14}$$

Now let $\delta \in [0, 1]$ and define $\Omega := \bar{\mathbb{B}}_V(z, \delta) \cap \Gamma \subset \Gamma$. Then, given any $l \in \{0, \dots, n\}$ and any $x, y \in \Omega$, (7.13) tells us that

$$\left\| F^{(l)}(x) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \min \left\{ A, A\delta^{\rho-l} + r_0 \sum_{s=0}^{n-l} \frac{\delta^s}{s!} \right\} \leq \min \{A, A\delta^{\rho-n} + r_0 e^\delta\}, \tag{7.15}$$

since $\delta \in [0, 1]$ means $\delta^{\rho-l} \leq \delta^{\rho-n}$ for every $l \in \{0, \dots, n\}$, whilst (7.14) tells us that

$$\left\| R_l^F(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq A(2\delta)^{\rho-\theta} \|y - x\|_V^{\theta-l}. \tag{7.16}$$

The estimates (7.15) and (7.16) allow us to conclude that $F \in \operatorname{Lip}(\theta, \Omega, W)$ with

$$\|F\|_{\operatorname{Lip}(\theta, \Omega, W)} \leq \max \left\{ (2\delta)^{\rho-\theta} A, \min \{A, A\delta^{\rho-n} + r_0 e^\delta\} \right\}$$

as claimed in (7.11). This completes the proof of Lemma 7.2. □

Extending the local Lipschitz estimates of Lemma 7.2 to the setting, in the notation of Lemma 7.2, that $\theta \leq n < \rho \leq n + 1$ is more challenging. We achieve this by combining the *Lipschitz Nesting* Lemma 6.1 from Section 6 with Lemma 7.2. The resulting local Lipschitz bounds are precisely recorded in the following result.

Lemma 7.3 (Local Lipschitz bounds II). Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Gamma \subset V$ is a non-empty closed subset with $z \in \Gamma$. Let $A > 0, r_0 \in [0, A), \rho > 1$ with $n \in \mathbb{Z}_{\geq 1}$ such that $\rho \in (n, n + 1]$, and $\theta \in (0, n]$ with $q \in \{0, \dots, n - 1\}$ such that $\theta \in (q, q + 1]$. Suppose that $F \in \operatorname{Lip}(\rho, \Gamma, W)$ satisfies that $\|F\|_{\operatorname{Lip}(\rho, \Gamma, W)} \leq A$, and that for every $j \in \{0, \dots, n\}$, we have the bound $\left\| F^{(j)}(z) \right\|_{\mathcal{L}(V^{\otimes j}; W)} \leq r_0$. Given any $\delta \in [0, 1]$, we have, for $\Omega := \bar{\mathbb{B}}_V(z, \delta) \cap \Gamma$, that

$$\left\| F_{[q]} \right\|_{\operatorname{Lip}(\theta, \Omega, W)} \leq \max \left\{ (2\delta)^{q+1-\theta} E_{n-b_q}, \min \left\{ E_{n-b_q}, \delta E_{n-b_q} + r_0 e^\delta \right\} \right\}, \tag{7.17}$$

where $F_{[q]} = (F^{(0)}, \dots, F^{(q)})$, $b_q := n - (q + 1)$, and for $s \in \{0, \dots, n - 1\}$ E_{n-s} is inductively defined by

$$E_{n-s} := \begin{cases} \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \max \left\{ (2\delta)^{\frac{\rho-n}{2}} A, \min \{A, \delta^{\rho-n} A + r_0 e^\delta\} \right\} & \text{if } s = 0 \\ \left(1 + \sqrt{2\delta}\right) \max \left\{ \sqrt{2\delta} E_{n-(s-1)}, \min \{E_{n-(s-1)}, \delta E_{n-(s-1)} + r_0 e^\delta\} \right\} & \text{if } s \geq 1. \end{cases} \tag{7.18}$$

Consequently, if $r_0 = 0$, we can conclude that

$$\|F_{[q]}\|_{\text{Lip}(\theta, \Omega, W)} \leq \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \left(1 + \sqrt{2\delta}\right)^{n-(q+1)} (2\delta)^{\frac{\rho-\theta}{2} + \frac{q+1-\theta}{2}} A. \tag{7.19}$$

If $0 < r_0 < A$, then there exists $\delta_* = \delta_*(A, r_0, \rho) > 0$ such that if we additionally impose that $\delta \in [0, \delta_*]$, then we may conclude that

$$\|F_{[q]}\|_{\text{Lip}(\theta, \Omega, W)} \leq \max \left\{ (2\delta)^{q+1-\theta} \mathcal{E}, \min \{ \mathcal{E}, \delta \mathcal{E} + r_0 e^\delta \} \right\} \tag{7.20}$$

for $\mathcal{E} = \mathcal{E}(A, r_0, \rho, \theta, \delta) > 0$ defined by

$$\mathcal{E} := \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \left(1 + \sqrt{2\delta}\right)^{n-(q+1)} \left(\delta^{\rho-(q+1)} A + r_0 \delta^{n-(q+1)} e^\delta\right) + \mathbf{X}_{n-(q+1)}(\delta), \tag{7.21}$$

where, for $t \in \{0, \dots, n-1\}$, the quantity $\mathbf{X}_t(\delta)$ is defined by

$$\mathbf{X}_t(\delta) := \begin{cases} 0 & \text{if } t = 0 \\ \left(1 + \sqrt{2\delta}\right) r_0 e^\delta \sum_{j=0}^{t-1} \delta^j \left(1 + \sqrt{2\delta}\right)^j & \text{if } t \geq 1. \end{cases} \tag{7.22}$$

Proof of Lemma 7.3. Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Gamma \subset V$ is a non-empty closed subset with $z \in \Gamma$. Let $A > 0$, $r_0 \in [0, A]$, $\rho > 1$ with $n \in \mathbb{Z}_{\geq 1}$ such that $\rho \in (n, n+1]$, and $\theta \in (0, n]$ with $q \in \{0, \dots, n-1\}$ such that $\theta \in (q, q+1]$. Suppose that $F \in \text{Lip}(\rho, \Gamma, W)$ satisfies that $\|F\|_{\text{Lip}(\rho, \Gamma, W)} \leq A$, and that for every $j \in \{0, \dots, n\}$, we have the bound $\|F^{(j)}(z)\|_{\mathcal{L}(V^{\otimes j}; W)} \leq r_0$. Fix $\delta \in [0, 1]$ and define $\Omega := \Gamma \cap \overline{\mathbb{B}}_V(z, \delta)$. For $s \in \{0, \dots, n-1\}$ inductively define

$$E_{n-s} := \begin{cases} \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \max \left\{ (2\delta)^{\frac{\rho-n}{2}} A, \min \{ A, A\delta^{\rho-n} + r_0 e^\delta \} \right\} & \text{if } s = 0 \\ \left(1 + \sqrt{2\delta}\right) \max \left\{ \sqrt{2\delta} E_{n-(s-1)}, \min \{ E_{n-(s-1)}, \delta E_{n-(s-1)} + r_0 e^\delta \} \right\} & \text{if } s \geq 1. \end{cases} \tag{7.23}$$

We first prove that each E_{n-s} is bounded from below by r_0 . This is the content of the following claim.

Claim 7.4. For every $s \in \{0, \dots, n-1\}$, we have

$$E_{n-s} \geq r_0. \tag{7.24}$$

Proof of Claim 7.4. The claim is proven via induction on $s \in \{0, \dots, n-1\}$. For $s = 0$, we have

$$E_n \stackrel{(7.23)}{\geq} \max \left\{ (2\delta)^{\frac{\rho-n}{2}} A, \min \{ A, A\delta^{\rho-n} + r_0 e^\delta \} \right\} \geq \min \{ A, A\delta^{\rho-n} + r_0 e^\delta \} \geq r_0, \tag{7.25}$$

where the last inequality uses that $r_0 \leq A$. If (7.24) is valid for $s \in \{0, \dots, n - 2\}$, then we compute that

$$\begin{aligned} E_{n-(s+1)} &\stackrel{(7.23)}{=} (1 + \sqrt{2\delta}) \max \left\{ \sqrt{2\delta}E_{n-s}, \min \{E_{n-s}, \delta E_{n-s} + r_0 e^\delta\} \right\} \\ &\geq \max \left\{ \sqrt{2\delta}E_{n-s}, \min \{E_{n-s}, \delta E_{n-s} + r_0 e^\delta\} \right\} \\ &\geq \min \{E_{n-s}, \delta E_{n-s} + r_0 e^\delta\} \geq r_0, \end{aligned}$$

where the last line uses that $E_{n-s} \geq r_0$ by the assumption that (7.24) is valid for s , and that $\delta E_{n-s} + r_0 e^\delta \geq r_0$ since $\delta \geq 0$. Thus, we have established that the estimate (7.24) for $s \in \{0, \dots, n - 2\}$ yields that the estimate (7.24) is true for $s + 1$. Since (7.25) establishes that (7.24) is true for $s = 0$, we may use induction to prove that (7.24) is, in fact, true for every $s \in \{0, \dots, n - 1\}$ as claimed. This completes the proof of Claim 7.4. \square

We now prove that, for each $s \in \{0, \dots, n - 1\}$, the $\text{Lip}(n - s, \Omega, W)$ -norm of $F_{[n-s-1]} = (F^{(0)}, \dots, F^{(n-s-1)})$ is bounded above by E_{n-s} . This is the content of the following claim.

Claim 7.5. For every $s \in \{0, \dots, n - 1\}$, we have that

$$\|F_{[n-(s+1)]}\|_{\text{Lip}(n-s, \Omega, W)} \leq E_{n-s}. \tag{7.26}$$

Proof of Claim 7.5. We will prove (7.26) via induction on $s \in \{0, \dots, n - 1\}$. We begin with the base case that $s = 0$. In this case, consider $\xi := \frac{\rho-n}{2} \in (0, \rho - n)$ so that $n + \xi \in (n, \rho)$ with $0 < \xi \leq 1/2$. An initial application of Lemma 7.2, with Γ, A, r_0, ρ and n here playing the same role and with the θ in Lemma 7.2 being $n + \xi$ here, yields that

$$\|F\|_{\text{Lip}(n+\xi, \Omega, W)} \leq \max \left\{ (2\delta)^{\rho-n-\xi} A, \min \{A, A\delta^{\rho-n} + r_0 e^\delta\} \right\}. \tag{7.27}$$

We next apply Lemma 6.1, with the Γ, ρ and θ of that result as $\Omega, n + \xi$ and n here, respectively, to obtain that

$$\|F_{[n-1]}\|_{\text{Lip}(n, \Omega, W)} \leq \min\{C_1, C_2\} \|F\|_{\text{Lip}(n+\xi, \Omega, W)}, \tag{7.28}$$

where (cf. (6.3) and recalling both that $\text{diam}(\Omega) \leq 2\delta \leq 2$ and that $0 < \xi \leq 1$)

$$C_1 = \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(\Omega)^\xi + \sum_{j=n}^n \frac{\text{diam}(\Omega)^{j-n}}{(j - (n - 1))!} \right\} \right\} = 1 + \text{diam}(\Omega)^\xi \leq 1 + (2\delta)^\xi, \tag{7.29}$$

and (cf. (6.4))

$$C_2 = \max \{1, \min \{1 + e, \text{diam}(\Omega)^{n-n}\}\} \left(1 + \min \{e, \text{diam}(\Omega)\}^\xi \right) (1 + \min \{e, \text{diam}(\Omega)\})^{n-n} \tag{7.30}$$

so that, since $\text{diam}(\Omega) \leq 2\delta \leq 2$ and $0 < \xi \leq 1$, we have

$$C_2 = 1 + \text{diam}(\Omega)^\xi \leq 1 + (2\delta)^\xi \tag{7.31}$$

The combination of (7.27), (7.28), (7.29), and (7.31) yields that

$$\begin{aligned} \|F_{[n-1]}\|_{\text{Lip}(n,\Omega,W)} &\leq \left(1 + (2\delta)^\xi\right) \max \left\{ (2\delta)^{\rho-n-\xi} A, \min \{A, A\delta^{\rho-n} + r_0 e^\delta\} \right\} \\ &= \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \max \left\{ (2\delta)^{\frac{\rho-n}{2}} A, \min \{A, A\delta^{\rho-n} + r_0 e^\delta\} \right\} \stackrel{(7.23)}{=} E_n. \end{aligned}$$

This completes the base case of our induction by verifying (7.26) when $s = 0$.

Now assume that $n - 1 \geq 1$, that $s \in \{1, \dots, n - 1\}$, and that (7.26) is true for $s - 1$. Consider $\xi := \frac{1}{2} \in (0, 1)$ so that $n - s + \xi \in (n - s, n - (s - 1))$. An initial application of Lemma 7.2, with Γ, A, r_0, ρ and θ of that result as $\Omega, E_{n-(s-1)}, r_0, n - (s - 1)$ and $n - s + \xi$ here, respectively, yields that

$$\|F_{[n-s]}\|_{\text{Lip}(n-s+\xi,\Omega,W)} \leq \max \left\{ (2\delta)^{1-\xi} E_{n-(s-1)}, \min \{E_{n-(s-1)}, E_{n-(s-1)}\delta + r_0 e^\delta\} \right\}. \tag{7.32}$$

We next apply Lemma 6.1, with the Γ, ρ and θ of that result as $\Omega, n - s + \xi$ and $n - s$ here, respectively, to obtain that

$$\|F_{[n-(s+1)]}\|_{\text{Lip}(n-s,\Omega,W)} \leq \min\{D_1, D_2\} \|F_{[n-s]}\|_{\text{Lip}(n-s+\xi,\Omega,W)} \tag{7.33}$$

where (cf. (6.3) and recalling that $\text{diam}(\Omega) \leq 2\delta \leq 2$ and $\xi := \frac{1}{2} \leq 1$)

$$\begin{aligned} D_1 &= \max \left\{ 1, \min \left\{ 1 + e, \text{diam}(\Omega)^\xi + \sum_{j=n-s}^{n-s} \frac{\text{diam}(\Omega)^{j-(n-s)}}{(j - (n - s - 1))!} \right\} \right\} \\ &= 1 + \text{diam}(\Omega)^\xi \leq 1 + (2\delta)^\xi, \end{aligned} \tag{7.34}$$

and (cf. (6.4))

$$D_2 = \max \left\{ 1, \min \{1 + e, \text{diam}(\Omega)^0\} \right\} \left(1 + \min \{e, \text{diam}(\Omega)\}^\xi \right) \left(1 + \min \{e, \text{diam}(\Omega)\} \right)^0 \tag{7.35}$$

so that, since $\text{diam}(\Omega) \leq 2\delta \leq 2$ and $\xi := \frac{1}{2} \leq 1$, we have

$$D_2 = 1 + \text{diam}(\Omega)^\xi \leq 1 + (2\delta)^\xi. \tag{7.36}$$

The combination of (7.32), (7.33), (7.34), and (7.36) yields that

$$\begin{aligned} \|F_{[n-(s+1)]}\|_{\text{Lip}(n-s,\Omega,W)} &\leq \left(1 + (2\delta)^\xi\right) \max \left\{ (2\delta)^{1-\xi} E_{n-(s-1)}, \min \{E_{n-(s-1)}, \delta E_{n-(s-1)} + r_0 e^\delta\} \right\} \\ &= \left(1 + \sqrt{2\delta}\right) \max \left\{ \sqrt{2\delta} E_{n-(s-1)}, \min \{E_{n-(s-1)}, \delta E_{n-(s-1)} + r_0 e^\delta\} \right\} \\ &\stackrel{(7.23)}{=} E_{n-s}. \end{aligned}$$

This completes the proof of the inductive step by establishing that if (7.26) is valid for $s - 1$ with $s \in \{1, \dots, n - 1\}$, then (7.26) is, in fact, valid for s .

Using the base case and the inductive step allows us to conclude that (7.26) is valid for every $s \in \{0, \dots, n - 1\}$. This completes the proof of Claim 7.5. \square

By appealing to Claim 7.5, we conclude that, for every $s \in \{0, \dots, n - 1\}$, we have that

$$\|F_{[n-(s+1)]}\|_{\text{Lip}(n-s,\Omega,W)} \leq E_{n-s}. \tag{7.37}$$

Let $b_q := n - (q + 1) \in \{0, \dots, n - 1\}$ so that $q + 1 = n - b_q$. Then, (7.37) for $s := b_q$ tells us that

$$\|F_{[q]}\|_{\text{Lip}(q+1,\Omega,W)} \leq E_{n-b_q}. \tag{7.38}$$

A final application of Lemma 7.2, with Γ, A, r_0, ρ and θ of that result as $\Omega, E_{n-b_q}, r_0, q + 1$ and θ here, yields that

$$\|F_{[q]}\|_{\text{Lip}(\theta,\Omega,W)} \leq \max \left\{ (2\delta)^{q+1-\theta} E_{n-b_q}, \min \left\{ E_{n-b_q}, \delta E_{n-b_q} + r_0 e^\delta \right\} \right\}, \tag{7.39}$$

which is precisely the bound claimed in (7.17).

Now suppose that $r_0 = 0$. Then, from (7.23), for $s \in \{0, \dots, n - 1\}$, we have that

$$E_{n-s} := \begin{cases} \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \max \left\{ (2\delta)^{\frac{\rho-n}{2}} A, \min \{A, \delta^{\rho-n} A\} \right\} & \text{if } s = 0 \\ \left(1 + \sqrt{2\delta}\right) \max \left\{ \sqrt{2\delta} E_{n-(s-1)}, \min \{E_{n-(s-1)}, \delta E_{n-(s-1)}\} \right\} & \text{if } s \geq 1. \end{cases} \tag{7.40}$$

Since $\delta \in [0, 1]$, we have both that $\delta \leq \sqrt{\delta} \leq 1$ and $\delta^{\rho-n} \leq \delta^{\frac{\rho-n}{2}} \leq 1$. Consequently, (7.40) yields that

$$E_{n-s} := \begin{cases} \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) (2\delta)^{\frac{\rho-n}{2}} A & \text{if } s = 0 \\ \left(1 + \sqrt{2\delta}\right) \sqrt{2\delta} E_{n-(s-1)} & \text{if } s \geq 1. \end{cases} \tag{7.41}$$

Proceeding inductively via (7.41), we establish that for any $s \in \{0, \dots, n - 1\}$, we have that

$$E_{n-s} = \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \left(1 + \sqrt{2\delta}\right)^s (2\delta)^{\frac{\rho-n+s}{2}} A. \tag{7.42}$$

Observing that $\delta \in [0, 1]$ means that $\delta \leq \delta^{q+1-\theta} \leq 1$, we may combine (7.39) and (7.42) for the choice $s := b_q = n - (q + 1)$ to obtain that

$$\|F_{[q]}\|_{\text{Lip}(\theta,\Omega,W)} \leq (2\delta)^{q+1-\theta} E_{n-b_q} \stackrel{(7.42)}{=} \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \left(1 + \sqrt{2\delta}\right)^{b_q} (2\delta)^{\frac{\rho-n+b_q}{2}+q+1-\theta} A. \tag{7.43}$$

Since $\rho - n + b_q = \rho - (q + 1)$, we see that (7.43) is precisely the estimate claimed in (7.19).

Now assume that $r_0 \in (0, A)$. We first let $\delta_* := 1$. To establish (7.20), we must reduce δ_* to a smaller constant. With the benefit of hindsight, it will suffice to reduce δ_* , depending only on A , r_0 and ρ , to ensure that whenever $\delta \in [0, \delta_*]$, we have the estimates

$$\left\{ \begin{array}{l} \text{(I)} \quad \max \left\{ 1 + \sqrt{2\delta}, 1 + (2\delta)^{\frac{\rho-n}{2}} \right\} < 2 \quad (\text{in particular } 2\delta < 1), \\ \text{(II)} \quad r_0 e^\delta \leq A(1 - \delta^{\rho-n}), \\ \text{(III)} \quad \left(2^{\frac{\rho-n}{2}} - \delta^{\frac{\rho-n}{2}} \right) \delta^{\frac{\rho-n}{2}} A \leq r_0 e^\delta, \\ \text{(IV)} \quad 2\sqrt{2\delta}(\delta^{\rho-n} A + r_0 e^\delta) \leq r_0 e^\delta \quad \text{and} \\ \text{(V)} \quad \sqrt{2\delta} \left[2^n (\delta^{\rho-n} A + r_0 \delta e^\delta) + 2r_0 e^\delta \left(\frac{1 - (2\delta)^n}{1 - 2\delta} \right) \right] \leq r_0 e^\delta. \end{array} \right. \tag{7.44}$$

We now consider a fixed choice of $\delta \in [0, \delta_*]$ and establish the estimate claimed in (7.20) for $\Omega := \overline{\mathbb{B}}_V(p, \delta) \cap \Gamma$. We begin by estimating the terms E_{n-s} for $s \in \{0, \dots, n - 1\}$. We first prove, for every $t \in \{0, \dots, n - 1\}$, that

$$E_{n-t} = \left(1 + (2\delta)^{\frac{\rho-n}{2}} \right) \left(1 + \sqrt{2\delta} \right)^t (\delta^{\rho-n+t} A + r_0 \delta^t e^\delta) + \mathbf{X}_t(\delta) \tag{7.45}$$

where $\mathbf{X}_t(\delta)$ is the quantity defined in (7.22). That is,

$$\mathbf{X}_t(\delta) := \begin{cases} 0 & \text{if } t = 0 \\ \left(1 + \sqrt{2\delta} \right) r_0 e^\delta \sum_{j=0}^{t-1} \delta^j \left(1 + \sqrt{2\delta} \right)^j & \text{if } t \geq 1. \end{cases} \tag{7.46}$$

We begin by considering $t := 0$. From (7.23), we have that

$$E_n = \left(1 + (2\delta)^{\frac{\rho-n}{2}} \right) \max \left\{ (2\delta)^{\frac{\rho-n}{2}} A, \min \{ A, \delta^{\rho-n} A + r_0 e^\delta \} \right\}. \tag{7.47}$$

A consequence of (II) in (7.44) is that $\delta^{\rho-n} A + r_0 e^\delta \leq A$ so that from (7.47), we see that

$$E_n = \left(1 + (2\delta)^{\frac{\rho-n}{2}} \right) \max \left\{ (2\delta)^{\frac{\rho-n}{2}} A, \delta^{\rho-n} A + r_0 e^\delta \right\}. \tag{7.48}$$

A consequence of (III) in (7.44) is that $(2\delta)^{\frac{\rho-n}{2}} A \leq \delta^{\rho-n} A + r_0 e^\delta$ so that from (7.48), we see that

$$E_n = \left(1 + (2\delta)^{\frac{\rho-n}{2}} \right) (\delta^{\rho-n} A + r_0 e^\delta), \tag{7.49}$$

which is the estimate claimed in (7.45) for $t = 0$ since $\mathbf{X}_0(\delta) := 0$.

Now consider $t \geq 1$ and assume that (7.45) is true for $t - 1$. From (7.23), we have that

$$E_{n-t} = \left(1 + \sqrt{2\delta} \right) \max \left\{ \sqrt{2\delta} E_{n-(t-1)}, \min \{ E_{n-(t-1)}, \delta E_{n-(t-1)} + r_0 e^\delta \} \right\}. \tag{7.50}$$

Since (7.45) is valid for $t - 1$, we have that

$$E_{n-(t-1)} = \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \left(1 + \sqrt{2\delta}\right)^{t-1} \left(\delta^{\rho-n+t-1}A + r_0\delta^{t-1}e^\delta\right) + \mathbf{X}_{t-1}(\delta), \tag{7.51}$$

We claim that $\delta E_{n-(t-1)} + r_0e^\delta \leq E_{n-(t-1)}$.

If $t = 1$, then (7.49) ensures that $(1 - \delta)E_n \geq (1 - \delta) \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) r_0e^\delta$. If we are able to conclude that $(1 - \delta) \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \geq 1$, then our desired estimate $\delta E_n + r_0e^\delta \leq E_n$ is true. The required lower bound $(1 - \delta) \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \geq 1$ is equivalent to $(2\delta)^{\frac{\rho-n}{2}} - \delta - \delta(2\delta)^{\frac{\rho-n}{2}} \geq 0$. A consequence of (I) in (7.44) is that $(2\delta)^{\frac{\rho-n}{2}} < 1$. This tells us that $\delta + \delta(2\delta)^{\frac{\rho-n}{2}} \leq 2\delta \leq (2\delta)^{\frac{\rho-n}{2}}$ where the latter inequality is true since $2\delta < 1$ and $\frac{\rho-n}{2} < 1$. Hence $(1 - \delta) \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \geq 1$, and so, we have that $\delta E_n + r_0e^\delta \leq E_n$ as required.

If $t > 1$, then (7.51) ensures that $(1 - \delta)E_{n-(t-1)} \geq (1 - \delta)\mathbf{X}_{t-1}(\delta) \geq (1 - \delta) \left(1 + \sqrt{2\delta}\right) r_0e^\delta$. If we are able to conclude that $(1 - \delta) \left(1 + \sqrt{2\delta}\right) \geq 1$, then our desired estimate $\delta E_{n-(t-1)} + r_0e^\delta \leq E_{n-(t-1)}$ is true. The required lower bound $(1 - \delta) \left(1 + \sqrt{2\delta}\right) \geq 1$ is equivalent to $\sqrt{2\delta} - \delta - \delta\sqrt{2\delta} \geq 0$. A consequence of (I) in (7.44) is that $\sqrt{2\delta} < 1$. This tells us that $\delta + \delta\sqrt{2\delta} \leq 2\delta \leq \sqrt{2\delta}$ where the latter inequality is true since $2\delta < 1$. Hence, $(1 - \delta) \left(1 + \sqrt{2\delta}\right) \geq 1$, and so, we have that $\delta E_{n-(t-1)} + r_0e^\delta \leq E_{n-(t-1)}$ as required.

Having established that $\delta E_{n-(t-1)} + r_0e^\delta \leq E_{n-(t-1)}$, (7.50) tells us that

$$E_{n-t} = \left(1 + \sqrt{2\delta}\right) \max \left\{ \sqrt{2\delta}E_{n-(t-1)}, \delta E_{n-(t-1)} + r_0e^\delta \right\}. \tag{7.52}$$

We claim that $\sqrt{2\delta}E_{n-(t-1)} \leq \delta E_{n-(t-1)} + r_0e^\delta$. If $t = 1$, then we compute, using (7.51) for $t = 1$, that

$$\begin{aligned} (\sqrt{2\delta} - \delta)E_n &= \sqrt{\delta} \left(\sqrt{2} - \sqrt{\delta}\right) \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \left(\delta^{\rho-n}A + r_0e^\delta\right) \\ &\stackrel{\text{(I) in (7.44)}}{\leq} 2\sqrt{2\delta} \left(\delta^{\rho-n}A + r_0e^\delta\right) \stackrel{\text{(IV) in (7.44)}}{\leq} r_0e^\delta. \end{aligned}$$

Consequently, we have $\sqrt{2\delta}E_n \leq \delta E_n + r_0e^\delta$ as claimed. If $t > 1$, then we compute, using (7.51) for $t > 1$, that

$$\begin{aligned} &(\sqrt{2\delta} - \delta)E_{n-(t-1)} \\ &= (\sqrt{2\delta} - \delta) \left(\left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \left(1 + \sqrt{2\delta}\right)^{t-1} \left(\delta^{\rho-n+t-1}A + r_0\delta^{t-1}e^\delta\right) + \mathbf{X}_{t-1}(\delta) \right) \\ &\stackrel{\text{(I) in (7.44)}}{\leq} 2^t \sqrt{2\delta} \left(\delta^{\rho-n+t-1}A + r_0\delta^{t-1}e^\delta\right) + \sqrt{2\delta}\mathbf{X}_{t-1}(\delta) \\ &= 2^t \sqrt{2\delta} \left(\delta^{\rho-n+t-1}A + r_0\delta^{t-1}e^\delta\right) + \sqrt{2\delta} \left(1 + \sqrt{2\delta}\right) r_0e^\delta \sum_{j=0}^{t-1} \delta^j \left(1 + \sqrt{2\delta}\right)^j \end{aligned}$$

$$\begin{aligned}
 & \stackrel{\text{(I) in (7.44)}}{\leq} 2^t \sqrt{2\delta} (\delta^{\rho-n} A + r_0 \delta e^\delta) + 2\sqrt{2\delta} r_0 e^\delta \sum_{j=0}^{t-1} (2\delta)^j \\
 & \stackrel{\text{(I) in (7.44)}}{=} \sqrt{2\delta} \left[2^t (\delta^{\rho-n} A + r_0 \delta e^\delta) + 2r_0 e^\delta \left(\frac{1 - (2\delta)^t}{1 - 2\delta} \right) \right] \\
 & \leq \sqrt{2\delta} \left[2^n (\delta^{\rho-n} A + r_0 \delta e^\delta) + 2r_0 e^\delta \left(\frac{1 - (2\delta)^n}{1 - 2\delta} \right) \right] \stackrel{\text{(IV) in (7.44)}}{\leq} r_0 e^\delta.
 \end{aligned}$$

Consequently, we have that $\sqrt{2\delta} E_{n-(t-1)} \leq \delta E_{n-(t-1)} + r_0 e^\delta$ as claimed.

Returning our attention to (7.52), the inequality $\sqrt{2\delta} E_{n-(t-1)} \leq \delta E_{n-(t-1)} + r_0 e^\delta$ means that

$$\begin{aligned}
 E_{n-t} &= (1 + \sqrt{2\delta}) (\delta E_{n-(t-1)} + r_0 e^\delta) \\
 & \stackrel{(7.51)}{=} (1 + \sqrt{2\delta}) \left((1 + (2\delta)^{\frac{\rho-n}{2}}) (1 + \sqrt{2\delta})^{t-1} (\delta^{\rho-n+t} A + r_0 \delta^t e^\delta) + \delta \mathbf{X}_{t-1}(\delta) + r_0 e^\delta \right) \\
 &= (1 + (2\delta)^{\frac{\rho-n}{2}}) (1 + \sqrt{2\delta})^t (\delta^{\rho-n+t} A + r_0 \delta^t e^\delta) + (1 + \sqrt{2\delta}) (r_0 e^\delta + \delta \mathbf{X}_{t-1}(\delta)).
 \end{aligned}$$

We observe that

$$\delta (1 + \sqrt{2\delta}) \mathbf{X}_{t-1}(\delta) \stackrel{(7.46)}{=} \begin{cases} 0 & \text{if } t - 1 = 0 \\ (1 + \sqrt{2\delta}) r_0 e^\delta \sum_{j=1}^t \delta^j (1 + \sqrt{2\delta})^j & \text{if } t - 1 \geq 1. \end{cases} \tag{7.53}$$

Via (7.53), we see that

$$\begin{aligned}
 (1 + \sqrt{2\delta}) (r_0 e^\delta + \delta \mathbf{X}_{t-1}(\delta)) &= \begin{cases} (1 + \sqrt{2\delta}) r_0 e^\delta & \text{if } t - 1 = 0 \\ (1 + \sqrt{2\delta}) r_0 e^\delta \sum_{j=0}^t \delta^j (1 + \sqrt{2\delta})^j & \text{if } t - 1 \geq 1 \end{cases} \\
 & \stackrel{(7.46)}{=} \begin{cases} \mathbf{X}_1(\delta) & \text{if } t = 1 \\ \mathbf{X}_t(\delta) & \text{if } t \geq 2 \end{cases} \\
 &= \mathbf{X}_t(\delta).
 \end{aligned}$$

Therefore, we have established that

$$E_{n-t} = (1 + (2\delta)^{\frac{\rho-n}{2}}) (1 + \sqrt{2\delta})^t (\delta^{\rho-n+t} A + r_0 \delta^t e^\delta) + \mathbf{X}_t(\delta), \tag{7.54}$$

which is the estimate claimed in (7.45) for t . Induction now allows us to conclude that the estimate (7.45) is valid for every $t \in \{0, \dots, n - 1\}$ as claimed.

To conclude, recall that $\theta \in (q, q + 1]$ and $b_q := n - (q + 1) \in \{0, \dots, n - 1\}$. Then, (7.39) yields that

$$\|F_{[q]}\|_{\text{Lip}(\theta, \Omega, W)} \leq \max \left\{ (2\delta)^{q+1-\theta} \mathcal{E}, \min \{ \mathcal{E}, \delta \mathcal{E} + r_0 e^\delta \} \right\}, \tag{7.55}$$

where, via (7.45) for $t := b_q$, $\mathcal{E} = \mathcal{E}(A, r_0, \rho, \theta, \delta) := E_{n-b_q}$, that is,

$$\mathcal{E} \stackrel{(7.45)}{=} \left(1 + (2\delta)^{\frac{\rho-n}{2}}\right) \left(1 + \sqrt{2\delta}\right)^{n-(q+1)} \left(\delta^{\rho-(q+1)} A + r_0 \delta^{n-(q+1)} e^\delta\right) + \mathbf{X}_{n-(q+1)}(\delta) \tag{7.56}$$

as claimed in (7.20) and (7.21). This completes the proof of Lemma 7.3. □

8 | PROOF OF THE POINTWISE LIPSCHITZ SANDWICH THEOREM

In this section, we use the local pointwise Lipschitz estimates established in Lemma 7.1 from Section 7 to establish the *Pointwise Lipschitz Sandwich* Theorem 3.11

Proof of Theorem 3.11. Assume that V and W are Banach spaces and that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume $\Sigma \subset V$ is a closed subset. Let $\varepsilon, \gamma > 0$ with $k \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$, $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$, and $0 \leq \varepsilon_0 < \min\{K_1 + K_2, \varepsilon\}$. Let $l \in \{0, \dots, k\}$ and define $\delta_0 = \delta_0(\varepsilon, \varepsilon_0, K_1 + K_2, \gamma, l) > 0$ by

$$\delta_0 := \sup \left\{ \theta > 0 : (K_1 + K_2)\theta^{\gamma-l} + \varepsilon_0 e^\theta \leq \min\{K_1 + K_2, \varepsilon\} \right\}. \tag{8.1}$$

A first consequence of (8.1) is that $\delta_0 \leq 1$, and so, for every $s \in \{0, \dots, l\}$, we have that $\delta_0^{\gamma-s} \leq \delta_0^{\gamma-l}$. A second consequence of (8.1) is that

$$(K_1 + K_2)\delta_0^{\gamma-l} + \varepsilon_0 e^{\delta_0} \leq \min\{K_1 + K_2, \varepsilon\} \leq \varepsilon. \tag{8.2}$$

Now assume that $B \subset \Sigma$ is a δ_0 -cover of Σ in the sense that

$$\Sigma \subset \bigcup_{x \in B} \overline{\mathbb{B}}_V(x, \delta_0) = B_{\delta_0} := \{v \in V : \exists z \in B \text{ such that } \|z - v\|_V \leq \delta_0\}. \tag{8.3}$$

Suppose $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$, $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfy the $\text{Lip}(\gamma, \Sigma, W)$ norm estimates $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Further suppose that for every $j \in \{0, \dots, k\}$ and every $x \in B$, the difference $\psi^{(j)}(x) - \varphi^{(j)}(x) \in \mathcal{L}(V^{\otimes j}; W)$ satisfies the bound

$$\left\| \psi^{(j)}(x) - \varphi^{(j)}(x) \right\|_{\mathcal{L}(V^{\otimes j}; W)} \leq \varepsilon_0. \tag{8.4}$$

Define $F \in \text{Lip}(\gamma, \Sigma, W)$ by $F := \psi - \varphi$ so that for every $j \in \{0, \dots, k\}$, we have $F^{(j)} := \psi^{(j)} - \varphi^{(j)}$. Then, $\|F\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1 + K_2$ and, for every integer $j \in \{0, \dots, k\}$ and every point $x \in B$, (8.4) gives that $\left\| F^{(j)}(x) \right\|_{\mathcal{L}(V^{\otimes j}; W)} \leq \varepsilon_0$.

Now fix $x \in \Sigma$ and $s \in \{0, \dots, l\}$. From (8.3), we conclude that there exists a point $z \in B$ with $\|z - x\|_V \leq \delta_0$. Then, apply Lemma 7.1, with $A := K_1 + K_2$, $r_0 := \varepsilon_0$, $\rho := \gamma$, $p := z$, $n := k$ and $q := k$, to conclude that (cf. (7.3))

$$\begin{aligned} \left\| F^{(s)}(x) \right\|_{\mathcal{L}(V^{\otimes s}; W)} &\leq \min \left\{ K_1 + K_2, (K_1 + K_2)\delta_0^{\gamma-s} + \varepsilon_0 \sum_{j=0}^{k-s} \frac{1}{j!} \delta_0^j \right\} \\ &\leq \min \left\{ K_1 + K_2, (K_1 + K_2)\delta_0^{\gamma-l} + \varepsilon_0 e^{\delta_0} \right\} \stackrel{(8.2)}{\leq} \varepsilon, \end{aligned} \tag{8.5}$$

where we have used that $\delta_0^{\gamma-s} \leq \delta_0^{\gamma-l}$. Since $F = \psi - \varphi$, the arbitrariness of $s \in \{0, \dots, l\}$ and of $x \in \Sigma$ ensure that (8.5) gives the bounds claimed in (3.9) This completes the proof of Theorem 3.11. \square

9 | PROOF OF THE SINGLE-POINT LIPSCHITZ SANDWICH THEOREM

In this section, we prove the *Single-Point Lipschitz Sandwich Theorem 3.8*. Our approach is to alter the constant δ_0 appearing in the *Pointwise Lipschitz Sandwich Theorem 3.11* in order to strengthen the conclusions to an estimate on the full $\text{Lip}(\eta)$ -norm of the difference.

To be more precise, recall that $\Sigma \subset V$ is closed and $\gamma > 0$ with $k \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$. Let $\eta \in (0, \gamma)$, $\varepsilon > 0$, $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$, and $0 \leq \varepsilon_0 < \min\{K_1 + K_2, \varepsilon\}$. Retrieve the constant $\delta_0 = \delta_0(K_1 + K_2, \varepsilon, \varepsilon_0, \gamma) > 0$ arising in Theorem 3.11 for the choice $l := k$. Given a point $p \in \Sigma$, define $\Omega := \Sigma \cap \overline{\mathbb{B}}_V(p, \delta_0)$.

Suppose $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$, $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfy the norm bounds $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Further suppose that for every $j \in \{0, \dots, k\}$, the difference $\psi^{(j)}(p) - \varphi^{(j)}(p) \in \mathcal{L}(V^{\otimes j}; W)$ satisfies $\|\psi^{(j)}(p) - \varphi^{(j)}(p)\|_{\mathcal{L}(V^{\otimes j}; W)} \leq \varepsilon_0$. Then, by applying Theorem 3.11, for the choices l, Σ and B , there as k, Ω and $\{p\}$ here, respectively, we may conclude that for every $s \in \{0, \dots, k\}$ and every $x \in \Omega$, we have $\|\psi^{(s)}(x) - \varphi^{(s)}(x)\|_{\mathcal{L}(V^{\otimes s}; W)} \leq \varepsilon$.

We will prove the *Single-Point Lipschitz Sandwich Theorem 3.8*, by establishing that after reducing the constant δ_0 , allowing it to additionally depend on η , we may strengthen these pointwise bounds into a bound on the full $\text{Lip}(\eta, \Omega, W)$ norm of $\psi - \varphi$. We do so by appealing to the local Lipschitz estimates established in Lemmas 7.2 and 7.3 in Section 7.

There is a natural dichotomy within this strategy between the case that $\eta \in (k, \gamma)$ and the case that $\eta \in (0, k]$. We first use Lemma 7.2 to establish the *Single-Point Lipschitz Sandwich Theorem 3.8* in the simpler case that $\eta \in (k, \gamma)$.

Proof of Theorem 3.8 (when $\eta \in (k, \gamma)$). Assume that V and W are Banach spaces and that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Let $\Sigma \subset V$ be non-empty and closed. Let $\varepsilon, \gamma > 0$ with $k \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$, $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$, and $0 \leq \varepsilon_0 < \min\{K_1 + K_2, \varepsilon\}$, and $\eta \in (k, \gamma)$. For notational convenience, we let $K_0 := K_1 + K_2$. With a view to later applying Lemma 7.2, define $\delta_0 = \delta_0(K_0, \varepsilon, \varepsilon_0, \gamma) > 0$ by

$$\delta_0 := \sup \left\{ \theta \in (0, 1] : K_0(2\theta)^{\gamma-\eta} \leq \min\{K_0, \varepsilon\} \text{ and } K_0\theta^{\gamma-k} + \varepsilon_0e^\theta \leq \min\{K_0, \varepsilon\} \right\} > 0. \tag{9.1}$$

It initially appears that δ_0 additionally depends on k . However, k is determined by γ , thus any dependence on k is really dependence on γ . We now fix the value of $\delta_0 > 0$ for the remainder of the proof. We record that (9.1) ensures that $\delta_0 \leq 1$ and

$$\text{(I)} \quad K_0(2\delta_0)^{\gamma-\eta} \leq \min\{K_0, \varepsilon\} \quad \text{and} \quad \text{(II)} \quad K_0\delta_0^{\gamma-k} + \varepsilon_0e^{\delta_0} \leq \min\{K_0, \varepsilon\}. \tag{9.2}$$

Now assume that $p \in \Sigma$ and that $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$ and $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)})$ are elements in $\text{Lip}(\gamma, \Sigma, W)$ with $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Further suppose that for every $l \in \{0, \dots, k\}$, the difference $\psi^{(l)}(p) - \varphi^{(l)}(p) \in \mathcal{L}(V^{\otimes l}; W)$ satisfies the bound

$$\left\| \psi^{(l)}(p) - \varphi^{(l)}(p) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \varepsilon_0. \tag{9.3}$$

Define $\Omega := \overline{\mathbb{B}}_V(p, \delta_0) \cap \Sigma$ and $F \in \text{Lip}(\gamma, \Sigma, W)$ by $F := \psi - \varphi$, so that for every $j \in \{0, \dots, k\}$, we have $F^{(j)} = \psi^{(j)} - \varphi^{(j)}$. We apply Lemma 7.2 to F , with $A := K_0 = K_1 + K_2$, $r_0 := \varepsilon_0$, $\rho := \gamma$, $n := k$, $z := p$, $\theta := \eta$ and $\delta := \delta_0$, to conclude both that $F \in \text{Lip}(\eta, \Omega, W)$ and (cf. (7.11)) that

$$\|F\|_{\text{Lip}(\eta, \Omega, W)} \leq \max \left\{ K_0(2\delta_0)^{\gamma-\eta}, \min \left\{ K_0, K_0\delta_0^{\gamma-k} + \varepsilon_0 e^{\delta_0} \right\} \right\} \stackrel{\text{(II) of (9.2)}}{\leq} \max \{K_0(2\delta_0)^{\gamma-\eta}, \varepsilon\} \stackrel{\text{(I) of (9.2)}}{\leq} \max \{\varepsilon, \varepsilon\} = \varepsilon.$$

Since $F = \psi - \varphi$ and $\Omega := \Sigma \cap \overline{\mathbb{B}}_V(p, \delta_0)$, this is precisely the estimate claimed in (3.5). This completes the proof of Theorem 3.8 for $\eta \in (k, \gamma)$. □

We now turn our attention to using Lemma 7.3 to establish the *Single-Point Lipschitz Sandwich Theorem 3.8* in the more challenging case that $\eta \in (0, k]$.

Proof of Theorem 3.8 (for $0 < \eta \leq k$). Assume that V and W are Banach spaces and that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Let $\Sigma \subset V$ be closed and non-empty. Let $\varepsilon, \gamma > 0$ with $k \in \mathbb{Z}_{\geq 1}$ such that $\gamma \in (k, k + 1]$, $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$ and $\eta \in (0, k]$. Observe that this requires $1 \leq k < \gamma$. Let $q \in \{0, \dots, k - 1\}$ such that $\eta \in (q, q + 1] \subset (0, k]$. Finally, let $0 \leq \varepsilon_0 < \min \{K_1 + K_2, \varepsilon\}$. For notational convenience, we set $K_0 := K_1 + K_2 > 0$ for the remainder of the proof

Our strategy is to establish the desired $\text{Lip}(\eta)$ -norm bounds via an application of Lemma 7.3. For this purpose, we retrieve the constant δ_* arising in Lemma 7.3 for $A := K_0 = K_1 + K_2$, $r_0 := \varepsilon_0$, $\rho := \gamma$ and $\theta := \eta$. Note that we are not actually applying Lemma 7.3, but simply retrieving a constant in preparation for its future application.

Let $\delta_0 := \min \{1, \delta_*\} > 0$, which depends only on $K_0 = K_1 + K_2, \varepsilon_0, \gamma$ and η . In order to ensure that applying Lemma 7.3 yields the desired $\text{Lip}(\eta)$ -norm estimate, we will allow ourselves to (potentially) further reduce δ_0 , additionally now depending on ε . With the benefit of hindsight, it will suffice to alter δ_0 to ensure that

$$\left\{ \begin{array}{l} \text{(A)} \quad \max \left\{ 1 + (2\delta_0)^{\frac{\gamma-k}{2}}, 1 + \sqrt{2\delta_0} \right\} < 2, \quad (\text{In particular, } 2\delta_0 < 1), \\ \text{(B)} \quad (2\delta_0)^{\frac{\gamma-\eta}{2} + \frac{q+1-\eta}{2}} \leq \frac{\varepsilon}{2^{k-q}K_0}, \\ \text{(C)} \quad \left(1 + (2\delta_0)^{\frac{\gamma-k}{2}} \right) \left(\delta_0^{\gamma-k} K_0 + \varepsilon_0 e^{\delta_0} \right) \leq \varepsilon, \\ \text{(D)} \quad 2^{k-q} \left(\delta_0^{\gamma-k} K_0 + \varepsilon_0 \delta_0 e^{\delta_0} \right) + \left(\frac{1 + \sqrt{2\delta_0}}{1 - 2\delta_0} \right) \varepsilon_0 e^{\delta_0} \leq \varepsilon \quad \text{and} \\ \text{(E)} \quad \varepsilon_0 e^{\delta_0} \leq (1 - \delta_0)\varepsilon. \end{array} \right. \tag{9.4}$$

We now fix the value of $\delta_0 = \delta_0(K_0, \gamma, \eta, \varepsilon_0, \varepsilon) > 0$ for the remainder of the proof. Since $K_0 := K_1 + K_2$, we note that δ_0 has the claimed dependencies.

Now let $p \in \Sigma$ and assume that $\psi = (\psi^{(0)}, \dots, \psi^{(k)})$ and $\varphi = (\varphi^{(0)}, \dots, \varphi^{(k)})$ are in $\text{Lip}(\gamma, \Sigma, W)$ with $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Suppose that for every integer $l \in \{0, \dots, k\}$, the difference $\psi^{(l)}(p) - \varphi^{(l)}(p) \in \mathcal{L}(V^{\otimes l}, W)$ satisfies the bound

$$\left\| \psi^{(l)}(p) - \varphi^{(l)}(p) \right\|_{\mathcal{L}(V^{\otimes l}, W)} \leq \varepsilon_0. \tag{9.5}$$

Define $\Omega := \overline{\mathbb{B}}_V(p, \delta_0) \cap \Sigma$ and $F \in \text{Lip}(\gamma, \Sigma, W)$ by $F := \psi - \varphi$ so that for every $j \in \{0, \dots, k\}$, we have $F^{(j)} = \psi^{(j)} - \varphi^{(j)}$.

We begin with the case that $\varepsilon_0 = 0$. Since $\delta_0 \leq 1$, the bounds (9.5) allow us to apply Lemma 7.3 to F , with $A := K_0 (= K_1 + K_2)$, $r_0 := \varepsilon_0$, $\rho := \gamma$, $\theta := \eta$ and $\delta := \delta_0$, to conclude that (cf. (7.19))

$$\left\| F_{[q]} \right\|_{\text{Lip}(\eta, \Omega, W)} \leq \left(1 + (2\delta_0)^{\frac{\gamma-k}{2}} \right) \left(1 + \sqrt{2\delta_0} \right)^{k-(q+1)} (2\delta_0)^{\frac{\gamma-\eta}{2} + \frac{q+1-\eta}{2}} K_0. \tag{9.6}$$

We compute that

$$\begin{aligned} \left\| F_{[q]} \right\|_{\text{Lip}(\eta, \Omega, W)} &\stackrel{(9.6)}{\leq} \left(1 + (2\delta_0)^{\frac{\gamma-k}{2}} \right) \left(1 + \sqrt{2\delta_0} \right)^{k-(q+1)} (2\delta_0)^{\frac{\gamma-\eta}{2} + \frac{q+1-\eta}{2}} K_0 \\ &\stackrel{\text{(A) in (9.4)}}{\leq} 2^{k-q} (2\delta_0)^{\frac{\gamma-\eta}{2} + \frac{q+1-\eta}{2}} K_0 \stackrel{\text{(B) in (9.4)}}{\leq} 2^{k-q} \frac{\varepsilon}{2^{k-q} K_0} K_0 = \varepsilon. \end{aligned}$$

Recalling that $\Omega := \Sigma \cap \overline{\mathbb{B}}_V(p, \delta_0)$ and $F := \psi - \varphi$, this is precisely the estimate claimed in (3.5), and our proof is complete for the case that $\varepsilon_0 = 0$.

Now consider the case that $\varepsilon_0 > 0$. Recalling how we chose δ_0 , the bounds (9.5) allow us to apply Lemma 7.3 to F , with $A := K_0 (= K_1 + K_2)$, $r_0 := \varepsilon_0$, $\rho := \gamma$, $\theta := \eta$ and $\delta := \delta_0$, to conclude via (7.20) that

$$\left\| F_{[q]} \right\|_{\text{Lip}(\eta, \Omega, W)} \leq \max \{ (2\delta_0)^{q+1-\eta} \mathcal{E}, \min \{ \mathcal{E}, \delta_0 \mathcal{E} + \varepsilon_0 e^{\delta_0} \} \} \tag{9.7}$$

for $\mathcal{E} = \mathcal{E}(K_0, \gamma, \eta, \varepsilon_0) > 0$ defined by (cf. (7.21))

$$\mathcal{E} := \left(1 + (2\delta_0)^{\frac{\gamma-k}{2}} \right) \left(1 + \sqrt{2\delta_0} \right)^{k-(q+1)} \left(\delta_0^{\gamma-(q+1)} K_0 + \varepsilon_0 \delta_0^{k-(q+1)} e^{\delta_0} \right) + \mathbf{X}_{k-(q+1)}(\delta_0) \tag{9.8}$$

where, for $t \in \{0, \dots, k-1\}$, the quantity $\mathbf{X}_t(\delta)$ is defined by (cf. (7.22))

$$\mathbf{X}_t(\delta) := \begin{cases} 0 & \text{if } t = 0 \\ \left(1 + \sqrt{2\delta_0} \right) \varepsilon_0 e^{\delta_0} \sum_{j=0}^{t-1} \delta_0^j \left(1 + \sqrt{2\delta_0} \right)^j & \text{if } t \geq 1. \end{cases} \tag{9.9}$$

We first prove that $(1 - \delta_0) \mathcal{E} \geq \varepsilon_0 e^{\delta_0}$.

If $k = q + 1$, then (9.8) ensures that $(1 - \delta_0) \mathcal{E} \geq (1 - \delta_0) \left(1 + (2\delta_0)^{\frac{\gamma-k}{2}} \right) \varepsilon_0 e^{\delta_0}$. If we are able to conclude that $(1 - \delta_0) \left(1 + (2\delta_0)^{\frac{\gamma-k}{2}} \right) \geq 1$, then our desired estimate $(1 - \delta_0) \mathcal{E} \geq \varepsilon_0 e^{\delta_0}$ is true. The required lower bound $(1 - \delta_0) \left(1 + (2\delta_0)^{\frac{\gamma-k}{2}} \right) \geq 1$ is equivalent to $(2\delta_0)^{\frac{\gamma-k}{2}} - \delta_0 - \delta_0 (2\delta_0)^{\frac{\gamma-k}{2}} \geq 0$.

A consequence of **(A)** in (9.4) is that $(2\delta_0)^{\frac{\gamma-k}{2}} < 1$. This tells us that $\delta_0 + \delta_0(2\delta_0)^{\frac{\gamma-k}{2}} \leq 2\delta_0 \leq (2\delta_0)^{\frac{\gamma-k}{2}}$ where the latter inequality is true since $2\delta_0 < 1$ and $\frac{\gamma-k}{2} < 1$. Hence, $(1 - \delta_0)(1 + (2\delta_0)^{\frac{\gamma-k}{2}}) \geq 1$, and so, we have $(1 - \delta_0)\mathcal{E} \geq \varepsilon_0 e^{\delta_0}$ as required.

If $k > q + 1$, then (9.8) and (9.9) yield $(1 - \delta_0)\mathcal{E} \geq (1 - \delta_0)\mathbf{X}_{k-(q+1)}(\delta_0) \geq (1 - \delta_0)(1 + \sqrt{2\delta_0})\varepsilon_0 e^{\delta_0}$. If we are able to conclude that $(1 - \delta_0)(1 + \sqrt{2\delta_0}) \geq 1$, then our desired estimate $(1 - \delta_0)\mathcal{E} \geq \varepsilon_0 e^{\delta_0}$ is true. The required lower bound $(1 - \delta_0)(1 + \sqrt{2\delta_0}) \geq 1$ is equivalent to $\sqrt{2\delta_0} - \delta_0 - \delta_0\sqrt{2\delta_0} \geq 0$. A consequence of **(A)** in (9.4) is that $\sqrt{2\delta_0} < 1$. This tells us that $\delta_0 + \delta_0\sqrt{2\delta_0} \leq 2\delta_0 \leq \sqrt{2\delta_0}$ where the latter inequality is true since $2\delta_0 < 1$. Hence, $(1 - \delta_0)(1 + \sqrt{2\delta_0}) \geq 1$, and so, we have $(1 - \delta_0)\mathcal{E} \geq \varepsilon_0 e^{\delta_0}$ as required.

Having established that $(1 - \delta_0)\mathcal{E} \geq \varepsilon_0 e^{\delta_0}$, we observe that (9.7) becomes

$$\|F_{[q]}\|_{\text{Lip}(\eta, \Omega, W)} \leq \max \{ (2\delta_0)^{q+1-\eta} \mathcal{E}, \delta_0 \mathcal{E} + \varepsilon_0 e^{\delta_0} \}, \tag{9.10}$$

We now prove the upper bound for \mathcal{E} that $\mathcal{E} \leq \varepsilon$. For this purpose, note that when $k = q + 1$ (9.8) yields that

$$\mathcal{E} = \left(1 + (2\delta_0)^{\frac{\gamma-k}{2}} \right) \left(\delta_0^{\gamma-k} K_0 + \varepsilon_0 e^{\delta_0} \right) \stackrel{\text{(C) in (9.4)}}{\leq} \varepsilon \tag{9.11}$$

since $\mathbf{X}_0(\delta_0) = 0$ from (9.9). If, however, $k > q + 1$, then $k - (q + 1) \geq 1$ and so, recalling that **(A)** in (9.4) ensures that $\delta_0(1 + \sqrt{2\delta_0}) < 2\delta_0 < 1$, we have

$$\mathbf{X}_{k-(q+1)}(\delta_0) \stackrel{\text{(9.9)}}{=} (1 + \sqrt{2\delta_0})\varepsilon_0 e^{\delta_0} \sum_{j=0}^{k-(q+1)} \delta_0^j (1 + \sqrt{2\delta_0})^j \leq \frac{(1 + \sqrt{2\delta_0})}{1 - 2\delta_0} \varepsilon_0 e^{\delta_0}. \tag{9.12}$$

Moreover, $2\delta_0 < 1$ ensures that $\delta_0^{\gamma-(q+1)} \leq \delta_0^{\gamma-k}$ and $\delta_0^{k-(q+1)} \leq \delta_0$. Hence, we can combine (9.8) and (9.12) to obtain that

$$\mathcal{E} \leq 2^{k-q} \left(\delta_0^{\gamma-k} K_0 + \varepsilon_0 \delta_0 e^{\delta_0} \right) + \frac{(1 + \sqrt{2\delta_0})}{1 - 2\delta_0} \varepsilon_0 e^{\delta_0} \stackrel{\text{(D) in (9.4)}}{\leq} \varepsilon.$$

Therefore, in both the case that $k = q + 1$ and the case that $k > q + 1$, we obtain that

$$\mathcal{E} \leq \varepsilon. \tag{9.13}$$

We complete the proof by using the upper bound in (9.13) to control $\|F_{[q]}\|_{\text{Lip}(\eta, \Omega, W)}$. Recalling that **(A)** in (9.4) means that $2\delta_0 < 1$, we have that

$$\text{(I)} \quad (2\delta_0)^{q+1-\eta} \mathcal{E} \leq \mathcal{E} \stackrel{\text{(9.13)}}{\leq} \varepsilon \quad \text{and} \quad \text{(II)} \quad \delta_0 \mathcal{E} + \varepsilon_0 e^{\delta_0} \stackrel{\text{(9.13)}}{\leq} \delta_0 \varepsilon + \varepsilon_0 e^{\delta_0} \stackrel{\text{(E) in (9.4)}}{\leq} \varepsilon. \tag{9.14}$$

Thus,

$$\|F_{[q]}\|_{\text{Lip}(\eta, \Omega, W)} \stackrel{\text{(9.10)}}{\leq} \max \{ (2\delta_0)^{q+1-\eta} \mathcal{E}, \delta_0 \mathcal{E} + \varepsilon_0 e^{\delta_0} \} \stackrel{\text{(I) in (9.14)}}{\leq} \max \{ \varepsilon, \delta_0 \varepsilon + \varepsilon_0 e^{\delta_0} \} \stackrel{\text{(II) in (9.14)}}{=} \varepsilon. \tag{9.15}$$

Recalling that $\Omega := \Sigma \cap \overline{B}_V(p, \delta_0)$ and $F := \psi - \varphi$, (9.15) is precisely the estimate claimed in (3.5), and our proof is complete for the case that $\varepsilon_0 > 0$. Having already established the conclusion for the case that $\varepsilon_0 = 0$, this completes the proof of Theorem 3.8 for the case that $\eta \in (0, k]$. \square

10 | PROOF OF THE LIPSCHITZ SANDWICH THEOREM 3.1

In this section, we establish the full *Lipschitz Sandwich Theorem 3.1*. Our strategy to prove this result is to patch together the local Lipschitz bounds achieved by the *Single-Point Lipschitz Sandwich Theorem 3.8* in a similar spirit to the patching of local Lipschitz bounds in Lemma 1.16 in [1]. We do not necessarily have local Lipschitz bounds on a small ball centred at any point in Σ ; we only have such estimates for points in the closed subset $B \subset \Sigma$, and we do not require that $B = \Sigma$. Consequently, our patching is more complicated than the patching used in Lemma 1.16 in [1].

To be more precise, recall that $\Sigma \subset V$ is closed and $\gamma > 0$ with $k \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$. Let $\eta \in (0, \gamma)$, $\varepsilon > 0$, $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$, and $0 \leq \varepsilon_0 < \min\{K_1 + K_2, \varepsilon\}$. Retrieve the constant $\delta_0 = \delta_0(K_1 + K_2, \varepsilon, \varepsilon_0, \gamma, \eta) > 0$ arising in the *Single-Point Lipschitz Sandwich Theorem 3.8*. Assume that $B \subset \Sigma$ is a δ_0 -cover of Σ in the sense that the δ_0 -fattening of B contains Σ .

Suppose $\psi = (\psi^{(0)}, \dots, \psi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ and $\varphi \in (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ both satisfy the norm bounds $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Further suppose that for every $j \in \{0, \dots, k\}$ and every $x \in B$, the difference $\psi^{(j)}(x) - \varphi^{(j)}(x) \in \mathcal{L}(V^{\otimes j}; W)$ satisfies $\|\psi^{(j)}(x) - \varphi^{(j)}(x)\|_{\mathcal{L}(V^{\otimes j}; W)} \leq \varepsilon_0$. Then, given any point $p \in B$, we can apply the *Single-Point Lipschitz Sandwich Theorem 3.8* to conclude that $\|\psi_{[q]} - \varphi_{[q]}\|_{\text{Lip}(\eta, \Omega_p, W)} \leq \varepsilon$ for $\Omega_p := \Sigma \cap \overline{B}_V(p, \delta_0)$ and $q \in \{0, \dots, k\}$ such that $\eta \in (q, q + 1]$.

It may initially appear that since $\Sigma = \cup_{p \in B} \Omega_p$, these local $\text{Lip}(\eta)$ -norm bounds should combine together to yield $\|\psi_{[q]} - \varphi_{[q]}\|_{\text{Lip}(\eta, \Sigma, W)} \leq \varepsilon$. However, this is not necessarily true. For example, given any $\alpha \in (0, 1)$, consider the function $F : [0, 1] \cup [1 + \alpha, 2] \rightarrow \mathbb{R}$ defined by $F(x) := 0$ if $x \in [0, 1]$ and $F(x) := \alpha$ if $x \in [1 + \alpha, 2]$. Then, $F \in \text{Lip}(1, [0, 1] \cup [1 + \alpha, 2], \mathbb{R})$ and we have that $\|F\|_{\text{Lip}(1, [0, 1], \mathbb{R})} = 0$ and $\|F\|_{\text{Lip}(1, [1 + \alpha, 2], \mathbb{R})} = \alpha$. But $|F(1 + \alpha) - F(1)| = \alpha = |1 + \alpha - 1|$ and so $\|F\|_{\text{Lip}(1, [0, 1] \cup [1 + \alpha, 2], \mathbb{R})} = 1 > \alpha$.

The main content of our proof of the *Lipschitz Sandwich Theorem 3.1* is to overcome this problem. We prove that, by requiring the constant ε_0 to be sufficiently small, depending only on $\varepsilon, K_1 + K_2, \gamma$ and η , rather than an arbitrary real number in the interval $[0, \min\{K_1 + K_2, \varepsilon\})$, we can patch together local Lipschitz estimates resulting from an application of the *Single-Point Lipschitz Sandwich Theorem 3.8* to yield global Lipschitz estimates throughout Σ . A key point is to ensure that the sets Ω_p on which the *Single-Point Lipschitz Sandwich Theorem 3.8* yields local Lipschitz estimates are not pairwise disjoint; that is, for each $p \in B$, there must be some $q \in B \setminus \{p\}$ such that the intersection $\Omega_p \cap \Omega_q$ is non-empty.

Proof of Theorem 3.1. Let V and W be Banach spaces, and assume that the tensor powers of V are all equipped with admissible norms (cf. Definition 2.1). Assume that $\Sigma \subset V$ is non-empty and closed. Let $\varepsilon, \gamma > 0$ with $k \in \mathbb{Z}_{\geq 0}$ such that $\gamma \in (k, k + 1]$, and $(K_1, K_2) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \setminus \{(0, 0)\}$. Further let $\eta \in (0, \gamma)$ with $q \in \{0, \dots, k\}$ such that $\eta \in (q, q + 1]$. It suffices to prove the theorem under the additional assumption that $\varepsilon \leq K_0 := K_1 + K_2$; the conclusion (3.3) being valid for ε immediately means it is also valid for any constant $\varepsilon' \geq \varepsilon$.

Define $\theta := \frac{1}{2(1+e)} > 0$ and retrieve the constant $\delta_0 > 0$ arising from Theorem 3.8 for the same constants K_1, K_2, γ and η as here, respectively, and with the choices of $\theta\varepsilon$ and $\frac{\theta}{2}\varepsilon$ here as the constants ε and ε_0 in Theorem 3.8, respectively. Note that we are not actually applying Theorem 3.8, but simply retrieving a constant in preparation for its future application. Examining the dependencies in Theorem 3.8 reveals that $\delta_0 > 0$ depends only on ε, K_0, γ and η . If necessary, we reduce δ_0 , without additional dependencies, so that $\delta_0 \leq 1$. Finally, we replace the resulting constant δ_0 by $\delta_0/2$.

Our choice of $\delta_0 > 0$ means that if $\psi = (\psi^{(0)}, \dots, \psi^{(k)}), \varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ with $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$, and if for a point $p \in \Sigma$ and every $l \in \{0, \dots, k\}$, we have the estimate $\|\psi^{(l)}(p) - \varphi^{(l)}(p)\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \frac{\theta}{2}\varepsilon$, then an application of Theorem 3.8 would allow us to conclude the estimate that $\|\psi_{[q]} - \varphi_{[q]}\|_{\text{Lip}(\eta, \Omega_p, W)} \leq \theta\varepsilon$ for $\Omega_p := \bar{\mathbb{B}}_V(p, 2\delta_0) \cap \Sigma$.

We now fix the value of $\delta_0 > 0$ for the remainder of the proof. Having done so, we define $\varepsilon_0 > 0$ by

$$\varepsilon_0 := \min \left\{ \theta, \frac{\delta_0^\eta}{e^{\delta_0}(1 + e^{\delta_0})} \right\} \frac{\varepsilon}{2} > 0. \tag{10.1}$$

Examining the dependencies in (10.1) reveals that ε_0 depends only on $\varepsilon, K_0 = K_1 + K_2, \gamma$ and η . We may now fix the value of $\varepsilon_0 > 0$ for the remainder of the proof.

Let $B \subset \Sigma$ satisfy that

$$\Sigma \subset \bigcup_{x \in B} \bar{\mathbb{B}}_V(x, \delta_0). \tag{10.2}$$

Suppose $\psi = (\psi^{(0)}, \dots, \psi^{(k)}), \varphi = (\varphi^{(0)}, \dots, \varphi^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfy the $\text{Lip}(\gamma, \Sigma, W)$ norm estimates $\|\psi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_1$ and $\|\varphi\|_{\text{Lip}(\gamma, \Sigma, W)} \leq K_2$. Further assume that whenever $l \in \{0, \dots, k\}$ and $x \in B$, we have the estimate $\|\psi^{(l)}(x) - \varphi^{(l)}(x)\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \varepsilon_0$. Let $p \in B$. Recalling how we chose the constant $\delta_0 > 0$ and that (10.1) means that $\varepsilon_0 \leq \frac{\theta}{2}\varepsilon$, we may appeal to Theorem 3.8 to conclude that

$$\|\psi_{[q]} - \varphi_{[q]}\|_{\text{Lip}(\eta, \Omega_p, W)} \leq \theta\varepsilon, \tag{10.3}$$

where $\Omega_p := \Sigma \cap \bar{\mathbb{B}}_V(p, 2\delta_0)$. The arbitrariness of $p \in B$ allows us to conclude that the estimate (10.3) is valid for every $p \in B$.

We complete the proof of Theorem 3.1 by establishing that having the bounds (10.3) for every $p \in B$ allows us to conclude that $\|\psi_{[q]} - \varphi_{[q]}\|_{\text{Lip}(\eta, \Sigma, W)} \leq \varepsilon$. This is proven in the following claim.

Claim 10.1. If $F = (F^{(0)}, \dots, F^{(k)}) \in \text{Lip}(\gamma, \Sigma, W)$ satisfies, for every $l \in \{0, \dots, k\}$ and every $z \in B$, that $\|F^{(l)}(z)\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \varepsilon_0$ and $\|F_{[q]}\|_{\text{Lip}(\eta, \Omega_z, W)} \leq \theta\varepsilon$, where $\Omega_z := \Sigma \cap \bar{\mathbb{B}}_V(z, 2\delta_0)$, then we have

$$\|F_{[q]}\|_{\text{Lip}(\eta, \Sigma, W)} \leq \varepsilon. \tag{10.4}$$

Proof of Claim 10.1. For each $l \in \{0, \dots, k\}$, let $R_l^F : \Sigma \times \Sigma \rightarrow \mathcal{L}(V^{\otimes l}; W)$ denote the remainder term associated to $F^{(l)}$. Therefore, whenever $l \in \{0, \dots, k\}$, $x, y \in \Sigma$ and $v \in V^{\otimes l}$, we have that (cf. (2.9))

$$R_l^F(x, y)[v] := F^{(l)}(y)[v] - \sum_{s=0}^{k-l} \frac{1}{s!} F^{(l+s)}(x)[v \otimes (y - x)^{\otimes s}]. \tag{10.5}$$

If $q = k$, then we may work with the unaltered remainder terms defined in (10.5). But if $q < k$, then we must first appropriately alter the remainder terms. For this purpose, for each $l \in \{0, \dots, q\}$, we define $\hat{R}_l^F : \Sigma \times \Sigma \rightarrow \mathcal{L}(V^{\otimes l}; W)$ for $x, y \in \Sigma$ and $v \in V^{\otimes l}$ by

$$\hat{R}_l^F(x, y)[v] := \begin{cases} R_l^F(x, y)[v] & \text{if } q = k \\ R_l^F(x, y)[v] + \sum_{s=q+1-l}^{k-l} \frac{1}{s!} F^{(l+s)}(x)[v \otimes (y - x)^{\otimes s}] & \text{if } q < k. \end{cases} \tag{10.6}$$

It follows from (10.5) and (10.6) that whenever $l \in \{0, \dots, q\}$, $x, y \in \Sigma$ and $v \in V^{\otimes l}$, we have

$$F^{(l)}(y)[v] = \sum_{s=0}^{q-l} F^{(l+s)}(x)[v \otimes (y - x)^{\otimes s}] + \hat{R}_l^F(x, y)[v]. \tag{10.7}$$

For each $z \in B$, the assumption that $\|F_{[q]}\|_{\text{Lip}(\eta, \Omega_z, W)} \leq \theta\varepsilon$ for $\Omega_z := \Sigma \cap \bar{\mathbb{B}}_V(z, 2\delta_0)$ tells us that for every $l \in \{0, \dots, q\}$ and any $x, y \in \Sigma \cap \bar{\mathbb{B}}_V(z, 2\delta_0)$, we have

$$\text{(I)} \quad \left\| F^{(l)}(x) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \theta\varepsilon \quad \text{and} \quad \text{(II)} \quad \left\| \hat{R}_l^F(x, y) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \theta\varepsilon \|y - x\|_V^{\eta-l}. \tag{10.8}$$

Consider $p \in \Sigma$. From (10.2), we know that $p \in \Sigma \cap \bar{\mathbb{B}}_V(z, \delta_0)$ for some $z \in B$. Consequently, the bound (I) in (10.8) holds for $x := p$. Since $p \in \Sigma$ was arbitrary, we conclude that for any $p \in \Sigma$, we have

$$\left\| F^{(l)}(p) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \theta\varepsilon. \tag{10.9}$$

Consider $l \in \{0, \dots, q\}$ and $p, w \in \Sigma$. If there exists $z \in B$ for which $p, w \in \bar{\mathbb{B}}_V(z, 2\delta_0)$, then (II) in (10.8) yields that

$$\left\| \hat{R}_l^F(p, w) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \theta\varepsilon \|w - p\|_V^{\eta-l}. \tag{10.10}$$

Now suppose that no single ball $\bar{\mathbb{B}}_V(z, 2\delta_0)$ contains both p and w . From (10.2), we know that $p \in \Sigma \cap \bar{\mathbb{B}}_V(z_i, \delta_0)$ and $w \in \Sigma \cap \bar{\mathbb{B}}_V(z_j, \delta_0)$ for some $z_i, z_j \in B$ which must be distinct. In fact, since $w \notin \Sigma \cap \bar{\mathbb{B}}_V(z_i, 2\delta_0)$, we can conclude that

$$\|w - p\|_V \geq \delta_0. \tag{10.11}$$

Observe that from (10.7), we have, for any $v \in V^{\otimes l}$, that

$$\hat{R}_l^F(p, w)[v] = F^{(l)}(w)[v] - \sum_{s=0}^{q-l} \frac{1}{s!} F^{(l+s)}(p)[v \otimes (w - p)^{\otimes s}]. \tag{10.12}$$

We may further use (10.7) to compute that

$$F^{(l)}(w)[v] = \sum_{u=0}^{q-l} \frac{1}{u!} F^{(l+u)}(z_j)[v \otimes (w - z_j)^{\otimes u}] + \hat{R}_l^F(z_j, w)[v]. \quad (10.13)$$

Since $w \in \bar{\mathbb{B}}_V(z_j, \delta_0)$, we may use (10.10) to conclude that

$$\left\| \hat{R}_l^F(z_j, w) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \theta \varepsilon \|w - z_j\|_V^{\eta-l} \leq \theta \varepsilon \delta_0^{\eta-l}. \quad (10.14)$$

Additionally, since $z_j \in B$, we may compute that

$$\sum_{u=0}^{q-l} \frac{1}{u!} \left\| F^{(l+u)}(z_j)[v \otimes (w - z_j)^{\otimes u}] \right\|_W \leq \varepsilon_0 \sum_{u=0}^{q-l} \frac{1}{u!} \|w - z_j\|_V^u \|v\|_{V^{\otimes l}} \leq \varepsilon_0 e^{\delta_0} \|v\|_{V^{\otimes l}}. \quad (10.15)$$

Combining (10.13), (10.14) and (10.15) yields the estimate

$$\left\| F^{(l)}(w)[v] \right\|_W \leq \left(\theta \varepsilon \delta_0^{\eta-l} + \varepsilon_0 e^{\delta_0} \right) \|v\|_{V^{\otimes l}}. \quad (10.16)$$

Turning our attention to the second term in (10.12), note that for any $s \in \{0, \dots, q-l\}$, we have via (10.7), for $v' := v \otimes (w - p)^{\otimes s} \in V^{\otimes(l+s)}$, that

$$F^{(l+s)}(p)[v'] = \sum_{u=0}^{q-l-s} \frac{1}{u!} F^{(l+s+u)}(z_i)[v' \otimes (p - z_i)^{\otimes u}] + \hat{R}_{l+s}^F(z_i, p)[v']. \quad (10.17)$$

Since $p \in \bar{\mathbb{B}}_V(z_i, \delta_0)$, we may use (10.10) to conclude that

$$\left\| \hat{R}_{l+s}^F(z_i, p) \right\|_{\mathcal{L}(V^{\otimes(l+s)}; W)} \leq \theta \varepsilon \|p - z_i\|_V^{\eta-(l+s)} \leq \theta \varepsilon \delta_0^{\eta-(l+s)}. \quad (10.18)$$

Via similar computations to those used to establish (10.15), the fact that $z_i \in B$ allows us to compute that

$$\sum_{u=0}^{q-l-s} \frac{1}{u!} \left\| F^{(l+u+s)}(z_i)[v' \otimes (p - z_i)^{\otimes u}] \right\|_W \leq \varepsilon_0 e^{\delta_0} \|v'\|_{V^{\otimes(l+s)}}. \quad (10.19)$$

Combining (10.17), (10.18) and (10.19) yields that

$$\left\| F^{(l+s)}(p)[v'] \right\|_W \leq \left(\varepsilon_0 e^{\delta_0} + \theta \varepsilon \delta_0^{\eta-(l+s)} \right) \|v'\|_{V^{\otimes(l+s)}} = \left(\varepsilon_0 e^{\delta_0} + \theta \varepsilon \delta_0^{\eta-(l+s)} \right) \|w - p\|_V^s \|v\|_{V^{\otimes l}}, \quad (10.20)$$

where the last equality uses that $v' := v \otimes (w - p)^{\otimes s}$ and that the tensor powers of V are equipped with admissible norms (cf. Definition 2.1). A consequence of (10.20) is that

$$\sum_{s=0}^{q-l} \frac{1}{s!} \left\| F^{(l+s)}(p)[v'] \right\|_W \leq \sum_{s=0}^{q-l} \frac{1}{s!} \left(\varepsilon_0 e^{\delta_0} + \theta \varepsilon \delta_0^{\eta-(l+s)} \right) \|w - p\|_V^s \|v\|_{V^{\otimes l}}. \quad (10.21)$$

Since from (10.11), we have that $\delta_0 \leq \|w - p\|_V$, we may multiply each term in the sum on the RHS of (10.21) by $\|w - p\|_V^{\eta-(l+s)} \delta_0^{-(\eta-(l+s))} \geq 1$ to conclude that

$$\sum_{s=0}^{q-l} \frac{1}{s!} \left\| F^{(l+s)}(p)[v'] \right\|_W \leq \sum_{s=0}^{q-l} \frac{1}{s!} \left(\varepsilon_0 e^{\delta_0} + \theta \varepsilon \delta_0^{\eta-(l+s)} \right) \delta_0^{-(\eta-l-s)} \|w - p\|_V^{\eta-l} \|v\|_{V^{\otimes l}}. \tag{10.22}$$

Combining (10.12), (10.16) and (10.22) allows us to deduce that

$$\left\| \hat{R}_l^F(p, w)[v] \right\|_W \leq \left(\theta \varepsilon \delta_0^{\eta-l} + \varepsilon_0 e^{\delta_0} + \sum_{s=0}^{q-l} \frac{1}{s!} \left(\varepsilon_0 e^{\delta_0} \delta_0^{-(\eta-l-s)} + \theta \varepsilon \right) \|w - p\|_V^{\eta-l} \right) \|v\|_{V^{\otimes l}}. \tag{10.23}$$

Observe that

$$\theta \varepsilon \delta_0^{\eta-l} \stackrel{(10.11)}{\leq} \theta \varepsilon \|w - p\|_V^{\eta-l}, \tag{10.24}$$

$$\varepsilon_0 e^{\delta_0} \stackrel{(10.11)}{\leq} \varepsilon_0 \delta_0^{-(\eta-l)} e^{\delta_0} \|w - p\|_V^{\eta-l}, \tag{10.25}$$

$$\varepsilon_0 e^{\delta_0} \delta_0^{-(\eta-l)} \sum_{s=0}^{q-l} \frac{1}{s!} \delta_0^s \leq \varepsilon_0 \delta_0^{-(\eta-l)} e^{2\delta_0} \quad \text{and} \tag{10.26}$$

$$\theta \varepsilon \sum_{s=0}^{q-l} \frac{1}{s!} \leq \theta e \varepsilon. \tag{10.27}$$

Combining (10.24)–(10.27) with (10.23) yields

$$\left\| \hat{R}_l^F(p, w)[v] \right\|_W \leq \left(\theta \varepsilon (1 + e) + \varepsilon_0 \delta_0^{-(\eta-l)} e^{\delta_0} (1 + e^{\delta_0}) \right) \|w - p\|_V^{\eta-l} \|v\|_{V^{\otimes l}}. \tag{10.28}$$

Taking the supremum over $v \in V^{\otimes l}$ with unit $V^{\otimes l}$ -norm in (10.28) yields

$$\left\| \hat{R}_l^F(p, w) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \left(\theta \varepsilon (1 + e) + \varepsilon_0 \delta_0^{-\eta} e^{\delta_0} (1 + e^{\delta_0}) \right) \|w - p\|_V^{\eta-l} \tag{10.29}$$

since $\delta_0 \leq 1$ means $\delta_0^{-(\eta-l)} \leq \delta_0^{-\eta}$ for every $l \in \{0, \dots, q\}$.

Together (10.10), (10.29) and the inequality $\theta \varepsilon < \theta \varepsilon (1 + e) + \varepsilon_0 \delta_0^{-\eta} e^{\delta_0} (1 + e^{\delta_0})$ mean that for any $l \in \{0, \dots, q\}$ and any $p, w \in \Sigma$, we have

$$\left\| \hat{R}_l^F(p, w) \right\|_{\mathcal{L}(V^{\otimes l}; W)} \leq \left(\theta \varepsilon (1 + e) + \varepsilon_0 \frac{e^{\delta_0} (1 + e^{\delta_0})}{\delta_0^\eta} \right) \|w - p\|_V^{\eta-l}. \tag{10.30}$$

The definitions (10.7), the bounds (10.9) and the Hölder estimates (10.30) tell us that

$$\begin{aligned} \left\| F_{[q]} \right\|_{\text{Lip}(\eta, \Sigma, W)} &\leq \theta \varepsilon (1 + e) + \varepsilon_0 \frac{e^{\delta_0} (1 + e^{\delta_0})}{\delta_0^\eta} \\ &\stackrel{(10.1)}{\leq} \frac{1}{2(1 + e)} \varepsilon (1 + e) + \frac{\varepsilon}{2} \frac{\delta_0^\eta}{e^{\delta_0} (1 + e^{\delta_0})} \frac{e^{\delta_0} (1 + e^{\delta_0})}{\delta_0^\eta} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

as claimed in (10.4). This completes the proof of Claim 10.1 □

Returning to the proof of Theorem 3.1 itself, we define $F := \psi - \varphi \in \text{Lip}(\gamma, \Sigma, W)$ so that for every $j \in \{0, \dots, k\}$, we have $F^{(j)} = \psi^{(j)} - \varphi^{(j)}$. Then, by assumption, we have for every $j \in \{0, \dots, k\}$ and every $z \in B$ that $\|F^{(j)}(z)\|_{\mathcal{L}(V^{\otimes j}; W)} \leq \varepsilon_0$. Moreover, (10.3) tells us that whenever $z \in B$, we have that $\|F_{[q]}\|_{\text{Lip}(\gamma, \Omega_z, W)} \leq \theta\varepsilon$ for $\Omega_z := \Sigma \cap \overline{\mathbb{B}}_V(z, 2\delta_0)$. Therefore, we can apply Claim 10.1 to F and conclude that $\|F_{[q]}\|_{\text{Lip}(\gamma, \Sigma, W)} \leq \varepsilon$. Since $F := \psi - \varphi$, this gives the estimate claimed in (3.3) and completes the proof of Theorem 3.1. \square

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